# CONGRUENCES IN MODULAR, JACOBI, SIEGEL, AND MOCK MODULAR FORMS WITH APPLICATIONS

BY

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DISSERTATION

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# ABSTRACT

We study congruences in the coefficients of modular and other automorphic forms. Ramanujan famously found congruences for the partition function like $p(5n+4) \equiv 0 \pmod{5}$. For a wide class of modular forms, we classify the primes for which there can be analogous congruences in the coefficients of the Fourier expansion. We have several applications. We describe the Ramanujan congruences in the counting functions for overparitions, overpartition pairs, crank differences, and Andrews' two-coloured generalized Frobenius partitions. We also study Ramanujan congruences in the Fourier coefficients of certain ratios of Eisenstein series. We also determine the exact number of holomorphic modular forms with Ramanujan congruences when the weight is large enough. In a chapter based on joint work with Olav Richter, we study Ramanujan congruences in the coefficients of Jacobi forms and Siegel modular forms of degree two. Finally, the last chapter contains a completely unrelated result about harmonic weak Maass forms.

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# INTRODUCTION

Imagine that you are at a dinner party making chit-chat with those around you. Someone will likely ask what you do. After you respond that you're a mathematician, there is a very short list of standard replies from which your interlocutors will choose. You had best have a mathematical gem prepared for this inevitable follow-up. The ideal gem for this situation will be simple to state, offer no obvious resolution, and be charmingly beautiful. Their momentary speechlessness will give you an opening to delve into mathematics and redeem the subject in the eyes of your companions. Your humble narrator suggests that you employ the following:

Let $p(n)$ denote the number of ways to write $n$ as a non-increasing sum of non-negative integers. Ramanujan famously established the congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \mod 5, \\ p(7n+5) &\equiv 0 \mod 7, \\ p(11n+6) &\equiv 0 \mod 11, \end{aligned} \tag{1.0.1}$$

and noted that there does not appear to be any other prime for which the partition function has equally simple congruences. However it took over eight decades until Ahlgren and Boylan [1] proved that (1.0.1) are indeed the only congruences of the form $p(\ell n+b) \equiv 0 \mod \ell$. The striking elegance of (1.0.1) makes one wonder if this phenomenon occurs elsewhere, and if so, how common it is.

## 1.1 Partitions and their variants

The partitions counted by $p(n)$ have been studied since Euler and continue to reveal their mysteries. A graceful tool in the study of partitions is the Ferrers diagram. For example, consider the partition $12 = 5+4+2+1$ whose Ferrers diagram is the left side of Figure 1.1. The *conjugate* of a partition is obtained by interchanging the rows and the columns of the Ferrers diagram. The two Ferrers diagrams in Figure 1.1 are conjugates of each other. Frobenius wanted a way to write partitions so that it was immediately obvious what the conjugate was. A *Frobenius partition* of $n$ is a sum

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$$

Figure 1.1: Ferrers diagrams of conjugate partitions

where

$$a_1 > a_2 > \cdots > a_r \geq 0,$$
$$b_1 > b_2 > \cdots > b_r \geq 0.$$

An alternative representation for a Frobenius partition is the *Frobenius symbol*

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}.$$

Figure 1.2 indicates a bijective construction of a Frobenius partition from a regular partition. The number of dots along the main diagonal becomes the number of columns $r$. The numbers of dots in each row to the right of the main diagonal become the $a_i$, while the numbers of dots in each column below the main diagonal become the $b_i$. Conjugating the original partition corresponds to inverting the rows of the Frobenius symbol. Thus, the regular partition $5+4+2+1$ has Frobenius symbol

$$\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$$

and the conjugate is

$$\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.$$

Andrews [4] adds an interesting twist to this construction. Let the $a_i$ and $b_i$ come from two copies of the integers where

$$\cdots > 4_2 > 4_1 > 3_2 > 3_1 > \cdots .$$

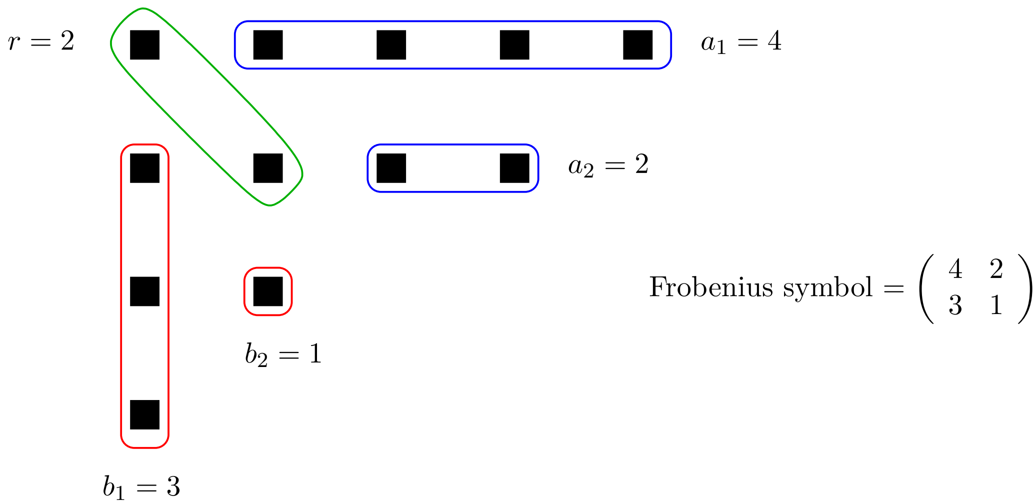


Figure 1.2: Constructing a Frobenius symbol from a partition

For example,

$$\begin{pmatrix} 4_1 & 2_1 \\ 3_1 & 1_1 \end{pmatrix}, \begin{pmatrix} 4_1 & 2_1 \\ 3_1 & 1_2 \end{pmatrix}, \text{ and } \begin{pmatrix} 4_2 & 2_1 \\ 2_2 & 2_1 \end{pmatrix}$$

are all examples of *two-coloured Frobenius partitions* of 12. Following Andrews, let $c\phi_2(n)$ denote the number of two-coloured Frobenius partitions of $n$. The only motivation for this construction which we offer is the following beautiful theorem of Andrews.

**Theorem 1.1** ([4] Corollary 10.1 and Theorem 10.2)**.** *For all $n$, we have*

$$c\phi_2(2n+1) \equiv 0 \mod 2 \tag{1.1.1}$$

$$c\phi_2(5n+3) \equiv 0 \mod 5. \tag{1.1.2}$$

In Chapter 4 we prove that these are the only simple congruences for $c\phi_2(n)$:

**Theorem 1.2.** *If $\ell$ is a prime, then the only congruences $c\phi_2(\ell n+b) \equiv 0 \mod \ell$ are (1.1.1) and (1.1.2).*

This thesis classifies congruences of this type for a wide class of combinatorial counting functions. The proof of Theorem 1.2 uses the theory of modular forms. We formally introduce modular forms in Chapter 2, but for now all that we need is that they have a Fourier series representation $\sum a(n)q^n$. We are only concerned with modular forms for which $a(n) \in \mathbb{Q}$. All of our applications will in fact have $a(n) \in \mathbb{Z}$. Since modular forms have bounded denominators, restricting attention to those with integral coefficients comes at no great price. A modular form $\sum a(n)q^n$ has a *Ramanujan* *congruence at* $b \mod \ell$ when, for all $n \in \mathbb{Z}$, we have

$$a(\ell n + b) \equiv 0 \mod \ell. \tag{1.1.3}$$

The statements that $p(n)$, $c\phi_2(n)$, or other partition-theoretic counting functions have Ramanujan congruences are equivalent to statements that certain associated modular forms have Ramanujan congruences. The specific association will be made clear through several examples in Chapter 4.

Ramanujan congruences at $0 \mod \ell$ in modular forms are very different from Ramanujan congruences at non-zero $b \mod \ell$. This thesis deals with both types. The former type of Ramanujan congruence is equivalent to the so-called $U_\ell$ congruences. The $U_\ell$-operator acts on modular forms via

$$\left(\sum a(n)q^n\right)\Big|U_\ell = \sum a(\ell n)q^n.$$

We say that a modular form satisfies a $U_\ell$-congruence when $\left(\sum a(n)q^n\right)\big|U_\ell \equiv 0 \mod \ell$, i.e. when it has a Ramanujan congruence at $0 \mod \ell$. On the other hand, Ramanujan congruences at $b \not\equiv 0 \mod \ell$ have been less commonly studied. Kiming and Olsson [26] proved an important theorem ruling them out for a particular modular form associated to the partition function. We prove:

**Theorem 1.3.** *Let* $f = \sum a(n)q^n \in M_k\left(\Gamma_1(N)\right)$ *where* $N = 1$ *or* $4$, $0 \leq k \in \mathbb{Z}$, *and all* $a(n) \in \mathbb{Z}$. *Then there are only finitely many primes* $\ell$ *for which* $f$ *has a Ramanujan congruence at* $b \not\equiv 0 \mod \ell$. *Moreover, such an* $\ell$ *satisfies* $\ell \leq 2k - 1$.

This theorem is interesting because many forms have, or are expected to have, infinitely many primes $\ell$ for which there is a Ramanujan congruence at $0 \mod \ell$. For example, Elkies proved that weight 2 newforms of conductor $N$ have infinitely many Ramanujan congruences at $0 \mod \ell$. In addition, if $\Delta \in S_{12}(\Gamma_1(1))$ has only finitely many Ramanujan congruences at $0 \mod \ell$, and if these $\ell$ were known, then Lehmer's conjecture on whether $\tau(n)$ is ever zero would be resolved.

We adapt the theory behind Theorem 1.3 to apply to functions which are not holomorphic modular forms, and to obtain better bounds on $\ell$. Nevertheless, in later chapters most our effort is spent on Ramanujan congruences at $0 \mod \ell$.

## 1.2 Quotients of Eisenstein series

Eisenstein series are basic building blocks of modular forms. Let $\sigma_m(n) := \sum_{d|n} d^m$ and define the Bernoulli numbers $B_k$ by $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$. Let $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$. For even $k \geq 2$, set

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

Table 1.1: Congruences of Berndt and Yee [7]

| $F(q)$ | $n \equiv 2 \mod 3$ | $n \equiv 4 \mod 8$ |
|---|---|---|
| $1/E_2$ | $a(n) \equiv 0 \mod 3^4$ | |
| $1/E_4$ | $a(n) \equiv 0 \mod 3^2$ | |
| $1/E_6$ | $a(n) \equiv 0 \mod 3^3$ | $a(n) \equiv 0 \mod 7^2$ |
| $E_2/E_4$ | $a(n) \equiv 0 \mod 3^3$ | |
| $E_2/E_6$ | $a(n) \equiv 0 \mod 3^2$ | $a(n) \equiv 0 \mod 7^2$ |
| $E_4/E_6$ | $a(n) \equiv 0 \mod 3^3$ | |
| $E_2^2/E_6$ | $a(n) \equiv 0 \mod 3^5$ | |

Note that $E_2 \equiv E_4 \equiv E_6 \equiv 1$ modulo 2 and 3. Berndt and Yee [7] prove congruences for the quotients of Eisenstein series in Table 1.1, where $F(q) := \sum a(n)q^n$. An obviously necessary requirement for the congruences in the $n \equiv 2 \mod 3$ column of Table 1.1 is that there are simple congruences of the form $a(3n+2) \equiv 0 \mod 3$. All but the first form in Table 1.1 are covered by the following theorem.

**Theorem 1.4.** *Let $r, s, t, b, \ell \in \mathbb{Z}$ where $r \geq 0$ and $\ell$ is prime. If $E_2^r E_4^s E_6^t = \sum a(n)q^n$ has a Ramanujan congruence $a(\ell n+b) \equiv 0 \mod \ell$, then either $\ell \leq 2r+8|s|+12|t|+21$ or $r = s = t = 0$.*

This theorem gives an explicit upper bound on primes $\ell$ for which there can be congruences of the form $a(\ell n+b) \equiv 0 \mod \ell^k$ as in the middle column of Table 1.1. See Remark 3.15 for a slight improvement of Theorem 1.4 in some cases.

**Example 1.5.** The form $E_6/E_4^{12}$ can only have simple congruences for $\ell \leq 129$. Of these, the primes $\ell = 2$ and 3 are trivial with $E_4 \equiv E_6 \equiv 1 \mod \ell$. For the remaining primes, the only congruences are

$$a(\ell n+b) \equiv 0 \mod 17, \text{ where } \left(\frac{b}{17}\right) = -1.$$

Mahlburg [35] shows that for each of the forms in Table 1.1 except $1/E_2$, there are infinitely many primes $\ell$ such that for any $i \geq 1$, the set of $n$ with $a(n) \equiv 0 \mod \ell^i$ has arithmetic density 1. On the other hand, our result shows that (for large enough $\ell$) every arithmetic progression modulo $\ell$ has at least one non-vanishing coefficient modulo $\ell$.

## 1.3 Forms with divisor supported at the cusps

We obtain precise results for meromorphic modular forms with divisor (i.e. the zeros and poles) supported at cusps. This additional technical condition gives us much better control on the possible Ramanujan congruences. Given a weakly holomorphic $f \in M_k^!(\Gamma_1(4)) \cap \mathbb{Z}[\![q]\!]$ with $k \in \frac{1}{2}\mathbb{Z}$ which is non-vanishing on the upper half plane, if $k \neq \frac{1}{2}$, then Corollary 4.19 shows there are only finitely many primes $\ell$ for which $f$ has a Ramanujan congruence at some $b \not\equiv 0 \mod \ell$. The situation for Ramanujan congruences at $0 \mod \ell$ is more intricate. We prove the finiteness of these congruences in three of the four cases below:

| | $k \in \mathbb{Z}$ | $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ |
|---|---|---|
| $k \leq 3/2$ | Theorem 4.14 | Theorem 4.16 |
| $k \geq 2$ | Open | Theorem 4.15 |

Theorems 4.14, 4.15, and 4.16 provide a method to find explicit bounds on the possible primes $\ell$ for which there could be Ramanujan congruences at $0 \mod \ell$. One may then simply check the finitely many possibilities to generate a list of all Ramanujan congruences for the power series in question. Seeking Ramanujan congruences in positive, integral weight modular forms includes hard problems such as determining when Ramanujan's $\tau(n)$ function satisfies $\tau(\ell) \equiv 0 \mod \ell$. We leave such problems open.

Theorem 4.14 overlaps with the conclusions of Sinick [42]. Theorem 4.16 is a generalization of Ahlgren and Boylan [1] and has the most involved proof of these three theorems. We provide several examples. Let $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ where $q = e^{2\pi i z}$. Then:

**Theorem 1.6.** *Define* $f := \frac{\eta^6(z)\eta^6(4z)}{\eta^3(2z)} \in S_{9/2}(\Gamma_1(4))$ *and let* $f^{-1} = \sum a(n)q^n$. *The Ramanujan congruences of* $f^{-1}$ *are exactly*

$$\begin{aligned}
a(2n+0) &\equiv 0 \mod 2\\
a(3n+0) &\equiv 0 \mod 3\\
a(3n+1) &\equiv 0 \mod 3\\
a(5n+2) &\equiv 0 \mod 5\\
a(5n+3) &\equiv 0 \mod 5.
\end{aligned}$$

**Theorem 1.7.** *Define* $f := \frac{\eta^{14}(z)\eta^6(4z)}{\eta^7(2z)} \in S_{13/2}(\Gamma_1(4))$ *and let* $f^{-1} = \sum b(n)q^n$. *The Ramanujan congruences of* $f^{-1}$ *are exactly*

$$\begin{aligned}
b(2n+0) &\equiv 0 \mod 2\\
b(7n+1) &\equiv 0 \mod 7\\
b(7n+2) &\equiv 0 \mod 7\\
b(7n+4) &\equiv 0 \mod 7.
\end{aligned}$$

Partition-theoretic functions like $c\phi_2(n)$ require a bit more care since their generating functions are not quite modular forms. In addition to 2-coloured Frobenius partitions, we also classify congruences in overpartitions, overpartition pairs, and crank differences, as described below.

An *overpartition* of $n$ is a sum of non-increasing positive integers in which the first occurrence of an integer may be overlined. Let $\overline{p}(n)$ count the number of such overpartitions and set $\overline{P}(z) =$

$\sum \overline{p}(n)q^n$. Background for overpartitions can be found in Corteel and Lovejoy [16]. Recently, Mahlburg [34] has shown that the set of integers $n$ with $\overline{p}(n) \equiv 0 \mod 64$ has arithmetic density 1, and Kim [25] has extended this result to modulus 128. For larger primes we have a very different situation.

**Theorem 1.8.** *Let $\ell$ be an odd prime and $b \in \mathbb{Z}$. Then there are no Ramanujan congruences $\overline{p}(\ell n + b) \equiv 0 \mod \ell$.*

An *overpartition pair* of $n$ is a decomposition $n = r + s$ and a pair of overpartitions, one for $r$ and one for $s$. Overpartition pairs have an important place in the theory of $q$-series and partitions [28, 11, 30]. Let $\overline{pp}(n)$ denote the number of overpartition pairs of $n$. Bringmann and Lovejoy [11] show that for all integers $n$,

$$\overline{pp}(3n+2) \equiv 0 \mod 3.$$

On the other hand, we show:

**Theorem 1.9.** *Let $\ell \geq 5$ be prime and $b \in \mathbb{Z}$. There are no Ramanujan congruences $\overline{pp}(\ell n+b) \equiv 0 \mod \ell$.*

If $\pi$ is a (regular) partition, define the *crank* by

$$\operatorname{crank}(\pi) := \begin{cases} \pi_1 & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi) & \text{if } \mu(\pi) > 0, \end{cases}$$

where $\pi_1$ denotes the largest part of $\pi$, $\mu(\pi)$ denotes the number of ones in $\pi$ and $\nu(\pi)$ denotes the number of parts of $\pi$ that are strictly larger than $\mu(\pi)$. The existence of non-Ramanujan congruences for the crank counting function is proven by Mahlburg [35]. Let $M_e(n)$ and $M_o(n)$ denote the number of partitions of $n$ with even and odd crank, respectively. Choi, Kang, and Lovejoy [15] studied the crank difference function $(M_e - M_o)(n)$ and found a Ramanujan congruence at $(M_e - M_o)(5n+4) \equiv 0 \mod 5$. They ask if the methods of [26] and [1] may be adapted to prove that there are no other Ramanujan congruences. We give a partial answer to their question.

**Theorem 1.10.** *Let $\ell \geq 5$ be prime, $\delta := \frac{\ell^2-1}{24}$ and $b \not\equiv -\delta \mod \ell$. The crank difference function has the Ramanujan congruence $(M_e - M_o)(\ell n - \delta) \equiv 0 \mod \ell$ if and only if $\ell = 5$. If for all integers $n$, $(M_e - M_o)(\ell n + b) \equiv 0 \mod \ell$, then for all $c$ satisfying $\left(\frac{b+\delta}{\ell}\right) = \left(\frac{c+\delta}{\ell}\right)$, we have $(M_e - M_o)(\ell n + c) \equiv 0 \mod \ell$.*

It is somewhat amusing that the unresolved Ramanujan congruences for crank differences are the "easy" congruences at $b \not\equiv 0 \mod \ell$.

## 1.4 Jacobi and Siegel forms

In joint work with Olav Richter [21], we generalize the notion of Ramanujan congruence to Jacobi forms and degree 2 Siegel forms. A Siegel form has a series representation indexed over matrices. Throughout Chapter 5 we will adopt the following notation. Let $Z := \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ be a variable in the Siegel upper half space of degree 2, $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi iz}$, $q' := e^{2\pi i\tau'}$, and $\mathbb{D} := (2\pi i)^{-2}\left(4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}\right)$ be the generalized theta operator, which acts on Fourier expansions of Siegel modular forms as follows:

$$\mathbb{D}\left(\sum_{\substack{T={}^tT\geq 0\\ T\,even}} a(T)e^{\pi i\,tr(TZ)}\right) = \sum_{\substack{T={}^tT\geq 0\\ T\,even}} \det(T)a(T)e^{\pi i\,tr(TZ)},$$

where $tr$ denotes the trace, and where the sum is over all symmetric, semi-positive definite, integral, and even $2\times 2$ matrices. Additionally, we always let $\ell \geq 5$ be a prime and (for simplicity) we always assume that the weight $k$ is an even integer.

**Definition 1.11.** A Siegel modular form $F = \sum a(T)e^{\pi i\,tr(TZ)}$ with $\ell$-integral rational coefficients has a Ramanujan congruence at $b \mod \ell$ if $a(T) \equiv 0 \mod \ell$ for all $T$ with $\det T \equiv b \mod \ell$.

**Theorem 1.12.** *Let* $F(Z) = \sum_{\substack{n,r,m\in\mathbb{Z}\\ n,m,4nm-r^2\geq 0}} A(n,r,m)q^n\zeta^r q'^m$ *be a Siegel modular form of degree 2 and even weight $k$ with $\ell$-integral rational coefficients and let $b \not\equiv 0 \mod \ell$. Then $F$ has a Ramanujan congruence at $b \mod \ell$ if and only if*

$$\mathbb{D}^{\frac{\ell+1}{2}}(F) \equiv -\left(\frac{b}{\ell}\right)\mathbb{D}(F) \mod \ell, \tag{1.4.1}$$

*where $\left(\frac{\cdot}{\ell}\right)$ is the Legendre symbol. Moreover, if $F$ has a Ramanujan congruence at $b \mod \ell$ and if there are $n, r, m$ such that $(4nm - r^2)a(n,r,m) \neq 0$, then either $\ell \leq k$ or $\ell \mid \gcd(n,m)(4nm - r^2)a(n,r,m)$.*

Note that such congruences at $0 \mod \ell$ have already been studied in [14] and the main result of Chapter 5 complements [14] by giving the case $b \not\equiv 0 \mod \ell$. Theorem 1.12 combines with a Sturm-bound type result of Poor and Yuen [37] to give an effective (i.e. finite) test for Ramanujan congruences in degree 2 Siegel forms. In Chapter 5 we list all degree 2 Siegel forms with Ramanujan congruences at $b \not\equiv 0 \mod \ell$, up to weight 20.

Theorem 1.12 follows from a study of Ramanujan congruences in Jacobi forms. See Chapter 5 for the notation.

**Theorem 1.13.** *Let $\phi \in \widetilde{J}_{k,m}$ where $k \geq 4$, $L_m(\phi) \not\equiv 0 \mod \ell$ and let $b \not\equiv 0 \mod \ell$. If $\ell > k$ and $\ell \nmid m$, then $\phi$ does not have a Ramanujan congruence at $b \mod \ell$.*

## 1.5 The rarity of Ramanujan congruences

A common theme in all of our main theorems so far is that Ramanujan congruences seem to be rare. Loosely speaking, if a modular form $f$ satisfies the hypotheses of one of our earlier theorems, then it has only finitely many Ramanujan congruences and there is a method to compute them. This motivates the following:

**Question 1.14.** Can one determine the precise number of modular forms which have Ramanujan congruences?

We answer this question in the affirmative. To state our main theorem, we need the following notation. Let $\ell \geq 5$ be prime, $k \in \mathbb{Z}$, and let $N = 1$ or $4$. Let $M_k$ be the $\mathbb{F}_\ell$-vector space obtained by coefficient-wise reduction modulo $\ell$ of all holomorphic modular forms on $\Gamma_1(N)$ with rational, $\ell$-integral coefficients. Recall that $\dim M_k$ is easily computed for any integer $k \geq 0$. Set

$$d_N := \begin{cases} \lfloor \frac{\ell}{12} \rfloor & \text{if } N = 1, \\ \lfloor \frac{\ell}{2} \rfloor & \text{if } N = 4. \end{cases}$$

For any integer $k \geq 2\ell$, write

$$k = C(\ell - 1) + D,$$

where

$$3 \leq D \leq \ell + 1,$$

and set

$$J := 1 + \left\lfloor \frac{C - D + 1}{\ell} \right\rfloor. \tag{1.5.1}$$

Let $\mathfrak{X} = \mathfrak{X}(N, \ell, k)$ be as in Definition 6.22. In Sections 6.5 and 6.6 we evaluate $\mathfrak{X}$ exactly. We will also show that:

- If $N = 4$ then $\mathfrak{X} = 0$.
- If $N = 1$ and $\ell \equiv 1 \mod 12$ then $\mathfrak{X} = 0$.
- If $N = 1$ and $\ell \equiv 5 \mod 12$ then $\frac{J}{3} - 1 \leq \mathfrak{X} \leq \frac{J}{3} + 1$.
- If $N = 1$ and $\ell \equiv 7 \mod 12$ then $\frac{J}{2} - 1 \leq \mathfrak{X} \leq \frac{J}{2} + 1$.
- If $N = 1$ and $\ell \equiv 11 \mod 12$ then $5\left(\frac{J}{6} - 1\right) \leq \mathfrak{X} \leq 5\left(\frac{J}{6} + 1\right)$.

Finally, let

$$\mathcal{P}(\ell, k, N) := \frac{|\{f \in M_k : f \text{ has a Ramanujan congruence at } 0 \mod \ell\}|}{|M_k|}$$

be the probability (with the uniform distribution) that $f \in M_k$ has a Ramanujan congruence at 0 mod $\ell$. The main result of this paper is that we can compute this probability exactly:

**Theorem 1.15.** *Let $\ell \geq 5$ be prime, $N = 1$ or 4, and $k \geq 2\ell$ be an integer. Let $M_k$, $d_N$, $C$, $D$, $J$, $\mathfrak{X}$, and $\mathcal{P}_\ell^k$ be as above. Then $\mathcal{P}(\ell, k, N) = \ell^{-d_N J - \dim M_D - \mathfrak{X}}$.*

*Proof.* This is a combination of Theorems 6.20 and 6.23. ∎

**Example 1.16.** Theorem 1.15 provides a context in which to understand results like Ahlgren and Boylan's [1] proof that (1.0.1) are the only Ramanujan congruences for $p(n)$. Let $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ denote the normalized, weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Kiming and Olsson [26] showed that the partition generating function $\sum p(n)q^n$ has a Ramanujan congruence at $b \mod \ell$ if and only if the holomorphic modular form $\Delta^{\frac{\ell^2-1}{24}}$ of weight $\frac{\ell^2-1}{2}$ has a Ramanujan congruence at $b + \left(\frac{\ell^2-1}{24}\right)$. Furthermore, Kiming and Olsson proved $\Delta^{\frac{\ell^2-1}{24}}$ can only have Ramanujan congruences at $0 \mod \ell$. Ahlgren and Boylan [1] later ruled out this last case. Theorem 1.15 above provides an interesting heuristic to judge how surprising the Ahlgren and Boylan result is. The probability that $g \in M_{\frac{\ell^2-1}{2}}$ has a Ramanujan congruence at $0 \mod \ell$ is given by Theorem 1.15 with $C = \frac{\ell-1}{2}$, $D = \ell - 1$, $J = 0$. Definition 6.22 will show that $\mathfrak{X} \geq 0$. Hence

$$\mathcal{P}\left(\ell, \frac{\ell^2-1}{2}, 1\right) \leq \ell^{-\left\lfloor \frac{\ell+11}{12} \right\rfloor}$$

for all primes $\ell \geq 5$. For example, $\mathcal{P}\left(\ell, \frac{\ell^2-1}{2}, 1\right) = \frac{1}{169}$. A very rough heuristic for an upper bound on the probability of $p(n)$ having a Ramanujan congruence at $0 \mod \ell$ for any prime $\ell \geq 13$ is

$$\sum_{\text{primes } \ell \geq 13} \mathcal{P}\left(\ell, \frac{\ell^2-1}{2}, 1\right) \approx 0.014\ldots.$$

We surmise that it would have been somewhat surprising if the Ahlgren and Boylan result had been false.

## 1.6 Applications of mock modular forms

A harmonic weak Maass form can be written as a sum of a holomorphic part and a nonholomorphic part, essentially an integral of a modular form which is called the shadow. Bringmann and Lovejoy [10], Bringmann, Ono, and Rhoades [13], and Bringmann [9] have found Maass forms whose holomorphic parts are related to the overpartition rank, the $M_2$-rank for partitions without repeated odd parts, and the full rank of 2-marked Durfee symbols. Zagier [47, Section 5] formulates a general principle (which is used in [10], [13] and [9]) to produce weakly holomorphic modular forms from Maass forms. Extracting an arithmetic progression of exponents which does not intersect the support of the shadow yields a modular form. The current work is focused on arithmetic progressions for which Zagier's principle does not apply. We study the Maass forms of [10], [13] and [9] and compute their nonholomorphic parts explicitly. Linear relations among these nonholomorphic parts imply that the corresponding generating functions are in fact weakly holomorphic modular forms. (Similar work was carried out in [3] for the rank of usual partitions.) This provides a framework for a general phenomenon, special cases of which are illustrated in recent works by Lovejoy and Osburn [31, 32] who showed that certain rank difference generating functions modulo $t = 3$ and 5 are weakly holomorphic modular forms. We determine the modularity properties of rank difference functions for all primes $t \geq 5$ (and in principle for most composites too) and for more complicated combinations of the rank functions.

Recall that an *overpartition* of $n$ is a partition in which the first appearance of a part may be overlined. The rank of an overpartition is the largest part minus the number of parts. Let $\overline{p}(n)$ be the number of overpartitions of $n$ and $\overline{N}(r,t,n)$ be the number of overpartitions of $n$ whose rank is congruent to $r \mod t$. Bringmann and Lovejoy [10] show that

$$\sum_{n=0}^{\infty} \left( \overline{N}(r,t,n) - \frac{1}{t}\overline{p}(n) \right) q^n \tag{1.6.1}$$

is the holomorphic part of a weak Maass form. Define the rank difference function

$$R_{rs}(d) = \sum_{n \equiv d(t)} \left( \overline{N}(r,t,n) - \overline{N}(s,t,n) \right) q^n. \tag{1.6.2}$$

Lovejoy and Osburn [31] compute closed forms of such functions for $t = 3$ and 5. From their computations, it is clear that some of these $R_{rs}(d)$ are weakly holomorphic modular forms. Using the fact that the nonholomorphic part corresponding to (1.6.2) is supported on terms whose exponents are negative squares, Bringmann and Lovejoy [10] show that $R_{rs}(d)$ is a weakly holomorphic modular form when $\left(\frac{-d}{t}\right) = -1$. We determine exactly when it is a modular form in the other half of the cases. (Recall that by conjugation [29], $\overline{N}(r,t,n) = \overline{N}(t-r,t,n)$.)

**Theorem 1.17.** *Let $t \geq 5$ be prime and $0 \leq s < r \leq \frac{t-1}{2}$. If $\left(\frac{-d}{t}\right) = -1$, then $R_{rs}(d)$ is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(16t^3)$. Otherwise, let $d'$ be such that $d'^2 \equiv -d \mod t$ and $0 \leq d' \leq \frac{t-1}{2}$. Then $R_{rs}(d)$ is a weakly holomorphic modular form exactly when one of the following is true:*

1. *$s > 2d'$ or $s > t - 2d'$,*

2. *$2 | r - s$, $r < 2d'$, and $r < t - 2d'$.*

In the cases $t = 3, 5$, Lovejoy and Osburn's [31] closed forms for those $R_{rs}(d)$ which are not modular contain (non-modular) Lambert series. For fixed $d$, these Lambert series are integer multiples of each other. We show that this is a general phenomenon. For any $t \geq 3$, in those cases when $R_{rs}(d)$ is not itself a weakly holomorphic modular form, it differs from one by a multiple of a fixed mock modular form which is independent of $r$ and $s$. By *mock modular form* we mean the holomorphic part of a weak Maass form.

**Theorem 1.18.** *Suppose that $t \geq 5$ is prime and that $0 \leq d < t$. There is a fixed mock modular form $F_{d,t}$ such that for every pair $(r,s)$ there is an integer $-4 \leq n \leq 4$ such that $R_{rs}(d) - nF_{d,t}$ is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(16t^3)$.*

As an example for $t = 17$, although neither $R_{26}(8)$ nor $R_{67}(8)$ are modular, their difference is.

