

Equilibrium Beam Distribution and Quantum Lifetime in the Presence of a Single Nonlinear Resonance*

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Abstract

In the proximity of a nonlinear resonance $\nu \approx \frac{m}{n}$, the beam distribution in a storage ring is distorted depending on how close by is the resonance and how strong is the resonance strength. In the 1-dimensional case, it is well known that the particle motion near the resonance can be described in a smooth approximation by a Hamiltonian of the form $(\nu - \frac{m}{n})J + D_\nu(J) + f_1(\phi, J)$, where (ϕ, J) are the phase space angle and action variables, D_ν is the detuning function, and f_1 is an oscillating resonance term. In a proton storage ring, the equilibrium beam distribution is readily solved to be any function exclusively of the Hamiltonian. For an electron beam, this is not true and the equilibrium distribution is more complicated. This paper solves the Fokker-Planck equation near a single resonance for an electron beam in a storage ring. The result is then applied to obtain the quantum lifetime of an electron beam in the presence of this resonance. Resonances due to multipole fields and due to the beam-beam force are considered as examples.

*Published Phys. Rev. Special Topics – Accelerators and Beams, Vol.6, 094001
(2003)*

*Work supported by Department of Energy contract DE-AC03-76SF00515.

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1 INTRODUCTION AND SUMMARY

Particle motion in a storage ring is seriously perturbed when the particle's tune ν (defined as particle's natural oscillation frequency divided by its revolution frequency around the ring) is close to a fractional value i.e., when a nonlinear resonance condition $\nu \approx \frac{m}{n}$ is fulfilled. In particular, the beam distribution in phase space can be significantly distorted depending on how close by is the resonance and how strong is the resonance strength.

In the 1-dimensional case, it is well known that the particle motion can be described in a smooth approximation [1] by a Hamiltonian of the form

$$K = \frac{1}{R} \left[\left(\nu - \frac{m}{n} \right) J + D_\nu(J) + f_1(\phi, J) \right] \quad (1)$$

where (ϕ, J) are the phase space angle and action variables, $2\pi R$ is the storage ring circumference, D_ν is the detuning function, and $f_1(\phi, J)$ is some resonance structure function which is purely oscillatory in the sense that it is periodic in ϕ with period $2\pi/n$ and its average over ϕ vanishes.

In a proton storage ring, the equilibrium beam distribution near a resonance is readily obtained once the smooth Hamiltonian is available; it is given by any

function exclusively of K , i.e.

$$\psi(\phi, J) = \text{any function } \psi_0(K) \quad (2)$$

where the function ψ_0 is determined by the initial condition of the beam. Any function of K is a possible equilibrium distribution for a proton beam.

For an electron beam, however, this is no longer the case and the equilibrium distribution is more complicated. In addition to the Hamiltonian dynamics, we will have also to include the slow radiation damping and quantum diffusion effects caused by synchrotron radiation [3]. In the absence of resonances, the equilibrium distribution is also well-known, and is given by the gaussian distribution

$$\psi(\phi, J) = e^{-J/J_0} \quad (3)$$

where J_0 is the nominal beam emittance determined by a balance between the damping and diffusion effects. (The distribution is exponential in J , but is gaussian in x and p . Its normalization is $\int_0^\infty dJ \int_0^{2\pi} d\phi \psi = 2\pi J_0$.) The question therefore remains as to what will the electron beam distribution become when there is a nonlinear resonance nearby. Simple physical arguments can be constructed to show that a simple gaussian of the new Hamiltonian such as $\psi \sim e^{-K}$ will not do the job. (For example, as D_ν is bound to dominate over the other terms in the Hamiltonian at large J , the distribution becomes divergent at large J and is unnormalizable if $D_\nu < 0$. Also, as another example, consider the case away from resonances, for which one has $f_1 = 0$. In that case, one would expect the beam distribution to be given by (3) regardless of D_ν .)

This expected behavior is not observed by $\psi \sim e^{-K}$.)

In this paper, we will first establish the Fokker-Planck equation near a single resonance for an electron beam in a storage ring. By solving this equation to first order in the resonance strength, we find that the equilibrium distribution is given by

$$\psi(\phi, J) = \exp \left[-\frac{1}{J_0} \left(J + \frac{f_1(\phi, J)}{\nu - \frac{m}{n} + D'_\nu(J)} \right) \right] \quad (4)$$

The three terms in the new Hamiltonian (1) take on different roles and appear in different places in the equilibrium beam distribution, reflecting their respective physical meanings.

