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# Quantum Gravity and Inflation

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## ABSTRACT

Using the Ashtekar-Sen variables of loop quantum gravity, a new class of exact solutions to the equations of quantum cosmology is found for gravity coupled to a scalar field, that corresponds to inflating universes. The scalar field, which has an arbitrary potential, is treated as a time variable, reducing the hamiltonian constraint to a time-dependent Schroedinger equation. When reduced to the homogeneous and isotropic case, this is solved exactly by a set of solutions that extend the Kodama state, taking into account the time dependence of the vacuum energy. Each quantum state corresponds to a classical solution of the Hamiltonian-Jacobi equation. The study of the latter shows evidence for an attractor, suggesting a universality in the phenomena of inflation. Finally, wavepackets can be constructed by superposing solutions with different ratios of kinetic to potential scalar field energy, resolving, at least in this case, the issue of normalizability of the Kodama state.

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# 1 Introduction

The inflationary scenario provides a framework for resolving the problems of the standard big bang (SBB) and, most importantly, provides a causal mechanism for generating structure in the universe. Currently, however, a completely satisfactory realization of inflation is still lacking. A hint for finding a concrete realization of inflation comes from the transplankian problem. Despite their successes in solving the formation of structure problem, most scalar field driven inflation models generically predict that the near scale invariant spectrum of quantum fluctuations which seeded structure were generated in the transplankian epoch. This, however, is inconsistent with the assumptions of a weakly coupled scalar field theory as well as the assumption that quantum gravity effects can be ignored in the model[2]. One can then view the transplankian problem as an indication, and hence an opportunity, that the concrete derivation of inflation should be embedded in quantum gravity.

There is also another interesting hint that suggests that quantum gravity must play a role in our understanding of inflation. Inflation addresses the issue of initial conditions in the SBB, but solutions to scalar field theory driven inflation suffer from geodesic incompleteness. This is an indication that inflation itself requires the specification of fine tuned initial conditions [6]. The issue of the robustness of the initial conditions necessary to start a phenomenologically acceptable period of inflation, and their sensitivity to the parameters of the underlying field theory, remain open questions which should in principle be addressed by quantum gravity.\* Therefore, a major goal of quantum gravity and cosmology is to find a quantum gravitational state which yields a consistent description of inflation. If this is accomplished then one may be in a better position to make observational predictions for CMB experiments.

The problem of inflation in quantum gravity has been much studied[3, 4, 5]. However in the past the poor understanding of quantum gravity necessitated that the study of inflation be restricted to the semiclassical approximation. This restriction makes it difficult to obtain reliable results about issues such as initial conditions and transplankian effects that involve the regime in which quantum gravitational effects will be significant.

In this light it is worth noting that in recent years a great deal of progress has been made in a non-perturbative approach to quantum gravity, called loop quantum gravity[7]. It is then appropriate to investigate whether these advances allow us to treat the problem of inflation within cosmology more precisely. Recent results of Bojowald and others[8] indicate that in loop quantum gravity one can find exact quantum states that allow us to investigate more precisely the role of quantum gravitational effects on issues in cosmology, including inflation and the fate of the initial singularity. Furthermore, for constant cosmological constant, there is an exact solution to the quantum constraints that define the full quantum general relativity, discovered by Kodama[9], which has both an exact Planck scale description and

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\*There are other motivations for expecting a quantum gravitational derivation of inflation but this is outside the scope of this paper. A good discussion of this issue is nicely covered in a review by Robert Brandenberger [1].

a semiclassical interpretation in terms of deSitter spacetime. While there are open issues of interpretation concerning this state[10, 11], it is also true that it can be used as the basis of both non-perturbative and semiclassical calculations[12, 13, 14]. Furthermore, exact results in the loop representation have made possible an understanding of the temperature and entropy of deSitter spacetime[12, 14] in terms of the kinematics of the quantum gravitational field.

Thus, there appears to be no longer any reason to restrict the study of quantum cosmology to the semiclassical approximation. In this paper we provide more evidence for this, by finding exact solutions to the equations of quantum cosmology that provide exact quantum mechanical descriptions of inflation.

In order to study the problem of inflation in quantum gravity we proceed by several steps. First, we couple general relativity to a scalar field,  $\phi$ , with an arbitrarily chosen potential,  $V(\phi)$ . We then choose a gauge for the Hamiltonian constraints in which this scalar field is constant on constant time hypersurfaces[16]. This is appropriate for the study of inflation, because it has been shown that, in terms of the standard cosmological time coordinates, inflation cannot occur unless the fluctuations of the scalar field on constant time surfaces are small[15]. There then always will exist a small, local rescaling of time that makes the scalar field constant<sup>†</sup>.

