ARITHMETIC OF PARTITION FUNCTIONS AND q-COMBINATORICS

BY

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DISSERTATION

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Abstract

Integer partitions play important roles in diverse areas of mathematics such as q-series, the theory of modular forms, representation theory, symmetric functions and mathematical physics. Among these, we study the arithmetic of partition functions and q-combinatorics via bijective methods, q-series and modular forms. In particular, regarding arithmetic properties of partition functions, we examine partition congruences of the overpartition function and cubic partition function and inequalities involving t-core partitions. Concerning q-combinatorics, we establish various combinatorial proofs for q-series identities appearing in Ramanujan's lost notebook and give combinatorial interpretations for third and sixth order mock theta functions.

To my parents

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Chapter 1

Introduction

1.1 Overview

After L. Euler originated the study of integer partitions, many great mathematicians such as P.A. McMahon, J.J. Sylvester, S. Ramanujan, and P. Erdös developed the theory of partitions. A *partition* of n is a weakly decreasing sequence of positive integers for which the sum is n, and we define the partition function $p(n)$ as the number of partitions of n. Surprisingly, this simple combinatorial object plays important roles in diverse areas of mathematics such as q-series, the theory of modular forms, representation theory, symmetric functions and mathematical physics. Among these, this thesis concerns the role of partitions in q-series and modular forms.

Since there was no prior reason to expect nice arithmetic properties of $p(n)$, Ramanujan's discovery of his famous congruences,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$

and

$$p(11n+6) \equiv 0 \pmod{11},$$

were ground-breaking results. Motivated by Ramanujan's congruences, there have been many studies on the arithmetic properties of the partition function or its variants. Since the generating function for $p(n)$ is essentially a modular form [6], the theory of partitions is naturally related to modular forms. Indeed, the theory of modular forms is a very powerful tool to investigate the arithmetic of $p(n)$ for the following reasons: (1) the space of modular forms is a vector space, (2) there are nice combinatorial operators such as Hecke operators, and (3) we know explicit bounds for the Fourier coefficients of certain types of modular forms. For example, in [97], K. Ono showed that there are infinitely many congruences for the partition function modulo every prime $p \geq 5$ by applying the theory of modular forms.

On the other hand, ever since Sylvester realized the importance of proving partition identities bijectively, finding a bijective or combinatorial proof for a partition identity or a q-series identity has been one of the main themes in the theory of partitions. By q-series, we mean a sum with summands containing expressions of the type $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, where we understand $(a;q)_0 := 1$ and $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n,\ |q|<1$ [27]. Since the building block of q-series, $(a;q)_n$, is a generating function for restricted partitions, it is not surprising that there is a close relation between q-series and partitions. Though there have been many improvements on combinatorial methods to prove q-series identities, it has still lagged behind the development of analytic methods. Therefore, it is important to provide more combinatorial arguments, since the combinatorial proofs are more illustrative and give more information on the identities, which enable us to generalize the identities or to prove new identities. Moreover, combinatorial proofs for q-series identities often lead to new information on partitions.

1.2 Arithmetic of partition functions

Even though there has been a dramatic improvement in our knowledge of $p(n)$ modulo primes ≥ 5, we still do not know well the behavior of $p(n)$ modulo 2 or 3. We even do not know whether, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{|\{n \leq x : p(n) \equiv 0 \pmod{2}\}|}{x^{\frac{1}{2}+\epsilon}} = \infty$$

or not, though the conjecture is that $\lim_{x\to\infty} \frac{|\{n\leq x : p(n)\equiv 0 \pmod{2}\}|}{x} = \frac{1}{2}$ [100]. The best known result in this direction is due to Serre [94, Appendix]. In this sense, finding partition congruences modulo 2 and 3 are always interesting problems and systematic approaches are desired. For example, K. Mahlburg [90] conjectured that $\overline{p}(n) \equiv 0 \pmod{2^k}$ for almost all integers n, where $\overline{p}(n)$ is the number of overpartitions of n, and k is a fixed positive integer. An *overpartition* of n is a partition in which the first occurrence of a number may be overlined. It is shown that this conjecture is true up to $k = 6$ by Mahlburg [90]. In Chapter 2, we will show this conjecture holds up to $k = 7$, and the proof given involves theta function identities, Dirichlet L-functions, and the arithmetic of quadratic forms.

To explain the Ramanujan congruences combinatorially, F. Dyson introduced the rank of a partition, which can explain the congruences modulo 5 and 7. Motivated by Dyson's rank, G.E. Andrews and F. Garvan defined the crank of a partition, which explains all three Ramanujan congruences [20], and F. Garvan, D. Kim and D. Stanton also introduced a new crank, which also explains all three Ramanujan congruences [53]. Therefore, it has been a natural and important question to seek rank or crank analogs once we have Ramanujan-type congruences. In Chapter 3, we will define Andrews and Garvan type cranks for certain types of partition functions, which explain the cubic partition congruences modulo 3. Generating functions for Andrews and Garvan type cranks are often essentially modular forms. In light of [91], we will also show that the crank for the cubic partition can explain infinitely many congruences of cubic partitions by using the theory of modular forms. We also define Garvan, Kim and Stanton type cranks to explain the congruences modulo 3 for linear combinations of mock theta functions in Chapter 7.

A partition is said to be a *p-core* if there are no hook numbers that are multiples of p. Due to its connection to representation theory and its independent interest, the arithmetic of the p-core partition function, $a_p(n)$, has been extensively investigated. Since the generating function for p-core partitions is essentially a modular form of weight $\frac{p-1}{2}$ on $\Gamma_0(p)$, we can apply various techniques in the theory of modular forms. In Chapter 4, by using the circle method, Petersson norm, Deligne's bound for the Fourier coefficients of cusp forms and automorphic L-functions, we give an explicit bound for $a_p(n)$. This answers an open question in the paper of A. Granville and Ono [59]. This result has many applications, such as establishing a lower bound for the number of p-blocks with defect zero in the symmetric group S_n and proving inequalities involving p-core partitions, which extends my previous results [75] dramatically. This also can be used to partly prove D. Stanton's conjecture [112], $a_t(n) \leq a_{t+1}(n)$ for all $n > t+1$.

1.3 Combinatorics of q-series identities

In his lost notebook, Ramanujan recorded many identities involving partial theta functions. For example, we can find the following two identities in [102, p. 10 and p. 28]

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n^2} = \frac{1}{(q;q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}, \tag{1.3.1}$$

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \sum_{n=0}^{\infty} \frac{(-q;q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n}, \tag{1.3.2}$$

where a *partial theta function* is a sum of the form

$$\sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} x^n, \quad |q| < 1.$$

The appearance of partial theta function in q-series identities makes them quite interesting combinatorially, since they indicate what remains after numerous cancellations of certain kinds of partitions. The above two identities are very interesting, since the left side of (1.3.1) is a generating function for *stacks with summits*, which were introduced by Andrews [13], and (1.3.2) can be used to evaluate certain Dirichlet L-function as in [89]. In Chapter 5, we establish combinatorial proofs for q-series identities in [102] including (1.3.2).

In Chapter 6, we introduce the subpartition and give properties of subpartitions. In particular, we establish combinatorial proofs for q-series identities including (1.3.1) and we find new q-series identities by using the notion of a subpartition of which analytic proofs remain open.

In his famous last letter to Hardy, Ramanujan introduced 17 examples of mock theta functions without giving an explicit definition. These functions have numerous partition theoretic implications. For example, the coefficients of a third order mock theta function $f(q)$ defined by

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_\infty^2},$$

can be interpreted as rank differences. Even equipped with harmonic Maass forms [99], this huge area of combinatorics of mock theta functions remains to be explored. In this direction, we give combinatorial interpretations of mock theta functions as generating functions for certain types of partitions and study their arithmetic properties in Chapter 7.

Chapter 2

Overpartition function mod 128

2.1 Introduction and statement of results

An *overpartition* of n is a weakly decreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. Let $\overline{p}(n)$ be the number of overpartitions of an integer n. For convenience, define $\overline{p}(0) = 1$. For example, $\overline{p}(3) = 8$ because there are 8 overpartitions of 3:

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

We observe that the overlined parts form a partition into distinct parts and that the unoverlined parts form an ordinary partition. Thus, the generating function for overpartitions is

$$\overline{P}(q) = \sum_{n\geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Here we use the following standard q-series notation:

$$\begin{aligned}
&(a;q)_0 := 1,\\
&(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \geq 1,
\end{aligned}$$

and

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n,\ |q| < 1.$$

The overpartition function was introduced in the work of S. Corteel and J. Lovejoy [49] and it has been used to interpret identities arising from basic hypergeometric series.

S. Treneer [115] showed that the coefficients of a wide class of weakly holomorphic modular forms have

[1]The content of this chapter is largely taken from my paper [73].

infinitely many congruence relations for powers of every prime p except 2 and 3. For example, Treneer showed that $\overline{p}(5l^3n) \equiv 0 \pmod 5$ for all n which are coprime to l, where l is a prime such that $l \equiv -1 \pmod 5$. However, much less is known modulo 2 and 3. For powers of 2, two different approaches have been used. One method is to find the generating function for an arithmetic progression by using q-series identities. For example, the identity

$$\sum_{n\geq 0} \overline{p}(8n+7)q^n = 64\frac{(q^2,q^2)_\infty^{22}}{(q,q)_\infty^{23}}$$

implies that $\overline{p}(8n+7) \equiv 0 \pmod{64}$. For more information, see J.-F. Fortin, P. Jacob and P. Mathieu [52] and M.D. Hirschhorn and J. Sellers [63]. Another way is to use relations between $\overline{p}(n)$ and the number of representations of n as a sum of squares. In this direction, K. Mahlburg [90] showed that $\overline{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1. This approach uses the fact that the generating function for the overpartition function can be represented by one of Ramanujan's classical theta functions. Here we will follow the method of Mahlburg [90] in order to prove the following.

Theorem 2.1.1. $\overline{p}(n) \equiv 0 \pmod{128}$ *for a set of integers of arithmetic density* 1.

Mahlburg conjectured that for all positive integers k, $\overline{p}(n) \equiv 0 \pmod{2^k}$ for almost all integers n. The method here, like Mahlburg's method, relies on an ad-hoc argument, and therefore seems unlikely to generalize to arbitrary powers of 2. In general, 2-adic properties of coefficients of modular form of half-integral weight are somewhat mysterious. For example, it has long been conjectured that

$$\frac{|\{n \leq x : p(n) \equiv 0 \pmod 2\}|}{x} \sim \frac{1}{2},$$

where $p(n)$ is the number of ordinary partitions of n. This stands in contrast to the behavior exhibited by $\overline{p}(n)$.

We will conclude this chapter with the following explicit example.

Proposition 2.1.2.

$$\overline{p}(10672200n + 624855) \equiv 0 \pmod{128}.$$

2.2 Proofs of Theorem 2.1.1 and Proposition 2.1.2

Let

$$\theta(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$\psi(q) = \sum_{n=0}^{\infty} q^{(n^2+n)/2},$$

and

$$\varphi(q) = \sum_{n=1}^{\infty} q^{n^2}.$$

Then, the coefficients $r_k(n)$ of $\theta(q)^k = \sum_{n\geq 0} r_k(n)q^n$ are the number of representations of n as the sum of k squares, where different orders and signs are counted as different. Similarly, the coefficients of $\varphi(q)^k = \sum_{n\geq 0} c_k(n)q^n$ are the number of representations of $n = n_1^2 + \cdots + n_k^2$ where each n_i is a positive integer.

From Mahlburg's paper [90, equation (5)], we have

$$\overline{P}(q) = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} (-1)^{n+k} c_k(n) q^n. \tag{2.2.1}$$

Reducing this expression modulo 128 we obtain

$$\begin{aligned}\overline{P}(q) \equiv& 1 + 2\sum_{n=1}^{\infty}(-1)^{n+1}c_1(n)q^n + 4\sum_{n=1}^{\infty}(-1)^{n}c_2(n)q^n + 8\sum_{n=1}^{\infty}(-1)^{n+1}c_3(n)q^n \\ &+ 16\sum_{n=1}^{\infty}(-1)^{n}c_4(n)q^n + 32\sum_{n=1}^{\infty}(-1)^{n+1}c_5(n)q^n + 64\sum_{n=1}^{\infty}(-1)^{n}c_6(n)q^n \pmod{128}.\end{aligned}$$

We will show that in each of the six sums, the coefficient of q^n is zero modulo 128 for a set of arithmetic density 1.

The following Lemma and its proof summarize results of Mahlburg [90].

Lemma 2.2.1. *For almost all integers n, $c_1(n)$, $c_2(n)$, $r_1(n)$ and $r_2(n)$ are zero. If k is a fixed positive integer, then $c_3(n)$, $c_4(n)$, $r_3(n)$ and $r_4(n)$ are almost always divisible by 2^k.*

Sketch of proof. First, note that by a simple combinatorial argument, we have

$$r_k(n) = 2^k c_k(n) + \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} r_{k-i}(n). \tag{2.2.2}$$

In [90], Mahlburg showed that $c_1(n)$, $c_2(n)$, $r_1(n)$ and $r_2(n)$ are almost always zero.

For $c_3(n)$ and $r_3(n)$, note that $c_3(n) = r_3(n)/8$ for almost all integers n by (2.2.2). By the famous result of Gauss [60], we have

$$r_3(n) = \begin{cases} 12H(-4n), & \text{if } n \equiv 1,\, 2,\, 5,\, 6 \pmod 8, \\ 24H(-n), & \text{if } n \equiv 3 \pmod 8, \\ r_3(n/4), & \text{if } n \equiv 0,\, 4 \pmod 8. \end{cases}$$

Here $H(-n)$ is the Hurwitz class number of positive definite binary quadratic forms. If $2^m \| H(-n)$, then m is at least or equal to the number of distinct odd primes dividing the squarefree part of n. Thus if n has at least l distinct odd primes in its squarefree part, then $r_3(n)$ is divisible by 2^l. By (8) of [90], if $\sigma_l(x)$ is the number of integers $n \leq x$ having at most l distinct odd prime factors, then asymptotically

$$\sigma_l(x) \sim \frac{x(\log\log x)^{(l-1)}}{(l-1)!\log x}. \tag{2.2.3}$$

Since $\sigma_l(x)/x$ tends to 0 as x tends to infinity, for a fixed positive integer k, $r_3(n)$ is divisible by 2^k for almost all n and so the same is true for $c_3(n)$, since $c_3(n) = r_3(n)/8$ for almost all integers n.

For $c_4(n)$ and $r_4(n)$, define $\sigma'(n) = \sum_{d|n, 4\nmid d} d$. Then, by [28], $r_4(n) = 8\sigma'(n)$. Since $r_3(n)$ is almost always divisible by 2^k, by (2.2.2), $c_4(n) \equiv \frac{1}{2}\sigma'(n) \pmod{2^k}$ for almost all integers n. By (11) of [90], we have

$$\sigma'(n) = C \cdot \sum_{i=0}^{a_1} p_1^i \cdots \sum_{i=0}^{a_m} p_m^i, \tag{2.2.4}$$

where $n = 2^{a_0} p_1^{a_1} \cdots p_m^{a_m}$ and $C = 1$ or 3 according to $a_0 = 0$ or not. Let $w(n) =$ the number of distinct prime factors of n with odd exponents. Then, by (2.2.4), we have $2^{w(n)} | \sigma'(n)$. Thus if there are at least l distinct odd primes with odd exponent in the factorization of n, then $2^l | \sigma'(n)$. Since the complement of this set is the set of integers whose squarefree parts have at most l distinct odd prime factors, by (2.2.3), for a fixed positive integer k, $2^k | \sigma'(n)$ for almost all integers n. This completes the proof.

□

By Lemma 2.2.1, it remains to show that $c_5(n) \equiv 0 \pmod 4$ and $c_6(n) \equiv 0 \pmod 2$ for almost all integers n. Let us show that $c_5(n) \equiv 0 \pmod 4$ for almost all integers n.

First, note that

$$\varphi^5(q) \equiv \varphi^2(q^2)\varphi(q) \pmod 4. \tag{2.2.5}$$

Let $\varphi^2(q^2)\varphi(q) = \sum_{n=1}^{\infty} R(n)q^n$. Then $R(n)$ is the number of representations of n of the form $n = x^2 + 2y^2 + 2z^2$ where x, y and z are positive integers and different orders are counted as different. It suffices to show that $R(n)$ is divisible by 4 for almost all integers n.

Before going further, we need the following lemma.

Lemma 2.2.2. *Let $r_{1,2}(n)$ be the number of representations of $n = x^2 + 2y^2$, where x and y are integers. Then, we have $r_{1,2}(n) = 0$ for almost all integers n.*

Since the proof is very similar to a proof by E. Landau [84] that $r_2(n)$ is almost always 0, and we will follow the idea of this proof as given in G.H. Hardy's book *Ramanujan* [61, Sect. 4.5 and Sect. 4.6], we give only a very brief sketch here. (As G. Hardy indicated, the idea of the proof is very similar to the proof of the prime number theorem.)

Sketch of proof. By [28, Theorem 3.7.3], we have

$$r_{1,2}(n) = 2(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)),$$

where $d_{j,8}(n)$, $j = 1, 3, 5, 7$, is the number of positive divisors d of n such that $d \equiv j \pmod 8$.

Thus $r_{1,2}(n) > 0$ if and only if $n = 2^a\mu\nu^2$, where μ is a product of primes congruent to 1 or 3 $\pmod 8$ and ν is a product of primes congruent to 5 or 7 $\pmod 8$. We denote primes $8n+1$, $8n+3$, $8n+5$ and $8n+7$ by q, r, u, and v, respectively.

Define $b(n)$ as 1 when $r_{1,2}(n) > 0$ and 0 otherwise. Then consider the functions:

$$\begin{aligned}
f(s) &= \sum \frac{b(n)}{n^s} = \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1-u^{-2s}} \prod_v \frac{1}{1-v^{-2s}}, \\
\zeta(s) &= \frac{1}{1-2^{-s}} \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1-u^{-s}} \prod_v \frac{1}{1-v^{-s}}, \\
L(s,\chi) &= \sum \frac{\chi(n)}{n^s} = \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1-r^{-s}} \prod_u \frac{1}{1+u^{-s}} \prod_v \frac{1}{1+v^{-s}},
\end{aligned}$$

where $\chi(n)$ is a Dirichlet character of conductor 8. Thus we have

$$f(s)^2 = \xi(s)\zeta(s)L(s,\chi),$$

where $\xi(s) = (1-2^{-s})^{-1}\prod_u(1-u^{-2s})^{-1}\prod_v(1-v^{-2s})^{-1}$.

It is well known that neither $\zeta(s)$ nor $L(s,\chi)$ vanishes in a region D, stretching to the left of $\sigma = 1$, of type

$$\sigma > 1 - \frac{B}{\{\log(|t|+2)\}^A},$$

where, as usual, $s = \sigma + it$ [85]. Finally, note that $\zeta(s)$ and $L(s,\chi)$ are $O((\log|t|)^A)$, as $|t|$ tends infinity in D and that $\xi(s)$ has no zeros for $\sigma > \frac{1}{2}$. It follows that $f(s) = (s-1)^{1/2} g(s)$, where $g(s)$ is analytic in D and $g(1) = (L(1,\chi)\xi(1))^{1/2}$. If $B(x) = \sum_{n\le x} b(n)$, then $B(x)$ is the number of representable numbers up to x, and we can approximate $B(x)$ by examining the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}ds$$

for $c > 1$. We can transform the path of integration into a path, stretching to the left of $\sigma = 1$ in the zero-free region D. By approximating this integral, we can conclude that

$$B(x) = \sum_{n\le x} b(n) \ll \frac{x}{\sqrt{\log x}}.$$

Therefore, $b(n) = 0$ for almost all n.

$\square$

Let $R(n,Q)$ be the number of essentially distinct representations of n by the quadratic form $Q = ax^2 + by^2 + cz^2$ where a, b and c are positive integers and x, y and z are integers. Let $r(n,Q)$ be the number of essentially distinct primitive representations of n by Q. Recall that essentially distinct representation means that different orders and signs are counted as the same and primitive representation means that the greatest common divisor of x, y and z is 1.

Then, by [101], we have

$$R(n,Q) = \sum_{d^2|n} r(\frac{n}{d^2}, Q). \tag{2.2.6}$$

Note that in $R(n, x^2+2y^2+2z^2)$, x, y or z could be 0 and a change of order between y and z is considered as the same. On the other hand, $R(n)$ is the number of representations of n of the form $n = x^2 + 2y^2 + 2z^2$, where x, y and z are positive integers and different orders are counted as different. By Lemma 2.2.1 and Lemma 2.2.2, we know that $R(n, 2x^2+2y^2)$, $R(n, x^2)$, $R(n, x^2+2y^2)$ and $R(n, 2x^2)$ are almost always 0. Therefore, we can conclude that $R(n) = 2R(n, x^2+2y^2+2z^2) - R(n, x^2+4y^2)$ for almost all n. By Lemma 2.2.2 again, we have

$$R(n) = 2R(n, x^2+2y^2+2z^2) \text{ for almost all integers } n, \tag{2.2.7}$$

because $x^2 + 4y^2 = x^2 + (2y)^2$.

For future use, we recall Theorem 86 of W. Jones [71].

Lemma 2.2.3. *Let Q be a ternary form in a genus consisting of a single class. Let d be the determinant of Q and Ω be the greatest common divisor of the two-rowed minor determinants of Q. Then, for all $n \neq \pm 1$ which are coprime to $2d$, we have*

$$r(n, Q) = h(-4nd/\Omega^2)2^{-t(d/\Omega^2)}\rho \text{ or } 0,$$

where $h(D)$ is the discriminant D class number, $t(\omega)$ is the number of odd prime factors of ω and $\rho = 1/8$, $1/6$, $1/4$, $1/3$, $1/2$, 1 or 2.

Remark. *For the criterion for the value of ρ, look at [71, Theorem 86].*

Let Q be the ternary quadratic form $x^2 + 2y^2 + 2z^2$. Then Q is a ternary form in a genus consisting of a single class. Thus, we can use Lemma 2.2.3 whenever n is an odd integer.

Therefore, for odd integer n, we have

$$r(n, Q) = h(-4n)\rho \text{ or } 0.$$

Let $t(n)$ be the number of odd primes dividing the squarefree part of n. Then, by simple genus theory, recall that the exponent of 2 in $h(-n)$ is greater than or equal to $t(n) - 1$. Thus if n has at least 5 distinct odd primes in its squarefree part, then $r(n, Q)$ is divisible by 2. Thus, by (2.2.6) and (2.2.7), we see that $R(n)$ is divisible by 4 for such n. By (2.2.3), $R(n) \equiv 0 \pmod 4$ for almost all n. Thus for odd integer n, we are done.

Let us consider the case $n \equiv 2 \pmod 4$. Since $n = x^2 + 2y^2 + 2z^2$ and n is divisible by 2, x must be an even number. Thus, we can write $n' = 2x'^2 + y^2 + z^2$, where $n = 2n'$ and $x = 2x'$. Set $Q = x^2 + y^2 + 2z^2$. Then Q is a ternary form in a genus consisting of a single class. Since n' is odd, the result follows again from Lemma 2.2.3.

For the case $n \equiv 0 \pmod 4$, we need the following identities.

$$\theta(q) = \theta(q^4) + 2q\psi(q^8), \tag{2.2.8}$$

$$\theta^2(q) = \theta^2(q^2) + 4q\psi^2(q^4). \tag{2.2.9}$$

Define U by

$$U\sum_{n\geq 0} a(n)q^n = \sum_{n\geq 0} a(4n)q^n.$$

Note that $\varphi(q) = \frac{1}{2}(\theta(q) - 1)$. Thus we have

$$\begin{aligned}
\varphi^2(q^2)\varphi(q) &= (\frac{1}{2}(\theta(q^2) - 1))^2\frac{1}{2}(\theta(q) - 1) \\
&= \frac{1}{8}(\theta^2(q^2) - 2\theta(q^2) + 1))(\theta(q) - 1) \\
&= \frac{1}{8}\{\theta^2(q^2)\theta(q) - 2\theta(q^2)\theta(q) + \theta(q) - \theta^2(q^2) + 2\theta(q^2) - 1\}.
\end{aligned}$$

Therefore, the coefficient of q^n in $\varphi^2(q^2)\varphi(q)$ is almost always the same as the coefficient of q^n in $\frac{1}{8}\theta^2(q^2)\theta(q)$ because the coefficients of the other terms are almost always 0 by Lemma 2.2.1 and Lemma 2.2.2.

Let $\theta^2(q^2)\theta(q) = \sum_{n\geq 0} R'(n)q^n$. Then, by (2.2.5), it suffices to show that $R'(n) \equiv 0 \pmod{32}$ for almost all integers n. By (2.2.8) and (2.2.9), we have

$$\begin{aligned}
U\theta^2(q^2)\theta(q) &= U((\theta^2(q^4) + 4q^2\psi^2(q^8))(\theta(q^4) + 2q\psi(q^8)) \\
&= U(\theta^3(q^4) + 2q\theta^2(q^4)\psi(q^8) + 4q^2\theta(q^4)\psi^2(q^8) + 8q^3\psi^3(q^8)) \\
&= \theta^3(q), \\
U\theta^3(q) &= \theta^3(q).
\end{aligned}$$

From this, we have $R'(4^\lambda n) = r_3(4^{\lambda-1}n) = \cdots = r_3(n)$, where n is not divisible by 4. Thus by Lemma 2.2.1, $c_5(n) \equiv 0 \pmod 4$ for almost all integers $n \equiv 0 \pmod 4$. From the last three cases, we conclude that $c_5(n) \equiv 0 \pmod 4$ for almost all integers n.

Finally, we need to show that $c_6(n)$ is almost always even. To show this, note that

$$\varphi^6(q) \equiv \varphi^3(q^2) \pmod 2.$$

Then,

$$c_6(n) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \text{ is odd}, \\ c_3(n/2) \pmod 2, & \text{if } n \text{ is even}. \end{cases} \tag{2.2.10}$$

Therefore, by Lemma 2.2.1 again, we conclude that $c_6(n)$ is almost always even. This completes the proof of Theorem 2.1.1.

$\square$

Now we turn to the proof for Proposition 2.1.2.

Proof of Proposition 2.1.2. When $n \equiv 7 \pmod 8$, $c_1(n), c_2(n), c_3(n)$ and $c_5(n)$ are zero. Moreover, $c_6(n)$ is always even by (2.2.10). Thus, if $c_4(n)$ is divisible by 8, then $\overline{p}(n) \equiv 0 \pmod{128}$ for such n. Recall the proof of Lemma 2.2.1. We have $c_4(n) = \frac{1}{2}\sigma'(n)$, where $\sigma'(n) = \sum_{d|n, 4\nmid d} d$. By (2.2.4), to guarantee that $\frac{1}{2}\sigma'(n)$ is divisible by 16, n must have at least four distinct odd prime factors with odd exponents.

We solve the following system of congruences:

$$
\begin{aligned}
n &\equiv 7 \pmod{8},\\
n &\equiv 3 \pmod{3^2},\\
n &\equiv 5 \pmod{5^2},\\
n &\equiv 7 \pmod{7^2},\\
n &\equiv 11 \pmod{11^2}.
\end{aligned}
$$

By a simple calculation, we find that $n \equiv 624855 \pmod{10672200}$. Since $3\|n$, $5\|n$, $7\|n$, $11\|n$ and $n \equiv 7 \pmod 8$, the proof of Theorem 2.1.1 implies that $\overline{p}(10672200n + 624855) \equiv 0 \pmod{128}$. $\square$

Remark. *We can completely determine the residue class of $\overline{p}(n)$ modulo 8. The following classification is given in [74], which answers an open question of Hirschhorn and Sellers [63]. Let n be a nonnegative integer. Then*

$$
\begin{aligned}
\overline{p}(n) &\equiv 2 \pmod{8}, && \textit{if } n \textit{ is a square of an odd number,}\\
\overline{p}(n) &\equiv 4 \pmod{8}, && \textit{if } n \textit{ is a double of a square,}\\
\overline{p}(n) &\equiv 6 \pmod{8}, && \textit{if } n \textit{ is a square of an even number,}\\
\overline{p}(n) &\equiv 0 \pmod{8}, && \textit{otherwise.}
\end{aligned}
$$

Chapter 3

Crank for the cubic partition function

3.1 Introduction and statement of results

In a series of papers ([38], [39], [37]) H.-C. Chan studied congruence properties for the partition function $a(n)$, which is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{3.1.1}$$

This partition function $a(n)$ arises from Ramanujan's cubic continued fraction. We can interpret $a(n)$ as the number of 2-color partitions of n with colors r and b subject to the restriction that the color b appears only in even parts. For example, there are 3 such partitions of 2:

$$2_r,\ \ 2_b,\ \ 1_r + 1_r.$$

Since $a(n)$ is closely related with Ramanujan's cubic continued fraction (see [38] for the relation), we will say that $a(n)$ is the number of cubic partitions of n.

In particular, by using identities for the cubic continued fraction, Chan found a result analogous to "Ramanujan's most beautiful identity" (in the words of G.H. Hardy [103, p. xxxv]), namely,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q)_\infty^6},$$

where $p(n)$ is the number of ordinary partitions of n. Chan's identity is given by

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q;q)_\infty^4 (q^2;q^2)_\infty}.$$

This implies immediately that

$$a(3n+2) \equiv 0 \pmod 3. \tag{3.1.2}$$

To give a combinatorial explanation of the famous Ramanujan partition congruences

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11},$$

G.E. Andrews and F.G. Garvan [20] introduced the crank of a partition. For a given partition λ, the crank $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \ell(\lambda), & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where r is the number of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r and $\ell(\lambda)$ is the largest part in λ.

Let $M(m,n)$ be the number of ordinary partitions of n with crank m. Andrews and Garvan showed that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) x^m q^n = (x-1)q + \frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty}. \tag{3.1.3}$$

Let $M(k,N,n)$ be the number of ordinary partitions of n with crank $\equiv k \pmod N$. In [20] and [54], Andrews and Garvan showed that for all $n \geq 0$,

$$M(i,5,5n+4) = M(j,5,5n+4), \text{ for all } 0 \leq i \leq j \leq 4,$$
$$M(i,7,7n+5) = M(j,7,7n+5), \text{ for all } 0 \leq i \leq j \leq 6,$$
$$M(i,11,11n+6) = M(j,11,11n+6), \text{ for all } 0 \leq i \leq j \leq 10.$$

These identities clearly imply Ramanujan's congruences.

As Chan mentioned in his paper [37], it is natural to seek an analog of the crank of the ordinary partition to give a combinatorial explanation of (3.1.2). In light of (3.1.3), it is natural to conjecture that

$$F(x,q) = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty (xq^2;q^2)_\infty (x^{-1}q^2;q^2)_\infty} \tag{3.1.4}$$

gives an analogous crank for cubic partitions. In Section 3.2, we will review the crank of Andrews and Garvan of the ordinary partition function and after that, by giving a combinatorial interpretation of (3.1.4), we will define a crank analog c_a that is a weighted count of cubic partitions, which we will call a cubic crank. By using basic q-series identities, we will prove our first theorem.

Theorem 3.1.1. *Let $M'(m,N,n)$ be the number of cubic partitions of n with cubic crank $\equiv m \pmod N$. Then, we have*

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3,$$

for all nonnegative integers n.

This immediately implies the following corollary.

Corollary 3.1.2. *For all nonnegative integers n, we have $a(3n+2) \equiv 0 \pmod 3$.*

Let us define c_k as

$$c_k := \begin{cases} \frac{7\cdot 3^n+1}{8}, & \text{if } k \text{ is even,} \\ \frac{5\cdot 3^n+1}{8}, & \text{if } k \text{ is odd.} \end{cases} \tag{3.1.5}$$

In [39], Chan proved the following congruences for cubic partitions.

Theorem 3.1.3 (Theorem 1 in [39])**.** *For all nonnegative n, $a(3^k n + c_k) \equiv \pmod{3^{2\lfloor k/2 \rfloor+1}}$.*

Surprisingly, our cubic crank can explain these congruences partially. To see this, we will prove the following theorem.

Theorem 3.1.4. *For all nonnegative n,*

$$M'(0,3,3^k n+c_k) - M'(1,3,3^k n+c_k) \equiv 0 \pmod{3^{\lfloor k/2 \rfloor+1}}.$$

By (3.2.7), Theorem 3.1.4 implies that

$$M'(0,3,3^k n+c_k) \equiv M'(1,3,3^k n+c_k) \equiv M'(2,3,3^k n+c_k) \pmod{3^{\lfloor k/2 \rfloor+1}}.$$

Moreover, from Theorem 3.1.3, we find that

$$M'(0,3,3^k n+c_k) \equiv M'(1,3,3^k n+c_k) \equiv M'(2,3,3^k n+c_k) \equiv 0 \pmod{3^{\lfloor k/2 \rfloor}}.$$

Therefore, we can see that the cubic crank gives a combinatorial explanation for the following congruences.

$$a(3^k n+c_k) \equiv 0 \pmod{3^{\lfloor k/2 \rfloor+1}},$$

for all nonnegative integers n. Though this cubic crank does not give a full explanation for Theorem 3.1.3, as far as the author's knowledge, this is the first crank which explains infinitely many congruences for a fixed arithmetic progression. In Section 3.3, we will review some basic properties of modular forms. With this equipment, we will prove Theorem 3.1.4 in Section 3.4.

