

© 2010 Jingjin Yu

RENDEZVOUS OF MULTIPLE DUBINS CAR AGENTS WITH  
MINIMAL SENSING AND CONTROL REQUIREMENTS

BY

JINGJIN YU

THESIS

Submitted in partial fulfillment of the requirements  
for the degree of Master of Science in Computer Science  
in the Graduate College of the  
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Adviser:

Professor Steven M. LaValle

# ABSTRACT

We study minimalism in sensing and control by considering a multi-agent system in which each agent moves like a Dubins car and has a limited sensor that reports only the presence of another agent within some sector of its windshield. Using a very simple quantized control law with three values, each agent tracks another agent assigned to it by maintaining that agent within this windshield sector. We use Lyapunov analysis to show that by acting autonomously in this way, the agents will achieve rendezvous given a connected initial assignment graph and a merge assumption. We then proceed to show that, with a slightly different control law, an initial assignment is not required and the sensing model can be weakened further. A distinguishing feature of our approach is that it does not involve any estimation procedure aimed at reconstructing coordinate information. Our scenario thus appears to be the first example in which an interesting task is performed with extremely coarse sensing and control, and without state estimation. The system was implemented in computer simulation, accessible through the Web, of which the results are presented in the thesis.

*To Yunxia and Mandy.*

# ACKNOWLEDGMENTS

This result presented here would not have been possible without the support of many people. Many thanks to my adviser, Steven M. LaValle, whose sole effort upon his students is to make them true researchers in every aspect (and paying them for doing that). I would also like to thank Daniel Liberzon, who is a great teacher, colleague and in many ways a co-advisor. My appreciation also goes to the Department of Computer Science and the Siebel Scholars Foundation for offering a Siebel Scholar Fellowship that partly supported me with the financial means to work on this project. And finally, thanks to my family, group members, and numerous other friends who endured this long process with me, always offering support and love.

# TABLE OF CONTENTS

LIST OF FIGURES . . . . .	vi
CHAPTER 1 INTRODUCTION . . . . .	1
CHAPTER 2 PROBLEM FORMULATION . . . . .	6
2.1 Vehicle Model . . . . .	6
2.2 Sensing Model And Control Law . . . . .	7
2.3 Possible Sensor Implementations . . . . .	8
2.4 Merging . . . . .	9
CHAPTER 3 THE ASSIGNMENT GRAPH AND LIVENESS CON- DITION . . . . .	10
CHAPTER 4 GUARANTEED RENDEZVOUS OF AGENTS . . . . .	13
4.1 A Candidate Lyapunov Function . . . . .	13
4.2 The Cyclic Case . . . . .	14
4.3 The Intree Case . . . . .	25
4.4 Unit Speed Cyclic Pursuit With Branches . . . . .	27
4.5 Direct Cyclic Pursuit . . . . .	28
CHAPTER 5 CONDITION ON ANGULAR VELOCITY AND EXTENSIONS . . . . .	30
5.1 Angular Velocity Condition . . . . .	30
5.2 Rendezvous Without Connectivity Or Distinguishability . . . . .	31
CHAPTER 6 IMPLEMENTATION AND SIMULATION RESULTS . . . . .	34
CHAPTER 7 CONCLUSION AND FUTURE DIRECTIONS . . . . .	37
REFERENCES . . . . .	39

# LIST OF FIGURES

1.1	Screen captures from simulation program of a seven agent cyclic pursuit with one agent running after another. a) - d) Four snapshots of the rendezvous process. e) Plot of the Lyapunov function over time during the rendezvous process. The seven lower valued curves are individual distances between agents and their targets; the higher valued curve is the sum of the seven distances (the Lyapunov function), with $a-d$ on the plot corresponding to the times when the four snapshots are taken. . . . .	4
2.1	a) An agent's vehicle. b) The agent's windshield, or field-of-view . . . . .	6
2.2	Three possible implementations of the sensing model. . . . .	9
3.1	An assignment graph, $G$ , with $n$ edges of which the underlying undirected graph is connected and contains a cycle (polygon) that encloses the shaded area. . . . .	11
4.1	Two Dubins car agents in a cyclic pursuit. . . . .	14
4.2	A regular 9-gon with agents on the vertices and moving along the lines tangent to the enclosing circle of the 9-gon. . .	16
4.3	$\Theta_{in}, \Theta_{out}$ and the slices cut by the $g(\Theta) - (n - 2)\pi = 0$ hyperplane, for the case of $n = 3$ and $\sum \phi_i > \pi/2$ . . . . .	20
4.4	(a) $\theta_i$ 's and $(\pi - \theta_i)$ 's for a self-intersecting polygon, as marked by the solid arcs and dashed arcs, respectively. The arrow marks the walker's walking direction and starting edge. (b) $\theta_i$ 's can be made arbitrarily small by "pressing" this polygon from left and right. . . . .	21
4.5	Bounded speed intree pursuit may not have $\dot{V} < 0$ at all times.	26
5.1	Dubins car agent $i$ that just loses its target $j$ at point $A$ . . . .	30

5.2	Agent at $A$ turns along a circle with center $O$ of radius $r$ , leaving the small solid circle as its trace. The two large dotted circles are of distance $\rho$ to the agent at $A$ and $A'$ . The shaded disc denotes the intersection of all discs with radius $\rho$ from the agent as it turns a full cycle. . . . .	32
6.1	a) Eight-agent cyclic pursuit, which converges, with $\phi = 0.35$ and $\omega = 2.0$ b) Plot of the Lyapunov function for (a), with numbered lines being individual $\ell_{i,i+1}$ terms. c) Diverging case with $\phi$ changed to 0.6 holding other conditions the same. The agents arrange themselves on a cycle quickly as they spin away from each other. Dots are the agents' positions as of the last snapshot. d) Plot of the Lyapunov function for (c). e) Eight-agent converging pursuit with agents 0, 4, 3 and 5 on a cycle; the branches are: $6 \rightarrow 2 \rightarrow 0, 7 \rightarrow 3, 1 \rightarrow 5$ . f) Plot of the Lyapunov function for (e). . . . .	35



# CHAPTER 1

## INTRODUCTION

A generally desirable feature in autonomous systems is to perform tasks with minimal information. For feedback systems, such information is captured in the form of sensing data and control law: a system completes a loop by taking in sensing data, estimating its state, making a control decision, and executing the control. More than often, however, it is the case that information being minimized is limited to only sensing or only control. For example, quantized control, as a branch of control theory, seeks to minimize the communication bandwidth between the plant and the controller. In doing so, full state estimation is usually assumed. On the other hand, sacrificing good state estimation usually demands more precise control laws. The seemingly inescapable trade-off between sensing and control capabilities prompts the question: Is any interesting task achievable with both coarse sensing and quantized control?

In this thesis we give a “yes” answer to the above question through investigating a multi-agent rendezvous problem with emphasis on minimalism in both sensing and control. The agents are Dubins car vehicles [9, 21] equipped with range sensors of limited field-of-view and bounded range. We show that the vehicles, each operating independently under a three-level feedback quantized control law, will rendezvous without ever obtaining coordinate data or state estimation. Initially we assume that: 1. Every agent is assigned a target agent or has no target, 2. An agent can track its target with its sensor in the presence of occlusion, 3. An agent and its target “merge” when they are sufficiently close by, and 4. An agent without target or two mutually merged agents will stop moving. Upon establishing that identical agents can be guaranteed to rendezvous in finite time, similar results are obtained for agents with different forward velocities. We then continue to show that,

with a slightly stronger switched control law, assumptions 1, 2, and 4 are not unnecessary for identical agents.

The rendezvous problem in the control context catches our attention as an actively pursued problem during the past decade. An early formulation and algorithmic solution of the multi-agent rendezvous problem is introduced in [1], in which agents have limited range sensing capabilities. *Stop-and-go* strategies extending the algorithm in [1] are proposed in [23] and [24], which cover various synchronous and asynchronous formulations. Later, an  $n$ -dimensional rendezvous problem was approached via proximity graphs in [6]. A research area that is closely related to rendezvous is *cyclic pursuit* or the *n-bug problem*, in which each agent pursues on an directed cycle its reachable neighboring agent. The mathematical study of pursuit curves and pursuit polygons with differential constraints originated this line of research [2, 5], with the focus being whether, when, where, and how the participating agents meet each other. Stable pursuit polygons in cyclic pursuit are also investigated in [35]. In the past few years, due partly to the realization that state estimation (sensing and computation) and control are both key but distinct components in practical systems, cyclic pursuit problems with feedback control have been further explored [26, 29, 36, 37], with [29] giving a history and review on cyclic pursuit.

In control theory, the problem of controlling a plant using coarse quantized measurements of its state (or output) has received much attention in recent years. Quantized control, an active branch of control theory, focuses particularly on minimal data rate control laws. The motivation for studying such control problems comes from situations in which the rate of information flow between the plant and the controller has to be minimized due to communication bandwidth constraints, shared network resources, security concerns, or other considerations. For some classes of systems, most notably linear ones, precise conditions have been obtained on how much information is needed for control; see, e.g., [4, 8, 10, 14, 15, 22, 31, 33, 38, 41]. However, in minimizing data rate, quantized control bases the control decisions on estimation of state coordinates, which requires significant computational resources as well as sophisticated analysis tools.

Minimalism also appears in robotics research that vies for simple abstract

sensors. Some earlier efforts are *bug algorithms* for navigation in environments with obstacles [20, 28] and algorithms for manipulating convex polygonal parts using sensorless robots [11, 13]. Recently, *gap navigation trees* for optimal navigation were introduced in [39]. In assuming weak sensors, these work typically equip the robots with sophisticated control laws. For example, [39] requires the robot to reliably move toward depth-map discontinuities.

In contrast to the afore mentioned studies, our work has the distinguishing feature that we engage quantized control without assuming the availability of perfect coordinate information within sensing range. The main contributions of this work are: 1) This thread of research bears fundamental importance as it promotes the understanding of the least amount of information or data rate required for a given task, which in turn offers better insights into the task’s inherent complexity. 2) From a practical standpoint, minimalism in sensing or control immediately leads to less complicated system design, improved robustness, lower production cost, and reduced energy consumption. 3) When physical size of agents is constrained, for instance in swarm robotics, sensing and computation become very limited, requiring a minimal design.

Moreover, a particular interesting element of our analysis is the Lyapunov function that we use, which assumes an unusual linear form (in distance) that simplifies the analysis. Lyapunov analysis has also been applied to study autonomous group coordination over graphs in [16], but that thesis’s approach requires the agents’ full awareness of their local environments. Besides [16], distributed consensus has also been the focus of [30, 34, 12, 7]. Our study is partly inspired by work on planning for a differential drive with a limited field-of-view [3], which does not, however, consider minimalism. This thesis extends the results found in [19]. In particular, finer rendezvous guarantee is achieved in some cases, sometimes with relaxed requirements on vehicle model and/or sensing capability. The results presented in the thesis are also demonstrated with simulation (see Fig. 1.1 and Section 6). To the best of our knowledge, there has been no attempt to reduce sensing and control, to the extremely limited combination considered in our thesis, to complete interesting tasks, such as rendezvous.