In this gauge the infinite number of Hamiltonian constraints are reduced to a single, time dependent Schroedinger equation[16]. This is then solved, for homogeneous, isotropic fields, as follows. The corresponding classical Hamilton evolution equations are solved exactly by a class of Hamilton-Jacobi functions. Each solution involves the numerical integration of an ordinary first order differential equation. These reduce, in the limit of vanishing slow role parameter,  $\dot{V}/V$ , to the Chern-Simons invariant of the Ashtekar connection. This is good, as the latter is known to be the Hamilton-Jacobi function for deSitter spacetime[9, 13, 12, 14]. By exponentiating the actions of these solutions, one obtains a semiclassical state that reduces in the same limit to the Kodama state. These new solutions are only good in the semiclassical approximation. However, in this case it is possible to find the corrections which make the wavefunctionals into exact solutions of the time-dependent Schrodinger equation.

The connection to the Kodama state allows us also to investigate issues regarding the physical interpretation of that state such as the normalizability of the wavefunction[10, 11]. In the case studied here, each exact quantum state we find is delta-function normalizable in the physical Hilbert space, corresponding to the reduced, homogeneous, isotropic degrees of freedom. It is then interesting to ask whether fully normalizable states can be constructed by superposing the different solutions. In fact, at a given time, defined by the value of  $\phi$ , the different solutions correspond to different ratios of  $\pi^2/V(\phi)$ , where  $\pi$  is the canonical momenta of the scalar field. It is then reasonable to superpose such solutions as there is no reason to believe that quantum state of the universe should at early times be an eigenstate of the ratio of kinetic to potential energy. When we do this we find wavepackets which are exact normalizable solutions. This suggests that the problem of normalizability of the

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<sup>†</sup>Technical subtleties regarding this choice of gauge are discussed below.

Kodama state in the exact theory may be resolved similarly by adding matter to the theory and then superposing extensions of the state corresponding to different eigenvalues of the matter energy momentum tensors.

In the next section we describe the scalar field general relativity system in the formalism of Ashtekar[7] together with the details of the procedure whereby the time gauge is fixed. Section 3 explains the reduction to homogeneous, isotropic fields in these variables, while section 4 describes the solutions to the resulting classical equations by means of a set of solutions to the Hamilton-Jacobi theory. The solutions are studied numerically and evidence for an attractor is found. In section 5 we quantize the homogeneous, isotropic system, discovering both semiclassical and exact solutions to the Schroedinger equation. Our conclusions and some directions for further research are described in the final section.

## 2 The Theory

We consider general relativity coupled to a scalar field,  $\phi$ , and additional fields  $\Psi$ , in the Ashtekar formulation of loop quantum gravity<sup>‡</sup>. Working in the canonical formalism, the Hamiltonian constraint is of the form,

$$\mathcal{H} = \mathcal{H}^{grav} + \frac{1}{2}\pi^2 + \frac{1}{2}E^{ai}E_i^b\partial_a\phi\partial_b\phi + \mathcal{H}^\Psi \quad (1)$$

where  $\pi$  is the conjugate momentum to  $\phi$  and  $E^{ai}$  is the conjugate momentum to the complex  $SO(3)$  connection  $A_{ai}$ . The latter couple satisfy the Poisson bracket relation

$$\{A_{ai}(\mathbf{x}), \mathbf{E}^{bj}(\mathbf{y})\} = \mathbf{i}G\delta_b^a\delta_i^j\delta^3(\mathbf{x}, \mathbf{y}) \quad (2)$$

where  $G$  is Newton's constant.

In (1) we use  $\mathcal{H}^\Psi$  to denote the Hamiltonian constraint for all other matter fields. Unless otherwise noted, we adopt the convention that lowercase latin indices  $a, b, c, \dots$  are spatial indices while lowercase latin indices  $i, j, k, \dots$  are internal  $SO(3)$  indices.

We include the scalar field potential  $V(\phi)$  in the gravitational term so that

$$\mathcal{H}^{grav} = \frac{1}{l_p}\epsilon_{ijk}E^{ai}E^{bj}(F_{ab}^k + \frac{GV(\phi)}{3}\epsilon_{abc}E^{ck}). \quad (3)$$

where  $F_{ab}^k$  is the curvature of the connection  $A_{ai}$ . Note that any bare cosmological constant,  $\Lambda$ , is included in  $V(\phi)$  and  $V(\phi) = \Lambda$  gives the Hamiltonian constraint for general relativity sourced only by  $\Lambda$  and no scalar field.

We also will impose the Gauss's law constraint, which enforces  $SO(3)$  gauge invariance

$$\mathcal{G}^i = D_a E^{ai} \quad (4)$$

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<sup>‡</sup>For an introduction to the Ashtekar formalism in the context of cosmology, see [14]. Other good, more general and complete reviews are in [7].

and the diffeomorphism constraint, that imposes spatial diffeomorphism invariance

$$\mathcal{D}_a = E^{bi} F_{abi} + \pi \partial_a \phi + \mathcal{D}_a^\Psi \quad (5)$$

where  $\mathcal{D}_a^\Psi$  contains the matter fields.

