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COMPUTATIONAL ELECTROMAGNETICS FOR MICROSTRIP AND MEMS
STRUCTURES

BY
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DISSERTATION

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ABSTRACT

In the first part of this thesis, the quasi-3D thin-stratified medium fast-multipole algorithm (TSM-FMA) will be introduced for the analysis of general microstrip structures. It is based on a newly developed matrix-friendly dyadic Green's function for layered media (DGLM), which is represented in terms of only two Sommerfeld integrals and is suitable for developing fast algorithms. The path deformation technique and the multipole-based acceleration are used to expedite the matrix-vector multiplication. Both the computation time per iteration and the memory requirement are $O(N \log N)$ in the quasi-3D TSM-FMA.

In the second part, an efficient and accurate way to evaluate the Casimir force between arbitrarily-shaped conducting objects in both 2D and 3D geometries will be presented. The Casimir force is the dominant force between charge-neutral objects when the separation is less than a micron. It is important in the design of micro-electromechanical systems (MEMS) and nano-electromechanical systems (NEMS). Our method casts the evaluation of the force as a series of traditional 2D or 3D electromagnetic scattering problems, which are formulated with integral equations and then solved using the method of moments. We demonstrate that this quantum electrodynamics phenomenon can be studied using the knowledge of classical electrodynamics.

To my parents and my husband

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CHAPTER 1

INTRODUCTION

1.1 Motivation

A large class of electromagnetic applications, such as geophysical prospecting and probing [1–3], remote sensing [4], and microwave circuits, antennas [5–7], and multilayer printed circuit boards (PCB) [8] can be cast as layered medium problems. For typical layered medium problems, the object is above or embedded in large dielectric substrates or half-space, sometimes with ground planes. In numerical analysis, it is preferable to put unknowns on the scatterers or radiating objects than to discretize the entire background, as the number of unknowns required is greatly reduced. Thus, the integral equation method is more efficient than other methods based on differential equations in solving these problems. By the method of moments (MOM) [9], the integral equation is then converted into a matrix equation.

Instead of the free space dyadic Green’s function [9, 10], the dyadic Green’s function of the layered medium (DGLM) needs to be derived to formulate the integral equation in layered media. Various forms of dyadic Green’s function for layered medium (DGLM) have been investigated in the literature [5, 11–22], both isotropic and anisotropic. Though the exact forms are different, the dyadic Green’s functions can only be written as Fourier

integrals expressed in terms of Sommerfeld integrals. In general, the Sommerfeld integrals are difficult to evaluate efficiently, due to the highly oscillatory and slowly decaying characteristics of the integrand. Thus, the dyadic Green's function should be formulated so that the integrand in Sommerfeld integral is least oscillatory.

Among the applications, microstrip structures have been one of the most rapidly developing and intensely studied subjects since the 1970s. Microstrip structures usually include thin dielectric substrates and ground plane. Their transverse dimensions are much larger than their vertical dimensions. Due to the richness of its configuration, a full wave solver designed specifically for these structures is demanded. Moreover, since the major advantages of microstrip antennas are realized in applications that require moderate size arrays, a full-wave solver with a fast algorithm that can handle large problems efficiently is mostly desired. In summary, the full-wave surface integral equation solver for microstrip structures should have the following characteristics:

- Ability to model arbitrarily shaped microstrip structures.
- Matrix-friendly forms.
- Efficient evaluation of Sommerfeld integral.
- Fast algorithm acceleration.

In this dissertation, RWG basis [10] is used in the surface integral equation formulation to characterize the surface current on the object. This basis function is flexible enough to model arbitrarily shaped structures. We introduce a new and elegant way of deriving the DGLM based on the pilot vector potential approach [23]. In the matrix representation of

this DGLM, the Sommerfeld integrals only involve zeroth order Bessel function and are less oscillatory than other forms.

The numerical evaluation of Sommerfeld integrals is usually time-consuming, because the integrand is both highly oscillatory and slowly decaying along the Sommerfeld integration path due to the closely located branch-point and pole singularities. One way to speed up the calculation of Sommerfeld integrals is to precompute these integrals on a grid of points in the solution domain and to interpolate with the tabulated integrals [24–26]. Another vastly popular approach is the discrete complex image method (DCIM) [17, 27–32]. The third class of method is to deform the original integration path to the steepest descent path (SDP), where the integrand converges exponentially, and include the contributions from any enclosed poles [18, 20, 21, 33, 34]. Specifically, the thin-stratified medium fast-multipole in reference [20] combines the path deformation and fast algorithm and is a good candidate for microstrip patch problems. However, in the early work, the algorithm is limited to planar microstrip structures only. In this thesis, we inherit the path deformation and fast-multipole algorithm techniques in reference [20] and apply them to the new form of DGLM. The new fast algorithm is called the quasi-3D thin-stratified medium fast-multipole algorithm (TSM-FMA) and it can handle arbitrary microstrip structures.

The quasi-3D thin-stratified medium fast-multipole algorithm is designed for but not limited to microstrip structures. In circuit analysis, the interconnects are often placed over conductive substrate and the substrate effect should be considered [35, 36]. The conductors of the interconnects are similar to microstrip structures in the sense that they are mostly planar. Thus the fast algorithm proposed in this thesis could be applied in circuit analysis as well, if two more features could be included:

- Include the conductor loss of the object.
- Overcome the low-frequency breakdown.

These will be left to future research.

1.2 Outline

The outline of this thesis is as follows: In Chapter 2, a new and succinct way of deriving dyadic Green’s function for layered media (DGLM) is introduced. The matrix representation of the DGLM for both embedded basis and straddling basis is presented. It is validated by comparing the numerical results with the analytical solutions in some cylindrically symmetric problems. In Chapter 3, when the observation point and the source point have relatively large horizontal separation (greater than $0.1 \lambda_0$), the Sommerfeld integrals are detoured from the Sommerfeld integration path (SIP) to the vertical steepest descent path (SDP) to accelerate its convergence. Then the implementation of the quasi-3D thin-stratified medium fast multipole algorithm (TSM-FMA) is described in detail. It speeds up the matrix-vector multiplication of the impedance matrix to $O(N \log N)$. Chapter 4 is devoted to developing a numerical method to evaluate Casimir force in both 2D and 3D structures. This is our first successful attempt to apply computational electromagnetic techniques in multi-physics simulation. Finally, the possible topics for future research are discussed.

CHAPTER 2

A NEWLY DEVELOPED MATRIX-FRIENDLY FORMULATION OF LAYERED MEDIUM GREEN'S FUNCTION

In this chapter, a new and succinct way of deriving the matrix representation of the dyadic Green's function for layered media (DGLM) is introduced. The derivation is based on the pilot vector potential approach, where two vector potentials in terms of TE and TM fields are presented. Then the matrix representation of this dyadic Green's function is given and it is validated for basis straddling across layers. Finally, numerical results from this formulation agree well with the analytical solutions of some cylindrically symmetric problems and validate the new form of dyadic Green's function for layered medium.

2.1 Introduction

Sommerfeld first solved the problem of a vertical electric dipole on top of a half space in 1909 using Hertzian potentials [37]. In 1972, Kong used the z components of the electromagnetic field instead of Hertzian potentials and extended to layered media [38]. The dyadic Green's function for layered media has been derived in various forms [5, 11–22].

As we know, in free space case, if the fields are expressed in terms of vector and scalar potentials, the kernel of the integral equation has a weaker singularity. The mixed-potential

integral equation (MPIE) is even more important for layered media, because the spectral Sommerfeld integrals in the potential form converge more rapidly, saving considerable computation resources. In potential expressions, part of the coordinate-space differential operator is moved to the current. Since the differential operators in coordinate-space translate into higher spectral components in the Fourier space, the Sommerfeld integrals converge more rapidly with these operators being transferred. Among the various forms of MPIE which have been derived [5, 39–42], the Michalski-Zheng formulation [5] is vastly popular and has been adopted by many workers [17, 32, 43–48].¹

Similar to the MPIE formulation, the dyadic Green’s function for layered media derived here is expressed in terms of two scalar potentials. Each scalar potential is a Sommerfeld integral involving only zeroth order Bessel function, which is convenient for designing acceleration techniques in the next chapter. The matrix representation of the DGLM is manipulated so that the coordinate-space singularities in the integrand are as weak as possible. As a result, the associated Sommerfeld integrals have fewer high frequency components and converge more rapidly.

2.2 Dyadic Green’s Function for Layered Media

We will start with a dyadic Green’s function that is succinctly derived in [18], then generalize it for layered medium and arrive at a form that is matrix-friendly for evaluating elements of its matrix representation.

For a homogeneous medium with wavenumber k_m , the dyadic Green’s function can be decomposed into TE_z and TM_z components. Using the z -directed pilot vector potential

¹This list is by no means complete.

approach [49], the dyadic Green's function for a homogeneous medium can be derived in terms of vector wave functions as [18, p. 411]

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') &= \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} \frac{d\mathbf{k}_s}{k_{mz}k_s^2} [\mathbf{M}(\mathbf{k}_s, \mathbf{r})\mathbf{M}(-\mathbf{k}_s, \mathbf{r}') \\ &\quad + \mathbf{N}(\mathbf{k}_s, \mathbf{r})\mathbf{N}(-\mathbf{k}_s, \mathbf{r}')] - \frac{\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_m^2} \delta(\mathbf{r} - \mathbf{r}')\end{aligned}\quad (2.1)$$

where

$$\mathbf{M}(\mathbf{k}_s, \mathbf{r})\mathbf{M}(-\mathbf{k}_s, \mathbf{r}') = (\mathbf{k}_s \times \hat{\mathbf{z}})(\mathbf{k}_s \times \hat{\mathbf{z}}) e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} e^{ik_{mz}|z - z'|} \quad (2.2)$$

$$\mathbf{N}(\mathbf{k}_s, \mathbf{r})\mathbf{N}(-\mathbf{k}_s, \mathbf{r}') = \frac{1}{k_m^2} (\mathbf{k}_{m\pm} \times \mathbf{k}_s \times \hat{\mathbf{z}})(\mathbf{k}_{m\mp} \times \mathbf{k}_s \times \hat{\mathbf{z}}) \cdot e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} e^{ik_{mz}|z - z'|} \quad (2.3)$$

$\mathbf{k}_{m\pm} = \mathbf{k}_s \pm \hat{\mathbf{z}}k_{mz}$, $\mathbf{k}_s = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y$, and the upper sign is chosen when $z > z'$ and the lower sign is chosen when $z < z'$. Here, $k_{mz} = \sqrt{k_m^2 - k_s^2}$ and $k_s^2 = \mathbf{k}_s \cdot \mathbf{k}_s$. The above integrals are ill-convergent when the source point and the field point are coplanar, viz., $z = z'$. This is because this plane contains the source point singularity whose Fourier transform is ill-convergent (see [44, 45, 48] for details). However, numerical evaluation of them can be avoided since they have closed form in coordinate space for the homogeneous medium case, i.e.,

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k_m^2} \right) \frac{e^{ik_m|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad (2.4)$$

In the above, the $\mathbf{M}$ function corresponds to TE waves while the $\mathbf{N}$ function corre-

sponds to TM waves. For a layered medium, we can replace the above by [18]

$$\mathbf{M}(\mathbf{k}_s, \mathbf{r})\mathbf{M}(-\mathbf{k}_s, \mathbf{r}') = (\nabla \times \hat{\mathbf{z}})(\nabla' \times \hat{\mathbf{z}})e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} F^{\text{TE}}(k_s, z, z') \quad (2.5)$$

$$\begin{aligned} \mathbf{N}(\mathbf{k}_s, \mathbf{r})\mathbf{N}(-\mathbf{k}_s, \mathbf{r}') = & - \left(\frac{\nabla \times \nabla \times \hat{\mathbf{z}}}{i\omega\epsilon_n} \right) \left(\frac{\nabla' \times \nabla' \times \hat{\mathbf{z}}}{i\omega\mu_m} \right) \\ & \cdot e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} F^{\text{TM}}(k_s, z, z') \end{aligned} \quad (2.6)$$

where we have assumed that the source point is in region m while the field point is in region n . The function $F^\alpha(k_s, z, z')$ above, where α is either TE or TM, satisfies the following ordinary differential equation [18, 50]:

$$\left[\frac{d}{dz} \frac{1}{p(z)} \frac{d}{dz} + \frac{1}{p(z)} k_z^2(z) \right] F^\alpha(k_s, z, z') = \frac{2i}{p(z)} k_z(z) \delta(z, z') \quad (2.7)$$

where $p = \mu$ for TE waves, and $p = \epsilon$ for TM waves. Moreover, $k_z = \sqrt{k^2(z) - k_s^2}$. When $k(z)$ is piecewise constant denoted by k_m for different regions, F^α can be found in closed form using a recursive procedure [18, p. 76]. Furthermore, F^α describes the propagation of B_z or D_z fields for TE and TM fields respectively in an inhomogeneous layer.

After using Equation (2.5) and (2.6) in Equation (2.1), and exchanging the order of derivatives and integrals, we can rewrite the DGLM as

$$\begin{aligned} \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = & (\nabla \times \hat{\mathbf{z}})(\nabla' \times \hat{\mathbf{z}})g^{\text{TE}}(\mathbf{r}, \mathbf{r}') \\ & + \frac{1}{k_{nm}^2} (\nabla \times \nabla \times \hat{\mathbf{z}})(\nabla' \times \nabla' \times \hat{\mathbf{z}})g^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \quad |z - z'| > 0 \end{aligned} \quad (2.8)$$

where $k_{nm}^2 = \omega^2 \epsilon_n \mu_m$. The above representation is valid only when $|z - z'| > 0$, because

when the derivative operators act on $\exp(ik_{mz}|z - z'|)$, a singularity is produced at $z = z'$ which is not the same as that in Equation (2.1). Here, $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$ and $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$ consist of the primary (direct) field terms and the secondary (reflected) field terms. The primary field term contains the source singularity, and hence, its representation in the Fourier space is inefficient. Since it is the same as the homogeneous medium case for which closed form exists, it is more expedient to express it in closed form in coordinate space (see (2.4)).

Another point to be noted is that in (2.8), the derivative operators are taken out of the spectral integrals. Hence, $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$ and $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$ in their coordinate-space representations are less singular than the dyadic Green's function itself. Hence, their spectral integrals are more rapidly convergent compared to the spectral integrals in (2.1).

The explicit expressions for $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$ and $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$ are

$$g^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} \frac{d\mathbf{k}_s}{k_{mz}k_s^2} \left[e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} F^{\text{TE}}(k_s, z, z') \right] \quad (2.9)$$

$$g^{\text{TM}}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi^2} \iint_{-\infty}^{\infty} \frac{d\mathbf{k}_s}{k_{mz}k_s^2} \left[e^{i\mathbf{k}_s \cdot (\mathbf{r}_s - \mathbf{r}'_s)} F^{\text{TM}}(k_s, z, z') \right] \quad (2.10)$$

The above Fourier integrals can be expressed in terms of Sommerfeld integrals involving a single integral and Bessel function (where $k_\rho = k_s$) using the techniques described in [18], namely

$$g^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho}{k_{mz}k_\rho} [J_0(k_\rho r_s) F^{\text{TE}}(k_\rho, z, z')] \quad (2.11)$$

$$g^{\text{TM}}(\mathbf{r}, \mathbf{r}') = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho}{k_{mz}k_\rho} [J_0(k_\rho r_s) F^{\text{TM}}(k_\rho, z, z')] \quad (2.12)$$

Due to the $1/k_s^2$ or the $1/k_\rho$ term in the integrands, these spectral integrals are more rapidly

convergent compared to those in (2.1). When F^{TM} and F^{TE} are defined in (2.11)-(2.12), the generalized reflection coefficients $\tilde{R}^{\text{TE}}$ and $\tilde{R}^{\text{TM}}$ [18, Chap. 2] distinguish polarization effects of the layered medium.

2.3 Propagation Factors

In the last section, we have given the ordinary differential equation that the propagation factors satisfy (2.7). In a layered medium where $p(z)$ and $k(z)$ are piecewise constant, the factor can be obtained through a recursive procedure [18, 50]. Though the other references already clearly present the generalized reflection coefficients ($G_{m,n}$) and propagation factor, a direct implementation may lead to overflow. Therefore, the expressions here have been manipulated to get a form which is friendlier to numerical implementation.

We will give the expression of $F_{nm}^\alpha(z, z')$ used in this thesis. Here α stands for either TE or TM, m is the index of the layer containing the source point z' and n is the index of the layer containing the testing point z . R_{nm}^α and T_{nm}^α are the Fresnel reflection coefficient and transmission coefficients from medium n to medium m . And G_{nm}^α are the generalized reflection coefficient:

$$G_{n,n+1} = R_{n,n+1} + \frac{T_{n,n+1}G_{n+1,n+2}T_{n+1,n} \exp[2ik_{n+1,z}(d_{n+2} - d_{n+1})]}{1 - G_{n+1,n+2}R_{n+1,n} \exp[2ik_{n+1,z}(d_{n+2} - d_{n+1})]} \quad (2.13)$$

$$G_{m,m-1} = R_{m,m-1} + \frac{T_{m,m-1}G_{m-1,m-2}T_{m-1,m} \exp[2ik_{m-1,z}(d_m - d_{m-1})]}{1 - G_{m-1,m-2}R_{m-1,m} \exp[2ik_{m-1,z}(d_m - d_{m-1})]} \quad (2.14)$$

If $m = n$, i.e. source point and field point are in the same layer, the propagation factor

can be summarized as

$$\begin{aligned}
F_{m,m}^\alpha(z, z') &= e^{ik_{mz}|z'-z|} + C_m G_{m,m-1} e^{ik_{mz}(z+z'-2d_m)} + C_m G_{m,m+1} e^{ik_{mz}(2d_{m+1}-z-z')} \\
&\quad + C_m G_{m,m-1} G_{m,m+1} e^{ik_{mz}(z'-z)+2ik_{mz}(d_{m+1}-d_m)} \\
&\quad + C_m G_{m,m-1} G_{m,m+1} e^{ik_{mz}(z-z')+2ik_{mz}(d_{m+1}-d_m)}
\end{aligned} \tag{2.15}$$

where

$$C_m = \frac{1}{1 - G_{m,m+1} G_{m,m-1} e^{2ik_{mz}(d_{m+1}-d_m)}}$$

The first term is the direct interaction. The other terms are the reflected terms and their physical meaning is clear through this rearrangement.

If $n > m$, i.e. observation layer is above the source layer,

$$\begin{aligned}
F_{nm}^\alpha(z, z') &= [e^{ik_{nz}(z-d_n)} + G_{n,n+1} e^{ik_{nz}(2d_{n+1}-z-d_n)}] D_n \prod_{j=m+1}^{n-1} D_j e^{ik_{j,z}(d_{j+1}-d_j)} \\
&\quad \times C_m [e^{ik_{mz}(d_{m+1}-z')} + G_{m,m-1} e^{ik_{mz}(d_{m+1}+z'-2d_m)}]
\end{aligned} \tag{2.16}$$

where

$$D_j = \frac{T_{j-1,j}}{1 - G_{j,j+1} R_{j,j-1} e^{2ik_{jz}(d_{j+1}-d_j)}} = \frac{1 - R_{j,j-1}}{1 - G_{j,j+1} R_{j,j-1} e^{2ik_{jz}(d_{j+1}-d_j)}} \tag{2.17}$$

If $n < m$, i.e. observation layer is below the source layer,

$$\begin{aligned}
F_{nm}^\alpha(z, z') &= [e^{ik_{nz}(d_{n+1}-z)} + G_{n,n-1} e^{ik_{nz}(d_{n+1}+z-2d_n)}] E_n \prod_{j=m-1,-1}^{n+1} E_j e^{ik_{j,z}(d_{j+1}-d_j)} \\
&\quad \times C_m [e^{ik_{mz}(z'-d_m)} + G_{m,m+1} e^{ik_{mz}(2d_{m+1}-z'-d_m)}]
\end{aligned} \tag{2.18}$$

where

$$E_j = \frac{T_{j+1,j}}{1 - G_{j,j-1}R_{j,j+1}e^{2ik_{jz}(d_{j+1}-d_j)}} = \frac{1 - R_{j,j+1}}{1 - G_{j,j-1}R_{j,j+1}e^{2ik_{jz}(d_{j+1}-d_j)}} \quad (2.19)$$

For the expressions above, if we choose the correct sign of k_{iz} so that its real part is positive, every exponential function is bounded and overflow would be prevented.

From (2.7), the boundary conditions for $F_{nm}^\alpha(z, z')$ across a discontinuity could be derived [18, p. 47]. Assume the source z' is fixed in layer m , and z is right at the interface of layer n and $n - 1$,

$$F_{nm}^{\text{TE}}(z, z') = F_{n-1,m}^{\text{TE}}(z, z'), \quad \text{when } z = d_m, \quad (2.20)$$

$$\frac{1}{\mu(z)} \frac{d}{dz} F_{nm}^{\text{TE}}(z, z') = \frac{1}{\mu(z)} \frac{d}{dz} F_{n-1,m}^{\text{TE}}(z, z') \quad (2.21)$$

$$F_{nm}^{\text{TM}}(z, z') = F_{n-1,m}^{\text{TM}}(z, z'), \quad (2.22)$$

$$\frac{1}{\epsilon(z)} \frac{d}{dz} F_{nm}^{\text{TM}}(z, z') = \frac{1}{\epsilon(z)} \frac{d}{dz} F_{n-1,m}^{\text{TM}}(z, z') \quad (2.23)$$

Now, we perform an asymptotic analysis for the propagation factors:

$$\lim_{k_\rho \rightarrow \infty} k_{mz} = \lim_{k_\rho \rightarrow \infty} \sqrt{k_m^2 - k_\rho^2} = ik_\rho \quad (2.24)$$

$$G_{md,\infty} \equiv \lim_{k_\rho \rightarrow \infty} G_{m,m-1} = \lim_{k_\rho \rightarrow \infty} R_{m,m-1} = \frac{p_{m-1} - p_m}{p_{m-1} + p_m} \quad (2.25)$$

$$G_{mu,\infty} \equiv \lim_{k_\rho \rightarrow \infty} G_{m,m+1} = \lim_{k_\rho \rightarrow \infty} R_{m,m+1} = \frac{p_{m+1} - p_m}{p_{m+1} + p_m} \quad (2.26)$$

$$T_{nd,\infty} \equiv \lim_{k_\rho \rightarrow \infty} D_n = 1 - \lim_{k_\rho \rightarrow \infty} R_{n,n-1} = \frac{2p_n}{p_{n-1} + p_n} \quad (2.27)$$

$$T_{nu,\infty} \equiv \lim_{k_\rho \rightarrow \infty} E_n = 1 - \lim_{k_\rho \rightarrow \infty} R_{n,n+1} = \frac{2p_n}{p_{n+1} + p_n} \quad (2.28)$$

Here p stands for μ for TE waves and ϵ for TM waves. Usually, dielectric substrate has a relative permeability of 1 and $G_{mu,\infty} = G_{md,\infty} = 0$. Using these results, the propagation factors could be approximated as follows when $k_\rho \rightarrow \infty$:

$$F_{m,m}(z, z') \approx G_{md,\infty} e^{ik_{mz}(z+z'-2d_m)} + G_{mu,\infty} e^{ik_{mz}(2d_{m+1}-z-z')} \\ + G_{md,\infty} G_{mu,\infty} [e^{ik_{mz}(z'-z)+2ik_{mz}(d_{m+1}-d_m)} + e^{ik_{mz}(z-z')+2ik_{mz}(d_{m+1}-d_m)}] \quad (2.29)$$

$$F_{m+1,m}(z, z') \approx T_{(m+1)d,\infty} e^{-k_\rho(z-z')} \quad (2.30)$$

$$F_{m-1,m}(z, z') \approx T_{(m-1)u,\infty} e^{-k_\rho(z'-z)} \quad (2.31)$$

These approximations would be useful for the quasi-static extraction that we perform later.

2.4 Matrix Representation

The primary field term is best expressed in coordinate space in closed form (see (2.4)). Hence, the matrix representation of the primary field term of the DGLM is straightforward and can be done following conventional methods. The reflected field terms do not contain singularities, and are regular.

In the following, the manipulation to obtain the matrix representation of the Green's function operator is intended for the secondary fields where the image source point and the field point do not coincide.

Focussing on the TM wave term for the dyadic Green's function, we have

$$\begin{aligned}
\overline{\mathbf{G}}^{\text{TM}}(\mathbf{r}, \mathbf{r}') &= \frac{1}{k_{nm}^2} (\nabla \times \nabla \times \hat{\mathbf{z}}) (\nabla' \times \nabla' \times \hat{\mathbf{z}}) g^{\text{TM}}(\mathbf{r}, \mathbf{r}') \\
&= \frac{1}{k_{nm}^2} (\nabla \nabla \cdot \hat{\mathbf{z}} + k_n^2 \hat{\mathbf{z}}) (\nabla' \nabla' \cdot \hat{\mathbf{z}} + k_m^2 \hat{\mathbf{z}}) g^{\text{TM}}(\mathbf{r}, \mathbf{r}') \\
&= \frac{1}{k_{nm}^2} (\nabla \nabla' \partial_z \partial_{z'} + k_n^2 \nabla'_z \nabla' + k_m^2 \nabla \nabla_z + k_n^2 k_m^2 \hat{\mathbf{z}} \hat{\mathbf{z}}) \cdot g^{\text{TM}}(\mathbf{r}, \mathbf{r}') \quad (2.32)
\end{aligned}$$

In the above, we have assumed that $(\nabla^2 + k_n^2)g^{\text{TM}}(\mathbf{r}, \mathbf{r}') = 0$ and $(\nabla'^2 + k_m^2)g^{\text{TM}}(\mathbf{r}, \mathbf{r}') = 0$, since we assume that the field point and the image source point do not coincide. Likewise, the TE wave term for the dyadic Green's function for layered media is

$$\overline{\mathbf{G}}^{\text{TE}}(\mathbf{r}, \mathbf{r}') = (\nabla \times \hat{\mathbf{z}}) (\nabla' \times \hat{\mathbf{z}}) g^{\text{TE}}(\mathbf{r}, \mathbf{r}') \quad (2.33)$$

The electric field due to an electric current can be expressed with the dyadic Green's function for an inhomogeneous medium as [18, p. 411]

$$\mathbf{E}(\mathbf{r}) = i\omega \int_V d\mathbf{r}' \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mu(\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') \quad (2.34)$$

Next, we evaluate the matrix element for the impedance matrix as

$$\begin{aligned}
Z_{ij} &= i\omega \mu_m \langle \mathbf{J}_{Ti}(\mathbf{r}), \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_j(\mathbf{r}') \rangle \\
&= i\omega \mu_m \int d\mathbf{r} \mathbf{J}_{Ti}(\mathbf{r}) \cdot \int d\mathbf{r}' \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_j(\mathbf{r}') \quad (2.35)
\end{aligned}$$

where we have assumed that the source basis function $\mathbf{J}_j(\mathbf{r}')$ is entirely in the m -th region. When a source basis straddles two or more regions, the above expression has to be

modified accordingly.

In the above, the term associated with TM waves is

$$\begin{aligned}
\langle \mathbf{J}_{Ti}(\mathbf{r}), \overline{\mathbf{G}}^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_j(\mathbf{r}') \rangle &= \frac{1}{k_{nm}^2} [\langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), \partial_z \partial_{z'} g^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad - k_n^2 \langle \mathbf{J}_{Ti}(\mathbf{r}) \cdot \hat{\mathbf{z}}, \partial_{z'} g^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad - k_m^2 \langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), \partial_z g^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad + k_n^2 k_m^2 \langle \hat{\mathbf{z}} \cdot \mathbf{J}_{Ti}(\mathbf{r}), g^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \rangle] \quad (2.36)
\end{aligned}$$

In deriving the above, we have made use of integration by parts as much as possible to move the ∇ operators away from g^{TM} . Notice that $\partial_z \partial_{z'} g^{\text{TM}}$ is the most singular function since derivatives enhance the order of the singularity. If the z derivative operators are brought into the spectral integral of g^{TM} , the resultant integral will be slowly convergent. The other integrals are more benign and are rapidly convergent when numerically integrated.

