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Benjamin–Bona–Mahony Equation with Variable Coefficients: Conservation Laws

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Abstract: This paper aims to construct conservation laws for a Benjamin–Bona–Mahony equation with variable coefficients, which is a third-order partial differential equation. This equation does not have a Lagrangian and so we transform it to a fourth-order partial differential equation, which has a Lagrangian. The Noether approach is then employed to construct the conservation laws. It so happens that the derived conserved quantities fail to satisfy the divergence criterion and so one needs to make adjustments to the derived conserved quantities in order to satisfy the divergence condition. The conservation laws are then expressed in the original variable. Finally, a conservation law is used to obtain exact solution of a special case of the Benjamin–Bona–Mahony equation.

Keywords: Benjamin–Bona–Mahony equation with variable coefficients; lagrangian; noether operators; conservation laws

1. Introduction

The Benjamin–Bona–Mahony equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \quad (1)$$

was, for the first time, studied by Benjamin *et al.* [1]. It is also known as the regularized long-wave equation and is applicable to shallow water waves and to the study of drift waves in plasma or the Rossby waves in rotating fluids. See, for example, [2] and references therein. It should be noted, however, that the BBM equation was also derived by Peregrine [3].

In the last two decades, various versions of the Benjamin–Bona–Mahony equation have been investigated in the literature. A general form of the Benjamin–Bona–Mahony equation is [2]

$$u_t + \alpha u_x + \beta uu_x + \delta u_{xxt} = 0 \quad (2)$$

where α, β and δ are constants with the nonlinear and dispersion coefficients $\beta \neq 0$ and $\delta < 0$, respectively. For different values of the constants α, β and δ , Equation (2), results in various types of nonlinear equations which are very useful in the study of various physical phenomena. See [2] and references therein. However, in certain cases, the physical situations of the problem dictate us to consider nonlinear equations with variable coefficients [2,4–8]. Recently, in [2] the variable coefficients version of the Benjamin–Bona–Mahony equation

$$u_t + \alpha(t)u_x + \beta(t)uu_x - \delta(t)u_{xxt} = 0 \quad (3)$$

was investigated and some exact solutions were obtained using the classical Lie group method [9].

In this study, we consider the Equation (3), but with δ being an arbitrary constant, namely

$$u_t + \alpha(t)u_x + \beta(t)uu_x + \delta u_{xxt} = 0 \quad (4)$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions of t . The objective of the study is to classify the Noether operators and to construct conservation laws for the Equation (4).

Conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, momentum and so on. They play a very important role in the solution and reduction of partial differential equations. Conservation laws have been widely used to investigate the existence, uniqueness and stability of solutions of nonlinear partial differential equations. This can be seen in the references [10–12]. They have also been employed in the development and use of numerical methods (see for example, [13,14]). Recently, conserved vectors associated with Lie point symmetries have been used to find exact solutions of some partial differential equations [15] and systems of ordinary equations [16]. Noether theorem [17] gives us an elegant way to construct conservation laws provided a Lagrangian is known for an Euler–Lagrange equation. Thus, the knowledge of a Lagrangian is essential in this case.

It is worth mentioning that no Lagrangian exists for Equation (4) and as a result one can not invoke Noether theorem. However, an interesting approach is employed to construct conservation laws for Equation (4). It should be noted that the present approach, which we use here, fails to construct the conservation laws for Equation (3), *i.e.*, when δ is a function of t .

The paper is organized as follows. In Section 2 we briefly give the fundamental notations and relations concerning the Noether symmetry approach. Section 3 obtains the Noether operators and the corresponding conservation laws for the Equation (4). In Section 4, a conservation law is used to obtain exact solution of a special case of the Benjamin–Bona–Mahony equation. Concluding remarks are presented in Section 5.

2. Fundamental Notations and Relations

Here we present some vital features of Noether operators concerning partial differential equations. These results will be used in Section 3. The reader is referred to [17,18] for further details.

