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TOA / FOA Geolocation Error Analysis

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ABSTRACT

This paper describes how confidence intervals can be calculated for radiofrequency emitter position estimates based on time-of-arrival and frequency-of-arrival measurements taken at several satellites. These confidence intervals take the form of 50th and 95th percentile circles and ellipses to convey horizontal error and linear intervals to give vertical error. We consider both cases where an assumed altitude is and is not used. Analysis of velocity errors is also considered. We derive confidence intervals for horizontal velocity magnitude and direction including the case where the emitter velocity is assumed to be purely horizontal, i.e., parallel to the ellipsoid. Additionally, we derive an algorithm that we use to combine multiple position fixes to reduce location error. The algorithm uses all available data, after more than one location estimate for an emitter has been made, in a mathematically optimal way.

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Introduction

Geolocation systems employing a constellation of satellite-based receivers can use time-of-arrival (TOA) and/or frequency-of-arrival (FOA) measurements to locate a radio-frequency (RF) emitter on or near the Earth. Alternatively the satellites can simply translate the uplink band to a different downlink frequency band so that the TOA and FOA measurements are made at a ground station. The TOAs and FOAs at the satellite are then inferred from the measurements at the ground using the known ground station and satellite positions and velocities so that the localization problem can proceed as before. The transmit time and transmit frequency are not known so the localization is based implicitly on the time-difference-of-arrival (TDOA) and frequency-difference-of-arrival (FDOA) of the signal at the satellites. In this report we derive the mathematical algorithms for calculating and presenting in convenient form the output location error levels given the input TOA and FOA measurement error variances. Algorithms for computing the location estimates themselves have been described in [1] and are not covered in this report, however we do describe a procedure for combining individual position fixes into the best overall position estimate.

Estimating Position Errors

In this section we will be concerned with using estimates of the TOA and FOA measurement errors and knowledge of the geometry to compute statistics of the errors in the computed position. The term geometry here refers to the positions and velocities of the satellites which are known and the position of the emitter which is also known following the geolocation calculation. We will represent the horizontal error for a computed location in several ways, for example by giving the size and orientation of an ellipse which is centered at the computed location and which is just large enough so that there is a 95% probability that the real emitter location is inside the ellipse. Following the discussion of horizontal errors we discuss vertical error estimation.

We begin the mathematical derivation of the error analysis algorithms by stating the TOA, FOA and ALT equations that are developed in [1]. The TOA equation that relates the emitter position $\mathbf{x}=[x\ y\ z]^T$, a column vector, at the time of transmission t, to the i^{th} satellite position $\mathbf{s}_i=[x_i\ y_i\ z_i]^T$, at the time of arrival t_i, is

$$\sqrt{(\mathbf{x}-\mathbf{s}_i)^T(\mathbf{x}-\mathbf{s}_i)} - ct_i + ct = 0 . \tag{1}$$

The FOA equation relates the transmitted frequency scaled by wavelength ν, and the Doppler-shifted FOA ν_i at satellite i. The scaled transmit frequency and FOAs are

$$\nu = \lambda_0 f ,\ \nu_i = \lambda_0 f_i \tag{2}$$

respectively where f is the unknown transmit frequency, f_i are FOAs and λ_0 is the nominal system wavelength. The FOA equation is

$$\frac{(\mathbf{x}-\mathbf{s}_i)^T(\dot{\mathbf{x}}-\dot{\mathbf{s}}_i)}{\sqrt{(\mathbf{x}-\mathbf{s}_i)^T(\mathbf{x}-\mathbf{s}_i)}}\left(\frac{\nu}{c}\right) - \nu - \nu_i = 0 \tag{3}$$

where $\dot{\mathbf{x}}$ and $\dot{\mathbf{s}}_i$ are emitter and satellite velocity respectively.

The assumed altitude (ALT) equation is obtained from the equation for an ellipsoid, or oblate spheroid more specifically,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = r_b^2 \tag{4}$$

where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_b^2/r_s^2 \end{bmatrix}, \tag{5}$$

and where r_b and r_s are the equatorial and polar radii respectively. This ellipsoid is related to the Earth ellipsoid having equatorial and polar radii of r_e and r_p respectively by $r_b = r_e + a$ and $r_s = r_p + a$ where a is the altitude of the emitter above the Earth ellipsoid. Here a can be obtained from a Digital Terrain Elevation Data (DTED) database after an approximate location is computed using a=0. Multiplying Eqn. (4) through by r_s/r_b and then taking the square root gives the ALT equation

$$\sqrt{\frac{r_s x^2}{r_b} + \frac{r_s y^2}{r_b} + \frac{r_b z^2}{r_s}} - \sqrt{r_b r_s} = 0. \tag{6}$$

We prefer to work with Eqn. (6) rather than Eqn. (4) directly because the units of Eqn. (6) are meters as in Eqn. (1) which facilitates the summing of total residual error.