The result (4) is then applied to obtain the quantum lifetime of an electron beam in the presence of a nearby single resonance. Quantum lifetime is the lifetime of the electron beam when particles are brought by the diffusion effect to a maximum allowed amplitude (representing an absorbing wall in the physical x -space) and are lost from the beam. We find that due to the proximity to the nonlinear resonance, the quantum lifetime is shortened by a factor

$$\exp \left(\frac{\hat{f}_1(\hat{J})}{J_0 \left| \nu - \frac{m}{n} + D'_\nu(\hat{J}) \right|} \right) \quad (5)$$

compared with the nominal expression [3]. In Eq.(5), $\hat{J} = A^2/2\beta$ with A the maximum allowed value of x and β the β -function at the aperture limit, and $\hat{f}_1(J)$ is the oscillation amplitude of $f_1(\phi, J)$.

In Section 2, we give a brief summary of the well-known smooth approximation in the 1-dimensional case. Section 3 derives the Fokker-Planck equation in the smooth approximation. Section 4 solves the Fokker-Planck equation for

the equilibrium beam distribution with weak resonances and slow damping and diffusion. Section 5 derives an expression for the quantum lifetime. Section 6 applies these results to specific examples, first to the case of a nonlinear magnet field error, and then to the case of a 1-dimensional beam-beam perturbation.

2 SMOOTH APPROXIMATION NEAR A SINGLE RESONANCE

Consider 1-dimensional dynamics of a particle in a storage ring. Let the particle motion be described by the Hamiltonian in the (x, p) phase space

$$H(x, p) = \frac{p^2}{2} + \frac{K(s)x^2}{2} + \frac{f(x, s)}{R} \quad (6)$$

where $K(s)$ specifies the linear focusing system of the storage ring, and $f(x, s)$ represents a weak nonlinear perturbation. Both $K(s)$ and $f(x, s)$ are periodic in s with period $2\pi R$.

In the presence of one and only one single resonance $\nu = \frac{m}{n}$, we follow Ref.[4] and make a canonical transformation similar to the Courant-Snyder transformation from (x, p) to (ϕ, J) . We use the generating function

$$G(x, \phi) = -\frac{x^2}{2\beta(s)} \left[\tan \phi_1 - \frac{\beta'(s)}{2} \right], \quad \phi_1 = \phi + \frac{m}{n} \frac{s}{R} + \int_0^s ds' \left(\frac{1}{\beta(s')} - \frac{\nu}{R} \right) \quad (7)$$

where $\beta(s)$ is the periodic Courant-Snyder β -function and $\beta'(s) = \frac{d\beta}{ds}$. With this generating function, we obtain

$$x = \sqrt{2J\beta(s)} \cos \phi_1$$

$$p = -\sqrt{\frac{2J}{\beta(s)}} \left[\sin \phi_1 - \frac{\beta'(s)}{2} \cos \phi_1 \right] \quad (8)$$

or

$$\begin{aligned} J &= \frac{x^2}{2\beta} + \frac{\beta}{2} \left(p - \frac{\beta'}{2\beta} x \right)^2 \\ \tan \phi_1 &= -\beta \frac{p}{x} + \frac{\beta'}{2} \end{aligned} \quad (9)$$

The new Hamiltonian is

$$\begin{aligned} K(\phi, J) &= H + \frac{\partial G}{\partial s} \\ &= \frac{J}{R} \left(\nu - \frac{m}{n} \right) + \frac{1}{R} f \left(\sqrt{2J\beta(s)} \cos \phi_1, s \right) \end{aligned} \quad (10)$$

where use has been made of $2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$ [4].