Consider the vector field

$$X = \tau(t, x, z) \frac{\partial}{\partial t} + \xi(t, x, z) \frac{\partial}{\partial x} + \eta(t, x, z) \frac{\partial}{\partial z} \quad (5)$$

which has the second-order prolongation $X^{[2]}$ given by

$$\begin{aligned} X^{[2]} = & \tau(t, x, z) \frac{\partial}{\partial t} + \xi(t, x, z) \frac{\partial}{\partial x} + \eta(t, x, z) \frac{\partial}{\partial z} + \zeta_t^1 \frac{\partial}{\partial z_t} + \\ & \zeta_x^2 \frac{\partial}{\partial z_x} + \zeta_{tt}^1 \frac{\partial}{\partial z_{tt}} + \zeta_{xx}^2 \frac{\partial}{\partial z_{xx}} + \zeta_{tx}^1 \frac{\partial}{\partial z_{tx}} + \cdots \end{aligned} \quad (6)$$

where the expressions for ζ_t^1 , ζ_x^2 , ζ_{tx}^1 and ζ_{xx}^2 are given in [5]. The Euler–Lagrange operator is defined by

$$\frac{\delta}{\delta z} = \frac{\partial}{\partial z} - D_t \frac{\partial}{\partial z_t} - D_x \frac{\partial}{\partial z_x} + D_t^2 \frac{\partial}{\partial z_{tt}} + D_x^2 \frac{\partial}{\partial z_{xx}} + D_x D_t \frac{\partial}{\partial z_{xt}} - \cdots \quad (7)$$

where the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + z_t \frac{\partial}{\partial z} + z_{tt} \frac{\partial}{\partial z_t} + z_{xt} \frac{\partial}{\partial z_x} + \cdots \quad (8)$$

$$D_x = \frac{\partial}{\partial x} + z_x \frac{\partial}{\partial z} + z_{xx} \frac{\partial}{\partial z_x} + z_{xt} \frac{\partial}{\partial z_t} + \cdots \quad (9)$$

Consider a partial differential equation of two independent variables, viz.,

$$E(t, x, z, z_x, z_t, z_{tt}, z_{xx}, \cdots) = 0 \quad (10)$$

which has a second-order Lagrangian L , i.e., Equation (10) is equivalent to the Euler–Lagrange equation

$$\frac{\delta L}{\delta z} = 0 \quad (11)$$

Definition 1. The vector field X , of the form Equation (5), is called a Noether operator corresponding to a second-order Lagrangian L of Equation (10) if

$$X^{[2]}(L) + \{D_t(\xi^1) + D_x(\xi^2)\}L = D_t(B^1) + D_x(B^2) \quad (12)$$

for some gauge functions $B^1(t, x, z)$ and $B^2(t, x, z)$.

We now recall the following theorem.

Theorem 1. (Noether [17]) *If X , as given in Equation (5), is a Noether point symmetry generator corresponding to a second-order Lagrangian L of Equation (10), then the vector $T = (T^1, T^2)$ with components*

$$T^1 = \tau L + W \frac{\delta L}{\delta z_t} + D_x(W) \frac{\delta L}{\delta z_{tx}} + D_t(W) \frac{\delta L}{\delta z_{tt}} - B^1 \quad (13)$$

$$T^2 = \xi L + W \frac{\delta L}{\delta z_x} + D_t(W) \frac{\delta L}{\delta z_{tx}} + D_x(W) \frac{\delta L}{\delta z_{xx}} - B^2 \quad (14)$$

is a conserved vector for Equation (10) associated with the operator X , where $W = \eta - z_t \tau - z_x \xi$ is the Lie characteristic function.

