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GRAPHS, CODES, AND COLORINGS

BY

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DISSERTATION

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Abstract

This thesis focuses on topics in extremal combinatorics.

Given an integer-valued function f defined on the vertices of a graph G , we say G is f -choosable if for every collection of lists with list sizes specified by f there is a proper coloring using the colors from the lists. The *sum choice number*, $\chi_{sc}(G)$, is the minimum of $\sum f(v)$ such that G is f -choosable. We show that if $q \geq 4a^2 \log a$, then there exist constants c_1 and c_2 such that

$$2q + c_1 a \sqrt{q \log a} \leq \chi_{sc}(K_{a,q}) \leq 2q + c_2 a \sqrt{q \log a}.$$

As a consequence, that $\chi_{sc}(G)/|V(G)|$ can be bounded while the minimum degree $\delta_{\min}(G) \rightarrow \infty$. This is not true for the list chromatic number, $\chi_\ell(G)$.

We further prove that, for fixed a , the limit $\lim_{q \rightarrow \infty} (\chi_{sc}(K_{a,q}) - 2q)/\sqrt{q}$ exists, and we give tight bounds for sum choice numbers of the graphs $G_{a,q}$, which are obtained from $K_{a,q}$ by adding all possible edges in the part of size a .

A pair of ordered k -tuples $(x_1, \dots, x_k), (y_1, \dots, y_k)$ is *reverse-free* if, for all $i < j$, (x_i, x_j) does not equal (y_j, y_i) . Let $F(n, k)$ be the maximum size of a pairwise reverse-free set of k -tuples with k distinct entries taken from $[n]$. Allowing repetitions within the k -tuples, we analogously define $\overline{F}(n, k)$.

We determine the asymptotics of $F(n, 3)$ and $\overline{F}(n, 3)$ as n approaches infinity, and we obtain exact formulas if n is a power of 3. We present some bounds for $F(n, k)$ for general k and for other related quantities. We also present upper and lower bounds for the important special case $F(n, n)$ (i.e., reverse-free permutations).

We also determine the order of magnitude of $\overline{F}(n, k)$ for n fixed and $k \rightarrow \infty$.

Finally, the *product dimension* $\dim(G)$ of a graph G is the minimum k such that G is an induced subgraph of the tensor product of k complete graphs. We focus on bounding this parameter on the family of trees. We improve the known upper and

lower bounds, and we determine the exact values of product dimension for infinitely many trees. We extend some of the results to *odd dimension* $\theta_{\text{odd}}(G)$, defined as the minimum k such that we can assign subsets of $[k]$ to the vertices of G in such a way that two vertices are adjacent if and only if the corresponding sets have odd-sized intersection.

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Chapter 1

Introduction

In this thesis, we present results in three areas of extremal combinatorics. Two of these areas relate graphs and codes in two different ways.

A *code* is a family of lists of the same (finite) length, with entries taken from a given underlying set. The elements of the family are called *codewords*. In Chapter 3 we define a graph F on the vertex set of all k -tuples with entries in $[n]$ by making (x_1, \dots, x_k) and (y_1, \dots, y_k) adjacent if there are indices i, j such that $x_i \neq x_j$, and $(x_i, x_j) = (y_j, y_i)$; such a pair of indices is a *reversed pair*. Our aim is to determine the maximum size of a *reverse-free code*, which is nothing more than an independent set in F . We also ask what the size of a maximum clique is in F . We are especially concerned with a special case of these problems that we get by considering the subgraph of F induced by k -tuples that have no repeated entries. These questions are similar in spirit to questions about intersecting permutations, in particular the analogue of the Erdős-Ko-Rado problem.

In Chapter 4, the interaction between graphs and codes is of a different kind. Motivated by applications in computer science, we seek an efficient representation of graphs in computers. One way to do that, inspired by the notion of dimension in partially ordered sets, is to embed the graph in a product of complete graphs. The resulting measure of “complexity” of the graph is called *product dimension*. It is defined as the minimum k such that it is possible to assign k -tuples to the vertices of the graph so that two vertices are adjacent if and only if their k -tuples differ in all k coordinates. We improve the current bounds on product dimension of trees, and we determine the exact values for trees that belong to certain classes.

Finally, the third area deals with a particular generalization of graph coloring. The *list chromatic number* of a graph G is the minimum k such that whenever lists of size k are assigned to the vertices of G , the graph can be properly colored with colors chosen from these sets. We get a parameter called *sum choice number* by allowing

different list sizes k_1, \dots, k_n , but keeping the aim of minimizing the sum $\sum k_i$ subject to the condition that for any list assignment with list sizes k_i , one can color the graph using the colors from the lists. Among other results, we determine the sum choice number for the complete bipartite graphs with one part much larger than the other. As a consequence, we observe that, unlike list chromatic number, sum choice number does not necessarily grow with average degree.

1.1 List colorings with unequal list sizes

Questions about graph coloring are among the most investigated in combinatorics, in part because they naturally model many engineering problems. For example, we need to assign frequencies to several transmitters in such a way that interfering transmitters always get different frequencies. Depending on the specific problem and the restrictions that we face, we get different variants of the coloring problem. If each transmitter has a set of admissible frequencies assigned to it, the appropriate model is a version of *list coloring*.

In the list coloring problem, each vertex v has a set $L(v)$ of available colors assigned to it (such L is called a *list assignment*), and the question is whether it is possible to color the graph properly with colors chosen from the lists. If such coloring exists, we call the list assignment *sufficient*, otherwise it is *insufficient*.

The *list chromatic number* $\chi_\ell(G)$, defined by Vizing in [73] and later independently by Erdős, Rubin, and Taylor in [26], is the minimum k such that every list assignment with lists of sizes at least k for every vertex of G is sufficient. This natural extension of graph coloring has been the subject of considerable attention in the last several decades. For every graph G , the chromatic number is a lower bound for list chromatic number. The complete bipartite graph $K_{3,3}$ is a well-known example of a graph for which the inequality is strict. The assignment of the sets $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ to the vertices of each part is insufficient, hence proving $\chi_\ell(K_{3,3}) \geq 3 > 2 = \chi(K_{3,3})$. Using similar insufficient list assignments for larger bipartite graphs, one can prove that there is a constant c such that $\chi_\ell(K_{n,n}) > c \log n$. On the other hand, one of the first proofs to use probability in combinatorics, inspiring the development of what came to be known as probabilistic method, was the upper bound $\chi_\ell(K_{n,n}) \leq \lceil \log_2 n \rceil + 1$,

which was proved in the seminal paper [26]. The exact value of $\chi_\ell(K_{n,n})$ is still unknown.

We consider a generalization of list chromatic number in which the vertices can have different list sizes. If f is an integer-valued function defined on the vertices of G , an f -assignment on G is a list assignment with list sizes specified by f . The function f itself is *sufficient* if every f -assignment L is sufficient. The *sum choice number* $\chi_{\text{sc}}(G)$ is the minimum of $\sum_{v \in V(G)} f(v)$ over all sufficient f .

Sum choice number was defined by Isaak in [39], and many results appear for example in [10, 36, 37]. The origins of the idea can be traced back to [26], however, where it was observed that a special f is sufficient; this fact was used to prove the analogue of Brooks' Theorem for list coloring.

For every graph G , the upper bound $\chi_{\text{sc}}(G) \leq |V(G)| \cdot \chi_\ell(G)$ is trivial. For balanced complete bipartite graphs, we have a lower bound of the same order of magnitude, i.e., allowing different list sizes does not alter the problem significantly. The situation is very different for unbalanced complete bipartite graphs. Our main result provides good bounds for $\chi_{\text{sc}}(K_{a,q})$. In particular, we prove that for some constants c_1 and c_2 , if $q \geq 4a^2 \log a$, then

$$2q + c_1 a \sqrt{q \log a} \leq \chi_{\text{sc}}(K_{a,q}) \leq 2q + c_2 a \sqrt{q \log a}.$$

We prove the lower bound by construction and the upper bound using the probabilistic method. It turns out that in both cases, the important functions f to consider are those with $f(v) = 2$ for all vertices v that belong to the larger part of $K_{a,q}$. A convenient way to think about the f -assignments with such f is to define a hypergraph with hyperedges corresponding to the lists of the vertices of the smaller part of $K_{a,q}$ and a graph with edges corresponding to the lists of the larger part. The question can then be rephrased as a question about hypergraph covering.

Alon proved in [4] that χ_ℓ is bounded below by a function of the average degree of the graph. This is in contrast to the behavior of chromatic number – the balanced complete bipartite graphs provide a sequence of graphs with average degree approaching infinity, while having fixed chromatic number (equal to 2). A natural question is whether allowing different list sizes changes the behavior of the parameter in this respect. The most interesting consequence of our main result is that the conclusion of Alon's theorem is no longer true in this case. We have a sequence of

graphs whose average degree tends to infinity, and a sequence of corresponding sufficient functions such that the average list sizes given by the functions do not exceed the value 3. We have

$$\lim_{a \rightarrow \infty, q \gg a^2 \log a} \frac{|E(K_{a,q})|}{a+q} = \infty, \quad \lim_{a \rightarrow \infty, q \gg a^2 \log a} \frac{\chi_{\text{sc}}(K_{a,q})}{a+q} = 2. \quad (1.1)$$

The average degree of $K_{a,q}$ tends to $2a$ as $q \rightarrow \infty$, but $\chi_{\text{sc}}(K_{a,q})/(a+q)$ tends to 2.

In Section 2.7 we prove that for fixed a , the limit $\lim_{q \rightarrow \infty} \frac{\chi_{\text{sc}}(K_{a,q}) - 2q}{\sqrt{q}}$ exists. We do this by again reducing the problem to considering only the functions f with $f(v) = 2$ for all v that belong to the larger part, and treating each such f as a point in a -dimensional Euclidean space. Roughly speaking, f is insufficient if the point is under some particular quadric surface. To find the limit, we slide a hyperplane until it touches the boundary of the set.

We also ask what happens when one adds edges to the smaller part of $K_{a,q}$. Let $G_{a,q}$ be the graph that we get by adding all $\binom{a}{2}$ such edges. For this graph, we have $2q + c_1 \sqrt{q(a-1)} \leq \chi_{\text{sc}}(G_{a,q}) \leq 2q + c_2 \sqrt{q(a-1)}$ for some constants c_1 and c_2 . The upper bound is established by proving a generalization of Turán's Theorem.

The results of this section are joint work with Zoltán Füredi.

1.2 Reverse-free codes and permutations

In Chapter 3 we consider a problem inspired by coding theory. Let X be an n -element underlying set. Two k -tuples $(x_1, \dots, x_k), (y_1, \dots, y_k)$ of elements from X are *reverse-free* if, for all pairs i, j such that $x_i \neq x_j$, (x_i, x_j) does not equal (y_j, y_i) . Let $F(n, k)$ be the maximum size of a (pairwise) reverse-free set of k -tuples in which each k -tuple has k distinct entries. Allowing repetitions within the k -tuples, we analogously define $\overline{F}(n, k)$. We also consider families of k -tuples such that no two of them are reverse-free. We call such family *flip-full* and denote its maximum size $G(n, k)$ if no repetitions are allowed, and $\overline{G}(n, k)$ otherwise.

For a reverse-free family \mathcal{F} , define a matrix $M(\mathcal{F})$ by listing the elements of \mathcal{F} as its rows. The problem of determining the maximum size of a reverse-free family is equivalent to determining the maximum number of rows of a k -column matrix that

has no submatrix $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a \neq b$. Many other coding theory problems can be formulated in terms of matrices with forbidden submatrices, or with certain submatrices required. The most notable example of the former is the work of Sauer [64], Shelah [65], and Vapnik and Chervonenkis [72] concerning VC-dimension. For an example of the latter, see [44].

All problems in Chapter 3 can be formulated in terms of independence and clique numbers of suitably defined graphs whose vertex set is either the set of k -tuples with entries in $[n]$, or permutations of $[n]$. Similar problems were considered in an information theoretic setting, for example in [48].

Perhaps the best known graph on the set of permutations of $[n]$ is the derangement graph. In this graph, two permutations π and σ are adjacent if they are not intersecting, i.e., $\pi(i) \neq \sigma(i)$ holds for all i . We say that two permutations π and σ are t -intersecting if there are at least t indices i such that $\pi(i) = \sigma(i)$. Extending the definition of derangement graph, we define a new graph by letting π and σ be adjacent if they are not t -intersecting. A clique in this graph is a family \mathcal{F} of permutations such that $M(\mathcal{F})$ does not contain the submatrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_t \\ a_1 & a_2 & \dots & a_t \end{pmatrix}$$

for any choice of $a_1, \dots, a_t \in [n]$. The independence number of this graph was determined recently in [23] to be equal to $(n-t)!$ whenever n is large enough with respect to t , confirming a conjecture by Deza and Frankl [30] and proving a version of the Erdős-Ko-Rado Theorem for t -intersecting permutations.

In our case, two k -tuples are adjacent if they are not reverse-free. The independence number and the clique number are equal to $F(n, k)$ or $\overline{F}(n, k)$, and $G(n, k)$ or $\overline{G}(n, k)$, depending on the choice of the vertex set. We will focus mostly on reverse-free families, but will prove some results about families that are flip-full as well.

The set of all increasing (or nondecreasing) k -tuples is reverse-free, and hence $F(n, k) \geq \binom{n}{k}$ and $\overline{F}(n, k) \geq \binom{n+k-1}{k}$. It is still an open problem to determine whether these inequalities are strict. We show however that the sequence $F(n, k)/\binom{n}{k}$ has a limit, if k is fixed and $n \rightarrow \infty$.

We determine the values of the parameter for small values of n and k . In particular, in Section 3.4 we show an iterative construction of a reverse-free family of triples,

providing a lower bound for $F(n, 3)$. In Section 3.5 we prove an upper bound with the same order of magnitude as the lower bound, and equal to it whenever n is a power of 3. We prove similar results for $\overline{F}(n, 3)$ in Section 3.6.

We provide bounds for $G(n, n)$ and $F(n, n)$. In this case, the k -tuples are actually permutations of n .

Finally, in Section 3.10 we consider the opposite problem: determining the behavior of $\overline{F}(n, k)$ with n fixed and $k \rightarrow \infty$. We use VC-dimension results to determine the asymptotics in this case.

The results of this section are joint work with, Zoltán Füredi, Angelo Monti, and Blerina Sinimeri.

1.3 Product dimension of trees

Many applications in computer science require a memory-efficient way of storing graphs and posets (viewed as directed acyclic graphs) that at the same time allows for fast retrieval of the structure, such as the edges of the graph or the poset relation. Dushnik and Miller defined the dimension of poset P in [19] as the minimum number t such that P can be embedded into the product of t chains. Using this embedding, we obtain a vector $\varphi(z) = (\varphi_1(z), \dots, \varphi_t(z))$ for every vertex z in a natural way, with the property that $x \leq y$ if and only if $\varphi_i(x) \leq \varphi_i(y)$ holds for all i . The vectors contain all information about the poset. If the dimension is small, then such a representation is efficient. The dimension can be thought of as a measure of non-linearity of P .

For graphs, dimension can be defined analogously, using an embedding into a product of cliques. Cliques are a natural analogue of chains, and are considered “simple” objects in this context. The *tensor product* $G_1 \times G_2$ of two graphs, G_1 and G_2 , has $V(G_1) \times V(G_2)$ as its vertex set, and the edges are the pairs $\{(u_1, v_1), (u_2, v_2)\}$ such that $u_1 u_2 \in E(G_1)$ and $v_1 v_2 \in E(G_2)$. The *product dimension* of a graph G , introduced by Nešetřil and Pultr in [56] and denoted $\dim(G)$, is the minimum t such that G is an induced subgraph of a direct product of t cliques. In Chapter 4 we improve the known upper and lower bounds on dimension of trees.

For many graphs, the best known lower bound on the dimension was given by a powerful theorem that was proved by Lovász, Nešetřil and Pultr in [52]. They proved that if we find distinct vertices x_1, \dots, x_k and some vertices y_1, \dots, y_k such

that $x_i y_i \in E(G)$ for all i , and $x_i y_j \notin E(G)$ whenever $i < j$, then $\dim(G) \geq \lceil \log_2 k \rceil$. This lower bound was the best known so far for trees as well.

In Section 4.3 we improve this lower bound. We prove that if P_1, \dots, P_m are vertex-disjoint paths of lengths s_1, \dots, s_m respectively in the tree T , then $\dim(T) \geq \lceil \log_2(\sum s_i) \rceil$.

The best known upper bound constructions were given by Alles in [2]. We give a construction that improves Alles' bound in many cases, including all trees that have no vertices of degree 2.

In Section 4.5 we determine the dimension exactly for some families of trees.

Finally, we prove analogous lower bound theorems for the *odd dimension*, $\theta_{\text{odd}}(G)$, defined in [20] as the minimum k such that we can assign subsets of $[k]$ to the vertices of G in such a way that $uv \in E(G)$ if and only if the intersection of the corresponding sets has odd size.

1.4 Basic definitions

A *graph* G consists of two sets, an underlying set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of distinct elements of $V(G)$ called *edges*. The *order* of a graph G is the number of vertices of G , while *size* is the number of edges. Throughout the thesis, we will reserve the letter n to denote the order of a graph.

We denote the edge $\{u, v\}$ simply by uv . The *endpoints* of an edge uv are the vertices u and v . When u and v are endpoints of some edge, they are *adjacent*. Pairs of vertices that do not belong to $E(G)$ are called *non-edges*. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v . The *degree* of a vertex v , denoted $d(v)$, is the size of its neighborhood. The *minimum degree* $\delta(G)$ of a graph G is the minimum of $d(v)$ over all vertices $v \in V(G)$. The *maximum degree* $\Delta(G)$ is defined analogously. The *average degree* $d_{\text{ave}}(G)$ is equal to $(\sum d(v))/|V(G)|$.

We say that a graph H is a *subgraph* of a graph G (or that G *contains* H) if there is an injection $\psi : V(H) \rightarrow V(G)$ such that $\psi(u)\psi(v) \in E(G)$ whenever $uv \in E(H)$. The subgraph is *induced* if ϕ satisfies the additional condition that $uv \notin E(H)$ implies $\psi(u)\psi(v) \notin E(G)$, for all $u, v \in V(H)$. A bijection $\psi : V(G) \rightarrow V(H)$ is an *isomorphism* if $uv \in E(G)$ if and only if $\psi(u)\psi(v) \in E(H)$.

Many isomorphism classes of graphs are important enough that they have names.

A *complete graph* K_n has n vertices, every two of which are adjacent. The vertices of a *bipartite graph* can be partitioned into two *parts* so that no edge has both endpoints in the same part. A *complete bipartite graph* $K_{s,t}$ is a bipartite graph with parts of sizes s and t , and st edges. A *cycle* C_n is any graph isomorphic to the graph with vertices v_1, \dots, v_n and edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$. A *path* P_n is isomorphic to the graph with vertices w_1, \dots, w_n and edges $w_iw_{i+1} \in E(G)$ for $i = 1, \dots, n-1$. The two vertices of degree 1 are its *endpoints*; the other vertices are *internal*. The *length* of a path is its number of edges.

A graph G is *connected* if, for every $u, v \in V(G)$, there is a path in G with endpoints u and v . A *component* of G is a maximal connected subgraph of G .

A *forest* is a graph that contains no cycle. A *tree* is a connected forest. In particular, forests and trees are examples of bipartite graphs, and paths are examples of trees. A *rooted tree* is a tree with one distinguished vertex, called the *root*. A *leaf* in a tree is a vertex of degree 1; every tree with at least two vertices has at least two leaves. An n -vertex *star* is a tree with $n-1$ leaves, all adjacent to a single vertex.

The vertices of the *Kneser graph* $K(n, k)$ are the k -element subsets of $[n]$. The edges are the pairs of disjoint sets. The graph $K(5, 2)$ is also called the *Petersen graph*. The *hypercube* Q_k has all binary k -tuples of length k as its vertices, with two of them adjacent when they differ only in one coordinate. A *matching* in a graph G is a set of disjoint edges. A matching is *perfect* if every vertex of G is contained in one of the edges of the matching.

A subset W of $V(G)$ is called an *independent set* if no two vertices in W are adjacent. A subset of pairwise adjacent vertices in a graph G is called a *clique*. The *independence number* $\alpha(G)$ is the maximum size of an independent set in G . Similarly, the *clique number* $\omega(G)$ is the maximum size of a clique in G . The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path with endpoints u and v in G . The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum of $d(v, w)$ over all vertices $w \in V(G)$. The minimum eccentricity over all vertices of G is called the *radius* of G . The *center* of G is the set of vertices with the least eccentricity. If G is a tree, then its center consists of one vertex or two adjacent vertices. The *diameter* of G , $\text{diam}(G)$, is the maximum of $d(u, v)$ over all pairs u, v . For a tree T with radius r , we have either $\text{diam}(T) = 2r$ or $\text{diam}(T) = 2r - 1$, depending whether the center of T has one or two vertices.

The *Cartesian product* of the sets A and B is defined as $\{(a, b) : a \in A, b \in B\}$.

Besides the tensor product, we will also need the *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 . The vertex set is again $V(G_1) \times V(G_2)$, and the edges are the pairs $\{(u_1, v_1), (u_2, v_2)\}$ such that $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$, or $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$.

A *proper coloring* of a graph G is an assignment of a label $c(v)$ to each vertex v such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The graph G is *k-colorable* if there is a proper coloring of G with labels $1, \dots, k$. The *chromatic number* of G , written $\chi(G)$, is the minimum k such that G is *k-colorable*. Given a graph G , we can form the *line graph* of G , denoted $L(G)$, with $V(L(G)) = E(G)$, by making $uv, xy \in E(G)$ adjacent if $\{u, v\} \cap \{x, y\} \neq \emptyset$. A *proper edge-coloring* of G is a proper coloring of the line graph of G . The *chromatic index* is the minimum k for which there exists a proper edge-coloring of G .

To emphasize that the edges are unordered pairs, a graph defined as we did in the first paragraph is sometimes called *undirected*. This is in contrast to *directed graphs* (or *digraphs*), whose edge sets consist of ordered pairs (u, v) of distinct vertices (*directed edges*). The vertex u in a directed edge (u, v) is called the *tail*, and the vertex v is the *head*. A cycle of length k in a directed graph G consists of k vertices v_1, \dots, v_k and the edges $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1)$. A directed graph G is *acyclic* if it has no cycle. A *tournament* is a directed graph that has exactly one of the two edges (u, v) and (v, u) for every pair $\{u, v\}$ of distinct vertices. A tournament on three vertices is called a *triangle*. A *directed triangle* is a cycle of length 3. A triangle that is not directed is *undirected*; sometimes it is called a *transitive triple*. An *out-neighbor* of u is a vertex v such that (u, v) is an edge. An *in-neighbor* of u is a vertex v such that (v, u) is an edge. The *out-degree* of u , denoted $d^+(u)$, is the number of out-neighbors. The *in-degree* $d^-(u)$ is the number of in-neighbors.

In a *multigraph* (directed or undirected), $E(G)$ is a multiset, that is, it can have multiple copies of each pair of vertices.