The thesis is organized as follows. Chapter 2 gives a precise formulation of the rendezvous problem. Chapter 3 introduces a graph structure with a

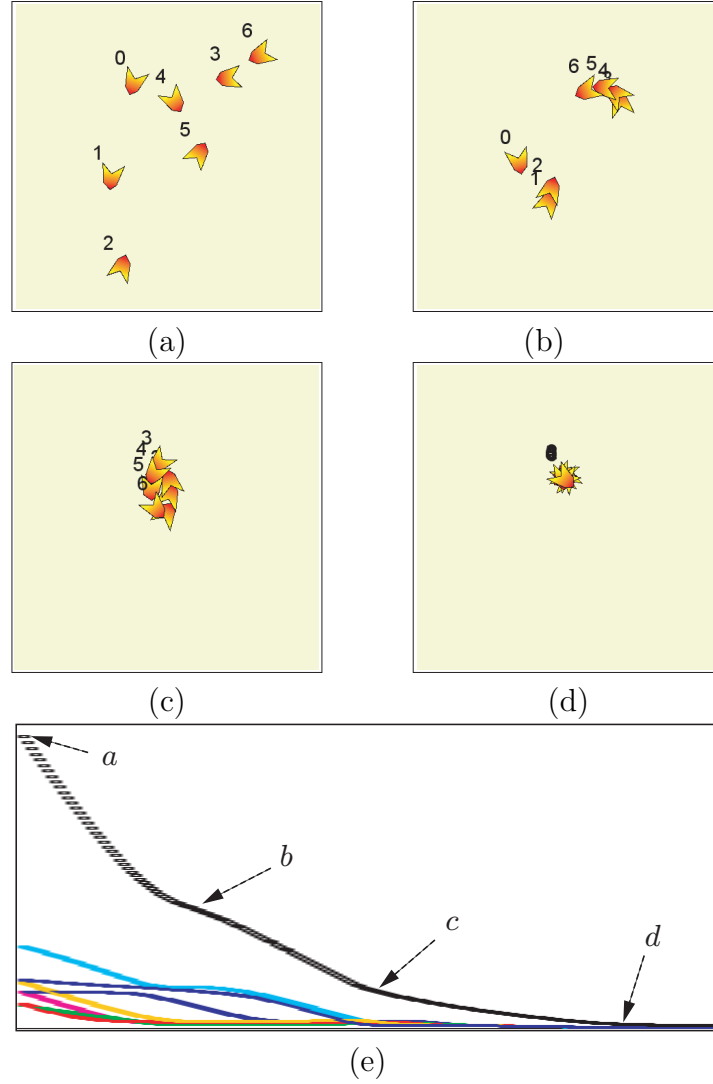


Figure 1.1: Screen captures from simulation program of a seven agent cyclic pursuit with one agent running after another. a) - d) Four snapshots of the rendezvous process. e) Plot of the Lyapunov function over time during the rendezvous process. The seven lower valued curves are individual distances between agents and their targets; the higher valued curve is the sum of the seven distances (the Lyapunov function), with  $a$ - $d$  on the plot corresponding to the times when the four snapshots are taken.

connectivity condition as the first ingredient of the sufficient conditions for rendezvous. Chapter 4 define the candidate Lyapunov function and give the second ingredient of the sufficient conditions, the windshield angle requirement, based on whether the agents are moving at the same constant speed or bounded, varying speeds. The remaining condition on the angular velocity requirement and several extensions are addressed in chapter 5. Simulation results are presented in chapter 6, after which we conclude with chapter 7.

# CHAPTER 2

## PROBLEM FORMULATION

### 2.1 Vehicle Model

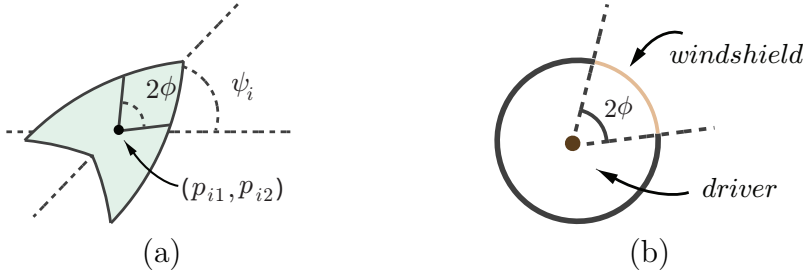


Figure 2.1: a) An agent's vehicle. b) The agent's windshield, or field-of-view

Consider a set of  $n$  *agents*, in which agent  $i$  is a point vehicle located at  $p_i = (p_{i1}, p_{i2})$  in the plane with orientation  $\psi_i$  (see Fig. 2.1(a)). Each vehicle moves as a Dubins car:

$$\dot{p}_{i1} = v_i \cos \psi_i, \quad \dot{p}_{i2} = v_i \sin \psi_i, \quad \dot{\psi}_i = u_i, \quad (2.1)$$

in which  $v_i$  is the forward speed, and the only control is  $u_i \in \{-\omega_i, 0, \omega_i\}$  for some fixed  $\omega_i > 0$ . For fixed  $v_i$ , such a vehicle will either move along a straight line ( $u_i = 0$ ) or turn clockwise/counterclockwise ( $u_i = \pm\omega_i$ ) along a circle with fixed radius. We use this fact later without further elaboration. Let  $X = SE(2)^n$  denote the *state space*, in which  $x \in X$  yields the position and orientation of all agents. Some of our results require that the agents are *identical*, which means  $v_i = v_j$  and  $\omega_i = \omega_j$  for all pairs,  $i, j$ , of agents.

## 2.2 Sensing Model And Control Law

The vehicle range sensor has a limited angular field-of-view, centered at  $\psi_i$  with a span  $(-\phi, \phi)$  for some given  $\phi \in (0, \pi]$ , which is the same and fixed for all agent in a system (see Fig. 2.1(b)). By imagining that one sits in the driver's seat, the field-of-view can be considered as a *windshield*. The sensor's sensing range is bounded as established in Proposition 12. Initially, we also assume that the sensor can follow a target in the windshield: an agent cannot occlude another in terms of the sensor view. This assumption will be lifted as we reach chapter 5.

In the previous chapter we mentioned that agents will try to maintain their targets in the windshield; this appears to require an initial condition in which each agent has in the windshield its target. Assume such an initial condition for the moment; we later (in chapter 5) show that this requirement is not necessary. For an agent  $i$ , let its target  $j$  initially reside in the  $(-\phi + \epsilon_\phi, \phi - \epsilon_\phi)$  sector of  $i$ 's windshield, with  $\epsilon_\phi$  be a tiny angle satisfying  $0 < \epsilon_\phi \ll \phi$ . The introduction of  $\epsilon_\phi$  provides a way to maintain  $j$  in  $i$ 's windshield of span  $(-\phi, \phi)$ : agent  $i$  can notice when  $j$  is about to disappear from  $i$ 's windshield and start turning to position  $j$  towards the center of  $i$ 's windshield. As long as  $i$  can turn fast enough,  $j$  will not leave  $i$ 's windshield. If  $i$  has a single target  $j$ , the observation space can be restricted as  $Y_i = \{-1, 0, 1\}$  and an observation  $y_i$  for agent  $i$  is obtained as

$$y_i = \begin{cases} 1 & \text{agent } j \text{ appears in } i\text{'s } [\phi - \epsilon_\phi, \phi) \text{ sector,} \\ 0 & \text{agent } j \text{ contained in } i\text{'s } (-\phi + \epsilon_\phi, \phi - \epsilon_\phi) \text{ sector,} \\ -1 & \text{agent } j \text{ appears in } i\text{'s } (-\phi, -\phi + \epsilon_\phi] \text{ sector,} \end{cases} \quad (2.2)$$

which defines a simple instantaneous mapping  $h : X \rightarrow Y_i$ . While the mapping given by (2.2) is a partial one, the part of  $X$  on which  $h$  is undefined is safe to ignore because of our temporary assumption that agents can keep targets in their windshield span. For each agent  $i$ , the sensor does not provide metric information, but instead indicates one of three simple quantized states with respect to some agent  $j$  and the windshield. Our sensor-feedback

control law  $k_i : Y_i \rightarrow U_i$  is then defined simply as

$$u_i = -\omega_i y_i. \quad (2.3)$$

Additionally, if an agent has no target, it does not move.

## 2.3 Possible Sensor Implementations

To implement the sensor based strictly on the instantaneous mapping, we may take continuous video of the entire  $(-\phi, \phi)$  windshield sector with markers at  $-\phi + \epsilon_\phi$  and  $\phi - \epsilon_\phi$ . It is then possible to directly locate agent  $j$  in the windshield from each snapshot. This implementation also enables  $i$  to track  $j$  via  $j$ 's continuous motion trajectory since  $j$  is always in the video sequences, even if all agents are indistinguishable based on appearance. Moreover, a sensor in reality does not need to decide  $\epsilon_\phi$ : a physical vehicle has an actual size hence will not disappear from the windshield at a single instant in time. Suppose agent  $j$  starts to move outside of  $i$ 's field-of-view;  $j$  will actually “stay” on  $i$ 's windshield boundary for some  $\Delta t$  time before it disappears from  $i$ 's view, which automatically provides  $\epsilon_\phi$  for agent  $i$ . This sensor implementation is illustrated in Fig. 2.2 (a): arc  $A_l, A_c, A_r$  corresponds to the  $(-\phi, -\phi + \epsilon_\phi), (-\phi + \epsilon_\phi, \phi - \epsilon_\phi), [\phi - \epsilon_\phi, \phi)$  sectors of the windshield, respectively. In fact, the left boundary of  $A_l$  and the right boundary of  $A_r$  in the sensor implementation are not essential; they can be extended and in the extreme case meet at the back of the vehicle (point  $B$  in Fig. 2.2(b)). In this case, another instantaneous sensor is obtained, with its  $2\pi$  angular field-of-view partitioned into three slices.

Alternatively, the sensor may be implemented as simple beam detectors placed at the two windshield boundaries,  $-\phi + \epsilon_\phi$  and  $\phi - \epsilon_\phi$ ; we may simply take  $\epsilon_\phi$  to be a fraction of  $\phi$  (see Fig. 2.2(c)). Except at the two boundaries, the agent is otherwise “blind”. Such a sensor, even simpler than previous ones, generates time based events and thus needs to have some immediate history of which sector the target is located in to produce the output. It also needs to have some additional mechanism to distinguish the agents that pass the beam detector, since the sensor no longer continuously tracks a target



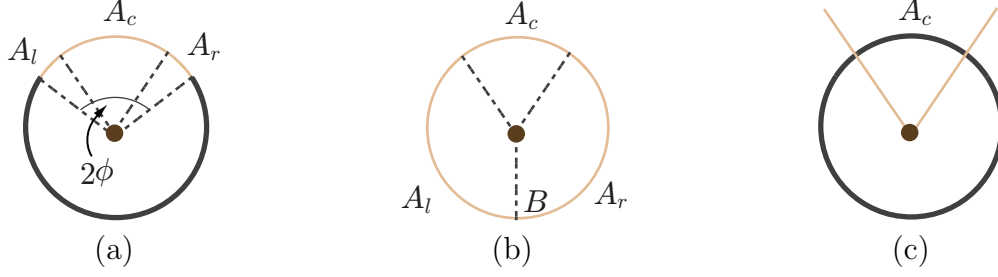


Figure 2.2: Three possible implementations of the sensing model.

agent.