3. Conservation Laws of Equation (4)

Consider the Benjamin–Bona–Mahony Equation (4). Here, we note that Equation (4) does not admit a Lagrangian. Nevertheless, we can transform Equation (4) into a variational form by setting $u = z_x$. Thus, Equation (4) becomes a fourth-order equation, namely

$$z_{tx} + \alpha(t) z_{xx} + \beta(t) z_x z_{xx} + \delta z_{xxxx} = 0 \quad (15)$$

It can easily be verified that a second-order Lagrangian of the Equation (15) is

$$L = \frac{1}{2} \left\{ \delta z_{xx} z_{xt} - \alpha(t) z_x^2 - z_t z_x - \frac{1}{3} \beta(t) z_x^3 \right\} \quad (16)$$

The substitution of L from Equation (16) into Equation (12) and splitting with respect to the derivatives of z , yields an overdetermined system of linear PDEs

$$\tau_z = 0, \quad \eta_z = 0, \quad \xi_z = 0, \quad \tau_x = 0, \quad \xi_t = 0, \quad \xi_x = 0, \quad (17)$$

$$\eta_{xx} = 0, \quad \eta_{xt} = 0, \quad \beta'(t) \tau + \beta(t) \tau_t = 0, \quad (18)$$

$$\alpha'(t) \xi^1 + \beta(t) \eta_x + \alpha(t) \tau_t = 0, \quad (19)$$

$$-\frac{1}{2} \eta_t - \alpha(t) \eta_x = B_z^2, \quad -\frac{1}{2} \eta_x = B_z^1, \quad (20)$$

$$B_t^1 + B_x^2 = 0. \quad (21)$$

After some lengthy calculations, the solution of the above system yields

$$\tau = a(t) \quad (22)$$

$$\xi = c_1 \quad (23)$$

$$\eta = c_2 x + d(t) \quad (24)$$

$$B^1 = -\frac{1}{2} c_2 z + f(t, x) \quad (25)$$

$$B^2 = -\frac{1}{2} d' z - \alpha(t) c_2 z + e(t, x) \quad (26)$$

$$f_t + e_x = 0 \quad (27)$$

$$\beta'(t)a(t) + \beta(t)a'(t) = 0 \quad (28)$$

$$\alpha'(t)a(t) + \alpha(t)a'(t) + \beta(t)c_2 = 0 \quad (29)$$

The analysis of Equations (28) and (29) prompts the following four cases:

Case 1. $\alpha(t)$ and $\beta(t)$ arbitrary but not of the form contained in Cases 2–4.

In this case, we obtain two Noether point symmetries. These are given below together with their corresponding gauge functions:

$$X_1 = \frac{\partial}{\partial x}, B^1 = f, B^2 = e, f_t + e_x = 0 \quad (30)$$

$$X_2 = d(t)\frac{\partial}{\partial z}, B^1 = f, B^2 = -\frac{1}{2}d'(t)z + e, f_t + e_x = 0 \quad (31)$$

Invoking Theorem 1, and reverting back to the original variables, the two nontrivial conserved vectors associated with these two Noether point symmetries are, respectively,

$$T_1^1 = \frac{u^2}{2} + \frac{\delta}{2}uu_{xx} - \frac{\delta}{2}u_x^2 - f \quad (32)$$

$$T_1^2 = \frac{1}{2}\alpha(t)u^2 + \frac{1}{3}\beta(t)u^3 + \delta uu_{xt} - \frac{\delta}{2}u_x u_t - e \quad (33)$$

and

$$T_2^1 = -\frac{1}{2}d(t)u - \frac{\delta}{2}d(t)u_{xx} - f \quad (34)$$

$$T_2^2 = -\alpha(t)d(t)u - \frac{1}{2}d(t) \int u_t dx - \frac{1}{2}\beta(t)d(t)u^2 - \delta d(t)u_{xt} \\ + \frac{1}{2}\delta d'(t)u_x + \frac{1}{2}d'(t) \int u dx - e \quad (35)$$

Here it can be seen that the above conserved vectors do not satisfy the divergence condition, viz., $D_i T^i|_{(4)} = 0$, as some excessive terms emerge that require some further analysis. By making a slight adjustment to these terms, it can be shown that these terms can be absorbed into the divergence condition. For,

$$D_t(T_1^1) + D_x(T_1^2) = \frac{\delta}{2}uu_{xxt} - \frac{\delta}{2}u_x u_{xt} - f_t - e_x \\ = D_t\left(\frac{\delta}{2}uu_{xx} - f\right) - D_x\left(\frac{\delta}{2}u_x u_t + e\right) \quad (36)$$

hence

$$D_t\left(T_1^1 - \frac{\delta}{2}uu_{xx} + f\right) + D_x\left(T_1^2 + \frac{\delta}{2}u_x u_t + e\right) = 0. \quad (37)$$

