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A LUMPED-PARAMETER MODEL OF TIRE-TERRAIN INTERACTIONS FOR
OFF-ROAD VEHICLES

BY

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THESIS

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Abstract

The inclusion of various electronic control units in modern vehicles, such as the anti-lock brake systems, the traction control systems, and the electronic stability control have improved vehicle performance and stability. Design and tuning of such control units requires large amounts of trial and error modifications as well as experiments. Therefore the need for accurate and computationally efficient models has increased to meet the demands of vehicle handling simulations for designing various control units. Tire-terrain model forms the center piece of such vehicle model development.

This thesis aims at developing a discrete, lumped-parameter, physical model for capturing transient interactions between a deformable tire and a deformable terrain. Specifically, the model accounts for the radial and the tangential deformation in the tire as well as the normal and the tangential deformation of terrain. Furthermore, the model captures onset and loss of contact, localized stick and slip phases for each of the discrete elements of the tire. The model also characterizes events that trigger generic transitions between the different phases of the tire elements. Moreover, special emphasis has been placed on establishing internal mathematical consistency of the formulation and energetic consistency of the model.

The thesis presents detailed analysis of two instances of tire-terrain interactions under steady state condition. The thesis further presents the application of a non-linear regression technique for identification of the seven model parameters and, in selected cases additional unknown kinematic variables. Specifically, the model is fit to experimental load-deflection, gross thrust, and net-pull data demonstrating good quantitative agreement.

The thesis further includes the preliminary efforts to extend the planar contact model to capture interactions in the lateral direction. Specifically, a method to uniquely resolve the time evolution of internal state variable during slip is presented. Furthermore, the issue of the motion of the contact points at the onset of contact is also addressed with some unanswered questions.

To Father and Mother.

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Chapter 1

Introduction

The tires on a vehicle play a significant role in dictating the vehicle behavior, since a significant contribution to the external forces on the vehicle is due to the interaction between the tires and the terrain. The tires serve various purposes; they support the entire vehicle load, provide the necessary traction to the vehicle, act as a secondary suspension system dampening the vibrational forces due to road irregularities and, last but not least, help the vehicle to steer in the required direction.

A reliable model of tire-terrain interactions, including the response to imposed deformations, is therefore vital for the fidelity of vehicle handling simulations [1]. Furthermore, application of such models in control design may be used to improve vehicle performance, for example braking and tracking capabilities.

Depending on the objectives of a computer simulation, a tire model needs to meet certain requirements. For handling simulations, the tire model is expected to accurately estimate forces and moments transferred from the terrain to the tire. For ride comfort simulations, the tire model should capture vibrational characteristics accounting for terrain and tire deformations. On the other hand, for design purposes, a relation between the tire forces and the tire design parameters should be reflected in the model. For a comprehensive tire-terrain model, it is therefore necessary to take into account effects due to deformations of the tire and the terrain, local stick-slip, and loss of contact.

The tire-terrain interactions are strongly non-linear, history dependent and potentially involving a large number of degrees of freedom. Due to current and future demands to enhance handling, ride comfort vehicle characteristics, it is becoming increasingly important to obtain high fidelity tire-terrain interactions models with low computational overhead.

The aim of this thesis is to formulate a physical model that will capture transient as well as steady state interactions between a deformable tire and a deformable terrain. The work presented here, in contrast to the existing literature, simultaneously considers compliances in the normal and the tangential directions of the tire and terrain respectively. The specific formulation thereby allows for the load interactions to be fully characterized by local deformation.

The remainder of this thesis is organized as follows. Chapter 2 explains the fundamental properties and definitions related to the characteristics of tires. and provides a brief review of existing tire models. Chapter 3 presents the formulation of a planar lumped parameter contact model for the tire-terrain interactions. The analysis establishes energetic and internal mathematical consistency of the model and explore model properties under two distinct steady state conditions. In particular, the net tire-terrain interactions forces and torques are found in the case of static load-deflection and steady rolling. In conjunction with experimental curve fits for these two steady state conditions, these results are subsequently used to provide a preliminary model parameter identification. Chapter 4 presents a preliminary efforts to extend the planar contact model to capture interactions in the lateral direction. For example, the model behavior during slip is uniquely resolved using the principle of maximal dissipation. A final discussion with suggestions for future work is included in Chapter 5.

The work presented in this thesis has been jointly developed with my adviser Prof. Harry Dankowicz and Bernard Romig at Deere & Co. In particular, the material in Chapter 3 is largely identical to the content of a journal manuscript jointly authored by me and Prof. Dankowicz and submitted to the journal of Multibody System Dynamics, and the content of a conference manuscript jointly authored by me, Prof. Dankowicz and Bernard Romig which will appear in the Proceedings of the ASME 2010 International Design Engineering Technical Conferences. Elements of the discussion of the rigid-on-compliant limit have been previously described in a conference manuscript authored by Prof. Dankowicz, Prof. Corina Sandu, Dr. Brendan Chan and Ms. Noelani Thompson.

Chapter 2

Background

2.1 Basic Tire Terminology

In this chapter, basic tire nomenclatures and definitions recommended by SAE [2] for vehicle dynamics simulations, are presented. The reference system described as per the SAE standard is: the longitudinal x -axis aligned with the wheel heading, the lateral y -axis perpendicular to the wheel, and the vertical z -axis pointing downwards as seen in Fig. (2.1). Some of the common terminologies used in vehicle dynamics (as defined by the SAE standard)

1. Slip angle (α): The angle between a rolling wheel's actual direction of travel and the direction towards which it is pointing (i.e., the angle of the vector sum of wheel translational velocity v_x and sideslip velocity v_y) with $\alpha < 90^\circ$. A positive slip angle corresponds to the tire moving to the right as it advances forward.
2. Translational Velocity V : The effective linear speed of the center of the wheel in the longitudinal direction.
3. Angular Velocity ω : It is the rotational velocity of the wheel measured at the center of wheel-tire assembly.
4. Longitudinal Slip ratio s : This is the mathematical quantity that captures the difference between the theoretical tire velocity and the actual tire velocity caused due to the slip phenomena at the tire-terrain interface. For given longitudinal linear V and rotational ω velocity of the tire, the slip ratio is defined as

$$s = 1 - \frac{V}{R\omega} \quad (2.1)$$

where $|s| < 1$ and R is the effective radius (or the moment arm) of the tire.

5. Camber angle γ : The camber angle is the inclination angle between the vertical (z -axis) and the wheel plane. A positive camber angle corresponds to the top of the tire being inclined outward from the vehicle.

6. Tire Forces:

F_x - Longitudinal force of the tire, acting in the direction of the forward motion of the wheel, in the plane of the road.

F_y - Lateral force of the tire, acting in the direction perpendicular to the longitudinal force, in the plane of the road.

F_z - Vertical force of the tire, acting in the direction of the vector normal to the road plane.

7. Tire moments:

M_x = Overturning moment of the tire, acting about the longitudinal x -axis.

M_y = Rolling resistance moment of the tire, acting about the lateral y -axis.

M_z = Aligning moment of the tire, acting about the axis parallel to the vertical force vector.

8. Coefficient of friction:

μ_x - Coefficient of friction in longitudinal direction, i.e., maximum ratio of longitudinal force to vertical force on the tire.

μ_y - Coefficient of friction in lateral direction, i.e., maximum ratio of lateral force to vertical force on the tire.

2.2 Literature Review of Tire-Terrain Models

There is an extensive literature in the field of tire-terrain contact models (see, for example, the review by Porcel *et al.* [3]). Tire-terrain models can be coarsely organized into three major categories, *viz.*, (i) '*knowledge models*' (physical) designed from first principles of mechanical interaction; (ii) '*representation models*' (empirical) based on experimental system identification of model parameters and (iii) '*hybrid models*' that combine elements of the former two to achieve balance between reliability and computation time.

It may be further useful to distinguish between those models, for which the tire-terrain interaction can be computed from the current state of tire entirely and those, for which the interaction depends on past history. An extreme case of the former is an empirical model in which a simple formula relates force interaction to such quantities as slip angle, camber angle and slip velocity. An example of the latter would be a physical model, which includes an internal state variable that captures an evolving property of the contact interface. A complication in this case is the need to initialize such internal state variables. This can be resolved in a unique way in steady state conditions. For transient handling it has to be addressed on a case-by-case basis.

This chapter gives a short review of several tire-terrain models that fall in the above-mentioned three categories.

2.3 Physical Models

Sharp *et al.* [4] proposed a physical **multi-spoke** model to predict shear forces generated at the contact interface for pneumatic tires under steady state conditions. The model accounts for the compliance of each of the spokes in the radial, circumferential and lateral directions respectively. As each spoke interacts with the rigid terrain it will undergo deformations. This will generate interaction forces and torques which are considered to be applied at the tip of each spoke. Following [5], for steady state conditions, the authors simply monitored the interactions on a single spoke during its motion when in contact with the terrain and then integrated the interactions to compute the net force and moment acting on the tire.

Specifically in [4], assuming stick with the terrain, the following two equations are used to find the internal state variables ρ and ζ describing the deformation of an individual spoke:

$$R \cos \epsilon = (R + \rho) \cos (\epsilon - \Omega t) - \zeta \sin (\epsilon - \Omega t) \quad (2.2)$$

$$R \sin \epsilon = U t + (R + \rho) \sin (\epsilon - \Omega t) + \zeta \cos (\epsilon - \Omega t) \quad (2.3)$$

where

U : Translational steady state tire velocity

Ω : Angular steady state tire velocity

ρ : Radial deformation of the spoke

ζ : Circumferential deformation of the spoke

ϵ : Angle made by the spoke with respect to downward vertical at onset of contact

R : Spoke length

The components of the force applied by the terrain to the spoke tip in the normal and the tangential directions to the terrain surface are then given by

$$f_x = \phi(\rho) \sin (\epsilon - \Omega t) + k_4 \zeta \cos (\epsilon - \Omega t) \quad (2.4)$$

$$f_z = \phi(\rho) \cos (\epsilon - \Omega t) - k_4 \zeta \sin (\epsilon - \Omega t) \quad (2.5)$$

where $\phi(\rho)$ is some nonlinear function of ρ representing the radial force on the spoke and k_4 is the constant circumferential spoke stiffness. If $|f_x| > -\mu f_z$, where μ represents the coefficient of friction, then sliding

have occurred and the Eqn. (eq: constraint2) no longer applies. Under these circumstances, the values of the deformations (ρ_0, ζ_0) are found by reducing the circumferential deformation until $|f_x| = -\mu f_z$.

A detailed analysis of the following procedure along with the extension of the model to include interaction in the lateral direction is presented in [4]. Zhou *et al.* [6] further modified this multi-spoke 3D model in order to capture transient interactions. This model was then tested under conditions corresponding to Furuichi and Sakais experiments [7]. The theoretical predictions were found to be in good qualitative agreement with the experimental data.

Kim *et al.* [8] proposed a analytical tire model that predicts a two-dimensional contact pressure distribution on even and uneven road surfaces along with the shear stresses generated at the contact interface. The basic structure of the model can be decomposed into four components; a rigid ring representing inertial properties of the tire, six-degrees-of-freedom spring/damper elements representing transient behavior of the tire, a static circular beam representing tire belts, and residual springs representing bending deformation of the belts in the radial direction.

Zegelaar *et al.* [9] proposed a flexible ring tire model in order to analyze high frequency linear in-plane dynamics of a tire. This model is able to address three effects in the tire dynamics: the vibration transmission of the tire, the existence of standing waves in the tread-band as well as the tire-terrain interactions.

The Ftire and CDtire models [10, 11] are commonly available for vehicle handling simulations. The Ftire model represents the tire with a collection of lumped-mass elements connected with springs and dampers. The tire-terrain contact model predicts contact pressure distribution and frictional forces. In contrast, the CDtire model is a mixture of a rigid ring model, a flexible belt model and a shell-type 3D model.

2.4 Empirical Models

The tire-terrain interactions can be decomposed into components in the normal and tangential directions respectively. For example, pressure-sinkage relations rely on semi-empirical equations that relate the vertical distribution of the pressure at the contact interface to the vehicle sinkage. Similarly, shear-tension-displacement relations rely on semi-empirical equations that relate the shear stress (based on the shear displacement) to the normal stresses generated at the contact interface.

One of the methods commonly used to predict the normal pressure acting on the tire due to its motion over a deformable ground is the **Bekker** formula [12] based on the bevameter technique (performed on

a sinkage plate of specified dimensions). The formula reads as:

$$\sigma_n = \left(\frac{k_c}{b} + k_\phi\right)z^n \quad (2.6)$$

where

σ_n : Pressure normal to the sinkage plate

k_c : Cohesion-dependent parameter

k_ϕ : Friction angle dependent parameter

n : Sinkage index

b : Plate width

z : Sinkage

Bekker also formulated an expression for the shear stress developed at the interface of the sinkage plate given by

$$\tau = \frac{K_3}{2k_1\sqrt{K_2^2-1}}(e^{(-K_2+\sqrt{K_2^2-1})K_1i} - e^{(-K_2-\sqrt{K_2^2-1})K_1i}) \quad (2.7)$$

where K_1 , K_2 , K_3 are empirical constants determined by experiments. Its evident that the shear deformation of the soil provided by Bekker's relationship is not dependent on the normal stresses generated at the soil-plate interface. The other most widely used shear stress formulation is based on the Janosi and Hanamoto approach [13]. After calculating the normal stress acting on the sinkage plate, the shear stress developed tangential to the surface can be expressed as a combination of Mohr-Coulomb failure criterion and Janosi-Hanamoto relation.

$$\tau = (c + \sigma_n \tan \phi)(1 - e^{\frac{j}{K}}) \quad (2.8)$$

where

c : Soil cohesion

j : Soil-plate interface shear displacement

K : Shear deformation modulus in direction of motion

ϕ : Internal friction angle, representing the slope of the Mohr-Coulomb Failure Envelope

Reece [14] proposed a new pressure-sinkage relationship based on the Meyerhof equation. The Meyerhof equation has the following two parts: one part pertaining to the cohesion of the soil and the other part to the surcharge and the soil weight.

$$\sigma_n = ck'_c\left(\frac{z}{b}\right)^n + \frac{\gamma bk'_\phi}{2}\left(\frac{z}{b}\right)^n \quad (2.9)$$

where

- k'_c : Cohesion-dependent parameter
- k'_ϕ : Friction angle dependent parameter
- γ : unit weight of the soil
- c : cohesion variable of the soil

A well established tire model used in ADAMS (Automatic Dynamic Analysis of Mechanical Systems) is known as the Fiala model [15, 16]. The model requires the following ten input parameters values

- R_1 : Unloaded tyre radius
- R_2 : Tyre carcass radius
- k_z : Tyre radial stiffness
- C_S : Longitudinal tyre stiffness, This is the slope at the origin of F_x against the slip ratio
- C_α : Lateral tyre stiffness due to the slip angle, This is the slope at the origin of F_y against α
- C_γ : Lateral tyre stiffness due to the camber angle, This is the slope at the origin of F_x against γ
- C_r : Rolling resistance moment coefficient
- ζ : Radial damping ratio
- μ_0 : Tyre-road coefficient of static friction
- μ_1 : Tyre-road coefficient of sliding friction

The first step in the model is to compute the effective coefficient of friction as a function of the comprehensive slip ratio $S_{L\alpha}$ (resultant of the longitudinal slip coefficient S_L and the lateral slip coefficient S_α). Specifically,

$$\mu = \mu_0 - S_{L\alpha}(\mu_0 - \mu_1) \quad (2.10)$$

where

$$S_{L\alpha} = \sqrt{S_\alpha^2 + S_L^2} \quad (2.11)$$

The next step is to obtain a critical value of the longitudinal slip ratio S_L^* , after which the tyre is sliding:

$$S_L^* = \left| \frac{\mu F_z}{2C_S} \right| \quad (2.12)$$

The longitudinal force is then given by

$$F_x = -C_S S_L, \text{ when } |S_L| < S_L^* \quad (2.13)$$

and,

$$F_x = -\text{sgn}(S_L) \left(\mu F_z - \left[\frac{(\mu F_z)^2}{4|S_L|C_S} \right] \right), \text{ when } |S_L| \geq S_L^* \quad (2.14)$$

Similarly, the lateral force is given by

$$F_y = -\mu|F_z|(1 - H^3)\text{sgn}(\alpha), \text{ when } |\alpha| < \alpha^* \quad (2.15)$$

and,

$$F_y = -\mu|F_z|\text{sgn}(\alpha), \text{ when } |\alpha| \geq \alpha^*, \quad (2.16)$$

where

$$H = 1 - \frac{C_\alpha |\tan \alpha|}{3\mu|F_z|} \quad (2.17)$$

and the critical slip angle α^* is obtained from

$$\alpha^* = \arctan \left| \frac{3\mu F_z}{C_\alpha} \right| \quad (2.18)$$

The Fiala model also formulates empirical equations for the aligning moments in all directions. As with any empirical model, there are certain limitations on its applicability. It is for example not suitable for combined braking and cornering.

There are other empirical methods developed for agricultural applications such as the Brixius equations [17]. Brixius formulated empirical expressions for the Gross Thrust Ratio (GTR) and the Net Pull Ratio (NPR) as a function of the mobility number (B_n) and the wheel slip ratio S ($= 1 - U/(R\omega)$), given by

$$GTR = c_1(1 - e^{-c_2 B_n})(1 - e^{-c_3 S}) + c_4 \quad (2.19)$$

$$NPR = c_1(1 - e^{-c_2 B_n})(1 - e^{-c_3 S}) + c_4 - (c_5/B_n + c_6 S/\sqrt{B_n}) \quad (2.20)$$

where,

$$B_n = \frac{CIbd}{W} \left(\frac{(1 + 5\delta/h)}{(1 + 3b/d)} \right), \quad (2.21)$$

b, d, h are tire dimensional properties, δ is tire deflection under static load, CI is the cone index of the terrain, and $c_1 - c_6$ are constants to be determined experimentally.