## 2.4 Merging

Since the Dubins car vehicle model has differential constraints which prevent the vehicle from turning arbitrarily fast, when two agents are chasing each other and get very close, they may not be able to keep each other in the windshield anymore. In such cases, they may keep turning to one side to circle each other forever. This phenomenon may also occur in general with  $n$  agents chasing one another in a cyclic fashion [29]. To resolve this, we introduce the assumption throughout the thesis that once agent  $i$  and its target agent  $j$  are within a small predetermined distance  $\rho$ , they combine into a single agent. We call this operation *merging* and  $\rho$  the *merging radius*. A formal definition will be given in the next chapter after the concept of *assignment graph* is defined. If two agents have each other as targets, then merging is mutual, in which case they as a whole are considered having no target and both stop moving. In practice, merging can be achieved by synchronizing the control for the involved agents.

With the Dubins car vehicle model, three possible sensing outputs, three control inputs, and the merging assumption, we want to determine conditions under which agents are guaranteed to rendezvous, involving no state estimation using coordinates.

# CHAPTER 3

## THE ASSIGNMENT GRAPH AND LIVENESS CONDITION

In this chapter we define a directed graph  $G$  called the *assignment graph*, in which each vertex is associated with an agent  $i$  having position  $p_i$  and orientation  $\psi_i$ . We then derive the connectivity condition that  $G$  must satisfy for the agents to rendezvous. The several basic graph properties being used in the discussion can be found in [40].

To rendezvous, agents must move relative to each other in some way. We formalize this relationship as *assignment*: we say that agent  $i$  is *assigned to* agent  $j$  if  $j$  is one of  $i$ 's targets. We define the assignment graph  $G$  in an obvious way:  $G$  initially has  $n$  vertices, one for each agent, and there is a direct edge  $e_{i,j}$  from  $i$  to  $j$  if and only if agent  $i$  is assigned to agent  $j$ . For an edge  $e_{i,j}$ , let  $\ell_{i,j}$  denote the distance between agents  $i, j$ 's positions in  $\mathbb{R}^2$ . With the introduction of the assignment graph, we define *merging* properly as: the merging of agent  $i$  into agent  $j$  is triggered if

$$e_{i,j} \in E(G) \quad \text{and} \quad \ell_{i,j} \leq \rho. \quad (3.1)$$

After a merge of agent  $i$  reaching agent  $j$ , vertex  $i$  and edge  $e_{i,j}$  are deleted from  $G$ . Any edge  $e_{k,i}$  that existed in  $E(G)$  before the merge is replaced by edge  $e_{k,j}$ , if such an edge does not already exist.

We say that an assignment graph  $G$  is *connected* (or weakly connected) if its underlying undirected graph has a single component. We say that  $G$  is *live* if it has at least one edge. If a graph only has a single vertex, we call it live by definition. It is desirable to maintain liveness in  $G$  at all times. If  $G$  is not live, then it has more than one vertex but no edges; it is then not possible for the system to rendezvous without additional assumptions. Since liveness is not preserved under merging, we need a stronger property from the initial graph.

**Lemma 1** *The assignment graph  $G$  is live for all  $t \geq 0$ , under arbitrary evolution of the agent positions and merging, if and only if it is connected at time  $t = 0$ .*

PROOF. “If” A connected graph is live and the connectivity of the graph is preserved under merging. (Indeed, connectivity means that there exists a path from any vertex to any other vertex, and merging vertices does not destroy any paths.)

“Only if” If  $G$  is not connected, then it has more than one connected component. By merging we can collapse each connected component to a single point. This results in a graph with more than one vertex and no edges, which is not live. ■

By Lemma 1, for a system of  $n$  agents to rendezvous, the assignment graph  $G$  for the system must have at least  $n - 1$  edges that connect all the vertices. If a connected graph with  $n$  nodes has  $n - 1$  edges, it is a tree. In particular, in an  $n - 1$  edge connected assignment graph, all vertices but one have a single outgoing edge, making the graph is an *intree*, a directed acyclic graph in which all maximal directed paths point to a single vertex, its *root*. The agent at the root of the tree has no assigned target; hence, it does not move by the control law.

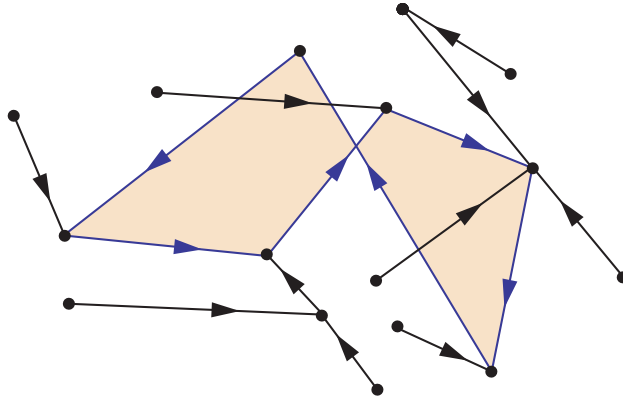


Figure 3.1: An assignment graph,  $G$ , with  $n$  edges of which the underlying undirected graph is connected and contains a cycle (polygon) that encloses the shaded area.

As we shall see later, the condition to guarantee rendezvous for an intree

assignment graph is quite trivial. However, such a formation is asymmetric in the sense that exactly one of the agent(the root) has no assignment. Such an assignment is not possible to achieve in a decentralized manner. The simplest symmetrical assignment graph  $G$  has  $n$  assignments, one for each agent. We do not investigate the case in which one agent has more than one assignment at a given instant, since more complicated control protocol will be required. Let us denote the assignment graph of our interest as *single-target* assignment graph. When  $G$  has  $n$  edges, the extra edge induces a cycle; therefore  $G$  has a single cycle on which each agent is assigned to the next one. The assignment also guarantees that any agent not on a cycle has a directed path to agents on the cycle (see Fig. 3.1). The behavior of the agents on the cycle is not affected by any agent not on the cycle. After defining the candidate Lyapunov function, the analysis of Chapter 4 first focuses on the case in which  $G$  is a single cycle; the general case in which  $G$  has a cycle plus some acyclic branches is then addressed. The proof for the intree case as well as some additional observations for the special case in which agents move directly towards each other will be briefly presented as well.

# CHAPTER 4

## GUARANTEED RENDEZVOUS OF AGENTS

### 4.1 A Candidate Lyapunov Function

Based on an assignment graph  $G$ , we define a candidate Lyapunov function  $V : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  as

$$V = \sum_{e_{i,j} \in E(G)} \ell_{i,j}, \quad (4.1)$$

in which  $p$  is the vector of agent positions and  $E(G)$  is the edge set of  $G$ .  $V$  is insensitive to edge directions in  $G$ . Unlike conventional quadratic Lyapunov functions, this specific  $V$  takes a form linear in  $\ell_{i,j}$ , which means that the corresponding time derivative is not linear in  $\ell_{i,j}$  or  $p_i$ . The reason for the choice of the Lyapunov function will become clear in the next section. Note that when a merge happens, there will be a small discontinuity in  $V$ ; but since a system with finitely many agents will only have finitely many merges, these discontinuities will not affect the overall rendezvous behavior of the system. Therefore, in our analysis, we ignore the discontinuities induced by merging.

We call a Lyapunov function a *graph-compatible Lyapunov function* if it is defined over some assignment graph  $G$ . For such a candidate Lyapunov function  $V$ , we say it is *rendezvous positive definite* if  $V = 0$  when all agents are in the same (unspecified) location, and  $V > 0$  otherwise. This leads to:

**Lemma 2** *A graph-compatible Lyapunov function  $V$  is rendezvous positive definite if and only if its assignment graph  $G$  is connected.*

PROOF. “If” It is clear that  $V \geq 0$ . Suppose that  $G$  is connected and  $V = 0$ . Since  $G$  contains a path connecting all agents, and since the lengths of all edges in this path must be zero, we have rendezvous.

“Only if” If  $G$  is not connected, then it has more than one connected component. If each connected component collapses to a single point,  $V$  becomes zero, even though we do not have rendezvous of all agents. ■

As an alternative to the above direct proof, we could deduce Lemma 2 from Lemma 1: by liveness, the graph will have at least one edge (and thus  $V$  will be positive) as long as rendezvous does not occur.  $V$  defined in (4.1) is clearly a graph-compatible Lyapunov function. By Lemma 2, it is rendezvous positive definite if and only if  $G$  is connected. We may then study the behavior of this Lyapunov function over single-target assignment graphs to derive sufficient conditions for rendezvous of the system. For moving agents,  $V$  can be considered as a function of time; its time derivative is then

$$\dot{V} = \sum_{e_{i,j} \in E(G)} \dot{\ell}_{i,j}. \quad (4.2)$$

## 4.2 The Cyclic Case

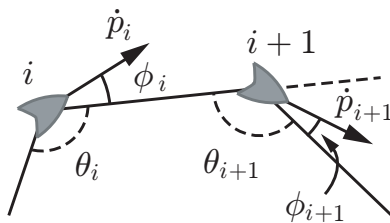


Figure 4.1: Two Dubins car agents in a cyclic pursuit.

We start the analysis with cyclic pursuit: the  $n$  edges of  $G$  form a single polygon. Given the Dubins car model (2.1) and our control law (2.3), for two consecutive agents  $i$  and  $i+1$  on the polygon, let  $\theta_i$  be either of the two angles of the polygon at vertex  $i$ . For example, we may use the internal angles for simple (non-self-intersecting) polygons. In the special case of  $n = 2$ ,  $\theta_i = 0$ . Let  $\phi_i$  be the angle between  $\dot{p}_i$  and the line segment from  $p_i$  to  $p_{i+1}$  (see Fig. 4.1). Note that  $\phi_i$  is not the same as  $\phi$ , half of the windshield span, which is fixed and the same for all  $i$ . The angle  $\phi_i$  is positive if  $\theta_i$  and  $\phi_i$  start from different sides with respect to the ray  $\overrightarrow{p_i p_{i+1}}$ .

For a specific  $\ell_{i,i+1}$ , agent  $i$ 's movement will cause it to shorten at a rate of  $v_i \cos \phi_i$ ; agent  $i+1$ 's movement will cause it to lengthen at a rate of  $v_{i+1} \cos(\pi - (\theta_i + \phi_i)) = -v_{i+1} \cos(\theta_i + \phi_i)$ . The derivative of  $\ell_{i,i+1}$  is then

$$\dot{\ell}_{i,i+1} = \left\langle \dot{p}_{i+1} - \dot{p}_i, \frac{p_{i+1} - p_i}{|p_{i+1} - p_i|} \right\rangle = -v_i \cos \phi_i - v_{i+1} \cos(\theta_{i+1} + \phi_{i+1}). \quad (4.3)$$

After summing up (4.3) for all  $i$  and rearranging, we have for the cyclic case

$$\dot{V} = \sum_i -v_i (\cos \phi_i + \cos(\theta_i + \phi_i)). \quad (4.4)$$

For identical agents with unit speed (this generalize easily to agents with identical constant speed), (4.4) becomes

$$\dot{V} = \sum_i -(\cos \phi_i + \cos(\theta_i + \phi_i)). \quad (4.5)$$

We want to keep  $\dot{V}$  negative at all times prior to rendezvous. From (4.5) we get

**Lemma 3** *For any integer  $n \geq 2$ , the windshield angle  $\phi > \pi/n$  permits trajectories for which  $\dot{V} > 0$ .*