We now redefine the conserved vectors in the parenthesis as:

$$\bar{T}_1^1 = T_1^1 - \frac{\delta}{2}uu_{xx} + f \\ = \frac{1}{2}u^2 - \frac{\delta}{2}u_x^2 \quad (38)$$

$$\bar{T}_1^2 = T_1^2 + \frac{\delta}{2}u_x u_t + e \\ = \frac{\alpha(t)}{2}u^2 + \frac{\beta(t)}{3}u^3 + \delta uu_{xt} \quad (39)$$

Thus, the modified conserved vectors \bar{T}_1^1 and \bar{T}_1^2 satisfy the divergence condition. We observe that the conserved vectors Equations (38) and (39) are local conserved vectors. Likewise, we can then construct the second pair of the conserved quantities associated with T_2^1 and T_2^2 as:

$$\bar{T}_2^1 = -\frac{d(t)}{2}u - \frac{\delta}{2}d(t)u_{xx} \quad (40)$$

$$\begin{aligned} \bar{T}_2^2 = & -\alpha(t)d(t)u - \frac{1}{2}d(t) \int u_t dx - \frac{\beta(t)}{2}d(t)u^2 - \frac{\delta}{2}d(t)u_{xt} + \frac{\delta}{2}d'(t)u_x \\ & + \frac{1}{2}d'(t) \int u dx \end{aligned} \quad (41)$$

Note that the conserved quantities Equations (40) and (41) are nonlocal conserved vectors, and since $d(t)$ is an arbitrary function of t , one obtains infinitely many nonlocal conserved vectors of Equation (4). A special case of Equations (40) and (41), when $d(t) = 1$, is

$$\bar{T}_2^1 = -\frac{u}{2} - \frac{\delta}{2}u_{xx} \quad (42)$$

$$\bar{T}_2^2 = -\alpha(t)u - \frac{1}{2} \int u_t dx - \frac{\beta(t)}{2}u^2 - \frac{\delta}{2}u_{xt} \quad (43)$$

Case 2. $\alpha(t) = \gamma$, $\beta(t) = \lambda$, where γ and λ are non-zero constants.

This case provides us with three Noether symmetry generators, namely, X_1 , X_2 given by the operators Equations (30) and (31) and X_3 given by

$$X_3 = \frac{\partial}{\partial t} \text{ with } B^1 = f, B^2 = e, f_t + e_x = 0 \quad (44)$$

The use of Noether conserved vector Equations (13) and (14) corresponding to X_3 yields

$$T_3^1 = -\frac{1}{2}\gamma u^2 - \frac{1}{6}\lambda u^3 + \frac{\delta}{2}u_{xx} \int u_t dx - f \quad (45)$$

$$\begin{aligned} T_3^2 = & \gamma u \int u_t dx + \frac{1}{2} \left[\int u_t dx \right]^2 + \frac{1}{2}\lambda u^2 \int u_t dx + \delta u_{xt} \int u_t dx \\ & - \frac{1}{2}\delta u_x \int u_{tt} dx - \frac{1}{2}\delta u_t^2 - e \end{aligned} \quad (46)$$

Again, the above conserved flow fails to satisfy the divergence criterion. Thus, by inheriting the same procedure as in Case 1, the nontrivial conserved flows associated with X_1 , X_2 and X_3 are

$$\bar{T}_1^1 = \frac{1}{2}u^2 - \frac{\delta}{2}u_x^2 \quad (47)$$

$$\bar{T}_1^2 = \frac{\gamma}{2}u^2 + \frac{\lambda}{3}u^3 + \delta u u_{xt} \quad (48)$$