When systems of Eqn. (1), Eqn. (3) and/or Eqn. (6) are solved then we can make a system of equations relating input errors and output errors by adding variation terms to the solution variables and to the data (measurement) variables in each equation. We can write Eqn. (1), Eqn. (3) and Eqn. (6) generically as $f_i(\mathbf{p}, \mathbf{d}) = 0$ where i is the equation number index such as i=1...4 in a 4-TOA problem, $\mathbf{p}$ is a vector of solution variables such as $\mathbf{p}=[\mathbf{x}\ t]$ and $\mathbf{d}$ is a vector of data variables such as $\mathbf{d}=[t_1\ t_2\ t_3\ t_4]$. The variational equations can then be written as

$$\mathbf{f}(\mathbf{p} + \delta\mathbf{p}, \mathbf{d} + \delta\mathbf{d}) = \mathbf{f}(\mathbf{p}, \mathbf{d}) + \mathbf{A}\delta\mathbf{p} + \mathbf{G}\delta\mathbf{d} \tag{7}$$

where $\mathbf{A}$ and $\mathbf{G}$ are the Jacobian matrices of partials $\mathbf{A} = \frac{\partial\mathbf{f}}{\partial\mathbf{p}}$, $\mathbf{G} = \frac{\partial\mathbf{f}}{\partial\mathbf{d}}$ evaluated at the solution and data values $\mathbf{p}$ and $\mathbf{d}$ respectively. Note that $\mathbf{G}$ is a square diagonal matrix since each measurement generates one independent equation. Since $\mathbf{f}(\mathbf{p}, \mathbf{d}) = \mathbf{0}$ Eqn. (7) can be written as

$$\mathbf{f}(\mathbf{p} + \delta\mathbf{p}, \mathbf{d} + \delta\mathbf{d}) = \mathbf{A}\delta\mathbf{p} + \mathbf{G}\delta\mathbf{d} \equiv \mathbf{e} \tag{8}$$

where $\mathbf{e}$ are errors resulting from the variations. For uniquely determined systems, as solved in [1], Eqn. (8) can be solved with $\mathbf{e} = \mathbf{0}$ giving the input/output error relation

$$\delta\mathbf{p} = -\mathbf{A}^{-1}\mathbf{G}\delta\mathbf{d}. \tag{9}$$

The output error covariance matrix then follows as

$$\Sigma_p = E\{\delta\mathbf{p}\delta\mathbf{p}^T\} = \mathbf{A}^{-1}\mathbf{G}\Sigma_d\mathbf{G}(\mathbf{A}^{-1})^T \tag{10}$$

where $\Sigma_d = E\{\delta\mathbf{d}\delta\mathbf{d}^T\}$ is the input data error covariance matrix. Σ_d is diagonal for the common situation of uncorrelated measurement errors so the diagonal holds our estimated data variances. These estimates are made by analyzing the TOA and FOA measurement processes and likely sources of error, a very difficult task that is beyond the scope of this report. Assumed altitude error variance is principally a function of the quality of the terrain elevation database i.e. the DTED being used for surface emitters.

If the system in Eqn. (8) is overdetermined then it can not generally be solved with $\mathbf{e} = \mathbf{0}$ and we choose instead to solve the system in a weighted least-squares sense. We minimize

$$\xi = \mathbf{e}^T\mathbf{W}\mathbf{e} \tag{11}$$

where

$$\mathbf{W} = (\mathbf{G}\Sigma_d\mathbf{G})^{-1}, \tag{12}$$

for maximum-likelihood weighting, by solving $\nabla_p\xi = \nabla_p\mathbf{e}^T\mathbf{W}\mathbf{e} = 0$. Using Eqn. (8) we have

$$\nabla_p(\mathbf{A}\delta\mathbf{p} + \mathbf{G}\delta\mathbf{d})^T\mathbf{W}(\mathbf{A}\delta\mathbf{p} + \mathbf{G}\delta\mathbf{d}) = 0 \tag{13}$$

giving

$$2\mathbf{A}^T\mathbf{W}\mathbf{A}\delta\mathbf{p} + 2\mathbf{A}^T\mathbf{W}\mathbf{G}\delta\mathbf{d} = 0. \tag{14}$$

Now output error covariance matrix follows as

$$\Sigma_p = E\{\delta\mathbf{p}\delta\mathbf{p}^T\} = (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{G}\Sigma_d\mathbf{G}\mathbf{W}\mathbf{A}(\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1} \tag{15}$$

which, using Eqn. (12) simplifies to

$$\Sigma_p = (\mathbf{A}^T(\mathbf{G}\Sigma_d\mathbf{G})^{-1}\mathbf{A})^{-1} \tag{16}$$

which agrees with Eqn. (10) when **A** and **G** are square.

Under the assumption that the measurement errors are small, zero and gaussian then the solution errors are linear combinations of the input errors and are gaussian. Assume further, for the moment, that the input errors are zero mean so that the output errors are also zero mean. Then the the (gaussian) probability density function (pdf) for a solution vector $\mathbf{p}$ when $\mathbf{p}_0$ is the computed position is

$$f(\mathbf{p}) = \frac{1}{(2\pi)^{m/2}|\Sigma_p|}\exp\left[-\frac{(\mathbf{p}-\mathbf{p}_0)^T\Sigma_p^{-1}(\mathbf{p}-\mathbf{p}_0)}{2}\right] \qquad (17)$$

where m is the number of solution variables in $\mathbf{p}$. For example in a TOA problem, m=4, since we solve for 3 position coordinates and transmit time. However the pdf in Eqn. (17) could just as easily represent the factored pdf for the (x,y,z) coordinates alone. In this case m=3 and the covariance matrix Σ_p is the upper 3x3 of the full covariance matrix. Since $(\mathbf{p}-\mathbf{p}_0)^T\Sigma_p^{-1}(\mathbf{p}-\mathbf{p}_0) = 1$ is the equation of an ellipsoid we see that there are constant probability density ellipsoids about the computed locations whose size and orientation are given by the output covariance matrix.