We assume that spacetime has topology  $\mathcal{M} = \mathcal{S} \times R$ , where  $\mathcal{S}$  is the spatial manifold. As we are interested in cosmology we assume  $\mathcal{S}$  has no boundary, so that the Hamiltonian is given by

$$H(N) = \int_{\mathcal{S}} N \mathcal{H} \quad (6)$$

which is defined for any lapse  $N$ . We note that  $N$  has density weight minus one.

## 2.1 Fixing the Time Gauge

In the Hamiltonian approach to general relativity, one is free to choose any slicing of spacetime into a one parameter family of spacelike surfaces, where that parameter can be considered to be a time coordinate. All such slicings are physically equivalent and the Hamiltonian constraint generates gauge transformations that take us from any one spatial slice to any other. The choice of a slicing is then a gauge choice.

In the usual treatments of inflation, the scalar field,  $\phi$ , is required to be homogeneous to a good approximation. As the deviations from homogeneity must be small for inflation to occur at all, in solutions to Einstein's equations in which inflation takes place we can assume that the surfaces of constant  $\phi$  are spacelike. The scalar field also varies as the universe expands, that is, it is “rolling down the hill”. It is then possible to measure time during inflation by the value of the  $\phi$  field, keeping in mind that the *forward* progression of time corresponds to  $\phi$  changing in the *negative* direction.

As a result, we will choose to gauge fix the action of the Hamiltonian constraint so that  $\phi$  is constant on constant time surfaces. We do this by imposing the gauge condition[16]

$$\partial_a \phi = 0. \quad (7)$$

We need to ensure that this condition is maintained by evolution generated by the Hamiltonian  $H(N)$ . That is, we demand

$$0 = \frac{d\partial_a \phi}{dt} = \{\partial_a \phi, H(N)\} = \partial_a (N\pi) \quad (8)$$

which tells us that, to ensure the gauge condition is preserved, we must use a lapse

$$N = k/\pi \quad (9)$$

where  $k$  is a constant.

The gauge condition (7) is not good on the whole configuration space as there are solutions to Einstein's equations for which none of the constant  $\phi$  surfaces are spacelike. Thus, (7) is

more than a gauge choice, it is also a restriction on the space of solutions. Nevertheless, it is a restriction which is appropriate to the study of inflation as there are results[] that indicate that, in models where the metric is approximately spatially homogeneous, inflation only takes place for solutions in which  $\phi$  is also, to a good approximation, spatially homogeneous.

However, for most initial data that satisfies (7), it is known that the gauge condition will not be preserved forever. The condition cannot be preserved if  $\pi$  becomes zero at any point on  $\mathcal{S}$ .

Of course,  $\pi$  is chosen on the initial data surface, and then evolves. Equation (9) tells us that an infinite lapse is required to preserve the gauge condition at points where  $\pi$  vanishes. So by fixing the gauge to (7) we will generally be able to study only a finite period in the evolution of the universe. The extent of the period in which the gauge choice is good depends on the initial values taken for  $\pi$  and the other fields. As we are interested in modeling inflation in which deviations from homogeneity must be assumed to be small, we will assume that the gauge choice remains good for the entire period of inflation. However, after we have built the quantum theory, we will have to be concerned with the extent to which these conditions are reliable.

As the hamiltonian constraint does not commute with the gauge condition, we have to solve all but one of the infinite number of Hamiltonian constraints for the conjugate variable  $\pi$ . The one that is not solved is the constraint whose lapse is inversely proportional to  $\pi$ , as that constraint commutes with the gauge condition.

We then find that

$$\pi = [-2\mathcal{H}^{grav} - 2\mathcal{H}^\Psi]^{1/2}. \quad (10)$$

There is one remaining Hamiltonian constraint which must be imposed which is

$$0 = \mathcal{H}(N = k/\pi) = \frac{k}{2} \int_{\mathcal{S}} \pi - \frac{1}{2} H \quad (11)$$

To get the dimensions right,  $N$  should be dimensionless so we pick  $k = 1/l_p^2$ . Then  $H$  is the Hamiltonian for evolution in the gauge we have picked. It is given by

$$H = \frac{\sqrt{2}}{l_p^2} \int_{\mathcal{S}} [-\mathcal{H}^{grav} - \mathcal{H}^\Psi]^{1/2}. \quad (12)$$

Finally, we define

$$P = \frac{1}{l_p^2} \int_{\mathcal{S}} \pi \quad (13)$$

which has dimensions of energy.

Thus, if we call the time  $T = l_p^2 \phi$ , where the factor of  $l_p^2$  is included so that  $T$  has dimensions of time, we have the Poisson bracket

$$\{T, P\} = 1. \quad (14)$$

We then have from (11)

$$-P + H = 0, \quad (15)$$



taking note of the fact that the Hamiltonian is time dependent because the potential term in  $\mathcal{H}^{grav}$  depends on  $T$ . Herein, we will use  $V(T)$  to denote the value of the original potential  $V(\phi)$  evaluated at  $\phi = T/l_p^2$ . We thus have reduced general relativity to an ordinary Hamiltonian system with a time dependent Hamiltonian.