In the smooth approximation, we drop fast oscillating terms from Eq.(10). This means the Hamiltonian is approximated by replacing the f term in Eq.(10) by its average over s over the range $(0, 2\pi nR)$. The new Hamiltonian therefore becomes

$$K(\phi, J) \approx \frac{J}{R} \left(\nu - \frac{m}{n} \right) + \frac{1}{R} \langle f \rangle_s \quad (11)$$

where

$$\langle f \rangle_s = \int_0^{2\pi nR} \frac{ds}{2\pi nR} f \left(\sqrt{2J\beta(s)} \cos \phi_1, s \right) \quad (12)$$

is a function of ϕ and J but is no longer a function of s . In Eq.(12), the integration over s is done by considering ϕ_1 as a function of s while holding ϕ fixed. The quantity $\langle f \rangle_s$ is a periodic function of ϕ with period $2\pi/n$.

The smooth approximation therefore allows the replacement of a time-dependent Hamiltonian (6) by a time-independent Hamiltonian (11). The particle motion

is described by applying the Hamilton's equations to the new Hamiltonian. Since this new Hamiltonian is independent of s , it is a constant of the motion. One may note that the Courant-Snyder transformation for a linear system [when $f(x, s) = 0$] is in fact a special case of smooth approximation in the sense that the s -dependent focusing function $K(s)$ has been replaced by the tune ν . The linear case, on the other hand, enjoys the distinction that the Courant-Snyder transformation is exact.

The function $\langle f \rangle_s$ is periodic in ϕ with period $2\pi/n$. As ϕ varies, $\langle f \rangle_s$ may be decomposed into a constant part given by

$$D_\nu(J) = \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \int_0^{2\pi} \frac{ds}{R} f \left(\sqrt{2J\beta(s)} \cos \left[\xi + \int_0^s ds' \left(\frac{1}{\beta(s')} - \frac{\nu}{R} \right) \right], s \right) \quad (13)$$

plus an oscillating part $f_1(\phi, J)$, i.e.

$$\langle f \rangle_s = D_\nu(J) + f_1(\phi, J) \quad (14)$$

which in turn gives Eq.(1). As we shall see, the two terms D_ν and f_1 play quite different roles in the beam dynamics.

3 FOKKER-PLANCK EQUATION

The Fokker-Planck equation in the (x, p) space is given by [5]

$$\begin{aligned} & \frac{\partial \psi}{\partial s} + \frac{\partial H}{\partial p} \frac{\partial \psi}{\partial x} - \left(\frac{\partial H}{\partial x} + 2\alpha p \right) \frac{\partial \psi}{\partial p} - \frac{D}{\beta(s)} \frac{\partial^2 \psi}{\partial p^2} - 2\alpha \psi \\ & = \frac{\partial \psi}{\partial s} + p \frac{\partial \psi}{\partial x} - \left(Kx + \frac{1}{R} \frac{\partial f}{\partial x} + 2\alpha p \right) \frac{\partial \psi}{\partial p} - \frac{D}{\beta(s)} \frac{\partial^2 \psi}{\partial p^2} - 2\alpha \psi = \quad (15) \end{aligned}$$

where α and D are the damping and diffusion coefficients. Note that a factor $\beta(s)$ is introduced in the diffusion constant. Strickly speaking, this form of Fokker-Planck equation assumes that quantum noise occurs in p and not in x . However, when the diffusion process is slow, our result does not change if the noise also occurs in x . What matters is only the effect of diffusion after averaging out the fast oscillating terms.

We then make the transformation to the (ϕ, J) space. The Fokker-Planck equation in the (ϕ, J) space becomes, after some algebra,

$$\begin{aligned} & \frac{\partial \psi}{\partial s} + \frac{1}{R} \left(\nu - \frac{m}{n} \right) \frac{\partial \psi}{\partial \phi} + \left(\frac{1}{R} \frac{\partial f}{\partial x} + 2\alpha p \right) \left(\sqrt{2J\beta} \sin \phi_1 \frac{\partial \psi}{\partial J} + \sqrt{\frac{\beta}{2J}} \cos \phi_1 \frac{\partial \psi}{\partial \phi} \right) \\ & - D \left(2J \sin^2 \phi_1 \frac{\partial^2 \psi}{\partial J^2} + \frac{\partial \psi}{\partial J} + \sin 2\phi_1 \frac{\partial^2 \psi}{\partial \phi \partial J} - \frac{1}{2J} \sin 2\phi_1 \frac{\partial \psi}{\partial \phi} + \frac{1}{2J} \cos^2 \phi_1 \frac{\partial^2 \psi}{\partial \phi^2} \right) - 2\alpha \psi = 0 \end{aligned} \quad (16)$$