For the term associated with TE waves, using (2.33), we can express it as

$$\langle \mathbf{J}_{Ti}(\mathbf{r}), \overline{\mathbf{G}}^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_j(\mathbf{r}') \rangle = \langle \mathbf{J}_{Ti}(\mathbf{r}), [(\nabla_s \times \hat{\mathbf{z}})(\nabla'_s \times \hat{\mathbf{z}})g^{\text{TE}}(\mathbf{r}, \mathbf{r}')], \mathbf{J}_j(\mathbf{r}') \rangle \quad (2.37)$$

We manipulate the expression by first showing that

$$\begin{aligned}
(\hat{\mathbf{z}} \times \nabla_s)(\hat{\mathbf{z}} \times \nabla'_s) &= k_s^2 \overline{\mathbf{I}}_s - \nabla_s \nabla'_s \\
&= \overline{\mathbf{I}}_s k_s^2 - (\nabla \nabla' - \nabla_z \nabla' - \nabla \nabla'_z + \nabla_z \nabla'_z) \quad (2.38)
\end{aligned}$$

(The first equality above can be easily proved in the spectral domain.) Then we can rewrite

(2.37) as

$$\begin{aligned}
\langle \mathbf{J}_{Ti}(\mathbf{r}), \overline{\mathbf{G}}^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_j(\mathbf{r}') \rangle &= \langle \mathbf{J}_{TSi}(\mathbf{r}), g_S^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_{Sj}(\mathbf{r}') \rangle \\
&\quad - \langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad - \langle \mathbf{J}_{Ti}(\mathbf{r}) \cdot \hat{\mathbf{z}}, \partial_z g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad - \langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), \partial_{z'} g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \rangle \\
&\quad - \langle \mathbf{J}_{Ti}(\mathbf{r}) \cdot \hat{\mathbf{z}}, \partial_z \partial_{z'} g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \rangle \quad (2.39)
\end{aligned}$$

In the above,

$$g_S^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho k_\rho}{k_{mz}} [J_0(k_\rho r_s) F^{\text{TE}}(k_\rho, z, z')] \quad (2.40)$$

and $\mathbf{J}_{TSi}$ is the part of $\mathbf{J}_{Ti}$ that is transverse to $\hat{\mathbf{z}}$, and similarly for $\mathbf{J}_{Sj}$.

We can combine the integrals in (2.39) with those in (2.36) to arrive at five basic terms. By combining the TM and the TE Green's functions together, and collecting like terms, we get

$$\begin{aligned}
\langle \mathbf{J}_{Ti}(\mathbf{r}), \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_j(\mathbf{r}') \rangle &= \left\langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), \frac{\partial_z \partial_{z'}}{k_{nm}^2} g^{\text{TM}}(\mathbf{r}, \mathbf{r}') - g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \right\rangle \\
&\quad - \left\langle \mathbf{J}_{Ti}(\mathbf{r}) \cdot \hat{\mathbf{z}}, \frac{\mu_n}{\mu_m} \partial_{z'} g^{\text{TM}}(\mathbf{r}, \mathbf{r}') + \partial_z g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \nabla' \cdot \mathbf{J}_j(\mathbf{r}') \right\rangle \\
&\quad - \left\langle \nabla \cdot \mathbf{J}_{Ti}(\mathbf{r}), \frac{\epsilon_m}{\epsilon_n} \partial_z g^{\text{TM}}(\mathbf{r}, \mathbf{r}') + \partial_{z'} g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \right\rangle \\
&\quad + \left\langle \hat{\mathbf{z}} \cdot \mathbf{J}_{Ti}(\mathbf{r}), k_{mn}^2 g^{\text{TM}}(\mathbf{r}, \mathbf{r}') - \partial_z \partial_{z'} g^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \hat{\mathbf{z}} \cdot \mathbf{J}_j(\mathbf{r}') \right\rangle \\
&\quad + \langle \mathbf{J}_{TSi}(\mathbf{r}), g_S^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_{Sj}(\mathbf{r}') \rangle \quad (2.41)
\end{aligned}$$

Evaluating (2.41) involves the computation of two basic Green's functions g^{TM} , g^{TE} and their z derivatives. The g_S^{TE} Green's function can be obtained from g^{TE} by taking r_s derivatives. It can be shown by using the property of the Bessel equation that

$$\frac{1}{r_s} \frac{\partial}{\partial r_s} r_s \frac{\partial}{\partial r_s} g^{\text{TE}} + g_S^{\text{TE}} = 0 \quad (2.42)$$

These Green's functions, g^{TM} , g^{TE} , can be computed and tabulated via Sommerfeld integrals, and their derivatives can be approximated by finite difference on tabulated values in order to evaluate the above five terms.

Alternatively, for higher accuracy, one may want to evaluate the z derivatives exactly by bringing them inside the spectral integrals to avoid the need for finite difference. To this end, the Green's functions sandwiched between the source basis and the test basis can be combined into one integral, and tabulated for efficiency. Also, in the special case when the source point and the field point are in the uppermost layer or the lowermost layer, the second and the third terms in (2.41) can be further combined, leaving only four integrals.

We consider the matrix representation when RWG basis functions [10] are used for expansion and testing. Using the vector notation shown in Figure 2.1, the matrix elements for the impedance matrix can be explicitly expressed as

$$Z_{jk} = i\omega\mu_m \sum_{p,q}^2 \frac{l_j l_k}{4S_j^p S_k^q} \int_{S_j^p} d\mathbf{r} \int_{S_k^q} d\mathbf{r}' \{ Z_{ss,jk}^{pq} + Z_{zz,jk}^{pq} + Z_{z_1,jk}^{pq} + Z_{z_2,jk}^{pq} + Z_{\Phi,jk}^{pq} \} \quad (2.43)$$

where

$$Z_{ss,jk}^{pq} = (\hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \boldsymbol{\xi}_j^p) \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \boldsymbol{\xi}_k^q) g_S^{\text{TE}}(\mathbf{r}, \mathbf{r}') \quad (2.44)$$

$$Z_{zz,jk}^{pq} = (\hat{\mathbf{z}} \cdot \boldsymbol{\xi}_j^p) (\hat{\mathbf{z}} \cdot \boldsymbol{\xi}_k^q) [k_{mn}^2 g^{\text{TM}} - \partial_z \partial_{z'} g^{\text{TE}}] \quad (2.45)$$

$$Z_{z_1,jk}^{pq} = -2\epsilon_q (\boldsymbol{\xi}_j^p \cdot \hat{\mathbf{z}}) \left[\frac{\mu_n}{\mu_m} \partial_{z'} g^{\text{TM}} + \partial_z g^{\text{TE}} \right] \quad (2.46)$$

$$Z_{z_2,jk}^{pq} = -2\epsilon_p (\boldsymbol{\xi}_k^q \cdot \hat{\mathbf{z}}) \left[\frac{\epsilon_m}{\epsilon_n} \partial_z g^{\text{TM}} + \partial_{z'} g^{\text{TE}} \right] \quad (2.47)$$

$$Z_{\Phi,jk}^{pq} = 4\epsilon_{pq} \left[\frac{\partial_z \partial_{z'}}{k_{nm}^2} g^{\text{TM}} - g^{\text{TE}} \right] \quad (2.48)$$

where $\epsilon_{p,q} = 1$ if $p = q$ or $\epsilon_{p,q} = -1$ if $p \neq q$, and $\epsilon_s = 1$ if $s = 1$ or $\epsilon_s = -1$ if $s = 2$, s represents p and q . The subscripts ss, z_1, z_2, zz, Φ of $Z_{\cdot,jk}^{pq}$ follow [5] for ease of comparison.

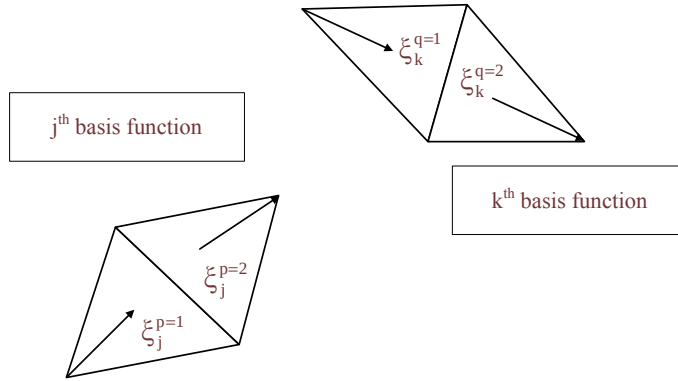


Figure 2.1: RWG basis and testing functions.

2.5 Quasi-Static Field Extraction

As we have pointed out in that last section, the first Sommerfeld integral converges slower than others. And quasi-static extractions could be used to accelerate the convergence rate because the slowly converging part of the integrand could be integrated analytically. We will discuss it for two different cases.

2.5.1 Source and field points in the same layer

In this case, we would make use of the Sommerfeld identity [18]

$$\int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z |z|} = -i \frac{e^{ik_0 r}}{r} \quad (2.49)$$

The TM part of the first Sommerfeld integral would converge more rapidly if we subtract the four possible quasi-image terms and integrate them analytically using Sommerfeld identity.

$$\begin{aligned} \int_0^\infty dk_\rho \partial_z \partial_{z'} g^{\text{TM}}(\mathbf{r}, \mathbf{r}') &= \int_0^\infty dk_\rho \frac{-k_m^2}{k_\rho k_{mz}} J_0(k_\rho \rho) F_0^{\text{TM}}(k_\rho, z, z') \\ &= \int_0^\infty dk_\rho \frac{k_\rho}{k_{mz}} J_0(k_\rho \rho) [F_0^{\text{TM}}(k_\rho, z, z') - F_0^{\text{TM}}(\infty, z, z')] \\ &\quad - i \sum_{p=1}^4 \frac{e^{ik_m r_p}}{r_p} F_{p,\infty}^{\text{TM}}(z, z') - \int_0^\infty \frac{dk_\rho k_m^2}{k_\rho k_{mz}} J_0(k_\rho \rho) F_0^{\text{TM}}(k_\rho, z, z') \end{aligned} \quad (2.50)$$

where

$$\begin{aligned}
F_0^{\text{TM}}(k_\rho, z, z') &= -\frac{\partial_z \partial_{z'}}{k_{mz}^2} F_{m,m}^{\text{TM}}(k_\rho, z, z') \\
&= C_m G_{m,m-1} e^{ik_{mz}(z+z'-2d_m)} + C_m G_{m,m+1} e^{ik_{mz}(2d_{m+1}-z-z')} \\
&\quad - C_m G_{m,m-1} G_{m,m+1} e^{ik_{mz}(z'-z)+2ik_{mz}(d_{m+1}-d_m)} \\
&\quad - C_m G_{m,m-1} G_{m,m+1} e^{ik_{mz}(z-z')+2ik_{mz}(d_{m+1}-d_m)}
\end{aligned} \tag{2.51}$$

and the distances from the images to the source point are

$$\begin{aligned}
r_1 &= \sqrt{\rho^2 + (z + z' - 2d_m)^2}, \quad r_3 = \sqrt{\rho^2 + [(z' - z) + 2(d_{m+1} - d_m)]^2}, \\
r_2 &= \sqrt{\rho^2 + (2d_{m+1} - z - z')^2}, \quad r_4 = \sqrt{\rho^2 + [(z - z') + 2(d_{m+1} - d_m)]^2}
\end{aligned} \tag{2.52}$$

The quasi-static reflection coefficients of each image are

$$\begin{aligned}
F_{1,\infty}^{\text{TM}} &= G_{md,\infty}^{\text{TM}}, \quad F_{2,\infty}^{\text{TM}} = G_{mu,\infty}^{\text{TM}}, \\
F_{3,\infty}^{\text{TM}} &= F_{4,\infty}^{\text{TM}} = -G_{md,\infty}^{\text{TM}} G_{mu,\infty}^{\text{TM}}
\end{aligned} \tag{2.53}$$

$$\begin{aligned}
F_0^{\text{TM}}(\infty, z, z') &= F_{1,\infty}^{\text{TM}} e^{ik_{mz}(z+z'-2d_m)} + F_{2,\infty}^{\text{TM}} e^{ik_{mz}(2d_{m+1}-z-z')} \\
&\quad + F_{3,\infty}^{\text{TM}} e^{ik_{mz}(z'-z)+2ik_{mz}(d_{m+1}-d_m)} + F_{4,\infty}^{\text{TM}} e^{ik_{mz}(z-z')+2ik_{mz}(d_{m+1}-d_m)}
\end{aligned} \tag{2.54}$$

2.5.2 Source and field points in different layers

We would only consider the case when the field and source points belong to continuous layers because the Sommerfeld integrals are affected most by quasi-static extraction in this

case. We use another identity for the following derivation:

$$\int_0^\infty dk_\rho J_0(k_\rho \rho) e^{-k_\rho z} = \frac{1}{\sqrt{\rho^2 + z^2}} \quad (2.55)$$

There are three terms that decay as $J_0(k_\rho \rho) e^{-k_\rho z}$ when k_ρ becomes large: the TM part of the first integral, the TE part of the fourth integral, and the last integral. We would look at them term by term. For the first term, we have

$$\begin{aligned} \int_0^\infty dk_\rho \partial_z \partial_{z'} g^{\text{TM}}(\mathbf{r}, \mathbf{r}') &= \int_0^\infty dk_\rho J_0(k_\rho \rho) \left[\frac{\partial_z \partial_{z'} F_{n,m}^{\text{TM}}(z, z')}{k_\rho k_{mz}} - iT_{n,\infty}^{\text{TM}} e^{-k_\rho |z-z'|} \right] \\ &\quad + \frac{iT_{n,\infty}^{\text{TM}}}{\sqrt{\rho^2 + (z-z')^2}} \end{aligned} \quad (2.56)$$

For the second term,

$$\begin{aligned} \int_0^\infty dk_\rho \partial_z \partial_{z'} g^{\text{TE}}(\mathbf{r}, \mathbf{r}') &= \int_0^\infty dk_\rho J_0(k_\rho \rho) \left[\frac{\partial_z \partial_{z'} F_{n,m}^{\text{TE}}(z, z')}{k_\rho k_{mz}} - iT_{n,\infty}^{\text{TE}} e^{-k_\rho |z-z'|} \right] \\ &\quad + \frac{iT_{n,\infty}^{\text{TE}}}{\sqrt{\rho^2 + (z-z')^2}} \end{aligned} \quad (2.57)$$

For the last term,

$$\begin{aligned} \int_0^\infty dk_\rho g_S^{\text{TE}}(\mathbf{r}, \mathbf{r}') &= \int_0^\infty dk_\rho J_0(k_\rho \rho) \left[\frac{k_\rho}{k_{mz}} F_{n,m}^{\text{TE}}(z, z') - iT_{n,\infty}^{\text{TE}} e^{-k_\rho |z-z'|} \right] \\ &\quad + \frac{iT_{n,\infty}^{\text{TE}}}{\sqrt{\rho^2 + (z-z')^2}} \end{aligned} \quad (2.58)$$

Here $T_{n,\infty} = T_{nd,\infty}$ when $n = m + 1$, and $T_{n,\infty} = T_{nu,\infty}$ when $n = m - 1$.

2.6 Straddling Basis

In this section, the new formulation of layered medium Green's function is extended to handle straddling objects. Michalski has pointed out that the scalar potential kernels have different continuity properties in different mixed-potential formulation [5]. Moreover, contour integrals exist in some formulation (Formulation B in [5]), while not in others (Formulation A and C). We have also noticed that some authors have claimed [51] that line integrals occur when the basis is bent at the interface. This is because the contribution of horizontal and vertical sources is considered separately in their formulation. Since the scalar potentials for horizontal and vertical sources are different in layered medium, the two line integrals do not cancel each other. For this formulation, we have been able to prove that no line integral is needed in the final expression after careful derivation.

Recall for (2.36), we have used integration by parts to move the ∇ operator from the kernel to the basis and testing function.

$$\begin{aligned} \iint_{\Delta} d\mathbf{r}' \nabla' g(\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') &= \iint_{\Delta} d\mathbf{r}' \{ \nabla' \cdot [g(\mathbf{r}') \mathbf{J}(\mathbf{r}')] - g(\mathbf{r}') \nabla' \cdot \mathbf{J}(\mathbf{r}') \} \\ &= \oint d\mathbf{r}' g(\mathbf{r}') \mathbf{J}(\mathbf{r}') \cdot \hat{\mathbf{n}} - \iint_{\Delta} d\mathbf{r}' g(\mathbf{r}') \nabla' \cdot \mathbf{J}(\mathbf{r}') \quad (2.59) \end{aligned}$$

As shown in Figure 2.2(a), the RWG basis is a class of vector basis function associated with an edge [10]:

$$\mathbf{f}_n(\mathbf{r}) = \begin{cases} \frac{l_n}{2a_n^+}(\mathbf{r} - \mathbf{r}_n^+), & \mathbf{r} \in T_n^+ \\ \frac{l_n}{2a_n^-}(\mathbf{r}_n^- - \mathbf{r}), & \mathbf{r} \in T_n^- \end{cases} \quad (2.60)$$

When $\mathbf{r}$ is along the common edge, we have the following identity:

$$\mathbf{f}_n^+ \cdot \hat{\mathbf{n}}^+ = 1, \quad \mathbf{f}_n^- \cdot \hat{\mathbf{n}}^- = -1, \quad (2.61)$$

If an RWG basis resides entirely in one layer, the line integrals on the positive and negative patches have the same value but opposite sign, thus cancelling each other. But when a basis straddles layers like in Figure 2.2(b), the expression of the integrand of the contour integral on each patch is different. And it is not self-evident that they will cancel each other. Before

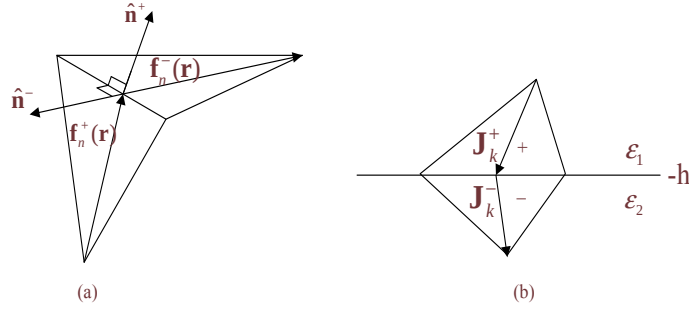


Figure 2.2: (a) RWG basis in free space. (b) RWG basis straddles layers.

starting the derivation, we first define the operator $\hat{B}$ as

$$\hat{B}f(k_\rho, z, z') = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho}{k_\rho} J_0(k_\rho r_s) f(k_\rho, z, z') \quad (2.62)$$

So the Green's function can be simply written as

$$g^\alpha = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho}{k_{mz} k_\rho} [J_0(k_\rho r_s) F^\alpha(k_\rho, z, z')] = \hat{B} \frac{F^\alpha(k_\rho, z, z')}{k_{mz}} \quad (2.63)$$

2.6.1 Case 1: only the testing basis straddles

Here, we assume the positive patch of testing basis lies in layer n and negative one in layer $n - 1$. The source basis resides entirely in layer m . There are two terms we need to consider:

- $\langle \mathbf{J}_{Ti}(\mathbf{r}), \nabla \nabla' \left(-\frac{\partial_z \partial_{z'}}{i\omega \epsilon_n} g^{\text{TM}} - i\omega \mu_m g^{\text{TE}} \right), \mathbf{J}_j(\mathbf{r}') \rangle$
- $-i\omega \langle \mathbf{J}_{Ti}(\mathbf{r}), \nabla \left(\frac{\epsilon_m \mu_m}{\epsilon_n} \partial_z g^{\text{TM}} + \mu_m \partial_{z'} g^{\text{TE}} \right), \hat{z} \cdot \mathbf{J}_j(\mathbf{r}') \rangle$

The contribution of the contour integral from the TM wave is proportional to

$$\begin{aligned}
 I_1^{\text{TM}} &= \hat{B} \int_{e_i} d\mathbf{r} \left[\frac{\partial_z F_{nm}^{\text{TM}}(k_\rho, z, z')}{\epsilon_n k_{mz}} (\mathbf{J}_{Ti}^+ \cdot \hat{\mathbf{n}}^+) + \frac{\partial_z F_{n-1,m}^{\text{TM}}(k_\rho, z, z')}{\epsilon_{n-1} k_{mz}} (\mathbf{J}_{Ti}^- \cdot \hat{\mathbf{n}}^-) \right] \\
 &= \hat{B} \int_{e_i} d\mathbf{r} \frac{1}{k_{mz}} \left[\frac{1}{\epsilon_n} \partial_z F_{nm}^{\text{TM}}(k_\rho, z, z') - \frac{1}{\epsilon_{n-1}} \partial_z F_{n-1,m}^{\text{TM}}(k_\rho, z, z') \right] \\
 &= 0
 \end{aligned} \tag{2.64}$$

The term e_j is the common edge of the j th basis and it lies on the interface between layer n and $n - 1$. The integral vanishes when we substitute the boundary condition (2.23) into it.

The TE part is proportional to

$$\begin{aligned}
 I_1^{\text{TE}} &= \hat{B} \int_{e_i} d\mathbf{r} \left[\frac{F_{nm}^{\text{TE}}(k_\rho, z, z')}{k_{mz}} (\mathbf{J}_{Tj}^+ \cdot \hat{\mathbf{n}}^+) + \frac{F_{n-1,m}^{\text{TE}}(k_\rho, z, z')}{k_{mz}} (\mathbf{J}_{Tj}^- \cdot \hat{\mathbf{n}}^-) \right] \\
 &= \hat{B} \int_{e_i} d\mathbf{r} \frac{1}{k_{mz}} [F_{nm}^{\text{TE}}(k_\rho, z, z') - F_{n-1,m}^{\text{TE}}(k_\rho, z, z')] \\
 &= 0
 \end{aligned} \tag{2.65}$$

From boundary condition (2.20), we know $F_{n-1,m}^{\text{TE}}$ is continuous across the interface and thus the integral also vanishes.

2.6.2 Case 2: only the source basis straddles

Next, we assume the positive patch of source basis lies in layer m and negative one in layer $m - 1$. The testing basis resides in layer n . To study this case, we need to derive the continuity condition of the propagation factor when we fix z and vary z' .

First the symmetry relation of the propagation factor [50] needs to be proved.

$$\left[\frac{d}{dz} \frac{1}{p(z)} \frac{d}{dz} + \frac{1}{p(z)} k_z^2(z) \right] F^\alpha(k_s, z, z_1) = \frac{2i}{p(z)} k_z(z) \delta(z - z_1) \quad (2.66)$$

$$\left[\frac{d}{dz} \frac{1}{p(z)} \frac{d}{dz} + \frac{1}{p(z)} k_z^2(z) \right] F^\alpha(k_s, z, z_2) = \frac{2i}{p(z)} k_z(z) \delta(z - z_2) \quad (2.67)$$

We multiply (2.66) by $F^\alpha(k_s, z, z_2)$, (2.67) by $F^\alpha(k_s, z, z_1)$ and then integrate both equations with respect to z .

$$-\int \frac{dz}{p(z)} \frac{d}{dz} F^\alpha(z, z_2) \frac{d}{dz} F^\alpha(z, z_1) + \int dz \frac{k_z^2(z)}{p(z)} F^\alpha(z, z_2) F^\alpha(z, z_1) = \frac{2i k_z(z_1)}{p(z_1)} F^\alpha(z_1, z_2) \quad (2.68)$$

$$-\int \frac{dz}{p(z)} \frac{d}{dz} F^\alpha(z, z_1) \frac{d}{dz} F^\alpha(z, z_2) + \int dz \frac{k_z^2(z)}{p(z)} F^\alpha(z, z_1) F^\alpha(z, z_2) = \frac{2i k_z(z_2)}{p(z_2)} F^\alpha(z_2, z_1) \quad (2.69)$$

The first terms are obtained from the application of integration by parts. Subtracting the

two equations above, we get the symmetry relation:

$$\frac{k_z(z_1)}{p(z_1)} F^\alpha(z_1, z_2) = \frac{k_z(z_2)}{p(z_2)} F^\alpha(z_2, z_1) \quad (2.70)$$

Again, $p(z)$ stands for $\epsilon(z)$ for TM wave and $\mu(z)$ for TE wave.

Use this relation, the boundary condition (2.20) becomes

$$\frac{\mu_m}{k_{mz}} F_{nm}^{\text{TE}}(z, z') = \frac{\mu_{m-1}}{k_{m-1,z}} F_{n,m-1}^{\text{TE}}(z, z') \quad (2.71)$$

$$\frac{1}{k_{mz}} \frac{d}{dz'} F_{nm}^{\text{TE}}(z, z') = \frac{1}{k_{m-1,z}} \frac{d}{dz'} F_{n,m-1}^{\text{TE}}(z, z') \quad (2.72)$$

$$\frac{\epsilon_m}{k_{mz}} F_{nm}^{\text{TM}}(z, z') = \frac{\epsilon_{m-1}}{k_{m-1,z}} F_{n,m-1}^{\text{TM}}(z, z') \quad (2.73)$$

$$\frac{1}{k_{mz}} \frac{d}{dz'} F_{nm}^{\text{TM}}(z, z') = \frac{1}{k_{m-1,z}} \frac{d}{dz'} F_{n,m-1}^{\text{TM}}(z, z') \quad (2.74)$$

The two terms contains ∇' operator are

- $\langle \mathbf{J}_{Ti}(\mathbf{r}), \nabla \nabla' \left(-\frac{\partial_z \partial_{z'}}{i\omega \epsilon_n} g^{\text{TM}} - i\omega \mu_m g^{\text{TE}} \right), \mathbf{J}_j(\mathbf{r}') \rangle$
- $-i\omega \langle \mathbf{J}_{Ti}(\mathbf{r}) \cdot \hat{z}, \nabla' (\mu_n \partial_{z'} g^{\text{TM}} + \mu_m \partial_z g^{\text{TE}}), \mathbf{J}_j(\mathbf{r}') \rangle$

The contour integral from the TM wave is proportional to

$$\begin{aligned} I_2^{\text{TM}} &= \hat{B} \int_{e_j} d\mathbf{r}' \left[\frac{\partial_{z'} F_{nm}^{\text{TM}}(k_\rho, z, z')}{k_{mz}} (\mathbf{J}_j^+ \cdot \hat{\mathbf{n}}^+) + \frac{\partial_{z'} F_{n,m-1}^{\text{TM}}(k_\rho, z, z')}{k_{m-1,z}} (\mathbf{J}_j^- \cdot \hat{\mathbf{n}}^-) \right] \\ &= \hat{B} \int_{e_j} d\mathbf{r}' \left[\frac{\partial_{z'} F_{nm}^{\text{TM}}(k_\rho, z, z')}{k_{mz}} - \frac{\partial_{z'} F_{n,m-1}^{\text{TM}}(k_\rho, z, z')}{k_{m-1,z}} \right] \\ &= 0 \end{aligned} \quad (2.75)$$

This term goes to zero when we use boundary condition (2.74).

The contribution from TE wave is proportional to

$$\begin{aligned}
I_2^{\text{TE}} &= \hat{B} \int_{e_j} d\mathbf{r}' \left[\frac{\mu_m F_{nm}^{\text{TE}}(k_\rho, z, z')}{k_{mz}} (\mathbf{J}_j^+ \cdot \hat{\mathbf{n}}^+) + \frac{\mu_{m-1} F_{n,m-1}^{\text{TE}}(k_\rho, z, z')}{k_{m-1,z}} (\mathbf{J}_j^- \cdot \hat{\mathbf{n}}^-) \right] \\
&= \hat{B} \int_{e_j} d\mathbf{r}' \left[\frac{\mu_m F_{nm}^{\text{TE}}(k_\rho, z, z')}{k_{mz}} - \frac{\mu_{m-1} F_{n,m-1}^{\text{TE}}(k_\rho, z, z')}{k_{m-1,z}} \right] \\
&= 0
\end{aligned} \tag{2.76}$$

Similarly, this is obtained by using (2.71). So far, we have shown that the line integral vanishes in this case too.

2.6.3 Case 3: both the testing basis and source basis straddle

The interaction of two RWG basis functions can always be split into the interactions of four triangle pairs. From the previous derivation, we know that no matter which pair straddles, there is no line integral. Thus, the expression of the matrix elements given by (2.41) is true not only for the basis which resides in a single layer, but for the straddling basis as well. This is because of the continuity property of the propagation factor at the interface of the layers.

2.6.4 Case 4: basis attached to the conducting ground

When we model microstrip structures, the coaxial feed is often modeled as a metallic via connecting the patch and the ground plane. A delta voltage source on the via serves as the excitation. In the layered medium Green's function approach, we would model it as a layered medium backed by a PEC plane and only the patch and the via parts are

discretized. The effect of the ground plane is included in the Green's function. Since the via is physically attached to the ground plane (Figure 2.3), the continuity condition of the current at the junction of the via and the ground plane should be satisfied. If we model the via with full RWG only, then the normal current is forced to zero at the boundary edge of the via and it would be an incorrect description of the problem. To establish a correct model, we need to take into account the non-zero normal current at the edge of the via.