PROOF. For  $n = 2$ ,  $\theta_1 = \theta_2 = 0$  and it is possible that  $\phi_1 = \phi_2 = \pi/2$  by assumption; substituting these into (4.4) gives  $\dot{V} = 0$ . For  $n \geq 3$ , since agents may take arbitrary initial formation, they may lie exactly on the vertices of a regular polygon and have the agents on the next vertex (say clockwise direction, see Fig. 4.2) as their targets; all agents can also have  $\phi_i = \pi/n < \phi$ . In this case, since  $\theta_i = \frac{n-2}{n}\pi$  for all  $i$ ,  $\theta_i + \phi_i = \frac{n-1}{n}\pi$ . Substituting these into (4.4) again gives

$$\dot{V} = \sum_i -v_i (\cos \frac{\pi}{n} + \cos \frac{n-1}{n}\pi) = 0. \quad (4.6)$$

$\dot{V}$  can be easily made strictly positive by letting  $\phi_i > \pi/n$  for all  $i$ . ■

By Lemma 3, for any  $n \geq 2$  and  $\phi > \pi/n$ , pursuit cycles exist for which  $\dot{V} \geq 0$ . We are now ready to give a sufficient condition for rendezvous over

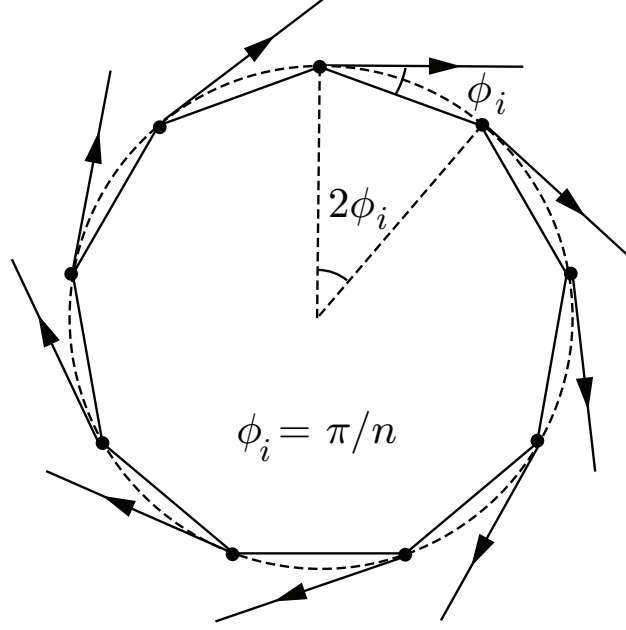


Figure 4.2: A regular 9-gon with agents on the vertices and moving along the lines tangent to the enclosing circle of the 9-gon.

the next five lemmas and theorems.

**Theorem 4** *Unit speed cyclic pursuit of  $n$  Dubins car agents will rendezvous if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with:*

$$0 < \phi \leq \begin{cases} \frac{\pi}{2}, & n = 2 \\ \cos^{-1} \frac{n-1}{n}, & n = 3, 4 \\ \frac{\pi}{n}, & n \geq 5 \end{cases} \quad (4.7)$$

The problem of how the agents can maintain their targets in a given windshield span, which is addressed with Proposition 19, is not part of this theorem. We proceed to prove Theorem 4 by first introducing several lemmas; essentially we want to show that if  $\phi$  satisfies (4.7), then  $\dot{V} < 0$ . We may then apply the Lyapunov analysis to conclude. To facilitate the discussion,



define the first and second terms of  $\dot{V}$  in (4.5) as

$$\begin{aligned} h(\Phi) &= h(\phi_1, \dots, \phi_n) := \sum_i -\cos \phi_i, \\ f(\Theta, \Phi) &= f(\theta_1, \dots, \theta_n, \phi_1, \dots, \phi_n) := \sum_i -\cos(\theta_i + \phi_i), \end{aligned} \quad (4.8)$$

and we have

$$\dot{V} = h(\Phi) + f(\Theta, \Phi). \quad (4.9)$$

For notational convenience, we use  $f$  in place of  $f(\Theta, \Phi)$  and  $h$  in place of  $h(\Phi)$  when it is appropriate to do so. By boundedness of the cosine function,  $-n \leq h, f \leq n$ . Since  $0 \leq |\phi_i| < \phi$  for all  $i$ ,  $h$  can be made arbitrarily close to  $-n$  by lowering  $\phi$ . Therefore, if for every fixed  $n$ , there exists some  $\delta_n > 0$  such that  $f < n - \delta_n$ , some small  $\phi$  can be chosen to make  $h < -n + \delta_n$  to obtain  $\dot{V} = h + f < 0$ . The bound  $\phi \leq \pi/2$  suffices for  $n = 2$ , which is straightforward to verify. Hence, we work with some  $n \geq 3$  and first consider the case in which the pursuit cycle of the agents is a simple (non-self-intersecting) polygon; the self-intersecting polygon case then follows similarly. Recall that Lemma 3 suggests that we need  $\phi$  to be no more than  $\pi/n$ . As we assume that  $|\phi_i| < \phi$  for all  $i$ , we need

$$-\pi/n < -\phi < \phi_i < \phi < \pi/n, \quad (4.10)$$

adding  $0 < \theta_i < 2\pi$  gives,

$$-\frac{\pi}{n} < (\theta_i + \phi_i) < 2\pi + \frac{\pi}{n}. \quad (4.11)$$

We partition the  $\theta_i$ 's satisfying (4.11) into two disjoint sets

$$\begin{aligned} \Theta_{out} &= \{(\theta_1, \dots, \theta_n) \mid \exists i, \theta_i + \phi_i \in (-\frac{\pi}{n}, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi + \frac{\pi}{n})\}, \\ \Theta_{in} &= \{(\theta_1, \dots, \theta_n) \mid \forall i, \theta_i + \phi_i \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}. \end{aligned} \quad (4.12)$$

On  $\Theta_{out}$ , for at least one  $i$ ,  $-\cos(\theta_i + \phi_i) < 0$ . We immediately have  $f < n - 1$  on  $\Theta_{out}$ . The following lemma tells us how  $f$  behaves on  $\Theta_{in}$ .

**Lemma 5** *Unit speed cyclic pursuit of  $n$  Dubins car agents with simple polygon pursuit cycle and satisfying (4.10) has the property that  $f(\Theta, \Phi)$  has a*

single stationary point on the interior of  $\Theta_{in}$ .

PROOF. If we fix  $\phi_i$ 's, then  $f$ , as a linear combination of cosine functions of  $\theta_i$ 's, is bounded and continuous. Recall that for simple polygons, we may use the internal angles as  $\theta_i$ 's. A simple polygon has the property that its internal angles sum up to  $(n-2)\pi$  (A proof of this property appears in the proof of Lemma 8). That is,

$$g(\Theta) = g(\theta_1, \dots, \theta_n) := \sum_i \theta_i = (n-2)\pi. \quad (4.13)$$

Again, we use  $g$  in place of  $g(\Theta)$  for notational convenience. Both  $f$  and  $g$  are analytic functions over the  $\theta_i$ 's. We may use  $g - (n-2)\pi = 0$  as the equality constraint to apply the method of Lagrange multipliers [27] over  $f$ . The method produces the Lagrangian,

$$\begin{aligned} \Lambda(\theta_1, \dots, \theta_n, \lambda) &= f(\theta_1, \dots, \theta_n, \phi_1, \dots, \phi_n) - \lambda(g(\theta_1, \dots, \theta_n) - (n-2)\pi) \\ &= -\sum_i \cos(\theta_i + \phi_i) - \lambda(\sum_i \theta_i - (n-2)\pi). \end{aligned} \quad (4.14)$$

The possible stationary points of  $f$  satisfy  $\nabla_{\theta_1, \dots, \theta_n, \lambda} \Lambda = 0$ , or equivalently,

$$\sin(\theta_i + \phi_i) = \lambda, \quad \text{for all } i. \quad (4.15)$$

From (4.10) and (4.13),

$$(n-3)\pi < \sum_i (\theta_i + \phi_i) < (n-1)\pi. \quad (4.16)$$

As a side note, the  $\Theta_{in}$  slice may be empty when  $n \leq 5$  by the first inequality of (4.16). For any  $n \geq 3$ , by the pigeonhole principle[40], for at least one  $i$ ,  $(\theta_i + \phi_i)$  must be in the range  $(0, \frac{n-1}{n}\pi)$ , which means that for that  $i$ ,  $\sin(\theta_i + \phi_i) = \lambda > 0$ . By (4.15), for all  $i$ ,  $\sin(\theta_i + \phi_i) = \lambda > 0$ . This forces  $f$  to have a single stationary point on the interior of  $\Theta_{in}$ . ■

**Lemma 6** *At the stationary point in Lemma 5,*

$$f(\Theta, \Phi) \leq -n \cos \frac{(n-2)\pi + n\phi}{n}. \quad (4.17)$$

PROOF. When all  $(\theta_i + \phi_i)$  are equal and in the interior of  $\Theta_{in}$ , we have  $(\theta_i + \phi_i) = \frac{\sum(\theta_i + \phi_i)}{n}$  for all  $i$ . Hence, at the stationary point,

$$\begin{aligned}
f(\Theta, \Phi) &= \sum_i -\cos(\theta_i + \phi_i) \\
&= -n \cos \frac{\sum_i (\theta_i + \phi_i)}{(n-2)\pi + \sum_i \phi_i} \\
&= -n \cos \frac{n}{n}.
\end{aligned} \tag{4.18}$$

By (4.18), this unique stationary point of  $f$  only depends on  $\sum \phi_i$  but not on the individual  $\phi_i$ 's. Also, fixing  $\theta_i$ 's, the maximum of the last expression in (4.18) increases as  $\sum \phi_i$  increases; hence, the upper bound of  $f$  is given by (4.17).  $\blacksquare$

**Lemma 7** *Unit speed cyclic pursuit of  $n$  Dubins car agents with simple polygon pursuit cycle has the property*

$$f(\Theta, \Phi) \leq \max \left\{ -n \cos \frac{(n-2)\pi + n\phi}{n}, n-1 \right\} \tag{4.19}$$

and  $\dot{V} < 0$ , if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with  $\phi$  satisfying (4.7).

