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BIJECTIVE PROOFS OF PARTITION IDENTITIES AND COVERING SYSTEMS

BY

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DISSERTATION

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Abstract

This dissertation involves two topics. The first is on the theory of partitions, which is discussed in Chapters $2-5$. The second is on covering systems, which are considered in Chapters $6-8$.

In 2000, Farkas and Kra used their theory of theta functions to establish a beautiful theorem on colored partitions, and they asked for a bijective proof of it. In Chapter 2, we give a bijective proof of a more general partition identity, with the Farkas and Kra partition theorem being a special case. We then derive three further general partition identities and give bijective proofs of these as well.

The quintuple product identity is one of the most famous and useful identities in the theory of theta functions and q-series, and dates back to 1916 or earlier. In his recent survey paper on this identity, Shaun Cooper remarked that there does not exist a bijective proof of it. In Chapter 3, employing bijective proofs of Jacobi's triple product identity and Euler's pentagonal number theorem, we provide the first bijective proof of the quintuple product identity.

In a recent paper, *The parity in partition identities*, George Andrews investigated parity questions in partition identities and listed 15 open problems at the end of his paper. In Chapter 4, we provide solutions to the first two open problems suggested by Andrews. More precisely, we provide combinatorial proofs of two partition identities which were derived by comparing Andrews' new identity with Göllnitz-Gordon identities or certain generalizations thereof.

In our last chapter on partitions, Chapter 5, we give a combinatorial proof of a companion to Euler's famous recurrence formula for the sum of divisors function $\sigma(n)$. Euler's recurrence formula had previously been combinatorially proved using a double counting argument, but its equally famous companion has not heretofore been established combinatorially. We not only provide such a combinatorial proof, but we also give a combinatorial proof of a vast generalization as

well.

M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu proved that if the least modulus N of a covering system is sufficiently large, then the sum of reciprocals of the moduli is bounded below by a function of N, tending to ∞ as $N \to \infty$, which confirms a conjecture of P. Erdős and J. L. Selfridge. They also showed that, for $K > 1$, the complement in $\mathbb{Z}$ of any union of residue classes $r(n) \pmod n$ with distinct $n \in (N, KN]$ has density at least d_K for N sufficiently large, which implies a conjecture of P. Erdős and R. L. Graham. In Chapter 6, we first define covering systems in number fields, and extend those results to arbitrary number fields.

In Chapter 7, we give an explicit version of their first theorem to provide a specific number for the least modulus of a covering system, where the reciprocal sum is strictly bigger than 1.

In the last chapter, Chapter 8, we consider exact covering systems in number fields. Motivated by the theorem of Davenport, Mirsky, Newman and Rado that there does not exist an exact covering system with distinct moduli, we raise the question whether or not this is true for covering systems in algebraic number fields. We provide affirmative answers for certain quadratic fields.

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Chapter 1

Introduction

The theory of partitions was initiated by L. Euler who proved many beautiful partition theorems, most notably the pentagonal number theorem and 'the number of partitions of n into distinct parts is equal to that into odd parts'. The theory has been developed by many great mathematicians– Gauss, Jacobi, Sylvester, Lebesque, MacMahon and Ramanujan for example– and has blossomed in the past few decades. Most partition identities were derived from generating functions in terms of q-series and hypergeometric series. Thus, they were first proved analytically, and some of them were proved combinatorially many years later. It is not obvious that finding a direct bijective proof of each partition identity is always a feasible task. We have few partition identities which have such proofs, and many partition theorems remain whose bijections are still obscure. Constructive partition theory is rich and powerful due to the ingenuity of bijective proofs, and also has the benefit that more general results can be sometimes derived from those proofs.

The topics discussed in Chapters $2, 3, 4$ and 5 are on partitions, mainly bijective proofs of certain partition identities. In Chapter 2, we establish four new partition identities and also give bijective proofs of them. H. M. Farkas and I. Kra [19, 20] seem to be the first mathematicians who related modular equations with partition theorems. The following theorem is the most elegant of their partition theorems, which is equivalent to a modular equation of degree 7, and Farkas asked for a bijective proof of it without the use of theta functions.

Theorem 1.1. *Consider the positive integers such that multiples of 7 occur in two copies, say $7k$ and $\overline{7k}$. Let $A(N)$ be the number of partitions of the even integer $2N$ into distinct even parts, and* *let $B(N)$ be the number of partitions of the odd integers $2N+1$ into distinct odd parts. Then*

$$A(N) = B(N).$$

For example, $A(8) = 7 = B(8)$, with the representations of 16 and 17 being given respectively by

$$16 = 14 + 2 = \overline{14} + 2 = 12 + 4 = 10 + 6 = 10 + 4 + 2 = 8 + 6 + 2,$$
$$17 = 13 + 3 + 1 = 11 + 5 + 1 = 9 + 7 + 1 = 9 + \overline{7} + 1 = 9 + 5 + 3 = 7 + \overline{7} + 3.$$

In [61], S. O. Warnaar mentioned that establishing a bijection between the partitions counted by $A(N)$ and $B(N)$ seems to be quite difficult. Although not finding a bijective proof of Theorem 2.1, he established a generalization of the modular equation of degree 7 which is the generating function identity of Theorem 2.1, and also gave a combinatorial proof of his generalized identity. Fortunately, we could establish a bijection for Theorem 2.1, which also works for the generalized theorem, by using Warnaar's aforementioned combinatorial proof. Besides, we derive a more generalized partition identity from Warnaar's generating function since his theorem is indeed a special case of it.

The bijection for Theorem 2.1 is so adjustable and powerful that we could construct three more generalized partition identities of similar kinds. We show one of them below, which will be mentioned again in Chapter 2.

Theorem 1.2. *Let m be a positive integer, and let α, β and γ be odd positive integers $\leq m$ with $\alpha + \beta + \gamma < 2m$. Consider the positive integers in which multiples of $2m$ occur in two copies, $2m$ and $\overline{2m}$. Let $A(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(\alpha+\beta+\gamma) \pmod{2m}$, and let $B(N)$ denote the number of partitions of $2N$ into parts congruent to $0, \overline{0}$, $\pm(\alpha+\beta)$, $\pm(\beta+\gamma)$, $\pm(\alpha+\gamma) \pmod{2m}$. Then, $A(N) = B(N)$.*

We close Chapter 2 with some applications of the four partition identities. Some of them are known from [8] and others are new.

In Chapter 3, we provide a combinatorial proof of the quintuple product identity

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n} - x^{-3n-1}) = \prod_{n=1}^{\infty}(1 - xq^n)(1 - q^n)(1 - x^{-1}q^{n-1})(1 - x^2q^{2n-1})(1 - x^{-2}q^{2n-1}).$$

The quintuple product identity is one of the most well known identities, and various applications can be found. For instance, in [10] B.C. Berndt proved many of Ramanujan's claims using the identity, and it can be applied to prove other identities such as Winquist's identity. The history of the quintuple product identity dates back to 1916, when R. Fricke presented it in terms of theta functions. It had been believed that Ramanujan also discovered the identity even though any general form is not found in his notebooks. In 1988, K. G. Ramanathan confirmed this belief by reporting that the identity appears in Ramanujan's Lost Notebook in a different form.

The quintuple product identity is often referred to as Watson's quintuple product identity, since in 1929 and 1938 G. N. Watson [59, 60] gave two proofs of the identity in proving some of Ramanujan's results. Among others, W. N. Bailey [7], D. B. Sears [51] and L. J. Slater [52] also gave proofs of the identity. O. L. Atkin and P. Swinnerton-Dyer [6] established the identity without knowing of its prior occurrence. Also, in 1961 B. Gordon [28] rediscovered the quintuple product identity. Since then, various proofs of the identity have been published. Recently, in his comprehensive survey paper on the quintuple product identity, S. Cooper [13] mentioned that it is interesting that no direct combinatorial proof has yet to be published, while at least 29 proofs of the identity are known. The key idea, in proving the quintuple product identity combinatorially in Chapter 3 is combining three known bijections. We apply two bijections of Jacobi's triple product identity in different forms, and in order to complete the proof, we also employ a bijective proof of Euler's pentagonal number theorem.

Parity has played a role in partition identities from the beginning. Most likely, the first theorem in the history of partitions is Euler's aforementioned famous discovery that the number of partitions of a positive integer n into distinct parts equals the number of partitions of n into odd parts.

Equivalently in terms of generating functions: for $|q| < 1$, [3, p. 5, eq. (1.2.5)]

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}.$$

In his recent paper [4], G. E. Andrews investigated a variety of parity questions in partition identities. At the end of the paper, he then listed 15 open problems. In Chapter 4, we provide answers to the first three problems from his list, which are related to the Göllnitz-Gordon identities and their generalizations. The famous first Rogers-Ramanujan identity (the number of partitions of n into nonconsecutive parts is equal to the number of partitions of n into parts congruent to 1 or 4 mod 5) does not involve parity. However, introducing a parity consideration to the identity yields a new partition identity, which is called the first Göllnitz-Gordon Identity. There are several results of this sort, especially related to Rogers-Ramanujan's identities. This motivated the deeper examination of parity in partition identities by Andrews.

Andrews derived a new partition identity by considering the parity restriction that even parts appear an even number of times in the celebrated Rogers-Ramanujan-Gordon identity [1, 30], which is a generalization of the Rogers-Ramanujan identities. He then compared two special cases with the first and second Göllnitz-Gordon identities to deduce a pair of new identities. The first and second questions from the list are to find bijective proofs of them. We provide answers to those questions in the second section of Chapter 4. The third problem is to prove bijectively the generalization of the first two problems, which was derived by comparing the aforementioned Andrews theorem and a generalization of the Göllnitz-Gordon identities, also by Andrews [2]. In the third section, we give an answer to the third question. This is joint work with Ae Ja Yee.

Next, we discuss Euler's recurrence formula for the sum of divisors $\sigma(n)$.

Theorem 1.3. *For every $n > 0$, we have*

$$\sum_{k=-\infty}^{\infty}(-1)^k\sigma\left(n-\frac{k(3k+1)}{2}\right) = \begin{cases} (-1)^{k-1}n, & \textit{if } n=\frac{k(3k+1)}{2},\, k\in\mathbb{Z}, \\ 0, & \textit{otherwise.}\end{cases}$$

Theorem 1.3 can be easily derived analytically from the pentagonal number theorem, and a combinatorial proof was also given in [45] and [58], which is based on a double counting argument. Now, we consider a companion of Theorem 1.3.

Theorem 1.4. *Let* $n \geq 1$. *Then,*

$$-\sigma(n) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{k(3k+1)}{2} p(n - \frac{k(3k+1)}{2}).$$

An analytic proof of Theorem 1.4 is elementary. However, a combinatorial proof does not seem to have been given. In Chapter 5, we generalize Theorem 1.4 and give a combinatorial proof of it, which is also based on a double counting argument. We also employ the quintuple product identity in a similar argument to derive another recurrence relation and its companion. We remark that combinatorial proofs of them can be given in a similar fashion.

The second topic of this dissertation is covering systems, which are discussed throughout Chapters $6, 7$ and 8. A finite collection of congruence classes, $\{a_1 \pmod{m_1}, \ldots, a_k \pmod{m_k}\}$ with $m_i > 1$ is called a covering system if each integer lies in at least one of them. The concept of a covering system was first introduced by P. Erdös in 1950, who answered, in the negative, Romanoff's question: Can every sufficiently large odd integer be expressed as the sum of a power of 2 and a prime? Erdös was particularly interested in covering systems with distinct moduli. We remark that the following is a covering system with the least number of distinct moduli :

$$0 \pmod 2, \quad 0 \pmod 3, \quad 1 \pmod 4, \quad 1 \pmod 6, \quad 11 \pmod{12}. \tag{1.1}$$

In his proof [15], he used (1.1) with the last two classes replaced by

$$3 \pmod 8, \quad 7 \pmod{12}, \quad 23 \pmod{24}.$$

Here are some famous conjectures concerning covering systems.

Least Modulus Problem (Erdös' conjecture): Can the least modulus in a covering system with distinct moduli be arbitrarily large?

D. J. Gibson [24] found a covering system with distinct moduli where the least modulus is 25, and a covering system with distinct moduli ≥ 40 has been recently discovered by P. Nielson [44].

Odd Moduli Problem : Is there a covering system with distinct odd moduli?

Schinzel's Conjecture : In every covering system, there is a modulus that divides one of the others.

In [18], J. Fabrykowski and T. Smotzer gave a simple proof showing that if Schinzel's Conjecture is false, then there exists an odd covering.

Recently, some conjectures of P. Erdös, J. L. Selfridge and R. L. Graham were confirmed by M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu [22]. Erdös and Selfridge [14] conjectured the following.

Conjecture I. *For any number B, there is a number N_B, such that in a covering system with distinct moduli greater than N_B, the sum of reciprocals of these moduli is greater than B.*

It's also interesting to study systems of residue classes where the moduli are distinct and come from an interval $(N, KN]$. Erdös and Graham [16] made the following conjecture.

Conjecture II. *For each number $K > 1$ there is a positive number d_K such that if N is sufficiently large, depending on K, and we choose arbitrary integers $r(n)$ for each $n \in (N, KN]$, then the complement in $\mathbb{Z}$ of the union of the residue classes $r(n) \pmod{n}$ has density at least d_K.*

In [22], stronger forms of these conjectures were proved by the aforementioned authors. In Chapter 6, we generalize the results from [22] to arbitrary number fields. We first define a covering system in a number field to be a finite set of cosets of ideals, whose union is the ring of the integers. Covering systems of groups by subgroups or cosets of subgroups, which is the most natural generalization of covering systems of $\mathbb{Z}$, have been investigated by B. H. Neumann [44] , M. M. Parmenter [46, 47] and Z. W. Sun [55, 56]. However, we restrict the moduli of covering systems in a number field to ideals, instead of arbitrary subgroups, in order to take advantage of various properties of ideals which are crucial in the proofs. We remark that the results in Chapter

6 are exactly as strong as those in [22]. Even though the methods are borrowed from [22], it is not straightforward that the arguments of [22] can be generalized to arbitrary number fields. The main difference and difficulty come from the function counting the number of ideals of norm n, which has irregular behavior. In particular, in [22], the authors applied the standard upper-bound estimates for the distribution of smooth numbers (numbers without large prime factors). However, in the number field setting, another method is required to understand the counting function, which is described in Lemma 6.9.

If a covering system covers every integer exactly once, then it is said to be an exact covering system. The following are two simple examples of exact covering systems:

$$\{0 \pmod 2, 1 \pmod 2\}, \qquad \{0 \pmod 2, \quad 1 \pmod 4, \quad 3 \pmod 4\}.$$

It is obvious that the sum of the reciprocals of the moduli of a covering system is at least 1. Furthermore, in an exact covering system, the reciprocal sum of the moduli is exactly 1, and by a density argument the reverse is also true. Here, one might ask if there is any exact covering system with distinct moduli. The Davenport-Mirsky-Newman-Rado result shows that there is no such exact covering system and in fact, the largest modulus must be repeated. We present their proof here.

Proposition 1.1. *If $\{r_i \pmod{n_i}\}_{i=1}^{l}$ is an exact covering systems with $n_1 \leq \cdots \leq n_l$, then we have $n_{l-1} = n_l$.*

Proof. Suppose $n_{l-1} < n_l$. We can assume that $0 \leq r_i < n_i$ for each i. Then,

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{i=1}^{l}(z^{r_i} + z^{r_i+n_i} + z^{r_i+2n_i} + \cdots) = \sum_{i=1}^{l} \frac{z^{r_i}}{1-z^{n_i}}$$

Letting z tend to a primitive n_l-th roof of unity, we have a pole on the right side of the equation, but not on the left side. Hence, $n_{l-1} = n_l$. $\square$

Thus, in other words, in a covering system with distinct moduli, the reciprocal sum of the moduli is strictly bigger than 1 and so there must be some overlap between the congruence classes. It is interesting to consider congruence classes that cover the set of integers with as little overlap as possible. Covering systems with distinct moduli are known with least modulus $2, 3$ and 4, where the reciprocal sum of the moduli can be arbitrarily close to 1 (see [32], §F13). As we have seen from the Conjecture I, which was confirmed in [22], this fails for all large enough choices of the least modulus. This motivates us to find a specific value of the least modulus. In Chapter 7, we prove an explicit version of the theorem from [22], which proves Conjecture I. This enables us to specify a number N such that if the least modulus of a covering system with distinct moduli is larger than N, then the reciprocal sum of the moduli is strictly larger than 1. We prove a slightly weaker form than the one in [22], due to the difficulty of obtaining estimates for the distribution of smooth numbers with explicit constants.

From Proposition 1.1, we have seen that there is no exact covering systems with distinct moduli and in fact, the largest modulus should be repeated. This leads naturally to a question:

Can we find an exact covering system with distinct moduli in a number field?

To begin with, we consider quadratic number fields whose rings of integers are principal ideal domains. In Chapter 8, we prove slightly stronger forms of the following result.

Theorem 1.5. *Let $S = \{r_1 + I_1, \ldots, r_k + I_k\}$ be an exact covering system in a quadratic number field $\mathbb{Q}(\sqrt{m})$, where the I_i's are principal ideals and I_k has the largest norm. Then, I_k must be repeated.*

The approach is somewhat analogous to that of the integer case shown above, but the argument is much more complex. We start with the two variable function

$$\frac{1}{1-z}\frac{1}{1-w} = \sum_{u,v\geq 0} z^u w^v. \tag{1.2}$$

By identifying $a + b\sqrt{m}$ with $(a, b) \in \mathbb{Z}^2$, we find a corresponding set $A_i \in \mathbb{Z}^2$ for each $r_i + I_i$, where each element in A_i is represented using the generator of I_i. After setting up an explicit identity starting from (1.2), we devise a similar argument to that in Proposition 1.1 involving double poles.

Ideally, one would like to prove the same results for all quadratic fields, or for all number fields. Actually, we conjecture that there is no exact covering system with distinct moduli in any number field. However, we could not even settle this conjecture for all quadratic number fields. In the rest of Chapter 8, we present a partial result, which proves that the above is true for certain imaginary quadratic fields with two ideal classes. Since ideals may not be principal in these settings, we classify the ideals according to the ideal classes of the field, and also we use some elementary algebraic facts. Furthermore, we need to take the coefficients of terms with double poles into consideration to establish such a result. In considering more general cases, it seems that we need to introduce new tools.

Chapter 2

Bijective Proofs of Partition Identities Arising from Modular Equations

2.1 Introduction

H.M Farkas and I. Kra [19], [20] established certain theta constant identities and observed that they are equivalent to partition identities. As we mentioned in Chapter 1, the following theorem is the most elegant of their partition theorems, and Farkas asked for a bijective proof of it without the use of theta functions.

Theorem 2.1. *Consider the positive integers such that multiples of 7 occur in two copies, say $7k$ and $\overline{7k}$. Let $A(N)$ be the number of partitions of the even integer $2N$ into distinct even parts, and let $B(N)$ be the number of partitions of the odd integers $2N+1$ into distinct odd parts. Then*

$$A(N) = B(N).$$

It is not hard to see that the generating function identity of Theorem 2.1 is

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty(q^7;q^{14})_\infty = 2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty, \tag{2.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

The first term on the left-hand side of (2.1) is the generating function of partitions into distinct odd parts with two copies of multiples of 7 allowed. Subtracting the same term with q replaced by $-q$ and dividing by 2, we suppress all even powers of q on the left-hand side. On the other hand, the right-hand side without the factor $2q$ gives the generating function of partitions into distinct even parts with two copies of multiples of 7 allowed. Thus, equating the coefficients of q^{2N+1} on both sides of (2.1) leads to Theorem 2.1.

Farkas and Kra proved (2.1) using the theory of theta functions. In [35], M.D. Hirschhorn gave a simple q-series proof of (2.1). The referee of [35] observed that (2.1) was equivalent to a modular equation of degree 7 in Ramanujan's notebooks [10, Chapter 19, Entry 19 (i)], but actually due to C. Guetzlaff [31] in 1834.

B.C. Berndt [9] observed that Ramanujan discovered five modular equations of this sort, and he gave partition-theoretic interpretations for each of them.

In [61], S. O. Warnaar established an extensive generalization of Theorem 2.1, which is the following.

Theorem 2.2. *Let α and β be even positive integers such that $\alpha < \beta$, and let γ be an odd positive integer. Fix an integer $m \geq \alpha + \beta + 2\gamma + 1$. Consider the positive integers in which multiples of $2m$ occur in two copies, $2m$ and $\overline{2m}$. Let $A(N)$ be the number of partitions of $2N$ with parts congruent to $0, \overline{0}, \pm\alpha, \pm\beta, \pm(\alpha+\beta+2\gamma) \pmod{2m}$, and let $B(N)$ be the number of partitions of $2N+\gamma$ with parts congruent to $\pm\gamma, \pm(\alpha+\gamma), \pm(\beta+\gamma), \pm(\alpha+\beta+\gamma) \pmod{2m}$. Then $A(N) = B(N)$.*

Warnaar mentioned that the conditions $\alpha < \beta$ and $m \geq \alpha+\beta+2\gamma+1$ can be replaced by the conditions that the sequences α, $2m-\alpha$, β, $2m-\beta$, $\alpha+\beta+2\gamma$, $2m-\alpha-\beta-2\gamma$ and γ, $2m-\gamma$, $\alpha+\gamma$, $2m-\alpha-\gamma$, $\beta+\gamma$, $2m-\beta-\gamma$, $\alpha+\beta+\gamma$, $2m-\alpha-\beta-\gamma$ are positive integers, and if some of the above integers coincide, then we introduce different copies of those numbers. Setting $(\alpha,\beta,\gamma) = (2,4,1)$ and $m = 7$, we can see that Theorem 2.2 implies Theorem 2.1. Note that the condition $\alpha+\beta+\gamma = 7 = 2m-\alpha-\beta-\gamma$ requires both of the numbers 7 and $\overline{7} \pmod{14}$.

In order to prove Theorem 2.2, Warnaar considerably generalized (2.1) to obtain

$$
\begin{aligned}
&(-c,-ac,-bc,-abc,-q/c,-q/ac,-q/bc,-q/abc;q)_\infty \\
&\quad - (c,ac,bc,abc,q/c,q/ac,q/bc,q/abc;q)_\infty
\end{aligned}
$$

$$= 2c(-a, -b, -abc^2, -q/a, -q/b, -q/abc^2, -q, -q; q)_\infty, \tag{2.2}$$

where

$$(a_1, \ldots, a_n; q)_\infty = (a_1; q)_\infty \cdots (a_n; q)_\infty,$$

and gave three different proofs of (2.2). Furthermore, N.D. Baruah and Berndt [8] observed that an equivalent formulation of (2.2) can be found in Ramanujan's notebooks.

However, one of the referees of our paper [37] observed that (2.2) is a special case of the addition formula

$$\begin{aligned}
&(ux, u/x, vy, v/y, q/ux, qx/u, q/vy, qy/v; q)_\infty \\
&\quad - (uy, u/y, vx, v/x, q/uy, qy/u, q/vx, qx/v; q)_\infty \\
&\qquad = v/x(uv, u/v, xy, x/y, q/uv, qv/u, q/xy, qy/x; q)_\infty,
\end{aligned}\tag{2.3}$$

which is given in [23, p. 52, Ex. 2. 16]. Setting $x = \sqrt{a}$, $y = -\sqrt{a}$, $u = -\sqrt{a}bc$ and $v = \sqrt{a}c$ in (2.3), we can derive (2.2).

Warnaar's proofs of (2.2) include a combinatorial one. However, he asked for a bijective proof of Theorem 2.1 without resorting to theta functions.

In Theorem 2.3 in Section 2.2, we derive a generalization of Theorem 2.2 from (2.2) and give a bijective proof of it, which naturally gives a bijective proof of Theorem 2.1. In order to establish a bijection of the generalization, we use Warnaar's bijection from [61]. The generalization of Theorem 2.2 also implies, in particular, two further partition identities derived from

$$(-q; q^2)_\infty^2(-q^3; q^6)_\infty^2 - (q; q^2)_\infty^2(q^3; q^6)_\infty^2 = 4q(-q^2; q^2)_\infty^2(-q^6; q^6)_\infty^2, \tag{2.4}$$

$$(-q; q^2)_\infty^8 - (q; q^2)_\infty^8 = 16q(-q^2; q^2)_\infty^8. \tag{2.5}$$

We remark that the identity (2.4) is equivalent to a modular equation of degree 3 and the identity (2.5) is called Jacobi's quartic identity. In [61], Warnaar showed that the identities (2.4) and (2.5) are specializations of the identity (2.2). However, he remarked that the partition theorems derived from (2.4) and (2.5) are not special cases of Theorem 2.2.

In Section 2.3, we establish three further identities in Theorems 2.6, 2.8 and 2.10 and give their partition theoretic consequences in Theorems 2.7, 2.9 and 2.11 together with bijective proofs. The referee also pointed out that the identities in Theorems 2.6, 2.8 and 2.10 can be proved using (2.3).

In [8], Baruah and Berndt derived partition identities associated with modular equations of degrees 3, 5 and 15. In Section 2.4, we show that some of them are special cases of the theorems from Sections 2.2 and 2.3. We also give new examples that follow from these theorems.

2.2 Generalizations of the Farkas and Kra partition theorem

In this section, we prove a generalization of Theorem 2.2 which implies not only Theorem 2.1, but the two partition theorems that can be derived from (2.4) and (2.5), respectively. We show that the generalization is an immediate consequence of (2.2) after some changes of variables, and we also give a bijective proof of the generalization.

Theorem 2.3. *Let m be a positive integer, and let α, β and γ be odd positive integers $\leq m$ such that $\alpha \leq \beta, \gamma$. Consider the positive integers in which multiples of $2m$ occur in two copies, $2m$ and $\overline{2m}$. Let $A(N)$ denote the number of partitions of $2N + \alpha$ into parts congruent to $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(-\alpha + \beta + \gamma) \pmod{2m}$, and let $B(N)$ denote the number of partitions of $2N$ into parts congruent to $0, \overline{0}, \pm(\beta-\alpha), \pm(\gamma-\alpha), \pm(\beta+\gamma) \pmod{2m}$. Let κ, $0 \leq \kappa \leq 3$, denote the number of elements from the set $\{\beta - \alpha, \gamma - \alpha, 2m - \beta - \gamma\}$ that are equal to 0. Then, $A(N) = 2^{\kappa} B(N)$.*

Here, if some of the integers above coincide in these congruences or are congruent to themselves with opposite sign $\pmod{2m}$, then we allow additional copies of those integers. For instance, if $(m, \alpha, \beta, \gamma) = (5, 1, 1, 5)$, then $(\pm\alpha, \pm\beta, \pm\gamma, \pm(-\alpha + \beta + \gamma)) = (\pm1, \pm1, \pm5, \pm5)$ and $(\pm(\beta-\alpha), \pm(\gamma-\alpha), \pm(\beta+\gamma)) = (\pm0, \pm4, \pm6)$. So, we consider the set consisting of two copies of the positive integers and two additional copies of multiples of 5.

Note that we obtain Theorem 2.2 by replacing α, β, γ by $\gamma, \alpha+\gamma, \beta+\gamma$, respectively. In particular, setting $(\alpha, \beta, \gamma) = (1, 3, 5)$ and $m = 7$ in Theorem 2.3 yields Theorem 2.1.

First proof of Theorem 2.3. Replacing q by q^{2m}, and then setting $a = q^{\beta-\alpha}$, $b = q^{\gamma-\alpha}$ and $c = q^{\alpha}$ in (2.2), we find that

$$\begin{aligned}
&(-q^{\alpha}, -q^{\beta}, -q^{\gamma}, -q^{\beta+\gamma-\alpha}, -q^{2m-\alpha}, -q^{2m-\beta}, -q^{2m-\gamma}, -q^{2m+\alpha-\beta-\gamma}; q^{2m})_\infty \\
&\quad + (q^{\alpha}, q^{\beta}, q^{\gamma}, q^{\beta+\gamma-\alpha}, q^{2m-\alpha}, q^{2m-\beta}, q^{2m-\gamma}, q^{2m+\alpha-\beta-\gamma}; q^{2m})_\infty \\
&\qquad = 2q^{\alpha}(-q^{\beta-\alpha}, -q^{\gamma-\alpha}, -q^{\beta+\gamma}, -q^{2m+\alpha-\beta}, -q^{2m+\alpha-\gamma}, -q^{2m-\beta-\gamma}, -q^{2m}, -q^{2m}; q^{2m})_\infty.
\end{aligned} \tag{2.6}$$

It is now easy to see that (2.6) has the partition-theoretic interpretation claimed in Theorem 2.3. $\square$

Second proof of Theorem 2.3. Let $\delta = -\alpha + \beta + \gamma$. Let π be a partition of $2N + \alpha$ into parts congruent to $\pm\alpha, \pm\beta, \pm\gamma, \pm\delta \pmod{2m}$. Then we can write

$$\pi = ((\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3), (\lambda_4, \mu_4)),$$

where $\lambda_1, \dots, \lambda_4$ are partitions into parts that are congruent to $\alpha, \beta, \gamma, \delta \pmod{2m}$, respectively, and $\mu_1, \dots, \mu_4$ are partitions with parts congruent to $-\alpha, -\beta, -\gamma, -\delta \pmod{2m}$, respectively. We obtain a new partition π' from π such that

$$\pi' = ((\lambda_1', \mu_1'), (\lambda_2', \mu_2'), (\lambda_3', \mu_3'), (\lambda_4', \mu_4')),$$

where $\lambda_1' = (\lambda_1 - \alpha)/2m, \dots, \lambda_4' = (\lambda_4 - \delta)/2m$ and $\mu_1' = (\mu_1 + \alpha)/2m, \dots, \mu_4' = (\mu_4 + \delta)/2m$. Note that $\lambda_i' \in D_0$ and $\mu_i' \in D$, where D_0 is the set of partitions with distinct non-negative parts and D is the set of partitions with distinct positive parts.

Let $d_i = \ell(\lambda_i) - \ell(\mu_i) = \ell(\lambda_i') - \ell(\mu_i')$ for $1 \le i \le 4$, where $\ell(\lambda)$ is the number of parts of the partition λ.

Note that $|\pi| = 2m|\pi'| + \alpha d_1 + \beta d_2 + \gamma d_3 + \delta d_4 = 2N + \alpha$, where $|\pi|$ is the sum of the parts of a partition π.

Since

$$\sum_{i=1}^{4} d_i = \sum_{i=1}^{4} \ell(\lambda_i) - \sum_{i=1}^{4} \ell(\mu_i) \equiv \sum_{i=1}^{4} \ell(\lambda_i) + \sum_{i=1}^{4} \ell(\mu_i) \equiv \ell(\pi) \equiv 1(\mathrm{mod}\ 2),$$

$\sum_{i=1}^{4} d_i = 2s+1$ for some $s \in \mathbb{Z}$. So, if we let $d_4 = n$, $d_2 = k - n$, $d_3 = l - n$, with n, k and $l \in \mathbb{Z}$, then $d_1 = 2s + 1 + n - k - l$.

Now, we introduce the bijection that Warnaar established in [61].

Let (λ, μ) be a partition pair such that $\lambda \in D_0$ and $\mu \in D$ and such that $\ell(\lambda) - \ell(\mu) = d$. Then we can draw a diagram of λ, say G, as follows. The diagram G consists of $\ell(\lambda)$ columns of nodes, and the ith column contains the nodes of the ith part of λ. Displace the $(i+1)$ th column one unit to the right and one unit down relative to the ith column. Similarly, draw a diagram H of μ, consisting of $\ell(\mu)$ rows of nodes where the ith row contains the nodes of the ith part of μ. Displace the $(i+1)$th row one unit to the right and one unit down relative to the ith row. For example (in examples, we may have subscripts on the aforementioned variables), if λ_1 = $(6, 4, 2, 1, 0)$ and μ_1 = $(3, 1)$, then

$$G_1: \begin{array}{cccc} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \qquad \text{and} \qquad H_1: \begin{array}{ccc} \bullet & \bullet & \bullet \\ & \bullet & \end{array}$$

and if λ_2 = $(4, 1, 0)$ and μ_2 = $(7, 5, 4, 2, 1)$, then

$$
G_2 : \begin{array}{ll} \bullet & \\ \bullet & \bullet \\ \bullet & \\ \bullet & \end{array}
\qquad \text{and} \qquad
H_2 : \begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & \bullet & \\ & & & \bullet & \bullet & & \\ & & & & \bullet & & \end{array}
$$

Now, if $d > 0$, concatenate the top row of H and the dth row of G, and if $d \leq 0$, concatenate the first column of G and the $(1-d)$ th column of μ to form a diagram K. For the examples above, the respective graphs K are

$$
K_1 : \begin{array}{llllll} \circ & & & & & \\ \circ & \circ & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & & & & \\ \bullet & & & & & \end{array}
\qquad \text{and} \qquad
K_2 : \begin{array}{lllllll} \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \circ & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & & \\ & & \bullet & & & & \\ & & \bullet & & & & \end{array}
$$

The graph K corresponds to an ordinary partition ν and a triangle of $\binom{d}{2}$ nodes with $|\nu| = |\lambda| + |\mu| - \binom{d}{2}$, and in the examples $\nu_1 = (6, 5, 2, 1)$ and $\nu_2 = (5, 4, 4, 3, 3, 1, 1)$, respectively.