2.5 Hybrid Models

Brush tire models are well-known examples of hybrid models of tire-terrain interactions [18, 19]. These describe the interactions by dividing the contact patch into regions of adhesion (stick) and sliding (slip). Under the assumption of a stiff tire carcass and a given normal pressure distribution (empirical approach)

across the contact interface, the rubber volume between the carcass and the terrain is divided into infinitesimal elements in the form of elastic rectangular blades, or bristles. Each infinitesimal element is assumed to have compliance in the lateral and the longitudinal directions. The deformations in the bristle are measured as the difference between the position of a contact point of the bristle with the terrain surface with co-ordinates $[x_r(x), y_r(x)]$ (which can be considered as internal variables) and the point where the bristle is attached to the carcass $[x, 0]$.

The bristle deformations in the longitudinal and lateral directions are

$$\delta_{xs}(x) = x_r(x) - x \quad (2.22)$$

$$\delta_{ys}(x) = y_r(x) \quad (2.23)$$

respectively. Let v_x and v_y denote the velocities in the longitudinal and lateral directions, respectively of the center of the tire. Let v_c denote the longitudinal direction of the carcass. Then assuming that the bristle remains in stick during the time interval $[0, t_c(x)]$ the contact point and bristle positions are given by

$$x_r(x) = a - \int_0^{t_c(x)} v_x dt \quad (2.24)$$

$$y_r(x) = - \int_0^{t_c(x)} v_y dt \quad (2.25)$$

$$x = a - \int_0^{t_c(x)} v_c dt, \quad (2.26)$$

With the assumption of linear elasticity, the component of the force acting at the tire-terrain interface are given by

$$dF_{ax}(x) = c_{px} dx \delta_{xs}(x) \quad (2.27)$$

$$dF_{ay}(x) = c_{py} dx \delta_{ys}(x) \quad (2.28)$$

where c_{px} and c_{py} are the longitudinal and lateral bristle stiffnesses per unit length. To accomodate transitions to slip additional constraints may be applied from which the motions of the contact points may be determined.

Mavros *et al.* [20, 21] have analyzed the mechanism of tire contact force generation under steady state conditions and transient operating conditions using the brush tire model, with both inertial and viscoelastic properties. Their analysis suggests that in order to accurately transient tire behaviour one has to consider viscoelastic circumferential connections between adjacent bristles. Additional applications of brush tire model may be found in [22, 23, 24, 25, 26, 27, 28].

El-Gawwad *et al.* [29, 30] developed a multi-spoke tire-terrain model, that mimics a tire with straight lugs, to predict the effects on interaction forces and moments due to various tire parameters. This hybrid

model uses combination of Eqn.(2.6) and Eqn. (2.8) to derive forces and moments acting on the tire as a function of the current values of tire parameters under steady state rolling and slip conditions. Guo *et al.* [31] proposed a non-linear transient semi-physical model that complies with the analytical boundary conditions of a simplified physical model. This hybrid model uses empirical expressions for the dynamic friction coefficient between the tire and the road surface as a function of the slip velocity and a assumes normalized pressure distribution along the contact interface.

Pacejka [32] introduced the well-known empirical steady state tire model known as the Magic tire Formula to derive a relatively simple hybrid model for longitudinal and lateral dynamics. Here, the longitudinal and lateral force respectively, are given by expressions of the form

$$Y(X) = y(x) + S_v, \quad (2.29)$$

where

$$y(x) = D \sin [C \arctan (Bx - E(Bx - \arctan Bx))], \quad (2.30)$$

$$x = X + S_h, \quad (2.31)$$

X is the input variable α or κ (lateral or longitudinal slip) and

B : stiffness factor

C : shape factor

D : peak value

E : curvature factor

S_h : horizontal shift

S_v : vertical shift

One of the commercially used hybrid tire models is the SWIFT model [33]. This model uses the 'magic formula' and a rigid ring model. This enables the model to achieve a balance between accurate prediction of the tire-terrain interactions for steady state cornering and representing the vibration characteristics in low-frequency ranges. Another commercially used hybrid tire model is the PAC2002 model [34]. This model is also based on the 'magic formula'. The model describes the steady state behavior using the modified 'magic formula' and the transient tire behavior is modeled by a physical approach, i.e., the stretched-string model or by explicit carcass compliances. The PAC2002 model is proposed to capture tire behavior during its motion over a smooth surface, dynamical behavior up to 12 Hz frequencies, parking behavior, low- and zero-speed applications, changing frictional properties as well as more dynamic analysis such as ABS braking.

One can find many other physical models [35, 36, 37, 38], empirical models [39, 40, 41, 42, 43] and hybrid models [44, 45, 46, 47, 48] in the literature for tire-terrain contact models. Even among these, models are developed for different operating condition. For example, a majority of representation models are developed for steady state conditions. It may therefore reasonable to question their reliability on transient handling conditions.

2.6 Figures

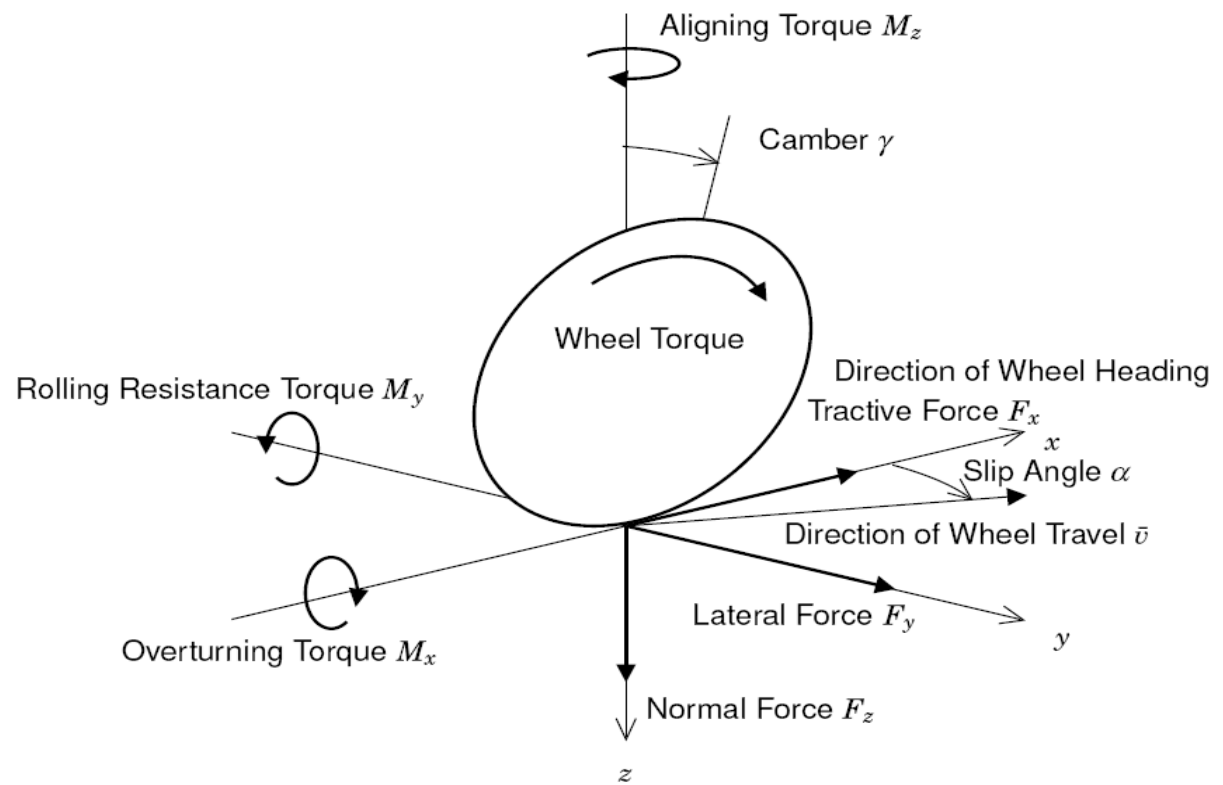


Figure 2.1: Tire terminology according to the SAE [2]

Chapter 3

2D Tire-Terrain Contact Model

In this chapter, a planar tire-terrain model formulated for capturing interactions between a deformable tire and a deformable terrain is presented (see also [49, 50, 51]). The analysis establishes the internal mathematical consistency, i.e, absence of singularities in the model description, the energetic consistency of the model, as well as the unique continuation of the model dynamics across generic transitions between various phases of motions. Furthermore, two instances of steady state interactions are studied in detail and a non-linear regression technique is applied in order to fit the model predicted data to experimental curve fit equations.

3.1 Planar Lumped Parameter Model

Consider the planar motion over level terrain of a circularly symmetric body consisting of M uniformly spaced straight line segments ('spokes'). By suitable scaling we will assume unit length for the spokes. Let $\mathcal{N}$ (see Fig. 3.1) be an inertial reference frame with reference point N located somewhere on the surface of the terrain and with reference basis n , such that $\mathbf{n}_1$ is parallel to the surface and $\mathbf{n}_2$ points in the upward vertical direction. Moreover, let $\mathcal{B}$ be a body-fixed reference frame with reference point B at the center of mass and reference basis b rotated relative to n by an angle $\varphi - \frac{\pi}{2}$ about the common 3-direction. Denote by σ and η the projected displacement of B relative to N along the surface and the elevation of B above the surface, respectively, such that

$$\mathbf{r}^{NB} = \sigma \mathbf{n}_1 + \eta \mathbf{n}_2. \quad (3.1)$$

Let $u_1 = \dot{\varphi}$, $u_2 = \dot{\sigma}$, and $u_3 = \dot{\eta}$ be the independent velocity coordinates describing the angular and translational speeds of the body, respectively. Finally, denote by P_i the end point of the i -th spoke, such that

$$\mathbf{r}^{BP_i} = \cos \theta_i \mathbf{b}_1 + \sin \theta_i \mathbf{b}_2, \quad (3.2)$$

where $\theta_i = \frac{2\pi}{M}(i-1)$.

We proceed to introduce forces and torques applied at each point P_i due to its interaction with the terrain. Following [4, 6] the net force and torque on the wheel is obtained by summation or, in the $M \rightarrow \infty$ limit, by integration over the contact interface. Specifically, in this model, each spoke can be in one of the following four phases: free flight, stick, slip in the direction of $\mathbf{n}_1$ (forward) and opposed to $\mathbf{n}_1$ (backward), respectively. In the discussion below, particular emphasis is placed on uniquely resolving generic transitions between these phases. These include the onset and termination of contact and transitions between stick and slip.

Each phase is associated with a distinct evolution of a set of internal state variables, which represent the loci of various fictitious points on or below the surface and dictate the force and torque acting on the spoke. To establish internal mathematical consistency, it is important to ascertain the uniform continuity of these variables over bounded intervals of time. It is further incumbent upon the analysis to establish that the proposed model is energetically consistent, i.e., that the total energy is a non-increasing function of time. These issues are addressed in a sequence of remarks below.

To accommodate a model of the interactions between a compliant body on a compliant surface, associate with each point P_i two contact points C_i and T_i (see Fig. 3.2), such that

$$\mathbf{r}^{NC_i} = \psi_i \mathbf{n}_1 \quad (3.3)$$

and

$$\mathbf{r}^{NT_i} = \tau_{1,i} \mathbf{n}_1 + \tau_{2,i} \mathbf{n}_2 \quad (3.4)$$

and such that the locations of the points C_i and T_i are instantaneously reset to coincide with P_i at the moment that contact is established between P_i and the surface. Denote by $t^{(i)}$ a triad rotated relative to b by the angle $\theta_i - \frac{\pi}{2}$ about the common 3-direction, such that $\mathbf{t}_2^{(i)}$ is parallel to and points in the direction of $\mathbf{r}^{BP_i}$. As long as the point P_i has a nonzero penetration below the surface, suppose that the contribution to the interactions between the surface and the rigid body mediated through the point P_i are equivalent to a contact force

$$\mathbf{F}_c^{P_i} = -k_{w,\parallel} \left(\mathbf{t}_1^{(i)} \bullet \mathbf{r}^{T_i P_i} \right) \mathbf{t}_1^{(i)} - k_{w,\perp} \left(\mathbf{t}_2^{(i)} \bullet \mathbf{r}^{T_i P_i} \right) \mathbf{t}_2^{(i)} \quad (3.5)$$

applied at the point P_i and a contact torque

$$\mathbf{T}_c^{P_i} = \mathbf{r}^{P_i T_i} \times \mathbf{F}_c^{P_i}, \quad (3.6)$$

Suppose, moreover, that as long as the point P_i has a nonzero penetration below the surface, the position of the corresponding contact point T_i is given by the solution to the force balance equation

$$\mathbf{F}_c^{T_i} - \mathbf{F}_c^{P_i} = \mathbf{0}, \quad (3.7)$$

where

$$\mathbf{F}_c^{T_i} = -k_{f,\parallel} (\mathbf{n}_1 \bullet \mathbf{r}^{C_i T_i}) \mathbf{n}_1 - k_{f,\perp} (\mathbf{n}_2 \bullet \mathbf{r}^{C_i T_i}) \mathbf{n}_2. \quad (3.8)$$

Here, $k_{f,\parallel}$ and $k_{f,\perp}$ are material parameters representing the linear compliance of the terrain in the lateral and the normal direction, respectively and $k_{w,\parallel}$ and $k_{w,\perp}$ are material parameters representing the compliance of the spokes in the circumferential and the radial direction, respectively.

Suppose that three different contact conditions may occur corresponding to three different motion constraints on the point C_i . In particular, the end point P_i is said to be in *stick* relative to the terrain when the contact point C_i is **stationary** relative to the inertial reference frame, i.e., when

$$\mathcal{N} \frac{dC_i}{dt} = \mathbf{0}, \quad (3.9)$$

and in *slip* relative to the boundary surface otherwise. In the latter case, the point P_i is said to be *slipping forward* when

$$\mathbf{n}_1 \bullet \mathcal{N} \frac{dC_i}{dt} > 0 \quad (3.10)$$

and *slipping backward* otherwise. Here, the notation

$$\mathcal{O} \frac{dA}{dt} \quad (3.11)$$

denotes the linear velocity of the point A relative to the reference frame $\mathcal{O}$.

To model friction-like bounds on the absolute value of the ratio between the tangential and normal components of $\mathbf{F}_c^{T_i}$, consider the quantities

$$\beta_{i,\pm} = \mu (\mathbf{n}_2 \bullet \mathbf{F}_c^{T_i}) \pm \mathbf{n}_1 \bullet \mathbf{F}_c^{T_i} \quad (3.12)$$

Specifically, suppose that stick is maintained as long as both remain positive. A transition to forward (backward) slip then occurs when $\beta_{i,+}(\beta_{i,-})$ changes sign from positive to negative, provided that this contact condition is consistent with the imposition of a corresponding constraint on the motion of the point C_i , which ensures that $\beta_{i,+}(\beta_{i,-})$ remains at zero. Furthermore, let $\dot{\beta}_{i,\pm} \Big|_{\text{stick}}$ denote the rate of change of $\beta_{i,\pm}$ under the assumption of stationary C_i . Forward (backward) slip is then maintained as long as $\dot{\beta}_{i,+} \Big|_{\text{stick}} < 0$ ($\dot{\beta}_{i,-} \Big|_{\text{stick}} < 0$) and $\beta_{i,-} > 0$ ($\beta_{i,+} > 0$). Finally, a transition back to stick occurs when $\dot{\beta}_{i,+} \Big|_{\text{stick}} (\dot{\beta}_{i,-} \Big|_{\text{stick}})$ changes sign.

Remark 1 Consistency of forward and backward slip: Suppose that $\phi_i = \varphi + \theta_i$ denotes the angle between the i -th spoke and the axis spanned by $\mathbf{n}_2$, such that

$$\pi_i = \cos \phi_i - \eta \quad (3.13)$$

denotes the penetration of the i -th spoke below the surface. Then,

$$\beta_{i,+} = 0 \Rightarrow \beta_{i,-} = \frac{4\mu k_{f,\perp} k_{w,\perp} k_{w,\parallel}}{a} \pi_i \quad (3.14)$$

and

$$\beta_{i,-} = 0 \Rightarrow \beta_{i,+} = \frac{4\mu k_{f,\perp} k_{w,\perp} k_{w,\parallel}}{b} \pi_i, \quad (3.15)$$

where

$$a = 2k_{w,\parallel} k_{w,\perp} + k_{f,\perp} (k_{w,\parallel} + k_{w,\perp}) + k_{f,\perp} (k_{w,\parallel} - k_{w,\perp}) (\cos 2\phi_i + \mu \sin 2\phi_i) \quad (3.16)$$

$$b = 2k_{w,\parallel} k_{w,\perp} + k_{f,\perp} (k_{w,\parallel} + k_{w,\perp}) + k_{f,\perp} (k_{w,\parallel} - k_{w,\perp}) (\cos 2\phi_i - \mu \sin 2\phi_i) \quad (3.17)$$

and, consequently,

$$a + b > 4 \min(k_{w,\parallel} (k_{f,\perp} + k_{w,\perp}), k_{w,\perp} (k_{f,\perp} + k_{w,\parallel})) > 0. \quad (3.18)$$

It follows that for $\pi_i > 0$ and $a > 0$ ($b > 0$), $\beta_{i,-}$ ($\beta_{i,+}$) is guaranteed to remain positive throughout forward (backward) slip. In this case, transitions from forward to backward slip (or vice versa) cannot occur without entering into stick first. In contrast, when $\pi_i > 0$, it is not possible to transition from stick to forward (backward) slip if $a < 0$ ($b < 0$).