Damping and diffusion effects are considered to be very slow. If we consider only this very slow evolution of ψ , then we can average the Fokker-Planck equation over s (or ϕ_1), i.e. average over the fast oscillation. This yields the Fokker-Planck equation under smooth approximation,

$$\begin{aligned} & \frac{\partial \psi}{\partial s} + \frac{\partial K}{\partial J} \frac{\partial \psi}{\partial \phi} - \frac{\partial K}{\partial \phi} \frac{\partial \psi}{\partial J} - 2\alpha \left(J \frac{\partial \psi}{\partial J} - \frac{\beta'}{4} \frac{\partial \psi}{\partial \phi} \right) \\ & - D \left(J \frac{\partial^2 \psi}{\partial J^2} + \frac{\partial \psi}{\partial J} + \frac{1}{4J} \frac{\partial^2 \psi}{\partial \phi^2} \right) - 2\alpha \psi = 0 \end{aligned} \quad (17)$$

When $f = 0$, the equilibrium beam distribution depends only on J , and indeed, by solving Eq.(17), we recover the distribution (3) with

$$J_0 = \frac{D}{2\alpha} \quad (18)$$

In terms of the unperturbed rms beam size σ_x , we have $J_0 = \sigma_x^2/\beta$.

4 EQUILIBRIUM DISTRIBUTION

We need to solve Eq.(17) for the equilibrium distribution when $f \neq 0$ in the smooth approximation. To do so, we make two assumptions. We first assume that the resonance is weak in such a way that f_1 is small. More specifically, as can be seen in the manipulations with Fokker-Planck equation, we require

$$\nu - \frac{m}{n} + D'_\nu(J) \gg \frac{1}{J} f_1(\phi, J) \quad (19)$$

The left hand side is the distance of the amplitude-dependent tune $\nu + D'_\nu(J)$ from the resonance. The right hand side can be considered a resonance width in tune units. This condition (19) applies when either $\nu - \frac{m}{n}$ or the detuning D'_ν is larger than the resonance width, but it does not require both to be larger. For example, the analysis will be applicable exactly on resonance $\nu = \frac{m}{n}$, provided that D'_ν is sufficiently large. On the other hand, the analysis breaks down when $\nu - \frac{m}{n} + D'_\nu(J) = 0$, i.e. when the amplitude-dependent tune crosses the resonance at some amplitude within the beam distribution.

We also assume that the damping and diffusion effects are very slow, so that

$$\nu - \frac{m}{n} + D'_\nu(J) \gg \alpha R = \frac{T_0}{2\pi\tau} \quad (20)$$

where $T_0 = 2\pi R/c$ = revolution period, and $\tau = 1/\alpha c$ = radiation damping time.

What assumptions (19) and (20) allow us to do is that, in the Fokker-Planck equation, we may keep only terms that are up to first order in f_1 , α , or D . Higher order terms in these parameters, such as terms proportional to αf_1 , can

be dropped. [Since f_1 , α and D are small parameters, this seems to be a reasonable procedure. Nevertheless, a more detailed justification of this procedure can be established by making another canonical transformation from (ϕ, J) coordinate system to a new system (Φ, K) , where the new momentum is just the Hamiltonian itself. The same exercise also provided the hint of the ansatz (21) below.]

With assumptions (19) and (20), it turns out that we may try the ansatz for the equilibrium beam distribution in the form

$$\psi = e^{-A(J) - B(J)f_1(\phi, J)} \quad (21)$$

Substituting Eq.(21) into (17), keeping terms first order in f_1 , α and D , yields the solution

$$\begin{aligned} A(J) &= \frac{J}{J_0} \\ B(J) &= \frac{1}{J_0[\nu - \frac{m}{n} + D'_\nu(J)]} \end{aligned} \quad (22)$$

where J_0 is given by Eq.(18). It is remarkable that this equilibrium distribution holds for arbitrary $\nu - \frac{m}{n}$, $D_\nu(J)$, and $f_1(\phi, J)$, and for all J . We have therefore arrived at Eq.(4). The term involving f_1 gives rise to island structure in the beam distribution. Near the resonance $\nu \approx \frac{m}{n}$, there will be n islands in the phase space.[2]

The normalization is such that, to first order in f_1 ,

$$\int_0^\infty dJ \int_0^{2\pi} d\phi \psi(\phi, J) = 2\pi J_0 \quad (23)$$

The distribution is not normalizable if f_1 overcomes the detuning, i.e. if the opposite of condition (19) holds.

