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NONLINEAR SCHWARZ–FAS METHODS FOR UNSTRUCTURED FINITE ELEMENT ELLIPTIC PROBLEMS

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ABSTRACT. This paper provides extensions of an element agglomeration AMG method to nonlinear elliptic problems discretized by the finite element method on general unstructured meshes. The method constructs coarse discretization spaces and corresponding coarse nonlinear operators as well as their Jacobians. We introduce both standard (fairly quasi–uniformly coarsened) and non–standard (coarsened away) coarse meshes and respective finite element spaces. We use both kind of spaces in FAS type coarse subspace correction (or Schwarz) algorithms. Their performance is illustrated on a number of model problems. The coarsened away spaces seem to perform better than the standard spaces for problems with nonlinearities in the principal part of the elliptic operator.

1. INTRODUCTION

We are interested in solving the nonlinear algebraic equations arising from finite element discretizations of nonlinear second order elliptic PDEs using finite elements. To be specific, consider the second order elliptic PDE,

$$-\nabla \cdot (a(x,\ u,\ \nabla u)\nabla u) + g(x,\ u,\ \nabla u)u = f, \tag{1.1}$$

posed on a polygonal domain $\Omega \in R^2$ with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$. The functions $a = a(x,\ u,\ v) > 0$, $g = g(x,\ u, v) \geq 0$, and $f = f(x)$ are given. In what follows, we assume that the functions a, g and their first partial derivatives can be analytically evaluated for any value of their arguments.

The remainder of this short paper is structured as follows. We first introduce the discretization scheme, then we derive the coarse (non-inherited) nonlinear operators based on agglomeration AMGe (as proposed in [6]). Finally we formulate a standard nonlinear Schwarz–FAS algorithm for solving the resulting system of nonlinear equations exploiting coarse subspace and respective coarse nonlinear operators. The performance of the method is illustrated in the final section.

2. DISCRETIZATION

The equation (1.1) posed variationally defines the following nonlinear operator $\mathcal{L}$

$$(\mathcal{L}u,\ \varphi) \equiv \int_\Omega [a(x,\ u,\ \nabla u)\nabla u \cdot \nabla\varphi + g(x,\ u,\ \nabla u)u\varphi]\ dx.$$

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Given a triangulation $\mathcal{T} = (T)$ of Ω and associated finite element space $V = V_h$, one can discretize $\mathcal{L}$, leading to a mapping of the form $F(\mathbf{u})\mathbf{u}$ which can be evaluated element-wise based on the weighted element matrices,

$$a\left((x)_T,\ (u)_T,\ \left(\frac{\partial u}{\partial x}\right)_T,\ \left(\frac{\partial u}{\partial y}\right)_T\right)\ A_T + g\left((x)_T,\ (u)_T,\ \left(\frac{\partial u}{\partial x}\right)_T,\ \left(\frac{\partial u}{\partial y}\right)_T\right)\ G_T.$$

Here $\{A_T\}$ and $\{G_T\}$ stand for the element matrices of the Laplace operator and the mass element matrices. Also, x_T, u_T, and $(\nabla u)_T$ stand for averaged values over every element T.

In order to compute the Jacobian of $F(\mathbf{v})\mathbf{v}$ at $\mathbf{v}_0$ (v_0), $J(\mathbf{v}_0)$, one can use the following formulas. Let, $a = a(v, v_x, v_y)$ and $g = g(v, v_x, v_y)$ and assume that one can analytically compute the partial derivatives

$$a' = \frac{\partial a}{\partial v},\ a'_x = \frac{\partial a}{\partial v_x},\ a'_y = \frac{\partial a}{\partial v_y},\ \text{and } g' = \frac{\partial g}{\partial v},\ g'_x = \frac{\partial g}{\partial v_x},\ g'_y = \frac{\partial g}{\partial v_y}.$$