We now discuss how to compute the horizontal ellipse centered at the computed location that has a 95% probability of containing the actual emitter location. We first extract the upper 3x3 section of the full covariance matrix Σ_p from Eqn. (16) giving the covariance matrix for the position coordinates in Earth-centered Earth-fixed (ECEF) coordinates. This matrix is pre and post multiplied by rotation matrices to change to local South-East-Zenith (SEZ) coordinates at which point the upper 2x2 matrix partition gives the South-East covariance matrix Σ_{SE}. This matrix is then diagonalized by an eigensolver with the resulting diagonal 2x2 matrix containing the 2 uncorrelated variances σ_1^2 and σ_2^2 with the eigenvector associated with the largest eigenvalue giving the direction of the ellipse semi-major axis in the South-East plane. In order to determine the ellipse semi-major and semi-minor axis lengths a and b we must determine the constant

$$k = \frac{a}{\sigma_1} = \frac{b}{\sigma_2} \qquad (18)$$

which gives the scale factor relating the size of the 95% ellipse to the “1-σ ellipse” which contains only 39% of the probability. The constant k is given by the equation

$$1 - \exp(-k^2/2) = p \qquad (19)$$

from [2] which can be solved for k using p=0.95 giving k=2.4477. Now the lengths of the 95% horizontal error ellipse axes can be determined from the eigenvalues σ_1^2 and σ_2^2 using Eqn. (18).

The plot in Fig. 1 shows constant time-difference-of-arrival and constant frequency-difference-of-arrival contours on Earth for a simulated emitter at 0 degrees latitude and longitude. The two satellites recording the TOAs and FOAs are in some random mid-earth orbit (MEO) positions visible to the emitter. The intersections of these two contours are the two possible Earth-bound emitter locations that could have produced the observed TOAs and FOAs, i.e. these are the two points that would be determined from the 2-sat solver in [1].

The plot in Fig. 2 shows the locations of 100 computed solutions for the emitter that generated Fig. 1, i.e. the ellipse in Fig. 2 is located at 0 degrees latitude and longitude.

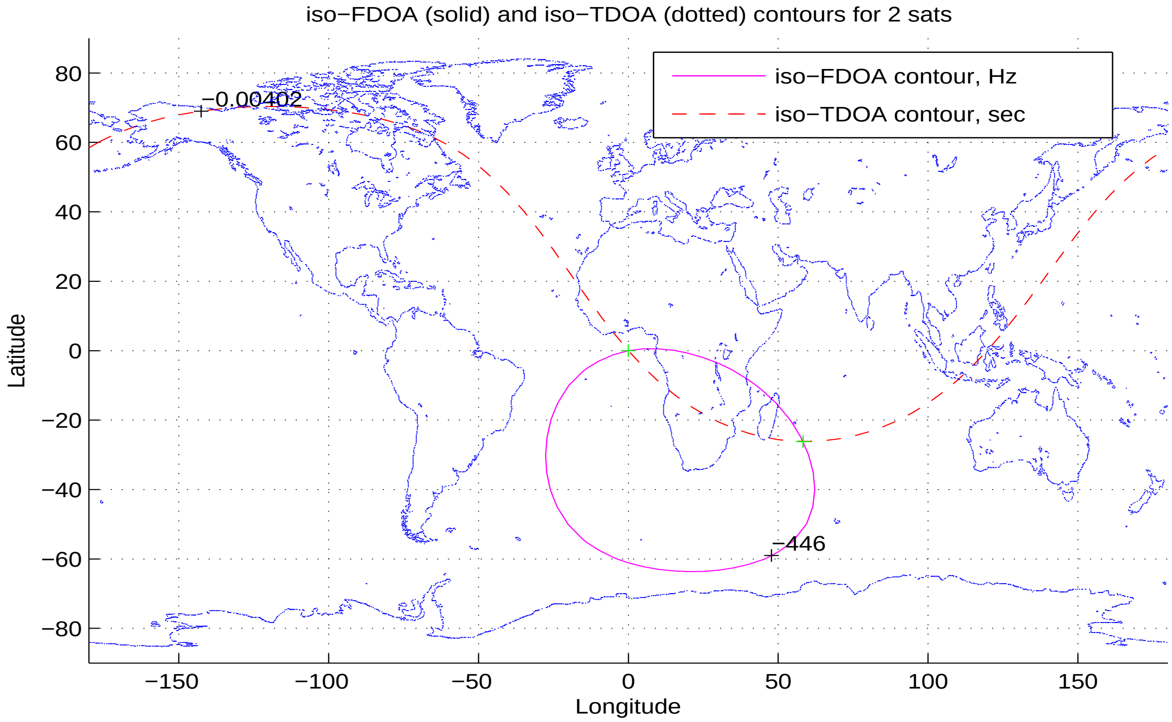

Figure 1. Iso-TDOA and Iso-FDOA contours for a simulated 2-sat example problem.