In [91], K. Mahlburg proved that there are infinitely many arithmetic progressions $An+B$ such that

$$M(m, \ell^j, An+B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \le m \le \ell^j - 1$, where $\ell \ge 5$ is a prime and τ, j are positive integers.

By using the theory of modular forms, in Section 3.4, we will prove our third theorem, which is analogous to Mahlburg's result.

Theorem 3.1.5. *There are infinitely many arithmetic progression $An+B$ such that*

$$M'(m, \ell^j, An+B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \le m \le \ell^j - 1$, where $\ell \ge 5$ is a prime and τ, j are positive integers.

3.2 A cubic crank for $a(n)$

We need to introduce some notation and review the definition of the crank of ordinary partitions. After Andrews and Garvan [20], we define that, for a partition λ, $\#(\lambda)$ is the number of parts in λ and $\sigma(\lambda)$ is the sum of the parts of λ with the convention $\#(\lambda) = \sigma(\lambda) = 0$ for the empty partition λ. Let $\mathcal{P}$ be the set of all ordinary partitions and $\mathcal{D}$ be the set of all partitions into distinct parts. We define

$$V = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 \in \mathcal{D}, \text{ and } \lambda_2, \lambda_3 \in \mathcal{P}\}.$$

For $\lambda = (\lambda_1,\ \lambda_2,\ \lambda_3)$, we define the sum of parts s, a weight w, and a crank t, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3), \\ w(\lambda) &= (-1)^{\#(\lambda_1)}, \\ t(\lambda) &= \#(\lambda_2) - \#(\lambda_3). \end{aligned}$$

We say λ is a vector partition of n if $s(\lambda) = n$. Let $N_V(m,n)$ denote the number of vector partitions of n (counted according to the weight w) with crank m, so that

$$N_V(m,n) = \sum_{\substack{\lambda \in V \\ s(\lambda)=n \\ t(\lambda)=m}} w(\lambda).$$

Then, we have

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_V(m,n) x^m q^n = \frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty}. \tag{3.2.1}$$

By putting $x = 1$ in (3.2.1) we find

$$\sum_{m=-\infty}^{\infty} N_V(m,n) = p(n).$$

Andrews and Garvan showed that this vector crank actually gives a crank for the ordinary partitions.

Theorem 3.2.1 (Theorem 1 in [20])**.** *For all $n > 1$, $M(m,n) = N_V(m,n)$.*

Now, we are ready to define a cubic crank for cubic partitions. For a given cubic partition λ, we define λ_r to be a partition that consists of parts with color r and λ_b to be a partition that is formed by dividing each of the parts with color b by 2. The generating function (3.1.4) suggests that it is natural to define a vector crank analog $N_V^a(m,n)$ as

$$N_V^a(m,n) = \sum_{\substack{\lambda_r, \lambda_b \in V \\ s(\lambda_r)+2s(\lambda_b)=n \\ t(\lambda_r)+t(\lambda_b)=m}} w(\lambda_r) w(\lambda_b).$$

Then, we have

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_V^a(m,n) x^m q^n = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty (xq^2;q^2)_\infty (x^{-1}q^2;q^2)_\infty}. \tag{3.2.2}$$

By putting $x = 1$ in (3.2.2), we find

$$\sum_{m=-\infty}^{\infty} N_V^a(m,n) = a(n).$$

From now on, if $\lambda = (1)$, then we will regard λ as an element of V with $s(\lambda) = 1$, and let us define the crank weight $wt(\lambda)$ for $\lambda \in \mathcal{P}$ as

$$wt(\lambda) = \begin{cases} 1, & \text{if } \lambda \neq (1), \\ w(\lambda), & \lambda = ((1), \emptyset, \emptyset),\ (\emptyset, (1), \emptyset) \text{ or } (\emptyset, \emptyset, (1)), \end{cases}$$

and the crank size $cs(\lambda)$ as

$$cs(\lambda) = \begin{cases} c(\lambda), & \text{if } \lambda \neq (1), \\ t(\lambda), & \text{if } \lambda = ((1),\emptyset,\emptyset),\ (\emptyset,(1),\emptyset) \text{ or } (\emptyset,\emptyset,(1)). \end{cases}$$

For a given cubic partition λ, we define a cubic crank $c_a(\lambda)$ as

$$c_a(\lambda) = (wt(\lambda_r)\cdot wt(\lambda_b),\ cs(\lambda_r)+cs(\lambda_b)).$$

For example, here are some $c_a(\lambda)$, where λ is a cubic partition:

$$\begin{aligned} c_a((1_r,1_r,1_r,2_b)) &= (1\cdot 1,-3+1),\ (1\cdot 1,-3-1),\ \text{and } (1\cdot(-1),-3+0), \\ c_a((1_r,1_r,2_r,2_b,2_b)) &= (1\cdot 1,-2-2). \end{aligned}$$

Let $M'(m,n)$ be the number of cubic partitions of n counted according to the weight, so that

$$M'(m,n) = \sum_{cs(\lambda_r)+cs(\lambda_b)=m} wt(\lambda_r)wt(\lambda_b).$$

Since

$$N_V(m,1) = \begin{cases} 1, & \text{if } m = 1 \text{ or } -1, \\ -1, & \text{if } m = 0, \\ 0, & \text{otherwise}, \end{cases}$$

by Theorem 3.2.1, we have

Theorem 3.2.2. *For all $n \geq 1$, we have $M'(m,n) = N_V^a(m,n)$.*

Therefore, we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M'(m,n)x^m q^n = F(x,q). \tag{3.2.3}$$

By an abuse of notation, we will say that $M'(m,n)$ is the number of cubic partitions of n with cubic crank m. Let $M'(m,N,n)$ be the number of cubic partitions of n with cubic crank $\equiv m \pmod N$. Now, we are ready to give a proof for Theorem 3.1.1.

Proof of Theorem 3.1.1. By a simple argument, we have

$$F(\zeta,q)=\frac{(q;q)_\infty(q^2;q^2)_\infty}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\sum_{k=0}^{2}M'(k,3,n)\zeta^k q^n,$$

where ζ is a primitive third root of unity.

To find the coefficient of q^{3n+2} of $F(\zeta,q)$, we multiply the numerator and the denominator by $(q;q)_\infty(q^2;q^2)_\infty$. Then, we have

$$\begin{aligned}F(\zeta,q)&=\frac{(q;q)_\infty^2(q^2;q^2)_\infty^2}{(q^3;q^3)_\infty(q^6;q^6)_\infty} &(3.2.4)\\ &=\frac{(q;q^2)_\infty^2(q^2;q^2)_\infty(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty} &(3.2.5)\\ &=\frac{\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}\right)\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}\right)}{(q^3;q^3)_\infty(q^6;q^6)_\infty}.\end{aligned}$$

For the last equality, we used the Jacobi triple product identity and Jacobi's identity. (See [28, pp. 12 – 14] for the proof of these identities.) Since $n^2\equiv 0$ or $1 \pmod 3$ and $m(m+1)\equiv 0$ or $2 \pmod 3$, the coefficient of q^{3n+2} of $F(\zeta,q)$ is the same as the coefficient of q^{3n+2} of

$$\frac{\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{9n^2}\right)\left(\sum_{m=0}^{\infty}(-1)^{3m+1}(6m+3)q^{9m^2+9m+2}\right)}{(q^3;q^3)_\infty(q^6;q^6)_\infty}. \qquad (3.2.6)$$

Note that the coefficients of (3.2.6) are multiples of 3. Thus, we have

$$\sum_{k=0}^{2}M'(k,3,3n+2)\zeta^k=3N,$$

for some integer N. Since $1+\zeta+\zeta^2$ is a minimal polynomial in $\mathbf{Z}[\zeta]$, we must have

$$M'(0,3,3n+2)\equiv M'(1,3,3n+2)\equiv M'(2,3,3n+2)\pmod 3.$$

This completes the proof of Theorem 3.1.1. □

Recall that

$$a(n)=\sum_{m=-\infty}^{\infty}M'(m,n).$$

Therefore, Theorem 3.1.1 immediately implies Corollary 2.

From (3.2.4), we see that

$$M'(1,3,n) = M'(2,3,n), \quad \text{for all } n \geq 1. \tag{3.2.7}$$

Therefore, by (3.2.5), we arrive at

$$\sum_{n=0}^{\infty} (M'(0,3,n) - M'(1,3,n))\, q^n = \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2}{(q^3;q^3)_\infty (q^6;q^6)_\infty}. \tag{3.2.8}$$

By (3.2.6) and the Jacobi triple product identity, we obtain that

$$\sum_{n=0}^{\infty} (M'(0,3,3n+2) - M'(1,3,3n+2))\, q^n = -3\frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{3.2.9}$$

Moreover, by using [51, (33.124)], we can see that

$$\sum_{n=0}^{\infty} (M'(0,3,9n+8) - M'(1,3,9n+8))\, q^n = -9\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}.$$

These identities illuminate the possibility that there are further congruences modulo powers of 3 for cubic crank differences.

3.3 Preliminary results

This section contains the basic definitions and properties of modular forms that we will use in Section 3.4. For additional basic properties of modular forms, see [98, Chaps. 1, 2, and 3].

Define $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$, and $\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N} \text{ and } c \equiv 0 \pmod{N} \right\}$. For a meromorphic function f on the complex upper half plane $\mathbb{H}$, define the slash operator by

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (cz+d)^{-k} f(\frac{az+b}{cz+d}).$$

Let $\mathcal{M}_k(\Gamma)$ (resp. $S_k(\Gamma)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) of weight k. Let $\mathcal{M}_k(\Gamma_0(N),\chi)$ (resp. $S_k(\Gamma_0(N),\chi)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. For a prime p and a positive integer m, we need to define the Hecke operators T_p, the U_m-operator and the V_m-operator on $\mathcal{M}_k(\Gamma_0(N),\chi)$. If $f(z)$ has a Fourier expansion $f(z) = \sum a(n)q^n$, then

$$f|T_p := \sum \left(a(pn) + \chi(p)p^{k-1}a(\frac{n}{p}) \right) q^n,$$

$$f|U_m := \sum a(mn)q^n = m^{\frac{k}{2}-1} \sum_{v=0}^{m-1} f|_k \begin{pmatrix} 1 & v \\ 0 & m \end{pmatrix},$$

$$f|V_m := \sum a(n)q^{mn}.$$

Recall that the Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = q^{\frac{1}{24}}(q;q)_\infty, \tag{3.3.1}$$

where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$, the upper complex half plane. For a fixed N and integers r_i, a function of the form

$$f(z) := \prod_{\substack{n|N \\ n>0}} \eta(nz)^{r_n}.$$

is called an η-quotient. The following theorem in [93] shows when an η-quotient becomes a modular form.

Theorem 3.3.1. *The η-quotient is in $\mathcal{M}_0(\Gamma_0(N))$ if and only if*

(1) $\sum_{n|N} r_n = 0$,

(2) $\sum_{n|N} nr_n \equiv 0 \pmod{24}$,

(3) $\sum_{n|N} \frac{N}{n} r_n \equiv 0 \pmod{24}$,

(4) $\prod_{n|N} n^{r_n}$ *is a square of a rational number.*

The following theorem in [87] gives the order of the η-quotient f at the cusps c/d of $\Gamma_0(N)$ provided $f \in \mathcal{M}_0(\Gamma_0(N))$.

Theorem 3.3.2. *If the η-quotient $f \in \mathcal{M}_0(\Gamma_0(N))$, then its order at the cusp c/d of $\Gamma_0(N)$ is*

$$\frac{1}{24} \sum_{n|N} \frac{N(d,n)^2 r_n}{(d, N/d)dn}.$$

Recall that if $p|N$ and $f \in \mathcal{M}_0(\Gamma_0(pN))$, then $f|U_p \in \mathcal{M}_0(\Gamma_0(N))$. Also, the following theorem in [57] gives bounds on the order of $f|U(p)$ at cusps of $\Gamma_0(N)$ in terms of the order of f at cusps of $\Gamma_0(pN)$.

Theorem 3.3.3. *Let p be a prime and $\pi(n)$ be the highest power of p dividing n. Suppose that $f \in \mathcal{M}_0(\Gamma_0(pN))$, where $p|N$ and $\alpha = c/d$ is a cusp of $\Gamma_0(N)$. Then,*

$$ord_\alpha f|U_p \geq \begin{cases} \frac{1}{p} ord_{\alpha/p} f, & \text{if } \pi(d) \geq \frac{1}{2}\pi(N), \\ ord_{\alpha/p} f, & \text{if } 0 < \pi(d) < \frac{\pi(N)}{2}, \\ \min_{0 \leq \beta \leq p-1} ord_{(\alpha+\beta)/p} f, & \text{if } \pi(d) = 0. \end{cases}$$

The following eta-quotient $E_{\ell,t}(z)$ will play an important role in our proof. Given a prime $\ell \geq 5$ and a positive integer t, we define

$$E_{\ell,t}(z) = \frac{\eta^{\ell^t}(z)}{\eta(\ell^t z)}.$$

The following lemma summarizes necessary and well-known properties of $E_{\ell,t}(z)$.

Lemma 3.3.4. *The eta-quotient $E_{\ell,t}$ satisfies the following:*

(i) For a prime $\ell \geq 5$,

$$E_{\ell,t}(z) \in \mathcal{M}_{(\ell^t-1)/2}\left(\Gamma_0(\ell^t), \chi_{\ell,t}\right),$$

where $\chi_{\ell,t}(\cdot) = \left(\frac{(-1)^{(\ell^t-1)/2}\ell^t}{\cdot}\right)$ denotes the Legendre-Jacobi symbol,

(ii) $E_{\ell,t}(z)^{\ell^j} \equiv 1 \pmod{\ell^{j+1}}$ for $j \geq 0$,

(iii) $E_{\ell,t}(z)$ vanishes at every cusp a/c with $\ell^t \nmid c$.

The following Theorem 3.3.5 is an integer weight version of Theorem 2.2 of [91].

Theorem 3.3.5. *For $0 \leq i \leq r$, let N_i and k_i be positive integers and let $g_i \in S_{k_i}(\Gamma_1(N_i))$, where the Fourier coefficients of g_i are algebraic integers. If $M \geq 1$, then a positive proportion of primes $p \equiv -1 \pmod{N_1 \cdots N_r M}$ have the property that for every i,*

$$g_i(z)|T_p \equiv 0 \pmod{M}.$$

If $\zeta = \exp(2\pi i/N)$, then for $1 \leq s \leq N-1$, we define the $(0, s)$-Klein form by

$$t_{0,s}(z) = \frac{\omega_s}{2\pi i} \frac{(\zeta^s q; q)_\infty (\zeta^{-s} q; q)_\infty}{(q;q)_\infty^2}, \text{ for } 1 \leq s \leq N-1, \tag{3.3.2}$$

where $\omega_s := \zeta^{s/2}(1 - \zeta^{-s})$.

The following proposition gives a transformation formula under $\Gamma_0(N)$.

Proposition 3.3.6 (Proposition 3.2 in [91], eqn. **K2** (p. 28) in [82])**.** *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$*, then*

$$t_{0,s}(z)|_{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \cdot t_{0,\overline{ds}}(z),$$

where β *is given by* $\exp\left(\frac{cs+(ds-\overline{ds})}{2N} - \frac{cds^2}{2N^2}\right)$.

For certain congruence subgroups, a Klein form is a weakly holomorphic modular form.

Proposition 3.3.7 (Corollary 3.3 of [91])**.** *If* $1 \le s \le N-1$*, then* $t_{0,s}(z) \in \mathcal{M}_{-1}(\Gamma_1(2N^2))$.

3.4 Proof of Theorem 3.1.4 and Theorem 3.1.5

Since we will follow the argument of B. Gordon and K. Hughes [57] for the proof of Theorem 3.1.4, we do not give every detail. Let us define

$$F(z) := \frac{\eta^2(z)\eta^2(2z)\eta(27z)\eta(54z)}{\eta(3z)\eta(6z)\eta^2(9z)\eta^2(18z)}, \tag{3.4.1}$$

$$G(z) := \frac{\eta(9z)\eta(18z)}{\eta(z)\eta(2z)}. \tag{3.4.2}$$

Then, by Theorem 3.3.1, $F(z) \in \mathcal{M}_0(\Gamma_0(54))$, and $G(z) \in \mathcal{M}_0(\Gamma_0(18))$. By Theorem 3.3.2, their orders as modular functions of level 54 at the cusps are as follows.

d	1	2	3	6	9	18	27	54
ord F	5	5	-1	-1	-2	-2	1	1
ord G	-3	-3	0	0	1	1	1	1

Table 3.1: Orders for $F(z)$ and $G(z)$ at the cusps for $\Gamma_0(54)$

Note that $G^i|U_3$, $FG^i|U_3$, and $G(z) \in \mathcal{M}_0(\Gamma_0(18))$. By Theorem 3.3.3, their orders at the cusps are as follows. By comparing the order at the cusps, we can see that $\frac{F|U_3}{G}$ is a holomorphic modular function, i.e.

d	1	2	3	6	9	18
ord G	-1	-1	0	0	1	1
ord $G^i\|U_3 \geq$	$-3i$	$-3i$	i	i	i	i
ord $FG^i\|U_3 \geq$	$\min\{5-3i,-1\}$	$\min\{5-3i,-1\}$	$\frac{i-2}{3}$	$\frac{i-2}{3}$	$\frac{i+1}{3}$	$\frac{i+1}{3}$

Table 3.2: Orders for $G(z)$, $G^i|U_3$, and $FG^i|U_3$ at the cusps for $\Gamma_0(18)$

a constant. Therefore, we can see that

$$F|_{U_3} = -3G.$$

Remark. *This can be proved by an elementary argument by using* (3.2.8) *and* (3.2.9)*.*

By using a similar argument, we can see the following:

$$\begin{aligned}
G|U_3 &= 3G + 9G^2 + 27G^3, \\
G^2|U_3 &= 2G + 33G^2 + 180G^3 + 729G^4 + 1458G^5 + 2187G^6, \\
G^3|U_3 &= G + 30G^2 + 414G^3 + 2916G^4 + 14580G^5 + 48114G^6 \\
&\quad + 118098G^7 + 177147G^8 + 177147G^9, \\
FG|U_3 &= -G, \\
FG^2|U_3 &= G, \\
FG^3|U_3 &= 3G^2 + 9G^3 + 27G^4.
\end{aligned}$$

By using Newton's formula, we obtain for $i \geq 3$, a recurrence formula for $G^i|U_3$,

$$G^i|U_3 = \sigma_1 G^{i-1}|U_3 + \sigma_2 G^{i-2}|U_3 + \sigma_3 G^{i-3}|U_3,$$

where $\sigma_1 = 9G + 27G^2 + 81G^3$, $\sigma_2 = -3G = 9G^2 - 27G^3$, and $\sigma_3 = G + 3G^2 + 9G^3$. Since $FG^i|U_3$ satisfies the same recurrence formula, for all $i \geq 1$, we can write $G^i|U_3$ and $FG^i|U_3$ as linear sums of G^i's, namely,

$$G^i|U_3 = \sum_{j=1}^{\infty} a_{i,j} G^j \quad \text{and} \quad FG^i|U_3 = \sum_{j=1}^{\infty} b_{i,j} G^j, \tag{3.4.3}$$

where $a_{i,j}$ and $b_{i,j}$ are integers. We define a sequence of functions L_k $(k \geq 0)$ inductively, by

$$L_0 := 1, \;\; L_{2k+1} = FL_{2k}|U_3, \;\; \text{and} \;\; L_{2k+2} = L_{2k+1}|U_3.$$

Be (3.4.3), each L_k for $k \geq 1$ is a linear sum of G^i. If $L_k = \sum_{j=1}^{\infty} l_j(k) G^j$, we will denote

$L_k = (l_1(k),\ l_2(k),\ l_3(k), \ldots)$. By setting $A := (a_{i,j})$ and $B := (b_{i,j})$, we obtain that

$$\begin{aligned} L_1 &= -3G = (-3,\ 0,\ 0, \ldots), \\ L_{2k+1} &= (-3,\ 0,\ 0, \ldots)(AB)^k, \\ L_{2k+2} &= (-3,\ 0,\ 0, \ldots)(AB)^k A. \end{aligned}$$

It is not hard to see that Theorem 3.1.4 is equivalent to the following for all $k \geq 0$:

$$\begin{aligned} \pi(l_j(2k+1) &\geq k + 1 + \left\lfloor \frac{j}{2} \right\rfloor, \\ \pi(l_j(2k+2) &\geq k + 1 + \left\lfloor \frac{j+1}{2} \right\rfloor, \end{aligned} \tag{3.4.4}$$

where $\pi(n)$ is the 3-adic order of n. By using recurrence formulas for $G^i|U_3$ and $FG^i|U_3$ and induction, we find that

$$\pi(a_{i,j}) \geq \left\lfloor \frac{3j - i + 1}{3} \right\rfloor \quad \text{and} \quad \pi(b_{i,j}) \geq \left\lfloor \frac{3j - i}{3} \right\rfloor.$$

From this, again by induction, we can derive (3.4.4), which completes the proof for Theorem 3.1.4.

Now we turn to the proof of Theorem 3.1.5. For the rest of this section, we define $N := \ell^j$, where ℓ is a fixed prime ≥ 5, and j is a fixed positive integer. Since our proof follows the works of K. Ono and S. Ahlgren ([7], [98]) and Mahlburg [91], we will not give every detail of each step.

Recall that

$$F(x, q) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M'(m, n) x^m q^n,$$

where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$. Then, by a simple argument,

$$\sum_{n=0}^{\infty} M'(m, N, n) q^n = \frac{1}{N} \sum_{s=0}^{N-1} F(\zeta^s, z) \zeta^{-ms}, \tag{3.4.5}$$

where $\zeta = \exp(2\pi i/N)$.

By (3.3.1) and (3.3.2), we deduce that

$$F(\zeta^s, z) = \frac{-\omega_s^2 q^{1/8}}{4\pi^2} \frac{1}{\eta(z)\eta(2z) t_{0,s}(z) t_{0,s}(2z)}. \tag{3.4.6}$$

Therefore, by (3.4.5) and (3.4.6),

$$\sum_{n=0}^{\infty} N \cdot M'(m,N,n)q^n = \frac{-1}{4\pi^2} \sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)} + \sum_{n=0}^{\infty} a(n)q^n.$$

Remark. *We have multiplied* (3.4.5) *by* N*, so as to ensure that the Fourier coefficients of*

$$\frac{-1}{4\pi^2} \sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)}$$

are algebraic integers with a view toward applying Theorem 3.3.5.

Define $\delta_\ell = \frac{\ell^2-1}{24}$, and $\overline{\delta_\ell} = 3\delta_\ell$. We also define

$$g_m(z) = \left(\sum_{n=0}^{\infty} N \cdot M'(m,N,n) q^{n+\overline{\delta_\ell}} \right) (q^\ell; q^\ell)_\infty^\ell (q^{2\ell}; q^{2\ell})_\infty^\ell. \tag{3.4.7}$$

Then, we have

$$\begin{aligned} g_m(z) &= \frac{-1}{4\pi^2} \sum_{s=1}^{N-1} \frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \frac{\omega_s^2 \zeta^{-ms}}{t_{0,s}(z)t_{0,s}(2z)} + \frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \\ &=: \frac{1}{4\pi^2} \sum_{s=1}^{N-1} G_{m,s}(z) + P(z). \end{aligned}$$

In [37], Chan proved that, for sufficiently large τ,

$$\left(\frac{P(z)|_{U_\ell}}{\eta^\ell(z)\eta^\ell(2z)} E_{\ell,1}^{\ell^\tau} \right) |_{V_8} \in S_k(\Gamma_0(128\ell, \chi), \tag{3.4.8}$$

for some positive integer k and Dirichlet character χ. Here, we prove a similar result.

Theorem 3.4.1. *For sufficiently large* τ*, there is a positive integer* k' *such that*

$$\left(\frac{G_{m,s}(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)} E_{\ell,j+1}^{\ell^\tau} \right) |_{V_8} \in S_{k'}(\Gamma_1(128N^2)), \text{ for all } 1 \le s \le N-1. \tag{3.4.9}$$

Throughout the proof, we will use the notation

$$q_m = e^{2\pi i z/m} = q^{1/m}, \;\text{ and }\; \lambda = e^{2\pi i/\ell}.$$

Proof. First, note that $\frac{\eta^\ell(\ell z)}{\eta(z)} \in \mathcal{M}_{(\ell-1)/2}(\Gamma_0(\ell), (\frac{\cdot}{\ell}))$. Thus, by Lemma 3.3.7, $G_{m,s}(z) \in M_{\ell+1}(\Gamma_1(4N^2))$. Since $\eta(8z)\eta(16z) \in S_1(\Gamma_1(128))$, the left side of (3.4.9) transforms correctly on $\Gamma_1(128N^2)$. By Lemma

3.3.4, if τ is sufficiently large, then we only need to show that $\frac{G_{m,s}(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)}$ vanishes at each cusp $\frac{a}{c}$ with $\ell N|c$. Since the Fourier expansion of $\eta(z)\eta(2z)$ at such cusps is of the form $B_0 q_2^{\ell/8} + \cdots$, where B_0 is a nonzero constant, it suffices to show that the Fourier expansion of $G_{m,s}|U_\ell$ at such cusps is of the form $B_1 q_2^h + \cdots$, where B_1 is a constant and $h > \ell/8$. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell N)$. Then,

$$G_{m,s}(z)|U_\ell|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ell^{(\ell-1)/2}\sum_{j=0}^{\ell-1} G_{m,s}(z)|_{\ell+1}\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix}|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.4.10}$$

Note that, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix},$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + cj & (-aj' - cjj' + b + dj)/\ell \\ c\ell & -cj' + d \end{pmatrix}.$$

By choosing $j' \in \{0, 1, \ldots, j-1\}$ such that $-aj' + b + dj \equiv 0 \pmod{\ell}$, we have $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(\ell N)$. Note that as j runs over a complete residue system modulo ℓ, j' does as well. Thus,

$$G_{m,s}(z)|U_\ell|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ell^{(\ell-1)/2}\sum_{j'=0}^{\ell-1} G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix}.$$

From the fact that

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 2a' & -a'v + b' \\ c' & (d' - c'v)/2 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 2 \end{pmatrix}, \tag{3.4.11}$$

where

$$v = \begin{cases} 0 \text{ , if } d' \text{ is even,} \\ 1 \text{ , if } d' \text{ is odd.} \end{cases}$$

we deduce that, by setting $u = (z+v)/2$,

$$G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \left(\frac{\eta^{\ell}(\ell z)\eta^{\ell}(2\ell z)}{\eta(z)\eta(2z)}\frac{\omega_s^2\zeta^{-ms}}{t_{0,s}(z)t_{0,s}(2z)}\right)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$= \chi(d')\chi((d'-c'v)/2)\frac{\eta^{\ell}(\ell z)}{\eta(z)}\frac{\eta^{\ell}(\ell u)}{\eta(u)}\frac{\omega_s^2\zeta^{-ms}}{\beta t_{0,\overline{d's}}(z)\beta' t_{0,\overline{(d'-c'v)s/2}}(u)},$$

where β and β' are the roots of unity defined in Proposition 3.3.6, and $\chi(d) = \left(\frac{\cdot}{\ell}\right)$. Since $\ell N|c$, after some calculation, we can check that β, β', $\chi(d')$ and $\chi((d'-c'v)/2)$ do not depend on j'. In summary, we obtain

$$G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A_1 q_2^{\overline{\delta_\ell}}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 0} c_1(n,j')q_2^n\right), \tag{3.4.12}$$

where A_1 is a nonzero constant not depending on j'.

Thus, we finally arrive at

$$G_{m,s}(z)|U_\ell|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_1\sum_{j'=0}^{\ell-1}\left(q_2^{\overline{\delta_\ell}}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 0}c_1(n,j')q_2^n\right)\right)|\begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix}$$
$$= A_2 q_{2\ell}^{\overline{\delta_\ell}}\sum_{j'=0}^{\ell-1}\lambda^{\overline{\delta_\ell}j'/2}(-1)^{\overline{\delta_\ell}v}\left(1+\sum_{n\geq 1}c_2(n,j')q_{2\ell}^n\right)$$
$$= q_{2\ell}^{\overline{\delta_\ell}}(\sum_{n\geq 1}c_3(n)q_{2\ell}^n),$$

since

$$\sum_{j'=0}^{\ell-1}\lambda^{\overline{\delta_\ell}j'/2}(-1)^{\overline{\delta_\ell}v} = 0,$$

by a simple calculation. Since $1+\overline{\delta_\ell}-\ell^2/8 > 0$, we are done.

$\square$

Now, we are ready to prove our Theorem 3.1.5. To that end,

$$g_m(z)|U_\ell = \left(\sum_{n=0}^{\infty} N\cdot M'(m,N,n)q^{n+\overline{\delta_\ell}}\right)|U_\ell(q;q)_\infty^\ell(q^2;q^2)_\infty^\ell$$

and so

$$\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)} = \sum_{n=0}^{\infty} N\cdot M'(m,N,\ell n-\overline{\delta_\ell})q^{n-\frac{\ell}{8}}.$$

Thus, by Theorem 3.4.1, for sufficiently large t,

$$\left(\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)}E_{\ell,j+1}^{\ell^t}\right)|_{V_8} \equiv \sum_{\substack{n\geq 0\\ \ell n\equiv -1 \pmod 8}} N\cdot M'(m,N,\frac{\ell n+1}{8})q^n \pmod{\ell^{\tau+j}},$$

$$\equiv H_1+H_2 \pmod{\ell^{\tau+j}},$$

where $H_1\in S_{k'}(\Gamma_1(128N^2))$ and $H_2\in S_k(\Gamma_0(128\ell),\chi)$. Then, by Theorem 3.3.5, a positive proportion of primes $Q\equiv -1 \pmod{128N^2}$ have the property that

$$H_1|_{T_Q}=H_2|_{T_Q}\equiv 0 \pmod{\ell^{\tau+j}}.$$

This implies that

$$N\cdot M'(m,N,\frac{\ell nQ+1}{8})\equiv 0 \pmod{\ell^{\tau+j}}, \text{ whenever } (n,Q)=1.$$

This completes the proof of Theorem 3.1.5.

3.5 Remarks

In [72], the overpartition analog of the cubic partition, namely the overcubic partition is introduced. An overcubic partition of n is a 2-color partition of n with colors r and b subject to the restriction that the color b appears only in even parts, and we may overline the first occurrence of a part. For example, there are 6 such partitions of 2:

$$2_r,\ \ \overline{2_r},\ \ 2_b,\ \ \overline{2_b},\ \ 1_r+1_r,\ \ \overline{1_r}+1_r.$$

Define $\overline{a}(n)$ as the number of overcubic partitions of n. Then, we can easily see that

$$\sum_{n=0}^{\infty}\overline{a}(n)q^n=\frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}.$$

In the same paper [72], the following Ramanujan type congruence is proven.

$$\overline{a}(3n+2)\equiv 0 \pmod 3. \tag{3.5.1}$$

One can also define an Andrews-Garvan type crank as follows. Let us define the crank for an overcubic partition of n as the number of even parts with color r minus the number of even parts with color b. Let

$CA(m,n)$ be the number of overcubic partitions of n with crank m. Then, from the fact that

$$\frac{1}{(zq;q)_\infty} = \sum_{m=0}^{\infty} p(m,n) z^m q^n$$

, where $p(m,n)$ denotes the number of partitions of n with the number of parts equaling m, we can easily derive that

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} CA(m,n) z^m q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q^2)_\infty(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}.$$

Let $CA(m,N,n)$ be the number of overcubic partitions of n with crank $\equiv m \pmod{N}$. Then, we can prove the following. For all nonnegative integers n,

$$M(0,3,3n+2) \equiv M(1,3,3n+2) \equiv M(2,3,3n+2) \pmod{3}.$$

Since $M(0,3,3n+2) + M(1,3,3n+2) + M(2,3,3n+2) = \overline{a}(3n+2)$, this immediately implies the congruence (3.5.1) for overcubic partitions.

Chapter 4

t-core partitions

4.1 Introduction and statement of results

A partition λ is a non-increasing sequence of natural numbers whose sum is n. Partitions are represented as Ferrers-Young diagrams, where the summands in the partition are arranged in rows. For example, the Ferrers-Young diagram for $12 = 5 + 4 + 2 + 1$ is below.

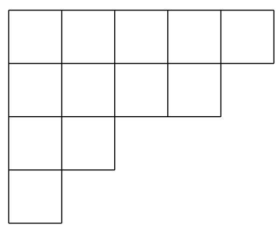

Figure 4.1: a Ferrers-Young diagram for $\lambda = (5, 4, 2, 1)$.