5 QUANTUM LIFETIME

When the storage ring imposes a maximum permitted aperture to the electron beam, the beam will suffer a continuous loss of particles due to quantum diffusion [3]. If the aperture is far from the beam core, this loss will be slow, and the corresponding quantum beam lifetime will be long. The beam distribution will still be accurately given by Eq.(4) except immediately near the aperture boundary [6]. In particular, the distribution will be a function of

$$J_1 = J + \frac{f_1(\phi, J)}{\nu - \frac{m}{n} + D'_\nu(J)} \quad (24)$$

which can be considered a distorted action variable and where the second term is purely oscillatory in ϕ with period $2\pi/n$. We shall write the beam distribution with an aperture limit as

$$\psi(\phi, J, s) = \psi_1(J_1) e^{-\frac{s}{\tau_q c}} \quad (25)$$

where τ_q is the quantum lifetime.

The aperture boundary is imposed on the physical amplitude x of particle motion. Let this limit be A , i.e. a particle is lost when its physical amplitude reaches $x \geq A$, or equivalently $J \geq A^2/2\beta$, with β the β -function at the aperture limit. This means that a particle is lost when its J_1 reaches $J_1 \geq \hat{J}_1$, where

$$\hat{J}_1 = \frac{A^2}{2\beta} - \frac{\hat{f}_1(\frac{A^2}{2\beta})}{\left| \nu - \frac{m}{n} + D'_\nu(\frac{A^2}{2\beta}) \right|} \quad (26)$$

$$\hat{f}_1(J) = \max. \text{ of } |f_1(\phi, J)| \text{ as } \phi \text{ is varied over the range } (0, \frac{2\pi}{n})$$

$$\text{under the condition that } [\nu - \frac{m}{n} + D'_\nu(\frac{A^2}{2\beta})] f_1 < 0 \quad (27)$$

A particle with $J_1 \geq \hat{J}_1$ will have during its motion a moment when its physical x reaches the aperture $x \geq A$.

The beam distribution still obeys the Fokker-Planck equation (17), except now that at the aperture boundary, the beam distribution will have to satisfy the additional condition

$$\psi_1(J_1 = \hat{J}_1) = 0 \quad (28)$$

With ψ_1 depending only on J_1 , and keeping terms linear in f_1 , α , and D , the Fokker-Planck equation reduces to

$$-\frac{1}{c\tau_q}\psi_1 - 2\alpha J_1\psi_1' - D(J_1\psi_1'' + \psi_1') - 2\alpha\psi_1 = 0 \quad (29)$$

where a prime on ψ_1 means taking derivative with respect to J_1 . Integrating Eq.(29) over J_1 from 0 to \hat{J}_1 and applying integration by parts, we obtain

$$\frac{1}{\tau_q c} \int_0^{\hat{J}_1} \psi_1 dJ_1 + D\hat{J}_1\psi_1'(\hat{J}_1) = 0 \quad (30)$$

When the aperture is far from the beam core, i.e. when $\hat{J}_1 \gg J_0$, ψ_1 will be accurately given by the unperturbed distribution (4) [3, 6]. Near the aperture boundary, ψ_1 will have to approach zero, but the derivative ψ_1' will remain very close to the derivative obtained using Eq.(4). (Physically this is because the diffusion rate outward from the beam through the aperture will be the same whether the aperture limit is there or not.) We therefore substitute $\psi_1(J_1) = e^{-J_1/J_0}$ into both terms in Eq.(30), and obtain

$$\frac{1}{\tau_q} = \frac{2\hat{J}_1}{\tau J_0} e^{-\hat{J}_1/J_0} \quad (31)$$

where $\tau = 1/\alpha c$ is the radiation damping time. When $f_1 = 0$, Eq.(31) reduces to the nominal expression in the absence of the nonlinear resonance [3],

$$\frac{1}{\tau_{q0}} = \frac{A^2}{\tau \sigma_x^2} e^{-A^2/2\sigma_x^2} \quad (32)$$

When $f_1 \neq 0$, one finds that the quantum lifetime in the presence of the resonance is shortened compared with the nominal expression by a factor (5). Note that the argument of the exponent in Eq.(5) does not have to be $\ll 1$ under the constraints (19) and (20).