A *hypergraph* \mathcal{H} consists of the *vertex set* $V(\mathcal{H})$ and a family $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$, called *hyperedges* or just *edges* for simplicity. We sometimes adopt the convention of identifying \mathcal{H} with its set of edges. The *size* of \mathcal{H} , denoted $|\mathcal{H}|$, is the number of edges of \mathcal{H} . A hypergraph is *k-uniform* if all its edges have size k . With this terminology, a graph is just a 2-uniform hypergraph. A *cover* (or *transversal*) of \mathcal{H} is a set of vertices that intersects all edges. The *covering number* $\tau(\mathcal{H})$ of \mathcal{H} is the minimum size of a cover of \mathcal{H} . A *multihypergraph* can have several copies of each edge.

A *partially ordered set* or a *poset* is a set P together with a reflexive, antisymmetric, and transitive binary relation \preceq on P . The elements $x, y \in P$ are *comparable* if $x \preceq y$ or $y \preceq x$. We say that y *covers* x if $x \preceq z \preceq y$ implies $x = y$ or $y = z$. The *cover graph* of a poset (P, \preceq) is the (undirected) graph G with $V(G) = P$ and an edge xy whenever y covers x . Another undirected graph based on the poset is its *comparability graph*. It has the same vertex set, but the edges are pairs xy such that $x \preceq y$. An *embedding* of (P, \preceq_1) into (Q, \preceq_2) is an injective function $\psi : P \rightarrow Q$ such that $x \preceq_1 y$ if and only if $\psi(x) \preceq_2 \psi(y)$. If there is an embedding of (P, \preceq_1) into (Q, \preceq_2) , we also say that (P, \preceq_1) is a *subposet* of (Q, \preceq_2) . The *product* of (P, \preceq_1) and (Q, \preceq_2) is the poset $(P \times Q, \preceq)$, with $(x_1, y_1) \preceq (x_2, y_2)$ whenever $x_1 \preceq_1 x_2$ and $y_1 \preceq_2 y_2$. A *chain* is a poset in which every two elements are comparable.

In several places, we use the “Big O notation” to describe the asymptotic behavior of functions. Let $f(x)$ and $g(x)$ be two functions defined on (some subset of) real numbers. We say that $f(x) = O(g(x))$ as $x \rightarrow \infty$ if there is a constant c such that $|f(x)| \leq c|g(x)|$ for x large enough. This is a common abuse of notation, since technically $O(g(x))$ is a set of functions. If there are two constants c_1 and c_2 such that $c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|$ for large enough x , we say that $f(x) = \Theta(g(x))$ as $x \rightarrow \infty$. We say that $f(x) = o(g(x))$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$.

In general, we write (x_1, \dots, x_k) for a k -tuple, and $\{x_1, \dots, x_k\}$ for a set or a multi-set. The exceptions are as follows: in Chapter 3 we drop the parentheses and commas and use $x_1 \dots x_k$ for a k -tuple in order to make the text easier to read. Also, as already mentioned, we adhere to the usual convention of writing uv for an edge of an undirected graph. These never appear together in the text, so there is little danger of confusion.

Let us finish with various miscellaneous pieces of notation. By deleting some rows and columns of a matrix A , we get a *submatrix* of A . In the context of a probability space, the probability measure will be denoted \mathbb{P} , and \mathbb{E} will be the expected value. The letter \mathbb{N} denotes the set of positive integers (in particular, \mathbb{N} does not include 0), and \mathbb{R} is the set of real numbers. The *characteristic vector* of the subset $A \subseteq [k]$ is the k -tuple (x_1, \dots, x_k) with $x_i = 1$ if $i \in A$ and $x_i = 0$ otherwise. The *weight* of such a vector is the number of 1s in it. We will switch between sets and their characteristic vectors as needed. The family of all subsets of X of size k is denoted $\binom{X}{k}$. The family of all subsets of X is denoted 2^X . For $k \in \mathbb{N}$, $[k]$ denotes the set $\{1, \dots, k\}$. If x is a real number, $\lceil x \rceil$ denotes the least integer y such that $y \geq x$, and $\lfloor x \rfloor$ is the greatest

integer y such that $y \leq x$. The natural logarithm of x is denoted $\ln x$. Additional notation will be defined as needed.

Chapter 2

List colorings with unequal list sizes

2.1 Choosability and balanced complete bipartite graphs

Many interesting real world problems can be modeled by some version of graph coloring. For example, in the most basic version of the frequency assignment problem, we are given a set of transmitters and we want to assign a frequency to each of them in such a way that if two transmitters interfere, they get different frequencies. Asking whether this can be done with k frequencies is equivalent to solving the k -colorability problem for a particular graph. If only some frequencies are available for each transmitter, the problem corresponds to the list coloring problem for a certain graph.

Definition 1. For a graph G , a *list assignment* is a function $L : V(G) \rightarrow 2^{\mathbb{N}}$ which assigns a list of *admissible colors* to each vertex v . If L is a list assignment such that G has a proper coloring c with $c(v) \in L(v)$ for all vertices v , we say that G is *L -colorable*, or that L is *sufficient*.

The *list coloring problem* is the following.

Given a graph G and a list assignment L on G , is G L -colorable?

For applications of graph coloring in frequency allocation, see for example [22, 62, 75].

The k -colorability problem with $k \geq 3$ is a well-known NP-complete problem. It is a simple special case of the list coloring problem, and hence the list coloring problem is NP-complete even if all lists have size 3. Let us remark that this problem is NP-complete even if various additional constraints are placed on the graph G and the list assignment L (see [49]).

The following notion was defined by Vizing in [73], and later independently by Erdős, Rubin and Taylor in [26].

Definition 2. A graph G is t -choosable if every list assignment L with $|L(v)| \geq t$ for all vertices $v \in V(G)$ is sufficient. The *list chromatic number* $\chi_\ell(G)$ is the minimum t such that G is t -choosable.

In the literature, the list chromatic number is sometimes called the *choosability* or the *choice number*.

While the main focus of this section is the more general notion of sum choice number, some techniques that have been developed while studying list chromatic number will be useful to us in the sum choice setting as well. In this section we will review the most important of these ideas.

The concept of choosability has been the focus of considerable attention and research in the last thirty years. In the seminal paper [26] Erdős, Rubin and Taylor characterized all 2-choosable graphs and provided good bounds for list chromatic number of balanced bipartite graphs. Our results on sum choice numbers of bipartite graphs use some of these ideas.

First, let us present a well-known example of an insufficient list assignment ([26]). Figure 2.1 shows this construction for $t = 3$.

Lemma 3 ([26]). *Fix $t \in \mathbb{N}$ and let $n = \binom{2t-1}{t}$. If L is the assignment of t -element subsets of $[2t-1]$ to the vertices of $K_{n,n}$ satisfying the condition that any two vertices that belong to the same part receive different sets, then $K_{n,n}$ is not L -colorable.*

Proof. Suppose to the contrary that there exists a proper coloring c with $c(v) \in L(v)$ for all vertices v . Let \mathcal{H} be the hypergraph with the vertex set $V(\mathcal{H}) = \cup_{v \in V(K_{n,n})} L(v)$ and the edge set $E(\mathcal{H}) = \{L(v) : v \in V(K_{n,n})\}$. The set $\{c(v) : v \in V(K_{n,n})\}$ is a cover of \mathcal{H} . It is easy to check that $\tau(\mathcal{H}) = t$, so the coloring uses at least t colors. But then there is at least one color that was used for both parts of $K_{n,n}$, a contradiction. \square

Corollary 4. $\chi_\ell(K_{n,n}) > \frac{1}{2} \log_2 n + \frac{1}{4} \log_2 \log_2 n - \frac{3}{4}$.

Proof. Let t be the largest integer such that $\binom{2t-1}{t} \leq n$. By Lemma 3, $\chi_\ell(K_{n,n}) > t$. We have $t > \frac{1}{2} \log n + \frac{1}{4} \log \log n - \frac{3}{4}$, because otherwise, using Stirling approximation

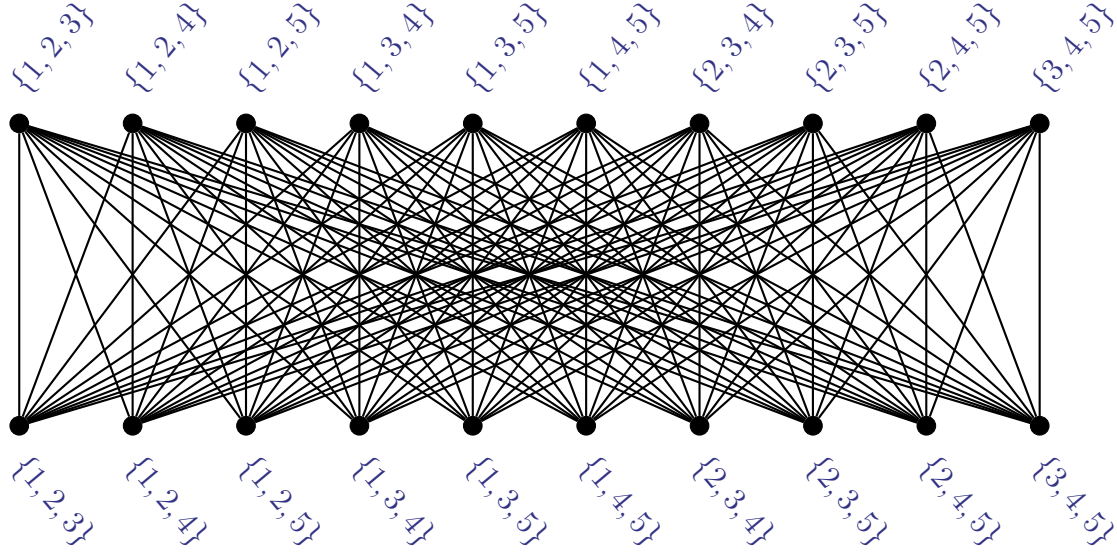


Figure 2.1: An insufficient list assignment for $K_{10,10}$.

and the fact that $4^t/\sqrt{t}$ is increasing for $t \geq 1$,

$$\binom{2(t+1)-1}{t+1} \leq 4 \binom{2t-1}{t} \leq 2 \binom{2t}{t} \leq 2 \frac{4^t}{\sqrt{t}} \leq 2 \frac{n 2^{-3/2} \sqrt{\log_2 n}}{\sqrt{\frac{1}{2} \log_2 n + \frac{1}{4} \log_2 \log_2 n - \frac{3}{4}}} \leq n,$$

and t was not the largest integer satisfying the inequality. \square

An upper bound for $\chi_\ell(K_{n,n})$ also appeared in [26].

Theorem 5 ([26]). $\chi_\ell(K_{n,m}) \leq \lceil \log_2(n+m) \rceil$.

Proof. Let X and Y be the two parts of the bipartite graph. Consider a list assignment L with $L(v) = k$ for all vertices $v \in V(K_{n,m})$. Construct a random subset T of $\cup_{v \in V(K_{n,m})} L(v)$ by including each color independently with probability $1/2$. Let C be the event that it is not possible to color X with colors from T and Y with the remaining colors. We have

$$\mathbb{P}(C) \leq (n+m) \frac{1}{2^k}. \quad (2.1)$$

If $k > \log_2(n+m)$, then the quantity in (2.1) is strictly less than 1, and hence there exists a choice of T such that we can color X with colors in T and Y with the remaining colors. \square

The exact value of $\chi_\ell(K_{n,n})$ is unknown for general n . It was already observed by Erdős, Rubin and Taylor in [26] that this problem has a strong connection to the problem of determining the least cardinality M_k of a k -uniform hypergraph \mathcal{H} that does not have property B . We say that a hypergraph has property B if it is 2-colorable, i.e., there is a set $T \subseteq V(\mathcal{H})$ that intersects all edges of \mathcal{H} but does not contain any edge of \mathcal{H} . In particular, if N_k is the minimum number of vertices of a complete bipartite graph that is not k -choosable, then

$$M_k \leq N_k \leq 2M_k.$$

To prove the first inequality, consider the smallest complete bipartite graph G that is not k -choosable, and a list assignment L with lists of size k such that G is not L -colorable. These lists form a k -uniform hypergraph that does not have property B . To prove the upper bound, take a k -uniform hypergraph that does not have property B , and assign the hyperedges as the lists to both parts of K_{M_k, M_k} . The graph K_{M_k, M_k} is not L -colorable for this L .

The problem of determining M_k is unsolved and is generally regarded as hard.

If G is L -colorable for every list assignment L with $|L(v)| \geq t$, then in particular this is true for the list assignment L defined as $L(v) = \{1, \dots, t\}$ for all v . In other words,

$$\chi(G) \leq \chi_\ell(G) \tag{2.2}$$

holds for all graphs G .

Since $\chi(K_{n,n}) = 2$ for all n , Corollary 4 provides evidence that $\chi_\ell(G)$ and $\chi(G)$ can be arbitrarily far apart. Surprisingly, no such examples are known for edge-colorings. We can define the *edge chromatic number* $\chi'(G)$ and the *list edge chromatic number* $\chi'_\ell(G)$ analogously to ordinary chromatic number and list chromatic number, as $\chi'(G) = \chi(L(G))$ and $\chi'_\ell(G) = \chi_\ell(L(G))$, where $L(G)$ is the line graph of G . Let us close this section by mentioning one of the most important conjectures concerning choosability. This conjecture is of unclear origin; for a discussion of its history, see [41].

Conjecture 6 (List coloring conjecture). *For all G , $\chi'_\ell(G) = \chi'(G)$.*

2.2 Average list sizes and planar graphs

It is natural to consider a generalization of list coloring where the sizes of the lists are prescribed by a non-constant function.

Definition 7. Consider a function $f : V(G) \rightarrow \mathbb{N}$. An f -assignment is a list assignment L with $|L(v)| = f(v)$ for all $v \in V(G)$. The graph G is f -choosable if all f -assignments are sufficient. If G is f -choosable, we say that the function f is *sufficient*.

A special case of this concept is already discussed in [26]. Let $\phi(G, H)$ be the set of all graphs that can be obtained by selecting a vertex from G and a vertex from H and identifying them. Let D be the minimal family of graphs that contains all cliques, all odd cycles, and with each pair G, H it contains all members of $\phi(G, H)$ as well.

Theorem 8 ([26]). *For a graph G , define $f(v) = \deg(v)$ for all vertices v . This f is sufficient if and only if G does not belong to D .*

This characterization of f -choosable graphs for this special f yields the following analogue of Brooks' Theorem [13] for list chromatic number.

Theorem 9 ([26]). *If G is a graph with maximum degree Δ , then $\chi_\ell(G) \leq \Delta$, unless G has a component $K_{\Delta+1}$, or $\Delta = 2$ and G has an odd cycle as a component.*

For a fixed ordering σ of the vertices of G , let $\hat{d}(v)$ be the number of vertices adjacent to v and preceding v in σ .

Proposition 10. *Let σ be an ordering of the vertices of G , and define $f(v) = \hat{d}(v) + 1$ with respect to σ . Such f is sufficient.*

Proof. Let L be an f -assignment. We color the vertices successively in the order specified by σ . When the time comes to color v , it has $\hat{d}(v)$ neighbors that are already colored. Since $|L(v)| = \hat{d}(v) + 1$, we still have an available color to assign to v . \square

Proof of Theorem 9. Without loss of generality, G is connected. If G is Δ -regular, then either it is f -choosable and we are done, or it is in D . The only regular graphs in D are cliques and odd cycles.

If G has a vertex v with $\deg(v) < \Delta$, define A_i for all $i \geq 0$ to be the set of vertices in distance i from v (and in particular $A_0 = \{v\}$). Fix an ordering σ such that $\sigma(u) < \sigma(v)$ whenever $u \in A_i$ and $v \in A_j$ with $i > j$. Each vertex other than v has a neighbor succeeding it in σ . The result is then a consequence of Proposition 10. \square

We will present two further examples of sufficient functions. As our first example, consider the complete graph K_n with vertices v_1, \dots, v_n , and the function $f(v_i) = i$ for all i . Consider an arbitrary f -assignment. Assign colors to v_1, v_2, \dots, v_n successively. At step i , $i - 1$ vertices already have colors, but since $|L(v_i)| = i$, it is possible to assign a color to v_i as well. This f is therefore sufficient.

The second example concerns planar graphs. It is a well-known fact, proved by Thomassen in [68], that

$$\chi_\ell(P) \leq 5 \tag{2.3}$$

for every planar graph P , and this is best possible ([74]).

But in fact, Thomassen's pretty argument for (2.3) proves a stronger statement.

Theorem 11 ([68]). *Let P be an n -vertex planar graph with external vertices v_1, \dots, v_t . If L is a list assignment with*

$$|L(v)| = \begin{cases} 1 & \text{for } v = v_1, \\ 1 & \text{for } v = v_2, \\ 3 & \text{for } v = v_3, \dots, v_t, \\ 5 & \text{otherwise} \end{cases} \tag{2.4}$$

and $L(v_1) \neq L(v_2)$, then L is sufficient.

It follows that the function f defined as

$$f(v) = \begin{cases} 1 & \text{for } v = v_1, \\ 2 & \text{for } v = v_2, \\ 3 & \text{for } v = v_3, \dots, v_t, \\ 5 & \text{otherwise} \end{cases} \tag{2.5}$$

is sufficient, for a planar graph with external vertices v_1, \dots, v_t .

2.3 Sum choice number, balanced complete bipartite graphs

In view of previous section, the following definition is natural.

Definition 12 ([39]). The *sum choice number* of G , denoted $\chi_{\text{sc}}(G)$, is the least k for which there exists a sufficient f with $\sum_{v \in V(G)} f(v) = k$.

Sum choice number was introduced in [39] by Isaak, who proved that if G is the line-graph of $K_{2,q}$, then $\chi_{\text{sc}}(G) = q^2 + \lceil 5q/3 \rceil$. Berliner, Bostelmann, Brualdi, and Deaett [10] determined the sum choice number of any graph that can be obtained as a union of two graphs sharing a single vertex, in terms of the sum choice numbers of the two graphs. They also found the value of the parameter for $K_{2,n}$. Heinold in [36] and [37] determined the sum choice number of Petersen graph, of $P_2 \square P_n$, and of $K_{3,n}$. Isaak in [40] concentrated on graphs whose every block is a complete graph and proved that the sum choice number equals $|V(G)| + |E(G)|$ for these graphs.

We will first provide several examples and observations.

Thomassen's Theorem, and in particular the observation that the function (2.5) is sufficient, implies that for planar graphs P , we have

$$\chi_{\text{sc}}(P) \leq 5n - 9. \quad (2.6)$$

Our second example is the clique K_n , with $V(K_n) = \{v_1, \dots, v_n\}$. We already showed that the function f defined as $f(v_i) = i$ is sufficient, providing an upper bound on the sum choice number. We will prove a matching lower bound, showing

$$\chi_{\text{sc}}(K_n) = \frac{n(n+1)}{2}. \quad (2.7)$$

To prove that $\chi_{\text{sc}}(G) \geq k$ for a given k , we need to show that no integer-valued function f with $\sum_{v \in V(G)} f(v) < k$ is sufficient. Consider an arbitrary integer-valued function f with $\sum_{v \in V(K_n)} f(v) < \frac{n(n+1)}{2}$. Assume without loss of generality that we have $f(v_1) \leq f(v_2) \leq \dots \leq f(v_n)$ for the vertices of K_n . Now construct a list assignment L by letting $L(v) = \{1, 2, \dots, f(v)\}$ for every vertex v . Let i be the least index such that $f(v_i) < i$. We have then $i - 1$ colors available to color v_1, \dots, v_i , so K_n is not L -colorable.

Next, let us mention two upper bounds for χ_{sc} . The first one is an obvious consequence of the definition of sum choice number.

Proposition 13. $\chi_{\text{sc}}(G) \leq |V(G)| \cdot \chi_{\ell}(G)$.

As seen in (2.6) and (2.7), planar graphs and cliques provide examples of graphs for which this bound is far from the truth, and provide motivation for our interest in this parameter. The average list size $\frac{1}{n} \sum f(v)$ of a sufficient function f can drop significantly if the function is allowed to be non-constant.

Proposition 14 ([40, 26]). $\chi_{\text{sc}}(G) \leq |V(G)| + |E(G)|$.

Proof. We have already shown in Proposition 10 that the function $f(v) = \hat{d}(v) + 1$ is sufficient, for any ordering of the vertices. For this f , we have $\sum f(v) = |V(G)| + |E(G)|$. \square

The papers [10] and [40] investigated graphs for which this bound holds with equality.

As an easy consequence, we get $\chi_{\text{sc}}(P) \leq 4n - 6$ for every planar graph. But in fact, we can get a slight improvement.

Theorem 15. *For every n -vertex planar graph P , there exists a sufficient function $f : V(P) \rightarrow \mathbb{N}$ such that $\sum f(v) = 4n - 6$ and $\max f(v) \leq 6$.* \square

Proof. Using Proposition 10, every ordering of $V(P)$ yields an upper bound on $\chi_{\text{sc}}(P)$. Since every planar graph has a vertex of degree at most 5, it is possible to order the vertices so that $\hat{d}(v) \leq 5$ for all v . \square

Problem 16. *Determine χ_{sc} for planar graphs.*

In particular, Thomassen's Theorem shows that for every planar graph there exists a sufficient function f with $f(v) \leq 5$ for all v , with average list size slightly less than 5. On the other hand, in Theorem 15 we show that there exists a sufficient function with average list size less than 4, but this improvement comes at the expense of raising the maximum list size to 6. Is it possible to combine the two? An affirmative answer to the following problem would be a strengthening of Thomassen's Theorem.

Problem 17. Does there exist a sufficient function f for every planar graph P , such that $f(v) \leq 5$ for all $v \in V(P)$ and $\sum f(v) \leq 4n$?

It seems that we cannot hope for any result better than this. There are planar graphs with list chromatic number 5, so the first condition cannot be improved. As for the second condition, chromatic number of planar graphs can be as high as 4, so it might be too much to require a smaller average list size.

We will state here explicitly, for future reference, two observations about monotonicity of sufficient functions and of χ_{sc} .

Lemma 18. Let G, H be graphs, and let f, g be integer-valued functions on $V(G)$.

- (i) If G is a subgraph of H , then $\chi_{\text{sc}}(G) \leq \chi_{\text{sc}}(H)$.
- (ii) If $f(v) \leq g(v)$ for all v , and g is insufficient, then so is f .

Now, let us anticipate the next section and turn our attention to complete bipartite graphs. Intuitively, the additional freedom of allowing a non-constant f should not help too much if the parts have the same size, i.e., $\chi_{\text{sc}}(K_{n,n})$ is likely close to the upper bound from Proposition 13. The following result supports our intuition.

Theorem 19. $\frac{1}{2}n \log n + \frac{2}{3}n < \chi_{\text{sc}}(K_{n,n}) \leq 2n(\lceil \log_2 n \rceil + 1)$.

Proof. The upper bound is a direct consequence of Proposition 13 together with Theorem 5.

To prove the lower bound, for every integer-valued function f with $\sum f(v) \leq \frac{1}{2}n \log n + \frac{2}{3}n$ we present an f -assignment L such that $K_{n,n}$ is not L -colorable. Fix such function f . Let $k_0 = \lfloor \log_4 n \rfloor + 1$. Let X and Y be the two parts of $K_{n,n}$, and let a_k and b_k respectively be the number of vertices in X and Y with list sizes at most k .