### 3 The Homogeneous Case

In this paper we will be concerned with the spatially homogeneous case, in order to be able to compare our approach to the standard results in inflationary cosmology. Thus, we now turn to the reduction of the Hamiltonian system just derived to the case of spatially homogeneous and isotropic universes. We will also consider from now on only the case in which the scalar field is the sole matter field.

The description of deSitter spacetime in Ashtekar-Sen variables is described in [14]. In a spatially flat slicing of deSitter spacetime, the  $SO(3)$  gauge can be chosen so that the solution is given by diagonal and homogeneous fields

$$A_{ai} = i\delta_{ai}A, \quad E^{ai} = \delta^{ai}E \quad (16)$$

where  $A$  and  $E$  are constant on each spatial slice  $\mathcal{S}$ .

DeSitter spacetime with cosmological constant,  $\Lambda$ , is given by

$$A = hf(t), \quad E = f^2, \quad f(t) = e^{ht} \quad (17)$$

where the Hubble parameter is  $h^2 = G\Lambda/3$  and  $t$  is the usual time coordinate defined so that the spacetime metric is given by

$$ds^2 = -dt^2 + f^2 (ds_3)^2 \quad (18)$$

where  $(ds_3)^2$  is the flat metric on  $\mathcal{S}$ .

We will consider the generalization of deSitter spacetime in which the homogeneous scalar field is used as the time coordinate, so that  $A$  and  $E$  are separately functions of  $T$ . In these coordinates, the spacetime metric is

$$ds^2 = -N^2 dT^2 + E(T) (ds_3)^2. \quad (19)$$

where  $N$  is the lapse (9).

The gauge and diffeomorphism constraints are solved automatically by the reduction to a homogeneous solution and the curvature is given by

$$F_{abi} = -A^2 \epsilon_{abi}. \quad (20)$$

The gravitational part of the Hamiltonian constraint in this reduced model is given by

$$\mathcal{H}^{grav} = \frac{6}{l_p^2} E^2 \left( -A^2 + \frac{l_p^2 V(T)}{3} E \right). \quad (21)$$

Given that  $\mathcal{S}$  is not compact and our fields homogeneous, we must give a well defined meaning to the integral over  $\mathcal{S}$ . As space is homogeneous, we can integrate over a compact region  $\Sigma \subset \mathcal{S}$  such that

$$\int_{\Sigma} = L^3 \quad (22)$$

where  $L$  is a fixed, non-dynamical length scale. In this way,  $\Sigma$  is a finite representative of the entire homogeneous space. We will use the dimensionless ratio  $R = L/l_p$ .  $R$  is a free parameter in the homogeneous cosmological model that is not part of the full field theory, but arises from the reduction from a field theory to a mechanical system.

If there are no matter fields, we then have the Hamiltonian

$$H(A, E, T) = R^3 \sqrt{12E^2 \left( A^2 - \frac{l_p^2 V(T)}{3} E \right)}. \quad (23)$$

In this way we have a finite dimensional Hamiltonian theory of cosmology with a spatially homogeneous scalar field.

## 4 Solution of the Hamiltonian System

In order to find a solution for  $A(T)$  and  $E(T)$  we must first determine their symplectic relationship. Integrating  $E^{bj}(\mathbf{y})$  over  $\Sigma$  in (2)<sup>§</sup> and substituting our homogeneous variables (16) gives the Poisson bracket

$$\{A, R^3 l_p E\} = 1 \quad (24)$$

where we recognize  $R^3 l_p E$  as the conjugate momentum to  $A$ .

### 4.1 Derivation of the Hamilton-Jacobi Function

We now proceed to solve our model using Hamilton-Jacobi (HJ) theory. We search for a Hamilton-Jacobi function  $S(A, T)$  such that<sup>¶</sup>

$$E = \frac{1}{R^3 l_p} \frac{\partial S}{\partial A}, \quad P = \frac{\partial S}{\partial T} \quad (25)$$

where the normalization of  $E$  is due to the relationship (24). Substituting these into equation (15) using the Hamiltonian (23) gives the HJ equation

$$\frac{\partial S}{\partial T} = \frac{1}{l_p} \sqrt{12 \left( \frac{\partial S}{\partial A} \right)^2 \left( A^2 - \frac{l_p V(T)}{3 R^3} \frac{\partial S}{\partial A} \right)}. \quad (26)$$

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<sup>§</sup> $E^{bj}$  has spatial density weight one and so can be integrated over the spatial manifold.

<sup>¶</sup>Other approaches to inflation which involve solutions to the Hamilton-Jacobi equations, in the “old” canonical variables, are described in [17].