The corresponding formula for $J(\mathbf{v}_0)$, for any $\mathbf{w}$ and $\mathbf{v}$, then reads,

$$\begin{aligned}
\mathbf{w}^T J(\mathbf{v}_0)\mathbf{v} \ &= \mathbf{w}^T F(\mathbf{v}_0)\mathbf{v} \qquad \text{(Picard linearization)}\\
&+ \Bigg\{ \sum_{T\in\mathcal{T}} \Bigg[(v)_T\, a'\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\\
&\qquad + \left(\tfrac{\partial v}{\partial x}\right)_T\, a'_x\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\\
&\qquad + \left(\tfrac{\partial v}{\partial y}\right)_T\, a'_y\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\Bigg]\ \mathbf{w}_T^T A_T \mathbf{v}_{0,\,T}\\
&+ \sum_{T\in\mathcal{T}} \Bigg[(v)_T\, g'\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\\
&\qquad + \left(\tfrac{\partial v}{\partial x}\right)_T\, g'_x\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\\
&\qquad + \left(\tfrac{\partial v}{\partial y}\right)_T\, g'_y\left((v_0)_T,\ \left(\tfrac{\partial v_0}{\partial x}\right)_T,\ \left(\tfrac{\partial v_0}{\partial y}\right)_T\right)\Bigg]\ \mathbf{w}_T^T G_T \mathbf{v}_{0,\,T}\Bigg\}.
\end{aligned}$$

Here, we need averaged values $(.)_T$ of any vector (function) and its derivatives.

3. Coarsening

A coarse nonlinear operator $\mathcal{L}_H$, for a coarse finite element space V_H (constructed by AMGe, in the form proposed in Jones and Vassilevski [6], for example) is defined, for $u, \varphi \in V_H$, by

$$\begin{aligned}
(\mathcal{L}_H u,\ \varphi) \ &\equiv \sum_{T\in\mathcal{T}_H} \Bigg[a(x_T,\ u_T,\ (\nabla u)_T) \int_T \nabla u \cdot \nabla\varphi\, dx\\
&\qquad + g(x_T,\ u_T,\ (\nabla u)_T) \int_T u\varphi\, dx \Bigg],
\end{aligned}$$

where x_T, u_T, and $(\nabla u)_T$ are averaged values over every element T. In matrix-vector form this reads, for $u, \varphi \in V_H$, and their respective coefficient vectors restricted to any element T, $\mathbf{u}_T$ and $\underline{\varphi}_T$,

$$(\mathcal{L}_H u,\ \varphi) \equiv \sum_{T\in\mathcal{T}_H} \mathbf{u}_T^T \left(a(x_T,\ u_T,\ (\nabla u)_T)A_T + g(x_T,\ u_T,\ (\nabla u)_T)G_T\right)\underline{\varphi}_T.$$

Here $\{A_T\}$ and $\{G_T\}$ stand for the coarse element matrices of the Laplace operator and the mass element matrices.

Thus one needs an **element averaging procedure** $(\cdot)_T$ on all grids.

3.1. **Computing derivatives on coarse grids by element averaging.** The following simple element-wise approximations to the derivatives are feasible,

$$\frac{\partial u}{\partial x}\Big|_T \simeq \frac{1}{|T|}\,\mathbf{X}^T A_T \mathbf{u}, \quad \frac{\partial u}{\partial y}\Big|_T \simeq \frac{1}{|T|}\,\mathbf{Y}^T A_T \mathbf{u},$$
$$\frac{\partial u}{\partial z}\Big|_T \simeq \frac{1}{|T|}\,\mathbf{Z}^T A_T \mathbf{u}.$$

This is motivated by the equalities,

$$\mathbf{X}^T A_T \mathbf{u} = \int_T \nabla u \cdot \nabla x = \int_T \nabla u \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \int_T \frac{\partial u}{\partial x} \simeq |T|\ \frac{\partial u}{\partial x}\Big|_T .$$

The relation ("$\simeq$") is actually equality if u is linear over T.