PROOF. We may fix  $\phi_i$ 's, in which case  $\Theta_{in}$  forms a  $n$ -dimensional cube (including the boundary) in the  $n$ -dimensional Euclidean space with  $\theta_1, \dots, \theta_n$  as the coordinates.  $\Theta_{in}$  is fully surrounded by  $\Theta_{out}$ . Together,  $\Theta_{out} \cup \Theta_{in}$  forms a larger  $n$ -dimensional cube. Since the  $\theta_i$ 's must also satisfy the constraint  $\sum \theta_i = (n-2)\pi$ , the valid  $\theta_i$ 's for simple polygon formation live on the slice of the  $\Theta_{out} \cup \Theta_{in}$  cube cut by the hyperplane  $\sum \theta_i - (n-2)\pi = 0$ . Geometrically, the slice through  $\Theta_{out}$  forms an annulus on the hyperplane  $\sum \theta_i - (n-2)\pi = 0$ , and the slice through  $\Theta_{in}$  is the inside of that annulus. Fig. 4.3 illustrates these for the case of  $n = 3$ . By continuity of  $f$ ,  $f \leq n-1$  on the boundary of the  $\Theta_{in}$  slice since the boundary is the inner part of the closure of the  $\Theta_{out}$  slice. On the other hand, the  $\Theta_{in}$  slice is closed and bounded hence compact. By the extreme value theorem,  $f$  must take both

maximum and minimum value on the  $\Theta_{in}$  slice. Therefore,  $f$  takes value no more than the larger of  $n - 1$  and (4.17) from Lemma 6. Combining the  $\Theta_{out}$

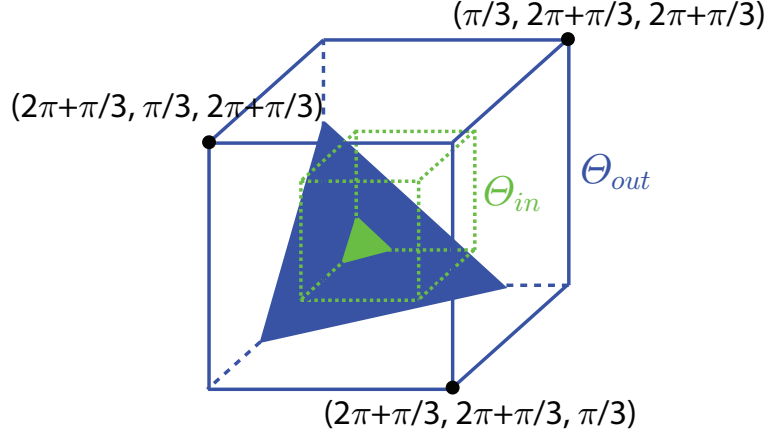


Figure 4.3:  $\Theta_{in}, \Theta_{out}$  and the slices cut by the  $g(\Theta) - (n - 2)\pi = 0$  hyperplane, for the case of  $n = 3$  and  $\sum \phi_i > \pi/2$ .

and  $\Theta_{in}$  slices, we obtain for any simple  $n$ -gon that (4.19) holds. To ensure  $\dot{V} = h + f < 0$ , we need  $-h$  to be more than the right hand side of (4.19) and rearranging the resulting inequality yields (4.7). We note that the easily verified fact that for all  $n \geq 5$ ,  $\pi/n < \cos^{-1}((n - 1)/n)$  is used. ■

The key property making the proof of Lemma 7 work is that the internal angles of any simple polygon in the plane sum up to  $(n - 2)\pi$ , which is less than  $n\pi$ . We can then choose  $\phi$  to make  $\sum(\theta_i + \phi_i)$  less than  $n\pi$  and the pigeonhole principle guarantees that some  $(\theta_i + \phi_i)$  will be less than  $\pi$ . In turn, it is guaranteed by the method of Lagrange multipliers that for all  $i$ ,  $(\theta_i + \phi_i)$  must take the same value  $\lambda$  and must be less than  $\pi$  for  $f$  to take extreme values on the  $\Theta_{in}$  slice. Combining the  $\Theta_{in}, \Theta_{out}$  slices then gives the result. The same technique can be applied when the polygon is not a simple one:

**Lemma 8** *Unit speed cyclic pursuit of  $n$  Dubins car agents with self-intersecting polygon pursuit cycle has the property  $\dot{V} < 0$  if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with  $\phi$  satisfying (4.7).*

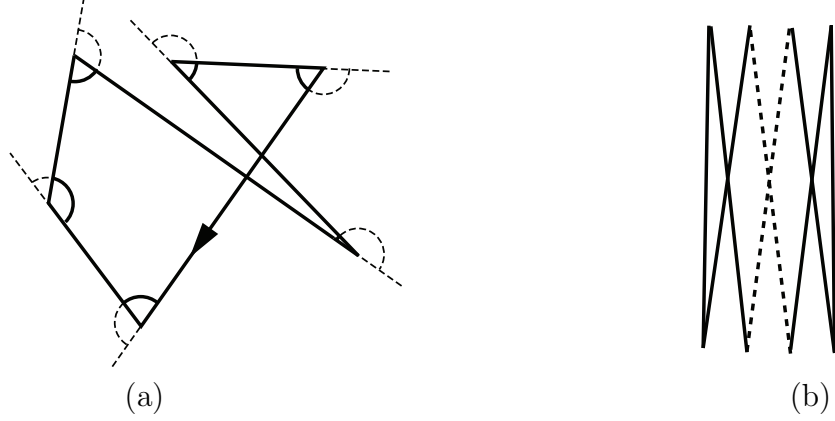


Figure 4.4: (a)  $\theta_i$ 's and  $(\pi - \theta_i)$ 's for a self-intersecting polygon, as marked by the solid arcs and dashed arcs, respectively. The arrow marks the walker's walking direction and starting edge. (b)  $\theta_i$ 's can be made arbitrarily small by "pressing" this polygon from left and right.

PROOF. When the polygon is self-intersecting, the internal angles are no longer well defined; therefore, the constraint,  $g$ , will change. However, for our purpose of calculating  $\dot{\ell}_{i,j}$ , we can always pick the smaller of the two angles at any vertex of the polygon, since such choice does not affect the value of the cosine function. We define these angles as  $\theta_i$  and look at  $\sum(\pi - \theta_i)$ . If we start from any edge of a polygon, self-intersecting or not, and walk along the edges to get back to the starting edge, all the  $(\pi - \theta_i)$ 's must add up to at least  $2\pi$  (see Fig. 4.4(a)). This is true since the walker must turn at least a cycle to get back and  $\sum(\pi - \theta_i)$  is exactly the sum of the angles turned. Thus  $\sum \theta_i \leq (n - 2)\pi$ . As a special case, this proves that the internal angles of a simple polygon sum up to  $(n - 2)\pi$ . The constraint function  $g(\Theta)$  can again be defined as that in (4.13). For self-intersecting polygon,  $g$  can be arbitrarily close to 0 (see Fig. 4.4(b)). Therefore,  $0 < g \leq (n - 2)\pi$  and (4.16) becomes

$$-\pi < \sum_i (\theta_i + \phi_i) < (n - 1)\pi. \quad (4.20)$$

As stated in Lemma 7, we only need to worry about the part of  $\{\theta_i + \phi_i\}$  that belongs to  $\Theta_{in}$ . That is, the interesting  $\theta_i$ 's must comply to

$$\frac{n\pi}{2} \leq \sum_i (\theta_i + \phi_i) < (n - 1)\pi. \quad (4.21)$$

We may fix  $g$  satisfying (4.21) as a constraint and apply the method of Lagrange multipliers similarly; bounds from Lemma 7 clearly hold. ■

**PROOF OF THEOREM 4.** Having proved that the agents may choose a windshield span satisfying (4.7) to ensure  $\dot{V} < 0$ , by the standard Lyapunov theorem on asymptotic stability with respect to a set (see, e.g., [25]), all agents will rendezvous. The attractive set here is the “diagonal” in  $\mathbb{R}^{2n}$ , in fact, its  $\rho$ -neighborhood in which  $\rho$  is the merging radius. Note that the introduction of  $\rho$  also addresses the issue that our  $V$  is not differentiable when some agents are in the same location; however, the result is valid even without this regularization. ■

Once  $\phi$  is fixed, the right side of (4.19) is determined, which leads easily to the existence of some  $\delta_\phi > 0$  for which  $\dot{V} < -\delta_\phi$  for all time  $t > 0$ . This yields:

**Corollary 9** *System in Theorem 4 achieves rendezvous in finite time.*

Finally in this section, we generalize Theorem 4 by removing the restriction that requires all  $v_i$ ’s to be identical. We say that the speed  $v_i$  of agent  $i$  is *bounded* if  $0 < v_{\min} \leq v_i < v_{\max} < +\infty$  for some constants  $v_{\min}$  and  $v_{\max}$ . The velocity  $v_i$  may change over time. When all agents’ speeds in a pursuit are bounded, we say the pursuit is a *bounded speed pursuit*.

**Theorem 10** *Bounded speed cyclic pursuit of  $n$  Dubins car agents will rendezvous if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with*

$$\phi < \cos^{-1} \frac{nv_{\max} - v_{\min}(1 - \cos \frac{\pi}{n})}{nv_{\max}}. \quad (4.22)$$

**PROOF.** For the proof we work with (4.4) and use the approach in the proof of Lemma 7. The simple polygon case is covered here; the proof for the self-intersecting polygon case then follows that of Lemma 8 similarly, which we do not repeat. Let  $\mathcal{V}$  represent the agents speeds  $v_1, \dots, v_n$  and define  $f, h$ ,

and  $g$  as

$$\begin{aligned} f(\Theta, \Phi, \mathcal{V}) &:= \sum_i -v_i \cos(\theta_i + \phi_i), \\ h(\Phi, \mathcal{V}) &:= \sum_i -v_i \cos \phi_i, \\ g(\Theta) &:= \sum_i \theta_i. \end{aligned} \tag{4.23}$$

The structures,  $\Theta_{out}$ ,  $\Theta_{in}$ , and the hyperplane  $g - (n-2)\pi = 0$  from Lemma 7 remain the same. For the  $\Theta_{in}$  slice by the hyperplane, applying the method of Lagrange multipliers to  $f$  as a function of  $\theta_i$ 's with constraint  $g - (n-2)\pi = 0$  yields that for all  $i$ ,

$$v_i \sin(\theta_i + \phi_i) = \lambda. \tag{4.24}$$

Once again, holding  $\phi_i$ 's fixed, for  $f$  to take maximum on the  $\Theta_{in}$  slice, for all  $i$ ,  $v_i \sin(\theta_i + \phi_i)$  must take the same value and therefore, must be positive if we keep  $\phi < 2\pi/n$ . Let us assume that we pick some  $\phi < \pi/n$ , then

$$\sum_i (\theta_i + \phi_i) < (n-2)\pi + n\frac{\pi}{n} = (n-1)\pi. \tag{4.25}$$

By the pigeonhole principle, for at least one  $k$ ,  $(\theta_k + \phi_k) < (n-1)\pi/n$ . Therefore,

$$\begin{aligned} f(\Theta, \Phi) &= \sum_i -v_i \cos(\theta_i + \phi_i), \\ &< \sum_{i \neq k} v_i - v_k \cos \frac{(n-1)\pi}{n} \\ &= \sum_i v_i - v_k \left(1 - \cos \frac{\pi}{n}\right) \\ &\leq \sum_i v_i - v_{\min} \left(1 - \cos \frac{\pi}{n}\right). \end{aligned} \tag{4.26}$$

To make  $f + h < 0$ , we need  $-h > f$ , which is true if

$$\sum_i v_i \cos \phi_i > \sum_i v_i - v_{\min} \left(1 - \cos \frac{\pi}{n}\right). \tag{4.27}$$

One way to satisfy this is to make sure that for each  $i$ ,

$$v_i \cos \phi > v_i - \frac{v_{\min}}{n} \left(1 - \cos \frac{\pi}{n}\right), \tag{4.28}$$

or equivalently,

$$\phi < \cos^{-1} \frac{v_i - \frac{v_{\min}}{n}(1 - \cos \frac{\pi}{n})}{v_i}. \quad (4.29)$$

The right side of (4.29) achieves the global minimum when  $v_i = v_{\max}$ , which gives us (4.22) as a sufficient condition for  $\dot{V} < 0$  on the  $\Theta_{in}$  slice. On the  $\Theta_{out}$  slice, we have that  $f \leq \sum v_i - \min\{v_i\}$ , which is less than the last expression in (4.26); therefore, (4.22) also works for the  $\Theta_{out}$  slice. ■

A parallel version of Corollary 9 immediately follows:

**Corollary 11** *System in Theorem 10 achieves rendezvous in finite time.*

It is natural to ask whether the system is stable: will some agents get arbitrarily far away from the rest during the converging process? The answer is no. Denoting  $V_0$  as the value of  $V$  at  $t = 0$ , we formalize the notion with the following proposition:

**Proposition 12** *For the system in Theorem 4 and Theorem 10,  $\dot{V} \leq 0$  ensures that all agents are inside a bounding disc with radius at most  $\frac{V_0}{2\sqrt{3}}$  for all time  $t \geq 0$ .*

PROOF. In a cyclic pursuit, the candidate Lyapunov function,  $V$ , is exactly the circumference of the pursuit cycle. Given any two different agents  $i, j$  on the pursuit cycle, there are two disjoint, undirected paths from  $i$  to  $j$ . One of these two paths must be no more than  $V/2$  in total length and by repeated application of the triangle inequality, the straight line distance between  $i, j$  cannot exceed  $V/2$ . The 2D version of Jung's theorem [17, 18] then tells us that there exists a bounding circle of the point set of all agents with radius no more than  $\frac{V}{2\sqrt{3}}$ . Since  $\dot{V} \leq 0$ ,  $V \leq V_0$  for all  $t \geq 0$ . Hence, there is no blowup prior to convergence (i.e., the system is stable in the sense of Lyapunov). ■



### 4.3 The Intree Case

For the case in which the assignment graph is an intree with one agent staying stationary, we again look at the agent system with unit speed first and then proceed to the system with varying but bounded speed. When agents are traveling with unit speed, we can guarantee that  $\dot{V} < 0$  by choosing an appropriate  $\phi$ :

**Lemma 13** *Unit speed pursuit of  $n$  Dubins car agents with an intree assignment graph has the property  $\dot{V} < 0$  and will rendezvous in finite time if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with  $0 < \phi < \cos^{-1} \frac{n-2}{n-1}$ .*

PROOF. When  $n = 2$ , one agent is stationary and one agent is moving;  $\phi < \pi/2$  guarantees  $\dot{V} < 0$ . When  $n \geq 3$ , for an edge  $e_{i,j}$  in an intree, we may express  $\dot{\ell}_{i,j}$  similarly as (4.3) (note the speeds are all 1):

$$\dot{\ell}_{i,j} = -\cos \phi_i - \cos(\theta_j + \phi_j). \quad (4.30)$$

When the target agent  $j$  is the root of the intree, the second term in (4.30) is zero since the root has no target and does not move. That is,  $\dot{\ell}_{i,j} = -\cos \phi_i$  for at least one edge  $e_{i,j}$  of the assignment graph. Let this edge be  $e_{k,m}$ . Summing over all assignment edges yields

$$\dot{V} = -\cos \phi_k + \sum_{e_{i,j} \in E(G), e_{i,j} \neq e_{k,m}} -(\cos \phi_i + \cos(\theta_j + \phi_j)). \quad (4.31)$$

$\dot{V} < 0$  is equal to

$$\cos \phi_k > \sum_{e_{i,j} \in E(G), e_{i,j} \neq e_{k,m}} -(\cos \phi_i + \cos(\theta_j + \phi_j)). \quad (4.32)$$

Assuming  $0 < \phi < \pi/2$ ,  $\cos \phi_k > \cos \phi$  and  $-\cos \phi + 1 > -(\cos \phi_i + \cos(\theta_j + \phi_j))$ , (4.32) is guaranteed by

$$\cos \phi > (n-2)(1 - \cos \phi), \quad (4.33)$$

which is equivalent to  $0 < \phi < \cos^{-1} \frac{n-2}{n-1}$ . The finite time guarantee the follows the argument from Corollary 9. ■

When the agents may have different but bounded speeds,  $\dot{V} < 0$  can no longer be guaranteed. An simple example is illustrated in Fig. 4.5. Agent  $r$  is at the root and does not move. Suppose agent  $i$  moves very fast and agents  $(j, k$  in the figure) following  $i$  barely move. Also assume that all agents are almost colinear. It is straightforward to see that, after a short period of time (the second drawing), the sum of the length of all edges, or  $V$ , increases. This suggests that  $\dot{V}$  must be positive at some point. However, such a system will still rendezvous. Supposing that the stationary agent is  $r$ , at least one agent, say  $i$ , is assigned to  $r$ . Thus,  $\dot{\ell}_{i,r} = -v_i \cos \phi_i < 0$  whenever  $\phi < \pi/2$ . Hence, agent  $i$  will merge into agent  $r$  in finite time, and all other agents will eventually follow. We have proved:

**Lemma 14** *Bounded speed pursuit of  $n$  Dubins car agents with an intree assignment graph will merge into the stationary agent in finite time if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with  $\phi < \pi/2$ .*

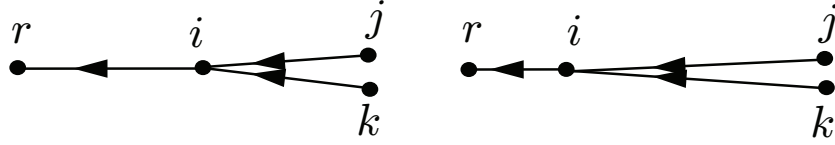


Figure 4.5: Bounded speed intree pursuit may not have  $\dot{V} < 0$  at all times.

Given an arbitrary connected, single-target assignment graph, there can be at most one cycle. By Corollary 11, rendezvous of agents on the cycle can be guaranteed when (4.22) is satisfied. These agents then merge into a single agent and stops moving as they do not have assignment any more. This leaves  $G$  as an intree. Since condition (4.22) is a superset of condition needed by Lemma 14, the whole system will achieve rendezvous as long as (4.22) is satisfied. Therefore, we have an overall sufficient condition:

**Theorem 15** *Bounded speed pursuit of  $n$  Dubins car agents with arbitrary connected, single-target assignment graph will rendezvous in finite time if the*

agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with fixed  $\phi$  satisfying (4.22).

Since rendezvous in Theorem 15 requires the agents on the cycle to achieve rendezvous first, we call this type of rendezvous *sequential rendezvous*.

## 4.4 Unit Speed Cyclic Pursuit With Branches

Although sequential rendezvous can be guaranteed for any connected assignment graph as shown above, there is no guarantee that the entire system has  $\dot{V} < 0$  all the time. Therefore, no equivalent of Lemma 12 can be stated for an arbitrary connected, single-target assignment graph, which means that the system may not be stable in the sense of Lyapunov. However, for the unit speed case, it is possible to guarantee  $\dot{V} < 0$  at all times.

**Theorem 16** *In unit speed pursuit of  $n$  unit speed Dubins car agents with arbitrary connected, single-target assignment graph, if the agents maintain their targets in the windshields of span  $(-\phi, \phi)$  with fixed  $\phi$  satisfying*

$$0 < \phi < \min \left\{ \frac{\pi}{5}, \frac{\pi}{n} \right\}, \quad (4.34)$$

*then  $\dot{V} < -\delta_\phi$  for some fixed  $\delta_\phi > 0$ .*

PROOF. We only need to show that  $\dot{V} < 0$  always holds under the condition (4.34), the same arguments from Corollary 9 then gives us the rest. Lemma 8 and Lemma 13 cover the single cycle and intree assignment graphs, which leaves only the case of a cycle plus some branches (see Fig. 3.1). For any connected, single-target assignment graph with  $n$  agents, there is a single cycle. Assume that there are  $k$  agents on that cycle. Let  $V_k$  denote the corresponding Lyapunov function for the  $k$ -cycle and let  $\phi < \pi/n$ . Then for the agents on the cycle,

$$\dot{V}_k = h_k + f_k < -\sum_{i=0}^k \cos \frac{\pi}{n} + \sum_{i=0}^k \cos \frac{\pi}{k} = k(\cos \frac{\pi}{k} - \cos \frac{\pi}{n}) \quad (4.35)$$

holds for  $k \geq 5$ . Here  $h_k, f_k$  are defined similarly as  $h, f$  before. For the agents not on the cycle, we have

$$\dot{\ell}_{i,j} = -\cos \phi_i - \cos(\theta_j + \phi_j) \leq 1 - \cos \frac{\pi}{n}. \quad (4.36)$$

Summing up all the terms for the cycle plus  $(n - k)$  other agents not on that cycle, we have

$$\dot{V} < k(\cos \frac{\pi}{k} - \cos \frac{\pi}{n}) + (n - k)(1 - \cos \frac{\pi}{n}) = -k(1 - \cos \frac{\pi}{k}) + n(1 - \cos \frac{\pi}{n}) \quad (4.37)$$

We want to show that the right hand side above is negative at all times. We may do this by showing that the function,

$$f_1(x) = x(1 - \cos \frac{\pi}{x}), \quad (4.38)$$

decreases monotonically for  $5 \leq x < \infty$ . This can be easily verified by showing that  $\dot{f}_1 < 0$  on the domain. In the cases in which  $k = 2, 3, 4$ , it can be numerically verified that  $\dot{V}_k$  is no more than  $-1$ , and we can choose  $\phi$  to be small enough to make the overall  $\dot{V} < 0$ . It can be hand checked that  $\pi/5$  works. Combining all the cases gives (4.34). ■

## 4.5 Direct Cyclic Pursuit

If the windshield of an agent is a single point ( $\phi = 0$ ), every agent moves directly toward its target, and we obtain a version of the classic cyclic pursuit, or  $n$ -bug problem. We call this variety the *direct cyclic pursuit* problem to distinguish it from the Dubins car cyclic pursuit. If we are not sure whether such pursuit is cyclic, we call it *direct pursuit*. We show that our approach again applies in this special case. Allowing the pursuit formation to be an arbitrary polygon and using the general form of  $\theta_i$  defined for self-intersecting polygon, (4.4) becomes:

$$\dot{V} = \sum_i -v_i(1 + \cos \theta_i), \quad (4.39)$$

with  $0 < \theta_i \leq \pi$  and  $\sum \theta_i \leq (n-2)\pi$ . In (4.39),  $\dot{V}$  is always less than zero. For fixed  $n$  and any set of  $v_i \geq \epsilon$  for some fixed  $\epsilon > 0$ , applying the method of Lagrange multipliers shows that there exists some  $\delta > 0$  for which  $\dot{V} \leq -\delta$  for all time. The steps are similar to those used in the proof of Theorem 10. The intree assignment graph case also follows. We obtain the following corollaries:

**Corollary 17** *Direct pursuit of  $n$  agents with lower bounded speed and arbitrary connected, single-target assignment graph will rendezvous in finite time.*

We can say a little more if  $v_i = 1$  (or any constant) holds for all agents:

**Corollary 18** *Unit speed direct pursuit of  $n$  agents with connected, single-target assignment graph has the property*

$$\dot{V} \leq \max \left\{ -1, -n(1 + \cos(\frac{n-2}{n}\pi)) \right\} =: -\delta, \quad (4.40)$$

*and will rendezvous in time no more than  $V_0/\delta$  (recall that  $V_0$  is the value of  $V$  at  $t = 0$ ).*

PROOF. Writing  $\dot{V} = h + f$  as that of (4.9), we readily see that  $h = -n$  for such systems. (4.40) is then easily obtained by substituting in (4.19). ■

## CHAPTER 5

# CONDITION ON ANGULAR VELOCITY AND EXTENSIONS

### 5.1 Angular Velocity Condition

Theorem 10 proves that, if the agents maintain their targets in some appropriate  $(-\phi, \phi)$  range, they will rendezvous. We show that agents do not need to rotate arbitrarily fast to achieve this.