A function  $S_{cs}(A, T)$  will have zero energy if it satisfies

$$\frac{\partial S_{cs}}{\partial A} = \frac{3R^3}{l_p V(T)} A^2 \quad (27)$$

so that  $H(A, \frac{\partial S_{cs}}{\partial A}, T) = 0$ . This implies

$$S_{cs}(A, T) = \frac{R^3}{l_p V(T)} A^3. \quad (28)$$

However, this does not solve the Einstein equations because

$$P = \frac{\partial S_{cs}}{\partial T} = -\frac{\dot{V}}{V} S_{cs} \neq 0 \quad (29)$$

in general.

We pause in our derivation to note that (28) is related to the Kodama solution of the full quantum theory [9], because  $S_{cs}$  is proportional to the Chern-Simons invariant

$$\int Y_{CS}(A) = \int \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = iR^3 A^3 \quad (30)$$

where the last equality comes from using the homogeneous variables (16).

We can understand why  $S_{cs}$  is not a solution to our model. Were  $V(T)$  constant so that  $\Lambda = V$  were the cosmological constant, (28) would be the Hamilton-Jacobi function for deSitter spacetime (see [14]). Were this the case, we would have to have  $\pi = 0$  so that the scalar field contributed no kinetic energy, but only the constant potential energy. As discussed above, this would violate our gauge condition (7).

The deviation from deSitter spacetime is then given by terms proportional to the ratio  $r = \pi^2/2V$ . This is proportional to the “slow roll parameter”

$$\eta = l_p \frac{\dot{V}}{V}. \quad (31)$$

When  $r$  and  $\eta$  are small, the kinetic energy of the scalar field is small compared to its potential energy.

To get a solution to our model, which requires  $\pi \neq 0$ , we need to modify deSitter spacetime by terms proportional to  $r$  and  $\eta$ . To do this, we modify the HJ function (28), which gives deSitter spacetime, by adding a new dimensionless function of time,  $u(T)$ , so that,

$$S_u(A, T) = \frac{R^3 A^3}{l_p V} (1 + u(T)). \quad (32)$$

We expect to find that  $u(T)$  scales with  $r$  and  $\eta$ . We plug this into the Hamilton-Jacobi equation, (26,) and we find an equation for  $u(T)$

$$\dot{u} = \frac{\dot{V}}{V} (1 + u) + \frac{6i}{l_p} (1 + u) \sqrt{3u}. \quad (33)$$

We note that the  $A$  decouples, as each term in (26) is proportional to  $A^3$ . The result is that for every solution to (33) we get a cosmological model.

Before looking at the consequences of our solution, we first show that, for  $u \neq 0$ , the spacetime deviates from deSitter spacetime. Recall that deSitter spacetime is the unique Lorentzian solution which satisfies the self-dual condition

$$J_{ab}^i = F_{ab}^i + \frac{l_p^2 \Lambda}{3} \epsilon_{abc} E^{ci} = 0 \quad (34)$$

where the proportionality can be taken to define  $\Lambda$ . In the homogeneous case we define  $J$  such that  $J_{ab}^i = J \epsilon_{ab}^i$  so the self-dual condition reads

$$J = -A^2 + \frac{l_p^2 \Lambda}{3} E = 0. \quad (35)$$

For solutions generated by (32), taking  $V(T) \equiv \Lambda$ , we have

$$J = uA^2 \quad (36)$$

which, for non-trivial spacetimes, only vanishes if  $u(T) \equiv 0$ .

## 4.2 Analysis of the Solutions

Given our HJ equation (32) we can calculate the lapse function (9) to be

$$N = -i \frac{V l_p}{6A^3(1+u)\sqrt{3u}}. \quad (37)$$

Introducing the integration constant  $\alpha$  that one obtains from integrating equation (33) and a further integration constant  $\beta$  (both dimensionless) we can derive a relationship between  $A$  and  $T$  by

$$\frac{\partial S_u}{\partial \alpha} = \beta. \quad (38)$$

This leads to the equation of motion

$$A(T) = \left[ \frac{l_p V(T) \beta}{R^3} \left( \frac{\partial u}{\partial \alpha} \right)^{-1} \right]^{\frac{1}{3}}. \quad (39)$$

In practice, the values of  $\alpha$  and  $\beta$  are determined by the initial  $A(T_0)$  and  $E(T_0)$ . Note that the partial derivative of  $u$  will, in general, add further  $T$  dependence to  $A(T)$ . We can then use equation (39) to derive the  $T$  dependence of the conjugate momentum to be

$$E(T) = \frac{3\beta^{2/3}(1+u)}{R^2 l_p^{4/3} V^{1/3}} \left( \frac{\partial u}{\partial \alpha} \right)^{-\frac{2}{3}} \quad (40)$$