The *hook number* $h_{i,j}$ of a node (i, j) in the Ferrers-Young diagram is the number of nodes in the hook containing that node. For example, the hook numbers of the nodes in the first row above are 8, 6, 4, 3, and 1, respectively. If t is a positive integer, a partition is called t-core if none of the hook numbers are multiples of t. For example, examine Figure 4.2.

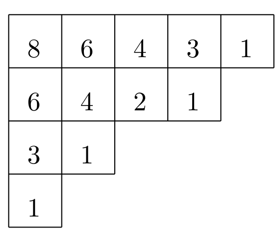

Figure 4.2: a 5-core partition $\lambda = (5, 4, 2, 1)$ with hook numbers.

[2]The content of this chapter is based upon a joint paper with Jeremy Rouse [77]. I am grateful to Dr. Rouse for his permission to include our joint work here.

If $pc_t(n)$ is the number of t-core partitions of n, then it is well-known [53] that

$$\sum_{n=0}^{\infty} pc_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}. \tag{4.1.1}$$

One motivation for studying t-core partitions comes from the representation theory of the symmetric group. Each partition α of n corresponds naturally to an irreducible representation $\rho : S_n \to \mathrm{GL}_d(\mathbb{C})$. Here the dimension d is given by the Frame-Robinson-Thrall hook formula (see [69], Theorem 2.3.21)

$$d = \frac{n!}{\prod_{i,j} h_{i,j}}, \tag{4.1.2}$$

where the denominator is the product of the hook numbers of the partition α. Alfred Young showed that a basis can be chosen for the d-dimensional space on which S_n acts so that the image of ρ lies in $\mathrm{GL}_d(\mathbb{Z})$ (see [69], Section 3.4). As a consequence, one obtains for each partition α a representation $S_n \to \mathrm{GL}_d(\mathbb{F}_p)$, by composing ρ with the natural map $\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{F}_p)$. This resulting p-modular representation is irreducible if and only if the power of p dividing d is equal to the power of p dividing $n!$. From (4.1.2), this occurs if and only if the original partition is a p-core partition.

A number of papers (see [33], [53], [55]) have investigated the combinatorial properties of $pc_t(n)$. Of particular note is the paper [53] of Garvan, Kim and Stanton, in which t-core partitions are used to produce cranks that combinatorially prove Ramanujan's congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

where $p(n)$ is the number of partitions of n.

Because of the connections with representation theory, the positivity of and asymptotics for $pc_t(n)$ have been extensively studied (see the papers by Ono [95, 96], Granville and Ono [59], and by Anderson [10]). In [112], Stanton stated (a slight variant) of the following conjecture.

Conjecture (Stanton's Conjecture)**.** *If $t \geq 4$ and $n \neq t+1$, then*

$$pc_{t+1}(n) \geq pc_t(n).$$

The restriction on n is necessary since $pc_{t+1}(t+1) = pc_t(t+1) - 1$. Motivated by this conjecture, Anderson [10] uses the circle method to establish asymptotics for $pc_t(n)$ and to verify that Stanton's conjecture is true for a fixed t provided n is sufficiently large. In [59], Granville and Ono prove that if $t \geq 4$, then $pc_t(n) > 0$ for all $n > 0$. When $t \geq 17$, Granville and Ono use an expression (due to Garvan, Kim, and Stanton) for $pc_t(n)$ as the number of representations of n by a particular quadratic form to prove positivity. The previous papers [95, 96] of Ono established positivity in all other cases $t \leq 16$ with the exception of $t = 13$. To describe Granville and Ono's approach in this case, we need some notation.

Let $p \geq 7$ be prime. Let $\chi_p(n) = \left(\frac{n}{p}\right)$. The modular form

$$f(z) := \frac{\eta^p(pz)}{\eta(z)} = \sum_{n=0}^{\infty} a_p(n)q^n = \sum_{n=0}^{\infty} pc_p(n)q^{n+\frac{p^2-1}{24}} \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$$

is essentially the generating function for $pc_p(n)$. Let

$$\sigma_{\frac{p-1}{2},\chi_p}(n) = \sum_{d|n} \chi_p\left(\frac{n}{d}\right) d^{\frac{p-3}{2}},$$

and

$$E_{\frac{p-1}{2}}(z) := \sum_{n=1}^{\infty} \sigma_{\frac{p-1}{2},\chi_p}(n)q^n$$

be one of the Eisenstein series of weight $\frac{p-1}{2}$ and level p. If e_p is the constant defined by

$$\frac{1}{e_p} = \frac{\left(\frac{p-3}{2}\right)!p^{\frac{p}{2}}}{(2\pi)^{\frac{p-1}{2}}} L\left(\frac{p-1}{2}, \chi_p\right), \tag{4.1.3}$$

then $f(z)$ can be decomposed as

$$f(z) = e_p E_{\frac{p-1}{2}}(z) + g(z)$$

where $g(z)$ is a cusp form in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. The form $g(z)$ can be expressed as a linear combination

$$g(z) = \sum_{i=1}^{s} r_i g_i(z), \tag{4.1.4}$$

of normalized Hecke eigenforms, where $s = \dim S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. As a consequence of the Weil conjectures, Deligne proved that the nth Fourier coefficient of $g_i(z)$ is bounded by $d(n)n^{\frac{p-3}{4}}$. To compute an explicit bound on $pc_p(n)$, the problem is therefore to bound the "cusp constant"

$$R(p) := \sum_{i=1}^{s} |r_i|. \tag{4.1.5}$$

In [108], J. Rouse found asymptotics for the cusp constants of powers of $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. The problem of bounding $R(p)$ is significantly more challenging for two reasons: (i) the levels of the forms in question are tending to infinity, and (ii) the form $f(z)$ is not a cusp form, and so we must understand the "size" of the difference between $f(z)$ and $e_p E_{\frac{p-1}{2}}(z)$. In [59], Granville and Ono explicitly calculate $R(p)$ for $p = 13$ by working in the 6-dimensional vector space $S_6(\Gamma_0(13), \chi_{13})$, and leave the remaining cases as an unsolved problem. We are able to determine an explicit upper bound on $R(p)$ valid for all primes p. As a consequence, we obtain the following explicit upper and lower bounds on $pc_p(n)$.

Theorem 4.1.1. *If $p \geq 7$ is an odd prime, $a_p(n) = pc_p\left(n - \frac{p^2-1}{24}\right)$, and e_p is defined by (4.1.3), then*

$$|a_p(n) - e_p\sigma_{\frac{p-1}{2},\chi_p}(n)| \leq \begin{cases} 98304e^{6\pi}p^4\log(p)\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}} d(n)n^{\frac{p-3}{4}}, & \text{if } p \equiv 1 \pmod 4, \\ 388535e^{6\pi}p^{\frac{9}{2}}\log(p)^{11/4}\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}} d(n)n^{\frac{p-3}{4}}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Noting that $e^{1.5} \approx 4.48 < 25.13 \approx 8\pi$, we immediately see the following.

Corollary 4.1.2. *Under the same assumptions as Theorem 4.1.1, $R(p) = \sum_{i=1}^{s}|r_i|$ tends to zero as p tends to infinity.*

Remark. *The bound in Theorem 4.1.1 is far from optimal. Numerical evidence suggests that $R(p)$ is not too far from the lower bound of about*

$$\frac{(2\pi^2/3)^{(p-3)/4}}{\left(\frac{p-3}{2}\right)!}.$$

We briefly describe our approach to the problem. First, we derive bounds on $pc_p(n)$ using the circle method. From these bounds, we derive an upper bound A on the Petersson inner product $\langle f, g_i\rangle$, defined by

$$\langle f, g_i\rangle := \frac{3}{\pi[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(p)]}\int_{\mathbb{H}/\Gamma_0(p)} f(z)\overline{g_i(z)}y^{\frac{p-1}{2}}\,\frac{dx\,dy}{y^2}.$$

It is known that the forms $\{E_{\frac{p-1}{2}}, g_1, \ldots, g_s\}$ are pairwise orthogonal, and so we have

$$\langle f, g_i\rangle = r_i\langle g_i, g_i\rangle.$$

Hence if B is a lower bound for $\langle g_i, g_i\rangle$, then $r_i \leq A/B$.

To derive a lower bound on $\langle g_i, g_i\rangle$, we use the fact that this quantity is essentially the special value at $s = 1$ of the adjoint square L-function associated to g_i. Goldfeld, Hoffstein, and Lieman showed in the appendix to [65] that this L-function has no Siegel zeroes, and we make their argument effective. An argument of Hoffstein [64] translates this zero-free region into a lower bound for the special value. In order to do this, we need to compute the local factors at p of the adjoint square and symmetric fourth powers of the g_i. This is done using the local Langlands correspondence.

As a consequence of Theorem 4.1.1, we obtain the following more precise version of [59, Theorem 4]. Recall that there is a bijection between the defect zero p-blocks of S_n and the p-core partitions of n.

Corollary 4.1.3. *Let $p \geq 7$ be an odd prime and let e_p and $R(p)$ be the constants defined by* (4.1.3) *and* (4.1.5)*. Then there are more than $\frac{2e_p}{5} n^{\frac{p-3}{2}}$ p-blocks with defect zero provided $n > (\frac{10R(p)}{e_p})^{\frac{4}{p-5}}$.*

Remark. *Note that $\left(\frac{10R(p)}{e_p}\right)^{\frac{4}{p-5}} \leq p^4$ for large primes p.*

As a second application, we will prove an inequality involving $pc_p(n)$. Recently, many interesting inequalities for the number of p-core partitions have been investigated using either modular equations or modular forms (see [26], [25], and [75]). The following inequality gives an explicit version of [75, Theorem 4].

Corollary 4.1.4. *Suppose that $p \geq 7$ is prime, t is a positive integer ≥ 2, and $k \geq 1$. Let $\delta_p = \frac{p^2-1}{24}$, and let e_p and $R(p)$ be the constants defined by* (4.1.3) *and* (4.1.5)*. Then for all*

$$n > \left(\frac{2\zeta\left(\frac{p-3}{2}\right)}{e_p} R(p) \left((k+1) t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1 \right) \right)^{\frac{4}{p-5}}$$

with $(n,t) = 1$, we have

$$pc_p(t^k n + \delta_p(t^k - 1)) > \left(\sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1 \right) pc_p(n). \tag{4.1.6}$$

Remark. *For large primes p, the bound on n in Corollary 4.1.4 is less than or equal to p^{4+k}.*

Finally, the bounds we obtain on $pc_t(n)$ using the circle method allow us to derive an explicit bound on possible counterexamples to Stanton's conjecture.

Theorem 4.1.5. *For all integers $t \geq 7$, if*

$$n \geq \begin{cases} \left(45503 t^{\frac{2t+1}{2}} \left(\frac{1}{2^7 \pi^3 \sqrt{e}} \right)^{\frac{t-1}{4}} \right)^{\frac{4}{t-4}}, & \text{if } t \geq 36, \\ \left(288305 t^{\frac{3t+7}{4}} \left(\frac{1}{4\pi^3 \sqrt{e}} \right)^{\frac{t-1}{4}} \right)^{\frac{4}{t-4}}, & \text{if } 7 \leq t \leq 35, \end{cases}$$

and $n \geq (t+1)^2$, then $pc_{t+1}(n) > pc_t(n)$.

Applying this theorem when $t \geq 12$, as well as more specialized arguments when $4 \leq t \leq 11$, we can verify Stanton's conjecture.

Corollary 4.1.6. *For $4 \leq t \leq 198$, Stanton's conjecture holds.*

This chapter is organized as follows. In Section 4.2, we will review basic facts on the circle method and modular forms. In Sections 4.3 the circle method is used to derive explicit bounds on $pc_t(n)$ and also on $\langle f, g_i \rangle$. In Section 4.4, the result of Goldfeld, Hoffstein and Lieman is made effective and a lower bound on $\langle g_i, g_i \rangle$ is computed. In Section 4.5, we will prove Theorem 4.1.1 and its corollaries. In Section 4.6, we will prove Theorem 4.1.5 and Corollary 4.1.6.

4.2 Preliminaries

In this section, we give a brief background on modular forms and basic tools for the circle method. For additional properties of modular forms, see [98, Chaps. 1, 2, and 3].

As usual, let $\eta(z)$ be Dedekind's eta function defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = \exp(2\pi i z)$ and z is in the complex upper half plane $\mathbb{H}$.

We define $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$. For a meromorphic function f on $\mathbb{H}$, we define the slash operator by

$$(f|_k\gamma)(z) := (\det \gamma)^{\frac{k}{2}} (cz + d)^{-k} f(\gamma z),$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. Suppose that f is a holomorphic function on $\mathbb{H}$ and χ is a Dirichlet character modulo N. We say f is a holomorphic modular (resp. cusp) form of weight k on $\Gamma_0(N)$ with character χ if f is holomorphic (resp. vanishing) at the cusps of $\Gamma_0(N)$ and $f|_k\gamma(z) = \chi(d)f(z)$ for all $\gamma \in \Gamma_0(N)$. Let $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) denote the vector space of holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. It is well-known that for primes $p \geq 5$, we have $\frac{\eta^p(pz)}{\eta(z)} \in M_{(p-1)/2}(\Gamma_0(p), \chi_p)$.

For each prime p, recall that the Hecke operator T_p is a linear operator on $S_k(\Gamma_0(N), \chi)$. If $f(z) \in S_k(\Gamma_0(N), \chi)$ has the Fourier expansion $f(z) = \sum_{n \geq 0} a(n) q^n$, then

$$f|T_p := \sum_{n \geq 0} \left(a(pn) + \chi(p) p^{k-1} a\left(\frac{n}{p}\right) \right) q^n.$$

We say that $f(z)$ is an eigenform of T_p if there is a $\lambda_p \in \mathbb{C}$ such that $f|T_p = \lambda_p f$. We call

$f(z) \in M_k(\Gamma_0(N), \chi)$ a Hecke eigenform if $f(z)$ is an eigenform of T_p for all primes p. It is well-known that $S_k(\Gamma_0(p), \chi_p)$ has basis of Hecke eigenforms (since in this case the old space is trivial), and these can be normalized so that the leading Fourier coefficient is 1. With this normalization, these forms are referred to as newforms. The Atkin-Lehner involution on $M_k(\Gamma_0(p), \chi_p)$ is defined by $f|_k \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$.

Now, we turn to the basic facts about the circle method. If $f(z) := \sum_{n=0}^{\infty} a(n)q^n$, then the residue theorem implies that

$$a(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{f(z)}{q^{n+1}} dq. \tag{4.2.1}$$

We choose $r = e^{\frac{-2\pi}{N^2}} := e^{-2\pi\rho}$ for a positive N to be determined later. By following the dissection given in [15, Chap. 5] or [41, pp. 115–117] and setting $z = k(\rho - i\varphi)$ and $\tau = \frac{h+iz}{k}$, we arrive at

$$a(n) = \sum_{1 \le k \le N} \sum_{\substack{0 \le h \le k \\ (h,k)=1}} e^{\frac{-2\pi i n h}{k}} \int_{\xi_{h,k}} f(\tau) e^{2\pi n \rho} e^{-2\pi i n \varphi} d\varphi, \tag{4.2.2}$$

where $\xi_{h,k} = [-\theta'_{h,k}, \theta''_{h,k}]$, and

$$\theta'_{h,k} = \frac{h}{k} - \frac{h_0 + h}{k_0 + k}$$
$$\theta''_{h,k} = \frac{h_1 + h}{k_1 + h} - \frac{h}{k}.$$

Here $\frac{h_0}{k_0}, \frac{h}{k}, \frac{h_1}{k_1}$ are three consecutive terms of the Farey sequence of order N. Note that each θ satisfies $\frac{1}{2kN} \le \theta \le \frac{1}{kN}$.

The following transformation formulas for the Dedekind η function will play an important role in the next section. For a proof of the transformation formulas, see [24, pp. 52–61].

Theorem 4.2.1. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have*

$$\eta(\gamma z) = e^{-\pi i s(d,c)} e^{\frac{\pi i (a+d)}{12c}} \sqrt{-i(cz+d)}\, \eta(z),$$

where $s(d, c)$ is the Dedekind sum defined by $s(d, c) = \sum_{r=1}^{c-1} \left(\frac{r}{c} - \left[\frac{r}{c}\right] - \frac{1}{2}\right)\left(\frac{dr}{c} - \left[\frac{dr}{c}\right] - \frac{1}{2}\right)$.

We prove the following two lemmas by using Theorem 4.2.1. We omit the proofs.

Lemma 4.2.2. *Let h, k be integers such that $k > 0$ and $(h, k) = 1$. Let $hh' \equiv -1 \pmod{k}$ and $z \in \mathbb{H}$.*

Then

$$\eta\left(\frac{h'+iz^{-1}}{k}\right)=e^{-\pi is(-h,k)}e^{\pi i\frac{h'-h}{12k}}\sqrt{z}\eta\left(\tau\right).$$

Lemma 4.2.3. *Let h, k be integers such that $k>0$ and $(h,k)=1$. Let $hh'\equiv-1 \pmod{k}$ and $thh''\equiv-(t,k) \pmod{k}$. Then*

$$\eta\left(\frac{(t,k)h''}{k}+i\frac{(t,k)^2}{ktz}\right)=e^{-\pi is(-\frac{-th}{(t,k)},\frac{k}{(t,k)})}e^{\pi i\frac{h''-\frac{th}{(t,k)}}{12\frac{k}{(t,k)}}}\sqrt{\frac{t}{(t,k)}z}\eta\left(t\tau\right).$$

We obtain the following lemma by modifying the argument in [41, Lemma 3.2].

Lemma 4.2.4. *Let*

$$I:=\int_{\xi_{h,k}}z^{-\frac{p-1}{2}}e^{2\pi n\rho}e^{-2\pi in\varphi}d\varphi.$$

Then we have

$$I=\frac{(2\pi)^{\frac{p-1}{2}}}{k^{\frac{p-1}{2}}\Gamma(\frac{p-1}{2})}n^{\frac{p-3}{2}}+E(I), \tag{4.2.3}$$

where $|E(I)|\leq 2^{\frac{p+1}{2}}N^{\frac{p-1}{2}}\frac{e^{2\pi n\rho}}{2\pi n}$.

The following estimate will play an important role in Sections 3 and 6. Let

$$F(q)=\prod_{n=0}^{\infty}(1-q^n)^{-1}=\sum_{n=0}^{\infty}p(n)q^n.$$

Then from the upper bound

$$p(n)<e^{\pi\sqrt{2n/3}}$$

(see Theorem 14.5 of [24, p. 316]), we have

$$|F(q)|\leq\sum_{n=0}^{\infty}p(n)|q|^n\leq\sum_{n=0}^{\infty}e^{\pi\sqrt{\frac{2n}{3}}}e^{-2\pi yn}. \tag{4.2.4}$$

It is easy to see that $\pi\sqrt{2n/3}-2\pi ny\leq-\pi ny$ if $n\geq\frac{2}{3y^2}$. It follows that

$$|F(q)|\leq\sum_{0\leq n<\frac{2}{3y^2}}e^{\frac{\pi}{12y}}+\sum_{n\geq\frac{2}{3y^2}}e^{-\pi yn}\leq\frac{2}{3y^2}e^{\frac{\pi}{12y}}+\frac{e^{-\frac{2\pi}{3y}}}{1-e^{-\pi y}}. \tag{4.2.5}$$

We will use this estimate with $y=\frac{1}{2t}$. When t is small, we will use the estimate

$$\sum_{n=0}^{\infty}p(n)e^{-2\pi yn}\leq\exp\left(\frac{e^{-2\pi y}}{(1-e^{-2\pi y})^2}\right), \tag{4.2.6}$$

given by Chan [41, Equation (3.19)].

4.3 An upper bound for $|\langle f, g\rangle|$

Recall that $p \geq 7$ is prime, $f(z) = \frac{\eta^p(pz)}{\eta(z)}$, and $g(z) \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ is a normalized Hecke eigenform. In this section, we will get an upper bound for

$$\langle f, g\rangle = \frac{3}{\pi}\frac{1}{[\Gamma : \Gamma_0(p)]}\sum_{j=1}^{[\Gamma:\Gamma_0(p)]}\int_F f|_{\alpha_j^{-1}}(z)\overline{g|_{\alpha_j^{-1}}(z)}y^{\frac{p-1}{2}}\frac{dx\,dy}{y^2}. \tag{4.3.1}$$

Here $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ is the union of right cosets $\Gamma = \bigcup_j \alpha_j\Gamma_0(p)$ and F is the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Note that

$$f|_{\frac{p-1}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z) = (-i)^{\frac{p-1}{2}}p^{\frac{-p}{2}}\frac{\eta^p(z)}{\eta(pz)} := (-i)^{\frac{p-1}{2}}p^{-\frac{p}{2}}\sum_{n=0}^{\infty} b_p(n)q_p^n. \tag{4.3.2}$$

Recall that $a_p(n)$ is the n-th Fourier coefficient of $f(z)$. Before calculating $|\langle f, g\rangle|$, we need to obtain an upper bound for $|a_p(n)|$ and $|b_p(n)|$.

Lemma 4.3.1. *For all integers $n \geq 1$ and odd primes $p \geq 7$, we have*

$$|a_p(n)| \leq A_\infty(p)n^{\frac{p-3}{2}} + B_\infty(p)n^{\frac{p-1}{4}}, \tag{4.3.3}$$

$$|b_p(n)| \leq A_0(p)n^{\frac{p-3}{2}} + B_0(p)n^{\frac{p-1}{4}}, \tag{4.3.4}$$

where $A_\infty(p)$, $A_0(p)$, $B_\infty(p)$, and $B_0(p)$ are constants (depending only on p) defined by

$$A_\infty(p) = p^{-\frac{p}{2}}\frac{(2\pi)^{\frac{p-1}{2}}\zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})}, \tag{4.3.5}$$

$$B_\infty(p) = e^{6\pi}\left(\frac{2e^{\frac{\pi}{p}}(C(p)-1)}{p^{\frac{p}{2}}}\left(\frac{p(p-1)}{8\pi e}\right)^{\frac{p-1}{4}} + 2.1\left(\frac{3}{e\pi(p+1)}\right)^{\frac{p-1}{4}} + p^{-\frac{p}{2}}\frac{2^{\frac{p-1}{2}}}{\pi}\right), \tag{4.3.6}$$

$$A_0(p) = \frac{(2\pi)^{\frac{p-1}{2}}\zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})}, \tag{4.3.7}$$

$$B_0(p) = e^{6\pi}\left(2C(p)\sqrt{p}\left(\frac{3(p-1)}{e\pi(p-\frac{1}{p})}\right)^{\frac{p-1}{4}} + 2.1\left(\frac{p-1}{8\pi e}\right)^{\frac{p-1}{4}} + \frac{2^{\frac{p-1}{2}}}{\pi}\right), \tag{4.3.8}$$

and $C(p) := \frac{8p^2}{3}e^{\frac{\pi p}{6}} + \frac{e^{\frac{-4\pi p}{3}}}{1-e^{\frac{-\pi}{2p}}}$.

We will prove this lemma at the end of the section.

Let $g(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. If $d(n)$ is the number of divisors of n, then Deligne's bound is

$$|c(n)| \leq d(n)n^{\frac{p-3}{4}}.$$

Note that g is an eigenform of the Atkin-Lehner involution with eigenvalue λ_p where $|\lambda_p| = 1$. Thus,

$$g|_{\frac{p-1}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (z) = \lambda_p p^{-\frac{p-1}{4}} \sum_{n=1}^{\infty} c(n) q_p^n.$$

Now we are ready to calculate an upper bound for $|\langle f, g\rangle|$. It is well known that $\alpha_j = I$ or $T^{-k}S$, where $k = 0, 1, \ldots, p-1$. When we set $\alpha_j = I$ in (4.3.1), we have

$$\begin{aligned}
&\left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right| \qquad (4.3.9)\\
&\leq \int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{k=2}^{\infty} \left(\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \right) e^{-2\pi k y} y^{\frac{p-5}{2}}\, dy \\
&\leq \frac{1}{(2\pi)^{\frac{p-3}{2}}} \int_{\pi\sqrt{3}}^{\infty} e^{-u} u^{\frac{p-5}{2}} \sum_{k \leq u} \frac{1}{k^{\frac{p-3}{2}}} \left(\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| \right) du.
\end{aligned}$$

By using the summation by parts formula and Lemma 4.3.1, we obtain

$$\begin{aligned}
\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| &\leq A_\infty(p) \sum_{n=1}^{k} d(n)(k-n)^{\frac{p-3}{2}} n^{\frac{p-3}{4}} + B_\infty(p) \sum_{n=1}^{k} d(n)(k-n)^{\frac{p-1}{4}} n^{\frac{p-3}{4}} \\
&\leq A_\infty(p) \frac{p-3}{2} k \int_1^k D(t) t^{\frac{p-7}{4}} (k-t)^{\frac{p-5}{2}}\, dt \\
&\quad + B_\infty(p) \frac{p-1}{4} k \int_1^k D(t) t^{\frac{p-7}{4}} (k-t)^{\frac{p-5}{4}}\, dt,
\end{aligned}$$

where $D(t) := \sum_{n \leq t} d(n)$. Therefore, by using the Beta integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and $D(t) \leq 1.8t^{5/4} + 3.6t^{1/4}$, we arrive at

$$\begin{aligned}
\sum_{n=1}^{k-1} |a_p(k-n)||c(n)| &\leq A_\infty(p) \frac{9(p-3)}{10} \left(k^{\frac{3p-4}{4}} \frac{\Gamma\left(\frac{p+2}{4}\right)\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{3p-4}{4}\right)} + 2k^{\frac{3p-8}{4}} \frac{\Gamma\left(\frac{p-2}{4}\right)\Gamma\left(\frac{p-3}{2}\right)}{\Gamma\left(\frac{3p-8}{4}\right)} \right) \\
&\quad + B_\infty(p) \frac{9(p-1)}{20} \left(k^{\frac{2p+1}{4}} \frac{\Gamma\left(\frac{p+2}{4}\right)\Gamma\left(\frac{p-1}{4}\right)}{\Gamma\left(\frac{2p+1}{4}\right)} + 2k^{\frac{2p-3}{4}} \frac{\Gamma\left(\frac{p-2}{4}\right)\Gamma\left(\frac{p-1}{4}\right)}{\Gamma\left(\frac{2p-3}{4}\right)} \right).
\end{aligned}$$

Note that for all real numbers $x \geq 1$, $\sum_{k \leq u} k^x \leq \frac{1}{x+1} u^{x+1} + u^x$. Applying this to (4.3.9), we arrive at

$$\begin{aligned}
&\left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right| \\
&\leq \frac{36 A_\infty(p) p}{5(2\pi)^{\frac{p-3}{2}}} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-3}{2}\right) + \frac{2 B_\infty(p) p^2}{7(2\pi)^{\frac{p-3}{2}}} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-1}{4}\right) \\
&:= U_\infty(p).
\end{aligned} \tag{4.3.10}$$

Similarly, for other α_j,

$$\begin{aligned}
&\left| \int_F f(z)|_{\alpha_j^{-1}} \overline{g(z)|_{\alpha_j^{-1}}} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right| \\
&\leq \frac{1}{(2\pi)^{\frac{p-3}{2}}} \left(\frac{1}{p}\right)^{\frac{p+5}{4}} \int_{\frac{\pi\sqrt{3}}{p}}^{\infty} e^{-u} u^{\frac{p-5}{2}} \sum_{k \leq u} \frac{1}{k^{\frac{p-3}{2}}} \left(\sum_{n=1}^{k} |b_p(k-n)||c(n)| \right) du.
\end{aligned}$$

By using a similar argument, we arrive at

$$\begin{aligned}
&\left| \int_F f(z)\overline{g(z)} y^{\frac{p-1}{2}} \frac{dx\,dy}{y^2} \right| \\
&\leq \frac{p^{-\frac{p+5}{4}}}{(2\pi)^{\frac{p-3}{2}}} \left(\frac{36 A_0(p) p}{5} \Gamma\left(\tfrac{p-2}{4}\right) \Gamma\left(\tfrac{p-3}{2}\right) + \frac{2 B_0(p) p^2}{7} \Gamma\left(\tfrac{p-2}{4}\right) \Gamma\left(\tfrac{p-1}{4}\right) + \Gamma\left(\tfrac{p-1}{2}\right) \right) \\
&:= U_0(p).
\end{aligned} \tag{4.3.11}$$

By using Lemma 4.3.1, (4.3.10) and (4.3.11), we obtain the following theorem.

Theorem 4.3.2. *Let $f(z) = \frac{\eta^p(pz)}{\eta(z)} \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$, and let $g(z)$ be a normalized newform in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. Then,*

$$\frac{\pi[\Gamma : \Gamma_0(p)]}{3} |\langle f, g \rangle| \leq 161.6 \cdot e^{6\pi} \Gamma\left(\frac{p-2}{4}\right) \Gamma\left(\frac{p-1}{4}\right) p^{\frac{7}{2}} \left(\frac{e^{1.5}}{32\pi^3}\right)^{\frac{p-1}{4}}.$$

Now we will prove Lemma 4.3.1 by using the circle method. This is very similar to the argument of Anderson in [10].

Proof of Lemma 4.3.1. By (4.2.2), we have

$$\begin{aligned}
a_p(n) &= \left(\sum_{\substack{1 \leq k \leq N \\ (k,p)=1}} + \sum_{\substack{1 \leq k \leq N \\ p|k}} \right) \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} e^{\frac{-2\pi i n h}{k}} \int_{\xi_{h,k}} \frac{\eta^p(p\tau)}{\eta(\tau)} e^{2\pi n \rho} e^{-2\pi i n \varphi} d\varphi \\
&=: S_1(A) + S_2(A),
\end{aligned}$$

where A is the integrand.

First, we consider $S_1(A)$. By (4.2.2) and (4.2.3), we have

$$A = p^{-\frac{p}{2}}\omega_{h,k} z^{-\frac{p-1}{2}} \frac{\eta^t\left(\exp\left(2\pi i\frac{h''}{k} - 2\pi\frac{1}{kpz}\right)\right)}{\eta\left(\exp\left(2\pi i\frac{h'}{k} - 2\pi\frac{1}{pz}\right)\right)}, \tag{4.3.12}$$

where $\omega_{h,k}$ is a constant depending on h and k with $|\omega_{h,k}| = 1$. Then,

$$\begin{aligned}
&S_1(A) = p^{-\frac{p}{2}} \sum_{\substack{1\le k\le N\\(k,p)=1}} \sum_{\substack{0\le h\le k\\(h,k)=1}} e^{\frac{-2\pi inh}{k}} \int_{\xi_{h,k}} \omega_{h,k} z^{-\frac{p-1}{2}} e^{2\pi n\rho} e^{-2\pi in\varphi} d\varphi \\
&+ p^{-\frac{p}{2}} \sum_{\substack{1\le k\le N\\(k,p)=1}} \sum_{\substack{0\le h\le k\\(h,k)=1}} e^{\frac{-2\pi inh}{k}} \int_{\xi_{h,k}} \omega_{h,k} \times \\
&\left(\frac{\eta^p\left(\exp\left(2\pi i\frac{h''}{k} - 2\pi\frac{1}{kpz}\right)\right)}{\eta\left(\exp\left(2\pi i\frac{h'}{k} - 2\pi\frac{1}{kz}\right)\right)} - 1\right) z^{-\frac{p-1}{2}} e^{2\pi n\rho} e^{-2\pi in\varphi} d\varphi \\
&:= T_1 + T_2.
\end{aligned}$$

By Lemma 4.2.4, we have

$$|T_1| \le p^{-\frac{p}{2}} \frac{(2\pi)^{\frac{p-1}{2}}\zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})} n^{\frac{p-3}{2}} + e^{6\pi} p^{-\frac{p}{2}} n^{\frac{p-1}{4}} \frac{2^{\frac{p-1}{2}}}{\pi},$$

by setting $N = [\sqrt{n}]$, because $\frac{n}{[\sqrt{n}]^2} \le 3$ for all $n \ge 1$.

For T_2, note that we can set $h' = th''$. Thus, if we set $\alpha = \frac{h''}{k} + 2\pi i\frac{1}{kpz}$, then we have, by (4.2.4) and (4.2.5),

$$\begin{aligned}
\left|\frac{\eta^p\left(\exp\left(2\pi i\frac{h''}{k} - 2\pi\frac{1}{kpz}\right)\right)}{\eta\left(\exp\left(2\pi i\frac{h'}{k} - 2\pi\frac{1}{kz}\right)\right)} - 1\right| &\le \left|\prod_{n=1}^{\infty} \frac{(1-q^n)^p}{1-q^{pn}} - 1\right| \le \sum_{n=1}^{\infty} aa(n)|q|^n \\
&\le |q| \sum_{n=1}^{\infty} p(n)|q|^{n-1} \le e^{-\frac{2\pi}{k}\mathrm{Re}\frac{1}{pz}} e^{\frac{\pi}{p}} \left(C(p) - 1\right).
\end{aligned}$$

Here, $q = \exp 2\pi i\alpha$ and $aa(n)$ is the number of partitions of n such that parts which are not multiples of p can be repeated up to p times and parts which are a multiple of p can be repeated at most $p-1$ times.