Conversely, for an ordinary partition ν and an integer d, add a triangle of $\binom{d}{2}$ nodes to the diagram ν to form a diagram K. When $d > 0$, place the triangle on top of the diagram of ν (so it is left-aligned), and when $d \leq 0$, place it to the left of ν (so it is top-aligned). Then K can be dissected into two diagrams G and H corresponding to a partition pair (λ, μ) such that $\lambda, \mu \in D$ and $|\lambda| + |\mu| = |\nu| + \binom{d}{2}$.

Note that $\ell(\lambda) - \ell(\mu)$ equals d or $d - 1$. When $\ell(\lambda) - \ell(\mu) = d - 1$, add the part 0 to λ. In summary, we always have the equality $\ell(\lambda) - \ell(\mu) = d$, where $\lambda \in D_0$ and $\mu \in D$.

This shows that there is a bijection between the set $\{(\lambda, \mu, d) : \lambda \in D_0, \mu \in D, \ell(\lambda) - \ell(\mu) =$

$d\}$ and the set $\{(\nu, d) : \nu$ is an ordinary partition and $d \in$Z$\}$, where $|\lambda| + |\mu| = |\nu| + \binom{d}{2}$.

Thus, $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4))$, with $\alpha_i \in D_0$, $\beta_i \in D$, $d_1 = 2s + 1 + n - k - l$, $d_2 = k - n$, $d_3 = l - n$ and $d_4 = n$, corresponds to $((\nu_1, d_1), (\nu_2, d_2), (\nu_3, d_3), (\nu_4, d_4))$, where the $\nu_i' s$ are ordinary partitions. Defining $d_1' = 2n + 1 + s - k - l, d_2' = k - s, d_3' = l - s$ and $d_4' = s$, we can check that

$$\sum_{i=1}^{4} \binom{d_i}{2} = \sum_{i=1}^{4} \binom{d_i'}{2},$$

and so

$$\sum_{i=1}^{4} (|\alpha_i| + |\beta_i|) = \sum_{i=1}^{4} \left(|\nu_i| + \binom{d_i}{2} \right) = \sum_{i=1}^{4} \left(|\nu_i| + \binom{d_i'}{2} \right).$$

Applying the inverse of the bijection with $(\nu_1, d_1'), (\nu_2, d_2'), (\nu_3, d_3'), (\nu_4, d_4')$ yields $\pi_1 = ((\omega_1, \tau_1),$ (ω_2, τ_2), (ω_3, τ_3), $(\omega_4, \tau_4))$ with $\ell(\omega_i) - \ell(\tau_i) = d_i'$ and $\omega_i \in D_0$, $\tau_i \in D$ for $1 \le i \le 4$.

Let us now use this bijection to set up a bijection between $A(n)$ and $B(n)$. We can apply it with

$$\pi' = ((\lambda_1', \mu_1'), (\lambda_2', \mu_2'), (\lambda_3', \mu_3'), (\lambda_4', \mu_4'))$$

to obtain the corresponding partition $\pi_1 = ((\omega_1, \tau_1), (\omega_2, \tau_2), (\omega_3, \tau_3), (\omega_4, \tau_4))$ with $\ell(\omega_i) - \ell(\tau_i) = d_i'$.

Now, multiply each part of the $\omega_i' s$ and $\tau_i' s$ by $2m$, and add $\beta - \alpha, \alpha - \beta$; $\gamma - \alpha, \alpha - \gamma$; and $\beta + \gamma, -\beta - \gamma$ to each part of $2m\omega_2$, $2m\tau_2$; $2m\omega_3$, $2m\tau_3$; $2m\omega_4$ and $2m\tau_4$, respectively. Then remove all the parts 0 from $2m\omega_1$, $2m\omega_2 + (\beta - \alpha)$, $2m\omega_3 + \gamma - \alpha$ and $2m\tau_4 - (\beta + \gamma)$. (Note that they can have part 0 if $\beta - \alpha = 0$, $\gamma - \alpha = 0$ or $\beta + \gamma = 2m$.) Then we obtain a new partition

$$\pi_1' = ((\omega_1', \tau_1'), (\omega_2', \tau_2'), (\omega_3', \tau_3'), (\omega_4', \tau_4')).$$

Since the sum of the numbers added to the $2m\omega_i' s$ and $2m\tau_i' s$ is

$$(\beta - \alpha) d_2' + (\gamma - \alpha) d_3' + (\beta + \gamma) d_4' = \alpha d_1 + \beta d_2 + \gamma d_3 + \delta d_4 - \alpha,$$

we have

$$\begin{aligned}|\pi_1'| &= 2m|\pi_1| + (\beta-\alpha)d_2' + (\gamma-\alpha)d_3' + (\beta+\gamma)d_4' \\ &= 2m|\pi'| + \alpha d_1 + \beta d_2 + \gamma d_3 + \delta d_4 - \alpha \\ &= |\pi| - \alpha = 2N.\end{aligned}$$

Thus, we can see that π' is a partition of $2N$ into parts congruent to $0, \overline{0}$, $\pm(\beta-\alpha)$, $\pm(\gamma-\alpha)$, $\pm(\beta+\gamma)$ $(\mathrm{mod}\ 2m)$.

However, consider two distinct partitions π_1' and π_1'' that both have the same copies of positive parts. If both of them have an even number of parts 0 or an odd number of parts 0, then they correspond to the same partition of $2N$ into parts congruent to $0, \overline{0}$, $\pm(\beta-\alpha)$, $\pm(\gamma-\alpha)$, $\pm(\beta+\gamma)$ $(\mathrm{mod}\ 2m)$. Note that if π_1' and π_1'' have an even (odd) number of parts 0, then they correspond to partitions of $2N$ with the number of parts odd (even) after removing parts 0. We can easily see that there are exactly $2^{\kappa+1}/2 = 2^{\kappa}$ partitions that have the same copies of positive parts and the same parity in the number of parts 0. Furthermore, the process above is reversible. Thus we can conclude that $A(N) = 2^{\kappa}B(N)$. □

From (2.4), Farkas and Kra [19], [20] infer the following theorem, which is an analogue of Theorem 2.1 for modulus 3.

Theorem 2.4. *Let S denote the set of positive integers in 4 distinct colors with two colors, say orange and blue, each appearing at most once, and the remaining two colors, say red and green, appearing at most once and only in multiples of 3. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S. Let $B(N)$ denote the number of partitions of $2N$ into even elements of S. Then,*

$$A(N) = 2B(N).$$

Proof. Let $m = 3$, $\alpha = \gamma = 1$ and $\beta = 3$ in Theorem 2.3. Then $A(N) = 2B(N)$ since $\kappa = 1$. □

Similarly, from (2.5), Farkas and Kra [19], [20] infer the following theorem.

Theorem 2.5. *Consider the positive integers such that each integer occurs in eight copies. Let $A(N)$ denote the number of partitions of $2N+1$ into distinct odd parts and $B(N)$ the number of partitions of $2N$ into distinct even parts. Then*

$$A(N) = 8B(N).$$

Proof. Let $m = \alpha = \beta = \gamma = 1$ in Theorem 2.3. Then $A(N) = 8B(N)$ since $\kappa = 3$.

□

2.3 Further general partition theorems

In this section, we establish three further identities that imply partition theorems with forms similar to that of Theorem 2.3.

First, we prove the following.

Theorem 2.6.

$$\begin{aligned}
&(-c, -a^2c, -b^2c, -d^2c, -q/c, -q/a^2c, -q/b^2c, -q/d^2c; q)_\infty \\
&\quad - (c, a^2c, b^2c, d^2c, q/c, q/a^2c, q/b^2c, q/d^2c; q)_\infty \\
&\qquad = c\{(-ab/d, -ad/b, -bd/a, -abdc^2, -qd/ab, -qb/ad, -qa/bd, -q/abdc^2; q)_\infty \\
&\qquad\qquad + (ab/d, ad/b, bd/a, abdc^2, qd/ab, qb/ad, qa/bd, q/abdc^2; q)_\infty\}. \qquad (2.7)
\end{aligned}$$

We remark that taking $d = ab$, and then replacing a^2 and b^2 by a and b, respectively, in Theorem 2.6 yields (2.2).

First Proof. Let $x = ra$, $y = b/d$, $u = rbcd$, and $v = ac$ in (2.3). Then we have

$$\begin{aligned}
&(r^2abcd, bcd/a, abc/d, acd/b, q/r^2abcd, qa/bcd, qd/abc, qb/acd; q)_\infty \\
&\quad - (rb^2c, rcd^2, ra^2c, c/r, q/rb^2c, q/rcd^2, q/ra^2c, qr/c; q)_\infty
\end{aligned}$$

$$= c/r(rab/d, rad/b, rabc^2d, rbd/a, qd/rab, qb/rad, q/rabc^2d, qa/rbd)_\infty. \tag{2.8}$$

Subtracting (2.8) with $r = -1$ from (2.8) with $r = 1$ yields

$$\begin{aligned}
&-(c, a^2c, b^2c, d^2c, q/c, q/a^2c, q/b^2c, q/d^2c; q)_\infty \\
&\quad + (-c, -a^2c, -b^2c, -d^2c, -q/c, -q/a^2c, -q/b^2c, -q/d^2c; q)_\infty \\
&\qquad = c(ab/d, ad/b, bd/a, abdc^2, qd/ab, qb/ad, qa/bd, q/abdc^2; q)_\infty \\
&\qquad\quad + c\{(-ab/d, -ad/b, -bd/a, -abdc^2, -qd/ab, -qb/ad, -qa/bd, -q/abdc^2; q)_\infty,
\end{aligned}$$

which completes the proof. □

Second Proof. We use the fact [3] that the coefficient of $a^k q^N$ in $(-a, -q/a; q)_\infty$ is the number of partition pairs (λ, μ), where $\lambda \in D_0$, $\mu \in D$, $|\lambda| + |\mu| = N$ and $\ell(\lambda) - \ell(\mu) = k$. Thus, the coefficient of $a^{2(k-n)}b^{2(l-n)}c^{2s+1}d^{2n}q^N$ on the left side of Theorem 2.6 divided by 2 is the cardinality of the set $\Gamma_{k,l,s,n}(N)$ consisting of four partition pairs $((\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3), (\lambda_4, \mu_4))$, where $\lambda_i \in D_0$, $\mu_i \in D$, $\sum_{i=1}^4 (|\lambda_i| + |\mu_i|) = N$, $d_1 = 2s + 1 - k - l + n$, $d_2 = k - n$, $d_3 = l - n$ and $d_4 = n$ with $d_i = \ell(\lambda_i) - \ell(\mu_i)$.

Similarly, the coefficient of $a^{2(k-n)}b^{2(l-n)}c^{2s+1}d^{2n}q^N$ on the right-hand side of Theorem 2.6 divided by 2 is the cardinality of the set $\Omega_{k,l,s,n}(N)$ consisting of four partition pairs (ω_1, τ_1), (ω_2, τ_2), (ω_3, τ_3), (ω_4, τ_4), where $\omega_i \in D_0$, $\tau_i \in D$, $\sum_{i=1}^4 (|\omega_i| + |\tau_i|) = N$, $d'_1 = -2n - s + l + k$, $d'_2 = k - s$, $d'_3 = l - s$ and $d'_4 = s$ with $d'_i = \ell(\omega_i) - \ell(\tau_i)$.

We can apply Warnaar's bijection described in the proof of Theorem 2.3 with $d_1, \ldots, d_4$ and $d'_1, \ldots, d'_4$ defined above, since

$$\sum_{i=1}^4 \binom{d_i}{2} = \sum_{i=1}^4 \binom{d'_i}{2}.$$

Now, it follows that $\Gamma_{k,l,s,n}(N)$ corresponds to $\Omega_{k,l,s,n}(N)$ bijectively. Hence, these sets have the same cardinality.

□

We deduce the following partition identity from Theorem 2.6, and, with the use of Warnaar's bijection, we also give a bijective proof of Theorem 2.7.

Theorem 2.7. *Let m be a positive integer, and let α, β, γ, and δ be nonnegative integers such that $\alpha+\beta+\gamma+\delta \equiv 1 \pmod 2$, $\alpha+\beta+\gamma+\delta+2 < 2m$ and $\alpha < \min\{\beta+\gamma-\delta, \beta-\gamma+\delta, -\beta+\gamma+\delta\}$. Let $A(N)$ denote the number of partitions of $2N+2\alpha+1$ into parts congruent to $\pm(2\alpha+1)$, $\pm(2\beta+1)$, $\pm(2\gamma+1)$, $\pm(2\delta+1) \pmod{2m}$, and let $B(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm(-\alpha+\beta+\gamma-\delta)$, $\pm(-\alpha+\beta-\gamma+\delta)$, $\pm(-\alpha-\beta+\gamma+\delta)$, $\pm(\alpha+\beta+\gamma+\delta+2) \pmod{2m}$. Then, $A(N) = B(N)$.*

Here, as before, if some of the integers above coincide, then we introduce additional copies of those numbers, while we allow only one copy of the remaining integers.

First Proof. Replacing q by q^{2m} and then replacing (a,b,c,d) by $(q^{\beta-\alpha}, q^{\gamma-\alpha}, q^{2\alpha+1}, q^{\delta-\alpha})$ in Theorem 2.6 yields

$$\begin{aligned}
&(-q^{2\alpha+1}, -q^{2\beta+1}, -q^{2\gamma+1}, -q^{2\delta+1}, -q^{2m-2\alpha-1}, -q^{2m-2\beta-1}, -q^{2m-2\gamma-1}, -q^{2m-2\delta-1}; q^{2m})_\infty \\
&\quad - (q^{2\alpha+1}, q^{2\beta+1}, q^{2\gamma+1}, q^{2\delta+1}, q^{2m-2\alpha-1}, q^{2m-2\beta-1}, q^{2m-2\gamma-1}, q^{2m-2\delta-1}; q^{2m})_\infty \\
&\quad\quad = q^{2\alpha+1}\{(-q^{-\alpha+\beta+\gamma-\delta}, -q^{-\alpha+\beta-\gamma+\delta}, -q^{-\alpha-\beta+\gamma+\delta}, -q^{\alpha+\beta+\gamma+\delta+2}, -q^{2m+\alpha-\beta-\gamma+\delta}, \\
&\quad\quad\quad\quad - q^{2m+\alpha-\beta+\gamma-\delta}, -q^{2m+\alpha+\beta-\gamma-\delta}, -q^{2m-\alpha-\beta-\gamma-\delta-2}; q^{2m})_\infty \\
&\quad\quad\quad + (q^{-\alpha+\beta+\gamma-\delta}, q^{-\alpha+\beta-\gamma+\delta}, q^{-\alpha-\beta+\gamma+\delta}, q^{\alpha+\beta+\gamma+\delta+2}, q^{2m+\alpha-\beta-\gamma+\delta}, \\
&\quad\quad\quad\quad q^{2m+\alpha-\beta+\gamma-\delta}, q^{2m+\alpha+\beta-\gamma-\delta}, q^{2m-\alpha-\beta-\gamma-\delta-2}; q^{2m})_\infty\}. \qquad (2.9)
\end{aligned}$$

It is now readily seen that Thoerem 2.7 follows from (2.9). $\square$

Second Proof. Let

$$\alpha' = -\alpha+\beta+\gamma-\delta, \qquad \beta' = -\alpha+\beta-\gamma+\delta,$$
$$\gamma' = -\alpha-\beta+\gamma+\delta, \qquad \delta' = \alpha+\beta+\gamma+\delta+2.$$

Similarly to the second proof of Theorem 2.3, for given a partition π of $2n + 2\alpha + 1$ into parts congruent to $\pm(2\alpha+1)$, $\pm(2\beta+1)$, $\pm(2\gamma+1)$, $\pm(2\delta+1)$ $(\mathrm{mod}\ 2m)$, we obtain a new partition

$$\pi' = ((\lambda_1', \mu_1'), (\lambda_2', \mu_2'), (\lambda_3', \mu_3'), (\lambda_4', \mu_4')),$$

where $\lambda_i \in D_0$, $\mu_i \in D$, and

$$|\pi| = 2m|\pi'| + (2\alpha+1)d_1 + (2\beta+1)d_2 + (2\gamma+1)d_3 + (2\delta+1)d_4,$$

where $d_i = \ell(\lambda_i') - \ell(\mu_i')$. We can write

$$d_1 = 2s + 1 + n - k - l, \quad d_2 = k - n, \quad d_3 = l - n, \quad d_4 = n,$$

since $\sum_{i=1}^4 d_i = 2s + 1$ for some integer s. Define

$$d_1' = -2n - s + k + l, \quad d_2' = k - s, \quad d_3' = l - s, \quad d_4' = s.$$

Note that $\sum_{i=1}^4 d_i'$ is even, and

$$\sum_{i=1}^{4} \binom{d_i}{2} = \sum_{i=1}^{4} \binom{d_i'}{2}.$$

Using Warnaar's bijection, we obtain the corresponding partition

$$\pi_1 = ((\omega_1, \tau_1), (\omega_2, \tau_2), (\omega_3, \tau_3), (\omega_4, \tau_4)),$$

where $\omega_i \in D_0$, $\tau_i \in D$, $|\pi'| = |\pi_1|$ and $d_i' = \ell(\omega_i) - \ell(\tau_i)$.

Now, multiplying all parts of π_1 by $2m$ and then adding $\alpha', \ldots, \delta'$, $-\alpha', \ldots, -\delta'$ to each part of $\omega_1, \ldots, \omega_4$, $\tau_1, \ldots, \tau_4$, respectively, yields a new partition, say π_1'. Since

$$|\pi_1'| = 2m|\pi_1| + \alpha' d_1' + \beta' d_2' + \gamma' d_3' + \delta' d_4'$$

$$= 2m|\pi'| + (2\alpha+1)d_1 + (2\beta+1)d_2 + (2\gamma+1)d_3 + (2\delta+1)d_4 - (2\alpha+1)$$
$$= 2N,$$

π_1' is a partition of $2N$ into parts congruent to $\alpha', \beta', \gamma', \delta' \pmod{2m}$.

Clearly, the process is reversible. Hence, $A(N) = B(N)$. □

Next, we show that the following identity is true.

Theorem 2.8.

$$\begin{aligned}
&(-c, -a^2c, -b^2c, -d^2c, -q/c, -q/a^2c, -q/b^2c, -q/d^2c; q)_\infty \\
&\quad + (c, a^2c, b^2c, d^2c, q/c, q/a^2c, q/b^2c, q/d^2c; q)_\infty \\
&\qquad = (-abc/d, -adc/b, -bdc/a, -abcd, -qd/abc, -qb/adc, -qa/bdc, -q/abdc; q)_\infty \\
&\qquad\quad + (abc/d, adc/b, bdc/a, abdc, qd/abc, qb/adc, qa/bdc, q/abdc; q)_\infty. \qquad (2.10)
\end{aligned}$$

We remark that the left-hand side of Theorem 2.8 has the opposite sign of the left-hand side of Theorem 2.6.

First Proof. Set $x = a$, $y = b/d$, $u = rac$, and $v = rbdc$ in (2.3) to deduce that

$$\begin{aligned}
&(ra^2c, rc, rb^2c, rd^2c, q/ra^2c, q/rc, q/rb^2c, q/rd^2c; q)_\infty \\
&\quad - (rabc/d, racd/b, rabcd, rbdc/a, qd/rabc, qb/racd, q/rabcd, qa/rbdc; q)_\infty \\
&\qquad = rbdc/a(ab/d, ad/b, r^2abdc^2, a/bd, qd/ab, qb/ad, q/r^2abdc^2, qbd/a; q)_\infty. \qquad (2.11)
\end{aligned}$$

Adding (2.11) with $r = -1$ and (2.11) with $r = 1$, we obtain (2.10). □

Second Proof. Repeating the same argument as in the second proof of Theorem 2.6, we see that the coefficient of $d^{2n}a^{2(k-n)}b^{2(l-n)}c^{2(k+l-n-s)}q^N$ on the left side of (2.10) divided by 2 is the cardinality of the set $\Gamma_{k,l,s,n}(N)$ consisting of four partition pairs $((\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3), (\lambda_4, \mu_4))$, where

$\lambda_i \in D_0$, $\mu_i \in D$, $\sum_{i=1}^{4}(|\lambda_i|+|\mu_i|) = N$, $d_2 = k-n$, $d_3 = l-n$, $d_4 = n$ and $d_1 = 2(k+l-n-s) - d_2 - d_3 - d_4 = -2s+k+l-n$, with $d_i = \ell(\lambda_i) - \ell(\mu_i)$.

Similarly, the coefficient of $d^{2n}a^{2(k-n)}b^{2(l-n)}c^{2(k+l-n-s)}q^N$ on the right-hand side of (2.10) divided by 2 is the cardinality of the set $\Omega_{k,l,s,n}(N)$ consisting of four partition pairs $((\omega_1,\tau_1)$, (ω_2,τ_2), (ω_3,τ_3), $(\omega_4,\tau_4))$, where $\omega_i \in D_0$, $\tau_i \in D$, $\sum_{i=1}^{4}(|\omega_i|+|\tau_i|) = N$, $d_1' = -2n+k+l-s$, $d_2' = k-s$, $d_3' = l-s$ and $d_4' = s$, with $d_i = \ell(\lambda_i) - \ell(\mu_i)$. (Note that (d_1', d_2', d_3', d_4') is a solution of (2.13).)

Using Warnaar's bijection with $d_1, \ldots, d_4$ and $d_1', \ldots, d_4'$ above, we deduce that

$$| \Gamma_{k,l,s,n}(N) | = | \Omega_{k,l,s,n}(N) |,$$

which completes the proof. □

The following partition theorem can be derived from Theorem 2.8.

Theorem 2.9. *Let m be a positive integer and α, β, γ, and δ be nonnegative integers satisfying the condition that any of them is at most the sum of the other three, $\alpha+\beta+\gamma+\delta \equiv 0 \pmod 2$, and $\alpha+\beta+\gamma+\delta+1 < 2m$. Let $A(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm(2\alpha+1), \pm(2\beta+1), \pm(2\gamma+1), \pm(2\delta+1) \pmod{2m}$ and $B(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm(\alpha+\beta+\gamma-\delta+1)$, $\pm(\alpha+\beta-\gamma+\delta+1)$, $\pm(\alpha-\beta+\gamma+\delta+1)$, $\pm(-\alpha+\beta+\gamma+\delta+1) \pmod{2m}$. Then, $A(N) = B(N)$.*

First Proof. Replacing q by q^{2m} and then letting $(a,b,c,d) = (q^{\beta-\alpha}, q^{\gamma-\alpha}, q^{2\alpha+1}, q^{\delta-\alpha})$ in Theorem 2.8 yields

$$\begin{aligned}
&(-q^{2\alpha+1}, -q^{2\beta+1}, -q^{2\gamma+1}, -q^{2\delta+1}, -q^{2m-2\alpha-1}, -q^{2m-2\beta-1}, -q^{2m-2\gamma-1}, -q^{2m-2\delta-1}; q^{2m})_\infty \\
&\quad + (q^{2\alpha+1}, q^{2\beta+1}, q^{2\gamma+1}, q^{2\delta+1}, q^{2m-2\alpha-1}, q^{2m-2\beta-1}, q^{2m-2\gamma-1}, q^{2m-2\delta-1}; q^{2m})_\infty \\
&\qquad = (-q^{\alpha+\beta+\gamma-\delta+1}, -q^{\alpha+\beta-\gamma+\delta+1}, -q^{\alpha-\beta+\gamma+\delta+1}, -q^{-\alpha+\beta+\gamma+\delta+1}, -q^{2m-\alpha-\beta-\gamma+\delta-1}, \\
&\qquad\qquad -q^{2m-\alpha-\beta+\gamma-\delta-1}, -q^{2m-\alpha+\beta-\gamma-\delta-1}, -q^{2m+\alpha-\beta-\gamma-\delta-1}; q^{2m})_\infty
\end{aligned}$$

$$+ (q^{\alpha+\beta+\gamma-\delta+1}, q^{\alpha+\beta-\gamma+\delta+1}, q^{\alpha-\beta+\gamma+\delta+1}, q^{-\alpha+\beta+\gamma+\delta+1}, q^{2m-\alpha-\beta-\gamma+\delta-1},$$
$$q^{2m-\alpha-\beta+\gamma-\delta-1}, q^{2m-\alpha+\beta-\gamma-\delta-1}, q^{2m+\alpha-\beta-\gamma-\delta-1}; q^{2m})_\infty. \tag{2.12}$$

It is now easy to see that (2.12) has the partition-theoretic interpretation given in the statement of Theorem 2.9. □

Second Proof. We repeat the same argument as in the proof of Theorem 2.7 with

$$\alpha' = \alpha+\beta+\gamma-\delta+1, \qquad \beta' = \alpha+\beta-\gamma+\delta+1,$$
$$\gamma' = \alpha-\beta+\gamma+\delta+1, \qquad \delta' = -\alpha+\beta+\gamma+\delta+1,$$

and $d_1 = -2s-n+k+l$ ($d_2, \ldots, d_4, d'_1, \ldots, d'_4$ remain unchanged). Using

$$(2\alpha+1)d_1 + (2\beta+1)d_2 + (2\gamma+1)d_3 + (2\delta+1)d_4 = \alpha' d'_1 + \beta' d'_2 + \gamma' d'_3 + \delta' d'_4,$$

we complete the proof. □

Lastly, we make a specialization of Theorem 2.8 and give another proof of it. Also, we deduce a partition identity from the specialization.

Theorem 2.10.

$$\begin{aligned}(-c, -ac, -bc, -abc^3, -q/c, -q/ac, -q/bc, -q/abc^3; q)_\infty \\ + (c, ac, bc, abc^3, q/c, q/ac, q/bc, q/abc^3; q)_\infty \\ = 2(-ac^2, -bc^2, -abc^2, -q/ac^2, -q/bc^2, -q/abc^2, -q, -q; q)_\infty.\end{aligned} \tag{2.13}$$

First Proof. Taking $d = abc$, then replacing (a^2, b^2) by (a, b) in Theorem 2.8, we obtain Theorem 2.10. □

Second proof. We can view both sides of (2.13) as functions of a. Let the left-hand side be denoted by $L(a)$ and the right-hand side by $R(a)$. Define $f(a) = L(a)/R(a)$. Then, we can easily see that $f(aq) = f(a)$, since $L(aq) = L(a)/a^2bc^4$ and $R(aq) = R(a)/a^2bc^4$. The values $a = -q^n/c^2, -q^n/bc^2$, $n \in \mathbb{Z}$, are simple zeroes of $R(a)$, provided that $b \neq 1$, and so are possible poles of $f(a)$. But, using

$$\begin{aligned}
(q^n/c, cq^{1-n}; q)_\infty &= (-1)^{n-1}c^{n-1}q^{-\binom{n}{2}}(c, q/c; q)_\infty, \\
(aq^n, q^{1-n}/a; q)_\infty &= (-1)^n a^{-n} q^{-\binom{n}{2}}(a, q/a; q)_\infty,
\end{aligned}$$

we obtain

$$\begin{aligned}
L(-q^n/c^2) &= (-c, q^n/c, -bc, q^n bc, -q/c, cq^{1-n}, -q/bc, q^{1-n}/bc; q)_\infty \\
&\qquad + (c, -q^n/c, bc, -q^n bc, q/c, -cq^{1-n}, q/bc, -q^{1-n}/bc; q)_\infty \\
&= -b^{-n}c^{-1}q^{-2\binom{n}{2}}(-c, c, -bc, bc, -q/c, q/c, -q/bc, q/bc; q)_\infty \\
&\qquad + b^{-n}c^{-1}q^{-2\binom{n}{2}}(c, -c, bc, -bc, q/c, -q/c, q/bc, -q/bc; q)_\infty \\
&= 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
L(-q^n/bc^2) &= (-c, q^n/bc, -bc, cq^n, -q/c, bcq^{1-n}, -q/bc, q^{1-n}/c; q)_\infty \\
&\qquad + (c, -q^n/bc, bc, -cq^n, q/c, -bcq^{1-n}, q/bc, -q^{1-n}/c; q)_\infty \\
&= -b^{n-1}c^{-1}q^{-2\binom{n}{2}}(-c, bc, -bc, c, -q/c, q/bc, -q/bc, q/c; q)_\infty \\
&\qquad + b^{n-1}c^{-1}q^{-2\binom{n}{2}}(c, -bc, bc, -c, q/c, -q/bc, q/bc, -q/c; q)_\infty \\
&= 0.
\end{aligned}$$

Thus, under the assumption that $b \neq 1$, f is an entire bounded function. By Liouville's theorem, f is a constant. Take $a = 1/c$; then

$$L(1/c) = 2(-c, -q, -bc, -bc^2, -q/c, -q, -q/bc, -q/bc^2; q)_\infty,$$

$$R(1/c) = 2(-c, -bc^2, -bc, -q/c - q/bc^2, -q/bc, -q, -q; q)_\infty.$$

Hence $f(a) = f(1/c) = 1$, which completes the proof when $b \neq 1$.

Since $L(a)$ and $R(a)$ can also be regarded as meromorphic functions of b, say, $L(a) = L^*(b)$ and $R(a) = R^*(b)$, then

$$f(a) = \frac{L(a)}{R(a)} = \frac{L^*(b)}{R^*(b)} =: f^*(b)$$

can also be considered as a meromorphic function of b, which is equal to 1 for $b \neq 1$. By analytic continuation, $f(a) = f^*(b) = 1$ at $b = 1$ as well. $\square$

Theorem 2.11. *Let m be a positive integer, and let α, β and γ be odd positive integers $\leq m$ with $\alpha + \beta + \gamma < 2m$. Consider the positive integers in which multiples of $2m$ occur in two copies, $2m$ and $\overline{2m}$. Let $A(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm\alpha$, $\pm\beta$, $\pm\gamma$, $\pm(\alpha+\beta+\gamma) \pmod{2m}$, and let $B(N)$ denote the number of partitions of $2N$ into parts congruent to $0, \overline{0}, \pm(\alpha+\beta), \pm(\beta+\gamma), \pm(\alpha+\gamma) \pmod{2m}$. Then, $A(N) = B(N)$.*

First Proof. Replacing q by q^{2m} and then letting $(a, b, c) = (q^\alpha, q^{\beta-\alpha}, q^{\gamma-\alpha})$ in Theorem 2.10, we obtain

$$\begin{aligned}
&(-q^\alpha, -q^\beta, -q^\gamma, -q^{\alpha+\beta+\gamma}, -q^{2m-\alpha}, -q^{2m-\beta}, -q^{2m-\gamma}, -q^{2m-\alpha-\beta-\gamma}; q^{2m})_\infty \\
&\quad + (q^\alpha, q^\beta, q^\gamma, q^{\alpha+\beta+\gamma}, q^{2m-\alpha}, q^{2m-\beta}, q^{2m-\gamma}, q^{2m-\alpha-\beta-\gamma}; q^{2m})_\infty \\
&\quad\quad = 2(-q^{\alpha+\beta}, -q^{\beta+\gamma}, -q^{\alpha+\gamma}, -q^{2m-\alpha-\beta}, -q^{2m-\beta-\gamma}, -q^{2m-\alpha-\gamma}, -q^{2m}, -q^{2m}; q^{2m})_\infty.
\end{aligned} \tag{2.14}$$

It is now easy to see that (2.14) has the partition-theoretic interpretation given in the statement of Theorem 2.11. $\square$

Second Proof. Let $\delta = \alpha+\beta+\gamma$. Let π be a partition of $2n$ into parts congruent to $\pm\alpha, \pm\beta, \pm\gamma, \pm\delta$ $\pmod{2m}$. Then, as in the proof of Theorem 2.3, we can obtain a new partition π' from π such that

$$\pi' = ((\lambda_1', \mu_1'), (\lambda_2', \mu_2'), (\lambda_3', \mu_3'), (\lambda_4', \mu_4')),$$

with $\lambda_i' \in D_0$, $\mu_i' \in D$ and $|\pi| = 2m|\pi'| + \alpha d_1 + \beta d_2 + \gamma d_3 + \delta d_4$, where $d_i = \ell(\lambda_i') - \ell(\mu_i')$.