Remark 2 Consistent motion of point C_i : Denote by $\dot{h}$ and $\dot{v}$ the tangential and normal speeds, respectively, of the end point P_i relative to the surface. Suppose that $\beta_{i,+} = 0$ ($\beta_{i,-} = 0$). It follows that

$$\dot{\beta}_{i,+} = \frac{2ak_{f,\parallel} k_{f,\perp} \mu (\dot{\psi}_i - \dot{h}) + k_{f,\perp} ((b-a)k_{f,\parallel} + c\mu^2) \dot{v}}{d\mu} + Au_1 \pi_i \quad (3.19)$$

and

$$\dot{\beta}_{i,-} = \frac{(-2bk_{f,\parallel} k_{f,\perp} \mu (\dot{\psi}_i - \dot{h}) + k_{f,\perp} ((a-b)k_{f,\parallel} + c\mu^2) \dot{v})}{d\mu} + Bu_1 \pi_i, \quad (3.20)$$

for some velocity-independent quantities A and B ,

$$c = k_{f,\parallel} (a + b - 4k_{f,\perp} k_{w,\parallel}) - 4(k_{f,\perp} k_{w,\parallel} + k_{f,\parallel} (k_{f,\perp} + k_{w,\parallel})) k_{w,\perp}, \quad (3.21)$$

and

$$\begin{aligned} d &= (a + b) (k_{f,\perp} - k_{f,\parallel}) + 4k_{f,\parallel} (k_{f,\perp} + k_{w,\parallel}) (k_{f,\perp} + k_{w,\perp}) \\ &> 4k_{f,\perp} \min((k_{f,\parallel} + k_{w,\parallel}) (k_{f,\perp} + k_{w,\perp}), (k_{f,\perp} + k_{w,\parallel}) (k_{f,\parallel} + k_{w,\perp})) \\ &> 0. \end{aligned} \quad (3.22)$$

If, in addition, $\dot{\beta}_{i,+}\big|_{stick} < 0 < a$ ($\dot{\beta}_{i,-}\big|_{stick} < 0 < b$), forward (backward) slip with

$$\dot{\psi}_i = -\frac{d}{2ak_{f,\parallel}k_{f,\perp}} \dot{\beta}_{i,+}\big|_{stick} > 0 \quad (\dot{\psi}_i = \frac{d}{2bk_{f,\parallel}k_{f,\perp}} \dot{\beta}_{i,-}\big|_{stick} < 0) \quad (3.23)$$

yields $\dot{\beta}_{i,+} = 0$ ($\dot{\beta}_{i,-} = 0$) and

$$\dot{\beta}_{i,-} = -\frac{4k_{f,\perp}k_{w,\parallel}k_{w,\perp}\mu}{a}\dot{v} + Cu_1\pi_i \quad \left(\dot{\beta}_{i,+} = -\frac{4k_{f,\perp}k_{w,\parallel}k_{w,\perp}\mu}{b}\dot{v} + Cu_1\pi_i \right), \quad (3.24)$$

for some velocity-independent quantity C .

Remark 3 Transition at the onset of contact: To determine the contact condition beyond the onset of contact, it is necessary to examine the signs of $\dot{\beta}_{i,\pm}\big|_{stick}$ at the moment when $\pi_i = 0$ under the assumption that $\dot{v} < 0$. If they are both positive, a self-consistent formulation is obtained by assuming that the end point is in stick immediately following the onset of contact. It follows from the previous remark that

$$-\frac{\dot{\beta}_{i,+}\big|_{stick}}{a} - \frac{\dot{\beta}_{i,-}\big|_{stick}}{b} = \frac{4k_{f,\perp}k_{w,\parallel}k_{w,\perp}\mu}{ab}\dot{v} \quad (3.25)$$

Now, suppose that $\dot{\beta}_{i,+}\big|_{stick} < 0$ ($\dot{\beta}_{i,-}\big|_{stick} < 0$) but $\dot{\beta}_{i,-}\big|_{stick} > 0$ ($\dot{\beta}_{i,+}\big|_{stick} > 0$). From Eqn. (3.25) it follows that $a > 0$ ($b > 0$). The analysis in the previous remark shows that a self-consistent formulation is obtained by assuming that the end point is in forward (backward) slip immediately following the onset of contact. Finally, if both $\dot{\beta}_{i,\pm}\big|_{stick} < 0$ and $b < 0 < a$ ($a < 0 < b$), a self-consistent formulation is obtained by assuming that the end point is in forward (backward) slip.

Remark 4 Uniform continuity of Points C_i and T_i : Suppose that

$$\rho \stackrel{def}{=} \frac{2k_{w,\parallel}k_{w,\perp} + k_{f,\perp}(k_{w,\perp} + k_{w,\parallel})}{k_{f,\perp}|k_{w,\parallel} - k_{w,\perp}|\sqrt{1+\mu^2}} < 1 \quad (3.26)$$

and let ϕ_i be chosen so that $a = 0$ ($b = 0$), in which case

$$\dot{a} = \mp 2k_{f,\perp}(k_{w,\parallel} - k_{w,\perp})\sqrt{1+\mu^2}\sqrt{1-\rho^2}u_1 (\dot{b} = \mp 2k_{f,\perp}(k_{w,\parallel} - k_{w,\perp})\sqrt{1+\mu^2}\sqrt{1-\rho^2}u_1) \quad (3.27)$$

where the two signs reflect the two possible values of ϕ_i . Now suppose that $\beta_{i,+} = 0$ ($\beta_{i,-} = 0$) and $\dot{a} < 0$ ($\dot{b} < 0$) and recall the definition of $A(B)$ in Eqn. (3.19) (Eqn. (3.20)). It is then straightforward to show that

$$\begin{aligned} \lim_{a \rightarrow 0} adA = & \sqrt{1+\mu^2}\sqrt{1-\rho^2} \left(4k_{f,\parallel}k_{f,\perp}^3(k_{w,\parallel} - k_{w,\perp})^2\sqrt{1+\mu^2}\sqrt{1-\rho^2} \right. \\ & \left. \pm 4k_{f,\perp}^3(k_{w,\parallel} - k_{w,\perp})(2k_{w,\parallel}k_{w,\perp} + k_{f,\parallel}(k_{w,\perp} + k_{w,\parallel})) \right) \mu \end{aligned} \quad (3.28)$$

$$\begin{aligned} \left(\lim_{b \rightarrow 0} bdB = & \sqrt{1+\mu^2}\sqrt{1-\rho^2} \left(-4k_{f,\parallel}k_{f,\perp}^3(k_{w,\parallel} - k_{w,\perp})^2\sqrt{1+\mu^2}\sqrt{1-\rho^2} \right. \right. \\ & \left. \left. \pm 4k_{f,\perp}^3(k_{w,\parallel} - k_{w,\perp})(2k_{w,\parallel}k_{w,\perp} + k_{f,\parallel}(k_{w,\perp} + k_{w,\parallel})) \right) \mu \right) \end{aligned} \quad (3.29)$$

is opposite in sign to $\dot{a}/u_1$ ($\dot{b}/u_1$), i.e., that if a (b) decreases toward zero during forward (backward) slip, then $\dot{\beta}_{i,+}|_{stick}$ ($\dot{\beta}_{i,-}|_{stick}$) must become positive prior to a (b) reaching zero. It follows that $\dot{\psi}_i$ remains finite during all phases of motion and thus that the positions of points C_i and T_i (the latter may be found from Eqn. (3.7)) are uniformly continuous functions of time on any compact interval of contact.

Remark 5 Transition to free flight: The construction in the previous remarks ensures that neither of the quantities $\beta_{i,\pm}$ will be negative during contact. It follows that

$$\mathbf{n}_2 \bullet \mathbf{F}_c^{T_i} = -k_{f,\perp} (\mathbf{n}_2 \bullet \mathbf{r}^{C_i T_i}) > 0 \quad (3.30)$$

during contact, i.e., that the point T_i lies below the surface of the incline during contact. For P_i on the surface of the incline

$$\beta_{i,+} = -\frac{2ak_{f,\parallel}k_{f,\perp}}{d} (\mathbf{n}_1 \bullet \mathbf{r}^{C_i P_i}) \quad (3.31)$$

and

$$\beta_{i,-} = \frac{2bk_{f,\parallel}k_{f,\perp}}{d} (\mathbf{n}_1 \bullet \mathbf{r}^{C_i P_i}). \quad (3.32)$$

Thus, in the case that $a, b > 0$, it follows that a slip phase typically precedes the loss of contact, so that C_i (and, therefore, T_i) coincides with P_i at the moment contact is lost. If, instead, $a > 0 > b$ or $b > 0 > a$, however, loss of contact may occur with a nonzero displacement between the points C_i and P_i .

Remark 6 Energetic Consistency: Let

$$V = \sum_j \frac{1}{2} \left(k_{f,\parallel} (\mathbf{n}_1 \bullet \mathbf{r}^{C_j T_j})^2 + k_{f,\perp} (\mathbf{n}_2 \bullet \mathbf{r}^{C_j T_j})^2 \right) \quad (3.33)$$

$$+ \sum_j \frac{1}{2} \left(k_{w,\parallel} (\mathbf{t}_1^{(i)} \bullet \mathbf{r}^{T_j P_j})^2 + k_{w,\perp} (\mathbf{t}_2^{(i)} \bullet \mathbf{r}^{T_j P_j})^2 \right), \quad (3.34)$$

where the summation is over those j for which the point P_j has nonzero penetration below the surface, denote the total potential energy stored in the elastic components of the contact interactions. It follows that

$$\begin{aligned} \frac{dK}{dt} &= \sum_j \left(\mathbf{F}_c^{P_j} \bullet \frac{dP_j}{dt} + \mathbf{T}_c^{P_j} \bullet \mathbf{n} \omega^b \right) \\ &= -\frac{dV}{dt} + \sum_j \mathbf{F}_c^{T_j} \bullet \frac{dC_j}{dt}, \end{aligned} \quad (3.35)$$

where the summand that again can be expressed as follow

$$\mathbf{F}_c^{P_j} \bullet \frac{dC_j}{dt} = \left(\frac{\beta_{i,+} - \beta_{i,-}}{2} \right) \dot{\psi}_i \quad (3.36)$$

vanishes when the point P_j is in stick and is negative otherwise. It follows that, in the absence of additional dissipative interactions, the total energy $K + V$ is a nonincreasing function of time and is constant only in the case that all points with nonzero penetration are in stick.

Remark 7 Limiting Cases: *In the limit that $k_{w,\parallel}, k_{w,\perp} \rightarrow \infty$ while $k_{f,\parallel}, k_{f,\perp}$ and the positions of C_i and T_i remain finite, it follows from the defining relation for the position of T_i in Eqn. (3.7) that $\mathbf{r}^{T_i P_i} = 0$. In this case, the force applied to the point P_i is given by $\mathbf{F}_c^{T_i}$ in Eqn. (3.8). In this limit, the lumped-parameter model captures interactions between a rigid tire and a deformable terrain. In the limit that $k_{f,\parallel}, k_{f,\perp} \rightarrow \infty$ while $k_{w,\parallel}, k_{w,\perp}$ and the positions of C_i and T_i remain finite, it follows from the defining relation for the position of T_i in Eqn. (3.7) that $\mathbf{r}^{C_i T_i} = 0$. In this case, the force applied to the point P_i is given by $\mathbf{F}_c^{P_i}$ in Eqn. (3.5) and $\beta_{i,\pm}$ are defined by replacing $F_c^{T_i}$ with $F_c^{P_i}$. In this limit, the lumped-parameter model captures interactions between a deformable tire and a rigid terrain.*

The Inclusion of Dissipation

During phases of slip, energy is dissipated through the motion of the contact points C_i . In addition, it is possible that contact terminates without the corresponding points P_i and C_i coinciding, in which case, potential energy stored in the elastic elements is lost instantaneously. To allow for additional dissipative mechanisms, we may consider the introduction of a damping force applied at the point P_i and given by

$$\mathbf{F}_d^{P_i} = -c_1(\mathbf{n}_1 \bullet \dot{\mathbf{r}}^{N P_i})\mathbf{n}_1 - c_2(\mathbf{n}_2 \bullet \dot{\mathbf{r}}^{N P_i})\mathbf{n}_2 \quad (3.37)$$

where $\dot{\mathbf{r}}^{N P_i}$ denotes the velocity of P_i relative to the inertial frame and c_1 and c_2 are material properties that representing the damping coefficients for the tangential and normal components of this velocity, respectively.

3.2 Steady State Analysis

Load-Deflection

Consider an initial configuration in which the body is at rest at some height above the surface of the terrain. We proceed to lower the body while monitoring the phase of motion of each individual spoke. It follows that $u_1 = u_2 = \dot{h}_i = 0$ and $u_3 = \dot{v}_i = \dot{v} < 0$.

Lemma 1 *Under the above assumptions the signs of $\dot{\beta}_{i,\pm} \Big|_{stick}$ are independent of the magnitude of $\dot{v}$ and the penetration π_i .*

Proof. Eqns. (3.19,3.20) reduce to

$$\dot{\beta}_{i,\pm} \Big|_{stick} = \frac{k_{f,\perp} (\pm (b-a) k_{f,\parallel} + c\mu^2) \dot{v}}{d\mu} \quad (3.38)$$

and the claim follows.

Corollary 1 *The phase of each spoke during contact is independent of the magnitude of $\dot{v}$ and the penetration π_i .*

Proof. The phase of an individual spoke at the onset of contact is dictated by the initial signs of $\dot{\beta}_{i,\pm}\big|_{\text{stick}}$. If these are both positive, the spoke is initially in stick. If either is negative, the spoke is initially in slip. Furthermore, all transitions between stick and slip are again dictated by the sign of $\dot{\beta}_{i,\pm}\big|_{\text{stick}}$. It follows from Lemma 1 that the signs of $\dot{\beta}_{i,\pm}\big|_{\text{stick}}$ remains the same for all nonzero penetration of the spoke end point and are independent of the vertical downward velocity. Therefore, if the spoke enters the terrain in stick it will remain in stick, since $\dot{\beta}_{i,\pm}\big|_{\text{stick}}$ will remain positive for all nonzero penetration. Similarly if the spoke enters in forward (backward) slip it will remain in forward (backward) slip, since $\dot{\beta}_{i,+}\big|_{\text{stick}}$ ($\dot{\beta}_{i,-}\big|_{\text{stick}}$) will remain negative for all nonzero penetration, and hence the claim follows.

Lemma 2 *The velocities of the contact points C_i and T_i are proportional to $\dot{v}$ with a piecewise constant coefficient.*

Proof. From Eqns. (3.23, 3.38) it follows that

$$\dot{\psi}_i = -\frac{(\pm(b-a)k_{f,\parallel} + c\mu^2)}{2ak_{f,\parallel}\mu}\dot{v}. \quad (3.39)$$

for a spoke in slip and $\dot{\psi}_i = 0$ otherwise. From Eqn. (3.7) it follows that the velocity of T_i is a linear combination of the velocities of P_i and C_i and the claim follows.

Corollary 2 *For a given deflection $\delta = 1 - \eta$, the net displacements of the contact points C_i and T_i are proportional to δ and independent of $\dot{v}$.*

Proof. From Corollary 1 and Eqn. (3.39) it follows that

$$\Delta\psi_i = -\frac{(\pm(b-a)k_{f,\parallel} + c\mu^2)}{2ak_{f,\parallel}\mu}\delta. \quad (3.40)$$

for a spoke in slip and $\Delta\psi_i = 0$ otherwise. From Eqn. (3.7) it follows that the displacement of T_i is a linear combination of the displacements of P_i and C_i and the claim follows.

Proposition 1 *For a given deflection, the normal load carried by a spoke is independent of the magnitude of $\dot{v}$.*

Proof. The normal load carried by an individual spoke is a function of the position of the contact points C_i and T_i . It follows from Corollary 2 that the position of the virtual contact points C_i and T_i

are uniquely determined by the phase and the depth of penetration of the spoke, independently of the vertical velocity, and hence the claim.

Following [4, 6], we consider each spoke as a representative differential circumferential element of the cylindrical body, such that the normal load carried across the contact interface is given by

$$W = 2 \int_0^\phi dW, \quad (3.41)$$

where

$$dW = -k_{f,\perp} (\mathbf{n}_2 \bullet \mathbf{r}^{C_i T_i}) d\phi_i, \quad (3.42)$$

, $\cos \phi = \eta$ and all compliance parameters are in units of stiffness per unit length. From the proposition it follows that dW is independent of $\dot{v}$. As shown by the next lemma, at most two distinct expressions for dW occur across the contact interface.

Lemma 3 *There exists a critical value μ^* such that all spokes are in stick during contact unless $0 \leq \mu < \mu^*$, in which case there exists two angles $0 \leq \phi_1^* < \phi_2^* \leq \pi/2$, such that all spokes with $\phi_1^* < \|\phi_i\| < \phi_2^*$ are in slip.*

Proof. Eqn. (3.38) implies that

$$\dot{\beta}_{i,\pm} \Big|_{\text{stick}} = -\frac{k_{f,\perp} k_{w,\perp} \mu}{k_{w,\perp} + k_{f,\perp}} \dot{v} > 0 \quad (3.43)$$

for $\phi_i = 0$ and

$$\dot{\beta}_{i,\pm} \Big|_{\text{stick}} = -\frac{k_{f,\perp} k_{w,\parallel} \mu}{k_{w,\parallel} + k_{f,\perp}} \dot{v} > 0 \quad (3.44)$$

for $\phi_i = \mp\pi/2$. Moreover,

$$\dot{\beta}_{i,\pm} \Big|_{\text{stick}} = 0 \Leftrightarrow \sin(2\phi_i \mp \arctan \mu) = \frac{\Delta \mu}{\sqrt{1 + \mu^2}}, \quad (3.45)$$

where

$$\Delta = \frac{k_{f,\parallel} (k_{w,\parallel} + k_{w,\perp}) + 2k_{w,\perp} k_{w,\parallel}}{k_{f,\parallel} (k_{w,\parallel} - k_{w,\perp})} \quad (3.46)$$

is a constant greater than one in magnitude. In particular, for $\mu = 0$, $\dot{\beta}_{i,\pm} \Big|_{\text{stick}} = 0$ provided that $\phi_i = 0$ or $\mp\pi/2$. From Eqn. (3.45) it follows that the corresponding root manifolds may be parametrized by ϕ_i for $0 \leq \mp\phi_i \leq \pi/2$. In each interval, $\mu(\phi_i)$ is a unimodal function with a global maximum equal to $\mu^* = 1/\sqrt{\Delta^2 - 1}$ at $\phi_i = \mp\frac{\pi}{4} - \frac{1}{2} \arctan \frac{1}{\sqrt{\Delta^2 - 1}}$ and the claim follows.