Therefore, we have

$$\begin{aligned}
&\left|\int_{\xi_{h,k}} \omega_{h,k}\left(\frac{\eta^p\left(\exp\left(2\pi i\frac{h''}{k}-2\pi\frac{1}{kpz}\right)\right)}{\eta\left(\exp\left(2\pi i\frac{h'}{k}-2\pi\frac{1}{kz}\right)\right)}-1\right)z^{-\frac{p-1}{2}}e^{2\pi n\rho}d\varphi\right|\\
&\le e^{\frac{\pi}{p}}\left(C(p)-1\right)\int_{\xi_{h,k}}|z|^{-\frac{p-1}{2}}e^{-\frac{2\pi}{k}\mathrm{Re}\frac{1}{pz}}e^{2\pi n\rho}d\varphi\\
&= e^{\frac{\pi}{p}}\left(C(p)-1\right)\int_{\xi_{h,k}}\left(\frac{p}{2\pi\rho}\right)^{\frac{p-1}{4}}\left(\frac{2\pi\rho}{pk^2(\rho^2+\varphi^2)}\right)^{\frac{p-1}{4}}\exp\left(\frac{-2\pi\rho}{pk^2(\rho^2+\varphi^2)}\right)e^{2\pi n\rho}d\varphi\\
&\le e^{\frac{\pi}{p}}\left(C(p)-1\right)\left(\frac{p(p-1)}{8\pi e}\right)^{\frac{p-1}{4}}n^{\frac{p-1}{4}}e^{2\pi n\rho}\frac{2}{kN},
\end{aligned}$$

where for the last inequality, we used the fact that the maximum of $x^{\frac{p-1}{4}}e^{-x}$ on $[0,\infty)$ is $\left(\frac{p-1}{4e}\right)^{\frac{p-1}{4}}$ and the length of path is at most $\frac{2}{kN}$. Thus, by setting $N=[\sqrt{n}]$, we arrive at

$$|T_2|\le\frac{2e^{\frac{\pi}{p}+6\pi}\left(C(p)-1\right)}{p^{\frac{p}{2}}}\left(\frac{p(p-1)}{8\pi e}\right)^{\frac{p-1}{4}}n^{\frac{p-1}{4}}.$$

Similarly, we obtain the following upper bound for $|S_2(A)|$:

$$|S_2(A)|\le 2.1e^{6\pi}\left(\frac{3}{e\pi(p+1)}\right)^{\frac{p-1}{4}}n^{\frac{p-1}{4}}.$$

In summary, we have deduced that

$$A_\infty(p)=p^{-\frac{p}{2}}\frac{(2\pi)^{\frac{p-1}{2}}\zeta(\frac{p-3}{2})}{\Gamma(\frac{p-1}{2})},$$

and

$$B_\infty(p)=e^{6\pi}\left(\frac{2e^{\frac{\pi}{p}}\left(C(p)-1\right)}{p^{\frac{p}{2}}}\left(\frac{p(p-1)}{8\pi e}\right)^{\frac{p-1}{4}}+2.1\left(\frac{3}{e\pi(p+1)}\right)^{\frac{p-1}{4}}+p^{-\frac{p}{2}}\frac{2^{\frac{p-1}{2}}}{\pi}\right),$$

as desired. The calculation of $A_0(p)$ and $B_0(p)$ is analogous, so we omit it. □

4.4 A lower bound for $\langle g,g\rangle$

In this section, we will derive a lower bound for $\langle g,g\rangle$, where $g\in S_{\frac{p-1}{2}}(\Gamma_0(p),\chi_p)$ is a normalized Hecke eigenform. Our approach is to use that the number $\langle g,g\rangle$ arises in a formula for the special value at $s=1$

of the adjoint square L-function $L(s, \mathrm{Ad}^2(g))$. In the appendix to [65], Goldfeld, Hoffstein and Lieman proved that this L-function has no zeroes close to $s = 1$. We make their argument effective, and use this to derive a lower bound on the special value at $s = 1$. In this section, we state all of our results at the beginning and provide proofs later in the section.

Write

$$g(z) = \sum_{n=1}^{\infty} a(n) q^n,$$

and for primes q, define $\alpha_q, \beta_q \in \mathbb{C}$ by

$$\alpha_q + \beta_q = a(q)/q^{\frac{p-1}{4}}, \quad \alpha_q \beta_q = \chi_p(q).$$

Define the adjoint square L-function by

$$L(s, \mathrm{Ad}^2(g)) = \prod_q (1 - \alpha_q^2 \chi_p(q) q^{-s})^{-1} (1 - q^{-s})^{-1} (1 - \beta_q^2 \chi_p(q) q^{-s})^{-1},$$

and define the completed L-function by

$$\Lambda(s, \mathrm{Ad}^2(g)) = p^s \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right) L(s, \mathrm{Ad}^2(g)).$$

In [56], Gelbart and Jacquet show that $L(s, \mathrm{Ad}^2(g))$ is the L-function of an automorphic form on GL(3), and hence that it has an analytic continuation and functional equation of the usual type. However, it is not immediately clear that the local factors at p and ∞ of Gelbart and Jacquet match the definition given above. The content of the next theorem is a computation of these local factors using the local Langlands correspondence.

Theorem 4.4.1. *Assume the notation above. Then, $L(s, \mathrm{Ad}^2(g))$ has an analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(s, \mathrm{Ad}^2(g)) = \Lambda(1 - s, \mathrm{Ad}^2(g)).$$

The fact that $L(s, g \otimes g) = \zeta(s) L(s, \mathrm{Ad}^2(g))$ and the classical Rankin-Selberg theory (see Chapter 13 of [67]) imply the following special value formula. Recall that

$$\langle g, g \rangle = \frac{3}{\pi[\Gamma : \Gamma_0(p)]} \int_{\mathbb{H}/\Gamma_0(p)} |g(x+iy)|^2 y^{\frac{p-1}{2}} \frac{dx\, dy}{y^2}.$$

Then

$$L(1, \mathrm{Ad}^2(g)) = \frac{\pi}{2}\left(1 + \frac{1}{p}\right)\frac{(4\pi)^{\frac{p-1}{2}}}{\left(\frac{p-3}{2}\right)!}\langle g, g\rangle. \tag{4.4.1}$$

We say that a modular form $g(z) = \sum_{n=1}^{\infty} a(n)q^n$ of weight $k \geq 2$ has complex multiplication (or CM) if there is a Hecke character ξ associated to a quadratic field K so that

$$g(z) = \sum_{\mathfrak{a} \subseteq O_K} \xi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

Equivalently, $g(z)$ has CM if and only if there is a discriminant D so that $a(p) = 0$ whenever $\left(\frac{D}{p}\right) = -1$.

In order to apply Goldfeld, Hoffstein, and Lieman's argument, we need information about the symmetric fourth power L-function attached to g. It is defined by

$$L(s, \mathrm{Sym}^4(g)) = \prod_q (1 - \alpha_q^4 q^{-s})^{-1}(1 - \alpha_q^2 \chi_p(q) q^{-s})^{-1}(1 - q^{-s})^{-1}(1 - \alpha_q^{-2}\chi_p(q)q^{-s})^{-1}(1 - \alpha_q^{-4}q^{-s})^{-1},$$

and the completed L-function is given by

$$\begin{aligned}\Lambda(s, \mathrm{Sym}^4(g)) :=& p^s \pi^{-3s/2} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s + \frac{p-3}{2}}{2}\right)\Gamma\left(\frac{s + \frac{p-1}{2}}{2}\right)\cdot \\ &\Gamma\left(\frac{s+p-3}{2}\right)\Gamma\left(\frac{s+p-1}{2}\right) L(s, \mathrm{Sym}^4(g)).\end{aligned}$$

In [78], H. Kim established the connection between this L-function and an automorphic form on GL(5). As a consequence, the symmetric fourth power L-function has the desired analytic properties. Again, we must compute the local factors at p and ∞ using the local Langlands correspondence.

Theorem 4.4.2. *Assume the notation above. Then $L(s, \mathrm{Sym}^4(g))$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies the functional equation*

$$\Lambda(s, \mathrm{Sym}^4(g)) = \Lambda(1 - s, \mathrm{Sym}^4(g)).$$

Moreover, if g does not have CM, then $L(s, \mathrm{Sym}^4(g))$ is entire.

Remark. *When g does have CM and corresponds to a Hecke character ξ, we have*

$$L(s, \mathrm{Sym}^4(g)) = \zeta(s) L(s, \xi^2) L(s, \xi^4).$$

Consequently, $L(s, \mathrm{Sym}^4(g))$ has a pole at $s = 1$.

The next result is an explicit version of the result of Goldfeld, Hoffstein, and Lieman.

Theorem 4.4.3. *Assume the notation above. If g does not have CM, then*

$$L(s, \mathrm{Ad}^2(g)) \neq 0$$

for s real with

$$s > 1 - \frac{7 - 4\sqrt{3}}{9\log(p)}.$$

We next translate this zero-free region into a lower bound on $L(1, \mathrm{Ad}^2(g))$.

Theorem 4.4.4. *Suppose that $g \in S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ is a normalized newform. If g has CM, then*

$$L(1, \mathrm{Ad}^2(g)) \geq \frac{1}{332\sqrt{p}\log(p)^{11/4}}.$$

If g does not have CM, then

$$L(1, \mathrm{Ad}^2(g)) \geq \frac{1}{84\log(p)}.$$

Remark. *There are CM forms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ if and only if $p \equiv 3 \pmod 4$.*

Proof of Theorem 4.4.1. The newform g in the statement of the theorem corresponds to an irreducible cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, where $\mathbb{A}_\mathbb{Q}$ is the adele ring of $\mathbb{Q}$ (for details about this correspondence, see [36], Chapter 7). The representation π admits a factorization

$$\pi = \oplus_{q \leq \infty} \pi_q,$$

where each π_q is a representation of the group $\mathrm{GL}_2(\mathbb{Q}_q)$. In [56], Gelbart and Jacquet prove that there is an automorphic representation $\mathrm{Ad}^2(\pi)$ of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ so that

$$\mathrm{Ad}^2(\pi) = \oplus_{q \leq \infty} \mathrm{Ad}^2(\pi_q).$$

The L-function $L(s, \mathrm{Ad}^2(\pi))$ is defined by $\prod_{q \leq \infty} L(s, \mathrm{Ad}^2(\pi_q))$. Let $\psi : \mathbb{A}_\mathbb{Q}/\mathbb{Q} \to \mathbb{C}^\times$ be a global additive character. The ϵ-factor is given by

$$\epsilon(s, \mathrm{Ad}^2(\pi), \psi) = \prod_{q \leq \infty} \epsilon(s, \mathrm{Ad}^2(\pi_q), \psi_q).$$

The above definition does not depend on the choice of ψ. Finally, the functional equation takes the form

$$L(s, \mathrm{Ad}^2(\pi)) = \epsilon(s, \pi) L(1-s, \mathrm{Ad}^2(\pi)),$$

since $\mathrm{Ad}^2(\pi)$ is self-contragredient.

The definition of $\mathrm{Ad}^2(\pi_q)$ is given by the local Langlands correspondence. If F is a local field, the local Langlands correspondence gives a bijection between the set of smooth, irreducible representations of $\mathrm{GL}_n(F)$, and the set of admissible degree n complex representations of W'_F, the Weil-Deligne group of F. For an introduction to the local Langlands correspondence, see [83], and Section 10.3 of [36]. The representations of W'_F that we consider will all be representations of the Weil group W_F, which is a quotient of W'_F.

The representation π_q corresponds to a representation $\rho_q : W'_F \to \mathrm{GL}_2(\mathbb{C})$. Using the embedding $\mathrm{Ad}^2 : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{C})$, one constructs $\mathrm{Ad}^2(\rho_q) : W'_F \to \mathrm{GL}_3(\mathbb{C})$. The local Langlands correspondence for GL_3 associates to $\mathrm{Ad}^2(\rho_q)$ a representation $\mathrm{Ad}^2(\pi_q)$. We shall now compute this in the cases $q = \infty$ and $q = p$.

When $q = \infty$, π_q is the discrete series of weight $k = \frac{p-1}{2}$ (we follow the normalization of Cogdell [47]). This corresponds by the local Langlands correspondence to a representation of the Weil group of $\mathbb{R}$. This is the group $\mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for z in $\mathbb{C}^\times$. The representation in question is

$$\rho_k(re^{i\theta}) = \begin{bmatrix} e^{i(k-1)\theta} & 0 \\ 0 & e^{-i(k-1)\theta} \end{bmatrix}, \quad \rho_k(j) = \begin{bmatrix} 0 & (-1)^{k-1} \\ 1 & 0 \end{bmatrix}.$$

The adjoint square lift of ρ is

$$\mathrm{Ad}^2(\rho_k)(re^{i\theta}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i(2k-2)\theta} & 0 \\ 0 & 0 & e^{-i(2k-2)\theta} \end{bmatrix}, \quad \mathrm{Ad}^2(\rho_k(j)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & (-1)^{k-1} \\ 0 & (-1)^{k-1} & 0 \end{bmatrix}.$$

One can see that $\mathrm{Ad}^2(\rho_q) = \rho_0^- \oplus \rho_{2k-1}$. Here ρ_0^- is the 1-dimensional representation given by $\rho_0^-(z) = 1$ and $\rho_0^-(j) = -1$. We have $L(s, D_0^-) = \pi^{-s/2}\Gamma\left(\frac{s+1}{2}\right)$, and $L(s, D_{2k-1}) = \pi^{-s}\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)$. The L and ϵ factors are defined so that they are inductive. In particular, $L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2)$, and $\epsilon(s, \rho_1 \oplus \rho_2, \psi_q) = \epsilon(s, \rho_1, \psi_q)\epsilon(s, \rho_2, \psi_q)$. It follows that

$$L(s, \mathrm{Ad}^2(\rho_\infty)) = L(s, D_0^-)L(s, D_{2k-1}).$$

The local root number is $\epsilon(\frac{1}{2}, D_0^-, \psi)\epsilon(\frac{1}{2}, D_{2k-1}, \psi) = i \cdot i^{2k-1} = (-1)^k$. Here, $\psi(x) = e^{2\pi i x}$ is the standard additive character.

When $q = p$, the local representation π_p has central character χ_p (the usual Dirichlet character thought of as a character of $\mathbb{Q}_p^\times$). The conductor of π_p is the power of $p^{s/2}$ that occurs in the functional equation for $L(s, \pi)$, and since the newform g has level p, the conductor of π_p is one. This can be determined from $\epsilon(s, \pi_p, \psi_p)$, and also from more intrinsic representation-theoretic data. For any representation σ, we will denote its conductor by $c(\sigma)$.

In Schmidt [110], a list of possibilities for local representations π together with their conductors is given. A simple calculation shows that the only possibility for a representation with conductor one and central character χ_p is a principal series $\pi(\chi_1, \chi_2)$, where χ_1 is unramified, and $\chi_2 = \chi_1^{-1}\chi_p$.

The Weil group $W_{\mathbb{Q}_p}$ can be taken to be the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ consisting of all elements restricting to some power of the Frobenius on $\overline{\mathbb{F}}_p$ (see [114]). Under the local Langlands correspondence, $\pi(\chi_1, \chi_2)$ corresponds to a two-dimensional representation of $W_{\mathbb{Q}_p}$ which is a direct sum of two characters. These characters ρ_1 and ρ_2 of $W_{\mathbb{Q}_p}$ are constructed so that

$$\rho_i(\sigma) = \chi_i(r(\sigma)),$$

where $r : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ is the reciprocity law isomorphism of local class field theory, normalized so that $r(\mathrm{Frob}_p) \in p^{-1}\mathbb{Z}_p$.

One can easily compute that $\mathrm{Ad}^2(\rho_1 \oplus \rho_2) = 1 \oplus \rho_1\rho_2^{-1} \oplus \rho_1^{-1}\rho_2$ (here 1 denotes the trivial character). Since $\rho_1\rho_2^{-1}$ and $\rho_1^{-1}\rho_2$ both have conductor 1, it follows that $c(\mathrm{Ad}^2(\pi_p)) = 2$. From the usual definition of the L-factors, and the compatibility with the local Langlands correspondence, we see that

$$L(s, \mathrm{Ad}^2(\pi_p)) = L(s, \mathrm{Ad}^2(\rho_1 \oplus \rho_2)) = (1 - p^{-s})^{-1}.$$

Moreover, we have

$$\epsilon\left(\frac{1}{2}, \mathrm{Ad}^2(\pi_p), \psi_p\right) = \epsilon\left(\frac{1}{2}, 1, \psi_p\right)\epsilon\left(\frac{1}{2}, \chi_1\chi_2^{-1}, \psi_p\right)\epsilon\left(\frac{1}{2}, \chi_1^{-1}\chi_2, \psi_p\right).$$

Equation 4 on page 117 of [110] states that if χ and ψ_p are unramified, then $\epsilon\left(\frac{1}{2}, \chi, \psi_p\right) = 1$. Equation 7 on page 118 of [110] implies that for any character χ,

we have $\epsilon\left(\frac{1}{2}, \chi, \psi_p\right)\epsilon\left(\frac{1}{2}, \chi^{-1}, \psi_p\right) = \chi(-1)$. It follows that the local root number of $\mathrm{Ad}^2(\pi_p)$ is $\chi_1\chi_2^{-1}(-1)$. Since χ_1 is unramified, $\chi_1(-1) = 1$, while $\chi_2^{-1}(-1) = \left(\frac{-1}{p}\right)$.

The global conductor of $\mathrm{Ad}^2(\rho)$ is therefore p^2 and the global root number is $(-1)^k \cdot \left(\frac{-1}{p}\right)$. Since $k = \frac{p-1}{2}$, the global root number is 1. These facts, combined with the meromorphic continuation and functional equation for L-functions of automorphic representations yield the desired result. If g does not have CM, then $\pi \otimes \chi \not\cong \pi$ for any character χ of $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$, and then Theorem 9.3 of [56] implies that $\mathrm{Ad}^2(\pi)$ is cuspidal, which implies that $L(s, \mathrm{Ad}^2(\pi))$ is entire. If g does have CM, then g corresponds to a Hecke character ξ, and one can check that

$$L(s, \mathrm{Ad}^2(\pi)) = L(s, \chi_p) L(s, \xi^2),$$

which is again entire. □

Proof of Theorem 4.4.2. This is entirely analogous to the case of the adjoint square lifting, thanks to the deep result of Henry Kim on the functoriality of the symmetric fourth power lifting [78]. The local factor at infinity is worked out in [47], with the desired result, and with the local root number equal to $(-1)^k$.

At $q = p$, $\mathrm{Sym}^4(\rho_1 \oplus \rho_2)$ is $\rho_1^4 \oplus \rho_1^3\rho_2 \oplus \rho_1^2\rho_2^2 \oplus \rho_1\rho_2^3 \oplus \rho_2^4$. Note that ρ_2 is ramified, but ρ_2^2 is not. Thus, the local L-factor has degree 3 and is given by

$$(1 - \alpha_p^4 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_p^{-4} p^{-s})^{-1},$$

where $\alpha_p = a(p)/p^{\frac{p-1}{4}}$. Similar to the above case, the conductor of $\mathrm{Sym}^4(\rho_1 \oplus \rho_2)$ is 2, and the local root number is $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = (-1)^k$. Thus, the global conductor is p^2 and the global root number is 1. Finally, we must show that under the stated hypotheses, $\mathrm{Sym}^4(\pi)$ is cuspidal. The main result of [79] is that $\mathrm{Sym}^4(\pi)$ is cuspidal unless π is monomial (equivalently g has CM), or π is of tetrahedral or octahedral type. This means that π arises from a representation of the global Weil group $W_{\mathbb{Q}}$, but this cannot be the case if the weight of g is greater than 1. The only case when the weight can be one is when $p = 3$. However in this case, any nonzero $g \in S_1(\Gamma_0(3), \chi_3)$ has $f^2 \in S_2(\Gamma_0(3))$, but since $\dim S_2(\Gamma_0(3)) = 0$, no such g exists. Thus, $\mathrm{Sym}^4(\pi)$ is cuspidal and $L(s, \mathrm{Sym}^4(\pi)) = L(s, \mathrm{Sym}^4(g))$ is entire. □

Proof of Theorem 4.4.3. Let

$$L(s) = \zeta(s)^2 L(s, \mathrm{Ad}^2(g))^3 L(s, \mathrm{Sym}^4(g)).$$

Let $k = \frac{p-1}{2}$ and

$$G(s) = p^{4s} \pi^{-8s} \Gamma\left(\tfrac{s}{2}\right)^3 \Gamma\left(\tfrac{s+1}{2}\right)^3 \Gamma\left(\tfrac{s+k-1}{2}\right)^4 \Gamma\left(\tfrac{s+k}{2}\right)^4 \Gamma\left(\tfrac{s+2k-2}{2}\right) \Gamma\left(\tfrac{s+2k-1}{2}\right).$$

If $\Lambda(s) = s^2(1-s)^2 G(s)L(s)$, then $\Lambda(s)$ is entire and $\Lambda(s) = \Lambda(1-s)$. One may verify from Theorems 4.4.1 and 4.4.2 that if $L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, then $b(n) \geq 0$ for all n. For the remainder of the proof, we will take s real and greater than 1. In this region, one has $L(s) > 0$ and $L'(s) = \sum_{n=2}^{\infty} \frac{-b(n)\log(n)}{n^s} < 0$. The function $\Lambda(s)$ is an entire function of order 1, and so admits a product expansion

$$\Lambda(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}.$$

Taking the logarithmic derivative gives

$$\sum_{\rho} \frac{1}{s-\rho} + \frac{1}{\rho} = \frac{2}{s} - \frac{2}{1-s} + \frac{L'(s)}{L(s)} + \frac{G'(s)}{G(s)} - B.$$

Taking the real part of both sides and using the relation $\mathrm{Re}\,(B) = -\sum_{\rho} \mathrm{Re}\left(\frac{1}{\rho}\right)$ (see Theorem 5.6, part 3 of [68]), we obtain

$$\sum_{\rho} \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \frac{2}{s} + \frac{2}{s-1} + \frac{G'(s)}{G(s)}.$$

Suppose that β is a real zero of $L(s, \mathrm{Ad}^2(g))$. Then we get

$$\frac{3}{s-\beta} \leq \frac{2}{s-1} + 2 + \frac{G'(s)}{G(s)}.$$

We have

$$\begin{aligned}\frac{G'(s)}{G(s)} &= 4\log(p) - 8\log(\pi) + 3/2\psi(s/2) + (3/2)\psi((s+1)/2) + 2\psi((s+k-1)/2) \\ &\quad + 2\psi((s+k)/2) + \frac{1}{2}\psi((s+2k-2)/2) + \frac{1}{2}\psi((s+2k-1)/2),\end{aligned}$$

where $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$. The formula $\psi(s) = \log(s) - \frac{1}{2s} - \int_0^{\infty} \frac{2t\,dt}{(s^2+t^2)(e^{2\pi t}-1)}$ (see for example [16], Exercise 1.43(b)) implies that $\psi(s)$ is increasing as a function of s, and that $\psi(s) \leq \log(s) - \frac{1}{2s}$. It follows that for $1 < s \leq 1.1$, $\frac{G'(s)}{G(s)} \leq 9\log(p) - 2$.

Now, set $s = 1+\alpha$, where $0 < \alpha \leq 0.1$. We obtain $\frac{3}{1+\alpha-\beta} \leq \frac{2}{\alpha} + 9\log(p)$. Solving for β and making the optimal choice of α gives $\alpha = \frac{\sqrt{6}-2}{9\log(p)}$, which is always less than 0.1. This yields

$$\beta \leq 1 - \frac{5-2\sqrt{6}}{9\log(p)}.$$

Note that $5 - 2\sqrt{6} > 7 - 4\sqrt{3}$, and so the desired result holds. □

Proof of Theorem 4.4.4. First, assume that g is a CM form corresponding to the Hecke character ξ. In this case, $L(s, \mathrm{Ad}^2(g)) = L(s, \chi_p)L(s, \xi^2)$. We derive a lower bound on $L(1, \xi^2)$ and apply the Dirichlet class number formula to bound $L(1, \chi_p)$. We wish to bound

$$\log L(1, \xi^2) = \int_1^\infty -\frac{L'(s, \xi^2)}{L(s, \xi^2)}.$$

We have the trivial bound

$$\left| \frac{L'(s, \xi^2)}{L(s, \xi^2)} \right| \leq -\frac{2\zeta'(s)}{\zeta(s)}. \tag{4.4.2}$$

Also, a virtually identical argument to that in the proof of Theorem 4.4.3 establishes a zero-free region for $L(s, \xi^2)$ and gives that

$$\sum_\rho \mathrm{Re}\left(\frac{1}{s-\rho}\right) \leq \frac{3}{s-1} + 9\log(p) \tag{4.4.3}$$

provided $1 \leq s \leq 1.1$, where the sum is over non-trivial zeroes of $\zeta(s)^3 L(s, \xi^2)^4 L(s, \xi^4)$. It follows from this, and the equation

$$\sum_\rho \frac{1}{s-\rho} = \frac{L'(s, \xi^2)}{L(s, \xi^2)} + \frac{G'(s)}{G(s)}, \tag{4.4.4}$$

where $G(s) = p^{s/2}(2\pi)^{-s}\Gamma\left(s + \frac{p-3}{2}\right)$, that

$$\left| \frac{L'(s, \xi^2)}{L(s, \xi^2)} \right| \leq \frac{3}{4(s-1)} + \frac{15}{4}\log(p) \tag{4.4.5}$$

for $1 \leq s \leq 1.1$. Finally, we must derive a bound on L'/L near $s = 1$.

To do this, we use (4.4.4) with $s = 2$ to derive the bound

$$\sum_\rho \mathrm{Re}\left(\frac{1}{2-\rho}\right) \leq \frac{3}{2}\log(p).$$

By pairing ρ with $1 - \rho$, we see that

$$\sum_{\substack{\rho \\ \gamma \geq \sqrt{3}/2}} \frac{1}{4+\gamma^2} + \frac{1/2}{1+\gamma^2} \leq \frac{3}{2}\log(p).$$

The equation (4.4.3) also implies that $L(s, \xi^2)$ has no zeroes in the region $\{\sigma + it : \sigma \geq \beta_0, |t| \leq s_0 - \beta_0\}$, where $s_0 = 1 + \frac{2\sqrt{3}-3}{9\log(p)}$ and $\beta_0 = 1 - \frac{7-4\sqrt{3}}{9\log(p)}$. Plugging this into (4.4.4) and using the bounds on sums over zeroes derived above, we obtain

$$\left|\frac{L'(s,\xi^2)}{L(s,\xi^2)}\right| \leq \frac{19}{6}\log(p) + \frac{30}{7-4\sqrt{3}}\log^2(p). \tag{4.4.6}$$

We apply (4.4.6) for $1 \leq s \leq 1 + \frac{7-4\sqrt{3}}{40\log^2(p)}$, (4.4.5) for s up to $1 + \frac{1}{3\log(p)}$ and (4.4.2) for the remaining s to derive a bound on $L(1,\xi^2)$. Combining this bound with the bound $L(1,\chi_p) \geq \frac{3\pi}{\sqrt{p}}$ when $p > 163$, we obtain

$$L(1, \mathrm{Ad}^2(g)) \geq \frac{1}{332\sqrt{p}\log(p)^{11/4}}.$$

One can verify that this bound is satisfied with $p \leq 163$ as well.

Now, we assume that g does not have CM. We mimic the argument of Lemma 3 of [108], which is in turn based on Hoffstein's argument for Dirichlet L-functions from [64]. Assume that $p \geq 17$ and set

$$L(s, g \otimes g) = \zeta(s)L(s, \mathrm{Ad}^2(g)) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

A careful inspection of the Euler factors given in Theorem 4.4.1 shows that $b(n) \geq 0$ for all n, and also that $b(n^2) \geq 1$ for all n. Let $\beta = 1 - \frac{7-4\sqrt{3}}{9\log(p)}$ and note that $3/4 < \beta < 1$. We set $x = p^A$ and choose A at the end of the proof (we will choose it to be equal to $16/5$). Consider

$$I = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{L(s+\beta, g\otimes g)x^s\, ds}{s\prod_{r=2}^{10}(s+r)}.$$

We use the fact that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s\, ds}{s\prod_{r=2}^{10}(s+r)} = \begin{cases} \frac{(x+9)(x-1)^9}{10!x^{10}} & \text{if } x > 1 \\ 0 & \text{if } x < 1. \end{cases}$$

This gives

$$I = \sum_{n \leq x} \frac{b(n)(x/n+9)(x/n-1)^9}{10!n^\beta (x/n)^{10}}.$$

We consider only those terms where $x/n \geq 44$. This gives

$$I \geq \frac{1}{10!}\frac{(44+9)(44-1)^9}{44^{10}} \sum_{n \leq \sqrt{x/44}} \frac{1}{n^2} \geq 1.54354$$

for $p \geq 17$. We move the contour in I to $\mathrm{Re}\,(s) = \alpha := -3/2 - \beta$ and pick up poles at $s = 1-\beta$, $s = 0$ and

$s = -2$. This gives

$$\begin{aligned} I =& \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(s+\beta, g\otimes g)x^s\,ds}{s\prod_{r=2}^{10}(s+r)} + \frac{L(1,\mathrm{Ad}^2(g))x^{1-\beta}}{(1-\beta)\prod_{r=2}^{10}(1-\beta+r)} \\ &+ \frac{L(\beta, g\otimes g)}{10!} - \frac{L(-2+\beta, g\otimes g)x^{-2}}{2\cdot 8!}. \end{aligned}$$

Since g does not have CM, Theorem 4.4.3 implies that $L(s, \mathrm{Ad}^2(g))$ has no real zeroes to the right of β. Therefore, we have $L(\beta, \mathrm{Ad}^2(g)) \geq 0$ and since $\zeta(\beta) < 0$, $L(\beta, g\otimes g) < 0$. Since $\beta < 1$, we have $-2+\beta < -1$ and so $L(s, \mathrm{Ad}^2(g)) < 0$. Since $\zeta(-2+\beta) < 0$, it follows that $L(-2+\beta, g\otimes g) > 0$. This gives

$$I - I_2 \leq \frac{L(1,\mathrm{Ad}^2(g))x^{1-\beta}}{(1-\beta)\prod_{r=2}^{10}(1-\beta+r)},$$

where

$$I_2 = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L(s+\beta, g\otimes g)x^s\,ds}{s\prod_{r=2}^{10}(s+r)}.$$

It suffices to bound I_2 in the above inequality. Using the functional equation for $L(s, g\otimes g)$, we have

$$\begin{aligned} |L(-3/2+it, g\otimes g)| =& p^4\pi^{-8}|1/4+it/2|^2|3/4+it/2|^2|k/2-1/4-it/2||k/2-5/4-it/2| \\ & |k/2+1/4-it/2||k/2-3/4-it/2||L(5/2-it, g\otimes g)|. \end{aligned}$$

We have $L(5/2-it, g\otimes g)| \leq \zeta(5/2)^4$, and $|x^s| = p^{A(-3/2-\beta)}$. Note that

$$\frac{1}{|-3/2-\beta+it|\prod_{r=2}^{10}|r-3/2-\beta+it|} \leq \frac{1}{|9/4+it||1/4+it|\prod_{r=3}^{10}|r-5/2+it|}.$$

Putting these estimates together, we get

$$\begin{aligned} |I_2| &\leq \frac{\zeta(5/2)^4 p^{8+A(-3/2-\beta)}}{2^{13}\pi^9}\cdot\int_{-\infty}^{\infty}\frac{|1/2+it|^2|3/2+it|^2|1+it|^4}{|1/4+it||9/4+it|\prod_{r=3}^{10}|r-5/2+it|}\,dt \\ &\leq \frac{0.011322p^{8+A(-3/2-\beta)}}{10!}. \end{aligned}$$

Thus, we have

$$L(1,\mathrm{Ad}^2(g)) \geq (1-\beta)\left(1.54354p^{A(\beta-1)} - 0.011322p^{8-5A/2}\right).$$

Setting $A = 16/5$, we obtain

$$L(1,\mathrm{Ad}^2(g)) \geq \frac{1}{84\log(p)},$$

Numerically, we evaluate $L(1, \mathrm{Ad}^2(g))$ for all newforms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ for $p < 17$ and check that the same relation holds. This completes the proof. $\square$

4.5 Proof of Theorem 4.1.1 and its corollaries

Recall that

$$f(z) = \frac{\eta^p(pz)}{\eta(z)} = \sum_{n=0}^{\infty} pc_p(n) q^{n+\frac{p^2-1}{24}}.$$

We decompose

$$f(z) = e_p E_{\frac{p-1}{2}}(z) + \sum_{i=1}^{s} r_i g_i(z)$$

where $g_i(z)$ are the normalized Hecke eigenforms in $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$. To bound $R(p) = \sum_{i=1}^{s} |r_i|$, we use that

$$r_i = \frac{\langle f, g_i \rangle}{\langle g_i, g_i \rangle}.$$

We derived an upper bound on the numerator in Section 4.3 and a lower bound on the denominator in Section 4.4. Now we prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Define

$$L(p) := \begin{cases} \frac{2}{\pi}\left(1+\frac{1}{p}\right)^{-1} \frac{\left(\frac{p-3}{2}\right)!}{(4\pi)^{\frac{p-1}{2}}} \frac{1}{84\log(p)}, & \text{if } p \equiv 1 \pmod 4, \\ \frac{2}{\pi}\left(1+\frac{1}{p}\right)^{-1} \frac{\left(\frac{p-3}{2}\right)!}{(4\pi)^{\frac{p-1}{2}}} \frac{1}{332\sqrt{p}\log(p)^{11/4}}, & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{4.5.1}$$

Then, Theorem 4.4.4 states that $\langle g_i, g_i \rangle \geq L(p)$ for all i. Sturm's theorem [113] states that a modular form $f \in M_k(\Gamma_0(N), \chi)$ is determined by its first $\frac{k}{12}[\Gamma : \Gamma_0(N)]$ Fourier coefficients. It follows that the dimension of $S_{\frac{p-1}{2}}(\Gamma_0(p), \chi) \leq \frac{p-1}{24}[\Gamma : \Gamma_0(p)]$. In summary, we have

$$\sum_{i=1}^{s} |r_i| \leq \frac{p-1}{24}[\Gamma : \Gamma_0(p)] \frac{A}{L(p)},$$

where A is an upper bound on $|\langle f, g \rangle|$.