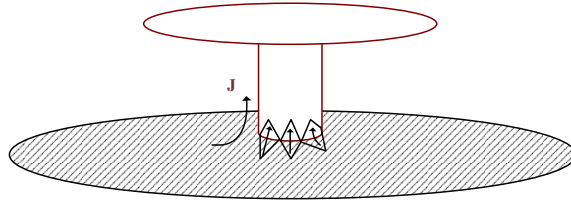


Figure 2.3: Current flow at the feed of a microstrip patch antenna.

The most intuitive solution is to include half-RWG basis on the boundary of the via, so that normal current is allowed to flow from the via to the ground. Recall from (2.59), if $\mathbf{J}(\mathbf{r}')$ is a half-RWG basis, a line integral would appear after performing integration by parts. However, the vector basis functions of full-RWG are defined so that their normal components are continuous across inner edges, and the current representation is free of line or point charges at subdomain boundaries according to the continuity equation [10]. In this case, the boundary of the via should be considered as inner edges, and it should be free of line or point charges at the edge of half-RWG basis as well. Thus we should discard the line integral that appeared in (2.59). As a conclusion, when the metallic objects are attached to the conducting ground plane, half-RWG basis should be constructed at the edges of the junction and the induced line integrals should be discarded.

2.7 Analytical Solution for Some Cylindrically Symmetric Problem

In order to validate the layered medium problem, it is desirable to have the analytical solution for some special cases. Here, we solve some cylindrically symmetric problems analytically. Later, the solutions can be used to verify the numerical results.

2.7.1 Current loop inside a cylindrical waveguide

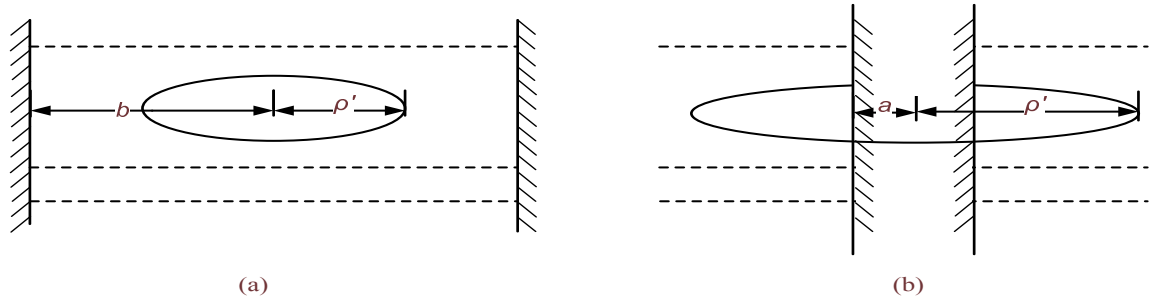


Figure 2.4: (a) Current loop inside a cylindrical waveguide. (b) Current loop in the vicinity of an impenetrable cylinder.

Assuming the electric current loop is at (ρ', z') inside a cylindrical waveguide as in Figure 2.4(a), the field it generated inside the waveguide is a TE to z field. Define a potential as

$$\psi(\rho, z) = \sum_i A(k_{\rho i}) J_0(k_{\rho i} \rho) e^{ik_{zi}|z-z'|} \quad (2.77)$$

The electric and magnetic field components of the TE field can be expressed by this

potential as

$$\begin{aligned} E_z &= E_\rho = 0 \\ E_\phi &= \frac{\partial \psi}{\partial \rho} = - \sum_i A(k_{\rho i}) k_{\rho i} J_1(k_{\rho i} \rho) e^{ik_{zi}|z-z'|} \end{aligned} \quad (2.78)$$

$$H_\rho = -\frac{1}{i\omega\mu} \frac{\partial^2 \psi}{\partial \rho \partial z} = \pm \frac{1}{\omega\mu} \sum_i A(k_{\rho i}) k_{\rho i} k_{zi} J_1(k_{\rho i} \rho) e^{ik_{zi}|z-z'|} \quad (2.79)$$

The unknown coefficients $A(k_{\rho i})$ and parameters $k_{\rho i}$ could be solved from the boundary conditions

$$E_\phi|_{\rho=b} = 0, \quad \Rightarrow \quad J_1(k_{\rho i} b) = 0, \quad i = 1, 2, 3 \dots \quad (2.80)$$

$$\hat{z} \times (H_\rho^+ - H_\rho^-) \hat{\rho}|_{z=z'} = \hat{\phi} I \delta(\rho - \rho'), \quad \Rightarrow \quad A(k_{\rho i}) = \frac{\omega\mu I}{k_{zi} k_{\rho i}} \frac{\rho'}{b^2} \frac{J_1(k_{\rho i} \rho')}{[J_1'(k_{\rho i} b)]^2} \quad (2.81)$$

Thus the analytical expression of the electric field is

$$E_\phi = -\omega\mu I \frac{\rho'}{b^2} \sum_i \frac{1}{k_{zi}} \frac{1}{[J_1'(k_{\rho i} b)]^2} J_1(k_{\rho i} \rho') J_1(k_{\rho i} \rho) e^{ik_{zi}|z-z'|} \quad (2.82)$$

If we let $b \rightarrow \infty$, the summation becomes an integral, and we could get the radiated field generated by a current loop in free space. For large arguments, the asymptotic form of the Bessel function is

$$J_1(k_{\rho i} b) \approx \sqrt{\frac{2}{\pi k_{\rho i} b}} \cos(k_{\rho i} b - \frac{3\pi}{4}), \quad (2.83)$$

$$\text{thus,} \quad J_1(k_{\rho i} b) = 0 \Rightarrow \cos(k_{\rho i} b - \frac{3\pi}{4}) = 0 \Rightarrow \Delta k_{\rho i} = \pi/b \quad (2.84)$$

In free space, the expression of the electric field then becomes

$$E_\phi = -\frac{\omega\mu I\rho'}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_1(k_\rho\rho') J_1(k_\rho\rho) e^{ik_z|z-z'|} \quad (2.85)$$

If the media inside the cylinder is a layered medium, then we only need to replace $e^{ik_z|z-z'|}$ with $F^{\text{TE}}(k_\rho, z, z')$.

2.7.2 Current loop in the vicinity of a cylindrical waveguide

The geometry is shown in Figure 2.4(b). First, we assume there exists another cylindrical waveguide of radius b . Later, we can let b go to infinity, and convert the summation into an integration. Define the potential ψ as

$$\psi = \sum_i A(k_{\rho i}) [J_0(k_{\rho i}\rho) - N_i Y_0(k_{\rho i}\rho)] e^{ik_{zi}|z-z'|} \quad (2.86)$$

The associated TE to z field is

$$E_z = E_\rho = 0$$

$$E_\phi = -\sum_i A(k_{\rho i}) k_{\rho i} [J_1(k_{\rho i}\rho) - N_i Y_1(k_{\rho i}\rho)] e^{ik_{zi}|z-z'|} \quad (2.87)$$

$$H_\rho = \pm \frac{1}{\omega\mu} \sum_i A(k_{\rho i}) k_{\rho i} k_{zi} [J_1(k_{\rho i}\rho) - N_i Y_1(k_{\rho i}\rho)] e^{ik_{zi}|z-z'|} \quad (2.88)$$

N_i and $k_{\rho i}$ can be solved from the boundary conditions that $E_\phi(\rho = a) = 0$ and $E_\phi(\rho = b) = 0$,

$$N_i = \frac{J_1(k_{\rho i}a)}{Y_1(k_{\rho i}a)} \quad (2.89)$$

$$J_1(k_{\rho i}b) - N_i Y_1(k_{\rho i}b) = 0, \quad i = 1, 2, 3 \dots \quad (2.90)$$

And from the boundary condition $(H_\rho^+ - H_\rho^-)|_{z=z'} = I\delta(\rho - \rho')$, we can solve for $A(k_{\rho i})$:

$$A(k_{\rho i}) = -\omega\mu I \frac{k_{\rho i}}{k_{zi}} \frac{\rho' F_1(k_{\rho i}\rho')}{F_1^2(k_{\rho i}b) [(k_{\rho i}b)^2 - 1] - F_1(k_{\rho i}a) [(k_{\rho i}a)^2 - 1]} \quad (2.91)$$

where $F_1(k_\rho \rho) = J_1(k_\rho \rho) - N_1(k_\rho a)Y_1(k_\rho \rho)$, $N_1(k_\rho a) = J_1(k_\rho a)/Y_1(k_\rho a)$.

When $b \rightarrow \infty$, the term $F_1^2(k_{\rho i}b)(k_{\rho i}b)^2$ dominates in the denominator. Using asymptotic analysis, we can obtain that

$$F_1^2(k_{\rho i}b)(k_{\rho i}b) = \frac{2k_{\rho i}b}{\pi} [1 + N_i^2(k_{\rho i}a)] \quad (2.92)$$

Moreover, the expression of the electric field becomes an integral,

$$E_\phi = -\frac{\omega\mu I \rho'}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} \frac{F_1(k_\rho \rho') F_1(k_\rho \rho)}{1 + N^2(k_\rho a)} e^{ik_z|z-z'|} \quad (2.93)$$

If we would like to generate a TM to z field, we can replace the electric current loop with a magnetic current loop. After similar derivation, the magnetic field component is given

by

$$H_\phi = -\frac{\omega \epsilon I \rho'}{2} \int_0^\infty dk_\rho \frac{k_\rho}{k_z} \frac{F_1(k_\rho \rho') F_1(k_\rho \rho)}{1 + N^2(k_\rho a)} e^{ik_z |z - z'|} \quad (2.94)$$

where $N(k_\rho a) = J_0(k_\rho a)/Y_0(k_\rho a)$ in this case. If the current loop and the cylindrical waveguide are located in a layered medium, we only need to replace $e^{ik_z |z - z'|}$ with $F^{\text{TE}}(k_\rho, z, z')$ or $F^{\text{TM}}(k_\rho, z, z')$.

2.8 Numerical Results

In this section, some numerical results are presented and compared with either analytical results or results from other formulation. The good agreement between them shows the validity of the new form of layered medium Green's function.

2.8.1 Scattering of a sphere over layered medium

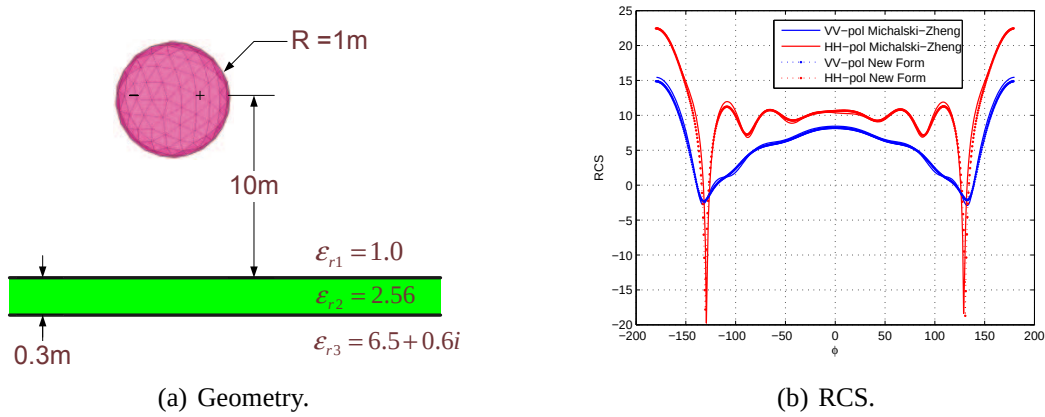


Figure 2.5: The radar cross section (RCS) of a sphere over a layered medium.

The first example studied is the scattering of a perfect electric conductor sphere above a two-layer medium, with the permittivity of the two layers as $\epsilon_{r,2} = 2.56$ and $\epsilon_{r,3} = 6.5 + 0.6i$. The thickness of the first dielectric layer is 0.3 m. The second layer is a lossy half space. The radius of the sphere is 1 m. The distance between the sphere and the interface is 10 m, as shown in Figure 2.5(a). The object is discretized by 432 small triangle patches at 300 MHz and the number of unknown is 647. The incident plane wave direction is $\theta^{inc} = 60^\circ$ and $\phi^{inc} = 0^\circ$. The bistatic RCS is computed at $\theta = 60^\circ$ and $\phi \in [-180^\circ, 180^\circ]$. The results calculated by the new form are compared with those calculated from the Michalski-Zheng formulation [5] and good agreement has been observed.

2.8.2 A cylinder straddles two half-spaces

In this part, two examples are constructed and compared with the analytical results from Section 1.5.2. The first one (Figure 2.6(a)) is a conducting cylinder straddling two lossy half-spaces, excited by an electric current loop. The radius of the cylinder is 0.1 m and its height is 1.2 m. It is discretized by 256 small triangles. Half of it is in the upper space ($\epsilon_{r,1} = 1 + 6i$) and the other half is buried in the lower space ($\epsilon_{r,2} = 16 + 6i$). The radius of the current loop is 0.2 m and it is 0.3 m above the interface of the two spaces.

In the analytical model, the cylindrical waveguide is infinitely long. Here we choose two highly lossy half-spaces so that we can use a truncated cylinder to approximate the infinitely long one. The near field distributions along both $\hat{\rho}$ and $\hat{z}$ directions obtained from numerical methods agree well with the analytical results (Figure 2.7). It validates the matrix expression of this new form for straddling objects.

The second example is similar to the first one, except that we have a perfect electric

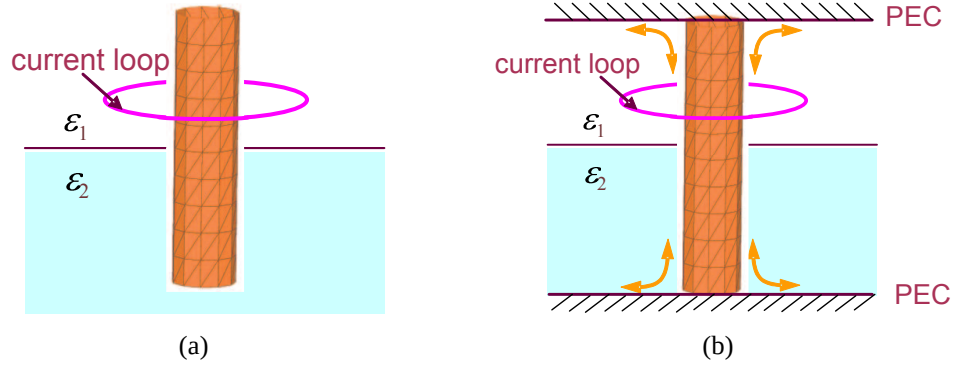
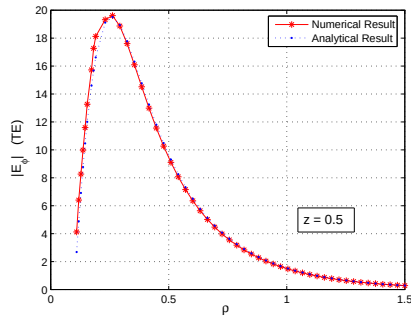
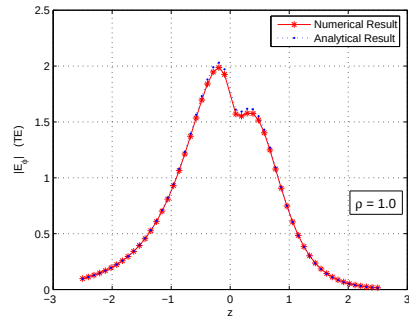


Figure 2.6: (a) Cylinder straddles two half-spaces. (b) With PEC ceiling and ground.



(a) Near field along $\hat{\rho}$



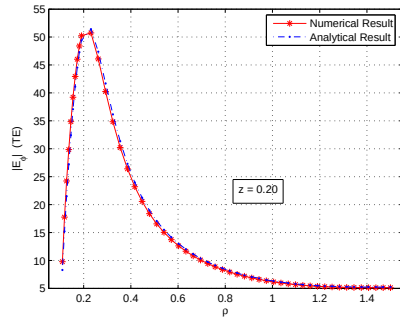
(b) Near field along $\hat{z}$

Figure 2.7: Near field in the geometry shown in Figure 2.6(a) (TE case).

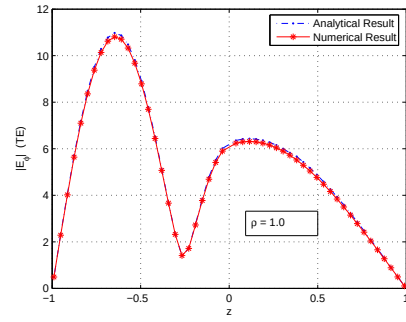
conductor (PEC) both as ground and ceiling in the structure. Here, the conducting cylinder is terminated by both ground and ceiling, and the model used in numerical analysis is the same as the one used in theoretical analysis in Section 1.5.2. The medium between the two PEC plates is an air substrate ($\epsilon_{r,1} = 1$) and a lossless dielectric substrate ($\epsilon_{r,2} = 16$). The thickness of both substrates is 1 m. The cylinder has a radius of 0.1 m, a height of 2 m and is discretized by 512 triangles. The radius of the current loop is 0.2 m and it is 0.3 m above the interface of the two substrates. The near field distributions from numerical methods again agree well with the analytical results (Figure 2.8). This example proves the validity of treating the boundary basis as half RWG basis when objects are connected to the ground or the ceiling. As shown in Figure 2.8(c), the current is not terminated at the top and bottom edge of the cylinder.

2.8.3 A cylindrical cavity in layered medium

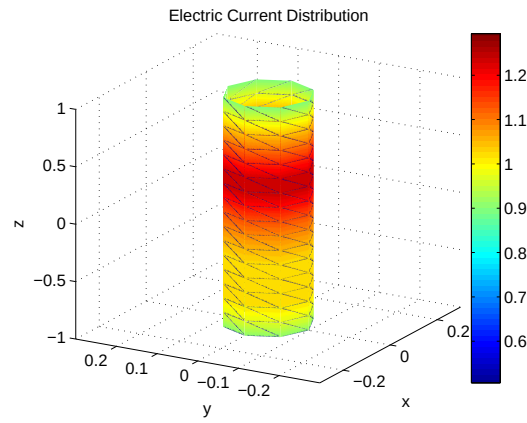
In this example, the electric current loop is placed inside a cylindrical cavity (Figure 2.9). It could be analyzed by the model of a current loop inside a cylindrical waveguide with PEC ground and ceiling. In our numerical model, only the top and the wall of the cavity are discretized. The top part lies on top of the dielectric substrate. The wall penetrates the substrate and is attached to the ground plane. The thickness of the substrate is 0.5 m and its relative permittivity is 2.6. The height of the cavity is 0.5 m and its radius is 2.5 m. The radius of the current loop is 2.0 m and it is 0.25 m above the ground plane. This structure contains both horizontal and vertical structures which straddle two layers; thus it serves as a good test case. The near field results are given in Figure 2.10.



(a) Near field along $\hat{\rho}$



(b) Near field along $\hat{z}$



(c) Current distribution on the cylinder.

Figure 2.8: Near field in the geometry shown in Figure 2.6(b) (TE case).

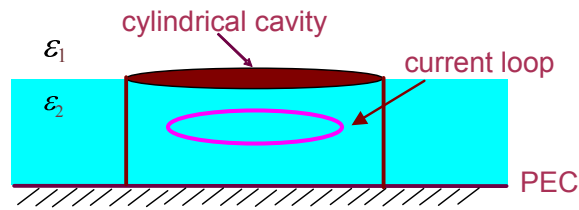
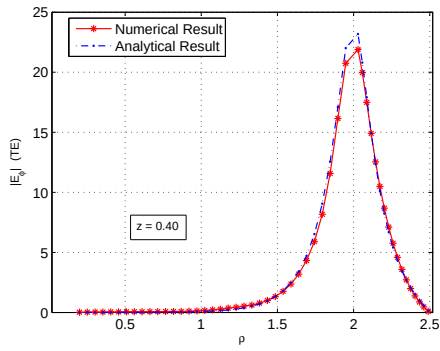
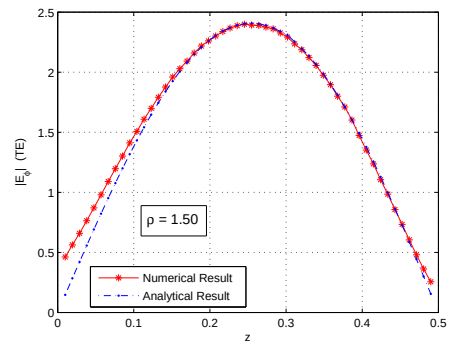


Figure 2.9: A cylindrical cavity resides in layered medium.



(a) Near field along $\hat{\rho}$



(b) Near field along $\hat{z}$

Figure 2.10: Near field of current loop inside a cylindrical cavity.

CHAPTER 3

QUASI-3D THIN-STRATIFIED MEDIUM FAST-MULTIPOLE ALGORITHM (TSM-FMA)

In this chapter, the key points to implement an quasi-3D thin-stratified medium fast-multipole algorithm are discussed. First, the evaluation of Sommerfeld integrals is expedited by deforming the integration path from Sommerfeld integration path (SIP) to the vertical steepest descent path. Then the deformed integral is approximated using standard Gaussian-Legendre quadratures. For each quadrature point, a two-dimensional multilevel fast-multipole algorithm is applied to accelerate the matrix vector multiplication process. The direct interaction term is treated similarly after some manipulation. Memory requirement analysis is presented and memory reduction techniques are discussed. Finally, numerical tests are performed and they confirm that both the computational complexity and the memory cost are of $O(N \log N)$.

3.1 Introduction

To accelerate the numerical integration of Sommerfeld integrals, there are three popular approaches. One is to pre-compute the integrals on a grid of points in the solution domain and to use interpolation techniques. For a strictly planar microstrip structures, only 1D interpolation on ρ is required. If the object is confined in a single layer, the DGLM could

be split into two parts, depending on $\rho, z - z'$ and $\rho, z + z'$ respectively. So 2D interpolation is needed. For an arbitrarily shaped object, the interpolation should be done with three variables: ρ, z and z' [2]. Another vastly popular approach is the discrete complex image method (DCIM) [17, 27–32]. The basic idea of the method is to extract the quasi-static image and guided waves from the spectral kernel, and approximate the remaining term by the sum of complex exponentials. Then the Sommerfeld integral could be evaluated in closed form via Sommerfeld identity [18] for each term, which is known as a complex image. This method is very efficient since it obviates numerical integration. The drawback is that it has no built-in convergence measure, and its accuracy can only be checked against conventional Sommerfeld integrations. In this dissertation, the third approach, the path deformation technique [18, 20, 21, 33, 34] has been adopted. The integrand is converging exponentially along the steepest descent path (SDP) and the deformed integral can be evaluated efficiently. Mathematically, this technique is rigid and the numerical error could be controlled by the truncation point and the order of Gaussian quadratures used in the numerical integration along the SDP. Usually it is difficult to find an SDP exactly, but a convenient approximate path exists for thin-stratified media.

The conjugate gradient fast Fourier transform (CG-FFT) [7, 35, 52, 53] method and the multilevel fast multipole algorithm based methods [54–57] are two commonly used methods for solving large microstrip problems. Both methods could significantly speed up the iterative solver by reducing the computational complexity of the matrix-vector multiplication to $O(N \log N)$. CG-FFT is more efficient than MLFMA based method when dealing with regular grid geometries (dense structure with rectilinear meshes), which are suitable for 2D-FFT, since the FFT algorithm has been studied for several decades and it

is highly mature. For irregular meshes or arbitrarily shaped structures, the meshes should be projected onto a regular grid to apply CG-FFT. When the structure is sparse (e.g. much empty space between microstrip lines), the efficiency of CG-FFT will deteriorate because the empty space has to be padded with zeros in 2D-FFT. On the other hand, the MLFMA based methods can handle triangle meshes and sparse structures easily without additional work or loss of efficiency. In this proposal, a fast algorithm will be developed based on path deformation technique and the principle of 2D MLFMA.

3.2 Path Deformation of Sommerfeld Integration

Recall from (2.41), a basic integral of this new dyadic Green's function for layered medium is

$$g_s^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{4\pi} \int_0^\infty dk_\rho \frac{k_\rho}{k_{mz}} [J_0(k_\rho r_s) F^{\text{TE}}(k_\rho, z, z')] \quad (3.1)$$

By using the fact that $J_0(k_\rho \rho) = 1/2[H_0^{(1)}(k_\rho \rho) + H_0^{(2)}(k_\rho \rho)]$ and the reflection formula that $H_0^{(1)}(-x) = -H_0^{(2)}(x)$, we get

$$g_s^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi} \int_{-\infty}^\infty dk_\rho \frac{k_\rho}{k_{mz}} [H_0^{(1)}(k_\rho r_s) F^{\text{TE}}(k_\rho, z, z')] \quad (3.2)$$

The evaluation of Sommerfeld-type integrals is very time-consuming because the integrands are both highly oscillatory and slowly decaying (see Figure 3.1). By virtue of Jordan's lemma and Cauchy's theorem, it is desirable to deform the integration path from the original Sommerfeld integration path (SIP) to the steepest descent path (SDP), where

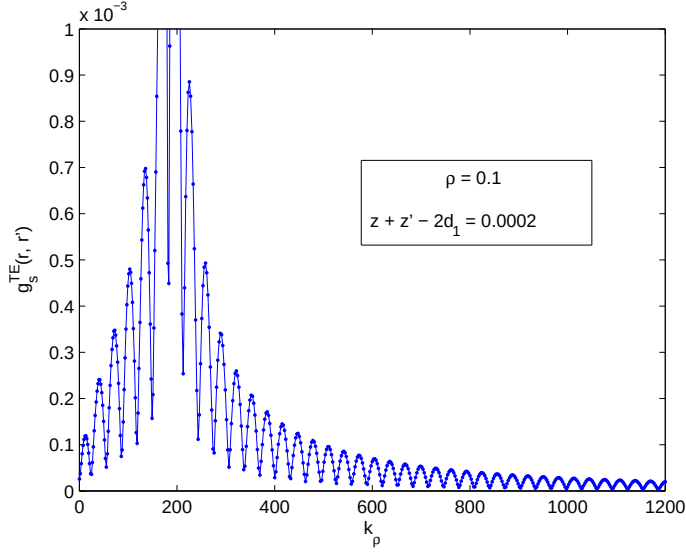


Figure 3.1: Integrand of (3.1) along the Sommerfeld integration path.

the integrand is decaying rapidly so that numerical integration is more efficient. However, the analytical expression of the SDP is not always available and it changes with source and observation points. It would be inefficient if we need to spend resources to find the SDP for each pair of $\mathbf{r}$ and $\mathbf{r}'$. Thus, a simple and geometry-independent path where the integrand also decays exponentially would be preferable. Moreover, in the region enclosed by SIP and the new path, the integrand should be analytic except for the pole singularities. In other words, the new path should enclose no singularity point except for the singularity from poles.