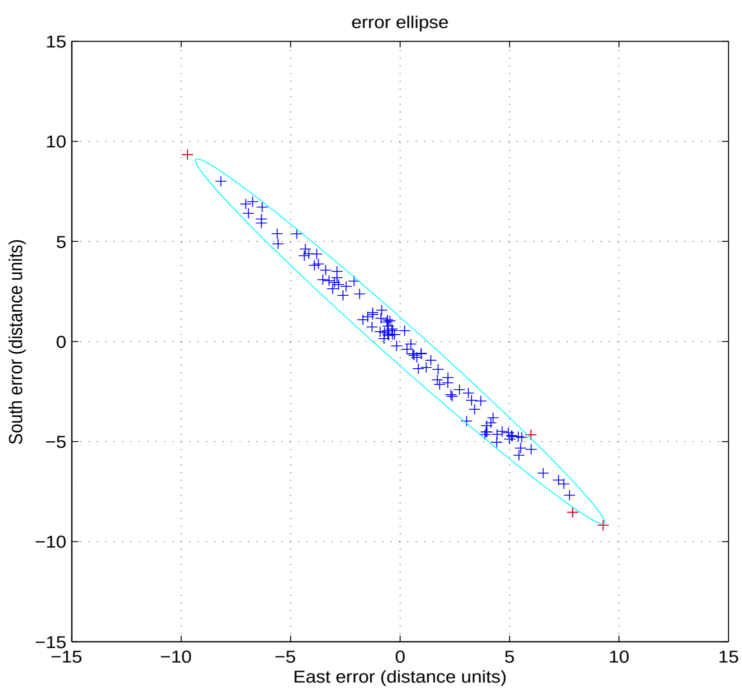

Figure 2. 100 computed locations of a simulated beacon located at center of the error ellipse.

The error ellipse does a good job of describing the spatial distribution of horizontal location errors but requires three parameters, 2 axis lengths and an orientation angle, to represent it. Therefore it is best drawn on a map. A simple way to convey error statistics verbally is the radius of a circle centered at the computed location that has a given likelihood of encompassing the real location. The 50th percentile radius is called the Circular Error Probable (CEP) and the 95th percentile radius is called the Horizontal Radial Error (HRE95). The observed and computed CEP and HRE95 for 100 locations estimates is shown in Fig. 3 below. The observed CEP is the minimum radius which contains exactly 50 of the computed locations. The estimated CEP is the radius that theory predicts should encompass half of the computed locations - or equivalently the radius about the computed point that encircles the true location half the time. We now describe a procedure to compute the CEP and HRE95 given the output error covariance matrix for a computed position.

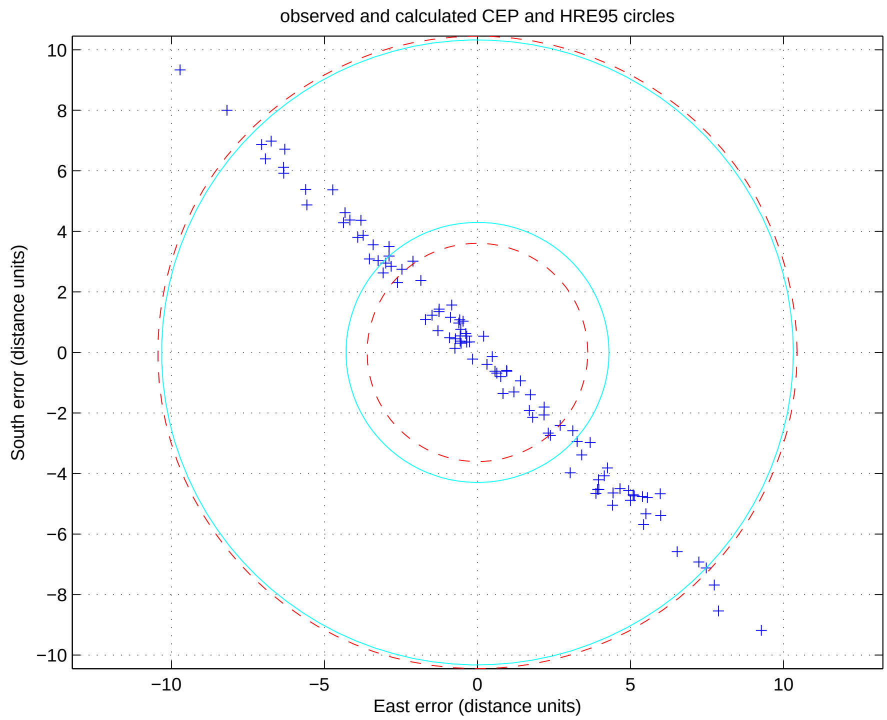

Figure 3. Observed (solid) and estimated (dashed) CEP and HRE95 error circles.

Using the two horizontal component error variances σ_1^2 and σ_2^2 referenced in Eqn. (18), the integral of the horizontal error pdf, Eqn. (17), over a circular region of radius R about the computed location can be written as

$$P = \int_0^{2\pi}\int_0^R \frac{r}{2\pi\sigma_1\sigma_2} \exp -\frac{1}{2}\left[\left(\frac{r\cos\theta}{\sigma_1}\right)^2 + \left(\frac{r\sin\theta}{\sigma_2}\right)^2\right] dr\,d\theta \qquad (20)$$

where P is the resulting accumulated probability. When $\sigma_1 > \sigma_2$ Eqn. (20) can be written in terms of $c = \sigma_2 / \sigma_1$ which is in the interval (0,1), as

$$P = \int_0^{2\pi}\int_0^{R/\sigma_1} \frac{r}{2\pi c}\exp-\frac{1}{2}\left[(r\cos\theta)^2 + \left(\frac{r\sin\theta}{c}\right)^2\right]dr d\theta . \quad (21)$$

When $\sigma_2 > \sigma_1$ Eqn. (20) can be written in terms of $c = \sigma_1 / \sigma_2$ as

$$P = \int_0^{2\pi}\int_0^{R/\sigma_2} \frac{r}{2\pi c}\exp-\frac{1}{2}\left[\left(\frac{r\cos\theta}{c}\right)^2 + (r\sin\theta)^2\right]dr d\theta . \quad (22)$$

By symmetry Eqn. (21) can be used for either case while letting c be the smaller standard deviation divided by the larger, so that $0 \le c \le 1$ and R is measured in lengths of the larger σ_k.