Analogous statements for non-prime $t$ are also possible. Our key Theorems 7.3, 7.5, and 7.8 hold for composite $t$. In addition, the modularity of arbitrary linear combinations of (1.6.1), along with (1.6.3) and (1.6.5) to follow, may be determined precisely using these key theorems.

The $M_2$*-rank* of a partition $\lambda$ without repeated odd parts is $\lceil l(\lambda)/2 \rceil - n(\lambda)$, where $l(\lambda)$ is the largest part and $n(\lambda)$ is the number of parts. Let $N_2(n)$ denote the number of such partitions and let $N_2(r,t,n)$ be the number of such partitions with rank congruent to $r \mod t$. Details of the $M_2$-rank can be found in [32]. It follows from a result of Bringmann, Ono and Rhoades [9, Theorem 4.2] that the $M_2$-rank generating function,

$$\sum_{n=0}^{\infty} \left( N_2(r,t,n) - \frac{1}{t} N_2(n) \right) q^{8n-1} \tag{1.6.3}$$

is the holomorphic part of a weak Maass form. We show that the nonholomorphic part differs from that corresponding to the usual partition rank generating function by a twist. Hence, we find relations analogous to [3]. Lovejoy and Osburn [32] have also found closed forms for the rank differences

$$T_{rs}(d) = \sum_{n \equiv d \mod t} \left( N_2(r,t,n) - N_2(s,t,n) \right) q^{8n-1} \tag{1.6.4}$$

for $t = 3$ and 5. The modularity of these functions for arbitrary $t$ is described by the following theorem, where $f_t := 2t/\gcd(t,4)$.

**Theorem 1.19.** *For any $t \geq 2$ and any $r$ and $s$, $T_{rs}(d)$ is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$ exactly when $8d - 1 \not\equiv -(2r \pm 1)^2, -(2s \pm 1)^2 \mod t$.*

There is also an analogue of Theorem 1.18.

**Theorem 1.20.** *Suppose that $t \geq 2$ is prime and that $0 \leq d < t$. There is a fixed mock modular form $F_{d,t}$ such that for every pair $(r,s)$ there is an integer $-3 \leq n \leq 3$ such that $T_{rs}(d) - nF_{d,t}$ is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$.*

For example, if $t = 17$ then $T_{01}(0)$ is not modular, but $T_{01}(0) + 3T_{15}(0)$ is. We may take $F_{0,17} = T_{15}(0)$.

To define the *2-marked Durfee symbol*, we first recall that the Durfee square of a partition is the largest square of nodes in the Ferrers graph. The Durfee symbol consists of two rows of numbers, plus a subscript. The first row describes the columns to the right of the Durfee square, while the second row describes the rows below the Durfee square. The subscript indicates the side length of the Durfee square. For example,

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & \end{pmatrix}_4$$

is a partition of $4^2 + 3 + 1 + 1 + 2 + 1 = 24$. In a 2-marked Durfee symbol each entry is labelled with a subscript of either 1 or 2 according to the rules:

1. The sequence of parts and the sequence of subscripts in each row are non-increasing.

2. The subscript 1 occurs in the first row.

3. If $M$ is the largest part in the first row with subscript 1, then all parts in the second row with subscript 1 lie in $[1,M]$, and with subscript 2 lie in $[M,S]$, where $S$ is the side length of the Durfee square.

For a 2-marked Durfee symbol $\delta$, define the full rank $FR(\delta)$ by

$$FR(\delta) := \rho_1(\delta) + 2\rho_2(\delta)$$

where

$$\rho_i(\delta) := \begin{cases} \tau_i(\delta) - \beta_i(\delta) - 1 & \text{for } i = 1, \\ \tau_i(\delta) - \beta_i(\delta) & \text{for } i = 2, \end{cases}$$

with $\tau_i(\delta)$ and $\beta_i(\delta)$ denoting the number of entries in the top and bottom rows, respectively, of $\delta$ with subscript $i$. Let $NF_2(m,n)$ denote the number of 2-marked Durfee symbols for $n$ with full rank $m$. Let $NF_2(r,t,n)$ denote the number of 2-marked Durfee symbols for $n$ with full rank congruent to $r \mod t$. Finally, let $\mathcal{D}_2(n)$ denote the number of 2-marked Durfee symbols related to $n$. Bringmann [9, Theorem 1.1] showed that there is a weak Maass form whose holomorphic part contains the generating function for 2-marked Durfee symbols. Using work of Bringmann and Ono on the partition function [12], in Section 5 we explicitly compute the nonholomorphic part of a Maass form whose holomorphic part is

$$\sum_{n=0}^{\infty} \left( NF_2(r,t,n) - \frac{1}{t}\mathcal{D}_2(n) \right) q^{24n-1}. \tag{1.6.5}$$

This is the most complicated example of the three we consider. The contrast between the examples in each of the last three sections of this thesis illustrates the varying complexity of some of these counting functions.

# CHAPTER 2

# PRELIMINARIES

Throughout this thesis, $N = 1$ or $4$. This will always indicate the level of a congruence subgroup. Furthermore, $\ell \in \mathbb{Z}$ will always denote a prime. Unless explicitly noted otherwise, we always take $\ell \geq 5$. This chapter contains definitions and background required in the rest of this thesis. It is adapted from [19, 20].

## 2.1 Modular forms over $\mathbb{C}$ and $\mathbb{Z}_{(\ell)}$

As usual,

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{array}{rl} a \equiv d \equiv 1 & \mod N, \\ c \equiv 0 & \mod N \end{array} \right\}.$$

Elements of $\mathrm{SL}_2(\mathbb{Z})$ act on $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ via fractional linear transformations:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \overline{\mathbb{H}} &\to \overline{\mathbb{H}} \\ \tau &\mapsto \frac{a\tau + b}{c\tau + d} \end{aligned}$$

and on meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ via

$$f(\tau)\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

**Definition 2.1.** A *meromorphic modular form of integral weight* $k \in \mathbb{Z}$ *on* $\Gamma_1(N)$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

and such that $f$ is meromorphic at all of the cusps of $\Gamma_1(N)$. A meromorphic modular form is *weakly holomorphic* if it is holomorphic at all $\tau \in \mathbb{H}$. A meromorphic modular form is a *holomorphic* *modular form*, or simply a *modular form*, if it is holomorphic at all $\tau \in \mathbb{H}$ and at all cusps. Let $M_k(\Gamma_1(N), \mathbb{C})$ denote the space of all weight $k$ (holomorphic) modular forms on $\Gamma_1(N)$. Let $M_k^!(\Gamma_1(N), \mathbb{C})$ denote the space of all weight $k$ weakly holomorphic modular forms on $\Gamma_1(N)$.

To recall the definition of half-integral weight modular forms, we need the following notation. Define $\left(\frac{c}{d}\right)$ as follows. If $d$ is an odd prime, then let $\left(\frac{c}{d}\right)$ be the usual Legendre symbol. For positive odd $d$, extend the definition of $\left(\frac{c}{d}\right)$ multiplicatively. For negative odd $d$, let

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c < 0. \end{cases}$$

Also, let $\left(\frac{0}{\pm 1}\right) = 1$. For odd $d$, define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ i & \text{if } d \equiv 3 \mod 4. \end{cases}$$

**Definition 2.2.** Suppose that $0 \leq \lambda \in \mathbb{Z}$. A *meromorphic modular form of half-integral weight* $\lambda + \frac{1}{2}$ on $\Gamma_1(4)$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda} (c\tau + d)^{\lambda + \frac{1}{2}} f(\tau)$$

and such that $f$ is meromorphic at the cusps 0, $\frac{1}{2}$ and 1. Let $M_{\lambda+\frac{1}{2}}(\Gamma_1(N), \mathbb{C})$ denote the space of all weight $\lambda + \frac{1}{2}$ (holomorphic) modular forms on $\Gamma_1(N)$.

Any $f(\tau) \in M_k(\Gamma_1(N), \mathbb{C})$ has a Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a(n) q^n$ where $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$, and $a(n) \in \mathbb{C}$. We identify a modular form with its Fourier expansion at infinity. For any prime $\ell$, let

$$\mathbb{Z}_{(\ell)} := \left\{ \frac{r}{s} \in \mathbb{Q} \middle| \ell \nmid s \right\}$$

denote the localization of $\mathbb{Z}$ at the prime ideal $\ell\mathbb{Z}$. We write

$$M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right) := M_k(\Gamma_1(N), \mathbb{C}) \cap \mathbb{Z}_{(\ell)}[\![q]\!]$$

for the $\mathbb{Z}_{(\ell)}$-module of level $N$, holomorphic modular forms with rational, $\ell$-integral coefficients. More generally, if $R$ is any subring of $\mathbb{C}$ then define

$$M_k(\Gamma_1(N), R) := M_k(\Gamma_1(N), \mathbb{C}) \cap R[\![q]\!].$$

**Definition 2.3.** Let $\ell$ be prime. We say that $a(n) : \mathbb{Z} \to \mathbb{Z}_{(\ell)}$ has a *Ramanujan congruence at b* mod $\ell$ if for all $n \in \mathbb{Z}$ we have

$$a(\ell n + b) \equiv 0 \mod \ell.$$

We say that a Laurent series $\sum a(n)q^n \in \mathbb{Z}_{(\ell)}[\![q]\!]$ has a *Ramanujan congruence at* $b \mod \ell$ if $a(n)$ has a Ramanujan congruence at $b \mod \ell$.

## 2.2 Basic examples of modular forms

We recall some well-known modular forms which we will need in the sequel. Let $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and define the Bernoulli numbers $B_k$ via $\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$. For $k \geq 4$ even, recall the Eisenstein series

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{k=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k\left(\Gamma_1(1), \mathbb{Q}\right).$$

Eisenstein series generate the space of level one modular forms, i.e.

$$M_k\left(\Gamma_1(1), \mathbb{C}\right) = \langle E_4^i E_6^j \rangle_{4i+6j=k}$$

The weight 2 Eisenstein series $E_2$ plays a special role in the theory. It is called quasi-modular and it satisfies the slightly different transformation rule

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau + d).$$

Let $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ and recall that

$$\begin{aligned}\Delta(\tau) &:= \frac{E_4^3(\tau) - E_6^2(\tau)}{1728} \\ &= \eta^{24}(\tau) \\ &= \sum_{n=1}^{\infty} \tau(n) q^n \in M_{12}\left(\Gamma_1(1), \mathbb{Z}\right).\end{aligned}$$

Level one modular forms of even integral weight $k \geq 0$ have a particularly nice basis. Write $k = 12r + s$ where $s = 0, 4, 6, 8, 10,$ or $14$. Then

$$M_k\left(\Gamma_1(1), \mathbb{C}\right) = \langle E_s E_6^{2r-2i} \Delta^i \rangle_{i=0}^{r}. \tag{2.2.1}$$

The salient features of the basis vectors

$$E_s E_6^{2r-2i} \Delta^i = q^i + \cdots \in \mathbb{Z}[\![q]\!]$$

are that they have distinct orders at $\infty$ and that the coefficients are all integral.

Three important modular forms of level four are

$$\begin{aligned}
E(\tau) &:= \frac{\eta^8(\tau)}{\eta^4(2\tau)} \in M_2\left(\Gamma_1(4), \mathbb{Z}\right), \\
F(\tau) &:= \frac{\eta^8(4\tau)}{\eta^4(2\tau)} = \sum_{n\geq 0} \sigma_1(2n+1) q^{2n+1} \in M_2\left(\Gamma_1(4), \mathbb{Z}\right), \\
\theta_0^2(\tau) &:= \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)} = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 \in M_1\left(\Gamma_1(4), \mathbb{Z}\right).
\end{aligned}$$

Let $\psi(\tau) = \sum_{j=0}^{\infty} q^{(j+1/2)^2}$. The expansions of $F$ and $\theta_0^2$ at the cusps $\frac{1}{2}$ and 0 are

$$\begin{aligned}
F(\tau)|_2 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= \theta_0^4(\tau) \in \mathbb{Z}_{(\ell)}[\![q]\!], \\
\theta_0^2(\tau)|_1 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= \psi^2(\tau) \in \mathbb{Z}_{(\ell)}[\![q^{1/2}]\!],
\end{aligned}$$

and

$$\begin{aligned}
F(\tau)|_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= -\frac{1}{64}\frac{\eta^8(\tau/4)}{\eta^4(\tau/2)} \in \mathbb{Z}_{(\ell)}[\![q^{1/4}]\!], \\
\theta_0^2(\tau)|_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= -\frac{i}{2}\theta_0^2(\tau/4) \in i\mathbb{Z}_{(\ell)}[\![q^{1/4}]\!],
\end{aligned}$$

*Remark* 2.4. Let $f \in M_k\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$ be non-zero where $k \in \mathbb{Z}$. Then

$$f \in M_k(\Gamma_1(4), \mathbb{C}) = M_k(\Gamma_0(4), \chi_{-1}^k, \mathbb{C})$$

and the valence formula for $\Gamma_0(4)$ shows that the total number of zeros of $f$ is

$$\frac{k}{12}[\Gamma_0(1) : \Gamma_0(4)] = \frac{k}{2}.$$

In particular $\mathrm{ord}_0 f + \mathrm{ord}_{1/2} f + \mathrm{ord}_\infty f \leq k/2$ with equality exactly when $f$ is non-vanishing on the upper half plane.

Note that $\mathrm{ord}_0(E) = 1$, $\mathrm{ord}_\infty(F) = 1$, $\mathrm{ord}_{1/2}(\theta_0^2) = 1/2$, and that these are the only zeros of these forms.

Since $\dim M_k\left(\Gamma_1(4), \mathbb{C}\right) = 1 + \lfloor k/2 \rfloor$, one sees that

$$\begin{aligned}
M_{2k}\left(\Gamma_1(4), \mathbb{C}\right) &= \langle E^{k-i} F^i \rangle_{i=0,1,\ldots,k}, \qquad (2.2.2)\\
M_{2k+1}\left(\Gamma_1(4), \mathbb{C}\right) &= \theta_0^2 \langle E^{k-i} F^i \rangle_{i=0,1,\ldots,k},
\end{aligned}$$

where for each $0 \le i \le k$ we have

$$\begin{aligned} E^{k-i}F^i &= q^i + \cdots \in \mathbb{Z}[\![q]\!], \\ \theta_0^2 E^{k-i}F^i &= q^i + \cdots \in \mathbb{Z}[\![q]\!]. \end{aligned}$$

In particular, these have the same salient properties (distinct orders at infinity and integral coefficients) as the basis (2.2.1). In Chapter 4 we shall construct more nuanced bases with specified orders of vanishing at the cusps of $\Gamma_1(4)$.

## 2.3 Modular forms over $\mathbb{F}_\ell$

Let $M_k(\Gamma_1(N), \mathbb{F}_\ell)$ be the $\mathbb{F}_\ell$-vector space obtained via coefficient-wise reduction modulo $\ell$ of every form in $M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. That is,

$$M_k(\Gamma_1(N), \mathbb{F}_\ell) := \left\{ \sum a(n)q^n \in \mathbb{F}_\ell[\![q]\!] \;\middle|\; \exists f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right) \text{ with } \sum a(n)q^n \equiv f \pmod{\ell} \right\}.$$

If $f \in \mathbb{Z}_{(\ell)}[\![q]\!]$, then denote its reduction modulo $\ell$ by

$$(f \mod \ell) \in \mathbb{F}_\ell[\![q]\!]$$

or

$$\overline{f} \in \mathbb{F}_\ell[\![q]\!].$$

Our point of view is that $M_k(\Gamma_1(N), \mathbb{F}_\ell)$ is a distinguished subset of $\mathbb{F}_\ell[\![q]\!]$. In other words, elements of $M_k(\Gamma_1(N), \mathbb{F}_\ell)$ do not "remember" which form they came from. For any $\overline{f} \in M_k(\Gamma_1(N), \mathbb{F}_\ell)$, there is an equivalence class of forms in $M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ which reduce to $\overline{f}$, however $\overline{f}$ is not itself that equivalence class.

**Lemma 2.5.** *For any $0 \le k \in \mathbb{Z}$ and any prime $\ell \ge 5$, we have*

$$\dim_{\mathbb{C}} M_k(\Gamma_1(N), \mathbb{C}) = \dim_{\mathbb{F}_\ell} M_k(\Gamma_1(N), \mathbb{F}_\ell).$$

*Proof.* Depending on the level $N = 1$ or $4$, the basis (2.2.1) or (2.2.2) reduces to a linearly independent set over $\mathbb{F}_\ell$. Hence $\dim_{\mathbb{C}} M_k(\Gamma_1(N), \mathbb{C}) \le \dim_{\mathbb{F}_\ell} M_k(\Gamma_1(N), \mathbb{F}_\ell)$. The reverse inequality is obvious from the definition of $M_k(\Gamma_1(N), \mathbb{F}_\ell)$. $\square$

For details on the statements contained in this paragraph, see Swinnerton-Dyer [45]. The

Kummer congruences imply that $E_{\ell-1}, E_{\ell+1} \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ and furthermore that

$$\overline{E_{\ell-1}} = 1,$$
$$\overline{E_{\ell+1}} = \overline{E_2}.$$

There are polynomials $A(Q,R), B(Q,R) \in \mathbb{Z}_{(\ell)}[Q,R]$ such that

$$A(E_4, E_6) = E_{\ell-1},$$
$$B(E_4, E_6) = E_{\ell+1}.$$

Reduce the coefficients of these polynomials modulo $\ell$ to get $\overline{A}, \overline{B} \in \mathbb{F}_\ell[Q,R]$. Then $\overline{A}$ has no repeated factor and is prime to $\overline{B}$. Furthermore, there is a natural isomorphism of $\mathbb{F}_\ell$-algebras

$$\frac{\mathbb{F}_\ell[Q,R]}{\overline{A}-1} \simeq \bigoplus_{k=0}^{\infty} M_k\left(\Gamma_1(1), \mathbb{F}_\ell\right) \tag{2.3.1}$$

via $Q \to E_4$ and $R \to E_6$.

In a similar fashion, Tupan [46] proves that there is a polynomial $C(X,Y) \in \mathbb{Z}_{(\ell)}[X,Y]$ such that $C(\theta_0^4, F) = E_{\ell-1}$, and further provides an explicit structural isomorphism showing

$$\frac{\mathbb{F}_\ell[X,Y]}{\overline{C}(X^4,Y)-1} \simeq \bigoplus_{0 \le k \in \frac{1}{2}\mathbb{Z}} M_k\left(\Gamma_1(4), \mathbb{F}_\ell\right) \tag{2.3.2}$$

via $X \to \theta_0$ and $Y \to F$. Combining these two situations, we see that in both level $N = 1$ or 4, if $\overline{f} \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$ then

$$\overline{f} = \overline{E_{\ell-1} f} \in M_{k+\ell-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right).$$

**Lemma 2.6.** *Suppose $\overline{f} \in M_{k_1}\left(\Gamma_1(N), \mathbb{F}_\ell\right)$ and $\overline{g} \in M_{k_2}\left(\Gamma_1(N), \mathbb{F}_\ell\right)$. If $\overline{f} = \overline{g} \neq 0$ then $k_1 \equiv k_2$* mod $\ell - 1$.

Thus multiplication by $E_{\ell-1}$ give a chain of vector space inclusions

$$M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le M_{k+\ell-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le M_{k+2(\ell-1)}\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le M_{k+3(\ell-1)}\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le \cdots.$$

When we would like to emphasize that $M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le M_{k+\ell-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right)$, we may write $\overline{E_{\ell-1}} M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right) \le M_{k+\ell-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right)$.

For $\overline{f} = \sum_{n=0}^{\infty} a(n)q^n \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$, we define the filtration

$$\omega\left(\overline{f}\right) := \inf\left\{k' : \overline{f} \in M_{k'}\left(\Gamma_1(N), \mathbb{F}_\ell\right)\right\}.$$

If $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ reduces to $\overline{f}$, then $\omega(f) := \omega\left(\overline{f}\right)$. For $\overline{f} = \sum_{n=0}^{\infty} a(n)q^n \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$, we also define the order at the infinite cusp

$$\operatorname{ord}_\infty\left(\overline{f}\right) := \inf\left\{n : a(n) \not\equiv 0 \mod \ell\right\}.$$

When $N = 4$, we define the order of $\overline{f}$ at the cusps $1/2$ and $0$ as follows. Choose any $f \in M_k\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$ such that $f$ reduces to $\overline{f}$. Write $f|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \sum_{n=0}^{\infty} b(n/2)q^{n/2}$ and $f|_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i^k \sum_{n=0}^{\infty} c(n)q^{n/4}$ and define

$$\begin{aligned}
\operatorname{ord}_{1/2}\left(\overline{f}\right) &:= \inf\left\{n/2 : b(n/2) \not\equiv 0 \mod \ell\right\} \\
\operatorname{ord}_{0}\left(\overline{f}\right) &:= \inf\left\{n : c(n) \not\equiv 0 \mod \ell\right\}.
\end{aligned}$$

It follows that for any of the cusps $s$ we have

$$\operatorname{ord}_s\left(\overline{f}\right) \geq \operatorname{ord}_s\left(f\right). \tag{2.3.3}$$

*Remark* 2.7. For any cusp $s$, $\operatorname{ord}_s\left(\overline{f}\right)$ is well-defined in the sense that if a power series $\sum a(n)q^n \in \mathbb{F}_\ell[\![q]\!]$ is congruent to both $f(\tau) \in M_k\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$ and $g(\tau) \in M_{k+m(\ell-1)}\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$, then by Lemma 2.6,

$$f(\tau)E_{\ell-1}^m(\tau) = g(\tau) + \ell h(\tau)$$

for some $h(\tau) \in M_{k+m(\ell-1)}\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$. Now

$$\begin{aligned}
f(\tau)E_{\ell-1}^m(\tau)|_{k+m(\ell-1)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= f(\tau)|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} E_{\ell-1}^m(\tau) \\
&\equiv f(\tau)|_k \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \mod \ell
\end{aligned}$$

and

$$\begin{aligned}
(g(\tau) + \ell h(\tau))|_{k+m(\ell-1)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= g(\tau)|_{k+m(\ell-1)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \ell h(\tau)|_{k+m(\ell-1)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\
&\equiv g(\tau)|_{k+m(\ell-1)} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \mod \ell.
\end{aligned}$$

The situation for the cusp 0 is similar.

Define $U_\ell$ on power series by

$$\left(\sum a(n)q^n\right)|U_\ell = \sum a(\ell n)q^n.$$

**Lemma 2.8.** *If $f \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$, then $f|U_\ell \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$*

*Proof.* Working modulo $\ell$, we have $\overline{f|U_\ell} = \overline{f|T_\ell}$ where $T_\ell$ is a Hecke operator which is well known to map $M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right) \to M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. $\square$

## 2.4 Ramanujan's differential operator

Define the operator

$$\Theta := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

Although it does not map modular forms to modular forms, if $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ then $12\Theta f - kE_2 f \in M_{k+2}\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. Along these lines, define

$$R(f) := \left(\Theta f - \frac{k}{12}E_2 f\right)E_{\ell-1} + \frac{k}{12}E_{\ell+1}f \in M_{k+\ell+1}\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right), \tag{2.4.1}$$

so that $\overline{R(f)} = \overline{\Theta f}$. The definition of $R(f)$ implicitly depends on the weight of $f$. We recursively define

$$\begin{aligned} R_1^f &:= R(f), \\ R_i^f &:= R(R_{i-1}^f) \in M_{k+i(\ell+1)}\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right), \end{aligned}$$

so that

$$\overline{R_i^f} = \overline{\Theta^i f}. \tag{2.4.2}$$

A short computation (for example [42] Lemma 4.2) shows that

$$\begin{aligned} R(f)|_{k+\ell+1}\gamma &= \left(\Theta(f|_k\gamma) - \frac{k}{12}E_2(f|_k\gamma)\right)E_{\ell-1} + \frac{k}{12}E_{\ell+1}(f|_k\gamma) \\ &= R(f|_k\gamma). \end{aligned} \tag{2.4.3}$$

**Lemma 2.9.** *If $f \in M_k\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$, then for every cusp $s \in \{0, 1/2, \infty\}$ and $i \geq 1$, we have* $\operatorname{ord}_s\left(\overline{R_i^f}\right) \geq \operatorname{ord}_s(f)$.

*Proof.* First recall that for $k \geq 2$, $E_k = 1 + O(q)$. Hence $\operatorname{ord}_\infty E_k = 0$. For the cusp $s = \infty$, by (2.4.1), we have

$$\begin{aligned} \operatorname{ord}_\infty(R(f)) &\geq \min\{\operatorname{ord}_\infty(\Theta f), \operatorname{ord}_\infty(f) + 1\} \\ &\geq \operatorname{ord}_\infty(f). \end{aligned}$$

For the cusp $s = 0$, set $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By (2.4.3), we have

$$\begin{aligned} \mathrm{ord}_0(R(f)) &= 4\ \mathrm{ord}_\infty \left(R(f)|_{k+\ell+1}\gamma\right) \\ &\geq 4 \min\{\mathrm{ord}_\infty(\Theta(f|_k\gamma)), \mathrm{ord}_\infty(f|_k\gamma) + 1\} \\ &\geq 4\ \mathrm{ord}_\infty(f|_k\gamma) \\ &= \mathrm{ord}_0(f). \end{aligned}$$

Similarly $\mathrm{ord}_{1/2}(R(f)) \geq \mathrm{ord}_{1/2}(f)$. For all cusps $s$, iteration yields $\mathrm{ord}_s(R_i^f) \geq \mathrm{ord}_s(f)$. Equation (2.3.3) gives the conclusion. ☐

Lemmas 2.10 and 2.11 below are due to Swinnerton-Dyer [45] who proved the statements for level $N = 1$. The generalization to level $N = 4$ may be found in, for example, [2].

**Lemma 2.10.** *Suppose $N = 1$ or 4, that $\ell \geq 5$ is prime, and $\overline{f} \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$. Then*

$$\omega\left(\Theta f\right) \leq \omega\left(f\right) + \ell + 1 \tag{2.4.4}$$

*with equality if and only if $\omega(f) \not\equiv 0 \mod \ell$. Furthermore, if $\Theta f \not\equiv 0 \mod \ell$ then there is an $s \geq 0$ such that*

$$\omega\left(\Theta f\right) = \omega\left(f\right) + \ell + 1 - s\left(\ell - 1\right). \tag{2.4.5}$$

*and we have $s = 0$ if and only if $\omega\left(\overline{f}\right) \equiv 0 \mod \ell$.*

*Proof.* By Equation (2.4.1) we see that (2.4.4) holds. The statement about equality follows from the explicit isomorphisms (2.3.1) and (2.3.2). Lemma 2.6 shows that the statement about $s$ holds. ☐

We also have:

**Lemma 2.11.** *Suppose $N = 1$ or 4. For all $i \geq 1$, we have $\omega\left(\overline{f}^i\right) = i\omega\left(\overline{f}\right)$.*

The following lemma follows from (2.3.1) and (2.3.2).

**Lemma 2.12.** *Suppose $\ell \geq 5$ is prime, $N = 1$ or 4, $k \in \mathbb{Z}$, $f, g \in M_k$, and $\omega(f) < \omega(g)$. Then $\omega(f + g) = \omega(g)$.*

# CHAPTER 3

# THE TATE CYCLE

In this chapter we work exclusively in characteristic $\ell \geq 5$. All equalities of (reduced) modular forms are in the ring $\mathbb{F}_\ell[\![q]\!]$. To ease the notation, we drop the tildes from $f \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$. The material in this chapter has appeared in [19] and [20]. Sections 3.1 and 3.2 contain technical machinery used in all of the author's work on Ramanujan congruences. Section 3.3 contains the main result of [20].

## 3.1 The Tate cycle

Consider the action of $\Theta$ on $f = \sum_{n\geq 0} a(n)q^n \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. We have

$$\Theta f \equiv \sum_{n\geq 0} a(n)nq^n \equiv \sum_{\ell\nmid n} a(n)nq^n \mod \ell.$$

Thus the coefficients of the image $\Theta f$ always vanish along the arithmetic progression $a(n\ell + 0) \equiv 0 \mod \ell$. For future reference we package this into a remark.

*Remark* 3.1. For any $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$, the form $\Theta f$ has a Ramanujan congruence at $0 \mod \ell$.

Fermat's little theorem easily implies that for any $f = \sum a(n)q^n \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$, we have

$$\Theta^\ell f \equiv \sum a(n)n^\ell q^n \equiv \sum a(n)nq^n \equiv \Theta f \mod \ell$$

and

$$\Theta^{\ell-1} f \equiv \sum a(n)n^{\ell-1}q^n \equiv \sum_{\ell\nmid n} a(n)q^n \mod \ell. \tag{3.1.1}$$

Thus for all $i \geq 1$, we have $\overline{\Theta^{i+\ell-1}f} = \overline{\Theta^i f}$. We say that the sequence $\overline{\Theta f}, \overline{\Theta^2 f}, \ldots, \overline{\Theta^{\ell-1} f}$ is the *Tate cycle* of $f$. Note that $f$ itself is not necessarily in its own Tate cycle. In light of Remark 3.1, the only way that we can have $\overline{f} \in \left\{\overline{\Theta f}, \overline{\Theta^2 f}, \ldots, \overline{\Theta^{\ell-1} f}\right\}$ is if $f$ has a Ramanujan congruence at $0 \mod \ell$. Furthermore, by (3.1.1) we see that $f$ will be in its Tate cycle if and only if $\overline{\Theta^{\ell-1} f} = \overline{f}$. We expand on these remarks slightly in the following lemma.

**Lemma 3.2.** *Let $N = 1$ or $4$ and $\ell \geq 5$ be prime. Let $f = \sum a(n)q^n \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. The following are equivalent:*

*(1) The form $f$ has a Ramanujan congruence at $0 \mod \ell$.*

*(2) The form $f$ is in its own Tate cycle.*

*(3) We have $\overline{\Theta^{\ell-1} f} = \overline{f}$.*

*(4) We have $\overline{f|U_\ell} = 0$.*

*Furthermore, $\overline{(f|U_\ell)^\ell} = \overline{f - \Theta^{\ell-1} f}$.*

*Proof.* Notice that

$$\begin{aligned} f - \Theta^{\ell-1} f &\equiv \sum_{n\in\mathbb{Z}} a(n)q^n - \sum_{\ell \nmid n} a(n)q^n \equiv \sum_{\ell | n} a(n)q^n \equiv \sum_{n\in\mathbb{Z}} a(n\ell)q^{n\ell} \mod \ell \\ &\equiv \left(\sum_{n\in\mathbb{Z}} a(n\ell)q^n\right)^\ell \equiv \left(f\middle|U_\ell\right)^\ell \mod \ell. \end{aligned}$$

Hence, (3) and (4) are equivalent and the "furthermore" statement is true. Moreover, (1), (2), and (3) are equivalent by the remarks in the paragraph before the statement of Lemma 3.2. ∎

With $s \geq 0$ as in Lemma 2.10, we have

$$\omega(\Theta f) \equiv \begin{cases} \omega(f) + 1 \mod \ell & \text{if } \omega(f) \not\equiv 0 \mod \ell \\ s + 1 \mod \ell & \text{if } \omega(f) \equiv 0 \mod \ell \end{cases}$$

and so by Lemma 2.10 the filtration usually rises by $\ell+1$ at each step of the Tate cycle. Occasionally, the filtration will fall. If $i$ is such that $\omega(\Theta^{i+1} f) < \omega(\Theta^i f) + \ell + 1$, then call $\Theta^i f$ a *high point* and $\Theta^{i+1} f$ a *low point* of the Tate cycle. An analysis as in Jochnowitz [24, Section 7] gives the following lemma which characterizes the rise-and-fall pattern of the filtration in the Tate cycle.