Since $\sum_{i=1}^4 d_i \equiv \ell(\pi) \equiv 0 \pmod 2$, we can find corresponding (s, n, k, l) such that

$$d_1 = -2s - n + k + l, \quad d_2 = k - n, \quad d_3 = l - n, \quad d_4 = n.$$

Defining

$$d_1' = -2n - s + k + l, \quad d_2' = k - s, \quad d_3' = l - s, \quad d_4' = s,$$

observing that $\sum_{i=1}^4 \binom{d_i}{2} = \sum_{i=1}^4 \binom{d_i'}{2})$, and using Warnaar's bijection with them, we obtain the corresponding partition $\pi_1 = ((\omega_1, \tau_1), (\omega_2, \tau_2), (\omega_3, \tau_3), (\omega_4, \tau_4))$ with $\ell(\omega_i) - \ell(\tau_i) = d_i'$.

Now, multiply each part of the $\omega_i's$ and $\tau_i's$ by $2m$ and add $\alpha + \beta, -\alpha - \beta$; $\beta + \gamma, -\beta - \gamma$; and $\alpha + \gamma, -\alpha - \gamma$ to each part of $2m\omega_2$, $2m\tau_2$; $2m\omega_3$, $2m\tau_3$; and $2m\omega_4$, $2m\tau_4$, respectively. Lastly, removing the part 0 from $2m\omega_1$ (if present), we can view π_1 as a partition of $2n$ into parts congruent to $0, \overline{0}, \pm(\alpha+\beta), \pm(\beta+\gamma), \pm(\alpha+\gamma) \pmod{2m}$, since the sum of the numbers added to the $2m\omega_i's$ and $2m\tau_i's$ is $(\alpha + \beta)d_2' + (\beta + \gamma)d_3' + (\alpha + \gamma)d_4' = \alpha d_1 + \beta d_2 + \gamma d_3 + \delta d_4$. We remark that if $2m\omega_1$ has a part 0, then π_1 (without the part 0) is a partition with an odd number of parts.

The converse is obvious, since we can add a part 0 to the first partition of π_1 if it has an odd number of parts. □

2.4 Applications

Baruah and Berndt [8] found new partition theorems associated with modular equations of degree 3, 5 and 15. In this section, we show that some of their partition theorems and their equivalent q-series identities are consequences of the results from Section 2.2 and Section 2.3. Also, we derive some new partition identities from our theorems.

From one of the modular equations of degree 3 [10, p. 230], Baruah and Berndt deduced that

[8, Eq. (6.6)]

$$(-q;q^2)_\infty^2(-q,-q^5;q^6)_\infty+(q;q^2)_\infty^2(q,q^5;q^6)_\infty=2(-q^2;q^2)_\infty^2(-q^2,-q^4;q^6)_\infty. \tag{2.15}$$

Replacing q by q^6 and then setting $(a,b,c)=(1,1,q)$ in Theorem 2.10 gives (2.15).

From (2.15), Baruah and Berndt derive the following theorem [8, Theorem 6.1].

Theorem 2.12. *Let S denote the set consisting of two copies of the positive integers and one additional copy of positive integers that are not multiples of 3. Let $A(N)$ and $B(N)$ denote the number of partitions of $2N$ into odd elements and even elements, respectively, of S. Then, for $n\geq 1$, $A(N)=B(N)$.*

Proof. Set $m=3$ and $\alpha=\beta=\gamma=1$ in Theorem 2.11. □

Next, Baruah and Berndt show that another modular equation of degree 3 [10, Entry 5 (viii), p. 231] implies the identities [8, Eqs. (6.18), (6.19)]

$$(-q.-q^5;q^6)_\infty^4-(q.q^5;q^6)_\infty^4=8q(-q^2;-q^2)_\infty(-q^6;-q^6)_\infty^5, \tag{2.16}$$

$$q\{(-q.-q^5;q^6)_\infty^4+(q.q^5;q^6)_\infty^4\}=(-q;q^2)_\infty(-q^3;q^6)_\infty^5-(q;q^2)_\infty(q^3;q^6)_\infty^5. \tag{2.17}$$

Replacing q by q^6 and then setting $a=b=1$, $c=q$ in (2.2) yields (2.16), and replacing q by q^6 and then letting $a=b=c=d=q$ in Theorem 2.6 gives (2.17).

From (2.16) and (2.17), the following theorem [8, Theorem 6.4] is deduced.

Theorem 2.13. *Let S denote the set consisting of one copy of positive integers and five additional copies of positive integers that are multiples of 3. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S, and let $B(N)$ denote the number of partitions of $2N$ into even elements of S. Furthermore, let T denote the set consisting of four copies of odd positive integers that are not multiples of 3 and let $C(N)$ denote the number of partitions of N into elements of T. Then, for $N\geq 1$, $C(2N)=A(N)$ and $C(2N+1)=4B(N)$.*

Proof. Letting $m = 3$, $\alpha = 0$ and $\beta = \gamma = \delta = 1$ in Theorem 2.7, we obtain $A(N) = C(2N)$. Set $m = 3$ and $\alpha = \beta = \gamma = 1$ in Theorem 2.3. Then, we have $C(2N + 1) = 4B(N)$, since $\kappa = 2$. $\square$

Baruah and Berndt also derived the following two identities [8, Eqs. (7.18), (7.19)] from a modular equation of degree 5 [10, Entry 13 (vii), p. 281], namely,

$$(-q.-q^3,-q^7,-q^9;q^{10})_\infty^2 - (q.q^3,q^7,q^9;q^{10})_\infty^2 = 4q(-q^2;-q^2)_\infty(-q^{10};-q^{10})_\infty^3, \tag{2.18}$$

$$\begin{aligned}(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 &- (q;q^2)_\infty(q^5;q^{10})_\infty^3\\ &= q\{(-q.-q^3,-q^7,-q^9;q^{10})_\infty^2 + (q.q^3,q^7,q^9;q^{10})_\infty^2\}.\end{aligned} \tag{2.19}$$

Replacing q by q^{10} and then setting $(a, b, c) = (q^2, q^6, q)$ in (2.2) yields (2.18). Also, replacing q by q^{10} and then letting $(a, b, c, d) = (q^2, q, q, q^2)$ in Theorem 2.6, we obtain (2.19).

Similarly, we deduce the following theorem [8, Theorem 7.4] from (2.18) and (2.19).

Theorem 2.14. *Let S denote the set consisting of one copy of the positive integers and three additional copies of the positive integers that are multiples of 5, and let T denote the set consisting of two copies of the odd positive integers that are not multiples of 5. Let $A(N)$ be the number of partitions of $2N + 1$ into odd elements of S, and let $B(N)$ be the number of partitions of $2N$ into even elements of S. Furthermore, let $C(N)$ be the number of partitions of N into elements of T. Then $C(2N) = A(N)$ and $C(2N + 1) = 2B(N)$ for $N \geq 1$.*

Proof. Let $(m, \alpha, \beta, \gamma, \delta) = (5, 0, 2, 1, 2)$ in Theorem 2.7. Then we have $A(N) = C(2N)$. Next, setting $(m, \alpha, \beta, \gamma) = (5, 1, 1, 3)$ in Theorem 2.3 implies $C(2N + 1) = 2B(N)$, since $\kappa = 1$. $\square$

The following two identities [8, Eqs. (8.10), (8.11)] were deduced from one of Ramanujan's modular equations of degree 15 [10, p. 383] :

$$\begin{aligned}(-q^3;q^6)_\infty(-q^5;q^{10})_\infty &+ (q^3;q^6)_\infty(q^5;q^{10})_\infty\\ &= (-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty\end{aligned}$$

$$+ (q, q^7, q^{11}, q^{13}, q^{17}, q^{19}, q^{23}, q^{29}; q^{30})_\infty, \tag{2.20}$$

$$2q(-q^6; q^6)_\infty(-q^{10}; q^{10})_\infty = (-q, -q^7, -q^{11}, -q^{13}, -q^{17}, -q^{19}, -q^{23}, -q^{29}; q^{30})_\infty$$
$$- (q, q^7, q^{11}, q^{13}, q^{17}, q^{19}, q^{23}, q^{29}; q^{30})_\infty. \tag{2.21}$$

Replacing q by q^{30} and then letting $(a, b, c, d) = (q, q^3, q^3, q^6)$ in Theorem 2.8 yields (2.20). Also, replacing q by q^{30} and then setting $(a, b, c) = (q^6, q^{10}, q)$ in (2.2), we obtain (2.21).

Equations (2.20) and (2.21) give the following partition-theoretic interpretations [8, Theorem 8.2].

Theorem 2.15. *Let S denote the set consisting of one copy of the positive integers that are multiples of 3 and another copy of the positive integers that are multiples of 5. Let $A(N)$ and $B(N)$ denote the number of partitions of $2N$ into, respectively, odd elements of S and even elements of S. Furthermore, let $C(N)$ denote the number of partitions of N into distinct odd parts that are not multiples of 3 or 5. Then, for $N \geq 6$, $C(2N) = A(N)$ and $C(2N+1) = B(N)$.*

Proof. Take $(m, \alpha, \beta, \gamma, \delta) = (15, 1, 2, 4, 7)$ in Theorem 2.9. Then we have $A(N) = C(2N)$. Next, letting $(m, \alpha, \beta, \gamma) = (15, 1, 7, 11)$ in Theorem 2.3, we obtain $C(2N+1) = B(N)$. □

Lastly, we show some new partition theorems.

Theorem 2.16. *Let S denote the set consisting of one copy of the positive integers, another copy of the positive integers that are either congruent to ± 2 or $\pm 3 \pmod{10}$ or are multiples of 5. Let $A(N)$ be the number of partitions of $2N+1$ into odd parts, and let $B(N)$ be the number of partitions of $2N$ into even parts. Then $A(N) = B(N)$.*

Proof. Set $m = 5$ and $(\alpha, \beta, \gamma) = (1, 3, 3)$ in Theorem 2.3. □

For example, $A(5) = 8 = B(5)$ with the representations

$$11 = 1 + 5 + \overline{5} = 1 + 3 + 7 = 1 + \overline{3} + 7 = 1 + 3 + \overline{7}$$
$$= 1 + \overline{3} + \overline{7} = 3 + \overline{3} + 5 = 3 + \overline{3} + \overline{5},$$

$$10 = \overline{10} = 2 + 8 = \overline{2} + 8 = 2 + \overline{8} = \overline{2} + \overline{8} = 2 + \overline{2} + 6 = 4 + 6.$$

Theorem 2.17. *Let S denote the set consisting of the odd positive integers that are not multiples of 17. Let $A(N)$ be the number of partitions of $2N$ into parts congruent to ± 3, ± 5, ± 9 or ± 15 $\pmod{34}$, and let $B(N)$ be the number of partitions of $2N$ into parts congruent to ± 1, ± 7, ± 11 or $\pm 13 \pmod{34}$. Then $A(N) = B(N)$.*

Proof. Take $m = 17$ and $(\alpha, \beta, \gamma, \delta) = (1, 2, 4, 7)$ in Theorem 2.9. □

For example, if $N = 20$, then $A(N) = 3 = B(N)$ with the relevant representations being

$$3 + 37 = 9 + 31 = 15 + 25,$$

$$7 + 33 = 13 + 27 = 1 + 7 + 11 + 21.$$

Theorem 2.18. *Let S denote the set consisting of one copy of the positive integers and another copy of the integers that are either congruent to ± 1 or $\pm 4 \pmod{10}$ or are multiples of 5. Let $A(N)$ be the number of partitions of $2N$ into odd parts, and let $B(N)$ be the number of partitions of $2N$ into even parts. Then $A(N) = B(N)$.*

Proof. Let $m = 5$ and $(\alpha, \beta, \gamma) = (1, 3, 5)$ in Theorem 2.11. □

For example, $A(4) = 4 = B(4)$, and we have the representations

$$1 + 7 = \overline{1} + 7 = 3 + 5 = 3 + \overline{5},$$

$$8 + 2 + 6 = 2 + \overline{6} = 4 + \overline{4}.$$

Chapter 3

A Bijective Proof of the Quintuple Product Identity

3.1 Introduction

The quintuple product identity is stated in the form

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n} - x^{-3n-1}) = \prod_{n=1}^{\infty}(1 - xq^n)(1 - q^n)(1 - x^{-1}q^{n-1})(1 - x^2q^{2n-1})(1 - x^{-2}q^{2n-1}). \tag{3.1}$$

It can be presented in many different forms and various proofs have been given, but, (3.1) seems to be the form that appears most frequently. S. Cooper [13] gave a comprehensive survey of the work on the quintuple product identity, and classified and discussed all known proofs. For historical notes and detailed proofs, the reader is directed to [13].

Although at least 29 proofs of the quintuple product identity have been given, no direct combinatorial proof has yet been shown. J. Lepowsky and S. Milne [39] set $q = uv^2$, $x = v^{-1}$ in (3.1) to obtain

$$\sum_{n=-\infty}^{\infty} u^{n(3n+1)/2}v^{n(3n-2)} - \sum_{n=-\infty}^{\infty} u^{n(3n+1)/2}v^{(n+1)(3n+1)} = \prod_{n=1}^{\infty}(1 - u^nv^{2n-1})(1 - u^{n-1}v^{2n-1})(1 - u^nv^{2n})(1 - u^{2n-1}v^{4n-4})(1 - u^{2n-1}v^{2n}),$$

and they gave the following combinatorial interpretation:

The excess of the number of partitions of (m, n) into an even number of distinct parts of the type $(a, 2a)$, $(b, 2b-1)$, $(c-1, 2c-1)$, $(2d-1, 4d-4)$, $(2e-1, 4e)$ over those into an odd number of parts is 1 or -1 if (m, n) is of the type $(r(3r+1)/2, r(3r-2))$ or $(r(3r+1)/2, (r+1)(3r+1))$, respectively, and 0 otherwise.

They remarked that a direct combinatorial proof of this interpretation can be given. However, Cooper [13] states that "this proof was never published and the notes are most likely now lost."

M. V. Subbaro and M. Vidyasagar [53] deduced the following identities from the quintuple product identity:

$$\begin{aligned}
1 + &\sum_{n=1}^{\infty} q^{3n^2} x^{3n-1} (xq^{2n} - x^{-1} q^{-2n}) \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n x^n q^n (1+qx)(1+q^3x)\cdots(1+q^{2n-1}x) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{n(n+1)}}{(1+qx)(1+q^3x)\cdots(1+q^{2n+1}x)}, \qquad (3.2)
\end{aligned}$$

and Subbarao [54] gave a combinatorial proof of (3.2). In [13], Cooper mentioned that this proof is not a completely combinatorial proof of the quintuple product identity because a lot of algebraic rearrangements are required to derive (3.2).

Thus, the goal of this chapter is to give a bijective proof of the quintuple product identity, especially in the form (3.1). We remark that the right hand side of (3.1) can be viewed as a product of two different forms of Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = \prod_{n=1}^{\infty} (1 + xq^{2n-1})(1 + x^{-1} q^{2n-1})(1 - q^{2n}). \qquad (3.3)$$

This naturally suggests that we can apply two bijections of (3.3) in different forms. In order to complete the proof, we also employ a bijective proof of Euler's pentagonal number theorem in the form

$$\prod_{n=1}^{\infty} (1-q^n)^{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = 1.$$

In the next section, we first derive a combinatorial interpretation from (3.1), and present the aforementioned three bijective proofs. Lastly, we give a bijective proof of the quintuple product identity using them.

3.2 A bijective proof of the quintuple product identity

Let $\mathcal{D}$ be the set of partitions into distinct positive parts, $\mathcal{D}_0$ be the set of partitions into distinct nonnegative parts and $\mathcal{O}$ be the set of partitions into distinct odd parts. The weight $|\pi|$ and the length $\ell(\pi)$ of a partition π denote the sum of the parts and the number of parts of π, respectively.

We can easily see that (3.1) has the following combinatorial interpretation by comparing the coefficients of $x^m q^N$ on each side of (3.1) :

Theorem 3.1. *The excess of the number of partitions of N into an even number of parts in the form*

$$N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2,$$

where $\pi_1, \pi_2 \in \mathcal{D}, \pi_3 \in \mathcal{D}_0, \sigma_1, \sigma_2 \in \mathcal{O}$ *and* $\ell(\pi_1) - \ell(\pi_3) + 2\ell(\sigma_1) - 2\ell(\sigma_2) = m$, *over those into an odd number of parts is* 1 *or* -1 *if* $(m, N) = (3n, n(3n+1)/2)$ *or* $(m, N) = (-3n-1, n(3n+1)/2)$, *respectively, and* 0 *otherwise.*

Before proving Theorem 3.1, we first introduce two combinatorial proofs of Jacobi's triple product identity. J. Zolnowsky [62] made the substitutions $q^2 = uv$, $x = -(u/v)^{1/2}$ in (3.3) to obtain

$$\prod_{n=1}^{\infty}(1 - u^n v^{n-1})(1 - u^{n-1}v^n)(1 - u^n v^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\big(u^{\frac{n(n+1)}{2}} v^{\frac{n(n-1)}{2}} + u^{\frac{n(n-1)}{2}} v^{\frac{n(n+1)}{2}}\big),$$

for which he gave a combinatorial proof. Using his bijection, we can also give a combinatorial proof of Jacobi's triple identity in the form

$$\sum_{n=-\infty}^{\infty} (-1)^n x^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1 - xq^n)(1 - q^n)(1 - x^{-1}q^{n-1}). \tag{3.4}$$

Comparing the coefficient of $x^m q^N$ on both sides of (3.4), we obtain the following combinatorial interpretation.

Theorem 3.2. *The excess of the number of partitions of N into an even number of parts in the form* $N = \tau_1 + \tau_2 + \tau_3$, *where* $\tau_1, \tau_2 \in \mathcal{D}$, $\tau_3 \in \mathcal{D}_0$ *and* $\ell(\tau_1) - \ell(\tau_3) = m$, *over those into an odd number of parts is* $(-1)^n$ *if* $(m, N) = (n, n(n+1)/2)$, *and* 0 *otherwise.*

For convenience, we follow Zolnowsky's notations and rules from [62]. We draw the Ferrars diagram of a partition placing parts left to right in decreasing order. For instance, the partitions $\pi = (6,5,4,2,1) \in \mathcal{D}$ and $\sigma = (5,4,3,2,1) \in \mathcal{D}$ are represented as the following.

$$\pi : \begin{array}{ccccc} \bullet & & & & \\ \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \sigma : \begin{array}{ccccc} \bullet & & & & \\ \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

We define the slope of the diagrams to be the portion consisting of $\circ$ in the following graphs.

$$\pi : \begin{array}{ccccc} \circ & & & & \\ \bullet & \circ & & & \\ \bullet & \bullet & \circ & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \sigma : \begin{array}{ccccc} \circ & & & & \\ \bullet & \circ & & & \\ \bullet & \bullet & \circ & & \\ \bullet & \bullet & \bullet & \circ & \\ \bullet & \bullet & \bullet & \bullet & \circ \end{array}$$

Thus, the length of the slope is equal to the number of consecutive parts starting from the largest one. We say that the slope of a partition in $\mathcal{D}$ is nondetachable if the largest part is the same as the number of parts as in the graph of σ and otherwise, we say that he slope is detachable, as in the graph of π. We define a slope of an empty partition to be nondetachable.

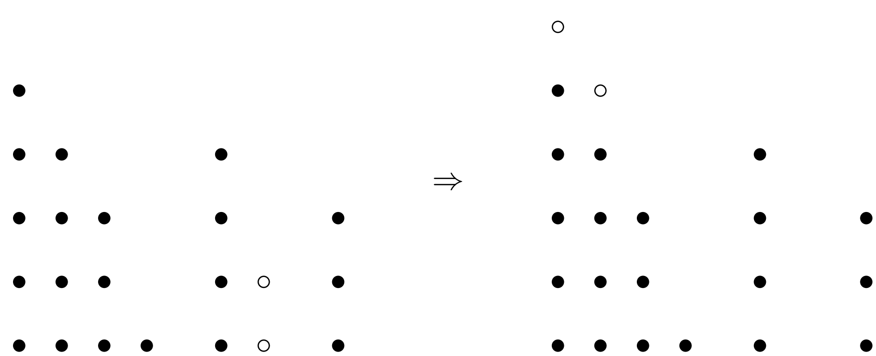
We can also define the slope of diagrams of partitions in $\mathcal{D}_0$ in a similar way. For example, the length of slope of $\pi = (5,4,3,1,0)$ is 3 and that of $\sigma = (4,3,2,1,0)$ is 5. Similarly, we say that the slope of π is detachable and the slope of σ is nondetachable. Note that if the slope of a partition $\in \mathcal{D}_0$ is nondetachable, then the largest part is the number of parts -1.

Proof of Theorem 3.2. First, we consider the case when $m \geq 0$, i.e., $\ell(\tau_1) \geq \ell(\tau_3)$.

Let LS denote the length of the slope of τ_1, HL designate the largest part of τ_1 (0 if τ_1 is empty) and HM and HR denote the smallest parts of τ_2 and τ_3, respectively (infinite if they are empty).

Case 1 : LS $\geq$ HM. (Note that τ_2 is not empty.)

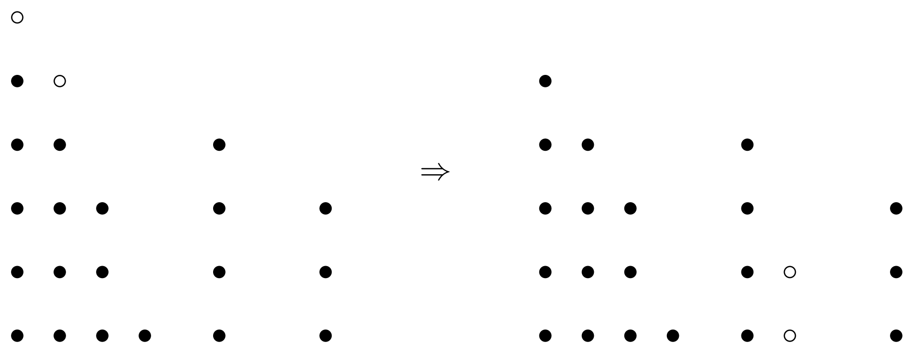
Move the least part of τ_2 onto the slope of τ_1 to create a new slope. For instance, $(5,4,3,1) + (4,2) + (3)$ corresponds to $(6,5,3,1) + (4) + (3)$.



Case 2 : LS $<$ HM, and the slope is detachable.

Remove the slope of τ_1 to create a new smallest part τ_2. For instance, $(6,5,3,1) + (4) + (3)$

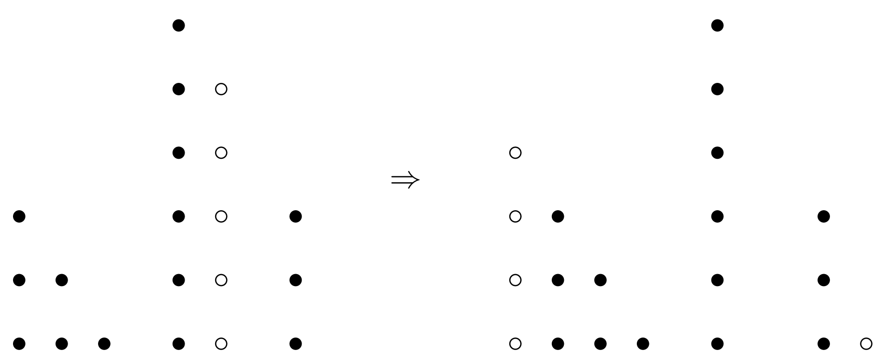
corresponds to $(5,4,3,1)+(4,2)+(3)$.



Note that Case 1 and Case 2 are inverses of each other.

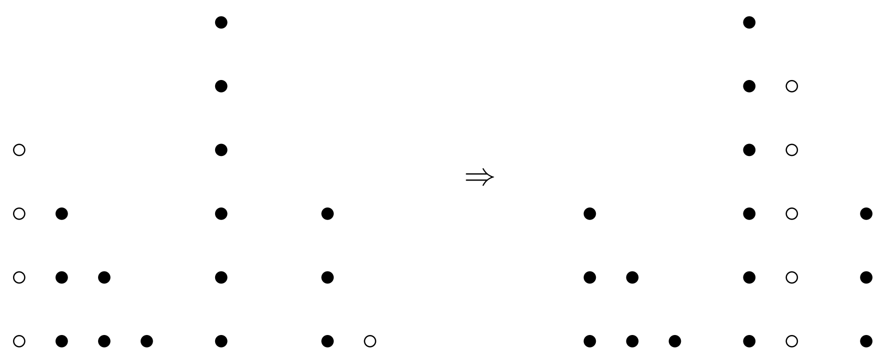
Case 3 : LS $<$ HM, the slope is nondetachable, and HM $\leq$ HL+HR with nonempty τ_2.

In this case, HM $>$ HL $=$ LS. Remove the smallest part of τ_2 to create a new largest part (=HL+1) and a new smallest part (since $0 \leq$ HM $-$ (HL $+1$) $<$ HR). For instance, $(3,2,1)+(6,5)+(3)$ corresponds to $(4,3,2,1)+(6)+(3,1)$.



Case 4 : LS $<$ HM, the slope is nondetachable, and HM $>$ HL+HR with nonempty τ_3. (Note that τ_1 is nonempty since $m \geq 0$.)

Add the largest part of τ_1 and the smallest part of τ_3 to form a new smallest part of τ_2. For instance, $(4,3,2,1)+(6)+(3,1)$ corresponds to $(3,2,1)+(6,5)+(3)$.



Note that Case 3 and Case 4 correspond to each other. Also, note that all four operations change the parity of partitions and none of the rules changes the condition $\ell(\tau_1)-\ell(\tau_3)=m$.

The bijection fails when the slope of τ_1 is nondetachable, and τ_2 and τ_3 are empty, i.e., for some $n \geq 0$,

$$N=\frac{n(n+1)}{2}, \qquad m=\ell(\tau_1)-\ell(\tau_3)=\ell(\tau_1)=n,$$

and the excess of the number of partitions of N into an even number of parts over those into an odd number parts is $(-1)^n$.

Now, consider the case when $m<0$. In this case, we switch the roles of τ_1 and τ_3. In other words, LS is the length of the slope of τ_3, HL denotes the largest part of τ_3 and HM and HR designate the smallest parts of τ_2 and τ_1, respectively. Recall that if the slope of τ_3 is nondetachable, then LS= HL $+1$ (so, in Case 3, HM-(HL+1)≥ 1). Similarly, the bijection fails when τ_1 and τ_2 are empty and the slope of τ_3 is nondetachable, i.e., for some negative integer n,

$$m=\ell(\tau_1)-\ell(\tau_3)=-\ell(\tau_3)=n, \qquad N=0+1+\cdots+(-n-1)=\frac{n(n+1)}{2},$$

and the excess of the number of partitions of N into an even number of parts over those into an odd number parts is $(-1)^n$. Hence, we complete the proof. $\square$

Next, we introduce another combinatorial proof of Jacobi's triple product identity in the form

$$\begin{aligned}\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1}) &= \Big(\sum_{n=-\infty}^{\infty} x^n q^{n^2}\Big)\Big(\prod_{n=1}^{\infty}(1-q^{2n})^{-1}\Big) \\ &= \Big(\sum_{n=-\infty}^{\infty} x^n q^{n^2}\Big)\Big(\sum_{n=0}^{\infty} p_e(2n)q^{2n}\Big), \end{aligned} \tag{3.5}$$

where $p_e(n)$ is the number of partitions of n into even parts. Comparing the coefficients of $x^k q^N$ on each side of (3.5), R. P. Lewis derived the following combinatorial interpretation and gave a bijective proof of it. We also present his proof here.

Theorem 3.3. *The number of partitions of N in the form $N = \pi + \sigma$, where $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$, is equal to $p_e(N-k^2)$..*

Remark : Lewis [40] proved Theorem 3.3 with $p((N-k^2)/2)$ instead of $p_e(N-k^2)$. Theorem 3.3 implies that given a partition of $N = \pi + \sigma$ with $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$, we can find a partition τ bijectively such that $N = k^2 + \tau$ and τ is a partition of $N - k^2$ into even parts.

Proof. Let us consider only the case when $k \geq 0$ since we can exchange π and σ. Given $N = \pi + \sigma$ with $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$, we draw the self-conjugate diagrams G_1 and G_2, respectively. For example, if $N = 38$, $\pi = (11, 9, 5, 1)$ and $\sigma = (9, 3)$, then

$$G_1 : \begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & & & & \end{array} \qquad\qquad G_2 : \begin{array}{lllll} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & \\ \circ & \circ & & & \\ \circ & & & & \\ \circ & & & & \end{array}$$

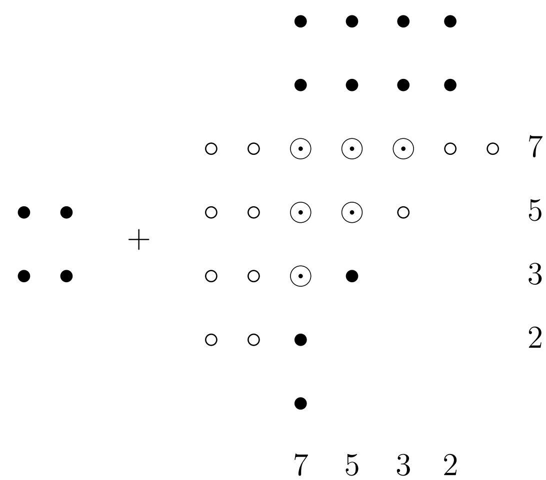
Now, superimpose G_2 on G_1 with the top left corner of G_2 over the point $k+1$ places down the diagonal of G_1. Then, remove the top left square of size k^2. For our example, since $k = 2$,

$$
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \odot & \odot & \odot & \circ & \circ \\
\bullet & \bullet & \odot & \odot & \circ & & \\
\bullet & \bullet & \odot & \circ & & & \\
\bullet & \bullet & \circ & & & & \\
 & & \circ & & & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet
\end{array}
\quad + \quad
\begin{array}{ccccccc}
 & & \bullet & \bullet & \bullet & \bullet & \\
 & & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \odot & \odot & \odot & \circ & \circ \\
\bullet & \bullet & \odot & \odot & \circ & & \\
\bullet & \bullet & \odot & \circ & & & \\
\bullet & \bullet & \circ & & & & \\
 & & \circ & & & &
\end{array}
$$

Lastly, switch $\bullet$ and $\circ$ below the diagonal of the diagram.

$$
\begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet
\end{array}
\quad + \quad
\begin{array}{ccccccc}
 & & \bullet & \bullet & \bullet & \bullet & \\
 & & \bullet & \bullet & \bullet & \bullet & \\
\circ & \circ & \odot & \odot & \odot & \circ & \circ \\
\circ & \circ & \odot & \odot & \circ & & \\
\circ & \circ & \odot & \bullet & & & \\
\circ & \circ & \bullet & & & & \\
 & & \bullet & & & &
\end{array}
$$

The new diagram is composed of the graph, drawn with $\bullet$, of a partition of $(N - k^2)/2$ with the graph of its conjugate, drawn with $\circ$, superimposed.



Since we have the same two partitions of $(N-k^2)/2$, by multiplying each part by 2, we obtain a partition of $N-k^2$ into even parts. Thus, for our example, we obtain a partition $14+10+6+4$ of $N-k^2 = 38-4 = 34$. This process is obviously reversible, so we complete the proof. $\square$

Lastly, we introduce a bijective proof of Euler's recurrence relation by D. M. Bressoud and D. Zeilberger [11]. From Euler's pentagonal number theorem in the form

$$\prod_{n=1}^{\infty}(1-q^n)^{-1}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = \sum_{n=0}^{\infty}p(n)q^n \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = 1 \tag{3.6}$$

we deduce the following theorem.

Theorem 3.4. *For $n \geq 1$,*

$$\sum_{i \text{ even}} p(n-i(3i+1)/2) = \sum_{i \text{ odd}} p(n-i(3i+1)/2),$$

where $i \in \mathbb{Z}$ is allowed to be negative.