3.2.1 Choose the path around the vertical branch cuts as the SDP

For the thin-stratified medium problems, such as microstrip structures, the horizontal dimension is usually much larger than the vertical dimension. In this case the path around

the vertical branch cut (Figure 3.2) would be an optimal choice [20, 54, 55]. We will take a

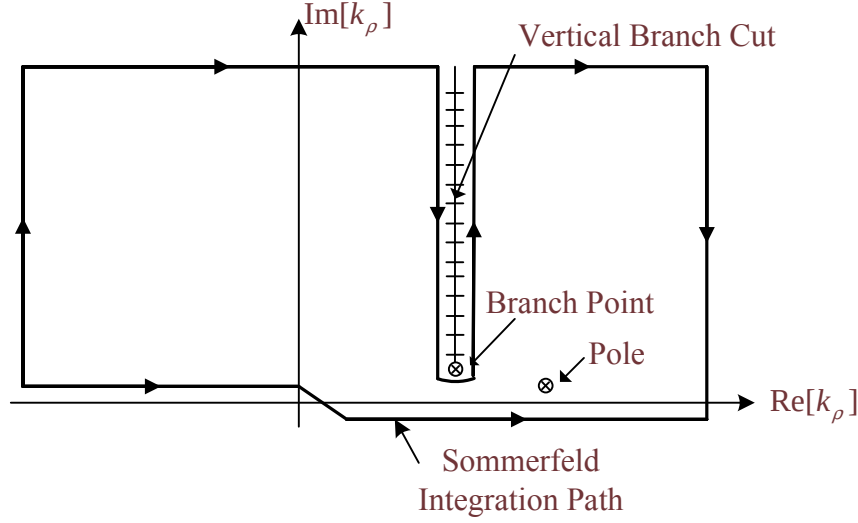


Figure 3.2: Deformation of the Sommerfeld integration path to the vertical branch cut.

simple example to illustrate the reason. If both the source point and the observation point are above the layered medium and close to the surface, (3.2) would become

$$g_s^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi} \int_{-\infty}^{\infty} dk_{\rho} \frac{k_{\rho}}{k_{1z}} H_0^{(1)}(k_{\rho}\rho) [e^{ik_{1z}|z-z'|} + \tilde{R}^{\text{TE}} e^{ik_{1z}(z+z'-d_1)}] \quad (3.3)$$

A large portion of the interacting elements have a greater separation in the ρ direction than in the z direction: $|\boldsymbol{\rho} - \boldsymbol{\rho}'| \gg |z - z'|$ and $|\boldsymbol{\rho} - \boldsymbol{\rho}'| \gg |z + z' - 2d_1|$. An asymptotic analysis of the expression is performed:

$$\begin{aligned} k_{\rho} \rightarrow \infty \quad \Rightarrow \quad k_{1z} &= \sqrt{k_1^2 - k_{\rho}^2} \rightarrow ik_{\rho}, \\ H_0^{(1)}(k_{\rho}\rho) &\rightarrow \sqrt{\frac{2}{\pi k_{\rho}\rho}} e^{i(k_{\rho}\rho - \pi/4)}, \\ e^{ik_{1z}h} &\rightarrow e^{-k_{\rho}h}, \quad h = |z - z'| \text{ or } |z + z' - 2d_1| \end{aligned} \quad (3.4)$$

Thus, the integrand decays as $e^{-k_\rho h} / \sqrt{k_\rho}$. When h is small, the exponential decay would be slow as k_ρ grows large compared to the case when h is large. Now if we evaluate the integrand along the vertical branch cut, the integration converges exponentially fast (see Figure 3.3) because of the factor $e^{-s\rho}$. The path also bypasses the branch point singularity by integrating around it.

$$\begin{aligned}
k_\rho = k_1 + is, \quad s \rightarrow \infty \Rightarrow \quad k_{1z} = \sqrt{k_1^2 - k_\rho^2} \rightarrow s, \\
H_0^{(1)}(k_\rho \rho) \rightarrow \sqrt{\frac{2}{\pi k_\rho \rho}} e^{i(k_1 \rho - \pi/4) - s\rho}, \\
e^{ik_{1z}h} \rightarrow e^{ish}, \quad h = |z - z'| \text{ or } |z + z' - 2d_1|
\end{aligned} \tag{3.5}$$

Another change of variable $s = u^2$ is made to eliminate the $1/\sqrt{s}$ singularity appearing

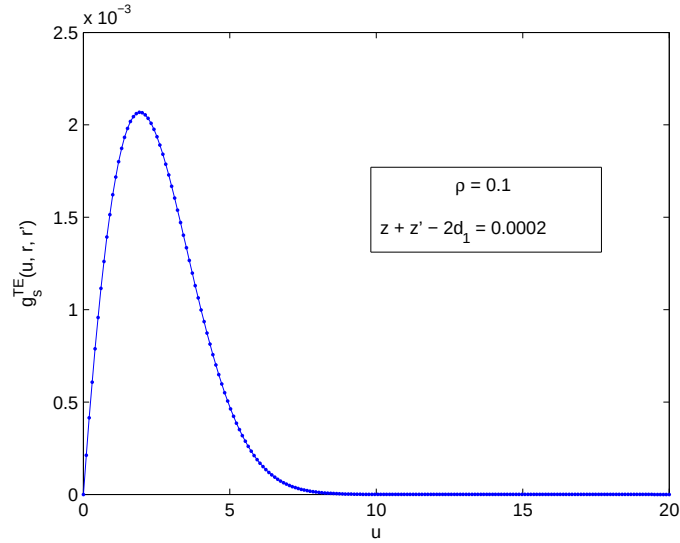


Figure 3.3: Integrand of (3.1) along one side of the vertical branch cut.

in the integrand.

$$k_\rho = k_1 + iu^2 \Rightarrow \quad dk_\rho = 2iudu, \quad k_{1z} = \pm u\sqrt{u^2 - 2ik_1}, \quad (3.6)$$

$$dk_\rho/k_{1z} = \pm \frac{2i}{\sqrt{u^2 - 2ik_1}}$$

Now the evaluation of the integral in (3.3) is done in two parts: the detoured path integral around vertical branch cut ($g_{s,BR}^{\text{TE}}$) and the contribution from the poles ($g_{s,P}^{\text{TE}}$) enclosed by the new path and SIP. Once the poles are found, the contribution from the poles is expressed as

$$g_s^{\text{TE}}(\mathbf{r}, \mathbf{r}') = g_{s,BR}^{\text{TE}}(\mathbf{r}, \mathbf{r}') + g_{s,P}^{\text{TE}}(\mathbf{r}, \mathbf{r}') \quad (3.7)$$

We discuss each of them in the following sections.

3.2.2 Integration around the vertical branch cut

For a general planar layered media, there are at most two branch points associated with the outermost layers on the top or bottom respectively (§2.7 [18]). The physical reason is that branch points are associated with lateral wave, which is a part of the spherical wave that is propagating along the interface of a substrate. Since a spherical wave exists only in unbounded regions, a lateral wave is always found within an unbounded region 1 or N . Thus, there are two branch points k_1 and k_N only when both the top and bottom layers are unbounded. If a conducting ground plane is added to either region, the associated branch point vanishes. If the whole structure is inside infinitely large parallel plates, then only

guided modes exist. Neither the branch point nor the contour integral exists any more.

In the following section, we will discuss the most complicated case in which two branch points exist. Since k_{1z} and k_{Nz} are multivalued functions associated with k_1 and k_N , it is very important to choose their signs correctly to ensure that the integrand is continuous along the integration path. In the complex k_ρ plane in Figure 3.4, there are two branch points and each is associated with a vertical branch cut. The imaginary parts of

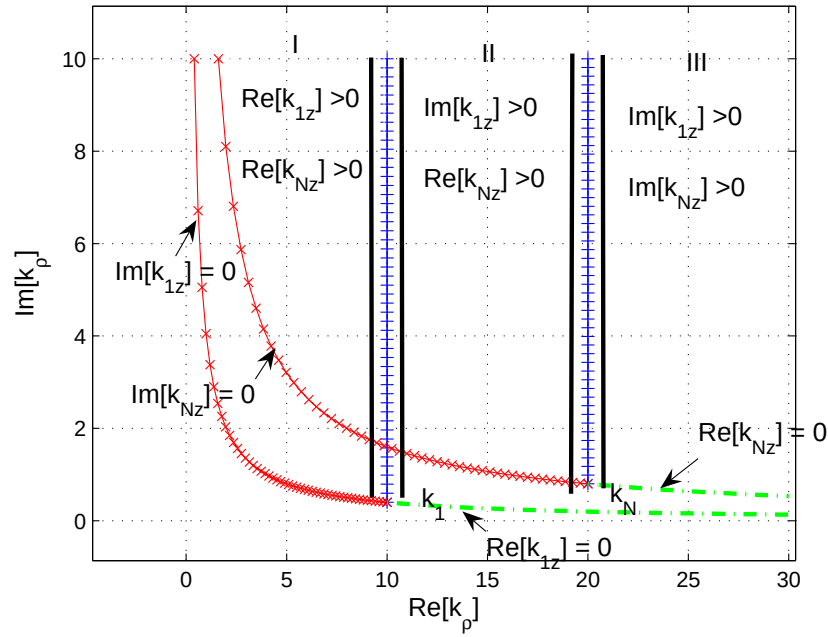


Figure 3.4: Sign choice for k_{1z} and k_{Nz} in complex k_ρ space.

both k_{1z} and k_{Nz} should be positive when $k_\rho \rightarrow \infty$ or $k_\rho \rightarrow -\infty$, so that the integrand vanishes at $|k_\rho| \rightarrow \infty$. To satisfy the continuity condition, the sign of the imaginary part of k_{1z} should change whenever the integration path goes across the red line representing $\text{Im}[k_{1z}] = 0$. Similarly, the sign of the real part of k_{1z} changes whenever the integration path goes across the green line representing $\text{Re}[k_{1z}] = 0$. The same thing is true for k_{Nz} .

Since region I is split by the $\text{Im}[k_{1z}] = 0$ and $\text{Im}[k_{Nz}] = 0$ lines, it is not efficient to determine the sign of k_{1z} and k_{Nz} based on their imaginary part. But the sign of the real part of k_{1z} and k_{Nz} should be consistent throughout region I and opposite to that in the upper part of region III.

To find the sign of the real part of k_{iz} , we can start from the upper part of region III. If we assume $k_i = k'_i + ik''_i$, we have $k'_1 k''_1 < k'_\rho k''_\rho$ in the upper part of region III (above the green line). In addition, $k''_{1z} > 0$. Thus, from (3.8) we conclude that $k'_{1z} < 0$ in this part. This suggests that both $\text{Re}[k_{1z}]$ and $\text{Re}[k_{Nz}]$ are positive in region I.

$$\begin{aligned} k_{1z}^2 &= (k'_{1z} + k''_{1z})^2 = k_{1z}'^2 - k_{1z}''^2 + 2ik'_{1z}k''_{1z} \\ &= (k_1'^2 - k_1''^2) - (k_\rho'^2 - k_\rho''^2) + 2i(k_1'k_1'' - k_\rho'k_\rho'') \end{aligned} \quad (3.8)$$

Similarly, in region III, $\text{Im}[k_{1z}]$ and $\text{Im}[k_{Nz}]$ are consistently positive. In region II, $\text{Im}[k_{1z}]$ and $\text{Re}[k_{Nz}]$ should be positive. Following this rule, we plot the integrand as a function of u along the path around the vertical branch cut in Figure 3.5. The integrand is continuous and it validates the rules for sign choice of k_{1z} and k_{Nz} .

The contribution from the path integral can be expressed as

$$\begin{aligned} g_{s,BR}^{\text{TE}}(\mathbf{r}, \mathbf{r}') &= -\frac{1}{4\pi} \int_0^\infty du (k_1 + iu^2) H_0^{(1)}((k_1 + iu^2)\rho) \left[-\frac{u}{k_{1z}} F^{\text{TE}}(k_\rho, z, z') \right]_{\text{I}} \\ &\quad + \frac{u}{k_{1z}} F^{\text{TE}}(k_\rho, z, z') \Big|_{\text{II}} - \frac{1}{4\pi} \int_0^\infty du (k_N + iu^2) H_0^{(1)}((k_N + iu^2)\rho) \\ &\quad \left[-\frac{u}{k_{1z}} F^{\text{TE}}(k_\rho, z, z') \right]_{\text{II}} + \frac{u}{k_{1z}} F^{\text{TE}}(k_\rho, z, z') \Big|_{\text{III}} \end{aligned} \quad (3.9)$$

The deformed integral is smooth and decays fast, so that it can be easily evaluated using

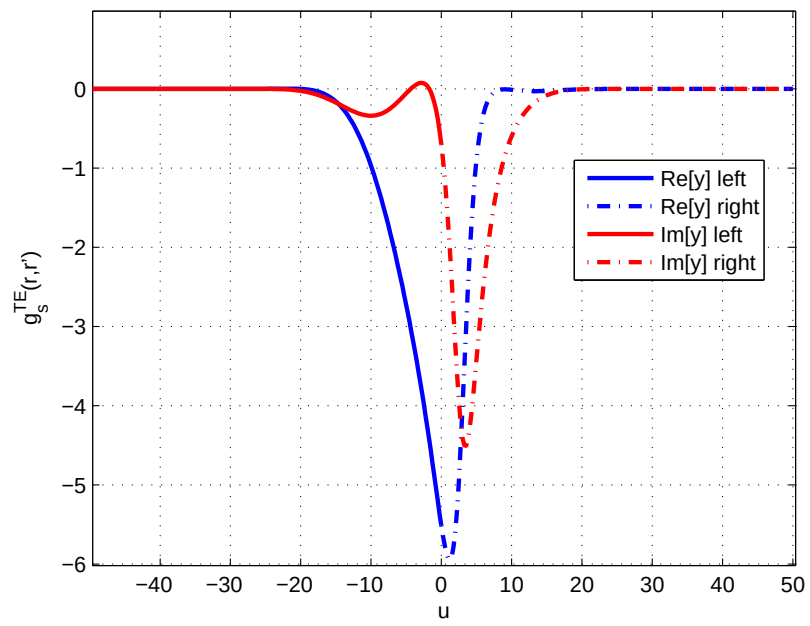


Figure 3.5: Integrand is continuous along the SDP path using the correct sign chosen rule.

Gaussian-Legendre quadrature. However, when poles exist near the branch point, it would cause the integrand to vary rapidly in the neighborhood of the pole. Hence, the integration range is first divided into two ranges. Starting from the branch point, the first range is short and more points are used in this range. The second range is integrated adaptively depending on the value of ρ . Recall (3.5), the oscillating factor of the integrand is $e^{i(k_{1z}h+k_1\rho)}$. Since $\rho \gg h$, the period of the oscillation is mainly determined by ρ . The larger the ρ is, the less the oscillation and the more rapid the decay. The adaptive rule would not divide the second range further and choose fewer quadrature points when ρ is relatively large compared to free space wavelength ($\rho > 0.1\lambda$). When $\rho < 0.1\lambda$, more subdivisions and higher order Gaussian quadrature rule are needed to get good accuracy. The discretized integral is expressed as:

$$\begin{aligned}
g_{s,BR}^{\text{TE}} = & -\frac{1}{4\pi} \sum_{p=1}^{N_q} w_p u_p k_{1\rho,p} H_0^{(1)}(k_{1\rho,p}\rho) \left[\frac{F^{\text{TE}}(k_{1\rho,p}, z, z')}{k_{1z,p}} \Big|_{\text{II}} - \frac{F^{\text{TE}}(k_{1\rho,p}, z, z')}{k_{1z,p}} \Big|_{\text{I}} \right] \\
& -\frac{1}{4\pi} \sum_{p=1}^{N_q} w_p u_p k_{N\rho,p} H_0^{(1)}(k_{N\rho,p}\rho) \left[-\frac{F^{\text{TE}}(k_{N\rho,p}, z, z')}{k_{1z,p}} \Big|_{\text{II}} + \frac{F^{\text{TE}}(k_{N\rho,p}, z, z')}{k_{1z,p}} \Big|_{\text{III}} \right] \\
& k_{1\rho,p} = k_1 + iu_p^2, \quad k_{N\rho,p} = k_N + iu_p^2
\end{aligned} \tag{3.10}$$

where N_q is the number of quadrature points, u_p are the quadrature points, and w_p are the weights.

3.2.3 Pole singularity contribution

The factor $F^{\text{TE}}(k_\rho, z, z')$ and $F^{\text{TM}}(k_{1\rho,p}, z, z')$ might have poles in complex k_ρ plane. Those poles correspond to the guided modes in the layered slab and their contribution should be

considered. Those modes are determined by the structure of the layered media and the working frequency. We have adopted the root solver from [33], which is efficient as well as accurate. The basic idea for this solver is that from the Cauchy theorem, the contour integral of any analytical function is zero. We take the entire complex plane as a rectangle box and perform a contour integral of a generalized reflection coefficient around it. If it is non-zero, which suggests poles exist, the big box is divided into four smaller boxes, and contour integrals are performed again. Each time, we ignore the boxes containing no poles and further divide the boxes which contain poles. In this manner, the locations of the poles can be estimated after a few more subdivisions and they are refined within the finest box.

$$g_{s,P}^{\text{TE}}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi} \sum_{p=1}^{N_p} \frac{k_{\rho,p}}{k_{1z,p}} H_0^{(1)}(k_{\rho,p}\rho) \text{Res}[F^{\text{TE}}(k_{\rho,p}, z, z')] \quad (3.11)$$

where $k_{\rho,p}$ are the poles and N_p is the number of poles.

So far, the Sommerfeld integral has been transformed to a summation at a discrete set of $k_{\rho,p}$. Next, we will discuss how to realize the fast matrix-vector multiplication with this discrete form.

3.3 Application of 2D MLFMA to DGLM

In this section, we will review the two-dimensional multilevel fast multipole algorithm first. Then the technique is incorporated with the discretized form of the Sommerfeld integral to accelerate the matrix-vector multiplication for the impedance matrix generated from the new dyadic Green's function for layered media.

3.3.1 Review of 2D MLFMA

Using the translational addition theorem [58, 59] and the integral representation of Bessel functions, Hankel functions can be decomposed as:

$$H_0^{(1)}(k\rho_{ji}) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \tilde{\beta}_{jl'}(\alpha) \tilde{\alpha}_{ll'}(\alpha) \tilde{\beta}_{li}(\alpha) \quad (3.12)$$

where

$$\tilde{\alpha}_{ll'}(\alpha) = \sum_{p=-P}^P H_p^{(1)}(k\rho_{ll'}) e^{-ip(\phi_{ll'} - \alpha + \pi/2)}, \quad (3.13)$$

$$\tilde{\beta}_{jl'}(\alpha) = e^{-ik\rho_{jl'} \cos(\alpha - \phi_{jl'})} = e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_{jl'}}, \quad \tilde{\beta}_{li}(\alpha) = e^{-ik\rho_{li} \cos(\alpha - \phi_{li})} = e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_{li}} \quad (3.14)$$

$\boldsymbol{\rho}_i$ is the source point, $\boldsymbol{\rho}_j$ is the field point, $\boldsymbol{\rho}_l$ the box center which contains the source point, and $\boldsymbol{\rho}_{l'}$ is the box center which contains the field point. This decomposition is true as long as $\rho_{ll'} > |\rho_{jl'} + \rho_{li}|$. If the elements of an impedance matrix are proportional to $H_0^{(1)}(k\rho_{ji})$, then the product of this matrix with a vector $[a_i]$ can be expressed as

$$b_j = \sum_i H_0^{(1)}(k\rho_{ji}) a_i = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \tilde{\beta}_{jl'}(\alpha) \sum_{l \neq l' + NN} \tilde{\alpha}_{ll'}(\alpha) \sum_{i \in \mathcal{G}_l} \tilde{\beta}_{li}(\alpha) a_i$$

+Near Interaction, $j \in \mathcal{G}_{l'}, l' = 1, 2, \dots$ (3.15)

$l \neq l' + NN$ means that $\mathcal{G}_l$ is not the self group $\mathcal{G}_{l'}$ or the near neighbor groups of it. The integral again can be replaced by a Q -point summation.

$$b'_j = \frac{1}{Q} \sum_{q=1}^Q \tilde{\beta}_{jl'}(\alpha_q) \sum_{l \neq l' + NN} \tilde{\alpha}_{ll'}(\alpha_q) \sum_{i \in \mathcal{G}_l} \tilde{\beta}_{li}(\alpha_q) a_i \quad (3.16)$$

This can be expressed as the products of matrix and vectors.

$$b'_j = \frac{1}{Q} \tilde{\beta}_{j l'}^t \cdot \sum_{l \neq l' + NN} \tilde{\alpha}_{l' l} \cdot \sum_{i \in \mathcal{G}_l} \tilde{\beta}_{li} a_i \quad (3.17)$$

where $\tilde{\alpha}_{l' l}$ is a diagonal matrix. This essentially illustrates a three-stage information transmission from source points to field points (see Figure 3.6) aggregate from each source point to the center of the groups, translate the information of each group to other groups who are not their near neighbors, and finally disaggregate the field information from group centers to each field point. It can be extended to a multilevel decomposition:

$$H_0^{(1)}(k \rho_{ji}) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \tilde{\beta}_{j J_1}(\alpha) \tilde{\beta}_{J_1 J_2}(\alpha) \tilde{\alpha}_{J_2 I_2}(\alpha) \tilde{\beta}_{I_2 I_1}(\alpha) \tilde{\beta}_{I_1 i}(\alpha) \quad (3.18)$$

where $\tilde{\beta}_{ab}(\alpha) = e^{-i\mathbf{k} \cdot \boldsymbol{\rho}_{ab}}$, and

$$b'_j = \frac{1}{Q} \tilde{\beta}_{j J_1}^t \cdot \tilde{\beta}_{J_1 J_2} \cdot \sum_{J_2 \neq I_2 + NN} \tilde{\alpha}_{J_2 I_2} \cdot \sum_{I_1 \in \mathcal{G}_{I_2}} \tilde{\beta}_{I_2 I_1} \cdot \sum_{i \in \mathcal{G}_{I_1}} \tilde{\beta}_{I_1 i} a_i \quad (3.19)$$

When the matrix-vector multiplication process is done through aggregation, translation and disaggregation, the computational complexity is reduced to $O(N \log N)$, and so is the memory requirement [59]. This would greatly accelerate the iterative solver.

This technique is first applied in the two-dimensional scattering problem, since the 2D scalar Green's function is $\frac{i}{4} H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$. But it is not limited to that simple case. Consider a Green's function of the following form:

$$g_s(\mathbf{r}_j, \mathbf{r}_i) = f_v(k, z_j) h(k_\rho) H_0^{(1)}(k|\boldsymbol{\rho}_j - \boldsymbol{\rho}'_i|) f_r(k, z_i) \quad (3.20)$$

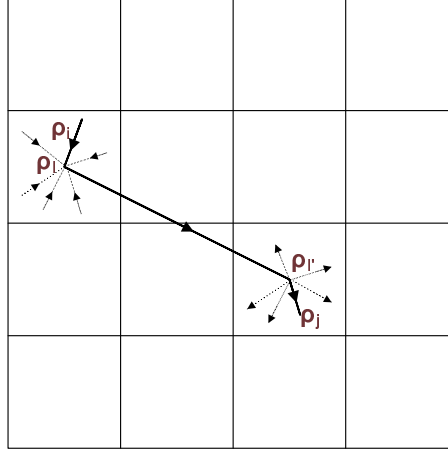


Figure 3.6: Graph of two-dimensional fast multipole algorithm.

It can also be decomposed as

$$g_s(\mathbf{r}_j, \mathbf{r}_i) = \frac{1}{Q} h(k_\rho) \sum_{q=1}^Q \underbrace{[f_v(k, z_j) e^{-i\mathbf{k}_q \cdot \boldsymbol{\rho}_{jv}}]}_{\text{receiving pattern}} \tilde{\alpha}_{ll}(\alpha_q) \underbrace{[e^{-i\mathbf{k}_q \cdot \boldsymbol{\rho}_{vi}} f_r(k, z_i)]}_{\text{radiation pattern}} \quad (3.21)$$

The matrix-vector product could be calculated efficiently by following the same process described earlier. Only the radiation pattern and receiving pattern have been changed.

In the next part, the dyadic Green's function for layered media would be cast into functions in the form of (3.20) and 2D multilevel fast multipole algorithm technique is used to accelerate the iterative solver.

3.3.2 Decomposition of DGLM

In the matrix representation of the DGLM (2.41), there are five Sommerfeld integrals, each associated with a different basis and testing function. It will be inefficient to decompose each of them. Instead, we start from (2.8), the dyadic Green's function for layered media

expressed in terms of two scalar potential integrals $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$, $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$. Now there are only two fundamental Sommerfeld integrals involved, much less than in (2.41).

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = (\nabla \times \hat{\mathbf{z}})(\nabla' \times \hat{\mathbf{z}})g^{\text{TE}}(\mathbf{r}, \mathbf{r}') + \frac{1}{k_{nm}^2}(\nabla \times \nabla \times \hat{\mathbf{z}})(\nabla' \times \nabla' \times \hat{\mathbf{z}})g^{\text{TM}}(\mathbf{r}, \mathbf{r}') \quad (3.22)$$

The expressions for $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$ and $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$ are

$$g^{\alpha}(\mathbf{r}, \mathbf{r}') = \frac{i}{4\pi} \int_0^\infty \frac{dk_\rho}{k_{mz}k_\rho} [J_0(k_\rho|\boldsymbol{\rho} - \boldsymbol{\rho}'|)F^\alpha(k_\rho, z, z')] \quad (3.23)$$

or

$$g^{\alpha}(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi} \int_{-\infty}^\infty \frac{dk_\rho}{k_{mz}k_\rho} \left[H_0^{(1)}(k_\rho|\boldsymbol{\rho} - \boldsymbol{\rho}'|)F^\alpha(k_\rho, z, z') \right] \quad (3.24)$$

The poles at $k_\rho = 0$ in (3.23) and (3.24) disappear when they are substituted into (3.22) because physically they are fictitious poles.

Using path deformation and Gaussian quadrature rule, $g^{\text{TE}}(\mathbf{r}, \mathbf{r}')$ and $g^{\text{TM}}(\mathbf{r}, \mathbf{r}')$ could be expressed as a discrete summation.

$$g^{\alpha}(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi} \sum_{p=1}^{N_q} \frac{w_p u_p}{k_{\rho,p}} H_0^{(1)}(k_{\rho,p}\rho) \left[\frac{F^\alpha(k_{\rho,p}, z, z')}{k_{mz,p}} \Big|_{\text{VB}^+} - \frac{F^\alpha(k_{\rho,p}, z, z')}{k_{mz,p}} \Big|_{\text{VB}^-} \right] \\ + \frac{i}{8\pi} \sum_{p=1}^{N_p} \frac{1}{k_{mz,p}k_{\rho,p}} H_0^{(1)}(k_{\rho,p}\rho) \text{Res}[F^\alpha(k_{\rho,p}, z, z')] \quad (3.25)$$

VB^+ represents a segment of path on the right side of the vertical branch cut, and VB^- represents the path on the left side of the branch cut. Comparing the terms in (3.25) with (3.21), we observe that the fast algorithm could be applied if the propagation factor

$F^\alpha(k_{\rho,p}, z, z')$ can be decomposed into the product or summation of products of functions depending solely on z or z' .

$$F^\alpha(k_{\rho}, z, z') = \sum_k f_{v,k}^\alpha(k_{\rho}, z) f_{r,k}^\alpha(k_{\rho}, z') \quad (3.26)$$

Given (3.26), $g^\alpha(\mathbf{r}, \mathbf{r}')$ could be expressed as

$$\begin{aligned} g^\alpha(\mathbf{r}, \mathbf{r}') &= \sum_{p,k} h(k_{\rho,p}) f_{v,k}^\alpha(k_{\rho,p}, z) H_0^{(1)}(k_{\rho,p} \rho) f_{r,k}^\alpha(k_{\rho,p}, z), z') / k_{mz,p} \\ &= \frac{2\pi}{Q} \sum_{p,k} \sum_q h(k_{\rho,p}) [f_{v,k}^\alpha(k_{\rho,p}, z) e^{-i\mathbf{k}_{p,q} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})}] \tilde{\alpha}_{l'l}(k_{\rho,p}, \alpha_q) \\ &\quad \cdot \left[\frac{f_{r,q}^\alpha(k_{\rho,p}, z')}{k_{mz,p}} e^{-i\mathbf{k}_{p,q} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho})} \right] \end{aligned} \quad (3.27)$$

where $\mathbf{k}_{p,q} = [k_{\rho,p} \cos(\alpha_q), k_{\rho,p} \sin(\alpha_q)]$. The factor inside the first bracket depends on $\mathbf{r}$ only, and the factor in the second bracket depends on $\mathbf{r}'$ only. The factor $1/k_{mz,p}$ is included in the second bracket, because m is the index of the layer where the source point resides. Substituting (3.27) into the TE part of the DGLM

$$\begin{aligned} \overline{\mathbf{G}}^{\text{TE}}(\mathbf{r}, \mathbf{r}') &= (\nabla \times \hat{z})(\nabla' \times \hat{z}) g^{\text{TE}}(\mathbf{r}, \mathbf{r}') \\ &= \frac{2\pi}{Q} \sum_u h(k_{\rho,u}) [(\nabla \times \hat{z}) f_{v,u}^{\text{TE}}(k_{\rho,u}, z) e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})}] \tilde{\alpha}_{l'l}(k_{\rho,u}) \\ &\quad \cdot [(\nabla' \times \hat{z}) \frac{f_{r,u}^{\text{TE}}(k_{\rho,u}, z')}{k_{mz,u}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')}] \end{aligned} \quad (3.28)$$

where u is the ensemble of $\{p, q, k\}$, $k_{\rho,u} = k_{\rho,p}$, $k_{mz,u} = \pm \sqrt{k_m^2 - k_{\rho,p}^2}$, and $\mathbf{k}_{\rho,u} = \mathbf{k}_{p,q}$. Now the dyadic Green's function's dependence and $\mathbf{r}$ ($\boldsymbol{\rho}, z$) and $\mathbf{r}'$ ($\boldsymbol{\rho}', z'$) are completely

decomposed, and this is the prerequisite for the implementation of the fast multipole algorithm. If we take the inner product of (3.28) with a basis function and a testing function, the following expression is obtained.

$$\begin{aligned} \langle \mathbf{J}_{Tj}, \bar{\mathbf{G}}^{\text{TE}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_i \rangle &= \frac{2\pi}{Q} \sum_u h(k_{\rho,u}) \left[\int_{\Delta_j} d\mathbf{r} \mathbf{J}_{Tj}(\mathbf{r}) \cdot (\nabla \times \hat{z}) f_{v,u}^{\text{TE}}(k_{\rho,u}, z) e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} \right] \\ &\quad \cdot \tilde{\alpha}_{l'l}(k_{\rho,u}, \rho_{l'}) \left[\int_{\Delta_i} d\mathbf{r}' \mathbf{J}_i(\mathbf{r}') \cdot (\nabla' \times \hat{z}) \frac{f_{r,u}^{\text{TE}}(k_{\rho,u}, z')}{k_{mz,u}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \right] \end{aligned} \quad (3.29)$$

For the TM part of the dyadic Green's function, the inner product could be expressed similarly,

$$\begin{aligned} \bar{\mathbf{G}}^{\text{TM}}(\mathbf{r}, \mathbf{r}') &= \frac{1}{k_{nm}^2} (\nabla \times \nabla \times \hat{z}) (\nabla' \times \nabla' \times \hat{z}) g^{\text{TM}}(\mathbf{r}, \mathbf{r}') \quad (3.30) \\ \langle \mathbf{J}_{Tj}, \bar{\mathbf{G}}^{\text{TM}}(\mathbf{r}, \mathbf{r}'), \mathbf{J}_i \rangle &= \frac{2\pi}{Qk_0^2} \sum_u h(k_{\rho,u}) \left[\int_{\Delta_j} d\mathbf{r} \mathbf{J}_{Tj}(\mathbf{r}) \cdot (\nabla \times \nabla \times \hat{z}) \frac{f_{v,u}^{\text{TM}}(k_{\rho,u}, z)}{\epsilon_{rn}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} \right] \\ &\quad \cdot \tilde{\alpha}_{l'l}(k_{\rho,p}, \alpha_q, \rho_{l'}) \left[\int_{\Delta_i} d\mathbf{r}' \mathbf{J}_i(\mathbf{r}') \cdot (\nabla' \times \nabla' \times \hat{z}) \frac{f_{r,u}^{\text{TM}}(k_{\rho,u}, z')}{\mu_{rm} k_{mz,u}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \right] \end{aligned} \quad (3.31)$$

In (3.29) and (3.31), the surface integrals inside the brackets are the radiation patterns and receiving patterns for thin-stratified medium fast-multipole algorithm (TSM-FMA). In the following section, we will give the explicit expression for the radiation and receiving patterns for each different case.