**Lemma 3.3.** *Let $\ell \geq 5$ be prime and $A, B \in \mathbb{Z}$ with $1 \leq B \leq \ell$. If $f \in M_{A\ell+B}\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ is in its own Tate cycle with $\omega(f) = A\ell + B \in \mathbb{Z}$ , then $\Theta f \neq 0$. Furthermore:*

*(1) We have $B \neq 1$.*

*(2) The Tate cycle has a single low point if and only if some term in the cycle (which will be the low point) has filtration congruent to 2 modulo $\ell$.*

*(3) Either there is one low point in the Tate cycle or there are two low points in the Tate cycle.*

*(4) For all $j \geq 1$ we have $\omega\left(\Theta^{j+1} f\right) \neq \omega\left(\Theta^j f\right) + 2$.*

(5) *If $f$ is a low point of its Tate cycle and if there are two low points, then the first high point has filtration*

$$\omega(\Theta^{\ell-B} f) = (A - B + \ell + 1)\ell,$$

*the other low point has filtration*

$$\omega\left(\Theta^{\ell-B+1} f\right) = A\ell + (\ell + 3 - B),$$

*and the last high point has filtration*

$$\omega\left(\Theta^{\ell-2} f\right) = (A + B - 2)\ell.$$

*Proof.* Since $\omega(f) = A\ell + B \neq -\infty$, we deduce $f \not\equiv 0$. Since $f$ is in its own Tate cycle, $0 \not\equiv f \equiv \Theta^{\ell-1} f \mod \ell$ and so $\Theta f \not\equiv 0 \mod \ell$.

(1) If $\omega(f) \equiv 1 \mod \ell$, then by Lemma 2.10, for $0 \leq i \leq \ell - 1$ we have

$$\omega(\Theta^i f) = \omega(f) + i(\ell + 1) \equiv 1 + i \mod \ell.$$

That is, $\omega(f) < \omega(\Theta f) < \cdots < \omega(\Theta^{\ell-1})$ and so $\overline{f} \neq \overline{\Theta^{\ell-1} f}$.

(2) If some point $\overline{g}$ of a Tate cycle has $\omega(g) \equiv 2 \mod \ell$, then by Lemma 2.10, for $0 \leq i \leq \ell - 2$ we have $\omega(\Theta^i g) = \omega(g) + i(\ell + 1) \equiv 2 + i \mod \ell$. Then $\overline{g}, \ldots, \overline{\Theta^{\ell-2} g}$ are $\ell - 1$ distinct elements of the cycle. Hence, the next iteration must be $\overline{\Theta^{\ell-1} g} = \overline{g}$. Therefore $\overline{g}$ is a low point and there are no other low points. Conversely, if there is only one drop, then there must be $\ell - 2$ increases in the filtration before the single fall. Then by Lemma 2.10 the low point must have filtration $2 \mod \ell$. Note that in the case of a single drop in filtration, the $s$ in (2.4.5) is $s = \ell + 1$.

(3) Suppose there is more than one high point. Let $g$ denote a low point of the Tate cycle of $f$ and label the high points $\Theta^{i_1} g, \ldots, \Theta^{i_t} g$ where $t \geq 2$. Then since $\overline{g} = \overline{\Theta^{\ell-1} g}$ is a low point, we have $i_t = \ell - 2$. In order to examine the change in filtration between consecutive high points, it is convenient to let $i_{t+1} = i_1 + \ell - 1$. By Lemma 2.10 and part (2) above, for each $1 \leq j \leq t$ we have $s_j \geq 2$ such that

$$\omega(\Theta^{i_j+1} g) = \omega(\Theta^{i_j} g) + \ell + 1 - s_j(\ell - 1) \equiv 1 + s_j \mod \ell.$$

Then $i_{j+1} - i_j \equiv -s_j \mod \ell$. Considering the full Tate cycle,

$$\omega(g) = \omega(\Theta^{\ell-1} g) = \omega(g) + (\ell - 1)(\ell + 1) - \sum_{j=1}^{t} s_j(\ell - 1)$$

and so we see that $\sum s_j = \ell + 1$. Since $t \geq 2$, for $1 \leq j \leq t$ we deduce $i_{j+1} - i_j = \ell - s_j$ from the previous congruence. Now $\ell - 1 = \sum_{j=1}^{t}(i_{j+1} - i_j) = t\ell - \sum s_j = t\ell - (\ell + 1)$ which implies $t = 2$.

(4) By Lemma 2.10, $\omega(\Theta^{j+1}f) = \omega(\Theta^j f) + 2$ implies $\omega(\Theta^j f) \equiv 0 \mod \ell$. Then $\omega(\Theta^{j+1}f) \equiv 2 \mod \ell$. As in the proof of part (2), the filtration increases $\ell - 2$ more times before falling. Hence $\omega(\Theta^{j+1+\ell-2}f) > \omega(\Theta^j f)$ and so $\overline{\Theta^j f} \neq \overline{\Theta^{j+\ell-1}f}$ which implies $\Theta^j f$ is not in its Tate cycle, a contradiction.

(5) This part simply collects what we already know. We use the notation from the proof of part (3) above. Since $\omega(f) \equiv B \mod \ell$, by Lemma 2.10, $i_1 = \ell - B$. The values of $s_j$ are found by recalling $s_1 + s_2 = \ell + 1$ and $i_2 - i_1 = \ell - s_1$ from the proof of part (3). Lemma 2.10 provides the filtrations. ☐

*Remark* 3.4. By part (5) of the above lemma, if $f$ is a low point of its Tate cycle, it will be the lowest of two low points exactly when $3 \leq B \leq \ell$ and

$$B < \ell + 3 - B$$

or equivalently when $3 \leq B < \frac{\ell+3}{2}$. If $f$ is a low point with $B = \frac{\ell+3}{2}$ then both low points have the same filtration. Conversely, if $f$ is one of two low points, each with the same filtration, then $B = \frac{\ell+3}{2}$.

## 3.2 A reformulation of Ramanujan congruences

The following wonderful lemma has been extracted from the proof of Proposition 3 of Kiming and Olsson [26].

**Lemma 3.5.** *Let $N = 1$ or $4$ and $\ell \geq 5$ be prime. A modular form $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ with $\overline{\Theta f} \neq 0$ has a congruence at $b \not\equiv 0 \mod \ell$ if and only if $\Theta^{\frac{\ell+1}{2}} f \equiv -\left(\frac{b}{\ell}\right) \Theta f \mod \ell$.*

*Proof.* Note that $\binom{\ell-1}{i} \equiv (-1)^{\ell-1-i} \mod \ell$. Since $\Theta$ satisfies the product rule,

$$\begin{aligned}
\Theta^{\ell-1}\left(q^{-b}f\right) &\equiv \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} (-b)^{\ell-1-i} q^{-b} \Theta^i f \mod \ell \\
&\equiv \sum_{i=0}^{\ell-1} b^{\ell-1-i} q^{-b} \Theta^i f \mod \ell \\
&\equiv b^{\ell-1} q^{-b} f + \sum_{i=1}^{\ell-1} b^{\ell-1-i} q^{-b} \Theta^i f \mod \ell.
\end{aligned}$$

A congruence at $b \not\equiv 0 \mod \ell$ is thus equivalent to $0 \equiv \sum_{i=1}^{\ell-1} b^{\ell-1-i} q^{-b} \Theta^i f \mod \ell$, and hence to $0 \equiv \sum_{i=1}^{\ell-1} b^{\ell-1-i} \Theta^i f \mod \ell$. By Lemma 2.10, for $1 \leq i \leq \frac{\ell-1}{2}$ we have

$$\omega(\Theta^i f) \equiv \omega(\Theta^{i+\frac{\ell-1}{2}} f) \equiv \omega(f) + 2i \mod \ell - 1.$$

By Lemma 2.10 and by (2.3.1) or (2.3.2) as appropriate, the only way for the given sum to be zero is if for all $1 \leq i \leq \frac{\ell-1}{2}$,

$$b^{\ell-1-i}\Theta^i f + b^{\ell-1-(i+\frac{\ell-1}{2})}\Theta^{i+\frac{\ell-1}{2}} f \equiv 0 \mod \ell,$$

which happens if and only if for each $i$

$$\Theta^{i+\frac{\ell-1}{2}} f \equiv -b^{\frac{\ell-1}{2}}\Theta^i f \equiv -\left(\frac{b}{\ell}\right)\Theta^i f \mod \ell$$

which happens if and only if

$$\Theta^{\frac{\ell+1}{2}} f \equiv -\left(\frac{b}{\ell}\right)\Theta f \mod \ell. \qquad \square$$

*Remark* 3.6. By the previous lemma, if $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ has a Ramanujan congruence at $b \mod \ell$, then it has a Ramanujan congruence at all $c \mod \ell$ such that $\left(\frac{c}{\ell}\right) = \left(\frac{b}{\ell}\right)$.

We now take a brief diversion from the main theory to explain a construction motivated by Lemma 3.5 and Remark 3.6. For any $f \in M_k\left(\Gamma_1(N), \mathbb{F}_\ell\right)$ and any prime $\ell \geq 5$, set

$$\begin{aligned} f_0 &:= f - \Theta^{\ell-1} f \in M_{k+\ell^2-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right), \\ f_{+1} &:= \frac{1}{2}\left(\Theta^{\ell-1} f + \Theta^{\frac{\ell-1}{2}} f\right) \in M_{k+\ell^2-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right), \\ f_{-1} &:= \frac{1}{2}\left(\Theta^{\ell-1} f - \Theta^{\frac{\ell-1}{2}} f\right) \in M_{k+\ell^2-1}\left(\Gamma_1(N), \mathbb{F}_\ell\right). \end{aligned} \tag{3.2.1}$$

Clearly $f = f_0 + f_{+1} + f_{-1}$ and if $f = \sum a(n)q^n$, then for $s = 0, \pm 1$, one finds that

$$f_s = \sum_{\left(\frac{n}{p}\right)=s} a(n)q^n. \tag{3.2.2}$$

Hence $f_s$ has Ramanujan congruences at all $b$ with $\left(\frac{b}{\ell}\right) \neq s$.

**Example 3.7.** Take $\ell = 11$ and $\Delta \in M_{12}\left(\Gamma_1(1), \mathbb{Z}\right)$. Recall $E_4E_6 \equiv 1 \mod 11$. Set

$$\begin{aligned} f_0 &:= E_4^{33} + 10E_6^{22} \in M_{132}\left(\Gamma_1(1), \mathbb{Z}\right), \\ f_{+1} &:= 5E_4^{33} + 5E_4^{24}E_6^{6} + 7E_4^{21}E_6^{8} + 5E_4^{15}E_6^{12} + 9E_4^{12}E_6^{14} + 2E_4^{9}E_6^{16} + 5E_4^{6}E_6^{18} + 6E_6^{22} \in M_{132}\left(\Gamma_1(1), \mathbb{Z}\right), \\ f_{-1} &:= 5E_4^{33} + 6E_4^{24}E_6^{6} + 4E_4^{21}E_6^{8} + 7E_4^{15}E_6^{12} + E_4^{12}E_6^{14} + 9E_4^{9}E_6^{16} + 6E_4^{6}E_6^{18} + 6E_6^{22} \in M_{132}\left(\Gamma_1(1), \mathbb{Z}\right). \end{aligned}$$

Although we have omitted the calculations which show that these $f_s$ match (3.2.1), it is easily checked that they sum to $\Delta$:

$$\begin{aligned}
&f_0 + f_{+1} + f_{-1} \\
&= 11E_4^{33} + 11E_4^{24}E_6^6 + 11E_4^{21}E_6^8 + 12E_4^{15}E_6^{12} + 10E_4^{12}E_6^{14} + 11E_4^9E_6^{16} + 11E_4^6E_6^{18} + 22E_6^{22} \\
&\equiv E_4^{15}E_6^{12} - E_4^{12}E_6^{14} \mod 11 \\
&\equiv E_4^3 - E_6^2 \mod 11 \\
&\equiv \Delta \mod 11.
\end{aligned}$$

Furthermore, the $f_s$ are supported on the appropriate arithmetic progressions:

$$\begin{aligned}
f_0 &\equiv q^{11} + 9q^{22} + \cdots \mod 11, \\
f_{+1} &\equiv q + 10q^3 + 2q^4 + q^5 + 9q^9 + 9q^{12} + 4q^{14} + 10q^{15} + 7q^{16} + 2q^{20} + 10q^{23} + \cdots \mod 11, \\
f_{-1} &\equiv 9q^2 + 2q^6 + 9q^7 + 9q^{10} + 4q^{13} + 9q^{17} + 4q^{18} + 2q^{21} + \cdots \mod 11.
\end{aligned}$$

Returning to the main line of development, the following lemmas illustrate how the existence of Ramanujan congruences constrains the structure of the Tate cycle.

**Lemma 3.8.** *Let $\ell \geq 5$ be prime, $b \not\equiv 0 \mod \ell$, and $k \in \mathbb{Z}$. Suppose $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ has a Ramanujan congruence at $b \mod \ell$ and $\Theta f \not\equiv 0 \mod \ell$. Then the Tate cycle of $f$ has two low points. Furthermore, if $\Theta^i f$ is a high point, then*

$$\omega(\Theta^{i+1} f) = \omega(\Theta^i f) + (\ell+1) - \left(\frac{\ell+1}{2}\right)(\ell-1) \equiv \frac{\ell+3}{2} \mod \ell.$$

*Proof.* By Lemma 3.5, $\omega\left(\Theta f\right) = \omega(\Theta^{\frac{\ell+1}{2}} f)$. Hence, the filtration is not monotonically increasing between $\Theta f$ and $\Theta^{\frac{\ell+1}{2}} f$, so there must be a fall in filtration (and hence a low point) somewhere in the first half of the Tate cycle. We also have $\omega(\Theta^{\frac{\ell+1}{2}} f) = \omega\left(\Theta f\right) = \omega\left(\Theta^\ell f\right)$ and so by the same reasoning there must be a low point somewhere in the second half of the Tate cycle. By Lemma 3.3, there are exactly two low points in the Tate cycle. Lemma 2.10 gives

$$\omega\left(\Theta f\right) = \omega\left(\Theta^{\frac{\ell+1}{2}} f\right) = \omega\left(\Theta f\right) + \left(\frac{\ell-1}{2}\right)(\ell+1) - s(\ell-1)$$

for some $s \geq 1$. Hence $s = \frac{\ell+1}{2}$. By the same reasoning, the fall in filtration for the second half of the Tate cycle must also have $s = \frac{\ell+1}{2}$. The lemma follows. ☐

**Lemma 3.9.** *Let $\ell \geq 5$ be prime and $k \in \mathbb{Z}$. Suppose $f \in M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$. If $\omega(f) = A\ell + B$ where $1 \leq B \leq \ell - 1$, then*

$$\frac{\ell+1}{2} \leq B \leq A + \frac{\ell+3}{2}.$$

*Proof.* Since $B \neq 0$, we have $\omega(\Theta f) = (A+1)\ell + (B+1)$. From the proof of Lemma 3.8, the Tate cycle has a high point before $\Theta^{\frac{\ell+1}{2}} f$. By Lemma 3.8, the high point is $\Theta^i f$ with $1 \leq i \leq \frac{\ell-1}{2}$. Hence we have

$$\omega(\Theta^i f) = A\ell + B + i(\ell+1) \equiv B + i \equiv 0 \mod \ell.$$

Together with the restrictions on $B$ and $i$, this congruence implies that $B + i = \ell$ and $B \geq \frac{\ell+1}{2}$. Also, by Lemma 2.10 the high point has filtration

$$\begin{aligned} \omega(\Theta^{\ell-B} f) &= \omega(f) + (\ell - B)(\ell+1) \\ &= (A + \ell - B + 1)\ell. \end{aligned}$$

Lemma 3.8 implies that the corresponding low point has filtration

$$\omega(\Theta^{\ell-B+1} f) = \left(A - B + \frac{\ell+3}{2}\right)\ell + \left(\frac{\ell+3}{2}\right).$$

The fact that $\omega(\Theta^{\ell-B+1} f) \geq 0$ implies the second inequality. $\square$

A consequence of the above lemma is that for any integral-weight, holomorphic modular form with integral coefficients, there are only finitely many primes $\ell$ for which there are Ramanujan congruences at some $b \mod \ell$.

*Proof of Theorem 1.3.* Suppose $f \in M_k(\Gamma_1(N), \mathbb{Z})$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$ where $\ell \geq 5$. Now $k \geq \omega(f) = A\ell + B \geq B$ for some $0 \leq B \leq \ell - 1$. By the first inequality of Lemma 3.9,

$$k \geq B \geq \frac{\ell+1}{2}.$$

The conclusion follows. $\square$

## 3.3 Ramanujan congruences in quotients of Eisenstein series

The theory of reduced modular forms can be applied to study congruences in certain Laurent series which are not the Fourier series of a holomorphic, integral weight modular form.

**Lemma 3.10.** *Suppose that $\ell$ is prime and that $f = \sum a(n)q^n$ and $g = \sum c(n)q^n \in \mathbb{Z}_{(\ell)}((q))$ with $g \not\equiv 0 \mod \ell$. The series $f$ has a Ramanujan congruence at $b \mod \ell$ if and only if the series $fg^\ell$ has a Ramanujan congruence at $b \mod \ell$.*

*Proof.* It suffices to consider the reductions modulo $\ell$ of the series

$$\left(\sum a(n)q^n\right)\left(\sum c(n)q^{\ell n}\right) \equiv \sum_n \left(\sum_m c(m)a(n-\ell m)\right) q^n \mod \ell.$$

If $a(n)$ vanishes when $n \equiv b \mod \ell$, then the inner sum on the right hand side will also vanish for $n \equiv b \mod \ell$. The converse follows via multiplication by $(\sum c(n)q^n)^{-\ell}$ and a repetition of this argument. $\square$

**Lemma 3.11.** *Let $a, b, c \geq 0$ be integers and let $\ell > 11$ be prime. Then*

$$\omega(E_{\ell+1}^a E_4^b E_6^c) = a\ell + a + 4b + 6c.$$

*Proof.* Recall the polynomials $\overline{A}$ and $\overline{B}$ from Section 2.3. Since $E_{\ell+1}^a E_4^b E_6^c \in M_{a\ell+a+4b+6c}\big(\Gamma_1(1), \mathbb{Z}_{(\ell)}\big)$, it suffices to show that $\overline{A}(Q,R)$ does not divide $\overline{B}(Q,R)^a Q^b R^c$. However $\overline{A}$ has no repeated factors and is prime to $\overline{B}$ and so it suffices to show that $\overline{A}$ does not divide $QR$. But $QR$ has weight 10 and $E_{\ell-1}$ has weight $\ell - 1 > 10$ so this is impossible. $\square$

If $\Theta f \equiv 0 \mod \ell$ then the Tate cycle is trivial and the lemmas from the previous section are not applicable. We dispense with this case now.

**Lemma 3.12.** *Let $f = E_2^r E_4^s E_6^t$ where $r \geq 0$ and $s, t \in \mathbb{Z}$. If $\ell$ is a prime such that $\Theta f \equiv 0 \mod \ell$ then either $\ell \leq 13$ or $r \equiv s \equiv t \equiv 0 \mod \ell$.*

**Example 3.13.** We have $\Theta(E_4 E_6) \equiv 0 \mod \ell$ for $\ell = 2, 3, 11$.

**Example 3.14.** We have $\Theta(E_2^{144} E_4^{-15} E_6^{-14}) \equiv 0 \mod \ell$ for $\ell = 2, 3, 5, 7, 13$.

Note that $\Theta f \equiv 0 \mod \ell$ is equivalent to $f$ having Ramanujan congruences at all $b \not\equiv 0 \mod \ell$.

*Proof of Lemma 3.12.* Assume $\ell \geq 17$ and expand $f$ as a power series to get

$$\begin{aligned} f = 1 &+ \big(-24r + 240s - 504t\big)q \\ &+ \big(288r^2 - 5760rs + 12096rt - 360r + 28800s^2 \\ &\qquad - 120960st - 26640s + 127008t^2 - 143640t\big)q^2 + \cdots . \end{aligned}$$

If $\Theta f \equiv 0 \mod \ell$, then the coefficients of $q$ and $q^2$ vanish modulo $\ell$. That is,

$$-24r + 240s - 504t \equiv 0 \mod \ell, \tag{3.3.1}$$

and

$$\begin{aligned} &288r^2 - 5760rs + 12096rt - 360r + 28800s^2 \\ &\qquad - 120960st - 26640s + 127008t^2 - 143640t \equiv 0 \mod \ell. \end{aligned} \tag{3.3.2}$$

The assumption $\Theta f \equiv 0 \mod \ell$ is equivalent to the statement that $f$ has Ramanujan congruences at all $b \mod \ell$. Thus by Lemma 3.10, we have that $E_2^r E_4^{s+\ell|s|} E_6^{t+\ell|t|}$ has Ramanujan congruences at all $b \not\equiv 0 \mod \ell$. Hence $\Theta E_2^r E_4^{s+\ell|s|} E_6^{t+\ell|t|} \equiv 0 \mod \ell$. By Lemmas 2.10 and 3.11 and the fact that $E_2 \equiv E_{\ell+1} \mod \ell$, we have

$$\omega(E_{\ell+1}^r E_4^{s+\ell|s|} E_6^{t+\ell|t|}) \equiv r + 4s + 6t \equiv 0 \mod \ell. \tag{3.3.3}$$

Solving the system of congruences given by (3.3.3) and (3.3.1) yields

$$7r \equiv -72t \mod \ell, \tag{3.3.4}$$

$$14s \equiv 15t \mod \ell. \tag{3.3.5}$$

Substituting (3.3.4) and (3.3.5) into 49 times (3.3.2) yields

$$-8255520t \equiv 0 \mod \ell.$$

Since $8255520 = 2^5 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$, the lemma follows. $\square$

*Proof of Theorem 1.4.* We begin with the trivial observation that $E_2^r E_4^s E_6^t = 1 + \cdots$ does not have a simple congruence at $0 \mod \ell$. Hence, we assume that $E_2^r E_4^s E_6^t$ has a simple congruence at $b \not\equiv 0 \mod \ell$, where $\ell \geq 5$. Since $E_2 \equiv E_{\ell+1} \mod \ell$, $E_{\ell+1}^r E_4^s E_6^t$ has a simple congruence at $b \mod \ell$. Recall that our goal is to show $\ell \leq 2r + 8|s| + 12|t| + 21$. Hence, if $\ell < |s|$ or $\ell < |t|$ then we are done. Thus we assume $\ell + s \geq 0$ and $\ell + t \geq 0$. We also assume $\ell > 11$. Lemma 3.12 allows us to take $\Theta(E_2^r E_4^s E_6^t) \not\equiv 0 \mod \ell$ (otherwise we are done). By Lemma 3.10 we see that

$$E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t} \in M_{(r+10)\ell+(r+4s+6t)}\left(\Gamma_1(1), \mathbb{Z}_{(\ell)}\right)$$

has a simple congruence at $b \mod \ell$. We work with the form $E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}$ because it is holomorphic (with positive weight) and so our filtration apparatus is applicable. By Lemma 3.11,

$$\omega(E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) = (r + 10)\ell + (r + 4s + 6t). \tag{3.3.6}$$

We break into four cases depending on the size of $r + 4s + 6t$:

1. If $\ell \leq |r + 4s + 6t|$ then we are done.

2. If $0 < r + 4s + 6t < \ell$ then by Equation (3.3.6) and the first inequality of Lemma 3.9, $\frac{\ell+1}{2} \leq r + 4s + 6t$ and we are done.

3. If $r + 4s + 6t = 0$, then by Lemma 2.10

$$\omega(\Theta E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) = (r + 11)\ell + 1 - s'(\ell - 1)$$

for some $1 \leq s'$. If $\ell \leq r + 13$ then we are done, so it suffices to consider $\ell > r + 13$. Now in order for the filtration above to be non-negative, $s' \leq r+11$. Now $\omega(\Theta E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) \equiv s'+1$

mod $\ell$. By Lemma 3.5, there must be a high point of the Tate cycle before $\Theta^{\frac{\ell+1}{2}} E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}$. Let $i$ be the index of the first high point, so $1 \leq i \leq \frac{\ell-1}{2}$. Then

$$\omega(\Theta^i E_{\ell+1}^r E_4^{\ell+s} E_6^{\ell+t}) \equiv s' + i \equiv 0 \mod \ell.$$

Together with the restrictions on $i$ and $s'$ (namely $s' \leq r + 11 < r + 13 < \ell$), this congruence implies that

$$s' \geq \frac{\ell+1}{2}.$$

That is, $\ell \leq 2s' - 1 \leq 2r + 21$ and we are done.

4. If $-\ell < r + 4s + 6t < 0$, then take $B = \ell + r + 4s + 6t$ and $A = r + 9$. Equation (3.3.6) and the second inequality of Lemma 3.9 give

$$\ell + r + 4s + 6t \leq r + 9 + \frac{\ell+3}{2}$$

which is equivalent to $\ell \leq 21 - 8s - 12t$ and we are done. $\square$

*Remark* 3.15. Combining these four cases and recalling that the proof assumed $\ell + s \geq 0$, $\ell + t \geq 0$ and $\ell > 11$, we can improve the bound in Theorem 1.4 slightly. In particular, if $r + 4s + 6t > 0$ then

$$\ell \leq \max\{|s| - 1, |t| - 1, 11, 2r + 8s + 12t - 1\},$$

and if $r + 4s + 6t \leq 0$ then

$$\ell \leq \max\{|s| - 1, |t| - 1, 11, 21 - 8s - 12t\}.$$

# CHAPTER 4

# FORMS WITH DIVISOR SUPPORTED AT THE CUSPS

This chapter is a mild reformulation of [19]. In this chapter we work exclusively with modular forms of level $N = 4$ and so we will write $M_k$ for $M_k\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$ and $\overline{M}_k$ for $M_k\left(\Gamma_1(4), \mathbb{F}_\ell\right)$. Similarly, we will write $M_k^!$ instead of $M_k^!\left(\Gamma_1(4), \mathbb{Z}_{(\ell)}\right)$.

A *divisor* of a modular form on $\Gamma_1(4)$ is a formal sum over the points of the compactified modular curve $X_1(4)$ where the coefficients are the orders of the zero or pole at the points:

$$\operatorname{div} f = \sum_{[x] \in X_1(4)} \operatorname{ord}_x f \cdot [x].$$

We restrict attention to meromorphic modular forms whose divisors are supported at the cusps 0, 1/2, and $\infty$. This technical condition provides key information about the Tate cycle. The most interesting (and the most computationally involved) case is when the meromorphic modular form has negative, half-integer weight. In the next section, we associate to any meromorphic modular form a holomorphic, integral weight modular form with equivalent Ramanujan congruences.

## 4.1 Examples of associated holomorphic, integral weight modular forms

In this section we associate modular forms to many common, combinatorial generating functions. The associated forms will have equivalent Ramanujan congruences. The method is quite general. Our key tool is Lemma 3.10. Recall that $q = e^{2\pi i \tau}$ and

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right).$$

For $d = 1, 2, 4$, we have $\eta(d\tau)^{24} = \Delta(d\tau) \in M_{12}$. Furthermore

$$\begin{aligned}
\operatorname{div} \Delta(\tau) &= 4 \cdot [0] + 1 \cdot \left[\frac{1}{2}\right] + 1 \cdot [\infty], \\
\operatorname{div} \Delta(2\tau) &= 2 \cdot [0] + 2 \cdot \left[\frac{1}{2}\right] + 2 \cdot [\infty], \\
\operatorname{div} \Delta(4\tau) &= 1 \cdot [0] + 1 \cdot \left[\frac{1}{2}\right] + 4 \cdot [\infty].
\end{aligned}$$

Since $24 \mid \ell^2-1$ when $\ell \geq 5$, the strategy is to use Lemma 3.10 to replace occurrences of $\eta(d\tau)^{-1}$ with $\eta(d\tau)^{\ell^2-1}$ and occurrences of $\eta(d\tau)$ with $\eta(d\tau)^{(\ell^2-1)(\ell-1)}$. This changes neither the filtration modulo $\ell$, nor the Ramanujan congruences. As illustrated in the examples below, since multiplication by powers of $q$ merely shifts the location of Ramanujan congruences, we can associate a holomorphic, integral weight modular form with equivalent Ramanujan congruences to any product of the form

$$q^r \prod_{n=1}^{\infty} (1-q^n)^s \left(1-q^{2n}\right)^t \left(1-q^{4n}\right)^u$$

where $r$, $s$, $t$, $u \in \mathbb{Z}$. Set

$$\delta = \delta_\ell := \frac{\ell^2-1}{24}.$$

**Example 4.1.** The overpartition generating function is

$$\overline{P}(\tau) = \sum \overline{p}(n)q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right) = \frac{\eta(2\tau)}{\eta(\tau)^2}.$$

By Lemma 3.10, $\overline{P}(\tau)$ has the same Ramanujan congruences as

$$f_{\overline{P}} := \eta(2\tau)^{(\ell-1)(\ell^2-1)}\eta(\tau)^{2(\ell^2-1)} = \Delta(2\tau)^{(\ell-1)\delta_\ell}\Delta(\tau)^{2\delta_\ell} \in M_{\frac{(\ell-1)(\ell+1)^2}{2}}.$$

Note that

$$\begin{aligned}\operatorname{div} f_{\overline{P}} &= (2(\ell-1)\delta_\ell + 8\delta_\ell)\cdot[0] + (2(\ell-1)\delta_\ell + 2\delta_\ell)\cdot\left[\frac{1}{2}\right] + (2(\ell-1)\delta_\ell + 2\delta_\ell)\cdot[\infty] \\ &= \delta_\ell\,(2\ell+6)\cdot[0] + \delta_\ell\,(2\ell)\cdot\left[\frac{1}{2}\right] + \delta_\ell\,(2\ell)\cdot[\infty].\end{aligned}$$

**Example 4.2.** The overpartition pair generating function is

$$\overline{PP}(\tau) = \sum \overline{pp}(n)q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^2 = \frac{\eta(2\tau)^2}{\eta(\tau)^4}.$$

By Lemma 3.10, $\overline{PP}(\tau)$ has the same Ramanujan congruences as

$$f_{\overline{PP}} := \eta(2\tau)^{2(\ell-1)(\ell^2-1)}\eta(\tau)^{4(\ell^2-1)} = \Delta(2\tau)^{2(\ell-1)\delta_\ell}\Delta(\tau)^{4\delta_\ell} \in M_{(\ell-1)(\ell+1)^2}.$$

Note that

$$\operatorname{div} f_{\overline{P}} = \delta_\ell\,(4\ell+12)\cdot[0] + \delta_\ell\,(4\ell)\cdot\left[\frac{1}{2}\right] + \delta_\ell\,(4\ell)\cdot[\infty].$$

**Example 4.3.** By [15], the crank difference generating function is

$$CD(\tau) := \sum_{n\geq 0} \left(M_e(n) - M_o(n)\right) q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{2n})^2}.$$

By Lemma 3.10, when $\ell \geq 5$ this has a congruence at $b \mod \ell$ if and only if

$$\begin{aligned}
&\prod_{n=1}^{\infty} (1-q^n)^{3(\ell^2-1)(\ell-1)} \left(1-q^{2n}\right)^{2(\ell^2-1)} \\
&= q^{\frac{3\left(\ell^2-1\right)(\ell-1)+4\left(\ell^2-1\right)}{24}} q^{-\frac{3\left(\ell^2-1\right)(\ell-1)+4\left(\ell^2-1\right)}{24}} \prod_{n=1}^{\infty} (1-q^n)^{3(\ell^2-1)(\ell-1)} \left(1-q^{2n}\right)^{2(\ell^2-1)} \\
&= q^{-\frac{3\left(\ell^2-1\right)(\ell-1)+4\left(\ell^2-1\right)}{24}} \Delta(\tau)^{3(\ell-1)\delta_\ell} \Delta(2\tau)^{2\delta_\ell} \\
&= q^{-3\delta_\ell(\ell-1)-4\delta_\ell} \Delta(\tau)^{3(\ell-1)\delta_\ell} \Delta(2\tau)^{2\delta_\ell}
\end{aligned}$$

has a congruence at $b \mod \ell$ which happens if and only if

$$f_{CD} := \Delta(\tau)^{3(\ell-1)\delta_\ell} \Delta(2\tau)^{2\delta_\ell} \in M_{\frac{(3\ell-1)(\ell^2-1)}{2}}$$

has a congruence at $b + 3\delta_\ell(\ell-1) + 4\delta_\ell \mod \ell$, which happens if and only if $f_{CD}$ has a congruence at $b + \delta_\ell \mod \ell$. Note that

$$\operatorname{div} f_{CD} = \delta_\ell (12\ell - 8) \cdot [0] + \delta_\ell (3\ell+1) \cdot \left[\frac{1}{2}\right] + \delta_\ell (3\ell+1) \cdot [\infty].$$

**Example 4.4.** Equation (10.6) of [4] says that the generating function of $c\phi_2(n)$ is

$$C\Phi_2(\tau) = \frac{\theta_0(\tau)}{q^{-1/12}\eta(\tau)^2}.$$

Now $C\Phi_2$ will have a congruence at $b \mod \ell$ if and only if $\left(q^{-1/12}\theta_0(\tau)^{\ell-1}\eta(\tau)^2\right)^{\ell^2-1}$ has a congruence at $b \mod \ell$. This happens if and only if $f_{C\Phi_2} := \theta_0(\tau)^{(\ell-1)(\ell^2-1)}\eta(\tau)^{2(\ell^2-1)} \in M_{(\ell-1)(\ell+1)^2/2}$ has a congruence at $b + 2\delta \mod \ell$.