Therefore, by Theorem 4.3.2 and (4.5.1) , we arrive at

$$\begin{aligned}
\sum_{i=1}^{s} |r_i| &\leq \frac{(p-1)\pi}{8} \frac{U_\infty(p) + pU_0(p)}{L(p)} \\
&\leq \begin{cases} 98304 \cdot e^{6\pi} p^4 \log p \left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}}, & \text{if } p \equiv 1 \pmod 4, \\ 388535 \cdot e^{6\pi} p^{\frac{9}{2}} (\log p)^{\frac{11}{4}} \left(\frac{e^{1.5}}{8\pi}\right)^{\frac{p-1}{4}}, & \text{if } p \equiv 3 \pmod 4, \end{cases}
\end{aligned}$$

where we have used the inequality $\frac{x^{x-\gamma}}{e^{x-1}} < \Gamma(x) < \frac{x^{x-\frac{1}{2}}}{e^{x-1}}$ from [86], and γ is the Euler constant. □

Before we prove our corollaries, note that for $p > 5$, we have

$$\sigma_{\frac{p-1}{2},\chi_p}(n) \geq n^{\frac{p-3}{2}} \zeta\left(\frac{p-3}{2}\right)^{-1}, \tag{4.5.2}$$

where $\zeta(s)$ is the Riemann zeta function.

Proof of Corollary 4.1.3. By Theorem 4.1.1,

$$\begin{aligned}
|a_p(n)| &\geq \frac{e_p}{\zeta\left(\frac{p-3}{2}\right)} n^{\frac{p-3}{2}} - R(p)d(n)n^{\frac{p-3}{4}} \\
&\geq e_p n^{\frac{p-3}{2}} \left(\frac{1}{\zeta\left(\frac{p-3}{2}\right)} - \frac{2R(p)}{e_p n^{\frac{p-5}{4}}} \right),
\end{aligned}$$

where we have used the fact that $d(n) \leq 2\sqrt{n}$. Since $\zeta(2) - \frac{1}{5} > \frac{2}{5}$, we arrive at $a_p(n) > \frac{2e_p}{5} n^{\frac{p-3}{2}}$ once $n \geq (\frac{10R(p)}{e_p})^{\frac{4}{p-5}}$, as desired. □

Proof of Corollary 4.1.4. Since $\sigma_{\frac{p-1}{2},\chi_p}$ is multiplicative and n is coprime to t, we have

$$\begin{aligned}
&a_p(t^k n) - (\sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1)a_p(n) \\
&\geq e_p \sigma_{\frac{p-1}{2},\chi_p}(n) - R(p)d(n)n^{\frac{p-3}{4}} \left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1 \right) \\
&\geq n^{\frac{p-1}{4}} \left(e_p \zeta\left(\frac{p-3}{2}\right)^{-1} n^{\frac{p-5}{4}} - 2R(p) \left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1 \right) \right).
\end{aligned}$$

Thus, for

$$n > \left(\frac{\zeta(\frac{p-3}{2})}{e_p} 2R(p) \left((k+1)t^{\frac{k(p-3)}{4}} + \sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1 \right) \right)^{\frac{4}{p-5}},$$

we have

$$a_p(t^k n) - (\sigma_{\frac{p-1}{2},\chi_p}(t^k) - 1)a_p(n) > 0.$$

$\square$

4.6 Proof of Theorem 4.1.5

Recall that $F(q) = \frac{q^{1/24}}{\eta(z)}$ and that the generating function $F_t(q)$ for the number of t-core partitions is $F_t(q) := \frac{F(q)}{F(q^t)^t}$. By using the transformation formula for the Dedekind eta function (Theorem 4.2.1) we can easily derive the transformation formula for $F(q)$:

$$F(e^{2\pi i\tau}) = e^{\pi i(\tau-\gamma\tau)/12}e^{-\pi i s(d,c)}e^{\pi i(a+d)/12c}\sqrt{-i(c\tau+d)}F(e^{2\pi i\gamma\tau}),\ \text{for } \gamma\in\Gamma.$$

Using this, we can derive a similar transformation formula for $F_t(q)$. By using this transformation formula and [10, Proposition 6], we can prove the following lemma.

Lemma 4.6.1. *Let $A(t)$ and $B(t)$ be the constants (depending only on t) defined by*

$$A(t) = \begin{cases} \frac{0.05\cdot(2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right)t^{\frac{t}{2}}}, & \text{if } t=6,\\ \frac{(2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right)t^{\frac{t}{2}}}\left(2-\zeta\left(\frac{t-3}{2}\right)\right), & \text{if } t\geq 7,\end{cases}$$

and

$$B(t) = \frac{(2\pi)^{\frac{t-1}{2}}}{\Gamma\left(\frac{t-1}{2}\right)t^{\frac{t}{2}}}\zeta\left(\frac{t-3}{2}\right).$$

Define $e^{-2\pi(1+\frac{2}{t})-\frac{\pi}{12}(1-\frac{1}{t^2})}E(t)$ by

$$\frac{2e^{\frac{\pi}{t}}\,(C(t)-1)}{t^{\frac{t}{2}}}\left(\frac{t(t-1)}{8\pi e}\right)^{\frac{t-1}{4}} + \left(\sum_{2\leq d|t}\left(\frac{1}{d^2-1}\right)^{\frac{t-1}{4}}d^{\frac{t}{2}}\right)\frac{2.1C(t)}{t^{\frac{t}{2}}}\left(\frac{3(t-1)}{e\pi}\right)^{\frac{t-1}{4}} + \frac{2^{\frac{t-1}{2}}}{\pi t^{\frac{t}{2}}}.$$

Then for all integers $n\geq t^2$ and $t\geq 6$, we have

$$A(t)\left(n+\frac{t^2-1}{24}\right)^{\frac{t-3}{2}} - E(t)n^{\frac{t-1}{4}} \leq pc_t(n) \leq B(t)\left(n+\frac{t^2-1}{24}\right)^{\frac{t-3}{2}} + E(t)n^{\frac{t-1}{4}}.$$

Since the proof of this lemma is identical to Lemma 4.3.1 (except for the estimate of S_2), we omit it. Now we are ready to prove Theorem 4.1.5.

Proof of Theorem 4.1.5. By Lemma 4.6.1, for all $n \geq (t+1)^2$, we have

$$\begin{aligned}
&pc_{t+1}(n) - pc_t(n) \\
&\geq A(t+1)\left(n + \frac{t^2+2t}{24}\right)^{\frac{t-2}{2}} - B(t)\left(n + \frac{t^2-1}{24}\right)^{\frac{t-3}{2}} - E(t+1)n^{\frac{t}{4}} - E(t)n^{\frac{t-1}{4}} \\
&\geq n^{\frac{t-3}{2}}\left(A(t+1)\sqrt{n} - B(t)\right) - n^{\frac{t}{4}}\left(E(t+1) + \frac{E(t)}{\sqrt{t}}\right).
\end{aligned}$$

Take n_1 such that if $n > n_1$, then $A(t+1)\sqrt{n} > 2B(t)$. We note that

$$\frac{2B(t)}{A(t+1)} \leq \frac{\zeta\left(\frac{t-3}{2}\right)}{\sqrt{\pi}(2-\zeta\left(\frac{t-3}{2}\right)} t\left(1+\frac{1}{t}\right)^{\frac{t}{2}}.$$

Therefore, we can choose $n_1 = 0.3 \cdot t^2$. Since $n > (t+1)^2$, we always have

$$pc_{t+1}(n) - pc_t(n) \geq B(t)n^{\frac{t-2}{2}} - n^{\frac{t}{4}}\left(E(t+1) + \frac{E(t)}{\sqrt{t}}\right).$$

We estimate $E(t)$ as follows

$$e^{-2\pi(1+\frac{2}{t})-\frac{\pi}{12}(1-\frac{1}{t^2})}E(t) \leq \begin{cases} 6t^{\frac{3}{2}}e^{\frac{\pi}{t}}\left(\frac{e^{1.5}}{8\pi}\right)^{\frac{t-1}{4}}, & \text{if } t \geq 36, \\ 9t^{\frac{5}{2}}\left(\frac{4e^{1.5}}{t\pi}\right)^{\frac{t-1}{4}}, & \text{if } 7 \leq t \leq 36. \end{cases}$$

Therefore, we have deduced that if

$$n \geq \begin{cases} \left(45503t^{\frac{2t+1}{2}}\left(\frac{1}{2^7\pi^3\sqrt{e}}\right)^{\frac{t-1}{4}}\right)^{\frac{4}{t-4}}, & \text{if } t \geq 36, \\ \left(288305t^{\frac{3t+7}{4}}\left(\frac{1}{4\pi^3\sqrt{e}}\right)^{\frac{t-1}{4}}\right)^{\frac{4}{t-4}}, & \text{if } 7 \leq t \leq 35, \end{cases}$$

and $n \geq (t+1)^2$, then $pc_{t+1}(n) > pc_t(n)$, as desired. □

Now we will prove Stanton's conjecture in the cases where $t \leq 198$. Since the bound in Theorem 4.1.5 is quite big for $t \leq 12$, we need to get sharper estimates for $pc_t(n)$ for $4 \leq t \leq 13$. We will achieve this goal by using various arguments. For $t = 4$, we will use the result of Ono and Sze [101], which relates $pc_4(n)$ to the class number of an imaginary quadratic field. For $t = 5$ and $t = 7$ we use that $R(5) = 0$ and $R(7) = 1/8$. For $t = 6$, we use that the generating function for $pc_6(n)$ is a weight $5/2$ modular form. For $8 \leq t \leq 13$, we will use the circle method as in Lemma 4.6.1, but we will set $N = \lceil\sqrt{2\pi n}\rceil$ and estimate $C(t)$ by (4.2.6) instead of (4.2.5) if $t \leq 11$.

For the $t = 4$ case, we first need to find an upper bound for class numbers.

Proposition 4.6.2. *For any discriminant $-D < 0$, we have*

$$h(-D) \leq \frac{w_{-D}}{\pi} \sqrt{D} \log(D).$$

Here w_{-D} is half the number of units in the imaginary quadratic order of discriminant $-D$. (Note that $w_{-D} = 1$ if $-D > 4$).

Proof. One can use the Dirichlet class number formula together with the elementary bound on the sum of a Dirichlet character mod q

$$\left| \sum_{n \leq x} \chi(n) \right| \leq \min(x \bmod q, q - x \bmod q)$$

to prove this result. $\square$

Now, from Ono and Sze [101], we have

$$pc_4(n) = \frac{1}{2} \sum_{d^2 | 8n+5} h\left(\frac{-32n - 20}{d^2} \right).$$

Note that $(-32n - 20)/d^2$ cannot be equal to -3 or -4, since it is always greater than or equal to 4, and $d^2 \neq 8n + 5$ since $d^2 \equiv 0, 1, 4 \pmod 8$. Thus, we have

$$\begin{aligned} pc_4(n) &\leq \frac{1}{2\pi} \sum_{d^2 | 8n+5} \sqrt{(32n + 20)/d^2} \log((32n + 20)/d^2) \\ &\leq \frac{1}{2\pi} \sqrt{32n + 20} \log(32n + 20) \sum_{d^2 | 8n+5} \frac{1}{d}. \end{aligned}$$

If $sq(8n + 5)$ is the largest positive integer so that $sq(8n + 5)^2 | 8n + 5$, then we have

$$\sum_{d^2 | 8n+5} \frac{1}{d} = \sum_{d | sq(8n+5)} \frac{1}{d} = \frac{\sigma(sq(8n + 5))}{sq(8n + 5)}.$$

Combining this with the result of Ivić [66] that $\sigma(n) < 2.59n \log(\log(n))$ for $n \geq 7$, we see that

$$pc_4(n) \leq \frac{2.59}{\pi} \sqrt{8n + 5} \log(32n + 20) \log(\log(8n + 5)).$$

Now, we have $pc_5(n) = \sigma_{2,\chi}(n+1) \geq n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \phi(n)$. Since $\frac{6}{\pi^2} < \frac{\phi(n)\sigma(n)}{n^2}$, we have

$$pc_5(n) > \frac{6}{2.59\pi^2} \frac{n+1}{\log(\log(n+1))}$$

for $n > 6$. It follows from these inequalities that $pc_4(n) < pc_5(n)$ provided $n \geq 1750513$. A computation verifies that the desired inequality holds if $n < 1750513$.

Now, we turn to the $t = 5$ case. Arguing as above, we have that

$$\sigma_{2,\chi}(n) \leq \frac{n^2}{\phi(n)} < \frac{\pi^2}{6}\sigma(n) \leq \frac{2.59}{6}\pi^2(n+1)\log(\log(n+1))$$

provided $n \geq 7$.

Now we will estimate $pc_6(n)$. Let

$$F(z) = \eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2 = \sum_{n=1}^{\infty} a(n)q^n \in S_4(\Gamma_0(6))$$

be the unique weight 4 newform of level 6. If D is a fundamental discriminant, define

$$L(F \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^{s+5/2}}.$$

Proposition 4.6.3. *Let n be a positive integer, and write $72n + 105 = Df^2$, where D is a fundamental discriminant and $f \geq 1$. Then,*

$$\begin{aligned} pc_6(n) = {} & \frac{D^{3/2}L(2,\chi_D)}{240\pi^2} \sum_{d|f} \mu(d)\chi_D(d)d\sigma_3(f/d) \\ & \pm \frac{(D/33)^{3/4}}{10} \sqrt{\frac{L(F \otimes \chi_D, 1/2)}{L(F \otimes \chi_{33}, 1/2)}} \sum_{d|f} \mu(d)d\chi_D(d)a(f/d). \end{aligned}$$

Proof. This follows from writing

$$\frac{\eta(144z)^6}{\eta(24z)} = q^{24} \sum_{n=0}^{\infty} pc_6(n)q^{24n+11},$$

a modular form of weight 5/2 on $\Gamma_0(576)$ with character χ_{12}, as the sum of an Eisenstein series and a cusp form. Cohen[48] has shown that the coefficients of the Eisenstein series involve the values at 2 of Dirichlet L-functions, and Waldspurger[116] has shown that the cusp form coefficients are essentially the square root of the twisted L-value $L(F \otimes \chi_D, 1/2)$. Combining these two results, we get the stated formula. $\square$

A simple estimate shows that the first term above is bounded below by $\frac{1}{40\pi^4}(72n+105)^{3/2}$, and that

$$\left|\sum_{d|f}\mu(d)d\chi_D(d)a(f/d)\right| \leq d(f)f^{3/2}\prod_{p|f}\left(1+\frac{1}{\sqrt{p}}\right).$$

Next, we need an upper bound on $L(F\otimes\chi_D,1/2)$. A variant of the standard convexity bound (see [62], Theorem F.4.1.9 for example) gives the following result.

Lemma 4.6.4. *Assume the notation above. Suppose that g is a newform in $S_k(\Gamma_0(N))$. Then*

$$L(g,1/2) \leq e^{1/2}\left(\frac{N}{2\pi}\right)^{1/4}\frac{\Gamma\left(\frac{k+1}{2}+\frac{1}{2\alpha}\right)}{\Gamma\left(\frac{k}{2}\right)}(1+2\alpha)^2,$$

where $\alpha = \log\left(\frac{N}{2\pi}\right)$.

Specializing to the case at hand, we have that $k=4$ and the conductor of $F\otimes\chi_D$ is bounded by $2D^2$. From this we get

$$|L(F\otimes\chi_D,1/2)| \leq 5.9(2D^2)^{1/4}\log^2(2D^2).$$

Combining this bound with the elementary bound $d(n) \leq \left(\frac{1536}{35}\right)^{1/3}n^{1/3}$, we obtain an upper bound on the second term in Proposition 4.6.3 of

$$0.744(72n+105)\log(72n+105).$$

We see from these bounds that $pc_6(n) > pc_5(n)$ provided $n \geq 58000548$. We refine this estimate by using the bounds $L(2,\chi_D) \geq \frac{6}{\pi^2}$, $L(F\otimes\chi_D,1/2) \leq 5.9(2D^2)^{1/4}\log^2(2D^2)$, and computing the rest of the terms in Proposition 4.6.3 exactly. This requires knowing the first 12000 coefficients of $F(z)$, and shows that $pc_6(n) > pc_5(n)$ for $n > 110868$. It is easy to check up to this bound and verify Stanton's conjecture in this case.

For $t=7$, one can compute that

$$\frac{\eta^7(7z)}{\eta(z)} = \frac{1}{8}E_3(z) - \frac{1}{8}\eta(z)^3\eta(7z)^3.$$

The latter form is a Hecke eigenform and from this it follows that

$$\frac{3}{4\pi^2}(n+2)^2 - \frac{1}{8}d(n)n \leq pc_7(n) \leq \frac{\pi^2}{48}(n+2)^2 + \frac{1}{8}d(n)n.$$

This bound makes it easy to check Stanton's conjecture.

For $8 \le t \le 13$, as we mentioned above, we can get sharper estimates than Lemma 4.6.1 by setting $N = \sqrt{2\pi n}$. For $n \ge t^2$,

$$A(t)\left(n+\frac{t^2-1}{24}\right)^{\frac{t-3}{2}} - E'(t)\lceil\sqrt{2\pi n}\rceil^{\frac{t-1}{2}} \le pc_t(n) \le B(t)\left(n+\frac{t^2-1}{24}\right)^{\frac{t-3}{2}} + E'(t)\lceil\sqrt{2\pi n}\rceil^{\frac{t-1}{2}},$$

where the $A(t)$ and $B(t)$ are the constants defined in Lemma 4.6.1 and

$$E'(t) := e^{\frac{25}{24}} e^{-2\pi(1+\frac{2}{t})-\frac{\pi}{12}(1-\frac{1}{t^2})} E(t).$$

By using the bounds mentioned above together with MAGMA, we can verify Stanton's conjecture for $t \le 198$.

Chapter 5

Combinatorial proofs for q-series identities in Ramanujan's lost notebook

5.1 Introduction

In [32], B. C. Berndt and A. J. Yee provided bijective proofs for several entries found in Ramanujan's lost notebook [102]. The entries for which combinatorial proofs were given arise from the Rogers–Fine identity and false theta functions, and are found in Chapter 9 of [17]. Although G. E. Andrews [12] had previously devised a combinatorial proof of the Rogers–Fine identity, the combinatorics of each of the identities proved in [32] is substantially different from that in Andrews's proof, so that even what might be considered small or subtle changes in an identity markedly alter the combinatorics. This chapter is a sequel to [32] in that we combinatorially prove further entries from Ramanujan's lost notebook. The entries to be examined in this chapter are connected with either Heine's transformation or partial theta functions. Readers may have difficulty discerning the connections of some of the entries with either Heine's transformation or partial theta functions. To see these relationships, consult the book [18] by Andrews and Berndt, where all of the identities established in this chapter are proved analytically.

Algorithm Z of D. Zeilberger plays an important role. Euler's partition identity and Sylvester's bijective proof of it also play leading roles. We will recall these and other bijections in Section 5.2. In Section 5.3, we present combinatorial proofs of some identities arising from Euler's identity. The next goal is to provide combinatorial proofs of entries that are related to Heine's ${}_2\phi_1$ transformation formula. Some of the proofs follow along the lines of Andrews's proof of Heine's ${}_2\phi_1$ transformation formula [11], but others do not. In Section 5.5, we introduce a new class of partitions, namely partitions with the parity sequence. We obtain the generating function of these partitions analytically and bijectively. Using this generating function, we give a combinatorial proof of an identity that is related to partial theta functions.

[3]This chapter is based on the joint paper with Bruce Berndt and Ae Ja Yee [30]. I am grateful to my coauthors, Berndt and Yee, for their permission to include our joint work here.

5.2 Preliminary results

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 + \cdots + \lambda_r = n$, and we shall write $\lambda \vdash n$ (see [15].) We relax our definition of a partition by including 0 as a part, if necessary. We denote the number of parts of a partition λ by $\ell(\lambda)$. As a convention, we denote the partition of 0 by $\emptyset$.

We recall some familiar bijections that are used in the sequel.

Sylvester's bijection. Sylvester's map for Euler's identity

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty \tag{5.2.1}$$

and many further contributions of Sylvester have been discussed by Andrews in [14]. We note here that Sylvester's bijection preserves the following statistic [50, 51, 109]:

$$\ell(\lambda) + (\lambda_1 - 1)/2 = \mu_1, \tag{5.2.2}$$

where λ is a partition into odd parts and μ is the partition into distinct parts associated with λ under Sylvester's bijection.

Franklin's involution. Recall that Franklin's involution provides a bijective proof of Euler's pentagonal number theorem [15, pp. 10–11]

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{5.2.3}$$

Wright's bijection. Recall that Wright's bijection [119] gives a bijective proof for the Jacobi triple product identity

$$(-zq;q)_\infty(-z^{-1};q)_\infty = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}. \tag{5.2.4}$$

Algorithm Z and its application. The following bijection is an application of Algorithm Z discovered by D. Zeilberger [19, 35]. It was first observed by J. T. Joichi and D. Stanton [70] that Algorithm Z can apply in this way to the q-binomial theorem and used by Yee in [120] to establish a combinatorial proof for Ramanujan's ${}_1\psi_1$ summation formula.

5.3 Bijective proofs of identities arising from the Euler identity

A combinatorial proof of the following theorem was given by Berndt and Yee in the process of combinatorially proving another entry from Ramanujan's lost notebook [32, p. 413]. We now provide a shorter proof.

Theorem 5.3.1. [102, p. 38], [18, Entry 1.6.4] *For each complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{5.3.1}$$

Proof. Replace a by $-a$ in (5.3.1). Then the left-hand side generates partitions λ into odd parts, and the exponent of a equals $\ell(\lambda) + (\lambda_1 - 1)/2$. The right-hand side of (5.3.1) generates partitions into distinct parts, and the exponent of a is the largest part. The identity now follows by Sylvester's bijection and its preserved statistic (5.2.2). □

Theorem 5.3.2. [102, p. 31], [18, Entry 6.5.1] *We have*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{12n^2+n}(1-q^{22n+11}) + q\sum_{n=0}^{\infty} q^{12n^2+7n}(1-q^{10n+5}) \tag{5.3.2}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{12n^2+5n}(1-q^{14n+7}) + q^2\sum_{n=0}^{\infty} q^{12n^2+11n}(1-q^{2n+1}). \tag{5.3.3}$$

Proof. We prove the first identity. The second one can be proved in a similar way and we omit its proof. Replacing q by q^2 in (5.3.2), we obtain the identity

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} = \sum_{n=0}^{\infty} q^{24n^2+2n}(1-q^{44n+22}) + q^2\sum_{n=0}^{\infty} q^{24n^2+14n}(1-q^{20n+10}). \tag{5.3.4}$$

The left-hand side generates partitions λ into an even number of odd parts with weight $(-1)^{(\lambda_1-1)/2}$. Clearly, λ is a partition of an even number $2N$. Thus, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} = \sum_{N=0}^{\infty} \sum_{\lambda\in O(2N)} (-1)^{(\lambda_1-1)/2} q^{2N}, \tag{5.3.5}$$

where $O(2N)$ is the set of partitions of $2N$ into odd parts. Let $D(2N)$ be the set of partitions of $2N$ into distinct parts. It follows from Euler's identity (5.2.1) that $O(2N)$ and $D(2N)$ are equinumerous. Let μ be the image of λ under Sylvester's bijection, which is a partition in $D(2N)$. Since λ is a partition of $2N$ into odd parts, $\ell(\lambda)$ is even. Thus we see from (5.2.2) that

$$(-1)^{(\lambda_1-1)/2} = (-1)^{\mu_1}.$$

It then follows that

$$\sum_{N=0}^{\infty} \sum_{\lambda \in O(2N)} (-1)^{(\lambda_1-1)/2} q^{2N} = \sum_{N=0}^{\infty} \sum_{\mu \in D(2N)} (-1)^{\mu_1} q^{2N}. \tag{5.3.6}$$

We now apply Franklin's involution for Euler's pentagonal number theorem (5.2.3), in which we compare the smallest part and the number of consecutive parts including the largest part. Note that in the pentagonal number theorem, partitions π have weight $(-1)^{\ell(\pi)}$. However, the involutive proof still works in our setting, since we move the smallest part to the right of the consecutive parts or subtract 1 from each of the consecutive parts in order to add the number of consecutive parts as a new part. Thus only the partitions of the even pentagonal numbers survive under the involution in our setting, too. Under the involution, only partitions λ of the form $(2n, 2n-1, \ldots, n+1)$ or $(2n-1, 2n-2, \ldots, n)$ survive. That is, $\lambda \vdash n(3n \pm 1)/2$. It is easy to see that

$$\begin{aligned}
n(3n+1)/2 &\equiv 0 \pmod 2, \quad \text{if } n \equiv 0, 1 \pmod 4, \\
n(3n-1)/2 &\equiv 0 \pmod 2, \quad \text{if } n \equiv 0, 3 \pmod 4.
\end{aligned}$$

When $n \equiv 0, 1 \pmod 4$, the surviving partition of $n(3n+1)/2$ has parts $2n, 2n-1, \ldots, n+1$. The largest part of the partition is even. When $n \equiv 0, 3 \pmod 4$, the largest part of the partition of $n(3n-1)/2$ is odd. Then

$$\begin{aligned}
\sum_{N=0}^{\infty} \sum_{\mu \in D(2N)} (-1)^{\mu_1} q^{2N} &= \sum_{\substack{n=0 \\ n(3n+1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n+1)/2} - \sum_{\substack{n=1 \\ n(3n-1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n-1)/2} \\
&= \sum_{n=0}^{\infty} q^{24n^2+2n}(1 - q^{44n+22}) + q^2 \sum_{n=0}^{\infty} q^{24n^2+14n}(1 - q^{20n+10}).
\end{aligned} \tag{5.3.7}$$

Hence, by (5.3.5), (5.3.6) and (5.3.7), we complete the proof of (5.3.4) and therefore also of Theorem 5.3.2. ☐

In the formulation of Ramanujan's next two identities, it will be convenient to use the notation for

Ramanujan's theta functions, namely,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Theorem 5.3.3. [102, p. 31], [18, Entry 6.5.2] *We have*

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n}} = \frac{f(q^5,q^3)}{(q;q)_\infty} \tag{5.3.8}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n+1}} = \frac{f(q^7,q)}{(q;q)_\infty}. \tag{5.3.9}$$

Proof. We prove the first identity. The second one can be proved in a similar way. In (5.3.8), replace q by q^2. Then we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_{2n}} = \frac{f(q^{10},q^6)}{(q^2;q^2)_\infty}. \tag{5.3.10}$$

The left-hand side generates partitions into an even number of odd parts. Equivalently, it generates partitions of an even number into odd parts. Thus, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_{2n}} = \sum_{N=0}^{\infty} \sum_{\lambda \in O(2N)} q^{2N} = \sum_{N=0}^{\infty} \sum_{\mu \in D(2N)} q^{2N},$$

where the second equality follows from Sylvester's bijection. By decomposing the parts of μ into even parts and odd parts, we obtain

$$\sum_{N=0}^{\infty} \sum_{\mu \in D(2N)} q^{2N} = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \sum_{\nu \in DO(2n)} q^{2n},$$

where $DO(2n)$ is the set of partitions on $2n$ into distinct odd parts. Let ν^1 and ν^3 be the partitions consisting of parts of ν congruent to 1 and 3 modulo 4, respectively. Note that since ν is a partition of $2n$, the number of parts of ν is even. Thus $\ell(\nu^1) \equiv \ell(\nu^3) \pmod 2$. We now use staircase 4-modular Ferrers diagrams for the partitions ν^1 and ν^3, in which the triangles on the main diagonal have the residue 1 or 3 and the remaining boxes have 4. We then apply Wright's bijection to the pair (ν^1, ν^3). Since $\ell(\nu^1) \equiv \ell(\nu^3)$ $\pmod 2$, we collect only even powers of z from the summation on the right-hand side of the Jacobi triple product identity (5.2.4). By substituting q^{-1} and q^4 for z and q, respectively, we obtain

$$\sum_{n=0}^{\infty} \sum_{\nu \in DO(2n)} q^{2n} = \frac{1}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{8k^2+2k}.$$

Thus it follows that

$$(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \sum_{\nu \in DO(2n)} q^{2n} = \frac{(-q^2;q^2)_\infty}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{8k^2+2k} = \frac{1}{(q^2;q^2)_\infty} f(q^{10},q^6).$$

This completes our bijective proof of (5.3.10). □

Corollary 5.3.4. *[102, p. 35], [18, Entry 1.7.7] We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n(1-q^{2n+1})} = qf(q,q^7).$$

Proof. By Theorem 5.3.3, it suffices to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n(1-q^{2n+1})} = q(q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q)_{2m+1}} = \sum_{m=0}^{\infty} q^{m+1}(q^{2m+2})_\infty.$$

Let λ be a partition arising from $(q^{2m+2})_\infty$. Then the parts of λ are distinct and larger than $2m+1$. Let $n = \ell(\lambda)$. Detach $2m$ from each of the n parts. By combining this with m from q^{m+1}, we have $(2n+1)m$, which is generated by $1/(1-q^{2n+1})$. The resulting parts of λ form a partition into distinct parts that are larger than 1 with weight $(-1)^n$. Such partitions are generated by

$$\frac{(-1)^n q^{2+3+\cdots+(n+1)}}{(q)_n}.$$

Combining them with q that was left from q^{m+1}, we arrive at

$$\frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n}.$$

This completes the proof. □

The following corollary can be proved by a similar argument, and so we omit the proof.

Corollary 5.3.5. [102, p. 35], [18, Entry 1.7.9] *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n(1-q^{2n+1})} = f(q^3,q^5).$$

5.4 Bijective proofs of identities arising from Heine's transformation

The identities in this section are proved in [18, Chapter 1] by appealing to Heine's transformation or some variant or generalization thereof.

Theorem 5.4.1. [102, p. 16], [18, Entry 1.4.8] *For arbitrary complex numbers a, b,*

$$\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n} = (-bq;q^2)_\infty \sum_{n=0}^{\infty}\frac{(aq)^{2n}}{(q;q)_{2n}(-bq;q^2)_n} + (-bq^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-bq^2;q^2)_n}. \tag{5.4.1}$$

Proof. Rewrite the left-hand side of (5.4.1) as

$$\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty}\frac{b^n q^{n^2}}{(aq^{n+1};q)_\infty (q^2;q^2)_n}. \tag{5.4.2}$$

The right-hand side is a generating function for vector partitions (π,ν) such that π is a partition into parts that are strictly larger than n, and ν is a partition into n distinct odd parts. We examine these partitions in two cases.

Case 1: π has an even number of parts. Let $2k$ be the number of parts in π. Detach n from each part of π and attach $2k$ to each part of ν. Denote the resulting partitions by σ and λ, respectively. It is clear that σ is a partition into $2k$ parts, and λ is a partition into distinct odd parts that are greater than or equal to $2k+1$. These are generated by

$$\sum_{k=0}^{\infty}\frac{(aq)^{2k}}{(q;q)_{2k}}(-bq^{2k+1};q^2)_\infty. \tag{5.4.3}$$

Case 2: π has an odd number of parts. Let $2k+1$ be the number of parts in π. Detach $2k+1$ from each part of π and attach $2k+1$ to each part of ν. By reasoning similar to that above, we can see that the resulting partition pairs are generated by

$$\sum_{k=0}^{\infty}\frac{(aq)^{2k+1}}{(q;q)_{2k+1}}(-bq^{2k+2};q^2)_\infty. \tag{5.4.4}$$

Combining the two generating functions (5.4.3) and (5.4.4) together with (5.4.2), we complete the proof. □

Theorem 5.4.2. [102, p. 10], [18, Entry 1.4.9] *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n^2} = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{5.4.5}$$

Proof. Multiplying both sides of (5.4.5) by $(q)_\infty$, we obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n}(q^{n+1};q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n}(-q^{n+1};q)_\infty, \tag{5.4.6}$$

since $(q^2;q^2)_\infty = (-q;q)_\infty (q;q)_\infty$. The left side of (5.4.6) is a generating function for the pair of partitions (π,ν), such that π is a partition into n distinct parts and ν is a partition into distinct parts that are strictly larger than n, and where the exponent of (-1) is the number of parts in ν. For a given partition pair (π, ν) generated by the left side of (5.4.6), let k be the number of parts in ν. Detach n from each part of ν and attach k to each part of π. Then we obtain partition pairs (σ, λ), such that σ is a partition into k distinct parts and λ is a partition into distinct parts that are strictly larger than k, and the exponent of (-1) is the number of parts in σ. These partitions are generated by the right side of (5.4.6). Since this process is easily reversible, our proof is complete. □

The identity in Theorem 5.4.2 is connected with the theory of gradual stacks with summits [13].

