ESSAYS ON ROBUST OPTIMIZATION, INTEGRATED INVENTORY AND PRICING, AND REFERENCE PRICE EFFECT

BY

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DISSERTATION

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Abstract

This dissertation consists of two distinct lines of research efforts. Chapter 2 proposes a general methodology to seek robust solution to multi-stage stochastic optimization problems. Chapters 3, 4 and 5 all deal with models that arise from inventory management and dynamic pricing.

Chapter 2 introduces the Extended Affinely Adjustable Robust Counterpart(EAARC). We first propose the general steps of extending affine decision rules via re-parameterizing the uncertainty set, then propose the example of splitting-based EAARC. We show that this approach extends the versatility of affine decision rules beyond what has been proposed by Ben-Tal et al. [9], while retaining tractability.

Chapter 3 looks at the classical joint inventory-and-pricing model (single product periodic-review) with concave ordering cost. Concave cost structures may often occur in settings with multiple sources of supply. For this model, assuming additive demand uncertainty, we show that a generalized (s, S, p) policy is optimal under certain conditions imposed on the distribution of the random perturbation.

Chapter 4 and 5 focus on the reference price effect in which the price impact on demand is no longer instantaneous, but history-dependent. Chapter 4 analyzes a joint inventory-and-pricing model with reference price effect. We prove that a reference price dependent base-stock policy is optimal even though the single period expected profit may not be concave. In the infinite horizon case, we further show that in the optimal trajectory, reference price converges to a steady state and provide a characterization. Finally, chapter 5 studies a continuous-time dynamic pricing problem under stochastic reference price effect. Stochastic optimal control theory is applied to the problem to derive an explicit solution. Various comparative statics are then conducted to benchmark our model against a few simplified models.

To my parents, for their love and support.

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Chapter 1

Introduction

1.1 Motivations and Philosophy

The intention of this thesis is twofolds: to develop general methodologies for multi-stage optimization problems under uncertainty; and to analyze specific models that arise from an operations management context.

Chapter 2 describes the first line of our efforts. We look at optimization problems under uncertainty, which means that certain parameters of the problem formulation might not be known in exact value. In most situations, explicitly assuming uncertainty leads to more realistic models. Because uncertainties arise from nearly every step in the modeling process. As an example, measurements from any physical process are subject to random noise, hence the true value that they are trying to measure would always be uncertain. Another example, the rainfall of a particular day in the future is uncertain. The difference between this and the previous example is that the true value for the rainfall will become revealed when that day comes(ignorning uncertainties from measuring rainfall), whereas in the measurement example, the true value will never be revealed. This difference will be crucial in multi-stage problems.

There are two approaches to uncertainty in a model. The first approach views any uncertain parameter as a random variable, with known probability distribution. For example observations from physical processes are usually assumed to be normally distributed, with some standard error implied by the accuracy of the measuring tool. In this approach, the decision maker's job is then to make his decision (or decisions contingent to outcomes of some uncertainties, if he is dealing with a multi-stage decision process) to maximize his expected gain, or minimize his expected loss. Here the expectation is of course taken over the random parameters. This approach, known as *Stochas-*

tic Programming, has been a fruitful area of research. One can consult Birge and Louveau [21] for an introductory treatment of the subject.

The second approach, to which our work belongs, takes a more conservative view of uncertainty. Namely it does not assume any known probability distribution of the uncertain parameters. Rather, it only assume a range of possible values for the uncertain parameter, usually in the form of an interval centered around some nominal value. Given these assumptions the decision maker's task is to make a decision that optimizes his worst case outcome, with respect to the ranged uncertainties. This approach is called *Robust Optimization*, the word "robust" in its name naturally follows from the fact that a decision maker is always trying to make a robust decision. The book by Nemirovski et al. [7] is an excellent reference on the subject.

The previous discussion has alluded to the major difficulty of multi-stage problems under uncertainty. When decision is made across time, those that are made at a later time may benefit from some uncertainties being revealed at that time (again, think of the rainfall example). Therefore decisions at a later time should be contingent on uncertainties revealed to the decision maker at that time. To be more precise, the decision maker would not be choosing scalar quantities for his decision variables. He will be choosing functions - functions of uncertainties that are realized before that decision takes place (known as *response functions*). This exponentiates the complexity of the decision process and, needless to say, is a daunting task. To regain tractability, the approach generally taken is to assume that the response functions take a certain form. Affin functions are common choices for this purpose, because the task of choosing an affine function boils down to choosing two quantities: the slope and the abscissas. Our approach takes one step beyond affine functions, and propose methodologies that will provide more versatile response functions, while retaining tractability.

Chapter 3 - 5 represent the second line of our efforts. In these chapters we analyze models related to inventory management, dynamic pricing, and reference price effects. The main task of classical inventory management problems, is to make more informed inventory ordering decisions in the presense of demand uncertainty, so that inventory cost is brought down, and drastic overstocks or understocks are avoided. On the contrary, dynamic pricing problems focus on the revenue side of profit. Dynamic pricing problems usually work around an established model for the price impact on demand. It then develops an optimal way to set different prices for a product across time, and across consumer segments. This would ensure that the firm's revenue is maximized.

The pair of problems described above both serve the purpose of increasing a firm's bottomline - one through the cost side and one through the revenue side. Traditionally, these two decisions are made separately. Marketing decision is made first, and then a rough sales target is formed. Inventory decisions are then made contingent on that sales target.

Recently there has been an increasing amount of work that tries to make an integrated decision. The benefit of an integrated decision is obvious: it is always better than two separate decisions, as it takes both sides into account simultaneously. This is especially important in the recent few decades, when electronic-based retailing has devoured much of the traditional brick-and-mortar stores, so that the complexities in product line, consumer demographics, supply chain structure have all grown drastically. With such a clear need for an integrated inventory-and-pricing decision process, there is of course the cost: additional complexity in the models makes them harder and harder to solve. That is why there is now a considerable amount of effort in the research community seeking to find theoretically guaranteed structures in optimal solution to these problems, so that the search for an optimal solution can be conducted on a smaller space, with complexity greatly reduced. Chapter 3 of this thesis belongs to this stream of theoretical analysis. It builds upon previously established results and generalize them to models with more general cost structure. Literature review on those previous results will be included in the next section and in later chapters.

Lastly, Chapter 4 and 5 focuses on what's called the *reference price effect* which was observed through empirical research in marketing science. The effect, which will be defined more precisely in chapter 4, is roughly as follows: When consumers make their purchasing decision of a product, they observe the current selling price, but they are usually aware of historical prices charged on this product. Therefore they are very likely to form their reference price based on those historical prices, and this reference price would in turn influence their purchasing decision at the present time. The implication of this effect is that, when a firm sets its price for its products, the price not only has impact on current demand, but also on future demands. Thus to make the optimal dynamic pricing decisions, the firm must balance instantaneous effects with future consequences.

To understand the implications of this effect on dynamic pricing decisions, chapter 4 analyzes a joint inventory-and-pricing model with reference price effects. Chapter 5 looks at the effect with slightly more scrutiny: reference price is formed by consumers' sentiments of historical prices, which is a very subjective process. Therefore it should be natural to assume that the evolution process of reference price is subject to randomness. By explicitly modeling reference price as a stochastic process, chapter 5 aims at characterizing the impact of this additional randomness on the optimal decision process, and tries to arrive at a more realistic and trustworthy model.

1.2 Literature Review

Detailed literature review will be provided at the beginning of each chapter. This section only describes in broad strokes some of the theories/methods that are already in existant in the literature, and are closely related to this thesis.

The book by Nemirovski et al. [7] contains most of the recent developments in robust optimization, together with some illustrative examples. The methodology that we will describe in chapter 2 is closely related to the paper by Ben-Tal et al. [9]. To deal with the explosive complexity in a multi-stage robust optimization problem, Ben-Tal et al. [9] develops the Affinely Adjustable Robust Counterpart(AARC). In chapter 2 we will extended their idea to include some non-affine response functions.

There is a vast pool of literature on the topic of joint inventory-and-pricing models, which we will analyze in chapters 3 and 4. One can consult Chen and Simchi-Levi [28] for a comprehensive review of this area. Some other excellent resources include Elmaghraby and Keskinocak [34], Federgruen and Heching [35] and Yano and Gilbert [59].

As for the topic of reference price effect, the majority of the literature belongs to marketing science. Mazumdar et al. [46] gives a good review of the many statistical models proposed for reference price effect. There are a few seminal papers trying to solve the problem of dynamic pricing in the presence of reference price effect. We will defer the introduction of those works until chapter 4.

1.3 Structure of the Thesis

Chapter 2 introduces our methodology to multi-stage robust optimization problems - the EAARC. Section 2.2 illustrates the limitation of the AARC and introduce the EAARC. Section 2.3 presents constraint reformulation of EAARC. Then section 2.4 describes a particular scheme of EAARC - the splitting based EAARC and identifies conditions under which the splitting based EAARC improves upon the AARC. Numerical experiments are described in section 2.5 and concluding remarks are given in section 2.6.

Chapter 3 describes our theoretical work on joint inventory-and-pricing models with general concave cost. Section 3.2 presents our main theoretical results with its proof included in Appendix A. Section 3.3 then applies the theoretical results to the joint inventory-and pricing model.

Chapter 4 analyzes the joint inventory-and-pricing model with reference price effects. Specifically section 4.2 presents the model for a finite horizon, and proves optimality of the base-stock policy. Section 4.3 then presents the model for infinite horizon and proves convergence results. Section 4.4 provides characterization of steady states and finally section 4.5 makes concluding remarks.

Chapter 5 studies the dynamic pricing problem with stochastic reference price effects. Section 5.2 presents the model. The optimal pricing policy is analyzed in section 5.3 and section 5.4 provides numerical experiments. Conclusions for this chapter are given in section 5.6.

Finally, Chapter 6 points out some potential directions for future research.