$$\bar{T}_2^1 = -\frac{d(t)}{2}u - \frac{\delta}{2}d(t)u_{xx} \quad (49)$$

$$\begin{aligned} \bar{T}_2^2 = & -\gamma d(t)u - \frac{1}{2}d(t) \int u_t dx - \frac{\lambda}{2}d(t)u^2 - \frac{\delta}{2}d(t)u_{xt} \\ & + \frac{\delta}{2}d'(t)u_x + \frac{1}{2}d'(t) \int u dx \end{aligned} \quad (50)$$

and

$$\bar{T}_3^1 = -\frac{1}{2}\gamma u^2 - \frac{1}{6}\lambda u^3 \quad (51)$$

$$\begin{aligned} \bar{T}_3^2 = & \gamma u \int u_t dx + \frac{1}{2} \left[\int u_t dx \right]^2 + \frac{1}{2} \lambda u^2 \int u_t dx + \delta u_{xt} \int u_t dx \\ & - \frac{1}{2} \delta u_t^2 \end{aligned} \quad (52)$$

respectively. The conserved vector Equations (47) and (48) is a local conserved vector whereas the conserved vectors Equations (49) and (52) are nonlocal conserved quantities. It should be noted that one can use Equations (49) and (50) to construct infinitely many nonlocal conserved vectors.

Case 3. $\alpha(t) = \gamma e^{\sigma t}$, $\beta(t) = \lambda e^{\sigma t}$, where γ, σ and λ are nonzero constants.

Here we get three Noether point symmetries operators, viz., X_1, X_2 given by the operators Equations (30) and (31) and X_3 given by

$$X_3 = e^{-\sigma t} \frac{\partial}{\partial t} \text{ with } B^1 = f, B^2 = e, f_t + e_x = 0 \quad (53)$$

The application of Theorem 1, with X_3 , gives

$$T_3^1 = -\frac{1}{2}\gamma u^2 - \frac{1}{6}\lambda u^3 + \frac{\delta}{2} e^{-\sigma t} u_{xx} \int u_t dx - f \quad (54)$$

$$\begin{aligned} T_3^2 = & \gamma u \int u_t dx + \frac{1}{2} e^{-\sigma t} \left[\int u_t dx \right]^2 + \frac{1}{2} \lambda u^2 \int u_t dx + \delta e^{-\sigma t} u_{xt} \int u_t dx \\ & - \frac{1}{2} \delta e^{-\sigma t} u_x \int u_{tt} dx + \frac{1}{2} \delta \sigma e^{-\sigma t} u_x \int u_t dx - \frac{1}{2} \delta e^{-\sigma t} u_t^2 - e \end{aligned} \quad (55)$$

and as before, the modified conserved vectors are given by

$$\begin{aligned} \bar{T}_3^1 = & T_3^1 - \frac{\delta}{2} e^{-\sigma t} u_{xx} \int u_t dx \\ = & -\frac{\gamma}{2} u^2 - \frac{\lambda}{6} u^3 \end{aligned} \quad (56)$$

$$\begin{aligned} \bar{T}_3^2 = & T_3^2 - \frac{1}{2} \delta \sigma e^{-\sigma t} u_x \int u_t dx + \frac{1}{2} \delta e^{-\sigma t} u_x \int u_{tt} dx \\ = & \gamma u \int u_t dx + \frac{1}{2} e^{-\sigma t} \left[\int u_t dx \right]^2 + \frac{1}{2} \lambda u^2 \int u_t dx + \delta e^{-\sigma t} u_{xt} \int u_t dx \\ & - \frac{1}{2} \delta e^{-\sigma t} u_t^2 \end{aligned} \quad (57)$$