Theorem 5.4.3. [102, p. 10], [18, Entry 1.4.12] *For each $n > 0$,*

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q)_m}(-bq^{nm+n};q^n)_\infty = \sum_{m=0}^{\infty} \frac{b^m q^{nm(m+1)/2}}{(q^n;q^n)_m}(-aq^{nm+1};q)_\infty.$$

Proof. First observe that $\frac{a^m q^{m(m+1)/2}}{(q)_m}$ generates partitions into m distinct parts, where the exponent of a is the number of parts. Second, $(-bq^{nm+n};q^n)_\infty$ generates partitions into distinct parts, where each part is at least $nm+n$, each part is a multiple of n, and the exponent of b equals the number of parts. Let (π, ν) be the partition pair generated by $\frac{a^m q^{m(m+1)/2}}{(q)_m}$ and $(-bq^{nm+n};q^n)_\infty$, respectively. Detach nm from each part of ν. The remaining partition is generated by $\frac{b^k q^{nk(k+1)/2}}{(q^n;q^n)_k}$. Attach mk to each part of π. Then the resulting partition is a partition into distinct parts that are greater than or equal to $nk+1$. Since this process is reversible, we are finished with the proof. □

Theorem 5.4.4. [102, p. 30], [18, Entry 1.4.17] *For each $n > 0$,*

$$(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}} = (-bq)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q)_m(-bq)_{nm}}. \tag{5.4.7}$$

Proof. Rewrite the left-hand side of (5.4.7) in the form

$$(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}} = \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m}(-aq^{mn+1})_\infty. \tag{5.4.8}$$

First, $\frac{b^m q^{m(m+1)/2}}{(q)_m}$ generates partitions into m distinct parts with the exponent of b keeping track of the number of parts. Second, $(-aq^{mn+1})_\infty$ generates partitions into distinct parts, each strictly larger than mn. Let (σ, ν) denote a pair of partitions generated by $\frac{b^m q^{m(m+1)/2}}{(q)_m}$ and $(-aq^{mn+1})_\infty$, respectively. Let k denote the number of parts in ν. Detach mn from each part of ν and denote the resulting partition by ν'. Attach kn to each part of σ and denote the resulting partition by σ'. Then ν' is a partition into k distinct parts, and σ' is a partition into distinct parts, each strictly larger than kn. Such partitions are generated by the right side of (5.4.8). Since the process is reversible, the proof is complete. □

Theorem 5.4.4 provides a generalization of a certain *Duality* that was utilized by D. M. Bressoud [34] in connecting the well-known identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{1}{(-q^2;q^2)_\infty (q;q^5)_\infty (q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{1}{(-q^2;q^2)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}$$

of L. J. Rogers [107] with the Rogers–Ramanujan identities. In particular, if we consider the case $n = 1$ in Theorem 5.4.4,

$$\sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}(-aq^{m+1})_\infty}{(q)_m} = \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}(-bq^{m+1})_\infty}{(q)_m}, \tag{5.4.9}$$

and replace q by q^2 and a by a/q in (5.4.9) we obtain the identity

$$F(a,b) := \sum_{m=0}^{\infty} \frac{a^m q^{m^2}(-bq^{2m+2};q^2)_\infty}{(q^2;q^2)_m} = \sum_{m=0}^{\infty} \frac{b^m q^{m^2+m}(-aq^{2m+1};q^2)_\infty}{(q^2;q^2)_m} = F(bq, a/q). \tag{5.4.10}$$

Note that the transformation T defined by

$$T(F(a,b)) = F(bq, aq^{-1})$$

is an involution. Thus (5.4.10) is a fixed point under this involution.

Bressoud [34] does not state this Duality explicitly but uses the underlying combinatorics in his paper [34]. K. Alladi [8] observed the involution (5.4.10) as Bressoud's Duality and used it to connect six identities of Rogers [107] with the Rogers–Ramanujan identities via the modified convergence of a certain continued fraction of Ramanujan, A. Selberg, and B. Gordon.

Similarly Theorem 5.4.3 is also a generalization of Bressoud's Duality.

Theorem 5.4.5. [102, p. 42], [18, Entry 1.5.1] *We have*

$$\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q)_n}=(-aq^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q^2;q^2)_n(-aq^2;q^2)_n} \tag{5.4.11}$$

$$=(-aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{a^nq^{n^2+n}}{(q^2;q^2)_n(-aq;q^2)_n}. \tag{5.4.12}$$

Proof. We prove (5.4.11). Moving $(-aq^2;q^2)_\infty$ inside the summation sign and using a corollary of the q-binomial theorem [15, p. 19, Eq. (2.2.6)], namely,

$$(-aq^{2n+2};q^2)_\infty=\sum_{m=0}^{\infty}\frac{a^mq^{m^2+m+2mn}}{(q^2;q^2)_m},$$

we find that it suffices to show that

$$\sum_{k=0}^{\infty}\frac{a^kq^{k^2}}{(q)_k}=\sum_{m,n=0}^{\infty}\frac{a^{m+n}q^{n^2+m^2+m+2mn}}{(q^2;q^2)_m(q^2;q^2)_n}. \tag{5.4.13}$$

Let us interpret the right side of (5.4.13). Consider a Durfee square of side $m+n$. Attach 1 to each of the first m rows. Append the 2-modular diagram of a partition generated by $\frac{1}{(q^2;q^2)_m}$ to the first m rows. Finally append the 2-modular diagram of a partition generated by $\frac{1}{(q^2;q^2)_n}$ to the next n rows. Then, it is clear that the resulting partition is generated by the sum on the left side of (5.4.13). For the reverse process, let π be a partition generated by the left side of (5.4.13). Then π has a Durfee square of side k, and below the Durfee square there are no parts. Let π_r be a partition to the right of the Durfee square in π. Let m be the number of odd parts in π_r. Rearrange the order of π_r so that the first m parts are odd. Detach 1 from each part of the first m parts of π_r. Then the first m parts are generated by $\frac{1}{(q^2;q^2)_m}$, and the remaining parts are generated by $\frac{1}{(q^2;q^2)_{k-m}}$. Setting $n=k-m$, we are done.

Since the proof of (5.4.12) is similar, we omit it. □

5.5 Partitions with a parity sequence

Let D_n be the set of partitions into n distinct parts less than $2n$ such that the smallest part of each partition is 1, and if $2k-1$ is the largest odd part, then all odd positive integers less than $2k-1$ occur as parts. For a partition $\lambda \in D_n$, we define the *parity sequence* as the longest sequence of decreasing consecutive numbers *containing* the largest odd part and denote its length by $c(\lambda)$. Thus, the largest part of the parity sequence might be even. For instance, when $n = 5$,

$$\begin{aligned} c((5,4,3,2,1)) &= 5, \\ c((8,6,5,4,3,1)) &= 4, \\ c((9,7,6,5,3,1)) &= 1. \end{aligned}$$

Let

$$\lambda = (\lambda_1, \dots, \lambda_s, \underline{\lambda_{s+1}, \dots, \lambda_{s+c}}, \lambda_{s+c+1}, \dots, \lambda_n) \in D_n,$$

where its parity sequence is underlined. By the definition of a parity sequence, we see that

(P1) $\lambda_1, \dots, \lambda_s$ are even;

(P2) all the positive odd integers less than or equal to λ_{s+1} occur in λ;

(P3) λ_{s+c} is odd and $\lambda_{s+c} = \lambda_{s+c+1} + 2$.

We now compute the generating function of D_n. For a partition $\lambda \in D_n$, let k be the number of odd parts of λ. Then it follows from the definition of D_n that the odd integers $1, 3, \dots, 2k-1$ occur in λ and the other $n-k$ parts are distinct even numbers. Note that the generating function of partitions into m distinct even parts less than $2n$ is [15, pp. 33–35]

$$q^{m(m+1)} \begin{bmatrix} n-1 \\ m \end{bmatrix}_{q^2},$$

as the q-binomial coefficient $\begin{bmatrix} a \\ b \end{bmatrix}_q$ generates partitions into at most b parts $\leq (a-b)$ for $0 \leq b \leq a$, where

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_a}{(q;q)_b (q;q)_{a-b}}, & \text{if } 0 \leq b \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{\lambda \in D_n} q^{\lambda_1+\cdots+\lambda_n} = \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}. \tag{5.5.1}$$

Lemma 5.5.1. *For any positive integer n,*

$$\sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} = (-q;q)_{n-1} q^{n(n+1)/2}. \tag{5.5.2}$$

Proof. Let $f_n(q) = (-q;q)_{n-1} q^{n(n+1)/2}$. Then, for $n \geq 1$,

$$f_{n+1}(q) = (q^{n+1} + q^{2n+1}) f_n(q).$$

We prove the lemma by showing that the left-hand side of (5.5.2) satisfies the same recurrence as $f_n(q)$. First of all, when $n = 1$, (5.5.2) holds true. For $n \geq 1$, using a familiar recurrence for $\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$ [15, Eq. (3.3.4)], we find that

$$\begin{aligned}
&\sum_{k=0}^{n} q^{(n+1-k)^2+k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n+1-k)^2+k^2+k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} + q^{n^2+n+1} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n+1-k)^2+k^2+k} \left(q^{2k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{q^2} \right) + q^{n^2+n+1} \\
&= q^{(n+1)^2} + \sum_{k=1}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-2} q^{(n-k)^2+(k+1)^2+k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + q^{n^2+n+1} \\
&= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+3k+2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} \\
&= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(k+1)^2+(n-k-1)^2+3(n-k-1)+2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}
\end{aligned}$$

$$= \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+2n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k+n+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2}$$

$$= (q^{2n+1} + q^{n+1}) \sum_{k=0}^{n-1} q^{(n-k)^2+k^2+k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2},$$

which completes the proof. □

The following proof is due to S.O. Warnaar [117].

Proof. Write

$$f_a(n) = \sum_{k=0}^{n-1} \frac{(-1)^k (q^{-n};q)_k (-q^{-n};q)_k q^{k^2+ak}}{(q)_k (-q)_k}.$$

Then Lemma 6.1 is $f_0(n) = (-q;q)_n q^{-n(n+1)/2}$. Change k to $n-k$. Then $f_a(n) = q^{(a-1)n} f_{2-a}(n)$. Therefore

$$\begin{aligned} f_0 = \frac{f_0(n) + q^{-n} f_2(n)}{2} &= \frac{1}{2} \sum_{k=0}^{n-1} \frac{(1+q^{2k-n})(-1)^k (q^{-n};q)_k (-q^{-n};q)_k q^{k^2}}{(q)_k (-q)_k} \\ &= \frac{1+q^{-n}}{2} \lim_{b,c\to\infty} {}_6W_5(-q^{-n}; b, c, q^{-n}; -q/bc) = (-q;q)_n q^{-n(n+1)/2}, \end{aligned}$$

where for the last identity we have used Rogers' identity,

$${}_6\phi_5 \left(\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n}; q, \frac{aq^{n+1}}{bc} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} \right) = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}.$$

□

We can prove the following theorem using (5.5.1) and (5.5.2). However, we provide a combinatorial proof.

Theorem 5.5.2. *For any positive integer n, the generating function of D_n is*

$$(-q;q)_{n-1} q^{n(n+1)/2}.$$

Proof. For a positive integer n, let $\tau = (n, n-1, \ldots, 2, 1)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be a partition into distinct parts less than n. We insert the parts μ_i in decreasing order into τ as follows.

Insertion: Let π be τ and begin with $i = 1$.

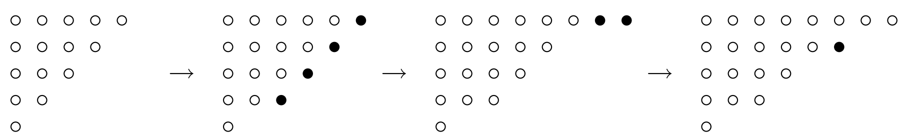

Figure 5.1: Insertion of $\mu = (4, 2, 1)$ into $\tau = (5, 4, 3, 2, 1)$.

(1) If $\pi_i + \mu_1$ is even, then add μ_1 to π_i, i.e., add μ_1 horizontally to π, and add 1 to i; if $\pi_i + \mu_1$ is odd, then add 1 to each of the $\pi_i, \ldots, \pi_{i+\mu_1-1}$, i.e., add μ_1 vertically down starting from π_i, and the i remains the same.

(2) By an abuse of notation, let us denote the resulting partition by π.

(3) Repeat the process with $\mu_2, \ldots, \mu_\ell$, i.e., until the parts of μ are depleted.

Figure 5.1 illustrates our insertion with an example.

Throughout the proof, we assume that $\pi_0 = \infty$. We first show that the final π is a partition in D_n with parity sequence $(\pi_{s+1}, \ldots, \pi_{s+c})$ such that if μ_ℓ was inserted horizontally, then

$$c \geq \mu_\ell \quad \text{and} \quad \pi_s - \pi_{s+1} - 1 = \mu_\ell; \tag{5.5.3}$$

and if μ_ℓ was inserted vertically, then

$$c = \mu_\ell \quad \text{and} \quad \pi_s - \pi_{s+1} - 1 > \mu_\ell. \tag{5.5.4}$$

We use induction on ℓ. If $\ell = 1$, then

$$\pi = \begin{cases} (n + \mu_1, \underline{n-1, n-2, \ldots, 2, 1}), & \text{if } n + \mu_1 \text{ is even,} \\ (\underline{n+1, n, \ldots, n - \mu_1 + 2}, n - \mu_1, \ldots, 2, 1), & \text{if } n + \mu_1 \text{ is odd,} \end{cases}$$

where in each case the parity sequence is underlined. Since $\mu_1 < n$, we see that $\pi \in D_n$ and the conditions in (5.5.3) and (5.5.4) are satisfied. Given $\tau = (n, n-1, \ldots, 1)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$, suppose that the partition π resulting from the insertion of $\mu_1, \ldots, \mu_{\ell-1}$ satisfies either (5.5.3) or (5.5.4). We denote

$$\pi = (\pi_1, \ldots, \pi_s, \underline{\pi_{s+1}, \ldots, \pi_{s+c}}, \pi_{s+c+1}, \ldots, \pi_n) \in D_n,$$

where its parity sequence is underlined. By (P1), we see that π_s is even. Since $\mu_j > 1$ for any $j < \ell$, it follows from the definition of insertion that the last horizontal insertion happened at the s-th part. Thus, in order to insert μ_ℓ, we need to examine the parity of $\pi_{s+1}+\mu_\ell$ by (P1). If $\pi_{s+1}+\mu_\ell$ is even, then we make a horizontal insertion; namely, the resulting partition is

$$\pi' = (\pi_1, \dots, \pi_s, \pi_{s+1}+\mu_\ell, \pi_{s+2}, \dots, \pi_{s+c}, \pi_{s+c+1}, \dots, \pi_n).$$

Since $\pi \in D_n$, all odd positive integers $\leq \pi_{s+1}$ occur in π, from which it follows that all odd positive integers $\leq \pi_{s+2}$ occur in π'. Also, since $\pi_s - \pi_{s+1} > \mu_{\ell-1}$ by (5.5.3) and (5.5.4), we see that

$$\pi'_s - \pi'_{s+1} = \pi_s - (\pi_{s+1}+\mu_\ell) > \mu_{\ell-1} - \mu_\ell \geq 1.$$

Thus $\pi' \in D_n$. We now show that π' satisfies (5.5.3). Since $c \geq \mu_{\ell-1}$ by (5.5.3) and (5.5.4), and $\mu_{\ell-1} > \mu_\ell$, we see that the parity sequence of π' is $(\pi_{s+2}, \dots, \pi_{s+c})$, which has length $c-1 \geq \mu_\ell$. Also, since $\pi_{s+1} = \pi_{s+2}+1$,

$$\pi'_{s+1} - \pi'_{s+2} = \pi_{s+1} + \mu_\ell - \pi_{s+2} = \mu_\ell + 1.$$

Therefore, π' is a partition in D_n satisfying (5.5.3). If $\pi_{s+1}+\mu_\ell$ is odd, then we make a vertical insertion; namely, the resulting partition is

$$\pi' = (\pi_1, \dots, \pi_s, \pi_{s+1}+1, \dots, \pi_{s+\mu_\ell}+1, \pi_{s+\mu_\ell+1}, \dots, \pi_n).$$

Since $c \geq \mu_{\ell-1}$ by (5.5.3) and (5.5.4), and $\mu_{\ell-1} > \mu_\ell$, we see that the parity sequence of π' is

$$(\pi_{s+1}+1, \dots, \pi_{s+\mu_{\ell+1}}+1),$$

whose length is μ_ℓ. Also, since $\pi_s - \pi_{s+1} > \mu_{\ell-1} > \mu_\ell$,

$$\pi'_s - \pi'_{s+1} = \pi_s - (\pi_{s+1}+1) > \mu_{\ell-1} - 1 \geq \mu_\ell.$$

Thus π' satisfies (5.5.4). We now show that $\pi' \in D_n$. Since $\pi_{s+1}+\mu_\ell$ is odd, we see that $\pi_{s+\mu_\ell}$ is even, so $\pi_{s+\mu_\ell}+1$ and $\pi_{s+\mu_\ell+1}$ are consecutive odd integers. Since $\pi \in D_n$, all odd positive integers $\leq \pi_{s+1}$ occur in π, from which it follows that all odd positive integers $\leq \pi_{s+1}+1$ occur in π'. Therefore, π' is a partition in D_n satisfying (5.5.4).

We now show that the map is bijective by defining its inverse. Let

$$\lambda = (\lambda_1, \dots, \lambda_s, \underline{\lambda_{s+1}, \dots, \lambda_{s+c}}, \lambda_{s+c+1}, \dots, \lambda_n) \in D_n,$$

where its parity sequence is underlined.

Deletion: We now compare c and $(\lambda_s - \lambda_{s+1} - 1)$.

(1) If there is no λ_s or $c < (\lambda_s - \lambda_{s+1} - 1)$, then we let $\sigma_1 = c$ and subtract 1 from each of $\lambda_{s+1}, \dots, \lambda_{s+c}$, i.e., subtract σ_1 vertically from λ; if $c \geq (\lambda_s - \lambda_{s+1} - 1)$, then we let $\sigma_1 = \lambda_s - \lambda_{s+1} - 1$ and subtract $(\lambda_s - \lambda_{s+1} - 1)$ from λ_{s-1}, i.e., subtract σ_1 horizontally from λ.

(2) By an abuse of notation, let us denote the resulting partition by λ.

(3) Repeat the process until we arrive at $\lambda = (n, n-1, \dots, 1)$; we record the amount we subtract in the i-th step as σ_i.

We now show that this process is well-defined, i.e., the resulting partition in each step is still in D_n and the sequence $\sigma_1, \sigma_2, \dots$ is strictly increasing with each part less than n. If σ_1 was subtracted vertically, then the resulting partition is

$$(\lambda_1, \dots, \lambda_s, \lambda_{s+1} - 1, \dots, \lambda_{s+c} - 1, \lambda_{s+c+1}, \dots, \lambda_n). \tag{5.5.5}$$

It follows from (P1), (P2), and (P3) that all the positive odd integers less than the largest odd part occur. If σ_1 was subtracted horizontally, then the resulting partition is

$$(\lambda_1, \dots, \lambda_{s-1}, \lambda_{s+1} + 1, \lambda_{s+1}, \dots, \lambda_{s+c}, \lambda_{s+c+1} \dots, \lambda_n). \tag{5.5.6}$$

The largest odd part of the resulting partition is either $\lambda_{s+1} + 1$ or λ_{s+1}. Again, by (P1), (P2), and (P3), the resulting partition is in D_n.

We now show that the sequence $\sigma_1, \sigma_2, \dots$ is strictly increasing with each part less than n. First of all, note that if $\lambda \neq (n, n-1, \dots, 1)$, then $c < n$. Thus we can easily see that $\sigma_1 < n$ since $\sigma_1 \leq c$. It now suffices to show that $\sigma_i > \sigma_{i+1}$ for $i = 1, 2, \dots$. Suppose that σ_1 was subtracted vertically from λ. Then, in (5.5.5), the length c^* of the parity sequence of the resulting partition is larger than c. Also,

$$\lambda_s - (\lambda_{s+1} - 1) - 1 = \lambda_s - \lambda_{s+1} > c.$$

Since σ_2 is the minimum of c^* and $\lambda_s - (\lambda_{s+1} - 1) - 1$, we see that $\sigma_2 > \sigma_1$. Suppose that σ_1 was subtracted horizontally from λ. Then, in (5.5.6), the length c^* of the parity sequence of the resulting partition is larger than c, which is larger than or equal to $(\lambda_s - \lambda_{s+1} - 1)$. Also,

$$\lambda_{s-1} - (\lambda_{s+1} + 1) - 1 = \lambda_{s-1} - \lambda_{s+1} - 2 \geq \lambda_s + 2 - \lambda_{s+1} - 2 > \lambda_s - \lambda_{s+1} - 1 = \sigma_1,$$

where the first inequality follows from (P1). Since σ_2 is the minimum of c^* and $\lambda_{s-1} - (\lambda_{s+1} + 1) - 1$, we see that $\sigma_2 > \sigma_1$.

We now show that the deletion map defined above is the inverse process of our insertion map. Let π be the partition resulting from the insertion of $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ into τ, namely

$$\pi = (\pi_1, \dots, \pi_s, \underline{\pi_{s+1}, \dots, \pi_{s+c}}, \pi_{s+c+1}, \dots, \pi_n) \in D_n.$$

If μ_ℓ was inserted horizontally, then we see that

$$c \geq \mu_\ell = \pi_s - \pi_{s+1} - 1,$$

by (5.5.3). Thus, by the map, we have to subtract μ_ℓ horizontally. If μ_ℓ was inserted vertically, then we see that

$$c = \mu_\ell \leq \pi_s - \pi_{s+1} - 1,$$

by (5.5.4). Thus, by the map, we have to subtract μ_ℓ vertically. □

Theorem 5.5.3. [102, p. 28], [18, Entry 1.6.2] *For any complex number a,*

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \sum_{n=0}^{\infty} \frac{(-q;q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n}. \tag{5.5.7}$$

Proof. Let E_n be the set of partitions into even parts less than or equal to $2n$. By Theorem 5.5.2, the right-hand side of (5.5.7) generates pairs of partitions (π, σ) with $\pi \in D_n$ and $\sigma \in E_n$, where the exponent of a denotes the number of parts of π plus the number of parts of σ, with the sign $(-1)^{\ell(\sigma)}$. Let π_e (resp. σ_e) be the largest even part in π (resp. σ). For convenience, we define $\pi_e = 0$ (resp. $\sigma_e = 0$) if there is no even part in π (resp. σ). Note that by the definition of D_n, the following are equivalent:

(i) $\pi = (2n-1, 2n-3, \dots, 3, 1)$;

(ii) $\pi_e = 0$;

(iii) $\pi_1 = 2n - 1$.

We now compare π_e and σ_e.

Case 1: If $\pi_e > 0$ and $\pi_e \geq \sigma_e$, then move π_e to σ. We denote by (π', σ') the resulting partition pair. Since $\pi \in D_n$ and $\pi_e > 0$, π has n parts $\leq 2n-2$. Thus, π' has $n-1$ parts $< 2n-2$ and σ'_e is still less than or equal to $2n-2$, from which it follows that $\pi' \in D_{n-1}$ and $\sigma' \in E_{n-1}$. The pair (π', σ') is generated by the right-hand side of (5.5.7), and it has the opposite sign.

Case 2: If $\sigma_e > 0$ and $\sigma_e > \pi_e$, then move σ_e to π. We denote by (π', σ') the resulting partition pair. Since $\pi \in D_n$, π has n parts $< 2n$. Also, since $\sigma \in E_n$, $\sigma_e \leq 2n$. Thus, π' has $n+1$ parts $\leq 2n$, from which it follows that $\pi' \in D_{n+1}$ and $\sigma' \in E_{n+1}$. The pair (π', σ') is generated by the right-hand side of (5.5.7), and it has the opposite sign.

Therefore, the partition pairs (π, σ) with $\pi_e > 0$ or $\sigma_e > 0$ are canceled, and there remain only $\pi = (2n-1, 2n-3, \ldots, 1)$ and $\sigma = \emptyset$, which are generated by the left-hand side of (5.5.7). □

Alladi [9] has devised a completely different proof of Theorem 5.5.3 and has also provided a number-theoretic interpretation of Theorem 5.5.3 as a weighted partition theorem. Although we have given a bijective proof of Theorem 5.5.3, we do not interpret Theorem 5.5.3 number-theoretically. On the other hand, even though Alladi interpreted Theorem 5.5.3 number-theoretically, his proof of Theorem 5.5.3 is q-theoretic. It would be worthwhile to see how our bijective proof of Theorem 5.5.3 translates into a combinatorial proof of Alladi's weighted partition theorem. This is explored by W. Y. C. Chen and E. H. Liu [43].

Recently, Yee [122] found another combinatorial proof of Theorem 5.5.3.

Chapter 6

Subpartitions

6.1 Introduction

Let $a_1 \geq a_2 \geq \cdots \geq a_\ell$ be an ordinary partition. In a recent paper [81], L. Kolitsch introduced the Rogers-Ramanujan subpartitions and established their connection to other partitions. The Rogers-Ramanujan subpartition is the longest sequence satisfying $a_1 > a_2 > \cdots > a_s$ and $a_s > a_{s+1}$, where $a_i - a_j \geq 2$ for all $i < j \leq s$. In this chapter, we will generalize his result with an arbitrary gap condition and will study connections between subpartitions and other partitions. Let us fix a positive integer d. Then, for a given partition, a subpartition with gap d is defined as the longest sequence satisfying $a_1 > a_2 > \cdots > a_s$ and $a_s > a_{s+1}$, where $a_i - a_j \geq d$ for all $i < j \leq s$. Note that Kolitsch's Rogers-Ramanujan subpartition is the case $d = 2$. For convenience, we will define the subpartition of the empty partition as the empty partition. We define the length of the subpartition with gap d as the number of parts in the subpartition. When the gap d is clear from context, we will say the subpartition instead of the subpartition with gap d. In the next section, we will give a generating function of the ordinary partition that keeps track of the length of the subpartition with gap d. We also study their connection to the partial theta function, which is of the form

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2} x^n,$$

by attaching a proper weight to the generating function. In Section 4, we will focus on the subpartition with gap 1. By using the properties of subpartitions, we will give combinatorial proofs of the identities:

$$\frac{1}{(q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{(n^2+n)/2} = \sum_{n=0}^{\infty}\frac{q^n}{(q)_n^2}, \tag{6.1.1}$$

$$\frac{1}{(q)_\infty^2}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{(n^2+n)/2}\right) = \sum_{n=0}^{\infty}\frac{q^{2n}}{(q)_n^2}, \tag{6.1.2}$$

[4]This chapter is largely taken from my paper [76].

which are entries in Ramanujan's lost notebook [102] [18, p. 19, Entry 1.4.10 and Entry 1.4.11].

6.2 Generating function for subpartitions

For a given partition λ, we always write it in the form $a_1 \geq a_2 \geq \cdots \geq a_\ell$. Before finding a generating function, we need to define some notation. Let us fix a positive integer d and define, for each nonnegative integer k,

$$S_{d,k} = \begin{cases} 1 + (1+d) + (1+2d) + \cdots (k-1)d + 1 = \frac{dk^2-(d-2)k}{2}, \text{ if } k \neq 0, \\ 0, \text{ if } k = 0. \end{cases}$$

Then for a given partition λ, there are three cases:

(I) There is no subpartition in λ.

(II) The subpartition of λ is λ. In this case, we will say the partition λ is a complete partition after Kolitsch.

(III) λ is not complete and it has a subpartition with length ℓ.

For the case (I), i.e. to have no subpartition in λ, we should have $a_1 = a_2$. By using a standard argument [15, chap. 1], we can easily see that

$$\sum_{i=1}^{\infty} \frac{q^{2i}}{(q)_i}. \tag{6.2.1}$$

generates such partitions.

For the case (II), i.e. λ is a complete partition, the gaps between successive parts of λ should be at least d. Such partitions are generated by

$$\sum_{\ell=0}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_\ell}. \tag{6.2.2}$$

Note that the length of the subpartition in the above is ℓ.

For the case (III), suppose that a given partition λ has the subpartition with length ℓ and $a_\ell = j$. Note that since λ is not a complete partition, there are at least $\ell+1$ parts in λ and, by definition, $a_\ell > a_{\ell+1}$. Then, there are two possibilities:

(i) $a_\ell - a_{\ell+1}$ is less than d.

(ii) $a_\ell - a_{\ell+1} \geq d$, but $a_{\ell+1} = a_{\ell+2}$.

For the case (i), we have the generating function

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_{\ell-1}} \left(\sum_{j=2}^{\infty} \frac{q^{(\ell+1)(j-1)}}{(q)_{j-1}} + \cdots + \sum_{j=d}^{\infty} \frac{q^{\ell(j-1)+(j-d+1)}}{(q)_{j-d+1}} \right) \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_{\ell-1}} \left(\sum_{n=1}^{\infty} \frac{q^{(\ell+1)n}}{(q)_n} + \cdots + q^{\ell(d-2)} \sum_{n=1}^{\infty} \frac{q^{(\ell+1)n}}{(q)_n} \right) \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\ell} \sum_{n=1}^{\infty} \frac{q^{(\ell+1)n}}{(q)_n} \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\ell} \left(\frac{1}{(q^{\ell+1})_\infty} - 1 \right) \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\infty} - \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\ell},
\end{aligned} \tag{6.2.3}$$

where in the penultimate line we used the q-binomial theorem [28, p. 8]. For the case (ii), we have the generating function

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_{\ell-1}} \sum_{j=d+1}^{\infty} q^{\ell(j-1)} \sum_{i=1}^{j-d} \frac{q^{2i}}{(q)_i} \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_{\ell-1}} \sum_{i=1}^{\infty} \sum_{j=d+i}^{\infty} q^{\ell(j-1)} \frac{q^{2i}}{(q)_i} \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}+\ell(d-1)}}{(q)_\ell} \sum_{i=1}^{\infty} \frac{q^{(\ell+2)i}}{(q)_i} \\
&= \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}+\ell(d-1)}}{(q)_\ell} \left(\frac{1}{(q^{\ell+2})_\infty} - 1 \right),
\end{aligned} \tag{6.2.4}$$

by the q-binomial theorem.

Since all partitions fall into one of the above three cases, the sum of the above generating functions ((6.2.1), (6.2.2), (6.2.3), and (6.2.4)) should be $\frac{1}{(q)_\infty}$. Thus, we have

$$\begin{aligned}
\frac{1}{(q)_\infty} &= \sum_{i=1}^{\infty} \frac{q^{2i}}{(q)_i} + \sum_{\ell=0}^{\infty} \frac{q^{S_{d,\ell}}}{(q)_\ell} \\
&\quad + \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\infty} - \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\ell} + \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}+\ell(d-1)}}{(q)_\ell} \left(\frac{1}{(q^{\ell+2})_\infty} - 1 \right) \\
&= \sum_{\ell=0}^{\infty} \frac{q^{S_{d,\ell}+\ell(d-1)}}{(q)_\ell (q^{\ell+2})_\infty} + \sum_{\ell=1}^{\infty} \frac{q^{S_{d,\ell}}(1-q^{\ell(d-1)})}{(q)_\infty},
\end{aligned}$$

since

$$\sum_{i=1}^{\infty} \frac{q^{2i}}{(q)_i} = \frac{1}{(q^2)_\infty} - 1,$$

by the q-binomial theorem. Thus, we have proved our first theorem.

