BUSCH'S THEOREM FOR MAPPINGS

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Abstract

For rotation-invariant Hamiltonian systems, canonical angular momentum is conserved. In beam optics, this statement is known as Busch's theorem. This theorem can be generalized to symplectic mappings; two generalizations are presented in this paper. The first one states that a group of rotation-invariant mappings is identical to a group of the angular-momentum preserving mappings, assuming both of them symplectic and linear. The second generalization of Busch's theorem claims that for any laminar beam which rotation symmetry happened to be preserved, an absolute value of the angular momentum of any particle from this beam is preserved as well; the linear symplectic mapping does not have to be rotation-invariant here.

1 INTRODUCTION

When a beam optics consists of non-dissipative axial-invariant elements, such as drifts, solenoids, round electrostatic lenses or axisymmetric RF fields, the canonical angular momentum (CAM) of any particle $M = xp_y - yp_x$ is preserved; here (x, p_x) and (y, p_y) are horizontal and vertical canonically conjugated pairs. This statement follows from the Hamilton equations applied to the canonical angular variables M , θ and is known as Busch's theorem (see, e. g. [1], p. 34). This theorem is extremely useful for such beam optics, referred to as *local-invariant* [2]. Local-invariant optics continuously preserves the CAM and beam axial symmetry.

However, the beam symmetry might also be preserved after a mapping which does not correspond to any sequence of the axial-invariant elements. If the mapping is rotation-invariant, or commute with rotations, it preserves the beam axial symmetry. This kind of mapping, though, can be constructed on a basis of non-invariant elements as dipoles, quadrupoles, non-symmetric RF fields, etc. The Busch's theorem says nothing about the CAM preservation by this *global-invariant* mapping. Thus, the question appears, whether the CAM is preserved by rotation-invariant symplectic mappings? (Here, only symplectic mappings are considered. A mapping is symplectic if particle motion is Hamiltonian, see e. g. Ref. [3] p. 51; also in the next section.) For linear transformations, this question is treated in the next section, and the positive answer is found. It is proved there that the group of rotation-invariant mappings is identical to the group of the CAM-preserving mappings, assuming both of these groups linear and symplectic.

However, the beam round symmetry might be restored even by non-invariant mapping. The first example is a mirror reflection. This kind of symplectic mappings does not commute with rotations, but preserves axial symmetry of any rotation-invariant initial beam distribution. Reflections do not preserve the CAM: they reverse its sign, but they still do preserve its absolute value. The second example shows that a mapping can preserve axial symmetry for a particular beam, but do not preserve for an arbitrary round beam. Indeed, imagine a symplectic $x-y$-uncoupled mapping, which acts vertically as the identity, and horizontally as a drift. Generally, this transformation does not preserve the beam symmetry: initially round beam is not round after that mapping. However, a round beam with zero momenta $p_x = p_y = 0$ is transformed into round beam again by this mapping. Thus, the beam rotation symmetry can be preserved even by a non-invariant mapping, either for any initial beam distribution (reflections) or for some distributions special for that mapping. Let it to be assumed now that a particular initially round beam distribution is mapped onto a round beam again. It is clear from the first example with reflections, that the sign of the CAM can be reversed without any damage to the beam axial symmetry. However, it can be suspected that if the beam symmetry is restored by a symplectic mapping, the absolute value of the CAM for every particle of this beam is restored as well. For laminar beams mooving in electro- and magneto-static fields, this statement is proven in Ref. [4]. For laminar beams and any kind of linear symplectic mappings, this statement is found to be true in the section 3.

2 MAPPING INVARIANCE AND ANGULAR MOMENTUM PRESERVATION

Group of rotations in the transverse plane through angles θ can be presented by matrices

$$\mathcal{R}(\theta) = \begin{pmatrix} \mathsf{cI} & \mathsf{sI} \\ -\mathsf{sI} & \mathsf{cI} \end{pmatrix} \tag{1}$$

with $\mathsf{c} = \cos\theta\,,\ \mathsf{s} = \sin\theta$ and I as the 2×2 identity matrix. Rotation invariance of a transformation $\mathcal{T}$ means that it commutes with the rotations:

$$\mathcal{R}\cdot\mathcal{T} - \mathcal{T}\cdot\mathcal{R} = 0\,. \tag{2}$$

This condition is equivalent to its particular case of an infinitesimal rotation by an angle $d\theta$ when

$$\mathcal{R} = \mathcal{I} + \mathcal{G}\cdot d\theta\,;\ \ \mathcal{G} = \begin{pmatrix} 0 & \mathsf{I} \\ -\mathsf{I} & 0 \end{pmatrix}\,;\ \ \mathcal{G}^2 = -\mathcal{I} \tag{3}$$

where $\mathcal{I}$ and I are 4×4 and 2×2 identity matrices correspondingly. Then, the invariance condition reduces to a

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commutation of the mapping $\mathcal{T}$ with the infinitesimal operator $\mathcal{G}$

$$\mathcal{G} \cdot \mathcal{T} - \mathcal{T} \cdot \mathcal{G} = 0\,. \tag{4}$$

The mapping $\mathcal{T}$ is assumed to be symplectic:

$$\mathcal{T}^T \mathcal{S} \mathcal{T} = \mathcal{S}\,, \tag{5}$$

where

$$\mathcal{S} = \begin{pmatrix} \mathsf{J} & 0 \\ 0 & \mathsf{J} \end{pmatrix} \; ; \; \mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \; \mathcal{S}^2 = -\mathcal{I} \tag{6}$$

is the symplectic unit matrix, $\mathcal{I}$ is the 4×4 identity matrix and the superscript ^{T} stands for the transposing.