Chapter 2

Uncertain Linear Programs: Extended Affinely Adjustable Robust Counterparts

2.1 Introduction

Decision making under uncertainty is the key ingredient in many operations research problems, for instance, supply chain management, revenue management and financial planning. One of the most important approaches for optimization under uncertainty is stochastic programming, in which objectives and constraints of optimization models are defined by averaging over possible outcomes or considering probabilities of events of interest. Over the past fifty years, a variety of stochastic programming theory and algorithms have been developed and some successful stochastic programming applications have also been reported (see, e.g. Ruszczynski and Shapiro [51], Birge and Louveau [21]).

However, despite its immense modeling potential, stochastic programming faces two significant challenges. First, stochastic programs, especially multistage problems, are notoriously difficult to solve to optimality and quite often, even finding a feasible solution is already a hard problem. Second, stochastic optimization problems require full distributional knowledge in each of the uncertain data. Unfortunately, such information may rarely be available in practice. The lack of tractable methodology and the full distributional requirement have restricted the applicability of stochastic programming in many practical settings.

To cope with some of the challenges faced by stochastic programming, robust optimization received considerable attention in recent years as an alternative approach to deal with optimization problems under uncertainty. The first step in this direction was taken by Soyster [55] who proposed a worst case model for linear optimization such that constraints are satisfied under all possible perturbations of the uncertain data of the the underlying model. Recent developments in robust optimization focused on more elaborate uncertainty sets of uncertain data in order to alleviate over-conservatism in worst case models, as well as to maintain computational tractability of the proposed approaches, (see, for example, Ben-Tal and Nemirovski [11, 12, 13], El-Ghaoui and Lebret [32], El-Ghaoui et al. [33], Goldfarb and Iyengar [40], Bertsimas and Sim [17, 18, 19, 20], Atamtürk [3]).

Most of the research on robust optimization focuses on static settings, in which all decisions must be made before the actual realization of the uncertain data (referred to as the *primitive uncertainties*). To extend the robust optimization methodology to dynamic settings, Ben-Tal et al. [9] proposed the *Adjustable Robust Counterpart* (ARC), in which the primitive uncertainties are assumed to vary within an uncertainty set while some decisions (recourse variables) can be made after the realization of the primitive uncertainties and be adjusted to its actual realization. A closely related approach was proposed by Bertsimas and Caramanis [14]. In this approach, they introduced the concept of finite adaptability, which is based on the selection of a finite number of (constant) contingency plans to incorporate the information revealed over time. Bertsimas and Caramanis [15] applied it to model the air traffic control. On the other hand, under the adjustable robust counterpart framework, Atamtürk and Zhang [4] analyzed network design problems under uncertainty.

Since the general adjustable robust counterpart is intractable, Ben-Tal et al. [9] proposed a tractable approach for solving fixed recourse instances using affine decision rules – restricting recourse variables as affine functions of the realization of the primitive uncertainties, referred to as the *Affinely Adjustable Robust Counterpart* (AARC). Even though the AARC has been successfully applied to inventory management (Ben-Tal et al. [9]) and supply contract problems (Ben-Tal et al. [8]), it is not surprising that the performance of the AARC may not be satisfactory under situations in which the recourse variables may exhibit high nonlinearity in terms of the primitive uncertainties.

The goal of this chapter is to illustrate that the potential of the AARC method is well beyond the one presented in Ben-Tal et al. [9]. Indeed, by re-parameterizing the primitive uncertainties and then applying the AARC method, we end up with a new model, which allows us to relax to certain degree the linearity restriction imposed by the AARC. Specifically, in our approach, we re-parameterize the primitive uncertainties by introducing auxiliary variables and represent the recourse variables as affine functions of the auxiliary variables. By using these auxiliary variables, the model can now capture certain nonlinear response of the recourse variables to the primitive uncertainties. In the sequel, we refer to the AARC as the AARC method directly applied on the primitive uncertainties while the *Extended Affinely Adjustable Robust Counterpart* (EAARC) as the AARC method applied to the re-parameterized model.

Since the primitive uncertainties can be re-parameterized in a variety of different ways, the EAARC is rather flexible and encompasses a broad class of decision rules. We analyze a specific EAARC - the splitting based EAARC in depth. In a simple setting, the splitting based EAARC essentially introduces auxiliary variables to represent the positive and negative parts of the primitive uncertainties. We demonstrate both theoretically and computationally that the splitting based EAARC may significantly improve upon the AARC.

The idea of re-parameterizing the original problem before applying the robust counterpart has been used in several papers for different purposes. For instance, to avoid the over-conservatism incurred by working directly on the primitive uncertainties, Ben-Tal et al. [10] re-parameterized the original multi-period portfolio selection problem and then apply the robust counterpart approach. In Ben-Tal et al. [6], the authors used a re-parametrization scheme in a linear control problem to avoid a non-convex robust counterpart. It is also common in the robust optimization literature, including Ben-Tal et al. [9] and our work here, to re-parameterize the uncertainty data by a vector of perturbations, referred to as the *primitive uncertainties*, varying in a nonempty convex compact perturbation set. Indeed, the AARC in Ben-Tal et al. [9] first applies the affine decision rule on the uncertain data, which are then re-parameterized in terms of the primitive uncertainties, while here we first re-parameterize the primitive uncertainties and then apply the AARC, which interestingly results in a more flexible AARC.

Our splitting based EAARC approach bears some similarity with the approach suggested by Chen et al. [24] and Chen et al. [25]. Specifically, both approaches are built upon the splitting of the primitive uncertainties to their negative parts and positive parts. In addition, the segregated linear decision rule proposed in [25] also represents the recourse response as affine functions of these negative parts and positive parts. Moreover, both approaches end up with second order conic programming problems.

However, the models analyzed in [24] and [25] are fundamentally different from the one proposed here. Indeed, [24] and [25] started with a (chance-constrained) stochastic program and proposed tractable (convex) approximations to the stochastic program, while here we start with an ARC and use the EAARC to approximate the ARC. Therefore, in this chapter, the primitive uncertainties are restricted to an uncertainty set and thus is non-stochastic, while in [24] and [25], the primitive uncertainties are stochastic with possibly known mean, support, and some deviation measures, which require totally different techniques for the analysis. Finally, even though both approaches end up with second order conic programming problems, the formulations are different.

The rest of the paper is organized as follows. In Section 2.2, we illustrate the limitation of the AARC and introduce the EAARC. In Section 2.3, we present equivalent formulations for constraints derived from the EAARC. In Section 2.4, we introduce and analyze the splitting based EAARC and identify conditions under which the splitting based EAARC improves upon the AARC. We then conduct numerical experiments to demonstrate the advantage of the splitting based EAARC over the AARC in Section 2.5. Finally, we provide some concluding remarks in Section 2.6.

2.2 Extended Affinely Adjustable Robust Counterpart

Consider the following two-stage uncertain linear programming problem [1]:

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{r\times n}$, $\boldsymbol{B} \in \mathbb{R}^{r\times m}$ and $\boldsymbol{b} \in \mathbb{R}^r$ are uncertain data, and $\boldsymbol{c}'$ denote the transpose of vector $\boldsymbol{c}$. In this problem, decision variables are classified into two groups. The first group, denoted by $\boldsymbol{x}$, represents "here and now" decisions, i.e., decisions made before the realization of uncertain

[1] The approach presented here can be straightforwardly extended to multi-stage uncertain linear programming problems.

data $(\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{b})$. The second group, denoted by $\boldsymbol{y}$, represents "wait and see" decisions, i.e., decisions that can be adjusted to the realization of uncertainty.

A framework for modeling the two-stage uncertain linear programs is two-stage stochastic programming. In such a framework, some stochastic structure is imposed on the uncertain data $(\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{b})$ and the objective is to minimize the expected cost such that the constraints are satisfied with high probability. Unfortunately, multi-stage stochastic programs are generally hard to solve to optimality. To make things worse, specifying the stochastic structure of the uncertain data may not be realistic in practice.

An alternative approach for modeling two-stage uncertain linear programs is the adjustable robust counterpart first introduced by Ben-Tal et al. [9]. In an ARC, the constraints are satisfied for all the uncertain data varying in a given uncertainty set, while the second stage decisions can be tuned to the realization of the uncertain data. Specifically, the two-stage ARC for the uncertain linear program can be written as follows.

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \forall (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{b}) \in \mathcal{U} \; \exists \boldsymbol{y} \; \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b}, \end{aligned} \tag{2.1}$$

where $\mathcal{U}$ is the uncertainty set. Here without loss of generality, we assume that the cost coefficient vector $\boldsymbol{c}$ is fixed.

If $\boldsymbol{B}$ is fixed in (2.1), then the above formulation defines the ARC to an uncertain linear program with *fixed recourse*. From now on, we focus on uncertain linear programs with fixed recourse. In particular, we assume that $\boldsymbol{B}$ is fixed and the uncertainty set can be parameterized affinely in terms of the primitive uncertainties $\boldsymbol{z} \in \mathbb{R}^N$.

$$\mathcal{U} = \left\{ (\boldsymbol{A}, \boldsymbol{b}) : \exists \boldsymbol{z} \in \Gamma, (\boldsymbol{A}, \boldsymbol{b}) = (\boldsymbol{A}^0, \boldsymbol{b}^0) + \sum_{j=1}^{N} (\boldsymbol{\Delta A}^j, \boldsymbol{\Delta b}^j) z_j \right\},$$

where $(\boldsymbol{A}^j, \boldsymbol{b}^j) \in \mathbb{R}^{r \times n} \times \mathbb{R}^r$, $j = 0, 1, \ldots, N$, are given, and Γ is a nonempty closed convex subset in $\mathbb{R}^N$. Notice that since $\boldsymbol{B}$ is fixed, we remove $\boldsymbol{B}$ from the representation of the uncertainty set.