Theorem 6.2.1. *Let ℓ be the length of the subpartition with gap d. Then we have*

$$\frac{1}{(q)_\infty} = \frac{1}{(q^2)_\infty} + \frac{1}{(q)_\infty}\sum_{\ell=1}^{\infty}\left(q^{S_{d,\ell}} - q^{S_{d,\ell+1}}\right). \tag{6.2.5}$$

Remark. *An analytic proof of Theorem 6.2.1 is very simple; thus we will omit it. Note that, by setting $d = 2$, we can recover Kolitsch's Theorem 1.*

Define $p(n, \ell, d)$ to be the number of partitions of n having a subpartition of length ℓ with gap d. Then, by observing coefficients of q^n in (6.2.5), we can easily deduce that

Corollary 6.2.2. *For all nonnegative integers n and ℓ and a positive integer d, we have*

$$p(n, \ell, d) = p(n - S_{d,\ell}) - p(n - S_{d,\ell+1}).$$

6.3 Subpartitions with parity condition

Let us define $p_e(n, d)$ to be the sum $\sum_{\ell\ even} p(n, \ell, d)$, i.e. the number of partitions of n that have subpartitions with even lengths, and similarly for $p_o(n, d)$. Then, we have

$$\begin{aligned}\sum_{n=0}^{\infty}(p_e(n,d) - p_o(n,d))q^n &= \frac{1}{(q^2)_\infty} + \frac{1}{(q)_\infty}\sum_{\ell=1}^{\infty}(-1)^{\ell}\left(q^{S_{d,\ell}} - q^{S_{d,\ell+1}}\right)\\ &= \frac{1}{(q)_\infty}\left(1 + 2\sum_{\ell=1}^{\infty}(-1)^{\ell}q^{S_{d,\ell}}\right)\\ &= \frac{1}{(q)_\infty}\left(2\sum_{\ell=0}^{\infty}(-1)^{\ell}q^{S_{d,\ell}} - 1\right).\end{aligned} \tag{6.3.1}$$

Note that when $d = 2$, $1 + 2\sum_{k=1}^{\infty}(-1)^k q^{S_{d,k}}$ becomes a theta function that recovers Kolitsch's Theorem 4. For other d's, $\sum_{k=0}^{\infty}(-1)^k q^{S_{d,k}}$ is a partial theta function of the form,

$$\sum_{k=0}^{\infty}(-1)^k q^{\frac{dk^2-(d-2)k}{2}}.$$

Since, $p_e(n, d) + p_o(n, d) = p(n)$, we have

$$\sum_{n=0}^{\infty} p_e(n,d)q^n = \frac{1}{(q)_\infty}\sum_{k=0}^{\infty}(-1)^k q^{S_{d,k}}. \tag{6.3.2}$$

By replacing a and q by q and q^d, respectively, in the identity [23, eqn (2.1b)],

$$\sum_{k=0}^{\infty}(-1)^k a^k q^{(k^2-k)/2} = (a)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n (a)_n},$$

we obtain

$$\sum_{n=0}^{\infty} p_e(n,d)q^n = \frac{(q;q^d)_\infty}{(q)_\infty} \sum_{m=0}^{\infty} \frac{q^{dm^2+m}}{(q^d;q^d)_m (q;q^d)_m}. \tag{6.3.3}$$

We have the following partition theoretic interpretation of (6.3.3).

Theorem 6.3.1. *In the case of $d \geq 2$, the number of partitions of n with an even length subpartition with gap d is the same as the number of partitions of n such that the parts which are congruent to 1 modulo d have the following property: Consider the d-modular diagram of the partition, which consists of such parts. If it has the Durfee square of a side k, then the largest part of the partition below the Durfee square should be less than or equal to $d(k-1)+1$. In the case of $d = 1$, the number of partitions of n with subpartitions of even length is the same as the number of partitions of n that have the following property: If it has the Durfee square of a side k, then the number of parts in the partition on the right side of the Durfee square is k.*

6.4 Subpartitions with gap 1

In this section, we will investigate the subpartitions with gap 1. By (6.3.1) and (6.3.2) in the previous section, we have

$$\sum_{n=0}^{\infty}(p_e(n,1)-p_o(n,1))q^n = \frac{1}{(q)_\infty}\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{(k^2+k)/2}\right),$$
$$\sum_{n=0}^{\infty} p_e(n,1)q^n = \frac{1}{(q)_\infty}\sum_{k=0}^{\infty}(-1)^k q^{(k^2+k)/2}.$$

Thus, Entry 1.4.10 and 1.4.11 of [18, p. 19] are equivalent to the following identities:

$$\sum_{n\geq 0} p_e(n,1)q^n = \sum_{n=0}^{\infty}\frac{q^n}{(q)_n}(q^{n+1})_\infty, \tag{6.4.1}$$

$$\sum_{n\geq 0}(p_e(n,1)-p_o(n,1))q^n = \sum_{n=0}^{\infty}\frac{q^{2n}}{(q)_n}(q^{n+1})_\infty. \tag{6.4.2}$$

Now we will give a combinatorial proof for the above identities. Throughout the proofs, $t(\lambda)$ denotes the number of parts in λ.

Proof of (6.4.1). Note that

$$\frac{q^n}{(q)_n}(q^{n+1})_\infty$$

generates partition pairs $(\pi(n), \sigma(n))$, where $\pi(n)$ is a partition with the largest part n, and $\sigma(n)$ is a partition into distinct parts such that the smallest part is larger than n, and the exponent of (-1) is $t(\sigma(n))$. For a given partition λ, suppose that λ has the subpartition of length ℓ. Then, λ is of the form $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > \lambda_{\ell+1} \geq \lambda_{\ell+2} \geq \cdots$. Thus, in the right hand side of (6.4.1), λ is generated $\ell+1$ times as $(\pi(\lambda_1), \emptyset)$, $(\pi(\lambda_2), \sigma(\lambda_2))$, $\ldots$, and $(\pi(\lambda_{\ell+1}),\sigma(\lambda_{\ell+1}))$. Note that, in fact, $\lambda_{\ell+1} = \lambda_{\ell+2}$. If not, the length of the subpartition should be bigger than ℓ. Thus, λ is not of the form ($\pi(\lambda_{\ell+2})$, $\sigma(\lambda_{\ell+2})$). Note also that the exponent of (-1) in the previous partition pairs is $(-1)^0$, $(-1)^1$, $\ldots$, $(-1)^\ell$, respectively. Thus, their sum is 1 if ℓ is even and is 0 if ℓ is odd. Thus, in the right side of (6.4.1), after cancellation, we are left with the partitions that have subpartitions with even length. $\square$

Proof of (6.4.2). Note that

$$\frac{q^{2n}}{(q)_n}(q^{n+1})_\infty$$

generates partition pairs $(\pi(n), \sigma(n))$, where $\pi(n)$ is a partition with $\pi_1(n) = \pi_2(n) = n$, $\sigma(n)$ is a partition into distinct parts such that the smallest part is larger than n, and the exponent of (-1) is $t(\sigma(n))$. For a given partition λ, suppose that λ has the subpartition with length ℓ. Then, as before, λ is of the form $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > \lambda_{\ell+1} \geq \lambda_{\ell+2} \geq \cdots$. Recall that $\lambda_{\ell+1} = \lambda_{\ell+2}$. Thus, in the right side, λ is generated as $(\pi(\lambda_{\ell+1}), \sigma(\lambda_{\ell+1}))$. Since the exponent of (-1) is ℓ in this partition pair, we are done. $\square$

Note that the right side of (6.1.1) is a generating function of the number of stacks with summit. For the definition of the stack of summit and its proof, consult the paper of Andrews [13]. For other combinatorial proofs of Entry 1.4.10 and Entry 1.4.11, examine the work of A.J. Yee [121] or the previous chapter.

Next, we will obtain another generating function for $p_e(n,1) - p_o(n,1)$ by using a simple Durfee square argument. For a given partition λ, let λ^r be the conjugate of the partition in the right side of the Durfee square and λ^b be the partition below the Durfee square. Let $s(\lambda)$ be the side of the Durfee square of λ. Then, the coefficient of q^n of

$$2\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k^2} - \frac{1}{(q)_\infty} = 2\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k^2} - \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q)_k^2} \tag{6.4.3}$$

is (the number of partitions of n such that $\lambda_1^b = s(\lambda)$) plus (the number of partitions of n such that $\lambda_1^r = s(\lambda)$) minus $p(n)$, by symmetry. Since λ with $\lambda_1^b < s(\lambda)$ is not generated by $\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k^2}$ and we count λ twice if $\lambda_1^r = \lambda_1^b = s(\lambda)$, we have

$$2\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{(q)_k^2} - \sum_{k=0}^{\infty}\frac{q^{k^2}}{(q)_k^2} = 1 + \sum_{k=1}^{\infty}\frac{q^{k^2+2k}}{(q)_k^2} - \sum_{k=1}^{\infty}\frac{q^{k^2}}{(q)_{k-1}^2}.$$

In summary, we have

$$\begin{aligned}\sum_{n\geq 0}(p_e(n,1)-p_o(n,1))q^n &= \frac{1}{(q)_\infty}\left(1+2\sum_{k=1}^{\infty}(-1)^k q^{(k^2+k)/2}\right),\\ &= 1+\sum_{k=1}^{\infty}\frac{q^{k^2+2k}}{(q)_k^2} - \sum_{k=1}^{\infty}\frac{q^{k^2}}{(q)_{k-1}^2},\\ &= (1-q)\sum_{k=0}^{\infty}\frac{q^{k^2+2k}}{(q)_k^2}.\end{aligned}$$

Therefore, we have proved the following theorem.

Theorem 6.4.1. *The difference between the number of partitions of n with subpartition (with gap 1) of even length and those of odd length is the number of partitions λ of n satisfying $\lambda_1^r = \lambda_1^b = s(\lambda)$ and 1 is not a part of λ^b.*

As an immediate corollary, we have

$$p_e(n,1) \geq p_o(n,1), \text{ for all } n \geq 2. \tag{6.4.4}$$

Note that equality holds if and only if $n = 2$.

Next, we will investigate the parity of $p_e(n,1)$. We see that

$$\begin{aligned}\sum_{n\geq 0} p_e(n,1)q^n &\equiv \frac{1}{(q)_\infty}\sum_{n\geq 0} q^{(n^2+n)/2} \pmod 2\\ &\equiv \frac{(q)_\infty^3}{(q)_\infty} \pmod 2\\ &\equiv (q^2;q^2)_\infty \pmod 2\\ &\equiv \sum_{m=-\infty}^{\infty} q^{m(3m-1)} \pmod 2,\end{aligned}$$

where for the second congruence, we used Jacobi's identity [28, p. 14], and for the last congruence, we used the pentagonal number theorem [28, p. 12]. Thus, we can conclude that $p_e(n,1)$ is almost always even. Hence, we have proved the following theorem.

Theorem 6.4.2. *For all nonnegative integers n, we have*

$$p_e(n,1) \equiv \begin{cases} 1 \pmod 2, & \textit{if } n \textit{ is of the form } m(3m \pm 1) \textit{ for some nonnegative integer } m, \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

6.5 Remark

We define the subpartition of a partition into distinct parts as the longest sequence where the gap between successive parts is 1. Then, by a similar argument that we used to prove Theorem 1, we can easily prove that

$$(-q)_\infty = \sum_{\ell=0}^{\infty} q^{(\ell^2+\ell)/2} + \sum_{\ell=1}^{\infty} q^{(\ell^2+\ell)/2} \sum_{i=1}^{\infty} q^{i\ell}(-q)_{i-1} \tag{6.5.1}$$

and

$$(-q)_\infty = 1 + \sum_{\ell=1}^{\infty} \frac{q^{(\ell^2+\ell)/2}}{1-q^\ell} + \sum_{\ell=1}^{\infty} q^{(\ell^2+\ell)/2} \sum_{i=1}^{\infty} q^{i\ell}\left((-q)_{i-1} - 1\right), \tag{6.5.2}$$

where ℓ is the length of the subpartition. It appears that (6.5.1) and (6.5.2) do not appear in the literature of q-series. Therefore, it is natural to ask whether we can prove (6.5.1) or (6.5.2) analytically.

Chapter 7

Mock theta functions via n-color partitions

7.1 Introduction

In his famous last letter to G. H. Hardy [31], S. Ramanujan introduced mock theta functions without giving an explicit definition. Ramanujan introduced 17 examples of mock theta functions in his letter. Among them, the third order mock theta functions are

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \quad \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}, \\ \psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \quad \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-q^m+q^{2m})}. \end{aligned} \tag{7.1.1}$$

Later, G. N. Watson [118] added three functions to the list of Ramanujan's third order mock theta functions. These are

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2}, \qquad \upsilon(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}, \quad \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\prod_{m=1}^{n+1}(1+q^{2m-1}+q^{4m-2})}. \tag{7.1.2}$$

These three third order mock theta functions are actually in Ramanujan's Lost Notebook [102]. In Ramanujan's Lost Notebook [102], we are also able to find Ramanujan's sixth and tenth order mock theta functions. Among them, Ramanujan's sixth order mock theta functions are

$$\begin{aligned} \Phi(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \quad \Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}, \\ \rho(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \quad \sigma(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}}, \\ \lambda(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n}, \quad \mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n}, \quad \nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}. \end{aligned} \tag{7.1.3}$$

G. E. Andrews and D. Hickerson [22] established the results for sixth order mock theta functions that

[5] This chapter is based upon a joint paper with Youn-Seo Choi [46]. I thank Y.S. Choi for his permission to include our joint work here.

are similar to the mock theta conjectures. Recently, B. C. Berndt and S. H. Chan [29], and R. J. McIntosh [92] independently discovered two new sixth order mock theta functions $\phi_{-}(q)$ and $\psi_{-}(q)$ which are

$$\Phi_{-}(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n} \quad \text{and} \quad \Psi_{-}(q) = \sum_{n=1}^{\infty} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n}. \tag{7.1.4}$$

To see the history of mock theta functions and their modern and classical developments, we recommend the survey papers [58] and [99]. Mock theta functions have numerous nontrivial connections to combinatorics, especially the theory of partitions [1], [21] and [51]. For example, a third order mock theta function $f(q)$ is a generating function for the number of partitions of n with even rank minus the number of partitions of n with odd rank, where the rank of a partition is defined to be its largest part minus the number of its parts.

The n-color partition and its overpartition analogue have been employed to understand q-series identities combinatorially. The n-color partition was introduced by A. K. Agarwal and G. E. Andrews [4], and its overpartition analogue was introduced by J. Lovejoy and O. Mallet [88]. The n-color partition and its overpartition analogue arise naturally, and have a connection to many other combinatorial objects [3], [2], [4] and [5]. An n-color partition of a positive integer v is a partition in which each part of size n may appear up to n different colors denoted by subscripts from 1 to n, and parts are ordered first by the size of part and then according to the color. Since we have n different copies of part n, we also call it a partition with "n copies of n". For example, there are 6 n-color partitions of 3;

$$3_3,\ 3_2,\ ,3_1,\ 2_2 1_1,\ 2_1 1_1,\ 1_1 1_1 1_1.$$

We define the weighted difference of two parts m_i, n_j denoted by $((m_i - n_j))$, as $m - n - i - j$ provided that $m \geq n$. An n-color overpartition of a positive integer v is an n-color partition of v in which we may overline the final occurrence of each part n_j. For example, the n-color overpartitions of 2 are

$$2_2,\ \overline{2_2},\ 2_1,\ \overline{2_1},\ 1_1 1_1,\ 1_1 \overline{1_1}.$$

We also define the weighted difference of two parts m_i, k_j in an n-color overpartition denoted also by $((m_i - k_j))$ as $m - k - i - j - \chi(m_i) - \chi(k_j)$ provided that $m \geq k$, where $\chi(k_j) = 1$ if k_j is an overlined part, and 0 otherwise. We note that this definition coincides with the definition of a weighted difference of n-color partition if there is no overlined part.

In [1], Agarwal interpreted a third order mock theta function $\psi(q)$ and three fifth order mock theta functions $F_0(q)$, $\Phi_0(q)$, $\Phi_1(q)$ as generating functions of certain kinds of n-color partitions by using

q-difference equations. His interpretation for $\psi(q)$ is as follows.

Theorem 7.1.1. *$\psi(q)$ generates n-color partitions satisfying*

(1) the weighted difference between two consecutive parts is always 0,

(2) the smallest part is of the form k_k,

(3) even parts have even colors and odd parts have odd colors.

In [45], Y. S. Choi showed a connection between bilateral basic hypergeometric series and mock theta functions, which leads to many new identities involving mock theta functions. This work is a sequel to [45], and the purpose of this chapter is to provide partition theoretic properties of third order mock theta functions $\phi(q)$, $\psi(q)$, $\upsilon(q)$ and sixth order mock theta functions $\Psi(q)$, $\Psi_{-}(q)$, $\rho(q)$, $\lambda(q)$. Our first goal of this chapter is to derive partition-theoretic interpretations for the mock theta functions above as generating functions of n-color partitions or n-color overpartitions. In particular, we will give a bijective proof of Theorem 7.1.1 in a constructive way, and describe similar partition-theoretic interpretation for the others. For example, the sixth order mock theta function $\Psi(q)$ can be interpreted as follows.

Theorem 7.1.2. *Let us define λ^1 to be the largest part in the partition λ, and let $c(\lambda^i)$ denote the color of λ^i. Then, $\Psi(q)$ generates n-color overpartitions satisfying*

(1) the smallest part is of the form k_k and not overlined,

(2) the weighted differences between two consecutive parts are even and ≥ 0, where the exponent of (-1) is given by $\frac{\lambda^1+c(\lambda^1)+\chi(\lambda^1)-2}{2}$.

From Theorem 7.1.1, we easily conclude the following corollary.

Corollary 7.1.3. *There is a bijection between n-color partitions described in Theorem 7.1.1 and partitions into odd parts without gaps. Moreover, if λ is an n-color partition corresponding to σ, a partition into odd parts without gaps, then $\sum_{i=1}^{\ell(\lambda)} c(\lambda^i) = \ell(\sigma)$, where $\ell(\lambda)$ is the number of parts in the partition λ. In other words, the sum of the subscripts (the colors) of each part of λ is the same as the number of parts in σ.*

Even though the first part of Corollary 7.1.3 was first observed by Agarwal [1], a bijective proof had been unknown.

The second goal of this chapter is to derive arithmetic properties from mock theta function identities. Every identity we examine is of the following form: a linear combination of two mock theta functions is equal to a theta function. These identities yield interesting combinatorial facts about the coefficients of mock theta functions. In [51, Chapters 2 and 3], N.J. Fine gave a partition theoretic interpretation for mock theta functions, and derived many interesting properties from various identities involving mock theta functions. In particular, Fine showed that

$$f(q) = \sum_{n=0}^{\infty} \left(p(n,0,2) - p(n,1,2)\right) q^n,$$
$$\phi(q) = \sum_{n=0}^{\infty} \left(p(n,0,4) - p(n,2,4)\right) q^n,$$

and

$$\chi(q) = \sum_{n=0}^{\infty} \left(p(n,0,6) + p(n,1,6) - p(n,2,6) - p(n,3,6)\right) q^n,$$

where $f(q)$, $\phi(q)$, and $\chi(q)$ are third order mock theta functions defined by (7.1.1), and $p(n,d,N)$ denotes the number of partitions of n with rank $\equiv d \pmod N$. By using a linear relation between third order mock theta functions, he proved that

$$\sigma(2n) = p(2n,1,4) - p(2n,2,4) \tag{7.1.5}$$

where $\sigma(n)$ denotes the number of partitions of n into distinct odd parts without gaps. Our theorems in this paper are inspired by Fine's work in [51, Chapters 2 and 3] even though we have to rely on the theory of modular forms to prove Theorems 7.3.5 and 7.6.1.

Theorem 7.1.4. *We define $\beta(n) := \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)}$, where the sum runs over partitions into distinct odd parts $\leq 2\ell(\lambda) - 1$ except that 1 can be repeated, and $\ell(\lambda)$ denotes the number of parts in a partition λ. Then, for all positive integers n, we have*

$$2\beta(2n) = p(2n,1,2) - 2p(2n,2,4),$$
$$2\beta(2n-1) = p(2n-1,1,2) - 2p(2n-1,0,4).$$

We also discuss Ramanujan type congruences and cranks by analyzing theta functions which are linear sums of mock theta functions. If we say that $A_f(n)$ is the number of partitions of n with the generating function f, then we have the following congruences.

Theorem 7.1.5. *For all $n \geq 0$, we have*

$$A_{\Psi}(3n+3) + 2A_{\Psi_-}(3n+3) \equiv 0 \pmod 9$$

and

$$2A_\rho(3n+2) + A_\lambda(3n+2) \equiv 0 \pmod 9.$$

This chapter is organized as follows. In Section 7.2, we introduce necessary definitions and theorems. In Section 7.3, we provide combinatorial interpretations for the third order mock theta functions $\phi(q)$, $\psi(q)$, $\nu(q)$, and study their arithmetic properties. In Section 7.4, we study the combinatorial properties of two sixth order mock theta function identities which are proved in Section 7.6, and give combinatorial interpretations for the sixth order mock theta functions $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$ and $\lambda(q)$ by using n-color overpartitions. In Section 7.5, we introduce Garvan-Kim-Stanton type crank functions for the congruences given in Section 7.4. In Section 7.6, we prove two identities involving sixth order mock theta functions.

7.2 Preliminaries

In this section, we summarize the basic definitions and theorems for partitions, q-series and modular forms.
Partitions. A partition of a positive integer n is a weakly decreasing sequence of positive integers $(\lambda^1, \ldots, \lambda^r)$ such that $\lambda^1 + \cdots + \lambda^r = n$. We denote the number being partitioned by $|\lambda|$. If λ is a partition of n, then we write $\lambda \vdash n$. Throughout this paper, we denote $A_f(n)$ be the coefficient of q^n in the q-expansion of f. If f is a generating function for certain partitions, then we regard $A_f(n)$ as the number of such partitions of n counted by f.
p-modular Ferrers diagram. We introduce a p-modular Ferrers diagram. For a partition λ into parts λ^i congruent to r modulo p where $0 < r \leq p$, its p-modular Ferrers diagram is the diagram in which the i-th row has $\lceil \lambda^i/p \rceil$ boxes, we denote r in the boxes in the last column, and denote p for the other boxes. It can easily be seen that the sum of the numbers in the boxes equals $|\lambda|$. We define the M_p-rank of the partition λ as $\lceil \frac{\lambda^1}{p} \rceil$ - $\ell(\lambda)$. In other words, the M_p-rank of the partition λ is the number of boxes in the largest part in the p-modular diagram minus the number of parts of λ. For example, examine Figure 7.1.

2	2	2	2	1
2	2	2	1	
2	2			
1				

Figure 7.1: 2-modular diagram of a partition $\lambda = (9, 7, 4, 1)$ with M_2-rank $= 1$.

t-residue diagram. In the Ferrers diagram of a partition λ, we color the box at row r and column c by $c - r \pmod{t}$. Thus, we have t different colors, denoted by 0, 1, …, $t-1$. We denote $r_j(\lambda)$ as the number of boxes with color j in the Ferrers diagram of a partition λ. For example, examine Figure 7.2.

0	1	2	0	1
2	0	1	2	
1	2			
0				

Figure 7.2: 3-residue diagram of a partition $\lambda = (9, 7, 4, 1)$ with $[r_0(\lambda), r_1(\lambda), r_2(\lambda)] = [4, 4, 4]$.

t-core partition. A partition λ is said to be a t-core if there are no hook numbers that are multiples of t. For example, in Figure 7.3, λ is a 5-core partition. Let $a_t(n)$ be the number of t-core partitions of n. Then,

8	6	4	3	1
6	4	2	1	
3	1			
1				

Figure 7.3: a 5-core partition $\lambda = (5, 4, 2, 1)$ with hook numbers.

it is well-known [53] that

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}. \tag{7.2.1}$$

q-series. We define Ramanujan's general theta function $f(a, b)$ as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Then, Jacobi's triple product identity [28, p. 10] asserts that

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \tag{7.2.2}$$

We also need Jacobi's identity [28, p. 14]

$$(q; q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{n(n+1)/2}. \tag{7.2.3}$$

We also introduce the following space saving notations:

$$(a_1, a_2, \ldots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

Modular forms. Now we give the basic properties of modular forms. For more details on this subject, consult [98], [104] and [106].

Definition 7.2.1. *For $z \in \mathbb{H}$ and any positive integers n, m, define*

$$\eta(nz) := \eta_n = q^{\frac{n}{24}} (q^n; q^n)_\infty \tag{7.2.4}$$

and

$$\eta_{n,m}(z) := \eta_{n,m} = q^{P_2(\frac{m}{n})\frac{n}{2}} \frac{f(-q^m, -q^{n-m})}{(q^n; q^n)_\infty}, \tag{7.2.5}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} := t - [t]$ is the fractional part of t.

Here we only consider the cases when $m \not\equiv 0 \pmod{n}$ for $\eta_{n,m}$.

Recall that the modular group $\Gamma = SL_2(\mathbb{Z})$ and its congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\} \text{ and } \Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv b \equiv 1 \pmod{N} \right\}.$$

For a fixed real number r, a function $F(z)$, defined and meromorphic in $\mathbb{H}$, is said to be a modular form of weight r with respect to Γ, with multiplier system v, if (a) $F(z)$ satisfies $F(Mz) = v(M)(cz + d)^r F(z)$ for any $z \in \mathbb{H}$ and $M \in \Gamma$, (b) there exists a standard fundamental region R such that $F(z)$ has at most finitely many poles in $\bar{R} \cap \mathbb{H}$, and (c) $F(z)$ is meromorphic at q_j, for each cusp q_j in $\bar{R}$.

Let $\{\Gamma, r, v\}$ denote the space of modular forms of weight r and multiplier system v on Γ, where Γ is a subgroup of $\Gamma(1)$ of finite index. When a multiplier system v is trivial, we denote $\{\Gamma, r, v\}$ by $M_r(\Gamma)$. Let $ord(f; z)$ denote the invariant order of a modular form f at z. If $z \in \mathbb{H}$, then $Ord_\Gamma(f; z) := \dfrac{1}{\ell} ord(f; z)$, where ℓ ($\ell = 1, 2,$ or 3) is the order of z as a fixed point of Γ. If z is a cusp with respect to Γ, $Ord_\Gamma(f; z) := N(\Gamma; z)\ ord(f; z)$, where $N(\Gamma; z)$ is the width of Γ at z.

Theorem 7.2.2. *The Dedekind eta-function $\eta(z)$ is a modular form of weight $\frac{1}{2}$ with multiplier system v_η on $\Gamma(1)$, where the multiplier system v_η is given by the following formula: for each*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1),$$

$$\upsilon_\eta(M) = \begin{cases} \left(\dfrac{d}{|\, c \,|}\right) \zeta_{24}^{bd(1-c^2)+c(a+d)-3c}, & \textit{if } c \textit{ is odd,} \\ \left(\dfrac{c}{|\, d \,|}\right) \zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \textit{if } d \textit{ is odd and either } c \geq 0 \textit{ or } d \geq 0, \\ -\left(\dfrac{c}{|\, d \,|}\right) \zeta_{24}^{ac(1-d^2)+d(b-c)+3(d-1)}, & \textit{if } d \textit{ is odd, } c < 0,\ d < 0, \end{cases}$$

and ζ_{24} is a primitive 24th root of unity.

Proof. See Theorem 2 of [80]. □

Theorem 7.2.3 (the valence formula). *If $f \in \{\Gamma, r, \nu\}$ and $f \neq 0$, then*

$$\sum_{z \in R} Ord_\Gamma(f; z) = \mu r,$$

where R is any fundamental region for Γ, and $\mu := \dfrac{1}{12}[\Gamma(1) : \Gamma]$.

Proof. See Theorem 4.1.4 in [104]. □

Lemma 7.2.4. *If $m_1, m_2, \ldots, m_{2n}$ are positive integers, n is a positive integer, N is a positive even integer, and the least common multiple of $m_1, m_2, \ldots, m_{2n}$ divides N, then, for $z \in \mathbf{H}$,*

$$\eta(m_1 z)\eta(m_2 z) \cdots \eta(m_{2n} z) \in \{\Gamma_1(N), n, \upsilon\},$$

, where the multiplier system υ is defined as follows: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, ζ_{24} is a primitive 24th root of unity, and

$$\upsilon(A) = \prod_{i=1}^{2n} \left(\frac{c/m_i}{|\, d \,|}\right) \zeta_{24}^{ac(1-d^2)/m_i + d(m_i b - c/m_i) + 3(d-1)}.$$

Proof. See Lemma 2.7 in [44]. □

Theorem 7.2.5. *For $z \in \mathbb{H}$, let $f(z) := \prod_{n|N,\ 0 \leq m < n} \eta_{n,m}^{r_{n,m}}(z)$, where $r_{n,m}$ are integers. If*

$$\sum_{n|N,\ 0 \leq m < n} n P_2\left(\frac{m}{n}\right) r_{n,m} \equiv 0 \pmod 2$$

and

$$\sum_{n|N,\ 0\le m<n} \frac{N}{n} P_2(0) r_{n,m} \equiv 0 \pmod 2,$$

then $f(z) \in \{\Gamma_1(N), 0, I\}$, *where for* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, $I(M) = 1$.

Proof. See Theorem 3 in [106, p. 126]. □

Lemma 7.2.6. *Let* ℓ, m *and* n *be positive integers. Then, for a cusp* $k = \frac{\lambda}{\mu\epsilon}$ *of* $\Gamma_1(N)$, *where* $\epsilon \mid N$ *and* $(\lambda, N) = (\lambda, \mu) = (\mu, N) = 1$,

$$ord(\eta_{n,m}; k) + ord(\eta_n; k) \ge 0, \quad ord(\eta_{\ell n,m}; k) + \ell\ ord(\eta_n; k) \ge 0,$$

and

$$ord(\eta_{n,m}; k) + \ell\ ord(\eta_{\ell n}; k) \ge 0.$$

Proof. See Lemma 2.10 in [44]. □

7.3 Third order mock theta function identities

The first identity we examine is

$$\phi(q) + 2\psi(q) = \frac{(q^2;q^2)_\infty^7}{(q)_\infty^3 (q^4;q^4)_\infty^3} = (-q;q^2)_\infty \sum_{n=-\infty}^{\infty} q^{n^2} \tag{7.3.1}$$

where $\phi(q)$ and $\psi(q)$ are third order mock theta functions. We are able to find the equations above in [51, p. 60].

In [1], Agarwal showed that $\psi(q)$ is a generating function for certain n-color partitions by using q-difference equations. Here, we obtain the same results in a constructive way. This will give a bijective proof for Theorem 7.1.1.

Proof of Theorem 7.1.1. In this proof, we always use 2-modular Ferrers diagrams. Recall that q^{n^2} generates the partition $\tau = (1, 3, \ldots, 2n-1)$. We assign to each part color 1. Note that the weight difference between two consecutive parts is 0. Recall that $\frac{1}{(q;q^2)_n}$ generates partitions λ into odd parts $\le 2n-1$. From the largest part of λ, we attach each part λ^i as follows. We first attach 2 from the first row to the $\frac{\lambda^i-1}{2}$-th row and attach 1 to the $\frac{\lambda^i+1}{2}$-th part. Then, we increase the color by 1 for the $\frac{\lambda^i+1}{2}$-th part of the resulting partition. For example, examine Figure 7.4. Note that during this process the weight difference between two consecutive parts remains the same. The second condition is clear from this construction. Since the color is increased by 1 when the parity of part is changed, the third condition holds. $\square$

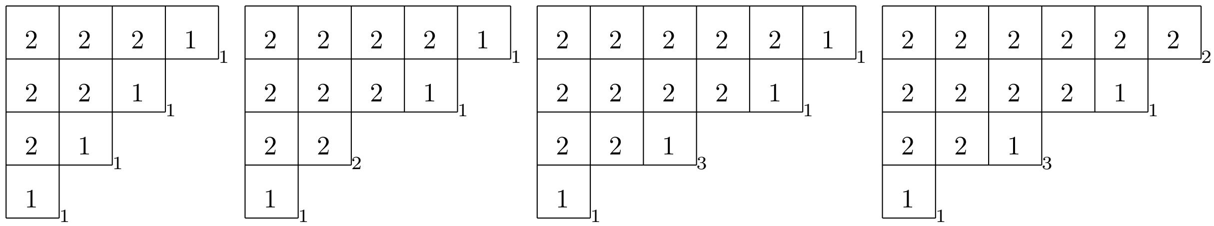

Figure 7.4: $\tau = (7,5,3,1)$ with $\lambda = (5,5,1)$.

Remark. *Actually, the last condition in Theorem 7.1.1 is not necessary. Since the weighted differences between two parts are always* 0 *and the smallest part is* k_k*, we can conclude that the parity of parts and their color should be the same.*

By using the bijection above, we are now ready to prove Corollary 7.1.3.

Proof of Corollary 7.1.3. For a given n-color partition σ enumerated by $\psi(q)$, we can easily recover the partitions τ and λ by reading the color of each part. By inserting parts in λ to τ in weakly decreasing order, we arrive at μ, a partition into odd parts without gaps. Since $\sum_{i=1}^{\ell(\sigma)} c(\sigma^i) = \ell(\tau) + \ell(\lambda) = \ell(\mu)$, this completes the proof. $\square$

Example. *An n-color partition* $(12_2, 9_1, 5_3, 1_1)$ *corresponds to the partition* $(7,5,5,5,3,1,1)$.

Analogously, we also can obtain an n-color partition theoretic interpretation for $\phi(q)$.