For instance, if $n=7$, then

$$\sum_{i \text{ even}} p(n-i(3i+1)/2) = p(7)+p(2)+p(0) = 15+2+1 = 18,$$

$$\sum_{i \text{ odd}} p(n - i(3i+1)/2) = p(6) + p(5) = 11 + 7 = 18.$$

Proof of Theorem 3.4. Let $a(i) = i(3i+1)/2$. Define the map γ by the following rule: for a partition $\lambda : n - a(i) = \lambda_1 + \lambda_2 + \cdots + \lambda_t$,

$$\gamma(\lambda) = \begin{cases} \lambda' : n - a(i-1) = (t + 3i - 1) + (\lambda_1 - 1) + \cdots + (\lambda_t - 1) & \text{if } t + 3i \geq \lambda_1, \\ \\ \lambda' : n - a(i+1) = (\lambda_2 + 1) + \cdots + (\lambda_t + 1) + \underbrace{1 + \cdots + 1}_{\lambda_1 - t - 3i - 1} & \text{if } t + 3i < \lambda_1. \end{cases}$$

It is not hard to see that γ is an involution, so we complete the proof. $\square$

Now, let us use the three bijections that we showed above to prove Theorem 3.1.

Proof of Theorem 3.1. First, fix $\sigma_1,\ \sigma_2 \in \mathcal{O}$, and say $|\sigma_1| + |\sigma_2| = M$. Now, consider all the partitions $\pi_1 + \pi_2 + \pi_3$ of $N - M$ with $\pi_1, \pi_2 \in \mathcal{D}$, $\pi_3 \in \mathcal{D}_0$ and $\ell(\pi_1) - \ell(\pi_3) = m - 2(\ell(\sigma_1) - \ell(\sigma_2))$, so that $N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2$. By the bijective proof of Theorem 3.2, the excess of the number of partitions of N into an even number of parts in the form $N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2$ over those into an odd number of parts (with fixed σ_1 and σ_2) is nonzero only when $\pi_1 = 1 + \cdots + t$, $t \geq 0$, and $\pi_2 = \pi_3 = \emptyset$, or $\pi_3 = 0 + 1 + + \cdots + (-t-1)$, $t < 0$, and $\pi_1 = \pi_2 = \emptyset$.

Thus, we only have to consider the partitions of the form

$$N = 1 + \cdots + t + \sigma_1 + \sigma_2,\ t \geq 0, \qquad N = 0 + 1 + \cdots + (-t-1) + \sigma_1 + \sigma_2,\ t < 0,$$

where $\sigma_1, \sigma_2 \in \mathcal{O}$ and $2(\ell(\sigma_1) - \ell(\sigma_2)) = m - t$. By the bijection described in Theorem 3.3, each pair (σ_1, σ_2) corresponds to $(\ell(\sigma_1) - \ell(\sigma_2))^2 + \tau$, where τ is a partition of $N - t(t+1)/2 - (\ell(\sigma_1) - \ell(\sigma_2))^2$ into even parts. Thus, each partition of N of the form

$$\mu : N = t(t+1)/2 + \sigma_1 + \sigma_2, \quad t \in \mathbb{Z}, \tag{3.7}$$

is bijectively associated with

$$\mu' : N = t(t+1)/2 + (\ell(\sigma_1) - \ell(\sigma_2))^2 + \tau.$$

We consider three different cases when $m \equiv 0, 1$ or $-1 \pmod 3$.

Case 1: $m = 3n, n \in \mathbb{Z}$.

Let $\ell(\sigma_1) - \ell(\sigma_2) = r$. Then we have $t + 2r = 3n$ and $\ell(\mu) = t + \ell(\sigma_1) + \ell(\sigma_2) \equiv t + r \equiv n - r \pmod 2$. Also,

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{n(3n+1)}{2} + 3(n-r)^2 + (n-r) + \tau. \tag{3.8}$$

So, if $N = n(3n+1)/2$, then we have $n = r = t$ and $|\tau| = 0$. Thus, the only possibilities for σ_1 and σ_2 for μ are $\sigma_1 = 1 + 3 + \cdots + 2n - 1$ and $\sigma_2 = \emptyset$ if $n \geq 0$, and $\sigma_1 = \emptyset$ and $\sigma_2 = 1 + 3 + \cdots + (-2n - 1)$ if $n < 0$, since $n = r = \ell(\sigma_1) - \ell(\sigma_2)$. Considering $\ell(\mu) \equiv 2n \pmod 2$, we can see that the excess of the number of partitions of N into an even number of parts over those into an odd number of parts in the form satisfying the condition of our theorem is 1.

Now, suppose $N \neq n(3n+1)/2$. Then, $L := N - n(3n+1)/2 \geq 1$ by (3.8). By the bijective relations between the solutions of μ and μ', the excess of the number of solutions of μ with $\ell(\mu)$ even over those with $\ell(\mu)$ odd is equal to the excess of the number of partitions of $L - (3(n-r)^2 + (n-r))$ into even parts with $n - r$ even over the number of partitions of $L - (3(n-r)^2 + (n-r))$ into even parts with $n - r$ odd since $\ell(\mu) \equiv n - r \pmod 2$. Using the fact that the number of partitions of a number a into even parts is equal to the number of partitions of $a/2$ and the bijection described in Theorem 3.4, we complete the proof of Case 1, because the previously described excess is equal to 0.

Case 2: $m = -3n - 1, n \in \mathbb{Z}$.

If $\ell(\sigma_1) - \ell(\sigma_2) = r$, then $t + 2r = -3n - 1$, $\ell(\mu) \equiv t + r \equiv n + r + 1 \pmod 2$ and

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{n(3n+1)}{2} + 3(n+r)^2 + (n+r) + \tau.$$

Similarly, if $N = n(3n+1)/2$, then we have $n = -r = -t-1$ and $|\tau| = 0$. Since $|\sigma_1| + |\sigma_2| = N - t(t+1)/2 = (t+1)^2$ by (3.7) and $\ell(\sigma_1) - \ell(\sigma_2) = t+1$, we have $\sigma_1 = 1+3+\cdots+2t+1$ and $\sigma_2 = \emptyset$ if $t \geq -1$, and $\sigma_1 = \emptyset$ and $\sigma_2 = 1+3+\cdots+(-2t-3)$ if $t < -1$. Considering $\ell(\mu) \equiv 2t+1 \pmod 2$, we complete the proof when $N = n(3n+1)/2$.

By the same argument as in Case 2, we can also prove the theorem when $N \neq n(3n+1)/2$. (The only difference is that $\ell(\mu)$ has the opposite parity of $n+r$.)

Case 3: $m = 3n+1$, $n \in \mathbb{Z}$.

Similarly, letting $\ell(\sigma_1) - \ell(\sigma_2) = r$, we have $t + 2r = 3n+1$, $\ell(\mu) \equiv t+r \equiv n-r+1 \pmod 2$ and

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{3n^2+3n+2}{2} + 3(n-r)(n-r+1) + \tau.$$

Let $L = N - (3n^2+3n+2)/2$. Then, the excess of the number of solutions of μ with $\ell(\mu)$ even over those with $\ell(\mu)$ odd is equal to the excess of the number of partitions of $L - (3(n-r)(n-r+1))$ into even parts with $n-r$ odd over those with $n-r$ even, which is 0, since $(n-r)(n-r+1) = \{-(n-r)-1\}\{(-(n-r)-1+1)\}$, and $n-r$ and $-(n-r)-1$ have opposite parity. Note that $L = 0$ is not an exceptional case since $(n-r)(n-r+1) = 0$ when $n-r = 0$ or $n-r+1 = 0$. $\square$

Chapter 4

Göllnitz-Gordon Identities and Parity Questions in Partitions

4.1 Introduction

Parity has played a role in additive number theory, in particular partition identities, from the beginning.

B. Gordon [29, 30] and H. Göllnitz [25, 26] independently considered parity as follows:

Theorem 4.1 (First Göllnitz-Gordon Identity)**.** *The number of partitions of n into distinct nonconsecutive parts with no even parts differing by exactly 2 equals the number of partitions of n into parts $\equiv 1, 4,$ or $7 \pmod 8$.*

The famous Rogers-Ramanujan identities do not immediately involve parity. However, several results related to the Rogers-Ramanujan identities concern parity. In particular, many q-series identities from Ramanujan's Lost Notebook raise parity questions.

These examples initiated the thorough examination of parity in partition identities by Andrews [4]. In a long recent paper [4], G. E. Andrews began a thorough study of parity questions arising from partition identities. At the end of his paper [4], he listed fifteen open problems, most of which ask for combinatorial and bijective proofs.

The purpose of this chapter is to provide answers to the first two problems of Andrews, which involve the celebrated Rogers-Ramanujan-Gordon Theorem [1, 30].

Theorem 4.2 (Rogers-Ramanujan-Gordon Identities)**.** *For $1 \le a \le k$, let $B_{k,a}(n)$ be the number of partitions of n of the form*

$$b_1 + b_2 + \cdots + b_j,$$

where $b_i \geq b_{i+1}$, $b_i - b_{i+k-1} \geq 2$, and at most $a-1$ of the b_i are equal to 1. Let $A_{k,a}$ be the number of partitions of n into parts $\not\equiv 0, \pm a \pmod{2k+1}$. Then for all $n \geq 0$,

$$A_{k,a}(n) = B_{k,a}(n).$$

We now add parity restrictions.

Theorem 4.3 (Andrews)**.** *Suppose $k \geq a \geq 1$ are integers with $k \equiv a \pmod 2$. Let $W_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with the added restriction that even parts appear an even number of times. If k and a are both even, let $G_{k,a}(n)$ denote the number of partitions of n in which no odd part is repeated and no even part is $\equiv 0, \pm a \pmod{2k+2}$. If k and a are both odd, let $G_{k,a}(n)$ denote the number of partitions of n into parts that are neither $\equiv 2 \pmod 4$ nor $\equiv 0, \pm a \pmod{2k+2}$. Then for all $n \geq 0$,*

$$W_{k,a}(n) = G_{k,a}(n).$$

It follows from a comparison of Theorem 4.3 with the Gollnitz-Gordon identity in Theorem 4.1 that $W_{3,3}(n)$ is equal to the number of partitions of n into parts that differ by at least 2 and by more than 2 if the parts are even. A bijective proof of this partition identity is the first problem in the list of Andrews [4]. The second problem is to show bijectively that $W_{3,1}(n)$ is equal to the number of partitions of n into parts (each > 2) that differ by at least 2 and by more than 2 if the parts are even.

A generalization of the Göllnitz-Gordon identities, the first of which is stated in Theorem 4.1, has been accomplished by Andrews [2] in the same manner that the Rogers-Ramanujan-Gordon identity stated in Theorem 4.2 generalizes the celebrated Rogers-Ramanujan identities.

Theorem 4.4 (Andrews)**.** *Let a and k be integers with $0 < a \leq k$. Let $C_{k,a}(n)$ be the number of partitions of n into parts that are neither $\equiv 2 \pmod 4$ nor $\equiv 0, \pm(2a-1) \pmod{4k}$. Let $D_{k,a}(n)$ denote the number of partitions of n of the form $n = \sum_{i\geq 1} f_i i$ with $f_1 + f_2 \leq a-1$ and for all*

$i \geq 1$,

$$f_{2i-1} \leq 1 \quad \textit{and} \quad f_{2i} + f_{2i+1} + f_{2i+2} \leq k-1,$$

where f_i denotes the number of appearances of i in the partition. Then $C_{k,a}(n) = D_{k,a}(n)$.

By comparing Theorems 4.3 and 4.4, we see that

$$W_{2k-1,2a-1}(n) = D_{k,a}(n). \tag{4.1}$$

In the third problem of Andrews, it is asked to prove (4.1) bijectively.

In Section 4.2, we prove combinatorially that

1. $W_{3,3}(n)$ is equal to the number of partitions of n into parts that differ by at least 2 and by more than 2 if the parts are even, namely $W_{3,3}(n) = D_{2,2}(n)$, and

2. $W_{3,1}(n)$ is equal to the number of partitions of n into parts (each > 2) that differ by at least 2 and by more than 2 if the parts are even, namely $W_{3,1}(n) = D_{2,1}(n)$.

4.2 Problems 1 and 2

Theorem 4.5. *For any positive integer n,*

$$W_{3,3}(n) = D_{2,2}(n)$$

Proof. Let $\pi = (\pi_1, \ldots, \pi_m)$ with $\pi_i \leq \pi_{i+1}$, be a partition counted by $W_{3,3}(n)$. By the definition of $W_{3,3}(n)$, we see that each part can be repeated at most twice and all the even parts appear exactly twice. We represent the partition π by an array with two rows (counted from bottom to top), where the first and the second rows consist of the first and second copies of the parts, respectively and each column has the same parts. For instance, if $\pi = (2, 2, 4, 4, 7, 9, 14, 14, 23, 23, 33)$ is counted by $W_{3,3}(135)$, then we write π as following.

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 2 & 4 & 7 & 9 & 14 & 23 & 33 \end{array}$$

We note that since $\pi_{i+2} - \pi_i \geq 2$ and even parts appear twice, the parts appearing only in the first row are odd and the parts from the first row differ by at least 2. Let $(\tau_1, \ldots, \tau_l)$ be the parts appearing in the first row. For each i with $1 \leq i \leq l$, subtract $2i-1$ from τ_i and add the parts in the same column. In the above example, we have $(\tau_1, \ldots, \tau_l) = (2, 4, 7, 9, 14, 23, 33)$, and we obtain

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 1 & 1 & 2 & 2 & 5 & 12 & 20 \\ \hline 3 & 5 & 2 & 2 & 19 & 35 & 20 \end{array}$$

We note that the sums of two parts from the same column are odd and the parts appearing only in the first row are even. Besides, since the parts from the second row differ by at least 2, all the odd parts in the resulting partition are distinct. Lastly, we rearrange the parts in weakly increasing order and add $2i-1$ to the $i-$th part for each $1 \leq i \leq l$. Then, the parts of the resulting partition differ by at least two and even parts differ by more than 2. Hence, the resulting partition is counted by $D_{2,2}(n)$. In the example, we obtain

$$\begin{array}{ccccccc} 2 & 2 & 3 & 5 & 19 & 20 & 35 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ \hline 3 & 5 & 8 & 12 & 28 & 31 & 48 \end{array}$$

and we see that $(3, 5, 8, 12, 28, 31, 48)$ is counted by $D_{2,2}(135)$.

Now, we show that the process is reversible. Let $\sigma = (\sigma_1, \ldots, \sigma_l)$ with $\sigma_i \leq \sigma_{i+1}$, be a partition counted by $D_{2,2}(n)$. We first subtract $2i-1$ from σ_i to obtain σ'. Since the even parts of σ differ by at least 4, σ' has distinct odd parts. For example, if $\sigma = (3, 5, 8, 12, 28, 31, 48)$, then $\sigma' = (2, 2, 3, 5, 19, 20, 35)$. Now, we rearrange the parts of σ' to obtain $w = (w_1, \ldots, w_l)$ as following. In order to select w_i from the parts of σ', we consider the remaining parts of σ' after removing $w_1, \ldots, w_{i-1}$ from σ', and choose the smallest odd and even parts among them, say σ'_o and σ'_e, respectively. If $(\sigma'_o - (2i-1))/2 \le \sigma'_e$, then let $w_i = \sigma'_o$, and otherwise, let $w_i = \sigma'_e$. We continue this process until we determine all of $w_1, \ldots, w_l$ (if we use all of odd parts or even parts of σ', then just arrange the remaining parts in weakly increasing order). In the same example, we have $\sigma'_o = 3$ and $\sigma_e = 2$. Since $(3-1)/2 \le 2$, we have $w_1 = 3$. For w_2, we have $\sigma'_o = 5$ and $\sigma'_e = 2$. Since $(5-3)/2 \le 2$, we have $w_2 = 5$. Similarly, since $\sigma'_o = 19$, $\sigma'_e = 2$ and $(19-5)/2 > 2$, we have $w_3 = 2$. By continuing this, we have $w = (3, 5, 2, 2, 19, 35, 20)$.

Now, if w_i is odd, then we split it into two parts $(w_i + (2i-1))/2$ and $(w_i - (2i-1))/2$, whose difference is $2i-1$. We write w by an array with two rows (counted from bottom to top), where the first and second rows of $i-$th column are $(w_i - (2i-1))/2$ and $(w_i + (2i-1))/2$ if w_i is odd, and if w_i is even, then place it in the first row of the $i-$th column. Thus, in the example we have

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 1 & 1 & 2 & 2 & 5 & 12 & 20 \end{array}$$

Lastly, we add $2i-1$ to the $i-$th part in the first row. Then, the columns with two parts will have the same parts. Since the parts in the first row differ by at least two and the parts appearing only in the first row are odd, the resulting partition is counted by $W_{3,3}(n)$. From the example, we obtain the following.

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 2 & 4 & 7 & 9 & 14 & 23 & 33 \end{array}$$

Note that the resulting partition is $(2, 2, 4, 4, 7, 9, 14, 14, 23, 23, 33)$, which is counted by $W_{3,3}(135)$. $\square$

Theorem 4.6. *For any positive integer* n*,*

$$W_{3,1}(n) = D_{2,1}(n)$$

Proof. By the definitions, 1 is not allowed in any partitions counted by $W_{3,1}(n)$, and none of 1 and 2 are allowed in partitions counted by $D_{2,1}(n)$. Thus, in the proof of Theorem 4.5, we add the constraints that the parts are greater than 1 and 2, respectively. Then, the rest of the proof is the same. We omit the details. $\square$

Chapter 5

A Combinatorial Proof of a Recurrence Relation for the Partition Function due to Euler

5.1 Introduction

One of Euler's famous identities is a recurrence formula for the sum of divisors $\sigma(n)$.

Theorem 5.1. *For every $n > 0$, we have*

$$\sum_{k=-\infty}^{\infty} (-1)^k \sigma\left(n - \frac{k(3k+1)}{2}\right) = \begin{cases} (-1)^{k-1} n & \text{if } n = \frac{k(3k+1)}{2},\ k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Euler [17, p. 234] derived the above result using logarithmic differentiation of Euler's pentagonal number theorem

$$F(x) := \prod_{k=1}^{\infty}(1 - x^k) = \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k+1)}{2}}. \tag{5.1}$$

The proof is elementary. We begin with

$$xF'(x) = F(x)\frac{xF'(x)}{F(x)} = -\sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k+1)}{2}} \sum_{j=1}^{\infty} \sigma(j) x^j.$$

On the other hand,

$$xF'(x) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{k(3k+1)}{2} x^{\frac{k(3k+1)}{2}}.$$

By equating coefficients of x^n, we obtain Theorem 5.1.

From (8.8), we can easily derive Euler's recurrence formula for $p(n)$,

$$\sum_{n=-\infty}^{\infty} (-1)^k p\left(n - \frac{k(3k+1)}{2}\right) = 0.$$

Thus, Theorem 5.1 indicates that a similar formula is also valid for $\sigma(n)$.

Considering $xF'(x)/F(x)$ instead of $xF'(x)$, we can deduce an identity, which is a companion of the identity from Theorem 5.1. From

$$-\sum_{j=1}^{\infty}\sigma(j)x^j = \prod_{k=1}^{\infty}(1-x^k)^{-1}\sum_{k=-\infty}^{\infty}(-1)^k\frac{k(3k+1)}{2}x^{\frac{k(3k+1)}{2}}$$

we obtain a formula for $\sigma(n)$ in terms of $p(n)$.

Theorem 5.2. *Let* $n \geq 1$. *Then,*

$$-\sigma(n) = \sum_{k=-\infty}^{\infty}(-1)^k\frac{k(3k+1)}{2}p(n-\frac{k(3k+1)}{2}).$$

Even though Theorems 5.1 and 5.2 are easily derived analytically, it is interesting to find combinatorial proofs. A combinatorial proof for Theorem 5.1, that is based on a double counting argument, can be found in [58, pp. 182–183] and [45, p. 53]. However, no such argument is known for Theorem 5.2.

In this chapter, we generalize Theorem 5.2 and give a combinatorial proof of this generalization, which is also based on a double counting argument. We can easily obtain the generalization by applying the previous argument to the general theta function, defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \tag{5.2}$$

where $|ab| < 1$. The latter equality is called Jacobi's triple product identity.

A generalization of Theorem 5.1 can also be found in a natural way. Moreover, the combinatorial argument from [58, pp. 182–183] used to prove Theorem 5.1 can be applied to the generalization of Theorem 5.1 with a little modification. Because the combinatorial proof of the more general theorem is similar to that for Theorem 5.1, we do not give it here.

Lastly, we show that the quintuple product identity can be employed to yield similar sorts of identities, and we remark that their combinatorial proofs can be given in a similar fashion.

5.2 A generalization of Theorem 5.2 and its combinatorial proof

Theorem 5.3. *Let n and m be positive integers, and put $n+m=L$. Then, for $N \geq 1$, we have*

$$-\sigma_{n,m}(N) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{k^2(n+m)+k(n-m)}{2} p_{n,m}(N - \frac{k^2(n+m)+k(n-m)}{2}), \quad (5.3)$$

where $p_{n,m}(N)$ is the number of partitions of N into parts congruent to n, m or $L \pmod{L}$ and

$$\sigma_{n,m}(k) = \sum_{\substack{d|k \\ d \equiv n,m,L \pmod{L}}} d.$$

Note that Theorem 5.3 with $n=1$, $m=2$ reduces to Theorem 5.2.

We can easily deduce Theorem 5.3 by taking the logarithmic derivative of $f(-q^n, -q^m)$. By (5.2), we have

$$\begin{aligned} f(-q^n, -q^m) &= \prod_{k=0}^{\infty} (1-q^{n+kL})(1-q^{m+kL})(1-q^{(k+1)L}) \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k q^{\frac{k^2(n+m)+k(n-m)}{2}} + \sum_{k=1}^{\infty} (-1)^k q^{\frac{k^2(n+m)-k(n-m)}{2}}. \end{aligned} \quad (5.4)$$

Consider

$$\begin{aligned} -q\frac{\frac{d}{dq} f(-q^n, -q^m)}{f(-q^n, -q^m)} &= \sum_{k=0}^{\infty} \frac{(n+kL)q^{n+kL}}{1-q^{n+kL}} + \sum_{k=0}^{\infty} \frac{(m+kL)q^{m+kL}}{1-q^{m+kL}} + \sum_{k=0}^{\infty} \frac{(k+1)Lq^{(k+1)L}}{1-q^{(k+1)L}} \\ &= \sum_{k=1}^{\infty} \sigma_{n,m}(k) q^k. \end{aligned}$$

On the other hand, we also have

$$-q\frac{\frac{d}{dq} f(-q^n, -q^m)}{f(-q^n, -q^m)} = \frac{-1}{f(-q^n, -q^m)} \sum_{k=-\infty}^{\infty} (-1)^k \frac{k^2(n+m)+k(n-m)}{2} q^{\frac{k^2(n+m)k+(n-m)}{2}}.$$

Thus, we obtain

$$-\sum_{k=1}^{\infty}\sigma_{n,m}(k)q^k = \frac{1}{f(-q^n,-q^m)}\sum_{k=1}^{\infty}(-1)^k\frac{k^2(n+m)\pm k(n-m)}{2}q^{\frac{k^2(n+m)\pm k(n-m)}{2}},$$

which implies Theorem 5.3.

A combinatorial proof of Theorem 5.3. Define the set

$$A_{n,m}(N) = \{(\pi,\lambda) : |\pi| + |\lambda| = N, \pi \in D_{n,m}, \lambda \in P_{n,m}\},$$

where $D_{n,m}$ is the set of partitions into distinct parts congruent to n, m or $L \pmod L$ and $P_{n,m}$ is the set of partitions into parts congruent to n, m or $L \pmod L$. Now let

$$B_{n,m}(N) = \sum_{(\pi,\lambda)\in A_{n,m}(N)} (-1)^{\ell(\pi)}|\pi|.$$

We show that $B_{n,m}(N)$ is equal to both sides of (5.3). The involution of the Jacobi triple product identity [62] implies a bijective proof of (5.4). Thus, bijectively, we have

$$\begin{aligned} B_{n,m}(N) &= \sum_{(\pi,\lambda)\in A_{n,m}(N)} (-1)^{\ell(\pi)}|\pi| = \sum_{b=0}^{N}(N-b)p_{n,m}(b)\sum_{\pi\in D_{n,m}(N-b)}(-1)^{\ell(\pi)} \\ &= \sum_{k=-\infty}^{\infty}(-1)^k\frac{k^2(n+m)+k(n-m)}{2}p_{n,m}(N-\frac{k^2(n+m)+k(n-m)}{2}), \end{aligned}$$

where $D_{n,m}(N-b)$ is the set of partitions of $N-b$ into distinct parts congruent to n, m or $L \pmod L$.

Next, we first show that the number of pairs $(\pi,\lambda) \in A_{n,m}(N)$ with $\ell(\pi)$ even is equal to the number of those with $\ell(\pi)$ odd. Let $s(\pi)$ be the smallest part of a partition π and define $s(\pi) = \infty$ if $\pi = \emptyset$. If $s(\pi) \le s(\lambda)$, then move $s(\pi)$ to the partition λ, and if $s(\pi) > s(\lambda)$, then move $s(\lambda)$ to the partition π. Obviously, this map is an involution, so we complete the proof of our claim. By this involution, we have

$$\begin{aligned}
B_{n,m}(N) = \sum_{(\pi,\lambda)\in A_{n,m}(N)} (-1)^{\ell(\pi)}|\pi| &= \sum_{s(\pi)\le s(\lambda)} (-1)^{\ell(\pi)}|\pi| + \sum_{s(\pi)>s(\lambda)} (-1)^{\ell(\pi)}|\pi| \\
&= \sum_{a=1}^{N} \sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)}(|\pi| - (|\pi| - a)) \\
&= \sum_{a=1}^{N} a \sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)}.
\end{aligned}$$

It suffices to show that

$$\sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)} = \begin{cases} -1, & \text{if } a \mid N, \\ 0, & \text{otherwise.} \end{cases}$$

Let $L(\pi)$ be the largest part of a partition π and define $L(\pi) = 0$ if $\pi = \emptyset$. Consider a pair $(\pi, \lambda) \in A_{n,m}(N)$ with $s(\pi) = a \le s(\lambda)$. Let $\pi = a + \mu$. If $L(\mu) \ge L(\lambda)$ and $\mu \ne \emptyset$, then move the largest part of μ to the partition λ, and if $L(\mu) < L(\lambda)$, except in the case when $L(\mu) = 0 < L(\lambda) = a$, then move the largest part of λ to the partition μ. We thus obtain a new partition (π', λ'). Then $s(\pi') = a \le s(\lambda')$ and $(-1)^{\ell(\pi)} = -(-1)^{\ell(\pi')}$. But, the above map fails when $\pi = a$ (i.e., $\mu = \emptyset$) and $\lambda = \emptyset$ or $\pi = a$ and $\lambda = a + \cdots + a$. So, we have $a \mid N$. Thus

$$\sum_{\substack{s(\pi)=a \\ s(\pi)\le s(\lambda)}} (-1)^{\ell(\pi)} = \begin{cases} -1, & \text{if } a \mid N, \\ 0, & \text{otherwise,} \end{cases}$$

whence

$$B_{n,m}(N) = \sum_{(\pi,\lambda)\in A_{n,m}(N)} (-1)^{\ell(\pi)}|\pi| = -\sum_{\substack{a\mid N \\ a\equiv n,m,L \pmod{L}}} a = -\sigma_{n,m}(N).$$

$\square$

5.3 Further results

Let k, l be positive integers. By the quintuple product identity [13],

$$F(q) := \prod_{n=0}^{\infty}(1 - q^{\frac{k+l}{2}+kn})(1 - q^{\frac{k+l}{2}+kn})(1 - q^{\frac{k-l}{2}+kn})(1 - q^{l+2kn})(1 - q^{2k-l+2kn})$$
$$= \sum_{n=-\infty}^{\infty} q^{\frac{3k}{2}n^2+\frac{2k-3l}{2}n} - q^{\frac{3k}{2}n^2+\frac{4k-3l}{2}n+\frac{k-l}{2}}.$$

Repeating the same argument as in Section 5.2, we have

$$-q\frac{\frac{d}{dq}F(q)}{F(q)} = \sum_{n=1}^{\infty} \epsilon_{k.l}(n)q^n, \tag{5.5}$$

where

$$\epsilon_{k,l}(n) = \sum_{\substack{d|n \\ d\equiv \frac{k+l}{2}, \frac{k-l}{2}, 0 \pmod k}} d + \sum_{\substack{d|n \\ d\equiv l, 2k-l \pmod{2k}}} d. \tag{5.6}$$

Also,

$$q\frac{d}{dq}F(q) = \sum_{n=-\infty}^{\infty} \left(\frac{3k}{2}n^2 + \frac{2k-3l}{2}n\right) q^{\frac{3k}{2}n^2+\frac{2k-3l}{2}n}$$
$$- \left(\frac{3k}{2}n^2 + \frac{4k-3l}{2}n + \frac{k-l}{2}\right) q^{\frac{3k}{2}n^2+\frac{4k-3l}{2}n+\frac{k-l}{2}}. \tag{5.7}$$

Equating the two expressions for $q(dF(q))/dq$ from (5.5) and (5.7) we conclude that

$$\left(\sum_{n=-\infty}^{\infty} q^{\frac{3k}{2}n^2+\frac{4k-3l}{2}n+\frac{k-l}{2}} - q^{\frac{3k}{2}n^2+\frac{2k-3l}{2}n}\right) \sum_{n=1}^{\infty} \epsilon_{k.l}(n)q^n$$
$$= \sum_{n=-\infty}^{\infty} \left(\frac{3k}{2}n^2 + \frac{2k-3l}{2}n\right) q^{\frac{3k}{2}n^2+\frac{2k-3l}{2}n} \tag{5.8}$$
$$- \left(\frac{3k}{2}n^2 + \frac{4k-3l}{2}n + \frac{k-l}{2}\right) q^{\frac{3k}{2}n^2+\frac{4k-3l}{2}n+\frac{k-l}{2}}. \tag{5.9}$$

The identity (5.8) has the following arithmetic interpretation.

Theorem 5.4. *For $N \geq 1$,*

$$\sum_{j+\frac{3k}{2}n^2+\frac{4k-3l}{2}n+\frac{k-l}{2}=N} \epsilon_{k,l}(j) - \sum_{j+\frac{3k}{2}n^2+\frac{2k-3l}{2}n=N} \epsilon_{k,l}(j) = \begin{cases} N, & \textit{if } N = \frac{3k}{2}n^2 + \frac{2k-3l}{2}n, \\ -N, & \textit{if } N = \frac{3k}{2}n^2 + \frac{4k-3l}{2}n + \frac{k-l}{2}, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\epsilon_{k,l}(j)$ are defined in (5.6).

Next, from the two different representations of $q(dF(q))/dqF(q)$ given in (5.5) and (5.7), we obtain the following identity.

Theorem 5.5. *For $N \geq 1$,*

$$\epsilon_{k,l}(N) = \sum_{n=-\infty}^{\infty} \left(\frac{3k}{2}n^2 + \frac{4k-3l}{2}n + \frac{k-l}{2}\right) \rho_{k,j}\left(N - \frac{3k}{2}n^2 - \frac{4k-3l}{2}n - \frac{k-l}{2}\right)$$
$$- \sum_{n=-\infty}^{\infty} \left(\frac{3k}{2}n^2 + \frac{2k-3l}{2}n\right) \rho_{k,j}\left(N - \frac{3k}{2}n^2 - \frac{2k-3l}{2}n\right),$$

where $\rho_{k,j}(n)$ is the number of partitions of n into parts congruent to $\frac{k+l}{2}, \frac{k-l}{2}, 0 \pmod{k}$ or $l, 2k-l \pmod{2k}$.

Using the bijective proof of the quintuple product identity from Chapter 3, we can give combinatorial proofs of Theorems 5.4 and 5.5 that are very similar to those for Theorems 5.1 and 5.3.

Chapter 6

Covering Systems in Number Fields

6.1 Introduction

As we mentioned earlier in the Introduction, Conjectures **I** and **II** were confirmed by M. Filaseta, K. Ford, S. Konyagin, C. Pomerance and G. Yu [22]. Their principal results are the following.

Let

$$L(N,s)=\exp\left(\log N\frac{\log\log(s\log N)}{\log(s\log N)}\right).$$

Theorem 6.1. *Suppose $0<b<\frac{1}{2}$, $0<c<\frac{1}{3}(1-4b^2)$ and let N be sufficiently large, depending on the choice of b and c. Suppose C is a finite set of congruence classes with moduli $>N$, each modulus appearing at most s times, where $s\le\exp\left(b\sqrt{\log N\log\log N}\right)$, and such that*

$$\sum_{(r \bmod n)\in C}\frac{1}{n}\le c\log L(N,s).$$

Then C is not a covering system.