## 4.2 Lifting data to characteristic zero

Consider the forms

$$\begin{aligned} E(\tau) &:= \frac{\eta^8(\tau)}{\eta^4(2\tau)} \in M_2, \\ F(\tau) &= \frac{\eta^8(4\tau)}{\eta^4(2\tau)} = \sum_{n\geq 0} \sigma_1(2n+1)q^{2n+1} \in M_2, \\ \theta_0^2(\tau) &= \frac{\eta^{10}(2\tau)}{\eta^4(\tau)\eta^4(4\tau)} = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2 \in M_1. \end{aligned}$$

Note that $\mathrm{ord}_0(E) = 1$, $\mathrm{ord}_\infty(F) = 1$, $\mathrm{ord}_{1/2}(\theta_0^2) = 1/2$, and that these are the only zeros of these forms. Recall that since $\dim M_k(\Gamma_1(4), \mathbb{C}) = 1 + \lfloor k/2 \rfloor$, we have

$$\begin{aligned} M_{2k} &= \langle E^{k-i}F^i \rangle_{i=0,1,\ldots,k}, \\ M_{2k+1} &= \theta_0^2 \langle E^{k-i}F^i \rangle_{i=0,1,\ldots,k}, \end{aligned} \tag{4.2.1}$$

as $\mathbb{Z}_{(\ell)}$-modules, where the basis vectors $E^{k-i}F^i = q^i + \cdots$ have rising orders at $\infty$. The following modification (partially) arranges for ascending orders at the other cusps as well. Set

$$G := \theta_0^4 = E + 16F \in M_2$$

and fix non-negative integers $m_\infty, m_0, m_{1/2}$ such that $m_\infty + m_0 + m_{1/2} \leq k$. Define the following submodules of $M_{2k}$ depending on $\overline{m} = (m_\infty, m_0, m_{1/2}, 2k)$:

$$\begin{aligned} V^{\overline{m}} &:= \{f \in M_{2k} | \text{ for all cusps } s, \mathrm{ord}_s f \geq m_s\} \\ &= E^{m_0}F^{m_\infty}G^{m_{1/2}}M_{2(k-m_0-m_\infty-m_{1/2})} \\ &= \langle E^{k-m_\infty-m_{1/2}-i}F^{m_\infty+i}G^{m_{1/2}} \rangle_{i=0,1,\ldots,k-m_0-m_\infty-m_{1/2}}, \\ W_\infty^{\overline{m}} &:= \langle E^{k-i}F^i \rangle_{i=0,1,\ldots m_\infty-1}, \\ W_0^{\overline{m}} &:= \langle E^iF^{k-i} \rangle_{i=0,1,\ldots m_0-1}, \\ W_{1/2}^{\overline{m}} &:= \langle E^{m_0}F^{k-m_0-i}G^i \rangle_{i=0,1,\ldots m_{1/2}-1}, \end{aligned} \tag{4.2.2}$$

so that each $W_s^{\overline{m}}$ has $m_s$ basis forms, each with distinct order at $s$. In particular,

$$W_s^{\overline{m}} \subseteq \{f \in M_{2k} | \, \mathrm{ord}_s f < m_s\}.$$

In addition, each form in (4.2.2) has a different order at $\infty$. It follows that (4.2.2) has $k$ linearly independent basis vectors and

$$M_{2k} = V^{\overline{m}} \oplus W_\infty^{\overline{m}} \oplus W_0^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$$

as a $\mathbb{Z}_{(\ell)}$-module. We have the following lifting result.

**Proposition 4.5.** *Let $m_\infty, m_0, m_{1/2}, k$ be non-negative integers satisfying $m_\infty + m_0 + m_{1/2} \le k$. Set $\overline{m} = (m_\infty, m_0, m_{1/2}, 2k)$. Let $V^{\overline{m}}$ and the $W_s^{\overline{m}}$ be submodules of $M_{2k}$ as in (4.2.2).*

*(a) If $f \in M_{2k}$ has $\operatorname{ord}_s(\overline{f}) \ge m_s$ for all cusps $s$, then we can write $f = g + \ell h$, where $g \in V^{\overline{m}}$ and $h \in W_0^{\overline{m}} \oplus W_\infty^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$.*

*(b) If $f' \in M_{2k+1}$ has $\operatorname{ord}_s(\overline{f'}) \ge m_s$ for all cusps $s$, then $f' = \theta_0^2 f$ for some $f \in M_{2k}$ with $\operatorname{ord}_s(\overline{f}) \ge m_s$ for all cusps $s$. (Recall $m_{1/2} \in \mathbb{Z}$.) There are $g \in V^{\overline{m}}$ and $h \in W_0^{\overline{m}} \oplus W_\infty^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$ such that $f' = \theta_0^2 g + \ell\theta_0^2 h$.*

*Proof.* Write $f = g + h_\infty + h_0 + h_{1/2}$, where $g \in V^{\overline{m}}$ and $h_s \in W_s^{\overline{m}}$. We show each $\overline{h_s} = 0$. (It is important to do this in the correct order.) Suppose $h_\infty = \sum_{i=0}^{m_\infty - 1} a_i E^{k-i} F^i$ with $a_i \in \mathbb{Z}_{(\ell)}$. If any $a_i \not\equiv 0 \mod \ell$, then let $t$ be the least such $i$. In this case, $h_\infty \equiv a_t q^t + \cdots \mod \ell$ has order $t$. By construction $V^{\overline{m}} \oplus W_0^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$ only contains forms of order at least $m_\infty$ at the infinite cusp. Hence

$$m_\infty \le \operatorname{ord}_\infty\left(\overline{f}\right) = \operatorname{ord}_\infty\left(\overline{h_\infty}\right) = t < m_\infty,$$

a contradiction. Thus $\overline{h_\infty} = 0$.

Now consider $h_0 = \sum_{i=0}^{m_0 - 1} b_i E^i F^{k-i}$ with $b_i \in \mathbb{Z}_{(\ell)}$. If any $b_i \not\equiv 0 \mod \ell$, then let $t$ be the least such $i$. Then $\operatorname{ord}_0(h_0) = t \le m_0 - 1$. Since $V^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$ only contains forms with order at least $m_0$ at zero and since $\overline{h_\infty} = 0$, we have

$$m_0 \le \operatorname{ord}_0\left(\overline{f}\right) = \operatorname{ord}_0\left(\overline{h_0}\right) = t < m_0,$$

a contradiction. Thus $\overline{h_0} = 0$. An analogous argument shows that if $\overline{h_{1/2}} \ne 0$, then

$$m_{1/2} \le \operatorname{ord}_{1/2}\left(\overline{f}\right) = \operatorname{ord}_{1/2}\left(\overline{h_{1/2}}\right) < m_{1/2},$$

another contradiction. For part (b), recall that any $f' \in M_{2k+1}$ must have $\operatorname{ord}_{1/2} f' \in \mathbb{Z} + \frac{1}{2}$ and hence is divisible by $\theta_0^2$. Apply part (a) to $f = f'/\theta_0^2 \in M_{2k}$. $\square$

We have the following Sturm-style result.

**Corollary 4.6.** *(a) Let $f \in M_{2k}$ and $\operatorname{ord}_0\left(\overline{f}\right) + \operatorname{ord}_\infty\left(\overline{f}\right) + \operatorname{ord}_{1/2}\left(\overline{f}\right) > k$. Then for all cusps $s$, $\operatorname{ord}_s\left(\overline{f}\right) = +\infty$ and $\overline{f} = 0$.*

*(b) Let $f \in M_{2k+1}$ and $\operatorname{ord}_0\left(\overline{f}\right) + \operatorname{ord}_\infty\left(\overline{f}\right) + \operatorname{ord}_{1/2}\left(\overline{f}\right) > k + 1/2$. Then for all cusps $s$, $\operatorname{ord}_s\left(\overline{f}\right) = +\infty$ and $\overline{f} = 0$.*

*Proof.* (a) Suppose $\overline{f} \ne 0$. For each cusp $s$, choose integers $0 \le m_s \le \operatorname{ord}_s\left(\overline{f}\right)$ such that $m_0 + m_\infty + m_{1/2} = k$. Set $\overline{m} = (m_\infty, m_0, m_{1/2}, 2k)$ and apply Proposition 4.5. Write $f = g + \ell h$, with $g \in V^{\overline{m}}$ and $h \in W_0^{\overline{m}} \oplus W_\infty^{\overline{m}} \oplus W_{1/2}^{\overline{m}}$. For the parameters in $\overline{m}$, $\dim V^{\overline{m}} = 1$. Therefore, $g = cE^{m_0} F^{m_\infty} G^{m_{1/2}} \in M_{2k}$, for some $c \in \mathbb{Z}_{(\ell)}$. We now have a contradiction since for any cusp $s$, $\operatorname{ord}_s(\overline{f}) = \operatorname{ord}_s(\overline{g}) = m_s$, contrary to our assumption that $\sum \operatorname{ord}_s(\overline{f}) > k$.

(b) Apply part (a) to $f/\theta_0^2 \in M_{2k}$. $\square$

In the next section we use the following proposition to lift a low point of a Tate cycle – a mod $\ell$ object – to a characteristic zero modular form with high orders of vanishing at the cusps.

**Proposition 4.7.** *Let $k'$ and $i$ be positive integers.*

*(a) Given $f \in M_{2k'}$, let $2k = \omega(\Theta^i f)$ and $m_s = \mathrm{ord}_s f$ for each cusp $s$. Set $\overline{m} = (m_\infty, m_0, m_{1/2}, 2k)$. Then there is $g \in V^{\overline{m}}$ such that $\overline{\Theta^i f} = \overline{g}$.*

*(b) Given $f \in M_{2k'+1}$, let $2k+1 = \omega(\Theta^i f)$ and $m_s = \lfloor \mathrm{ord}_s f \rfloor$ for each cusp $s$. Set $\overline{m} = (m_\infty, m_0, m_{1/2}, 2k)$. Then there is $g \in V^{\overline{m}}$ such that $\overline{\Theta^i f} = \overline{\theta_0^2 g}$.*

*Proof.* Lemma 2.9 implies that for each cusp $s$, $\mathrm{ord}_s\left(\overline{R_i^f}\right) \geq \mathrm{ord}_s(f) \geq m_s$. In the even weight case, apply Proposition 4.5 (a) to deduce $\Theta^i f \equiv R_i^f \equiv g \mod \ell$ for some $g \in V^{\overline{m}}$. In the odd weight case use Proposition 4.5 (b). ☐

## 4.3 Congruences in holomorphic forms which vanish only at the cusps

This section considers modular forms which vanish only at the cusps. This condition implies a lot about the Tate cycle. To begin with, if $f \in M_k$, $\overline{\Theta f} \neq 0$, and $f$ vanishes only at the cusps but is not congruent to a cusp form, then $\overline{f|U_\ell} \neq 0$. This follows from the more general proposition below:

**Proposition 4.8.** *Let $k \in \mathbb{Z}$, let $f \in M_k$ be non-zero, and suppose that for some cusp $s$, $\mathrm{ord}_s(\overline{f}) \equiv 0 \mod \ell$. Then $\overline{f|U_\ell} \neq 0$.*

*Proof.* Since $\mathrm{ord}_s\left(\overline{f}\right) \equiv 0 \mod \ell$, we have that $\mathrm{ord}_s(\overline{\Theta f}) > \mathrm{ord}_s\left(\overline{f}\right)$ because $\Theta$ kills the leading term in the Fourier expansion at $s$. To be more precise, let $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ depending on whether $s = \infty, 0$ or $1/2$, respectively. Set $c = 4$ if $s = 0$ and $c = 1$ otherwise. (Thus $c$ is the width of the cusp $s$.) By examining the orders of the summands in (2.4.3), we have

$$\mathrm{ord}_s\left(\overline{R_1^f}\right) = c \cdot \mathrm{ord}_\infty\left(\overline{R_1^f|_{k+\ell+1}\gamma}\right) \geq 1 + \mathrm{ord}_s \overline{f}.$$

By the proof of Lemma 2.9, $\mathrm{ord}_s\left(\overline{R_{\ell-1}^f}\right) \geq \mathrm{ord}_s\left(\overline{R_1^f}\right) \geq 1 + \mathrm{ord}_s \overline{f}$. Thus by Remark 2.7 it is impossible for $\overline{R_{\ell-1}^f} = \overline{f}$. That is, $\overline{(f|U_\ell)^\ell} = \overline{f} - \overline{\Theta^{\ell-1} f} \neq 0$. ☐

**Proposition 4.9.** *Suppose that $k \in \mathbb{Z}$, that $f \in M_k$, that $f$ vanishes only at the cusps, and that $\overline{\Theta f} \neq 0$. Then for $i \geq 0$, we have $\omega(\Theta^i f) \geq \omega(f) = k$. In particular, if $f$ is a member of its own Tate cycle, then $f$ is a low point. If $f$ is not a member of its own Tate cycle, then $\Theta f$ is a low point.*

*Proof.* Since $f \in M_k$, obviously $\omega(f) \leq k$. By Remark 2.4, we have

$$\mathrm{ord}_0 f + \mathrm{ord}_\infty f + \mathrm{ord}_{1/2} f = k/2.$$

Thus by Corollary 4.6, we have $\omega(f) \geq k$ and equality follows. For any $i \geq 1$ and for all cusps $s$, by Lemma 2.9, $\mathrm{ord}_s\left(\overline{R_i^f}\right) \geq \mathrm{ord}_s(f)$. Hence $\mathrm{ord}_0\left(\overline{R_i^f}\right) + \mathrm{ord}_\infty\left(\overline{R_i^f}\right) + \mathrm{ord}_{1/2}\left(\overline{R_i^f}\right) \geq k/2$. By Corollary 4.6 we must have $\omega(\Theta^i f) \geq k$.

Suppose $f$ is not a member of its own Tate cycle and, for the sake of contradiction, that $\overline{\Theta f} = \overline{\Theta^\ell f}$ is not a low point. There are two possibilities: either $\omega(f) \equiv 0 \mod \ell$ or $\omega(f) \not\equiv 0 \mod \ell$.

If $\omega(f) \equiv 0 \mod \ell$, then we have $\omega(\Theta f) = \omega(f)+\ell+1-s(\ell-1)$ with $s \geq 1$. Since $\omega(\Theta f) \geq \omega(f)$, we deduce that $s = 1$ and $\omega(\Theta f) = \omega(f) + 2 \equiv 2 \mod \ell$. By Lemma 3.3 (2) the Tate cycle has a single low point with filtration $2 \mod \ell$ and the low point must then be $\Theta f$.

On the other hand, if $\omega(f) \not\equiv 0 \mod \ell$, then since $\Theta^\ell f$ is not a low point, we have

$$\omega(f) + \ell + 1 = \omega(\Theta f) = \omega(\Theta^\ell f) = \omega(\Theta^{\ell-1} f) + \ell + 1.$$

In particular $\omega(\Theta^{\ell-1} f) = \omega(f) = k$. However in this case $\dim V^{\overline{m}} = 1$. Therefore $\Theta^{\ell-1} f$ is a constant multiple of $f$ which contradicts the assumption that $f$ is not in its Tate cycle (since $\Theta$ commutes with scalar multiplication). ☐

The following two corollaries show the differences between congruences at $b \not\equiv 0 \mod \ell$ and at $0 \mod \ell$.

**Corollary 4.10.** *Suppose that $k \in \mathbb{Z}$, that $f \in M_k$, and that $f$ vanishes only at the cusps. Suppose further that $\overline{\Theta f} \neq 0$ and that $\omega(f) = A\ell + B$, with $1 \leq B \leq \ell$. If $f$ has a congruence at $b \not\equiv 0 \mod \ell$, then either*

1. *$B = \frac{\ell+1}{2}$ and $f$ does not have a congruence at $0 \mod \ell$, or*
2. *$B = \frac{\ell+3}{2}$ and $f$ does have a congruence at $0 \mod \ell$.*

*Proof.* If $f$ does not have a congruence at $0 \mod \ell$, then by Lemma 3.2, $f$ is not a member of its Tate cycle. By Proposition 4.9, $\omega(\Theta f) = (A+1)\ell + (B+1)$ is a low point. By Lemma 3.8, $B + 1 \equiv \frac{\ell+3}{2} \mod \ell$.

Similarly, if $f$ does have a congruence at $0 \mod \ell$, it is a low point of its Tate cycle by Proposition 4.9. Now by Lemma 3.8, $B \equiv \frac{\ell+3}{2} \mod \ell$. ☐

**Corollary 4.11.** *Suppose that $k \in \mathbb{Z}$, that $f \in M_k$, that $f$ vanishes only at the cusps, and that $\overline{\Theta f} \neq 0$. Suppose further that $\omega(f) = A\ell + B$ where $1 \leq B \leq \ell$. If $B \geq \frac{\ell+5}{2}$, then $\overline{f|U_\ell} \neq 0$.*

*Proof.* If $\overline{f|U_\ell} = 0$, then $f$ is a member of its Tate cycle. Proposition 4.9 implies $f$ is the lowest low point of its cycle, but Remark 3.4 shows that the lowest low point must have $1 \leq B \leq \frac{\ell+3}{2}$. ☐

The following two corollaries eliminate the chance for Ramanujan congruences at all but finitely many primes $\ell$ in half-integral weight forms vanishing only at the cusps, and in the inverses of integral-weight forms vanishing only at the cusps, respectively.

**Corollary 4.12.** *Let $\lambda \in \mathbb{N}$, let $f \in M_{\lambda+1/2}$, and suppose that $f$ vanishes only at the cusps. If $\lambda \geq 1$, then $f$ has no congruences for $\ell > 2\lambda+1$. If $\lambda = 0$, then $f$ is a scalar multiple of $\theta_0 = \sum q^{n^2}$ and clearly has congruences at $b \mod \ell$ where $\left(\frac{b}{\ell}\right) = -1$.*

*Proof.* In the case $\lambda \geq 2$, by Lemma 3.10 it suffices to show $f^{\ell+1} \in M_{(\lambda+1/2)(\ell+1)}$ has no congruences. Since $f^{\ell+1}$ vanishes only at the cusps and has integer weight, Proposition 4.9 implies that $\omega(f^{\ell+1}) = \left(\frac{\ell+1}{2}\right)(2\lambda+1)$. It follows that $\omega(f^{\ell+1}) \equiv \frac{\ell+2\lambda+1}{2} \mod \ell$. Now if $\ell > 2\lambda+1$, then it suffices to take $B = \frac{\ell+2\lambda+1}{2} < \ell$ in Corollaries 4.10 and 4.11.

If $\lambda = 0$ or 1, then $f$ is not a cusp form and Proposition 4.8 precludes congruences at $0 \mod \ell$. By Corollary 4.10, in the subcase $\lambda = 1$ there are no congruences at all. The subcase $\lambda = 0$ is obvious. $\square$

**Corollary 4.13.** *Let $k \in \mathbb{Z}$ and let $f \in M_k$. If $f$ vanishes only at the cusps, then $f^{-1}$ has no congruences for any prime $\ell > 2k+3$.*

*Proof.* By Lemma 3.10, the power series $f^{-1}$ has the same congruences as $f^{\ell-1} \in M_{k(\ell-1)}$. Since $f^{\ell-1}$ vanishes only at the cusps, Proposition 4.9 guarantees that its weight and filtration agree. That is, $\omega(f^{\ell-1}) = k(\ell-1) \equiv \ell - k \mod \ell$. Now if we assume that $\ell > 2k+3$, then we get $\frac{\ell+3}{2} < \ell - k < \ell$. Take $B = \ell - k$ in Corollaries 4.10 and 4.11. $\square$

The congruences of the inverse of a half-integral weight modular form are a bit trickier to find, but will always yield to an extension of the Ahlgren-Boylan technique which we illustrate in the following section.

## 4.4 Ramanujan congruences in weakly holomorphic forms with divisor supported at the cusps

Let $k \in \frac{1}{2}\mathbb{Z}$. Suppose $f \in M_k^!(\Gamma_1(4), \mathbb{Z})$ has divisor supported at the cusps. That is

$$\operatorname{div} f = m_0 \cdot [0] + m_\infty \cdot [\infty] + m_{1/2} \cdot \left[\frac{1}{2}\right]$$

where

$$m_0 = \operatorname{ord}_0 f \in \mathbb{Z},$$
$$m_\infty = \operatorname{ord}_\infty f \in \mathbb{Z},$$
$$m_{1/2} = \operatorname{ord}_{1/2} f \in \frac{1}{4}\mathbb{Z}.$$

In fact, there is some $c \in \mathbb{Z}$ such that

$$f = cE^{m_0} F^{m_\infty} \theta_0^{4m_{1/2}}. \tag{4.4.1}$$

Without loss of generality, we assume $c = 1$. Note that $k = 2m_0 + 2m_\infty + 2m_{1/2} \in \frac{1}{2}\mathbb{Z}$. Define

$$\delta := \begin{cases} 0 & \text{if } k \in \mathbb{Z} \\ 1 & \text{if } k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z} \end{cases} \tag{4.4.2}$$

and

$$k' := k + \frac{10-\delta}{2} \cdot \ell \in \mathbb{Z}_{>0}. \tag{4.4.3}$$

For a prime $\ell$, set

$$f_\ell := f\left(EF\theta_0^{2-\delta}\right)^\ell \in M_{k'}^!. \tag{4.4.4}$$

By Lemma 3.10, the forms $f$ and $f_\ell$ have the same Ramanujan congruences for the prime $\ell$. If $\ell > \max\left\{|m_0|, |m_\infty|, 4|m_{1/2}|\right\}$, then $f_\ell \in M_{k'}$ is holomorphic and our Tate cycle machinery is applicable.

We will now prove the finiteness of the primes $\ell$ for which $f$ has a Ramanujan congruence at $0 \mod \ell$ for three cases which depend on $k$.

**Theorem 4.14.** *Let $\ell \geq 5$ be prime, $1 \geq k \in \mathbb{Z}$, and $m_0, m_\infty, 4m_{1/2} \in \mathbb{Z}$. Let $f := E^{m_0}F^{m_\infty}\theta_0^{4m_{1/2}} \in M_k^!\left(\Gamma_1(4), \mathbb{Z}\right)$. If $f$ has a Ramanujan congruence at $0 \mod \ell$, then*

$$\ell \leq \max\left\{|m_0|, |m_\infty|, 4|m_{1/2}|, |2k-3|, 3\right\}.$$

*Proof.* Assume $\ell > \max\left\{|m_0|, |m_\infty|, 4|m_{1/2}|, |2k-3|, 3\right\}$ and let $f_\ell \in M_{k'}$ be as in (4.4.2-4.4.4). Then $k' = k + 5\ell$. Since $-\ell < 2k - 3$, we have

$$-\left(\frac{\ell-3}{2}\right) < k \leq 1$$

and hence

$$4\ell + \frac{\ell+3}{2} < k' \leq 5\ell + 1. \tag{4.4.5}$$

If $f$ has a Ramanujan congruence at $0 \mod \ell$ then by Lemma 3.10 so does $f_\ell$. Since $f_\ell$ has divisor supported at the cusps, by Proposition 4.9, we have that $f_\ell$ is the lowest low point of its Tate cycle and $\omega(f_\ell) = k'$. By Lemma 3.3 (1), we have $k' \neq 5\ell + 1$. Write $k' = A'\ell + B'$ where $1 \leq B' \leq \ell$. Then by (4.4.5) we have $\frac{\ell+3}{2} < B \leq \ell$, contrary to Remark 3.4. □

**Theorem 4.15.** *Let $\ell \geq 5$ be prime, $\frac{5}{2} \leq k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$, and $m_0, m_\infty, 4m_{1/2} \in \mathbb{Z}$. Let $f := E^{m_0}F^{m_\infty}\theta_0^{4m_{1/2}} \in M_k^!\left(\Gamma_1(4), \mathbb{Z}\right)$. If $f$ has a Ramanujan congruence at $0 \mod \ell$, then*

$$\ell \leq \max\left\{|m_0|, |m_\infty|, 4|m_{1/2}|, |2k-3|, 3\right\}.$$

*Proof.* Assume $\ell > \max\{|m_0|, |m_\infty|, 4|m_{1/2}|, |2k-3|, 3\}$ and let $f_\ell \in M_{k'}$ be as in (4.4.2-4.4.4). Then $k' = k + \frac{9}{2}\ell$. Since $2k - 3 < \ell$, we have

$$\frac{5}{2} \leq k < \frac{\ell+3}{2}$$

and hence

$$4\ell + \left(\frac{\ell+3}{2}\right) < k' \leq 5\ell + 1.$$

Continue as in the proof of Theorem 4.14. $\square$

The proof of the next theorem is more involved.

**Theorem 4.16.** *Let $\frac{3}{2} \geq k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$, and $m_0, m_\infty, 4m_{1/2} \in \mathbb{Z}$. Let*

$$f := E^{m_0} F^{m_\infty} \theta_0^{4m_{1/2}} \in M_k^! \left(\Gamma_1(4), \mathbb{Z}\right).$$

*Then there are only finitely many primes $\ell$ such that $f$ has a Ramanujan congruence at $0 \mod \ell$.*

Moreover, the provides a method to find all such $\ell$. The method is illustrated through several examples in the next section.

*Proof.* Assume $\ell > \max\{|m_0|, |m_\infty|, 4|m_{1/2}|, 5-2k, 3\}$ and let $f_\ell \in M_{k'}$ be as in (4.4.2-4.4.4). Here $k' = k + \frac{9}{2}\ell$. Assume that $f$ has a Ramanujan congruence at $0 \mod \ell$. If we also had $\Theta f \equiv 0 \mod \ell$ then we would have $f \equiv 0 \mod \ell$, contrary to the choice of $f$. By Lemma 3.10 we have that $f_\ell$ has a congruence at $0 \mod \ell$ and $\Theta f_\ell \not\equiv 0 \mod \ell$. Since $f_\ell$ has divisor supported at the cusps, by Proposition 4.9 we have that $f_\ell$ is the lowest low point of its Tate cycle and that $\omega(f_\ell) = k'$.

Since $-\ell < 2k - 5$, we have

$$-\left(\frac{\ell-5}{2}\right) < k \leq \frac{3}{2}$$

and hence

$$4\ell + \frac{5}{2} < k' \leq 4\ell + \frac{\ell+3}{2}. \tag{4.4.6}$$

Since $k' \in \mathbb{Z}$, we have $4\ell + 3 \leq k' \leq 4\ell + \frac{\ell+3}{2}$. Define $B'$ by the equation

$$k' = 4\ell + B'.$$

By Lemma 3.3, the other low point is

$$\omega\left(\Theta^{\ell-B'+1} f'\right) = 4\ell + \left(\ell + 3 - B'\right) = k' + (\ell + 3 - 2B').$$

By Proposition 4.7, there is $g \in M_{4\ell+(\ell+3-B')}$ such that $\Theta^{\ell-B'+1} f_\ell \equiv g \mod \ell$ and such that for all cusps $s$ we have $\operatorname{ord}_s g \geq \operatorname{ord}_s f_\ell$. In particular, $g/f_\ell \in M_{\ell+3-2B'}$. Now

$$2k' \equiv 2B' \equiv 2k \mod \ell.$$

Hence by (4.4.6), we have $B' = \frac{\ell+2k}{2}$ and so $\ell + 3 - 2B' = 3 - 2k \in 2\mathbb{Z}_{\geq 0}$. Therefore, $g/f_\ell$ is in the module $M_{3-2k}$ of rank $\frac{3-2k}{2} + 1$. The basis (2.2.2) shows that there exist $a_i \in \mathbb{Z}_{(\ell)}$ such that

$$\begin{aligned} \Theta^{\frac{\ell+2-2k}{2}} f_\ell &= \Theta^{\ell-B'+1} f_\ell \\ &\equiv g \mod \ell \\ &\equiv f_\ell \left(\frac{g}{f_\ell}\right) \mod \ell \\ &\equiv f_\ell \left(\sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i\right) \mod \ell. \end{aligned} \tag{4.4.7}$$

Since we work modulo $\ell$, we may actually take $a_i \in \mathbb{Z}$ in (4.4.7).

Write

$$f = \sum_{n=m_\infty}^{\infty} b_n q^n \in \mathbb{Z}[\![q]\!].$$

Since

$$f_\ell = f\,(EF\theta_0)^\ell = f\Big(q + O(q^2)\Big)^\ell = q^\ell f + O\left(q^{2\ell+m_\infty}\right), \tag{4.4.8}$$

by the usual rules for differentiation and (4.4.8), we have

$$\Theta^{\frac{\ell+2-2k}{2}} f_\ell \equiv q^\ell \Theta^{\frac{\ell+2-2k}{2}} f + O\left(q^{2\ell+m_\infty}\right) \mod \ell. \tag{4.4.9}$$

By (4.4.7) and (4.4.8), we have

$$\begin{aligned} \Theta^{\frac{\ell+2-2k}{2}} f_\ell &\equiv \left(q^\ell f + O\left(q^{2\ell+m_\infty}\right)\right)\left(\sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i\right) \mod \ell \\ &\equiv q^\ell f \sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i + O\left(q^{2\ell+m_\infty}\right) \mod \ell. \end{aligned} \tag{4.4.10}$$

Combine (4.4.9) and (4.4.10) to get

$$\Theta^{\frac{\ell+2-2k}{2}} f \equiv f \sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i + O\left(q^{\ell+m_\infty}\right) \mod \ell. \tag{4.4.11}$$

An essential part of the hypothesis is the assumption $k \leq \frac{3}{2}$. This permits the following manipulation:

$$\begin{aligned}
\Theta^{\frac{\ell+2-2k}{2}} f &= \Theta^{\frac{3-2k}{2}} \Theta^{\frac{\ell-1}{2}} f \\
&= \Theta^{\frac{3-2k}{2}} \sum_{n=m_\infty}^{\infty} b_n n^{\frac{\ell-1}{2}} q^n \\
&\equiv \Theta^{\frac{3-2k}{2}} \sum_{n=m_\infty}^{\infty} b_n \left(\frac{n}{\ell}\right) q^n \mod \ell \\
&\equiv \sum_{n=m_\infty}^{\infty} b_n n^{\frac{3-2k}{2}} \left(\frac{n}{\ell}\right) q^n \mod \ell.
\end{aligned} \tag{4.4.12}$$

This is a key point in the argument. The dependence on $\ell$ (for which there are infinitely many choices) in the number of applications of $\Theta$ has been exchanged for a dependence on finitely many Legendre symbols.