**Proposition 19** *In bounded speed pursuit of  $n$  Dubins car agents with windshield span  $(-\phi, \phi)$ , for any  $\rho > 0$ , selecting*

$$\omega_i > \frac{v_{\max}}{\rho} \quad (5.1)$$

is sufficient for an agent to either keep its target within its windshield or merge with its target.

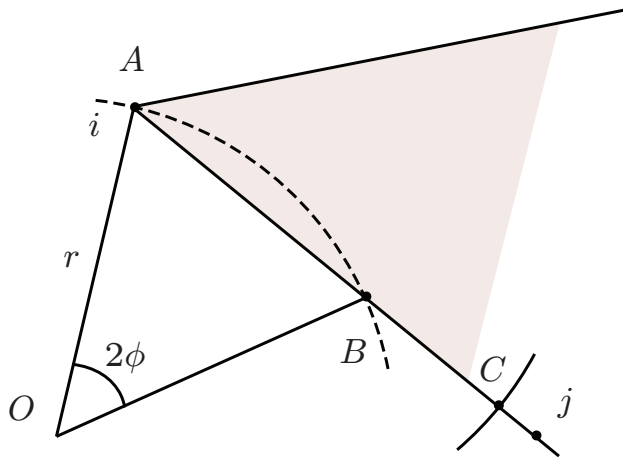


Figure 5.1: Dubins car agent  $i$  that just loses its target  $j$  at point  $A$ .

PROOF. We look at the instant  $t$  when agent  $i$  just loses sight of agent  $j$ . If agent  $j$  is within distance  $\rho$  of  $i$  then they must have merged; suppose not. The setting is shown in Fig. 5.1: Agent  $i$  is initially located at  $A$  and the ray  $\overrightarrow{AC}$  can be imagined as the right boundary of agent  $i$ 's field of view (shaded area in figure); it cannot see anything to the right of  $\overrightarrow{AC}$ . Let  $|AC| = \rho$ , the merging radius. At that instant, agent  $i$ 's field-of-view cone is rotating around  $A$  with angular velocity  $\omega_i$ . To stay out of  $i$ 's windshield, agent  $j$ 's velocity projection on the direction perpendicular to  $AC$  must be greater than  $\rho\omega_i$ . If  $\rho\omega_i > v_j$ , then once  $v_i$  starts to rotate, it will be able to get  $j$  back into its windshield again. This condition is equivalent to (5.1). ■

## 5.2 Rendezvous Without Connectivity Or Distinguishability

In the set of sufficient conditions stated in Theorems 4, 10, 15, and 16 that guarantee rendezvous, we have assumed that the assignment graph  $G$  has a single connected component initially. We now show that the assumption of single connected assignment graph and agents stopping are not necessary by: 1. Allowing agents to perform *reassignment*. 2. Allowing agent without targets to merge with other agents within distance  $\rho$ . Initially, the assignment graph  $G$  has  $n$  components: each component contains a single agent. Look at one such component  $G_i$  with a single agent  $i$ . If there are other agents within distance  $\rho$  of  $i$ , they will merge with  $i$ , therefore, we may assume that there are no other agents within distance  $\rho$  of agent  $i$ . To rendezvous with other agents, let agent  $i$  pick any target in its windshield and if it cannot find any, let it start turning. We will show that with two full turns, it can find agents at least  $\rho$  away from it or confirm that there are no such agents to conclude rendezvous is achieved. We call this procedure of finding new targets *reassignment*. We have the following:

**Proposition 20** *By allowing merging and reassignment for agents that have no targets, bounded speed pursuit of  $n$  Dubins car agents will achieve rendezvous without initial connectivity requirement, provided that the system will*

rendezvous if given a connected initial assignment graph and

$$\omega_i \geq \frac{3v_{max}}{\rho}. \quad (5.2)$$

PROOF. First let us assume that  $i$  travels at constant speed  $v_i$ . In Fig. 5.2,

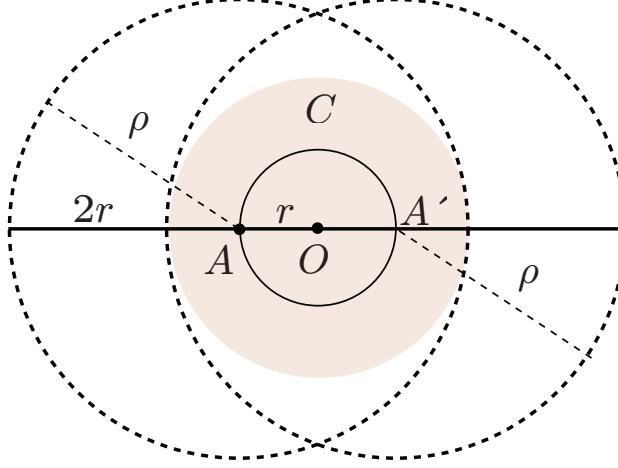


Figure 5.2: Agent at  $A$  turns along a circle with center  $O$  of radius  $r$ , leaving the small solid circle as its trace. The two large dotted circles are of distance  $\rho$  to the agent at  $A$  and  $A'$ . The shaded disc denotes the intersection of all discs with radius  $\rho$  from the agent as it turns a full cycle.

assume that agent  $i$  is initially located at  $A$  and turns clockwise around  $O$  with radius  $r$ . As it turns around, the intersection of all discs of radius  $\rho$  with  $i$  as the moving center is again a disc (the shaded area  $C$ ) of radius  $\rho - r$  centered at  $O$  (This can be easily shown: any point inside  $C$  is in all of the moving discs of radius  $\rho$  and any point outside  $C$  is not in at least one of the moving discs). Since any agent falling into  $C$  will merge with  $i$ , to avoid being found by  $i$ , an agent, say  $j$ , must move outside  $C$  and as  $i$  turns two full cycles,  $j$  must turn at least one full cycle to avoid appearing in  $i$ 's windshield. Hence, if it takes less time for  $i$  to turn two cycles than for  $j$  to turn one cycle,

$$\frac{2 \times 2\pi r}{v_i} < \frac{2\pi(\rho - r)}{v_j}, \quad (5.3)$$



agent  $i$  will then find an existing agent or merge with it. The inequality (5.3) yields the condition

$$\omega_i = \frac{v_i}{r} > \frac{2v_j + v_i}{\rho}. \quad (5.4)$$

Selecting (5.2) is then enough to guarantee that  $G$  regains liveness and the system will rendezvous. If  $v_i$  is not constant, since  $\omega_i$  remains a constant by assumption, the intersection must have a minimum radius of  $v_{\min}/\omega_i$  from  $O$ . In this case, we need

$$\frac{2 \times 2\pi}{\omega_i} < \frac{2\pi(\rho - v_{\min}/\omega_i)}{v_j}, \quad (5.5)$$

which is satisfied by (5.2). Lastly, (5.1) is also satisfied. ■

Recall that our sensing model may do without agents being distinguishable if agents are capable to take continuous videos of their targets in windshield. For agents moving at unit speed, such capability is not necessary provided that when an agent is confused by two possible targets that are colinear with it, it can follow the target that is closer. Such switching is a natural one since a closer target will block a more distant one, effectively removing the assumption requiring no visibility occlusion. Alternatively, if all agents simply send out a beacon signal, a closer agent will have a stronger signal than a more distant one. We have:

**Proposition 21** *By allowing merging and reassignment for agents that have no targets, as well as the ability for an agent to choose a closer agent as target when two possible targets become colinear with it, unit speed pursuit of  $n$  Dubins car agents will achieve rendezvous with  $\omega_i \geq 3/\rho$  and windshield angle satisfying (4.34), without initial connectivity or agent distinguishability.*

PROOF. Denoting agent  $i$  and its two possible targets  $t_{near}$  and  $t_{far}$ , when such switching of targets happens,  $V$  of the system cannot get larger since the affected terms in  $V$  during a target switching will have  $l_{i,t_{near}} \leq l_{i,t_{far}}$  and  $l_{i,t_{near}}$  is always chosen. At the same time, Theorem 16 guarantees that  $\dot{V}$  remains negative during a target switching. ■

## CHAPTER 6

# IMPLEMENTATION AND SIMULATION RESULTS

A Java implementation was written to verify the theoretical developments, which focuses on the two key scenarios of rendezvous: assignment graph with a single cycle and assignment graph with a cycle plus branches. The program<sup>1</sup> was developed adhering to the Java 5 language standard under the Eclipse environment.

Discretization is necessary to implement the continuously evolving system. Sensing needs to be discretized because the Dubins car model allows jumps in angular velocity; therefore, there could be a large number of control switches within a short time interval. In our implementation, we divide the windshield into a left sector and a right sector. The triggering event is simulated by remembering the latest observation of the target agent  $j$ 's relative position in pursuing agent  $i$ 's windshield at time step  $k$ ,

$$y_i[k] = \begin{cases} -1 & \text{agent } j \text{ is in the left sector} \\ 0 & \text{agent } j \text{ is out of view} \\ 1 & \text{agent } j \text{ is in the right sector,} \end{cases} \quad (6.1)$$

and comparing that with the next observation. If the agent is near the center of the windshield, we can give it 1 or  $-1$ . The quantized control can be encoded as

$$u_i[k] = \begin{cases} \omega & y_i[k] = 0, y_i[k-1] = -1 \\ 0 & y_i[k] \in \{-1, 1\}, y_i[k-1] = 0 \\ -\omega & y_i[k] = 0, y_i[k-1] = 1 \\ u_i[k-1] & \text{otherwise,} \end{cases} \quad (6.2)$$

---

<sup>1</sup>The simulation program is fully accessible as a Java applet through Java enabled web browsers at <http://msl.cs.uiuc.edu/~jyu18/pe/rendezvous.html>

with  $u_i[0] = 0$  by assumption. This control law can be expressed as an equivalent four state automaton. The implementation of an agent is therefore also quite simple, which enables the simulation to scale well with respect to the number of agents in the system. Plotting the Lyapunov function for systems with more than 10,000 agents can be handled quickly on our computer with a small memory footprint; smooth animation is possible for 100 agents, with the Lyapunov function and each  $\ell_{i,i+1}$  plotted simultaneously in a separate window.