Theorem 6.2. *Suppose $0<\varepsilon<1/2$, $0<b<\frac{1}{2}\sqrt{\varepsilon}$ and $N\ge 100$. Suppose that C is a finite set of congruence classes with moduli from $(N,KN]$, each modulus appearing at most s times, where $s\le\exp\left(b\sqrt{\log N\log\log N}\right)$ and $K=L(N,s)^{(1/2-\varepsilon)/s}$. Then the density of the integers not covered by C is*

$$\ge\left(1+O\left(\frac{1}{(\log N)^{\lambda}}\right)\right)\prod_{(r \bmod n)\in C}\left(1-\frac{1}{n}\right),$$

where λ is a positive constant depending only on ε and b.

It is not hard to see that Theorems 6.1 and 6.2 imply Conjectures I and II, respectively, by setting $s = 1$.

In this chapter, we generalize Theorems 6.1 and 6.2 to arbitrary number fields. We take advantage of the facts that all the ideals in the ring of integers of a number field have unique factorization into prime ideals, the greatest common divisor and the least common multiple of ideals are defined, and the Chinese Remainder Theorem holds. These properties are necessary in the proofs of the results from [22].

Now, we introduce a concept of covering systems in number fields. For example, consider the field of Gaussian rationals $\mathbb{Q}(i)$ with ring of integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, which is the set of Gaussian integers. Let $I = (1 + i)$, which is the ideal in $\mathbb{Z}[i]$ generated by $1 + i$. Then we can see that $\mathbb{Z}[i] = I \bigcup \{1 + I\}$. In other words, $\{0 \pmod{I}, 1 \pmod{I}\}$ covers $\mathbb{Z}[i]$. We say that $\{0 \pmod{I}, 1 \pmod{I}\}$ is a covering system of $\mathbb{Z}[i]$ (or in $\mathbb{Q}(i)$). More generally, let $F/\mathbb{Q}$ be a number field of degree d with ring of integers $\mathcal{O}_F$. We call $\{r_1 \pmod{I_1}, \ldots, r_k \pmod{I_k}\}$ a covering system in F (or of $\mathcal{O}_F$) if for each $i \leq k$, $r_i \in \mathcal{O}_F$, I_i is an ideal in $\mathcal{O}_F$, and $\mathcal{O}_F = \bigcup_{i=1}^{k} \{r_i + I_i\}$. Furthermore, if a covering system covers every element of $\mathcal{O}_F$ exactly once, then it is said to be an exact covering system. Thus, in fact, $\{0 \pmod{I}, 1 \pmod{I}\}$ is an exact covering system of $\mathbb{Z}[i]$.

We remark that a covering system in a number field can be identified with a covering system of $\mathbb{Z}^d$ by cosets of subgroups, where d is the degree of the number field, since the ring of integers of a number field with degree d is isomorphic to $\mathbb{Z}^d$ as an additive group. However, the moduli from a covering system in a number field are ideals in the ring of integers. Thus, the covering systems in a number field are more restrictive than those of $\mathbb{Z}^n$ by cosets of any subgroups.

Here, note that if $\{r_1 \pmod{I}, \ldots, r_k \pmod{I}\}$ is an exact covering system, then we must have $k = |\mathcal{O}_F/I| = \|I\|$, which is the norm of I. Analogous to the notion of density for sets of integers, we say that the density of each $r_i \pmod{I}$ is $1/\|I\|$.

In order to handle ideals in the ring of integers, we introduce the functions $f(n)$ and $g(n)$, which denote the number of ideals of norm n and the number of prime ideals of norm n, respectively. In particular, we use the following key propositions.

Proposition 6.1 ([41], Corollary of Theorem 39)**.** *Let $F/\mathbb{Q}$ be a number field of degree d. Then,*

$$\sum_{n\leq x} f(n) = c_F x + O(x^{1-\frac{1}{d}}),$$

where c_F is a constant depending on F.

Proposition 6.2 ([38], page 670)**.** *Let $F/\mathbb{Q}$ be a number field of degree d. Then,*

$$\sum_{n\leq x} g(n) = Li(x) + O\left(x \exp\left(-(\log x)^{1/13}\right)\right), \quad \textit{where} \quad Li(x) = \int_2^x \frac{dt}{\log t}.$$

Here and throughout this chapter, constants implied by the $O-$symbol may depend on the field F. Dependence on any other quantity will be indicated by a subscript. We remark that a stronger version of Proposition 6.2 is possible using Theorem 5.33 of [36] (the term involving a possible exceptional zero is absorbed into the error estimate at the cost of the $O-$constant, which thus depends on the field F in an inexplicit way). We can easily see that $g(p^n) \leq d$ and $g(p^n) = 0$ if $n > d$, where p is a prime.

We adopt the following notation which is analogous to that in [22]. We call a finite collection of congruence classes $C = \{r_1 \pmod{I_1}, \ldots, r_k \pmod{I_k}\}$ in a number field $F/\mathbb{Q}$ a residue system. We let $S(C)$ be the multiset $\{I_1, \ldots, I_k\}$ and we say that the multiplicity of I_i is the number of times that I_i appears in $S(C)$. By $\delta(C)$ we denote the density of the elements of the ring of integers not covered by C, and we also set

$$\alpha(C) = \prod_{I\in S(C)} \left(1 - \frac{1}{\|I\|}\right).$$

The goal of this chapter is to derive analogues of all the lemmas and theorems of [22], including Theorems 6.1 and 6.2, in the number field setting. In Section 6.2, we present analogues of many preparatory lemmas from [22]. Most of the proofs are very similar to those of [22]. A notable exception is Lemma 6.11 below. In Section 6.3, we prove our main theorems, which are analogues of Theorems 2, 3 and 4 of [22]. Let us state three of our results, the first and the third being analogues of Theorems 6.1 and 6.2, respectively.

Theorem 6.3. *Suppose* $0 < b < \frac{1}{2}$, $0 < c < \frac{1}{3}(1 - 4b^2)$. *Let* $F/\mathbb{Q}$ *be a number field of degree* $d \geq 1$. *Let* N *be sufficiently large, depending on the choice of* b, c *and* F. *Suppose* C *is a residue system in* $F/\mathbb{Q}$ *with* $S(C)$ *consisting of ideals* $\|I\| > N$, *each having multiplicity at most* s, *where* $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$, *and such that*

$$\sum_{I \in S(C)} \frac{1}{\|I\|} \leq c \log L(N, s). \tag{6.1}$$

Then $\delta(C) > 0$.

Theorem 6.4. *Suppose that* C *is a residue system of a number field* $F/\mathbb{Q}$ *of degree* d. *Suppose* $0 < \varepsilon < \ (1 - \log 2)^{-1}$, $b < \frac{1}{2}\sqrt{(1 - \log 2)\varepsilon}$, N *is sufficiently large, depending on the choice of* ε, b *and* F, *and* $S(C)$ *consists of ideals whose norms are in* $(N, KN]$ *with multiplicity at most* s, *where* $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$ *and* $K = L(N, s)^{((1-\log 2)^{-1}-\varepsilon)/c_F s}$. *Then* $\delta(C) > 0$.

As in [22], the following theorem shows that if K is a bit smaller than in Theorem 6.4, then we have

$$\delta(C) \geq (1 + o(1))\alpha(C).$$

Theorem 6.5. *Suppose* $0 < \varepsilon < 1/2$, $0 < b < \frac{1}{2}\sqrt{\varepsilon}$ *and* $N \geq 100$. *Suppose that* C *is a residue system of* $F/\mathbb{Q}$ *with* $S(C)$ *consisting of ideals whose norms are in* $(N, KN]$ *with multiplicity at most* s, *where* $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$ *and* $K = L(N, s)^{(1/2-\varepsilon)/c_F s}$. *Then*

$$\delta(C) \geq \left(1 + O_{\varepsilon,b}\left(\frac{1}{(\log N)^{\lambda}}\right)\right)\alpha(C),$$

where λ *is a positive constant depending only on* ε *and* b.

We remark that we obtain our main theorems under the same conditions on b, c and ε as in Theorems $2, 3$ and 4 of [22]. Additional theorems which are analogues of those from [22] will be given later in Section 6.4 and Section 6.5. In Section 6.4, we construct an exact covering system in a number field with the multiplicity of each modulus $\leq \exp(\sqrt{\log N \log\log N})$ and we also show that the density $\delta(C)$ can be considerably smaller than that of Theorem 6.5 provided K is sufficiently large. In Section 6.5, we study normal behaviors of $\delta(C)$ over random residue systems C with fixed $S(C)$.

6.2 Preliminary Lemmas

In this section, we present lemmas that are analogues of all the lemmas in [22]. Throughout this chapter, n denotes a positive integer and p represents a prime. We use the Vinogradov notation $A \ll B$, which is the same as $A = O(B)$, and constants implied by the notation $\ll$, as with the notation O, may depend on the field F.

Let $F/\mathbb{Q}$ be a number field of degree d and let $\mathcal{O}_F$ be the ring of integers of F. Let C be a finite set of ordered pairs (I, r), which is a set of residue classes $r \pmod{I}$, where I is an ideal of $\mathcal{O}_F$ and $r \in \mathcal{O}_F$. We say such a set is a *residue system of* F. Let $S = S(C)$ denote the multiset of the moduli I appearing in C, and we call the number of times an ideal I appears in S the *multiplicity* of I. By $R(C)$ we denote the set of elements of $\mathcal{O}_F$ *not* congruent to $r \pmod{I}$ for any $(I, r) \in C$, and we denote the asymptotic density of $R(C)$ by $\delta(C)$. For $C = \{(I_1, r_1), \ldots, (I_l, r_l)\}$, we let

$$\alpha(C) = \prod_{I \in S(C)} \left(1 - \frac{1}{\|I\|}\right) = \prod_{j=1}^{l} \left(1 - \frac{1}{\|I_j\|}\right), \quad \beta(C) = \sum_{\substack{i<j \\ \|\gcd(I_i, I_j)\|>1}} \frac{1}{\|I_i\|\|I_j\|},$$

where $\|I\|$ is the norm of the ideal I. We also let $f(n)$ and $g(n)$ denote the number of ideals of norm n and the number of prime ideals of norm n, respectively, as in Introduction.

Lemma 6.6. *For an arbitrary residue system C of a number field $F/\mathbb{Q}$, we have $\delta(C) \geq \alpha(C) - \beta(C)$.*

Proof. Let $\alpha = \alpha(C)$ and $\beta = \beta(C)$. We set

$$C' = \{(I_1, r_1), \ldots, (I_{l-1}, r_{l-1})\}, \quad C'' = \{(I_j, r_j) : j < l, \ \|\gcd(I_j, I_l)\| = 1\},$$

$$\alpha' = \alpha(C') = \prod_{j=1}^{l-1} \left(1 - \frac{1}{\|I_j\|}\right) \text{ and } \beta' = \beta(C') = \sum_{\substack{i<j\le l-1, \\ \|\gcd(I_i, I_j)\|>1}} \frac{1}{\|I_i\|\|I_j\|}.$$

Now, follow the proof of Lemma 2.1 of [22] replacing (n_i, r_i), $1/n_i$ and $\gcd(n_j, n_l)$ by (I_i, r_i), $1/\|I_i\|$ and $\|\gcd(I_j, I_l)\|$, respectively. □

We can factor each modulus I as $I_{\underline{Q}} I_{\overline{Q}}$, where $I_{\underline{Q}}$ is the smallest ideal dividing I composed solely of prime ideals that lie over prime numbers in $[1, Q]$ with $Q \geq 1$, and $I_{\overline{Q}} = I / I_{\underline{Q}}$.

Lemma 6.7. *Let C be a residue system of a number field $F/\mathbb{Q}$. Let $Q \geq 2$ be arbitrary, and set*

$$M = \operatorname{lcm}\{I_{\underline{Q}} : I \in S(C)\}.$$

Let $\{(M, h_i) : 1 \leq i \leq \|M\|\}$ be a covering system of $\mathcal{O}_F$. For each h_i, let C_{h_i} be the set

$$C_{h_i} = \left\{ \left(I_{\overline{Q}}, r\right) : (I, r) \in C, \ r \equiv h_i \pmod{I_{\underline{Q}}} \right\}.$$

Then

$$\delta(C) = \frac{1}{\|M\|} \sum_{i=1}^{\|M\|} \delta(C_{h_i}).$$

Proof. Using the Chinese Remainder Theorem, we can follow the same argument as in the proof of Lemma 3.1 of [22] (replacing $M, n_{\underline{Q}}$ and $n_{\overline{Q}}$ by $\|M\|, I_{\underline{Q}}$ and $I_{\overline{Q}}$, respectively). □

Now, we use the fact that $\|I_{\overline{Q}}\|$ has no prime factors $\leq Q$ to get an upper bound for the sum of $\beta(C_{h_i})$ as in the proof of Lemma 3.2 of [22].

Lemma 6.8. *Let $K > 1$, and suppose C is a residue system of a number field $F/\mathbb{Q}$ of degree d with $S(C)$ consisting of ideals whose norms are in the interval $(N, KN]$, each with multiplicity at* *most s. Let $Q \geq 2$, and define M and C_{h_i} as in Lemma 6.7. Then*

$$\frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \ll \frac{s^2(1+\log K)^2\log^2 Q}{Q}. \tag{6.2}$$

Proof. For $J \mid M$, let S_J be the set of distinct ideals $I_{\overline{Q}} = I/\gcd(I,M)$, where $I \in S(C)$ and $I_{\underline{Q}} = \gcd(I,M) = J$. For $J, J' \mid M$, let

$$G(r,J,r',J') = \#\{1 \leq i \leq \|M\| : h_i \equiv r \text{ (mod } J),\ h_i \equiv r' \text{ (mod } J')\}.$$

Then

$$\frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \leq \frac{1}{\|M\|}\sum_{\substack{J|M\\J'|M}}\ \sum_{\substack{I\in S_J\\I'\in S_{J'}\\\|\gcd(I,I')\|>1}}\frac{1}{\|I\|\|I'\|}\sum_{\substack{(IJ,r)\in C\\(I'J',r')\in C}}G(r,J,r',J').$$

We can see that $G(r,J,r',J')$ is either 0 or $\|M\|/\|\mathrm{lcm}[J,J']\|$, so the inner sum is at most

$$s^2\frac{\|M\|}{\|\mathrm{lcm}[J,J']\|}.$$

Next, let $\mathcal{P}$ denote a prime ideal, and let $P(n)$ and $P^-(n)$ denote the largest prime factor and the least prime factor of $n \geq 1$, respectively. Then

$$\begin{aligned}\sum_{\substack{I\in S_J\\I'\in S_{J'}\\\|\gcd(I,I')\|>1}}\frac{1}{\|I\|\|I'\|} &\leq \sum_{P(\|\mathcal{P}\|)>Q}\ \sum_{\substack{I\in S_J\\I'\in S_{J'}\\\mathcal{P}|I,\ \mathcal{P}|I'}}\frac{1}{\|I\|\|I'\|}\\ &= \sum_{P(\|\mathcal{P}\|)>Q}\frac{1}{\|\mathcal{P}\|^2}\left(\sum_{\substack{N/\|\mathcal{P}J\|<\|I\|\leq KN/\|\mathcal{P}J\|\\P^-(\|I\|)>Q}}\frac{1}{\|I\|}\right)\left(\sum_{\substack{N/\|\mathcal{P}J'\|<\|I'\|\leq KN/\|\mathcal{P}J'\|\\P^-(\|I'\|)>Q}}\frac{1}{\|I'\|}\right).\end{aligned}$$

Using Proposition 6.1 and partial summation, we obtain

$$\sum_{y<n\leq x}\frac{f(n)}{n} = \frac{1}{x}\sum_{n\leq x}f(n) - \frac{1}{y}\sum_{n\leq y}f(n) + \int_y^x \frac{1}{t^2}\sum_{n\leq t}f(n)dt$$

$$= \quad c_F \log \frac{x}{y} + O(y^{-\frac{1}{d}}) \ll \log \frac{x}{y} + 1. \tag{6.3}$$

Thus,

$$\sum_{\substack{N/\|\mathcal{P}J\| < \|I\| \leq KN/\|\mathcal{P}J\| \\ P^-(\|I\|) > Q}} \frac{1}{\|I\|} \leq \sum_{N/\|\mathcal{P}J\| < n \leq KN/\|\mathcal{P}J\|} \frac{f(n)}{n} \ll \log K + 1$$

and similarly with J', I' replacing J, I. We have the estimate

$$\sum_{P(\|\mathcal{P}\|) > Q} \frac{1}{\|\mathcal{P}\|^2} = \sum_{\substack{n \geq 1 \\ p > Q}} \frac{f(p^n)}{p^{2n}} \leq d \sum_{p > Q} \left(\frac{1}{p^2} + \frac{1}{p^4} + \cdots \right) \ll \frac{d}{Q \log Q},$$

which follows from the prime number theorem and partial summation.

Thus,

$$\sum_{\substack{I \in S_J \\ I' \in S_{J'} \\ \|\gcd(I, I')\| > 1}} \frac{1}{\|I\| \|I'\|} \ll \frac{(1 + \log K)^2}{Q \log Q},$$

so that

$$\frac{1}{\|M\|} \sum_{i=1}^{\|M\|} \beta(C_{h_i}) \ll \frac{s^2 (1 + \log K)^2}{Q \log Q} \sum_{\substack{J|M \\ J'|M}} \frac{1}{\|\text{lcm}[J, J']\|} = \frac{s^2 (1 + \log K)^2}{Q \log Q} \sum_{u|M} \sum_{\text{lcm}[J, J'] = u} \frac{1}{\|u\|}.$$

Let $\tau(I)$ denote the number of divisors of an ideal I. Then

$$\begin{aligned}
\sum_{u|M} \sum_{\text{lcm}[J, J'] = u} \frac{1}{\|u\|} &= \sum_{u|M} \frac{\tau(u^2)}{\|u\|} \leq \prod_{P(\|\mathcal{P}\|) \leq Q} \left(1 + \frac{3}{\|\mathcal{P}\|} + \frac{5}{\|\mathcal{P}\|^2} + \cdots \right) \\
&= \prod_{n=1}^{d} \prod_{p \leq Q} \left(1 + \frac{3}{p^n} + \frac{5}{p^{2n}} + \cdots \right)^{g(p^n)} \\
&= \prod_{p \leq Q} \left(1 + \frac{3}{p} + \frac{5}{p^2} + \cdots \right)^{g(p)} \prod_{n=2}^{d} \prod_{p \leq Q} \left(1 + \frac{3}{p^n} + \frac{5}{p^{2n}} + \cdots \right)^{g(p^n)} \\
&\ll \prod_{p \leq Q} \left(1 + \frac{3}{p} + \frac{5}{p^2} + \cdots \right)^{g(p)}
\end{aligned}$$

$$\le \exp\left(\sum_{p\le Q} g(p)\left(\frac{3}{p}+\frac{5}{p^2}+\cdots\right)\right) \ll \exp\left(3\sum_{p\le Q}\frac{g(p)}{p}\right)$$
$$\le \exp\left(3\sum_{n\le Q}\frac{g(n)}{n}\right) \ll \log^3 Q,$$

since $\sum_{\|\mathcal{P}\|\le Q} 1/\|\mathcal{P}\| = \sum_{n\le Q} g(n)/n = \log\log Q + O(1)$ by Proposition 6.2 and partial summation. This completes the proof. $\square$

We can also obtain a lower bound for the sum of $\alpha(C_{h_i})$ using those I's in $S(C)$ for which $P(\|I\|) \le Q$.

Lemma 6.9. *Let C be an arbitrary residue system of $F/\mathbb{Q}$. For $Q \ge 2$, define M and C_{h_i} as in Lemma 6.7. Let $C' = \{(I, r) \in C : I|M\} = \{(I, r) \in C : P(\|I\|) \le Q\}$ and suppose $\delta(C') > 0$. Then*

$$\frac{1}{M}\sum_{i=1}^{\|M\|}\alpha(C_{h_i}) \ge (\alpha(C))^{(1+1/Q)/\delta(C')}.$$

Proof. Note that $\mathcal{O}_L \in S(C_{h_i})$ if and only if there is a pair $(I, r) \in C'$ with $h_i \equiv r \pmod{I}$. Let

$$\mathcal{M}' = \{1 \le i \le \|M\| : \mathcal{O}_F \notin S(C_{h_i})\}, \quad M' = |\mathcal{M}'|.$$

Then

$$\frac{M'}{\|M\|} = \delta(C'). \tag{6.4}$$

Now, follow the same argument as in the proof of Lemma 3.3 of [22] replacing M, $1/n$, $1/n'$, $1/n_{\underline{Q}}$ and $1/n_{\overline{Q}}$ by $\|M\|$, $1/\|I\|$, $1/\|I'\|$, $1/\|I_{\underline{Q}}\|$ and $1/\|I_{\overline{Q}}\|$, respectively.

$\square$

Now, combining the above two lemmas yields the following.

Lemma 6.10. *Suppose $K > 1$, N is a positive integer, and C is a residue system of $F/\mathbb{Q}$ with $S(C)$ consisting of ideals whose norms are in $(N, KN]$, each with multiplicity at most s. Let $Q \ge 2$, and* *as in Lemma 6.9, let $C' = \{(I, r) \in C : P(\|I\|) \leq Q\}$. If $\delta(C') > 0$, then*

$$\delta(C) \geq \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2(1+\log K)^2 \log^2 Q}{Q}\right),$$

where the implied constant depends on F only.

Proof. Using the same definition of M and C_{h_i} as in Lemma 6.7 and by Lemmas 6.6, 6.7, 6.8, and 6.9, we have

$$\begin{aligned}\delta(C) = \frac{1}{M}\sum_{i=1}^{\|M\|}\delta(C_{h_i}) &\geq \frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\alpha(C_{h_i}) - \frac{1}{\|M\|}\sum_{i=1}^{\|M\|}\beta(C_{h_i}) \\ &\geq \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2(1+\log K)^2 \log^2 Q}{Q}\right).\end{aligned}$$

Thus, we complete the proof of the lemma. □

Next, we show an analogue of Lemma 4.1 of [22] which is about smooth numbers.

Our result is more complicated to prove because we need to understand $f(n)$ at smooth arguments n.

Lemma 6.11. *Let $F/\mathbb{Q}$ be a number field of degree d with the ring of integers $\mathcal{O}_F$ Suppose $Q \geq 2$ and $Q < N \leq \exp(\exp(\log^{2/5} Q))$. If $f(n)$ is the number of ideals of norm n in $\mathcal{O}_F$, then*

$$\sum_{\substack{n>N \\ P(n)\leq Q}} \frac{f(n)}{n} \ll (\log Q)e^{-u\log u}, \quad \textit{where } u = \frac{\log N}{\log Q}.$$

Proof. We use Corollary 2.3 of [57] with $\kappa = 1$, and $L_{1/5}(z) = \exp\{(\log z)^{2/5}\}$.

Using Proposition 6.2, we have

$$\begin{aligned}\sum_{p\leq z} f(p)\log p &= \sum_{\|\mathcal{P}\|\leq z}\log\|\mathcal{P}\| - \sum_{\substack{\|\mathcal{P}\|\leq z \\ \|\mathcal{P}\|=q^l \\ l\geq 2}}\log\|\mathcal{P}\| \\ &= \sum_{\|\mathcal{P}\|\leq z}\log\|\mathcal{P}\| + O(\sqrt{z}) = z + O\left(\frac{z}{\exp(\log z)^{1/13}}\right).\end{aligned}$$

Thus, for some constant C, if $z > 1$, then

$$|\sum_{p\le z} f(p)\log p - z| \le Cz/L_{1/5}(z),$$

which is (2.1) of [57]. Since (1.8) of [57] also holds for some $A > 0$ and $\eta \in (0, 1/2)$, $f \in \mathscr{M}_1(A, C, \eta, L_{1/5})$. By Corollary 2.3 of [57],

$$\sum_{\substack{n\le t\\ P(n)\le Q}} f(n) \ll \frac{t}{u_t^{u_t}},$$

where $u_t = \log t/\log Q$, provided $Q \le t \le t_0 = Q^{L_{1/5}(Q)}$, since $\rho_1(u) = \rho(u) \ll u^{-u}$ (Corollary 2.3 of [34]).

Let $Q_1(t) = \exp\{(\log\log t)^{5/2}\}$. Note that if $t > t_0$, then $Q_1(t) \ge Q$. Thus, for $t > t_0$

$$\sum_{\substack{n\le t\\ P(n)\le Q}} f(n) \le \sum_{\substack{n\le t\\ P(n)\le Q_1(t)}} f(n) \ll t/v^v, \quad \text{where } v = v(t) = \frac{\log t}{\log Q_1(t)},$$

since $1 \le v(t) \le L_{1/5}(Q_1)$. Let i_0 be the largest integer such that $NQ^{i_0} \le t_0$. Then

$$\begin{aligned}
\sum_{\substack{n>N\\ P(n)\le Q}} \frac{f(n)}{n} &= \int_N^\infty \frac{1}{t^2} \sum_{\substack{N<n\le t\\ P(n)\le Q}} f(n)\,dt \\
&\le \sum_{i=0}^{i_0-1} \int_{NQ^i}^{NQ^{i+1}} \frac{1}{t^2} \sum_{\substack{n\le t\\ P(n)\le Q}} f(n)\,dt + \int_{NQ^{i_0}}^{t_0} \frac{1}{t^2} \sum_{\substack{n\le t\\ P(n)\le Q}} f(n)\,dt + \int_{t_0}^\infty \frac{1}{t^2} \sum_{\substack{n\le t\\ P(n)\le Q}} f(n)\,dt \\
&\ll \sum_{i\ge 0} \frac{\log Q}{(u+i)^{u+i}} + \int_{NQ^{i_0}}^{t_0} \frac{1}{tu_t^{u_t}}\,dt + \int_{t_0}^\infty \frac{1}{t\log^2 t}\cdot\frac{\log^2 t}{v^v}\,dt \\
&\ll \frac{\log Q}{u^u} + \int_{NQ^{i_0}}^{NQ^{i_0+1}} \frac{1}{tu_t^{u_t}}\,dt + \frac{\log t_0}{v(t_o)^{v(t_0)}} \ll \frac{\log Q}{u^u},
\end{aligned}$$

which implies the lemma. □

Lastly, we present a lemma that will be needed in the proof of Theorem 6.4 in Section 6.3.

Lemma 6.12. *Suppose s is a positive integer and C is a residue system of a number field $F/\mathbb{Q}$ with $S(C)$ consisting of ideals whose norms are in $(1, B]$ with multiplicity at most s. Let*

$$C_0 = \{(I, r) \in C : \mathcal{P}|\ I \Rightarrow\ \|\mathcal{P}\| \leq \sqrt{s\nu B}\},$$

where ν is a constant depending on F such that $\sum_{n \leq x} f(n) \leq \nu x$ for all x. (Note that Proposition 6.1 guarantees that such a constant ν exists, and $\nu \geq c_F$). If $\delta(C_0) > 0$, then $\delta(C) > 0$.

Proof. Suppose $\delta(C_0) > 0$. Let ***P*** be the set of prime ideals whose norms are in $(\sqrt{s\nu B}, B]$ and let l be the least common multiple of all $I \in S(C_0)$. Let $\mathcal{P} \in$ ***P***. Since the number of ideals $I \in S(C)$ such that $\mathcal{P}|\ I$ is $\leq s \sum_{n \leq B/\|\mathcal{P}\|} f(n) \leq s\nu B/\|\mathcal{P}\| < \|\mathcal{P}\|$, there are at most $\|\mathcal{P}\| - 1$ ideals I such that $\mathcal{P}|\ I$. Call them $I_1, \ldots, I_t$, and let $r_1, \ldots, r_t$ be the corresponding residue classes. Then there is a choice for $b = b(\mathcal{P})$ such that if $x \equiv b \pmod{\mathcal{P}}$, then x is not covered by any of the congruences $x \equiv r_j \pmod{I_j}$ with $1 \leq j \leq t$.

By assumption, there is a residue class $a \bmod l$ in $R(C_0)$. Let A be a solution to the system $A \equiv a \pmod{l}$ and $A \equiv b(\mathcal{P}) \pmod{\mathcal{P}}$ for each prime ideal $\mathcal{P} \in$ ***P***. Such A exists via the Chinese Remainder Theorem. Then we have $A \not\equiv r \pmod{I}$ for each $(I, r) \in C_0$. Furthermore, for each prime $\mathcal{P} \in$ ***P*** and $(I, r) \in C$ with $\mathcal{P}|\ I$, $A \not\equiv r \pmod{I}$. Since this exhausts the pairs $(I, r) \in C$, we have $A \in R(C)$, and this completes the proof. □

6.3 Proof of Theorems 6.1, 6.2 and 6.3

Proof of Theorem 6.1. We can repeat the proof of Theorem 2 of [22] using Lemmas 6.11 and 6.10 (instead of Lemma 4.1 and Lemma 3.4 of [22]). □

Proof of Theorem 6. 2. We can suppose that $\varepsilon > 0$ is sufficiently small and $K \geq 2$. Let

$$C_0 = \{(I, r) \in C : \mathcal{P}|\ I \Rightarrow\ \|\mathcal{P}\| \leq \sqrt{s\nu KN}\},$$

with ν as in Lemma 6.12. Then, by (6.3),

$$\begin{aligned}
\sum_{I\in S(C_0)} \frac{1}{\|I\|} &\leq s \sum_{N<\|I\|\leq KN} \frac{1}{\|I\|} - s \sum_{\substack{N<\|I\|\leq KN \\ \exists\ \mathcal{P}|I:\ \|\mathcal{P}\|>\sqrt{s\nu B}}} \frac{1}{\|I\|} \\
&= s \sum_{N<n\leq KN} \frac{f(n)}{n} - s \sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\|<\|I'\|\leq KN/\|\mathcal{P}\|} \frac{1}{\|I'\|} \\
&= sc_F \log K + O\left(\frac{s}{N^{1/d}}\right) - s \sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\|<n\leq KN/\|\mathcal{P}\|} \frac{f(n)}{n}.
\end{aligned}$$

Now,

$$\sum_{N/\|\mathcal{P}\|<n\leq KN/\|\mathcal{P}\|} \frac{f(n)}{n} = \begin{cases} c_F \log K + O((\|\mathcal{P}\|/N)^{\frac{1}{d}}), & \|\mathcal{P}\| \leq N \\ c_F \log(KN/\|\mathcal{P}\|) + O(1), & N < \|\mathcal{P}\| \leq KN. \end{cases}$$

Thus,

$$\begin{aligned}
&\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\|<n\leq KN/\|\mathcal{P}\|} \frac{f(n)}{n} \\
&= \sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq N} \left(\frac{c_F \log K}{\|\mathcal{P}\|} + O\left(\frac{1}{N^{\frac{1}{d}}\|\mathcal{P}\|^{1-\frac{1}{d}}}\right)\right) + c_F \sum_{N<\|\mathcal{P}\|\leq KN} \frac{\log K}{\|\mathcal{P}\|} \\
&\qquad + c_F \sum_{N<\|\mathcal{P}\|\leq KN} \frac{\log N - \log\|\mathcal{P}\| + O(1)}{\|\mathcal{P}\|} \\
&= c_F \log K \sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq KN} \frac{1}{\|\mathcal{P}\|} + c_F \sum_{N<\|\mathcal{P}\|\leq KN} \frac{\log N}{\|\mathcal{P}\|} - c_F \sum_{N<\|\mathcal{P}\|\leq KN} \frac{\log\|\mathcal{P}\|}{\|\mathcal{P}\|} \\
&\qquad + O\left(\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\leq N} \frac{1}{N^{\frac{1}{d}}\|\mathcal{P}\|^{1-\frac{1}{d}}}\right) + O\left(\sum_{N<\|\mathcal{P}\|\leq KN} \frac{1}{\|\mathcal{P}\|}\right).
\end{aligned}$$

By Proposition 6.2 and partial summation,

$$\sum_{y<\|\mathcal{P}\|\leq x} \frac{1}{\|\mathcal{P}\|} = \sum_{y<n\leq x} \frac{g(n)}{n} = \log\log x - \log\log y + O\left(\frac{1}{\log y}\right),$$

$$
\begin{aligned}
\sum_{y<\|\mathcal{P}\|\le x} \frac{\log \|\mathcal{P}\|}{\|\mathcal{P}\|} &= \sum_{y<n\le x} \frac{g(n)\log n}{n} = \log \frac{x}{y} + O\left(\frac{1}{\log y}\right), \\
\sum_{y<\|\mathcal{P}\|\le x} \frac{1}{\|\mathcal{P}\|^{1-1/d}} &= \sum_{y<n\le x} \frac{g(n)}{n^{1-1/d}} = O\left(\frac{x^{\frac{1}{d}}}{\log y}\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sum_{\sqrt{s\nu KN}<\|\mathcal{P}\|\le KN} \frac{1}{\|\mathcal{P}\|} \sum_{N/\|\mathcal{P}\|<n\le KN/\|\mathcal{P}\|} \frac{f(n)}{n} \\
&= c_F \log K \left(\log\log KN - \log\log \sqrt{s\nu KN}\right) + c_F \log N \left(\log\log KN - \log\log N\right) \\
&\qquad - c_F \log K + O(\log\log KN - \log\log N) + O(1) \\
&= c_F \log K \left(\log 2 - \log\left(1 + \frac{\log s\nu}{\log KN}\right)\right) + c_F \log N \log\left(1 + \frac{\log K}{\log N}\right) - c_F \log K \\
&\qquad + O\left(\log\left(1 + \frac{\log K}{\log N}\right)\right) + O(1) \\
&= c_F \log 2 \log K + O\left(\frac{\log K \log s\nu}{\log KN}\right) + O(1) = (c_F \log 2 + o(1)) \log K.
\end{aligned}
$$

Thus,

$$
\sum_{I\in S(C_0)} \frac{1}{\|I\|} \le s c_F ((1-\log 2) + o(1)) \log K.
$$

Since

$$
-\log \alpha(C_0) \le \sum_{I\in S(C_0)} \frac{1}{\|I\|} + O\left(s \sum_{n>N} \frac{f(n)}{n^2}\right) = \sum_{I\in S(C_0)} \frac{1}{\|I\|} + O\left(\frac{s}{N}\right),
$$

we have

$$
-\log \alpha(C_0) \le s c_F \big(1 - \log 2 + o(1)\big) \log K \le \big(1 - (1-\log 2)\varepsilon + o(1)\big) \log L(N,s).
$$

Let $Q = L(N,s)^{1-\lambda}$, where $\lambda = \frac{1}{4}((1-\log 2)\varepsilon - 4b^2)$, and let $C' = \{(n,r) \in C_0 : P(n) \le Q\}$.