It follows from the lemma that, depending on the model parameters and the deflection δ , there are three possibilities for the distribution of the phases of motion across the contact interface: stick throughout; a central stick region surrounded symmetrically about the vertical by an annular slip region;

or a central stick region surrounded symmetrically about the vertical by an annular slip region, which in turn is surrounded symmetrically about the vertical by an annular stick region. Consequently, the total normal load is given by

$$W = 2 \left(\int_0^\phi dW_{\text{stick}} \right) \quad (3.47)$$

in the case of stick throughout;

$$W = 2 \left(\int_0^{\phi_1^*} dW_{\text{stick}} + \int_{\phi_1^*}^\phi dW_{\text{slip}} \right) \quad (3.48)$$

in the case of a central stick region and a surrounding slip region; and

$$W = 2 \left(\int_0^{\phi_1^*} dW_{\text{stick}} + \int_{\phi_1^*}^{\phi_2^*} dW_{\text{slip}} + \int_{\phi_2^*}^\phi dW_{\text{stick}} \right) \quad (3.49)$$

in the case of an additional stick region.

Explicit expressions for dW_{stick} and dW_{slip} are given in the next section.

Differential Normal Load

It follows from Eqns. (3.5,3.7,3.8), that the differential normal load is given by

$$dW = \frac{k_{f,\perp} ((1 - \delta)c\mu + k_{f,\parallel}(a - b)\psi_i + 4k_{f,\perp}k_{w,\perp}(k_{w,\parallel} + k_{f,\parallel})\cos\phi)}{d\mu} d\phi \quad (3.50)$$

where a, b, c, d are given in Eqns. (3.16, 3.17, 3.21, 3.22). Here $k_{w,\parallel}$ and $k_{w,\perp}$ represent spoke compliance in the circumferential and radial directions, respectively, in units of stiffness per unit circumferential length. Similarly, $k_{f,\parallel}$ and $k_{f,\perp}$ represent terrain compliance in tangential and normal direction, respectively, in units of stiffness per unit linear length.

From the analysis in the previous section, it follows that dW_{stick} is obtained by substitution into Eqn.(3.50) of

$$\psi_i = \sin\phi_i, \quad (3.51)$$

where, without loss of generality, we have assumed that B lies on a vertical line through N. Similarly, dW_{slip} is obtained by substitution into Eqn.(3.50) of

$$\psi_i = \sin\phi_i - \frac{(\pm(b - a)k_{f,\parallel} + c\mu^2)}{2ak_{f,\parallel}\mu} \delta. \quad (3.52)$$

where the $+$ sign applies to points in forward slip and the $-$ sign applies to points in backward slip.

Remark 8 For $k_{w,\parallel}, k_{w,\perp} \gg k_{f,\parallel}, k_{f,\perp}$, it follows from Eqn. (3.46) that $\Delta \gg 1$, i.e, all spokes remain in stick during contact. In this case, the differential normal load dW given in Eqn. (3.50) simplifies to

$$dW = k_{f,\perp}\pi_i d\phi_i = k_{f,\perp}(\cos\phi_i - 1 + \delta)d\phi_i \quad (3.53)$$

Integrating across the contact interface yields the expression

$$W = 2k_{f,\perp}(\sin \phi - \phi \cos \phi) \quad (3.54)$$

for the total normal load carried by the spokes.

Similarly, for $k_{f,\parallel}, k_{f,\perp} \gg k_{w,\parallel}, k_{w,\perp}$, it follows that Eqn. (3.46) simplifies to

$$\Delta = \frac{k_{w,\parallel} + k_{w,\perp}}{k_{w,\parallel} - k_{w,\perp}} \quad (3.55)$$

In this case Eqn. (3.56) reduces to

$$dW = \frac{(1 - \delta)c\mu + (a - b)\tau_i + 4k_{w,\perp} \cos \phi}{4\mu} d\phi \quad (3.56)$$

where

$$\begin{aligned} a &= k_{w,\parallel} + k_{w,\perp} + (k_{w,\parallel} - k_{w,\perp})(\cos 2\phi_i + \mu \sin 2\phi_i) \\ b &= k_{w,\parallel} + k_{w,\perp} + (k_{w,\parallel} - k_{w,\perp})(\cos 2\phi_i - \mu \sin 2\phi_i) \\ c &= a + b - 4(k_{w,\parallel} + k_{w,\perp}) \end{aligned} \quad (3.57)$$

and τ_i is given by the right-hand side of Eqn. (3.51) in stick and Eqn. (3.52) in slip, respectively.

Integration across the contact interface then yields the total normal load carried by the spokes.

Computational Steps

1. Calculate the value of μ^* mentioned in Lemma 3. It follows that for $\mu \geq \mu^*$, all the spokes that are in contact with the terrain remain in stick throughout the entire motion.
2. For $\mu < \mu^*$, compute ϕ_1^* and ϕ_2^* corresponding to the roots of the Eqn. (3.45) and compare it with the value of ϕ corresponding to a given deflection of the body.
3. The total normal load acting on the body can then be obtained from Eqn. (3.47), Eqn. (3.48) or Eqn. (3.49) after substituting ψ_i from Eqn. (3.51) for differential elements in stick and Eqn. (3.52) for differential elements in slip into Eqn. (3.50).

A numerical illustration of the results of this analysis is given in Figs. (3.7, 3.8), obtained for $k_{w,\perp} = 1$, $k_{w,\parallel} = 5$, $k_{f,\parallel} = 5$, $k_{f,\perp} = 1$ in a set of consistent units. In particular, Fig. (3.7) shows the root manifold of Eqn. (3.45) showing the absence of a slip region for sufficiently large μ and the presence of a stick-slip-stick phase distribution across the contact interface for sufficient large values of δ and $\mu < \mu^*$. Fig. (3.8) shows the dependence of the normal load W (in a consistent set of units) on the deflection δ for a collection of discrete values of μ . For $\mu < \mu^*$, the load-deflection curves coincide with that for $\mu = \mu^*$ until the value of δ corresponds to a locus on the root manifold in Fig. (3.7)

Steady Rolling

Let us consider the case of steady rolling in which the body is translating with a linear speed $u_2 = \dot{\sigma} = U$ in the direction of $\mathbf{n}_1$ and rotating with an angular speed $-u_1 = \omega$ about the negative $\mathbf{n}_3$ direction. It follows that

$$\mathbf{r}^{NB} = Ut\mathbf{n}_1 + \cos\phi\mathbf{n}_2 \quad (3.58)$$

The steady-state assumption implies that each spoke undergoes an identical interaction with the terrain albeit shifted in time. It follows that the totality of the interaction between the body and the terrain can be equivalently captured by investigating the time history of forces and torques applied at the end point P of a single spoke throughout its duration of contact. To this end, let

$$\mathbf{r}^{BP} = \sin(\phi - \omega t)\mathbf{n}_1 - \cos(\phi - \omega t)\mathbf{n}_2 \quad (3.59)$$

denote the position vector from the center of the body to the end point P as a function of time t , such that $t = 0$ corresponds to the onset of contact and $t = 2\phi/\omega$ corresponds to the termination of contact.

In order to obtain the net force and torque applied to the body, it is necessary to track the time histories of the internal state variables on which the individual forces and torques depend. Each of the following three sections outlines a method of obtaining this information under certain assumption on the stiffnesses.

Rigid on Compliant

As discussed previously, in the limit that $k_{w,||}, k_{w,\perp} \gg k_{f,||}, k_{f,\perp}$, the points T and P coincide during contact. It follows from Eqn. (3.8) that the differential normal load is proportional to the penetration of the spoke as per the following formula

$$dF_n = k_{f,\perp}(\cos(\phi - \omega t) - \cos\phi)\omega dt \quad (3.60)$$

Integrating with respect to time from 0 to $2\phi/\omega$ then yields,

$$F_n = (2\sin\phi - 2\phi\cos\phi) * k_{f,\perp} \quad (3.61)$$

for the total normal load acting on the body. Furthermore from Eqn. (3.8), it follows that the differential lateral force is given by

$$dF_l = -k_{f,||}(Ut + \sin(\phi - \omega t) - \psi)\omega dt \quad (3.62)$$

where ψ describes the location of the contact point C . During forward (backward) slip this is given by

$$\psi = Ut + \sin(\phi - \omega t) \pm \mu(k_{\perp}/k_{||})(\cos(\phi - \omega t) - \cos\phi) \quad (3.63)$$

while, in stick, ψ remains equal to its value at the onset of stick.

Compliant on Rigid

Similarly, in the limit that $k_{f,||}, k_{f,\perp} \gg k_{w,||}, k_{w,\perp}$, the points T and C coincide during contact. Let

$$\mathbf{r}^{TP} = \zeta \mathbf{t}_1 + \rho \mathbf{t}_2 \quad (3.64)$$

It follows from Eqn. (3.5) that the differential normal and lateral loads equal

$$dF_n = (k_{w,||} \zeta \sin(\phi - \omega t) + k_{w,\perp} \rho \cos(\phi - \omega t)) \omega dt \quad (3.65)$$

$$dF_l = (k_{w,||} \zeta \cos(\phi - \omega t) - k_{w,\perp} \rho \sin(\phi - \omega t)) \omega dt \quad (3.66)$$

where

$$\rho = 1 - \cos(\phi) \cos(\phi - \omega t) + (Ut - \tau) \sin(\phi - \omega t) \quad (3.67)$$

$$\zeta = -\cos(\phi) \sin(\phi - \omega t) - (Ut - \tau) \cos(\phi - \omega t)$$

and τ describes the location of the contact point T . During forward (backward) slip this is obtained from

$$\begin{aligned} \tau - Ut &= \left[\mu(k_{w,||} - k_{w,\perp}) \sin(\phi - \omega t) \cos(\phi - \omega t) \pm k_{w,||} \cos(\phi - \omega t)^2 \pm k_{w,\perp} \sin(\phi - \omega t)^2 \right]^{-1} \\ &\quad \left[\mu(k_{w,||} \sin(\phi - \omega t)^2 + k_{w,\perp} \cos(\phi - \omega t)^2) \cos \phi - \mu k_{w,\perp} \cos(\phi - \omega t) \right. \\ &\quad \left. \pm k_{w,\perp} \sin(\phi - \omega t) + (\pm k_{w,||} \mp k_{w,\perp}) \cos(\phi - \omega t) \cos \phi \sin(\phi - \omega t) \right] \end{aligned} \quad (3.68)$$

while, in stick, τ remains equal to its value at the onset of stick.

Compliant on Compliant

Finally, consider the general case. Let

$$\mathbf{r}^{NP} = p_1 \mathbf{n}_1 + p_2 \mathbf{n}_2 \quad (3.69)$$

It follows from Eqn. (3.8) that the differential normal and lateral loads equal

$$dF_n = -k_{f,\perp} \tau_2 \omega dt \quad (3.70)$$

$$dF_l = -k_{f,||} (\tau_1 - \psi) \omega dt \quad (3.71)$$

where ψ , τ_1 and τ_2 describe the locations of contact points C and T . Here,

$$\tau_1 = \left[(k_{f,\perp} - k_{f,\parallel})(k_{w,\perp} - k_{w,\parallel}) \sin^2(\phi - \omega t) + (k_{f,\perp} + k_{w,\perp})(k_{f,\parallel} + k_{w,\parallel}) \right]^{-1} \quad (3.72)$$

$$\left[(\psi k_{f,\parallel} - p_1 k_{f,\perp})(k_{w,\parallel} - k_{w,\perp}) \sin^2(\phi - \omega t) \right. \\ \left. + p_2 k_{f,\perp}(k_{w,\parallel} - k_{w,\perp}) \sin(\phi - \omega t) \cos(\phi - \omega t) + k_{f,\parallel} \psi (k_{f,\perp} + k_{f,\parallel}) + p_1 k_{w,\parallel}(k_{w,\perp} + k_{f,\perp}) \right]$$

$$\tau_2 = \left[(k_{f,\perp} - k_{f,\parallel})(k_{w,\perp} - k_{w,\parallel}) \sin^2(\phi - \omega t) + (k_{f,\perp} + k_{w,\perp})(k_{f,\parallel} + k_{w,\parallel}) \right]^{-1} \quad (3.73)$$

$$\left[p_2 k_{w,\perp}(k_{f,\parallel} + k_{w,\parallel})(k_{w,\perp} - k_{w,\parallel})(\psi - p_1) k_{f,\parallel} \sin(\phi - \omega t) \cos(\phi - \omega t) \right. \\ \left. - p_2 k_{f,\parallel}(k_{w,\perp} - k_{w,\parallel}) \sin^2(\phi - \omega t) \right]$$

Moreover, during forward (backward) slip

$$\psi = \frac{k_{f,\parallel} \tau_1 \pm k_{f,\perp} \tau_2}{k_{f,\parallel}} \quad (3.74)$$

while, in stick, ψ remains equal to its value at the onset of stick.

Simulation

Let Q be the total reaction torque about the $-n_3$ direction through the center of the tire from the net interactions between the spokes and the terrain. Let P and W be the total tangential and normal force in the directions of n_1 and n_2 from these interactions, respectively. Further let us consider three functions $\Pi(t)$, $\Lambda(t)$ and $\Gamma(t)$ such that $\Pi(0) = \Lambda(0) = \Gamma(0) = 0$ and

$$\dot{\Pi} = \frac{dF_n}{dt} \quad (3.75)$$

$$\dot{\Lambda} = \frac{dF_l}{dt} \quad (3.76)$$

$$\dot{\Gamma} = -\omega(\mathbf{T}_c^P + \mathbf{r}^{BP} \times \mathbf{F}_c^T) = -\mathbf{r}^{BT} \times \left(\frac{dF_l}{dt} \mathbf{n}_1 + \frac{dF_n}{dt} \mathbf{n}_2 \right) \quad (3.77)$$

where dF_n and dF_l are expressed in Eqns. (3.60,3.62) for rigid on compliant case, in Eqns. (3.65,3.66) for compliant on rigid case and Eqns. (3.70,3.71) for compliant on compliant. It follows that $\Pi(2\pi/\omega) = W$, $\Lambda(2\pi/\omega) = P$ and $\Gamma(2\pi/\omega) = Q$. Let us further describe the four phases of motions by the following modes for the spoke

$$\mathbf{m} = \begin{cases} 1(\text{free flight}) & \text{if } \mathbf{n}_2 \bullet \mathbf{r}^{NP} > 0 \\ 2(\text{backward slip}) & \text{if } \mathbf{n}_2 \bullet \mathbf{r}^{NP} \leq 0 \text{ and } \beta_- = 0 \\ 3(\text{stick}) & \text{if } \mathbf{n}_2 \bullet \mathbf{r}^{NP} \leq 0 \text{ and } \beta_{\pm} > 0 \\ 4(\text{forward slip}) & \text{if } \mathbf{n}_2 \bullet \mathbf{r}^{NP} \leq 0 \text{ and } \beta_+ = 0 \end{cases} \quad (3.78)$$

In addition,

$$h_{\text{contact}} = \mathbf{t}_2 \bullet \mathbf{r}^{NP_i}, \quad (3.79)$$

$$h_{\text{slip}\pm} = \beta_{i,\pm}, \quad (3.80)$$

$$h_{\text{stick}\pm} = -\dot{\beta}_{\pm} \Big|_{\text{stick}}, \quad (3.81)$$

represents the five co-dimension-one events that trigger transitions between these distinct phases of motion (or the modes). To compute P , Q and W , a matlab code was written, which solves the differential equations given in Eqn. (3.75-3.77) for the time interval $t = 0$ to $t = 2\pi/\omega$ with the given initial conditions and while monitoring changes in the signs of the event functions in Eqn. (3.79-3.81). The transition diagram in Table (3.4) represents the changes in $\mathbf{m}$ (from $\mathbf{m}_b$ to $\mathbf{m}_a$) that result from transversal intersections with the corresponding zero-level surfaces.

Recall from Chapter 2 the definitions of the gross thrust ratio

$$GTR = \frac{Q}{W} \quad (3.82)$$

and the net pull ratio

$$NPR = \frac{P}{W} \quad (3.83)$$

Figs. (3.3-3.6) show the gross thrust ratio, net pull ratio, normal load and the percentage of the duration of contact corresponding to an initial stick phase, respectively, as a function of deflection and slip ratio $(1 - U/\omega)$ for $k_{w,\parallel} = 60$, $k_{w,\perp} = 15$, $k_{f,\parallel} = 1$, $k_{f,\perp} = 15$, $\mu = 0.7$ in a set of consistent units. Figs. (3.3) and Figs. (3.4) are in qualitative agreement with experimental curve fits given in [17]. Further as $\delta \rightarrow 0$, the graph in Fig. (3.6) approaches a discontinuous function which equals 1 for zero slip ratio and 0 otherwise. This further supports the known fact that for $\delta = 0$, zero slip ratio indicates the tire is in stick with the terrain and in slip otherwise.

3.3 System Identification

The model description given above includes seven physical parameters governing the compliance properties of the tire and terrain, respectively, as well as friction-like limits on the ratio between tangential and normal forces, and damping $(k_{f,\parallel}, k_{f,\perp}, k_{w,\parallel}, k_{w,\perp}, c_1, c_2, \mu)$. It follows from Proposition (1) that the load-deflection relationship is independent of c_1 and c_2 . In contrast, the steady rolling results shown above were obtained assuming $c_1 = c_2 = 0$. For non-zero c_1 and c_2 it would be necessary to modify Eqns. (3.75-3.77) to account for the contributions from the damping force $\mathbf{F}_d^P$.