Letting P=0.5 Eqn. (21) can now be used to find a polynomial approximation for CEP, R, for various values of σ_1 and σ_2, or equivalently various values of c and $\sigma = max(\sigma_1, \sigma_2)$. The polynomial in c for R/σ is found by fitting the polynomial to data pairs determined from Eqn. (21). For a given value of c Eqn. (21) is numerically integrated with an increasing radius R/σ until the desired value of P is reached. For P=0.5 the follow data pairs were obtained in this way:

Table 1: Data used to make CEP polynomial: R/σ vs. *c* for *P*=0.5

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
R/σ	.6745	.682	.706	.750	.808	.870	.934	.996	1.058	1.118	1.1774

The data in Table 1 was found to be approximated well by

$$R/\sigma = 0.6745 + 0.118c + 1.0133c^2 - 0.5235c^3 . \quad (23)$$

In summary, to calculate CEP at a given location from the diagonalized horizontal location error covariance matrix, we divide the smaller standard deviations by the larger to get c, which is put into Eqn. (23) to get R/σ which is then multiplied by the larger of the standard deviations to give CEP.

Similarly the following HRE95 data pairs were obtained from Eqn. (21) for P=0.5:

Table 2: Data used to make HRE95 polynomial: R/σ vs. *c* for *P*=0.95

0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.96	1.963	1.97	1.984	2.005	2.036	2.081	2.146	2.23	2.332	2.488

The data in Table 2 is approximated well by

$$R/\sigma = 1.9576 + 0.0846c - 0.1648c^2 + 0.6053c^3 . \quad (24)$$

Therefore to calculate HRE95 at a given location from the diagonalized horizontal location error covariance matrix, we divide the smaller of the standard deviations by the larger to get c, which is put into Eqn. (24) to get R/σ which is then multiplied by the larger standard deviations to give HRE95.

The 50th percentile Vertical Error Probable (VEP) is found by scaling the Zenith standard deviation, the square-root of the (3,3) element of the SEZ covariance matrix, by the factor 0.6745 which is the absolute value of the two abscissa values between which a unit gaussian pdf integrates to 0.5. Since the unit gaussian integrates to 0.5 between -0.6745 and +0.6745 it follows that a zero-mean gaussian pdf with a standard deviation of σ_Z integrates to 0.5 between $-0.6745\sigma_Z$ and $0.6745\sigma_Z$.

Using the arguments of the previous paragraph the 95th percentile vertical error (VRE95) is found by scaling the Zenith standard deviation σ_Z by 1.96, i.e. the vertical error pdf integrates to 0.95 between $\pm 1.96\sigma_Z$.

Estimating Velocity Errors

Systems $\mathbf{f}$, of FOA equations, Eqn. (3), can be solved for velocity $\mathbf{v} = \dot{\mathbf{x}}$ and scaled transmit frequency ν when emitter position $\mathbf{x}$ is known, typically from solving the TOA problem. The system of equations can optionally include a surface motion equation of the form

$$\mathbf{x}^T\mathbf{Q}\mathbf{v} = 0 \tag{25}$$

which is derived by differentiating Eqn. (4). Eqn. (25) is required for the 3-satellite problem to achieve the minimum number of 4 equations required to solve for the 4 unknowns, $\mathbf{v}$ and ν.

The derivation of the output error covariance matrix Eqn. (16) in the position error analysis section is still valid when we are solving for velocity and transmit frequency, however we now define our vector of solution variables as $\mathbf{p}$=[$\mathbf{v}$ ν], velocity and transmit frequency, and the vector of data variables is $\mathbf{d}$=[ν_1 ν_2 ν_3 ...], the scaled FOAs defined in Eqn. (2). For example in the 4-satellite velocity problem $\mathbf{d} = \begin{bmatrix}\nu_1 & \nu_2 & \nu_3 & \nu_4\end{bmatrix}$. The Jacobians $\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{p}}$, $\mathbf{G} = \frac{\partial \mathbf{f}}{\partial \mathbf{d}}$ are then computed from $\mathbf{f}$ the vector of FOA equations, Eqn. (3), and Σ_d is formed from the FOA measurement variance estimates to form Σ_p using Eqn. (16).

The velocity estimate could be reported as the 3 components of $\mathbf{v}$ however often a more convenient form is in terms of magnitude and azimuth angle of horizontal velocity and a signed vertical velocity. To implement this we rotate $\mathbf{v}$ into SEZ components $\begin{bmatrix}v_S & v_E & v_Z\end{bmatrix}$ and then compute the azimuth angle as the arctangent of the East over the North velocity component. In the following we work with the supplement of the velocity azimuth angle,θ, given by

$$\theta = \operatorname{atan}\left(\frac{v_E}{v_S}\right). \tag{26}$$

This angle can now be used to make the 2×2 rotation matrix $R = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$ that rotates the horizontal velocity into a frame with axes in the direction of the horizontal velocity and perpendicular to it given

$$R\begin{bmatrix}v_S \\ v_E\end{bmatrix} = \begin{bmatrix}v_A \\ v_B\end{bmatrix} = \begin{bmatrix}v_A \\ 0\end{bmatrix} \tag{27}$$

where v_A and v_B are the names will now use for these horizontal velocity components. We will assume that $|v_A| » 0$. When this is not the case, i.e. when the horizontal velocity is near zero, the velocity direction error analysis will itself be imprecise but this is acceptable since the velocity azimuth angle becomes meaningless as the velocity magnitude goes to 0.