6 EXAMPLE APPLICATIONS

We now apply the results obtained so far to specific examples. Consider a perturbation of the form

$$f(x, s) = \epsilon g(x) \delta_P(s), \quad \delta_P(s) = \text{periodic } \delta\text{-function} \quad \sum_{k=-\infty}^{\infty} \delta\left(\frac{s}{R} - 2\pi k\right) \quad (33)$$

with arbitrary $g(x)$. This $f(x, s)$ represents a nonlinear perturbation located at $s = 0$ that gives δ -function kicks to the particle as it circulates around the ring.

Following the smooth approximation, we first obtain

$$\langle f \rangle_s = \frac{\epsilon}{2\pi n} \sum_{k=1}^n g\left(\sqrt{2J\beta} \cos\left(\phi + 2\pi \frac{km}{n}\right)\right) \quad (34)$$

where $\beta = \beta(0)$. We then decompose $\langle f \rangle_s$ into a detuning term plus an oscillating term according to Eq.(14). The detuning term is found using Eq.(13) to be

$$D_\nu(J) = \frac{\epsilon}{4\pi^2} \int_0^{2\pi} d\xi g(\sqrt{2J\beta} \cos \xi) \quad (35)$$

6.1 Multipole Magnet Field Error

When the perturbation is given by a magnet multipole, we take

$$\begin{aligned} g(x) &= x^n \\ \epsilon &= \frac{\ell}{n!(B\rho)} \frac{\partial^{n-1} B_y}{\partial x^{n-1}} \end{aligned} \quad (36)$$

where $(B\rho) = P/e$ is the magnetic rigidity of the particle (P is the particle momentum, e is its electric charge), ℓ is the length of the multipole magnet. For a nonlinear multipole, we assume $n \geq 3$. Applying the smooth approximation, we obtain

$$\begin{aligned} D_\nu(J) &= \frac{\epsilon}{\pi} \left(\frac{J\beta}{2} \right)^{n/2} \begin{cases} \frac{n!}{2[(n/2)!]^2}, & \text{if } n = \text{even} \\ 0, & \text{if } n = \text{odd} \end{cases} \\ f_1(\phi, J) &= \frac{\epsilon}{\pi} \left(\frac{J\beta}{2} \right)^{n/2} \cos n\phi \end{aligned} \quad (37)$$

where β is the β -function evaluated at the multipole.

In this example, we see that the detuning function $D_\nu(J)$ is provided by the multipole if n is even. When n is odd, however, the multipole contribution to the detuning vanishes, and to stabilize the beam distribution, it is necessary that an additional detuning is provided by some other source of nonlinearity, and this additional detuning must be such that $D_\nu(J)$ grows with J at least as fast as $J^{n/2}$ to overcome the resonance term.

Substituting Eq.(37) into Eq.(4) gives (take n to be even) the equilibrium beam distribution

$$\psi(\phi, J) = \exp \left[-\frac{J}{J_0} - \frac{\frac{\epsilon}{\pi J_0} \left(\frac{J\beta}{2} \right)^{\frac{n}{2}} \cos n\phi}{\nu - \frac{m}{n} + \frac{\epsilon}{4\pi J} \frac{n n!}{[(n/2)!]^2} \left(\frac{J\beta}{2} \right)^{\frac{n}{2}}} \right] \quad (38)$$

Substituting into Eq.(5) gives the quantum lifetime shortening factor

$$\exp \left[\frac{|\epsilon| A^n}{\left| \pi 2^n \left(\nu - \frac{m}{n} \right) \frac{\sigma_x^2}{\beta} + \frac{1}{2} \frac{n n!}{[(n/2)!]^2} \epsilon \sigma_x^2 A^{n-2} \right|} \right] \quad (39)$$

If operated exactly on resonance, $\nu = \frac{m}{n}$, it follows that the quantum lifetime is shortened by a factor independent of the multipole strength. For an octupole magnet, for example, $n = 4$, and the factor is $e^{A^2/12\sigma_x^2}$.