In this section, we report on efforts to identify appropriate values for the model parameters to achieve satisfactory agreement with data obtained from experimental curve-fit models. The results of preliminary parameter identification obtained by seeking to fit model predictions to the predictions from experimental curve fits for the two steady-state conditions discussed above are presented.

Specifically, a non-linear regression routine is employed to find an optimal choice of parameter values that minimizes the sum of the squares of the differences between model predictions and predictions from the experimental curve-fits. For each set of data, we consider three different optimization runs, capturing three different conditions of relative compliance of the tire and the terrain. In particular, results identified as those for a rigid-on-compliant model refer to the $k_{w,\parallel}, k_{w,\perp} \rightarrow \infty$ limit, in which case only $k_{f,\parallel}, k_{f,\perp}, c_1, c_2, \mu$ remain to be identified. Similarly, results identified as those for a compliant-on-rigid model refer to the $k_{f,\parallel}, k_{f,\perp} \rightarrow \infty$ limit, in which case only $k_{w,\parallel}, k_{w,\perp}, c_1, c_2, \mu$ remain to be identified. Finally, the compliant-on-compliant case corresponds to fitting the full model to the available data.

Load Deflection

As an alternative to using the explicit formulae for the differential load elements derived in a previous section, we here seek to obtain the load-deflection relationship by numerically simulating the experiment described at the onset of that section. In particular, consider an initial configuration of the body at rest with its center of mass at a unit distance above the nominal surface of the terrain. We then proceed to lower the tire, until the required deflection δ is achieved, monitoring the phase of each spoke coming in contact with the terrain. Since the dependence of the normal load on the tire-deflection is independent of the vertical velocity used in loading the tire, this experiment only serves to identify the compliance parameters and μ , respectively. In addition, in the case of the rigid-on-compliant model, W is independent of $k_{f,\parallel}$ and μ as follows from Eqn. (3.54).

The numerical predictions for the load deflection relationship may be compared with prediction from experimental curve fits for Bridgestone/Firestone tires (and obtained from Bernard Romig at Deere & Co) under three experimental conditions. The load-deflection curve fits are given by

$$\delta = C_1 + C_2 W + C_3 \ln(W + 1) + C_4 e^{C_5 W} \quad (3.84)$$

where C_1 through C_5 are given in Table 3.1 for three different tire conditions. Here W is measured in units of 1000 lbs and the deflection δ is measured in inches.

Specifically, a discrete set of data points were computed using Eqn. (3.84) for each of the three tire conditions. For each model and each tire condition, a single run of the MATLAB `lsqnonlin` function

was performed starting at some set of initial guesses for the parameter values, and for a fixed number of iterations. The results of this analysis are shown in Figs. (3.9-3.11) . Here, solid curves represent the graphs of the experimental curve fits and open circles represent the final predicted values from each model for the total normal load at the corresponding load deflection.

It is interesting to observe that, while relatively good fits are obtained, the agreement is better for the compliant-on-rigid and compliant-on-compliant models than for the rigid-on-compliant model. This is to be expected, since the load-deflection data used corresponds to experiments performed on cement.

Steady Rolling

Let us again consider the case of steady rolling in which the tire is translating with a linear speed $u_2 = \dot{\sigma} = U$ in the direction of $\mathbf{n}_1$ and rotating with an angular speed $-u_1 = \omega$ about the negative $\mathbf{n}_3$ direction. Let us proceed to fit the predicted relationships from each of the three models between the gross-thrust and net-pull ratios, respectively and the slip ratio to the numerical predictions of experimental curve fits obtained from [17]. Specifically, [17] parametrizes this dependence on a non-dimensional experimentally measured quantity B_n , known as the mobility number, which depends on the material parameters, the load, and the resultant static tire deflection. Consequently, we seek optimal values for all model parameters, i.e., including the damping coefficients and the angle ϕ . In particular, for a given mobility number, an optimal set of parameter values is sought that minimizes the prediction error for both the gross- thrust and net-pull ratios simultaneously.

A discrete set of data points were computed from the curve fits. For each model and each mobility number, a single run of the MATLAB lsqnonlin function was performed, starting at some set of initial guesses for the parameter values, and for a fixed number of iterations. The results of this analysis are shown in Figs. (3.12-3.14). Here, solid curves represent the graphs of the experimental curve fits and open circles represent the final predicted values from each model for the gross-thrust and net-pull ratios, respectively at the corresponding slip ratio.

It is interesting to observe that, while relatively good fits are obtained, this time the agreement is better for the rigid-on-compliant and compliant-on-compliant models than for the compliant-on-rigid model. This is again to be expected, since the curve-fit data used corresponds to experiments performed on soil.

3.4 Tables

	C_1	C_2	C_3	C_4	C_5
14.9 R 30 12 psi	-9.09942E-03	5.90454E-01	7.67057E-01	-1.21418E-02	3.037079E-01
14.9 R 30 36 psi	-6.90761E-03	2.22804E-01	8.47334E-01	7.16841E-03	3.37079E-01
710/70 R 38 24 psi	7.80306E-03	4.05438E-01	1.38573E-01	-3.028890E-02	2.14266E-01

Table 3.1: Coefficients for Bridgestone/Firestone Load-deflection curve-fits

	$k_{f,\perp}(klb/(inch - rad))$	$k_{f,\parallel}(klb/(inch - rad))$	μ
14.9 R 30 12 psi	1.2489	100	0.7
14.9 R 30 36 psi	2.4043	100	0.7
710/70 R 38 24 psi	4.0267	100	0.7

Table 3.2: Load-Deflection relationship: Locally optimal value of system parameters for rigid-on-compliant model

	$k_{w,\parallel}(klb/(inch - rad))$	$k_{w,\perp}(klb/(inch - rad))$	μ
14.9 R 30 12 psi	0	1.4291	0.19437
14.9 R 30 36 psi	0	2.5342	0.7
710/70 R 38 24 psi	0	3.974	0.7

Table 3.3: Load-Deflection relationship: Locally optimal value of system parameters for compliant-on-rigid model

	$k_{w,\parallel}$	$k_{w,\perp}$	$k_{f,\parallel}$	$k_{f,\perp}$	μ
14.9 R 30 12 psi	0.27271	3.2779	0.00026131	3.3657	0.7
14.9 R 30 36 psi	0.88302	5.2675	0.11951	5.3785	0.7
710/70 R 38 24 psi	0.0027228	18.46	0.10124	18.399	0.7

Table 3.4: Load-Deflection relationship: Locally optimal value of system parameters (all compliance parameters are in $klb/(inch - rad)$) for compliant-on-compliant model

$m_b \setminus m_a$	1	2	3	4
1	★	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} < 0$ $h_{\text{stick}+} < 0$ $h_{\text{stick}-} > 0$	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} < 0$ $h_{\text{stick}+} < 0$ $h_{\text{stick}-} < 0$	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} < 0$ $h_{\text{stick}+} > 0$ $h_{\text{stick}-} < 0$
2	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} > 0$	★	$h_{\text{stick}-} = 0$ $\dot{h}_{\text{stick}-} < 0$	★
3	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} > 0$	$h_{\text{slip}-} = 0$ $\dot{h}_{\text{slip}-} < 0$	★	$h_{\text{slip}+} = 0$ $\dot{h}_{\text{slip}+} < 0$
4	$h_{\text{contact}} = 0$ $\dot{h}_{\text{contact}} > 0$	★	$h_{\text{stick}+} = 0$ $\dot{h}_{\text{stick}+} < 0$	★

Table 3.5: Transition diagram showing changes from m_b to m_a that result from transversal intersections with the zero-level surfaces

B_n	$k_{f,\perp}$	$k_{f,\parallel}$	c_2	c_1	μ	ϕ
10	13.059	21.423	18.573	1.497E-08	1.0884	3.6287
15	16.097	28.974	22.404	1.8689E-08	2.0555	1.9752
25	21.248	35.83	22.211	0.052995	2.6454	1.0891
60	36.037	36.66	16.37	0.086793	2.0112	0.58649

Table 3.6: Gross thrust and net pull ratio relationship: Locally optimal value of system parameters (in a set of consistent units) for rigid-on-compliant model

B_n	$k_{w,\parallel}$	$k_{w,\perp}$	μ	c_1	c_2	ϕ
10	126.17	3.4421	1.6902	0.25676	7.9976	0.50446
15	126.17	3.443	1.6908	0.25708	7.997	0.50446
25	126.92	3.8967	1.6478	0.30457	7.7276	0.51101
60	130.93	4.1229	1.6793	0.40278	7.5674	0.52982

Table 3.7: Gross thrust and net pull ratio relationship: Locally optimal value of system parameters (in a set of consistent units) for compliant-on-rigid model

B_n	$k_{w, }$	$k_{w,\perp}$	$k_{f, }$	$k_{f,\perp}$	μ	c_1	c_2	ϕ
10	1.1913	73.256	102.44	0.79271	1.0553	2.36E-08	1.019	0.50088
15	1.3977	67.188	92.804	0.98716	1.4787	0.0019693	0.94316	0.39202
25	1.4731	67.309	89.412	1.1844	1.757	0.0050316	0.78547	0.29866
60	1.3826	66.721	25.153	1.3912	2.051	0.0045626	0.66404	0.20458

Table 3.8: Gross thrust and net pull ratio relationship: Locally optimal value of system parameters (in a set of consistent units) for compliant-on-compliant model

3.5 Figures

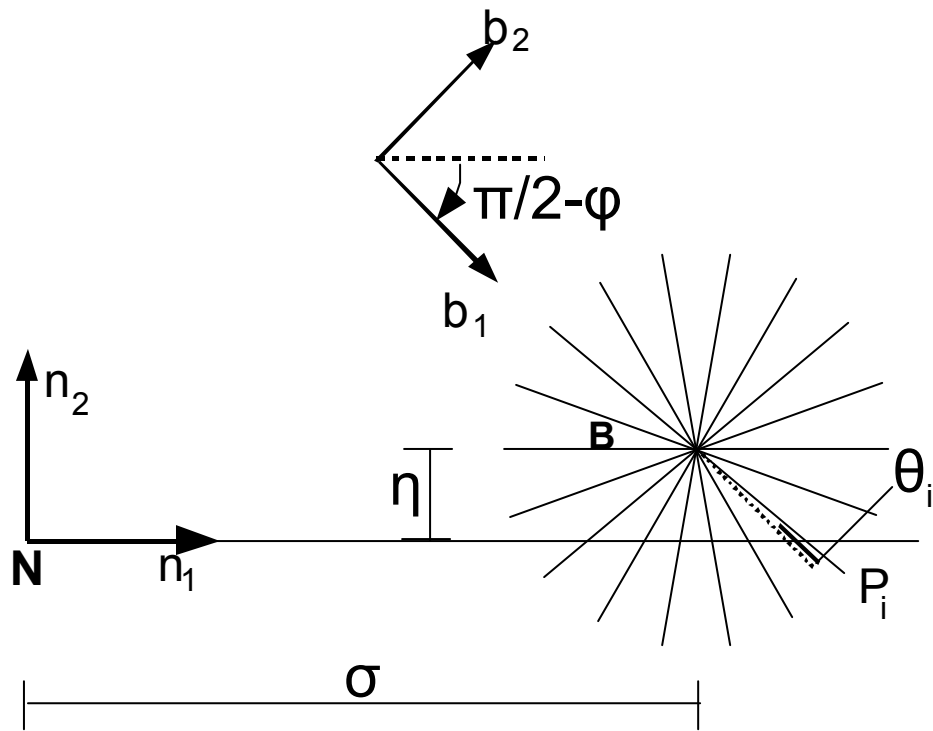


Figure 3.1: Schematic diagram illustrating the overall body geometry.

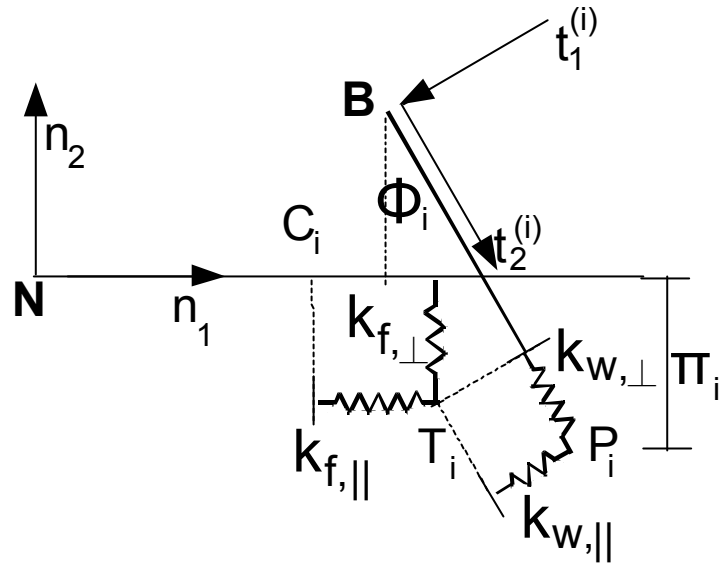


Figure 3.2: Schematic of contact geometry for model of deformable body on a deformable terrain.

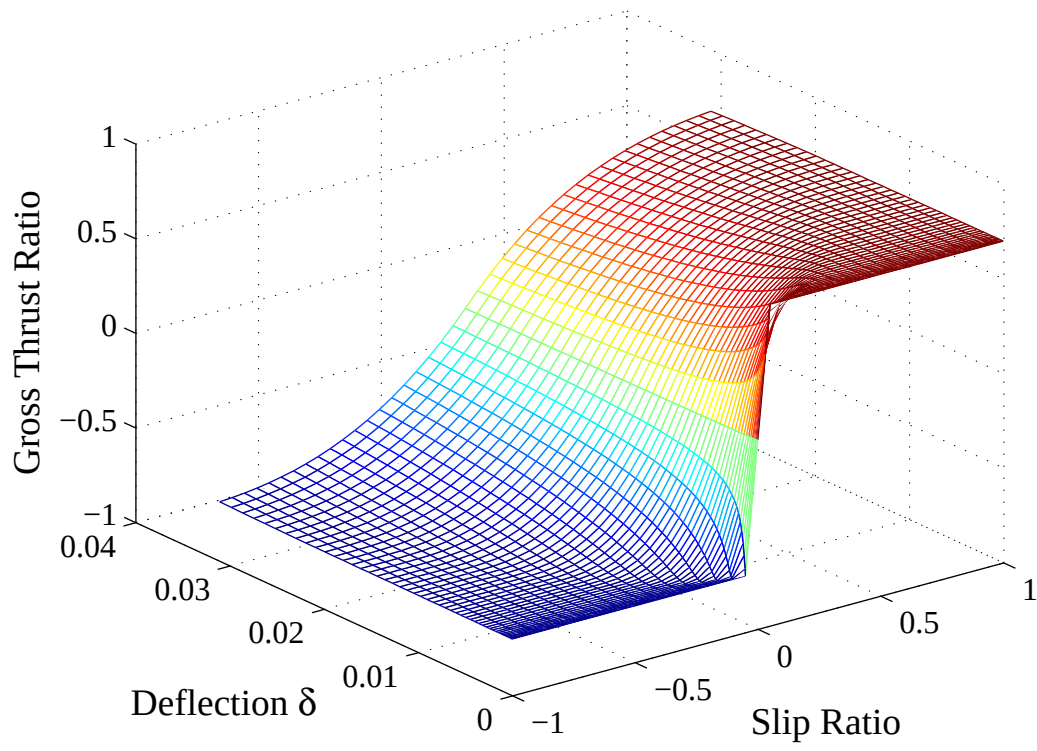


Figure 3.3: Gross Thrust ratio parametrized by the deflection δ and the slip ratio $1 - U/\omega$ for the parameter values given in the text.

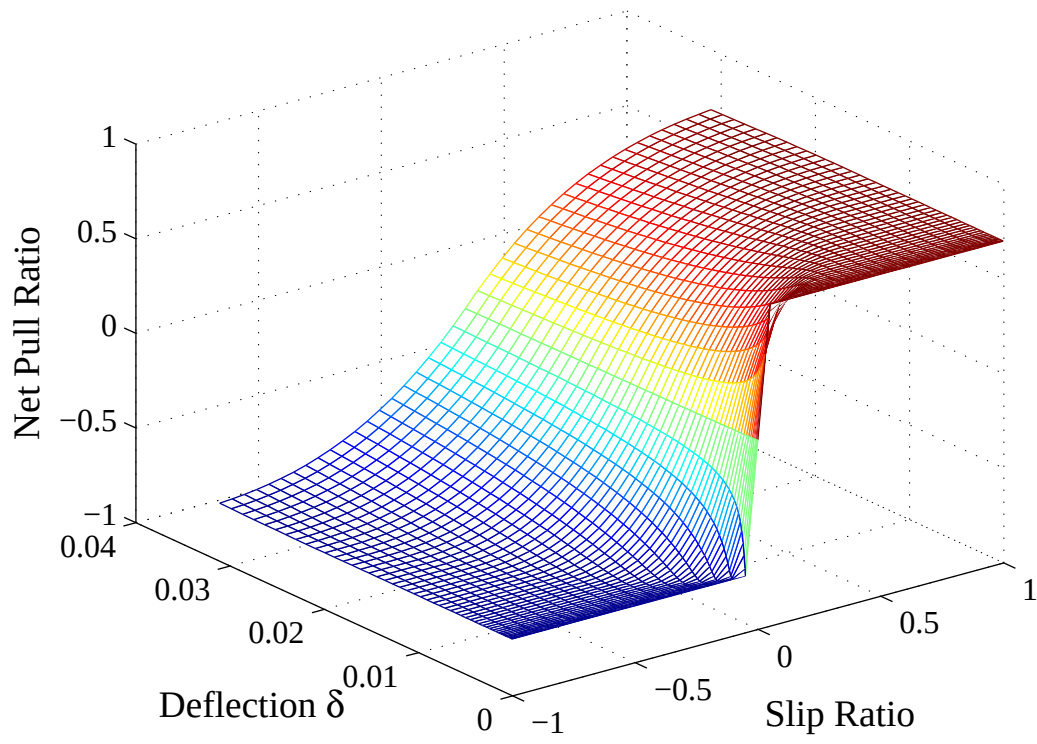


Figure 3.4: Net Pull ratio parametrized by the deflection δ and the slip ratio $1 - U/\omega$ for the parameter values given in the text.