The standard deviation of the horizontal velocity magnitude is now available simply as the standard deviation of v_A while deviations in the angle θ are related to deviations in v_A and v_B by

$$\delta\theta = \operatorname{atan}\left(\frac{\delta v_B}{v_A + \delta v_A}\right) \approx \frac{\delta v_B}{v_A} \tag{28}$$

where we have used a small angle approximation. Therefore the standard deviation of velocity angle is linearly related to the standard deviation of v_B by $\sigma_\theta = \frac{\sigma_{v_B}}{v_A}$. Since v_A and v_B are linear combinations of normal random variables they are normally distributed and therefore the velocity magnitude and angle errors are normal as well. The 50% and 95% magnitude and angle errors are therefore easily computed from the standard deviations using the same factors of 0.6745 and 1.9600 that were used for VEP and VRE95 respectively in the vertical position error analysis, which was also a univariate normal random variable.

In the 3-FOA case we must solve for 4 variables, 3 emitter velocity components and the transmit frequency, using 3 FOA equations and the surface motion equation which is derived by differentiating Eqn. (4) with respect to time giving

$$\mathbf{x}^T\mathbf{Q}\mathbf{v} = r_b\dot{a} \tag{29}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the emitter velocity, $r_b = r_e + a$ as defined following Eqn. (5), while $\dot{a}$ is the altitude rate of change which we assume is 0. One could attempt to estimate $\dot{a}$ using the horizontal velocity estimate and a DTED database but this small refinement has not been adopted. The addition of the surface motion equation makes for several differences between the present 3-FOA case and the 4-FOA case above. The new model equation in **f** produces a new corresponding row of matrix **A** and a new row of matrix **G** which are partials of Eqn. (29) with respect to the solution variables and data variables respectively. Another difference is the data vector **d** now has 3 scaled FOAs and the assumed altitude rate, which we take to be 0, giving

$$\mathbf{d} = \begin{bmatrix}\nu_1 & \nu_2 & \nu_3 & \dot{a}\end{bmatrix} = \begin{bmatrix}\nu_1 & \nu_2 & \nu_3 & 0\end{bmatrix}. \tag{30}$$

The variance of the new data variable $\dot{a}$, the last element of the diagonal of the data covariance matrix Σ_d, must be estimated. At present we assume the standard deviation of the grade of the surface at the emitter location is 1% so that the standard deviation of $\dot{a}$ is 0.01 times the horizontal velocity, i.e. 1-σ vertical velocity $\sigma_{\dot{a}}$, is assumed to be 1% of the horizontal velocity.

In other words we wish to use our a priori knowledge that motion is commonly parallel to Earth, while our mathematical model actually has the motion parallel to an ellipsoid Earth model. So to compute the output variable error statistics based on the input variable error statistics we note that even a vehical travelling on Earth is not moving parallel to the ellipsoid, but is changing altitude relative to the ellipsoid at a rate that depends on the grade of the local surface and the horizontal velocity. The horizontal velocity is known approximately and we use a guess of 1% for the standard deviation of road grade for roads of interest.

Other methods of estimating $\sigma_{\dot{a}}$ and even (non-zero) $\dot{a}$ are feasible using a DTED database but have not been developed yet.

After Σ_d is formed from $\sigma^2_{\nu_i}$ and $\sigma^2{}_{\dot{a}}$, Σ_p is computed using Eqn. (16) and the procedure continues as for the 4-FOA case above.

Combining Position Estimates of a Stationary Emitter

The following procedure combines a time sequence of location estimates to give an optimal composite estimate. This estimate is optimal in the sense that it minimizes the variance of the composite estimate given covariance matrices of the individual estimates. The procedure is also attractive in that it is computationally efficient. It is computed recursively so that the cost of computing the composite solution remains fixed even as the number of incorporated estimates grows.

The procedure is as follows. After the first beacon transmission we obtain a set of TOA and FOA measurements and compute an estimate of the beacon's location. When the beacon transmits again we receive a second set of TOA and FOA data and compute a second estimate of the beacons location. If the computed velocity of the emitter is small enough that the emitter is deemed to be stationary then we combine the two position estimates into a single composite estimate using the formula given below. When the beacon transmits a third time we again use the same procedure to update the composite solution with the third estimate. This procedure continues to update the composite location using newly computed locations as they become available so long as the emitter appears to remain stationary.

The minimum variance composite position estimate, $\hat{\mathbf{x}}$, given two constituent position vectors, $\mathbf{x}_1$ and, $\mathbf{x}_2$ with respective covariance matrices Σ_1 and Σ_2 is

$$\hat{\mathbf{x}} = \mathbf{x}_2 + \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}(\mathbf{x}_1 - \mathbf{x}_2) \qquad (31)$$

while the covariance of $\hat{\mathbf{x}}$ is given by

$$\Sigma = (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}. \qquad (32)$$

These equations are applied recursively so that outputs $\hat{\mathbf{x}}$ and Σ become inputs $\mathbf{x}_1$ and Σ_1 in the next iteration while $\mathbf{x}_2$ and Σ_2 are the new calculated emitter location and associated error covariance matrix to be added to the composite solution. We call this the Running Minimum Variance (RMV) algorithm.