Note that $\binom{2k-1}{k} \leq 4^{k-1}$. If, for some $k \leq k_0$, X and Y each have at least 4^{k-1} vertices v with $f(v) \leq k$, then we can construct a list assignment as in Lemma 3, with the help of Lemma 18. If f is sufficient, then

$$\min\{a_k, b_k\} \leq 4^{k-1} - 1 \tag{2.8}$$

for every $k \leq k_0$. We have $\sum f(v) = n + (n - a_1) + (n - a_2) + \cdots + (n - a_{k_0}) + n + (n - b_1) + (n - b_2) + \cdots + (n - b_{k_0})$. Under the conditions (2.8), this is minimized if

$a_k = 4^{k-1} - 1$ and $b_k = n$ holds for all k , i.e., if $f(v) = 1$ for $v \in Y$. We then have

$$\begin{aligned} \sum f(v) &\geq n + n + (n - 4^0 + 1) + (n - 4^2 + 1) + \cdots + (n - 4^{k_0-1} + 1) \\ &\geq 2n + k_0(n + 1) - \sum_{k=0}^{k_0-1} 4^k \\ &> \frac{1}{2}n \log_2 n + \frac{2}{3}n. \end{aligned}$$

This is a contradiction. \square

On the other hand, $\chi_\ell(K_{n,n}) = \Theta(\log n)$.

Theorem 20 ([26]). $\lfloor \log_2 n - 2 \log_2 \log_2 n \rfloor - 1 \leq \chi_\ell(K_{n,n}) \leq \lceil \log_2 n \rceil + 1$.

Proof. Let C_k be the minimum n such that $\chi_\ell(K_{n,n}) > k$. Recalling that M_k is the minimum size of a k -uniform hypergraph without property B , we have $M_k \leq 2C_k \leq 2M_k$. Also, it is known (see [26]) that $2^{k-1} < M_k < k^2 2^{k+1}$. Hence if $n \geq k^2 2^{k+1}$, then $\chi_\ell(K_{n,n}) \geq k$ and the lower bound follows easily. \square

The sum choice number of a balanced complete bipartite graph divided by $2n$ is therefore within a constant multiple of the list chromatic number of the same graph. Theorem 19 shows that this constant is at most 4. It is obvious from the proof that this is not best possible, but even this rough estimate suffices to demonstrate that χ_{sc} normalized by the number of vertices does not differ significantly from χ_ℓ for balanced complete bipartite graphs.

As we will see in the next sections, the situation is radically different for unbalanced complete bipartite graphs. If q is sufficiently large with respect to a , then $\chi_\ell(K_{a,q}) > a$. To see that, assign disjoint a -element sets to the part of size a . The Cartesian product of these sets has a^a members. Let these be the lists assigned to the q -element part. No matter what labels we choose for the a -element part, there will always be a vertex in the larger part that we cannot color. On the other hand, Theorem 21 in Section 2.4 shows that for $q \rightarrow \infty$ and a fixed, $\chi_{\text{sc}}(K_{a,q})/|V(K_{a,q})| \sim 2$.

2.4 Unlike χ_ℓ , χ_{sc} does not grow with average degree

As we already mentioned, Erdős, Rubin and Taylor showed in [26] that

$$\chi_\ell(K_{q,q}) = \Theta(\log q). \quad (2.9)$$

If one of the parts is substantially smaller than the other one, then allowing different list sizes again results in smaller average lists. Theorem 21 is one of the main theorems that we prove in this chapter.

Theorem 21. *There exist constants c_1 and c_2 such that for all $a \geq 2$ and $q \geq 4a^2 \log a$,*

$$2q + c_1 a \sqrt{q \log a} \leq \chi_{\text{sc}}(K_{a,q}) \leq 2q + c_2 a \sqrt{q \log a}.$$

We have seen that in some cases, the list chromatic number behaves very similarly to chromatic number (Brooks' Theorem), or is at least conjectured to (list coloring conjecture). In some other contexts, however, the two parameters behave very differently. An example of this is the behavior of the two parameters with respect to the average degree. It is easy to construct a sequence of graphs G_i for which the average degree $\deg_{\text{ave}}(G_i)$ approaches infinity, but $\chi(G_i)$ is bounded by a constant – just take the complete bipartite graphs $K_{n,n}$. On the other hand, Alon has shown in [5] that χ_ℓ depends heavily on the average degree.

Theorem 22 ([5]). *If G has average degree d , then $\chi_\ell(G) \geq (\frac{1}{2} - o(1)) \log_2 d$.*

When we relax the definition of choosability to allow different list sizes, a natural question that immediately arises in the light of the above is, how does this relaxation affect the behavior of the parameter with respect to the average degree? Does an analogue of Alon's theorem still hold? Or can we construct a sequence of graphs with average degree growing and χ_{sc} bounded by a constant?

One of the most interesting consequences of our Theorem 21 is that it answers this question. If different list sizes are allowed, the conclusion of Theorem 22 is no longer true. The sum choice number depends on the structure of the graph much more than the list chromatic number, whose value can be estimated solely based on the value of a global parameter such as average degree. In this respect, sum choice number resembles chromatic number more than list chromatic number.

With q tending to infinity, the average degree of $K_{a,q}$ approaches $2a$. We obtain

$$\lim_{a \rightarrow \infty, q \gg a^2 \log a} \frac{|E(K_{a,q})|}{a+q} = \infty, \quad \lim_{a \rightarrow \infty, q \gg a^2 \log a} \frac{\chi_{\text{sc}}(K_{a,q})}{a+q} = 2. \quad (2.10)$$

Using Theorem 21, we can therefore provide an infinite sequence of graphs with average degree approaching infinity and sum choice number bounded by a constant, in complete analogy with the situation concerning chromatic number.

For $a = 2$ and $a = 3$, the value of $K_{a,q}$ was determined by Berliner, Bostelmann, Brualdi, and Deaett in [10], and by Heinold in [37].

Theorem 23 ([10]). $\chi_{\text{sc}}(K_{2,q}) = 2q + 1 + \lfloor \sqrt{4q + 1} \rfloor$.

Theorem 24 ([37]). $\chi_{\text{sc}}(K_{3,q}) = 2q + 1 + \lfloor \sqrt{12q + 4} \rfloor$.

Our main result (Theorem 21) extends Theorems 23 and 24 to $K_{a,q}$ with arbitrary a . Let us remark here that we need the full generality of Theorem 21 to draw the conclusions about the behavior of $\chi_{\text{sc}}(G)$ with respect to average degree; Theorems 23 and 24 do not suffice.

* * *

Throughout this chapter, the two parts of the complete bipartite graph $K_{a,q}$ will be called A and Q , with $|A| = a$ and $|Q| = q$.

2.5 Upper bound, there are sufficient short lists

A variation of the proof of Theorem 5 yields a good upper bound on $\chi_{\text{sc}}(K_{a,q})$.

Theorem 25. *For every constant $c > 1$, there is an a_0 such that if $q > a > a_0$, then*

$$\chi_{\text{sc}}(K_{a,q}) \leq 2q + ca\sqrt{q} \ln a.$$

Proof. We present a function f with $\sum_{v \in A \cup Q} f(v) \geq 2q + ca\sqrt{q} \ln a$ such that every f -assignment is sufficient.

Define f as

$$f(v) = \begin{cases} r & \text{for } v \in A; \\ 2 & \text{for } v \in Q \end{cases}$$

where r will be defined later.

Fix a list assignment L with $L(v) = f(v)$ for all v . Construct a random subset T of $\cup_{v \in A \cup Q} L(v)$ by including each color independently with probability p . We will attempt to color the vertices of A with the colors in T and the vertices of Q with the remaining colors. Let C be the event that this is not possible. We have

$$\mathbb{P}(C) \leq a(1-p)^r + qp^2. \quad (2.11)$$

If the quantity on the right side of (2.11) is less than 1, then in particular $p < 1/\sqrt{q}$. Let $\alpha(q)$ be a sequence with $\alpha(q) < 1$ for all q , and set

$$p = \frac{\alpha(q)}{\sqrt{q}}.$$

Using that $1 - x \leq \exp(-x)$ for all x , we get

$$\mathbb{P}(C) \leq a \cdot \exp\left(-\frac{r\alpha(q)}{\sqrt{q}}\right) + (\alpha(q))^2.$$

For this to be less than 1, we need

$$r > \sqrt{q} \left(\frac{\ln a}{\alpha(q)} - \frac{\ln(1 - \alpha(q)^2)}{\alpha(q)} \right). \quad (2.12)$$

For a large enough, we can find $\alpha(q)$ (sufficiently close to 1) so that the right side of (2.12) is bounded from above by $c\sqrt{q} \log a$. Therefore, if a is large enough and $r > c\sqrt{q} \log a$, then $\mathbb{P}(C) < 1$ and there exists a choice of T such that we can color A with colors in T and Q with the remaining colors. \square

However, it is clear that to improve this bound further, additional ideas are needed. Using a more complicated argument, we can improve the $\ln a$ in the second term to $\sqrt{\ln a}$.

Theorem 26. *If $a, q \in \mathbb{N}$ with $q > a > 3$, then*

$$\chi_{\text{sc}}(K_{a,q}) \leq 2q + a \lceil \sqrt{32q(1 + \ln a)} \rceil.$$

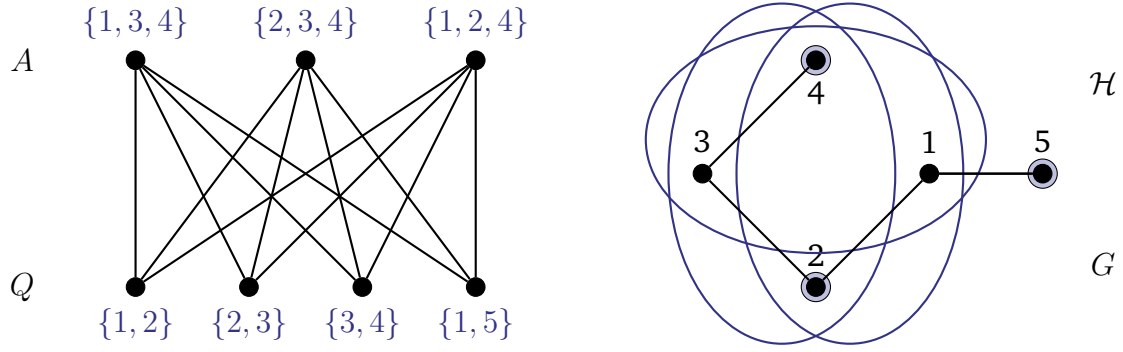


Figure 2.2: An example of a bipartite graph with a list assignment (left) and the corresponding graph G and hypergraph \mathcal{H} (right). The circled vertices in the right part of the picture show a possible choice of the set T of colors used for coloring part A of the bipartite graph.

Proof. We again present a sufficient function f

$$f(v) = \begin{cases} r & \text{for } v \in A; \\ 2 & \text{for } v \in Q, \end{cases}$$

where r will be defined later in (2.15), such that $\sum_{v \in A \cup Q} f(v) \geq 2q + a\sqrt{32q(1 + \ln a)}$.

Let L be an arbitrary f -assignment, i.e., $|L(v)| = f(v)$ for all v . Let S be the union of all the lists, $S = \bigcup_{v \in A \cup Q} L(v)$. The list assignment L yields a (multi)hypergraph and a multigraph with the same vertex set S and with edge sets $\mathcal{L}_A = \{L(u) : u \in A\}$ and $\mathcal{L}_Q = \{L(v) : v \in Q\}$, respectively. The sufficiency of L means that one can find a set $T \subset S$ intersecting all hyperedges of \mathcal{L}_A such that $S \setminus T$ intersects all edges of \mathcal{L}_Q , so T is an independent set in the graph \mathcal{L}_Q . (See Figure 2.2 for an example.) We can color each vertex u with a color $c(u)$ such that

$$c(u) \in L(u) \cap T \text{ for } u \in A \quad \text{and} \quad c(v) \in L(v) \cap (S \setminus T) \text{ for } v \in Q.$$

We are going to construct such T by a two-step random process.

Let us pick, randomly and independently, each element of S with probability p . Let B be the random set of all elements picked. Define a random variable X_u for each $u \in A$ by $X_u = |L(u) \cap B|$, and the random variable Y by

$$Y = |\{v \in Q : L(v) \subseteq B\}|,$$

so Y is the number of edges of \mathcal{L}_Q spanned by B . If we remove an element $\ell(v) \in L(v)$ for each edge of \mathcal{L}_Q spanned by B , the remaining set $T \subset B$ is certainly independent in \mathcal{L}_Q . If $Y < X_u$ for each $u \in A$, then T intersects all $L(u) \in \mathcal{L}_A$ and we are done.

The expected value of Y is p^2q , so the Markov inequality gives

$$\mathbb{P}(Y < 2p^2q) > \frac{1}{2}. \quad (2.13)$$

The expected value of X_u is pr , so the Chernoff inequality gives

$$\mathbb{P}(X_u < \mathbb{E}X_u - t) < e^{-t^2/2rp},$$

for any $t > 0$. Hence

$$\mathbb{P}(X_u \geq pr - t \text{ for all } u \in A) \geq 1 - ae^{-t^2/2rp}, \quad (2.14)$$

and this again exceeds $1/2$ for $t^2 > 2rp \ln(2a)$. The sum of the probabilities in (2.13) and (2.14) is larger than 1, so there is an appropriate choice of B (and then T) if $t^2 = 2rp(1 + \ln a)$ and $pr - t > 2p^2q$. This is true for

$$p = \sqrt{\frac{2(1 + \ln a)}{q}} \quad \text{and} \quad r \geq 4pq = \sqrt{32(1 + \ln a)q}. \quad (2.15)$$

□

2.6 Lower bound, much shorter lists are not sufficient

To prove that $\chi_{\text{sc}}(G) \geq k$ for a particular k , we need to show that for every f with $\sum_{v \in G} f(v) = k$, there exists an insufficient f -assignment.

We will first show how to construct such a list assignment for a very special f . Then we will consider a general f and show that we can apply this construction to a subgraph of $K_{a,q}$, provided that $\sum f(v)$ is sufficiently small.

Lemma 27. *Let $t \geq 2$ and $l \geq 1$. For $a = 2^t$ and $q = t\ell^2$, there exists a list assignment L*

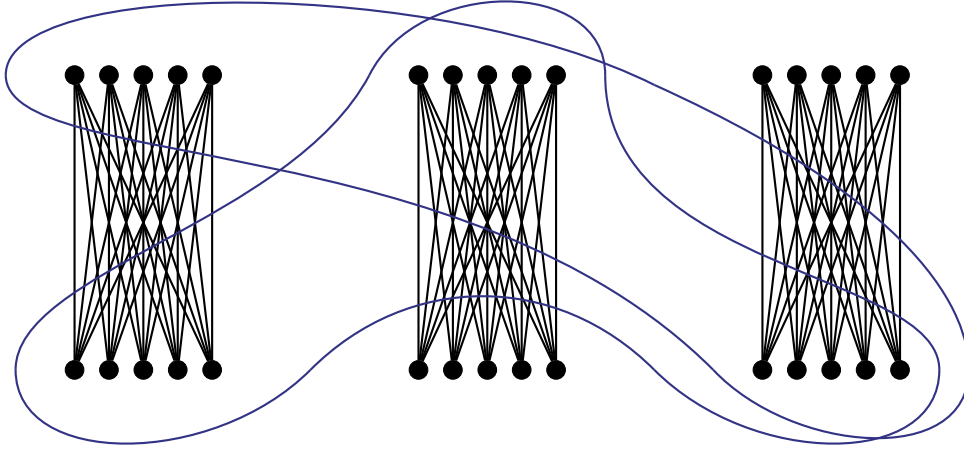


Figure 2.3: An example of a hypergraph \mathcal{H} and a graph G that correspond to an insufficient list assignment for $K_{a,q}$ with $a = 2^3$ and $q = 3 \cdot 5^2$. This list assignment uses $6 \cdot 5$ colors. Only two hyperedges out of 8 are shown in the picture.

with

$$L(v) = \begin{cases} 2 & \text{for } v \in Q; \\ t\ell & \text{for } v \in A. \end{cases}$$

such that $K_{a,q}$ is not L -colorable.

Proof. Take the complete t -partite t -uniform hypergraph with parts of size 2, i.e., the family of all sets $A \subseteq \{u_1, v_1, u_2, v_2, \dots, u_t, v_t\}$ that contain exactly one of $\{u_i, v_i\}$ for all i . Replace the vertices u_i, v_i with disjoint vertex sets U_i, V_i of size ℓ . Let \mathcal{H} be the resulting $t\ell$ -uniform hypergraph. That is, $V(\mathcal{H}) = \bigcup_{i=1}^t (U_i \cup V_i)$ and $E(\mathcal{H}) = \{(\bigcup_{i \in I} U_i) \cup (\bigcup_{j \notin I} V_j) : I \subseteq \{1, \dots, t\}\}$. The covering number of this hypergraph is $\tau(\mathcal{H}) = t + 1$.

Define a graph G on the vertex set $\bigcup_{i=1}^t (U_i \cup V_i)$ by setting $E(G) = \bigcup_{i=1}^t (U_i \times V_i)$. Every minimum cover of \mathcal{H} contains at most one vertex from each U_i and each V_i . It has $t + 1$ vertices, so by the pigeonhole principle, it contains both endpoints of some edge in G . See Figure 2.3 for an example of \mathcal{H} and G .

Define the lists of the vertices $v \in A$ to be the sets in $E(\mathcal{H})$ and the two element sets in $E(G)$ to be the lists of the vertices $v \in Q$. The number of edges of G is $t \cdot \ell^2 = q$, so this can be done. We have shown in the previous paragraph that this list assignment is not sufficient. \square

Note that with this choice of a and q , we have $|L(v)| \geq \sqrt{q \log_2 a}$ for $v \in A$.

Theorem 28. *If $a \geq 2$ and $q > 4a^2 \ln a$, then*

$$\chi_{\text{sc}}(K_{a,q}) \geq 2q + 0.061a\sqrt{q \ln a}.$$

Proof. Let s be such that $\sum_{v \in A \cup Q} f(v) = 2q + as$. We will prove that, as long as $s \leq 0.061\sqrt{q \ln a}$, the function f is not sufficient. Suppose to the contrary that every f -assignment is sufficient.

Let q_1 , q_2 , and q_3 be the numbers of vertices $v \in Q$ with $f(v) = 1$, $f(v) = 2$, and $f(v) \geq 3$ respectively. Since $f(u) \geq q_1$ for each $u \in A$ (otherwise f is not sufficient),

$$2q + as = \sum_{v \in A \cup Q} f(v) \geq aq_1 + (q_1 + 2q_2 + 3q_3) \geq 2q + q_1 + q_3.$$

It follows that $q_1 + q_3 \leq as$ and Q has at least $q - as$ vertices with lists of size 2.

Let $q^* = q_2$ and let a^* be a power of 2 as large as possible such that $a^* \leq \frac{a}{2}$.

If there are at least a^* vertices $u \in A$ with $f(u) \leq \sqrt{q^* \log_2 a^*}$, we can use Lemma 27 together with both parts of Lemma 18 to construct an insufficient list assignment.

If this is not true, then A has more than $\frac{a}{2}$ vertices with lists of sizes greater than $\sqrt{(q - as) \log_2 a^*}$. If q satisfies the hypothesis of the theorem and $s \leq 0.061\sqrt{q \ln a}$, then

$$\sum_{v \in A \cup Q} f(v) \geq \frac{1}{2}a \cdot \sqrt{(q - as) \log_2 a^*} + 2(q - as) > 2q + as. \quad (2.16)$$

To get the last inequality for $a \geq 5$, we observe that $a^* \geq \frac{a}{4}$. The cases $a = 2$ and $a = 3$ follow from Theorems 23 and 24, and the case $a = 4$ is not difficult to handle separately.

Equation (2.16) contradicts the assumptions. \square

The proof of Theorem 28 is fairly intuitive and it provides a lower bound which differs from our upper bound only in the value of the multiplicative constant appearing in the second term. The value 0.061 works for all $a \geq 2$, even if q is relatively not too large. But even for q very large with respect to a , and a itself approaching infinity, the constant that we obtain in the above proof does not exceed $1/2$. This value can be improved significantly by the use of a randomized construction in place

of Theorem 27. This proof is interesting in its own right and substantially different from the proof of Theorem 28, so we will present it here as well, albeit with less detail.

First let us make two technical observations. They can be proved easily by algebraic manipulation; the proof is omitted here. We follow the convention that $\binom{n}{k}$ is defined to equal 0 for $n < k$.

If $2 + c \leq d \leq 2t$, then

$$\frac{\binom{t}{c+1}}{\binom{2t}{c+1}} + \frac{\binom{t}{d-1}}{\binom{2t}{d-1}} < \frac{\binom{t}{c}}{\binom{2t}{c}} + \frac{\binom{t}{d}}{\binom{2t}{d}}. \quad (2.17)$$

Also, if $c < d \leq t$, then

$$\frac{\binom{t}{c}}{\binom{2t}{c}} > \frac{\binom{t}{d}}{\binom{2t}{d}}. \quad (2.18)$$

Theorem 29. *For every sufficiently large a there exists a constant $q_0(a)$ such that if $q > q_0(a)$, then $\chi_{\text{sc}}(K_{a,q}) \geq 2q + 0.849a\sqrt{q \ln a}$.*

Proof. Let $f(v)$ be the given list sizes for $v \in A \cup Q$ and suppose first that $f(v) = 2$ for all $v \in Q$. We seek an insufficient list assignment L with the appropriate list sizes. We will use $2t$ colors, where t will be determined later. For the sake of readability, we henceforth omit most floors and ceilings. If a and q are very large, taking floors or ceilings changes the quantities very little in all of these cases. Omitting them will greatly simplify our computations, while not making any difference in the results.

First, let us remove the influence of q by setting $s_i = f(v_i) \cdot \sqrt{t/q}$. Using random choice, we construct a family $\mathcal{A} = \{A_i : i = 1, \dots, a\}$ of subsets of the vertex set $\{x_1, y_1, x_2, y_2, \dots, x_t, y_t\}$ with $|A_i| = s_i$, such that $\tau(\mathcal{A}) > t$ holds for the covering number of \mathcal{A} . Replace the vertices x_i, y_i by disjoint sets X_i, Y_i of size $\sqrt{q/t}$. The resulting sets of size $s_i \cdot \sqrt{q/t} = f(v_i)$ will be the color sets assigned to the vertices of A . Place complete bipartite graphs between X_i and Y_i for all i . Altogether, these have $t \cdot (\sqrt{q/t})^2 = q$ edges. Use the two-element sets corresponding to these edges as the lists for the vertices of Q . Since the covering number of the hypergraph is more than t , this list assignment is not sufficient.