Thus, the nontrivial conserved flows associated with X_1, X_2 and X_3 are, respectively,

$$\bar{T}_1^1 = \frac{1}{2} u^2 - \frac{\delta}{2} u_x^2 \quad (58)$$

$$\bar{T}_1^2 = \frac{\gamma}{2} e^{\sigma t} u^2 + \frac{\lambda}{3} e^{\sigma t} u^3 + \delta u u_{xt} \quad (59)$$

$$\bar{T}_2^1 = -\frac{1}{2} d(t) u - \frac{1}{2} \delta d(t) u_{xx} \quad (60)$$

$$\begin{aligned}\bar{T}_2^2 &= -\gamma e^{\sigma t} d(t) u - \frac{1}{2} d(t) \int u_t dx - \frac{\lambda}{2} e^{\sigma t} d(t) u^2 - \frac{\delta}{2} d(t) u_{xt} \\ &\quad + \frac{\delta}{2} d'(t) u_x + \frac{1}{2} d'(t) \int u dx\end{aligned}\quad (61)$$

$$\bar{T}_3^1 = -\frac{\gamma}{2} u^2 - \frac{\lambda}{6} u^3 \quad (62)$$

$$\begin{aligned}\bar{T}_3^2 &= \gamma u \int u_t dx + \frac{1}{2} e^{-\sigma t} \left[\int u_t dx \right]^2 + \frac{1}{2} \lambda u^2 \int u_t dx + \delta e^{-\sigma t} u_{xt} \int u_t dx \\ &\quad - \frac{1}{2} \delta e^{-\sigma t} u_t^2\end{aligned}\quad (63)$$

Case 4. $\alpha(t) = \gamma/a(t)$, $\beta(t) = \lambda/a(t)$, where γ, λ are constants, with $\gamma, \lambda \neq 0$ and $a(t)$ an arbitrary function of t .

In this case, we obtain three Noether point symmetry generators, viz., X_1, X_2 given by Equations (30) and (31) and X_3 given by

$$X_3 = a(t) \frac{\partial}{\partial t} \text{ with } B^1 = f, B^2 = e, f_t + e_x = 0 \quad (64)$$

Following the above procedure, the nontrivial local and nonlocal conserved quantities corresponding to X_1, X_2 and X_3 , in this case, are

$$\bar{T}_1^1 = \frac{\delta}{2} u^2 - \frac{\delta}{2} u_x^2 \quad (65)$$

$$\bar{T}_1^2 = \frac{\gamma}{2} a(t) u^2 + \frac{\lambda}{3} a(t) u^3 + \delta u u_{xt} \quad (66)$$

$$\bar{T}_2^1 = -\frac{d(t)}{2} u - \frac{\delta}{2} d(t) u_{xx} \quad (67)$$

$$\begin{aligned}\bar{T}_2^2 &= -\frac{\gamma}{a(t)} d(t) u - \frac{1}{2} d(t) \int u_t dx - \frac{\lambda}{a(t)} d(t) u^2 - \frac{\delta}{2} d(t) u_{xt} \\ &\quad + \frac{\delta}{2} d'(t) u_x + \frac{1}{2} d'(t) \int u dx\end{aligned}\quad (68)$$

$$\bar{T}_3^1 = -\frac{1}{2} \gamma u^2 - \frac{1}{6} \lambda u^3 \quad (69)$$

$$\begin{aligned}\bar{T}_3^2 &= \gamma u \int u_t dx + \frac{1}{2} a(t) \left[\int u_t dx \right]^2 + \frac{1}{2} \lambda u^2 \int u_t dx + \delta a(t) u_{xt} \int u_t dx \\ &\quad - \frac{1}{2} \delta a(t) u_t^2\end{aligned}\quad (70)$$

respectively.

Remark 1. Remark: It should be noted that since the Lagrangian Equation (16) is invariant under the spatial translation symmetry, this will give rise to the linear momentum conservation laws. Moreover, if $\alpha(t) = \gamma$, $\beta(t) = \lambda$, where γ and λ are non-zero constants, then the corresponding Lagrangian Equation (16) is also invariant under the time translation symmetry and thus the linear momentum and energy are both conserved.

4. Exact Solution of Equation (4) for a Special Case Using Conservation Laws

First we recall a definition and theorem from [15] that we utilize in this Section.

Definition 2. Suppose that X is a symmetry of Equation (10) and T a conserved vector of Equation (10). Then if X and T satisfy

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, 2 \quad (71)$$

then X is said to be associated with T .