6.2 Beam-beam Perturbation

Now we apply our results to the beam-beam perturbation. In a colliding-beam storage ring, the head-on collisions of the colliding beams produce a perturbation of the form (33). Consider only x -motion (or y -motion as the case may be). Assuming no dispersion at the interaction point, the beam-beam interaction can be described by [7]

$$\begin{aligned} \epsilon &= \frac{N r_0}{\gamma} \\ g(x) &= \int_0^\infty dt \frac{1 - \exp\left(-\frac{x^2}{2\sigma_x^2 + t}\right)}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \end{aligned} \quad (40)$$

where r_0 is the classical radius of the particles, γ is the relativistic Lorentz factor, and N is the number of particles in a colliding beam bunch. Again following the smooth approximation procedure, we obtain

$$\begin{aligned} \langle f \rangle_s &= \frac{\epsilon}{2\pi n} \int_0^\infty \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \sum_{k=1}^n \left[1 - e^{-2X \cos^2(\phi + 2\pi \frac{km}{n})} \right] \\ &= \frac{\epsilon}{2\pi} \int_0^\infty \frac{dt}{\sqrt{(2\sigma_x^2 + t)(2\sigma_y^2 + t)}} \left[1 - e^{-X} \sum_{\ell=-\infty}^\infty (-1)^{\ell n'} I_{\ell n'}(X) e^{2i\ell n' \phi} \right] \end{aligned}$$

where $X = \frac{J\beta}{2\sigma_x^2+t}$, and

$$n' = \begin{cases} n, & \text{if } n = \text{odd} \\ \frac{n}{2}, & \text{if } n = \text{even} \end{cases} \quad (42)$$

Note that the results are independent of m . We can set $m = 1$.

Decomposing $\langle f \rangle_s$ into detuning and oscillating terms,

$$\begin{aligned} D_\nu(J) &= \frac{\epsilon}{2\pi} \int_0^\infty \frac{dt}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}} [1 - e^{-X} I_0(X)] \\ f_1(\phi, J) &= -\frac{\epsilon}{\pi} \sum_{\ell=1}^\infty (-1)^{\ell n'} \cos 2\ell n' \phi \int_0^\infty \frac{dt e^{-X} I_{\ell n'}(X)}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}} \end{aligned} \quad (43)$$

The beam distribution is given by substituting Eq.(43) into Eq.(4).

The extreme value of f_1 occurs at $\phi = \pi/2$, i.e. $\hat{f}_1(J) = |f_1(\frac{\pi}{2}, J)|$. (This is because the integral over t is positive; the extreme value occurs when $\cos 2\ell n' \phi = (-1)^{\ell n'}$. Also, for $\hat{f}_1(J)$, we take the absolute value of $f_1(\frac{\pi}{2}, J)$ because we are interested in obtaining the maximum oscillation amplitude of f_1 and we assumed $\nu - \frac{m}{n} + D'_\nu(\frac{A^2}{2\beta}) > 0$.) If we designate

$$\begin{aligned} D_\nu &= \frac{\epsilon}{2\pi} h_0 \left(\frac{J\beta}{2\sigma_x^2}, \frac{\sigma_y}{\sigma_x} \right) \\ \hat{f}_1(J) &= \frac{\epsilon}{2\pi} h_n \left(\frac{J\beta}{2\sigma_x^2}, \frac{\sigma_y}{\sigma_x} \right) \end{aligned} \quad (44)$$

then

$$\begin{aligned} h_0(x, r) &= \int_0^1 \frac{du}{u\sqrt{1+(r^2-1)u}} [1 - e^{-xu} I_0(xu)] \\ h_n(x, r) &= \int_0^1 \frac{du e^{-xu}}{u\sqrt{1+(r^2-1)u}} \left[\frac{1}{n} \sum_{k=1}^n e^{xu \cos(4\pi \frac{k}{n})} - I_0(xu) \right] \end{aligned} \quad (45)$$