Invert $f$ as a Laurent series over $\mathbb{Z}$ and write $f^{-1} = \sum_{n=-m_\infty}^{\infty} c_n q^n \in \mathbb{Z}[\frac{1}{q}][\![q]\!]$. Hence by (4.4.11) and (4.4.12),

$$\sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i \equiv \left(\sum_{n=-m_\infty}^{\infty} c_n q^n\right) \left(\sum_{n=m_\infty}^{\infty} b_n n^{\frac{3-2k}{2}} \left(\frac{n}{\ell}\right) q^n\right) + O\left(q^\ell\right) \mod \ell \tag{4.4.13}$$

Truncate the series above to keep only the first $\frac{5-2k}{2}$ terms. The truncation of the right hand side of (4.4.13) will have finitely many Legendre symbols. For each tuple of possible choices for the Legendre symbols, there are unique integers $a_i$ which give equality in the truncation

$$\sum_{i=0}^{\frac{3-2k}{2}} a_i E^{\frac{3-2k}{2}-i} F^i = \left(\sum_{n=-m_\infty}^{\infty} c_n q^n\right) \left(\sum_{n=m_\infty}^{\infty} b_n n^{\frac{3-2k}{2}} \left(\frac{n}{\ell}\right) q^n\right) + O\left(q^{\frac{5-2k}{2}}\right).$$

Lemma 4.17 (to follow) proves that there must be some coefficient of $q$ at which $\Theta^{(\ell+2-2k)/2} f_\ell$ and $g$ from (4.4.7) are not equal, only congruent. The difference between these two coefficients must be divisible by $\ell$. (The prime $\ell$ must also satisfy the choices for the Legendre symbols.) Hence, there can only be finitely many primes $\ell$ such that $f$ has a Ramanujan congruence at $0 \mod \ell$. In the next section, the proofs of Theorems 1.6, 1.7, 1.10, and 1.2 give explicit examples of these types of calculations. $\square$

**Lemma 4.17.** *Let $3/2 \geq k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ and $\ell > 5 - 2k$ be prime. For any non-zero $f \in M_{k+\frac{9}{2}\ell}$ and non-zero $g \in M_{3-2k}$, we have $\Theta^{(\ell+2-2k)/2} f \neq g$.*

*Proof.* We adapt Atkin and Garvan's [6] Proposition 3.3 to suit our specific needs. The quasi-modular form $\Theta^{(\ell+2-2k)/2}f$ is of the form

$$\Theta^{(\ell+2-2k)/2}f(\tau)=\sum_{j=0}^{\frac{\ell+2-2k}{2}} f_j(\tau)E_2^j(\tau),$$

where $f_j\in M_{\frac{11}{2}\ell-k+2-2j}$. Assume $g(\tau)=\sum f_j(\tau)E_2^j(\tau)$ and apply $\tau\mapsto\frac{\tau}{4\tau+1}$. Recall $E_2(\frac{\tau}{4\tau+1})=(4\tau+1)^2E_2(\tau)-\frac{24i}{\pi}(4\tau+1)$. Letting $\alpha:=-\frac{24i}{\pi}$, we have for all $\tau\in\mathbb{H}$,

$$(4\tau+1)^{3-2k}g(\tau)=\sum_{j=0}^{\frac{\ell+2-2k}{2}}(4\tau+1)^{\frac{11}{2}\ell-k+2-2j}f_j(\tau)\Big((4\tau+1)^2E_2(\tau)+\alpha(4\tau+1)\Big)^j,$$

and hence for all $\tau\in\mathbb{H}$,

$$0=(4\tau+1)^{3-2k}g(\tau)-\sum_{m=5\ell+1}^{\frac{11\ell}{2}+2-k}(4\tau+1)^m\left(\sum_{\substack{0\le j\le\frac{\ell+2-2k}{2}\\0\le s\le j\\j=\frac{11\ell}{2}+2-k+s-m}}\binom{j}{s}\alpha^{j-s}f_j(\tau)E_2^s(\tau)\right).$$

Since $g(\tau),f_j(\tau)$ and $E_2(\tau)$ are all invariant under $\tau\mapsto\tau+1$, the polynomial

$$z^{3-2k}g(\tau)-\sum_{m=5\ell+1}^{\frac{11\ell}{2}+2-k}z^m\left(\sum_{\substack{0\le j\le\frac{\ell+2-2k}{2}\\0\le s\le j\\j=\frac{11\ell}{2}+2-k+s-m}}\binom{j}{s}\alpha^{j-s}f_j(\tau)E_2^s(\tau)\right)$$

has infinitely many zeros $z=4\tau+1,4\tau+5,4\tau+9,\dots$. Therefore the coefficients must be zero. By the assumption $\ell>5-2k$, we have $3-2k<5\ell+1$ and hence the index $m$ is never $3-2k$. Hence $g(\tau)=0$ contrary to assumption. □

We now turn to Ramanujan congruences at $b\not\equiv 0 \mod \ell$.

**Lemma 4.18.** *Let $\ell\ge 5$ be prime, $b\not\equiv 0 \mod \ell$, $k\in\frac{1}{2}\mathbb{Z}$, and $m_0,m_\infty,4m_{1/2}\in\mathbb{Z}$. Let $f:=E^{m_0}F^{m_\infty}\theta_0^{4m_{1/2}}\in M_k^!(\Gamma_1(4),\mathbb{Z})$. Then $f$ has a Ramanujan congruence at $b \mod \ell$ only if:*

- *$f$ also has a Ramanujan congruence at $0 \mod \ell$ and $\ell\mid 2k(2k-3)$, or*
- *$f$ does not have a Ramanujan congruence at $0 \mod \ell$ and $\ell\mid 2k(2k-1)$.*

*Moreover, if $k=0$ and $f$ has a Ramanujan congruence at $b \mod \ell$, then $\ell$ divides $\gcd\big(m_0,m_\infty,4m_{1/2}\big)$.*

*Proof.* Assume $f$ has a Ramanujan congruence at $b \mod \ell$. By Lemma 3.10, we know that

$$g := fE^{|m_0|\ell}F^{|m_\infty|\ell}\theta_0^{4|m_{1/2}|\ell} \in M^!_{k+\ell\left(|m_0|+|m_\infty|+4|m_{1/2}|\right)}$$

has the same Ramanujan congruences modulo $\ell$ as $f$. Note that

$$g = E^{|m_0|(\ell\pm1)}F^{|m_\infty|(\ell\pm1)}E^{4|m_{1/2}|(\ell\pm1)}$$

where the signs of the $\pm1$ terms depend on the signs of the corresponding $m$. Thus

$$g \in M_{k+\ell\left(|m_0|+|m_\infty|+4|m_{1/2}|\right)}$$

is indeed holomorphic and of integral weight. For convenience, denote the weight of $g$ by

$$k' := k + \ell\left(|m_0| + |m_\infty| + 4|m_{1/2}|\right).$$

Notice that $g \equiv q^{|m_\infty|(\ell\pm1)} + \cdots \not\equiv 0 \mod \ell$. Thus, as in the proof of Proposition 4.9, by Remark 2.4 and Corollary 4.6 we deduce that $\omega(g) \geq k'$. Clearly $k' \geq \omega(g)$ and so $\omega(g) = k'$. If $\ell | k'$ then $\ell | 2k$ and the bulleted conclusions are true. Thus, we assume $\ell \nmid k' = \omega(g)$. Hence by Lemma 2.10, we deduce that $\Theta g \not\equiv 0 \mod \ell$ and Corollary 4.10 applies. Since

$$k' \equiv \frac{\ell+3}{2} \mod \ell \iff 2k' \equiv 2k \equiv \ell + 3 \mod \ell \iff \ell \mid 2k-3,$$

and

$$k' \equiv \frac{\ell+1}{2} \mod \ell \iff 2k' \equiv 2k \equiv \ell + 1 \mod \ell \iff \ell \mid 2k-1,$$

the bulleted conclusions follow by Corollary 4.10.

If $k = 0$, then $k' \equiv 0 \mod \ell$. If $\Theta g \not\equiv 0 \mod \ell$, then by Corollary 4.10 we have $\ell | 3$, contrary to choice of $\ell \geq 5$. Thus $\Theta g \equiv 0 \mod \ell$. However, by (2.4.3) this implies that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have $\Theta(g|\gamma) \equiv 0 \mod \ell$. Hence,

$$\Theta\left(g\Big|_{k'}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\right) \equiv \Theta\left(q^{m_{1/2}} + \cdots\right) \equiv 0 \mod \ell,$$
$$\Theta\left(g\Big|_{k'}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) \equiv \Theta\left(q^{\frac{m_0}{4}} + \cdots\right) \equiv 0 \mod \ell.$$

Thus $\ell$ divides each of $m_0$, $m_\infty$, and $4m_{1/2}$. □

**Corollary 4.19.** *Let $\frac{1}{2} \neq k \in \frac{1}{2}\mathbb{Z}$. Suppose $1 \neq f := E^{m_0}F^{m_\infty}\theta_0^{4m_{1/2}} \in M^!_k(\Gamma_1(4), \mathbb{Z})$. Then there are only finitely many primes $\ell$ for which $f$ has a Ramanujan congruence at some $b \not\equiv 0 \mod \ell$.*

*Proof.* Suppose $k \neq 0, \frac{1}{2}, \frac{3}{2}$. If $\ell$ is prime and $f$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$, then by Lemma 4.18, we deduce that $\ell$ is one of the finitely many prime divisors of $6k(2k-3)(2k-1) \neq 0$.

Suppose $k = 0$. Since $f \neq 1$, at least one of $m_0$, $m_\infty$, and $m_{1/2}$ is non-zero. Hence

$$\gcd\left(m_0, m_\infty, 4m_{1/2}\right) \in \mathbb{Z}\backslash\{0\}.$$

If $\ell$ is prime and $f$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$, then by Lemma 4.18 we deduce that $\ell$ is one of the finitely many prime divisors of $6\gcd\left(m_0, m_\infty, 4m_{1/2}\right)$.

Suppose $k = 3/2$. By Theorem 4.16, there are only finitely many primes for which there is a Ramanujan congruence at $0 \mod \ell$. For any other prime $\ell$, if $f$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$ then by Lemma 4.18 we deduce that $\ell$ is one of the finitely many prime divisors of $6k(2k-1)$. □

## 4.5 Proofs of Theorems 1.6-1.10 and 1.3

*Proof of Theorem 1.6.* The cusp forms of least weight on $\Gamma_1(4)$ are scalar multiples of

$$f := \theta_0 F E \in S_{9/2}(\Gamma_1(4)). \tag{4.5.1}$$

By Lemma 3.10 the series $f^{-1}$ will have a congruence at $b \mod \ell$ if and only if $f^{\ell-1}$ has one at $b \mod \ell$. Since $\omega(f^{\ell-1}) = \frac{9}{2}(\ell-1) \equiv \frac{\ell-9}{2} \mod \ell$, by Corollary 4.10 there can be congruences at $b \not\equiv 0 \mod \ell$ only if $\ell = 3$ or $5$.

In the first case, the Sturm bound [44] implies that only a short computation is needed to see that $f^2 \equiv -\Theta f^2 \mod 3$ and so $f^2 \equiv \Theta^2 f^2 \mod 3$. By Lemma 3.5, $f^{-1}$ has congruences at 0 mod 3 and $1 \mod 3$. In the second case, a finite computation shows that $f^{-1}$ only has congruences for $\ell = 5$ at $2 \mod 5$ and $3 \mod 5$. Although our machinery does not apply for $\ell = 2$, a short calculation shows $f^{-1}$ has a congruence at $0 \mod 2$. An inspection of the coefficients of $q^7, q^{13}$ and $q^{22}$ in $f^{-1}$ shows there are no congruences for $\ell = 7, 11, 13$. We now move on to $\ell \geq 17$.

Suppose $f^{\ell-1}$ has a congruence at $0 \mod \ell$. The rest of this proof follows the proof of Theorem 4.16 and so we only provide the explicit calculations. Now $f^{\ell-1}$ is a low point of its Tate cycle and, by Lemma 3.3(5), the other low point is $\omega(\Theta^{\frac{\ell+11}{2}} f^{\ell-1}) = \omega(f^{\ell-1}) + 12$. Hence

$$\Theta^{\frac{\ell+11}{2}} f^{\ell-1} \equiv f^{\ell-1}\left(\sum_{i=0}^{6} a_i E^{6-i} F^i\right) \mod \ell,$$

implying

$$\Theta^{\frac{\ell+11}{2}} f^{-1} \equiv f^{-1}\left(\sum_{i=0}^{6} a_i E^{6-i} F^i\right) + O(q^{\ell-1}) \mod \ell. \tag{4.5.2}$$

Invert $f$ as a power series with integer coefficients to get

$$f^{-1} = q^{-1} + 6 + 24q + 80q^2 + 240q^3 + 660q^4 + 1696q^5 + 4128q^6 + 9615q^7 + 21560q^8 + O(q^9).$$

We compute

$$\begin{aligned}\Theta^{\frac{\ell+11}{2}} f^{-1} &\equiv \left(\frac{\cdot}{\ell}\right) \otimes \Theta^6 f^{-1} \mod \ell \\ &\equiv \left(\frac{-1}{\ell}\right) q^{-1} + 24q + \left(\frac{2}{\ell}\right) 5120q^2 + \left(\frac{3}{\ell}\right) 174960q^3 + 2703360q^4 \\ &\quad + \left(\frac{5}{\ell}\right) 26500000q^5 + O(q^6) \mod \ell.\end{aligned} \tag{4.5.3}$$

For each of the $2^4$ choices of signs for the Legendre symbols, a computer can easily compute the integers $a_i$ in Equation (4.5.2). Comparing the coefficients of $q^6, q^8$, and $q^9$ in Equation (4.5.2) leads to a contradiction. For example, suppose $\ell$ satisfies $\left(\frac{-1}{\ell}\right) = \left(\frac{2}{\ell}\right) = -\left(\frac{3}{\ell}\right) = -\left(\frac{5}{\ell}\right) = 1$. One computes that $a_0 = 1$, $a_1 = 42$, $a_2 = 612$, $a_3 = 8656$, $a_4 = -76608$, $a_5 = 1074912$, $a_6 = -15155584$. Hence the right side of Equation (4.5.2) is

$$\begin{aligned}&q^{-1} + 24q + 5120q^2 - 174960q^3 + 2703360q^4 - 26500000q^5 - 29891712q^6 - 911605665q^7 \\ &\quad - 2744268800q^8 - 18190442184q^9 - 59662291200q^{10} - 254616837584q^{11} + O(q^{12}),\end{aligned}$$

whereas the left side may be computed as in Equation (4.5.3):

$$\begin{aligned}&q^{-1} + 24q + 5120q^2 - 174960q^3 + 2703360q^4 - 26500000q^5 - 192595968q^6 \pm 1131195135q^7 \\ &\quad + 5651824640q^8 + 24858684216q^9 - 98592000000q^{10} \pm 358875741136q^{11} + O(q^{12}).\end{aligned}$$

The $\pm$ come from $\left(\frac{7}{\ell}\right)$ and $\left(\frac{11}{\ell}\right)$. Since these power series are congruent modulo $\ell$, so are the coefficients of $q^6$ and $q^8$. But $-29891712 \equiv -192595968 \mod \ell$ implies $\ell = 2, 3, 11, 13$ or $2963$, while $-2744268800 \equiv 5651824640 \mod \ell$ implies $\ell = 2, 5, 7$ or $117133$. Since we've assumed $\ell \geq 17$, we have reached a contradiction. $\square$

*Proof of Theorem 1.7.* Let $g = \theta_0 E^2 F \in S_{13/2}(4)$. Now $g^{-1}$ will have a congruence at $b \mod \ell$ if and only if $g^{\ell-1}$ does. Since $\omega(g^{\ell-1}) \equiv \frac{\ell-13}{2} \mod \ell$, Corollary 4.10 implies there can only be congruences with $b \not\equiv 0 \mod \ell$ if $\ell = 2$ or $7$. For $\ell = 7$, one checks that $\Theta^4 g^6 \equiv -\Theta g^6$ and by Lemma 3.5, $g^6$ and hence $g^{-1}$ have congruences at $1, 2, 4 \mod 7$.

Elementary calculations show no congruences for $0 \mod \ell$ when $3 \leq \ell \leq 13$. For $l \geq 17$, if $g^{\ell-1}$ has a congruence at $0 \mod \ell$, then it is the lowest low point of its Tate cycle and the other low point is $\omega(\Theta^{\frac{\ell+15}{2}} g^{\ell-1}) = \omega(g^{\ell-1}) + 16$. Analogously to Theorem 1.6, we have

$$\Theta^{\frac{\ell+15}{2}} g^{-1} \equiv g^{-1} \left( \sum_{i=0}^{8} b_i E^{8-i} F^i \right) + O(q^\ell) \mod \ell.$$

In the case where $\left(\frac{-1}{\ell}\right) = \left(\frac{2}{\ell}\right) = \left(\frac{3}{\ell}\right) = \left(\frac{5}{\ell}\right) = \left(\frac{7}{\ell}\right) = -1$, solving for the $b_i$ yields $b_0 = -1$, $b_1 = -50$, $b_2 = -788$, $b_3 = -175024$, $b_4 = -26446064$, $b_5 = 539142592$, $b_6 = -13397175040$, $b_7 = 271206416128$, and $b_8 = -5171059369600$. Examining the coefficients of $q^8, \dots, q^{12}$ in both sides of the previous equivalence precludes all possible primes $\ell \geq 17$. The situation for each of the $2^5$ choices for the Legendre symbols is similar. □

*Proof of Theorem 1.8.* The prime 3 may be checked by direct computation and so we let $\ell \geq 5$ be prime. Recall $f_{\overline{P}}$ from Example 4.1. Since $\operatorname{ord}_\infty f_{\overline{P}} = \operatorname{ord}_\infty \overline{f_{\overline{P}}} = 2\ell\delta_\ell \equiv 0 \mod \ell$, by Proposition 4.8 there is no congruence at $0 \mod \ell$. Since $\omega(f_{\overline{P}}) \equiv \frac{\ell-1}{2} \mod \ell$, by Corollary 4.10 there can only be congruences at $a \mod \ell$ if $\frac{\ell-1}{2} \equiv \frac{\ell+1}{2} \mod \ell$ which never happens for $\ell \geq 5$. □

*Proof of Theorem 1.9.* Let $\ell \geq 5$ be prime. Recall $f_{\overline{PP}}$ from Example 4.2. Since $\operatorname{ord}_\infty f_{\overline{PP}} = \operatorname{ord}_\infty \overline{f_{\overline{PP}}} = 4\ell\delta_\ell \equiv 0 \mod \ell$, by Proposition 4.8 there is no congruence at $0 \mod \ell$. Since $\omega(f_{\overline{PP}}) \equiv -1 \mod \ell$, by Corollary 4.10 there can only be congruences at $a \mod \ell$ if $-1 \equiv \frac{\ell+1}{2} \mod \ell$ which never happens for $\ell \geq 5$. □

*Proof of Theorem 1.10.* Recall $f_{CD}$ from Example 4.3. Since $f_{CD}$ vanishes only at the cusps, by Proposition 4.9, $\omega(f) = \frac{(\ell^2-1)(3\ell-1)}{2} \equiv \frac{\ell+1}{2} \mod \ell$.

The fact that $\omega(f_{CD}) \equiv \frac{\ell+1}{2} \mod \ell$ is unfortunate. This is the only time that Corollary 4.10 does not rule out congruences at $b \not\equiv 0 \mod \ell$. However, Lemma 3.5 guarantees that if $CD(z)$ has a congruence at $b \mod \ell$, then in fact $CD(z)$ has a congruence at all $c \mod \ell$ such that $\left(\frac{b+\delta}{\ell}\right) = \left(\frac{c+\delta}{\ell}\right)$.

We now apply the method of the proof of Theorem 4.16 to find all $\ell$ such that $f_{CD}$ has a congruence at $0 \mod \ell$. Assume $f_{CD}|U_\ell \equiv 0 \mod \ell$. Then $f_{CD}$ is a low point of its Tate cycle and by Lemma 3.3, the other low point has filtration $\omega(f_{CD}) + 2$. Hence by Proposition 4.7, $(\Theta^{\frac{\ell+1}{2}} f_{CD})/f_{CD} \in \overline{M}_2$. Since

$$f_{CD} \equiv q^{\frac{\ell^3-\ell}{8}} \left(q^\delta \prod \frac{(1-q^n)^3}{(1-q^{2n})^2}\right) + O\left(q^{\ell+\delta+\frac{\ell^3-\ell}{8}}\right) \mod \ell,$$

and since $\Theta$ is linear and satisfies the product rule, we obtain

$$\Theta^{\frac{\ell+1}{2}} f_{CD} \equiv q^{\frac{\ell^3-\ell}{8}} \Theta^{\frac{\ell+1}{2}} \left(q^\delta \prod \frac{(1-q^n)^3}{(1-q^{2n})^2}\right) + O\left(q^{\ell+\delta+\frac{\ell^3-\ell}{8}}\right) \mod \ell.$$

Thus $(\Theta^{\frac{\ell+1}{2}} f_{CD})/f_{CD}$ is congruent to

$$\begin{aligned} &\Theta^{\frac{\ell+1}{2}}(q^\delta - 3q^{\delta+1} + 2q^{\delta+2} + \cdots) \cdot \left(q^\delta - 3q^{\delta+1} + 2q^{\delta+2} + \cdots\right)^{-1} \mod \ell \\ &\equiv \delta^{\frac{\ell+1}{2}} + \left(3\delta^{\frac{\ell+1}{2}} - 3(\delta+1)^{\frac{\ell+1}{2}}\right) q + \\ &\qquad \left(7\delta^{\frac{\ell+1}{2}} - 9(\delta+1)^{\frac{\ell+1}{2}} + 2(\delta+2)^{\frac{\ell+1}{2}}\right) q^2 + \cdots \mod \ell. \end{aligned} \tag{4.5.4}$$

Since this is congruent to a weight two form, and since the basis form $F = q + 4q^3 + \cdots$ , lacks a $q^2$ term, we compare the coefficients of $q^2$ in $\delta^{\frac{\ell+1}{2}} E = \delta^{\frac{\ell+1}{2}}(1 - q + 24q^2 + \cdots)$ and in Equation (4.5.4) to deduce $24\delta^{\frac{\ell+1}{2}} \equiv 7\delta^{\frac{\ell+1}{2}} - 9(\delta+1)^{\frac{\ell+1}{2}} + 2(\delta+2)^{\frac{\ell+1}{2}} \mod \ell$. Multiplying by $24^{\frac{\ell+1}{2}}$, we find

$$-17\left(\frac{-1}{\ell}\right) \equiv -207\left(\frac{23}{\ell}\right) + 94\left(\frac{47}{\ell}\right) \mod \ell. \tag{4.5.5}$$

That is, $17 \equiv \pm 207 \pm 94 \mod \ell$. If $\ell \geq 5$, then this implies that $\ell$ is one of $5, 13, 53$ and $71$. However, only 5 and 53 satisfy (4.5.5). By the equivalences above, $f$ having a congruence at $0 \mod \ell$ is equivalent to the crank difference function having a congruence at $b \mod \ell$ with $24b \equiv 1 \mod \ell$. For the primes 5 and 53, this means $b = 4$ and $42$, respectively. We have recovered the congruence at $4 \mod 5$ of [15]. Calculations reveal that the coefficient of $q^{42}$ precludes a congruence at $42 \mod 53$. $\square$

*Proof of Theorem 1.2.* Calculations show there is no congruence for $\ell = 3$. Thus we take $\ell \geq 5$ prime. Recall $f_{C\Phi_2}$ from Example 4.4. Since $f_{C\Phi_2}$ vanishes only at the cusps, Proposition 4.9 implies that $\omega(f_{C\Phi_2}) = \frac{(\ell-1)(\ell+1)^2}{2} \equiv \frac{\ell-1}{2} \mod \ell$. By Corollary 4.10, there are no congruences at $b \not\equiv 0 \mod \ell$ when $\ell \geq 5$.

Suppose $f_{C\Phi_2}$ has a congruence at $0 \mod \ell$. Then by Proposition 4.9, $f_{C\Phi_2}$ is a low point of its Tate cycle and by Lemma 3.3 the other low point has filtration $\omega(f_{C\Phi_2}) + 4$. Hence $(\Theta^{\frac{\ell+3}{2}} f_{C\Phi_2})/f_{C\Phi_2} \in \overline{M}_4$ by Proposition 4.7. We compute

$$\begin{aligned}
f_{C\Phi_2} &\equiv q^{2\delta}\theta_0(z)\prod(1-q^{2n})^{-2} + O\left(q^{\ell+2\delta}\right) \mod \ell \\
&\equiv q^{2\delta} + 4q^{2\delta+1} + 9q^{2\delta+2} + 20q^{2\delta+3} + \cdots \mod \ell \\
f_{C\Phi_2}^{-1} &\equiv q^{-2\delta} - 4q^{-2\delta+1} + 7q^{-2\delta+2} - 12q^{-2\delta+3} + \cdots \mod \ell
\end{aligned}$$

and

$$\Theta^{\frac{\ell+3}{2}} f_{C\Phi_2} \equiv (2\delta)^{\frac{\ell+3}{2}} q^{2\delta} + 4(2\delta+1)^{\frac{\ell+3}{2}} q^{2\delta+1} + 9(2\delta+2)^{\frac{\ell+3}{2}} q^{2\delta+2} + 20(2\delta+3)^{\frac{\ell+3}{2}} q^{2\delta+3} + \cdots \mod \ell.$$

Hence we compute

$$\begin{aligned}
\left(\Theta^{\frac{\ell+3}{2}} f_{C\Phi_2}\right) f_{C\Phi_2}^{-1} \equiv (2\delta)^{\frac{\ell+3}{2}} &+ \left(-4(2\delta)^{\frac{\ell+3}{2}} + 4(2\delta+1)^{\frac{\ell+3}{2}}\right) q \\
&+ \left(7(2\delta)^{\frac{\ell+3}{2}} - 16(2\delta+1)^{\frac{\ell+3}{2}} + 9(2\delta+2)^{\frac{\ell+3}{2}}\right) q^2 \\
&+ \left(-12(2\delta)^{\frac{\ell+3}{2}} + 28(2\delta+1)^{\frac{\ell+3}{2}} - 36(2\delta+2)^{\frac{\ell+3}{2}} + 20(2\delta+3)^{\frac{\ell+3}{2}}\right) q^3 \\
&+ \cdots \mod \ell.
\end{aligned} \tag{4.5.6}$$

Recalling our basis (4.2.1), we conclude

$$\left(\Theta^{\frac{\ell+3}{2}} f_{C\Phi_2}\right) f_{C\Phi_2}^{-1} \equiv (2\delta)^{\frac{\ell+3}{2}} E^2 + \left(12(2\delta)^{\frac{\ell+3}{2}} + 4(2\delta+1)^{\frac{\ell+3}{2}}\right) EF$$
$$+ \left(-9(2\delta)^{\frac{\ell+3}{2}} + 16(2\delta+1)^{\frac{\ell+3}{2}} + 9(2\delta+2)^{\frac{\ell+3}{2}}\right) F^2. \qquad (4.5.7)$$

Multiplying the coefficients of $q^3$ in both (4.5.6) and (4.5.7) by $12^{\frac{\ell+3}{2}}$ leads to

$$0 \equiv 100(-1)^{\frac{\ell+3}{2}} - 84(11)^{\frac{\ell+3}{2}} - 36(23)^{\frac{\ell+3}{2}} + 20(35)^{\frac{\ell+3}{2}} \mod \ell$$
$$\equiv 100\left(\frac{-1}{\ell}\right) - 10164\left(\frac{11}{\ell}\right) - 19044\left(\frac{23}{\ell}\right) + 24500\left(\frac{35}{\ell}\right) \mod \ell \qquad (4.5.8)$$
$$\equiv \pm 100 \pm 10164 \pm 19044 \pm 24500 \mod \ell. \qquad (4.5.9)$$

The only primes $\ell \geq 5$ satisfying (4.5.9) are $5, 13, 19, 31, 59, 97, 131, 601,$ and $6701$. It is easily checked that only $\ell = 5$ satisfies (4.5.8). That is, we have recovered the congruence (1.1.2) and proved there are no others. $\square$

# CHAPTER 5

# RAMANUJAN CONGRUENCES IN SIEGEL AND JACOBI FORMS

This chapter represents joint work with Olav Richter. It appears in essentially the same form in [21], although Theorem 1.12 and its proof have been rephrased.

## 5.1 Congruences and filtrations of Jacobi forms

A *Jacobi form* on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $\phi(\tau, z) : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the transformations

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \, e^{\frac{2\pi imcz}{c\tau+d}} \, \phi(\tau, z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi im\left(\lambda^2\tau+2\lambda z\right)} \phi(\tau, z) \qquad \forall \lambda, \mu \in \mathbb{Z}$$

and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2 \le 4mn}} c(n, r) e^{2\pi i(n\tau+rz)}.$$

The numbers $k$ and $m$ are non-negative integers called the *weight* and *index*, respectively. Write $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$. Let $J_{k,m}$ be the vector space of Jacobi forms of even weight $k$ and index $m$. For details on Jacobi forms, see Eichler and Zagier [22].

The theory of reduced Jacobi forms is analogous to the theory of reduced modular forms that we have been using thus far. The heat operator

$$L_m := (2\pi i)^{-2}\left(8\pi im\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)$$

is a natural tool in the theory of Jacobi forms and plays an important role in this section. In particular, if $\phi = \sum c(n, r) q^n \zeta^r$, then

$$L_m\phi := L_m(\phi) = \sum (4nm - r^2) c(n, r) q^n \zeta^r. \tag{5.1.1}$$

Set

$$\widetilde{J}_{k,m} := \left\{\phi \mod \ell \,:\, \phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}_{(\ell)}[\zeta,\zeta^{-1}][[q]]\right\},$$

where $\mathbb{Z}_{(\ell)} := \mathbb{Z}_\ell \cap \mathbb{Q}$ denotes the local ring of $\ell$-integral rational numbers. If $\phi \in \widetilde{J}_{k,m}$, then we denote its filtration modulo $\ell$ by

$$\Omega\big(\phi\big) := \inf\left\{k \,:\, \phi \mod \ell \in \widetilde{J}_{k,m}\right\}.$$

Recall the following facts on Jacobi forms modulo $\ell$:

**Proposition 5.1** (Sofer [43]). *Let $\phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$ and $\psi(\tau,z) \in J_{k',m'} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$ such that $0 \not\equiv \phi \equiv \psi \mod \ell$. Then $k \equiv k' \mod \ell-1$ and $m = m'$.*

**Proposition 5.2** ([40]). *If $\phi(\tau,z) \in J_{k,m} \cap \mathbb{Z}[\zeta,\zeta^{-1}][[q]]$, then $L_m\phi \mod \ell \in \widetilde{J}_{k+\ell+1,m}$. Moreover, we have*

$$\Omega\left(L_m\phi\right) \le \Omega\left(\phi\right) + \ell + 1,$$

*with equality if and only if $\ell \nmid \left(2\Omega\left(\phi\right) - 1\right)m$.*

We will now explore Ramanujan congruences for Jacobi forms.

**Definition 5.3.** For $\phi(\tau,z) = \sum c(n,r)q^n\zeta^r \in \widetilde{J}_{k,m}$, we say that $\phi$ has a Ramanujan congruence at $b \mod \ell$ if $c(n,r) \equiv 0 \mod \ell$ whenever $4nm - r^2 \equiv b \mod \ell$.