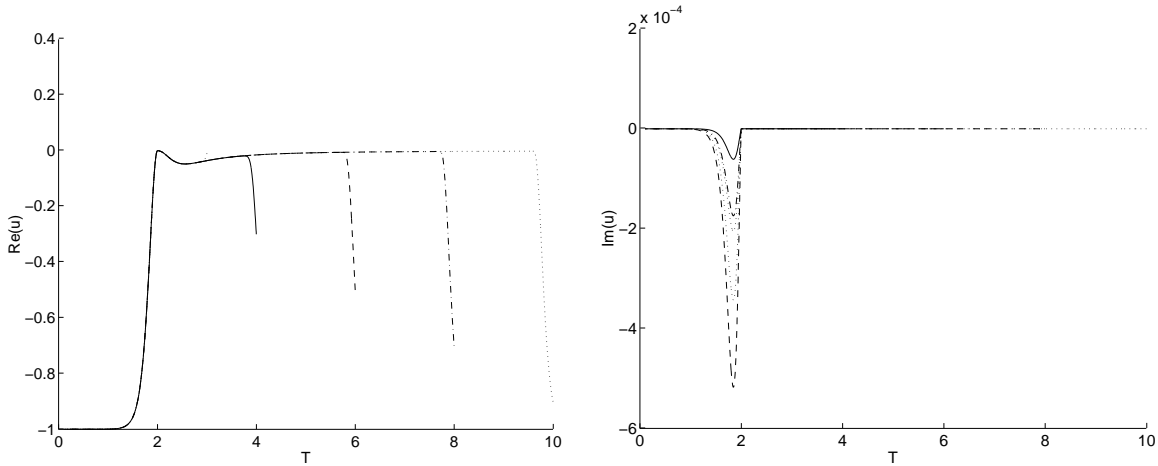


Figure 1: Numerical solutions of  $u(T)$  using the potential (42) for  $\lambda = 1$ ,  $m = 2$  and  $V_{min} = 5$  and various initial conditions  $T_0, u(T_0)$  satisfying  $T_0 > 2$  and  $-1 < u(T_0) < 0$ . The real part of  $u$  is plotted on the left while the imaginary part is on the right and the different line types on the left and right correspond to the same solution. The time variable  $T$  is in units of  $l_p/c$ . Recall that *forward* time corresponds to  $T$  evolving in the *negative* direction.

Given the form of the metric in these coordinates (19) we see that the  $TT$  component of the metric is given by  $-N^2$ . In order for the metric to remain real and Lorentzian, we must have that  $N^2$  be positive and real. Looking at (37) we see that this requires that  $u(T)$  be negative and real. Furthermore, for the metric to remain Lorentzian we require that  $E > 0$ . Equation (40) then requires that  $u > -1$ . Hence, our gauge condition will break down unless

$$-1 < u(T) < 0. \quad (41)$$

This then places a strong restriction on the value of  $u$ .

In order to get a sense of the behaviour of  $u(T)$ , we have solved (33) numerically for the quartic potential

$$V(T) = \lambda (T^2 - m^2)^2 + V_{min} \quad (42)$$

for several initial conditions  $u(T_0)$  consistent with the restrictions discussed above. Potentials of this sort are renormalizable and satisfies the slow roll condition.

Unless otherwise noted, we always take  $T_0 > m$  so that  $T = l_p^2 \phi$  proceeding in the negative direction corresponds to  $\phi$  “rolling down the hill”. These solutions can be seen in Figure 1 for  $\lambda = 1$ ,  $m = 2$  and  $V_{min} = 5$ . These plots show the generic behaviour of  $u$  for a wide range of parameters in (42).

The most noticeable characteristic is the apparent attractor in Figure 1. That is, all initial conditions satisfying our physical condition (41) merge to the same solution for some  $T < T_0$ .

The significance of this apparent robustness to initial conditions suggests a universality in the phenomena of inflation, which should be further investigated.

Furthermore, in Figure 1 we see that, for  $T > m$ , all of our solutions remain within the bounds  $-1 < u < 0$ . However, for  $T < m$ ,  $u$  quickly becomes positive and takes on an imaginary component and the metric becomes unphysical.  $T = m$  corresponds to the scalar field having its minimum potential energy and is traditionally the end of the inflationary period. Hence, it is not surprising that our gauge condition should break down at this point, as discussed in section 2.1. This break-down of the gauge condition at the minimum of the inflaton potential is a common feature for all of the different parameters in (42) that were attempted. Of note is the case where  $V_{min} = 0$  in which the evolution equation (33) becomes singular for  $T = m$ .

While we defer a full numerical analysis of our model to a later time, we briefly discuss the behavior of  $u$  for “unphysical” initial conditions that do not satisfy (41). It is clear from (33) that an initial condition  $u(T_0) > 0$  forces  $u$  to have an imaginary component and hence give an unphysical metric. For all of the initial conditions  $u(T_0) < -1$  that were attempted we found that  $u$  rapidly diverged to  $-\infty$ . Hence, the physical conditions imposed on  $u$  seem to be reflected in its functional behavior governed by (33).

## 5 Quantum Mini-Superspace

We now proceed to build a quantum theory from our classical Hamiltonian theory and find a full solution to the resulting Schrödinger equation.