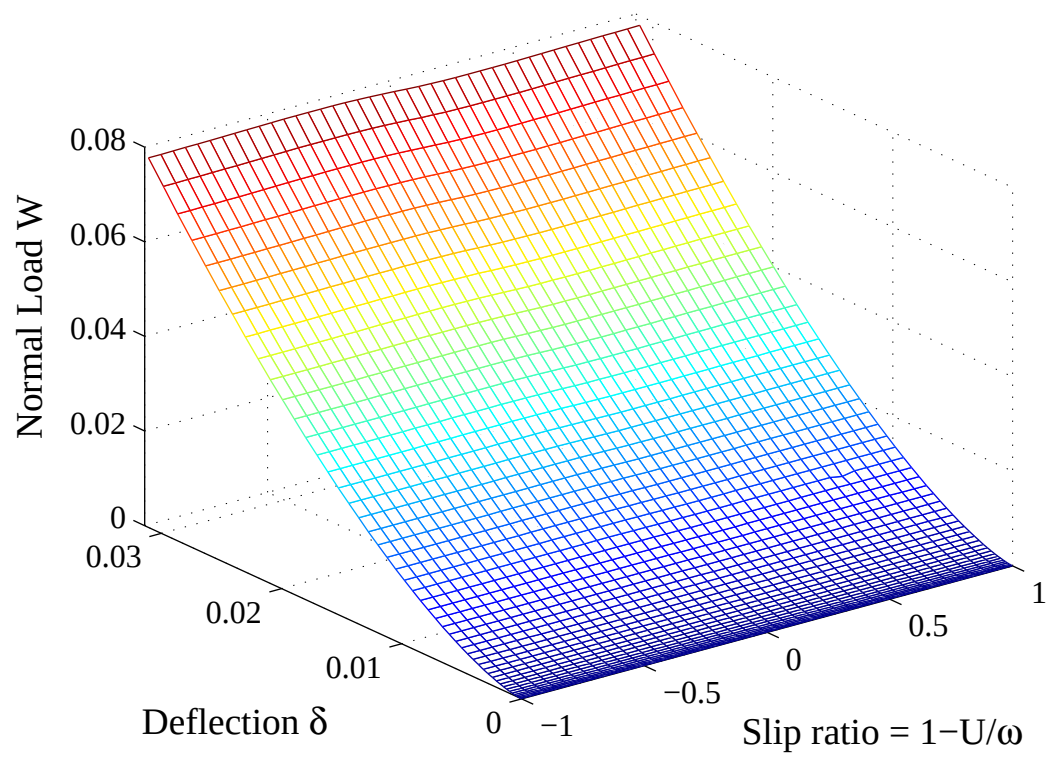


Figure 3.5: Normal Load parametrized by the deflection δ and the slip ratio $1 - U/\omega$ for the parameter values given in the text.

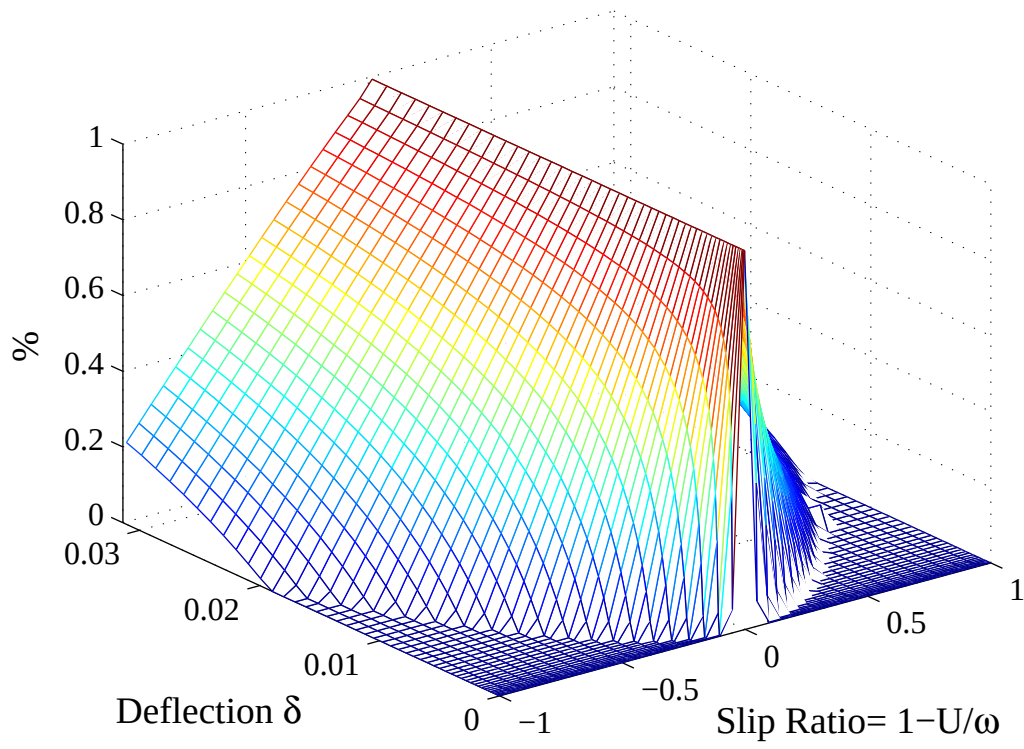


Figure 3.6: Percentage of the duration of contact corresponding to an initial stick phase parametrized by the deflection δ and the slip ratio $1 - U/\omega$ for the parameter values given in the text.

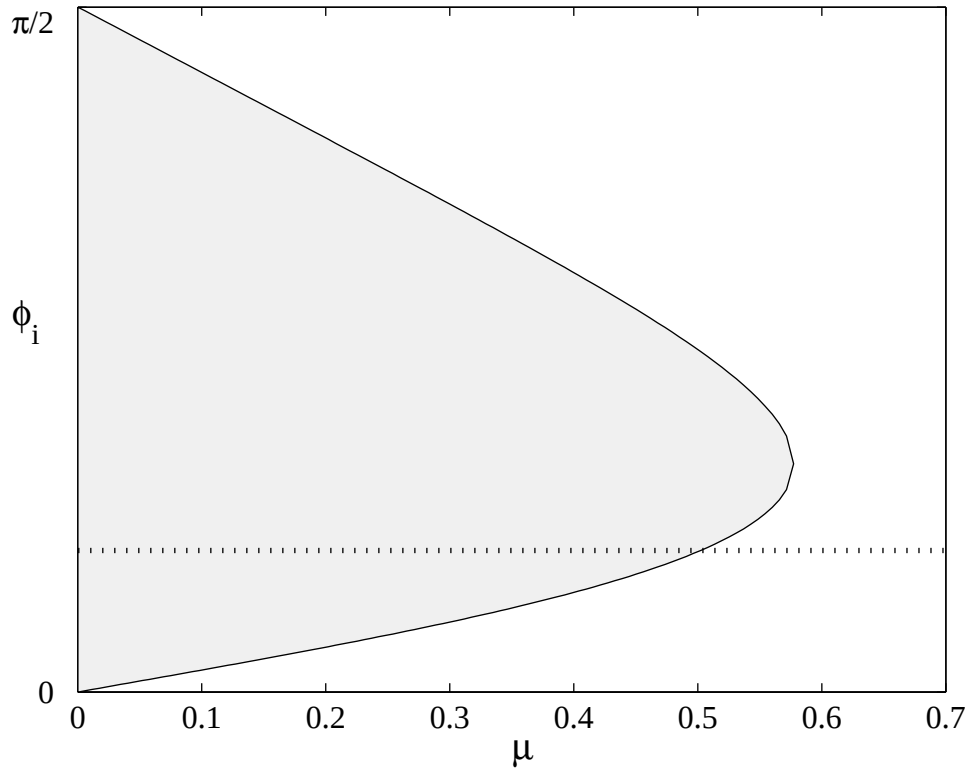


Figure 3.7: Root manifold of Eqn. (3.45) (solid) separating combinations of μ and ϕ_i corresponding to stick (white) and slip (shaded) for the parameter values given in the text. Here, the dashed horizontal line corresponds to a particular deflection δ .

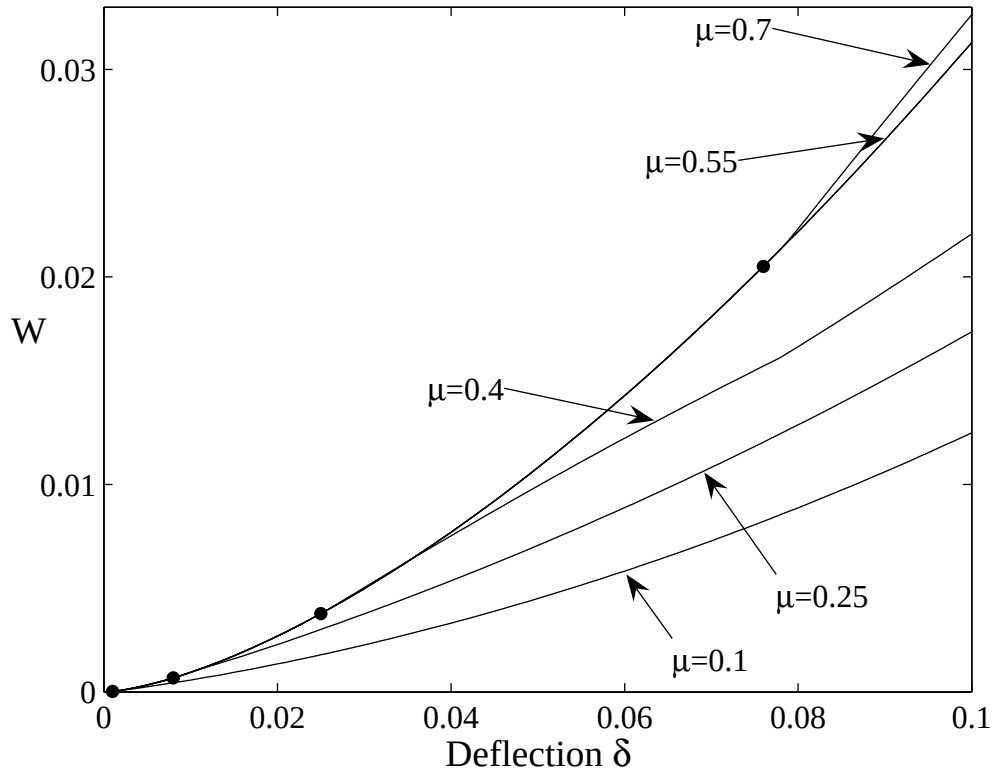


Figure 3.8: The normal load W as a function of the deflection δ for different values of μ . For $\mu < \mu^*$, the load-deflection curves coincide with that for $\mu = \mu^*$ until the value of δ corresponds to a locus on the root manifold in Fig. (3.7)

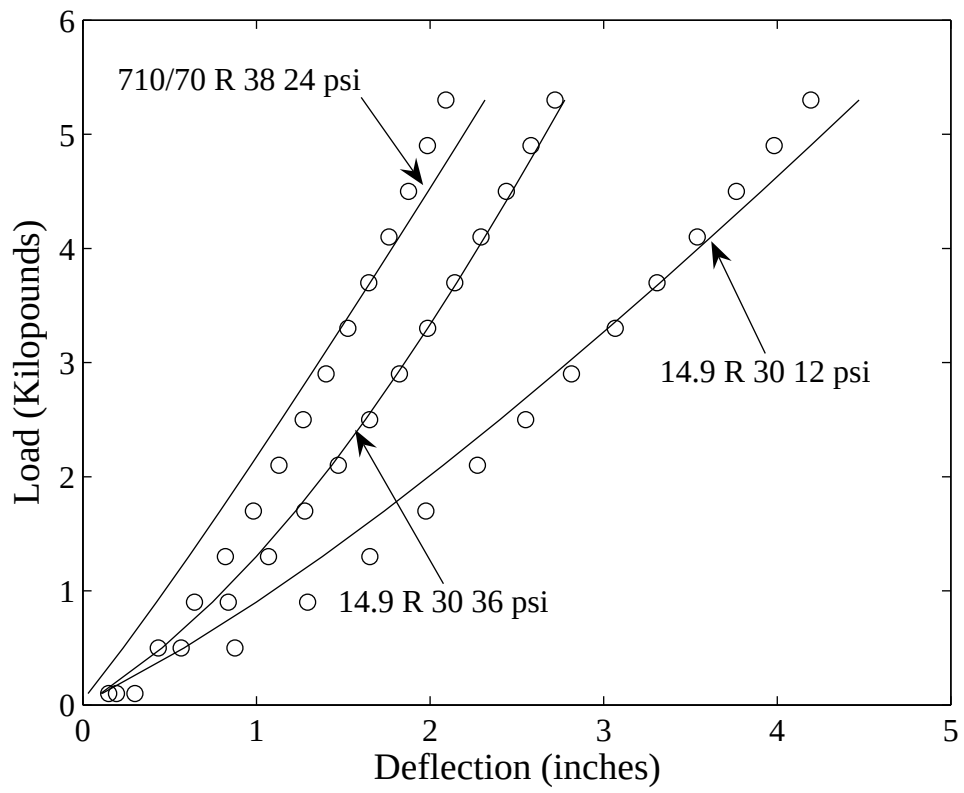


Figure 3.9: Experimental curve fits (Solid lines) for different tire conditions and predictions of the rigid-on-compliant model (Circles) for a locally optimal value of system parameters.

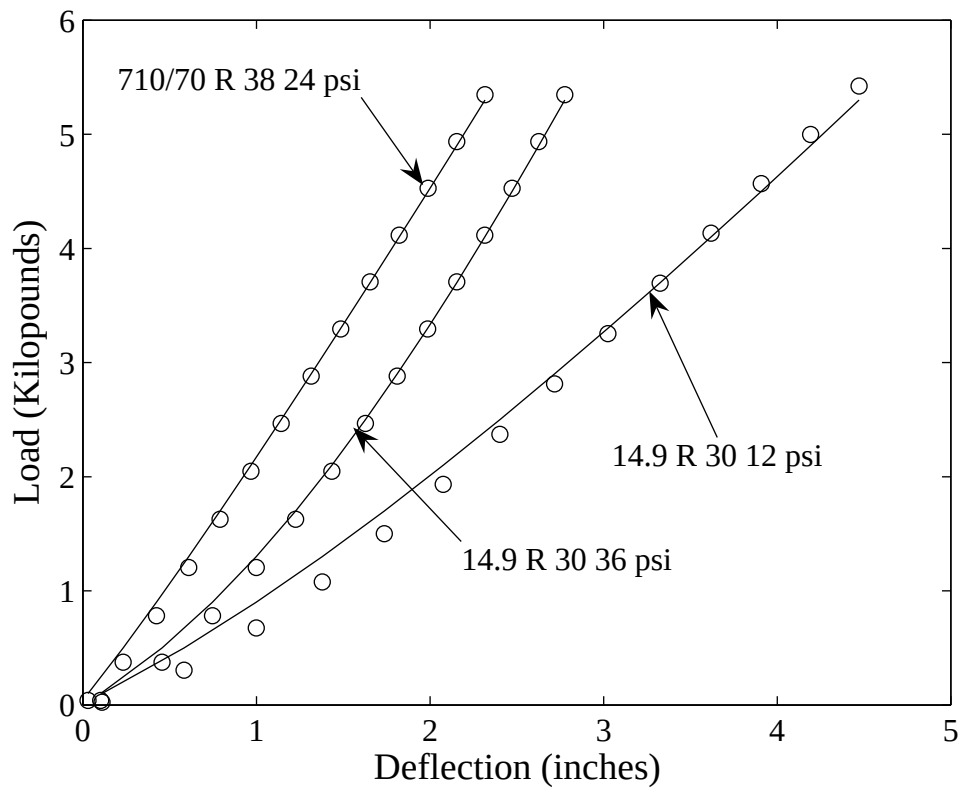


Figure 3.10: Experimental curve fits (Solid lines) for different tire conditions and predictions of the compliant-on-rigid model (Circles) for a locally optimal value of system parameters.