Define $\boldsymbol{m}^0(\boldsymbol{x}) = \boldsymbol{A}^0\boldsymbol{x} - \boldsymbol{b}^0$. For a given $\boldsymbol{x}$, let $\boldsymbol{M}(\boldsymbol{x})$ be a matrix in $\mathbb{R}^{r \times N}$ with the jth column given by $\boldsymbol{\Delta A}^j\boldsymbol{x} - \boldsymbol{\Delta b}^j$. The feasible set of the first stage decision in problem (2.1) can be written as

$$X_0 = \left\{ \boldsymbol{x} : \forall \boldsymbol{z} \in \Gamma, \exists \boldsymbol{y}, \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{0} \right\}.$$

In general, the ARC problem (2.1) is intractable (See Ben-Tal et al. [9]). To overcome this difficulty, Ben-Tal et al. [9] proposed the affinely adjustable robust counterpart (AARC) assuming that the "wait and see" (or recourse) variables are affinely dependent on the primitive uncertainties. That is,

$$\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{y}^j z_j,$$

which will render the problem tractable. In this case, the feasible set X_0 is approximated by

$$X_{AARC} = \left\{ \boldsymbol{x} : \exists \boldsymbol{y}^0, \boldsymbol{y}^j, \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{B}\boldsymbol{y}^j z_j \leq \boldsymbol{0} \ \forall \boldsymbol{z} \in \Gamma \right\}.$$

It is clear that $X_{AARC} \subseteq X_0$.

The AARC is motivated by the belief that the change in recourse variables is often linear to small changes in data uncertainty. However, the AARC may be too restrictive, particularly in cases where linear dependency fails to be a good approximation, as illustrated in the following example.

Example 2.2.1. *Consider the following ARC.*

$$\begin{array}{ll} \min & x \\ \forall \|\boldsymbol{z}\|_1 \leq 1 \exists \ \boldsymbol{y} \ s.t. & -y_i \leq z_i, -y_i \leq -z_i, i = 1, \ldots, N \\ & \sum_{i=1}^{N} y_i \leq x. \end{array}$$

The example implies that $|z_i| \leq y_i$ and hence $x \geq \sum_{i=1}^{N} y_i \geq \|\boldsymbol{z}\|_1$. Therefore, the optimal objective value of the ARC is 1*.*

If we employ the linear decision rule $\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{y}^j z_j$, then the AARC is as follows.

$$\begin{array}{ll} \min & x \\ s.t. & -(y_i^0 + \sum_{j=1}^{N} y_i^j z_j) \leq z_i, \\ & -(y_i^0 + \sum_{j=1}^{N} y_i^j z_j) \leq -z_i, i = 1, \ldots, N, \forall \|\boldsymbol{z}\|_1 \leq 1 \\ & \sum_{i=1}^{N} (y_i^0 + \sum_{j=1}^{N} y_i^j z_j) \leq x. \end{array}$$

The first two constraints imply that $|z_i| \leq y_i^0 + \sum_{j=1}^N y_i^j z_j$ for all $\|\boldsymbol{z}\|_1 \leq 1$. In particular, it is true for $\boldsymbol{z} = \pm \boldsymbol{e}^i$ where $\boldsymbol{e}^i$ is the unit vector with 1 at its ith component. Therefore,

$$1 \leq y_i^0 + y_i^i, 1 \leq y_i^0 - y_i^i,$$

which implies that $y_i^0 \geq 1$. In addition, if we let $\boldsymbol{z} = \boldsymbol{0}$, the last constraint then implies that $x \geq N$. Finally, if $\boldsymbol{y}^j = 0$ for $j = 1, \ldots, N$ and $\boldsymbol{y}^0 = \boldsymbol{e}$ where $\boldsymbol{e}$ is the all ones vector, then we know the optimal objective value of the AARC is N.

The purpose of this chapter is to relax the restriction of the AARC proposed in Ben-Tal et al. [9], in which the recourse variables depend on the *primitive uncertainties* in an affine manner. Specifically, we introduce auxiliary variables $\boldsymbol{u} \in \mathbb{R}^K$ for some dimension K, such that the recourse variables can be represented as affine functions of the *auxiliary variables* $\boldsymbol{u}$ in addition to the primitive uncertainties $\boldsymbol{z}$.

$$\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{j=1}^{N} \boldsymbol{y}^j z_j + \sum_{j=N+1}^{N+K} \boldsymbol{y}^j u_j, \tag{2.2}$$

where $(\boldsymbol{z}, \boldsymbol{u}) \in \Lambda$, u_j is the jth component of $\boldsymbol{u}$ and Λ, referred to as the *extended uncertainty set*, is a nonempty closed convex set in $\mathbb{R}^{N+K}$ to be specified later.

We now provide some motivation for using the above formulation (2.2). As we illustrated in Example 2.2.1, it may be too restrictive in certain settings to require the recourse variables to be affine functions of the realization of the primitive uncertainties. By introducing the new variables $\boldsymbol{u}$, we hope that they would capture certain nonlinearity of the response functions. Ideally, it would be nice to represent $\boldsymbol{u}$ as some nonlinear functions of $\boldsymbol{z}$, say piecewise linear functions of $\boldsymbol{z}$, which, however, usually leads to intractable formulations. Thus, instead of representing $\boldsymbol{u}$ directly as some nonlinear functions of $\boldsymbol{z}$, we impose the constraint $(\boldsymbol{z}, \boldsymbol{u}) \in \Lambda$.

Letting $\boldsymbol{y}$ take the form in (2.2), we can write down an approximation to problem (2.1) as follows.

$$
\begin{aligned}
\min \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \exists \boldsymbol{y}^0, \boldsymbol{y}^j : \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{y}^0 + \textstyle\sum_{j=1}^{K} \boldsymbol{B}\boldsymbol{y}^j z_j + \sum_{j=N+1}^{N+K} \boldsymbol{B}\boldsymbol{y}^j u_j \\
& \quad \leq \boldsymbol{0} \ \forall (\boldsymbol{z}, \boldsymbol{u}) \in \Lambda.
\end{aligned}
\tag{2.3}
$$

We call the problem the *extended affinely adjustable robust counterpart*. In this problem, the feasible set of the first stage decision is

$$
\begin{aligned}
X_{EAARC} := \Bigg\{ \boldsymbol{x} : \exists \boldsymbol{y}^0, \boldsymbol{y}^j, \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{y}^0 + \sum_{j=1}^{K} \boldsymbol{B}\boldsymbol{y}^j z_j \\
+ \sum_{j=N+1}^{N+K} \boldsymbol{B}\boldsymbol{y}^j u_j \leq \boldsymbol{0} \ \forall (\boldsymbol{z}, \boldsymbol{u}) \in \Lambda \Bigg\},
\end{aligned}
$$

which can be considered as an approximation of the feasible set X_0. When necessary, we will also use $X_{EAARC}(\Lambda)$ to emphasize the extended uncertainty set Λ.

It is straightforward to show that if $\Gamma \subseteq \text{Proj}_{\boldsymbol{z}}(\Lambda)$, where $\text{Proj}_{\boldsymbol{z}}(\Lambda)$ is the projection of Λ into the $\boldsymbol{z}$ space, then $X_{EAARC} \subseteq X_0$.

As there are many different ways of choosing the extended uncertainty set Λ, the EAARC is rather flexible. For instance, the AARC is a special case of the EAARC. On the other hand, if the uncertainty set Γ itself is defined through some auxiliary variables, then there is a natural way of defining the extended uncertainty set Λ. Specifically, consider the uncertainty set analyzed in Ben-Tal et al. [9]:

$$\Gamma = \{\boldsymbol{z} : \exists \boldsymbol{U} : \boldsymbol{Z}\boldsymbol{z} + \boldsymbol{U}\boldsymbol{u} \preceq_{\mathcal{K}} \boldsymbol{d}\},$$

where $\mathcal{K}$ is a nonempty convex cone and $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{y} - \boldsymbol{x} \in \mathcal{K}$. In this case, it is natural to define

$$\Lambda = \{(\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{Z}\boldsymbol{z} + \boldsymbol{U}\boldsymbol{u} \preceq_{\mathcal{K}} \boldsymbol{d}\}.$$

Finally, if we choose Λ appropriately, X_{EAARC} may recover the feasible set X_0. Indeed, assume that the set Γ is a polytope with extreme points, $\boldsymbol{z}^1, \ldots, \boldsymbol{z}^M$. That is,

$$\Gamma = \left\{ \boldsymbol{z} : \boldsymbol{z} = \sum_{j=1}^{M} \boldsymbol{z}^j u_j, \sum_{j=1}^{M} u_j = 1, u_j \geq 0, \forall j = 1, \ldots, M \right\}. \qquad (2.4)$$

Choose $K = M$ and let

$$\Lambda = \left\{ (\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{z} = \sum_{j=1}^{M} \boldsymbol{z}^j u_j, \sum_{j=1}^{M} u_j = 1, u_j \geq 0, \forall j = 1, \ldots, M \right\}. \qquad (2.5)$$

In this case, we have

$$X_{EAARC} = \Big\{ \boldsymbol{x} : \exists \boldsymbol{y}^0, \boldsymbol{y}^j, \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{y}^0 + \sum_{j=1}^{M} (\boldsymbol{M}(\boldsymbol{x})\boldsymbol{z}^j + \boldsymbol{B}\boldsymbol{y}^j) u_j \leq \boldsymbol{0} \ \forall \boldsymbol{e}'\boldsymbol{u} = 1, \boldsymbol{u} \geq \boldsymbol{0} \Big\}.$$

In this following, we show that $X_{EAARC} = X_0$.

Theorem 2.2.1. *If* Γ *and* Λ *are given by* (2.4) *and* (2.5) *respectively, then* $X_{EAARC} = X_0$.

Proof. From the definition of Γ and Λ, we have $\Gamma = \mathrm{Proj}_{\boldsymbol{z}}(\Lambda)$. Thus, $X_{EAARC} \subseteq X_0$. It remains to show $X_0 \subseteq X_{EAARC}$.