First, we take quantum states to be functions  $\Psi(A, T)$ . We then define  $\hat{A}$  to be a multiplicative operator and define the operators

$$\hat{P}\Psi = i\hbar \frac{\partial \Psi}{\partial T} \quad \hat{E}\Psi = \frac{i\hbar}{l_p R^3} \frac{\partial \Psi}{\partial A}. \quad (43)$$

Again, the normalization of  $\hat{E}$  stems from the relation (24). Herein, we will work in units such that  $\hbar = 1$ .

The evolution equation becomes a time dependent Schrödinger equation,

$$i \frac{\partial \Psi}{\partial T} = \hat{H} \Psi \quad (44)$$

where we choose the ordering of the quantum Hamiltonian to be

$$\hat{H} = \frac{\sqrt{12}i}{l_p} \frac{\partial}{\partial A} \left( A \sqrt{\hat{J}} \right) \quad (45)$$

where we define  $\hat{J}$  as

$$\hat{J} := 1 - \frac{iVl_p}{3R^3 A^2} \frac{\partial}{\partial A}. \quad (46)$$

We then take a semi-classical quantum state

$$\Psi_u(A, T) = e^{-iS_u(A, T)} = e^{-i\frac{A^3 R^3}{V l_p^3}(1+u)} \quad (47)$$

and note that it is an eigenfunction of  $\hat{J}$  with a time dependent eigenvalue  $-u(T)$

$$\hat{J}\Psi_u = -u(T)\Psi_u \quad (48)$$

which implies

$$\sqrt{\hat{J}}\Psi_u = i\sqrt{u(T)}\Psi_u. \quad (49)$$

Hence, the action of the Hamiltonian on our semi-classical state is

$$\hat{H}\Psi_u = \left[ \frac{6iA^3 R^3(1+u)\sqrt{3u}}{V l_p^2} - \frac{\sqrt{12u}}{l_p} \right] \Psi_u. \quad (50)$$

We now show that  $\Psi_u$  is, indeed, an approximate solution to the Schrödinger equation (44) by computing the time derivative

$$\frac{\partial \Psi_u}{\partial T} = \frac{6A^3 R^3(1+u)\sqrt{3u}}{V l_p^2} \Psi_u. \quad (51)$$

where we have used the fact that  $u(T)$  is a solution of (33). Comparing (50) and (51) we see that, provided

$$\frac{3A^3 R^3(1+u)}{V l_p} \gg 1, \quad (52)$$

our semi-classical state (47) is indeed an approximate solution to the Schrödinger equation.

To find a full solution to (44), we take the ansatz

$$\Psi(A, T) = \Psi_u(A, T)\chi(T) \quad (53)$$

where  $\chi$  is an arbitrary function of  $T$  alone. Substitution of (53) into the Schrödinger equation yields the equation

$$\frac{\partial \chi}{\partial T} = \frac{\sqrt{12u}}{l_p} \chi \quad (54)$$

which can be immediately integrated to give

$$\chi(T) = e^{\frac{\sqrt{12}}{l_p} \int_{T_0}^T \sqrt{u(t)} dt}. \quad (55)$$

This yields

$$\Psi(A, T) = e^{\frac{i\sqrt{12}}{l_p} \int_{T_0}^T \sqrt{-u(t)} dt - i\frac{A^3 R^3}{V l_p^3}(1+u)} \quad (56)$$

which is a full solution to the Schrödinger equation (44). We have factored out an  $i$  from the  $\sqrt{u(t)}$  in the integral to make manifest the pure phase character of the wave function for solutions of  $u$  that satisfy the physical condition (41).

To summarize, given a solution of (33) we have found a complete solution (56) to the quantum theory of a homogeneous cosmology coupled to a scalar field in a potential.

## 5.1 Wavepackets and Normalizable States

Finally, we describe the physical inner product and show how to construct exact, normalizable, quantum states of the universe. In quantum gravity, the physical inner product is determined by the reality conditions for physical observables. In the present case, in which all gauge degrees of freedom are fixed by either gauge conditions or the reduction to homogeneous, isotropic solutions,  $A(T)$  and  $E(T)$  are physical degrees of freedom, and they are indeed real. In the representation we are using in which states are functionals of  $A$  and  $T$  and the latter is the time coordinate, the physical inner product is hence,

$$\langle \Psi(T) | \Psi(T) \rangle = \int dA |\Psi(A, T)|^2. \quad (57)$$

Our exact solutions (56) are phases, so long as  $u(T)$  is real and negative, corresponding to real solutions to the classical Einstein equations. Hence, with this restriction the solutions are all delta function normalizable.