Let $s = \frac{1}{a} \sum_{v_i \in A} s_i$. We need to prove that if s is small enough, then there exists an integer t such that the family \mathcal{A} with properties described above exists. Let $B \subseteq A$

be the set of vertices with $s_i \leq 2t$, and let $b = |B|$. For every $v_i \in B$, let A_i be a set chosen randomly, uniformly from $\binom{[2t]}{s_i}$ (the collection of subsets of $[2t]$ of size s_i). Let \mathcal{A} be the resulting family. Assign lists arbitrarily to the vertices of $A \setminus B$; these vertices do not play any role in our construction.

$$\begin{aligned} \mathbb{P}(\tau(\mathcal{A}) \leq t) &\leq \sum_{T \subseteq [2t], |T|=t} \mathbb{P}(\text{for all } i, A_i \cap T \neq \emptyset) \\ &= \binom{2t}{t} \cdot \prod_{v_i \in B} \left(1 - \frac{\binom{t}{s_i}}{\binom{2t}{s_i}}\right) \\ &\leq \binom{2t}{t} \cdot \exp\left(-\sum_{v_i \in B} \frac{\binom{t}{s_i}}{\binom{2t}{s_i}}\right). \end{aligned} \quad (2.19)$$

In the last line, we used the fact that $(1 - x) \leq \exp(-x)$ for all x . If some s_i are not equal to $\lfloor (\sum_{v_i \in B} s_i)/b \rfloor$ or $\lceil (\sum_{v_i \in B} s_i)/b \rceil$, then find s_i and s_j such that $s_j \geq s_i + 2$. Replace s_i with $s_i + 1$ and s_j with $s_j - 1$. By (2.17), the sum in (2.19) decreases. Repeat until $s_i \in \{\lfloor (\sum_{v_i \in B} s_i)/b \rfloor, \lceil (\sum_{v_i \in B} s_i)/b \rceil\}$ for all i .

For the next step, we want t to satisfy $t \geq s$. For this, it is sufficient for t to satisfy

$$t \geq \frac{1}{q} \left(\frac{1}{a} \sum_{v_i \in A} f(v_i) \right)^2. \quad (2.20)$$

Note that also $s \geq \frac{1}{b} \sum_{v_i \in B} f(v_i)$. We can use (2.18) with $c = \lfloor (\sum_{v_i \in B} s_i)/b \rfloor$ or $c = \lceil (\sum_{v_i \in B} s_i)/b \rceil$ and $d = t$ to get

$$\sum_{v_i \in B} \frac{\binom{t}{s_i}}{\binom{2t}{s_i}} \geq \frac{b}{\binom{2t}{t}}.$$

We have

$$\mathbb{P}(\tau(\mathcal{A}) \leq t) \leq \exp\left(t \cdot 2 \ln 2 - \frac{b}{\binom{2t}{t}}\right). \quad (2.21)$$

This is less than 1 if $t \cdot 2 \ln 2 \lesssim \ln b - \ln \ln b$. It is easy to observe that $b \geq a/2$. Hence, in order for (2.21) to be less than 1, it suffices to have

$$t \leq \frac{\ln a}{2 \ln 2} - o(\ln a). \quad (2.22)$$

Combining (2.20) with (2.22), if a is large enough, we get

$$\sum_{v_i \in A} f(v_i) \leq \sqrt{\frac{q \ln a}{2 \ln 2}} < 0.849 \sqrt{q \ln a}.$$

Whenever this is true (and a is large enough), there is a t such that the random choice described above results in an insufficient list assignment, with nonzero probability. With q vertices in Q that all have list sizes equal to 2, there exists the desired list assignment with list sizes $f(v_i)$ for $v_i \in A$, as long as $\sum_{v_i \in A} f(v_i) \leq 0.849 \sqrt{q \ln a}$.

Now let us consider a general list size function f , where we do not necessarily have $f(v) = 2$ for all $v \in Q$. The above proof shows that, as long as there are at least q_2 vertices in Q with list sizes equal to 2, then whenever the average list size on A is at most $0.849 \sqrt{q_2 \ln a}$, we can find an insufficient list assignment.

For an arbitrary list assignment f , we have proved in the proof of Theorem 28 that the number q_2 of vertices $v \in Q$ with $f(v) = 2$ is at least $q - o(q)$. A short computation suffices to show that, by making a and q large enough, we can get arbitrarily close to the constant 0.849 in this general case as well. \square

Let us remark that if we choose the constants in the proof of Theorem 26 more carefully, we can improve the constant $\sqrt{32}$ in the upper bound to approximately 3.67.

2.7 For fixed a , the limit as $q \rightarrow \infty$ exists

We have proved that $(\chi_{\text{sc}}(K_{a,q}) - 2q)/\sqrt{q}$ is bounded from above and below by constants. Now we show that in fact the limit exists, as q tends to ∞ .

Theorem 30. *For fixed a , the limit $\lim_{q \rightarrow \infty} \frac{\chi_{\text{sc}}(K_{a,q}) - 2q}{\sqrt{q}}$ exists.*

First we consider a simpler problem and define $\chi_{\text{sc}2}(K_{a,q})$ to be the least k for which there exists a sufficient f with $\sum_{A \cup Q} f(v) = k$ and with $f(v) = 2$ for all $v \in Q$. Obviously $\chi_{\text{sc}2}(K_{a,q}) \geq \chi_{\text{sc}}(K_{a,q})$. We will identify functions f with $f(v) = 2$ for $v \in Q$ with vectors $\mathbf{f} = (f_1, \dots, f_a)$.

Theorem 31. *For fixed a , the limit $\lim_{q \rightarrow \infty} \frac{\chi_{\text{sc}2}(K_{a,q}) - 2q}{\sqrt{q}}$ exists.*

If f is not sufficient and $f(v) = 2$ for all $v \in Q$, then there exists an insufficient f -assignment L such that $K_{a,q}$ is not L -colorable. We will identify such L with the pair (\mathcal{L}, G) , where \mathcal{L} is a multihypergraph \mathcal{L} with edges L_i satisfying $|L_i| = f_i$ for $1 \leq i \leq a$, and a G is a multigraph on $V(\mathcal{L})$ with at most q edges, such that no transversal of \mathcal{L} is an independent set in G .

For $I \subseteq [a]$, define $X_I = \cap_{i \in I} L_i$. An insufficient list assignment is *symmetric* if, for all pairs $I \neq J$, the bipartite subgraph of G induced by X_I and X_J is either empty or complete, and for each I , X_I induces the empty graph. Using the following lemma, we can without loss of generality assume that an insufficient list assignment is symmetric.

Lemma 32. *Fix a, f and q . A function f is insufficient if and only if there exists a symmetric insufficient list assignment.*

Proof. Suppose that L is an insufficient list assignment, and let \mathcal{L} and G be the corresponding hypergraph and graph. If u and v belong to the same X_I , then no minimal transversal of \mathcal{L} contains both of them. We can therefore delete all edges induced by X_I .

Now suppose that $u, v \in X_I$ and $|N(u)| \leq |N(v)|$. Replace the neighborhood of v by the neighborhood of u . It is still true that every transversal of \mathcal{L} induces an edge of G . Repeated application of this procedure eventually produces a symmetric insufficient f -assignment L such that $K_{a,q}$ is not L -colorable. \square

Proof of Theorem 31. Consider a symmetric f -assignment L . Let \mathcal{L}, G and X_I (for $I \subseteq [a]$) be as before. Let $x_I = |X_I|$ and let $\mathbf{x} = (x_\emptyset, x_{\{1\}}, \dots)$ be the vector of the x_I 's, ordered in some way. Replace each X_I with a vertex v_I . Let R be the *reduced graph* of the symmetric list assignment, i.e., the graph with $V(R) = \{v_I : x_I \neq 0\}$ and whose edges correspond to the complete bipartite subgraphs of G . Similarly, the hypergraph \mathcal{L} turns into the *reduced hypergraph* on the same vertex set, $V(R)$. Note that every cover of the reduced hypergraph induces an edge of R . We will call such graphs R *persistent*. The vector \mathbf{x} satisfies $\sum_{I, J \in E(R)} x_I x_J \leq q$, and moreover $x_I = 0$ whenever $v_I \notin V(R)$. The set A_R^q of all such \mathbf{x} is the body bounded by a quadric surface which depends on R and q .

Define the linear map $\varphi : \mathbb{R}^{2^{[a]}} \rightarrow \mathbb{R}^a$ by $\varphi(\mathbf{x}) = (f_1, \dots, f_a) = \mathbf{f}$ where $f_i = \sum_{i \in I} x_I$. The function f is insufficient for this q if and only if \mathbf{f} is the image of some integer point \mathbf{x} that is in A_R^q for some persistent R .

If there exists an insufficient f -assignment L for every integer vector \mathbf{f} such that $\sum f_i = k$, then we have $\chi_{\text{sc2}}(K_{a,q}) - 2q > k$. We are therefore looking for the maximum k such that every integer point on the hyperplane $\sum f_i = k$ is the image (under φ) of some integer point in $\bigcup A_R^q$, where the union is taken over all persistent R .

Let us normalize everything by \sqrt{q} . For every persistent R , define

$$A_R = \{\mathbf{x} : \sum_{IJ \in E(R)} x_I x_J \leq 1, \text{ and } x_I = 0 \text{ for } v_I \notin V(R)\} \quad \text{and} \quad B_R = \varphi(A_R).$$

For every R we now have only one quadric surface, independent of q . We say that a vector \mathbf{v} is a q -grid point if $\sqrt{q} \cdot \mathbf{v}$ is an integer point.

For every q , define k_q to be the maximum k such that $\sqrt{q} \cdot k \in \mathbb{N}$ and every q -grid point on the hyperplane $\sum f_i = k$ is the image of some q -grid point in $\bigcup A_R$. If this is true, then the same is true for q -grid points with $\sum f_i \leq k$, as well, by the second part of Lemma 18. Hence k_q is equal to the maximum k with $\sqrt{q} \cdot k \in \mathbb{N}$ such that every q -grid point f with $\sum f_i \leq k$ is the image of some q -grid point in $\bigcup A_R$.

Also, define b to be the maximum k such that the simplex $C_k = \{\mathbf{f} : \sum f_i \leq k\}$ is a subset of $\bigcup B_R$.

We want to prove that the limit $\lim k_q$ exists and equals b . That is, we want to prove that for every ε , if q is large enough,

- every q -grid point in $C_{b-\varepsilon}$ is the image under φ of some q -grid point in $\bigcup A_R$ (then $b \leq \liminf k_q$), and
- there is a q -grid point in $C_{b+\varepsilon}$ which is not the image of any q -grid point in $\bigcup A_R$ (then $b \geq \limsup k_q$).

To prove the first claim, fix q and let \mathbf{f} be a point on the hyperplane $\sum f_i = b$. The point \mathbf{f} is in $\bigcup B_R$, so it is the image of some $\mathbf{x} \in \bigcup A_R$. Each set A_R is a *downset* in the sense that with every \mathbf{x} it also contains all points \mathbf{z} such that $z_i \leq x_i$ for all i . It follows that \mathbf{y} defined as $\frac{\lfloor \sqrt{q} \cdot \mathbf{x} \rfloor}{\sqrt{q}}$ is a q -grid point in $\bigcup A_R$. Each entry of \mathbf{y} differs by at most $\frac{1}{\sqrt{q}}$ from the corresponding entry of \mathbf{x} , and a simple computation suffices to show that the distance of $\varphi(\mathbf{y})$ and \mathbf{f} is at most $\frac{c}{\sqrt{q}}$, where $c \leq \sqrt{a} \cdot 2^{a-1}$ is a constant dependent only on a . That is, for each point \mathbf{f} on the hyperplane $\sum f_i = b$ we have found, in distance at most ε_q , where $\varepsilon_q = \frac{c}{\sqrt{q}}$, an image of a q -grid point from $\bigcup A_R$. Call this point \mathbf{f}' .

Note that, by definition of χ_{sc} (and χ_{sc2}), whenever a q -grid point \mathbf{f} is the image of a q -grid point in $\bigcup A_R$, the same is true for all q -grid points in the box $D_{\mathbf{f}}$, defined as $\{\mathbf{g} : g_i \leq f_i \text{ for all } i\}$. This is the second part of Lemma 18.

Let \mathbf{h} be a q -grid point such that $\sum h_i \leq b - a\varepsilon_q$. Let \mathbf{f} be its perpendicular projection on the hyperplane $\sum f_i = b$ and find the corresponding \mathbf{f}' . Since the distance of \mathbf{f} and \mathbf{f}' is at most ε_q , the point \mathbf{h} belongs to $D_{\mathbf{f}'}$, and hence it is the image of a q -grid point in $\bigcup A_R$. Choosing q large enough so that $a\varepsilon_q \leq \varepsilon$ for our given ε concludes the proof.

To prove the second claim, let \mathbf{f} be a point outside $\bigcup B_R$, but within the distance ε from C_b . Take a bounded cube $Q \subseteq \mathbb{R}^a$ that contains \mathbf{f} . Now take a bounded cube S in \mathbb{R}^{2a} which contains all points \mathbf{x} such that $\varphi(\mathbf{x}) \in Q$. The set T defined as $T = S \cap (\bigcup A_R)$ is compact, so φ maps it to a compact set. The complement of $\varphi(T)$ in $\varphi(S)$ is open, and contains \mathbf{f} . Note that $(\bigcup B_R) \cap Q \subseteq \varphi(T)$.

Therefore, for some small δ , the δ -ball around \mathbf{f} is outside $\bigcup B_R$. If q is large enough, the ball contains some q -grid point. This point not only has no q -grid preimages in $\bigcup A_R$, it has no preimages in $\bigcup A_R$ whatsoever, and the claim is proven. \square

Let $Q = Q_1 \cup Q_2$, with q_1 denoting the cardinality of Q_1 . Let f be an integer-valued function defined on $A \cup Q$, such that $f(v) > q_1$ for $v \in A$, $f(v) \geq 2$ for $v \in Q_2$, and $f(v) = 1$ for $v \in Q_1$.

Given this f , define a function \hat{f} on $A \cup Q_2$, by

$$f(v) = \begin{cases} f(v) - q_1 & \text{for } v \in A; \\ f(v) & \text{for } v \in Q_2. \end{cases}$$

Lemma 33. *The function f is sufficient for $K_{a,q}$ if and only if \hat{f} is sufficient for $K_{a,q-q_1}$.*

Proof. Consider an \hat{f} -assignment L such that K_{a,q_2} is not L -colorable. Let $1, \dots, q_1$ be labels that do not appear in any of the sets of L . Assign these labels as lists to the vertices of Q_1 , and add all of them to each list $L(v)$ for $v \in A$. The result is an f -assignment L' such that $K_{a,q}$ is not L' -colorable.

Now consider an f -assignment L' such that $K_{a,q}$ is not L' -colorable. Delete the labels of the vertices in Q_1 from all the lists $L'(v)$ for $v \in A$. Delete additional colors from the lists of the vertices $v \in A$, until each has list of size $\hat{f}(v)$. The restriction

of the resulting list assignment to $A \cup Q_2$ is an \hat{f} -assignment such that K_{a,q_2} is not L -colorable. \square

Proof of Theorem 30. Define sequences $\{a_q\}_{q=1}^\infty$ and $\{b_q\}_{q=1}^\infty$ by

$$a_q = \frac{\chi_{\text{sc}}(a, q) - 2q}{\sqrt{q}} \quad \text{and} \quad b_q = \frac{\chi_{\text{sc}2}(a, q) - 2q}{\sqrt{q}}.$$

It was already mentioned that $a_q \leq b_q$ for all q .

Let f be a sufficient function on $K_{a,q}$ with $\sum_{v \in A \cup Q} f(v) = \chi_{\text{sc}}(K_{a,q})$. Again, let q_1 be the number of vertices $v \in Q$ such that $f(v) = 1$. We have $f(v) > q_1$ for all $v \in A$. Let $d(q)$ be the number of vertices $v \in Q$ for which $f(v) \neq 2$. Delete these vertices. The restriction of \hat{f} to the resulting graph $K_{a,q-d(q)}$ is sufficient, by Lemma 33, so we have $\chi_{\text{sc}2}(K_{a,q-d(q)}) \leq 2(q - d(q)) + \sum_{v \in A} \hat{f}(v)$. We get

$$\begin{aligned} a_q \sqrt{q} &= \chi_{\text{sc}}(K_{a,q}) - 2q \geq \chi_{\text{sc}}(K_{a,q}) - q_1 - \sum_{v \in Q} f(v) \geq \chi_{\text{sc}}(K_{a,q}) - aq_1 - \sum_{v \in Q} f(v) = \\ &= \sum_{v \in A} f(v) - aq_1 \geq \chi_{\text{sc}2}(K_{a,q-d(q)}) - 2(q - d(q)) = b_{q-d(q)} \sqrt{q - d(q)}. \end{aligned}$$

We get the following relationship between a_q and b_q :

$$b_q \geq a_q \geq \frac{\sqrt{q - d(q)}}{\sqrt{q}} b_{q-d(q)}.$$

We have proved that $\sum_{v \in A \cup Q} f(v) \leq 2q + ca\sqrt{q \ln a}$ for some constant c . If any $f(v)$ decreases by 1, the function becomes insufficient. The argument in the proof of Theorem 28 shows that for all but at most $ca\sqrt{q \ln a}$ vertices in Q , we have $f(v) = 2$. In other words, $d(q) \leq O(\sqrt{q})$. The limit $\lim_{q \rightarrow \infty} b_q$ exists by Theorem 31. We have

$$\lim_{q \rightarrow \infty} \frac{\sqrt{q - d(q)}}{\sqrt{q}} b_{q-d(q)} = \lim_{q \rightarrow \infty} b_q,$$

which proves the claim. \square

2.8 Graphs with large independent sets

It is natural to consider adding edges to the set A of $K_{a,q}$. If we add all $\binom{a}{2}$ possible edges, all vertices of A have to receive distinct colors in any proper coloring of the resulting graph. How restrictive is really this additional requirement? Theorem 34 shows that it significantly alters the problem.

Let $G_{a,q}$ be the graph that we get from $K_{a,q}$ by inserting the edge uv for every pair of distinct $u, v \in A$.

Theorem 34. *There exist constants c_1 and c_2 , independent of q , such that*

$$2q + c_1\sqrt{q(a-1)} \leq \chi_{\text{sc}}(G_{a,q}) \leq 2q + c_2\sqrt{q(a-1)}$$

We first prove a generalization of Turán's Theorem.

For fixed s and t , let $t(s, k) = \min \sum_{i=1}^k \binom{x_i}{2}$, where the minimum is taken over all (x_1, \dots, x_k) such that $\sum x_i = s$.

Theorem 35. *Let $s, a \geq 2$ be integers. Let L_1, \dots, L_a be sets of size s and G a graph with less than $t(s, a-1)$ edges. There exists a system of distinct representatives $\{u_1, \dots, u_a\}$ of L_1, \dots, L_a , such that $\{u_1, \dots, u_a\}$ is an independent set in G .*

If $L_1 = \dots = L_a$, this is indeed equivalent to Turán's Theorem.

Let us also remark that the result is sharp: if G is allowed to have $t(s, a-1)$ edges, then taking $L_1 = \dots = L_a$ provides a family that violates the claim, with G being the disjoint union of $a-1$ cliques of almost equal sizes.

Proof. We want a set $\{u_1, \dots, u_a\}$ of distinct vertices, such that $u_k \in L_{i_k}$, where $\{i_1, \dots, i_a\}$ is some permutation of $\{1, \dots, a\}$.

Define the vertices u_1, \dots, u_a one by one. Let $V_1 = L_1 \cup \dots \cup L_a$ and $G_1 = G[V_1]$ (the restriction of G to V_1). Let u_1 be the vertex of minimum degree in G_1 , D_1 the closed neighborhood of u_1 in G_1 , and $d_1 = |D_1|$. Let L_{i_1} be one of the hyperedges containing u_1 .

Suppose that u_j, D_j , and L_{i_j} are already defined for $j = 1, \dots, k$. We define $V_{k+1} = (\bigcup_{i \notin \{i_1, \dots, i_k\}} L_i) \setminus (D_1 \cup \dots \cup D_k)$, $G_{k+1} = G[V_{k+1}]$, and let u_{k+1} be a vertex of minimum degree in G_{k+1} , D_{k+1} its closed neighborhood in G_{k+1} , $d_{k+1} = |D_{k+1}|$, and $L_{i_{k+1}}$ one of the hyperedges containing u_{k+1} , different from L_{i_1}, \dots, L_{i_k} .

If we cannot define u_{k+1} , then V_{k+1} is empty, and $|D_1| + \dots + |D_k| \geq |L_{k+1}| = s$. We have $k < a$ and D_1, \dots, D_k are non-empty, disjoint sets, so

$$2|E(G)| \geq \sum_{u \in D_1 \cup \dots \cup D_k} \deg(u) \geq d_1(d_1 - 1) + \dots + d_k(d_k - 1) \geq 2t(s, k) \geq 2t(s, a - 1).$$

□

Proof of Theorem 34. Again, we will prove the upper bound by presenting a sufficient function f . Let

$$f(v) = \begin{cases} 2 & \text{for } v \in Q; \\ s & \text{for } v \in A. \end{cases}$$

The result is the essence of Theorem 35 and some algebraic manipulation.

For the lower bound, we imitate the method of Section 2.6. Let $\{v_1, \dots, v_a\}$ be the vertices of A and let

$$f(v) = \begin{cases} 2 & \text{for } v \in Q; \\ s_i & \text{for } v_i. \end{cases}$$

Without loss of generality, suppose $s_1 \leq s_2 \leq \dots \leq s_a$. We claim that if f is sufficient, then $q < t(s_i, i - 1)$ for all $i \geq 2$, and $q < \binom{s_1}{2} \cdot (s_2 - s_1)$.

Suppose that $q \geq t(s_i, i - 1)$ for some $i \geq 2$. Let $L(v_1), \dots, L(v_i)$ be nested sets of the appropriate sizes. Let G be the graph on the vertex set $L(v_i)$, consisting of $i - 1$ disjoint cliques with sizes as equal as possible. Assign the pairs corresponding to the edges of G as the lists for the vertices of Q . This list assignment is not sufficient. Likewise, if $q \geq \binom{s_1}{2} \cdot (s_2 - s_1)$, let G be the graph that we get by taking a clique of order s_2 and deleting edges of a clique of order $s_2 - s_1$. Assign its edges as the lists of Q , and let $L(v_2) = V(G)$.

We use arguments analogous to the proof of Theorem 28 to conclude the proof for general lists. □

We did not try to achieve the best constants possible with this method, since the method does not give a tight result. We conjecture that the lower bound holds with equality, at least when the lists of the vertices of Q have size 2.

Conjecture 36. *Let*

$$f(v) = \begin{cases} 2 & \text{for } v \in Q; \\ s_i & \text{for } v_i. \end{cases}$$

If $q < t(s_i, i - 1)$ for all i and $q < \binom{s_1}{2} \cdot (s_2 - s_1)$, then $G_{a,q}$ is f -choosable.

Chapter 3

Reverse-free codes and permutations

3.1 Motivation and main concepts

Let k and n be natural numbers and X an n -element underlying set. The set of k -tuples with entries in X is denoted by X^k , its cardinality is n^k . The set of k -tuples with distinct entries is denoted by $X_{(k)}$, with cardinality $|X_{(k)}| = n(n-1)\dots(n-k+1) = k!\binom{n}{k}$. We have $X_{(k)} \subset X^k$. The set X is often identified with $[n]$. In this chapter we omit the parentheses when speaking about k -tuples, e.g., $x_1 \dots x_k$ is the shorthand for (x_1, \dots, x_k) .

A code \mathcal{C} is simply a subset of X^k , k is called its length, $|\mathcal{C}|$ is its size. Its elements are called the *codewords*, and X is sometimes called the *alphabet*. A typical problem in coding theory is to determine the maximum size of a code satisfying some local condition. For example, define the Hamming distance of two codewords to be the number of coordinates where they differ. The following problem is fundamental in the theory of error-correcting codes:

Given n, q and d , what is the maximum size of a code $\mathcal{C} \subseteq [q]^n$ such that every two codewords have Hamming distance at least d ?