Recall the definition of $X_0 = \{ \boldsymbol{x} : \forall \boldsymbol{z} \in \Gamma, \exists \boldsymbol{y}, \boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{0} \ \}$. Hence, for a given $\boldsymbol{x} \in X_0$, there exists $\boldsymbol{y}^j$ such that

$$\boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{M}(\boldsymbol{x})\boldsymbol{z}^j + \boldsymbol{B}\boldsymbol{y}^j \leq \boldsymbol{0}, \ \forall \ j = 1, 2, \ldots, M,$$

which implies that

$$\boldsymbol{m}^0(\boldsymbol{x}) + \sum_{j=1}^{M} (\boldsymbol{M}(\boldsymbol{x})\boldsymbol{z}^j + \boldsymbol{B}\boldsymbol{y}^j) u_j \leq \boldsymbol{0}, \ \forall \boldsymbol{e}'\boldsymbol{u} = 1, \boldsymbol{u} \geq \boldsymbol{0}.$$

Thus, $x \in X_{EAARC}$ and $X_0 \subseteq X_{EAARC}$. $\square$

2.3 Constraint Reformulation

In the previous section, we show that the EAARC is rather flexible. In fact, if we define the extended uncertainty set Λ using the extreme points of the original uncertainty set Γ, we can recover the feasible set of the ARC. Unfortunately, in general, for a polyhedral set defined by linear equalities and inequalities, the number of extreme points is exponential in terms of the number of constraints of the polyhedral set and we may end up with an intractable formulation.

Since the EAARC is essentially the application of the AARC on a re-parameterized model, all the theoretical results for the AARC method carry over to the EAARC verbatim as long as the extended uncertainty set is chosen appropriately. In the following, we present equivalent formulations for the robust constraints when the extended uncertainty set Λ is defined as follows:

$$\Lambda = \{(\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{Z}\boldsymbol{z} + \boldsymbol{U}\boldsymbol{u} \leq \boldsymbol{d}\}.$$

Under this assumption and using the constraint reformulation result of the AARC method (see Ben-Tal et al. [9] for more details), we have that $x \in X_{EAARC}$ if and only if there exist $\boldsymbol{W}, \boldsymbol{y}^0, \boldsymbol{Y}^{\boldsymbol{z}}, \boldsymbol{Y}^{\boldsymbol{u}}$ such that the following linear inequalities hold.

$$\begin{aligned} &\boldsymbol{m}^0(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{y}^0 + \boldsymbol{W}\boldsymbol{d} \leq \boldsymbol{0} \\ &\boldsymbol{W}\boldsymbol{Z} = \boldsymbol{M}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{Y}^{\boldsymbol{z}} \\ &\boldsymbol{W}\boldsymbol{U} = \boldsymbol{B}\boldsymbol{Y}^{\boldsymbol{u}} \\ &\boldsymbol{W} \geq \boldsymbol{0}. \end{aligned} \tag{2.6}$$

Thus, problem (2.2) can be equivalently reformulated as the following linear program.

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & (2.6) \text{ holds.} \end{array}$$

If the set Λ has a polynomial size representation in terms of the input data, then the above linear program and hence its associated EAARC is tractable.

In the remainder of this section, we derive the dual of the feasibility problem (2.6). This dual is useful when we compare the feasibility set based on the EAARC and the one based on the AARC for the extended uncertainty sets proposed in the next section.

Lemma 2.3.1. *The dual of problem* (2.6) *is given as follows:*

$$\begin{array}{ll} min & -\langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha} \rangle - \langle \boldsymbol{M(x)}, \boldsymbol{\beta} \rangle \\ s.t. & \boldsymbol{Z\beta}' + \boldsymbol{U\gamma}' \leq \boldsymbol{d\alpha}' \\ & \boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0} \\ & \boldsymbol{B}'\boldsymbol{\beta} = \boldsymbol{0} \\ & \boldsymbol{B}'\boldsymbol{\gamma} = \boldsymbol{0} \\ & \boldsymbol{\alpha} \geq \boldsymbol{0}, \end{array} \tag{2.7}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^r$, $\boldsymbol{\beta} \in \mathbb{R}^{r \times N}$ *and* $\langle \cdot, \cdot \rangle$ *denotes the inner product (specifically,* $\langle \boldsymbol{M(x)}, \boldsymbol{\beta} \rangle = trace(\boldsymbol{M(x)}'\boldsymbol{\beta})$ *denotes the inner product of the two matrices). In addition,* $\boldsymbol{x} \in X_{EAARC}$ *if and only if the optimal value of* (2.7) *is* 0.

Proof. Define the Lagrangian function of the feasibility problem (2.6):

$$\begin{aligned} & L(\boldsymbol{y}^0, \boldsymbol{Y^z}, \boldsymbol{W}, \boldsymbol{Y^u}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \\ =& \langle -\boldsymbol{m}^0(\boldsymbol{x}) - \boldsymbol{B}\boldsymbol{y}^0 - \boldsymbol{W}\boldsymbol{d}, \boldsymbol{\alpha} \rangle + \langle \boldsymbol{WZ} - \boldsymbol{M}(\boldsymbol{x}) - \boldsymbol{B}\boldsymbol{Y^z}, \boldsymbol{\beta} \rangle \\ & + \langle \boldsymbol{WU} - \boldsymbol{B}\boldsymbol{Y^u}, \boldsymbol{\gamma} \rangle \\ =& - \langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha} \rangle - \langle \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{\beta} \rangle + \langle \boldsymbol{W}, \boldsymbol{Z\beta}' + \boldsymbol{U}\gamma' - d\alpha' \rangle \\ & - \langle \boldsymbol{y}^0, \boldsymbol{B}'\boldsymbol{\alpha} \rangle - \langle \boldsymbol{Y^z}, \boldsymbol{B}'\boldsymbol{\beta} \rangle - \langle \boldsymbol{Y^u}, \boldsymbol{B}'\boldsymbol{\gamma} \rangle \end{aligned}$$

Consider the dual function defined by

$$Q(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \max_{\boldsymbol{W} \geq 0, \boldsymbol{y}^0, \boldsymbol{Y^z}, \boldsymbol{Y^u}} L(\boldsymbol{y}^0, \boldsymbol{Y^z}, \boldsymbol{W}, \boldsymbol{Y^u}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}).$$

The Lagrangian dual of the feasibility problem (2.6) is given as

$$\min_{\boldsymbol{\alpha} \geq 0, \boldsymbol{\beta}, \boldsymbol{\gamma}} Q(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}),$$

which is equivalent to (2.7). It is clear that the feasibility problem (2.6) is feasible if and only if its dual has an optimal objective value zero. □

The sets Γ and Λ can be extended to incorporate conic constraints. We have a result parallel to Lemma 2.3.1. Since its proof is similar to the one for Lemma 2.3.1 and follows directly from the conic programming duality theory, we omit its proof.

Lemma 2.3.2. *Assume that*

$$\Lambda = \{(\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{Z}\boldsymbol{z} + \boldsymbol{U}\boldsymbol{u} \preceq_{\mathcal{K}} \boldsymbol{d}\},$$

and there exists $(\boldsymbol{z}, \boldsymbol{u})$ *such that* $\boldsymbol{d} - \boldsymbol{Z}\boldsymbol{z} - \boldsymbol{U}\boldsymbol{u}$ *lies in the interior of* $\mathcal{K}$. *Then,* $\boldsymbol{x} \in X_{EAARC}$ *if and only if* 0 *is the optimal value of the following problem*

$$\begin{array}{ll} \textit{min} & -\langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha}\rangle - \langle \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{\beta}\rangle \\ \textit{s.t.} & \boldsymbol{Z}\boldsymbol{\beta}' + \boldsymbol{U}\boldsymbol{\gamma}' \preceq_{\mathcal{K}} \boldsymbol{d}\boldsymbol{\alpha}' \\ & \boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0} \\ & \boldsymbol{B}'\boldsymbol{\beta} = \boldsymbol{0} \\ & \boldsymbol{B}'\boldsymbol{\gamma} = \boldsymbol{0} \\ & \boldsymbol{\alpha} \geq \boldsymbol{0}. \end{array}$$

2.4 The Splitting Based EAARC

In this section, we propose one way of choosing the extended uncertainty set Λ. To illustrate the basic idea, we consider a simple setting in which the uncertainty set Γ is the intersection of a polyhedral set and a norm constrained set, that is,

$$\Gamma = \{\boldsymbol{z} : \boldsymbol{L}\boldsymbol{z} \leq \boldsymbol{l}\} \cap \{\boldsymbol{z} : \|\boldsymbol{z}\| \leq \Omega\}. \tag{2.8}$$

for some norm $\|\cdot\|$. The idea is essentially to split $\boldsymbol{z}$ into two parts, which can be thought of as the positive part and the negative part of $\boldsymbol{z}$. Specifically, we let $\boldsymbol{u} = (\boldsymbol{v}', \boldsymbol{w}')'$ and $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}$. That is, $\boldsymbol{z}$ is defined as the difference of two auxiliary variables $\boldsymbol{v}$ and $\boldsymbol{w}$, which represents the positive part and the negative part of $\boldsymbol{z}$ respectively. The extended uncertainty set can be naturally defined as

$$\begin{aligned} \Lambda = \{(\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{L}\boldsymbol{z} \leq \boldsymbol{l}\} \cap \{(\boldsymbol{z}, \boldsymbol{u}) : \\ \boldsymbol{u} = (\boldsymbol{v}', \boldsymbol{w}')', \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \|\boldsymbol{v} + \boldsymbol{w}\| \leq \Omega, \boldsymbol{v} \geq \boldsymbol{0}, \boldsymbol{w} \geq \boldsymbol{0}\}. \end{aligned}$$

Thus, instead of using affine decision rules in terms of $\boldsymbol{z}$, we consider decision rules that are affine in $\boldsymbol{v}$ and $\boldsymbol{w}$ (obviously the affine part in $\boldsymbol{z}$ is automatically subsumed in this case), namely,

$$\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{j=1}^{N} (\boldsymbol{r}^j v_j + \boldsymbol{s}^j w_j),$$

where $\boldsymbol{y}^0, \boldsymbol{r}^j$ and $\boldsymbol{s}^j$ are vectors to be determined. The resulting EAARC is referred to as the *splitting based EAARC*.