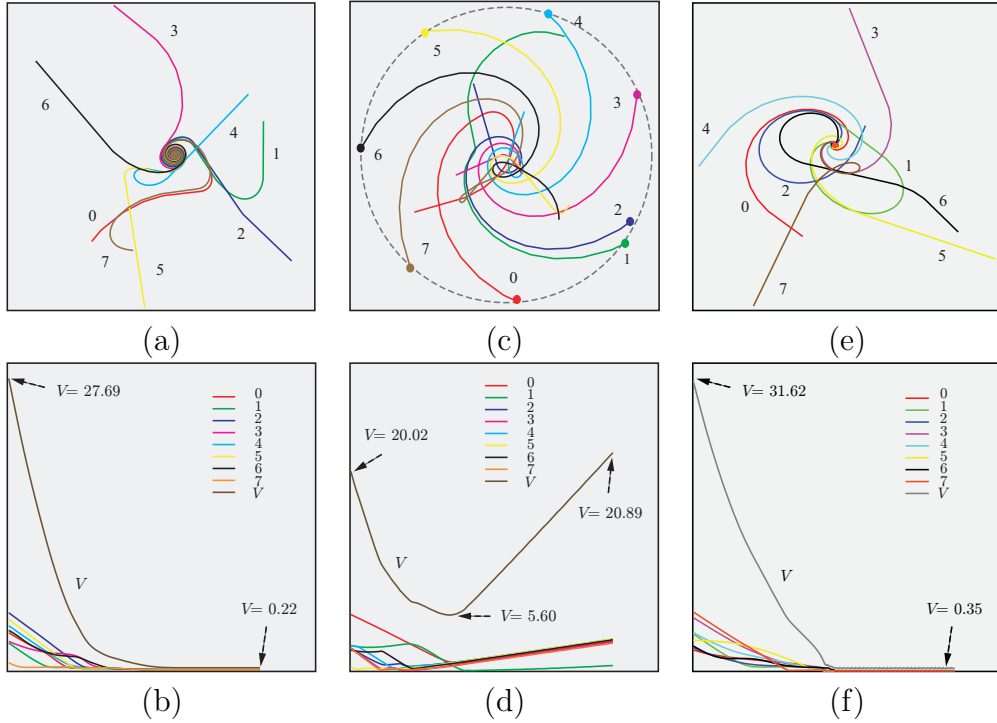


Figure 6.1: a) Eight-agent cyclic pursuit, which converges, with  $\phi = 0.35$  and  $\omega = 2.0$  b) Plot of the Lyapunov function for (a), with numbered lines being individual  $\ell_{i,i+1}$  terms. c) Diverging case with  $\phi$  changed to 0.6 holding other conditions the same. The agents arrange themselves on a cycle quickly as they spin away from each other. Dots are the agents' positions as of the last snapshot. d) Plot of the Lyapunov function for (c). e) Eight-agent converging pursuit with agents 0, 4, 3 and 5 on a cycle; the branches are:  $6 \rightarrow 2 \rightarrow 0$ ,  $7 \rightarrow 3$ ,  $1 \rightarrow 5$ . f) Plot of the Lyapunov function for (e).

To verify the cyclic pursuit theorems for the Dubins car model, each agent

is assigned a unique integer identifier. For the single cycle case, we initialize the system by positioning the agents randomly and assigning each agent to the one with a preceding identifier; the first agent is assigned to the last one. The behavior of the cyclic pursuit is then observed by watching the agents' trajectories and a simultaneously drawn plot of the Lyapunov function, under different parameter settings. Since we want to observe the behavior of the cyclic pursuit with all agents, the merging radius  $\rho$  is set to near zero to inhibit merging. For every attempted set of  $n$  agents, the simulation indicates that  $\phi = \pi/n$  is a tight rendezvous bound. As predicted, randomly positioned agents rendezvous when  $\phi < \pi/n$ , but they will eventually diverge if  $\phi$  is only 1% above  $\pi/n$ . In the latter case, the agents get closer quickly in the beginning because the random arrangement induces a negative  $\dot{V}$ . Interestingly, the agents then arrange themselves on a circle and eventually form a regular polygon, making  $\dot{V}$  positive. We speculate that the continuous system has the same behavior, which is shown in [32](under different motion primitive and control law). Fig. 6.1 a) - d) shows both a converging and a diverging case with  $n = 8$ . In the rendezvous case,  $V$  does not go to zero because  $\rho$  is set to be very small. The case with cycle plus branches is implemented similarly: we make a random smaller cycle (less than  $n$  agents) and then attach the rest of the agents to the cycle in a random way. Its simulation behavior (a converging example is shown in Fig. 6.1 e) - f)) is very similar to the single cycle case.

# CHAPTER 7

## CONCLUSION AND FUTURE DIRECTIONS

We have pursued a minimalist approach to a multi-agent rendezvous problem. Using a simple agent model with a ternary output sensor, a three level quantized control, and a Dubins car model, we have shown that a group of these constrained agents can still achieve guaranteed rendezvous behavior without coordinates, communication, or even distinguishability. Furthermore, by exploiting the geometric constraints on internal angles of polygons, we have shown that the convergence behavior of the system is precisely predictable based on the parameters of the individual agents. Our simulation indicates that the bounds derived in the theorems may be close to tight. Moreover, when simulated sensing error is introduced by providing the vehicles with randomly wrong feedback 20% of the time, the simplistic system still behave as predicted in simulation.

An interesting problem arises from from our investigation of the rendezvous problem: For agents on a pursuit cycle, a regular polygon appears to be the “preferred” formation (the agents seem to form it without much effort). Such formation is also confirmed for unicycles with finer feedback control in [29]. Although we are able to avoid global dynamics arguments, a better understanding of how the system evolves over time will help explain why this is the case. This understanding could also lead to more accurate lower bounds on the time that rendezvous takes for a given arrangement of agents. Another related open problem is prescribing the location of rendezvous, which is theoretically appealing and useful for practical purposes.

Going beyond the thesis, we want to approach the following questions: 1) Is it possible for an even simpler agent model to rendezvous? By simpler we mean that one or more of sensing and control are strictly less powerful, holding the rest of the agent model unchanged. 2) Are there any other tasks

achievable with similar simple agents? For example, we see that it is possible for the agents to get into clusters; can they form a regular lattice structure? Can we get them to follow prescribed paths up to homotopy?

Even though we focus on the rendezvous task in this thesis, our motivation in this work lies with a more general goal: Investigating what task classes are possible with minimal amount of information. For a given task, there seems to be an intrinsic relation among the required strengths of the sensors and the controller of an agent. For example, an agent can move to and touch an object to learn its shape; alternatively, it can take a picture and extract the same information. Thus, there must be some equivalence between those two agent models. A firm grasp of this relation will not only help pin down the most basic requirements for a given task, but also offer powerful design guidance for better autonomous systems.

## REFERENCES

- [1] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita. Distributed memoryless point convergence algorithm for mobile robots with limited visibility. *IEEE Trans. Robot. Automat.*, 15(5):818–828, October 1999.
- [2] A. Bernhart. Polygons of pursuit. *Scripta Math.*, 24:23–50, 1959.
- [3] S. Bhattacharya, R. Murrieta-Cid, and S. Hutchinson. Optimal paths for landmark-based navigation by differential-drive vehicles with field-of-view constraints. *IEEE Transactions on Robotics*, 23(1):47–59, February 2007.
- [4] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Autom. Control*, 45(7):1279–1289, 2000.
- [5] A. M. Bruckstein, N. Cohen, and A. Efrat. Ants, crickets and frogs in cyclic pursuit. Technical Report CIS Report #9105, Computer Science Department, Israel Institute of Technology, July 1991.
- [6] J. Cortés, S. Martínez, and F. Bullo. Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Trans. Autom. Control*, 51(8):1289–1298, August 2006.
- [7] A. T. Şamiloğlu, V. Gazi, and A. B. Koku. Asynchronous cyclic pursuit. In *SAB06*, volume 4095 of *Lecture Notes in Artificial Intelligence*, pages 667–678. Springer, 2006.
- [8] D. F. Delchamps. Stabilizing a linear system with quantized output record. *IEEE Transactions on Automatic Control*, 35(8):916–926, 1990.
- [9] L. E. Dubins. On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79:497–516, 1957.
- [10] N. Elia and S. K. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Autom. Control*, 46(9):1384–1400, 2001.

- [11] M. A. Erdmann and M. T. Mason. An exploration of sensorless manipulation. *IEEE Transactions on Robotics & Automation*, 4(4):369–379, August 1988.
- [12] V. Gazi. Stability of a discrete-time asynchronous swarm with time-dependent communication links. *IEEE Transactions on Systems, Man, and Cybernetics: Part B*, 38(1):267–274, February 2008.
- [13] K. Y. Goldberg. Orienting polygonal parts without sensors. *Algorithmica*, 10:201–225, 1993.
- [14] J. P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. in *Proc. 15th Int. Symp. Math. Theory Networks and Syst.*, 2002.
- [15] H. Ishii and B. A. Francis. Stabilizing a linear system by switching control with dwell time. *IEEE Trans. Autom. Control*, 47(12):1962–1973, December 2002.
- [16] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automat. Control*, 48:988–1001, 2003.
- [17] H. Jung. Über den kleinsten kreis, der eine ebene figur einschließt. *J. Reine Angew. Math.*, 137.
- [18] H. Jung. Über die kleinste kugel, die eine räumliche figur einschließt. *J. Reine Angew. Math.*, 123:241, 1901.
- [19] J. Yu, S. M. LaValle, and D. Liberzon. Rendezvous without coordinates. *Proc. IEEE Conf. Decision and Control*, pages 1803–1808, 2008.
- [20] I. Kamon, E. Rivlin, and E. Rimon. Range-sensor based navigation in three dimensions. In *Proceedings IEEE International Conference on Robotics & Automation*, 1999.
- [21] S. M. LaValle. *Planning Algorithms*. Cambridge University Press, Cambridge, U.K., 2006. Also available at <http://planning.cs.uiuc.edu/>.
- [22] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, MA, 2003.
- [23] J. Lin, A. S. Morse, and B. D. O. Anderson. The multi-agent rendezvous problem. part 1: The synchronous case. *SIAM Journal on Control and Optimization*, 46(6):2096–2119, November 2007.



- [24] J. Lin, A. S. Morse, and B. D. O. Anderson. The multi-agent rendezvous problem. part 2: The asynchronous case. *SIAM Journal on Control and Optimization*, 46(6):2120–2147, November 2007.
- [25] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.*, 34:124–160, 1996.
- [26] Z. Lin, M. Broucke, and B. Francis. Local control strategies for groups of mobile autonomous agents. *IEEE Trans. Autom. Control*, 49(4):622–629, April 2004.
- [27] D. G. Luenberger. *Linear and Nonlinear Programming*. Addison Wesley, 2nd edition, 1984.
- [28] V. J. Lumelsky and A. A. Stepanov. Path planning strategies for a point mobile automaton moving amidst unknown obstacles of arbitrary shape. *Algorithmica*, 2:403–430, 1987.
- [29] J. A. Marshall, M. E. Broucke, and B. A. Francis. Formations of vehicles in cyclic pursuit. *IEEE Trans. Autom. Control*, 49(11):1963–1974, July 2004.
- [30] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Autom. Control*, 50(2):169–182, February 2005.
- [31] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Control Optim.*, 43:413–436, 2004.
- [32] M. Pavone and E. Frazzoli. Decentralized policies for geometric pattern formation and path coverage. *Journal of Dynamic Systems, Measurement, and Control*, 129(5):633–643, 2007.
- [33] I. R. Petersen and A. V. Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. in *Proc. 40th IEEE Conf. Decision Control*, pages 304–309, 2001.
- [34] W. Ren and R. W. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control*, 50(5):655–661, May 2005.
- [35] T. Richardson. Stable polygons of cyclic pursuit. *Annals of Mathematics and Artificial Intelligence*, 31:147–172, 2001.

- [36] A. Sinha and D. Ghose. Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents. *IEEE Trans. Autom. Control*, 51(11):1819–1824, November 2006.
- [37] S. L. Smith, M. E. Broucke, and B. A. Francis. Curve shortening and the rendezvous problem for mobile autonomous robots. *IEEE Trans. Autom. Control*, 52(6):1154–1159, June 2007.
- [38] S. Tatikonda and S. K. Mitter. Control under communication constraints. *IEEE Trans. Autom. Control*, 49(7):1056–1068, July 2004.
- [39] B. Tovar, R. Murrieta, and S. M. LaValle. Distance-optimal navigation in an unknown environment without sensing distances. *IEEE Transactions on Robotics*, 23(3):506–518, June 2007.
- [40] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2nd edition, 2001.
- [41] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback. *IEEE Trans. Autom. Control*, 44(5):1049–1053, May 1999.