Using Lemma 6.11 yields

$$\delta(C') = 1 + O\Bigg(s \sum_{\substack{n>N \\ P(n)\le Q}} \frac{f(n)}{n} \Bigg) = 1 + o(1) \qquad (N \to \infty).$$

Thus,

$$\alpha(C_0)^{(1+1/Q)/\delta(C')} \gg L(N,s)^{-1+(1-\log 2)\varepsilon-\lambda}.$$

On the other hand,

$$\frac{s^2(1+\log K)^2 \log^2 Q}{Q} \ll L(N,s)^{-1+\lambda} s^2 (\log L(N,s))^4 \ll L(N,s)^{-1+4b^2+2\lambda}.$$

By Lemma 6.10, we have $\delta(C_0) > 0$ for N sufficiently large. Hence $\delta(C) > 0$ by Lemma 6.12. $\square$

Proof of Theorem 6.3. By (6.3), we have

$$-\log \alpha(C) \le s \sum_{N<n\le KN} \left(\frac{f(n)}{n} + \frac{f(n)}{n^2} \right) \le s \left(c_F \log K + O\left(\frac{1}{N^{1/d}} \right) \right).$$

So,

$$\alpha(C) \gg K^{-sc_F} = L(N,s)^{-1/2+\varepsilon}.$$

Let $\lambda = \frac{1}{3}(\varepsilon - 4b^2)$ and $Q = L(N,s)^{1/2-\lambda}$. Let $u = \log N / \log Q$, and let C' be as in Lemma 6.10. By Lemma 6.11, we have

$$1 - \delta(C') \ll \frac{s \log Q}{u^u} \ll \frac{s \log N}{(s \log N)^{2+\lambda}} = \frac{1}{(s \log N)^{1+\lambda}},$$

so that $1/\delta(C') = 1 + O\left((s\log N)^{-1-\lambda}\right)$. Using $|\log \alpha(C)| \le \log N$, we have

$$\alpha(C)^{(1+1/Q)/\delta(C')} = \left(1 + O\left(\frac{1}{(\log N)^{\lambda}} \right) \right) \alpha(C).$$

By Lemma 6.10 it suffices to show that

$$\frac{s^2(1+\log K)^2\ \log^2 Q}{Q} = O\left(\frac{\alpha(C)}{(\log N)^\lambda}\right).$$

But, for large N we have $s^2\log^4 L(N,s) \le L(N,s)^{4b^2+\lambda}$. Thus,

$$\begin{aligned}\frac{s^2(1+\log K)^2\log^2 Q}{Q} &\ll \frac{s^2\log^4 L(N,s)}{L(N,s)^{1/2-\lambda}} \ll \frac{1}{L(N,s)^{1/2-2\lambda-4b^2}}\\ &\ll \frac{1}{L(N,s)^{1/2-\varepsilon+\lambda}} \ll \frac{\alpha(C)}{L(N,s)^\lambda}.\end{aligned}$$

□

6.4 Exact coverings and near coverings in number fields

In this section, we prove analogues of Theorem 5 and Theorem 6 of [22]. As in [22], they imply that there is an exact covering system of an arbitrary number field, where each modulus I has norm $\ge N$ and multiplicity near the upper bound given in Theorems 1, 2 and 3, and the density $\delta(C)$ can be considerably smaller than that given in Theorem 6.5 if we allow K to be sufficiently large.

Theorem 6.13. *Let $F/\mathbb{Q}$ be a number field with the ring of integers $\mathcal{O}_F$. For sufficiently large N and $s=\exp(\sqrt{\log N\log\log N})$, there exists an exact covering system of F with squarefree moduli whose norm is greater than N, such that the multiplicity of each modulus does not exceed s.*

Proof. We follow the key idea from the proof of Theorem 5 of [22] to construct the desired covering system, and we also use the method from an older preprint version of [22] based on the Remark 4 of [22] to complete the proof.

Let $\mathcal{P}$ denote a prime ideal of $\mathcal{O}_F$, and define a sequence $\{X_j\}$ by

$$X_0 = 1 \quad\text{and}\quad X_{j+1} = \min\left\{x : \sum_{X_j<\|\mathcal{P}\|\le x}\left[\frac{x}{\|\mathcal{P}\|}\right] \ge X_j\right\} \quad\text{with}\ \ j\ge 0,$$

where $[x]$ denotes the greatest integer which is $\leq x$. Let

$$P_j = \{\mathcal{P} : X_{j-1} < \|\mathcal{P}\| \leq X_j\}.$$

First, for $J \geq 1$ and $s = X_J$, we construct an exact covering system C_J with squarefree moduli of the form $\mathcal{P}_1 \cdots \mathcal{P}_J$ with $\mathcal{P}_i \in P_i$ with the multiplicity of each modulus $\leq s$. Note that such moduli have norms greater than

$$N_J = \prod_{j=0}^{J-1} X_j.$$

We construct C_J through induction on J. Choose any prime ideal $\mathcal{P}$ in P_1. Then, we can find r_i's from $\mathcal{O}_F$ such that $C_1 = \{(\mathcal{P}, r_1), \ldots, (\mathcal{P}, r_{\|\mathcal{P}\|})\}$ is an exact covering system of F. We can see that C_1 satisfies the above conditions with $J = 1$.

Now, suppose that we have C_J as above for some $J \geq 1$. Fix a modulus $I = \mathcal{P}_1 \cdots \mathcal{P}_J$, and let $(I, r_1), \ldots, (I, r_t)$ be all the pairs in C_J corresponding to I. Note that $t \leq X_J$. Let $P_{J+1} = \{\mathcal{Q}_1, \ldots, \mathcal{Q}_m\}$. Replace each (I, r_i), $i \leq [X_{J+1}/\|\mathcal{Q}_1\|]$, by the $\|\mathcal{Q}_1\|$ pairs $(I\mathcal{Q}_1, r_i + a_k)$, where $I = \bigcup_{k=1}^{\|\mathcal{Q}_1\|}(a_k + I\mathcal{Q}_1)$. Note that the multiplicity of the modulus $I\mathcal{Q}_1$ is $[X_{J+1}/\|\mathcal{Q}_1\|]\|\mathcal{Q}_1\| \leq X_{J+1}$ and $r_i + I = \bigcup_{k=1}^{\|\mathcal{Q}_1\|}(r_i + a_k + I\mathcal{Q}_1)$.

Next, replace each (I, r_i), $[X_{J+1}/\|\mathcal{Q}_1\|] < i \leq [X_{J+1}/\|\mathcal{Q}_1\|] + [X_{J+1}/\|\mathcal{Q}_2\|]$, with the $\|\mathcal{Q}_2\|$ pairs $(I\mathcal{Q}_2, r_i + b_k)$, where $I = \bigcup_{k=1}^{\|\mathcal{Q}_2\|}(b_k + I\mathcal{Q}_2)$. Similarly, the multiplicity of the modulus $I\mathcal{Q}_2$ is $\leq X_{J+1}$ and $r_i + I = \bigcup_{k=1}^{\|\mathcal{Q}_2\|}(r_i + b_k + I\mathcal{Q}_2)$. Continuing this construction, all the pairs $(I, r_1), \ldots, (I, r_t)$ can be replaced with sets of residue classes with moduli of the form $I\mathcal{Q}_i$, since

$$t \leq X_J \leq \sum_{X_J < \|\mathcal{P}\| \leq X_{J+1}} [\, X_{J+1}/\|\mathcal{P}\| \,].$$

Applying this procedure for each $I \in S(C_J)$ completes the inductive construction of C_{J+1}.

In order to complete the proof, it suffices to show that for sufficiently large N, we can take J such that

$$N \leq N_J \quad \text{and} \quad \log^2 s = \log^2 X_J \leq \log N \log\log N.$$

We begin by showing that if $\varepsilon \in (0,1)$ and j is sufficiently large, say $j \geq j(\varepsilon)$, then

$$X_{j+1} \leq X_j \frac{(1+\varepsilon)\log X_j}{\log\log X_j}. \tag{6.5}$$

Set

$$x = \left[X_j \frac{(1+\varepsilon)\log X_j}{\log\log X_j} \right].$$

By Proposition 6.2, we have

$$\begin{aligned}
\sum_{X_j < \|\mathcal{P}\| \leq x} [\, x/\|\mathcal{P}\| \,] &\geq x \sum_{X_j < n \leq x} \frac{g(n)}{n} - \sum_{X_j < n \leq x} g(n) \\
&= x(\log\log x - \log\log X_j + O(1/\log X_j)) + O(x/\log x) \\
&= x \log\left(1 + \frac{\log\log X_j + O(\log\log\log X_j)}{\log X_j}\right) + O(x/\log X_j) \\
&\geq x\left(\frac{(1-\varepsilon/3)(\log\log X_j)}{\log X_j}\right) + O(x/\log X_j) \\
&\geq x\left(\frac{(1-\varepsilon/2)(\log\log X_j)}{\log X_j}\right) > X_j.
\end{aligned}$$

This completes the proof of (6.5).

Next, we show that for every $\varepsilon \in (0,1)$, J sufficiently large (depending on ε), and $j \leq J$, we have

$$\log X_j \geq \log X_J - (J-j)(\log\log X_J - \log\log\log X_J + \varepsilon). \tag{6.6}$$

We consider first the case that $j \geq j(\varepsilon)$, where we choose $j(\varepsilon)$ such that $X_{j(\varepsilon)} \geq e^e$. It follows from (6.5) that in the case that $j \in [j(\varepsilon), J)$ we have

$$\begin{aligned}
\log X_{j+1} &\leq \log X_j + \log\log X_j - \log\log\log X_j + \log(1+\varepsilon) \\
&\leq \log X_j + \log\log X_J - \log\log\log X_J + \varepsilon.
\end{aligned}$$

Therefore, (6.6) holds for all $j \in [j(\varepsilon), J)$. On the other hand, if J is large, the left side of (6.6) decreases by a smaller amount as j changes from $j(\varepsilon)$ to 1 by comparison to the amount of decrease on the right side of (6.6). Hence, (6.6) in fact holds for all $j \le J$ provided J is sufficiently large.

Now, we complete the proof of the theorem. Fix $\varepsilon \in (0,1)$. Let N be large, and take J so that

$$N_{J-1} = \prod_{j=1}^{J-2} X_j < N \le N_J = \prod_{j=1}^{J-1} X_j.$$

Let $s = X_J$, and set

$$\Delta = \log\log X_J - \log\log\log X_J + \varepsilon = \log\log s - \log\log\log s + \varepsilon.$$

From (6.6), we have

$$\log N_{J-1} \ge \sum_{\substack{i\ge 2 \\ i\Delta \le \log s}} (\log s - i\Delta) \ge \frac{\log^2 s}{2\Delta + \varepsilon}.$$

Let Y denote the expression on the right side above. Then

$$\log Y = 2\log\log s - \log\log\log s + O(1) > 2\Delta + \varepsilon.$$

Thus,

$$\log N \ge \log N_{J-1} \ge \frac{\log^2 s}{\log Y} \ge \frac{\log^2 s}{\log\log N},$$

and this completes the proof of the theorem. $\square$

Let C be a residue system of a number field $F/\mathbb{Q}$, where $S(C)$ consists of distinct ideals whose norms are in $(N, KN]$. Then, using (6.3), we have

$$\begin{aligned} \alpha(C) &= \prod_{I\in S(C)} \left(1 - \frac{1}{\|I\|}\right) \ge \prod_{N<n\le KN} \left(1 - \frac{1}{n}\right)^{f(n)} \\ &\ge \exp\left(-\sum_{N<n\le KN} \left(\frac{f(n)}{n} + \frac{f(n)}{n^2}\right)\right) \\ &= \exp\left(-c_F \log K + O(N^{-1/d})\right). \end{aligned}$$

Thus, if K is not too large, then Theorem 6.5 implies that $\delta(C)$ has a lower bound approximately $1/K^{c_F}$.

The following theorem shows that, when we allow K to be much larger than N, C can be chosen so that $\delta(C)$ is considerably smaller than $1/K^{c_F}$.

Theorem 6.14. *Suppose N and K are integers sufficiently large depending on $F/\mathbb{Q}$. Then there is some residue system C consisting of distinct moduli whose norms are from $(N, KN]$ such that*

$$\delta(C) \leq \frac{1}{K^{c_F}} \exp\left(-c_F^2 \frac{\log K}{3N}\right).$$

Before proving Theorem 6.14, we present a lemma about the expected value of $\delta(C)$. Let T be a set of ideals and let $\mathcal{C}(T)$ be the set of residue systems C with $S(C) = T$. Define

$$W_0(T) = \prod_{I \in T} I \qquad \text{and} \qquad W(T) = \#\mathcal{C}(T) = \prod_{I \in T} \|I\| = \|W_0(T)\|.$$

Lemma 6.15. *Let T be a set of distinct ideals. Then the expected value of $\delta(C)$ over $C \in \mathcal{C}(T)$, denoted by $\mathbf{E}\delta(C)$, is $\prod_{I \in T}(1 - 1/\|I\|)$.*

Proof. Put $W = W(T)$ and $W_0 = W_0(T)$. Let $(W_0, m_1), \ldots, (W_0, m_{\|W_0\|})$ be an exact covering system. Since the number of systems $C \in \mathcal{C}(T)$ with $m_i \in R(C)$ is $\prod_{I \in T}(\|I\| - 1)$, we have

$$\sum_{C \in \mathcal{C}(T)} \delta(C) = \sum_{C \in \mathcal{C}(T)} \frac{1}{W} \sum_{\substack{i=1 \\ m_i \in R(C)}}^{W} 1 = \frac{1}{W} \sum_{i=1}^{W} \sum_{\substack{C \in \mathcal{C}(T) \\ m_i \in R(C)}} 1 = \frac{1}{W} \sum_{i=1}^{W} \prod_{I \in T}(\|I\| - 1) = \prod_{I \in T}(\|I\| - 1).$$

Dividing the equations above by W, we complete the proof. $\square$

Proof of Theorem 6.14. We follow the construction of covering systems described in the proof of Theorem 6 of [22] : We will randomly choose the values of $r(I)$ for I with $N < \|I\| \leq 2N$ so that each residue class modulo I is taken with the same probability $1/\|I\|$ and the variables $r(I)$ are independent. We then select the remaining values of $r(I)$ for I with $2N < \|I\| \leq KN$ via a greedy algorithm. It suffices to show that, under our construction, the expected value of $\delta(C)$ over all randomly chosen values of $r(I)$ for I with $N < \|I\| \le 2N$ is

$$\le \frac{1}{K^{c_F}} \exp\left(-c_F^2 \frac{\log K}{3N}\right).$$

Let $C_{2N} = \{(I, r(I)) : N < \|I\| \le 2N\}$, where each $r(I)$ is selected randomly. Using Lemma 6.15 and (6.3), we have

$$\begin{aligned}\mathbf{E}\delta(C_{2N}) = \prod_{I \in T} (1 - 1/\|I\|) &\le \exp\left(-\sum_{N<n\le 2N} \frac{f(n)}{n}\right) \\ &= \exp\left(-c_F \log 2 + O(N^{-1/d})\right).\end{aligned}$$

Thus, by the arithmetic mean–geometric mean inequality, it follows that

$$\mathbf{E} \log \delta(C_{2N}) \le -c_F \log 2 + O(N^{-1/d}). \tag{6.7}$$

Now, we describe how to choose $r(J)$, where $2N < \|J\| \le KN$. First, let $C_j = \{(I, r(I)) : N < \|I\| \le j\}$ and let $I_{j,1}, \ldots, I_{j,f(j)}$ be the ideals whose norm is j. Here, if $f(j) = 0$, then $C_j = C_{j-1}$. Note that the residue class $r(J) \pmod J$ contains $r \pmod{I_{j,i}}$ when $J | I_{j,i}$ and $r \equiv r(J) \pmod J$. Thus, if $I_{j,i}$ has a divisor J with $N < \|J\| \le 2N$, then there are residue classes modulo J not intersecting $R(C_{j-1})$. Let

$$D(j,i) = \{J : J | I_{j,i}, N < \|J\| \le 2N\}, \quad \widetilde{C}_{j,i} = \{(J, r(J)) : J \in D(j,i)\}.$$

Let $h(j,i)$ be the number of residue classes $r \pmod{I_{j,i}}$ for which $r \not\equiv r(J) \pmod J$ for each $J \in D(j,i)$. Note that if $h(j,i) = 0$ or 1 for some i, then we have $R(C_{j-1}) = \emptyset$ or $R(C_j) = \emptyset$. Thus, we assume that $h(j,i) > 1$ for all i. Then, we can select $r(J)$ from the $h(j,i)$ choices so that

$$\delta(C_j) \le \prod_{i=1}^{f(j)} \left(1 - \frac{1}{h(j,i)}\right) \delta(C_{j-1}). \tag{6.8}$$

Using linearity of expectation, we obtain

$$\mathbf{E}\log\delta(C_j)-\mathbf{E}\log\delta(C_{j-1})\le\mathbf{E}\sum_{i=1}^{f(j)}\log\left(1-\frac{1}{h(j,i)}\right)\le-\sum_{i=1}^{f(j)}\mathbf{E}\left(\frac{1}{h(j,i)}\right). \tag{6.9}$$

Also, Lemma 6.15 implies

$$\mathbf{E}\delta\big(\widetilde{C}_{j,i}\big)=\prod_{J\in D(j,i)}\left(1-\frac{1}{\|J\|}\right).$$

We can see that $\delta\big(\widetilde{C}_{j,i}\big)=h(j,i)/j$, since $I_{j,i}$ is a common multiple of the members of $D(j,i)$. Thus,

$$\mathbf{E}h(j,i)=j\mathbf{E}\delta\big(\widetilde{C}_{j,i}\big)=j\prod_{J\in D(j,i)}\left(1-\frac{1}{\|J\|}\right),$$

and using the arithmetic mean-harmonic mean inequality, we also have

$$\mathbf{E}\left(\frac{1}{h(j,i)}\right)\ge j^{-1}\prod_{J\in D(j,i)}\left(1-\frac{1}{\|J\|}\right)^{-1}\ge\frac{1}{j}+\sum_{J\in D(j,i)}\frac{1}{\|J\|j}.$$

From (6.9), we obtain

$$\mathbf{E}\log\delta(C_j)-\mathbf{E}\log\delta(C_{j-1})\le-\frac{f(j)}{j}-\sum_{i=1}^{f(j)}\sum_{J\in D(j,i)}\frac{1}{\|J\|j}.$$

Thus, by (6.3),

$$\begin{aligned}\mathbf{E}\log\delta(C)-\mathbf{E}\log\delta(C_{2N})&\le-\sum_{j=2N+1}^{KN}\frac{f(j)}{j}-\sum_{j=2N+1}^{KN}\sum_{i=1}^{f(j)}\sum_{J\in D(j,i)}\frac{1}{\|J\|j}\\&=-\sum_{j=2N+1}^{KN}\frac{f(j)}{j}-\sum_{N<\|J\|\le 2N}\sum_{2N/\|J\|<\|J'\|\le KN/\|J\|}\frac{1}{\|J\|^2\|J'\|}\\&=-\sum_{j=2N+1}^{KN}\frac{f(j)}{j}-\sum_{N<n\le 2N}\frac{f(n)}{n^2}\sum_{2N/\|J\|<n\le KN/\|J\|}\frac{f(n)}{n}\\&=-c_F\log(K/2)+O(1/N^{1/d})-(c_F\log K+O(1))\sum_{N<n\le 2N}\frac{f(n)}{n^2}.\end{aligned}$$

For sufficiently large N, we have

$$\sum_{N<n\le 2N} \frac{f(n)}{n^2} = \frac{c_F}{2N} + O(1/N^{1+1/d}) \ge \frac{c_F}{2.9N}.$$

Hence, by (6.7),

$$\begin{aligned}\mathbf{E}\log\delta(C) &\le -c_F\log K + O(1/N^{1/d}) - c_F^2\frac{\log K + O(1)}{2.9N}\\ &\le -c_F\log K - c_F^2\frac{\log K}{3N},\end{aligned}$$

for sufficiently large N and K. □

6.5 Normal value of $\delta(C)$

In this section, we estimate the variance of $\delta(C)$ over $C \in \mathcal{C}(T)$, where $\mathcal{C}(T)$ is the set of residue systems C in a number field $F/\mathbb{Q}$ with $S(C) = T$. As in the case of the integers, we can expect $\delta(C) \approx \alpha(C)$ for almost all $C \in \mathcal{C}(T)$. In fact, we can establish the same result for the variance of $\delta(C)$ as in [22].

Theorem 6.16. *Let T be a set of distinct ideals with minimum norm $N \ge 3$. Let α be the common value of $\alpha(C)$ for $C \in \mathcal{C}(T)$. Then,*

$$\frac{1}{W(T)}\sum_{C\in\mathcal{C}(T)} |\delta(C)-\alpha|^2 \ll \frac{\alpha^2\log N}{N^2}.$$

Proof. Let $\alpha = \alpha(C)$, $W = W(T)$ and $W_0 = W_0(T)$. By Lemma 6.15,

$$\frac{1}{W}\sum_{C\in\mathcal{C}(T)} |\delta(C)-\alpha|^2 = \frac{1}{W}\sum_{C\in\mathcal{C}(T)} \big(\delta(C)^2-\alpha^2\big). \tag{6.10}$$

Put $u = \sum_{I \in T} 1/\|I\|^2$, and also define

$$\ell(m_i, m_j) = \prod_{\substack{I \in T \\ m_i - m_j \in I}} \frac{\|I\| - 1}{\|I\| - 2},$$

where $(W_0, m_1), \ldots, (W_0, m_W)$ is an exact covering system in $F/\mathbb{Q}$. By an argument similar to that in the proof of Theorem 7 of [22], we obtain

$$\begin{aligned}\sum_{C \in \mathcal{C}(T)} \delta(C)^2 &= \frac{\alpha^2}{W}\left(1 - u + O\left(\sum_{n \geq N} \frac{f(n)}{n^3}\right)\right) \sum_{1 \leq i,j \leq W} \ell(m_i, m_j). \\ &= \frac{\alpha^2}{W}\left(1 - u + O\left(\frac{1}{N^2}\right)\right) \sum_{1 \leq i,j \leq W} \ell(m_i, m_j). \end{aligned} \tag{6.11}$$

Let $M(S) = \prod_{I \in S}(\|I\| - 2)$, where S is a finite set of ideals whose norms are ≥ 3,and let $L(S)$ denote the least common multiple of the members of S. Then

$$\ell(m_i, m_j) = \prod_{\substack{I \in T \\ m_i - m_j \in I}} \left(1 + \frac{1}{\|I\| - 2}\right) = \sum_{\substack{S \subseteq T \\ m_i - m_j \in L(S)}} \frac{1}{M(S)},$$

and thus

$$\sum_{1 \leq i,j \leq W} \ell(m_i, m_j) = \sum_{S \subseteq T} \frac{1}{M(S)} \sum_{\substack{1 \leq i,j \leq W \\ m_i - m_j \in L(S)}} 1 = W^2 \sum_{S \subseteq T} \frac{1}{M(S)\|L(S)\|}. \tag{6.12}$$

First, considering the case when $\#S \leq 1$, we have

$$\sum_{\substack{S \subseteq T \\ \#S \leq 1}} \frac{1}{M(S)\|L(S)\|} = 1 + \sum_{I \in T} \frac{1}{(\|I\| - 2)\|I\|} = 1 + u + O\left(1/N^2\right). \tag{6.13}$$

On the other hand, if $S \subseteq T$ and $\#S \geq 2$, let J_1, J_2 be two members of S such that for $I \in S$,

$\|J_1\| \geq \|J_2\| \geq \|I\|$. Then $\|L(S)\| \geq \|\mathrm{lcm}[J_1, J_2]\| = \|J_1\|\|J_2\|/\|\gcd(J_1, J_2)\|$, so that

$$\begin{aligned} E := & \sum_{\substack{S \subseteq T \\ \#S \geq 2}} \frac{1}{M(S)\|L(S)\|} \\ \leq & \sum_{\|J_1\| \geq \|J_2\| \geq N} \frac{\|\gcd(J_1, J_2)\|}{(\|J_1\|-2)(\|J_2\|-2)\|J_1\|\|J_2\|} \sum_{U \subseteq \{I : N \leq \|I\| \leq \|J_2\|\}} \frac{1}{M(U)}. \end{aligned}$$

Since the inner sum is equal to

$$\begin{aligned} \prod_{N \leq \|I\| \leq \|J_2\|} \left(1 + \frac{1}{\|I\|-2}\right) &= \prod_{N \leq n \leq \|J_2\|} \left(1 + \frac{1}{n-2}\right)^{f(n)} \\ &= \exp\left(\sum_{N \leq n \leq \|J_2\|} f(n) \log\left(1 + \frac{1}{n-2}\right)\right) \\ &\leq \exp\left(\sum_{N \leq n \leq \|J_2\|} \frac{f(n)}{n-2}\right) \ll \frac{\|J_2\|}{N}, \end{aligned}$$

by Proposition 6.1, we have

$$\begin{aligned} E &\ll \frac{1}{N} \sum_{\|J_1\| \geq \|J_2\| \geq N} \frac{\|\gcd(\|J_1\|, \|J_2\|)\|}{\|J_1\|^2\|J_2\|} \leq \frac{1}{N} \sum_{\|J\| \geq 1} \sum_{\substack{\|J_1\| \geq \|J_2\| \geq N \\ J|J_1,\ J|J_2}} \frac{\|J\|}{\|J_1\|^2\|J_2\|} \\ &= \frac{1}{N} \sum_{\|J\| \geq 1} \sum_{\|V\| \geq \|V'\| \geq N/\|J\|} \frac{1}{\|V\|^2\|V'\|\|J\|^2} = \frac{1}{N} \sum_{\|J\| \geq 1} \sum_{\|V'\| \geq N/\|J\|} \frac{1}{\|V'\|\|J\|^2} \sum_{\|V\| \geq \|V'\|} \frac{1}{\|V\|^2} \\ &\ll \frac{1}{N} \sum_{\|J\| \geq 1} \sum_{\|V'\| \geq N/\|J\|} \frac{1}{\|V'\|^2\|J\|^2} \ll \frac{1}{N} \left(\sum_{\|J\| \leq N} \sum_{\|V'\| \geq N/\|J\|} \frac{1}{\|V'\|^2\|J\|^2} + \sum_{\|J\| > N} \frac{1}{\|J\|^2} \right) \\ &\ll \frac{1}{N^2} \left(\sum_{\|J\| \leq N} \frac{1}{\|J\|} + 1 \right) \ll \frac{\log N}{N^2}. \end{aligned} \tag{6.14}$$

Combining (6.13) and (6.14), and using (6.12), we obtain

$$\sum_{1 \leq i,j \leq W} \ell(m_i, m_j) = W^2 \left(1 + u + O((\log N)/N^2)\right).$$

Hence, from (6.11) and $u \ll_F 1/N$, we have

$$\sum_{C \in \mathcal{C}(T)} \delta(C)^2 = \alpha^2 W \left(1 + O((\log N)/N^2)\right).$$

By (6.10), we complete the proof. $\square$

Chapter 7

On the Efficiency of Covering Systems

7.1 Introduction

A famous problem of Erdös from 1950, the least modulus problem, is to determine whether the least modulus in a covering system with distinct moduli can be arbitrarily large. As mentioned in the Introduction, P. Nielson has recently constructed a covering system with distinct moduli ≥ 40, which stands as the largest known least modulus. It is widely believed that the least modulus in Erdös' problem can be arbitrary large. If we assume that this is true, then we can consider the "efficiency" of covering systems with distinct moduli and a given least modulus. Let

$$g(N) = \inf_{C(N)} \sum_{(r \bmod n) \in C(N)} \frac{1}{n},$$

where $C(N)$ is the set of covering systems of the set of integers with distinct moduli and least modulus N. That is, we are given the least modulus N, and we select congruence classes with little overlap. Thus, $g(N)$ measures the maximum efficiency of covering systems in $C(N)$. It is known that $g(2) = g(3) = g(4) = 1$, and we note that Theorem 6.1 from Chapter 6 implies that $g(N) > 1$ if N is sufficiently large. We can restate Theorem I with $s = 1$, using the function $g(N)$.

Theorem 7.1. *Suppose $0 < c < \frac{1}{3}$ and let N be sufficiently large, depending on the choice of c. Then*

$$g(N) \geq c\frac{\log N \log\log\log N}{\log\log N}.$$

It is natural to try to find the least N for which $g(N) > 1$. Motivated by this question, in this chapter, we prove an explicit version of Theorem 7.1 (but without the factor $\log\log\log N$).

Theorem 7.2. *Let $N \geq 3$. Then,*

$$g(N) > 0.056413 \frac{\log N}{\log \log N}.$$

We remark that this bound implies

$$g(N) > 1$$

if $N \geq 2.759 \times 10^{33}$. For the proof of Theorem 7.2, we follow the key ideas from [22] and use some approximate formulas for prime numbers from [50]. In Section 7.2, we prove explicit versions of some Lemmas from [22]. In Section 7.3, we give a proof of Theorem 7.2.

7.2 Preliminary results

Throughout this chapter, n is a positive integer and p represents a prime. We let $P(n)$ and $P^{-}(n)$ denote the largest prime factor and the least prime factor of $n \geq 1$, respectively. We use the notations from Chapter 6 below, but now restricted to the case of the set of integers. If $C = \{(n_1, r_1), \ldots, (n_l, r_l)\}$ is a set of congruence classes, then we call such a set a residue system. Let $S(C) = \{n_1, \ldots, n_l\}$ be a multiset of the moduli of C. By $\delta(C)$ we denote the density of integers not covered by congruence classes from C. We also set

$$\alpha(C) = \prod_{n \in S(C)} \left(1 - \frac{1}{n}\right) = \prod_{j=1}^{l} \left(1 - \frac{1}{n_j}\right), \quad \beta(C) = \sum_{\substack{i<j \\ \gcd(n_i, n_j) > 1}} \frac{1}{n_i n_j}.$$

Proposition 7.1 ([22], Lemma 2.1)**.** *For any residue system C, we have $\delta(C) \geq \alpha(C) - \beta(C)$.*

We can factor each modulus $n = n_{\underline{Q}} n_{\overline{Q}}$, where $Q \geq 1$, $n_{\underline{Q}}$ is the largest divisor of n composed solely of primes $\leq Q$, and $n_{\overline{Q}} = n / n_{\underline{Q}}$. A positive integer n is called $Q-$smooth if $P(n) \leq Q$. So, $n_{\underline{Q}}$ is the largest divisor of n that is $Q-$smooth.