It can be shown now that symplectic invariant transformations $\mathcal{T}$ preserve the CAM

$$M \equiv x p_y - y p_x \equiv \frac{1}{2} \mathbf{x}^T \cdot \mathcal{L} \cdot \mathbf{x} \tag{7}$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & \mathsf{J} \\ -\mathsf{J} & 0 \end{pmatrix} \; ; \; \mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \; \mathcal{L}^2 = \mathcal{I}\,. \tag{8}$$

Note that the CAM matrix $\mathcal{L}$ is rotation-invariant:

$$\mathcal{G} \cdot \mathcal{L} - \mathcal{L} \cdot \mathcal{G} = 0\,. \tag{9}$$

In terms of its matrix $\mathcal{L}$, CAM preservation at the mapping $\mathcal{T}$ can be expressed as

$$\mathcal{T}^T \mathcal{L} \mathcal{T} = \mathcal{L}. \tag{10}$$

To prove that this is true when conditions (5, 4) are provided, it is convenient to use the relation between the infinitesimal operator $\mathcal{G}$, the symplectic unit matrix $\mathcal{S}$, and the CAM matrix $\mathcal{L}$:

$$\mathcal{S} \cdot \mathcal{L} = \mathcal{L} \cdot \mathcal{S} = -\mathcal{G}\,. \tag{11}$$

which is straightforward to prove. It means that the matrices $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{G}$ form an algebra: any their product returns one of them. From (9, 11) the symplecticity matrix can be presented as

$$\mathcal{S} = -\mathcal{L} \cdot \mathcal{G}\,. \tag{12}$$

Being substituted in the symplecticity condition (5), after the commutation (4), it leads to the CAM preservation (10). Thus, the invariant transformations preserve the CAM.

The reverse statement can be proven as well: if a symplectic mapping preserves the CAM of any initial state, it is rotationally invariant. Indeed, with the matrix $\mathcal{T}^T$ expressed from the symplecticity condition (5) and substituted in the CAM preservation (10), it leads to what can be seen as the invariance property (4) when Eq. (11) is used. Thus, mapping invariance gives rise to CAM preservation and vice versa, so these properties are absolutely equivalent.

A general form of the CAM-preserving matrices was found by E. Pozdeev [5] and E. Perevedentsev [6] (see details in [7]):

$$\mathcal{T} = \begin{pmatrix} \mathsf{T} \cdot \cos\theta & \mathsf{T} \cdot \sin\theta \\ -\mathsf{T} \cdot \sin\theta & \mathsf{T} \cdot \cos\theta \end{pmatrix} \equiv \mathcal{R}(\theta) \begin{pmatrix} \mathsf{T} & 0 \\ 0 & \mathsf{T}\,. \end{pmatrix} \tag{13}$$

Merged with the reflections, it leads to a wider group:

$$\mathcal{T} = \mathcal{R}(\theta) \begin{pmatrix} \mathsf{T} & 0 \\ 0 & \pm\mathsf{T}\,. \end{pmatrix} \tag{14}$$

As it is shown in the next section, this group includes all the mappings that preserve the axial symmetry for any initial round beam distribution.

3 BEAM SYMMETRY AND ANGULAR MOMENTUM PRESERVATION

As it was discussed in the section 1, the round symmetry of a particular beam might be preserved by a non-invariant mapping as well. The question is, whether the CAM is preserved together with the beam symmetry even in this case?

For hydrodynamic, or laminar beams, a positive answer to this question follows from the generalized Busch's theorem [4]. The theorem states that when such a beam is transported by means of arbitrary static electric and magnetic fields, the contour integral

$$\oint_\Gamma \vec{p} d\vec{l} = \oint_\Gamma \vec{k} d\vec{l} - e\Phi/c \tag{15}$$

is conserved. Here the contour Γ bounds an arbitrary tube of trajectories in the 3D coordinate space x, y, z and Φ is the magnetic flux through the contour. If the initial and final beam states are rotationally invariant, the contour Γ is a circumference in the transverse plane, and the CAM preservation follows. Note that the field linearity is not required here.

Below, this theorem is extended from the electro- and magneto-static fields to arbitrary Hamiltonian systems. This extension, however, requires to assume the linearity of the transformation. Thus, the statement to be proved claims following: if a particular laminar round beam is transformed by a symplectic linear mapping into a round state again, the CAM of every particle is restored.