3.4 Radiation and Receiving Patterns

In this section, we will discuss the radiation and receiving pattern for four different cases.

3.4.1 Patterns for transmitted field

When the source point and field point are located in different layers, the received field is a transmitted field from the source in layer m to layer n . The propagation factor for the transmitted field, (2.16) and (2.18), is naturally decomposed into the product of a z dependent factor and a z' dependent factor. As a result, there is only one set of radiation pattern and receiving pattern needed for each mode. For the TE mode, when the field point is above the source point ($n > m$), the detailed expressions of radiation and receiving patterns are given below.

$$\begin{aligned}
 I_{rad,m}^{TE1} &= \int_{\Delta_i} d\mathbf{r}' \mathbf{J}_i(\mathbf{r}') \cdot (\nabla' \times \hat{z}) \frac{f_{r,u}^{TE}(z')}{k_{mz,u}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \\
 &= \frac{i}{k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \\
 &\quad \cdot C_m \left[e^{ik_{mz,u}(d_{m+1}-z')} + G_{m,m-1} e^{ik_{mz,u}(d_{m+1}+z'-2d_m)} \right] \quad (3.32)
 \end{aligned}$$

$$\begin{aligned}
 I_{rev,n}^{TE,m} &= \int_{\Delta_j} d\mathbf{r} \mathbf{J}_j(\mathbf{r}) \cdot (\nabla \times \hat{z}) f_{v,u}^{TE}(z) e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} \\
 &= (-i) \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} \\
 &\quad \cdot D_{mn} \left[e^{ik_{nz,u}(z-d_n)} + G_{n,n+1} e^{ik_{nz,u}(2d_{n+1}-z-d_n)} \right] \quad (3.33)
 \end{aligned}$$

Here $I_{rad,m}^{TE,1}$ is the first radiation pattern of a TE wave associated with a source in layered m , and it represents the upgoing wave generated by the source within this layer. And $I_{rev,n}^{TE,m}$ is the receiving pattern of an observation point in layer n when the source radiates from

layer m . When the field point is below the source point ($n < m$), the patterns become:

$$I_{rad,m}^{TE2} = \frac{i}{k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \cdot C_m \left[e^{ik_{mz,u}(z' - d_m)} + G_{m,m+1} e^{ik_{mz,u}(2d_{m+1} - z' - d_m)} \right] \quad (3.34)$$

$$I_{rev,n}^{TE,m} = (-i) \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} \cdot E_{mn} \left[e^{ik_{nz,u}(d_{n+1} - z)} + G_{n,n-1} e^{ik_{nz,u}(d_{n+1} + z - 2d_n)} \right] \quad (3.35)$$

Similarly, $I_{rad,m}^{TE,2}$ is the second TE radiation pattern associated with a source in layered m , and it represents the downgoing wave in this layer.

For TM mode, the $\nabla \times \nabla \times$ operator makes the expression of the patterns more complicated. Note that $\nabla e^{i\mathbf{k} \cdot \mathbf{r}} = (i\mathbf{k})e^{i\mathbf{k} \cdot \mathbf{r}}$, the ∇ operator becomes a vector $i\mathbf{k}$ for a plane wave. The radiation and receiving patterns for TM waves are summarized below. When $n > m$,

$$I_{rad,m}^{TM1} = \frac{1}{\mu_{rm}} \int_{\Delta_i} d\mathbf{r}' \mathbf{J}_i(\mathbf{r}') \cdot (\nabla' \times \nabla' \times \hat{z}) \frac{f_{r,u}^{TE}(z')}{k_{mz,u}} e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} \\ = -\frac{C_m}{\mu_{rm} k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' \left\{ [(\mathbf{k}_{mu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} e^{ik_{mz,u}(d_{m+1} - z')} \right. \\ \left. + [(\mathbf{k}_{mu}^+ \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} G_{m,m-1} e^{ik_{mz,u}(d_{m+1} + z' - 2d_m)} \right\} \quad (3.36)$$

$$I_{rev,n}^{TM,m} = \frac{1}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} \mathbf{J}_j(\mathbf{r}) \cdot (\nabla \times \nabla \times \hat{z}) f_{v,u}^{TE}(z) e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} \\ = -\frac{D_{mn}}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} \left\{ [(\mathbf{k}_{nu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} e^{ik_{nz,u}(z - d_n)} \right. \\ \left. + [(\mathbf{k}_{nu}^+ \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} G_{n,n+1} e^{ik_{nz,u}(2d_{n+1} - z - d_n)} \right\} \quad (3.37)$$

where $\mathbf{k}_{iu}^\pm = \mathbf{k}_{\rho,u} \pm \hat{z}k_{iz,u}$. When $n < m$,

$$I_{rad,m}^{TM2} = -\frac{C_m}{\mu_{rm}k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' \left\{ [(\mathbf{k}_{mu}^+ \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} e^{ik_{mz,u}(z' - d_m)} \right. \\ \left. + [(\mathbf{k}_{mu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} G_{m,m+1} e^{ik_{mz,u}(2d_{m+1} - z' - d_m)} \right\} \quad (3.38)$$

$$I_{rev,n}^{TM,m} = -\frac{E_{mn}}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} \left\{ [(\mathbf{k}_{nu}^+ \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} e^{ik_{nz,u}(d_{n+1} - z)} \right. \\ \left. + [(\mathbf{k}_{nu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} G_{n,n-1} e^{ik_{nz,u}(d_{n+1} + z - 2d_n)} \right\} \quad (3.39)$$

3.4.2 Patterns for reflected field

When the observation point and the source point belong to the same layer, the received field has two parts: the direct field which propagates directly from the source to field point and the reflected field, which reaches the field point after being reflected by other layers. The first term of the propagation factor in (2.15) represents the direct field. The remaining four terms represent the reflected field and they could be rewritten as the summation of decomposed functions of z and z' .

$$F_{m,m}^{\alpha,R}(z, z') = C_m \left[e^{ik_{mz}(d_{m+1} - z')} + G_{m,m-1} e^{ik_{mz}(d_{m+1} + z' - 2d_m)} \right] G_{m,m+1} e^{ik_{mz}(d_{m+1} - z)} \\ + C_m \left[e^{ik_{mz}(z' - d_m)} + G_{m,m+1} e^{ik_{mz}(2d_{m+1} - z' - d_m)} \right] G_{m,m-1} e^{ik_{mz}(z - d_m)} \quad (3.40)$$

Since the factors containing z' are the same as the propagation factors of the transmitted field case, there is no need to calculate new radiation patterns. Only a new receiving pattern

is needed.

$$I_{rev,m}^{TE,mI} = (-i) \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}') } G_{m,m+1} e^{ik_{mz,u}(d_{m+1}-z)} \quad (3.41)$$

$$I_{rev,m}^{TE,mII} = (-i) \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}') } G_{m,m-1} e^{ik_{mz,u}(z-d_m)} \quad (3.42)$$

$I_{rev,m}^{TE,mI}$ is the receiving pattern in layer m associated with the radiation pattern $I_{rad,m}^{TE1}$ from layer m . $I_{rev,m}^{TE,mII}$ is associated with the radiation pattern $I_{rad,m}^{TE2}$.

Similarly, the receiving pattern for the reflected field of TM mode is given by

$$I_{rev,m}^{TM,mI} = -\frac{G_{m,m+1}}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{mu}^+ \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}') } e^{ik_{mz}(d_{m+1}-z)} \quad (3.43)$$

$$I_{rev,m}^{TM,mII} = -\frac{G_{m,m-1}}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{mu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}') } e^{ik_{mz}(z-d_m)} \quad (3.44)$$

3.4.3 Patterns for direct field

The propagation factor for the direct field is $e^{ik_{mz}|z-z'|}$ and the modulus sign makes it impossible to be decomposed as two independent functions of z and z' respectively. The Sommerfeld integral needs to be changed to remove the modulus sign over $z - z'$. The TE part of a dyadic Green's function in homogenous space is given by:

$$\overline{\mathbf{G}}^{TE,d} = (\nabla \times \hat{z})(\nabla' \times \hat{z}) \frac{i}{4\pi} \int_0^\infty dk_\rho \frac{1}{k_\rho k_z} J_0(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) e^{ik_z |z-z'|} \quad (3.45)$$

By deforming the integration path and using Gaussian quadrature rule, the term could be expressed as a discrete summation:

$$\begin{aligned} \overline{\mathbf{G}}^{TE,d} = & -\frac{1}{4\pi} \sum_{p=1}^{N_q} \frac{w_p u_p}{k_{\rho,p}} \left[(\nabla \times \hat{z})(\nabla' \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \frac{e^{ik_{z,p}|z-z'|}}{k_{z,p}} \Big|_{\text{VB}^+} \right. \\ & \left. - (\nabla \times \hat{z})(\nabla' \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \frac{e^{ik_{z,p}|z-z'|}}{k_{z,p}} \Big|_{\text{VB}^-} \right] \end{aligned} \quad (3.46)$$

where $k_{z,p} = \sqrt{k^2 - k_{\rho,p}^2}$, $k_\rho = k$ is the branch point. Here $k_{z,p}$ on the right side of the vertical branch cut (VB⁺) and the $k_{z,p}$ on the left side of the vertical branch cut (VB⁻) are of opposite signs.

$$\begin{aligned} \overline{\mathbf{G}}^{TE,d} = & -\frac{1}{4\pi} \sum_{p=1}^{N_q} \frac{w_p u_p}{k_{\rho,p}} (\nabla_s \times \hat{z})(\nabla'_s \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \left[\frac{e^{ik_{z,p}|z-z'|}}{k_{z,p}^+} - \frac{e^{ik_{z,p}|z-z'|}}{-k_{z,p}^+} \right] \\ = & -\frac{1}{4\pi} \sum_{p=1}^{N_q} \frac{w_p u_p}{k_{\rho,p}} (\nabla_s \times \hat{z})(\nabla'_s \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \left[\frac{e^{ik_{z,p}(z-z')}}{k_{z,p}^+} - \frac{e^{ik_{z,p}(z-z')}}{-k_{z,p}^+} \right] \end{aligned} \quad (3.47)$$

Here, $\nabla_s = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}$, $k_{z,p}^+$ is the value of k_z on VB⁺. From the factor in the brackets containing z and z' , it is obvious that $\overline{\mathbf{G}}^{TE,d}$ is an even function of $|z - z'|$, so we can remove the modulus sign over $z - z'$ and get the same result. The radiation and receiving patterns for the TE part of the direct field in the m th layer are

$$\begin{aligned} I_{rad,m}^{TE3} &= \frac{i}{k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho}')} e^{-ik_{mz,u} z'} \\ I_{rev,m}^{TE,m\text{III}} &= (-i) \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,p} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_l')} e^{ik_{mz,u} z} \end{aligned} \quad (3.48)$$

Now we move on to study the TM part of the dyadic Green's function in homogenous

media. Its discrete form is

$$\begin{aligned} \overline{\mathbf{G}}^{TM,d} = & -\frac{1}{4\pi k_{nm}^2} \sum_{p=1}^{N_q} \frac{w_p u_p}{k_{\rho,p}} \left[(\nabla \times \nabla \times \hat{z})(\nabla' \times \nabla' \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \frac{e^{ik_{z,p}|z-z'|}}{k_{z,p}} \Big|_{\text{VB}^+} \right. \\ & \left. - (\nabla \times \nabla \times \hat{z})(\nabla' \times \nabla' \times \hat{z}) H_0^{(1)}(k_{\rho,p} \rho) \frac{e^{ik_{z,p}|z-z'|}}{k_{z,p}} \Big|_{\text{VB}^-} \right] \end{aligned} \quad (3.49)$$

We expand the operator $(\nabla \times)(\nabla' \times)$ as

$$\begin{aligned} (\nabla \times)(\nabla' \times) &= [(\nabla_s + \frac{\partial}{\partial z} \hat{z}) \times][(\nabla'_s + \frac{\partial}{\partial z'} \hat{z}) \times] \\ &= (\nabla_s \times)(\nabla'_s \times) + (\nabla_s \times)(\frac{\partial}{\partial z'} \hat{z} \times) + (\frac{\partial}{\partial z} \hat{z} \times)(\nabla'_s \times) + (\frac{\partial}{\partial z} \hat{z} \times)(\frac{\partial}{\partial z'} \hat{z} \times) \end{aligned} \quad (3.50)$$

Substituting (3.50) into (3.49), we found the first and last terms are even functions of $|z - z'|$ while the second and third terms are not. However, the first term dominates because $|k_\rho| \gg |k_{mz}|$ for the significant quadrature points. Thus, the modulus sign here could be removed as well.

Similarly, the radiation and receiving patterns of the TM part of the direct field are

$$\begin{aligned} I_{rad,m}^{TM3} &= -\frac{1}{\mu_{rm} k_{mz,u}} \int_{\Delta_i} d\mathbf{r}' [(\mathbf{k}_{mu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_i(\mathbf{r}')] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho}_l - \boldsymbol{\rho})} e^{-ik_{mz,u} z'} \\ I_{rev,m}^{TM,m\text{III}} &= -\frac{1}{\epsilon_{rn}} \int_{\Delta_j} d\mathbf{r} [(\mathbf{k}_{mu}^- \times \mathbf{k}_{\rho,u} \times \hat{z}) \cdot \mathbf{J}_{Tj}(\mathbf{r})] e^{-i\mathbf{k}_{\rho,u} \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_{l'})} e^{ik_{mz,u} z} \end{aligned} \quad (3.51)$$

The physical meanings of the radiation and receiving patterns are summarized in Figure 3.7.

$m+2$		$\nabla I_{rev,m}^{mode,m+2}$
$m+1$		$\nabla I_{rev,m}^{mode,m+1}$
m	$I_{rad,m}^{mode,1}$ 	$\nabla I_{rev,m}^{mode,m1}$
$m-1$		
$m-2$		

(a) Upgoing waves.

$m+2$		
$m+1$		
m	$I_{rad,m}^{mode,2}$ 	$\nabla I_{rev,m}^{mode,mII}$
$m-1$		$\nabla I_{rev,m}^{mode,m-1}$
$m-2$		$\nabla I_{rev,m}^{mode,m-2}$

(b) Downgoing waves.

$m+2$		
$m+1$		
m	$I_{rad,m}^{mode,3}$ 	$\nabla I_{rev,m}^{mode,mIII}$
$m-1$		
$m-2$		

(c) Direct waves

Figure 3.7: Physical meanings of radiation and receiving patterns.

3.4.4 Assembling of radiation and receiving patterns

For every source, there are three sets of radiation patterns that exist for each mode for each $k_{\rho,u}$: $I_{rad,m}^{TEi}$, $I_{rad,m}^{TMi}$, $i = 1, 2, 3$, where m is the index of the layer where the source point resides. For each observation point, it needs $N + 2$ sets of receiving patterns to receive the sources radiated from layer 1 to N : $I_{rev,n}^{TM,m}$, $m = 1, 2, \dots, N, m \neq n$, $I_{rev,n}^{TM,mI}$, $I_{rev,n}^{TM,mII}$, $I_{rev,n}^{TM,mIII}$. The assembly process is summarized in Figure 3.8.

3.5 Numerical Results

In this section, some numerical examples are calculated and compared with their references. The quasi-3D TSM-FMA has a better modeling capability, while it retains the $O(N \log N)$ computational complexity.

3.5.1 Scattering from patch antennas

To start with, two two-dimensional scattering problems are analyzed and validated against the results from two-dimensional TSM-FMA.

The first one is the scattering from a circular patch on top of a microstrip structure. The radius of the patch is 0.65 cm. The thickness of the dielectric substrate is 0.07874 cm and its relative permittivity is 2.33. A plane wave is incident at $(\theta^i, \phi^i) = (60^\circ, 180^\circ)$ with θ polarization. The patch is discretized into 1,198 small triangles at 15 GHz. The monostatic RCS shown in Figure 3.9 agrees well with the results given in [20, 60].

The second example is the scattering from a 30×30 microstrip array with rectangular patch elements. The size of the rectangular patch is 3.66 cm $\times$ 2.6 cm. The separation

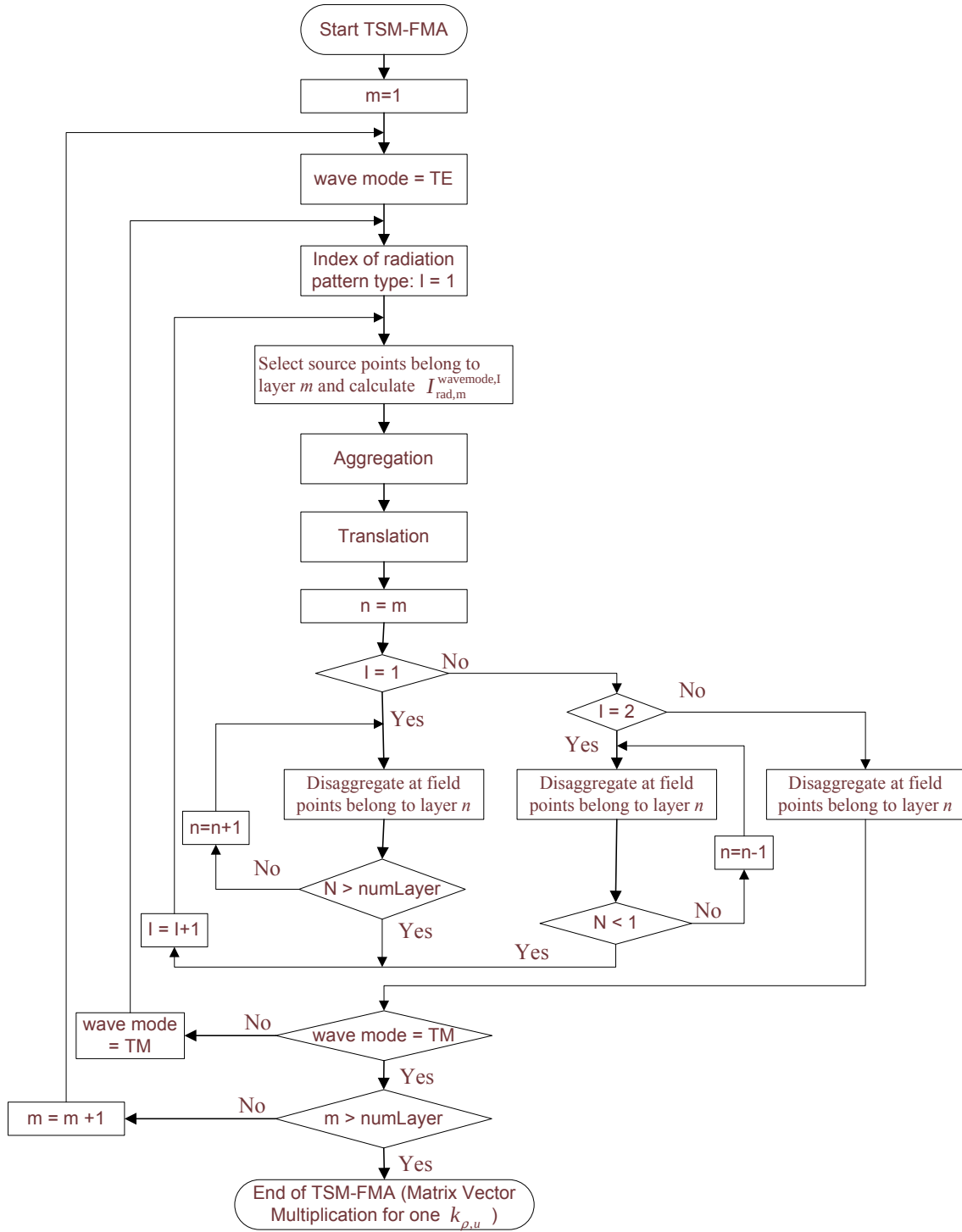


Figure 3.8: Diagram of TSM-FMA assembling process.

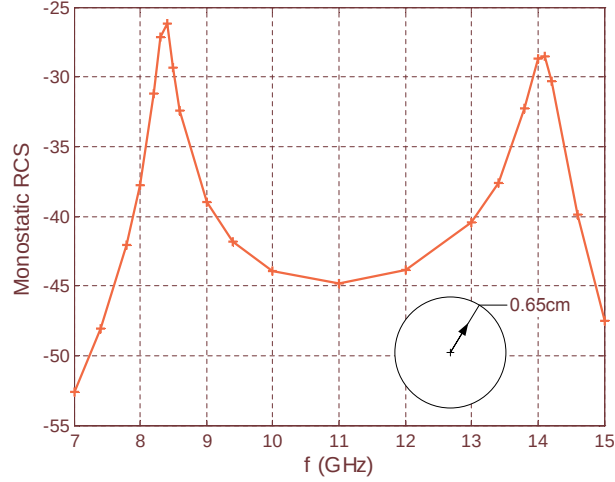


Figure 3.9: Monostatic RCS versus frequency of a circular patch.

of the patches is 2.34 cm in the x direction and 3.4 cm in the y direction. The dielectric substrate of the microstrip has a thickness of 0.158 cm and a relative permittivity of 2.17. The incident plane wave is at 2.2 GHz, with incident angle of $(\theta^i, \phi^i) = (60^\circ, 45^\circ)$ and θ polarization. The total number of unknowns is 117,000 and memory usage is 234 MB. The matrix solving time is 70 minutes and time used per iteration is 75 seconds on an Intel 3.0 GHz processor. Again, the bistatic RCS agrees with the result from 2D TSM-FMA [20] as shown in Figure 3.10.

3.5.2 Input impedance of probe-fed microstrip patches

The new fast algorithm is designed to handle microstrip structures that have vertical variations, for example, patch antennas fed by probes. A probe is modeled by a vertical via connecting the patch and the ground with a delta-gap source on it. The input impedances of several different probe-fed microstrip patches are calculated and presented.

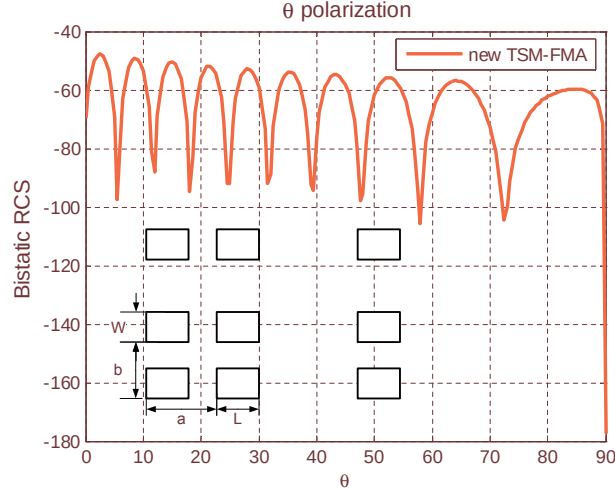


Figure 3.10: Bistatic RCS for a 30×30 microstrip antenna array. $a = 6$ cm, $b = 6$ cm, $L = 3.66$ cm, $W = 2.6$ cm.

The first example is a probe-fed circular patch as shown in Figure 3.11. The radius of the patch is 1.88 cm. The vertical probe is 0.94 cm away from the center of the patch. The substrate has a thickness of 0.16 cm and a relative permittivity of 2.6. The current distribution at 2.7 GHz is shown in Figure 3.12(a). The input impedance at the probe versus frequency is presented in Figure 3.12(b). The calculation shows that the resonance frequency of the patch antenna is 2.8 GHz. It agrees with the theoretical analysis given in [61].

The second example is a patch antenna that straddles multilayer substrates. Its configuration is given in Figure 3.13(a). All the dimensions are in millimeters. The input impedance is shown on a Smith chart in Figure 3.13(b) and it agrees with reference [62]. This example demonstrates the modeling capability of the new fast algorithm: it can handle more complicated patch antennas compared to the earlier version of TSM-FMA.

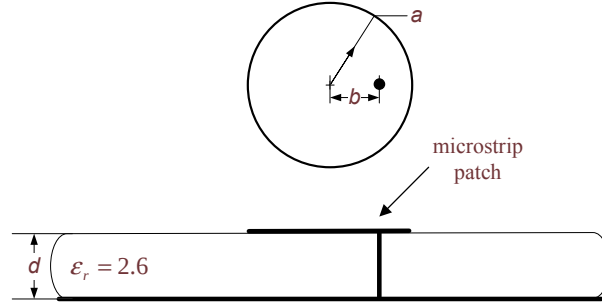
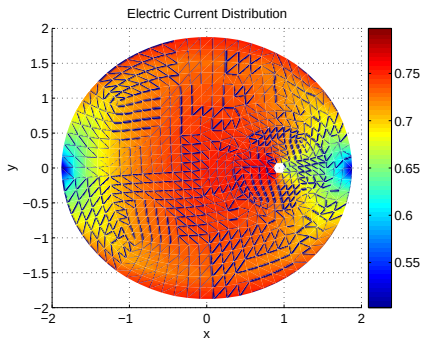
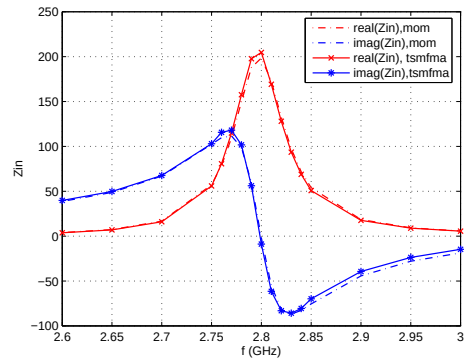


Figure 3.11: Geometrical configurations of the circular patch. $a = 1.88$ cm, $b = 0.94$ cm, $d = 0.16$ cm.



(a) Current distribution.



(b) Input impedance of the circular patch.

Figure 3.12: Input impedance of a circular patch.