If the codewords from such a code are sent over a noisy channel and less than $d/2$ entries are changed in each of them during the transmission, we can still recover the original codewords at the end, correcting the error.

In this chapter we consider a similar problem, with a different local condition.

Definition 37. Let a and b be two distinct integers. The pair $\{a, b\}$ is a *reversed pair* for a pair of sequences $\mathbf{x} = x_1 \dots x_k$ and $\mathbf{y} = y_1 \dots y_k$ if there are two coordinates $i, j \in [k]$ such that $(x_i, x_j) = (y_j, y_i) = (a, b)$. If \mathbf{x} and \mathbf{y} have no reversed pair, they are *reverse-free*.

A code \mathcal{F} is called (*pairwise*) *reverse-free* if any two of its members are reverse-free.

Let $\overline{F}(n, k)$ be the maximum cardinality of a reverse-free code $\mathcal{F} \subset [n]^k$, and $F(n, k)$ the maximum cardinality of a reverse-free code $\mathcal{F} \subset [n]_{(k)}$. That is, in the former we allow repetitions of symbols, while in the latter we do not.

A natural companion notion is that of a *flip-full* code, where every two codewords are required to have a reversed pair.

Given a family \mathcal{F} , we can define a $|\mathcal{F}| \times k$ matrix $M(\mathcal{F})$ in a natural way, by listing the the sequences in \mathcal{F} as its rows. The family \mathcal{F} is then reverse-free if $M(\mathcal{F})$ has distinct rows and does not contain a submatrix of type $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a \neq b$. Similarly, \mathcal{F} is full of flips if every two rows contain such submatrix.

Many coding theory problems can be formulated in this way, as extremal problems with forbidden submatrices, or with certain submatrices required for every pair (triple, quadruple...) of rows. For example, if \mathcal{A} is a family of sets of ordered triples, Körner in [44] defines $N(\mathcal{A}, k)$ as the maximum number of rows in a k -column matrix such that whenever $\mathbf{w}, \mathbf{x}, \mathbf{y}$ are three rows and $A \in \mathcal{A}$, there exists an index i such that the triple (w_i, x_i, y_i) belongs to A . He reformulates a number of well-known coding theory problems as special cases of this problem.

A substantial body of work has been also developed for the problem of maximizing the number of rows of a matrix with a forbidden submatrix, most notable being the results of Sauer [64], Shelah [65], and Vapnik and Chervonenkis [72]. For a survey of further results, see [7].

Similar problems have also been investigated in an information-theoretic setting. The notion of *robust capacity* was introduced by Körner and Simonyi ([48]) in an attempt to solve Renyi's problem of qualitatively independent partitions. Let G be an undirected graph. We say that two sequences $\mathbf{x}, \mathbf{y} \in V(G)^k$ are *robustly G -different* if they have a reversed pair $\{x_i, x_j\}$ such that $\{x_i, x_j\} \in E(G)$. The largest cardinality of a pairwise robustly G -different subset of $V(G)^k$ is denoted $R(G, k)$.

The authors of [12] derived bounds for $R(L, k)$ for the semi-infinite path L and used them to obtain results concerning graph-different permutations.

For $G = K_n$, $R(G, k)$ is equal to the maximum size $G(n, k)$ of a flip-full code. This problem is considered in section 3.7.

It is not hard to see that $R(G, k)$ is the clique number of a suitably defined graph. Within the framework of information theory, one is not interested in determining the

behavior of $R(G, k)$ precisely, but only in a somewhat rough estimate, captured in the notion of *robust capacity*, defined as

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log R(G, k).$$

By substituting the independence number for the clique number in this problem, we get a similar notion of *forbiddance*.

All problems considered in this chapter can be interpreted as the problem of determining the independence number (or the clique number) of suitably defined graphs whose vertices correspond to permutations of n or to k -tuples with entries in $[n]$ (see Section 3.8 for more detail). However, unlike the information-theoretic setting, we are interested in precise values of these parameters.

Similar problems, namely clique and independence numbers of other graphs defined on the set of permutations of $[n]$, have also been considered outside information theory. We say that two permutations are t -intersecting if there exist at least t indices i such that $\pi(i) = \rho(i)$. The *derangement graph* has permutations of $[n]$ as its vertices, and two of them are adjacent if and only if they are not 1-intersecting. It is easy to see that the independence number of the derangement graph is $(n-1)!$. For every $i, j \in [n]$, the set of permutations π such that $\pi(i) = i$ is a trivial example of a maximum independent set. It is nontrivial to prove that all maximum independent sets are of this form. This was done in [16] and independently in [51], thus proving a version of the Erdős-Ko-Rado theorem for permutations.

Similarly, for every pair of t -tuples (i_1, \dots, i_t) and (j_1, \dots, j_t) with distinct entries in $[n]$, the set of permutations π such that $\pi(i_m) = j_m$ for all $m = 1, \dots, t$ is a t -intersecting set of permutations of size $(n-t)!$. Extending the definition of derangement graph, we define a new graph by letting π and σ be adjacent if they are not t -intersecting. Using the terminology that we introduced for reverse-free families, we can characterize the cliques of this graph in terms of forbidden submatrices. Namely, a clique in this graph is a family \mathcal{F} of permutations such that $M(\mathcal{F})$ does not contain the submatrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_t \\ a_1 & a_2 & \dots & a_t \end{pmatrix}$$

for any choice of $a_1 \dots a_t \in [n]$. Deza and Frankl conjectured in [30] that if n is sufficiently large with respect to t , then $(n-t)!$ is the maximum size of a t -intersecting

set of permutations, and they proved a number of special cases. This well-known conjecture was very recently verified in its full generality in [23].

We will focus mostly on reverse-free codes in this chapter, but we will present some results about flip-full codes as well. It seems to be difficult to determine the exact value of $F(n, k)$ and $\overline{F}(n, k)$, so we concentrate on estimating their asymptotic behavior with k fixed and $n \rightarrow \infty$. We also solve the first non-trivial cases and determine the asymptotic behavior of $F(n, 3)$ and $\overline{F}(n, 3)$. Moreover we establish the order of magnitude of $\overline{F}(3, k)$.

3.2 Recurrences

The set of all increasing k -tuples from $[n]$ forms a set of pairwise reverse-free k -tuples with distinct entries. We obtain a lower bound for $F(n, k)$

$$\binom{n}{k} \leq F(n, k). \quad (3.1)$$

There are $k!$ permutations of k elements, so we have the upper bound

$$F(n, k) \leq k! \binom{n}{k}.$$

Easy as both of these bounds are, they nevertheless show that $F(n, k) = \Theta(n^k)$ if k is fixed and $n \rightarrow \infty$.

One can easily see that in (3.1) equality holds for $k = 1$ and $k = 2$

$$F(n, 1) = n, \quad F(n, 2) = \binom{n}{2}. \quad (3.2)$$

Let us define the asymptotic density

$$f(k) = \lim_{n \rightarrow \infty} \frac{F(n, k)}{k! \binom{n}{k}}. \quad (3.3)$$

We first show that the sequence $F(n, k)/(k! \binom{n}{k})$ is monotonically non-increasing in n and hence the quantity $f(k)$ is well-defined.

Lemma 38. Fix k . If $n_1 < n_2$, then

$$\frac{F(n_2, k)}{k! \binom{n_2}{k}} \leq \frac{F(n_1, k)}{k! \binom{n_1}{k}}. \quad (3.4)$$

Proof. Let \mathcal{F} be a pairwise reverse-free set of k -tuples with distinct entries from $[n_2]$ attaining the maximum cardinality $|\mathcal{F}| = F(n_2, k)$. The restriction of \mathcal{F} to $A \subset [n_2]$, defined as $\mathcal{F}[A] = \{\mathbf{x} \in \mathcal{F} : \text{all } x_i \in A\}$, is again a reverse-free family. Counting all pairs (\mathbf{x}, A) where A is an n_1 -subset of $[n_2]$ and \mathbf{x} is a k -tuple of elements of A that belongs to \mathcal{F} , we obtain

$$\begin{aligned} \text{the number of } (\mathbf{x}, A) \text{ pairs} &= \sum_{\mathbf{x} \in \mathcal{F}} |\{A : \mathbf{x} \in \mathcal{F}[A]\}| = F(n_2, k) \binom{n_2 - k}{n_1 - k} \\ &= \sum_{A \in \binom{[n_2]}{n_1}} |\{\mathbf{x} : \mathbf{x} \in \mathcal{F}[A]\}| = \sum |\mathcal{F}[A]| \leq \binom{n_2}{n_1} F(n_1, k). \end{aligned}$$

Noting that $\binom{n_2 - k}{n_1 - k} \binom{n_2}{k} = \binom{n_2}{n_1} \binom{n_1}{k}$, the conclusion follows. \square

Consider the matrix M of a reverse-free code $\mathcal{F} \subset [n]_{(k)}$ and restrict its columns to a subset $I \subset [k]$ of size $|I| = i$. After removing the repeated rows we obtain a smaller reverse-free code. Thus $M|I$ contains at most $F(n, i)$ distinct rows. Let \mathbf{y} be a row of $M|I$ and consider the family $\mathcal{F}' \subseteq \mathcal{F}$ of k -tuples \mathbf{x} such that the restriction of \mathbf{y} to I is equal to \mathbf{y} . The restrictions of these \mathbf{x} to $[k] \setminus I$ form a reverse-free family of length $k - i$, with alphabet $[n] \setminus \{y_1, \dots, y_i\}$. We obtain the following lemma.

Lemma 39. $F(n, k) \leq F(n, i)F(n - i, k - i)$. \square

Using Lemma 39 and the observations in (3.2), we have

$$\begin{aligned} F(n, k) &\leq F(n, 2)F(n - 2, k - 2) \leq F(n, 2)F(n - 2, 2) \dots F(n - 2i, k - 2i) \\ &\leq \left(\prod_{0 \leq i \leq \lfloor k/2 \rfloor - 1} \binom{n - 2i}{2} \right) F(n - 2\lfloor k/2 \rfloor, k - 2\lfloor k/2 \rfloor) = \frac{k! \binom{n}{k}}{2^{\lfloor \frac{k}{2} \rfloor}}. \quad (3.5) \end{aligned}$$

And finally, putting together the inequalities (3.1) and (3.5),

$$\frac{1}{k!} \leq f(k) \leq \frac{1}{2^{\lfloor \frac{k}{2} \rfloor}}. \quad (3.6)$$

If $k = 2$, these bounds are tight and we have $f(2) = 1/2$. We determine $f(3)$ in Sections 3.4 and 3.5.

We will list one more useful lower bound formula for $F(n, k)$. Take two reverse-free codes, $\mathcal{F}_1 \subset X_{(k_1)}$ and $\mathcal{F}_2 \subset Y_{(k_2)}$. If X and Y are disjoint, then we can make a product code \mathcal{F} with alphabet $X \cup Y$ by concatenating every $\mathbf{x} \in \mathcal{F}_1$ with every $\mathbf{y} \in \mathcal{F}_2$. Then \mathcal{F} is reverse-free, and we obtain

$$F(n_1, k_1)F(n_2, k_2) \leq F(n_1 + n_2, k_1 + k_2). \quad (3.7)$$

Let us turn our attention briefly to codes with repetitions allowed. All nondecreasing sequences of k elements from $[n]$ form a reverse-free set in this case as well. We have

$$\binom{n+k-1}{k} \leq \overline{F}(n, k), \quad \overline{F}(n, 1) = n \quad \text{and} \quad \overline{F}(n, 2) = \binom{n+1}{2}. \quad (3.8)$$

Let \mathcal{F} be a reverse-free code and for each $\mathbf{x} \in \mathcal{F}$, define its i -support to be the set $\text{supp}_i(\mathbf{x}) = \{\ell : x_\ell = i\}$. If the alphabet only consists of two symbols, say 1 and 2, then the sets $\text{supp}_1(\mathbf{x})$ form a chain, thus

$$\overline{F}(1, k) = 1 \quad \text{and} \quad \overline{F}(2, k) = k + 1. \quad (3.9)$$

3.3 Small constructions

In this section we collect a few small optimal constructions. The ideas of the constructions and the proofs here are used again later. We have

$$\begin{aligned} F(3, 3) = 3, \quad F(4, 3) = 6, \quad F(5, 3) = 15, \\ \text{and} \\ F(4, 4) = 5, \quad F(5, 5) = 13, \quad \overline{F}(3, 3) = 11. \end{aligned} \quad (3.10)$$

For $n = k = 3$ it is hard not to see that two ordered triples on three elements are reverse-free if and only if one is a cyclic shift of the other. We have $F(3, 3) = 3$, and the optimal constructions are $\{abc, cab, bca\}$ and $\{cba, acb, bac\}$.

Lower bounds.

The lower bounds for $F(4, 3)$ and $F(5, 3)$ can be obtained by considering the set of directed triangles of particular directed graphs on $n = 4$ and $n = 5$ vertices, respectively. Namely, one can consider the directed graph with edges $(1, 2)$, $(2, 3)$, $(3, 1)$, $(1, 4)$ and $(4, 3)$ (then the directed triangles are 123, 231, 312 and 143, 431, 314). For $n = 5$ we take the directed triangles of the tournament with the edge-set $\{(i, i + 1), (i, i + 2)\}$ where all numbers are taken modulo 5.

The lower bound constructions for $F(4, 4)$, $F(5, 5)$ and $\overline{F}(3, 3)$ are given in Figure 3.1.

Upper bounds.

The case $F(4, 3)$ is easy, and $F(5, 3) \leq (5/2) \cdot F(4, 3) = 15$ follows from Lemma 38.

Concerning a reverse-free code $\mathcal{F} \subset [4]_{(4)}$, suppose that there exists an element appearing at least twice in the same position. Without loss of generality, the element 1 is in the first position twice, say $1234 \in \mathcal{F}$ and $1423 \in \mathcal{F}$. If another shift 1342 belongs to \mathcal{F} , then there are no more triples that we can add, $|\mathcal{F}| = 3$ and we are done. Otherwise, the only other possible members of \mathcal{F} are 2341 and 2431, 3124 and 3142, and 4213 and 4312. However, each of these three pairs has a reversed pair, so only one of each pair can belong to \mathcal{F} . This implies $|\mathcal{F}| \leq 5$.

The case $F(5, 5) = 13$ was proved by a computer search.

Suppose L is a reverse-free code of length 3, on three symbols A , B and C . We can have 3 sequences using one symbol (AAA , BBB and CCC), and 2 sequences using both A and B and not using C , so altogether L can have $3 + 3 \times 2 = 9$ members using at most two symbols and three sequences using all three symbols. These are $9 + 3 = 12$ sequences, but if L contains ABC and two cyclic shifts, then it has no member with exactly two symbols. This implies $\overline{F}(3, 3) < 9 + 3$.

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 2 & 5 & 4 & 1 & 3 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 3 & 5 & 2 & 1 & 4 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 4 & 1 & 3 & 2 \\ 5 & 4 & 2 & 1 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \quad L = \begin{cases} AAA \\ BBB \\ CCC \\ AAB \\ AAC \\ ABB \\ CBB \\ CAC \\ CBC \\ ABC \\ CAB \end{cases}$$

Figure 3.1: Constructions for $F(3, 3)$, $F(4, 4)$, $F(5, 5)$ and $\overline{F}(3, 3)$.

3.4 An iterated construction of reverse-free triple systems

We have shown already in (3.6) that $\frac{1}{6} \leq f(3) \leq \frac{1}{2}$. In fact, both of these bounds are far from the truth. In this section and the next one we show that the exact value is $f(3) = 5/24$. In other words, we show that

$$F(n, 3) = \left(\frac{5}{4} + o(1) \right) \binom{n}{3}.$$

Theorem 40. *For any $n \in \mathbb{N}$, we have*

$$F(n, 3) > \frac{5}{24}n^3 - \frac{1}{2}n^2 - O(n \log n).$$

If n is a power of 3, we have

$$F(n, 3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n.$$

Proof. Our aim is to build a large reverse-free family \mathcal{F} of triples with entries in $[n]$. Let A, B, C be disjoint sets of sizes a, b , and c respectively, such that $[n] = A \cup B \cup C$. Take three reverse-free families, \mathcal{F}_A , \mathcal{F}_B , and \mathcal{F}_C , on the sets A, B, C , of sizes $F(a, 3)$,

$F(b, 3)$, and $F(c, 3)$. The underlying sets of these families are disjoint, so the union $\mathcal{F}_A \cup \mathcal{F}_B \cup \mathcal{F}_C$ is also reverse-free. We include all these triples in \mathcal{F} .

We will add more triples to \mathcal{F} . To this end, let us introduce the *type* of the triple $\mathbf{x} = x_1x_2x_3$ to be the triple $X_1X_2X_3$, where $X_i \in \{A, B, C\}$ is such that $x_i \in X_i$. All triples that we will add will be of one of the 11 types specified by L in Figure 3.1. The family L is reverse-free, so adding triples in this way generates only a few conflicts. The only way a triple can have a pair that is reversed in another triple is if both elements of the pair belong to the same set in the decomposition A, B, C . For example, there can be a triple $x_1x_2x_3 \in ABB$ whose last two elements are reversed in another triple of type ABB , in a triple of type CBB , or in a triple in \mathcal{F}_B (i.e., type BBB).

Define a directed graph G_A on the vertex set A by putting

$$E(G_A) = \{(x, y) : xyz \in \mathcal{F}_A \text{ for some } z \in A\}.$$

If G_A is not a tournament, add some edges with arbitrary orientations to turn it into one. Take all triples of type AAB that are *consistent* with G_A , that is, all triples $x_1x_2x_3$ such that $(x_1, x_2) \in E(G_A)$ and $x_3 \in B$, and add them to \mathcal{F} . There are $\binom{a}{2}b$ such triples. Similarly, add all $\binom{a}{2}c$ triples of type AAC consistent with G_A .

In an analogous way, define a tournament G_B with $V(G_B) = B$ and $E(G_B) = \{(x, y) : zxy \in \mathcal{F}_B \text{ for some } z \in B\}$, and add all triples of types ABB and CBB consistent with G_B to \mathcal{F} . There are $\binom{b}{2}(a + c)$ of those.

And finally, add all $\binom{c}{2}(a + b)$ triples of types CAC and CBC , consistent with the tournament G_C on C , with $E(G_C) = \{(x, y) : yzx \in \mathcal{F}_C \text{ for some } z \in C\}$.

This leaves only the types ABC, CAB . Include all the $2abc$ triples of these types in \mathcal{F} .

The family \mathcal{F} that we constructed is reverse-free. Altogether, we get

$$\begin{aligned} F(a + b + c, 3) \geq |\mathcal{F}| = & F(a, 3) + F(b, 3) + F(c, 3) + \\ & + \binom{a}{2}(b + c) + \binom{b}{2}(a + c) + \binom{c}{2}(a + b) + 2abc \end{aligned}$$

or equivalently, by rearranging the above formula,

$$\begin{aligned} \left(F(a+b+c, 3) - \binom{a+b+c}{3} \right) &\geq \\ &\geq \left(F(a, 3) - \binom{a}{3} \right) + \left(F(b, 3) - \binom{b}{3} \right) + \left(F(c, 3) - \binom{c}{3} \right) + abc. \end{aligned} \quad (3.11)$$

In particular, if n is a power of 3, we can split the underlying set in three equal parts in each step, and the recurrence (3.11) together with the starting value $F(3, 3) = 3$ yields

$$F(n, 3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n, \quad (3.12)$$

proving the second part of our theorem.

For general n , in order to give a lower bound for $F(n, 3)$, we seek an upper bound on the remainder term $r(n)$ defined by

$$F(n, 3) = \binom{n}{3} + \frac{1}{24}n^3 + \frac{7}{24}n - r(n).$$

Note that the quantity in (3.12) equals $\binom{n}{3} + \frac{1}{24}n^3 + \frac{7}{24}n$. Theoretically, the remainder $r(n)$ might be negative, but that would only improve the lower bound. Substituting in (3.11), we obtain

$$\frac{(a+b+c)^3}{24} - r(a+b+c) \geq \frac{a^3+b^3+c^3}{24} + abc - r(a) - r(b) - r(c).$$

Letting a, b and c be the appropriate numbers with pairwise differences not exceeding 1, we have

$$\begin{aligned} r(3n) &\leq 3r(n) \\ r(3n-1) &\leq 2r(n) + r(n-1) + \frac{1}{4}n \\ r(3n+1) &\leq 2r(n) + r(n+1) + \frac{1}{4}n. \end{aligned}$$

It easily follows by induction that, for some constant C ,

$$r(n) \leq Cn \log n,$$

completing the proof of Theorem 40. □

3.5 An upper bound on reverse-free triple systems

Theorem 41.

$$F(n, 3) \leq \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd,} \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even.} \end{cases}$$

In order to give an upper bound on the number of triples in a reverse-free set, we group them according to their underlying 3-element sets. We define the *underlying set* of an ordered triple to be the set of its three elements. We say that the set $\{u, v, w\}$ has i occurrences in \mathcal{F} if \mathcal{F} contains i ordered triples with this underlying set. The family of three-element sets having exactly i occurrences in \mathcal{F} will be denoted by T_i . In this section, we refer to a fixed reverse-free family \mathcal{F} whenever we talk about T_i .

Lemma 42. $|T_0| + |T_1| + |T_2| + |T_3| = \binom{n}{3}$.

Proof. The sets T_i form a partition of $\binom{[n]}{3}$. Two ordered triples with the same underlying set are reverse-free if and only if one is a cyclic shift of the other, so any three-element set has at most three occurrences in \mathcal{F} . □

Lemma 43. If $0 \leq d_1 \leq d_2 \leq \dots \leq d_n \leq n-1$ are integers such that

$$\sum_{i=1}^n d_i = \frac{n(n-1)}{2},$$

then

$$\sum_{i=1}^n \binom{d_i}{2} \geq \begin{cases} n \binom{\frac{n-1}{2}}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} \binom{\frac{n-2}{2}}{2} + \frac{n}{2} \binom{\frac{n}{2}}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let d_1, \dots, d_n be a sequence that minimizes $\sum_{i=1}^n \binom{d_i}{2}$ under the conditions given by the hypothesis. The average of the d_i 's is equal to $\frac{n-1}{2}$. It follows that if n is even and $d_i \in \{\frac{n}{2}, \frac{n-2}{2}\}$ for all i , then $d_i = \frac{n}{2}$ for exactly half of the d_i 's, and we are done. If n is odd and $d_i = \frac{n-1}{2}$ for all i , then the situation is even simpler; there is nothing to prove.

If, on the other hand, $d_1 < \frac{n-2}{2}$, then $d_n > d_1 + 1$. A simple computation shows that in this case we have

$$\binom{d_1 + 1}{2} + \binom{d_n - 1}{2} < \binom{d_1}{2} + \binom{d_n}{2}$$

while preserving the sum of the d_i 's, which is a contradiction. \square

For a directed graph G , let $c(G)$ be the number of directed triangles in G .