Proposition 7.2 ([22], Lemma 3.1)**.** *Let C be an arbitrary residue system. Let $Q \geq 2$ be arbitrary,* *and set*

$$M = \operatorname{lcm}\{n_{\underline{Q}} : n \in S(C)\}.$$

For $0 \le h \le M-1$, *let* C_h *be the set*

$$C_h = \left\{ \left(n_{\overline{Q}}, r\right) : (n,r) \in C,\ r \equiv h \pmod{n_{\underline{Q}}} \right\}.$$

Then

$$\delta(C) = \frac{1}{M} \sum_{h=0}^{M-1} \delta(C_h).$$

Proposition 7.3. *Suppose that* C *is a residue system,* $Q \ge 2$*, and define* M *and* C_h *as in Proposition 7.2. Also let* $C' = \{(n,r) \in C : n|M\} = \{(n,r) \in C : P(n) \le Q\}$ *and suppose* $\delta(C') > 0$*. Then*

$$\frac{1}{M} \sum_{h=0}^{M-1} \alpha(C_h) \ge (\alpha(C))^{(1+1/Q)/\delta(C')}.$$

We present some approximate formulas for some functions of prime numbers from [50].

Proposition 7.4 ([50], (3.5), (3.6) and (3.7))**.** *For* $x \ge 17$,

$$\pi(x) > \frac{x}{\log x}. \tag{7.1}$$

For $x > 1$,

$$\pi(x) < 1.25506 \frac{x}{\log x}. \tag{7.2}$$

For $1 < x < 113$ *and* $x \ge 113.6$,

$$\pi(x) < 1.25 \frac{x}{\log x}. \tag{7.3}$$

Proposition 7.5 ([50], (3.18))**.** *For* $x \geq 286$,

$$\sum_{p \leq x} \frac{1}{p} < \log\log x + B + \frac{1}{2\log^2 x},$$

where $B = 0.2614972....$

Proposition 7.6 ([50], (3.21) and (3.22))**.** *For* $x > 1$,

$$\sum_{p \leq x} \frac{\log p}{p} > \log x + E + \frac{1}{2\log x},$$

and for $x \geq 319$,

$$\sum_{p \leq x} \frac{\log p}{p} < \log x + E + \frac{1}{2\log x},$$

where $E = -1.3325822....$

Proposition 7.7 ([50], (3.30))**.** *For* $x > 1$,

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \log x \left(1 + \frac{1}{\log^2 x}\right).$$

We now give an explicit upper bound for the average of $\beta(C_h)$ (this is an explicit version of Lemma 3.2 with $s = 1$ in [22]).

Lemma 7.3. *Let* $K \geq 17$ *and* $Q \geq 17$. *Suppose* C *is a residue system with distinct moduli in* $[N, KN]$. *If we define* M *and* C_h *as in Proposition 7.2, then*

$$\frac{1}{M} \sum_{h=0}^{M-1} \beta(C_h) \leq 0.1725 \frac{(\log K + 44.5)^2 \log^2 Q}{Q} \Big(1 + \frac{1}{\log^2 Q}\Big)^3.$$

Proof. As in [22], for each $m|M$, we let S_m be the set of distinct numbers $n_{\overline{Q}} = n/\gcd(n, M)$, where $n \in S(C)$ and $n_{\underline{Q}} = \gcd(n, M) = m$. For $m, m' \mid M$, we define

$$F(r, m, r', m') = \#\{0 \leq h \leq M - 1 : h \equiv r \text{ (mod } m),\ h \equiv r' \text{ (mod } m')\}.$$

Then, we have

$$\begin{aligned}
\frac{1}{M}\sum_{h=0}^{M-1}\beta(C_h) &\le \frac{1}{M}\sum_{\substack{m|M\\ m'|M}}\ \sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ \gcd(n,n')>1}} \frac{1}{nn'} \sum_{\substack{(nm,r)\in C\\ (n'm',r')\in C}} F(r,m,r',m') \\
&\le \sum_{\substack{m|M\\ m'|M}} \frac{1}{\operatorname{lcm}[m,m']} \sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ \gcd(n,n')>1}} \frac{1}{nn'} \\
&\le \sum_{\substack{m|M\\ m'|M}} \frac{1}{\operatorname{lcm}[m,m']} \sum_{p>Q} \sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ p|n,\ p|n'}} \frac{1}{nn'} \\
&= \sum_{\substack{m|M\\ m'|M}} \frac{1}{\operatorname{lcm}[m,m']} \sum_{p>Q} \Bigg(\sum_{\substack{N/m\le n\le KN/m\\ p|n,\ P^-(n)>Q}} \frac{1}{n} \Bigg) \Bigg(\sum_{\substack{N/m'\le n'\le KN/m'\\ p|n',\ P^-(n')>Q}} \frac{1}{n'} \Bigg).
\end{aligned}$$

Next,

$$\sum_{\substack{N/m\le n\le KN/m\\ p|n,\ P^-(n)>Q}} \frac{1}{n} = \frac{1}{p} \sum_{\substack{\frac{N}{pm}\le t\le \frac{KN}{pm}\\ P^-(t)>Q}} \frac{1}{t} \le \frac{1}{p}\Big(\sum_{\substack{t\le \frac{KN}{pm}\\ P^-(t)>17}} \frac{1}{t} - \sum_{\substack{t< \frac{N}{pm}\\ P^-(t)>17}} \frac{1}{t}\Big).$$

In order to find an estimate of the above, we consider

$$\begin{aligned}
\sum_{n\le x}\frac{1}{n} &= \frac{[x]}{x} + \int_1^x \frac{[t]}{t^2}\,dt \\
&= 1 - \frac{\{x\}}{x} + \int_1^x \frac{1}{t}\,dt - \int_1^x \frac{\{t\}}{t^2}\,dt \\
&= \log x + \gamma - \frac{\{x\}}{x} + \int_x^\infty \frac{\{t\}}{t^2}\,dt,
\end{aligned}$$

so that

$$\log x + \gamma - \frac{1}{x} \ \le\ \sum_{n\le x}\frac{1}{n} \ \le\ \log x + \gamma + \frac{1}{x}. \tag{7.4}$$

Let $P = \prod_{p\le 17} p$. Then, by (7.4)

$$\sum_{\substack{t\le x\\ P^-(t)>17}} \frac{1}{t} = \sum_{t\le x}\frac{1}{t}\sum_{d|(t,P)}\mu(d) = \sum_{d|P}\frac{\mu(d)}{d}\sum_{m\le x/d}\frac{1}{m}$$
$$\le \sum_{d|P}\frac{\mu(d)}{d}(\log x - \log d + \gamma) + \frac{1}{x}\sum_{d|P}1$$
$$= \frac{3072}{17017}(\log x+\gamma) - \sum_{d|P}\frac{\mu(d)}{d}\log d + \frac{128}{x}.$$

Similarly, we also have

$$\sum_{\substack{t\le x\\ P^-(t)>17}} \frac{1}{t} \ge \frac{3072}{17017}(\log x+\gamma) - \sum_{d|P}\frac{\mu(d)}{d}\log d - \frac{128}{x}.$$

Thus, if $N/pm \ge 17$, then

$$\sum_{\substack{t\le \frac{KN}{pm}\\ P^-(t)>17}} \frac{1}{t} - \sum_{\substack{t< \frac{N}{pm}\\ P^-(t)>17}} \frac{1}{t} \le \frac{3072}{17017}\log K + 128\frac{pm}{KN} + 129\frac{pm}{N}$$
$$\le \frac{3072}{17017}\log K + 8.032 \le 0.18053(\log K + 44.5).$$

If $N/pm < 17$, then

$$\sum_{\substack{t\le \frac{KN}{pm}\\ P^-(t)>17}} \frac{1}{t} - \sum_{\substack{t< \frac{N}{pm}\\ P^-(t)>17}} \frac{1}{t} \le \sum_{\substack{t\le 17K\\ P^-(t)>17}} \frac{1}{t} \le 0.18053(\log 17K+\gamma) + 0.47 + \frac{128}{17K}$$
$$\le 0.18053(\log K + 8.5).$$

So, we have

$$\sum_{\substack{t\le \frac{KN}{pm}\\ P^-(t)>17}} \frac{1}{t} - \sum_{\substack{t< \frac{N}{pm}\\ P^-(t)>17}} \frac{1}{t} \le 0.18053(\log K + 44.5).$$

Thus,

$$\sum_{p>Q}\Bigg(\sum_{\substack{N/m\le n\le KN/m\\ p|n,\ P^-(n)>Q}}\frac{1}{n}\Bigg)\Bigg(\sum_{\substack{N/m'\le n'\le KN/m'\\ p|n',\ P^-(n')>Q}}\frac{1}{n'}\Bigg)\le 0.18053^2(\log K+44.5)^2\sum_{p>Q}\frac{1}{p^2}.$$

Using partial summation, (7.1) and (7.2), we obtain

$$\sum_{p>Q}\frac{1}{p^2}=-\frac{\pi(Q)}{Q^2}+2\int_Q^\infty\frac{\pi(x)}{x^3}\,dx<-\frac{Q}{\log Q}+2.52\int_Q^\infty\frac{1}{x^2\log x}\,dx\le\frac{1.52}{Q\log Q}.$$

Also, in [22], it was shown that

$$\sum_{\substack{m|M\\ m'|M}}\frac{1}{\mathrm{lcm}[m,m']}\le\prod_{p\le Q}\frac{1+1/p}{(1-1/p)^2}.$$

Thus,

$$\begin{aligned}\frac{1}{M}\sum_{h=0}^{M-1}\beta(C_h)&\le 0.04954\frac{(\log K+44.5)^2}{Q\log Q}\sum_{\substack{m|M\\ m'|M}}\frac{1}{\mathrm{lcm}[m,m']}\\&\le 0.04954\frac{(\log K+44.5)^2}{Q\log Q}\prod_{p\le Q}\frac{1+1/p}{(1-1/p)^2}\\&=0.04954\frac{(\log K+44.5)^2}{Q\log Q}\prod_{p\le Q}(1-\frac{1}{p^2})\prod_{p\le Q}\left(\frac{p}{p-1}\right)^3.\end{aligned}$$

Since, by Proposition 7.7,

$$\prod_{p\le Q}\left(\frac{p}{p-1}\right)^3<e^{3\gamma}\log^3 Q\left(1+\frac{1}{\log^2 Q}\right)^3$$

and

$$\prod_{p\le Q}(1-\frac{1}{p^2})\le\prod_{p\le 17}(1-\frac{1}{p^2})\le 0.616,$$

we have

$$\frac{1}{M}\sum_{h=0}^{M-1}\beta(C_h) \le 0.1725\frac{(\log K+44.5)^2\log^2 Q}{Q}\big(1+\frac{1}{\log^2 Q}\big)^3,$$

which completes the proof. □

Lemma 7.4. *Suppose $K \ge 17$, N is a positive integer, and C is a residue system with distinct moduli from the interval $[N, KN]$. Let $Q \ge 17$ and $C' = \{(n,r) \in C : P(n) \le Q\}$. If $\delta(C') > 0$, then*

$$\delta(C) \ge \alpha(C)^{(1+1/Q)/\delta(C')} - 0.1725\frac{(\log K+44.5)^2\log^2 Q}{Q}\big(1+\frac{1}{\log^2 Q}\big)^3,$$

Proof. By combining Propositions 7.1, 7.2, 7.3 and Lemma 7.3, we complete the proof. □

Next, we give an explicit estimate for sums of reciprocals of smooth numbers.

Lemma A ([22], Lemma 4.1). *Suppose $Q \ge 2$ and $Q < N \le \exp(\sqrt{Q})$. Then*

$$\sum_{\substack{n>N\\P(n)\le Q}}\frac{1}{n} \ll (\log Q)e^{-u\log u}, \quad \textit{where } u = \frac{\log N}{\log Q}.$$

For the proof of Lemma A, the authors of [22] applied standard upper-bound estimates for the distribution of smooth numbers: The number of Q-smooth numbers at most t is $\ll t/u_t^{u_t}$, where $u_t = \log t/\log Q$, provided $Q \le t \le \exp\left(Q^{1-\varepsilon}\right)$ ([34], Theorem 1.2 and Corollary 2.3). We use a technique called "Rankin's Method" (see [34, p. 414] or [49]) to derive an explicit version of a slightly weaker form of Lemma A.

Lemma 7.5. *If $Q \ge 319$, then*

$$\sum_{\substack{n\ge N\\P(n)\le Q}}\frac{1}{n} \le e^{5.8417+A(\sigma)}\frac{\log Q}{e^{2.5u}},$$

where $u = \log N/\log Q$, $A(\sigma) = \sum_{k=2}^{\infty}\sum_{p\le Q} 1/(kp^{k\sigma})$ and $\sigma = 1-2.5/\log Q$.

Proof. We begin with

$$\sum_{\substack{n\geq N\\ P(n)\leq Q}} \frac{1}{n} \leq \frac{1}{N^{1-\sigma}} \sum_{P(n)\leq Q} \frac{1}{n^{\sigma}} = \frac{1}{N^{1-\sigma}} \prod_{p\leq Q} \left(1-\frac{1}{p^{\sigma}}\right)^{-1}$$
$$= \frac{1}{N^{1-\sigma}} \exp\Big\{ \sum_{p\leq Q} -\log\left(1-\frac{1}{p^{\sigma}}\right) \Big\}$$
$$= \frac{1}{e^{2.5u}} \exp\Big\{ \sum_{p\leq Q} \frac{1}{p^{\sigma}} + \sum_{k=2}^{\infty} \sum_{p\leq Q} \frac{1}{kp^{k\sigma}} \Big\}.$$

Next,

$$\sum_{p\leq Q} \frac{1}{p^{\sigma}} = \sum_{p\leq Q} \frac{p^{\frac{2.5}{\log Q}}}{p} = \sum_{p\leq Q} \frac{e^{2.5\frac{\log p}{\log Q}}}{p}.$$

If $x \leq 2.5$, then we obtain

$$e^{x} \leq 1 + \sum_{n=1}^{10} \frac{x^{n}}{n!} + 3.149\times 10^{-8} x^{11}. \tag{7.5}$$

Also, by partial summation and Proposition 7.6, we have

$$\sum_{p\leq Q} \frac{\log^2 p}{p} = \log Q \sum_{p\leq Q} \frac{\log p}{p} - \int_2^Q \frac{1}{t} \sum_{p\leq t} \frac{\log p}{p}\, dt$$
$$\leq \log^2 Q + E\log Q + \frac{1}{2} - \int_2^Q \frac{\log t}{t} + \frac{E}{t} - \frac{1}{2t\log t}\, dt$$
$$\leq \frac{1}{2}\log^2 Q + \frac{1}{2}\log\log Q + \frac{1}{2} + \frac{\log^2 2}{2} + E\log 2 - \frac{\log\log 2}{2}$$
$$\leq \frac{1}{2}\log^2 Q + \frac{1}{2}\log\log Q - 0.00019. \tag{7.6}$$

Similarly, we can derive

$$\sum_{p\leq Q} \frac{\log^n p}{p} \leq \frac{1}{n}\log^n Q + \frac{2n-3}{2n-4}\log^{n-2} Q + \frac{n-1}{n}\log^n 2$$
$$+ E\log^{n-1} 2 - \frac{n-1}{2n-4}\log^{n-2} 2, \quad \text{for} \quad n\geq 3$$

$$= \frac{1}{n}\log^n Q + \frac{2n-3}{2n-4}\log^{n-2} Q + T(n), \quad \text{for} \quad n \geq 3. \tag{7.7}$$

Thus, by (7.5), (7.6), (7.7) and Proposition 7.5, we have

$$\begin{aligned}
\sum_{p\leq Q}\frac{1}{p^\sigma} &\leq \sum_{p\leq Q}\frac{1}{p} + \sum_{n=1}^{10}\frac{2.5^n}{n!\log^n Q}\sum_{p\leq Q}\frac{\log^n p}{p} + \frac{3.149\times 10^{-8}\times 2.5^{11}}{\log^{11} Q}\sum_{p\leq Q}\frac{\log^{11} p}{p} \\
&\leq \log\log Q + B + \frac{1}{2\log^2 Q} \\
&\quad + \sum_{n=1}^{10}\frac{2.5^n}{n!\log^n Q}\sum_{p\leq Q}\frac{\log^n p}{p} + \frac{3.149\times 2.5^{11}}{10^8\times\log^{11} Q}\sum_{p\leq Q}\frac{\log^{11} p}{p} \\
&\leq \log\log Q + B + \sum_{n=1}^{10}\frac{2.5^n}{n!n} + \frac{3.149\times 2.5^{11}}{10^8\times 11} + \frac{2.5E+1.25}{\log Q} + \frac{2.5^2\log\log Q}{4\log^2 Q} \\
&\quad + \frac{1-2.5^2\times 0.00019}{2\log^2 Q} + \sum_{n=3}^{10}\frac{2.5^n}{n!n}\left(\frac{2n-3}{(2n-4)\log^2 Q} + \frac{T(n)}{\log^n Q}\right) \\
&\leq \log\log Q + B + \sum_{n=1}^{10}\frac{2.5^n}{n!n} + \frac{3.149\times 2.5^{11}}{10^8\times 11} \\
&\leq \log\log Q + 5.84177.
\end{aligned}$$

□

Corollary 7.6. *If* $Q \geq 10^8$, *then*

$$\sum_{\substack{n\geq N\\ P(n)\leq Q}}\frac{1}{n} \leq 539.365\frac{\log Q}{e^{2.5u}},$$

where $u = \log N/\log Q$.

Proof. Using $\sigma = 1 - 2.5/\log Q \geq 0.86428$, we have by (7.3),

$$\begin{aligned}
A(\sigma) &\leq \sum_{p\leq 200}\sum_{k=2}^{\infty}\frac{1}{kp^{0.86428k}} + \frac{1}{2}\sum_{p\geq 211}\frac{1}{p^\sigma(p^\sigma-1)} \\
&\leq \sum_{p\leq 200}\left(-\log\left(1-\frac{1}{p^{0.86428}}\right) - \frac{1}{p^{0.86428}}\right) + \frac{1.01}{2}\sum_{p\geq 211}\frac{1}{p^{2\sigma}}
\end{aligned}$$

$$\leq 0.443228 + \frac{2.5 \cdot 1.01\sigma}{2} \int_{211}^{\infty} \frac{1}{x^{2\sigma} \log x} \, dx$$

$$\leq 0.443228 + 0.005394$$

$$\leq 0.448622.$$

Since $e^{5.8417+0.448622} \leq 539.365$, we complete the proof. $\square$

7.3 Proof of Theorem 7.2

Proof of Theorem 7.2. Let C be a covering system with $S(C)$ consisting of distinct integers $n \geq N$. Define

$$f(N) = \frac{0.05416 \log N}{0.96 \log \log N}.$$

We shall show that for $N \geq e^{77}$,

$$\sum_{n \in S(C)} \frac{1}{n} > f(N). \tag{7.8}$$

Since $0.05416/0.96 > 0.056413$ and

$$0.056413 \frac{77}{\log 77} < 1,$$

(7.8) implies Theorem 7.2. Suppose

$$\sum_{n \in S(C)} \frac{1}{n} \leq f(N).$$

Then, we have

$$-\log \alpha(C) \leq \sum_{n \in S(C)} \left(\frac{1}{n} + \frac{1}{n^2} \right) \leq \left(1 + \frac{1}{N} \right) \sum_{n \in S(C)} \frac{1}{n} \leq \left(1 + \frac{1}{N} \right) f(N).$$

Set $a = 0.216035$, $b = 0.461$ and $d = 0.076885$. Define, for $j \geq 1$,

$$Q_0 = \exp\Big(\frac{\log N}{0.96 \log\log N}\Big), \qquad Q_j = \exp(Q_{j-1}^a), \qquad K_j = \exp(Q_{j-1}^b / \log Q_{j-1}).$$

Let

$$C_j = \{(n, r) \in C : P(n) \leq Q_j\},$$

and define

$$\delta_0 = 0.70443, \qquad \delta_j = e^{-f(N)(1+1/N)(1+1/Q_0)/\delta_{j-1}}.$$

We will show that

$$\delta(C_0) \geq \delta_0, \qquad \delta(C_j) \geq 0.0001\delta_j > 0 \quad (j \geq 1).$$

First, by Corollary 7.6,

$$1 - \delta(C_0) \leq 539.365 \frac{\log Q_0}{\exp\{2.5 \frac{\log N}{\log Q_0}\}} = \frac{539.365}{0.96 \log\log N (\log N)^{1.4}} \leq 0.29557.$$

Thus, we have

$$\delta(C_0) \geq 0.70443 = \delta_0.$$

Next, suppose $j \geq 1$, and $\delta(C_{j-1}) \geq \delta_{j-1}$. Let

$$C_j' = \{(n, r) \in C_j : n \leq K_j\}, \qquad C_j'' = \{(n, r) \in C_j : n > K_j\}.$$

Then, by Lemma 7.4, we have

$$\begin{aligned}
\delta(C_j') &\geq \alpha(C_j')^{(1+1/Q_0)/\delta_{j-1}} - 0.1725 \frac{(\log K_j / N + 44.5)^2 \log^2 Q_{j-1}}{Q_{j-1}} \big(1 + \frac{1}{\log^2 Q_{j-1}}\big)^3 \\
&\geq e^{-f(N)(1+1/N)(1+1/Q_0)/\delta_{j-1}} - 0.1725 \big(1 + \frac{1}{\log^2 Q_{j-1}}\big)^3 Q_{j-1}^{-1+2b} \\
&\geq \delta_j - 0.1725 \Big(1 + \frac{1}{\log^2 Q_{j-1}}\Big)^3 Q_{j-1}^{-1+2b}.
\end{aligned}$$

Also, by Corollary 7.6,

$$1-\delta(C_j'')\leq \sum_{\substack{n>K_j\\ P(n)\leq Q_j}}\frac{1}{n}\leq 539.365\frac{\log Q_j}{\exp\{\frac{2.5Q_{j-1}^{b-a}}{\log Q_{j-1}}\}}$$
$$=539.365\frac{Q_{j-1}^{a}}{\exp\{\frac{2.5Q_{j-1}^{b-a}}{\log Q_{j-1}}\}}.$$

If $N\geq e^{77}$, then $Q_{j-1}\geq Q_0>10^8$. So, we have

$$0.1725\Big(1+\frac{1}{\log^2 Q_{j-1}}\Big)^3 Q_{j-1}^{-1+2b}+539.365\frac{Q_{j-1}^{a}}{\exp\{\frac{2.5Q_{j-1}^{b-a}}{\log Q_{j-1}}\}}\leq 0.9999Q_{j-1}^{-d},$$

so that

$$\delta(C_j)\geq \delta(C_j')-(1-\delta(C_j''))\geq \delta_j-0.9999Q_{j-1}^{-d}.$$

Thus, it suffices to prove that

$$Q_{j-1}^{-d}\leq \delta_j, \tag{7.9}$$

or equivalently,

$$\begin{aligned} d\log Q_{j-1} &\geq f(N)(1+1/N)(1+1/Q_{j-1})/\delta_{j-1}\\ &=0.05416\log Q_0(1+1/N)(1+1/Q_{j-1})/\delta_{j-1}. \end{aligned} \tag{7.10}$$

We have

$$d\log Q_0\geq 0.05416\log Q_0(1+1/N)(1+1/Q_0)/\delta_0,$$

since $0.05416(1+1/N)(1+1/Q_0)/\delta_0<d$. This proves (7.10) when $j=1$. Now suppose that (7.9) and (7.10) hold for some $j\geq 1$. We show that

$$d\log Q_j\geq 0.05416\log Q_0(1+1/N)(1+1/Q_j)/\delta_j.$$

By (7.9), it suffices to show that

$$d \log Q_j \geq 0.05416 \log Q_0 (1 + 1/N)(1 + 1/Q_j) Q_{j-1}^d$$

or equivalently,

$$dQ_{j-1}^{a-d} \geq 0.05416 \log Q_0 (1 + 1/N)(1 + 1/Q_j). \tag{7.11}$$

Since (7.11) holds for $Q_0 \geq 10^8$, we deduce

$$Q_j^{-d} \leq \delta_{j+1}.$$

By induction, (7.9) holds for all $j \geq 1$. This completes the proof that for $N \geq e^{77}$,

$$\sum_{n \in S(C)} \frac{1}{n} > f(N).$$

$\square$

Chapter 8

Exact Covering Systems in Quadratic Number Fields

8.1 Introduction

In this chapter, we consider exact covering systems in quadratic number fields. As mentioned earlier in the Introduction, we give a partial answer to the question:

Does there exist an exact covering system with distinct moduli in a number field?

Let $\mathbb{Q}(\sqrt{m})$ be a quadratic field. Then, the ring of integers is ([41])

$$\mathbb{Z}[\sqrt{m}] \quad \text{if } m \equiv 2,3 \pmod 4, \text{ and } \quad \mathbb{Z}\Big[\frac{1+\sqrt{m}}{2}\Big] \quad \text{if } m \equiv 1 \pmod 4.$$

For each ideal I, we write $I = [\alpha, \beta]$ where $\{\alpha, \beta\}$ is an integral basis of I. We first present the results for imaginary quadratic fields. In the proof, we use the following result.

Proposition 8.1 ([42], page 300)**.** *In an imaginary quadratic field with discriminant $-\Delta$, every ideal is equivalent to one and only one of the ideals $[a, (b+\sqrt{-\Delta})/2]$ such that $b^2+\Delta$ is divisible by $4a$, and either $-a < b \le a$, $b^2+\Delta > 4a^2$, or $0 \le b \le a$, $b^2+\Delta = 4a^2$.*

Theorem 8.1. *Let $S = \{r_1 + I_1, \ldots, r_k + I_k\}$ be an exact covering system of the ring of integers of an imaginary quadratic field $\mathbb{Q}(\sqrt{-m})$ with $\|I_1\| \le \cdots \le \|I_k\|$. If I_k is principal, then it must be repeated.*

We next consider imaginary quadratic number fields with two ideal classes. The complete list of imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$ with class number two is the following [12].

$$m = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427$$

By straightforward calculation, we can see that $\mathbb{Q}(\sqrt{-m})$ has an reduced ideal $[a, (b+\sqrt{-\Delta})/2]$ with $b^2 + \Delta = 4a^2$ precisely when $m = 15, 35, 91, 187$ and 403.

Theorem 8.2. *Let $\mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic fields with class number two and $m \neq 15, 35, 91, 187, 403$. If $S = \{r_1 + I_1, \ldots, r_k + I_k\}$ is an exact covering system of the of ring of the integers, then the moduli can not be distinct.*

The key idea of the proofs is analogous to that of Proposition 1.1. We first express each element in a coset $r_i + I_i$ using the basis of I_i, in order to find an explicit identity for

$$\frac{1}{1-z}\frac{1}{1-w} = \sum_{u,v\geq 0} z^u w^v$$

After some manipulation, we consider the terms with double poles when z and w tend to some $\|I_k\|-$th roots of 1, where $\|I_k\|$ is the largest norm. Since there is no pole at these points on the left-hand side, we must have at least two terms with double poles from the other side. If all the ideals are principal, then any ideal I_j that produces a term with a double pole must be equal to I_k, which proves Theorem 8.1. On the other hand, if some of the moduli are not principal as in Theorem 8.2, then we need to classify the ideals according to the ideal classes of the field and also, we should consider the coefficients of terms with double poles.

It turns out that the case of real quadratic fields is more difficult. We could succeed only when we assume all the ideals are principal.

Theorem 8.3. *Let m be a positive integer. If $S = \{r_1 + I_1, \ldots, r_k + I_k\}$ is an exact covering system of the ring of integers of $\mathbb{Q}(\sqrt{m})$ and all the moduli are principal, then any modulus of the largest norm must be repeated.*

We remark that Theorems 8.1 and 8.3 imply that if the ring of integers of a quadratic field is a PID, then there is no exact covering systems with distinct moduli in this number field. We also note that there are only finitely many imaginary quadratic fields with class number one (for precisely following values $m = -1, -2, -3, -7, -11, -19, -43, -67, -163$), while Gauss' conjecture that there are infinitely many real quadratic fields with class number one remains open. More precisely, H. Cohen and H. Lenstra [12] predict that about 75.446% of real quadratic fields will have class number one.

In Section 8.2, we provide proofs of Theorems 8.1, 8.2 and 8.3. Since the arguments in Theorems 8.2 and 8.3 are similar to that of Theorem 8.1, we skip some details in their proofs.

The methods described above also work for some higher degree fields. For example, we also examined the number field $\mathbb{Q}(\sqrt[3]{2})$, whose ring of integers is a PID, and confirmed that any ideal with the largest norm in an exact covering system must be repeated. We conjecture that there is no exact covering system with distinct moduli in any number field. This is the subject of ongoing investigations.

8.2 Proofs of Theorems 8.1, 8.2 and 8.3

Proof of Theorem 8.1. We first consider the case $-m \equiv 2, 3 \pmod 4$. Note that b is even in Proposition 8.1. Thus, for each i, I_i is equivalent to $[\eta_i, \tau_i + \sqrt{-m}]$. Since $\eta_i^2 \leq \tau_i^2 + m \leq \eta_i^2/4 + m$, we have $\eta_i \leq \sqrt{4m/3}$. Also, if an ideal I is principal, then I is equivalent to $[1, \sqrt{-m}] = \mathbb{Z}[\sqrt{-m}]$.

For each i, there exist x_i, y_i and z_i with $\gcd(x_i, y_i, z_i) = 1$ such that

$$\begin{aligned} I_i &= \frac{y_i + z_i\sqrt{-m}}{x_i}[\eta_i, \tau_i + \sqrt{-m}] \\ &= \left[\frac{\eta}{x_i}(y_i + z_i\sqrt{-m}), \frac{y_i\tau_i - mz_i}{x_i} + \frac{z_i\tau_i + y_i}{x_i}\sqrt{-m}\right] \\ &= \left[\alpha_i + \beta_i\sqrt{-m}, X_i + Y_i\sqrt{-m}\right], \end{aligned}$$

where $\alpha_i = \eta_i y_i/x_i$, $\beta_i = \eta_i z_i/x_i$, $X_i = (\alpha_i\tau_i - m\beta_i)/\eta_i$ and $Y_i = (\alpha_i + \tau_i\beta_i)/\eta_i$. Note that

$$\|I_i\| = \begin{vmatrix} \alpha_i & \beta_i \\ X_i & Y_i \end{vmatrix} = \frac{{\alpha_i}^2 + m{\beta_i}^2}{\eta_i}.$$

Letting $r_i = \gamma_i + \delta_i\sqrt{-m}$, we have

$$r_i + I_i = \Big\{a\alpha_i + bX_i + \gamma_i + (a\beta_i + bY_i + \delta_i)\sqrt{-m} : a, b \in \mathbb{Z}\Big\}.$$

We define

$$D_1 = \{1 \le i \le k : \alpha_i = 0, \beta_i > 0\}, \qquad D_2 = \{1 \le i \le k : \alpha_i > 0, \beta_i = 0\},$$
$$D_3 = \{1 \le i \le k : \alpha_i > 0, \beta_i > 0\}, \qquad D_4 = \{1 \le i \le k : \alpha_i > 0, \beta_i < 0\}.$$

Then, note that for each $1 \le i \le k$, there exists j with $1 \le j \le 4$, such that $i \in D_j$.