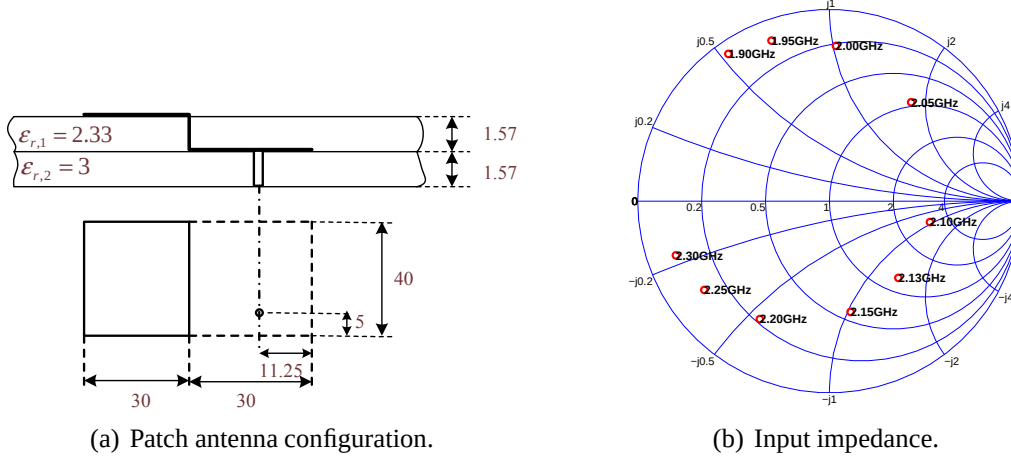


Figure 3.13: The two-layer microstrip patch antenna and its input impedance.

3.5.3 Radiation pattern of microstrip patch arrays

In this part, several types of microstrip patch arrays with vertical probe feed are simulated.

The first example is a planar Yagi-Uda patch array [63, 64]. Similar to the Yagi-Uda dipole array [65, 66], the array is fed at the driven element, and there is a reflector patch on its left, and two director patches on its right. The structure directs the main beam of the radiation pattern at 30° away from the broadside direction. The geometrical configuration and dimensions of a single row Yagi-Uda patch array are given in Figure 3.14. To increase the gain of the array, four rows of Yagi-Uda antennas could be aligned on the microstrip substrate. The voltage source at the two probes of the driven element has the same magnitude but a phase difference of 90° , so that a circularly polarized field could be generated. The working frequency is 1.552 GHz. The number of unknowns for the four-row case is 6,332. If we use the full impedance matrix of MOM, it needs at least 320 MB of memory. However, with TSM-FMA, the total memory usage is reduced to 36 MB.

The radiation pattern of one row and four rows of Yagi-Uda arrays are shown and

compared in Figure 3.15. In the elevation plane, the main beam has been tilted by about 30° for both cases. By deploying a four-row array, the gain has been increased by almost 10 dB. In the azimuth plane, the radiation pattern of the four-row array is better focused because of mutual coupling.

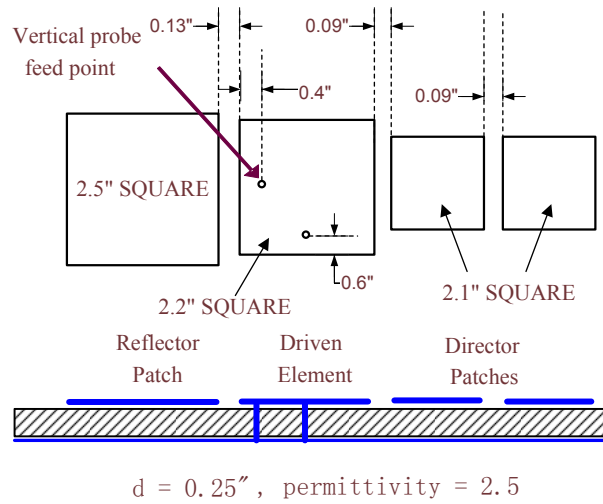
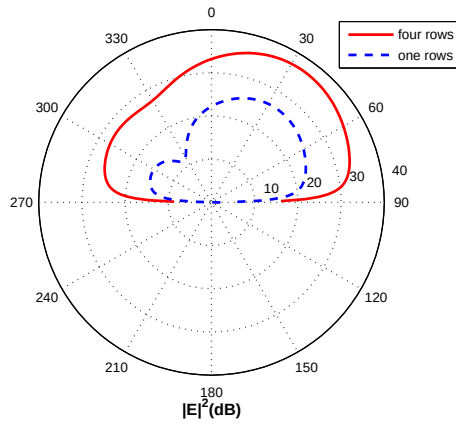


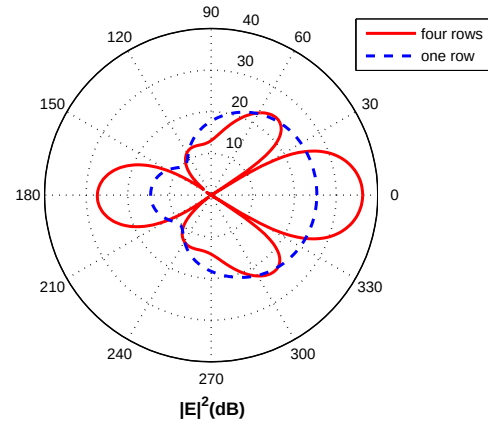
Figure 3.14: Geometrical configurations of a single row of Yagi-Uda patch array.

A circularly polarized field can be generated by a single-feed microstrip patch antenna as well [67]. For example, a nearly square diagonal fed antenna can produce a circularly polarized field if it is excited at a frequency in between the resonant frequencies of its two resonance modes, which correspond to the adjacent sides of the rectangle. The dimensions of the patch and the position of the feed are given in Figure 3.16 with a unit of centimeter. The microstrip substrate has a thickness of 0.3175 cm and a relative permittivity of 2.52. The feeding voltage source is working at 3.15 GHz.

A 12×12 patch array made up of this type of patch is simulated and the radiation patterns are given in Figure 3.17. In the broadside direction, the field is almost circular



(a) Radiation pattern in the elevation plane.
($\phi = 0^\circ$)



(b) Radiation pattern in the azimuth plane.
($\theta = 50^\circ$)

Figure 3.15: The radiation pattern of planar Yagi-Uda patch arrays.

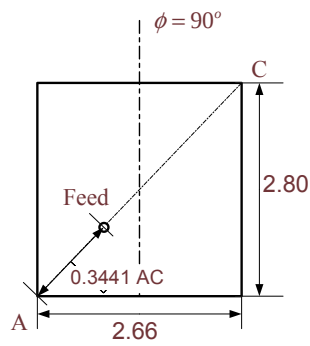


Figure 3.16: Nearly square diagonal fed antennas.

($|E_\theta| \approx |E_\phi|$). The sidelobes are due to the mutual coupling of the patches.

In this example, the total number of unknowns is 80,316. The memory used for setup is 358 MB and the additional memory needed for one matrix-vector multiplication is 33 MB. Time per iteration is 61 seconds on an Intel 3.0 GHz CPU.

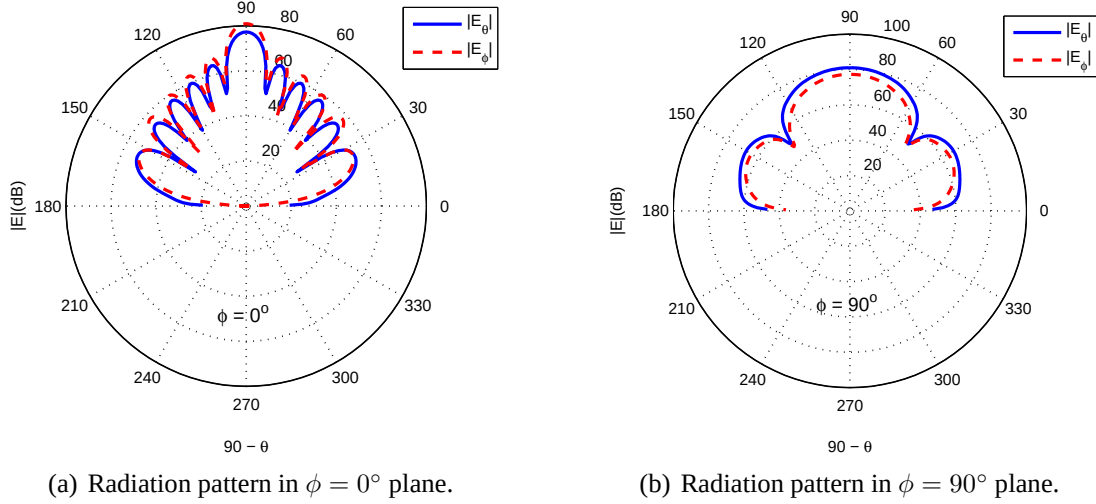
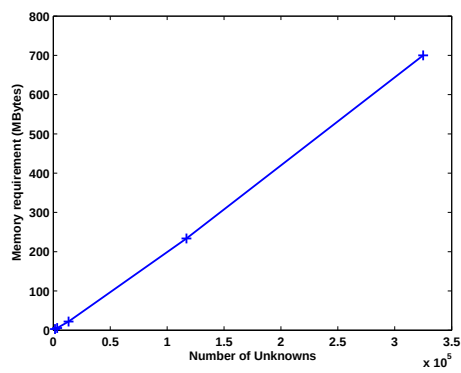


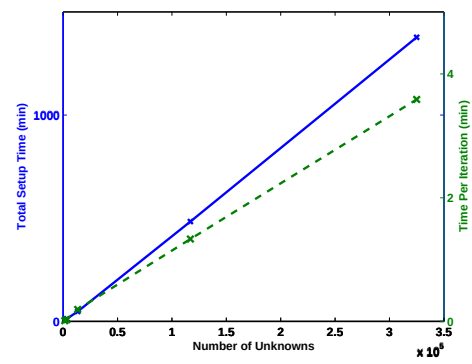
Figure 3.17: The radiation pattern of 12×12 microstrip patch array of nearly square patch elements.

3.5.4 Computational complexity and memory usage

We increase the number of elements in a patch array, and record the memory usage and time usage versus the number of unknowns. They have a nearly linear relationship as shown in Figure 3.18(a) and 3.18(b). It confirms that the algorithm has a $O(N \log N)$ computational complexity in both time and memory.



(a) Memory versus number of unknowns.



(b) Time versus number of unknowns

Figure 3.18: Memory requirement and time usage versus the number of unknowns.

CHAPTER 4

EFFICIENT NUMERICAL EVALUATION OF CASIMIR FORCE BY INTEGRAL EQUATION METHOD

In this chapter, Casimir force is introduced and expressed in terms of dyadic Green's functions and their derivatives. Those quantities could be calculated numerically by solving integral equations, with a given set of sources and field testing operators.

4.1 Introduction

Casimir force is a tiny force which exists among charge-neutral bodies due to the quantum fluctuation of the electromagnetic fields in vacuum [68]. This force exists even when there is zero field in the classical sense between the bodies. The classical zero field case corresponds to the ground state of a quantum electromagnetic field. The ground state of a quantum electromagnetic field has zero mean field, but nonzero mean square field. In infinite free space, the fluctuating waves can have any frequency, but between two conducting plates, their frequencies become a set of discretized values, which form the eigenmodes of the parallel plate waveguide. Thus, the energy of the fluctuating field between the plates is less than that outside. Each side of the plates would experience a different pressure from the fluctuating field and this induces an attractive force between the two plates. Casimir first predicted the strength of the force between two infinitely large, perfect conducting

plates. Evgeny Lifshitz and colleagues have generalized Casimir's work to real materials: nonperfect conductors and dielectrics [69]. They also considered the situation that the plates were separated by a fluid, not a vacuum. The force is also known as the Casimir-Lifshitz (CCL) force.

The Casimir force is a close cousin of the van der Waals force between non-polar molecules or large objects. The van der Waals force is the intermolecular attractive force generated by two dipoles, when the ever-moving charging in one molecule creates a momentary electric dipole and they induce a dipole in a nearby molecule. In the van der Waals limit, when the distance between molecules is so small that the electric field propagates between them much faster than the charges can oscillate, the interaction between two dipoles happens almost instantaneously. But when the objects are farther apart and the propagation time becomes important, the dependence of the force on the object's separation is modified. This is known as the retardation effect on the van der Waals force due to the finite speed of light [70]. For example, Casimir has shown that for large separation R , the interacting energy between a neutral atom and a conducting wall is proportional to R^{-4} rather than to R^{-3} . The interacting energy between two neutral atoms is proportional to R^{-6} at short distances, whereas for large distances the energy is proportional to R^{-7} .

For two perfectly conducting parallel plates, the attractive force between the plates has the magnitude (per unit surface area)

$$F/A = \frac{\pi^2}{240} \frac{\hbar c_0}{a^4} = 0.016 \frac{1}{a^4} \text{dyn}(\mu m)^4/\text{cm}^2 \quad (4.1)$$

Usually it is too small to be observed. For example, two plates with an area of 1 cm^2 separated by a distance of $1 \mu m$ have an attractive force of 10^{-7} N . However, when the

distance is further reduced, the Casimir force becomes the dominant force between neutral objects. For example, at the separation of 10 nm, the pressure produced by the Casimir force is about 1 atmosphere. As a result, the Casimir force is negligible above micron scale since it decreases rapidly with separation between the interacting bodies. It dominates the interaction between neutral objects at submicron distance.

This quantum electrodynamical phenomenon has been studied for decades in physics for theoretical interests. But accurate measurement emerged just a decade ago. The first modern measurement of the Casimir force was reported in 1997 by Lamoreaux using a torsional pendulum and sphere-plate configuration [71]. When Lamoreaux brought the sphere and the plate together to within several microns, the Casimir force pulled the two objects together and caused the pendulum to twist. His experimental measurement agrees with theoretical values to an accuracy of 5%. Inspired by this work, Mohideen and his group [72] attached a sphere coated with gold to the tip of an atomic force microscope (AFM). When the sphere is brought close to a large flat disk coated with gold or aluminum, the attractive force between them causes the cantilever to bend, and this was detected by the reflection of a laser beam. This improves the accuracy to within 1% of the expected theoretical value. These two experiments are carried out when the size of the sphere is much greater than the distance from sphere to plate, and the theoretical value is obtained from proximity force approximation (PFA). When the separation is comparable to the size of the sphere, the result from PFA is no longer accurate. Krause and his co-workers have conducted experiments to explore the correction to PFA formula [73].

Since then, Casimir force has been measured in a wide variety of experimental geometries. For example, Ederth has also used an atomic force microscope to measure two gold-

coated crossed cylinders and the results agreed to within 1% with theoretical value [74]. Later, Chan and his group measured the Casimir force between a gold sphere and a silicon surface with an array of nanoscale, rectangular corrugations using a micromechanical torsional oscillator [75]. This experiment demonstrates the strong dependence of the Casimir force on the shape of the interacting bodies. Although a wide variety of geometries have been used, very few experiments measured the original configuration of two parallel mirror planes. The reason is that it is difficult to keep two planes perfectly parallel in the experiment. It is much easier to bring a sphere close to a plate. One recent experiment on two parallel plates was carried out by Bressi's group, and the accuracy is within 15% compared to theoretical values. This relatively poor fit reflects the technical difficulty in this kind of setup. In 2007, the most precise measurement of the Casimir force between parallel plates by using a micromachined torsional device was reported [76].

As recent advances in miniaturization lead to the development of micro- and nano-electromechanical systems (MEMS and NEMS), it is also increasingly of practical interest to study the role and influence of Casimir force in these small devices now. Micro-electromechanical systems (MEMS) are micron-sized devices in which mechanical elements and moving parts, such as tiny sensors and actuators, are carved into a silicon substrate. As further miniaturization takes place, the devices may reduce to nano-scale and become nanoelectromechanical systems (NEMS). They have a wide class of applications. For example, the release of the airbag in cars is controlled by a MEMS-based accelerometer. One of the principal causes of malfunctioning in MEMS is stiction, i.e., the collapse of movable elements into nearby surfaces, resulting in their permanent adhesion. The Casimir effect is often an important underlying mechanism causing this phenomenon [77]. On the

other hand, it could be put to good use. Capasso and his group showed that the force can be used to control the actuation of a micromachined torsional device [78]. This group also showed that the Casimir attraction can be used to make a nonlinear oscillator. The force influences the dynamical properties of a micromachined device, changing its resonance frequency, hysteretic behavior, and bistability in its frequency response to an ac excitation. They proposed that this device could serve as a nanometric position sensor [79, 80].

The general attractive behavior of the Casimir force has been a drawback rather than advantage for MEMS systems. If the Casimir force becomes repulsive, it could allow quantum levitation of objects in a fluid and lead to a new class of nano-scale device with ultra-low friction. Lifshitz has predicted in theory that repulsive force exists when the permittivity of the fluid medium (ϵ_2) satisfies the relationship: $\epsilon_1 > \epsilon_2 > \epsilon_3$, where ϵ_1 and ϵ_3 are the permittivities of the two solid objects [69]. An important breakthrough on this subject was made by Capasso's group in early 2009 [81]. They show experimentally that the sign of the force can be changed from attractive to repulsive by a suitable choice of interacting materials immersed in a fluid in accordance with theoretical prediction. The measured repulsive interaction is found to be weaker than the attractive.

From the fluctuation dissipation theorem, any vacuum system in thermal equilibrium with another system will be filled with electromagnetic field. At zero temperature, the fluctuating field is at the ground state and the Casimir force can be predicted precisely without any ambiguity. When the objects are not ideal metals, the calculation of the thermal correction to the Casimir force at non-zero temperature is not uniquely formulated and has been hotly debated since 2000 [82]. For ideal metal bodies, the thermal correction can be neglected when $T \ll \hbar c_0 / k_B d$, where c_0 is the speed of light, k_B is the Boltzmann

constant and d is the smallest separation between the interacting objects [83]. For real metal bodies described by the Drude model, the thermal correction at room temperature is rather large even below $1\ \mu\text{m}$ [84]. All the calculations performed in this paper assume zero temperature and ideal metals.

There are different ways to quantitatively predict the Casimir force. For relatively simple geometries, approximate methods such as the proximity force approximation (PFA) [85], and optical approach in terms of virtual photons moving along ray optical paths [86] are quite intuitive and efficient. But when the geometry becomes complicated as in future MEMS devices, an exact and geometry-independent numerical method would be preferable. There are two popular general methods: the path integral method (also known as EGJK method) and the Maxwell stress-tensor method.

The path integral method was proposed by Büscher [87]. It begins by expressing the Casimir energy as the logarithm of a functional integral over all field fluctuations constrained by the boundary conditions on a set of surfaces. Then the functional integral is performed over the field. It has been generalized in [88] (EGJK) for compact objects of arbitrary shape and separation and has been applied to predict the force between a cylinder and a plate [89], and between two spheres [88]. EGJK's algorithm is efficient to predict the Casimir energy between compact 3D objects of spheroidal or nearly spheroidal shape, or even leads to analytically tractable series solution. But it is not of practical use for general geometries due to poor convergence rate for objects with corners or cusps. This limitation occurs since they used spherical basis functions to expand the current distribution. It could be overcome by expanding the current with basis functions defined on a pair of adjacent triangles, which is known as RWG basis [90]. The RWG basis is widely used in surface

integral method in computational electromagnetics. After some manipulation, the evaluation of Casimir energy depends only on solving for all the eigenvalues of the impedance matrix obtained from integral equation by method of moments.

The theoretical foundation for the Maxwell stress tensor was built by Lifshitz and his coworkers [69]: the net Casimir force on a body can be expressed as an integral over any closed surface around the body of the mean electromagnetic stress tensor, integrated over all frequencies. And the fluctuation-dissipation theorem states that the mean fluctuating field, which forms the Maxwell stress tensor, is related to the imaginary part of the dyadic Green's function of the same geometry. The dyadic Green's function can be evaluated by existing numerical methods in computational electromagnetics (electrodynamics) [91]. Rodriguez has demonstrated the applicability of this method by using a simple finite-difference frequency-domain (FDTD) method to calculate the dyadic Green's function in z -invariant structure [92,93]. This approach relates the quantum electrodynamic phenomena with classical electromagnetics and it motivates our piece of work: to evaluate the Casimir force among arbitrary objects using integral equation method. With the surface integral equation method, the number of unknowns can be greatly reduced. Moreover, fast algorithms [59] are available to further improve the efficiency of the method. The computational complexity of both the stress tensor approach and the path integral approach can be reduced to $O(N \log N)$ at best, N being the number of surface unknowns. Thus their efficiencies are comparable. The main difference between them is the physical quantity being calculated directly: the path integral method calculates the energy while the stress tensor approach calculates the force distribution (pressure).

In this thesis, we propose to use the integral equation method to calculate the dyadic

Green's function of a general geometry and its derivatives in order to get the Casimir force. The dyadic Green's function is obtained by solving the electromagnetic problem with a monopole point source placed very close to the surface of the object, and then evaluating the scattered field at the same location. Compared to finite difference methods, the integral equation method involves a discretization only at the surface of the coupled objects, thus leading to fewer unknowns.

First, geometries that are z -invariant are studied. In this case, the field can be completely decomposed into TE (transverse electric) to z and TM (transverse magnetic) to z modes, and the problem can be cast into two separate two-dimensional (2D) scattering problems. Then, arbitrary 3D geometries are studied. The integral equation method significantly reduces the computational complexity of the problem. Using dipole point sources in different directions, the derivatives of the dyadic Green's function could be calculated from the integral equation with the same kernel. In other words, when the integral equation is transformed into a matrix equation, different components and derivatives of the dyadic Green's function are obtained from the same matrix equation with different right-hand sides. This would further simplify the numerical procedure.

Now, we review the fundamental theory and key steps of calculating Casimir force from the Maxwell stress tensor and dyadic Green's function, which has been given in [91].

4.2 Casimir Force Represented by 3D Dyadic Green's Function

The net Casimir force acting on the surface S of an object is given by [94]

$$\mathbf{F} = \oint_S \langle \overline{\mathbf{T}}(\mathbf{r}') \rangle \cdot d\mathbf{s}' \quad (4.2)$$

where $\overline{\mathbf{T}}$ is the Maxwell stress tensor defined as [95]:

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} \left(\epsilon_0 \sum_{k=1}^3 E_k^2 + \frac{1}{\mu_0} \sum_{k=1}^3 B_k^2 \right) \delta_{ij} \quad (4.3)$$

The average of the fluctuating electric and magnetic field is obtained from fluctuation-dissipation theorem [96]:

$$\langle 0 | \hat{E}_i(\mathbf{r}, t) \hat{E}_j(\mathbf{r}', t) | 0 \rangle = \frac{\hbar}{\pi} \text{Im} \int_0^\infty \omega^2 G_{ij}(\mathbf{r}, \mathbf{r}', \omega) d\omega \quad (4.4)$$

$$\langle 0 | \hat{B}_i(\mathbf{r}, t) \hat{B}_j(\mathbf{r}', t) | 0 \rangle = \frac{\hbar}{\pi} \text{Im} \int_0^\infty (\nabla \times)_{il} (\nabla' \times)_{jm} G_{lm}(\mathbf{r}, \mathbf{r}', \omega) d\omega \quad (4.5)$$

where the dyadic Green's function $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega)$ satisfies the following equation:

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) - k^2 \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}', \omega) = \mu_0 \overline{\mathbf{I}} \delta(\mathbf{r}, \mathbf{r}') \quad (4.6)$$

For an arbitrary 3D object, we define $\hat{u}$ and $\hat{v}$ be the tangent vector at the surface, and $\hat{n}$ be the normal direction of the surface. They satisfy the relationship $\hat{u} \times \hat{v} = \hat{n}$. The stress tensor and dyadic Green's function at point $\mathbf{r}_i$ are conveniently expressed in terms

of $\hat{u}$, $\hat{v}$ and $\hat{n}$ at this point as

$$\mathbf{F}(\mathbf{r}_i) = \langle T_{nn}(\mathbf{r}_i) \rangle \hat{n}_i + \langle T_{un}(\mathbf{r}_i) \rangle \hat{u}_i + \langle T_{vn}(\mathbf{r}_i) \rangle \hat{v}_i \quad (4.7)$$

If the object is a perfect conductor, the tangential electric field E_u , E_v and the normal magnetic field B_n vanish at the surface of the object. The stress tensor could be simplified as follows:

$$T_{nn}(\mathbf{r}) = \frac{1}{2} \epsilon_0 E_n^2(\mathbf{r}) - \frac{1}{2\mu_0} B_u^2(\mathbf{r}) - \frac{1}{2\mu_0} B_v^2(\mathbf{r}) \quad (4.8)$$

$$T_{un}(\mathbf{r}) = \epsilon_0 E_u(\mathbf{r}) E_n(\mathbf{r}) + \frac{1}{\mu_0} B_u(\mathbf{r}) B_n(\mathbf{r}) = 0 \quad (4.9)$$

$$T_{vn}(\mathbf{r}) = \epsilon_0 E_v(\mathbf{r}) E_n(\mathbf{r}) + \frac{1}{\mu_0} B_v(\mathbf{r}) B_n(\mathbf{r}) = 0 \quad (4.10)$$

Thus Casimir force is nonzero only in the normal direction.

The closed form solution of dyadic Green's function is not available for an arbitrary geometry. Usually, $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ could only be obtained numerically through calculating the scattered field generated by the objects at $\mathbf{r}$ due to the delta source at $\mathbf{r}'$. Their derivatives could be calculated by applying another set of source and field operators. In the next two sections, the expression of Casimir force in 2D and 3D structures will be given respectively, followed by the integral equation formulation to evaluate it.

4.3 Integral Equation Formulation for Casimir Force in 2D Structures

In the last section, the general expression of Casimir force in terms of 3D dyadic Green's function was given. If the geometry is a z -invariant 2D structure, the expression could be further simplified.

Now it is convenient to define the stress tensor in another set of local coordinates as in Figure 4.1. Here, $\hat{t}$ is the tangent of the contour of the object's cross section, and $\hat{n}$ is

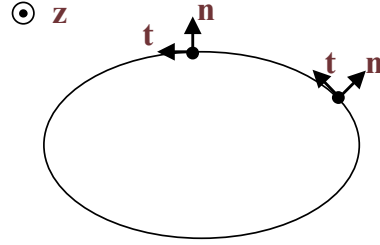


Figure 4.1: Local coordinate for z -invariant objects.

the normal direction of the contour. Correspondingly, here $\hat{u} = \hat{t}$ and $\hat{v} = \hat{z}$. The relation $\hat{n} \times \hat{t} = \hat{z}$ still holds for every segment as in the general local coordinate system.

4.3.1 3D dyadic Green's function and its Fourier transform

In a three-dimensional geometry, the electric field is expressed by the 3D dyadic Green's function as

$$E_{\alpha}(\mathbf{r}) = i\omega \int_{-\infty}^{\infty} dz' \int_S d\rho' G_{\alpha\beta}(k, \mathbf{r}, \mathbf{r}') J_{\beta}(\mathbf{r}') \quad (4.11)$$

In z -invariant structures, the inhomogeneity is two-dimensional. The fields are three-dimensional because of the point nature of the source. As a result, the fields and the source can be represented in terms of their Fourier transforms in the z direction:

$$J_\beta(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{J}_\beta(k_z, \boldsymbol{\rho}) e^{ik_z z} \quad (4.12)$$

$$E_\alpha(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{E}_\alpha(k_z, \boldsymbol{\rho}) e^{ik_z z} \quad (4.13)$$

$$G_{\alpha\beta}(k, \mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \tilde{G}_{\alpha\beta}(k, k_z, \boldsymbol{\rho}, \boldsymbol{\rho}') e^{ik_z(z-z')} \quad (4.14)$$

Their Fourier transforms satisfy the following relationship:

$$\tilde{E}_\alpha(k_z, \boldsymbol{\rho}) = i\omega \int_S d\rho' \tilde{G}_{\alpha\beta}(k, k_z, \boldsymbol{\rho}, \boldsymbol{\rho}') \tilde{J}_\beta(k_z, \boldsymbol{\rho}') \quad (4.15)$$

$$\tilde{H}_z(k_z, \boldsymbol{\rho}) = \frac{1}{\mu_0} \int_S d\rho' \left[\frac{\partial}{\partial n} \tilde{G}_{t\beta}(k, k_z, \boldsymbol{\rho}, \boldsymbol{\rho}') - \frac{\partial}{\partial t} \tilde{G}_{n\beta}(k, k_z, \boldsymbol{\rho}, \boldsymbol{\rho}') \right] \tilde{J}_\beta(k_z, \boldsymbol{\rho}') \quad (4.16)$$

In the above, $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial t}$ are normal and tangential derivatives respectively. From (4.15) and (4.16), it is evident that the Fourier transform of the dyadic Green's function and its derivatives could be obtained by calculating the electric or magnetic field excited by a delta point source.