Corollary 44. *If D is a tournament on n vertices, then*

$$c(D) = \binom{n}{3} - \sum_{v \in V(G)} \binom{d^+(v)}{2} \leq \begin{cases} \frac{n^3 - n}{24} & \text{for } n \text{ odd,} \\ \frac{n^3 - 4n}{24} & \text{for } n \text{ even.} \end{cases}$$

Proof. In a tournament, each triangle is either directed or undirected. The undirected triangles of D are in one-to-one correspondence with pairs of edges sharing a tail. There are $\sum_{v \in V(G)} \binom{d^+(v)}{2}$ such pairs. The out-degree sequence of D satisfies the hypotheses of Lemma 43. \square

Given the code \mathcal{F} , we create a directed graph G_1 on the vertex set $[n]$ by putting $uv \in E(G_1)$ whenever there exists a vertex x such that $uvx \in \mathcal{F}$. Note that $uv \in E(G_1)$ implies $vu \notin E(G_1)$. If the resulting graph is not a tournament, we add edges arbitrarily to turn it into one. Similarly, let G_2 be a tournament that contains all edges uv such that, for some vertex x , we have $xuv \in \mathcal{F}$, and G_3 a tournament containing all edges uv such that $vxu \in \mathcal{F}$ for some x .

Define two additional directed graphs, \mathcal{D} and \mathcal{M} , by putting

$$\begin{aligned} E(\mathcal{M}) &= \{uv : uv \in E(G_i) \cap E(G_j) \text{ for some pair } i, j \in \{1, 2, 3\}\} \\ E(\mathcal{D}) &= \{uv : uv \in E(G_1) \cap E(G_2) \cap E(G_3)\}. \end{aligned} \tag{3.13}$$

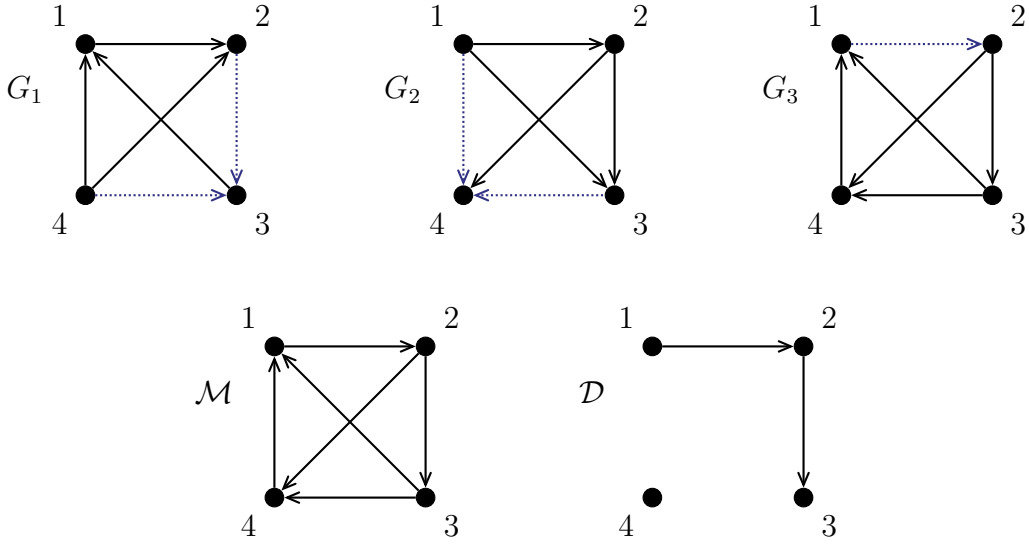


Figure 3.2: The graphs G_1 , G_2 , G_3 , \mathcal{M} and \mathcal{D} for the family $\mathcal{F} = \{123, 124, 312, 412, 413, 423\}$. The directions of the dotted edges were chosen arbitrarily. The graphs \mathcal{M} and \mathcal{D} reflect this choice.

Figure 3.2 shows what these graphs might look like for a particular family \mathcal{F} .

Lemma 45.

$$|T_2| + |T_3| \leq \begin{cases} \frac{n^3 - n}{24} & \text{if } n \text{ is odd,} \\ \frac{n^3 - 4n}{24} & \text{if } n \text{ is even.} \end{cases}$$

Proof. If uvw and vwu both belong to \mathcal{F} , then $uv \in E(G_1)$ and $uv \in E(G_3)$, so $uv \in E(\mathcal{M})$. Similarly, $vw, wu \in E(\mathcal{M})$. We therefore have $|T_2| + |T_3| = |T_2 \cup T_3| \leq c(\mathcal{M})$, and the result follows. \square

Lemma 46. $|T_3| - |T_0| \leq \frac{n}{3}$.

Proof. For any vertex u of the graph \mathcal{D} , define the family

$$S_u = \{\{u, v, w\} : uv \in E(\mathcal{D}) \text{ and } uw \in E(\mathcal{D})\}.$$

It is easy to see that $\bigcup_{u \in [n]} S_u \subseteq T_0$. The family S_u consists of all three-element sets containing u and two of its out-neighbors in \mathcal{D} , hence $|S_u| = \binom{d^+(u)}{2}$ (the out- and

in-degree of the vertex u refer to the directed graph \mathcal{D}). For $u \neq v$ the families S_u and S_v are disjoint, so $|T_0| \geq \sum \binom{d^+(u)}{2}$. Similarly, $|T_0| \geq \sum \binom{d^-(u)}{2}$. It follows that

$$|T_0| \geq \frac{1}{2} \sum_{u \in [n]} \left[\binom{d^+(u)}{2} + \binom{d^-(u)}{2} \right]. \quad (3.14)$$

Each three-element set from T_3 induces a directed triangle in \mathcal{D} . A vertex u in $D(\mathcal{F})$ is in at most $d^+(u) \cdot d^-(u)$ directed triangles and hence

$$|T_3| \leq c(\mathcal{D}) \leq \frac{1}{3} \sum_{u \in [n]} d^+(u) \cdot d^-(u). \quad (3.15)$$

Subtract (3.14) from (3.15). Use the fact that for non-negative integers p, q we have $\frac{1}{3}pq - \frac{1}{4}(p^2 - p) - \frac{1}{4}(q^2 - q) \leq \frac{1}{3}$. We obtain

$$|T_3| - |T_0| \leq \sum_{u \in [n]} \left[\frac{1}{3}d^+(u) \cdot d^-(u) - \frac{1}{2} \binom{d^+(u)}{2} - \frac{1}{2} \binom{d^-(u)}{2} \right] \leq \frac{n}{3}.$$

This concludes the proof of the claim. \square

Proof of Theorem 41. We add the three inequalities that we obtained in Lemmas 42, 45 and 46.

$$\begin{aligned} |\mathcal{F}| &= |T_1| + 2|T_2| + 3|T_3| \\ &= \binom{n}{3} + (|T_2| + |T_3|) + (|T_3| - |T_0|) \\ &\leq \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd} \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

\square

3.6 Reverse-free triple systems: The case of repetitions

In this section we determine the asymptotics of $\overline{F}(n, 3)$. We already know that $\binom{n}{k} \leq F(n, k) \leq k! \binom{n}{k}$ and $\binom{n+k-1}{k} \leq \overline{F}(n, k) \leq n^k$. It follows that if k is fixed and $n \rightarrow \infty$, the two quantities have the same order, namely $\Theta(n^k)$.

For $k = 3$ we again have an exact result.

Theorem 47. *For any $n \in \mathbb{N}$, we have*

$$\frac{5}{24}n^3 + \frac{1}{2}n^2 - O(n \log n) < \overline{F}(n, 3) \leq \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{7}{24}n,$$

and when n is a power of 3, equality holds in the upper bound.

Proof of the lower bound. The proof follows the outline of the proof of Theorem 40. We want to build a large reverse-free family $\mathcal{F}' \subseteq [n]^3$. Partition $[n]$ into three non-empty sets A, B , and C of sizes a, b , and c . Again, all the ordered triples in \mathcal{F}' will have a type from the set L defined in Figure 3.1. As for the types AAA, BBB , and CCC , we have three reverse-free families on A, B , and C respectively, of sizes $\overline{F}(a, 3), \overline{F}(b, 3)$, and $\overline{F}(c, 3)$. Put all these triples into \mathcal{F}' . The three families define graphs G_A, G_B , and G_C . As in the proof of Theorem 40, take all the triples in $[n]_{(3)}$ (i.e., without repetition) of types AAB and AAC that are consistent with G_A , and include them in \mathcal{F}' . Furthermore, put into \mathcal{F}' all triples xyy and xxz for $x \in A, y \in B$, and $z \in C$. Altogether, we added $\binom{a}{2}(b+c) + ab + ac = \binom{a+1}{2}(b+c)$ triples. Proceed analogously with G_B and the types CBB, ABB , and with G_C and the types CAC, CBC . Conclude with adding the $2abc$ triples of types ABC and CAB . The resulting set is reverse-free. We have

$$\begin{aligned} \overline{F}(a+b+c, 3) &\geq |\mathcal{F}'| = \overline{F}(a, 3) + \overline{F}(b, 3) + \overline{F}(c, 3) + \\ &\quad + \binom{a+1}{2}(b+c) + \binom{b+1}{2}(a+c) + \binom{c+1}{2}(a+b) + 2abc. \end{aligned} \quad (3.16)$$

If n is a power of 3, the inequality (3.16) together with the base case $\overline{F}(3, 3) = 11$ yield the lower bound

$$\overline{F}(n, 3) \geq \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{7}{24}n. \quad (3.17)$$

For a general n we proceed as in the proof of Theorem 40 to prove that (3.17) is at most $O(n \log n)$ far from $\overline{F}(n, 3)$. \square

Proof of the upper bound. Let $\mathcal{F} \subseteq [n]^3$ be a reverse-free code of the maximum size, $\overline{F}(n, 3)$. As in the proof of Theorem 41, each triple in \mathcal{F} has an *underlying set* associated with it, i.e., the set of its symbols. Let \mathcal{F}_i be the set of all ordered triples of \mathcal{F} whose underlying set has cardinality i . Clearly, $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|$, and $|\mathcal{F}_1| \leq n$. Moreover, the triples with underlying sets of size 3 do not have repeated elements, so $|\mathcal{F}_3| \leq F(n, 3)$. It remains to bound $|\mathcal{F}_2|$. Given any two distinct elements x, y from $[n]$, at most one triple from each of the sets $\{xxxy, xyxx, yxxx\}$ and $\{yyxy, yxyx, xyyy\}$ belongs to \mathcal{F}_2 , so at most two triples with underlying set $\{x, y\}$ appear in \mathcal{F}_2 , and we have $|\mathcal{F}_2| \leq 2\binom{n}{2}$. Define the graph \mathcal{D} with respect to \mathcal{F}_3 in the same way as in (3.13). If $xy \in E(\mathcal{D})$, then none of the six triples with underlying set $\{x, y\}$ can be in \mathcal{F}_3 . Hence $|\mathcal{F}_2| \leq 2\binom{n}{2} - 2E(\mathcal{D})$. As in the proof of Theorem 41, we get

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \\ &\leq n + 2\binom{n}{2} - 2E(\mathcal{D}) + \binom{n}{3} + (|T_2| + |T_3|) + (|T_3| - |T_0|). \end{aligned} \quad (3.18)$$

Using (3.14) and (3.15) we have

$$\begin{aligned} |T_3| - |T_0| - 2E(\mathcal{D}) &\leq \\ &\leq \sum_{u \in [n]} \left[\frac{1}{3}d^+(u) \cdot d^-(u) - \frac{1}{2}\binom{d^+(u)}{2} - \frac{1}{2}\binom{d^-(u)}{2} - d^+(u) - d^-(u) \right] \leq 0 \end{aligned} \quad (3.19)$$

where in the last inequality we use the fact that we have $\frac{1}{3}pq - \frac{1}{4}(p^2 - p) - \frac{1}{4}(q^2 - q) - p - q \leq 0$ for any p, q . From (3.18) using Lemma 45 and (3.19) we have

$$|\mathcal{F}| \leq \begin{cases} \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{7}{24}n & \text{for } n \text{ odd} \\ \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n & \text{for } n \text{ even.} \end{cases}$$

This concludes the proof. \square

The argument for the upper bound sheds some additional light on the lower bound

as well. Notice that the value in (3.17) equals $F(n, 3) + n + 2\binom{n}{2} - \frac{n}{3}$. Indeed, the lower bound construction may be obtained from the construction for $F(n, 3)$ by adding triples xxx for all x , two triples with support $\{x, y\}$ for each pair x, y (one from $\{xxy, xyx, yxx\}$ and one from $\{yyx, yxy, xyy\}$), and removing a triple xyz of three distinct elements if $\{x, y, z\}$ was an underlying set for three triples (there were $n/3$ of those). Notice that the induction step does not introduce any edges of \mathcal{D} , so the lastly mentioned operation removed all of them.

3.7 A related problem: Codes with many flips

A code $\mathcal{G} \subset [n]^k$ is *flip-full* if there is a reversed pair of distinct symbols for every pair of its members. Let $G(n, k)$ be the maximum size of a flip-full code $\mathcal{G} \subset [n]_{(k)}$, and let $\overline{G}(n, k)$ be the analogous quantity for $[n]^k$.

Obviously $G(2, 2) = G(3, 3) = 2$. We have

$$G(4, 4) = 4 \quad \text{and} \quad G(5, 5) = 8. \quad (3.20)$$

The lower bounds are given by the following constructions

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 5 & 2 & 1 & 4 & 3 \\ 5 & 3 & 4 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 4 & 1 & 2 & 3 & 5 \\ 4 & 2 & 1 & 5 & 3 \\ 4 & 3 & 5 & 1 & 2 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix}, \quad (3.21)$$

and computer searches show that these lower bounds are tight.

Lemma 48. $G(n_1, k_1)G(n_2, k_2) \leq G(n_1 + n_2, k_1 + k_2)$.

Proof. Take two flip-full codes, $\mathcal{G}_1 \subset X_{(k_1)}$ and $\mathcal{G}_2 \subset X_{(k_2)}$, for disjoint sets X and Y . Let \mathcal{G} be the product code on the alphabet $X \cup Y$ obtained by concatenating each $\mathbf{x} \in \mathcal{G}_1$ and each $\mathbf{y} \in \mathcal{G}_2$. Such \mathcal{G} is flip-full. \square

Corollary 49. $\frac{1}{8}(1.515\dots)^n < G(n, n)$.

Proof. $G(n, n) \geq G(5, 5) \cdot G(n-5, n-5) \geq \dots \geq G(5, 5)^{\lfloor n/5 \rfloor} G(n-5\lfloor n/5 \rfloor, n-5\lfloor n/5 \rfloor) \geq 8^{\lfloor n/5 \rfloor} > \frac{1}{8}(1.515\dots)^n$. \square

3.8 Long permutations

Define a graph P_n by setting

$$V(P_n) = \{\pi : \pi \text{ is a permutation of } [n]\} \quad (3.22)$$

$$E(P_n) = \{\{\pi, \rho\} : \pi \text{ and } \rho \text{ have a reversed pair}\}. \quad (3.23)$$

The quantities $F(n, n)$ and $G(n, n)$ correspond to the independence number and the clique number of G_n , respectively. The graph P_n is vertex-transitive, so we can use the clique-coclique bound

$$G(n, n)F(n, n) \leq n!.$$

It is easy to prove this bound directly for this particular graph. In the next section we prove a stronger version of the inequality, so we omit the proof here.

Let us also remark here that, if we define the graph $P_{n,k}$ on the set of ordered k -tuples out of $[n]$ in an analogous way, the clique-coclique bound becomes

$$G(n, k)F(n, k) \leq k! \binom{n}{k}. \quad (3.24)$$

Lower bounds on $G(n, n)$ obtained in Section 3.7 translate into upper bounds on $F(n, n)$.

Corollary 50. $F(n, n) \leq \frac{n!}{G(n, n)} \leq \frac{8 \cdot n!}{(1.515\dots)^n}$.

For $G(6, 6)$, the lower bound from Lemma 48 reduces to the rather trivial $G(6, 6) \geq G(5, 5) = 8$. With this value, the upper bound from Corollary 50 is

$$F(6, 6) \leq \frac{6!}{8} = 90. \quad (3.25)$$

Since the graphs P_n are vertex transitive, it seems natural to try to use eigenvalue techniques to estimate the independence number. This approach worked for other graphs defined on the set of permutations. For example, Renteln in [61] completely determined the eigenvalues of the derangement graph Q_n (where two permutations π and ρ are adjacent if there is no i such that $\pi(i) = \rho(i)$). And Godsil and Meagher in [34] used methods of representation theory to give a new proof of the Erdős-Ko-Rado Theorem for permutations. They used group characters to characterize the extremal cases, i.e., to show that every clique of order $(n-1)!$ in Q_n is trivial. Their work was extended in [23] by Ellis, Friedgut and Pilpel, who used spectral methods and representations of the symmetric group to prove the analogue of the Erdős-Ko-Rado Theorem for t -intersecting permutations – they proved that pairwise t -intersecting family of permutations of $[n]$ has no more than $(n-t)!$ members, if n is sufficiently large.

Using computer, we determined the eigenvalues of P_n for $n = 1, \dots, 6$. In particular, the distinct eigenvalues of P_6 are $-27, -21, -15, -3, 0, 3, 5, 9, 25, 285$, with multiplicities 25, 100, 26, 25, 256, 100, 81, 25, 81, 1 respectively.

A well-known result due to Hoffman ([38]) is that if G is a d -regular graph and τ its least eigenvalue, then

$$\alpha(G) \leq \frac{|V(G)|}{1 + \frac{d}{|\tau|}}. \quad (3.26)$$

Proposition 51. $F(6, 6) \leq 62$.

Proof of Proposition 51. The graph P_6 is 285-regular with $6!$ vertices, and the smallest eigenvalue is $\tau = -27$. Substituting these values in the Hoffman bound (3.26), we have $F(6, 6) \leq \frac{6!}{1 + \frac{285}{|-27|}} \leq 62.31$. \square

This is an improvement over 3.25. However, it seems hard to derive a formula for the least eigenvalue of P_n for general n (or even just a lower estimate).

To obtain a lower bound for $F(n, n)$, we introduce one more recurrence.

Lemma 52. $F(n, a)^b F(b, b) \leq F(nb, ba)$.

Proof. Take the reverse-free codes $\mathcal{F}^i \subset X_{(a)}^i$ where X^1, \dots, X^b are disjoint n -sets and $|\mathcal{F}^i| = F(n, a)$, and consider a reverse-free code $\mathcal{G} \subset [b]_{(b)}$ of size $F(b, b)$. One can create a reverse-free code \mathcal{F} of size $(\prod |\mathcal{F}^i|)|\mathcal{G}|$ with underlying set $\cup X^i$ and of length

ab by taking a codeword \mathbf{x}_i from each \mathcal{F}^i and a member $\sigma \in \mathcal{G}$ and creating all the ab -tuples of the form $(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(b)})$. \square

Corollary 53. $F(n, n) \geq \frac{1}{n^2}(1.898\dots)^n$.

Proof. Starting with $F(5, 5) = 13$, induction gives $F(5^t, 5^t) \geq 13^{\frac{5^t}{4} - \frac{1}{4}}$.

Write n as $n = \sum_{i=0}^{\lfloor \log_5 n \rfloor} c_i 5^i$ with $0 \leq c_i \leq 4$. Note that $\sum_{i=0}^{\lfloor \log_5 n \rfloor} c_i \leq 4 \log_5 n$. Using (3.7) we obtain

$$F(n, n) \geq \prod_{i=0}^{\lfloor \log_5 n \rfloor} F(5^i, 5^i)^{c_i} \geq 13^{\frac{n}{4} - \frac{1}{4} \sum_{i=0}^{\lfloor \log_5 n \rfloor} c_i} \geq \frac{1}{n^2} 13^{n/4} \geq \frac{1}{n^2} (1.898\dots)^n.$$

\square

3.9 Better upper bounds for $F(n, k)$, $\overline{F}(n, k)$

The following lemma generalizes the clique-coclique bound from section 3.8.

Lemma 54. $G(k, k)F(n, k) \leq k! \binom{n}{k}$.

Proof. Consider a reverse-free code $\mathcal{F} \subset [n]_{(k)}$ and a flip-full code $\mathcal{G} \subset [k]_{(k)}$. Consider all k -tuples of the form $(x_{\sigma(1)}, \dots, x_{\sigma(k)})$ where $\mathbf{x} \in \mathcal{F}$ and $\sigma \in \mathcal{G}$. We claim these are all distinct. Indeed, suppose that $(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = (y_{\tau(1)}, \dots, y_{\tau(k)})$, $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, $\sigma, \tau \in \mathcal{G}$, and $(\mathbf{x}, \sigma) \neq (\mathbf{y}, \tau)$. Since \mathbf{x} has no repeated symbols, $\mathbf{x} \neq \mathbf{y}$. But then also $\sigma \neq \tau$. By the definition of \mathcal{G} there are $i \neq j$ with $\sigma(i) = \tau(j)$ and $\sigma(j) = \tau(i) \neq \alpha$. Let $\alpha = \sigma(i)$ and $\beta = \sigma(j)$. We have $x_{\sigma(i)} = y_{\tau(i)}$, $x_{\sigma(j)} = y_{\tau(j)}$, and $\alpha \neq \beta$, implying that (x_α, x_β) and (y_α, y_β) are reversed pairs, a contradiction. \square

Using Lemma 54 together with Corollary 49, we can improve the bounds (3.5) and (3.6).

$$F(n, k) \leq \frac{k! \binom{n}{k}}{G(k, k)} \quad \text{and} \quad f(k) \leq \frac{1}{G(k, k)} \leq \frac{8}{(1.515\dots)^k}.$$

However, this upper bound would be more useful if we had stronger lower bound on $G(n, n)$. As it is, we get better results using Theorem 41 and Lemma 39.

Proposition 55. $F(n, k) \leq \frac{k! \binom{n}{k}}{(1.686 \dots + o(1))^k}$.

Proof. In Theorem 41 we proved that $F(n, 3) = \left(\frac{5}{4} + o(1)\right) \binom{n}{3}$. We also know that

$$F(n - 3\lfloor k/3 \rfloor, 1) = n - 3\lfloor k/3 \rfloor \quad \text{and} \quad F(n - 3\lfloor k/3 \rfloor, 2) = \binom{n - 3\lfloor k/3 \rfloor}{2}.$$

Using Lemma 39 repeatedly together with the above, we get

$$\begin{aligned} F(n, k) &\leq F(n, 3)F(n - 3, 3)F(n - 6, 3) \dots \\ &\leq \left(\frac{5}{4} + o(1)\right)^{\lfloor k/3 \rfloor} \frac{n(n-1) \dots (n - k + 1)}{6^{\lfloor k/3 \rfloor}} \\ &\leq \left(\frac{5}{24} + o(1)\right)^{k/3} k! \binom{n}{k} = \frac{k! \binom{n}{k}}{(1.686 \dots + o(1))^k}. \end{aligned}$$

□

The proof of the following inequality is the same as the proof of Lemma 39

$$\overline{F}(n, k) \leq \overline{F}(n, i) \overline{F}(n, k - i). \quad (3.27)$$

Combine Theorem 47 and (3.27) to obtain

Proposition 56. $\overline{F}(n, k) \leq \overline{F}(n, 3)^{\lfloor k/3 \rfloor} n^{k - 3\lfloor k/3 \rfloor} \leq \left(\frac{5}{24} + o(1)\right)^{k/3} n^k = \frac{n^k}{(1.686 \dots + o(1))^k}$.