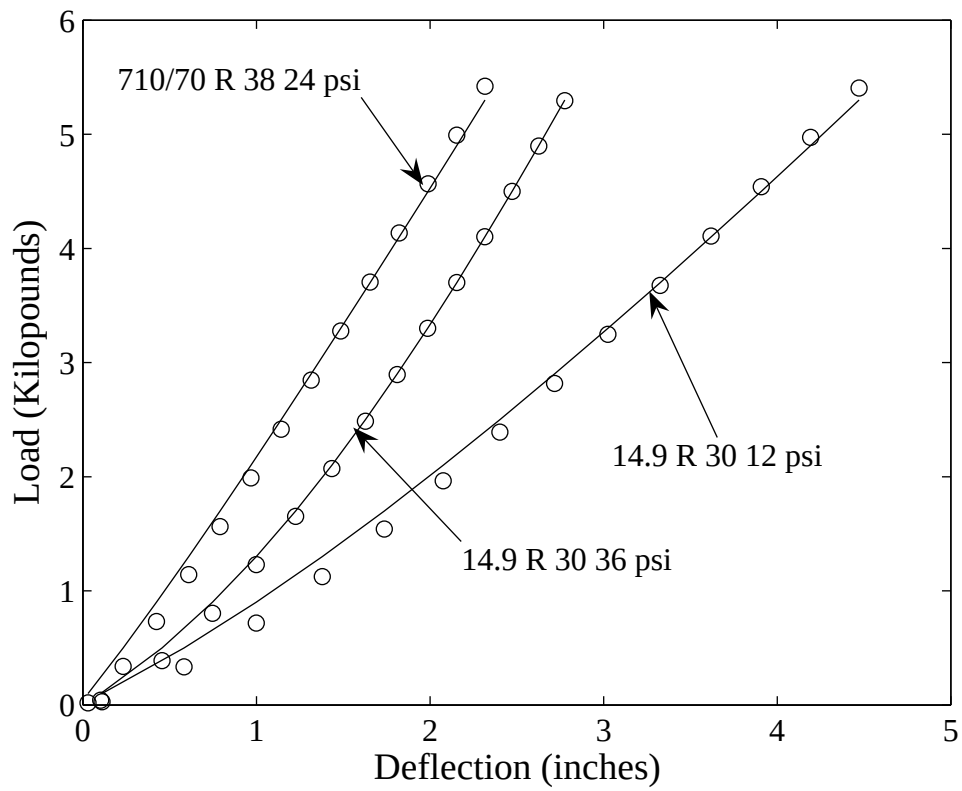


Figure 3.11: Experimental curve fits (Solid lines) for different tire conditions and predictions of the compliant-on-compliant model (Circles) for a locally optimal value of system parameters.

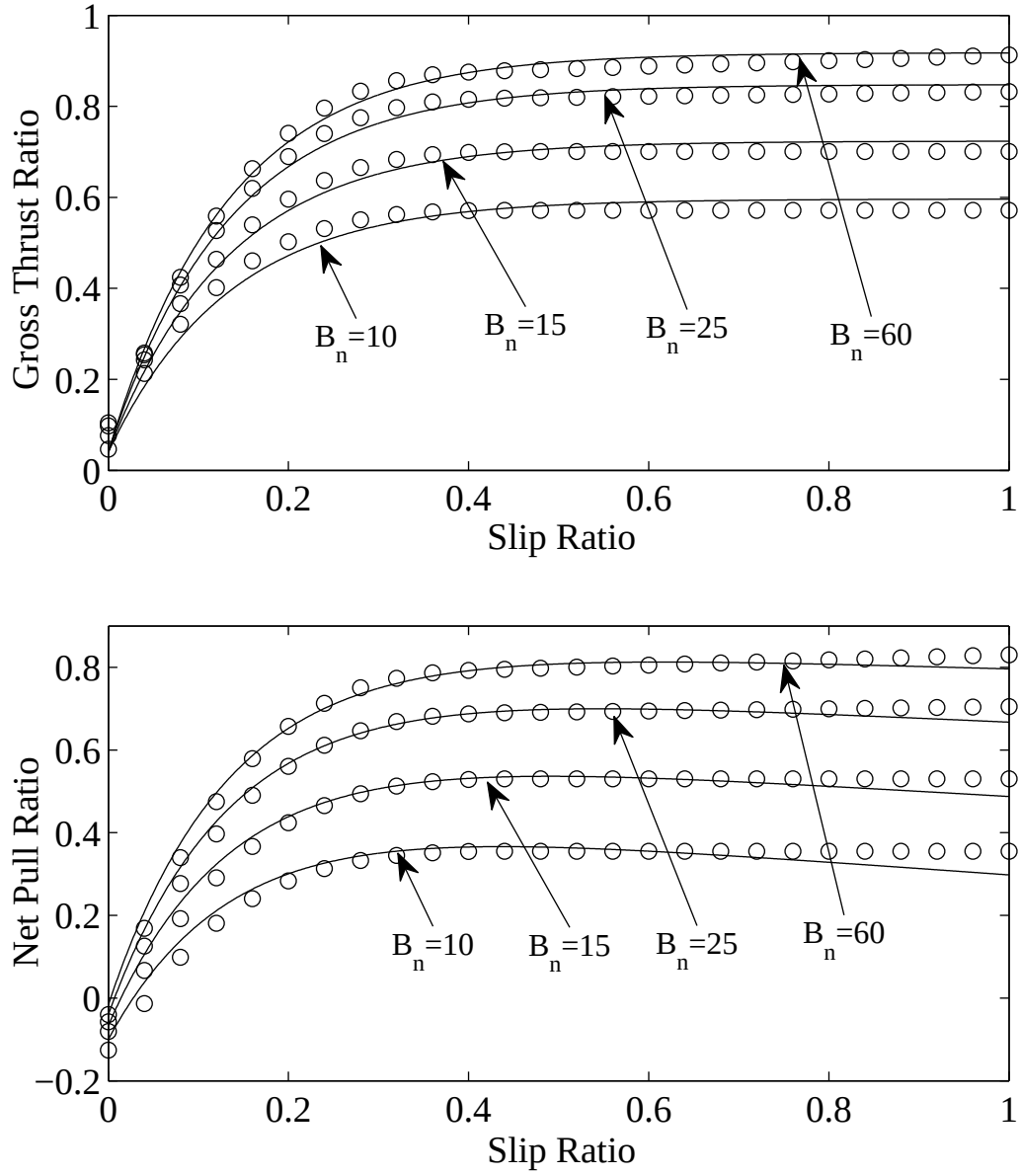


Figure 3.12: Experimental curve fits (Solid lines) at different mobility number [17] and predictions of the rigid-on-compliant model (Circles) for a locally optimal value of system parameters.

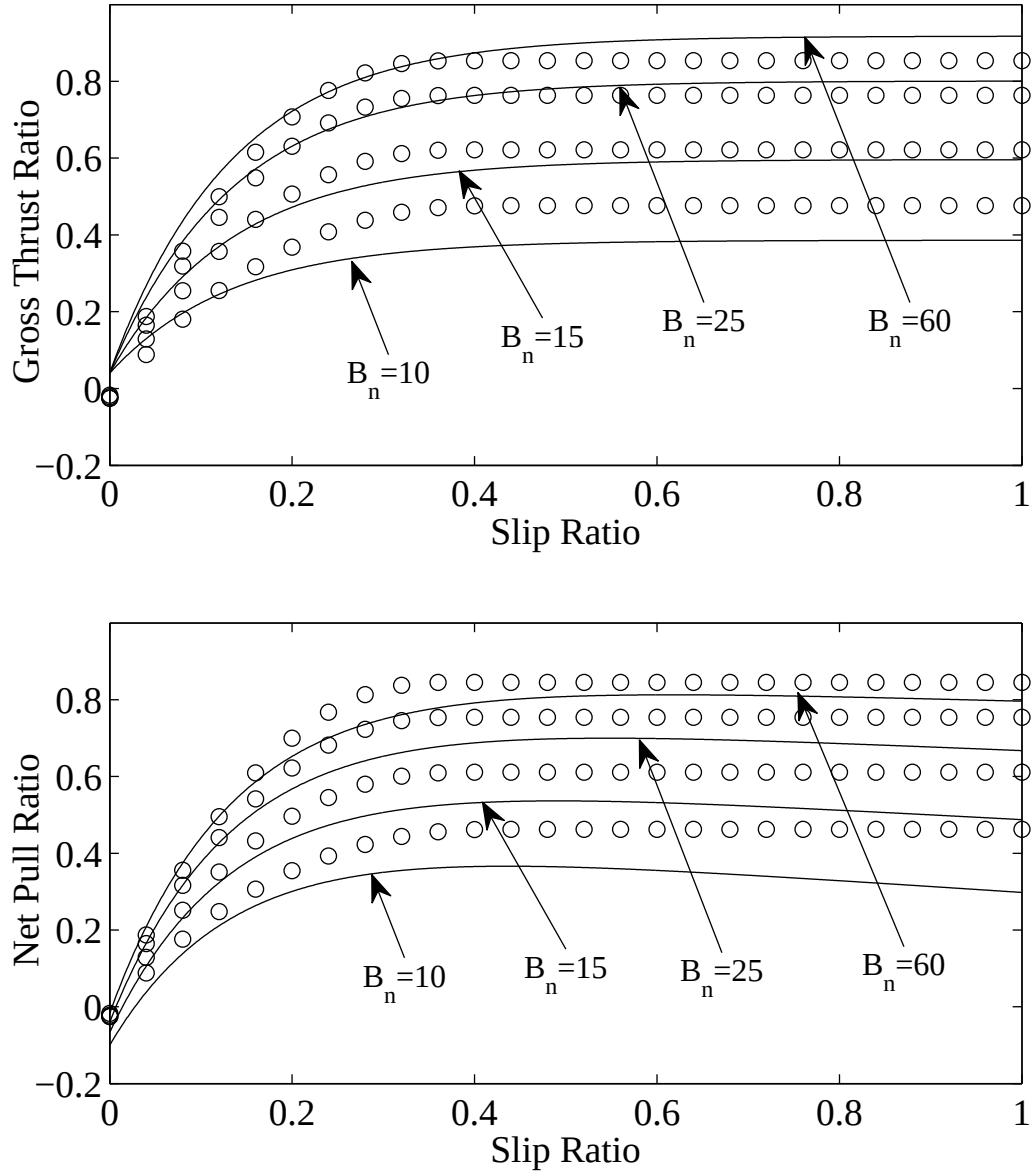


Figure 3.13: Experimental curve fits (Solid lines) at different mobility number [17] and predictions of the compliant-on-rigid model (Circles) for a locally optimal value of system parameters.

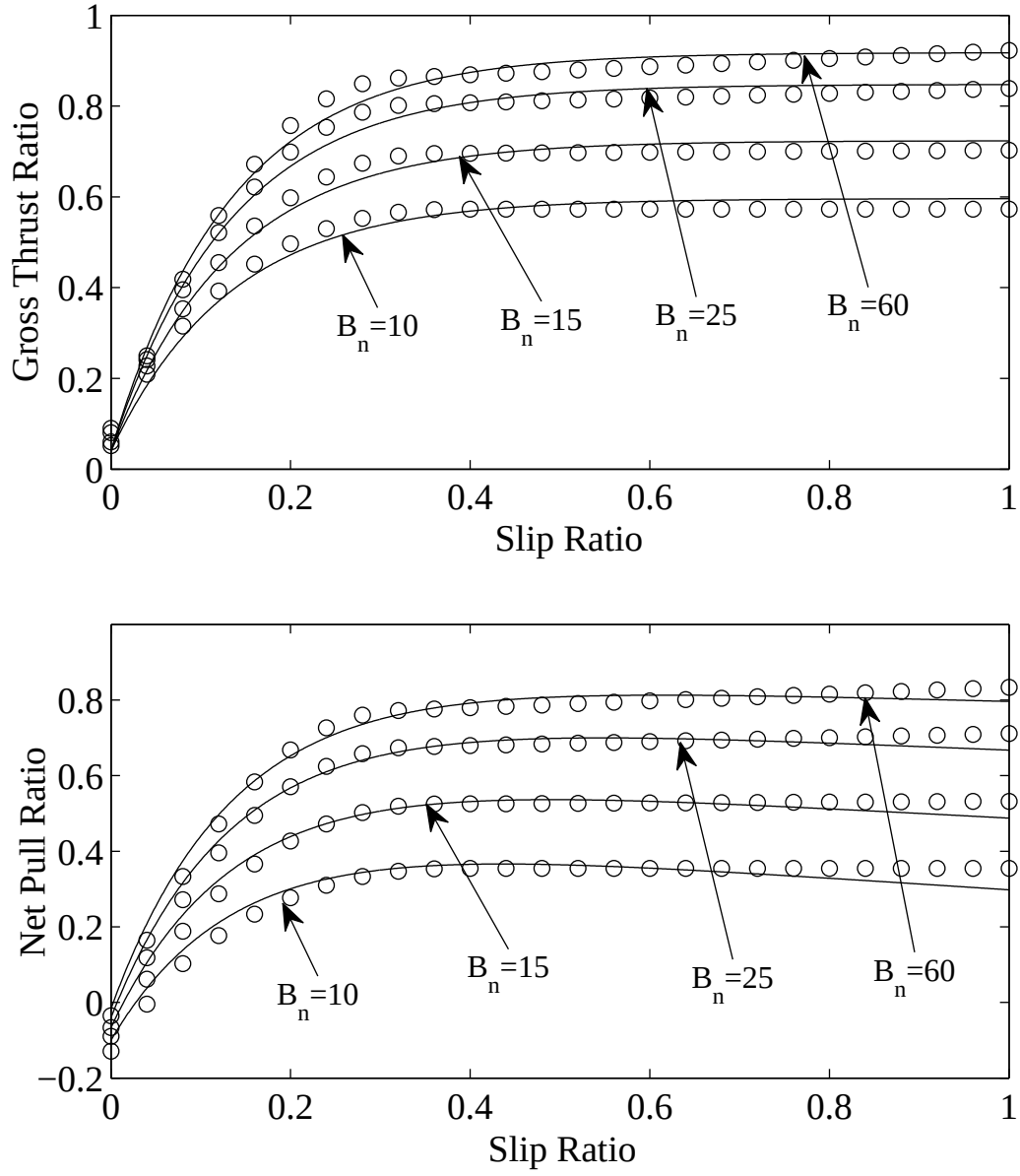


Figure 3.14: Experimental curve fits (Solid lines) at different mobility number [17] and predictions of the compliant-on-compliant model (Circles) for a locally optimal value of system parameters.

Chapter 4

3D Tire-Terrain Contact Model

We explore an extension of the model formulated in Chapter 3 to a two-dimensional contact condition. Specifically, let $\mathcal{N}$ again, denote the inertial reference system with reference point N located somewhere on the surface of the terrain and with reference basis n , such that $\mathbf{n}_1$ and $\mathbf{n}_2$ is in the plane of the surface, and $\mathbf{n}_3$ points in the upward vertical direction. Let the displacement of the center of mass of the body B with respect to N be given by

$$\mathbf{r}^{NB} = \sigma \mathbf{n}_1 + \eta \mathbf{n}_2 + \rho \mathbf{n}_3. \quad (4.1)$$

Consider a basis h ($\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$) rotated relative to n by an angle ν about the common 3-direction, such that $\mathbf{h}_1$ is parallel to the intersection of the plane of the body with the surface. The rotation matrix

$$R_h^n = \begin{bmatrix} \cos \nu & -\sin \nu & 0 \\ \sin \nu & \cos \nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.2)$$

transforms the coordinates of a vector in the basis h to the corresponding coordinates in the basis n . Furthermore, consider the body-fixed basis b ($\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$) rotated relative to h by an angle γ about the $\mathbf{h}_1$ direction, followed by another rotation by an angle φ about the $\mathbf{b}_2$ direction, such that the plane of the body is spanned by b . The rotation matrix

$$R_b^h = \begin{bmatrix} \cos \varphi & 0 & \sin \varphi \\ \sin \varphi \sin \gamma & \cos \gamma & -\cos \varphi \sin \gamma \\ -\cos \gamma \sin \varphi & \sin \gamma & \cos \varphi \cos \gamma \end{bmatrix} \quad (4.3)$$

transforms the coordinates of a vector in the basis b to the corresponding coordinates in the basis h . Finally consider a triad t rotated relative to b by an angle θ_i about the $\mathbf{b}_2$ direction, such that $\mathbf{t}_1$ is parallel to $\mathbf{r}^{BP_i}$. The rotation matrix

$$R_t^b = \begin{bmatrix} \cos \theta_i & 0 & \sin \theta_i \\ 0 & 1 & 0 \\ -\sin \theta_i & 0 & \cos \theta_i \end{bmatrix} \quad (4.4)$$

transforms the coordinates of a vector in the basis t to the corresponding coordinates in the basis b . It follows that

$$\begin{aligned}
\mathbf{r}^{NP_i} &= (\cos \nu \cos [\varphi + \theta_i] - \sin \nu \sin \gamma \sin [\varphi + \theta_i] + \sigma) \mathbf{n}_1 \\
&+ (\cos [\varphi + \theta_i] \sin \nu + \cos \nu \sin \gamma \sin [\varphi + \theta_i] + \rho) \mathbf{n}_2 \\
&- (\cos \gamma \sin [\varphi + \theta_i] + \eta) \mathbf{n}_3 \\
&= P_1 \mathbf{n}_1 + P_2 \mathbf{n}_2 + P_3 \mathbf{n}_3
\end{aligned} \tag{4.5}$$

To accommodate a model of the interactions between a deformable body on a deformable terrain, associate with each point P_i two contact points C_i and T_i , such that

$$\begin{aligned}
\mathbf{r}^{NT_i} &= \tau_{1,i} \mathbf{n}_1 + \tau_{2,i} \mathbf{n}_2 + \tau_{3,i} \mathbf{n}_3 \\
\mathbf{r}^{NC_i} &= \psi_{1,i} \mathbf{n}_1 + \psi_{2,i} \mathbf{n}_2
\end{aligned} \tag{4.6}$$

and such that the locations of the points C_i and T_i are instantaneously reset to coincide with P_i at the moment that contact is established between P_i and the surface. As long as the point P_i has a nonzero penetration below the surface, suppose again that the contribution to the interactions between the surface and the rigid body mediated through the point P_i is equivalent to a contact force

$$\mathbf{F}_c^{P_i} = -k_{w,1}(\mathbf{t}_1 \cdot \mathbf{r}^{T_i P_i}) \mathbf{t}_1 - k_{w,2}(\mathbf{t}_2 \cdot \mathbf{r}^{T_i P_i}) \mathbf{t}_2 - k_{w,3}(\mathbf{t}_3 \cdot \mathbf{r}^{T_i P_i}) \mathbf{t}_3 \tag{4.7}$$

applied at the point P_i and a contact torque

$$\mathbf{T}_c^{P_i} = \mathbf{r}^{P_i T_i} \times \mathbf{F}_c^{P_i}, \tag{4.8}$$

where $k_{w,1}$, $k_{w,2}$ and $k_{w,3}$ are material properties representing linear compliance of the body in the radial and the two tangential directions respectively. Suppose, moreover that, as long as the point P_i has a nonzero penetration below the surface, the position of the corresponding contact point T_i is given by the solution to the force balance equation

$$\mathbf{F}_c^{T_i} - \mathbf{F}_c^{P_i} = \mathbf{0}, \tag{4.9}$$

where

$$\mathbf{F}_c^{T_i} = -k_{f,1}(\mathbf{h}_1 \cdot \mathbf{r}^{C_i T_i}) \mathbf{h}_1 - k_{f,2}(\mathbf{h}_2 \cdot \mathbf{r}^{C_i T_i}) \mathbf{h}_2 - k_{w,3}(\mathbf{h}_3 \cdot \mathbf{r}^{C_i T_i}) \mathbf{h}_3 \tag{4.10}$$

Here, $k_{f,1}$, $k_{f,2}$ and $k_{f,3}$ are material properties representing linear compliance of the terrain in the longitudinal, lateral and normal directions, respectively.