In a z -invariant structure, only two components of the electromagnetic field are independent and the field can be represented completely in terms of E_z and H_z . In other words, the field in such a geometry can be decomposed into TM ($E_z \neq 0, H_z = 0$) and TE ($H_z \neq 0, E_z = 0$) fields [18, §8.1.4]. Therefore, the average Maxwell stress tensor (4.3)

for each mode could be expressed as

$$\begin{aligned} \langle T_{nn}^{\text{TM}}(\boldsymbol{\rho}) \rangle &= \frac{\hbar c_0}{2\pi^2 \mu_0} \text{Im} \int_0^\infty dk \int_{-\infty}^\infty dk_z \left[\left(k^2 - \frac{\partial^2}{\partial z \partial z'} \right) \tilde{G}_{nn} \right. \\ &\quad \left. - ik_z \frac{\partial}{\partial n} \tilde{G}_{zn} + ik_z \frac{\partial}{\partial n'} \tilde{G}_{nz} - \frac{\partial^2}{\partial n \partial n'} \tilde{G}_{zz} \right] \end{aligned} \quad (4.17)$$

$$\begin{aligned} \langle T_{nn}^{\text{TE}}(\boldsymbol{\rho}) \rangle &= \frac{\hbar c_0}{2\pi \mu_0} \text{Im} \int_0^\infty dk \int_{-\infty}^\infty dk_z \left[\left(k^2 - \frac{\partial^2}{\partial z \partial z'} \right) \tilde{G}_{nn} \right. \\ &\quad \left. + \frac{\partial^2}{\partial n \partial n'} \tilde{G}_{tt} - \frac{\partial^2}{\partial t \partial n'} \tilde{G}_{nt} - \frac{\partial^2}{\partial n \partial t'} \tilde{G}_{tn} + \frac{\partial^2}{\partial t \partial t'} \tilde{G}_{nn} \right] \end{aligned} \quad (4.18)$$

If we substitute (4.15) and (4.16) into (4.17) and (4.18), the mean stress tensor could be further simplified as

$$\langle T_{nn}^{\text{TM}}(\boldsymbol{\rho}) \rangle = \frac{\hbar c_0}{2\pi^2} \text{Im} \int_0^\infty \frac{dk}{i\omega \mu_0} \int_{-\infty}^\infty dk_z \frac{k^2}{k_\rho^2} \frac{\partial}{\partial n} \tilde{E}_z(\boldsymbol{\rho}, s_1) \quad (4.19)$$

$$\langle T_{nn}^{\text{TE}}(\boldsymbol{\rho}) \rangle = \frac{\hbar c_0}{2\pi^2} \text{Im} \int_0^\infty dk \int_{-\infty}^\infty dk_z \left[\frac{k_\rho^2}{i\omega \mu_0} \tilde{E}_n(\boldsymbol{\rho}, s_2) - \tilde{H}_z(\boldsymbol{\rho}, s_3) + \tilde{H}_z(\boldsymbol{\rho}, s_4) \right] \quad (4.20)$$

where $\tilde{E}_k(\boldsymbol{\rho}, s_i)$ and $\tilde{H}_k(\boldsymbol{\rho}, s_i)$ are Fourier transforms of the electric/magnetic field generated by the source of type i in the geometry. They could be solved from two independent integral equations [18] using the standard method of moments [9]. The evaluation of the mean stress tensor has been converted to a scattering problem with a set of specific delta point sources. The required field components and their corresponding excitations are summarized in Table 4.1.

Table 4.1: Source and field operators for each term in (4.19) and (4.20).

Wave Mode	Field Component	Source ($\rho_i = \rho$)
TM	$\frac{\partial}{\partial n} \tilde{E}_z(\rho, s_1)$	(1) $\hat{z} \frac{\partial}{\partial n'} \delta(\rho' - \rho_i)$
TE	$\tilde{E}_n(\rho, s_2)$	(2) $\hat{n} \delta(\rho' - \rho_i)$
TE	$\tilde{H}_z(\rho, s_3)$	(3) $\hat{t} \frac{\partial}{\partial n'} \delta(\rho' - \rho_i)$
TE	$\tilde{H}_z(\rho, s_4)$	(4) $\hat{n} \frac{\partial}{\partial n'} \delta(\rho' - \rho_i)$

4.3.2 2D integral equations for z -invariant structures

In a z -invariant structure, only two components of the electromagnetic field are independent and the field can be represented completely in terms of E_z and H_z . In other words, the field in such a geometry can be decomposed into TM(E_z) and TE(H_z) fields [18, §8.1.4]. They satisfy the following two independent equations:

$$\tilde{E}_z(\rho) = \tilde{E}_z^i(\rho) - \int_S dS' \hat{n}' \cdot \left[g(\rho, \rho') \nabla'_s \tilde{E}_z(\rho') - \tilde{E}_z(\rho') \nabla'_s g(\rho, \rho') \right] \quad (4.21)$$

$$\tilde{H}_z(\rho) = \tilde{H}_z^i(\rho) - \int_S dS' \hat{n}' \cdot \left[g(\rho, \rho') \nabla'_s \tilde{H}_z(\rho') - \tilde{H}_z(\rho') \nabla'_s g(\rho, \rho') \right] \quad (4.22)$$

where $g(\rho, \rho')$ is the two-dimensional homogeneous-medium Green's function,

$$g(\rho, \rho') = \frac{i}{4} H_0^{(1)}(k_\rho |\rho - \rho'|) \quad (4.23)$$

On the surface of the perfectly conducting object, E_z satisfies the Dirichlet boundary condition and H_z satisfies the Neumann boundary condition. Enforcing (4.21) and (4.22)

on the surface with the boundary conditions, they could be simplified to two 2D integral equations, one for the TM case, the other for the TE case.

$$0 = \tilde{E}_z^i(\boldsymbol{\rho}) - \int_S dS' g(\boldsymbol{\rho}, \boldsymbol{\rho}') \hat{n}' \cdot \nabla'_s \tilde{E}_z(\boldsymbol{\rho}') \quad (4.24)$$

$$\tilde{H}_z(\boldsymbol{\rho}) = \tilde{H}_z^i(\boldsymbol{\rho}) + \int_S dS' \hat{n}' \cdot \nabla'_s g(\boldsymbol{\rho}, \boldsymbol{\rho}') \tilde{H}_z(\boldsymbol{\rho}') \quad (4.25)$$

In the following sections, the tilde symbol would be dropped from $\tilde{E}_\alpha^i(\boldsymbol{\rho})$ and $\tilde{H}_\alpha^i(\boldsymbol{\rho})$. Since their argument is $\boldsymbol{\rho}$ instead of $\mathbf{r}$, they would not be confused with the 3D field notations.

4.3.3 Matrix equation for TM wave

In this part, we will formulate the matrix equation to find E_z at any point in space. Assuming the surface contour C is divided into N segments ΔC_n , the pulse function is defined as [9]:

$$f_n(\boldsymbol{\rho}) = \begin{cases} 1 & \text{on } \Delta C_n \\ 0 & \text{on all other } \Delta C_m, \quad m \neq n \end{cases} \quad (4.26)$$

We let $\frac{\partial}{\partial n} E_z(\boldsymbol{\rho}) = \sum a_n f_n, \boldsymbol{\rho}$ on C and use Galerkin's method on each segment ΔC_m ; a matrix equation is then obtained from Eqn. (4.24):

$$[l_{mn}][a_n] = [g_m] \quad (4.27)$$

where

$$g_m = \int_{C_m} E_z^i(\boldsymbol{\rho}) dl \quad (4.28)$$

$$l_{mn} = \frac{i}{4} \int_{C_m} dl \int_{C_n} dl' H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.29)$$

When $m = n$, the Hankel function has an integrable singularity which must be evaluated analytically:

$$l_{nn} = \frac{i}{4} (\Delta C_n)^2 \left[1 + i \frac{2}{\pi} \ln \frac{\Delta C_n k e^\gamma}{4e} \right] \quad (4.30)$$

4.3.4 Matrix equation for TE wave

In the integral equation for TE waves, the integral on the right-hand side of (4.25) should be evaluated as a sum of the principal value integral and its residue. And the integral equation becomes

$$-H_z^i(\boldsymbol{\rho}) = -\frac{1}{2} H_z(\boldsymbol{\rho}) + p.v. \int_S dS' \hat{n}' \cdot \nabla'_s g(\boldsymbol{\rho}, \boldsymbol{\rho}') H_z(\boldsymbol{\rho}'), \quad \boldsymbol{\rho} \text{ on } S \quad (4.31)$$

If the magnetic field on the surface is discretized, $H_z = \sum b_n f_n$, the resulting matrix equation should be $[l_{mn}][b_n] = [h_m]$, where

$$h_m = - \int_{C_m} dl H_z^i(\boldsymbol{\rho}) \quad (4.32)$$

$$l_{mn} = -\frac{C_m}{2} \delta_{mn} + \frac{i}{4} \int_{C_m} dl \int_{C_n} dl' \hat{n}' \cdot \nabla'_s H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.33)$$

The diagonal elements l_{nn} are equal to $-\frac{1}{2}C_n$.

4.3.5 Evaluating the incident field generated by different sources

From Table 4.1, it is clear that four kinds of source should be employed in order to calculate the stress tensor. In this section, we will study the expression of the incident field generated by each of them, which are the right-hand-side vectors of the matrix equations.

The general formula for the incident field from a specific source is

$$E_z^i(k_z, \boldsymbol{\rho}) = -\frac{\omega\mu_0}{4k_0^2} \int_S dr' \left(k_0^2 \delta_{z\alpha} + ik_z \frac{\partial}{\partial \alpha} \right) H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) J_\alpha(\boldsymbol{\rho}') \quad (4.34)$$

$$\begin{aligned} H_z^i(k_z, \boldsymbol{\rho}) &= \frac{i}{4} \int_S d\boldsymbol{\rho}' H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) (\nabla'_s \times \mathbf{J}_s(\boldsymbol{\rho}')) \cdot \hat{z} \\ &= \frac{i}{4} \int_S d\boldsymbol{\rho}' H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \left[\frac{\partial}{\partial n'} J_t(\boldsymbol{\rho}') - \frac{\partial}{\partial t'} J_n(\boldsymbol{\rho}') \right] \end{aligned} \quad (4.35)$$

We have assumed that the delta source is located near the center of segment C_i , and $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_i + \hat{n}_i \delta$ ($\hat{n}_i$ is the normal direction at $\boldsymbol{\rho}_i$).

- First kind of source (TM mode): $\hat{z} \delta'(\rho_n - \rho_{n0}) \delta(\rho_t - \rho_{t0})$

$$g_m = \int_{C_m} E_z^i(\boldsymbol{\rho}) dl = -\frac{\omega\mu_0 k_\rho^2}{4k_0^2} \int_{C_m} dl \frac{\partial}{\partial \hat{n}_i} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \quad (4.36)$$

- Second kind of source (TE mode): $\hat{n} \delta(\boldsymbol{\rho} - \boldsymbol{\rho}_0)$

$$h_m = - \int_{C_m} dl H_z^i(\boldsymbol{\rho}) = -\frac{i}{4} \int_{C_m} dl \frac{\partial}{\partial t'_i} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \Big|_{\boldsymbol{\rho}'=\boldsymbol{\rho}_0} \quad (4.37)$$

- Third kind of source (TE mode): $\hat{n}_i \delta(\rho_n - \rho_{n0}) \delta'(\rho_t - \rho_{t0})$

$$h_m = - \int_{C_m} dl H_z^i(\boldsymbol{\rho}) = \frac{i}{4} \int_{C_m} dl \frac{\partial^2}{\partial \hat{t}_i^2} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \quad (4.38)$$

- Fourth kind of source: $\hat{t}\delta'(\rho_n - \rho_{n0})\delta(\rho_t - \rho_{t0})$

$$h_m = - \int_{C_m} dl H_z^i(\boldsymbol{\rho}) = -\frac{i}{4} \int_{C_m} dl \frac{\partial^2}{\partial \hat{n}_i^2} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \quad (4.39)$$

4.3.6 Evaluating the field testing operators

The z components of the electric and magnetic field are given by

$$E_z(\boldsymbol{\rho}) = - \int_S dS' g(\boldsymbol{\rho}, \boldsymbol{\rho}') \hat{n}' \cdot \nabla'_s E_z(\boldsymbol{\rho}') = -\frac{i}{4} \sum a_m \int_{C_m} dl' H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.40)$$

$$H_z(\boldsymbol{\rho}) = \int_S dS' \hat{n}' \cdot \nabla'_s g(\boldsymbol{\rho}, \boldsymbol{\rho}') H_z(\boldsymbol{\rho}') = \frac{i}{4} \sum b_m \int_{C_m} dl' \frac{\partial}{\partial \hat{n}'_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.41)$$

The transverse electric field is given by [18]

$$\mathbf{E}_s = \frac{i}{k_\rho^2} [k_z \nabla_s E_z + \omega \mu \nabla_s \times \mathbf{H}_z] \quad (4.42)$$

Now we will study the detailed form of the four field testing operators. Similarly, the observation point should be near the center of segment C_i where the source is located.

- First Field Testing Operator (TM mode): $(\partial/\partial \hat{n}_i) \hat{z}$.

$$\frac{\partial}{\partial \hat{n}_i} E_z(\boldsymbol{\rho}) = -\frac{i}{4} \sum a_m \int_{C_m} dl' \frac{\partial}{\partial \hat{n}_i} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad \boldsymbol{\rho} = \boldsymbol{\rho}_i + \hat{n}_i \delta \quad (4.43)$$

- Second Field Testing Operator (TE mode): $\hat{n}$.

$$E_{n_i}(\boldsymbol{\rho}) = \frac{i\omega\mu}{k_\rho^2} \frac{\partial}{\partial \hat{t}_i} H_z = -\frac{\omega\mu}{4k_\rho^2} \sum b_m \int_{C_m} dl' \frac{\partial^2}{\partial \hat{t}_i \partial \hat{n}'_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.44)$$

- Third Field Testing Operator: $\hat{z}$.

$$H_z(\boldsymbol{\rho}) = \frac{i}{4} \sum b_m \int_{C_m} dl' \frac{\partial}{\partial \hat{n}'_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}'|) \quad (4.45)$$

The integrals involved in the incident field and testing field evaluation can be cast into two general integrals:

$$\int_{C_m} dl \frac{\partial}{\partial \mu} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|), \quad \int_{C_m} dl \frac{\partial^2}{\partial \mu \partial \nu} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|)$$

As the order of the derivative operators becomes higher, the integrand becomes more singular. One needs to be careful if an accurate evaluation is desired. The details of the numerical integration of Hankel functions are discussed in Appendix.

4.4 Integral Equation Formulation for Casimir Force in 3D Structures

First, we would like to cast the evaluation of dyadic Green's function and its derivatives into solving a set of scattering problems as we did in the 2D structures.

4.4.1 Casimir force represented by field components

In 3D structures, the field components and the dyadic Green's function satisfy the following relationship:

$$E_\alpha(\mathbf{r}) = i\omega \int_S d\mathbf{r}' G_{\alpha\beta}(k, \mathbf{r}, \mathbf{r}') J_\beta(\mathbf{r}') \quad (4.46)$$

$$H_u(\mathbf{r}) = \frac{1}{\mu_0} \int_S d\mathbf{r}' \left[\frac{\partial}{\partial v} G_{n\beta}(k, \mathbf{r}, \mathbf{r}') - \frac{\partial}{\partial n} G_{v\beta}(k, \mathbf{r}, \mathbf{r}') \right] J_\beta(\mathbf{r}') \quad (4.47)$$

Substituting (4.46) and (4.47) into (4.4) and (4.5), the average of each term in the stress tensor could be represented by

$$\begin{aligned} \frac{1}{2} \epsilon_0 \langle E_n(\mathbf{r}) E_n(\mathbf{r}') \rangle &= \frac{\hbar \epsilon_0}{2\pi} \text{Im} \int_0^\infty \omega^2 G_{nn}(\mathbf{r}, \mathbf{r}', \omega) d\omega = \frac{\hbar c_0}{2\pi \mu_0} \text{Im} \int_0^\infty \frac{k_0^2}{i\omega} E_n(k_0, s_1) dk_0 \\ \frac{1}{2\mu_0} \langle B_u(\mathbf{r}) B_u(\mathbf{r}') \rangle &= \frac{\hbar c_0}{2\pi \mu_0} \text{Im} \int_0^\infty dk_0 \left[\frac{\partial^2}{\partial n \partial n'} G_{vv}(\mathbf{r}, \mathbf{r}') - \frac{\partial^2}{\partial v \partial n'} G_{nv}(\mathbf{r}, \mathbf{r}') \right. \\ &\quad \left. - \frac{\partial^2}{\partial n \partial v'} G_{vn}(\mathbf{r}, \mathbf{r}') + \frac{\partial^2}{\partial v \partial v'} G_{nn}(\mathbf{r}, \mathbf{r}') \right] \\ &= \frac{\hbar c_0}{2\pi} \text{Im} \int_0^\infty d(k_0) [-H_u(k_0, \mathbf{r}, s_2) + H_u(k_0, \mathbf{r}, s_3)] \end{aligned} \quad (4.48)$$

$$\frac{1}{2\mu_0} \langle B_v(\mathbf{r}) B_v(\mathbf{r}') \rangle = \frac{\hbar c_0}{2\pi} \text{Im} \int_0^\infty d(k_0) [-H_v(k_0, \mathbf{r}, s_4) + H_u(k_0, \mathbf{r}, s_5)] \quad (4.49)$$

Here, $\mathbf{E}(k_0, \mathbf{r}, s_i)$ and $\mathbf{H}(k_0, \mathbf{r}, s_i)$ are the electric/magnetic field at location $\mathbf{r}$ in the geometry under the excitation of source s_i . Their corresponding relationship is summarized in Table 4.2.

Substituting (4.8) and (4.10) into (4.7), the normal component of the Casimir force can

Table 4.2: Source and field operators for each term in the stress tensor for 3D geometry.

	Field Component	Source ($\mathbf{r}_i = \mathbf{r}$)
(1)	$E_n(\mathbf{r}, s1)$	$\hat{n}\delta(\mathbf{r}' - \mathbf{r}_i)$
(2)	$H_u(\mathbf{r}, s2)$	$\hat{v} \frac{\partial}{\partial n'} \delta(\mathbf{r}' - \mathbf{r}_i)$
(3)	$H_u(\mathbf{r}, s3)$	$\hat{n} \frac{\partial}{\partial v'} \delta(\mathbf{r}' - \mathbf{r}_i)$
(4)	$H_v(\mathbf{r}, s4)$	$\hat{n} \frac{\partial}{\partial u'} \delta(\mathbf{r}' - \mathbf{r}_i)$
(5)	$H_v(\mathbf{r}, s5)$	$\hat{u} \frac{\partial}{\partial n'} \delta(\mathbf{r}' - \mathbf{r}_i)$

be expressed as:

$$P(\mathbf{r}) = T_{nn}(\mathbf{r}) = \frac{\hbar c_0}{2\pi} \text{Im} \int_0^\infty dk \left[\frac{k^2}{i\omega\mu_0} E_n(\mathbf{r}, s_1) + H_u(\mathbf{r}, s_2) - H_u(\mathbf{r}, s_3) \right. \\ \left. + H_v(\mathbf{r}, s_4) - H_v(\mathbf{r}, s_5) \right]_{\mathbf{r}'=\mathbf{r}} \quad (4.50)$$

4.4.2 Matrix equation for arbitrary 3D objects

If the object is made of perfect conducting material, there is only electric current $\mathbf{J}$ on the surface. The electric field and magnetic field in the free space region could be obtained from the electric current by $\mathcal{L}$ and $\mathcal{K}$ operators respectively.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + \mathcal{L}\mathbf{J}(\mathbf{r}) \quad (4.51)$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^i(\mathbf{r}) - \mathcal{K}\mathbf{J}(\mathbf{r}) \quad (4.52)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the total field in the space and $\mathbf{E}^i$ and $\mathbf{H}^i$ are the incident field generated by an external source. The operators are defined as

$$\mathcal{L}\mathbf{X}(\mathbf{r}) = i\omega\mu_0 \int_s \left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_0^2} \right] g(\mathbf{r}, \mathbf{r}') \cdot \mathbf{X}(\mathbf{r}') d\mathbf{r}' \quad (4.53)$$

$$\mathcal{K}\mathbf{X}(\mathbf{r}) = \int_s \mathbf{X}(\mathbf{r}') \times \nabla g(\mathbf{r}, \mathbf{r}') d\mathbf{r}' \quad (4.54)$$

where

$$g(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (4.55)$$

The total tangential electric field vanishes on the surface of the PEC object. Enforcing Eqn. (4.53) on the surface we obtain the electric field integral equation (EFIE):

$$-\hat{n} \times \mathbf{E}^i(\mathbf{r}) = \hat{n} \times \mathcal{L}\mathbf{J}(\mathbf{r}) \quad (4.56)$$

By expanding the unknown current with RWG basis [10], ($\mathbf{J}(\mathbf{r}') = \sum_n a_n \mathbf{J}_n(\mathbf{r}')$) and using Galerkin's method, the integral equation (4.56) could be cast into the following matrix equation:

$$[Z_{mn}][a_n] = [g_m] \quad (4.57)$$

where

$$g_m = \int_{T_m} d\mathbf{r} \mathbf{J}_m(\mathbf{r}) \cdot \mathbf{E}^i(\mathbf{r}) \quad (4.58)$$

$$Z_{mn} = i\omega\mu_0 \int_{T_m} d\mathbf{r} \int_{T_n} d\mathbf{r}' \mathbf{J}_m \cdot \left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k_0^2} \right] g(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_n(\mathbf{r}') \quad (4.59)$$

4.4.3 Evaluating the incident field generated by different sources

The field components in Table 4.2 could be readily obtained by using (4.53) and (4.54). Meanwhile, the incident field generated by those five different sources should be derived carefully. In this subsection, we will study the expression of the incident field generated by each of them, which form the right-hand-side vectors of the matrix equations.

The general formula for the incident field from a specific source is

$$\mathbf{E}_\alpha^i(k_0, \mathbf{r}) = \frac{i\omega\mu_0}{4\pi k_0^2} \int_S d\mathbf{r}' \left(k_0^2 \delta_{\alpha\beta} + \frac{\partial}{\partial\alpha\partial\beta} \right) \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} J_\beta(\mathbf{r}') \quad (4.60)$$

Assume that the delta source is located near the center of patch T_i , and $\mathbf{r}_0 = \mathbf{r}_{ci} + \hat{n}_i \delta$ ($\hat{n}_i$ is the directional vector normal to patch T_i).

- First kind of source: $\hat{n}\delta(\mathbf{r} - \mathbf{r}_0)$

$$\begin{aligned} g_m &= \int_{T_m} \mathbf{J}_m(\mathbf{r}) \cdot \mathbf{E}^i(\mathbf{r}, \mathbf{r}_0) d\mathbf{r} \quad \mathbf{R} = \mathbf{r} - \mathbf{r}_0 \quad (4.61) \\ &= \int_{T_m} d\mathbf{r} (\mathbf{J}_m(\mathbf{r}) \cdot \hat{n}_i) \frac{e^{ik_0|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} + \frac{1}{k_0^2} \int_{T_m} d\mathbf{r} (\nabla \cdot \mathbf{J}_m(\mathbf{r})) \left[\nabla' \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cdot \hat{n}_i \right]_{\mathbf{r}'=\mathbf{r}_0} \\ &= \sum_{q=1}^2 \frac{\epsilon_q l_m}{2A_m^q} \int_{T_m^q} d\mathbf{r} \hat{n}_i \cdot (\mathbf{r} - \mathbf{r}_{vm}^q) \frac{e^{ik_0 R}}{R} - \sum_{q=1}^2 \frac{\epsilon_q l_m}{A_m^q k_0^2} \int_{T_m^q} d\mathbf{r} \left[\nabla \frac{e^{ik_0 R}}{R} \cdot \hat{n}_i \right] \end{aligned}$$

- Second kind of source: $\hat{v}_i \frac{\partial}{\partial n'_i} \delta(\mathbf{r}' - \mathbf{r}_0)$

$$\begin{aligned}
g_m &= \int_{T_m} (\mathbf{J}_m(\mathbf{r}) \cdot \hat{v}_i) \left[\nabla \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cdot \hat{n}_i \right] d\mathbf{r} + \\
&\quad \frac{1}{k_0^2} \int_{T_m} (\nabla \cdot \mathbf{J}_m(\mathbf{r})) \left[\frac{\partial}{\partial \hat{n}'_i} \left(\nabla' \frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \cdot \hat{v}_i \right) \right]_{\mathbf{r}'=\mathbf{r}_0} d\mathbf{r} \\
&= \sum_{q=1}^2 \frac{\epsilon_q l_m}{2A_m^q} \int_{T_m^q} d\mathbf{r} \hat{v}_i \cdot (\mathbf{r} - \mathbf{r}_{vm}^q) \left[\nabla \frac{e^{ik_0 R}}{R} \cdot \hat{n}_i \right] + \\
&\quad \sum_{q=1}^2 \frac{\epsilon_q l_m}{A_m^q k_0^2} \int_{T_m^q} d\mathbf{r} \left[\hat{n}_i \cdot \nabla \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{v}_i \right] \quad (4.62)
\end{aligned}$$

- Third kind of source: $\hat{n}_i \frac{\partial}{\partial v'_i} \delta(\mathbf{r}' - \mathbf{r}_0)$

$$g_m = \sum_{q=1}^2 \frac{\epsilon_q l_m}{2A_m^q} \left\{ \int_{T_m^q} d\mathbf{r} \hat{n}_i \cdot (\mathbf{r} - \mathbf{r}_{vm}^q) \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{v}_i + \frac{2}{k_0^2} \int_{T_m^q} d\mathbf{r} \left[\hat{n}_i \cdot \nabla \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{v}_i \right] \right\}$$

- Fourth kind of source: $\hat{n}_i \frac{\partial}{\partial u'_i} \delta(\mathbf{r}' - \mathbf{r}_0)$

$$g_m = \sum_{q=1}^2 \frac{\epsilon_q l_m}{2A_m^q} \left\{ \int_{T_m^q} d\mathbf{r} \hat{n}_i \cdot (\mathbf{r} - \mathbf{r}_{vm}^q) \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{u}_i + \frac{2}{k_0^2} \int_{T_m^q} d\mathbf{r} \left[\hat{n}_i \cdot \nabla \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{u}_i \right] \right\}$$

- Fifth kind of source: $\hat{u}_i \frac{\partial}{\partial n'_i} \delta(\mathbf{r}' - \mathbf{r}_0)$

$$g_m = \sum_{q=1}^2 \frac{\epsilon_q l_m}{2A_m^q} \left\{ \int_{T_m^q} d\mathbf{r} \hat{u}_i \cdot (\mathbf{r} - \mathbf{r}_{vm}^q) \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{n}_i + \frac{2}{k_0^2} \int_{T_m^q} d\mathbf{r} \left[\hat{u}_i \cdot \nabla \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{n}_i \right] \right\}$$

In summary, there are three typical integrals to handle:

$$\begin{aligned}
I_a &= \int_T d\mathbf{r} \nabla g \cdot \hat{n} = \int_T d\mathbf{r} \frac{\mathbf{R}}{R^3} (ikR - 1) e^{ikR} \cdot \hat{n} \\
I_b &= \int_T d\mathbf{r} [\hat{u} \cdot (\mathbf{r} - \mathbf{r}_{vm})] (\nabla g \cdot \hat{n}) \\
I_c &= \int_{T_m} \left[\hat{n}_i \cdot \nabla \nabla \frac{e^{ik_0 R}}{R} \cdot \hat{u}_i \right] d\mathbf{r}
\end{aligned}$$

The first integral is similar to the $\mathcal{K}$ operator and can be handled similarly. The second integral is not as singular as the first when $\mathbf{r}'$ is close to $\mathbf{r}$ and could be evaluated numerically. The third type of integral is super-hyper singular and is the most difficult one. Fortunately, its treatment has been discussed in detail in [97].

Now, we will discuss the challenges we face in this project and the possible solutions. First, the dyadic Green's function $G_{nn}(k, \mathbf{r}, \mathbf{r}')$ on the surface of a perfect electric conductor is singular when $\mathbf{r}' = \mathbf{r}$. Part of the singularity is due to the free-space dyadic Green's function and part of it is because of the contribution from the primary image. The stress tensor related to the free space dyadic Green's function is uniformly distributed on the surface and the net Casimir source from it is zero.

$$\mathbf{F}_0 = \langle T \rangle \oint_S 1 d\mathbf{s}' = \langle T \rangle \oint_S \sqrt{1 + y'^2} \left(\frac{y'^2 dx}{\sqrt{1 + y'^2}} \hat{x} + \frac{dx}{\sqrt{1 + y'^2}} \hat{y} \right) = 0 \quad (4.63)$$

Thus, we will use only the scattered field in dyadic Green's functions calculation. When the source is placed on the surface of the conductor, its primary image coincides with the source itself, giving rise to a singularity as well. This singularity could be removed by lifting the source from the surface in the normal direction for a small distance δ . Then the

source and its image has a separation of 2δ and the singularity is replaced by a finite large number. However, there is a compromise between accuracy and smoothness. If the source is too far away from the surface, the accuracy of the dyadic Green's function suffers. But if it is too close, the dyadic Green's function is nearly singular and overflow might occur.