Equation (5.1.1) implies that a Jacobi form $\phi$ has a Ramanujan congruence at $0 \mod \ell$ if and only if $L_m^{\ell-1}\phi \equiv \phi \mod \ell$. More generally, $\phi$ has a Ramanujan congruence at $b \mod \ell$ if and only if

$$L_m^{\ell-1}\left(q^{-\frac{b}{4m}}\phi\right) \equiv q^{-\frac{b}{4m}}\phi \mod \ell.$$

Ramanujan congruences at $0 \mod \ell$ for Jacobi forms have been considered in [39, 40]. The following proposition determines when Ramanujan congruences at $b \not\equiv 0 \mod \ell$ for Jacobi forms exist. Compare the next proposition with Lemma 3.5.

**Proposition 5.4.** *Let $\phi \in \widetilde{J}_{k,m}$ and $b \not\equiv 0 \mod \ell$. Then $\phi$ has a Ramanujan congruence at $b \mod \ell$ if and only if $L_m^{\frac{\ell+1}{2}}\phi \equiv -\left(\frac{b}{\ell}\right)L_m\phi \mod \ell$.*

*Proof.* If $\phi \in \mathbb{Z}_{(\ell)}[\zeta,\zeta^{-1}][[q]]$ and $f \in \mathbb{Z}_{(\ell)}[[q]]$, then $L_m(f\phi) = L_m(f)\phi + fL_m(\phi)$. This implies

$$\begin{aligned}
L_m^{\ell-1}\left(q^{-\frac{b}{4m}}\phi\right) &= \sum_{i=0}^{\ell-1}\binom{\ell-1}{i}L_m^{\ell-1-i}\left(q^{-\frac{b}{4m}}\right)L_m^i\phi \\
&= \sum_{i=0}^{\ell-1}\binom{\ell-1}{i}\left(-b\right)^{\ell-1-i}q^{-\frac{b}{4m}}L_m^i\phi \\
&\equiv q^{-\frac{b}{4m}}\sum_{i=0}^{\ell-1}b^{\ell-1-i}L_m^i\phi \mod \ell.
\end{aligned}$$

In particular, $\phi$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$ if and only if

$$0 \equiv \sum_{i=1}^{\ell-1} b^{\ell-1-i} L_m^i \phi \mod \ell. \tag{5.1.2}$$

We now rewrite the $L_m^i \phi$ appearing in (5.1.2) using a standard decomposition of even weight Jacobi forms. See §8 and §9 of [22] for full details and also for the corresponding result for Jacobi forms of odd weight. Every even weight $\phi \in J_{k,m}$ can be written as

$$\phi = \sum_{j=0}^{m} f_j (\phi_{-2,1})^j (\phi_{0,1})^{m-j}, \tag{5.1.3}$$

where

$$\phi_{-2,1}(\tau, z) := \left(\zeta - 2 + \zeta^{-1}\right) + \left(-2\zeta^2 + 8\zeta - 12 + 8\zeta^{-1} - 2\zeta^{-2}\right) q + \cdots$$

and

$$\phi_{0,1}(\tau, z) := \left(\zeta + 10 + \zeta^{-1}\right) + \left(10\zeta^2 - 64\zeta + 108 - 64\zeta^{-1} + 10\zeta^{-2}\right) q + \cdots$$

are weak Jacobi forms with integer coefficients of index 1 and weights $-2$ and 0, respectively, and where each $f_j \in M_{k+2j}(\Gamma_1(1), \mathbb{C})$ is uniquely determined. For any $m \geq 1$, the set $\mathcal{T} := \left\{\phi_{-2,1}^j \phi_{0,1}^{m-j}\right\}_{j=0}^m$ is linearly independent over $\mathbb{F}_\ell$. In fact, the coefficients of $q^0$ of the elements of $\mathcal{T}$ are linearly independent for the following reason: Let $X := \zeta - 2 + \zeta^{-1}$. It suffices to show that $\mathcal{S} := \left\{X^{m-j}(X+12)^j\right\}_{j=0}^m$ is linearly independent over $\mathbb{F}_\ell$. But $X^{m-j}(X+12)^j = X^m + \cdots + 12^j X^{m-j}$, and one finds that $\mathcal{S}$ is linearly independent over $\mathbb{F}_\ell$ since 12 is invertible. Returning to (5.1.3), if $\phi$ has $\ell$-integral rational coefficients, then so do all of the $f_j$'s, since otherwise there is some $t \geq 1$ such that $0 \equiv \ell^t \phi \equiv \sum_{j=0}^m \left(\ell^t f_j\right) (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \mod \ell$ is a non-trivial linear independence relation for $\mathcal{T}$, contrary to what we have just shown.

By Proposition 5.2, for every $i$ there exists $\psi_i \in J_{k+i(\ell+1),m}$ such that $L_m^i \phi \equiv \psi_i \mod \ell$. Hence there exist $F_{i,j} \in M_{k+i(\ell+1)+2j}\left(\Gamma_1(1), \mathbb{Z}_{(\ell)}\right)$ such that

$$L_m^i \phi \equiv \psi_i \equiv \sum_{j=0}^{m} F_{i,j} (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \mod \ell$$

and hence (5.1.2) is equivalent to

$$0 \equiv \sum_{j=0}^{m} \left(\sum_{i=1}^{\ell-1} b^{\ell-1-i} F_{i,j}\right) (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \mod \ell.$$

By the linear independence of the elements of $\mathcal{T}$, we deduce that (5.1.2) is equivalent to

$$\sum_{i=1}^{\ell-1} b^{\ell-1-i} F_{i,j} \equiv 0 \mod \ell$$

for every $j$. Elliptic modular forms modulo $\ell$ have a natural direct sum decomposition (see Section 3 of [45] or Theorem 2 of [41]) graded by their weights modulo $\ell - 1$. Thus (5.1.2) is equivalent to

$$0 \equiv b^{\ell-1-i} F_{i,j} + b^{(\ell-1)/2-i} F_{i+(\ell-1)/2,j} \mod \ell$$

and hence also

$$F_{i+(\ell-1)/2,j} \equiv -\left(\frac{b}{\ell}\right) F_{i,j} \mod \ell$$

for all $0 \leq j \leq m$ and $1 \leq i \leq \frac{\ell-1}{2}$. This implies, for all $1 \leq i \leq \frac{\ell-1}{2}$,

$$\begin{aligned} L_m^{i+\frac{\ell-1}{2}} \phi &\equiv \sum_{j=0}^{m} F_{i+\frac{\ell-1}{2},j} (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \\ &\equiv \sum_{j=0}^{m} -\left(\frac{b}{\ell}\right) F_{i,j} (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \\ &\equiv -\left(\frac{b}{\ell}\right) L_m^i \phi \mod \ell. \end{aligned}$$

We conclude that

$$L_m^{\frac{\ell+1}{2}} \phi \equiv -\left(\frac{b}{\ell}\right) L_m \phi \mod \ell,$$

which completes the proof. $\square$

By (5.1.1), $L_m^\ell \phi \equiv L_m \phi \mod \ell$. We call $L_m \phi, L_m^2 \phi, \ldots, L_m^{\ell-1} \phi$ the *heat cycle* of $\phi$ and we say that $\phi$ is in its own heat cycle whenever $L_m^{\ell-1} \phi \equiv \phi \mod \ell$. Assume $L_m \phi \not\equiv 0 \mod \ell$ and $\ell \nmid m$. By Proposition 5.2, applying $L_m$ to $\phi$ increases the filtration of $\phi$ by $\ell + 1$ except when $\Omega(\phi) \equiv \frac{\ell+1}{2}$ mod $\ell$. If $\Omega\left(L_m^i \phi\right) \equiv \frac{\ell+1}{2} \mod \ell$, then call $L_m^i \phi$ a *high point* and $L_m^{i+1} \phi$ a *low point* of the heat cycle. By Propositions 5.1 and 5.2,

$$\Omega\left(L_m^{i+1} \phi\right) = \Omega\left(L_m^i \phi\right) + \ell + 1 - s(\ell - 1) \tag{5.1.4}$$

where $s \geq 1$ if and only if $L_m^i \phi$ is a high point and $s = 0$ otherwise. The structure of the heat cycle of a Jacobi form is similar to the structure of the theta cycle of a modular form (see Lemma 3.3). We will now prove a few basic properties:

**Lemma 5.5.** *Let $\phi \in \widetilde{J}_{k,m}$ with $\ell \nmid m$ a prime such that $L_m \phi \not\equiv 0 \mod \ell$.*

1. *If $j \geq 1$, then $\Omega\left(L_m^j \phi\right) \not\equiv \frac{\ell+3}{2} \mod \ell$.*

2. *The heat cycle of $\phi$ has a single low point if and only if there is some $j \geq 1$ with $\Omega\left(L_m^j \phi\right) \equiv \frac{\ell+5}{2} \mod \ell$. Furthermore, $L_m^j \phi$ is the low point.*

*3. If $j \geq 1$, then $\Omega\left(L_m^{j+1}\phi\right) \neq \Omega\left(L_m^j\phi\right) + 2$.*

*4. The heat cycle of $\phi$ either has one or two high points.*

*Proof.* 1. If $\Omega\left(L_m^j\phi\right) \equiv \frac{\ell+3}{2} \mod \ell$, then by (5.1.4) for $1 \leq n \leq \ell - 1$ we have

$$\Omega\left(L_m^{j+n}\phi\right) = \Omega\left(L_m^j\phi\right) + n(\ell+1).$$

In particular, $L_m^{j+\ell-1}\phi \not\equiv L_m^j\phi \mod \ell$, which is impossible.

2. If $\Omega\left(L_m^j\phi\right) \equiv \frac{\ell+5}{2} \mod \ell$, then by (5.1.4), for $1 \leq n \leq \ell - 2$ we have

$$\Omega\left(L_m^{j+n}\phi\right) = \Omega\left(L_m^j\phi\right) + n(\ell+1)$$

and

$$\Omega\left(L_m^j\phi\right) = \Omega\left(L_m^{j+\ell-1}\phi\right) = \Omega\left(L_m^j\phi\right) + (\ell-1)(\ell+1) - s(\ell-1)$$

where $s$ must be $\ell+1$ and there can be no other low point. On the other hand, if there is a single low point, then the filtration must increase $\ell - 2$ consecutive times. The only way this is possible is if the low point has filtration $\frac{\ell+5}{2} \mod \ell$.

3. By Proposition 5.2, $\Omega\left(L_m^{j+1}\phi\right) = \Omega\left(L_m^j\phi\right) + 2$ can only happen when $\Omega\left(L_m^j\phi\right) \equiv \frac{\ell+1}{2} \mod \ell$. Suppose $\Omega\left(L_m^{j+1}\phi\right) = \Omega\left(L_m^j\phi\right) + 2 \equiv \frac{\ell+5}{2} \mod \ell$. By part (2), this implies that the filtration increases $\ell - 2$ more times before falling. Hence $L_m^{j+\ell-1}\phi \not\equiv L_m^j\phi \mod \ell$, which is impossible.

4. Suppose there are $t \geq 2$ high points $L_m^{i_j}\phi$ where $1 \leq i_1 < \cdots < i_t \leq \ell - 1$. By (5.1.4) and part (3) above, there are $s_j \geq 2$ such that

$$\Omega\left(L_m^{i_j+1}\phi\right) = \Omega\left(L_m^{i_j}\phi\right) + \ell + 1 - s_j(\ell-1). \tag{5.1.5}$$

Hence

$$\Omega\left(L_m\phi\right) = \Omega\left(L_m^\ell\phi\right) = \Omega\left(L_m\phi\right) + (\ell-1)(\ell+1) - \sum_{j=1}^{t} s_j(\ell-1),$$

and so $\sum s_j = \ell + 1$. By (5.1.5), $\Omega\left(L_m^{i_j+1}\phi\right) \equiv \frac{\ell+1}{2} + 1 + s_j \mod \ell$ and so there will be $\ell - 1 - s_j$ increases before the next fall. That is, for $1 \leq j \leq t$, $i_{j+1} - i_j = \ell - s_j$ where we take $i_{t+1} = i_1 + \ell - 1$ for convenience. Thus

$$\ell - 1 = i_{t+1} - i_1 = \sum_{j=1}^{t}(i_{j+1} - i_j) = \sum_{j=1}^{t}(\ell - s_j) = t\ell - (\ell+1),$$

i.e., $t = 2$. We conclude that the heat cycle of $\phi$ has at most two (i.e., one or two) high points. $\square$

The following Corollary of Proposition 5.4 is a key ingredient in the proof of Proposition 1.13 below.

**Corollary 5.6.** *If $\phi \in \widetilde{J}_{k,m}$ has a Ramanujan congruence at $b \not\equiv 0 \pmod{\ell}$ and $L_m\phi \not\equiv 0 \pmod{\ell}$, then the heat cycle of $\phi$ has two low points which both have filtration congruent to $2 \pmod{\ell}$.*

*Proof.* Since $L_m^{\frac{\ell+1}{2}}\phi \equiv -\left(\frac{b}{\ell}\right) L_m\phi \pmod{\ell}$, we have $\Omega\left(L_m^{\frac{\ell+1}{2}}\phi\right) = \Omega\left(L_m\phi\right) = \Omega\left(L_m^{\ell}\phi\right)$. Hence there is a fall in the first half of the heat cycle and in the second half of the heat cycle. Furthermore, after a low point, the filtration increases $\frac{\ell-3}{2}$ times and then falls once. Thus, the filtration of the low points is $2 \pmod{\ell}$. $\square$

We now prove our main theorem for Jacobi forms.

*Proof of Theorem 1.13.* Assume that $\phi$ has a Ramanujan congruence at $b \pmod{\ell}$. First suppose $k = \frac{\ell+1}{2}$. Then $\Omega\left(\phi\right) = \frac{\ell+1}{2}$ and so we must have $s \geq 1$ in (5.1.4). Since we need $\Omega\left(L_m\phi\right) \geq 0$, we must have $s = 1$ and hence $\Omega\left(L_m\phi\right) = \frac{\ell+5}{2}$. But by Lemma 5.5 (2), this implies there is only one low point, contrary to Corollary 5.6.

Now suppose $k \neq \frac{\ell+1}{2}$. Then $\Omega\left(L_m\phi\right) = k + \ell + 1$. There must be a low point of the heat cycle with filtration either $k + \ell + 1$ or $k$. By Corollary 5.6, either $k + 1 \equiv 2 \pmod{\ell}$ or $k \equiv 2 \pmod{\ell}$. Both of these alternatives are impossible since $\ell > k \geq 4$. $\square$

## 5.2 Proof of Theorem 1.12

We employ the Fourier-Jacobi expansion of a Siegel modular form (as in [14]) to prove Theorem 1.12. Let $M_k^{(2)}$ denote the vector space of Siegel modular forms of degree 2 and even weight $k$ (for details on Siegel modular forms, see for example Freitag [23] or Klingen [27]). Set

$$\widetilde{M}_k^{(2)} := \left\{F \mod \ell \,:\, F(Z) = \sum a(T)e^{\pi i\, tr(TZ)} \in M_k^{(2)} \text{ where } a(T) \in \mathbb{Z}_{(\ell)}\right\}.$$

Recall the following two theorems on Siegel modular forms modulo $\ell$:

**Theorem 5.7** (Nagaoka [36]). *There exists an $E \in M_{\ell-1}^{(2)}$ with $\ell$-integral rational coefficients such that $E \equiv 1 \pmod{\ell}$. Furthermore, if $F_1 \in M_{k_1}^{(2)}$ and $F_2 \in M_{k_2}^{(2)}$ have $\ell$-integral rational coefficients where $0 \not\equiv F_1 \equiv F_2 \pmod{\ell}$, then $k_1 \equiv k_2 \pmod{\ell - 1}$.*

**Theorem 5.8** (Böcherer and Nagaoka [8]). *If $F \in \widetilde{M}_k^{(2)}$, then $\mathbb{D}(F) \in \widetilde{M}_{k+\ell+1}^{(2)}$.*

Thus, the reduced Siegel forms have an arithmetic analogous to reduced modular and Jacobi forms.

*Proof of Theorem 1.12.* Let $F \in M_k^{(2)}$ be as in Theorem 1.12 with Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'},$$

i.e., $\phi_m \in J_{k,m}$. Let $b \not\equiv 0 \mod \ell$. Then $F$ has a Ramanujan congruence at $b \mod \ell$ if and only if for all $m$, $\phi_m$ has a Ramanujan congruence at $b$. By Proposition 5.4, it is equivalent that for all $m$, we have

$$L_m^{\frac{\ell+1}{2}} \phi_m \equiv -\left(\frac{b}{\ell}\right) L_m \phi_m \mod \ell,$$

which is equivalent to (1.4.1), since

$$\mathbb{D}(F) = \sum_{m=0}^{\infty} L_m \left(\phi_m(\tau, z)\right) e^{2\pi i m \tau'}.$$

Now we turn to the second part of Theorem 1.12. Suppose $F$ has a congruence at $b \not\equiv 0 \mod \ell$, $\ell > k$, and $\ell \nmid \gcd(n,m)(4nm - r^2)A(n,r,m)$ for some fixed $n, r, m$. Note that $k \geq 4$, since $F$ is non-constant.

If $\ell \nmid m$, then by Proposition 1.13, $L_m \phi_m \equiv 0 \mod \ell$. But this contradicts the fact that $L_m \phi_m$ has a coefficient $(4nm - r^2)A(n,r,m) \not\equiv 0 \mod \ell$.

On the other hand, if $\ell \nmid n$, then since $F(\tau, z, \tau') = F(\tau', z, \tau)$ we have $A(n,r,m) = A(m,r,n)$. But now $L_n \phi_n$ has a coefficient $(4nm - r^2)A(n,r,m) \not\equiv 0 \mod \ell$, contrary to Proposition 1.13. $\square$

Theorems 5.7 and 5.8 imply that for any $F \in \widetilde{M}_k^{(2)}$, we have

$$G := \mathbb{D}^{\frac{\ell+1}{2}}(F) + \left(\frac{b}{\ell}\right) \mathbb{D}(F) \in \widetilde{M}^{(2)}_{k + \frac{(\ell+1)2}{2}}. \tag{5.2.1}$$

Theorem 1.12 states that $F \in \widetilde{M}_k^{(2)}$ has a Ramanujan congruence at $b \not\equiv 0 \mod \ell$ if and only if $G \equiv 0 \mod \ell$ in (5.2.1). One can apply the following analog of Sturm's theorem for Siegel modular forms of degree 2 to verify that $G \equiv 0 \mod \ell$ in (5.2.1) for concrete examples of Siegel modular forms.

**Theorem 5.9** (Poor and Yuen [37])**.** *Let $F = \sum a(T) e^{\pi i\, tr(TZ)} \in M_k^{(2)}$ be such that for all $T$ with dyadic trace $w(T) \leq \frac{k}{3}$ one has that $a(T) \in \mathbb{Z}_{(\ell)}$ and $a(T) \equiv 0 \mod \ell$. Then $F \equiv 0 \mod \ell$.*

*Remark* 5.10. If $T = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) > 0$ is Minkowski reduced (i.e., $2|b| \leq a \leq c$), then $w(T) = a + c - |b|$. For more details on the dyadic trace $w(T)$, see Poor and Yuen [38].

The following table gives all Ramanujan congruences at $b \not\equiv 0 \mod \ell$ for Siegel cusp forms of weight 20 or less when $\ell \geq 5$. Let $E_4, E_6, \chi_{10}$, and $\chi_{12}$ denote the usual generators of $M_k^{(2)}$ of weights 4, 6, 10, and 12, respectively, where the Eisenstein series $E_4$ and $E_6$ are normalized by $a\left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = 1$ and where the cusp forms $\chi_{10}$ and $\chi_{12}$ are normalized by $a\left(\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)\right) = 1$. Cris Poor and David Yuen kindly provided Fourier coefficients up to dyadic trace $w(T) = 74$ of the basis vectors for $M_k^{(2)}$ with $k \leq 20$. We used Magma to check that $G \equiv 0 \mod \ell$ in (5.2.1) for each of the forms in Table 5.1 below. It is not difficult to verify that (up to scalar multiplication) no further Ramanujan congruences at $b \not\equiv 0 \mod \ell$ exist for Siegel cusp forms of weights 20 or less.

Table 5.1: Siegel forms of weight $\leq 20$ with Ramanujan congruences at $b \not\equiv 0 \mod \ell$

| | $b \not\equiv 0 \mod \ell$ |
|---|---|
| $\chi_{12}$ | $b \equiv 1, 4 \mod 5$ and $b \equiv 2, 6, 7, 8, 10 \mod 11$ |
| $E_4\chi_{12}$ | $b \equiv 1, 4 \mod 5$ |
| $E_4\chi_{12} - E_6\chi_{10}$ | $b \equiv 3, 5, 6 \mod 7$ |
| $E_6\chi_{12}$ | $b \equiv 1, 4 \mod 5$ |
| $E_4^2\chi_{10} + 7E_6\chi_{12}$ | $b \equiv 1, 2, 4, 8, 9, 13, 15, 16 \mod 17$ |
| $E_4^2\chi_{12}$ | $b \equiv 1, 4 \mod 5$ |
| $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ | $b \equiv 2, 3, 8, 10, 12, 13, 14, 15, 18 \mod 19$ |

*Remark* 5.11. For $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ modulo 19 we have $G \in \widetilde{M}_{220}^{(2)}$ in (5.2.1) and we really do need Fourier coefficients up to dyadic trace $w(T) = \frac{220}{3}$, i.e., up to 74 in Theorem 5.9 to prove that $G \equiv 0 \mod 19$.

*Remark* 5.12. For Siegel modular forms in the *Maass Spezialschar* one could decide the existence and non-existence of their Ramanujan congruences also using Propositions 5.4 and 1.13 in combination with Maass' lift [33] (see also §6 of [22]). However, Theorem 1.12 is an essential tool in establishing such results for Siegel modular forms that are not in the Maass Spezialschar, such as $E_4^2\chi_{12}$ and $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ for example.

# CHAPTER 6

# THE RARITY OF RAMANUJAN CONGRUENCES

Throughout this section, work exclusively in characteristic $\ell$. To ease the notation, we write $M_k$ to denote $M_k\left(\Gamma_1(N), \mathbb{Z}_{(\ell)}\right)$. Unless explicitly stated otherwise, all of the lemmas and statements in this chapter are valid for $N = 1$ or 4. The author expects to publish the material in this chapter as [18].

## 6.1 The plan for the proof of Theorem 1.15

The subset of $M_k$ consisting of forms with Ramanujan congruences at $0 \mod \ell$ is in fact a subspace of $M_k$. For reasons to be explained in Section 6.2, we denote this subspace by $IT_k$. Since $M_k$ is in fact a finite set, we have

$$\mathcal{P}\left(\ell, k, N\right) = \frac{|IT_k|}{|M_k|} = \frac{\ell^{\dim IT_k}}{\ell^{\dim M_k}}. \tag{6.1.1}$$

The key is to determine the dimension, or more precisely the codimension, of $IT_k$. This turns out to be surprisingly intricate and employs several main ideas. An important tool is Ramanujan's $\Theta = q\frac{d}{dq}$ operator which encapsulates the notion of Ramanujan congruences via the so-called Tate cycle. In Sections 6.2 and 6.3, we use the $\Theta$ operator to decompose $M_k$ into many pieces of known dimension, most of which are either disjoint from $IT_k$ or contained in $IT_k$. One of these pieces can only be understood after a detailed study of the kernel of $\Theta$; see Section 6.4. Finally, Sections 6.5 and 6.6 contain the dimension calculations required to compute $\mathcal{P}\left(\ell, k, N\right)$ exactly.

## 6.2 Fundamental subspaces

The following lemma is an elementary fact from linear algebra.

**Lemma 6.1.** *Let $V \leq W \leq M_k$ be a chain of subspaces. Then there exists a space $V^{\perp}$ such that $W = V \oplus V^{\perp}$.*

By Lemmas 2.6 and 2.10, we have the following fundamental linear maps:

$$E_{\ell-1} : M_k \hookrightarrow M_{k+\ell-1} \tag{6.2.1}$$

$$\Theta : M_k \to M_{k+\ell+1}. \tag{6.2.2}$$

It is no exaggeration to say that this chapter is devoted to studying the images of these two maps. Consider first (6.2.1). If $k \geq \ell - 1$ then $E_{\ell-1}M_{k-\ell+1}$ is an intrinsic[1] subspace of $M_k$. That is,

$$E_{\ell-1}M_{k-\ell+1} = \{f \in M_k : \omega(f) \leq k - \ell + 1\} \leq M_k.$$

The $E_{\ell-1}$ notation is merely a bookkeeping device to remind us that $M_{k-\ell+1} \leq M_k$. Occasionally we dispense with writing $E_{\ell-1}$. This subspace inclusion is so important for us that we reiterate it in the following remark.

*Remark* 6.2. Recall that Lemma 6.1 guarantees the existence of a subspace $W$ such that

$$M_k = E_{\ell-1}M_{k-\ell+1} \oplus W. \tag{6.2.3}$$

Furthermore, for any $f \in M_k$, we have $f \in E_{\ell-1}M_{k-\ell+1}$ if and only if $f$ has filtration $\omega(f) < k$. Hence, if $0 \not\equiv f \in W$, then $\omega(f) = k$. The converse is of course false.

The subspace $W$ from Remark 6.2 is not intrinsic. The key Lemmas 6.3 and 6.4 below show that we can always choose $W$ in (6.2.3) so that it has nice properties related to the image of $\Theta$.

Recall (6.2.2) and define the intrinsic subspaces

$$K_k := \ker\left(\Theta : M_k \to M_{k+\ell+1}\right) = \left\{ f \in M_k : f = \sum_{\ell \mid n} a_n q^n \right\}, \tag{6.2.4}$$

$$IT_k := \ker\left(\Theta^{\ell-1} - E_{\ell-1}^{\ell+1} : M_k \to M_{k+\ell^2-1}\right) = \left\{ f \in M_k : f = \sum_{\ell \nmid n} a_n q^n \right\}. \tag{6.2.5}$$

By Lemma 3.2, $IT_k$ is the set of all forms with a Ramanujan congruence at $0 \mod \ell$. Equivalently, it is the set of all forms in their own Tate cycle. We refer to it as the "In Tate" space. It is clear from the definitions that for any $k$ we have

$$\Theta : M_k \to IT_{k+\ell+1}$$

and

$$K_k \cap IT_k = 0.$$

The next two lemmas relate the images of the maps $E_{\ell-1}$ and $\Theta$.

[1]By *intrinsic* subspace, we mean a space which is uniquely and canonically defined. In practice, this means we do not appeal to Lemma 6.1 to define the subspace.

**Lemma 6.3.** *Suppose $2\ell \leq k \not\equiv 1 \mod \ell$. For any $W \leq M_{k-\ell-1}$ such that $M_{k-\ell-1} = E_{\ell-1}M_{k-2\ell} \oplus W$, we have $M_k = E_{\ell-1}M_{k-\ell+1} \oplus \Theta W$. Moreover, $\Theta W \leq IT_k$.*

*Proof.* Consider the commutative diagram below.

$$\begin{tikzcd}
M_{k-2\ell} \arrow[r, "E_{\ell-1}"] \arrow[d, "\Theta"'] & M_{k-\ell-1} \arrow[r, equal] \arrow[d, "\Theta"] & E_{\ell-1}M_{k-2\ell} \oplus W \\
M_{k-\ell+1} \arrow[r, "E_{\ell-1}"'] & M_k &
\end{tikzcd}$$

Suppose $0 \neq f \in W$. By Remark 6.2, $\omega(f) = k - \ell - 1 \not\equiv 0 \mod \ell$. Hence Lemma 2.10 implies that $\omega(\Theta f) = k$. Thus $\Theta f \notin E_{\ell-1}M_{k-\ell+1}$. We conclude that $E_{\ell-1}M_{k-\ell+1} \cap \Theta W = 0$, and hence that we have a direct sum $E_{\ell-1}M_{k-\ell+1} \oplus \Theta W \leq M_k$. We have also shown that $\Theta|_W$ is injective. It is clear that $E_{\ell-1}$ is injective. Now

$$\begin{aligned}
\dim (E_{\ell-1}M_{k-\ell+1} \oplus \Theta W) &= \dim E_{\ell-1}M_{k-\ell+1} + \dim \Theta W \\
&= \dim M_{k-\ell+1} + \dim W \\
&= \dim M_{k-\ell+1} + (\dim M_{k-\ell-1} - \dim M_{k-2\ell}) \\
&= \dim M_k,
\end{aligned}$$

where the last equality follows from an elementary calculation (see Lemmas 6.19 and 6.21). It follows that $M_k = E_{\ell-1}M_{k-\ell+1} \oplus \Theta W$. The last statement of the lemma is immediate since the image of $\Theta$ is always contained in an $IT$ space. □

**Lemma 6.4.** *Suppose $k \equiv 1 \mod \ell$. For any $W \leq M_k$ such that $M_k = E_{\ell-1}M_{k-\ell+1} \oplus W$, we have $W \cap IT_k = 0$.*

*Proof.* By Remark 6.2, any $0 \neq f \in W$ has $\omega(f) = k \equiv 1 \mod \ell$. By Lemma 3.3 (1), we know that $f \notin IT_k$. □

## 6.3 The main decomposition

We now have the tools to give our main decomposition of $M_k$ into subspaces with specified Tate cycle structures.

**Definition 6.5.** If $0 \leq k < 2\ell$ or $k \equiv 1 \mod \ell$, then define $M_{k*}$ to be any subspace $M_{k*} \leq M_k$ such that

$$M_k = E_{\ell-1}M_{k-\ell+1} \oplus M_{k*}.$$

If $k \geq 2\ell$ and $k \not\equiv 1 \mod \ell$ then recursively define $M_{k*} := \Theta M_{k-\ell-1*}$.

Note that for $0 \leq k < \ell - 1$ we have $M_{k*} = M_k$ since the only negative weight holomorphic modular form is 0. When $\ell - 1 \leq k < 2\ell$ or $k \equiv 1 \mod \ell$, the choice of $M_{k*}$ is not canonical. For $k \geq 2\ell$ and $k \not\equiv 1 \mod \ell$, the space $M_{k*}$ is uniquely determined by the lower weight "starred" spaces $M_{j*}$ with $j < k$.

By this definition and Lemma 6.3, for all $k$ we have

$$M_k = E_{\ell-1} M_{k-\ell+1} \oplus M_{k*}. \tag{6.3.1}$$

In particular,

$$\dim M_{k*} = \dim M_k - \dim M_{k-\ell+1}, \tag{6.3.2}$$

which allows us to compute $\dim M_{k*}$.