Following the usual procedure in quantum theory, we can construct normalizable solutions by constructing wavepackets. We may note that at a given  $T$ , different classical solutions, with different values of  $u(T)$  correspond to cases in which the scalar field  $\phi \approx T$  is moving at different rates. Thus, while  $V(T)$  is fixed at the same  $T$ ,  $\pi(T)$  can vary, leading to different ratios of  $\pi(T)^2/V(T)$ . However, quantum mechanically we would expect that this ratio would not be a sharp observable. Thus the quantum state of the expanding universe should not physically be built from a single solution to the classical equations,  $u(T)$ . As there is no reason to expect that the initial conditions for  $u(T)$  are fixed classically in the very early universe, we should expect the inflating universe to be described by a wavepacket corresponding to superposing over solutions that differ in the value of  $u(T)$  at fixed  $T$ .

To do this, let us fix  $T = T_0$ , and consider initial values for  $u(T_0)$ . To construct a wavepacket we consider a central value  $u_0$  and define  $v = u(T_0) - u_0$ . We then label

$$\Psi(A, T_0)_v = e^{-i \frac{A^3 R^3}{V(T_0) l_p^3} (1+u_0+v)} \quad (58)$$

Given a normalizable function  $f(v)$  we may then define an exact solution determined by the initial conditions

$$\Psi(A, T_0)_f = \int dv f(v) \Psi(A, T_0)_v. \quad (59)$$

By a suitable choice of  $f(v)$ , with support in the physical interval,  $u = u_0 + v \in (-1, 0)$ , the initial state is normalizable.

Let us then define  $\mathcal{E}(T, v) = \int_{T_0}^T \sqrt{-u(t)} dt$  with initial condition  $u(T_0) = u_0 + v$  and  $u_v(T)$  to be the value of  $u$  at time  $T$  with the same initial condition. Then the wavepacket at later time is given by

$$\Psi(A, T)_f = \int dv f(v) e^{i \frac{\sqrt{12}}{l_p} \mathcal{E}(T, v) - i \frac{A^3 R^3}{V l_p^3} (1+u_v(T))}. \quad (60)$$



## 6 Conclusions and Discussion

The results of this paper represent a step towards a detailed study of the very early universe beyond the semiclassical approximation, in which quantum gravitational effects are treated in a non-perturbative and background independent manner. For each potential  $V(\phi)$  and classical slow roll solution  $u(T)$  consistent with inflation, we have found a quantum state given by (56) which is an exact solution to the quantum equations of motion, but has a classical limit given by that classical solution. Furthermore, we can construct normalizable states which are wavepackets around the initial conditions that generate that classical solution. Thus, inflation is here described in terms of exact quantum states.

As a byproduct, the simplicity of the Hamilton-Jacobi solutions to the coupled Einstein-scalar field problem, using the Ashtekar formulation, may provide a new, simplified approach to studying inflation classically. In our first investigation of the problem we found an attractor, suggesting, at least in this case, a possible universality in the dynamics of the very early universe. This deserves more investigation, as it may provide an understanding of the hypothesis of chaotic inflation[5].

A number of very interesting questions remain, which these results suggest can now be approached.

- It would be very interesting to understand the relationship of these results to those of Bojowald and others, in the context of loop quantum cosmology[8]. In that case a similar reduction is used, and solutions to quantum cosmology are found which are exact and non-perturbative. However Bojowald's results obtained using a representation conjugate to that used here, which is roughly a dimensionally reduced spin network basis. A number of interesting results are obtained, including indications that the initial singularity is removed. It would then be very useful to establish a homogeneous version of the loop transform, to express the states studied here in Bojowald's representation.
- The ordering of the Hamiltonian used here is not Hermitian. It is easy in the dimensional reduction to find Hermitian orderings for the Hamiltonian. The exact quantum states found here will solve the Hermitian ordering of the Schrodinger equation at the semiclassical approximation. It is challenging, but not impossible, that exact results can be obtained also for the case of Hermitian ordering. If not, at least a semiclassical expansion could be constructed that would be reliable above the Planck scale, whose leading order terms would be given by the states found here.
- It will be also interesting to incorporate the inhomogeneous modes of the gravitational and matter fields by means of a perturbative expansion around the states constructed here. The aim here will be first principles predictions for transplankian effects in the spectra of scalar and tensor perturbations, as well as polarizations, detectible in cosmological observations.
- We note that the validity of our gauge condition ends roughly when inflation ends, as surfaces of constant  $\phi$  no longer track surfaces of constant scale factor, once the

universe enters the stage where, in terms of the latter, the scalar field oscillates around the minimum of  $V(\phi)$ . We see that at this point in the classical dynamics,  $u(T)$  becomes complex, leading to complex values of the spacetime metric. To study the problem of exiting from inflation it will then be necessary to choose another time parameter, which is good throughout the exit from inflation, and use the wavefunction generated here as initial conditions for evolution in that parameter.

- There exist extensions of the Kodama state to supergravity with  $N = 1, 2$  [18]. It is then likely that the results of the present paper can be extended to supersymmetric models of inflation.
- It will be interesting to investigate whether the mechanism used here to construct normalizable states will work in the full theory, perhaps resolving the issue of the normalizability of the Kodama state.

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