Eqn. (31) and Eqn. (32) are derived as follows. We begin with two noisy vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ having zero-mean errors with covariance matrices Σ_1 and Σ_2. In our application these vectors are the 1-by-3 coordinate partitions of geolocation solution vectors, such as $\mathbf{p}_n=[\mathbf{x}_n\ t_n]$, and Σ_1 and Σ_2 are upper 3-by-3 partitions of solution error covariance matrices as in Eqn. (16). Let $\mathbf{x}$ represent the true emitter location that $\mathbf{x}_n$ are noisy estimates of then we say that the expectation of $\mathbf{x}_n$ is $\mathbf{x}$

$$E\{\mathbf{x}_1\} = E\{\mathbf{x}_2\} = \mathbf{x}. \qquad (33)$$

Although the measurement errors are uncorrelated the resulting solution errors are correlated as shown by Eqn. (16), the errors in different solutions however are uncorrelated, that is we can write

$$E\{(\mathbf{x}_n - \mathbf{x})(\mathbf{x}_m - \mathbf{x})^T\} = \begin{cases} \Sigma_n & n = m \\ \mathbf{0} & n \neq m \end{cases}. \qquad (34)$$

We wish to find the combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ that minimum-error estimate, $\hat{\mathbf{x}}$, of x:

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x}_1 + \mathbf{Q}\mathbf{x}_2, \tag{35}$$

i.e. we wish to find matrices P and Q that minimize the variance of the composite estimate

$$\Sigma = E\{(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T\}. \tag{36}$$

We also want $E\{\hat{\mathbf{x}}\} = \mathbf{x}$ which gives $E\{\hat{\mathbf{x}}\} = \mathbf{P}E\{\mathbf{x}_1\} + \mathbf{Q}E\{\mathbf{x}_2\} = \mathbf{P}\mathbf{x} + \mathbf{Q}\mathbf{x} = \mathbf{x}$ or

$$\mathbf{P} + \mathbf{Q} = \mathbf{I}. \tag{37}$$

Putting Eqn. (35) into Eqn. (36) gives $\Sigma = E\{(\mathbf{P}\mathbf{x}_1 + \mathbf{Q}\mathbf{x}_2 - \mathbf{x})(\mathbf{P}\mathbf{x}_1 + \mathbf{Q}\mathbf{x}_2 - \mathbf{x})^T\}$ which becomes, using Eqn. (37),

$$\Sigma = E\{(\mathbf{P}\mathbf{x}_1 + \mathbf{Q}\mathbf{x}_2 - (\mathbf{P} + \mathbf{Q})\mathbf{x})(\mathbf{P}\mathbf{x}_1 + \mathbf{Q}\mathbf{x}_2 - (\mathbf{P} + \mathbf{Q})\mathbf{x})^T\} \tag{38}$$

or

$$\Sigma = E\{(\mathbf{P}(\mathbf{x}_1 - \mathbf{x}) + \mathbf{Q}(\mathbf{x}_2 - \mathbf{x}))(\mathbf{P}(\mathbf{x}_1 - \mathbf{x}) + \mathbf{Q}(\mathbf{x}_2 - \mathbf{x}))^T\}. \tag{39}$$

Using Eqn. (34) this gives

$$\Sigma = \mathbf{P}\Sigma_1\mathbf{P}^T + \mathbf{Q}\Sigma_2\mathbf{Q}^T \tag{40}$$

or using Eqn. (37) an expression involving **P** alone is $\Sigma = \mathbf{P}\Sigma_1\mathbf{P}^T + (\mathbf{I} - \mathbf{P})\Sigma_2(\mathbf{I} - \mathbf{P})^T$ or

$$\Sigma = \mathbf{P}(\Sigma_1 + \Sigma_2)\mathbf{P}^T - \mathbf{P}\Sigma_2 - \Sigma_2\mathbf{P}^T + \Sigma_2. \tag{41}$$

A necessary condition that **P** minimizes Σ is that the variation of Σ with **P** vanishes, i.e. we put $\mathbf{P} + \delta\mathbf{P}$ into Eqn. (41) to obtain $\Sigma + \delta\Sigma$ and then find **P** such that $\delta\Sigma = 0$ for all $\delta\mathbf{P}$:

$$\Sigma + \delta\Sigma = (\mathbf{P} + \delta\mathbf{P})(\Sigma_1 + \Sigma_2)(\mathbf{P} + \delta\mathbf{P})^T - (\mathbf{P} + \delta\mathbf{P})\Sigma_2 - \Sigma_2(\mathbf{P} + \delta\mathbf{P})^T + \Sigma_2. \tag{42}$$