We now recursively decompose the original, un-starred space $M_k$ into a direct sum of starred spaces. Write

$$k = C(\ell - 1) + D$$

where

$$3 \leq D \leq \ell + 1$$

and iteratively apply (6.3.1) to get

$$\begin{aligned} M_k &= M_{C(\ell-1)+D} \\ &= M_{C(\ell-1)+D*} \oplus E_{\ell-1} M_{(C-1)(\ell-1)+D} \\ &= M_{C(\ell-1)+D*} \oplus E_{\ell-1} M_{(C-1)(\ell-1)+D*} \oplus E_{\ell-1}^2 M_{(C-2)(\ell-1)+D} \\ &= \cdots \\ &= \left( \bigoplus_{i=1}^{C} E_{\ell-1}^{C-i} M_{i(\ell-1)+D*} \right) \oplus E_{\ell-1}^{C} M_D. \end{aligned} \tag{6.3.3}$$

For each of the $M_{i(\ell-1)+D*}$ terms in (6.3.3), if $i(\ell-1)+D \equiv 1 \mod \ell$, then by Lemma 6.4 we have $M_{i(\ell-1)+D*} \cap IT_k = 0$. If $i(\ell-1)+D \not\equiv 1 \mod \ell$ and $i(\ell-1)+D \geq 2\ell$, then by Lemma 6.3 and the map (6.2.1) we have that $M_{i(\ell-1)+D*} \leq IT_{i(\ell-1)+D} \hookrightarrow IT_k$. This motivates the following regrouping of the summands from (6.3.3):

**Definition 6.6.** Let $k = C(\ell-1)+D$ with $3 \leq D \leq \ell+1$ as above. Define

$$W_1^k := \bigoplus_{\substack{1 \leq i \leq C \\ i(\ell-1)+D \equiv 1 \mod \ell \\ 2\ell \leq i(\ell-1)+D}} E_{\ell-1}^{C-i} M_{i(\ell-1)+D*},$$

$$W_2^k := \bigoplus_{\substack{1 \le i \le C \\ i(\ell-1)+D \not\equiv 1 \mod \ell \\ 2\ell \le i(\ell-1)+D}} E_{\ell-1}^{C-i} M_{i(\ell-1)+D*},$$

$$W_3^k := \left( \bigoplus_{\substack{1 \le i \le C \\ i(\ell-1)+D < 2\ell}} E_{\ell-1}^{C-i} M_{i(\ell-1)+D*} \right) \oplus E_{\ell-1}^{C} M_D,$$

so that by (6.3.3), we have

$$M_k = W_1^k \oplus W_2^k \oplus W_3^k. \tag{6.3.4}$$

*Remark* 6.7. Before continuing, we sketch the proof of our main theorem. Section 6.4 will define a space $OT_{\ell+D-1}$ ("Out Tate") such that $W_3^k = IT_{\ell+D-1} \oplus OT_{\ell+D-1}$. Then we will use Lemmas 6.8 and 6.9 below to show that

$$M_k = \underbrace{W_1^k \oplus \overbrace{OT_{\ell+D-1}}}_{\text{complementary to } IT_k} \oplus \underbrace{\overbrace{IT_{\ell+D-1}}^{W_3^k} \oplus W_2^k}_{IT_k}.$$

Proving Theorem 1.15 requires computing $\mathcal{P}_\ell^k$ which, by the previous equation and (6.1.1), means that we need to compute $\dim M_k - \dim IT_k = \dim W_1^k + \dim OT_{\ell+D-1}$. Lemma 6.16 gives $\dim OT_{\ell+D-1}$. Sections 6.5 and 6.6 use (6.3.2) to compute $\dim W_1^k$. For the remainder of this section we study the $W_i^k$.

**Lemma 6.8.** *Let $k = C(\ell-1)+D$ with $3 \le D \le \ell+1$ and let $W_1^k$ be as in Definition 6.6. There are $1 + \lfloor \frac{C-D+1}{\ell} \rfloor$ direct summands in $W_1^k$. Furthermore, $W_1^k \cap IT_k = 0$.*

*Proof.* Let $\mathcal{J}$ be the set of subscripts appearing in the definition of $W_1^k$. That is, $\mathcal{J}$ is defined by the equation (ignoring $E_{\ell-1}$)

$$W_1^k = \bigoplus_{j \in \mathcal{J}} M_{j*}.$$

Thus

$$\mathcal{J} = \{ j \in \mathbb{Z} | \exists i \in \mathbb{Z}, 1 \le i \le C, j = i(\ell-1)+D, j \ge 2\ell, j \equiv 1 \mod \ell \}.$$

Since $i \equiv D-1 \mod \ell$, we see that the possible $i$ are of the form $i = D-1+t\ell$. In particular,

$$\mathcal{J} = \left\{ j \in \mathbb{Z} \middle| \exists t \in \mathbb{Z}, 0 \le t \le \frac{C-D+1}{\ell}, j = \ell\Big(t(\ell-1)+D-1\Big)+1 \right\}. \tag{6.3.5}$$

Hence we have $|\mathcal{J}| = 1 + \lfloor \frac{C-D+1}{\ell} \rfloor$. (We remark that the quantity $J = |\mathcal{J}|$ appears in (1.5.1) and the statement of Theorem 1.15.)

Now suppose $0 \neq f \in W_1^k$. Write $f = \sum_{j \in \mathcal{J}} f_j$ where $f_j \in M_{j*}$. Let $j_0$ be the largest index such that $f_{j_0} \neq 0$. By Lemma 2.12, we have $\omega(f) = \omega(f_{j_0}) = j_0 \equiv 1 \mod \ell$. By Lemma 3.3(1), we have $f \notin IT_k$ and hence we conclude that $IT_k \cap W_1^k = 0$. □

**Lemma 6.9.** *Let $W_2^k$ be as in Definition 6.6. Then $W_2^k \leq IT_k$.*

*Proof.* Recalling that $\Theta : M_{k-\ell-1} \to IT_k$, we deduce by Definition 6.5 that each of the summands in $W_2^k$ is contained in $IT_k$. □

Our last major challenge is to compute $\dim\left(W_3^k \cap IT_k\right)$. We study $W_3^k$ until the end of Section 6.4.

**Lemma 6.10.** *Let $k$ and $W_3^k$ be as in Definition 6.6. If $3 \leq D \leq \ell$, then $W_3^k = E_{\ell-1}^{C-1} M_{\ell+D-1}$. If $D = \ell + 1$ then $W_3^k = E_{\ell-1}^{C} M_{\ell+1}$ and $W_3^k \cap IT_k = 0$.*

*Proof.* If $3 \leq D \leq \ell$, then $W_3^k = E_{\ell-1}^{C-1} M_{(\ell-1)+D*} \oplus E_{\ell-1}^{C} M_D$. By (6.3.1), we have $W_3^k = E_{\ell-1}^{C-1} M_{(\ell-1)+D}$ as desired.

If $D = \ell + 1$, then $i(\ell - 1) + D \geq 2\ell$ for all $1 \leq i \leq C$. Thus the direct sum indexed by $i$ in the definition of $W_3^k$ is empty and $W_3^k = M_{\ell+1}$. Suppose that $f \in M_{\ell+1} \cap IT_k$. Then by Lemma 3.3(1), we cannot have $\omega(f) = \ell + 1$. Thus $f \in M_2$. If the level is $N = 1$, then $M_2 = 0$ and the conclusion holds. Otherwise, $N = 4$ and $M_2$ is spanned by

$$E := \left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)^4 = 1 + 8q + \cdots$$

and

$$F := \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1} = q + \cdots + (\ell+1) q^{\ell} \cdots .$$

If $f = aE + bF$, then since $f \in IT_k$, we must have $a = 0$. But then we must also have $b = 0$. Thus $f = 0$ and the conclusion that $W_3^k \cap IT_k = 0$ follows. □

## 6.4 The kernel of $\Theta$

In this section we study the spaces $W_3^k$ via an in-depth examination of the kernel of $\Theta$. In particular, we will decompose $W_3^k$ into two subspaces, one contained in $IT_k$ and the other (which we will call $OT_{\ell+D-1}$) having trivial intersection with $IT_k$. We will determine the dimension of each of these subspaces. The case when $D = \ell + 1$ is a bit unusual and has been dealt with in Lemma 6.10. For the remainder of this section, we assume that $3 \leq D \leq \ell$ so that $W_3^k = M_{\ell+D-1}$.

By Lemma 2.10 the map $\left(E_{\ell-1}^{\ell+1} - \Theta^{\ell-1}\right)$ takes $M_{\ell+D-1}$ into $M_{(\ell+D-1)+(\ell-1)(\ell+1)}$. In fact, the image is contained in a much smaller (and lower weight) subspace.

**Lemma 6.11.** *Let $3 \le D \le \ell$. Recall the notation from (6.2.1), (6.2.2), (6.2.4), and (6.2.5). Then*

$$\left(E_{\ell-1}^{\ell+1} - \Theta^{\ell-1}\right) : M_{\ell+D-1} \to K_{\ell D}.$$

*Proof.* Let $f \in M_{\ell+D-1}$. Since $\Theta^{\ell} f \equiv \Theta f \mod \ell$, we have

$$\Theta\left(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f\right) = \Theta f - \Theta^{\ell} f = 0$$

and hence $\left(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f\right) \in K_{(\ell+D-1)+(\ell-1)(\ell+1)}$. Suppose $\left(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f\right) \neq 0$. Then by Lemma 2.6, $\omega(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) = (\ell + D - 1) + (\ell - 1)(\ell + 1) - t(\ell - 1)$ for some $t \geq 0$. By Lemma 2.10, the fact that $E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f$ is in the kernel of $\Theta$ implies that

$$(\ell + D - 1) + (\ell - 1)(\ell + 1) - t(\ell - 1) \equiv 0 \mod \ell.$$

Thus $t \equiv 2 - D \mod \ell$. Since $3 \le D \le \ell$, we deduce that $t = \ell + 2 - D + s\ell$, for some $s \geq 0$. Now

$$\begin{aligned} \omega(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) &= \ell + D - 1 + (\ell - 1)(\ell + 1) - t(\ell - 1) \\ &= \ell\left(D - (\ell - 1)s\right). \end{aligned} \tag{6.4.1}$$

Filtrations of non-zero forms are non-negative and so $D \geq (\ell - 1)s$. But $D \leq \ell$ and so there are three cases:

- If $s = 0$, then $\omega(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) = \ell D$.
- If $s = 1$ and $D = \ell - 1$, then $\omega(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) = 0$.
- If $s = 1$ and $D = \ell$, then $\omega(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) = \ell$.

In any case, $(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f) \in K_{\ell D}$. □

**Definition 6.12.** For $D$ in the range $3 \le D \le \ell$, let $OT_{\ell+D-1}$ be any complementary subspace such that

$$M_{\ell+D-1} = IT_{\ell+D-1} \oplus OT_{\ell+D-1}. \tag{6.4.2}$$

By Lemma 6.11 we have an injection

$$\left(E_{\ell-1}^{\ell+1} - \Theta^{\ell-1}\right) : OT_{\ell+D-1} \hookrightarrow K_{\ell D}.$$

**Lemma 6.13.** *Let $3 \le D \le \ell$. The injection $E_{\ell-1}^{\ell+1} - \Theta^{\ell-1} : OT_{\ell+D-1} \to K_{\ell D}$ factors as*

$$\left(E_{\ell-1}^{\ell+1} - \Theta^{\ell-1}\right)(\cdot) = (\cdot|U_\ell)^{\ell}$$

*where* $U_\ell : OT_{\ell+D-1} \to M_D$ *and* $(\cdot)^\ell : M_D \to K_{\ell D}$*. Hence*

$$\dim OT_{\ell+D-1} \le \dim M_D \le \dim K_{\ell D}. \tag{6.4.3}$$

*Proof.* Suppose $0 \ne f \in OT_{\ell+D-1}$. The factorization $E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f = (f|U_\ell)^\ell$ is immediate from Lemma 3.2. By Lemma 2.8, we have that $U_\ell : M_{\ell+D-1} \to M_{\ell+D-1}$. By Lemma 2.11,

$$\omega\left(f|U_\ell\right) = \frac{1}{\ell}\omega\Big(\left(f|U_\ell\right)^\ell\Big) = \frac{1}{\ell}\omega\left(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f\right).$$

By the computation of $\omega\left(E_{\ell-1}^{\ell+1} f - \Theta^{\ell-1} f\right)$ at the end of the proof of Lemma 6.11, we deduce that $\omega\left(f|U_\ell\right) \le D$ and hence $f|U_\ell \in M_D$.

If $f \in M_D$, then $f^\ell \in M_{\ell D}$. By considering the action on coefficients, we see that the map $(\cdot)^\ell$ in fact takes $f \in M_D$ to $K_{\ell D}$.

The statement about the dimensions is true because these maps are injective. ☐

We will now show that we have equality in (6.4.3) by decomposing $M_{\ell D}$ as in (6.3.3):

$$M_{\ell D} = \bigoplus_{i=0}^{\left\lfloor \frac{\ell D}{\ell-1} \right\rfloor} E_{\ell-1}^i M_{\ell D - i(\ell-1)*} = W_1^{\ell D} \oplus W_2^{\ell D} \oplus W_3^{\ell D}. \tag{6.4.4}$$

**Lemma 6.14.** *Suppose that* $3 \le D \le \ell$*. Then*

$$W_1^{\ell D} = E_{\ell-1} M_{(D-1)\ell+1*} \tag{6.4.5}$$

$$W_2^{\ell D} \le IT_{\ell D} \tag{6.4.6}$$

$$W_3^{\ell D} = E_{\ell-1}^{D-1} M_{\ell+D-1}. \tag{6.4.7}$$

*Proof.* Since $\ell D = D(\ell-1) + D$, we have that $i = D-1$ is the only index which appears in the direct sum defining $W_1^{\ell D}$ and so $W_1^{\ell D} = E_{\ell-1} M_{(D-1)\ell+1*}$. This proves (6.4.5). Now (6.4.6) and (6.4.7) are immediate from Lemmas 6.9 and 6.10. ☐

**Lemma 6.15.** *Let* $3 \le D \le \ell$*. Then* $\left(W_1^{\ell D} \oplus IT_{\ell D}\right) \cap K_{\ell D} = 0$.

*Proof.* Since any two of $W_1^{\ell D}$, $IT_{\ell D}$, and $K_{\ell D}$ have trivial intersection, it suffices to show that

$$W_1^{\ell D} \cap \left(IT_{\ell D} \oplus K_{\ell D}\right) = 0.$$

Suppose $0 \ne f \in W_1^{\ell D} = E_{\ell-1} M_{(D-1)\ell+1*}$. Then $\omega(f) = (D-1)\ell + 1$, and by Lemma 2.10 we have

$$\omega(\Theta f) = \omega(f) + \ell + 1 = \ell D + 2.$$

If $f = g + h$ where $g \in IT_{\ell D}$ and $h \in K_{\ell D}$, then by Lemmas 2.10 and 3.3(4), we have

$$\omega(\Theta f) = \omega(\Theta g + \Theta h) = \omega(\Theta g) \neq \ell D + 2$$

which contradicts the previous equation. $\square$

**Lemma 6.16.** *Let $3 \leq D \leq \ell$. Then* $\dim OT_{\ell+D-1} = \dim M_D = \dim K_{\ell D}$.

*Proof.* Recall (6.4.2), the decomposition (6.4.4) and Lemma 6.14, which give

$$\begin{aligned} M_{\ell D} &= W_1^{\ell D} \oplus W_2^{\ell D} \oplus W_3^{\ell D} \\ &= W_1^{\ell D} \oplus W_2^{\ell D} \oplus E_{\ell-1}^{D-1} M_{\ell+D-1} \\ &= \underbrace{W_1^{\ell D} \oplus \underbrace{W_2^{\ell D} \oplus E_{\ell-1}^{D-1} IT_{\ell+D-1}}_{\leq IT_{\ell D}}}_{\cap K_{\ell D}=0} \oplus E_{\ell-1}^{D-1} OT_{\ell+D-1}. \end{aligned}$$

In the last equation above, all but the last summand $E_{\ell-1}^{D-1} OT_{\ell+(D-1)}$ has trivial intersection with the kernel of $\Theta$ by Lemma 6.15. Hence, $\dim K_{\ell D} \leq \dim OT_{\ell+(D-1)}$. Therefore we have equality throughout (6.4.3). $\square$

**Corollary 6.17.** *Let $3 \leq D \leq \ell$. Then $K_{\ell D} = (M_D)^\ell$.*

*Proof.* By Lemma 6.16, we have equality in (6.4.3) and hence the maps appearing in Lemma 6.13 are all bijections. $\square$

**Proposition 6.18.** *Let $\ell \geq 5$ be prime and $k = C(\ell-1) + D \geq 2\ell$ where $3 \leq D \leq \ell + 1$. Suppose $N = 1$ or $4$. If $W_1^k$ is as in Definition 6.6, then*

$$\mathcal{P}(\ell, k, N) = \ell^{-\dim W_1^k - \dim M_D}.$$

*Proof.* Suppose $3 \leq D \leq \ell$. Then by Lemma 6.10 and Definition 6.12, we have

$$W_3^k = M_{\ell+D-1} = IT_{\ell+D-1} \oplus OT_{\ell+D-1}.$$

By (6.3.4), we thus have

$$M_k = \left(W_1^k \oplus OT_{\ell+D-1}\right) \oplus \left(IT_{\ell+D-1} \oplus W_2^k\right).$$

The $(C-1)$st iterate of the inclusion map (6.2.1) shows $IT_{\ell+D-1} \leq IT_{C(\ell-1)+D} = IT_k$ and hence by Lemma 6.9 we have $IT_{\ell+D-1} \oplus W_2^k \leq IT_k$.

We now prove that we actually have the equality $IT_{\ell+D-1} \oplus W_2^k = IT_k$: Suppose

$$0 \neq f \in \left(W_1^k \oplus OT_{\ell+D-1}\right) \cap IT_k.$$

Then $f = g + h$ for some $g \in W_1^k$ and $h \in OT_{\ell+D-1}$. If $g \neq 0$, then by Lemma 2.12, we have $\omega(f) = \omega(g) \equiv 1 \mod \ell$. But now Lemma 3.3(1) implies that $f \notin IT_k$, a contradiction. On the other hand, if $g = 0$, then $f = h \in OT_{\ell+D-1}$. By the definition of $OT_{\ell+D-1}$, we have $f \notin IT_{\ell+D-1} \leq IT_k$. We conclude that

$$\left(W_1^k \oplus OT_{\ell+D-1}\right) \cap IT_k = 0.$$

So

$$\begin{aligned}
\dim M_k &\geq \dim\left(W_1^k \oplus OT_{\ell+D-1}\right) + \dim IT_k \\
&\geq \dim\left(W_1^k \oplus OT_{\ell+D-1}\right) + \dim\left(IT_{\ell+D-1} \oplus W_2^k\right) \\
&\geq M_k,
\end{aligned}$$

and hence $\left(IT_{\ell+D-1} \oplus W_2^k\right) = IT_k$.

Now recall that by (6.1.1), we have

$$\mathcal{P}\left(\ell, k, N\right) = \frac{|IT_k|}{|M_k|} = \ell^{-(\dim M_k - \dim IT_k)} = \ell^{-\dim W_1^k - \dim OT_{\ell+D-1}}.$$

Lemma 6.16 yields the desired conclusion.

The case when $D = \ell + 1$ is similar. By Lemma 6.10 we have $0 = W_3^k \cap IT_k = IT_{\ell+D-1}$. In the proof above, replace "$OT_{\ell+D-1}$" with $W_3^k = M_{\ell+1}$. □

## 6.5 Dimension counts for level $N = 4$

In this section we assume the level is $N = 4$ and we determine $\dim M_{k*}$ for any $k$, and $\dim W_1^k$ for $k \geq 2\ell$. In the next section we will compute the more complicated case $N = 1$. Recall that for $N = 4$, we have $\dim M_k = \left\lfloor \frac{k}{2} \right\rfloor + 1$ for all $k \geq 0$.

**Lemma 6.19.** *Let $N = 4$ and $\ell \geq 5$ be prime. For $k \geq \ell - 1$, we have $\dim M_{k*} = \frac{\ell-1}{2}$.*

*Proof.* By (6.3.2), for $k \geq \ell - 1$ we have

$$\begin{aligned}
\dim M_{k*} &= \dim M_k - \dim M_{k-\ell+1} \\
&= \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) - \left(\left\lfloor \frac{k-\ell+1}{2} \right\rfloor + 1\right) \\
&= \frac{\ell - 1}{2},
\end{aligned}$$

which is independent of $k$. □

**Theorem 6.20** (Main Theorem for $N = 4$.)**.** *Let $\ell \geq 5$ be prime, $N = 4$, and $k = C(\ell-1)+D \geq 2\ell$ where $3 \leq D \leq \ell + 1$. The probability $\mathcal{P}\left(\ell, k, 4\right)$ that $f \in M_k$ has a Ramanujan congruence at* 0 mod $\ell$ *is*

$$\mathcal{P}(\ell, k, 4) = \ell^{-\left(\frac{\ell-1}{2}\right)\left(1+\left\lfloor \frac{C-D+1}{\ell} \right\rfloor\right)-\left\lfloor \frac{D}{2} \right\rfloor-1}.$$

*Proof.* Since $\dim M_D = \left\lfloor \frac{D}{2} \right\rfloor + 1$, in light of Proposition 6.18 it suffices to compute $\dim W_1^k$. By Lemma 6.8, there are $J = 1 + \left\lfloor \frac{C-D+1}{\ell} \right\rfloor$ direct summands in the definition of $W_1^k$. By Lemma 6.19, each summand has dimension $\frac{\ell-1}{2}$. Hence $\dim W_1^k = J\left(\frac{\ell-1}{2}\right)$. □

## 6.6 Dimension counts for level $N = 1$

The idea behind the dimension computations in this section is simple. In level $N = 1$, if $k \geq \ell - 1$ then by (6.3.2) we have that

$$\dim M_{k*} = \dim M_k - \dim M_{k-\ell+1} \approx \frac{k}{12} - \frac{k-\ell+1}{12} = \frac{\ell-1}{12}.$$

Hence,

$$\dim M_{k+\ell+1*} - \dim M_{k*} \approx 0.$$

The exact value of $\dim M_{k*}$ will depend on $k \mod 12$ and $\ell \mod 12$. Write

$$\begin{aligned} k &= 12k_0 + k_1, \\ \ell &= 12\ell_0 + \ell_1, \end{aligned}$$

where $k_1 \in \{0, 2, 4, 6, 8, 10\}$ and $\ell_1 \in \{1, 5, 7, 11\}$. Table 6.1 lists $\dim M_{k*}$ for each of the resulting 24 cases. We illustrate with one example: Suppose $k_1 = 6$ and $\ell_1 = 5$. Then

$$\begin{aligned} \dim M_{k*} &= \dim M_k - \dim M_{k-\ell+1} \\ &= \dim M_{12k_0+6} - \dim M_{12(k_0-\ell_0)+2} \\ &= (k_0 + 1) - (k_0 - \ell_0) \\ &= \ell_0 + 1. \end{aligned}$$

Table 6.1: Dimension of $M_{k*}$ when $k \geq \ell - 1$ and $N = 1$

| $\ell_1 \backslash k_1$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 1 | $\ell_0$ | $\ell_0$ | $\ell_0$ | $\ell_0$ | $\ell_0$ | $\ell_0$ |
| 5 | $\ell_0 + 1$ | $\ell_0$ | $\ell_0$ | $\ell_0 + 1$ | $\ell_0$ | $\ell_0$ |
| 7 | $\ell_0 + 1$ | $\ell_0$ | $\ell_0 + 1$ | $\ell_0$ | $\ell_0 + 1$ | $\ell_0$ |
| 11 | $\ell_0 + 2$ | $\ell_0$ | $\ell_0 + 1$ | $\ell_0 + 1$ | $\ell_0 + 1$ | $\ell_0$ |

**Lemma 6.21.** *For $N = 1$, $\ell \geq 5$ prime, and $k \geq 2\ell$, we have* $\dim M_{k*} = \dim M_{k-\ell-1*}$.

*Proof.* A case by case analysis using Table 6.1 shows that this is true. ☐

By Lemma 6.8, $W_1^k$ is a direct sum of $J = 1 + \left\lfloor \frac{C-D+1}{\ell} \right\rfloor$ spaces of the form $M_{i(\ell-1)+D*}$. By Table 6.1,

$$\begin{aligned}\dim M_{i(\ell-1)+D*} &= \ell_0 + \{0, 1, \text{ or } 2\} \\ &= \left\lfloor \frac{\ell}{12} \right\rfloor + \{0, 1, \text{ or } 2\}.\end{aligned}$$

Hence,

$$\dim W_1^k \geq J \left\lfloor \frac{\ell}{12} \right\rfloor.$$

This motivates the following definition of the quantity $\mathfrak{X}$ which appears in the statement of Theorem 1.15.

**Definition 6.22.** Let $\ell \geq 5$ be prime and $k = C(\ell - 1) + D \geq 2\ell$ where $3 \leq D \leq \ell + 1$. Suppose $N = 1$ or 4. If $J$ is as in (1.5.1), and $W_1^k$ is as in Definition 6.6, then set

$$\begin{aligned}\mathfrak{X} :=& \mathfrak{X}(N, \ell, k) \\ :=& \dim W_1^k - \left(1 + \left\lfloor \frac{C - D + 1}{\ell} \right\rfloor\right) \left\lfloor \frac{\ell}{12} \right\rfloor \\ =& \dim W_1^k - J \left\lfloor \frac{\ell}{12} \right\rfloor.\end{aligned}$$

The proof of Theorem 6.20 showed that $\mathfrak{X}(4, \ell, k) = 0$.

**Theorem 6.23** (Main Theorem for $N = 1$)**.** *Let $\ell \geq 5$ be a prime and let $k = C(\ell - 1) + D \geq 2\ell$ be even, where $3 \leq D \leq \ell + 1$. For $N = 1$ and $J = 1 + \left\lfloor \frac{C-D+1}{\ell} \right\rfloor$, we have*

$$\mathcal{P}(\ell, k, 1) = \ell^{-J\left\lfloor \frac{\ell}{12} \right\rfloor - \mathfrak{X} - \dim M_D},$$

*where*

1. *if* $\ell \equiv 1 \mod 12$ *then* $\mathfrak{X} = 0$,
2. *if* $\ell \equiv 5 \mod 12$ *then* $\mathfrak{X} = \left\lfloor \frac{J}{3} \right\rfloor + \delta$ *with*

$$\delta = \begin{cases} 1 & \text{if } J \equiv 1 \mod 3 \text{ and } D \equiv 2 \mod 6 \\ 1 & \text{if } J \equiv 2 \mod 3 \text{ and } D \equiv 2, 4 \mod 6 \\ 0 & \text{otherwise} \end{cases}$$

*3. if* $\ell \equiv 7 \mod 12$ *then* $\mathfrak{X} = \lfloor \frac{J}{2} \rfloor + \delta$ *with*

$$\delta = \begin{cases} 1 & \textit{if } J \equiv 1 \mod 2 \textit{ and } D \equiv 2 \mod 4 \\ 0 & \textit{otherwise} \end{cases}$$

*4. if* $\ell \equiv 11 \mod 12$ *then* $\mathfrak{X} = 5 \lfloor \frac{J}{6} \rfloor + \delta$ *with* $\delta \in \{0, 1, 2, 3, 4, 5\}$*. The term* $\delta$ *is computable in terms of* $J \mod 6$ *and* $D \mod 12$*, but is omitted for the sake of brevity.*

*Proof.* In light of Proposition 6.18, we only need to compute $\dim W_1^k$. We only do the case $\ell \equiv 5 \mod 12$ since the rest are analogous. Recall from the proof of Lemma 6.8 that $\mathcal{J}$ denotes the set of indices appearing in the definition of $W_1^k$. From (6.3.5), every $j \in \mathcal{J}$ is of the form

$$j(t) = \ell\Big(t(\ell - 1) + D - 1\Big) + 1$$

for $0 \leq t \leq \lfloor \frac{C-D+1}{\ell} \rfloor$. Since $\ell \equiv 5 \mod 12$, we get

$$j(t) \equiv 8(t + 1) + 5D \mod 12. \tag{6.6.1}$$

We see from Table 6.1 that for any $t$,

$$\dim M_{j(t)*} = \ell_0 + \{0 \text{ or } 1\}.$$

Notice that for any two consecutive $t, t + 1$, we have

$$\dim M_{j(t)*} + \dim M_{j(t+1)*} = 2\ell_0 + \{0 \text{ or } 1\}$$

and for any three consecutive $t, t + 1, t + 2$, we have

$$\dim M_{j(t)*} + \dim M_{j(t+1)*} + \dim M_{j(t+2)*} = 3\ell_0 + 1.$$

Thus,

$$\begin{aligned} \dim W_1^k &= \left(1 + \left\lfloor \frac{C - D + 1}{\ell} \right\rfloor\right) \ell_0 + \left\lfloor \frac{\left(1 + \lfloor \frac{C-D+1}{\ell} \rfloor\right)}{3} \right\rfloor + \{0 \text{ or } 1\} \\ &= J \left\lfloor \frac{\ell}{12} \right\rfloor + \left\lfloor \frac{J}{3} \right\rfloor + \{0 \text{ or } 1\}. \end{aligned} \tag{6.6.2}$$

Furthermore, from Table 6.1 and (6.6.1) we see that the $\{0 \text{ or } 1\}$ in (6.6.2) is 1 exactly when either of the following occur:

- $\left(1 + \lfloor \frac{C-D+1}{\ell} \rfloor\right) \equiv 1 \mod 3$ and $j(0) \equiv 8 + 5D \equiv 0 \mod 6$.
- $\left(1 + \lfloor \frac{C-D+1}{\ell} \rfloor\right) \equiv 2 \mod 3$ and $j(0) \equiv 8 + 5D \equiv 0$ or $4 \mod 6$.

The conclusion follows by Proposition 6.18 and Definition 6.22. $\square$

# CHAPTER 7

# APPLICATIONS OF MOCK MODULAR FORMS

This chapter has been previously published in [17].

## 7.1 Notations

We recall the definition of a harmonic weak Maass form of half-integral weight $k \in \frac{1}{2}\mathbb{Z}$. Letting $z = x + iy \in \mathbb{C}$, the hyperbolic Laplacian of weight $k$ is

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For $d$ odd, define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ i & \text{if } d \equiv 3 \mod 4. \end{cases}$$

A harmonic weak Maass form of weight $k$ on the congruence subgroup $\Gamma \subset \Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k f(z)$.
2. $\Delta_k f = 0$.
3. $f(z)$ has at most linear exponential growth at all of the cusps.

For a positive integer $N \equiv 0 \mod 4$, the $\mathbb{C}$-vector space of harmonic weak Maass forms of weight $k$ on $\Gamma_1(N)$ is denoted $\widetilde{\mathcal{M}}_k(N)$.

A harmonic weak Maass form is the sum of a holomorphic part and a nonholomorphic part. See Zagier [47] for a nice overview. The harmonic weak Maass forms that we will consider have nonholomorphic parts given by the integral of a cusp form (the *shadow* of the Maass form). Thus, the Fourier expansions for the nonholomorphic parts will only have negative powers of $q$. Recalling that the incomplete Gamma function is defined by

$$\Gamma(\alpha, x) := \int_x^\infty e^{-t} t^{\alpha - 1} dt,$$

the Fourier expansion for a weak Maass form $f(z)$ of the type we consider is

$$f(z) = \sum_{n=n_0}^{\infty} a(n)q^n + \sum_{n=1}^{\infty} b(n)\Gamma(1-k, 4\pi ny)q^{-n},$$

where the first (resp. second) summand is called the holomorphic (resp. nonholomorphic) part of $f(z)$.

We will need some fundamental operators on these forms. For any positive integer $\ell$, define the $U(\ell)$ operator by its action on the Fourier coefficients:

$$f(z)|U(\ell) := \sum a(\ell n)q^n + \sum b(\ell n)\Gamma(1-k, 4\pi ny)q^{-n}.$$

**Lemma 7.1** ([3], Lemma 2.1)**.** *Suppose that $N, \ell$ are positive integers with $4|N$. Define $\ell_0 := \prod_{p|\ell \text{ prime}} p$, let $\ell_1$ be the conductor of $\mathbb{Q}(\sqrt{\ell})$, and set $N' := \mathrm{lcm}(N, \ell_0, \ell_1)$. Then the operator $U(\ell)$ maps $\widetilde{\mathcal{M}}_k(N)$ to $\widetilde{\mathcal{M}}_k(N')$.*

We may also twist a Maass form by a Dirichlet character $\chi$. The effect in terms of the Fourier expansion is

$$f(z) \otimes \chi := \sum \chi(n)a(n)q^n + \sum \chi(-n)b(n)\Gamma(1-k, 4\pi ny)q^{-n}.$$

**Lemma 7.2** ([3], Lemma 2.2)**.** *Suppose that $N$ is a positive integer with $4|N$, that $f(z) \in \widetilde{\mathcal{M}}_k(N)$, and that $\chi$ is a Dirichlet character modulo $r$. Set $N' := \mathrm{lcm}(Nr, r^2)$. Then $f \otimes \chi \in \widetilde{\mathcal{M}}_k(N')$.*

We will frequently transform a Maass form by taking the subseries whose powers of $q$ lie in an arithmetic progression $d \mod t$. This returns a Maass form by the previous lemma since this subseries is given by

$$\frac{1}{\phi(t)} \sum_{\chi \mod t} \overline{\chi}(d) f(z) \otimes \chi,$$

where $\phi(t)$ is Euler's totient function.