Similarly, one can perform element averaging of u based on G_T,

$$u|_T \simeq \frac{1}{|T|}\int_T 1.u\ dx = \frac{1}{|T|}(\mathbf{1})^T G_T \mathbf{u}_T.$$

Here, X, Y, Z and $\mathbf{1}$ stand for the vector representations of the linear functions x, y, z, and 1.

4. Schwarz–FAS AMGe algorithm

Consider the fine grid nonlinear problem

$$F(\mathbf{u})\mathbf{u} = \mathbf{f},$$

and let $\mathbf{u}_0$ be a given initial approximation. Also, let the coarse subspaces $\mathbf{V}_k$ and the respective interpolation matrices $P_k :\ \mathbf{V}_k \mapsto \mathbf{V}$ be given. Finally, let I_k be a simple (injection) operator which restricts a fine-grid vector to a coarse grid one in $\mathbf{V}_k$.

Then one performs the following steps to get a next approximation to $\mathbf{u}$.

Algorithm 4.1 (Subspace Correction–FAS Method).

- *For $k = 1, \ldots, p$ loop over the coarse subspaces V_k:*
  (1) *restrict global residual $\mathbf{r} = \mathbf{f} - F(\mathbf{u})\mathbf{u}$ to $\mathbf{V}_k$, i.e., $\mathbf{r}_k = P_k^T \mathbf{r}$.*
  (2) *solve (e.g., using Newton's method) the coarse nonlinear problem*

  $$F_k(\mathbf{u}_k)\mathbf{u}_k = \mathbf{f}_k,$$

  with an initial approximation $\mathbf{u}_k^0 = I_k \mathbf{u}$ and a right hand side $\mathbf{f}_k = \mathbf{r}_k + F_k(\mathbf{u}_k^0)\mathbf{u}_k^0$.
  (3) *interpolate the coarse grid correction $\mathbf{u}_k - \mathbf{u}_k^0$ and update the fine-grid approximation; that is,*

  $$\mathbf{u} := \mathbf{u} + P_k(\mathbf{u}_k - \mathbf{u}_k^0).$$

- *an optional "post-smoothing" step; for example, a few nonlinear Gauss Seidel iterations.*
- *the new nonlinear FAS iterate is* $\mathbf{u} := \mathbf{u} + \underline{\xi}$.

For details on theory regarding the structured finite element case, cf. Dryja and Hackbusch [5], Xu [11], Tai and Espedal [9], Tai and Xu [10], and for the classical FAS method, cf. Brandt [3] and Briggs et al. [4].

For some other multigrid approaches for nonlinear diffusion equations on unstructured meshes, cf., Mavriplis [7].

5. A NONLINEAR TEST PROBLEM

Consider a nonlinear elliptic problem with more general nonlinearity:

$$-\nabla \cdot (a(u,\ \nabla u)\ \nabla u) + u^3 = f,$$

in $\Omega = (0,1)^2$ with Dirichlet boundary conditions. Here, $a(u,\ \nabla u) = \frac{1}{\sqrt{\epsilon + u^2 + |\nabla u|^2}}$, $\epsilon = 0.001$. The r.h.s. function f is chosen such that $u = x(1-x)y(1-y)$ is the exact solution.

The initial iterates for the FAS subspace correction method were chosen 0.9 times the true solution. The iterations are terminated after relative residual ℓ^2-norm reduction of the initial residual by a factor of 10^{-6} has been achieved.

5.1. Coarsened away meshes. The coarse spaces corresponded to our AMGe-constructed ones. The coarse agglomerated elements are obtained by first partitioning the initial set of fine elements using METIS into "# domains" subsets, then the elements in a given subdomain were selected (fixed) allowing for agglomeration only the elements that are more than one layer away from the fixed set of elements on the previous level. Thus one ends up with algebraically constructed finite element spaces of small dimension but of global nature. This is illustrated in Fig. 1 and Fig. 2.