Note that $h_n(x, r) = h_{2n}(x, r)$ if $n = \text{odd}$. Note also that the beam-beam tune shift parameter [7] is given by

$$\Delta\nu_{\text{BB}} = D'_\nu(0) = \frac{Nr_0}{2\pi\gamma} \frac{\beta}{2\sigma_x^2} \frac{\partial h_0(x, r)}{\partial x} \Big|_{x=0} = \frac{Nr_0\beta}{2\pi\gamma\sigma_x(\sigma_x + \sigma_y)} \quad (46)$$

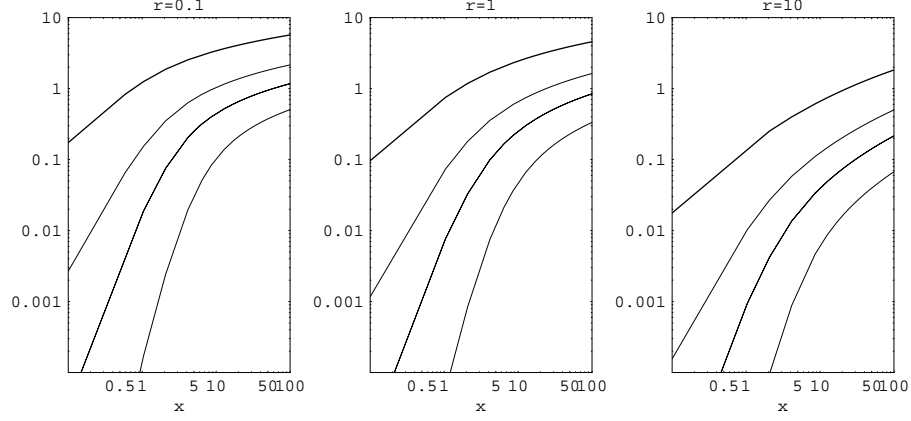


Figure 1: Functions $h_0(x, r)$ and $h_n(x, r)$ versus x .

Figure 1 shows $h_0(x, r)$ and $h_n(x, r)$ as functions of x for three cases $r = \sigma_y/\sigma_x = 0.1, 1$, and 10 , respectively. In each sub-figure, the top thick curve is h_0 . The three thinner curves, ordered from top to bottom, are h_4 , $h_3 = h_6$, and $h_5 = h_{10}$, respectively. The case $r = 10$ can be alternatively considered to be applied to the case $r = 0.1$ but for the y -motion. Figure 2 shows the detuning function $\partial h_0(x, r)/\partial x$ as a function of x for the cases (from top to bottom) $r = 10, 1$, and 0.1 .

Quantum lifetime is reduced by a factor

$$\exp \left(\frac{\Delta\nu_{\text{BB}}(1+r)h_n}{\left| \nu - \frac{m}{n} + \frac{1}{2}\Delta\nu_{\text{BB}}(1+r)h'_0 \right|} \right) \quad (47)$$

where $h'_0 = \frac{\partial h_0(x, r)}{\partial x}$, $h_n = h_n(x, r)$ with $x = \frac{A^2}{4\sigma_x^2}$, $r = \frac{\sigma_y}{\sigma_x}$. As a numerical example, take $\Delta\nu_{\text{BB}} = 0.05$, $n = 4$, $\nu - \frac{m}{4} = 0.01$, $A = 6\sigma_x$, $r = 0.1$, we have $h_4 = 0.99$, $h'_0 = 0.11$ and the quantum lifetime is shortened by a factor of 70 compared with the value without the nearby resonance.

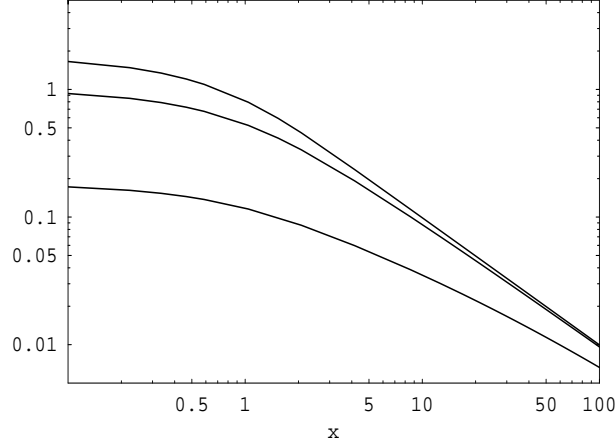


Figure 2: Function $\partial h_0(x, r)/\partial x$ versus x .

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