We now apply the splitting based EAARC to Example 2.2.1 to illustrate that the EAARC may significantly improve upon the AARC.

Example 2.4.1. *Consider the adjustable robust counterpart presented in Example 2.2.1. In the EAARC decision rule, we have* $\boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \boldsymbol{v} \geq 0, \boldsymbol{w} \geq 0$ *and* $\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{j=1}^{N} (\boldsymbol{r}^j v_j + \boldsymbol{s}^j w_j)$. *Then the EAARC is defined as follows:*

$$\begin{array}{ll}
\min & x \\
s.t. & -(y_i^0 + \sum_{j=1}^{N} (r_i^j v_j + s_i^j w_j)) \leq v_i - w_i, \\
 & -(y_i^0 + \sum_{j=1}^{N} (r_i^j v_j + s_i^j w_j)) \leq -(v_i - w_i), i = 1, \ldots, N, \\
 & \qquad \forall \|\boldsymbol{v} + \boldsymbol{w}\|_1 \leq 1, \boldsymbol{v} \geq 0, \boldsymbol{w} \geq 0 \\
 & \sum_{i=1}^{N} (y_i^0 + \sum_{j=1}^{N} (r_i^j v_j + s_i^j w_j)) \leq x.
\end{array}$$

The first two constraints imply that

$$|v_i - w_i| \leq y_i^0 + \sum_{j=1}^{N} (r_i^j v_j + s_i^j w_j).$$

It is clear that $\boldsymbol{y} = \sum_{j=1}^{N} \boldsymbol{e}^j (v_j + w_j)$ *satisfies the first two constraints. Furthermore, in this case,*

$$x \geq \sum_{i=1}^{N} (v_i + w_i) = \|\boldsymbol{v} + \boldsymbol{w}\|_1.$$

Hence the optimal objective value of the EAARC is 1. *This is exactly the optimal objective value of the ARC, while the optimal objective value of the AARC is* N.

Observe that in the above example, the optimal response function is $y_j(\boldsymbol{z}) = |z_j|$, which is nonlinear in $\boldsymbol{z}$. By introducing the positive and negative parts of $\boldsymbol{z}$, we are able to capture the nonlinearity in this specific example.

We now extend the splitting idea to more general uncertainty sets. Specifically, we focus on the uncertainty set

$$\Gamma = \{\boldsymbol{z} : \exists((\boldsymbol{u}^1)', \dots, (\boldsymbol{u}^\tau)')' \in \mathbb{R}^{K_1} \times \dots \mathbb{R}^{K_\tau} \ \boldsymbol{Z}\boldsymbol{z} + \sum_{t=1}^{\tau} \boldsymbol{U}^t \boldsymbol{u}^t \leq \boldsymbol{d},$$
$$\|\boldsymbol{z}\|_{(0)} \leq \Omega_0, \|\boldsymbol{u}^t\|_{(t)} \leq \Omega_t, t = 1, \dots, \tau\},$$

where $\boldsymbol{d} \in \mathbb{R}^\ell$, $\boldsymbol{Z} \in \mathbb{R}^{\ell \times N}$ and $\boldsymbol{U}_t \in \mathbb{R}^{\ell \times K_t}$. Here $\|\cdot\|_{(t)}$, $t = 0, 1, \dots, \tau$ are vector norms. In this chapter, all the vector norms $\|\cdot\|_{(t)}$ in the uncertainty set satisfy the following condition:

$$\|\boldsymbol{u}^t\|_{(t)} = \||\boldsymbol{u}^t|\|_{(t)},$$

where $|\boldsymbol{u}^t|$ is the vector with the jth component equal to $|u_j|\ \forall j \in \{1, \dots, N\}$. For technical reasons, we assume that the Slater condition holds. That is, there exists $\boldsymbol{u}^t$, $t = 0, 1, \dots, \tau$, such that $\boldsymbol{Z}\boldsymbol{u}^0 + \sum_{t=1}^{\tau} \boldsymbol{U}^t \boldsymbol{u}^t \leq \boldsymbol{d}$ with $\|\boldsymbol{u}^t\|_{(t)} < \Omega_t$ if $\|\cdot\|_{(t)}$ is not a polyhedral norm. This assumption would allow us to employ Lemma 2.3.2 in the following analysis.

The representation of our uncertainty set is broad enough to include many uncertainty sets commonly used in the robust optimization literature. Obviously, the uncertainty set (2.8) is a special case. More importantly, it also includes the intersection of several general ellipsoids as a special case.

We now propose a specific extended uncertainty set Λ by splitting $(\boldsymbol{z}, \boldsymbol{u})$ into its positive and negative parts. Specifically, let $\boldsymbol{v} = ((\boldsymbol{v}^0)', \dots, (\boldsymbol{v}^\tau)')' \in \mathbb{R}^{K_0} \times \dots \mathbb{R}^{K_\tau}$, $\boldsymbol{w} = ((\boldsymbol{w}^0)', \dots, (\boldsymbol{w}^\tau)')' \in \mathbb{R}^{K_0} \times \dots \mathbb{R}^{K_\tau}$ with $K_0 = N$ and define the extended uncertainty set as follows.

$$\Lambda = \{(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{Z}\boldsymbol{z} + \sum_{t=1}^{\tau} \boldsymbol{U}^t(\boldsymbol{v}^t - \boldsymbol{w}^t) \leq \boldsymbol{d}, \boldsymbol{z} = \boldsymbol{v}^0 - \boldsymbol{w}^0,$$
$$\boldsymbol{v}^t \geq \boldsymbol{0}, \boldsymbol{w}^t \geq \boldsymbol{0}, \|\boldsymbol{v}^t + \boldsymbol{w}^t\|_{(t)} \leq \Omega_t, t = 0, 1, \dots, \tau\}. \quad (2.9)$$

Now instead of using affine decision rules in terms of the primitive uncertainties $\boldsymbol{z}$, we represent the recourse decision $\boldsymbol{y}$ affinely in $\boldsymbol{v}$ and $\boldsymbol{w}$ (again the affine part in $\boldsymbol{z}$ is automatically subsumed in this case), i.e.,

$$\boldsymbol{y} = \boldsymbol{y}^0 + \sum_{t=0}^{\tau} \sum_{i=1}^{K_t} (\boldsymbol{r}^{t,i} v_i^t + \boldsymbol{s}^{t,i} w_i^t). \tag{2.10}$$

We now formulate the EAARC as an equivalent conic programming problem, whose proof follows from Theorem 3.2 in Ben-Tal et al. [9] and thus is omitted.

Theorem 2.4.1. *The splitting based EAARC with the extended uncertainty set* (2.9) *and the affine decision rule* (2.10) *is equivalent to the following conic programming problem.*

$$\begin{array}{ll} min & \boldsymbol{c'x} \\ s.t. & \boldsymbol{m}^0(\boldsymbol{x}) + By^0 - \boldsymbol{\Phi d} \le \boldsymbol{0} \\ & (\boldsymbol{\mu}^t)' \ge \Omega_t \|(\boldsymbol{h}^t)'\|^*_{(t)}, t = 0, 1, \dots, \tau \\ & \boldsymbol{H}^0 \ge \boldsymbol{Br}^0 + \boldsymbol{\Phi Z} + \boldsymbol{M}(\boldsymbol{x}) \\ & \boldsymbol{H}^0 \ge \boldsymbol{Bs}^0 - \boldsymbol{\Phi Z} - \boldsymbol{M}(\boldsymbol{x}) \\ & \boldsymbol{H}^t \ge \boldsymbol{Br}^t + \boldsymbol{\Phi U}^t, t = 1, 2, \dots, \tau \\ & \boldsymbol{H}^t \ge \boldsymbol{Bs}^t - \boldsymbol{\Phi U}^t, t = 1, 2, \dots, \tau \\ & \boldsymbol{H}^t \ge \boldsymbol{0}, t = 0, 1, \dots, \tau \\ & \boldsymbol{\Phi} \ge \boldsymbol{0}, \end{array} \tag{2.11}$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{r\times\ell}$, $\boldsymbol{H}^t \in \mathbb{R}^{r\times K_t}$ *and* $\boldsymbol{\mu}^t \in \mathbb{R}^r$. *In addition,* $\|(\boldsymbol{H}^t)'\|^*_{(t)}$ *is an* r*-dimensional row vector with its* j*th entry equal to the conjugate norm of* $\|\cdot\|_{(t)}$ *taken over the* j*th row of* $\boldsymbol{H}^t$.

Note that when all the vector norms $\|\cdot\|_{(t)}$ are 2-norms, the above problem (2.11)) becomes a second order conic program.

We are interested in identifying conditions under which $X_{EAARC}(\Lambda)$ improves upon X_{AARC} when Λ is given in (2.9). However, rather than directly comparing $X_{EAARC}(\Lambda)$ and X_{AARC}, we will compare $X_{EAARC}(\Lambda)$ and another extended uncertainty set $X_{EAARC}(\Lambda^0)$, where

$$\Lambda^0 = \{(\boldsymbol{z}, \boldsymbol{u}) : \boldsymbol{Zz} + \sum_{t=1}^{\tau} \boldsymbol{U}^t \boldsymbol{u}^t \le \boldsymbol{d}, \|\boldsymbol{z}\|_{(0)} \le \Omega_0, \|\boldsymbol{u}^t\|_{(t)} \le \Omega_t, t = 1, 2, \dots, \tau\}$$

is the natural extension of the original uncertainty set.