The second challenge is the numerical integration of (4.50). We would study the property of this integral in the case of parallel plates, since analytical solution is available for this example. For parallel plates separated by a distance of a , the analytic expression of the stress tensor is

$$F = \frac{\hbar c_0}{2\pi^2} \text{Im} \int_0^\infty dk_0 \mathcal{F}(k_0) = \frac{\hbar c_0 \pi^2}{240a^4}$$

where $\mathcal{F}(k_0) = \sum_{m=0}^\infty \left(-\frac{2k_0^2}{r_r} - \frac{4ik_0}{r_r^2} + \frac{4}{r_r^3} \right) e^{ik_0 r_r}, \quad r_r = 2(m+1)a \quad (4.64)$

The integrand of the equation above is plotted in Figure 4.2. It is both oscillatory and growing with k_0 . So it is impossible to evaluate this integral numerically along this integral path. From causality, we know that the Green's function has no poles in the upper-half plane. So we can perform a *Wick rotation* [91] to transform the integral path from the real axis to the imaginary axis. The value of the integrand along the imaginary axis is shown in Figure 4.3. After the Wick rotation, the integral converges rapidly and it can be calculated accurately with only a few quadrature points.

$$F = \frac{\hbar c_0}{2\pi^2} \int_0^\infty dk_0 \mathcal{F}(ik_0) = \frac{\hbar c_0}{2\pi^2} \sum_i w_i \mathcal{F}(ik_{0i}) \quad (4.65)$$

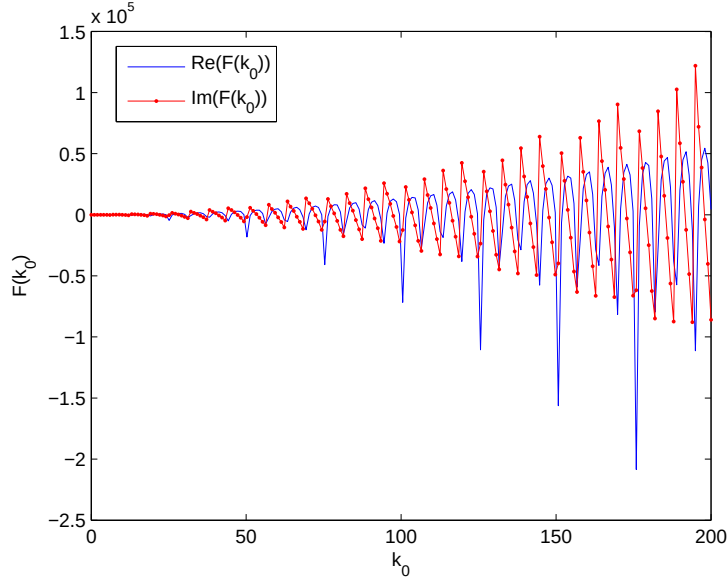


Figure 4.2: The function $\mathcal{F}(k_0)$.

4.5 Numerical Results

Now we give the numerical results obtained from this method and validate them against analytical results or other calculations. The perfect conducting parallel plate case is relatively simple and has been used as a basic test.

4.5.1 Casimir force between 2D objects

A more complicated geometry is an infinitely large plate near an infinitely long circular cylinder. The exact Casimir force between the two objects has been derived in [89] and used to validate the numerical calculation in [91]. The normalization factor $F_{\text{PFA}} = \frac{\pi^3}{384} \hbar c_0 L \sqrt{R/2a^7}$ is the force calculated by proximity-force approximation in reference

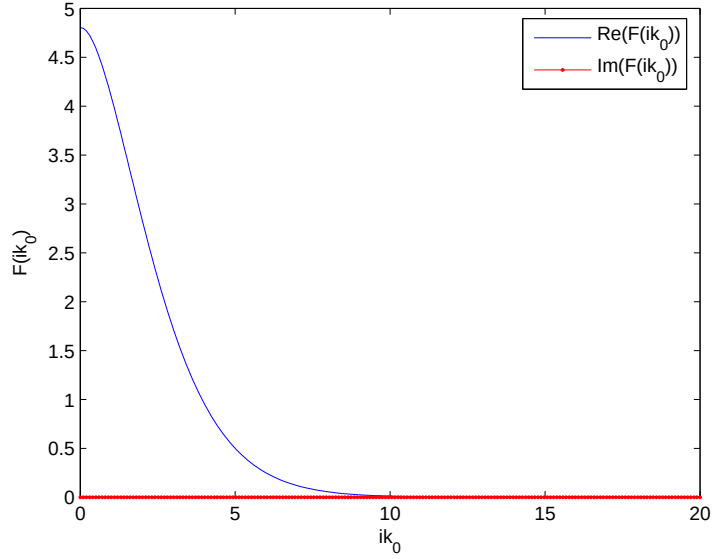


Figure 4.3: The function $\mathcal{F}(ik_0)$.

[98]. Notice that reference [98] also gives the asymptotic analytical results of this geometry at small separations. Compared with the exact solution, the relative error of the asymptotic results at $a/R = 0.2$ is well below one percent for TM mode, but it is about six percent for the TE mode. By using the integral equation method given in this paper, the relative error is always below one percent for different separation (Figure 4.4). The reason why the Casimir force from the TM polarization increases with separation a/R is because the normalization factor F_{PFA} decreases more rapidly with separation.

The second geometry includes two square cylinders sandwiched between two parallel plates. We calculate the attractive force between the two cylinders while changing the separation between the cylinders and the parallel plates. The results are compared with the data given in [91] obtained using finite difference method (Figure 4.5).

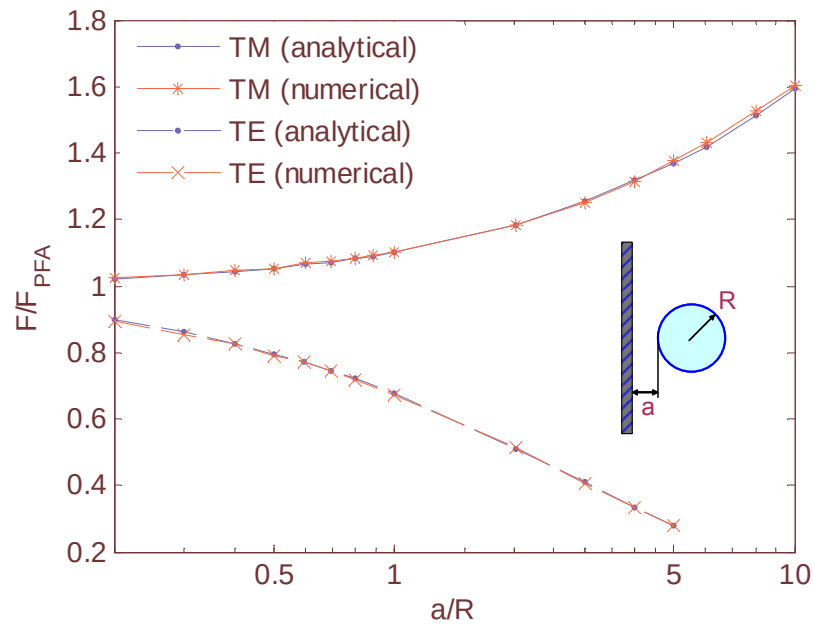


Figure 4.4: Casimir force between a cylinder of radius R and a large plate of distance a to the cylinder, relative to the proximity-force approximation.

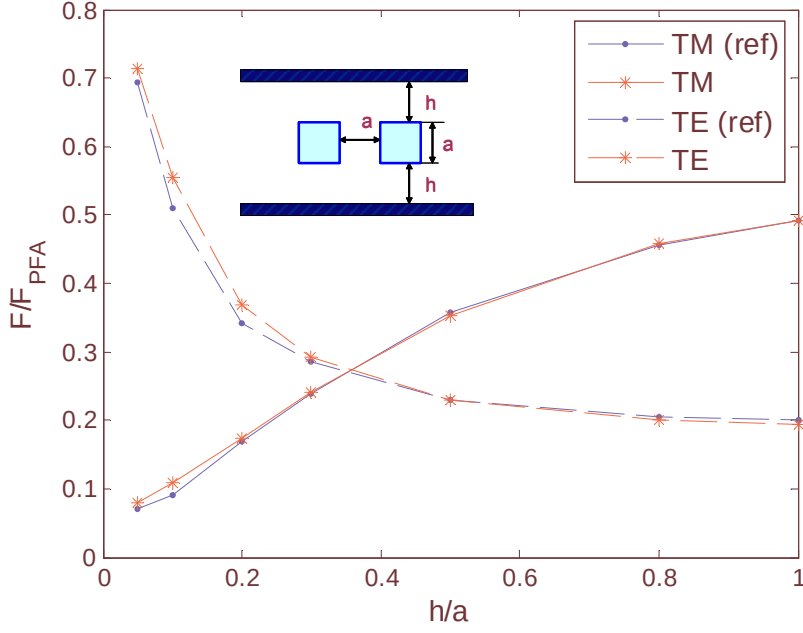


Figure 4.5: Casimir force between two square cylinders of size a located between two large parallel plates, relative to the proximity-force approximation $F_{\text{PFA}} = \hbar c_0 \pi^2 a / 480 a^4$.

4.5.2 Casimir force between 3D objects

As in the 2D case, first the force between two parallel plates is evaluated and compared with the well-established analytical solution. Then we start to evaluate the Casimir force between two spheres, since they are the simplest 3D objects and the analytical expression of the Casimir energy is available in [88]. We assume that the radius of both spheres is R and their minimum distance is Z ; the product of the force and the square of the radius ($F * R^2 / \hbar c_0$) is a dimensionless and scale-invariant quantity. So the result is given by their product ($F * R^2 / \hbar c_0$) versus the normalized separation (Z/R). In the numerical process, first the Casimir force per unit area (pressure) is calculated at different points on the surface, and then the total force is obtained by integrating the pressure over the

entire surface. Each sphere is discretized into 286 triangles and each data point can be obtained within 3 to 4 minutes on a 3.0 GHz computer. The maximum relative error is about 5% percent. The average relative error is 3%. In Figure 4.6, the numerical results are compared with the analytical result obtained by interpolating the data given in [88].

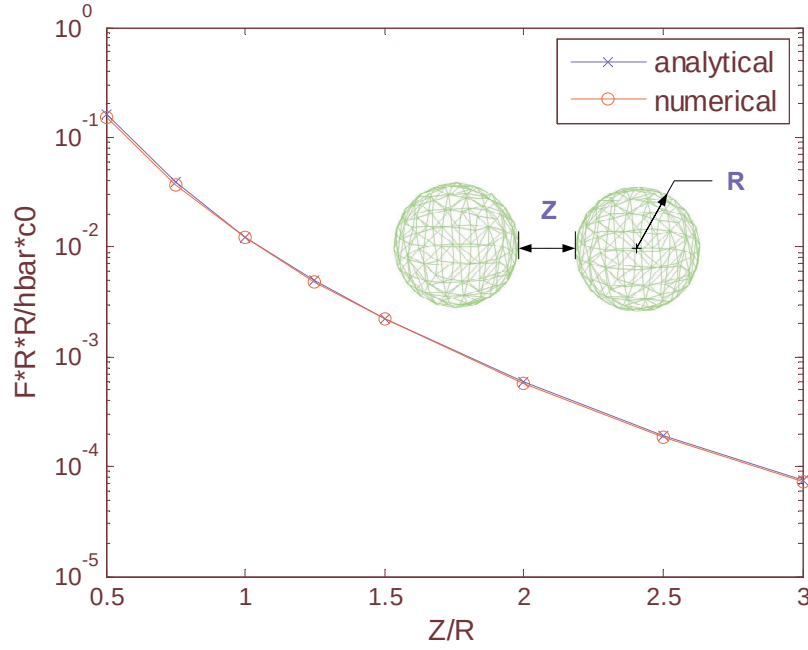


Figure 4.6: Casimir force between two spheres of radius R and a separation of Z .

Compared to the method presented in [88] and [90], our method has the advantage that we can obtain the pressure distribution on the surface of the object. The pressure distribution on one of the spheres in this example is shown in Figure 4.7.

The geometry of the second example contains two identical capsules. A capsule is a cylinder of radius R with hemispherical end caps. Its total length is denoted by L . The force between two capsules is evaluated for two cases: they are parallel or they are perpendicular. Each capsule is discretized into 852 triangles. The reference data is obtained by

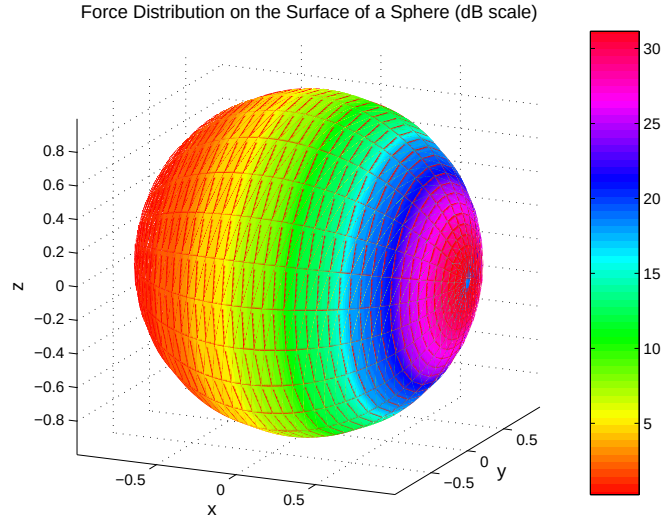


Figure 4.7: Casimir pressure distribution on the surface of one of the spheres ($Z = R$). The force is shown in log scale.

interpolating the numerical results given in [90]. The comparison is presented in Figure 4.8.

4.6 Conclusion

Compared to the finite difference method, the integral equation method has reduced the computation cost significantly. First, the discretization of the computational domain occurs only at the surface, so we need many fewer unknowns. Second, now the Casimir force is a smooth function along the smooth surface of the geometry. When this force is integrated to obtain the total force, a few quadrature points on the surface are sufficient. The number of unknowns can be increased to enhance the accuracy of the integral equation method.

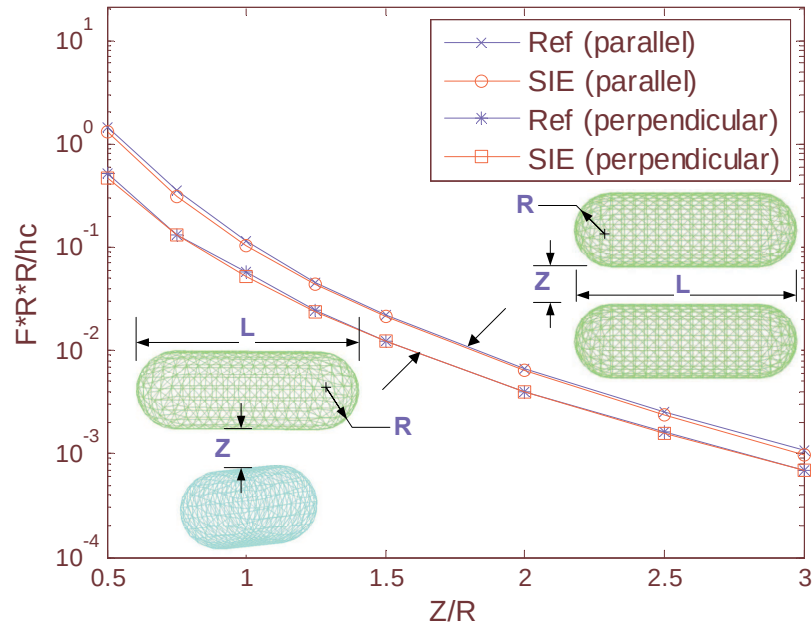


Figure 4.8: Casimir force for two pairs of perfectly conducting objects: identical capsules with parallel axes, and identical capsules with perpendicular axes. Here $L = 6R$. SIE refers to the surface integral equation method derived in this thesis.

But both the coordinate space integration (4.2) and the spectral space integration (4.17) and (4.18) are independent of the number of unknowns. For solving the integral equation with iterative methods, the computational complexity could be reduced to $O(N \log N)$ per iteration by using the multilevel fast multipole algorithm [59]. Thus the overall efficiency of this numerical method could be of $O(N \log N)$ per iteration at best, where N is the number of surface unknowns.

So far, the calculation has assumed an ideal condition: perfect reflectors, perfectly smooth surfaces, zero-temperature. For a more realistic calculation, one needs to account for imperfect reflection of the object and its frequency-dependent reflection coefficients. The measurement is usually carried out at room temperature and there exist thermal fluctuations superimposed on vacuum fluctuations. Thermal correction must be included in calculations of the force at separations greater than $1 \mu\text{m}$. Finally, real mirrors are not perfectly smooth. The roughness affects measurements of the Casimir force, which is very sensitive to small changes in distance.

CHAPTER 5

CONCLUSION AND FUTURE WORK

Layered media Green's function along with surface integral equation is a powerful method to model the full wave behavior of metallic structures residing in layered media, for example, microstrip structures.

In this thesis, a new and elegant approach to derive the dyadic Green's function for layered medium (DGLM) has been proposed. This new form of DGLM could be expressed in term of two Sommerfeld integrals involving zeroth order Bessel function, one for TE waves and the other for TM waves. This form would be amenable for designing acceleration techniques. The matrix representation of the DGLM using MOM is manipulated so that the coordinate-space singularities in the integrand are as weak as possible. As a result, the associated Sommerfeld integrals have fewer high frequency components and converge rapidly. We also call it a matrix-friendly formulation.

In order to accelerate the solution of the MOM equation, a quasi-3D thin-stratified medium fast-multipole algorithm (quasi-3D TSM-FMA) is proposed. It is designed for 3D structures whose horizontal dimension is much greater than the thickness of the layered medium, which is true for all microstrip and circuit applications. The quasi-3D TSM-FMA is able to handle objects which straddle the layers of the medium. Memory requirement analysis is presented and memory reduction techniques are discussed. Finally, numeri-

cal tests are performed and they confirm that both the computational complexity and the memory cost are of $O(N \log N)$.

So far, this algorithm still suffers low-frequency breakdown as a conventional integral equation method. When the size of the element is less than 1/1000 of wavelength, the algorithm fails to produce the correct result. In many circuit problems, for example, the interconnects on chip or package, fall into the domain of the low-frequency problem. In order to overcome this drawback, a loop-tree decomposition method or augmented surface integral equation method should be incorporated with the current algorithm. Then it would be useful in a larger scope.

In the second part of this thesis, an efficient numerical method to evaluate the Casimir force between perfect conductors has been proposed. The Casimir force is important in MEMS and NEMS devices and a general numerical method to predict it efficiently is desired. The force can be expressed in terms of the average of Maxwell stress tensor, which is related to the dyadic Green's function and its derivatives from fluctuation-dissipation theorem. The dyadic Green's function and its derivatives in an arbitrary geometry are evaluated numerically by applying a specific set of point sources and then solving a conventional scattering problem. The formulations for 2D and 3D structures have both been derived and validated against analytical results. The computational cost of this algorithm depends only on the complexity of the matrix-equation solver. If multilevel fast-multipole algorithm (MLFMA) is used to solve the integral equation, the computational complexity could be reduced to $O(N \log N)$ per iteration. Thus the overall efficiency of this numerical method could be of $O(N \log N)$ per iteration at best, where N is the number of surface unknowns.

The current integral equation method for Casimir force is valid only for interactions between perfect conductors in zero-temperature. The next stage would be to extend it to dielectric objects. In the future, the method also needs to account for the effect caused by finite conductivities of metals, thermal fluctuation pressure and roughness of the surface.

APPENDIX A

NUMERICAL INTEGRALS OF HANKEL FUNCTION AND ITS DERIVATIVES

A.1 Coordinate Transform

Assume that there are two sets of coordinate system as shown in Figure A.1:

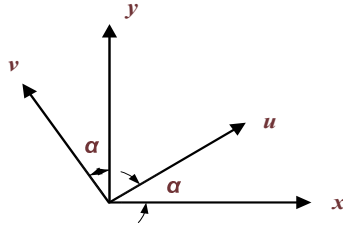


Figure A.1: Coordinate transform.

From the graph, the relationship between the two set of coordinates is

$$\begin{aligned}\hat{u} &= \cos \alpha \hat{x} + \sin \alpha \hat{y} \\ \hat{v} &= -\sin \alpha \hat{x} + \cos \alpha \hat{y} \\ \boldsymbol{\rho} &= x\hat{x} + y\hat{y} = u\hat{u} + v\hat{v} \\ &= (u \cos \alpha - v \sin \alpha)\hat{x} + (u \sin \alpha + v \cos \alpha)\hat{y}\end{aligned}\tag{A.1}$$

Thus, we have the following relationship:

$$\frac{\partial x}{\partial u} = \cos \alpha, \quad \frac{\partial x}{\partial v} = -\sin \alpha, \quad \frac{\partial y}{\partial u} = \sin \alpha, \quad \frac{\partial y}{\partial v} = \cos \alpha, \quad \frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 y}{\partial u \partial v} = 0 \quad (\text{A.2})$$

The first order directional derivative can be obtained from the two basic derivatives,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} (\hat{x} \cdot \hat{u}) + \frac{\partial f}{\partial y} (\hat{y} \cdot \hat{u}) \quad (\text{A.3})$$

The general second order directional derivative is given by

$$\begin{aligned} \frac{\partial^2 f}{\partial u \partial v} &= \frac{\partial}{\partial u} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \right) = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial v} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial u \partial v} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &\quad + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \end{aligned} \quad (\text{A.4})$$

Substituting the derivatives of x and y , we have

$$\frac{\partial^2 f}{\partial u \partial v} = \begin{bmatrix} \hat{x} \cdot \hat{v} & \hat{y} \cdot \hat{v} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} \hat{x} \cdot \hat{u} \\ \hat{y} \cdot \hat{u} \end{bmatrix} \quad (\text{A.5})$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial u} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial u} \right)^2 \\ \frac{\partial^2 f}{\partial v^2} &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial v} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial v} \frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial v} \right)^2 \end{aligned} \quad (\text{A.6})$$

A.2 Integration of the First Order Derivative of Hankel Function

The first general integral is the integration of the gradient of the Bessel function,

$$\int_{C_m} dl \nabla H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) = g_n \hat{n}_m + g_t \hat{t}_m \quad (\text{A.7})$$

When the $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_0$ are not close to each other, this integral could be accurately evaluated numerically. When they are close, we need to extract the singular term in the gradient and integrate it analytically. Now,

$$\begin{aligned} g_n &= -k_\rho \int_{C_m} dl H_1^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \hat{n}_m \cdot \frac{\boldsymbol{\rho} - \boldsymbol{\rho}_0}{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|} \\ &= -k_\rho \int_{y_1}^{y_2} dy \left(\frac{k_\rho \sqrt{y^2 + \delta^2}}{2} - \frac{2i}{k_\rho \pi} \frac{1}{\sqrt{y^2 + \delta^2}} \right) \frac{\delta}{\sqrt{y^2 + \delta^2}} \text{sgn}((\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \hat{n}_m) \\ &= -k_\rho \left[k_\rho \delta / 2 - \frac{2i}{k_\rho \pi} (\tan^{-1} \frac{y_2}{\delta} - \tan^{-1} \frac{y_1}{\delta}) \right] \text{sgn}((\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \hat{n}_m) \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned} g_t &= -k_\rho \int_{C_m} dl H_1^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \hat{t}_m \cdot \frac{\boldsymbol{\rho} - \boldsymbol{\rho}_0}{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|} \\ &= -k_\rho \int_{y_1}^{y_2} dy \left(\frac{k_\rho \sqrt{y^2 + \delta^2}}{2} - \frac{2i}{k_\rho \pi} \frac{1}{\sqrt{y^2 + \delta^2}} \right) \frac{y}{\sqrt{y^2 + \delta^2}} \\ &= -\frac{k_\rho^2}{4} (y_2^2 - y_1^2) + \frac{i}{\pi} \ln \frac{y_2^2 + \delta^2}{y_1^2 + \delta^2} \end{aligned} \quad (\text{A.9})$$

A.3 Integration of the Second Order Derivative of Hankel Function

Now we are going to evaluate the second important integral,

$$\begin{aligned}
I_{nt} &= \int_{C_m} dl \frac{\partial^2}{\partial \hat{n}_m \partial \hat{t}_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \\
&= \frac{\partial}{\partial \hat{n}_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \Big|_{\boldsymbol{\rho}=\boldsymbol{\rho}_m^+} - \frac{\partial}{\partial \hat{n}_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \Big|_{\boldsymbol{\rho}=\boldsymbol{\rho}_m^-} \\
&= -k_\rho H_1^{(1)}(k_\rho |\boldsymbol{\rho}_m^+ - \boldsymbol{\rho}_0|) (\boldsymbol{\rho}_m^+ - \boldsymbol{\rho}_0) \cdot \hat{n}_m + k_\rho H_1^{(1)}(k_\rho |\boldsymbol{\rho}_m^- - \boldsymbol{\rho}_0|) (\boldsymbol{\rho}_m^- - \boldsymbol{\rho}_0) \cdot \hat{n}_m
\end{aligned} \tag{A.10}$$

Next,

$$\begin{aligned}
I_{tt} &= \int_{C_m} dl \frac{\partial^2}{\partial \hat{t}_m^2} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \\
&= \frac{\partial}{\partial \hat{t}_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \Big|_{\boldsymbol{\rho}=\boldsymbol{\rho}_m^+} - \frac{\partial}{\partial \hat{t}_m} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \Big|_{\boldsymbol{\rho}=\boldsymbol{\rho}_m^-} \\
&= -k_\rho H_1^{(1)}(k_\rho |\boldsymbol{\rho}_m^+ - \boldsymbol{\rho}_0|) (\boldsymbol{\rho}_m^+ - \boldsymbol{\rho}_0) \cdot \hat{t}_m + k_\rho H_1^{(1)}(k_\rho |\boldsymbol{\rho}_m^- - \boldsymbol{\rho}_0|) (\boldsymbol{\rho}_m^- - \boldsymbol{\rho}_0) \cdot \hat{t}_m
\end{aligned} \tag{A.11}$$

Finally,

$$\begin{aligned}
I_{nn} &= \int_{C_m} dl \frac{\partial^2}{\partial \hat{n}_m^2} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \\
&= \int_{C_m} dl \left[\frac{\partial^2}{\partial \rho^2} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \left(\frac{\partial \rho}{\partial \hat{n}_m} \right)^2 + \frac{\partial}{\partial \rho} H_0^{(1)}(k_\rho |\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \frac{\partial^2 \rho}{\partial \hat{n}_m^2} \right] \\
&= \int_{C_m} dl \left[\frac{k_\rho}{\rho} \left(\frac{2x^2}{\rho^2} - 1 \right) H_1^{(1)}(k_\rho \rho) - k_\rho^2 \frac{x^2}{\rho^2} H_0^{(1)}(k_\rho \rho) \right] \\
&\quad \text{where } \rho = |\boldsymbol{\rho} - \boldsymbol{\rho}_0|, \quad x = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \hat{n}_m
\end{aligned} \tag{A.12}$$

When $k_\rho \rho$ is small, we can use the small argument approximation and evaluate the integral analytically.

$$\begin{aligned}
I_{nn} &= \int_{C_m} dl \left[\frac{k_\rho}{\rho} \left(\frac{2x^2}{\rho^2} - 1 \right) \left(\frac{k_\rho \rho}{2} - \frac{2i}{\pi k_\rho} \frac{1}{\rho} \right) - k_\rho^2 \frac{x^2}{\rho^2} \left(1 + \frac{2i}{\pi} \ln \frac{k_\rho \rho e^\gamma}{2} \right) \right] \\
&= \int_{C_m} dl \left[-\frac{4ix^2}{\pi} \frac{1}{\rho^4} + k_\rho^2 \frac{x^2}{\rho^2} + \frac{2i}{\pi} \frac{1}{\rho^2} - \frac{k_\rho^2}{2} - k_\rho^2 \frac{x^2}{\rho^2} \left(1 + \frac{2i}{\pi} \ln \frac{k_\rho \rho e^\gamma}{2} \right) \right] \quad (\text{A.13})
\end{aligned}$$

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