Define a nonlocal variable v by $T^t = v_x$, $T^x = -v_t$. Then using the similarity variables r, s, w with the generator $X = \frac{\partial}{\partial s}$, we have $T^r = v_s$, $T^s = -v_r$, and the conservation law is then rewritten as

$$D_r T^r + D_s T^s = 0,$$

where

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)} \quad (72)$$

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)} \quad (73)$$

Theorem 2. An n -th order partial differential equation with two independent variables, which admits a symmetry X that is associated with a conserved vector T , is reduced to an ordinary differential equation of order $n - 1$; namely, $T^r = k$, where T^r is defined by Equation (72) for solutions invariant under X .

We now use the above definition and theorem to obtain an exact solution of one special case of Equation (4) by making use of its conservation law. We consider Case 2 of Section 3.

Recall that the Equation (4) with $\alpha(t) = \gamma$, $\beta(t) = \lambda$, where γ and λ are non-zero constants, admits (among others)

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \quad (74)$$

and possesses the conservation law with conserved vector

$$T = \left(\frac{1}{2}u^2 - \frac{\delta}{2}u_x^2, \frac{\gamma}{2}u^2 + \frac{\lambda}{3}u^3 + \delta uu_{xt} \right) \quad (75)$$

that is associated with both X_1 and X_2 . We now define the combination of X_1 and X_2 by $X = \rho X_1 + X_2$. Thus, the canonical coordinates of X are given by

$$s = x, \quad r = \rho x - t, \quad u = u(r) \quad (76)$$

where $u = u(r)$ is the invariant solution under X if u satisfies Equation (4) with $\alpha(t) = \gamma$, $\beta(t) = \lambda$. Employing Equation (72), the r -component of T given in Equation (75) is

$$T^r = \frac{1}{2}u^2 - \frac{\alpha\gamma}{2}u^2 - \frac{\delta\alpha^2}{2}u_r^2 - \frac{\alpha\lambda}{3}u^3 + \delta\alpha^2 uu_{rr} \quad (77)$$

Using Theorem 2 above, Equation (77) can be written as

$$\frac{1}{2}u^2 - \frac{\alpha\gamma}{2}u^2 - \frac{\delta\alpha^2}{2}u_r^2 - \frac{\alpha\lambda}{3}u^3 + \delta\alpha^2uu_{rr} = k \quad (78)$$

By letting $u_r = p(u)$, we have $u_{rr} = p\frac{dp}{du}$. Then Equation (78) reduces to the first order ordinary differential equation

$$p' - \frac{1}{2u}p = \frac{1}{p}\left(\frac{\lambda}{3\delta\alpha}u^2 - \frac{1}{2\delta\alpha^2}u + \frac{\gamma}{2\delta\alpha}u + \frac{k}{\delta\alpha^2u}\right) \quad (79)$$

The integration of Equation (79) leads to the four parameter family of solutions

$$\pm \int (\lambda u^3/3\delta\alpha - 2k/\alpha^2\delta - u^2/\alpha^2\delta + \gamma u^2/\delta\alpha + k_1 u^2)^{-\frac{1}{2}} du = \rho x - t + k_2 \quad (80)$$

of Equation (4) invariant under $X = \rho X_1 + X_2$.

5. Concluding Remarks

In this paper we derived the conservation laws for the Benjamin–Bona–Mahony equation with variable coefficients Equation (4). This equation does not have a Lagrangian. In order to use Noether theorem, we transformed Equation (4) to a fourth-order Equation (15) which admits a Lagrangian. We then employed the Noether theorem to derive the conservation laws for Equation (15). The derived conserved vectors needed further adjustment to satisfy the divergence criterion. By reverting back to the original variable u , we constructed the conserved vectors for our third-order Benjamin–Bona–Mahony equation with variable coefficients Equation (4). The conserved vectors consisted of some local and infinite number of nonlocal conserved vectors. Finally, we obtained an exact solution for the Benjamin–Bona–Mahony equation using the double reduction theory. The importance of finding conservation laws and their physical applications are mentioned in the paper.

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Author Contributions

Ben Muatjetjeja and Chaudry Masood Khalique worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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