Notice that Lemma 2.3.2 implies that $\boldsymbol{x} \in X_{EAARC}(\Lambda^0)$ if and only if 0 is the optimal value of the following problem.

$$
\begin{aligned}
\min \quad & -\langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha} \rangle - \langle \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{\beta} \rangle \\
\text{s.t.} \quad & \boldsymbol{Z}\boldsymbol{\beta}' + \textstyle\sum_{t=1}^{\tau} \boldsymbol{U}^t (\boldsymbol{\gamma}^t)' \leq \boldsymbol{d}\boldsymbol{\alpha}' \\
& \boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0} \\
& \boldsymbol{\beta} = \boldsymbol{\gamma}^0 \\
& \boldsymbol{B}'\boldsymbol{\gamma}^t = \boldsymbol{0}, t = 0, 1, \ldots, \tau \\
& \|(\boldsymbol{\gamma}^t)'\|_{(t)} \leq \Omega_t \boldsymbol{\alpha}', t = 0, 1, \ldots, \tau \\
& \boldsymbol{\alpha} \geq \boldsymbol{0}.
\end{aligned} \tag{2.12}
$$

Here $\boldsymbol{\alpha} \in \mathbb{R}^r$, $\boldsymbol{\beta} \in \mathbb{R}^{r \times N}$, $\boldsymbol{\gamma}^t \in \mathbb{R}^{r \times K_t}$ and $\|(\boldsymbol{\gamma}^t)'\|_{(t)}$ is an r dimensional row vector with each entry equal to the norm of the corresponding column in $(\boldsymbol{\gamma}^t)'$.

Similarly, $\boldsymbol{x} \in X_{EAARC}$ if and only if 0 is the optimal value of the following problem.

$$
\begin{aligned}
\min \quad & -\langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha} \rangle - \langle \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{\beta} \rangle \\
\text{s.t.} \quad & \boldsymbol{Z}\boldsymbol{\beta}' + \textstyle\sum_{t=1}^{\tau} \boldsymbol{U}^t (\boldsymbol{\eta}^t - \boldsymbol{\delta}^t)' \leq \boldsymbol{d}\boldsymbol{\alpha}' \\
& \boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0} \\
& \boldsymbol{\beta} = \boldsymbol{\eta}^0 - \boldsymbol{\delta}^0 \\
& \boldsymbol{B}'\boldsymbol{\eta}^t = \boldsymbol{0}, t = 0, 1, \ldots, \tau \\
& \boldsymbol{B}'\boldsymbol{\delta}^t = \boldsymbol{0}, t = 0, 1, \ldots, \tau \\
& \|(\boldsymbol{\eta}^t)' + (\boldsymbol{\delta}^t)'\|_{(t)} \leq \Omega_t \boldsymbol{\alpha}', t = 0, 1, \ldots, \tau \\
& \boldsymbol{\alpha}, \boldsymbol{\eta}^t, \boldsymbol{\delta}^t \geq \boldsymbol{0}, t = 0, 1, \ldots, \tau.
\end{aligned} \tag{2.13}
$$

Here $\boldsymbol{\eta}^t, \boldsymbol{\delta}^t \in \mathbb{R}^{r \times K_t}$.

It is straightforward to see that for any given feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ of problem (2.13), $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma}' = \boldsymbol{\eta}' - \boldsymbol{\delta}'$ is feasible for problem (2.12). If in addition, for any given feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ of problem (2.12), we can find $(\boldsymbol{\eta}, \boldsymbol{\delta})$ such that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ is feasible for problem (2.13), then $X_{EAARC}(\Lambda) = X_{EAARC}(\Lambda^0)$. However, if the projection of the feasible set of problem (2.13) onto the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ space is a true subset of the projection of the feasible set of problem (2.12) onto the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ space, then it is possible that $X_{EAARC}(\Lambda^0)$ and thus X_{AARC} are true subsets of $X_{EAARC}(\Lambda)$. In the following we compare the two sets under several different cases. First, we assume that Ω_t is finite. The following theorem illustrates that when we use the infinity norm in the uncertainty set, the EAARC with the extended uncertainty set Λ does not improve upon the EAARC with the extended uncertainty set Λ^0.

Theorem 2.4.2. *If $\|\cdot\|_{(t)} = \|\cdot\|_\infty$ and $\Omega_t < \infty$ for all $t = 0, 1, \ldots, \tau$, then the projection of the feasible set of problem* (2.13) *onto the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ space coincides with the projection of the feasible set of problem* (2.12) *onto the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ space. Thus, in this case, $X_{EAARC}(\Lambda) = X_{EAARC}(\Lambda^0)$.*

Proof. For any feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ of problem (2.12), we have that $\|(\boldsymbol{\gamma}^t)'\|_\infty \le \Omega_t \boldsymbol{\alpha}'$. Define for $t = 0, 1, \ldots, \tau$,

$$\boldsymbol{\eta}^t = \frac{1}{2}(\Omega_t \boldsymbol{e}\boldsymbol{\alpha}' + \boldsymbol{\gamma}^t)$$

and

$$\boldsymbol{\delta}^t = \frac{1}{2}(\Omega_t \boldsymbol{e}\boldsymbol{\alpha}' - \boldsymbol{\gamma}^t).$$

It is straightforward to check that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ is feasible for problem (2.13) and gives the same objective value. Thus, $X_{EAARC}(\Lambda) = X_{EAARC}(\Lambda^0)$. $\square$

The above proof can be easily extended to the case in which $\Omega_t = \infty$ for any $t = 0, 1, \ldots, \tau$.

Theorem 2.4.3. *Assume $\Omega_t = \infty$ for all $t = 0, 1, \ldots, \tau$. If Λ^0 is bounded, then $X_{EAARC}(\Lambda) = X_{EAARC}(\Lambda^0)$.*

We now present an example to show that Theorem 2.4.3 may fail if Λ^0 is not bounded.

Example 2.4.2. *Let $r = N = 2$, $\Lambda^0 = \{(z_1, z_2, u_1, u_2) : z_1 - z_2 \le 1, \boldsymbol{z} = \boldsymbol{u}\}$ and*

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In this case, it is clear that $\boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0}, \boldsymbol{\alpha} \ge \boldsymbol{0}$ implies that $\boldsymbol{\alpha} = 0$. Thus, problem (2.13) *has a unique solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta}) = \boldsymbol{0}$. However, in addition to the feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \boldsymbol{0}$, problem* (2.12) *has a nonzero feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, in which $\boldsymbol{\alpha} = \boldsymbol{0}$, $\boldsymbol{\beta} = \boldsymbol{\gamma}$ and*

$$\boldsymbol{\beta} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Therefore, the feasible set of problem (2.12) *is a strict subset of the feasible set of problem* (2.13) *and thus Theorem 2.4.3 does not hold.*

We now show that Theorem 2.4.2 fails if the norm is different from the infinity norm. For this purpose, we need the following result.

Lemma 2.4.1. *Given a norm* $\|\cdot\|$ *with the property that* $\|\boldsymbol{u}\| = \||\boldsymbol{u}|\|$ *for any* $\boldsymbol{u}$*,*

$$\|\boldsymbol{u}\| \leq \|\boldsymbol{v}\|, \forall\ \mathbf{0} \leq \boldsymbol{u} \leq \boldsymbol{v}.$$

Proof. Let $\boldsymbol{e}^i$ be the unit vector with its ith component being one. It suffices to show that

$$\|\boldsymbol{u}\| \leq \|\boldsymbol{v}\|,$$

for any $\boldsymbol{u} \geq \mathbf{0}$ and $\boldsymbol{v} = \boldsymbol{u} + \gamma \boldsymbol{e}^i$ for any i and $\gamma \geq 0$.

Define a new vector $\hat{\boldsymbol{v}}$ such that

$$\hat{\boldsymbol{v}} = \boldsymbol{u} - (2u_i + \gamma)\boldsymbol{e}^i.$$

It is clear that $\|\hat{\boldsymbol{v}}\| = \|\boldsymbol{v}\|$. In addition, $\boldsymbol{u}$ lies within the line segment between $\hat{\boldsymbol{v}}$ and $\boldsymbol{v}$. Thus,

$$\|\boldsymbol{u}\| \leq \max\{\|\hat{\boldsymbol{v}}\|, \|\boldsymbol{v}\|\} = \|\boldsymbol{v}\|.$$

□

In the following, we further assume that the vector norms $\|\cdot\|_{(t)}$ satisfies the following conditions:

$$\|\boldsymbol{e}^i\|_{(t)} = 1, \forall i.$$

Theorem 2.4.4. *If* $\|\cdot\|_{(t)} \neq \|\cdot\|_\infty$ *and* Ω_t *is finite for some* t*, then* $X_{EAARC}(\Lambda)$ *may be a true subset of* $X_{EAARC}(\Lambda^0)$*. In this case, the EAARC based on the extended uncertainty set* Λ *may provide a strict improvement upon the AARC.*

Proof. We prove this result by constructing an example. Specifically, we construct an example in which for a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ of problem

(2.12), we cannot find $(\boldsymbol{\eta}, \boldsymbol{\delta})$ such that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ is feasible for problem (2.13). We consider the basic setting in which

$$\Lambda^0 = \{(\boldsymbol{z}, \boldsymbol{u}) : L\boldsymbol{z} \leq \boldsymbol{l}, \boldsymbol{z} = \boldsymbol{u}, \|\boldsymbol{u}\| \leq \Omega\}.$$

In this case, we have for a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ of problem (2.12), $\boldsymbol{\gamma} = \boldsymbol{\beta}$. Thus, it suffices to talk about the feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of problem (2.12).