Using Eqn. (41) gives

$$\delta\Sigma = \mathbf{P}(\Sigma_1 + \Sigma_2)\delta\mathbf{P}^T + \delta\mathbf{P}(\Sigma_1 + \Sigma_2)\mathbf{P}^T - \delta\mathbf{P}\Sigma_2 - \Sigma_2\delta\mathbf{P}^T \tag{43}$$

where we have neglected the smaller term that is quadratic in $\delta\mathbf{P}$. Note that Eqn. (43) is of the form $\mathbf{B}^T + \mathbf{B} = \mathbf{A}\delta\mathbf{P}^T + \delta\mathbf{P}\mathbf{A}^T$ where $\mathbf{A} = \mathbf{P}(\Sigma_1 + \Sigma_2) - \Sigma_2$ so that for the diagonal elements of $\delta\Sigma = \mathbf{B}^T + \mathbf{B}$ to each be 0, the diagonal elements of **B** must each be 0, for all $\delta\mathbf{P}$ therefore **A** must be zero giving

$$\mathbf{P} = \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}. \tag{44}$$

Using Eqn. (35) and Eqn. (37) the minimum variance estimate can be written as

$$\hat{\mathbf{x}} = \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\mathbf{x}_1 + (\mathbf{I} - \Sigma_2(\Sigma_1 + \Sigma_2)^{-1})\mathbf{x}_2 \tag{45}$$

or simply as

$$\hat{\mathbf{x}} = \mathbf{x}_2 + \mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) \tag{46}$$

which is Eqn. (31).

The variance of this estimate is given by Eqn. (41) which can be expressed in a simpler form by putting Eqn. (44)into Eqn. (41) giving

$$\Sigma = \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}(\Sigma_1 + \Sigma_2)(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2 - \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2 - \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2 + \Sigma_2$$

which immediately reduces to

$$\Sigma = \Sigma_2 - \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\Sigma_2. \tag{47}$$

This however can still be reduced to Eqn. (32) by the following sequence of steps.

$$\Sigma = \Sigma_2[\mathbf{I} - (\Sigma_1 + \Sigma_2)^{-1}\Sigma_2] \tag{48}$$

$$= \Sigma_2[(\Sigma_1 + \Sigma_2)^{-1}(\Sigma_1 + \Sigma_2) - (\Sigma_1 + \Sigma_2)^{-1}\Sigma_2] \tag{49}$$

$$= \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}[(\Sigma_1 + \Sigma_2) - \Sigma_2] \tag{50}$$

$$= \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\Sigma_1 \tag{51}$$

$$= \Sigma_2(\Sigma_1\Sigma_1^{-1} + \Sigma_2\Sigma_1^{-1})^{-1} \tag{52}$$

$$= (\Sigma_2^{-1}\Sigma_1\Sigma_1^{-1} + \Sigma_2^{-1}\Sigma_2\Sigma_1^{-1})^{-1} \tag{53}$$

$$= (\Sigma_2^{-1} + \Sigma_1^{-1})^{-1}. \tag{54}$$

Eqn. (54) is the form given in Eqn. (32), but we note that Eqn. (44) and Eqn. (51) can be used to put Eqn. (31) and Eqn. (32) into the alternative simple form $\hat{\mathbf{x}} = \mathbf{x}_2 + \mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2)$ and $\Sigma = \mathbf{P}\Sigma_1$. This completes the derivation of the RMV algorithm.

Figure 4 below shows the generally decreasing location error as the RMV processes 100 input location estimates. Also shown in the same figure is the HRE95 at each RMV step.

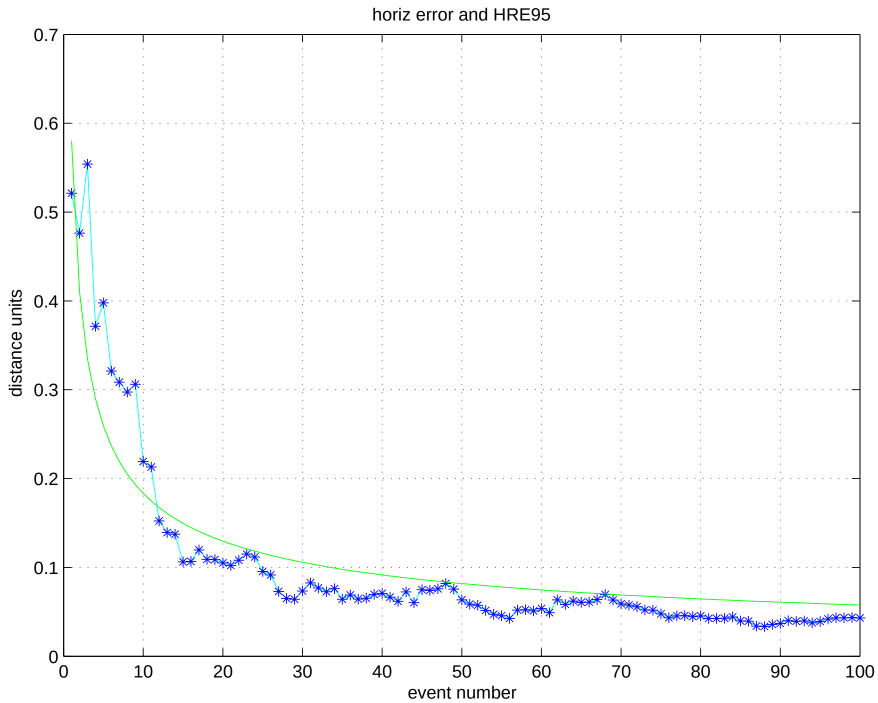

Figure 4. Location error for 100 RMV outputs and HRE95.

The solution accuracy metrics described in the previous section, such CEP, can be calculated for RMV results just as easily as for individual geolocations since the RMV provides an output error covariance matrix Σ_p for each RMV solution.

References

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