To prove this, a property of the symplectic transformations to conserve skew-scalar products can be used (see e. g. [8]). The skew-scalar product of two vectors in the 4D transverse phase space $\mathbf{x_1} = (x_1, p_{x1}, y_1, p_{y1})$ and $\mathbf{x_2} = (x_2, p_{x2}, y_2, p_{y2})$ is an antisymmetric bilinear form $[\mathbf{x_1}, \mathbf{x_2}]$. Expressed in terms of the usual scalar product, it can be written as $[\mathbf{x_1}, \mathbf{x_2}] = (\mathbf{x_1}, \mathcal{S}\mathbf{x_2})$ with $\mathcal{S}$ as a rotation by 90^o in each of the phase planes, or

$$[\mathbf{x_1}, \mathbf{x_2}] = -x_1 p_{x2} - y_1 p_{y2} + x_2 p_{x1} + y_2 p_{y1}.$$

Let $\mathbf{x_{1i}}$ and $\mathbf{x_{2i}}$ be two arbitrary vectors of the initial state finally transformed into $\mathbf{x_{1f}}$ and $\mathbf{x_{2f}}$. Due to the symplecticity,

$$[\mathbf{x_{1i}}, \mathbf{x_{2i}}] = [\mathbf{x_{1f}}, \mathbf{x_{2f}}] \tag{16}$$

for any choice of $\mathbf{x_1}$ and $\mathbf{x_2}$. It can be seen that for laminar beams, the angles between their 2D $x - y$ components are conserved by the transformation. This property is an obvious consequence of the rotation invariance of the both states: without it, there would be an angular asymmetry of the final beam density distribution. However, the sign of this angle can be changed that would not contradict the angular symmetry of the final state. The two initial vectors can be taken as 2D-orthogonal:

$$\begin{aligned} \mathbf{x_{1i}} = &\quad (r_i, p_{ir}, 0, p_{it}) \\ \mathbf{x_{2i}} = &\quad \mathbf{\tilde{x}_{1i}} \equiv (0, -p_{it}, r_i, p_{ir}) \end{aligned} \tag{17}$$

having the angular momentum $M_i = r_i p_{it}$ where r_i is the initial beam radius. Because of the angle conservation, these two vectors are 2D-orthogonal again after the transformation. Without a lack of generality, the x-axis can be assumed to go along the vector $\vec{x}_1$ both for the initial and the final states; this follows from symplecticity of the rotations. So the final states can be presented as

$$\begin{aligned} \mathbf{x_{1f}} = &\quad (r_f, p_{fr}, 0, p_{ft}) \\ \mathbf{x_{2f}} = &\quad \pm\mathbf{\tilde{x}_{1f}} \equiv \pm(0, -p_{ft}, r_f, p_{fr}) \end{aligned} \tag{18}$$

with $M_f = \pm r_f p_{ft}$ as the final angular momentum.

In fact, the symplecticity condition (16) for a given vector $\mathbf{x_1}$ and arbitrary $\mathbf{x_2}$ is equivalent to the particular choice (17, 18). Indeed, for a given $\mathbf{x_1}$, any $\mathbf{x_2}$ can be expanded over the two orthogonal vectors: $\mathbf{x_1}$ and its orthogonal counterpart $\mathbf{\tilde{x}_1}$. Then, the part of $\mathbf{x_2}$ parallel to $\mathbf{x_1}$ gives an identical zero for the both sides of the symplecticity condition (16), while the component along $\mathbf{\tilde{x}_1}$ gives the same result as (17).

Conservation of the skew-scalar product

$$[\mathbf{x_{1i}}, \mathbf{x_{2i}}] = [\mathbf{x_{1f}}, \mathbf{x_{2f}}]$$

for the orthogonal pare $\mathbf{x_1}$, $\mathbf{x_2}$ immediately yields

$$M_i = \pm M_f \tag{19}$$

as was to be shown.

Thus, no transformation can change an absolute value of the canonical angular momentum of a particle without breaking the rotational symmetry of the laminar beam, which this particle belongs to.

Actually, the statement just been proven means that the property of the canonical momentum conservation goes beyond the mapping (or Hamiltonian) invariance. For the invariant mappings, any initially symmetric state of beam transforms into a symmetric state again. It was shown above that the mapping invariance does not follow from the fact that one particular symmetric state was eventually transformed into other, also symmetric, state, which property can be referred to as the projective invariance. It was proved in fact that the mapping invariance is a somewhat surplus requirement for the momentum conservation; the projective invariance for a laminar beam is sufficient to claim that every particle of this beam restores the absolute value of its CAM.

It follows from above that if a mapping preserves rotation symmetry of any initial beam, than, it preserves the absolute value of the CAM by itself; thus such a mapping has a form of Eq. (14).

4 SUMMARY

Two statements are proved above, showing a deep relation between rotation symmetry and angular momentum preservation. First, it is proved that a group of rotation-invariant mappings is identical to a group of the angular-momentum preserving mappings. Second, it is shown that if a laminar round beam is symplectically mapped onto a round beam again, an absolute value of the canonical angular momentum for any particle of this beam is preserved - even if the mapping is not rotation-invariant.

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