For convenience, we assume

$$0 \le \beta_i\gamma_i - \alpha_i\delta_i < \|I_i\|$$

for $i \in D_j$, $j = 1, 3, 4$, and if $i \in D_2$, we assume

$$-\|I_i\| < \beta_i\gamma_i - \alpha_i\delta_i = -\alpha_i\delta_i \le 0.$$

Note that the above is possible since we can replace $\gamma_i+\delta_i\sqrt{-m}$ by $\gamma_i+\delta_i\sqrt{-m}+n(X_i+Y_i\sqrt{-m})$ for $n \in \mathbb{Z}$ and

$$\beta_i(\gamma_i + nX_i) - \alpha_i(\delta_i + nY_i) = \beta_i\gamma_i - \alpha_i\delta_i + n\|I_i\|.$$

Define

$$A(i) = \{(a, b) \in \mathbb{Z} : a\alpha_i + bX_i + \gamma_i \ge 0, a\beta_i + bY_i + \delta_i \ge 0\}.$$

Since S is an exact covering system of $\mathbb{Z}[\sqrt{-m}]$, we have

$$\frac{1}{1-z}\frac{1}{1-w} = \sum_{u,v \ge 0} z^u w^v$$

$$= \sum_{i=1}^{k} \sum_{(a,b)\in A(i)} z^{a\alpha_i+bX_i+\gamma_i} w^{a\beta_i+bY_i+\delta_i}.$$

We define for each $1 \le i \le k$ and $\alpha_i, \beta_i \neq 0$,

$$\lambda_i(t) = \left\lceil \frac{-tX_i-\gamma_i}{\alpha_i} \right\rceil, \quad \mu_i(t) = \left\lceil \frac{-tY_i-\delta_i}{\beta_i} \right\rceil, \quad \mu_i'(t) = \left\lfloor \frac{-tY_i-\delta_i}{\beta_i} \right\rfloor.$$

If $i \in D_1$, then

$$\begin{aligned}
\sum_{(a,b)\in A(i)} z^{a\alpha_i+bX_i+\gamma_i} w^{a\beta_i+bY_i+\delta_i} &= z^{\gamma_i} w^{\delta_i} \sum_{b \le \frac{\gamma_i\eta_i}{m\beta_i} = \frac{\gamma_i\beta_i}{\|I_i\|}} (z^{X_i} w^{Y_i})^b \sum_{a \ge \frac{-bY_i-\delta_i}{\beta_i}} (w^{\beta_i})^a \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{1-w^{\beta_i}} \sum_{b\le 0} (z^{X_i} w^{Y_i})^b (w^{\beta_i})^{\left\lceil \frac{-bY_i-\delta_i}{\beta_i} \right\rceil} \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{1-w^{\beta_i}} \sum_{t=1-\beta_i}^{0} \sum_{c\beta_i+t\le 0} z^{cX_i\beta_i+X_it} w^{tY_i+\beta_i\mu_i(t)} \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{1-w^{\beta_i}} \sum_{c\le 0} (z^{X_i\beta_i})^c \sum_{t=1}^{\beta_i} (z^{X_i} w^{Y_i})^t w^{\beta_i\mu_i(t)} \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{1-w^{\beta_i}} \sum_{c\ge 0} (z^{-X_i\beta_i})^c \sum_{t=1}^{\beta_i} (z^{X_i} w^{Y_i})^t w^{\beta_i\mu_i(t)} \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{(1-w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=1}^{\beta_i} (z^{X_i} w^{Y_i})^t w^{\beta_i\mu_i(t)}.
\end{aligned}$$

Similarly, for $i \in D_2$, we have

$$\begin{aligned}
\sum_{(a,b)\in A(i)} z^{a\alpha_i+bX_i+\gamma_i} w^{a\beta_i+bY_i+\delta_i} &= z^{\gamma_i} w^{\delta_i} \sum_{b \ge \frac{-\delta_i\alpha_i}{\|I_i\|}} (z^{X_i} w^{Y_i})^b \sum_{a \ge \frac{-bX_i-\gamma_i}{\alpha_i}} (z^{\alpha_i})^a \\
&= \frac{z^{\gamma_i} w^{\delta_i}}{(1-z^{\alpha_i})(1-w^{\|I_i\|})} \sum_{t=0}^{\alpha_i-1} (z^{X_i} w^{Y_i})^t z^{\alpha_i\lambda_i(t)}.
\end{aligned}$$

Now, let $i \in D_3$. Observe that

$$a\alpha_i + bX_i + \gamma_i \ge 0 \;\Leftrightarrow\; a \ge \frac{-bX_i-\gamma_i}{\alpha_i},$$

$$a\beta_i + bY_i + \delta_i \geq 0 \;\Leftrightarrow\; a \geq \frac{-bY_i - \delta_i}{\beta_i},$$

and

$$\frac{-bX_i - \gamma_i}{\alpha_i} > \frac{-bY_i - \delta_i}{\beta_i} \;\Leftrightarrow\; b > \frac{\beta_i\gamma_i - \alpha_i\delta_i}{\|I_i\|}.$$

Thus, we have

$$\begin{aligned}
&\sum_{(a,b)\in A(i)} z^{a\alpha_i+bX_i+\gamma_i} w^{a\beta_i+bY_i+\delta_i} \\
&= z^{\gamma_i}w^{\delta_i} \sum_{b>\frac{\beta_i\gamma_i-\alpha_i\delta_i}{\|I_i\|}} (z^{X_i}w^{Y_i})^b \sum_{a\geq\frac{-bX_i-\gamma_i}{\alpha_i}} (z^{\alpha_i}w^{\beta_i})^a + z^{\gamma_i}w^{\delta_i} \sum_{b\leq\frac{\beta_i\gamma_i-\alpha_i\delta_i}{\|I_i\|}} (z^{X_i}w^{Y_i})^b \sum_{a\geq\frac{-bY_i-\delta_i}{\beta_i}} (z^{\alpha_i}w^{\beta_i})^a \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{b\geq 1} (z^{X_i}w^{Y_i})^b (z^{\alpha_i}w^{\beta_i})^{\left\lceil \frac{-bX_i-\gamma_i}{\alpha_i} \right\rceil} + \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{b\leq 0} (z^{X_i}w^{Y_i})^b (z^{\alpha_i}w^{\beta_i})^{\left\lceil \frac{-bY_i-\delta_i}{\beta_i} \right\rceil} \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{c\geq 0} (w^{\alpha_iY_i-\beta_iX_i})^c \sum_{t=1}^{\alpha_i} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\qquad + \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{c\leq 0} (z^{\beta_iX_i-\alpha_iY_i})^c \sum_{t=1-\beta_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i(t)} \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-w^{\|I_i\|})} \sum_{t=1}^{\alpha_i} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\qquad + \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=1-\beta_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i(t)}.
\end{aligned}$$

Lastly, let $i \in D_4$. Then,

$$\begin{aligned}
a\alpha_i + bX_i + \gamma_i \geq 0 \;&\Leftrightarrow\; a \geq \frac{-bX_i - \gamma_i}{\alpha_i}, \\
a\beta_i + bY_i + \delta_i \geq 0 \;&\Leftrightarrow\; a \leq \frac{-bY_i - \delta_i}{\beta_i}.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
\sum_{(a,b)\in A(i)} z^{a\alpha_i+bX_i+\gamma_i} w^{a\beta_i+bY_i+\delta_i} &= z^{\gamma_i} w^{\delta_i} \sum_{b\leq \frac{\beta_i\gamma_i-\alpha_i\delta_i}{\|I_i\|}} (z^{X_i}w^{Y_i})^b \sum_{\frac{-bX_i-\gamma_i}{\alpha_i}\leq a\leq \frac{-bY_i-\delta_i}{\beta_i}} (z^{\alpha_i}w^{\beta_i})^a \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{b\leq 0} (z^{X_i}w^{Y_i})^b \Big\{ (z^{\alpha_i}w^{\beta_i})^{\left\lceil \frac{-bX_i-\gamma_i}{\alpha_i}\right\rceil} - (z^{\alpha_i}w^{\beta_i})^{\left\lfloor \frac{-bY_i-\delta_i}{\beta_i}\right\rfloor+1} \Big\} \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{c\leq 0} (w^{\alpha_iY_i-\beta_iX_i})^c \sum_{t=1-\alpha_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\qquad - \frac{z^{\alpha_i+\gamma_i}w^{\beta_i+\delta_i}}{1-z^{\alpha_i}w^{\beta_i}} \sum_{c\geq 0} (z^{\beta_iX_i-\alpha_iY_i})^c \sum_{t=0}^{\beta_i-1} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i'(t)} \\
&= \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-w^{\|I_i\|})} \sum_{t=1-\alpha_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\qquad - \frac{z^{\alpha_i+\gamma_i}w^{\beta_i+\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=0}^{\beta_i-1} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i'(t)}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{1}{1-z}\frac{1}{1-w} &= \sum_{i\in D_1} \frac{z^{\gamma_i}w^{\delta_i}}{(1-w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=1}^{\beta_i} (z^{X_i}w^{Y_i})^t w^{\beta_i\mu_i(t)} \\
&\quad + \sum_{i\in D_2} \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i})(1-w^{\|I_i\|})} \sum_{t=0}^{\alpha_i-1} (z^{X_i}w^{Y_i})^t z^{\alpha_i\lambda_i(t)} \\
&\quad + \sum_{i\in D_3} \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-w^{\|I_i\|})} \sum_{t=1}^{\alpha_i} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\quad + \sum_{i\in D_3} \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=1-\beta_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i(t)} \\
&\quad + \sum_{i\in D_4} \frac{z^{\gamma_i}w^{\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-w^{\|I_i\|})} \sum_{t=1-\alpha_i}^{0} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \\
&\quad - \sum_{i\in D_4} \frac{z^{\alpha_i+\gamma_i}w^{\beta_i+\delta_i}}{(1-z^{\alpha_i}w^{\beta_i})(1-z^{\|I_i\|})} \sum_{t=0}^{\beta_i-1} (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\mu_i'(t)}. \qquad (8.1)
\end{aligned}
$$

Now, we show that I_k must repeated. First, we find some $p > 0$ such that

$$\alpha_k \pm p\beta_k \neq 0.$$

Next, in (8.1) we let

$$z = (1-\epsilon)e^{2\pi i \frac{\beta_k}{\|I_k\|}}, \qquad w = (1-p\epsilon)e^{-2\pi i \frac{\alpha_k}{\|I_k\|}} \quad \text{and} \quad \epsilon \to 0.$$

Then we can see that the terms with $i = k$ tend to

$$\begin{aligned}
&\frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)}}{1-(1-\epsilon)^{\alpha_k}(1-p\epsilon)^{\beta_k}}\Big\{\frac{\alpha_k}{1-(1-p\epsilon)^{\|I_k\|}} \pm \frac{\beta_k}{1-(1-\epsilon)^{\|I_k\|}}\Big\} \\
&\approx \frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)}}{1-(1-\alpha_k\epsilon)(1-\beta_k p\epsilon)}\Big\{\frac{\alpha_k}{1-(1-p\epsilon\|I_k\|)} \pm \frac{\beta_k}{1-(1-\epsilon\|I_k\|)}\Big\} \\
&\approx \frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)}}{(\alpha_k + p\beta_k)\epsilon}\Big\{\frac{\alpha_k}{p\epsilon\|I_k\|} \pm \frac{\beta_k}{\epsilon\|I_k\|}\Big\} \\
&= \frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)}(\alpha_k \pm p\beta_k)}{p\|I_k\|(\alpha_k + p\beta_k)}\frac{1}{\epsilon^2} \neq 0.
\end{aligned}$$

Thus, we can see that the above has a pole of order 2 as $\epsilon \to 0$. Since the left-hand side of (8.1) cannot have a pole of order 2, there must be more terms on the right -hand side that also have a pole of order 2, say $i = j$. In other words, both of $z^{\alpha_j}w^{\beta_j}$ and $w^{\|I_j\|}$ tend to 1 and the corresponding inner sum on t does not tend to 0 or both of $z^{\alpha_j}w^{\beta_j}$ and $z^{\|I_j\|}$ tend to 1 and the corresponding inner sum on t does not tend to 0 as $\epsilon \to 0$. We first assume the first case. Then we have

$$\|I_k\| \,\Big|\, \alpha_j\beta_k - \beta_j\alpha_k, \quad \text{and} \quad \|I_k\| \,\Big|\, \alpha_k\|I_j\|.$$

Note that

$$\sum_t (z^{X_i}w^{Y_i})^t (z^{\alpha_i}w^{\beta_i})^{\lambda_i(t)} \to \sum_t e^{\frac{2\pi i}{\|I_k\|}(\beta_k X_j - \alpha_k Y_j)t},$$

and $e^{2\pi i(\beta_k X_j - \alpha_k Y_j)/\|I_k\|}$ is an α_j-th root of unity since

$$(\beta_k X_j - \alpha_k Y_j)\alpha_j = X_j(\alpha_j\beta_k - \beta_j\alpha_k) + \alpha_k\|I_j\|.$$

Thus we have

$$\|I_k\| \,\Big|\, \beta_k X_j - \alpha_k Y_j$$

so that

$$\sum_t (z^{X_i} w^{Y_i})^t (z^{\alpha_i} w^{\beta_i})^{\lambda_i(t)} \to \sum_t 1 = \alpha_j.$$

In a very similar way, we can show that the second case also implies $\|I_k\| \,\Big|\, \beta_k X_j - \alpha_k Y_j$ using $\|I_k\| \,\Big|\, \alpha_j\beta_k - \beta_j\alpha_k$ and $\|I_k\| \,\Big|\, \beta_k\|I_j\|$.

Now, let A and B be integers such that

$$\beta_k\alpha_j - \alpha_k\beta_j = A\|I_k\| \quad \text{and} \quad \beta_k X_j - \alpha_k Y_j = B\|I_k\|. \tag{8.2}$$

Solving (8.2), we obtain

$$\begin{aligned} \alpha_j + \beta_j\sqrt{-m} &= \frac{\alpha_k + \beta_k\sqrt{-m}}{\eta_k}\left(\tau_j A - \eta_j B - A\sqrt{-m}\right) \\ &= \left(\alpha_k + \beta_k\sqrt{-m}\right)\left(\tau_j A - \eta_j B - A\sqrt{-m}\right). \end{aligned} \tag{8.3}$$

Note that $\eta_k = 1$ since I_k is principal. Taking norms, we derive

$$\|I_j\| = \|I_k\|\frac{(\tau_j A - \eta_j B)^2 + mA^2}{\eta_j} \leq \|I_k\|,$$

and so $(\tau_j A - \eta_j B)^2 + mA^2 \leq \eta_j \leq \sqrt{4m/3}$ by the remark below Proposition 8.1. If $m = 1$, then $\mathbb{Z}[\sqrt{-m}] = \mathbb{Z}[\sqrt{-1}]$ is a PID and $\eta_j = 1$. Then, since $\tau_j A - \eta_j B - A\sqrt{-m}$ is a unit, $I_j = I_k$.

Suppose $m \geq 2$. Then $A = 0$ and we have $0 < \eta_j B^2 \leq 1$, and so $\eta_j = |B| = 1$. Hence $I_j = I_k$, which completes the proof of the case $-m \equiv 2, 3 \pmod 4$.

Now, consider the case $-m \equiv 1 \pmod 4$. We note that b is odd in Proposition 8.1. Let $M = \frac{1+\sqrt{-m}}{2}$. Then, each ideal I_i, $1 \le i \le k$, is equivalent to $[\eta_i, \tau_i + M]$, and we can show that $\eta_i \le \sqrt{m/3}$ from the conditions of Proposition 8.1. Also, any principal ideal is equivalent to $[1, M]$. Similarly, we have, for each i,

$$\begin{aligned} I_i &= \frac{y_i + z_i\sqrt{-m}}{x_i}[\eta_i, \tau_i + M] \\ &= [\frac{\eta}{x_i}(y_i + z_i M), \frac{y_i\tau_i}{x_i} - \frac{(1+m)z_i}{4}x_i + \frac{z_i(1+\tau_i)+y_i}{x_i}M] \\ &= [\alpha_i + \beta_i M, X_i + Y_i M] \end{aligned}$$

where $\alpha_i = \eta_i y_i / x_i$, $\beta_i = \eta_i z_i / x_i$, $X_i = (\alpha_i \tau_i - \frac{1+m}{4}\beta_i)/\eta_i$ and $Y_i = (\alpha_i + \beta_i(1+\tau_i))/\eta_i$.

Also,

$$\|I_i\| = \begin{vmatrix} \alpha_i & \beta_i \\ X_i & Y_i \end{vmatrix} = \frac{1}{\eta_i}({\alpha_i}^2 + \alpha_i\beta_i + \frac{1+m}{4}{\beta_i}^2).$$

By repeating the same argument, we can show that for some j, we have

$$\begin{aligned} \alpha_j + \beta_j M &= \frac{\alpha_k + \beta_k M}{\eta_k}((1+\tau_j)A - \eta_j B - AM) \\ &= (\alpha_k + \beta_k M)((1+\tau_j)A - \eta_j B - AM). \end{aligned} \tag{8.4}$$

Since $\|I_j\| \le \|I_k\|$, we obtain

$$((1+\tau_j)A - \eta_j B - \frac{1}{2}A)^2 + \frac{m}{4}A^2 \le \eta_j \le \sqrt{\frac{m}{3}}.$$

If $m = 3$, then $(1+\tau_j)A - \eta_j B - AM$ is a unit and I_j is also principal. Thus, we have $I_k = I_j$. And, if $m \ge 7$, then $A = 0$, so $\eta_j = |B| = 1$. Hence, we have $I_k = I_j$, which completes the proof. $\square$

Proof of Theorem 8.2. We suppose that S is an exact covering system with distinct moduli and I_k is an ideal of the largest norm. In this proof, we follow the notations and the proof of Theorem 8.1

except for the choice of p. We first consider the case when $-m \equiv 2, 3 \pmod 4$. We choose $p > 0$ such that for all the pairs $(e_1, e_2) = (1, 0), (0, 1)$ and $(1, 1)$,

$$|\alpha_k \pm p\beta_k| \neq 2|e_1\alpha_k \pm e_2 p\beta_k|. \tag{8.5}$$

Repeating the argument from Theorem 8.1, by (8.3), we have

$$\alpha_j + \beta_j\sqrt{-m} = \frac{\alpha_k + \beta_k\sqrt{-m}}{\eta_k}\left(\tau_j A - \eta_j B - A\sqrt{-m}\right), \tag{8.6}$$

for some $j \neq k$, and some integers A and B. Since

$$\|I_j\| = \frac{\|I_k\|}{\eta_k\eta_j}\left((\tau_j A - \eta_j B)^2 + mA^2\right) \leq \|I_k\|,$$

we have

$$(\tau_j A - \eta_j B)^2 + mA^2 \leq \eta_k\eta_j.$$

It is easy to see that the ideals $[2, \sqrt{-m}]$ and $[2, 1 + \sqrt{-m}]$ satisfy the condition described in Proposition 8.1 when m is even and odd, respectively. Thus, $\eta_i = 1$ or 2 for each $1 \leq i \leq k$. By Theorem 8.1, we can assume that I_k is not principal, i.e., $\eta_k = 2$.

We first assume that $\eta_j = \eta_k = 2$. Since $m \geq 5$ and

$$(\tau_j A - 2B)^2 + mA^2 \leq 4,$$

we see that $A = 0$ and $|B| = 1$, whence $I_j = I_k$ by (8.6). Thus, we can assume that $\eta_k = 2$ and $\eta_j = 1$. Then we have

$$(\tau_j A - B)^2 + mA^2 \leq 2,$$

which implies that $A = 0$ and $|B| = 1$. Thus,

$$\alpha_j + \beta_j\sqrt{-m} = \frac{1}{2}(\alpha_k + \beta_k\sqrt{-m}), \tag{8.7}$$

and so

$$\alpha_j = \frac{1}{2}\alpha_k, \quad \text{and} \quad \beta_j = \frac{1}{2}\beta_k.$$

Also,

$$\|I_j\| = {\alpha_j}^2 + m{\beta_j}^2 = \frac{{\alpha_k}^2 + m{\beta_k}^2}{4} = \frac{1}{2}\|I_k\|.$$

If there exists $t \neq j, k$ such that the term with $i = t$ has a pole of order 2 as $\epsilon \to 0$, then by repeating the same argument, we have $I_t = I_k$ or $I_t = I_j$. Therefore the terms can have a pole of order 2 only when $i = j$ and $i = k$. Now we consider the coefficients of the poles of order 2. Since $(1-z)^{-1}(1-w)^{-1}$ has no such pole, the sum of the coefficients of ϵ^{-2} of the terms when $i = j$ and $i = k$ should be 0 as $\epsilon \to 0$. However, from the proof of Theorem 8.1, the sum of the coefficients is

$$\begin{aligned}
&\frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)}(\alpha_k \pm p\beta_k)}{p\|I_k\|(\alpha_k + p\beta_k)} + \frac{e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_j - \alpha_k\delta_j)}(e_1\alpha_j \pm e_2 p\beta_j)}{p\|I_j\|(\alpha_j + p\beta_j)} \\
&\quad = \frac{1}{p\|I_k\|(\alpha_k + p\beta_k)}\Big\{(\alpha_k \pm p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)} + 2(e_1\alpha_k \pm e_2 p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_j - \alpha_k\delta_j)}\Big\} \\
&\quad \neq 0
\end{aligned}$$

by (8.5), which is a contradiction. Hence, we complete the proof of the case when $-m \equiv 2, 3 \pmod 4$.

Next, if $-m \equiv 1 \pmod 4$, then by (8.4), we have

$$\alpha_j + \beta_j M = \frac{\alpha_k + \beta_k M}{\eta_k}\Big((1+\tau_j)A - \eta_j B - AM\Big)$$

for some $j \neq k$, and some integers A and B. Similarly, we can assume that $\eta_k > 1$ by Theorem 8.1. By Proposition 8.1, we can see that $\eta_k = 3$ if $m = 51, 123, 267$, $\eta_k = 5$ if $m = 115, 235$ and $\eta_k = 7$ if $m = 427$.

Using $\|I_j\| \leq \|I_k\|$, we obtain

$$\Big((1+\tau_j)A - \eta_j B - \frac{1}{2}A\Big)^2 + \frac{m}{4}A^2 \leq \eta_j \eta_k.$$

We first assume that $\eta_j = \eta_k$. Since $m/4 > \eta_k^2$, we have $A = 0$ and $|B| = 1$, whence $I_j = I_k$.

If we assume that $\eta_j = 1$, then since $\eta_k \leq \sqrt{m/3}$, we see that $A = 0$, $B^2 \leq \eta_k$ and

$$\alpha_j + \beta_j M = -\frac{B}{\eta_k}(\alpha_k + \beta_k M).$$

If $m = 51, 123$ or 267, then since $|B| = 1$,

$$\alpha_j = \frac{1}{3}\alpha_k, \quad \text{and} \quad \beta_j = \frac{1}{3}\beta_k.$$

We choose $p > 0$ such that for all the pairs $(e_1, e_2) = (1,0), (0,1)$ and $(1,1)$,

$$|\alpha_k \pm p\beta_k| \neq 3|e_1\alpha_k \pm e_2 p\beta_k|.$$

Then, similarly, the sum of the coefficients of ϵ^{-2} in (8.1) is

$$\frac{1}{p\|I_k\|(\alpha_k + p\beta_k)}\Big\{(\alpha_k \pm p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k - \alpha_k\delta_k)} + 3(e_1\alpha_k \pm e_2 p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_j - \alpha_k\delta_j)}\Big\} \neq 0,$$

which is a contradiction.

Now, if $m = 115, 235$ or 427, then since $|B| = 1$ or 2, we have two possibilities for I_j, say I_{j1} and I_{j2}. And, we have

$$\alpha_{j1} = \frac{1}{\eta_k}\alpha_k, \quad \beta_{j1} = \frac{1}{\eta_k}\beta_k \quad \text{and} \quad \alpha_{j2} = \frac{2}{\eta_k}\alpha_k \quad \beta_{j2} = \frac{2}{\eta_k}\beta_k.$$

Let κ_i with $i = 1, 2$ and e_i with $1 \leq i \leq 4$ to be 0 or 1. Then the sum of the coefficients of ϵ^{-2} in

(8.1) is

$$
\begin{aligned}
&\frac{1}{p\|I_k\|(\alpha_k+p\beta_k)}\Big\{(\alpha_k \pm p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k-\alpha_k\delta_k)} + \kappa_1\eta_k(e_1\alpha_k \pm e_2 p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j1}-\alpha_k\delta_{j1})} \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{4}\kappa_2\eta_k(e_3\alpha_k \pm e_4 p\beta_k)e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j2}-\alpha_k\delta_{j2})}\Big\} \\
&= \frac{1}{p\|I_k\|(\alpha_k+p\beta_k)}\Big\{\alpha_k\Big(e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k-\alpha_k\delta_k)} + \kappa_1 e_1\eta_k\epsilon^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j1}-\alpha_k\delta_{j1})} + \frac{1}{4}\kappa_2 e_3\eta_k e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j2}-\alpha_k\delta_{j2})}\Big) \\
&\qquad + p\beta_k\Big(e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k-\alpha_k\delta_k)} + \kappa_1 e_2\eta_k e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j1}-\alpha_k\delta_{j1})} + \frac{1}{4}\kappa_2 e_4\eta_k e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j2}-\alpha_k\delta_{j2})}\Big)\Big\}
\end{aligned}
\tag{8.8}
$$

Since for $(s,t)=(1,3)$ and $(2,4)$,

$$
e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_k-\alpha_k\delta_k)} + \kappa_1 e_s\eta_k e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j1}-\alpha_k\delta_{j1})} + \frac{1}{4}\kappa_2 e_t\eta_k e^{\frac{2\pi i}{\|I_k\|}(\beta_k\gamma_{j2}-\alpha_k\delta_{j2})} \neq 0,
$$

we can take p so that both sides of (8.8) are nonzero, which completes the proof. $\square$

Proof of Theorem 8.3. We adopt the same notation and use a similar argument from Theorem 8.1. We can assume that $\|I_1\| \leq \cdots \leq \|I_k\|$. We first consider the case : $m \equiv 2,3 \pmod 4$. Since all the ideals are principal, we put $\eta_i = 1$ and $\tau_i = 0$ for all $1 \leq i \leq k$ in the proof of Theorem 8.1, and also replace $-m$ by m. Then, we have

$$
I_i = \Big[\alpha_i + \beta_i\sqrt{m}, m\beta_i + \alpha_i\sqrt{m}\Big],
$$

with $\alpha_i \geq 0$. Also, we set $r_i = \gamma_i + \delta_i\sqrt{m}$. Now, we define

$$
\begin{aligned}
D_1 &= \{1 \leq i \leq k : \alpha_i = 0, \beta_i > 0\}, \\
D_2 &= \{1 \leq i \leq k : \alpha_i > 0, \beta_i = 0\}, \\
D_3 &= \{1 \leq i \leq k : \alpha_i > 0, \beta_i > 0, {\alpha_i}^2 - m{\beta_i}^2 > 0\}, \\
D_4 &= \{1 \leq i \leq k : \alpha_i > 0, \beta_i > 0, {\alpha_i}^2 - m{\beta_i}^2 < 0\}, \\
D_5 &= \{1 \leq i \leq k : \alpha_i > 0, \beta_i < 0, {\alpha_i}^2 - m{\beta_i}^2 > 0\},
\end{aligned}
$$

$$D_6 = \{1 \leq i \leq k : \alpha_i > 0, \beta_i < 0, {\alpha_i}^2 - m{\beta_i}^2 < 0\}.$$

Then, for each $1 \leq i \leq k$, we have exactly one l, $1 \leq l \leq 6$, such that $i \in D_l$. Also, for each i, we have the redefined functions

$$\lambda_i(t) = \left\lceil \frac{-m\beta_i t - \gamma_i}{\alpha_i} \right\rceil, \quad \mu_i(t) = \left\lceil \frac{-\alpha_i t - \delta_i}{\beta_i} \right\rceil, \quad \mu_i'(t) = \left\lfloor \frac{-\alpha_i t - \delta_i}{\beta_i} \right\rfloor$$

Using the argument from Theorem 8.1, we obtain

$$\begin{aligned}
\frac{1}{1-z}\frac{1}{1-\omega} = &\sum_{i \in D_1} \frac{z^{\gamma_i} w^{\delta_i + \beta_i \left\lfloor \frac{-\delta_i}{\beta_i} \right\rfloor}}{(1 - z^{m\beta_i})(1 - w^{\beta_i})} + \sum_{i \in D_2} \frac{z^{\gamma_i + \alpha_i \left\lfloor \frac{-\gamma_i}{\alpha_i} \right\rfloor} w^{\delta_i}}{(1 - z^{\alpha_i})(1 - w^{\alpha_i})} \\
&+ \sum_{i \in D_3} \frac{z^{\gamma_i}\omega^{\delta_i}}{1 - z^{\alpha_i}\omega^{\beta_i}} \Bigg\{ \frac{1}{1 - w^{\|I_i\|}} \sum_{t=0}^{\alpha_i - 1} z^{m\beta_i t + \alpha_i \lambda_i(t)} \omega^{t\alpha_i + \beta_i \lambda_i(t)} \\
&\qquad\qquad + \frac{1}{1 - z^{\|I_i\|}} \sum_{t=-\beta_i}^{-1} z^{m\beta_i t + \alpha_i \mu_i(t)} \omega^{t\alpha_i + \beta_i \mu_i(t)} \Bigg\} \\
&+ \sum_{i \in D_4} \frac{z^{\gamma_i}\omega^{\delta_i}}{1 - z^{\alpha_i}\omega^{\beta_i}} \Bigg\{ \frac{1}{1 - w^{\|I_i\|}} \sum_{t=1-\alpha_i}^{0} z^{m\beta_i t + \alpha_i \lambda_i(t)} \omega^{t\alpha_i + \beta_i \lambda_i(t)} \\
&\qquad\qquad + \frac{1}{1 - z^{\|I_i\|}} \sum_{t=1}^{\beta_i} z^{m\beta_i t + \alpha_i \mu_i(t)} \omega^{t\alpha_i + \beta_i \mu_i(t)} \Bigg\} \\
&+ \sum_{i \in D_5} \frac{z^{\gamma_i}\omega^{\delta_i}}{1 - z^{\alpha_i}\omega^{\beta_i}} \Bigg\{ \frac{1}{1 - w^{\|I_i\|}} \sum_{t=0}^{\alpha_i - 1} z^{m\beta_i t + \alpha_i \lambda_i(t)} \omega^{t\alpha_i + \beta_i \lambda_i(t)} \\
&\qquad\qquad - \frac{z^{\alpha_i}\omega^{\beta_i}}{1 - z^{\|I_i\|}} \sum_{t=-\beta_i+1}^{0} z^{m\beta_i t + \alpha_i \mu_i'(t)} \omega^{t\alpha_i + \beta_i \mu_i'(t)} \Bigg\} \\
&+ \sum_{i \in D_6} \frac{z^{\gamma_i}\omega^{\delta_i}}{1 - z^{\alpha_i}\omega^{\beta_i}} \Bigg\{ \frac{1}{1 - w^{\|I_i\|}} \sum_{t=1-\alpha_i}^{0} z^{m\beta_i t + \alpha_i \lambda_i(t)} \omega^{t\alpha_i + \beta_i \lambda_i(t)} \\
&\qquad\qquad - \frac{z^{\alpha_i}\omega^{\beta_i}}{1 - z^{\|I_i\|}} \sum_{t=0}^{\beta_i - 1} z^{m\beta_i t + \alpha_i \mu_i'(t)} \omega^{t\alpha_i + \beta_i \mu_i'(t)} \Bigg\}.
\end{aligned}$$

Here, we note that

$$\sum_{i\in D_1} \frac{z^{\gamma_i} w^{\delta_i+\beta_i\left\lfloor \frac{-\delta_i}{\beta_i}\right\rfloor}}{(1-z^{m\beta_i})(1-w^{\beta_i})} + \sum_{i\in D_2} \frac{z^{\gamma_i+\alpha_i\left\lfloor \frac{-\gamma_i}{\alpha_i}\right\rfloor} w^{\delta_i}}{(1-z^{\alpha_i})(1-w^{\alpha_i})}$$
$$= \sum_{i\in D_1} \frac{z^{\gamma_i}\omega^{\delta_i}}{1-\omega^{\beta_i}} \frac{1}{1-z^{\|I_i\|}} \sum_{t=0}^{\beta_i-1} z^{m\beta_i t}\omega^{\beta_i\mu_i(t)} + \sum_{i\in D_3} \frac{z^{\gamma_i}\omega^{\delta_i}}{1-z^{\alpha_i}} \frac{1}{1-w^{\|I_i\|}} \sum_{t=0}^{\alpha_i-1} z^{\alpha_i\lambda_i(t)}\omega^{t\alpha_i}.$$

Now, we repeat the same argument regarding poles of order 2 by letting

$$z = (1-\epsilon)e^{2\pi i\frac{\beta_k}{\|I_k\|}}, \qquad w = (1-p\epsilon)e^{-2\pi i\frac{\alpha_k}{\|I_k\|}} \quad \text{and} \quad \epsilon \to 0,$$

where p is a nonzero number such that $\alpha_k \pm p\beta_k \neq 0$. Then, for some $j \neq k$, we have

$$\alpha_j + \beta_j\sqrt{m} = (\alpha_k + \beta_k\sqrt{m})(-B - A\sqrt{m}),$$

where A, B are integers. Since

$$\|I_j\| = \|I_k\||B^2 - mA^2| \leq \| I_k\|,$$

$|B^2 - mA^2| = 1$, which implies that $-B - A\sqrt{m}$ is a unit. Hence, we have $I_j = I_k$.

Now, consider the case $m \equiv 1 \pmod 4$. Similarly, we can let, for each i,

$$I_i = \left[\alpha_i + \beta_i M, \frac{m-1}{4}\beta_i + (\alpha_i+\beta_i)M\right], \quad \text{where} \quad M = \frac{1+\sqrt{m}}{2}.$$

Repeating the same argument again, we obtain

$$\alpha_j + \beta_j\sqrt{m} = (\alpha_k + \beta_k\sqrt{m})(A - B - AM),$$

for some j and integers A, B. Similarly, since the norm of $A - B - AM$ is 1, it is a unit. Hence, $I_k = I_j$, which completes the proof. □

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