Then the thus constructed coarse spaces, operators and their Jacobians are used in Algorithm 4.1 and in a non-linear preconditioned GCG method (cf. [1]) where the fine-grid Jacobian is preconditioned with additive Schwarz preconditioner coming from the coarsened away spaces. The results are found in Tables 1 and 2.

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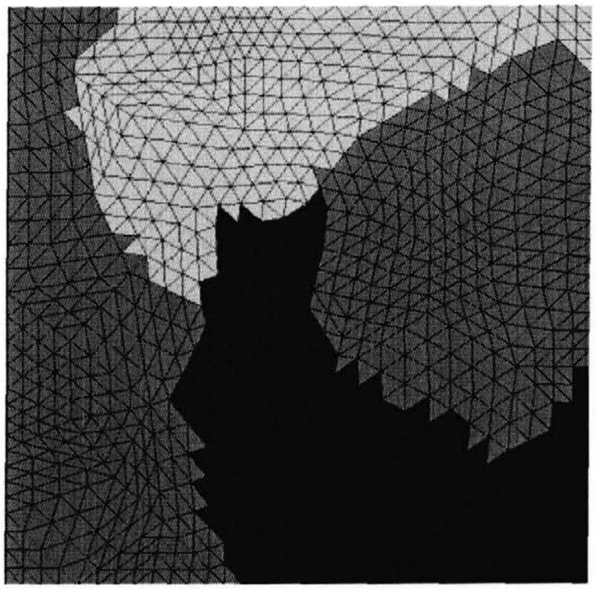

FIGURE 1. *Fine mesh partitioned into four mesh subdomains: 1,600 fine elements, 2460 fine degrees of freedom. Each subdomain consists of 400 elements and is represented by a single color.*

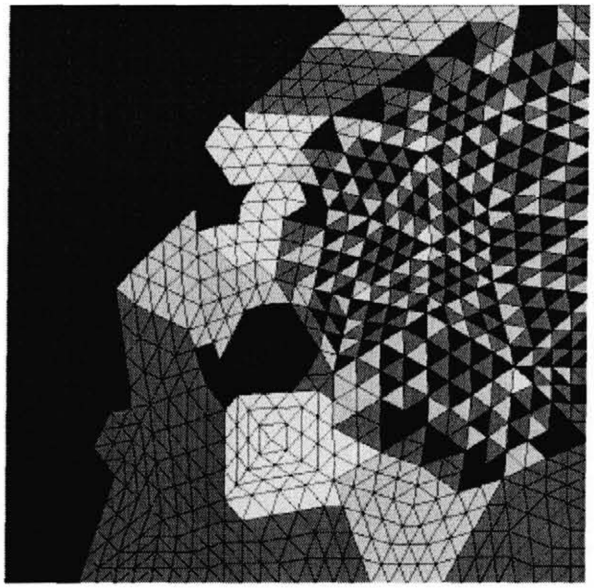

FIGURE 2. *An agglomeration based coarsened away mesh: 1,600 fine elements, 466 agglomerated elements, 400 subdomain elements, 270 coarse degrees of freedom. Each color represents an agglomerate.*

TABLE 1. Unstructured triangular grids; number of iterations of FAS subspace correction method with smoothing.

# domains	400 elements	1600 elements	6400 elements	25600 elements
4	10	10	12	14
8	9	11	16	17
16	12	15	15	⋆

TABLE 2. Unstructured triangular grids; number of iterations of preconditioned nonlinear GCG method. Each second row shows the total number of preconditioned (with additive Schwarz Jacobian preconditioner) GCG iterations for inexact solving with fine-grid Jacobians for achieving relative tolerance 0.0001.

# domains	400 elements	1600 elements	6400 elements	25600 elements
4	3	3	3	4
	47	67	92	196
8	3	3	3	3
	59	72	102	136
16	3	3	3	4
	73	100	149	204

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