Let $\Omega = \|[1\ 1]'\|$. Choose $\boldsymbol{B}$ such that the null space of $\boldsymbol{B}'$ is spanned by $\boldsymbol{\alpha} = [1\ 1\ 1\ 1\ 1]'$, $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$, where

$$\boldsymbol{\beta} = [\boldsymbol{\beta}^1, \boldsymbol{\beta}^2] = \begin{bmatrix} 1 & 1 \\ \Omega & 0 \\ 0 & \Omega \\ -\Omega & 0 \\ 0 & -\Omega \end{bmatrix}.$$

It is easy to verify that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is feasible for problem (2.12). Now assume that there exist $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$ such that $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ is feasible for problem (2.13). Since $\boldsymbol{B}'\boldsymbol{\delta} = 0$, we have that

$$\boldsymbol{\delta}^j = \omega_j \boldsymbol{\alpha} + \mu_j \boldsymbol{\beta}^1 + \nu_j \boldsymbol{\beta}^2, j = 1, 2,$$

for some scalars ω_j, μ_j and ν_j. Since $\boldsymbol{\beta}' = \boldsymbol{\eta}' - \boldsymbol{\delta}'$, we have that $\boldsymbol{\eta} = [\boldsymbol{\eta}^1\ \boldsymbol{\eta}^2]$ with

$$\boldsymbol{\eta}^1 = \omega_1 \boldsymbol{\alpha} + (1 + \mu_1)\boldsymbol{\beta}^1 + \nu_1 \boldsymbol{\beta}^2,$$

and

$$\boldsymbol{\eta}^2 = \omega_2 \boldsymbol{\alpha} + \mu_2 \boldsymbol{\beta}^1 + (1 + \nu_2)\boldsymbol{\beta}^2.$$

Thus, the jth column of $\boldsymbol{\eta}' + \boldsymbol{\delta}'$ is given by

$$\begin{bmatrix} 2\omega_1 + 2\mu_1 \beta_j^1 + 2\nu_1 \beta_j^2 + \beta_j^1 \\ 2\omega_2 + 2\mu_2 \beta_j^1 + 2\nu_2 \beta_j^2 + \beta_j^1 \end{bmatrix}.$$

Since $\|\boldsymbol{\eta}_j' + \boldsymbol{\delta}_j'\| \leq \Omega \boldsymbol{\alpha}_j$ and $\boldsymbol{\delta} \geq 0$, letting $j = 1$ implies that

$$\omega_j + \mu_j + \nu_j = 0, \text{ for } j = 1, 2.$$

Similarly, letting $j = 2, 3$ implies that

$$\omega_1 + \Omega\mu_1 = 0, \omega_2 + \Omega\nu_2 = 0.$$

The above equalities imply that

$$\omega_1 = -\Omega\mu_1, \nu_1 = (\Omega - 1)\mu_1,$$

and

$$\omega_2 = -\Omega\nu_2, \mu_2 = (\Omega - 1)\nu_2.$$

Letting $j = 4, 5$, $\boldsymbol{\eta} \geq 0$ implies that $-\mu_1 \geq 1/2$ and $-\nu_2 \geq 1/2$. In addition, we have that

$$\left\| \begin{bmatrix} -4\mu_1\Omega - \Omega \\ -2\Omega^2\nu_2 \end{bmatrix} \right\| \leq \Omega, \left\| \begin{bmatrix} -2\Omega^2\mu_1 \\ -4\nu_2\Omega - \Omega \end{bmatrix} \right\| \leq \Omega \tag{2.14}$$

However, $-4\mu_1\Omega - \Omega \geq \Omega$ and $-2\Omega^2\nu_2 \geq \Omega^2$. The above inequalities together with Lemma 2.4.1 imply that $\|[1\ \Omega]'\| \leq 1$. Hence, $\Omega = 1$ and $\|[1\ 1]'\| = 1$. Again, this together with Lemma 2.4.1 implies that $\|(u_1, u_2)'\| = 1$ if and only if $\|(u_1, u_2)'\| = \|(u_1, u_2)'\|_\infty$ for any u_1 and u_2. $\square$

In the construction of the uncertainty set of the EAARC, we essentially split the primitive uncertainty $\boldsymbol{z}$ and the auxiliary variable $\boldsymbol{u}$ to the positive parts and negative parts. We may generalize the idea by further splitting $\boldsymbol{u}$ to more parts as follows.

$$\begin{aligned}\Lambda(K) = \{(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{Z}\boldsymbol{z} + \sum_{t=1}^{\tau}\sum_{k=1}^{K} \boldsymbol{U}^t(\boldsymbol{v}^{t,k} - \boldsymbol{w}^{t,k}) \leq \boldsymbol{d}, \\ \boldsymbol{z} = \sum_{k=1}^{K}(\boldsymbol{v}^{0,k} - \boldsymbol{w}^{0,k})\} \cap \left(\bigcap_{t=0}^{\tau} \mathcal{G}_t(K)\right)\end{aligned}$$

where $\boldsymbol{v}^t = [(\boldsymbol{v}^{t,1})' \ldots (\boldsymbol{v}^{t,K})']'$, $\boldsymbol{w}^t = [(\boldsymbol{w}^{t,1})' \ldots (\boldsymbol{w}^{t,K})']'$ and for $t = 0, 1, \ldots, \tau$,

$$\mathcal{G}_t(K) = \{(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w}) : \| \sum_{k=1}^{K} (\boldsymbol{v}^{t,k} + \boldsymbol{w}^{t,k}) \|_{(t)} \le \Omega_t, \\ \boldsymbol{0} \le \boldsymbol{v}^{t,k} \le \boldsymbol{a}^{t,k}, \boldsymbol{0} \le \boldsymbol{w}^{t,k} \le \boldsymbol{b}^{t,k}, k = 1, \ldots, K\}.$$

One may conjecture that by introducing more flexibility into the uncertainty set, we can make further improvement. For a fair comparison, we require that for $t = 0, 1, \ldots, \tau$,

$$\{\boldsymbol{\xi} = \sum_{k=1}^{K} (\boldsymbol{v}^{t,k} - \boldsymbol{w}^{t,k}) : (\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w}) \in \mathcal{G}_t(K)\} = \{\boldsymbol{\xi} : \|\boldsymbol{\xi}\|_{(t)} \le \Omega_t\}. \qquad (2.15)$$

Unfortunately, under these assumptions, $\Lambda(K)$ may not provide any improvement over $\Lambda(1)$. To see this, we consider the dual associated with the uncertainty set $\Lambda(K)$, which can be written as follows.

$$\begin{array}{lll} P(K): & \min & -\langle \boldsymbol{m}^0(\boldsymbol{x}), \boldsymbol{\alpha} \rangle - \langle \boldsymbol{M}(\boldsymbol{x}), \boldsymbol{\beta} \rangle \\ & \text{s.t.} & \boldsymbol{Z}\boldsymbol{\beta}' + \sum_{t=1}^{\tau} \sum_{k=1}^{K} \boldsymbol{U}^t (\boldsymbol{\eta}^{t,k} - \boldsymbol{\delta}^{t,k})' \le \boldsymbol{d}\boldsymbol{\alpha}' \\ & & \boldsymbol{B}'\boldsymbol{\alpha} = \boldsymbol{0} \\ & & \boldsymbol{\beta} = \sum_{k=1}^{K} (\boldsymbol{\eta}^{0,k} - \boldsymbol{\delta}^{0,k}) \\ & & \boldsymbol{B}'\boldsymbol{\eta}^{t,k} = \boldsymbol{0}, t = 0, 1, \ldots, \tau, k = 1, 2, \ldots, K \\ & & \boldsymbol{B}'\boldsymbol{\delta}^{t,k} = \boldsymbol{0}, t = 0, 1, \ldots, \tau, k = 1, 2, \ldots, K \\ & & \| \sum_{k=1}^{K} ((\boldsymbol{\eta}^{t,k})' + (\boldsymbol{\delta}^{t,k})') \|_{(t)} \le \Omega_t \boldsymbol{\alpha}', t = 0, 1, \ldots, \tau \\ & & \boldsymbol{0} \le \boldsymbol{\eta}^{t,k} \le \boldsymbol{a}^{t,k} \boldsymbol{\alpha}', t = 0, 1, \ldots, \tau, k = 1, 2, \ldots, K \\ & & \boldsymbol{0} \le \boldsymbol{\delta}^{t,k} \le \boldsymbol{b}^{t,k} \boldsymbol{\alpha}', t = 0, 1, \ldots, \tau, k = 1, 2, \ldots, K \\ & & \boldsymbol{\alpha} \ge \boldsymbol{0}. \end{array} \qquad (2.16)$$

It is obvious that for any feasible solution of problem (2.16) for general K, we can construct a feasible solution for problem (2.13) with the same objective value.

On the other hand, since (2.15) holds, we claim that

$$\sum_{k=1}^{K} \boldsymbol{a}^{t,k} \ge \Omega_t \boldsymbol{e}, \sum_{k=1}^{K} \boldsymbol{b}^{t,k} \ge \Omega_t \boldsymbol{e}.$$

Indeed, since $\|\Omega_t \boldsymbol{e}^i\|_{(t)} = \Omega_t$, there exist $\boldsymbol{v}^{t,k}$ and $\boldsymbol{w}^{t,k}$ such that

$$\boldsymbol{0} \le \boldsymbol{v}^{t,k} \le \boldsymbol{a}^{t,k}, \boldsymbol{0} \le \boldsymbol{w}^{t,k} \le \boldsymbol{b}^{t,k}, \|\sum_{k=1}^{K}(\boldsymbol{v}^{t,k} + \boldsymbol{w}^{t,k})\|_{(t)} \le \Omega_t,$$

and $\sum_{t=1}^{K}(\boldsymbol{v}^{t,k} - \boldsymbol{w}^{t,k}) = \Omega_t \boldsymbol{e}^i$. Since $\boldsymbol{v}^{t,k}, \boldsymbol{w}^{t,k} \ge 0$,

$$\sum_{k=1}^{K} a_i^{t,k} \ge \sum_{k=1}^{K} v_i^{t,k} \ge \Omega_t.$$

Thus, $\sum_{k=1}^{K} \boldsymbol{a}^{t,k} \ge \Omega_t \boldsymbol{e}$. Similarly, we can show that $\sum_{k=1}^{K} \boldsymbol{b}^{t,k} \ge \Omega_t \boldsymbol{e}$. For any feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ of problem (2.13), Lemma 2.4.1 implies that

$$\boldsymbol{0} \le (\boldsymbol{\eta}^j)' \le \Omega \boldsymbol{\alpha}', \boldsymbol{0} \le (\boldsymbol{\delta}^j)' \le \Omega \boldsymbol{\alpha}'.$$

Therefore, there exists $\rho_k^i \ge 0$ and $\phi_k^i \ge 0$ such that

$$0 \le (\boldsymbol{\eta}_i^{t,k})' = \psi_i^{t,k}(\boldsymbol{\eta}_i^t)' \le \psi_i^{t,k}\Omega_t \boldsymbol{\alpha}' \le a_i^{t,k}\boldsymbol{\alpha}', \sum_{k=1}^{K} \psi_i^{t,k} = 1,$$

and

$$0 \le (\boldsymbol{\delta}_i^{t,k})' = \phi_i^{t,k}(\boldsymbol{\delta}_i^t)' \le \boldsymbol{\phi}_i^{t,k}\Omega_t \boldsymbol{\alpha}' \le b_i^{t,k}\boldsymbol{\alpha}', \sum_{k=1}^{K} \phi_i^{t,k} = 1.$$

Hence, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\delta})$ is feasible for problem (2.16) for general K, which implies that further splitting $(\boldsymbol{z}, \boldsymbol{u})$ does not provide an improvement.