Suppose that two different contact conditions may occur corresponding to two different motion constraints on the point C_i . In particular, the end point P_i is said to be in *stick* relative to the terrain when

the contact point C_i is **stationary** relative to the inertial reference frame, i.e., when

$$\frac{{}^{\mathcal{N}}dC_i}{dt} = \mathbf{0}, \quad (4.11)$$

and in *slip* relative to the boundary surface otherwise. To model friction-like bounds on the absolute value of the ratio between the tangential and normal components of $\mathbf{F}_c^{T_i}$ relative to the surface, consider the quantity

$$\beta_i = (\mathbf{F}_c^{T_i} \cdot \mathbf{h}_3) - \sqrt{\frac{(\mathbf{F}_c^{T_i} \cdot \mathbf{h}_1)^2}{\mu_x^2} + \frac{(\mathbf{F}_c^{T_i} \cdot \mathbf{h}_2)^2}{\mu_y^2}} \quad (4.12)$$

Specifically, suppose that stick is maintained as long as this remains positive. A transition to slip then occurs when β_i changes sign, provided that this contact condition is consistent with the imposition of a constraint on the motion of the point C_i , which ensures that β_i remains at zero. Furthermore, again let $\dot{\beta}_i|_{\text{stick}}$ denote the rate of change of β_i under the assumption of stationary C_i . Slip is then maintained as long as the rate of change of $\dot{\beta}_i|_{\text{stick}}$ remains negative. Finally, a transition back to stick occurs when the rate of change of $\dot{\beta}_i|_{\text{stick}}$ changes sign.

As in the case of the analysis in Chapter 3, a consistent formulation requires the contact point T to remain below the surface, as long as P has a non-zero penetration, since otherwise the sign of the vertical component of $\mathbf{F}_c^T$ corresponds to an adhesive interaction with the terrain. In addition, in slip, it is required that the time history of the contact point C not have any finite time singularities. The analysis to prove such requirements will be a topic of future work.

4.1 Unique Motion of Contact points

In the analysis of Chapter 3, the condition that $\beta_{i,\pm}$ remain equal to zero during slip resulted in a unique trajectory of the contact point C_i . In the analysis of the present chapter, the location of the contact point C_i is given by two coordinates. The condition $\dot{\beta}_{i,\pm}$ equal to zero must therefore be accompanied by an additional condition to yield a unique time history.

To this end, let

$$\dot{\psi}_x = \dot{\psi}_1 \cos \nu + \dot{\psi}_2 \sin \nu \quad (4.13)$$

$$\dot{\psi}_y = -\dot{\psi}_1 \sin \nu + \dot{\psi}_2 \cos \nu \quad (4.14)$$

corresponding to the components of the velocity relative to $\mathcal{N}$ in the h basis. Consider the following three conditions (cf. Svendenius [52])

$$F_y \dot{\psi}_x - F_x \dot{\psi}_y = 0 \quad (4.15)$$

corresponding to the tangential force being parallel to the velocity of the contact point C_i ;

$$F_y \mu_x \dot{\psi}_x - F_x \mu_y \dot{\psi}_y = 0; \quad (4.16)$$

and

$$F_y \mu_x^2 \dot{\psi}_x - F_x \mu_y^2 \dot{\psi}_y = 0. \quad (4.17)$$

where $F_x = \mathbf{F}_c^T \cdot \mathbf{h}_1$ and $F_y = \mathbf{F}_c^T \cdot \mathbf{h}_2$. Each of these, under suitable non-singularity conditions, yield a second condition under which a unique time history for the evolution of the contact point C_i may be determined.

To explain the significance of the third condition, consider the power of the tangential contact force acting on the point C_i

$$F_x \dot{\psi}_x + F_y \dot{\psi}_y. \quad (4.18)$$

From the condition that $\beta_i = 0$, it follows that

$$F_x = \mu_x F_z \cos \lambda \quad (4.19)$$

$$F_y = \mu_y F_z \sin \lambda \quad (4.20)$$

for some λ . Substituting these into Eqn. (4.18) and differentiating with respect to λ , shows that Eqn. (4.17) is a necessary condition for the power to be maximally dissipative. This condition is consequently often referred to as the maximal dissipation principle [53].

We proceed to apply this condition to the model of contact between a deformable tire and a deformable terrain introduced in this chapter. Specifically, it follows from Eqns. (4.7, 4.9 4.10),

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = (D_2 + D_1)^{-1} (D_1 \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} + D_2 \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \end{bmatrix}) \quad (4.21)$$

where

$$D_1 = R_t^n \begin{bmatrix} -k_{w,1} & 0 & 0 \\ 0 & -k_{w,2} & 0 \\ 0 & 0 & -k_{w,3} \end{bmatrix} (R_t^n)^{-1}, \quad D_2 = R_h^n \begin{bmatrix} -k_{f,1} & 0 & 0 \\ 0 & -k_{f,2} & 0 \\ 0 & 0 & -k_{f,3} \end{bmatrix} (R_h^n)^{-1} \quad (4.22)$$

and $R_t^n = R_h^n R_b^h R_t^b$. The component of the force acting on the point T are then given by

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = K \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} + L \begin{bmatrix} \psi_1 \\ \psi_2 \\ 0 \end{bmatrix} \quad (4.23)$$

where

$$K = (R_h^n)^{-1} D_2 (D_2 + D_1)^{-1} D_1, L = (R_h^n)^{-1} D_2 ((D_2 + D_1)^{-1} D_2 - I) \quad (4.24)$$

When $\beta = 0$, the condition that $\dot{\beta} = 0$ is equivalent to

$$G\dot{\psi}_x + H\dot{\psi}_y = F \quad (4.25)$$

where,

$$\begin{aligned} G &= \frac{F_x l_{11}}{\mu_x^2} + \frac{F_y l_{21}}{\mu_y^2} - F_z l_{31} \\ H &= \frac{F_x l_{13}}{\mu_x^2} + \frac{F_y l_{23}}{\mu_y^2} - F_z l_{33} \\ F &= \frac{F_x B}{\mu_x^2} + \frac{F_y C}{\mu_y^2} - F_z E \\ B &= \dot{k}_{11} P_1 - \dot{l}_{11} \psi_1 + \dot{k}_{12} P_2 + \dot{k}_{13} P_3 - \dot{l}_{12} \psi_2 + k_{11} \dot{P}_1 + k_{12} \dot{P}_2 + k_{13} \dot{P}_3 \\ C &= \dot{k}_{31} P_1 - \dot{l}_{31} \psi_1 + \dot{k}_{32} P_2 + \dot{k}_{33} P_3 - \dot{l}_{32} \psi_2 + k_{31} \dot{P}_1 + k_{32} \dot{P}_2 + k_{33} \dot{P}_3 \\ E &= \dot{k}_{21} P_1 - \dot{l}_{21} \psi_1 + \dot{k}_{22} P_2 + \dot{k}_{23} P_3 - \dot{l}_{22} \psi_2 + k_{21} \dot{P}_1 + k_{22} \dot{P}_2 + k_{23} \dot{P}_3 \end{aligned} \quad (4.26)$$

where $k_{i,j}$ and $l_{i,j}$ represents i-th row and j-th column element in K and L , respectively. On simultaneously solving linear Eqns. (4.17, 4.25)

$$\dot{\psi}_1 = \frac{F}{G + HR} \quad (4.27)$$

$$\dot{\psi}_2 = \frac{FR}{G + HR} \quad (4.28)$$

where

$$R = \frac{\sin \nu \mu_y^2 F_x + \cos \nu \mu_x^2 F_y}{\cos \nu \mu_y^2 F_x - \sin \nu \mu_x^2 F_y} \quad (4.29)$$

Remark 9 : In the limit that $k_{w,\parallel}, k_{w,\perp} \rightarrow \infty$ while $k_{f,\parallel}, k_{f,\perp}$ remain finite, after carrying out similar analysis, it follows that the components of the force acting on the point T in the h basis are given by Eqn. (4.23), where

$$K = -L = (R_h^n)^{-1} D_2 \quad (4.30)$$

Similarly, in the limit that $k_{f,\parallel}, k_{f,\perp} \rightarrow \infty$ while $k_{w,\parallel}, k_{w,\perp}$ remain finite, these components are again given by Eqn. (4.23) with τ_1 and τ_2 in the place of ψ_1 and ψ_2 ($\tau_3 = 0$ in this limit), where

$$K = -L = (R_t^n)^{-1} R_t^n \begin{bmatrix} -k_{w,1} & 0 & 0 \\ 0 & -k_{w,2} & 0 \\ 0 & 0 & -k_{w,3} \end{bmatrix} (R_t^n)^{-1} = D. \quad (4.31)$$

In this case,

$$\begin{aligned}
G &= \frac{F_x d_{11}}{\mu_x^2} + \frac{F_y d_{21}}{\mu_y^2} - F_z d_{31} \\
H &= \frac{F_x d_{13}}{\mu_x^2} + \frac{F_y d_{23}}{\mu_y^2} - F_z d_{33} \\
F &= \frac{F_x B}{\mu_x^2} + \frac{F_y C}{\mu_y^2} - F_z E \\
B &= \dot{d}_{11}(P_1 - \tau_1) + \dot{d}_{12}(P_2 - \tau_2) + \dot{d}_{13}P_3 + d_{11}\dot{P}_1 + d_{12}\dot{P}_2 + d_{13}\dot{P}_3 \\
C &= \dot{d}_{31}(P_1 - \tau_1) + \dot{d}_{32}(P_2 - \tau_2) + \dot{d}_{33}P_3 + d_{31}\dot{P}_1 + d_{32}\dot{P}_2 + d_{33}\dot{P}_3 \\
E &= \dot{d}_{21}(P_1 - \tau_1) + \dot{d}_{22}(P_2 - \tau_2) + \dot{d}_{23}P_3 + d_{21}\dot{P}_1 + d_{22}\dot{P}_2 + d_{23}\dot{P}_3
\end{aligned} \tag{4.32}$$

respectively, where $d_{i,j}$ represents i -th row and j -th column element in D .

We finally note an alternative to the above treatment for uniquely resolving the time history of the contact point C_i based on the approach used by Sharpet *al.* [4]. Let $\tilde{K}$ be given by the first two rows of the matrix K in Eqn. (4.24). Similarly, let $\tilde{L}$ be given by the upper left two-by-two submatrix of L in Eqn. (4.24). Then

$$\begin{bmatrix} \psi_{1,0} \\ \psi_{2,0} \end{bmatrix} = \tilde{L}^{-1} \tilde{K} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{4.33}$$

equals the position of the contact point C_i that would result in zero tangential force. Denote this position by C_0 . Now consider the conditions

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \kappa \begin{bmatrix} \psi_1 - \psi_{1,0} \\ \psi_2 - \psi_{2,0} \end{bmatrix} \tag{4.34}$$

where κ is initialized at 1 at the onset of slip. To first order in Δt , it follows that the position of C_i at time $t + \Delta t$ lies on the straight line segment between the position of C_i at time t and the position of C_0 at time $t + \Delta t$. Together with $\dot{\beta} = 0$, these conditions may be solved to yield $\dot{\psi}_1$ and $\dot{\psi}_2$.

4.2 Motion of Contact points at the onset of contact

As in the case of the analysis in Chapter 3, it is necessary to determine the initial phase of motion following the onset of contact. Recall that at the onset of contact, $F_x = F_y = F_z = 0$. Suppose that the onset of contact results in stick. In this case, from the expansion

$$\beta = \left(\dot{F}_z \Big|_{\text{stick}} - \sqrt{\frac{\dot{F}_x^2 \Big|_{\text{stick}}}{\mu_x^2} + \frac{\dot{F}_y^2 \Big|_{\text{stick}}}{\mu_y^2}} \right) \Delta t + \mathcal{O}(t^2), \tag{4.35}$$

where $\dot{F}_x|_{\text{stick}}$, $\dot{F}_y|_{\text{stick}}$ and $\dot{F}_z|_{\text{stick}}$ are evaluated at the onset of contact, this state of motion is consistent with the condition on β provided that

$$\dot{F}_z|_{\text{stick}} > \sqrt{\frac{\dot{F}_x^2|_{\text{stick}}}{\mu_x^2} + \frac{\dot{F}_y^2|_{\text{stick}}}{\mu_y^2}} \quad (4.36)$$

otherwise, the spoke must enter the terrain in slip.

In the case of slip, $\beta = 0$ implies that

$$F_x - \mu_x F_z \cos \lambda = 0 \quad (4.37)$$

$$F_y - \mu_y F_z \sin \lambda = 0 \quad (4.38)$$

for some λ , from which it follows that

$$\dot{F}_x + \mu_x \dot{\lambda} F_z \sin \lambda - \mu_x \dot{F}_z \cos \lambda = 0 \quad (4.39)$$

$$\dot{F}_y - \mu_y \dot{\lambda} F_z \cos \lambda - \mu_y \dot{F}_z \sin \lambda = 0 \quad (4.40)$$

and consequently,

$$\dot{\lambda} F_z = \frac{\mu_y \dot{F}_x \sin \lambda - \mu_x \dot{F}_y \cos \lambda}{\mu_x \mu_y} \quad (4.41)$$

$$\dot{F}_z = \frac{\dot{F}_x \mu_y \cos \lambda + \dot{F}_y \mu_x \sin \lambda}{\mu_x \mu_y} \quad (4.42)$$

In the limit as F_z goes to zero, λ relaxes at an infinite rate to a value for which the right-hand side of the Eqn. (4.41) equals zero. It follows that at the onset of contact, λ should be initiated to such a value for which $\dot{F}_z > 0$. Denote this value by λ_0 . It follows that

$$\dot{F}_x - \mu_x \dot{F}_z \cos \lambda_0 = 0 \quad (4.43)$$

$$\dot{F}_y - \mu_y \dot{F}_z \sin \lambda_0 = 0 \quad (4.44)$$

$$\mu_x \sin \lambda_0 \dot{\psi}_x - \mu_y \cos \lambda_0 \dot{\psi}_y = 0 \quad (4.45)$$

where the last equation is obtained by differentiating the maximal dissipation principle, and all derivatives are evaluated at the onset of contact. Using Eqn. (4.23), one can eliminate $\dot{\psi}_1$ and $\dot{\psi}_2$ and come up with a single non-linear equation in λ_0 .

There remains to prove that this non-linear equation has exactly one solution for which $\dot{F}_z > 0$ (but see [54] for similar discussion in the context of impact models). This is the topic for further research.

Chapter 5

Conclusions

A physical planar multi-spoke contact model to capture transient interactions between a deformable tire and a deformable terrain has been developed. The analysis has established the mathematical and energetical consistency of the model, for example by guaranteeing that all internal variables are uniformly continuous functions on bounded intervals of time. Special emphasis has been placed on uniquely resolving any ambiguities associated with such transitions.

A detailed analysis have been carried out to thoroughly understand the tire-terrain interactions under two steady state conditions, namely the static load-deflection and the steady state non-zero motion of the tire. Under this steady state operating conditions, a non-linear regression (`lsqnonlin` in Matlab) was applied to seek optimal model parameters to fit the predicted data to the Bridgestone/Firestone load-deflection and the brixius gross thrust ratio and net pull ratio experimental curve fits. The analysis demonstrated a good quantitative agreement of the predicted data to the experimental curve fits. Although the non-linear regression analysis applied for parameter identification appears to have yielded a relatively good fit between the model predictions and the experimental curve fits, a more satisfying approach would be to base this identification on direct experimental data. It might also be appropriate to consider large numbers of separate optimization runs to avoid convergence onto local minima.

The preliminary research done to extend the planar model capture tire-terrain interactions in the lateral direction is also discussed. For example, the issue of uniquely resolving the evolution of the internal state variable is been ensured by satisfying the maximal dissipation principle for localized slip.

Future Work

1. Identification of model parameters to fit transient data.

In this thesis, efforts to seek optimal model parameters to fit the predicted data to steady state experimental curve fits is presented. The next step in this area would be to fit predicted data to steady state experimental data instead of curve fits. In the literature there exists a varieties of model that is able to capture tire-terrain interactions under steady state circumstances. Most of

these models are incapable of accurately predicting interactions in transient and cornering conditions. Therefore it becomes essential to perform identification of model parameters to fit model predictions to experimental data for transient conditions like acceleration, deceleration, turning and etc.

2. Complete analysis of 3D model

There are certain issues, as discussed in Chapter 4, yet to be resolved in the formulation of the 3D model. Further one can then perform identification of model parameters to fit model predictions to experimental steady state as well as transient data in 3D. Moreover one can then perform detailed analysis of the model behaviour under conditions like turning, acceleration and braking.

3. Implementation of the model for vehicle handling simulations.

Currently the model is simulated by solving piecewise continuous differential equations and detecting events that triggers transitions between the various phases of motions. This approach is time consuming. Various efforts have been put in to come up with more efficient way of simulating such models. One such method is that of the set value functions proposed in [53]. There is scope of improvement when model simulation is concerned and efforts to reformulate the model in terms of set value functions could prove to be of worthy.

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