Theorem 7.3.1. $\phi(q)$ *generates n-color partitions λ satisfying*

(1) the smallest part is of the form $(2k-1)_k$*,*

(2) the color of λ_i is given by $\frac{\lambda_i - \lambda_{i+1}}{2}$ *except the smallest part, and the exponent of* (-1) *is given by M_2-rank of λ.*

Remark. *Since the color of each part is an integer, the conditions above imply that all parts are odd.*

Proof. The constructive proof is very similar to the proof of Theorem 7.1.1, so we omit it. Alternatively, by splitting the partition counted by $A_\phi(m,v)$ into two classes: partitions having 1_1 as a part and the partitions without 1_1, we can see that

$$A_\phi(m,v) = A_\phi(m-1, v-2m+1) - A_\phi(m, v-2m), \tag{7.3.2}$$

where $A_\phi(m,v)$ is the number of n-color partitions of v with m parts. If we define

$$f(z,q) := \sum_{v,m=0}^{\infty} A_\phi(m,v) z^m q^v,$$

then, by using (7.3.2), we can deduce that

$$f(z,q) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(-q^2;q^2)}.$$

By setting $z=1$, we complete the proof. $\square$

By (7.3.1), it is clear that $A_\phi(\nu) + 2A_\psi(\nu) \geq 0$ for all $\nu \geq 1$. Now we show that

$$A_\phi(\nu) + A_\psi(\nu) \geq 0,$$

for all $\nu \geq 1$. To this end, we introduce a new function $\phi^*(q)$, which is defined by

$$\phi^*(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}.$$

Note that $\phi^*(q)$ generates n-color partitions described in Theorem 7.3.1 except that the weight is always 1. Let λ be a partition enumerated by $\phi^*(q)$. We subtract $c(\lambda^i)-1$ from λ^i if $c(\lambda^i) > 1$, and denote the resulting partition as μ. Let r be the sum

$$\sum_{c(\lambda^i)>1} \big(c(\lambda^i)-1\big) = \left(\sum_{1\leq i\leq \ell(\lambda)} c(\lambda^i)\right) - \ell(\lambda).$$

We attach r to the largest part of μ, and also increase the color by r. Then, we observe that the resulting partition σ is an n-color partition counted by $\psi(q)$. Since each λ corresponds to a different σ, we have proven that

$$A_\phi(\nu) + A_\psi(\nu) \geq 0.$$

Example. *An n-color partition $\lambda = (13_2, 9_1, 7_3, 1_1)$ corresponds to $\mu = (12_2, 9_1, 5_3, 1_1)$ with $r=3$. Then, the resulting partition $\sigma = (15_5, 9_1, 5_3, 1_1)$ satisfies the conditions in Theorem 7.1.1 as desired.*

The second identity we investigate is

$$\upsilon(q) + \upsilon_3(q,q,;q) = 2\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} \tag{7.3.3}$$

where $\upsilon(q)$ is defined by (7.1.2) and

$$\upsilon_3(q,q;q) = \frac{1}{1+q}\sum_{n=1}^{\infty} q^n(-q^{-1};q^2)_n$$

is the function defined by Choi [45]. We easily obtain (7.3.3) by replacing α and z by q and q respectively in Theorem 1 of [45].

Recall that the generating function of t-core partitions is (7.2.1). Note also that

$$\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}(-q^2;q^2)_\infty.$$

Thus, the product on the right side of (7.3.3) generates partition pairs (λ,σ) where λ is a 2-core partition of even parts and σ is a partition into distinct even parts.

Remark. *By Gauss identity [51, p. 6],*

$$\frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)}.$$

Therefore, every 2*-core partition consisting of even parts is of the form* $(2k, 2k-2, \dots, 2)$.

Let $b(n)$ be the number of such partition pairs. Then, we can prove the following congruence.

Theorem 7.3.2. *For all nonnegative integers n,*

$$b(5n+3) \equiv 0 \pmod 5. \tag{7.3.4}$$

Proof. By using Jacobi's identity, we arrive at

$$\begin{aligned}\frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2} &= \frac{(q^4;q^4)_\infty^3(q^2;q^2)_\infty^3}{(q^2;q^2)_\infty^5}\\ &\equiv \frac{\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{2m(m+1)}\right)\left(\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)}\right)}{(q^{10};q^{10})_\infty} \pmod 5.\end{aligned}$$

Since $2m(m+1)+k(k+1) \equiv 3 \pmod 5$ holds only if $m \equiv 2 \pmod 5$ and $k \equiv 2 \pmod 5$, the coefficient of q^{5n+3} is divisible by 5 as desired. $\square$

We can also find an exact formula for the generating function of $b(5n+3)$ by using modular functions.

Theorem 7.3.3.

$$\sum_{n=0}^{\infty} b(5n+3)q^n = 5q\frac{(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^2(q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4}. \tag{7.3.5}$$

We will follow the argument in [57] to prove (7.3.5).

Proof. Define $F(z)$ as

$$F(z) := \frac{\eta^3(4z)\eta^2(10z)\eta(100z)}{\eta^2(2z)\eta^4(20z)}.$$

By Theorem 3.3.1, we have $F(z) \in \mathcal{M}_0(\Gamma_0(100))$. Note that $f(z)|U_5 \in \mathcal{M}_0(\Gamma_0(20))$. Let us define $G(z)$ as

$$G(z) := \frac{\eta^2(10z)\eta^2(20z)}{\eta^2(2z)\eta^2(4z)}.$$

Then, by Theorem 3.3.1, we have $G(z) \in M_0(\Gamma_0(20))$. From the order at each cusp by employing Theorems 3.3.2 and 3.3.3, we see that

$$\frac{F|U_5}{G}$$

is a holomorphic modular function, namely, a constant. From this, we can easily deduce that $F(z)|U_5 = 5G(z)$. Recall that $(f(pz)g(z))|U_p = f(z)g(z)|U_p$. Thus, we arrive at

$$\frac{(q^{20};q^{20})_\infty (q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4} \left(\sum_{n=0}^{\infty} b(n) q^{n+2} \right) |U_5 = 5q^2 \frac{(q^{20};q^{20})_\infty^2 (q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2}$$

or

$$\sum_{n=0}^{\infty} b(5n+3) q^n = 5q \frac{(q^4;q^4)_\infty^2 (q^{10};q^{10})_\infty^2 (q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4},$$

as desired. $\square$

Remark. *Theorem 7.3.3 and* (7.3.3) *imply that*

$$\sum_{n=1}^{\infty} \left(A_\upsilon(5n+3) + A_{\upsilon_3}(5n+3) \right) q^n = 10q \frac{(q^4;q^4)_\infty^2 (q^{10};q^{10})_\infty^2 (q^{20};q^{20})_\infty}{(q^2;q^2)_\infty^4}.$$

Now, we will show that the left side of (7.3.3) generates a certain type of n-color partitions.

First, we note that

$$\upsilon(q) = \upsilon_+(q) + \upsilon_-(q),$$

where

$$\upsilon_+(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_n} \quad \text{and} \quad \upsilon_-(q) := \sum_{n=1}^{\infty} \frac{(-1)q^{n(n-1)} q^{2n-1}}{(-q;q^2)_n}.$$

We see that $\upsilon_+(q)$ generates n-color partitions satisfying the following properties:

(1) the smallest part is of the form $(k+1)_k$,

(2) the weighted difference between any two consecutive parts is 0, where the exponent of (-1) is $k-1$, namely the color of the smallest part minus 1.

Remark. *From the condition above, we observe that odd parts have even colors, and even parts have odd colors.*

Similarly, we observe that $\upsilon_-(q)$ generates n-color partitions satisfying the following properties:

(1) the smallest part is of the form k_k,

(2) the weighted difference between any two consecutive parts not containing the smallest part is 0, and 1 otherwise, where the exponent of (-1) is k, namely the color of the smallest part.

Remark. *We can see that odd parts have even colors and even parts have odd colors, except for the smallest part.*

In summary, $\upsilon(q)$ generates n-color partitions satisfying the following conditions.

(1) the smallest part is of the form $(k+1)_k$ or k_k.

(2) the weighted difference between any two consecutive parts is 0 except that the weighted difference involving the smallest part of the form k_k is 1,

where the exponent of (-1) is $c(\lambda^{\ell(\lambda)}-1$ if the smallest part is $(k+1)_k$ or $c(\lambda^{\ell(\lambda)}$ if the smallest part is k_k. Now we turn to $\upsilon_3(q,q;q)$. Let us define $\upsilon^*(q)=(1+q)\upsilon_3(q,q;q)$. If we allow 0_0 as a part, then $\upsilon^*(q)$ generates n-color partitions satisfying the following properties;

(1) the smallest part is of the form 1_1 or 0_0,

(2) the weighted difference for two consecutive parts is -2 except that the weight difference involving the part 0_0 is 0.

Let $A_{\upsilon^*}(\nu)$ denote the number of such n-color partitions of ν. Then, we have

$$\begin{aligned}\upsilon_3(q,q;q) &= \frac{1}{1+q}\sum_{\nu=0}^{\infty} A_{\upsilon^*}(\nu)q^{\nu}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}A_{\upsilon^*}(k)\right)q^n.\end{aligned}$$

Since it is clear that $b(2n+1)=0$, we have

$$A_\upsilon(2\nu+1) = -\sum_{k=0}^{2\nu+1}(-1)^{2\nu+1-k}A_{\upsilon^*}(k) = \sum_{k=0}^{2\nu+1}(-1)^k A_{\upsilon^*}(k),$$

where $A_\upsilon(\nu)$ is the number of n-color partitions of ν generated by $\upsilon(q)$. We easily see that $A_{\upsilon_3}(\nu) > 0$, for all $\nu \geq 1$. Thus, by (7.3.3), $A_\upsilon(2\nu+1) < 0$ for all nonnegative integers ν.

We turn to prove Theorem 7.1.4.

Proof of Theorem 7.1.4. Replacing q by $-q$ and setting $\alpha = -q$ and $z = q$ in the first identity of Theorem 1 in [45], we arrive at

$$2\sum_{n=1}^{\infty}(-q)^n(-q^2;q^2)_{n-1} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n} = \frac{f(-q,-q)}{(-q;q)_\infty}. \tag{7.3.6}$$

Note that the first sum generates partitions into n odd parts $\leq 2n-1$ such that

(1) the only repeatable part is 1,

(2) the exponent of (-1) is the number of parts.

Let $\mathcal{O}_1$ be the set of partitions λ into distinct odd parts $\leq 2\ell(\lambda)-1$ except that 1 can be repeated. Recall that

$$\beta(n) = \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{O}_1}} (-1)^{\ell(\lambda)}.$$

Note that the second sum is $\phi(-q)$. Thus, from the equation (26.66) in [51], we have

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \phi(-q) + 2\sum_{n=1}^{\infty}\beta(n)q^n. \tag{7.3.7}$$

Recall that $\phi(q) = \sum_{n=0}^{\infty}(p(n,0,4)-p(n,2,4))q^n$. Therefore, we arrive at

$$2\beta(n) = (-1)^n p(n,0,4) - (-1)^n p(n,2,4) - p(n,0,2) + p(n,1,2).$$

Using the fact that $p(n,0,2) = p(n,0,4) + p(n,2,4)$, we deduce that

$$2\beta(2n) = p(2n,1,2) - 2p(2n,2,4)$$

and

$$2\beta(2n-1) = p(2n-1,1,2) - 2p(2n-1,0,4),$$

which completes the proof of Theorem 7.1.4. $\square$

Let $\mathcal{O}$ be the set of partitions into odd parts without gaps. Then, $\psi(-q) = \sum_{n=0}^{\infty} \gamma(n) q^n$ where

$$\gamma(n) := \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{O}}} (-1)^{\ell(\lambda)}.$$

Therefore, by (7.3.7), we are able to derive the following theorem.

Theorem 7.3.4. *For all positive integers n, $\gamma(n) = \beta(n)$.*

We provide a bijective proof.

Proof. Let λ be a partition in $\mathcal{O}$. Let σ be a partition consisting of parts $\lambda^i - 1$ for all $1 \le i \le \ell(\lambda)$. Let σ' be the partition obtained by conjugating the 2-modular diagram of σ. We attach 1 from the first part to $\ell(\lambda)$-th part of σ'. Then, the resulting partition μ is in $\mathcal{O}_1$. Since the number of parts of λ and that of μ are the same, this completes the proof. $\square$

7.4 Sixth order mock theta function identities

In this section, we discuss the following two identities involving sixth order mock theta functions

$$\Psi(q) + 2\Psi_-(q) = 3\frac{q(q^6;q^6)_\infty^3}{(q)_\infty (q^2;q^2)_\infty} \tag{7.4.1}$$

and

$$2\rho(q) + \lambda(q) = 3\frac{(q^3;q^3)_\infty^3}{(q)_\infty (q^2;q^2)_\infty}. \tag{7.4.2}$$

We will prove these identities in Section 6.

First, note that the right sides of (7.4.1) and (7.4.2) generate partitions analogous to the partitions defined by

$$\frac{1}{(q)_\infty (q^2;q^2)_\infty},$$

which have been studied by H.-C. Chan [38], [39], [37]. This partition function satisfies many congruences [40], [42], but there is only one simple congruence [111]. A crank function for this partition was studied in Chapter 3.

Here, we study two analogous partition functions defined by

$$\sum_{n=1}^{\infty} c(n)q^n = \frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty (q^2;q^2)_\infty} \tag{7.4.3}$$

and

$$\sum_{n=0}^{\infty} d(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{7.4.4}$$

Remark. *From the generating function for* t*-core partition* (7.2.1)*, we can regard these partitions as* 3*-core partition analogues of H.-C. Chan's partitions.*

We can easily prove that these two partition functions satisfy the following congruences.

Theorem 7.4.1.

$$c(3n) \equiv 0 \pmod 3, \tag{7.4.5}$$

$$d(3n+2) \equiv 0 \pmod 3. \tag{7.4.6}$$

Now we obtain exact generating functions for these arithmetic progressions. Since $\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}$ is a Hecke eigenform in $M_2(\Gamma_0(6))$, we see that

$$\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}|U_3 = 3\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}. \tag{7.4.7}$$

Remark. *A proof of* (7.4.7) *without using the theory of modular forms can be found in Fine's book [51, (33.124)].*

Proof of Theorem 7.4.1. By (7.4.3), (7.4.4) and (7.4.7), we see that

$$\left(\sum_{n=1}^{\infty} c(3n)q^n\right)(q)_\infty^3 = 3q\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q)_\infty (q^2;q^2)_\infty}$$

and

$$\left(\sum_{n=1}^{\infty} d(3n-1)q^n\right)(q^2;q^2)_\infty^3 = 3q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty},$$

which implies that

$$\sum_{n=1}^{\infty} c(3n)q^n = 3q\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty^4(q^2;q^2)_\infty}$$

and

$$\sum_{n=0}^{\infty} d(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty^4}.$$

□

Now, we will give combinatorial interpretations for the sixth order mock theta functions $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$ and $\lambda(q)$ by using n-color overpartitions.

Proof of Theorem 7.1.2. We rewrite $\Psi(q)$ as

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q^2)_{n+1}(-q^2;q^2)_n}.$$

Recall that $(n+1)^2$ generates partition into odd parts from 1 to $2n+1$. We assign the color 1 to each part. Then, the weight difference of two consecutive parts is 0. We attach each part λ^i in λ generated by $\frac{1}{(q^2;q^2)_n}$ as follows. We attach 2 from the first row to the $\frac{\lambda^i}{2}$-th row. Then, we can see that the weighted difference between the $\frac{\lambda^i}{2}$-th part and the $\frac{\lambda^i}{2}+1$-th part of the resulting partition increases by 2. We also attach each part σ^j in σ generated by $\frac{1}{(q;q^2)_{n+1}}$ as follows. We attach 2 from the first row to the $\frac{\sigma^j-1}{2}$-th row and attach 1 to the $\frac{\sigma^j+1}{2}$-th row. Then, we increase the color of the $\frac{\sigma^j+1}{2}$-th part of the resulting partition by 1. We can observe that this does not affect the weight difference. Finally, we attach each part of μ^k in μ generated by $(q;q^2)_n$ as we did for σ^j, and overline the $\frac{\mu^k+1}{2}$-th part of the resulting partition. We see that this also does not affect the weight difference. By tracking the exponent of (-1), we complete the proof. □

By employing a similar argument, we can prove the following theorem.

Theorem 7.4.2. $\Psi_-(q)$ *generates n-color overpartitions satisfying*

(1) the smallest part is of the form k_k, which cannot be overlined,

(2) the weighted difference between two consecutive parts is 0 *or* -2.

$\rho(q)$ *generates* n*-color overpartitions satisfying that*

(1) the smallest part is of the form k_k *or* $\overline{(k+1)_k}$,

(2) the weighted difference of two consecutive part is -2 *if the smaller part is overlined and* 0 *or* -1 *if it involves the unoverlined smallest part and* -1*, otherwise.*

And $\lambda(q)$ *generates* n*-color overpartitions* λ *satisfying*

(1) the smallest part is of the form k_k *or* $\overline{(k+1)_k}$,

(2) the weighted difference of two consecutive parts forms a non-decreasing sequence of which sum equals $-2(\ell(\lambda)-1)$*, where the exponent of* (-1) *is the sum of colors plus the number of overlined parts.*

Now we are ready to prove Theorem 7.1.5.

Proof of Theorem 7.1.5. Combining Theorems 7.1.2, 7.4.1, and 7.4.2, we can derive the following congruences. For all $n \geq 0$, we have

$$\begin{aligned} A_{\Psi}(3n+3) + 2A_{\Psi_-}(3n+3) &\equiv 0 \pmod{9}, \\ 2A_{\rho}(3n+2) + A_{\lambda}(3n+2) &\equiv 0 \pmod{9}. \end{aligned}$$

We have completed the proof of Theorem 7.1.5. □

7.5 Crank analogues for $c(n)$ and $d(n)$

Recall that $c(n)$ and $d(n)$ are partition functions defined by (7.4.3) and (7.4.4), respectively. We find a Garvan-Kim-Stanton type crank [53] for $c(n)$ and $d(n)$ by modifying a crank given in Z. Reti's thesis [105]. Since Reti's result has not been published and is not well-known, we give details from his thesis, and show how this crank can be extended to $c(n)$ and $d(n)$. Interested readers should consult [53] and [105]. The following lemma enables us to extend a crank for t-core partitions to a crank for ordinary partitions. Here and in the sequel, $\mathcal{P}$ denotes the set of ordinary partitions and $\mathcal{P}_t^*$ is the set of t-core partitions.

Lemma 7.5.1 (Bijection 1 of [53])**.** *There is a bijection between* $\pi \in \mathcal{P}$ *and* $[\pi_0, \dots, \pi_{t-1}, \pi^*] \in \mathcal{P} \times \cdots \times \mathcal{P} \times \mathcal{P}_t$*, which satisfies*

$$|\pi| = t\sum_{j=0}^{t-1} |\pi_j| + |\pi^*|.$$

Let us define the set

$$S^*(n) := \{[\pi(1), \pi(2)] \in \mathcal{P}_3^* \times \mathcal{P}_3^* : |\pi(1)| + 2|\pi(2)| = n\}.$$

Recall that $r_j(\pi)$ is the number of dots colored j in the 3-residue diagram of π. We define a coordinate system $\underline{a}$ by

$$\underline{a} := [r_0(\pi(1)) - r_1(\pi(1)), r_0(\pi(1)) - r_2(\pi(1)), r_0(\pi(2)) - r_1(\pi(2)), r_0(\pi(2)) - r_2(\pi(2))],$$

where $[\pi(1), \pi(2)] \in S^*(n)$. We understand $\#S^*(n, A)$ as the number of elements in the set S^* satisfying the property A. Now we are ready to give cranks for $S^*(3n+2)$.

Lemma 7.5.2 (Theorem 5 of [105])**.** *The following two vectors are cranks for $S^*(3n+2)$:*

$$\underline{f}(1) := [-1, 1, -1, 1] \text{ and } \underline{f}(2) := [-1, 1, 1, -1],$$

in the sense of

$$\#S^*(3n+2, \underline{f} \cdot \underline{a} \equiv k \pmod 3) = \frac{\#S^*(3n+2)}{3},$$

for all $0 \le k \le 2$, where $\#(S)$ is the number of element in the set S.

Even though the two cranks above are defined only on the set of $S^*(n)$, we can extend these cranks to $S_1(n)$ (resp. $S_2(n)$) by using Lemma 7.5.1, where $S_1(n)$ (resp. $S_2(n)$) is the set of partitions enumerated by $c(n)$ (resp. $d(n)$). In the next proposition, we give such an extension in the spirit of [53, Proposition 1].

Proposition 7.5.3. *Let $[\pi(1), \pi(2)]$ be a partition in $S_1(n)$ or $S_2(n)$ and $r_j(\pi)$ be the number of j-colored boxes in the 3-residue diagram of π. Then, the following two linear combinations*

$$r_1(\pi(1)) - r_2(\pi(1)) + r_1(\pi(2)) - r_2(\pi(2)) \text{ and } r_1(\pi(1)) - r_2(\pi(1)) - r_1(\pi(2)) + r_2(\pi(2)),$$

are crank statistics for $S_1(n)$ and $S_2(n)$.

The proof of the above proposition is analogous to that of Proposition 1 in [53]. The key idea is that the above statistics are invariant under the removal of 3-rim hooks. By using Proposition 7.5.3, we can deduce the crank statistics, which can be calculated from the Ferrers diagram in the spirit of Theorem 3 in [53].

Theorem 7.5.4. *For all partitions $[\pi(1), \pi(2)] \in S_1(n)$ (or $S_2(n)$), we can define a crank from $f(1)$ by*

$$\sum_{j=1}^{2} \sum_{i=1}^{\ell(\pi(j))} \left(\delta(\pi(j)_i - i) - \delta(-i)\right),$$

where $\delta(x) = 1$ for $x \equiv 1 \pmod 3$ and 0, otherwise, and $\ell(\pi)$ is the number of parts in π. We can also define a crank from $f(2)$ by

$$\sum_{i=1}^{\ell(\pi(1))} \left(\delta(\pi(1)_i - i) - \delta(-i)\right) - \sum_{i=1}^{\ell(\pi(2))} \left(\delta(\pi(2)_i - i) - \delta(-i)\right).$$

The proof of Theorem 7.5.4 is easily obtained by calculating the contribution of each row to the crank from Proposition 7.5.3, so we omit it.

7.6 Proof of two sixth order mock theta function identities

In this section, we prove the following two identities which played an important role in Section 4.

Theorem 7.6.1. *For $|q| < 1$,*

$$\Psi(q) + 2\Psi_-(q) = 3\frac{q(q^6;q^6)_\infty^3}{(q)_\infty (q^2;q^2)_\infty}, \tag{7.6.1}$$

$$2\rho(q) + \lambda(q) = 3\frac{(q^3;q^3)_\infty^3}{(q)_\infty (q^2;q^2)_\infty}, \tag{7.6.2}$$

where $\Psi(q)$, $\Psi_-(q)$, $\rho(q)$, and $\lambda(q)$ are the sixth order mock theta functions defined by (7.1.3).

Before proving these identities, we need to prove the following two eta function identities. Throughout the proof, we let E_N be a complete set of inequivalent cusps for $\Gamma_1(N)$ and η_n and $\eta_{i,j}$ are functions defined by (7.2.4) and (7.2.5).

Theorem 7.6.2. *For $z \in \mathbb{H}$,*

$$-\eta_2^4\eta_4^2\eta_6^6\eta_{12}^2\eta_{4,2}^2\eta_{6,2}^6\eta_{12,2}^2 + 4\eta_1^2\eta_3^2\eta_4^8\eta_6^2\eta_{3,1}^2\eta_{6,1}^2 = 3\eta_1^8\eta_4^2\eta_6^2\eta_{12}^2. \tag{7.6.3}$$

Proof. For $1 \le i \le 3$, let f_i^1 be the product of eta-functions in each of the 3 products in (7.6.3), and for $1 \le i \le 2$, g_i^1 be the product of the generalized eta-functions in each of the 2 products in (7.6.3). Each f_i^1 is the product of 14 eta-functions, and by Lemma 7.2.4 and a straightforward calculation, each f_i^1 is a modular form of weight 7 on $\Gamma_1(72)$ with the multiplier system υ_1, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(72)$, $\upsilon_1(A) = \zeta_{24}^{4bd}$. By Theorem 7.2.5 and a straightforward calculation, each g_i^1 is a modular form of weight 0 on $\Gamma_1(72)$ with the multiplier system I. Therefore, $f_1^1 g_1^1$, $f_2^1 g_2^1$, and f_3^1 are modular forms of weight 7 on $\Gamma_1(72)$ with multiplier system υ_1.

Recall that $[\Gamma(1) : \Gamma_1(72)] = 3456$. Let F_1 denote the difference of the left and right sides of (7.6.3). By applying the three equations in Lemma 7.2.6 to F_1 and a straightforward calculation, we find that for each $k \in E_{72}$, $k \neq \infty$,

$$ord(F_1; k) \geq 0. \tag{7.6.4}$$

Applying Theorem 7.2.3 for a fundamental region R for $\Gamma_1(72)$, and using (7.6.4), we deduce that, for F_1,

$$\sum_{z \in R} Ord_{\Gamma_1(72)}(F_1; z) = \frac{7 \cdot 3456}{12} = 2016 \geq ord(F_1; \infty), \tag{7.6.5}$$

since both sides of (7.6.3) are analytic on R. Using *Maple*, we calculated the Taylor series of F_1 about $q = 0$ (or about the cusp $z = \infty$) and found that $F_1 = O(q^{2017})$. Unless F_1 is a constant, we have a contradiction to (7.6.5). We have thus completed the proof of Theorem 7.6.2. □

Theorem 7.6.3. *For $z \in \mathbb{H}$,*

$$-\eta_1^{16}\eta_4^4\eta_6^4\eta_{12}^4 + \eta_2^{16}\eta_3^8\eta_{12}^4\eta_{3,1}^4\eta_{12,2}^2 = 12\eta_1^{10}\eta_2^2\eta_3^2\eta_4^6\eta_6^2\eta_{12}^6. \tag{7.6.6}$$

Proof. For $1 \leq i \leq 3$, let f_i^2 be the product of eta-functions in each of the 3 products in (7.6.6), and $g^2 := \eta_{3,1}^4\eta_{12,2}^2$ be the product of the generalized eta-functions in the second term in (7.6.6). Each f_i^2 is the product of 28 eta-functions, and by Lemma 7.2.4 and a straightforward calculation, each f_i^2 is a modular form of weight 14 on $\Gamma_1(24)$ with the multiplier system υ_2, where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(24)$, $\upsilon_2(A) = \zeta_{24}^{8bd}$. By Theorem 7.2.5 and a straightforward calculation, g^2 is a modular form of weight 0 on $\Gamma_1(24)$ with the multiplier system I. Therefore, f_1^2, $f_2^2 g^2$, f_3^2 are modular forms of weight 14 on $\Gamma_1(24)$ with multiplier system υ_2.

Recall that $[\Gamma(1) : \Gamma_1(24)] = 384$. Let F_2 denote the difference of the left and right sides of (7.6.6). By applying the three equations in Lemma 7.2.6 to F_2 and a straightforward calculation, we find that for each

$k \in E_{24}$, $k \neq \infty$,

$$ord(F_2; k) \geq 0. \tag{7.6.7}$$

By using an argument similar to that in the proof of Theorem 7.6.2, we can see that $F_2 = 0$ by checking the first 449 terms, which was done by *Maple*. $\square$

We now derive two theta function identities from the previous eta function identities.

Theorem 7.6.4. *For* $|\, q \,| < 1$,

$$-f(q, q^5)^6 f(q^3, q^3)^2 + f(q^2, q^4)^6 f(1, q^6)^2 = 3\frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty^8}{(q^2;q^2)_\infty^4}.$$

Proof. By the Jacobi triple product identity,

$$f(q, q^5) = (-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty = \frac{(q^2;q^{12})_\infty (q^{10};q^{12})_\infty}{(q;q^6)_\infty (q^5;q^6)_\infty} (q^6;q^6)_\infty, \tag{7.6.8}$$

$$f(q^3, q^3) = (-q^3;q^6)_\infty^2 (q^6;q^6)_\infty = \frac{(q^6;q^{12})_\infty^2}{(q^3;q^6)_\infty^2} (q^6;q^6)_\infty, \tag{7.6.9}$$

$$f(1, q^6) = 2(-q^6;q^6)_\infty^2 (q^6;q^6)_\infty = 2\frac{(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty}, \tag{7.6.10}$$

and by Euler's identity,

$$f(q^2, q^4) = (-q^2;q^6)_\infty (-q^4;q^6)_\infty (q^6;q^6)_\infty = \frac{(q^4;q^{12})_\infty (q^8;q^{12})_\infty}{(q^2;q^6)_\infty (q^4;q^6)_\infty} (q^6;q^6)_\infty. \tag{7.6.11}$$

Dividing both sides of (7.6.3) by $q^2 \frac{\eta_2^4 \eta_4^2 \eta_{12}^2 \eta_{6,1}^6 \eta_{6,2}^6 \eta_{6,3}^2}{\eta_6^2}$, using $\eta_{2\ell,\ell}\eta_{2\ell}^2 = \eta_\ell^2$ and $\eta_{3\ell,\ell}\eta_{3\ell} = \eta_\ell$ frequently, and employing (7.6.8)–(7.6.11), we get the identity in Theorem 7.6.4. $\square$

Theorem 7.6.5. *For* $|\, q \,| < 1$,

$$-(q;q^2)_\infty^6 f(q^3, q^3) + (-q;q^2)_\infty^6 f(-q^3, -q^3) = 12q\frac{(q^6;q^6)_\infty (q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^2 f(-q^2, -q^{10})^2}.$$

Proof. By Jacobi triple product identity, we can derive that

$$f(q^3,q^3) = (-q^3;q^6)_\infty^2(q^6;q^6)_\infty = \frac{(q^6;q^{12})_\infty}{(q^3;q^6)_\infty}(q^6;q^6)_\infty \tag{7.6.12}$$

and

$$(-q;q^2)_\infty = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty}. \tag{7.6.13}$$

Now, dividing both sides of (7.6.6) by $q^{-\frac{1}{4}}\eta_1^4\eta_2^{10}\eta_4^6\eta_6^3\eta_{12}^4\eta_{2,1}^3\eta_{6,3}\eta_{12,2}^2$, using $\eta_{2\ell,\ell}\eta_{2\ell}^2 = \eta_\ell^2$ and $\eta_{3\ell,\ell}\eta_{3\ell} = \eta_\ell$ frequently, and employing (7.6.12) and (7.6.13), we derive the identity in Theorem 7.6.5. □

Finally, we are ready to prove Theorem 7.6.1.

Proof of Theorem 7.6.1. First, we prove (7.6.1). Replacing z by q in Theorem 4 [45] and using Theorem 7.6.4, we deduce that

$$\begin{aligned}
&\frac{1+q}{q}\left(\psi(q)+2\psi_-(q;q)\right)\\
&= -\frac{q^2}{2}\frac{(-q^{-1},-q^{-1},-q^3,-q,-q;q^2)_\infty}{(q,-q^2,q,q,q;q^2)_\infty}f(1,q^6) + \frac{1}{2}\frac{(-1,-1,-q^2,-q^2,-1;q^2)_\infty}{(q,-q^3,q,q,q;q^2)_\infty}f(q^3,q^3)\\
&= -\frac{(1+q)(-1;q)_\infty^3}{128}\left(8(-q;q^2)_\infty^6 f(1,q^6) - (-1;q^2)_\infty^6 f(q^3,q^3)\right)\\
&= -\frac{(1+q)(-1;q)_\infty^3}{128}\frac{16(-q^3;q^3)_\infty^2}{(q^6;q^6)_\infty^7}\left\{f(q,q^5)^6 f(q^3,q^3)^2 - f(1,q^6)^2 f(q^2,q^4)^6\right\}\\
&= 3\frac{(1+q)(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.
\end{aligned}$$

Multiplying both sides of the above equation by $\frac{q}{1+q}$, we conclude

$$\psi(q)+2\psi_-(q;q) = 3\frac{q(q^6;q^6)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.$$

Now we turn to (7.6.2). Using Theorem 7.6.5 with q replaced by $q^{1/2}$, we obtain

$$\begin{aligned}
2\rho(q)+\lambda(q;q) &= -\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty^6 f\left(q^{\frac{3}{2}},q^{\frac{3}{2}}\right)-(-q^{\frac{1}{2}};-q^{\frac{1}{2}})_\infty^6 f\left(-q^{\frac{3}{2}},-q^{\frac{3}{2}}\right)}{4q^{\frac{1}{2}}(q;q)_\infty^3(q^2;q^2)_\infty^3}\\
&= -\frac{(q^{\frac{1}{2}};q)_\infty^6 f(q^{\frac{3}{2}},q^{\frac{3}{2}})-(-q^{\frac{1}{2}};q)_\infty^6 f(-q^{\frac{3}{2}},-q^{\frac{3}{2}})}{4q^{\frac{1}{2}}(-q;q)_\infty^3}\\
&= 3\frac{(q;q)_\infty(q^3;q^3)_\infty(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^3 f(-q,-q^5)^2}\\
&= 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty(q^2;q^2)_\infty}.
\end{aligned}$$

$\square$

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