3.10 Small alphabets

When n is fixed and the length of the codewords k tends to ∞ , then the order of magnitude of the maximum size code is polynomial in k .

Theorem 57. *If $n \geq 2$, $k \geq 2$, then*

$$\left(\frac{k}{\binom{n}{2}}\right)^{\binom{n}{2}} \leq \overline{F}(n, k) \leq \binom{k}{\leq n-1} \binom{k}{\leq n-2} \dots \binom{k}{\leq 1} = O\left(k^{\binom{n}{2}}\right),$$

where $\binom{k}{\leq \ell}$ stands for $\sum_{0 \leq i \leq \ell} \binom{k}{i}$.

Proof of the upper bound. We fix k and use induction on n , with $\overline{F}(1, k) = 1$ and $\overline{F}(2, k) = k + 1$ as the base cases.

Let $\mathcal{F} \subset [n]^k$ be a reverse-free code, $n, k \geq 3$. Let $\mathbf{x} \in [n]^k$ be an arbitrary codeword. Define its i -support, $\text{supp}_i(\mathbf{x})$, as the subset of the coordinates where \mathbf{x} takes the value i ,

$$\text{supp}_i(\mathbf{x}) = \{\ell : x_\ell = i\}.$$

Let \mathcal{F}_1 be the family of the 1-supports and for each $A \in \mathcal{F}_1$ let

$$\mathcal{F}_A = \{\mathbf{y} \in \mathcal{F} : \text{supp}_1(\mathbf{y}) = A\}.$$

Our first observation is that A cannot appear as a 1-support too many times,

$$|\mathcal{F}_A| \leq \overline{F}(n-1, k-|A|). \quad (3.28)$$

Indeed, the projection $\mathcal{F}_A|([k] \setminus A)$ is a reverse-free code of length $k-|A|$ using $n-1$ symbols (namely $\{2, 3, \dots, n\}$), so (3.28) follows.

Hence by induction hypothesis

$$|\mathcal{F}_A| \leq \overline{F}(n-1, k) \leq \binom{k}{\leq n-2} \cdots \binom{k}{\leq 1}.$$

We claim that

$$|\mathcal{F}_1| \leq \binom{k}{n-1} + \binom{k}{n-2} + \cdots + \binom{k}{0} = \binom{k}{\leq n-1}. \quad (3.29)$$

The upper bound then follows.

Our main tool for the proof of (3.29) is the following theorem which was discovered independently and about the same time by Sauer [64], Shelah [65] and Vapnik and Chervonenkis [72], and became an important result in different contexts. Let M be a $m \times k$ matrix of 0's and 1's with distinct rows. We say that the *VC-dimension* of M is at least s if one can find s columns S such that $M|_S$, the matrix restricted to these columns, contains all the 2^s possible 0-1 rows. It is known ([64, 65, 72]) that if the VC-dimension is at most s , then

$$m \leq \sum_{0 \leq i \leq s} \binom{k}{i}. \quad (3.30)$$

We claim that the VC-dimension of \mathcal{F}_1 is at most $n - 1$. Thus (3.30) implies (3.29). Suppose to the contrary that there exists an n -set $S \subset [k]$, $S = \{s^{(1)}, \dots, s^{(n)}\}$, such that \mathcal{F} induces all possible traces, $\mathcal{F}|_S = 2^S$. Then there are members $F^{(i)} \in \mathcal{F}_1$ such that $F^{(i)} \cap S = S \setminus \{s^{(i)}\}$, $1 \leq i \leq n$. This means that there exist $\mathbf{x}^{(i)} \in \mathcal{F}$ with coordinates

$$\mathbf{x}_s^{(i)} = \begin{cases} 1 & \text{for } s \in S, s \neq s^{(i)}, \\ \alpha_i & \text{for } s = s^{(i)}, \end{cases}$$

where $\alpha_i \in \{2, 3, \dots, n\}$, $i \in [n]$. By the pigeonhole principle we obtain indices $i, j \in [n]$ such that $i \neq j$ but $\alpha_i = \alpha_j$. Then $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ contain a reversed pair (at coordinates $s^{(i)}$ and $s^{(j)}$). This final contradiction completes the proof of the upper bound. \square

Proof of the lower bound. We explicitly construct a reverse-free code $\mathcal{F} \subset [n]^k$ of the desired size.

Split $[k]$ into $n(n-1)/2$ almost equal parts, $[k] = \cup_{1 \leq i < j \leq n} V_{i,j}$. Here $|V_{i,j}| \geq \lfloor k / \binom{n}{2} \rfloor$. Take a reverse-free family $\mathcal{F}_{i,j}$ of $|V_{i,j}| + 1$ vectors with coordinates $V_{i,j}$ such that each $\mathbf{x} \in \mathcal{F}_{i,j}$ takes values i and j only. Define \mathcal{F} as the product of all of these families

$$\mathcal{F} = \{\mathbf{y} \in [n]^k : \mathbf{y}|_{V_{i,j}} \in \mathcal{F}_{i,j} \text{ for all } 1 \leq i < j \leq n\}.$$

This \mathcal{F} is reverse-free. Suppose to the contrary that there exist $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ such that $\{x_\alpha, x_\beta\} = \{y_\alpha, y_\beta\}$ is a reversed pair. If i denotes $x_\alpha = y_\beta$ and j denotes $x_\beta = y_\alpha$ with $j > i$, then $x_\alpha, x_\beta \in V_{i,j}$, but $\mathcal{F}_{i,j}$ has no reversed pairs, a contradiction. \square

3.11 Hypergraph problems

Throughout this chapter we have dealt with a problem concerning k -tuples from a set of n elements. However, it is worthwhile to look at this problem also from some different perspectives.

We can consider it in the spirit of Turán type problems. Turán problems [71] deal with questions of the following type: Given a family of graphs \mathcal{H} , what is the maximum number of edges that an n -vertex graph may have if it does not contain any

of the graphs in \mathcal{H} as a subgraph? This type of problems has been studied extensively and has been generalized to different types of combinatorial structures. As we move from ordinary graphs to directed graphs, these problems become much more complicated, and for hypergraphs they are notoriously difficult. In this case, exact results are rare and even the asymptotic behavior is poorly understood (see, e.g., [31]). In particular the case of uniform hypergraphs has drawn considerable attention [18, 33, 43, 42]. The problem of pairwise reverse-free k -tuples is a Turán type problem for a particular generalization of this structure, directed uniform hypergraphs. For the sake of completeness we state the definition of a directed k -uniform hypergraph according to [14].

Definition 58. A k -uniform directed hypergraph H is a pair (V, E) , where V is a finite set of vertices and E is a family of ordered k -tuples of vertices (all vertices in each k -tuple must be distinct, i.e., we do not allow loops).

Therefore in the light of this reformulation it would be interesting to determine at least the asymptotic behavior of this non-trivial problem.

3.12 Traces

We can consider our problem from another point of view. Note that the pairwise reverse-free property of a set of k -tuples trivially implies that any projection of such a set on two coordinates contains at most $\binom{n}{2}$ different pairs. Thus, loosely speaking, we ask how large a set of k -tuples can be if its projections (in this case on two coordinates) are somehow “small”. This question is very much in the spirit of Vapnik-Chervonenkis dimension type problems. The result (3.30) was generalized by Frankl [29] and Alon [3] as follows.

Let $\mathcal{F} \subset [n]^k$ be a set of sequences. We say that $\mathcal{F} \rightarrow (s, r)$, or that \mathcal{F} is (s, r) -dense, if there exists an s -set $S \subset [k]$ such that it *induces* at least r traces, i.e., $|\mathcal{F}|_S| \geq r$, where $\mathcal{F}|_S$ is the projection of \mathcal{F} to the coordinates from S , $\mathcal{F}|_S = \{\mathbf{x}|_S : \mathbf{x} \in \mathcal{F}\}$. The family \mathcal{F} is *monotone* if $\mathbf{x} \in \mathcal{F}$, $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{y} \in [n]^k$, $1 \leq y_i \leq x_i$ for $1 \leq i \leq k$ imply $\mathbf{y} \in \mathcal{F}$. The theorem by Frankl and Alon says that if $\mathcal{F} \not\rightarrow (s, r)$, then one can find a monotone family $\mathcal{F}' \subset [n]^k$ of the same size $|\mathcal{F}'| = |\mathcal{F}|$, such that $\mathcal{F}' \not\rightarrow (s, r)$.

Further information on matrices with forbidden configurations can be found in the papers of Anstee and Sali, see [7, 6].

If we restrict ourselves to the quantitative version of the pairwise reverse-free property, it is not difficult to see that the exact asymptotics can be determined. Indeed, a pairwise reverse-free family induces at most $\binom{n}{2}$ pairs on a projection on any two coordinates, in other words $\mathcal{F} \not\rightarrow (2, \binom{n}{2} + 1)$.

Let \mathcal{F}' be the monotone family such that $|\mathcal{F}| = |\mathcal{F}'|$ and $\mathcal{F}' \not\rightarrow (2, \binom{n}{2} + 1)$ whose existence is guaranteed by the Alon–Frankl result, and let $y_i = \max\{x_i : \mathbf{x} \in \mathcal{F}'\}$ for all i . Suppose that $|\mathcal{F}'| > \binom{n}{2}^{k/2}$, and let i, j be indices such that $y_i \geq y_j \geq y_\ell$ for all $\ell \notin \{i, j\}$. Since $|\mathcal{F}'| = \prod_{i=1}^n y_i$, we have $|\mathcal{F}'|_{\{i, j\}} = y_i y_j > \binom{n}{2}$, which is a contradiction.

It follows that the maximum size of such a family of ordered k -tuples is

$$\max |\mathcal{F}| = \left(\frac{n}{\sqrt{2}} \right)^k + O(n^{k-1}).$$

Another proof for this upper bound can be given by applying Shearer’s lemma [17]. It is trivial to find a construction achieving it, namely the family $[m]^k$ with $m = \lfloor n/\sqrt{2} \rfloor - 1$.

This result suggests that the difficulty of the problem of the pairwise reverse-free sets stems from the fact that this property settles some kind of qualitative requirement regarding the structure of the projections. Several papers [8, 9, 57, 58] dealt with this kind of requirements regarding set families (or equivalently binary strings representing their characteristic vectors), and it would be interesting to consider this type of problems on families of ordered sets, too.

3.13 Conclusion, more problems

We determined the asymptotic behavior for $F(n, 3)$ as $n \rightarrow \infty$, and even an exact formula whenever n is a power of 3. The order of $F(n, k)$ for k fixed and $n \rightarrow \infty$ is trivially n^k . We proved some upper and lower bounds for $F(n, k)$, but the asymptotic, let alone the exact formula, is still unknown.

Problem 59. *Determine the asymptotic behavior of $F(n, k)$ for fixed k and $n \rightarrow \infty$.*

Similarly, we can use Lemma 48 with $G(n, 2) = 2$ for a lower bound, and (3.24) together with the trivial estimate $F(n, k) \geq \binom{n}{k}$ for an upper bound on $G(n, k)$, the

maximum size of a flip-full code,

$$2^{k/2} \leq G(n, k) \leq \frac{k! \binom{n}{k}}{\binom{n}{k}} = k!$$

but the exact asymptotic behavior is again unknown.

Even more interesting open problems concern permutations. We have proved the following bounds, which we believe are far from the truth.

$$\begin{aligned} \frac{1}{n^2} (1.898 \dots)^n &\leq F(n, n) \leq \frac{n!}{(1.686 \dots + o(1))^n}, \\ \frac{1}{8} (1.515 \dots)^n &\leq G(n, n) \leq \frac{n^2 n!}{(1.898 \dots)^n}. \end{aligned}$$

Problem 60. Determine the asymptotic behavior of $F(n, n)$ and $G(n, n)$ as $n \rightarrow \infty$.

Open problems also remain in the area of codes with repetitions allowed. We established the order of magnitude of $\overline{F}(n, k)$ for n fixed and $k \rightarrow \infty$, but it would be interesting to investigate this quantity further.

Problem 61. Determine the asymptotic behavior of $\overline{F}(n, k)$ for n fixed and $k \rightarrow \infty$.

Finally, we would like to mention two further seemingly related open problems.

Two permutations σ and τ of $[n]$ are *colliding* if there is an i with $|\sigma(i) - \tau(i)| = 1$. Let $\rho(n)$ be the maximum cardinality of a set of pairwise colliding permutations. Körner and Malvenuto [46] proved the lower bound $\rho(n) > c^n$ for $c = 1.661 \dots$ and also the upper bound $\rho(n) \leq \binom{n}{\lfloor n/2 \rfloor}$, and showed that equality holds in the upper bound for $n = 1, 2, \dots, 7$. They conjecture that equality holds for all n . The lower bound was improved in [12], where it is proved that $\rho(n) > c^n$ with $c = 1.8155 \dots$

Sperner's theorem states that $\overline{G}(2, n) = \binom{n}{\lfloor n/2 \rfloor}$. Körner conjectures [45] that $\overline{G}(3, n) = \overline{G}(2, n)$, i.e., the maximum number of n -tuples such that any two have a reversed pair is the same for the ternary and the binary cases. Using Sperner capacities, an upper bound 2^n is known (by Blokhuis [11] and by Calderbank, Frankl, Graham, Li, and Shepp [15]).

Chapter 4

Product dimension of trees

4.1 Motivation and definition of product dimension

Efficient encoding (representation) of partially ordered sets is important in many areas of computer science, particularly in object-oriented languages, databases, machine learning and knowledge representation ([54, 66, 1]). We need to represent a poset (P, \preceq) in a way that is both space efficient and makes it possible to quickly answer queries such as “is $x \preceq y$?”

A popular method of storing a poset is to view it as a directed acyclic graph and store its adjacency matrix, or adjacency lists ([55]). However, depending on the class of posets in question, one can often find an encoding function φ that makes it possible to answer queries more efficiently than if the poset is stored in the straightforward way described above.

As an example of such encoding, we can assign a vector $(\varphi_1(x), \dots, \varphi_k(x))$ to each $x \in P$ in such a way that $\varphi_i(x) \leq \varphi_i(y)$ holds for all i if and only if $x \preceq y$. In other words, we embed (P, \preceq) into a product of k chains. The well-known (Dushnik-Miller) dimension of the partially ordered set (P, \preceq) , defined in [19], is the minimum k for which this is possible.

Similar notions of dimension arise naturally in other categories as well. Often we have a class \mathcal{C} of objects (e.g., posets, digraphs, graphs, or some subfamilies of these), a product-like operation \otimes (for example, the usual categorical product), and a subclass $\mathcal{B} \subset \mathcal{C}$ of objects that we regard as “simple”, such that every object $C \in \mathcal{C}$ can be embedded in a product \otimes of elements of \mathcal{B} . The number of elements of \mathcal{B} that are needed for this embedding can be regarded as a measure of complexity of C . For examples of situations of this kind, and the different notions of dimension that arise following this general recipe, see [56, 35, 69, 70].

In this chapter, we deal with the same scenario in the context of undirected graphs. The following definition was first introduced by Nešetřil and Pultr in [56].

Definition 62. [56] The *product dimension* (or *Prague dimension*) of a graph G , denoted $\dim(G)$, is the minimum number k such that G is an induced subgraph of a tensor product of k complete graphs.

Equivalently, it is the minimum number of proper colorings $\varphi_1, \varphi_2, \dots, \varphi_k$ such that for every non-edge uv there is an index i with $\varphi_i(u) = \varphi_i(v)$, and the colorings distinguish the vertices in the sense that for every pair u, v there exists an index i such that $\varphi_i(u) \neq \varphi_i(v)$. And yet another way to define the same parameter is as the minimum t such that we can assign distinct vectors $\tau(u)$ of length t to the vertices $u \in V(G)$ in such a way that $uv \in E(G)$ if and only if $\tau(u)$ and $\tau(v)$ differ in all coordinates. We will switch between these as needed. Figure 4.1 shows two examples of such encoding.

This definition is not isolated, but rather a part of an active field of graph representations (see [67] for a monograph on graph representations written from both the mathematical and the computer science perspective). Especially various versions of intersection representation received a great deal of attention (see, for example, [53, 25, 63]). Generally speaking, the aim is to assign sets of a specified kind (such as intervals, boxes, or spheres in Euclidean space, etc.) to the vertices of the graph in such a way that two vertices are adjacent if and only if the corresponding sets intersect.

A concept of a different kind that is especially relevant in our context is that of *modular representation* [24]. A *representation modulo k* is an assignment of a label $\tau(v) \in \{0, 1, \dots, k-1\}$ to each vertex v so that $uv \in E(G)$ if and only if $\gcd(\tau(u) - \tau(v), k) = 1$. The *representation number* $\text{rep}(G)$ of a graph G is the minimum k such that G has a representation modulo k . If G is reduced (i.e., no two vertices have the same neighborhood), then $\text{rep}(G)$ is a product of distinct primes ([27]). If these are p_1, \dots, p_t , then G can be embedded as an induced subgraph in $K_{p_1} \times \dots \times K_{p_t}$, so $\dim(G) \leq t$. On the other hand, if we know a product representation, we can construct a modular representation using the Chinese remainder theorem. For many reduced graphs, $\dim(G)$ is equal to the number of prime divisors of $\text{rep}(G)$, but it is not known whether this is always true. For details, see [28].

And finally, several natural generalizations of the concept of product dimension

were introduced in [47] by Körner and Monti, who also proved many results about them.

4.2 Some previously known results

Each coloring φ_i in a product representation of a graph G forms an equivalence relation E_i on the vertices of G , and $\cup E_i$ equals the set of non-edges of G . In addition, since the mapping into the product of complete graphs is injective, the relations satisfy the additional requirement that for every pair (u, v) , there is an i such that $(u, v) \notin E_i$. The product dimension is the minimum t such that there is a set of t equivalences with these properties. An easy example of such a set of equivalences on a graph G with $V(G) = \{w_1, \dots, w_n\}$ is obtained by introducing a proper coloring φ_{uv} for each pair $\{u, v\} \notin E(G)$ by

$$\varphi_{uv}(w_i) = \begin{cases} 0 & \text{if } w_i = u \text{ or } w_i = v, \\ i & \text{otherwise.} \end{cases}$$

This proves that the dimension is well defined for every graph, although the upper bound obtained here, $\dim(G) \leq \binom{n}{2} - |E(G)|$, is generally not very good. For every one of these equivalences, only one class has more than one element. One would expect to be able to do better by using equivalences with more than one non-trivial class, and with non-trivial classes that contain more than two elements. In fact, Lovász, Nešetřil, and Pultr [52] showed that

$$\dim(G) \leq |V(G)| - 1.$$

Graphs with dimension equal to 1 are precisely the complete graphs. Nešetřil and Pultr in [56] provided a polynomial-time characterization of graphs with dimension 2. However, for $k \geq 3$ they provided a polynomial reduction of the problem “Is $\dim(G) \leq k$?” to the k -colorability problem, proving the *NP*-completeness of the former.

Let us note for future reference that the dimension of any star is equal to 2 (see Figure 4.1 for an example of an encoding).

There are only very few classes of graphs for which the dimension is either known

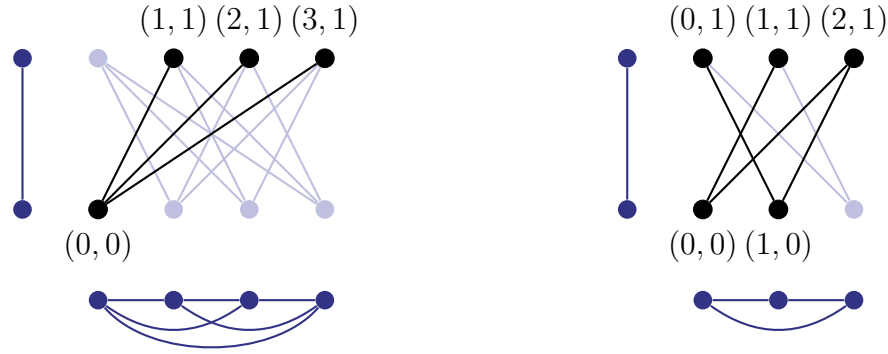


Figure 4.1: Examples of product representation: an embedding of the star with 3 leaves into the product $K_4 \times K_2$ and an embedding of path of length 4 into $K_3 \times K_2$.

exactly or we at least have good estimates. In [52], Lovász, Nešetřil, and Pultr used ingenious linear algebraic reasoning to prove the exact values for matchings, paths, and some cycles. Our proof in Section 4.3 is modeled on their approach. Křivka in [50] determined the dimension of the cycles C_ℓ for some more values of ℓ , and of the hypercubes Q_k . And finally, Poljak, Pultr, and Rödl in [60] showed that $\dim(K(n, k)) \sim \log \log n$ holds for the Kneser graphs $K(n, k)$ for k fixed and $n \rightarrow \infty$.

The connection to the dimension of posets not only provides motivation for the concept, but is of a practical use as well. Füredi exploits it in [32] to give a short proof of a result that originally appeared in [60] – an upper bound on the product dimension of Kneser graphs. He provides a construction of the colorings $\varphi_1, \varphi_2, \dots$, using previously known bound on the dimension of the poset whose vertices are the 1- and $(2k - 2)$ -subsets of $[n]$, and the relation is inclusion.

Studying some simple graphs, such as trees, is a natural starting point in investigating the behavior of any graph parameter. It is a well-known fact that if P is a tree-like poset, i.e., its covering graph is a tree, then $\dim(P) = 2$ [76]. The situation is more complicated in our case.

The dimension of trees was first investigated by Poljak, and Pultr in [59] and later by Alles in [2]. In both papers, the lower bounds use Theorem 63, proved by Lovász, Nešetřil, and Pultr in [52].

Theorem 63 ([52]). *If x_1, \dots, x_k are distinct elements of $V(G)$ such that for some $y_1, \dots, y_k \in V(G)$, $\{x_i, y_i\} \in E(G)$ for all i , and $\{x_i, y_j\} \notin E(G)$ for $i < j$, then $\dim G \geq \log_2 k$.*

When applied to trees, this yields $\dim(T) \geq \log_2(n - l + 1)$, where n is the number of vertices and l is the number of leaves. In Section 4.3 we will improve this bound by proving a stronger version of Theorem 63.

The best upper bounds were proved in [2]. For a fixed tree T , let x be one of the vertices in its center. Define $S_i = \{y : d(x, y) = i\}$ for $i > 0$, and put $d_i = \max\{\deg(y) : y \in S_i\}$ if $S_i \neq \emptyset$, and $d_i = 1$ otherwise. That is, d_i is the maximum degree in the ball of radius i around one of the vertices in the center of T .

Theorem 64 ([2]). *If T is a tree, then*

$$\dim(T) \leq \lceil \log_2 \text{diam } T \rceil - 1 + \lceil \log_2 d_0 \rceil + \sum_{i \geq 1} \lceil \log_2 (d_i - 1) \rceil.$$

Instead of the balls S_i around the center, one can use “generalized balls”, and get a better upper bound. Define $S'_0(x) = \{x\}$. If $S'_0(x), \dots, S'_{k-1}(x)$ are already defined, let $S'_k(x)$ be the set of $y \in V(T)$ such that $\deg(y) > 2$, and if there is a vertex $z \in V(T)$ with $d(x, z) < d(x, y)$, then either $\deg(z) = 2$, or $z \in S'_j(x)$ for some $j < k$. For $i \geq 0$, let $d'_i = \max\{\deg(y) : y \in S'_i(x)\}$ if $S'_i(x) \neq \emptyset$, and 1 otherwise.