## 7.2 Overpartitions

We compute the nonholomorphic part of the Maass form of Bringmann and Lovejoy [10].

**Theorem 7.3.** *Let $t$ be odd. The function (1.6.1) is the holomorphic part of a weight $\frac{1}{2}$ weak Maass form on $\Gamma_1(16t^2)$ whose nonholomorphic part is*

$$-\sqrt{\pi} \sum_{n=1}^{\infty} A(r, t, n)\Gamma\left(\frac{1}{2}, 4\pi yn^2\right) q^{-n^2},$$

*where* $A(r,t,0)=0$ *and for* $0 \le r \le \frac{t-1}{2}$, $0 < n \le \frac{t-1}{2}$,

$$A(r,t,n) = \begin{cases} (-1)^{n+r} & \text{if } r = 2n \text{ or } r = t-2n, \\ (-1)^{n+r}2 & \text{if } r < 2n \text{ and } r < t-2n, \\ 0 & \text{if } r > 2n \text{ or } r > t-2n, \end{cases}$$

*and for all* $r$, $t$, *and* $n$,

$$A(r,t,n) = -A(r,t,n+t) = -A(r,t,-n) = A(r+t,t,n) = A(-r,t,n). \tag{7.2.1}$$

*Remark* 7.4. Using (7.2.2) below, an equivalent formulation of this theorem is to say that the shadow corresponding to (1.6.1) is $-\pi i \sum_{n=1}^{\infty} A(r,t,n) n q^{n^2}$. Theorems 7.5 and 7.8 to follow also have a similar reformulation.

*Proof:* Define $\mathcal{O}(w,q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} \overline{N}(m,n) w^m q^n$ and let $\zeta_t = \exp(2\pi i/t)$. Orthogonality of roots of unity implies that

$$\frac{1}{t} \sum_{j=0}^{t-1} \zeta_t^{-rj} \mathcal{O}(\zeta_t^j, q) = \sum_{n=0}^{\infty} \overline{N}(r,t,n) q^n.$$

Hence $\sum_{n=0}^{\infty} \left( \overline{N}(r,t,n) - \frac{\overline{p}(n)}{t} \right) q^n = \frac{1}{t} \sum_{j=1}^{t-1} \zeta_t^{-rj} \mathcal{O}(\zeta_t^j, q)$. Bringmann and Lovejoy [10, Theorem 1.1] show $\mathcal{O}(\zeta_t^j, q)$ is the holomorphic part of a weak Maass form on $\Gamma_1(16t^2)$ whose nonholomorphic part is given as an integral of theta functions. Using this theorem, the definition [10, Equation (1.7)], the transformation law [10, Equation (3.4)], and some algebraic manipulations we find that the nonholomorphic part is

$$-\frac{\pi\sqrt{2}}{t} \sum_{j=1}^{t-1} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n \zeta_t^{-rj} \zeta_{2t}^{n(4j+t)} \tan\left(\frac{j\pi}{t}\right) \int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i \tau n^2}}{\sqrt{-i(\tau+z)}} d\tau.$$

The integral may be evaluated (via the changes of variable $\tau' = \tau + z$ and $\tau' = -2\pi i n^2 \tau$) as

$$\int_{-\overline{z}}^{i\infty} \frac{e^{2\pi i \tau n^2}}{\sqrt{-i(\tau+z)}} d\tau = \frac{q^{-n^2} i}{\sqrt{2\pi n^2}} \int_{4\pi n^2 y}^{\infty} e^{-\tau} \tau^{-1/2} dt = \frac{q^{-n^2} i}{\sqrt{2\pi}|n|} \Gamma\left(\frac{1}{2}, 4\pi y n^2\right). \tag{7.2.2}$$

Hence the nonholomorphic part corresponding to (1.6.1) is

$$
\begin{aligned}
&-\frac{i\sqrt{\pi}}{t}\sum_{n\neq 0}\frac{n}{|n|}\left(\sum_{j=1}^{t-1}\zeta_t^{-rj}\zeta_{2t}^{n(4j+t)}\tan\left(\frac{j\pi}{t}\right)\right)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2}\\
&=-\frac{i\sqrt{\pi}}{t}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{t-1}\zeta_t^{-rj}\left(\zeta_{2t}^{n(4j+t)}-\zeta_{2t}^{-n(4j+t)}\right)\tan\left(\frac{j\pi}{t}\right)\right)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2}\\
&=-\sqrt{\pi}\sum_{n=1}^{\infty}A(r,t,n)\Gamma\left(\frac{1}{2},4\pi yn^2\right)q^{-n^2},
\end{aligned}
$$

where

$$
A(r,t,n):=(-1)^{n+1}\frac{2}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\sin\left(\frac{4jn\pi}{t}\right)\tan\left(\frac{j\pi}{t}\right).
$$

The periodicity claimed in (7.2.1) follows from that of the summands of $A(r,t,n)$. In addition, clearly $A(r,t,0)=0$. We now have

$$
\begin{aligned}
A(r,t,n)&=(-1)^n\frac{1}{t}\sum_{j=0}^{t-1}\zeta_t^{-rj}\left(\frac{\zeta_{2t}^{4nj}-\zeta_{2t}^{-4nj}}{\zeta_{2t}^{j}+\zeta_{2t}^{-j}}\right)\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\\
&=(-1)^n\frac{1}{t}\sum_{j=0}^{t}\zeta_t^{-rj}\left(\zeta_{2t}^{(4n-1)j}-\zeta_{2t}^{(4n-3)j}+\cdots-\zeta_{2t}^{(-4n+1)j}\right)\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\\
&=(-1)^n\frac{1}{t}\sum_{j=0}^{t}\zeta_t^{-rj}\left(\zeta_t^{2nj}+2\sum_{k=-2n+1}^{2n-1}(-1)^k\zeta_t^{kj}+\zeta_t^{-2nj}\right).
\end{aligned}
$$

We count a contribution of $(-1)^n$ whenever $2n\equiv\pm r \mod t$ and $(-1)^{k+n}\cdot 2$ when $-2n+1\le k\le 2n-1$ with $k\equiv r \mod t$. That is, we must examine how frequently $r+mt\in[-2n,2n]$ for $m\in\mathbb{Z}$. By the assumptions $0\le r\le\frac{t-1}{2}$ and $0<n\le\frac{t-1}{2}$, only $r$ and $r-t$ possibly lie in this interval. If $r\ge 2n$, then $n\le\frac{t-1}{4}$ so $r-t<-2n$ and we only get a contribution when $r=2n$. Otherwise, $r<2n$ and we always get $2(-1)^{r+n}$ plus possibly a contribution depending on the size of $r-t$ relative to $-2n$. For example, if also $r-t=-2n$ then (in addition to the contribution of $2(-1)^{r+n}$ from $0\le r\le 2n$) we also get $(-1)^n$. So here $A(r,t,n)=2(-1)^{r+n}+(-1)^n=-2(-1)^n+(-1)^n=-(-1)^n=(-1)^{r+n}$, since $t$ is odd and so $r$ must be too. The other cases $r-t>-2n$ and $r-t<-2n$ are similar. $\square$

The behavior of $A(r,t,n)$ is illustrated in Table 1 for the values of $A(r,17,n)$.

*Example:* We have $A(2,17,3)-2A(6,17,3)+A(7,17,3)=0$. Recall that we can sift out coefficients which lie in an arithmetic progression. Then $R_{26}(8)-R_{67}(8)$ is a weakly holomorphic modular form since its nonholomorphic part only has terms with $q^{-n^2}$ where $-n^2\equiv 8 \mod 17$, i.e. $n\equiv\pm 3 \mod 17$, and these terms vanish. In fact, $R_{26}(-9)-R_{67}(-9)$ is modular for any prime $t\ge 17$.

*Proof of Theorem 1.17:* $R_{rs}(d)$ is the holomorphic part of a Maass form whose nonholomorphic

Table 7.1: The values of $A(r, 17, n)$.

| $n$ | $r$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -2 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2 | -2 | 2 | -2 | 1 | 0 | 0 | 0 | 0 |
| 3 | -2 | 2 | -2 | 2 | -2 | 2 | -1 | 0 | 0 |
| 4 | 2 | -2 | 2 | -2 | 2 | -2 | 2 | -2 | 1 |
| 5 | -2 | 2 | -2 | 2 | -2 | 2 | -2 | 1 | 0 |
| 6 | 2 | -2 | 2 | -2 | 2 | -1 | 0 | 0 | 0 |
| 7 | -2 | 2 | -2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 8 | 2 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

part is

$$\begin{aligned} &- \sqrt{\pi} \sum_{\substack{n=0 \\ -n^2 \equiv d \mod t}}^{\infty} [A(r,t,n) - A(s,t,n)] \, \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2} \\ &= -\sqrt{\pi} \sum_{\substack{n=0 \\ n \equiv \pm d' \mod t}}^{\infty} \pm \left[A(r,t,d') - A(s,t,d')\right] \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2} \end{aligned} \tag{7.2.3}$$

By Theorem 7.3, in the first case $A(r,t,d') = A(s,t,d') = 0$ and the second case is exactly when $A(r,t,d') = A(s,t,d') = \pm 2$. $\square$

*Proof of Theorem 1.18:* Assume $\left(\frac{-d}{t}\right) = 1$ and let $d'^2 \equiv -d \mod t$ with $0 \le d' \le \frac{t-1}{2}$. Consider Equation (7.2.3). If $d' < \frac{t-1}{4}$, then $A(2d', t, d') - A(2d'+1, t, d') = \pm 1 - 0$, whereas $A(r,t,d') - A(s,t,d') \in [-4, 4]$. Take $F_{d,t} = R_{2d',2d'+1}(d)$. The other cases $d' = \frac{t-1}{4}, d' = \frac{t+1}{4}$ and $d' > \frac{t+1}{4}$ are similar. $\square$

## 7.3 $M_2$-rank of partitions with distinct odd parts

The nonholomorphic part related to $M_2$-rank is given by the following theorem which uses $f_t = 2t/\gcd(t,4)$.

**Theorem 7.5.** *Let $t \ge 2$. The function (1.6.3) is the holomorphic part of a weight $\frac{1}{2}$ weak Maass form on $\Gamma_1(2^{10} f_t^4)$ whose nonholomorphic part is*

$$\frac{-1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \chi(n) B(r,t,n) \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2},$$

*where*

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1,7 \mod 8, \\ -1 & \text{if } n \equiv 3,5 \mod 8, \\ 0 & \text{else,} \end{cases}$$

*and*

$$B(r,t,n) = \begin{cases} \epsilon & \text{if } 2r \equiv 0 \mod t,\ n \equiv 2r+\epsilon \mod 2t,\ \text{with } \epsilon \in \{\pm 1\}, \\ \epsilon/2 & \text{if } 2r \not\equiv 0, \pm 1 \mod t,\ n \equiv \pm 2r+\epsilon \mod 2t,\ \text{with } \epsilon \in \{\pm 1\}, \\ 0 & \text{else.} \end{cases}$$

*Proof:* Theorem 1.2 of [29] specializes to a statement about the $M_2$-rank for partitions without repeated odd parts by restricting to $r = 0$ and $\chi(\lambda) = 0$ in the notation of [29]. Hence we take $a = 0$ and $b = c = 1$ in that theorem to get that

$$\mathcal{N}(w,q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} N_2(m,n) w^m q^n = \sum_n \frac{q^{n^2}(-q;q^2)_n}{(wq^2, q^2/w; q^2)_n}$$

is the $M_2$-rank generating function for partitions without repeated odd parts. Replacing $q$ with $-q$ gives the function which [9, Equation (1.8)] denotes as $\mathcal{K}'(w,z)$, i.e. $\mathcal{N}(w,-q) = \mathcal{K}'(w,z)$. As in the proof of Theorem 7.3, we sum over roots of unity and see that

$$\sum_{n=0}^{\infty} \left( N_2(r,t,n) - \frac{1}{t} N_2(n) \right) (-q)^n = \sum_{j=1}^{t-1} \zeta_t^{-rj} \mathcal{K}'(\zeta_t^j; z).$$

Theorem 4.2 of [13] and the equation at the top of page 12 of [13] show that

$$\sum_{n=0}^{\infty} \left( N_2(r,t,n) - \frac{1}{t} N_2(n) \right) (-1)^n q^{2f_t^2 n - f_t^2/4}$$

is the holomorphic part of a weak Maass form on $\Gamma_1(64f_t^4)$ and expresses the nonholomorphic part in terms of an integral of a theta function. Following the method of the proof of Theorem 7.3, we use [9, Equation 4.6], the formula for $T$ on page 21 of [13] and a series of manipulations to compute that the nonholomorphic part is

$$-\frac{1}{\sqrt{\pi}} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} B(r,t,n) \Gamma\left(\frac{1}{2}, \pi y n^2 f_t^2\right) q^{-n^2 f_t^2/4},$$

where

$$B(r,t,n) := \frac{2}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj} \sin\left(\frac{j\pi}{t}\right)\sin\left(\frac{nj\pi}{t}\right).$$

Apply the $U(f_t^2/4)$ operator to get the weak Maass form

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n) - \frac{1}{t}N_2(n)\right)(-1)^n q^{8n-1} - \frac{1}{\sqrt{\pi}}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} B(r,t,n)\Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2}.$$

To eliminate the $(-1)^n$ in the holomorphic part, twist out the arithmetic progression $15 \mod 16$ and subtract from it the progression $7 \mod 16$. This produces the character $\chi(n)$ in the nonholomorphic part. That is,

$$\sum_{n=0}^{\infty}\left(N_2(r,t,n) - \frac{1}{t}N_2(n)\right) q^{8n-1} - \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} \chi(n) B(r,t,n)\Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2}$$

is a weak Maass form on $\Gamma_1(2^{10}f_t^4)$.

Finally, we may redefine $B(r,t,n) = 0$ for $n$ even. Otherwise for odd $n$,

$$\begin{aligned} B(r,t,n) &= -\frac{1}{2t}\sum_{j=0}^{t-1} \zeta_t^{-rj}\left(\zeta_{2t}^{j} - \zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{jn} - \zeta_{2t}^{-jn}\right) \\ &= \frac{1}{2t}\sum_{j=0}^{t-1} \zeta_{2t}^{j(-n+1-2r)} + \zeta_{2t}^{j(n-1-2r)} - \zeta_{2t}^{j(n+1-2r)} - \zeta_{2t}^{j(-n-1-2r)}. \end{aligned}$$

Since the exponents are even, we have complete sums of $t$th roots of unity. We count contributions exactly when $2t | n \pm 1 \pm 2r$. Elementary considerations show that we have at most two such contributions, that $B = 0, \pm\frac{1}{2}, \pm 1$, and that $B = \pm 1$ implies $2r \equiv 0 \mod t$. If $r \equiv 0 \mod t$ then $B = \pm 1$ exactly when $n \equiv \pm 1 \equiv 2r \pm 1 \mod 2t$. If $r \equiv \frac{t}{2} \mod t$, then $B = \pm 1$ exactly when $n \equiv t \pm 1 \equiv 2r \pm 1 \mod 2t$. If $2r \equiv \pm 1 \mod t$, then $B = 0$ because the contributions will cancel. Otherwise, $B = \pm\frac{1}{2}$ whenever $n \equiv \pm 2r \pm 1 \mod 2t$. $\square$

Using our notation, the corresponding result for the usual partition function computed in [3] is that

$$\sum_{n=0}^{\infty}\left(N(r,t,n) - \frac{1}{t}p(n)\right) q^{24n-1} - \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} \psi(n) B(r,t,n)\Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2} \tag{7.3.1}$$

is a weak Maass form, where

$$\psi(n) = \begin{cases} 1 & \text{if } n \equiv 1, 11 \mod 12 \\ -1 & \text{if } n \equiv 5, 7 \mod 12 \\ 0 & \text{else.} \end{cases} \tag{7.3.2}$$

The Maass forms of Theorem 7.5 and (7.3.1) have very similar nonholomorphic parts and as $r$ varies they will satisfy the same linear relations. Hence, theorems analogous to those in [3] hold for the $M_2$-rank generating function. For example, compare the following with Corollary 1.5 of that paper.

*Example:* For $t$ prime and $2 \leq r \leq t-2$,

$$\sum_{8n-1 \not\equiv -9, -(2r\pm 1)^2 \mod t} \left(N_2(0,t,n) + 2N_2(1,t,n) - 3N_2(r,t,n)\right) q^{8n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$ since

$$\begin{aligned} &B(0,t,n) + 2B(1,t,n) - 3B(r,t,n) \\ &\qquad = \begin{cases} (\pm 1) + 2(\mp \frac{1}{2}) + 0, & \text{if } n \equiv \pm 1 \mod 2t \\ 0 + 2(0) + 0, & \text{if } n \not\equiv \pm 1, \pm 3, \pm 2r \pm 1 \mod 2t. \end{cases} \end{aligned}$$

A useful corollary of Theorem 7.5 is

**Corollary 7.6.** *If $t \geq 2$, then $1 - 8d \not\equiv (2r \pm 1)^2 \mod t$ if and only if*

$$\sum_{\substack{n=0 \\ n \equiv d \mod t}} \left(N_2(r,t,n) - \frac{1}{t} N_2(n)\right) q^{8n-1} \tag{7.3.3}$$

*is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(2^{10} f_t^4 t)$.*

*Proof:* By Theorem 7.5, (7.6) is the holomorphic part of a Maass form whose nonholomorphic part is supported on $q^{-n^2}$ where $-n^2 \equiv d \mod t$. The given parameters are exactly where $B$ vanishes. □

*Proof of Theorem 1.19:* Immediate from Corollary 7.6. □

*Proof of Theorem 1.20:* Analogous to Theorem 1.18. □

If we take the primitive character $\phi(n) = \chi^{-1}(n)\psi(n)$ with conductor 24 then we have the following amusing theorem.

**Theorem 7.7.** *Let $t$ be odd with $3 \nmid t$. Then*

$$\sum_{n=0}^{\infty} \left(N_2(r,t,3n) - N(r,t,n) - \frac{N_2(3n) - p(n)}{t}\right) q^{24n-1}$$

*is a weight $\frac{1}{2}$ weakly holomorphic modular form on $\Gamma_1(2^{16}3^3 f_t^4)$.*

*Proof:* Take the subseries of the Maass form of Theorem 7.5 supported on $q$ with exponents $\equiv 23 \mod 24$ and then twist by $\phi(n)$. This has the same nonholomorphic part as (7.3.1). ☐

## 7.4 2-marked Durfee symbols

Our final object of study has a nonholomorphic part whose coefficients are more complicated to describe.

**Theorem 7.8.** *If $0 \leq r < t$ are integers with $2, 3 \nmid t$ then (1.6.5) is the holomorphic part of a weight $\frac{1}{2}$ weak Maass form on $\Gamma_1(576t^4)$ whose nonholomorphic part is given by*

$$-\frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \psi(n) C(r,t,n) \Gamma\left(\frac{1}{2}, 4\pi y n^2\right) q^{-n^2},$$

*where $\psi$ is as in (7.3.2) and $C(r,t,n)$ is a function defined by the following properties. For all odd $n$ and all $r$,*

$$C(r,t,n) = C(r+t,t,n) = C(t-r,t,n) = C(r,t,n+2t) = -C(r,t,2t-n). \tag{7.4.1}$$

*For all $\overline{r} \in [0, t/2]$ and odd $\overline{n} \in [1,t]$, $C(\overline{r},t,\overline{n}) - \frac{\overline{n}}{t} \in \{-2,-1,0,1\}$. Moreover, Table 2 allows one to determine the exact value of this quantity according to the following instructions.*

Table 7.2: The function $C(r,t,n)$ is defined using the instructions following Theorem 7.8.

| $\overline{r} \mod 3$ | $\overline{n} \mod 3$ | | |
|---|---|---|---|
| | 0 | 1 | 2 |
| 0 | $\boxed{\overline{n} \geq 2\overline{r}+3}$ | $\overline{n} \geq \overline{r}+1$ | $\overline{n} \geq \overline{r}+2$ |
| 1 | $\overline{n} \geq \overline{r}+2$ | | $\frac{\overline{n}+3}{2} \leq \overline{r} \leq \overline{n}-1$ |
| 2 | $\overline{n} \geq \overline{r}+1$ | $\frac{\overline{n}+3}{2} \leq \overline{r} \leq \overline{n}-2$ | |
| t-1 | $\overline{n} \geq t-\overline{r}+2$ | | $\frac{\overline{n}+3}{2} \leq t-\overline{r} \leq \overline{n}-1$ |
| t | | $\overline{n} \geq t-\overline{r}+1$ | $\overline{n} \geq t-\overline{r}+2$ |
| t+1 | $\overline{n} \geq t-\overline{r}+1$ | $\frac{\overline{n}+3}{2} \leq t-\overline{r} \leq \overline{n}-2$ | |

Find the appropriate column and the two appropriate rows based on the congruence classes mod 3. For each of the corresponding table entries, if there is a set of inequalities listed, and if $\overline{n}, \overline{r}, t$ satisfy those inequalities, count a contribution of -1. If the entry is blank, there is no contribution. The only exception is $\overline{n} \equiv \overline{r} \equiv 0 \mod 3$ which counts +1 when $\overline{n} \geq 2\overline{r}+3$. Consider for example Table 3 which shows $C(\overline{r}, 29, \overline{n}) - \frac{\overline{n}}{29}$.

Table 7.3: The values of $C(\overline{r}, 29, \overline{n}) - \frac{\overline{n}}{29}$.

| $\overline{n}$ | $\overline{r}$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | -1 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | -1 | 0 | 0 | -1 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 1 | -1 | -1 | 1 | -1 | -1 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | -1 | 0 | -1 | -1 | 0 | 0 | 0 | 0 |
| 13 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | -1 | -1 | 0 | -1 | -1 | 0 | 0 |
| 15 | 1 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | -1 | 0 | -1 | -1 | 0 | -1 | -1 |
| 17 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | -1 | 0 | -1 | -2 | -1 |
| 19 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | -2 | -2 | 0 | -2 |
| 21 | 1 | -1 | -1 | 1 | -1 | -1 | 1 | -1 | -1 | 0 | -2 | -1 | -1 | -2 | -1 |
| 23 | -1 | 0 | 0 | -1 | 0 | 0 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -2 | -1 |
| 25 | -1 | 0 | 0 | -1 | 0 | -1 | -2 | 0 | -1 | -2 | 0 | -1 | -2 | 0 | -2 |
| 27 | 1 | -1 | -1 | 0 | -2 | -1 | 0 | -2 | -1 | 0 | -2 | -1 | 0 | -2 | -1 |
| 29 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |

*Proof:* Define the full rank generating function

$$\mathcal{R}_2(w, q) := \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} NF_2(m, n) w^m q^n.$$

Andrews [5] showed that for $w^3 \neq 1$,

$$\mathcal{R}_2(w, q) = \frac{w^2}{(1-w)(w^3-1)} (\mathcal{R}(w, q) - \mathcal{R}(w^2, q)), \tag{7.4.2}$$

where

$$\mathcal{R}(w, q) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} N(m, n) w^m q^n$$

is the usual partition rank generating function. By (7.4.2),

$$\sum_{n=0}^{\infty}\left(NF_2(r,t,n)-\frac{1}{t}\mathcal{D}_2(n)\right)q^n = \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\mathcal{R}_2(\zeta_t^j,q)$$
$$= \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\left(\frac{\zeta_t^{2j}}{(1-\zeta_t^j)(\zeta_t^{3j}-1)}\right)\left(\mathcal{R}(\zeta_t^j;q)-\mathcal{R}(\zeta_t^{2j};q)\right)$$
$$= \frac{1}{4t}\sum_{j=1}^{t-1}\left(\frac{\zeta_t^{-rj}}{\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{3\pi j}{t}\right)}\right)\left(\mathcal{R}(\zeta_t^j;q)-\mathcal{R}(\zeta_t^{2j};q)\right).$$

By Theorem 1.2 of [12], $\mathcal{R}(\zeta_t^j;q)$ is essentially the holomorphic part of a weak Maass form. Continuing as in the proof of Theorem 7.3, we find the nonholomorphic part is

$$-\frac{1}{2\sqrt{\pi}}\sum_{n\equiv 1 \mod 6}(-1)^{\frac{n-1}{6}}\frac{n}{|n|}C(r,t,n)\Gamma\left(\frac{1}{2},4\pi y n^2\right)q^{-n^2}$$
$$= -\frac{1}{2\sqrt{\pi}}\sum_{n=1}^{\infty}\psi(n)C(r,t,n)\Gamma\left(\frac{1}{2},4\pi y n^2\right)q^{-n^2},$$

where

$$C(r,t,n) := \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}\frac{\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi n j}{t}\right)-\sin\left(\frac{2\pi j}{t}\right)\sin\left(\frac{2\pi n j}{t}\right)}{\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{3\pi j}{t}\right)}.$$

The periodicity claimed in (7.4.1) follows easily. Now for $r=\overline{r}\in[0,t/2]$ and odd $n=\overline{n}\in[1,t]$ we have

$$C(r,t,n) = \frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\left(\zeta_{2t}^{2j}-\zeta_{2t}^{-2j}\right)\left(\zeta_{2t}^{2nj}-\zeta_{2t}^{-2nj}\right)}{\left(\zeta_{2t}^{j}-\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}\right)}\right]$$
$$= \frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\left(\zeta_{2t}^{j}+\zeta_{2t}^{-j}\right)\left(\zeta_{2t}^{2nj}-\zeta_{2t}^{-2nj}\right)}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}\right]$$
$$= \frac{1}{t}\sum_{j=1}^{t-1}\zeta_{2t}^{-2rj}\left[\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}-\zeta_{2t}^{(2n+1)j}+\zeta_{2t}^{(-2n+1)j}-\zeta_{2t}^{(2n-1)j}+\zeta_{2t}^{(-2n-1)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}\right].$$

For each congruence class of $n$ mod 3, there is an appropriate grouping of the numerator terms allowing the $\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}$ to cancel. For example, if $n\equiv 0 \mod 3$,

$$C(r,t,n) = \frac{1}{t}\sum_{j=1}^{t-1}-\frac{\zeta_{2t}^{(2n+1-2r)j}-\zeta_{2t}^{(-2n+1-2r)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}-\frac{\zeta_{2t}^{(2n-1-2r)j}-\zeta_{2t}^{(-2n-1-2r)j}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}+\frac{\zeta_{2t}^{nj}-\zeta_{2t}^{-nj}}{\zeta_{2t}^{3j}-\zeta_{2t}^{-3j}}.$$

After dividing, all of the resulting terms will have an even exponent. Hence we'll have a collection of $n$ incomplete sums of $t$th roots of unity. Completing these sums will require adding in the $j = 0$ terms. In effect, we subtract off $1/t$ for each of the $n$ sums. Continuing with the $n \equiv 0 \mod 3$ case, $C(r, t, n) - n/t$ will get a contribution of $-1$ each time

$$t | n - r - 1, n - r - 4, \ldots, -n - r + 2 \tag{7.4.3}$$

$$t | n - r - 2, n - r - 5, \ldots, -n - r + 1 \tag{7.4.4}$$

and get a contribution of 1 each time

$$t | \frac{n-3}{2} - r, \frac{n-3}{2} - 3 - r, \ldots, -\frac{n-3}{2} - r. \tag{7.4.5}$$

By hypotheses on $n, r$ we have

$$t > n - r - 1 > \cdots > -n - r + 2 > -2t$$

and so one of the conditions in (7.4.3) will occur when both $n - r - 1 \equiv 0 \mod 3$ and $n - r - 1 \geq 0$ or when both $n - r - 1 \equiv -t \mod 3$ and $-n - r + 2 \leq -t$. This gives the table entry for $n \equiv 0 \mod 3$, $r \equiv 2 \mod 3$ and the entry for $n \equiv 0 \mod 3$, $r \equiv t - 1 \mod 3$. The rest of the cases are similar. □

The restriction $2 \nmid t$ in this theorem may be removed by taking a different congruence subgroup using Theorem 1.1 of [12]. As a general indication of the utility of Theorem 7.8, we provide two examples.

*Example:* Since $2C(3, 29, 25) - C(6, 29, 25) - C(7, 29, 25) = 2(-1) - (-2) - (0) = 0$, we deduce that

$$\sum_{n \equiv 3 \mod 29} \left[2NF_2(3, 29, n) - NF_2(6, 29, n) - NF_2(7, 29, n)\right] q^{24n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.

*Example:* Since

$$3C(6, 29, 21) + C(8, 29, 21) + C(10, 29, 21) - 5C(9, 29, 21) = 3(1) + (-1) + (-2) - 5(0),$$

we deduce that

$$\sum_{n \equiv 1 \mod 29} \left[3NF_2(6, 29, n) + NF_2(8, 29, n) + NF_2(10, 29, n) - 5NF_2(9, 29, n)\right] q^{24n-1}$$

is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.

Analogously with overpartitions, we define the generating functions of the full rank differences:

$$S_{rs}(d) = \sum_{n \equiv d \mod t} \left[ NF_2\left(r, t, \frac{n+1}{24}\right) - NF_2\left(s, t, \frac{n+1}{24}\right) \right] q^n.$$

This is the holomorphic part of a Maass form supported on $q^{-n^2}$ with $-n^2 \equiv d \mod t$. As noted before, when $\left(\frac{-d}{t}\right) = -1$, $S_{rs}(d)$ is a weakly holomorphic modular form. When $\left(\frac{-d}{t}\right) \neq -1$, the nonholomorphic part may still be zero. The exact situation is quite complicated and it is difficult to express general theorems that are aesthetically pleasing. However, the following corollaries give some idea of the types of possible conclusions.

**Corollary 7.9.** *Let $t \geq 5$ be prime. For all $r, s$, $S_{rs}(0)$ is a weakly holomorphic modular form on $\Gamma_1(576t^5)$.*

*Proof:* A case by case analysis of Theorem 7.8 reveals that regardless of the congruence class of $r \mod 3$, $C(r, t, t) = 0$. Hence

$$\sum_{n \equiv 0 \mod t} \left[ NF_2\left(r, t, \frac{n+1}{24}\right) - \frac{1}{t}\mathcal{D}_2\left(\frac{n+1}{24}\right) \right] q^n$$

is a weakly holomorphic modular form, and so $S_{rs}(0)$ must be too. □

**Corollary 7.10.** *If $t = 7$ then $S_{rs}(d)$ is a weakly holomorphic modular form exactly when one of the following is true:*

1. *$d = 0, 1, 2, 4$, or*
2. *$d = 3, 5$ and $r, s \in \{1, 2, 5, 6\}$, or*
3. *$d = 3, 5$ and $r, s \in \{3, 4\}$.*

**Corollary 7.11.** *If $t = 7$ then*

$$\sum_{n \equiv 5 \mod 7} \left[ NF_2\left(0, 7, \frac{n+1}{24}\right) + NF_2\left(1, 7, \frac{n+1}{24}\right) - 2NF_2\left(3, 7, \frac{n+1}{24}\right) \right] q^n$$

*is a weakly holomorphic modular form.*

**Corollary 7.12.** *If $3 \nmid t$ then*

$$\sum_{n=0}^{\infty} \left[ NF_2\left(1, t, \frac{n+1}{24}\right) - NF_2\left(2, t, \frac{n+1}{24}\right) \right] q^n$$

*is a weakly holomorphic modular form.*

A similar statement can be made about the generating function of $NF_2(r, t, \frac{n+1}{24}) - NF_2(r + 1, t, \frac{n+1}{24})$ where $r \equiv 1 \mod 3$, except that we must twist out some arithmetic progressions as per Theorem 7.8.

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