Theorem 65 ([2]). *If T is a tree, then*

$$\dim(T) \leq \lceil \log_2 \text{diam}(T) \rceil - 1 + \lceil \log_2 d'_0 \rceil + \sum_{i \geq 1} \lceil \log_2 (d'_i - 1) \rceil.$$

In Section 4.4 we will provide another upper bound which is significantly better for some trees.

4.3 Lower bound for trees, using linear algebra

For trees, we can somewhat relax the condition $\{x_i, y_j\} \notin E(G)$ for $i < j$. In particular, we prove the following improvement of Theorem 63.

Theorem 66. *Let P_1, \dots, P_m be vertex disjoint paths in a tree T . If l_i is the length of P_i and $k = \sum_{i=1}^m l_i$, then $\dim T \geq \lceil \log_2 k \rceil$.*

The proof is similar to the proof of Theorem 63 found in [52]. The matrix M is constructed in the same way, but in this case we need more work to show that it is nonsingular.

Proof. Assign the labels x_i and y_i for $1 \leq i \leq k$ to the vertices of the paths in the following way. Let $P_1 = z_1 \dots z_{l_1+1}$ and let $x_i = z_i$ and $y_i = z_{i+1}$ for $1 \leq i \leq l_1$. Continue with labeling P_2 . If $P_2 = z_{l_1+2} \dots z_{l_1+l_2+2}$, let $x_i = z_{i+1}$ and $y_i = z_{i+2}$ for $l_1+1 \leq i \leq l_1+l_2+1$. Proceed analogously for P_3 , etc. See Figure 4.2 for an example of such labeling.

Let $\dim(T) = d$. Consider a representation $\varphi_1, \dots, \varphi_d$, and for a vertex u , put $\varphi(u) = (\varphi_1(u), \dots, \varphi_d(u))$. Let $S(d)$ be the family of all subsets of $[d]$. For a vector $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{N}^d$ define vectors $\bar{\mathbf{v}}, \tilde{\mathbf{v}} \in \mathbb{N}^{S(d)}$ whose entries are indexed by elements of $S(d)$ by putting

$$\bar{\mathbf{v}}(A) = \prod_{i \in A} v_i \quad \text{and} \quad \tilde{\mathbf{v}}(A) = \prod_{i \notin A} (-v_i).$$

Let M be a matrix whose (i, j) -th entry M_{ij} is equal to $\overline{\varphi(x_i)} \cdot \widetilde{\varphi(y_j)}$, where the dot indicates the inner product. We have

$$\overline{\varphi(x_i)} \cdot \widetilde{\varphi(y_j)} = \sum_{A \subseteq [d]} \left(\prod_{\ell \in A} \varphi_\ell(x_i) \prod_{\ell \notin A} \varphi_\ell(y_j) \right) = \prod_{\ell=1}^d (\varphi_\ell(x_i) - \varphi_\ell(y_j)),$$

so $M_{ij} \neq 0$ if and only if x_i and y_j are adjacent.

Let α_i be coefficients such that $\sum_{i=1}^k \alpha_i \overline{\varphi(x_i)} = \mathbf{0}$. Then, for a fixed j ,

$$\sum_{i=1}^k \alpha_i (\overline{\varphi(x_i)} \cdot \widetilde{\varphi(y_j)}) = \left(\sum_{i=1}^k \alpha_i \overline{\varphi(x_i)} \right) \cdot \widetilde{\varphi(y_j)} = 0.$$

That is, the linear combination of the rows of M with the coefficients α_i equals $\mathbf{0}$. If M is nonsingular, then $\alpha_i = 0$ for all i and thus $\overline{\varphi(x_1)}, \dots, \overline{\varphi(x_k)}$ are linearly independent in $\mathbb{N}^{S(d)}$, and hence

$$k \leq 2^d.$$

To prove that M is nonsingular, it suffices to prove the following claim. The biadjacency matrix of a bipartite graph with parts X and Y is a the $(0, 1)$ -matrix with the rows indexed by the elements of X and the columns indexed by the elements of Y , whose (i, j) -th entry equals 1 if and only if the corresponding vertices are adjacent.

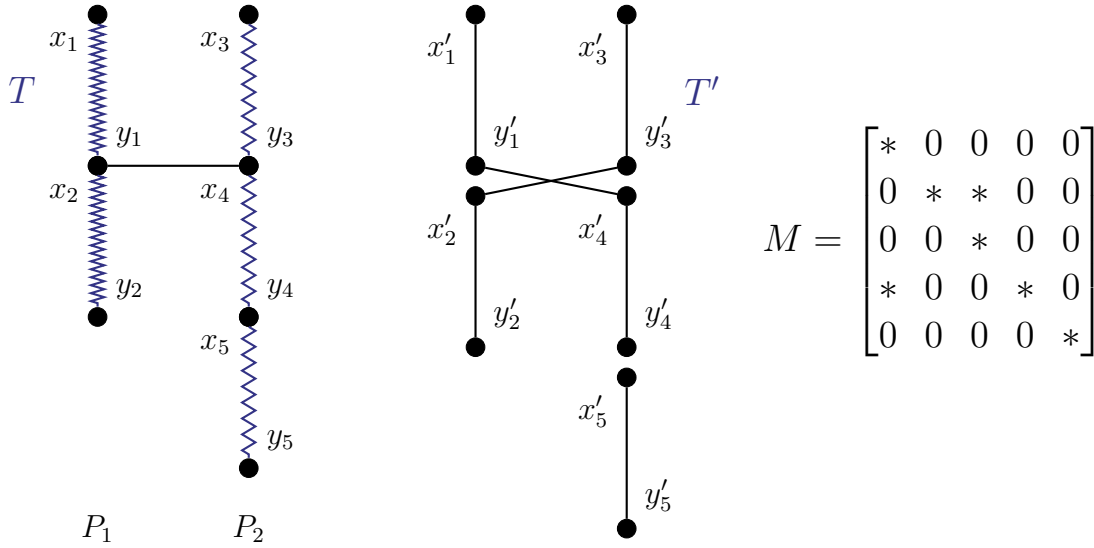


Figure 4.2: An example illustrating the proof of Theorem 66. The zigzag edges highlight the paths P_1 and P_2 . Nonzero entries in the matrix M are denoted by stars.

Claim. There is exactly one way to pick k nonzero entries of M so that each row and each column contains exactly one of them.

To prove this claim, replace the nonzero entries of M by 1's. The new matrix is the biadjacency matrix of a bipartite graph T' , with $V(T') = X \cup Y$ for $X = \{x'_1, \dots, x'_k\}$ and $Y = \{y'_1, \dots, y'_k\}$, and with $E(T') = \{x'_i y'_j : x_i y_j \in E(T)\}$. The edges of T' are in one-to-one correspondence with the nonzero positions of M . The problem is equivalent to proving that T' has exactly one perfect matching.

Suppose that T' has a cycle $D' = v_1 v_2 \dots v_s$, where $v_s = v_1$. We may choose the labels so that $v_i \in X$ for odd i , and $v_i \in Y$ for even i . Every vertex in T corresponds to at most two vertices of T' , one in each part. Let D be the subgraph of T that we obtain by identifying the pairs of vertices of D' that correspond to a single vertex in T . Since i and j must have different parity whenever we identify v_i and v_j , D cannot be a tree, a contradiction.

It follows that T' is a forest. It has at least one perfect matching – take the edges that correspond to the elements on the diagonal of M . On the other hand, if a forest has a perfect matching, it is unique. \square

Example 67. Consider the tree T depicted in Figure 4.2. By Theorem 66, $\dim(T) \geq \lceil \log_2 5 \rceil = 3$, but Theorem 63 only tells us that $\dim(T) \geq \log_2 4 = 2$.

4.4 Upper bound for trees, using reflections

Let G_1 be a subgraph of H , and let G_2 be the graph induced by $V(H) \setminus V(G_1)$. Let v_1, \dots, v_t be the vertices of G_2 . Create a new graph H' by taking another copy of G_2 and attaching it to G_1 the same way as the original G_2 is attached. Formally, $V(H') = V(H) \cup W$, where $W = \{w_1, \dots, w_t\}$ is a set of vertices disjoint from $V(H)$, and the edges of H' are given by the following:

- H is an induced subgraph of H' ,
- $w_i w_j \in E(H')$ whenever $v_i, v_j \in V(G_2)$ and $v_i v_j \in E(H)$, and
- $w_i u \in E(H')$ whenever $v_i \in V(G_2)$, $u \in V(G_1)$, and $v_i u \in E(H)$.

We will call the resulting graph H' the *reflection* of H around G_1 .

The following lemma is a generalization of the construction used in [52] to provide an upper bound for the dimension of paths. The upper bound from [52] is obtained by repeated reflections with G_1 being an endpoint of the path.

Lemma 68. *Let G_1 be a subgraph of H such that the graph $(V(H), E(H) \setminus E(G_1))$ is a forest. If H' is the reflection of H around G_1 , then $\dim(H') \leq \dim(H) + 1$.*

Proof. We use the notation introduced in the first paragraph of this section. Find an encoding of H . Assign the code of the corresponding v_i to each w_i , and add an extra coordinate to all codes: for w_i , append the code by 0 if the distance of w_i from G_1 is even, and by 1 otherwise. For v_i , add 1 if the distance is even, and 0 otherwise. For the vertices u_i , put distinct numbers other than 0 and 1 at the end of each code. This is a valid encoding of H' . \square

Theorem 69 ([52]). *If P is a path of length k , then $\dim(P) = \lceil \log_2 k \rceil$.*

Proof. The encoding of the path of length 4 by vectors with two coordinates is given in Figure 4.1. Reflect it around its endpoints and use Lemma 68 repeatedly until we

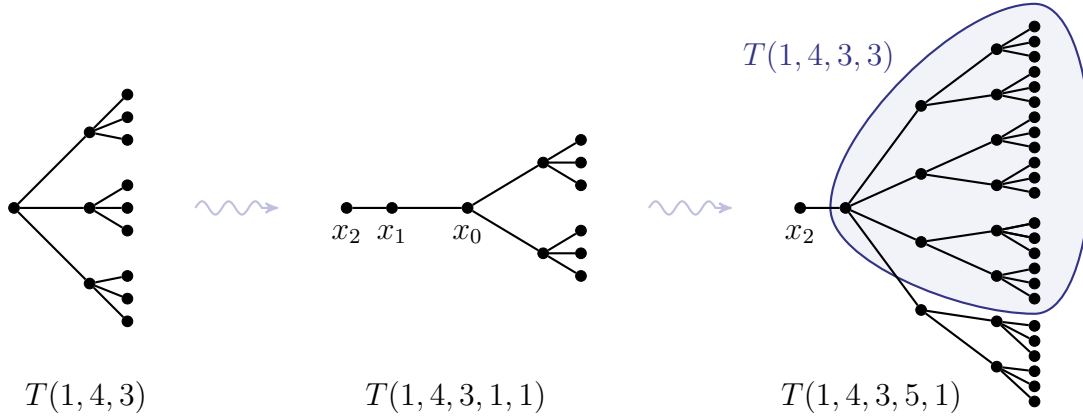


Figure 4.3: An example of the process described in the proof of Lemma 71. To obtain an encoding of $T(1, 4, 3, 3)$, start with an encoding of $T(1, 4, 3)$ and use two reflections around the path x_1x_2 to obtain an encoding of $T(1, 4, 3, 5, 1)$. This graph has $T(1, 4, 3, 3)$ as an induced subgraph.

have a path of the required length. This proves the upper bound. The lower bound is an easy consequence of Theorem 66. \square

For a tree T , let x be one of the vertices in its center. Let r be the radius of T . For $0 \leq i \leq r$, let δ_i be the maximum degree among all vertices u such that $d(u, x) = r - i$. That is, for d_j defined in Section 4.2, $\delta_i = d_{r-i}$. Also, let $S = \{2^i : i \in \mathbb{N}\}$.

Theorem 70. $\dim(T) \leq 2 + \lceil \log_2(\delta_r) \rceil + \sum_{i \in S, 2 \leq i < r} \lceil \log_2(\delta_i) \rceil + \sum_{i \notin S, 3 \leq i < r} \lceil \log_2(\delta_i - 1) \rceil$.

If G is an induced subgraph of a graph H , then $\dim(G) \leq \dim(H)$. We will use this trivial but useful fact multiple times, sometimes without referring to it explicitly.

In particular, it suffices to prove Lemma 71 in order to prove Theorem 70. In the following paragraphs, $T(\delta_0, \delta_1, \dots, \delta_r)$ is a rooted tree with the root x , all other vertices in distance at most r from x , and with all vertices in distance $r - i$ from x having degree δ_i (hence in particular, $\delta_0 = 1$).

Lemma 71. *If $r \in \mathbb{N}$, and $\delta_0, \delta_1, \dots, \delta_r$ is a list of positive integers with $\delta_0 = 1$ and $\delta_1, \dots, \delta_{r-1} \neq 1$, then*

$$\dim(T(\delta_0, \delta_1, \dots, \delta_r)) \leq 2 + \lceil \log_2(\delta_r) \rceil + \sum_{i \in S, 2 \leq i < r} \lceil \log_2(\delta_i) \rceil + \sum_{i \notin S, 3 \leq i < r} \lceil \log_2(\delta_i - 1) \rceil.$$

Proof. We will use induction on r . For $r = 1$, the tree in question is isomorphic to the star K_{1,δ_1} . We can encode it with two colorings.

Let s be the largest power of 2 such that $s < r$. By induction hypothesis, we have an encoding of $T(\delta_0, \delta_1, \dots, \delta_s)$ with $\sum_{j \in S, j \leq s} \lceil \log_2(\delta_j) \rceil + \sum_{j \notin S, 3 \leq j \leq s} \lceil \log_2(\delta_j - 1) \rceil + 2$ coordinates. The center of $T(\delta_0, \delta_1, \dots, \delta_s)$ consists of one vertex; call it x_0 . Let $x_0 \dots x_s$ a path from x_0 to one of the leaves. Delete all the vertices $y \notin \{x_1, \dots, x_s\}$ for which the path from y to x_0 goes through the vertex x_1 . Now we have a tree T' with center $\{x_0\}$, one path of length s starting at x_0 , and $\delta_s - 1$ pairwise isomorphic subtrees incident to x_0 . Using our notation and designating x_s as the root, we have $T' = T(\delta_0, \dots, \delta_s, 1, 1, \dots, 1)$, where the number of the 1's following δ_s is s .

We will successively reflect the tree around portions of the fixed path $x_0 \dots x_s$ to encode first $T(\delta_0, \dots, \delta_s, \delta_{s+1}, 1, 1, \dots, 1)$, then $T(\delta_0, \dots, \delta_s, \delta_{s+1}, \delta_{s+2}, 1, \dots, 1)$, etc.

First reflect T' around the subpath $x_1 \dots x_s$. This increases the degree of x_1 to 3. A total number of $\lceil \log_2(\delta_{s+1} - 1) \rceil$ reflections suffices to make the degree of x_1 greater than or equal to δ_{s+1} , and the same number of additional colorings suffices to color the resulting tree, by Lemma 68. Let T'' be the resulting tree, $T'' = T(\delta_0, \dots, \delta_s, \delta_{s+1}, 1, \dots, 1)$, where the number of 1's at the end is $s-1$. Use $\lceil \log_2(\delta_{s+2} - 1) \rceil$ reflections to get an encoding of $T(\delta_0, \dots, \delta_s, \delta_{s+1}, \delta_{s+2}, 1, \dots, 1)$. Continue in a similar fashion until we have an encoding of $T(\delta_0, \dots, \delta_{r-1}, 1, \dots, 1)$.

In the last step, we again reflect the graph sufficiently many times to increase the degree of the appropriate vertex (in this case, it is the vertex x defined above), and then delete the remaining vertices of the path $x_0 \dots x_s$ that still have degrees 1 or 2 (if there are any left). This might decrease the degree of x by 1, so in this step we need $\lceil \log_2(\delta_r) \rceil$ reflections. \square

This bound is at least as good as the one given by Theorem 64 whenever $\delta_1 > 2$, or for some $i \in S$, $\delta_i \notin \{2^j + 1 : j \in \mathbb{N}\}$. The bounds in Theorems 64 and 65 are the same whenever T has no vertices of degree 2. So in particular, if T has no vertices of degree 2, the bound from Theorem 70 is at least as good as both aforementioned bounds.

In many cases, the improvement is as large as $\lceil \log_2 r \rceil$ (where r is again the radius of T), e.g., for the balanced 4-regular tree $T(4, 4, \dots, 4)$.

4.5 Exact values

In [52], Theorem 63 was used together with the upper bound obtained by multiple reflections around endpoints to determine the dimension of paths (see Theorem 69). These methods can be used to determine the dimension of many other graphs as well.

As an example of a family where we can successfully apply the same ideas, let $R_2 = \{P_5\}$, where P_5 is the path of length 4. For $k > 2$, the set R_k will be defined inductively as the set of all trees T that are not stars and that can be produced by reflecting some $T' \in R_{k-1}$ around one of its vertices.

Theorem 72. *If $T \in R_k$, then $\dim(T) = k$.*

Proof. Let $n(G)$ and $\ell(G)$ be the number of vertices and the number of leaves of G respectively. Let $c_k = \min\{n(G) - \ell(G) + 1 : G \in R_k\}$. We have $c_k \geq 2c_{k-1} - 2 = 2^{k-1} + 2$. By Theorem 63, $\dim(G) \geq \lceil \log c_k \rceil = k$. The upper bound follows from Lemma 68. \square

For other trees, Theorem 63 is not powerful enough to determine the exact values of product dimension, but Theorem 66 is. For an example of such family, fix a $k \geq 2$ and take two vertex-disjoint paths P_1 and P_2 of lengths s_1 and s_2 respectively, so that $s_1 \geq 2$ and $s_2 \geq 2$, and $s_1 + s_2 = 2^k + 1$. Select internal vertices $u \in P_1$ and $v \in P_2$ and connect them with an edge. Let D_k be the family of all the graphs that we can obtain in this way.

Proposition 73. *If $T \in D_k$, then $\dim(T) = k + 1$.*

Proof. The vertex u divides P_1 in two portions. Let $u_1u_2 \dots u$ be the longer one. Similarly, let $v_1v_2 \dots v$ be the longer portion of P_2 . The path $u_1u_2 \dots uv \dots v_2v_1$ has length at most 2^k , so we can encode it with k colorings, by Theorem 69. If we reflect it around the edge uv , the resulting graph has dimension at most $k + 1$, by Lemma 68, and has T as an induced subgraph.

The lower bound follows from Theorem 66. \square

4.6 Odd dimension

Let us finish the chapter with a slight detour to another related parameter, the *odd dimension*.

Definition 74. An *odd representation* is an assignment of sets to the vertices of G such that $uv \in E(G)$ for $u \neq v$ if and only if the corresponding sets have an odd-sized intersection. The *odd dimension* $\theta_{\text{odd}}(G)$ of a graph G is the minimum t such that there exists an odd representation of G with subsets of $[t]$.

It is convenient to consider the characteristic vectors of the sets instead of the sets themselves. An odd representation is then an assignment of binary vectors $\varphi(x)$ to the vertices x in such a way that $\varphi(x) \cdot \varphi(y) = 1$ if and only if $xy \in E(G)$. Here the dot denotes the dot product taken modulo 2. Unlike product representation, we do not require that the vectors be unique.

This parameter was defined in [21] by Eaton and Grable, who proved that

$$n - \sqrt{2n} - \lceil \log n \rceil < \theta_{\text{odd}}(G) \leq n - 1$$

holds for almost all graphs, while the upper bound holds for all graphs.

Both Theorem 63 and 66 can be adapted for odd dimension as well.

Theorem 75. If x_1, \dots, x_k are distinct vertices of G and y_1, \dots, y_k are some vertices such that $x_i y_i \in E(G)$ for all i , $x_i y_j \notin E(G)$ for $i < j$, and moreover $x_i \neq y_j$ for $i < j$, then $\theta_{\text{odd}}(G) \geq k$.

Proof. Suppose that $\theta_{\text{odd}}(G) = t$, and for every vertex x , let $\varphi(x) = (\varphi_1(x), \dots, \varphi_t(x))$ be the characteristic vector of the odd representation. Let M be the matrix defined by $M_{i,j} = \varphi(x_i) \cdot \varphi(y_j)$, where the dot denotes the dot product modulo 2. We have $M_{i,j} = 1$ if and only if either $x_i y_j \in E(G)$, or $x_i = y_j$ and $\varphi(x_i)$ has an odd number of 1's.

We will show that the vectors $\varphi(x_i)$ are linearly independent. As a consequence, we will get $k \leq t$. Let α_i be coefficients such that $\sum \alpha_i \varphi(x_i) = \mathbf{0}$. For any fixed j , we have

$$0 = \left(\sum_i \alpha_i \varphi(x_i) \right) \cdot \varphi(y_j) = \sum_i \alpha_i (\varphi(x_i) \cdot \varphi(y_j)).$$

That is, $\sum_i \alpha_i M_{i,j} = 0$ for any fixed j . But M is lower diagonal, and hence its rows are linearly independent. It follows that $\alpha_i = 0$ for all i . \square

Corollary 76. *For every k , $\theta_{\text{odd}}(P_{k+1}) = k$. For the matching kK_2 of size k , we have $\theta_{\text{odd}}(kK_2) = k$.*

Proof. The statement about paths follows from Theorem 75 together with the fact that $\theta_{\text{odd}}(G) \leq n - 1$ for all graphs.

The lower bound for the matchings follows from Theorem 75 by labeling one endpoint of the i -th edge of the matching x_i and the other endpoint y_i . For the upper bound, assign the set $\{i\}$ to both endpoints of the i -th edge, for all i . \square

Theorem 77. *If T is a tree and P_1, \dots, P_k are vertex disjoint paths in T of lengths s_1, \dots, s_k respectively, then $\theta_{\text{odd}}(T) \geq \sum s_i$.*

Proof. Let $s = \sum s_i$. Label the paths with the labels x_1, \dots, x_s and y_1, \dots, y_s like in the proof of Theorem 66. Define a matrix M by $M_{i,j} = \varphi(x_i) \cdot \varphi(y_j)$ (again, the dot product is modulo 2). Like in the proof of Theorem 66, we consider the bipartite graph T' defined by the biadjacency matrix M , and wish to prove that T' has only one perfect matching. We have to be a little more careful here, since it is well possible that x_i and y_j are two labels for the same vertex x , and $\varphi(x)$ has an odd weight. In that case the corresponding vertices x'_i and y'_j of the bipartite graph are adjacent. It is therefore no longer true that T' is a forest. We will prove that there is only one perfect matching, nevertheless.

We will proceed by induction on s . The subgraph of T induced by all vertices that have been labeled (by some x_i , or y_j , or both) is a forest. Find a leaf u in this forest. Because of the way we assigned the labels, u has only one label assigned to it, say x_i (the situation is completely analogous if it is y_j for some j). Any perfect matching then contains the edge $x_i y_i$, and it does not contain any other edge incident with y_i . We can therefore delete the i -th row and column of M . By induction hypothesis, there is only one perfect matching on the rest of T' . \square

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