2.5 Numerical Experiment

In the previous section, we proposed one way of choosing the uncertainty set in the EAARC and identified conditions under which the EAARC improves upon the AARC. In this section, we conduct numerical experiments to illustrate the improvement on a project management problem.

A project management problem can be represented by a directed graph with m arcs and n nodes. Each node on the graph represents an event marking the completion of a particular subset of activities. We denote the set of directed arcs on the graph as E. Hence, an arc $(i, j) \in E$ is an activity that connects event i to event j. By convention, we use node 1 as the start event and the last node n as the end event.

We consider a project with several activities. The completion of activities must satisfy precedent constraints. For example, activity e_1 precedes activity e_2 if activity e_1 must be completed before starting activity e_2.

Each activity $(i,j) \in E$ has an uncertain duration $t_{ij} + z_{ij}\epsilon_{ij}$ in which t_{ij} and ϵ_{ij} are constants, and $z_{ij} \in [-1,1]$ is the primitive uncertainty. The value of z_{ij} is realized after event i is completed. But before this realization, certain resources can be allocated to the activity to shorten its duration. Specifically, we assume that if y_{ij} units of resource is allocated to activity $(i,j) \in E$, then the duration of activity (i,j) would become $t_{ij} + z_{ij}\epsilon_{ij} - y_{ij}$. Let b_{ij} be the cost of using each unit of resource for the activity on the arc (i,j). Our goal is to find a tradeoff of the completion time of the project and the total cost of resource allocations.

Mathematically, our project management problem can be formulated as a multi-stage uncertain linear program:

$$\left\{\begin{array}{lll} \min & \sum_{ij} b_{ij}y_{ij} + Cx_n & \\ \text{s.t.} & x_j - x_i + y_{ij} - z_{ij}\epsilon_{ij} \geq t_{ij} & \forall (i,j) \in E \\ & y_{ij} \geq 0 & \forall (i,j) \in E \\ & t_{ij} + z_{ij}\epsilon_{ij} - y_{ij} \geq M_{ij} & \forall (i,j) \in E \\ & x_1 = 0 & \\ & x_n \leq D & \end{array}\right\}, \forall\, z_{ij}.$$

In this model, C is the per unit cost on the completion time, and x_i denotes the completion time of event i. The first constraint implies that the completion time of event j is no less than the completion time of event i plus the completion time of activity (i,j). The third constraint requires that the reduction of the completion time of an activity cannot be arbitrarily large. In particular, in our experiment, we assume that the minimum duration of project $(i,j) \in E$ must be at least M_{ij}. We also require that the completion time of the entire project meets a strict deadline D.

If the distributional information of the uncertain data is available, one would formulate the problem as a multi-stage stochastic programming problem. Unfortunately, analysis of the project management problem within the stochastic programming framework, such as determining the expected completion time and quantile of completion time, is notoriously difficult (see

Hagstrom [42]). A tractable approximation is proposed in Chen et al. [24] to a two-stage project management problem with uncertainty, which requires mild distributional knowledge of the uncertain completion time t_{ij}.

In our experiment, instead of imposing distributional assumptions on the uncertain data, we assume that the uncertain data are restricted within some uncertainty set and formulate the project management problem within the adjustable robust counterpart framework. Specifically, the uncertainty set Γ is defined as follows:

$$\Gamma = \{\boldsymbol{z} = (z_{ij})_{(i,j)\in E} : -\bar{\boldsymbol{w}} \le \boldsymbol{z} \le \bar{\boldsymbol{v}}, \|\boldsymbol{z}\|_2 \le \Omega\}.$$

We will compare the performance of the AARC and the splitting based EAARC on the multi-stage project management problem. In the splitting based EAARC, we define the uncertainty set Λ as follows:

$$\begin{aligned}\Lambda = \{(\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w}) = (z_{ij}, v_{ij}, w_{ij})_{(i,j)\in E} &: \boldsymbol{z} = \boldsymbol{v} - \boldsymbol{w}, \\ &- \bar{\boldsymbol{w}} \le \boldsymbol{v} - \boldsymbol{w} \le \bar{\boldsymbol{v}}, \|\boldsymbol{v} + \boldsymbol{w}\|_2 \le \Omega, (\boldsymbol{v}, \boldsymbol{w}) \ge \boldsymbol{0}\}.\end{aligned}$$

In addition, the decision variables x_i and y_{ij} are represented as

$$\begin{aligned} x_i &= x_i^0 + \sum_{(k,l)\in I_i} x_i^{kl,v} v_{kl} + \sum_{(k,l)\in I_i} x_i^{kl,w} w_{kl}, \\ y_{ij} &= y_{ij}^0 + \sum_{(k,l)\in I_i} y_{ij}^{kl,v} v_{kl} + \sum_{(k,l)\in I_i} y_{ij}^{kl,w} w_{kl}, \end{aligned} \tag{2.17}$$

where v_{kl} and w_{kl} can be regarded as the positive part and negative part of the primitive uncertainty z_{kl} respectively. It is clear that when we impose the constraints $x_i^{kl,v} = -x_i^{kl,w}$ and $y_{ij}^{kl,v} = -y_{ij}^{kl,w}$, the EAARC reduces to the AARC.

Note that in the above formulations, instead of summing (k, l) across all arcs, we only take the summation in a selected set I_i. The set I_i is called the *information set* for decision variable x_i. By choosing the information set properly, we ensure that at any stage of decision, the system only takes into account information from previous realized uncertainties. Furthermore, based on our assumption that resource on an arc is allocated before the primitive uncertainty on that arc is realized, arc (i, j) should always have the same information set as node i.

Employing Theorem 2.4.1, the splitting based EAARC can be reformulated as a second order conic program. However, it seems more convenient to carry out the reformation using the following result which is shown in Chen et al. [24].

Lemma 2.5.1. *For a given scalar α and a vector $\boldsymbol{a}$, the robust constraint*

$$\begin{aligned}\alpha + \boldsymbol{a}'\boldsymbol{v} + \boldsymbol{b}'\boldsymbol{w} \leq 0, \forall(\boldsymbol{v},\boldsymbol{w}) \in \{(\boldsymbol{v},\boldsymbol{w}) :& \\ -\bar{\boldsymbol{w}} \leq \boldsymbol{v} - \boldsymbol{w} \leq \bar{\boldsymbol{v}}, \|\boldsymbol{v}+\boldsymbol{w}\|_2 \leq \Omega, (\boldsymbol{v},\boldsymbol{w}) \geq \boldsymbol{0}\}&\end{aligned}$$

can be equivalently written as

$$\begin{aligned}&\alpha + \Omega\|\boldsymbol{u}\|_2 + \boldsymbol{r}'\bar{\boldsymbol{v}} + \boldsymbol{s}'\bar{\boldsymbol{w}} \leq 0\\ &u_j \geq a_j - r_j + s_j, \forall\ j\\ &u_j \geq b_j + r_j - s_j, \forall\ j\\ &\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{s} \geq 0.\end{aligned}$$

Since all the constraints in the EAARC have the same form as the robust constraint in Lemma 2.5.1, they are referred to as robust constraints in the sequel and we will use Lemma 2.5.1 to reformulate all the robust constraints into their equivalent second order conic constraints. But before we do this, note that any primitive uncertainty not in the information set should not have an influence on the corresponding decision variable, therefore we have the following constraints:

$$\begin{aligned}x_i^{kl,v} &= x_i^{kl,w} = 0 \quad \forall(k,l) \notin I_i\\ y_{ij}^{kl,v} &= y_{ij}^{kl,w} = 0 \quad \forall(k,l) \notin I_{ij}\end{aligned}$$

Similarly the constraint $x_1 = 0$ means $x_0^{kl,v} = x_0^{kl,w} = 0\ \forall(k,l)$.

The objective of minimizing $\sum\limits_{ij} b_{ij}y_{ij} + Cx_n$ can be written as minimizing a new variable τ, subject to the robust constraint $\tau \geq \sum\limits_{ij} b_{ij}y_{ij} + Cx_n$. Given the representation of x_i and y_{ij} in (2.17) and the extended uncertainty set Λ, this robust constraint, together with the four other sets of robust constraints, is turned into their equivalent second order conic constraints. The reformulations are presented below for convenience.

First set:

$$\tau \geq \sum_{ij} b_{ij} y_{ij} + C x_n$$

becomes

$$\begin{aligned}
&\tau \geq C x_n^0 + \sum_{(i,j)\in E} b_{ij} y_{ij}^0 + \Omega \|\boldsymbol{t}\|_2 + \sum_{(k,l)\in E} (t_v^{kl} \bar{v}_{kl} + t_w^{kl} \bar{w}_{kl}) \\
&t^{kl} \geq \sum_{(i,j)\in E} b_{ij} y_{ij}^{kl,v} + C x_n^{kl,v} - t_v^{kl} + t_w^{kl} \quad \forall (k,l) \in E \\
&t^{kl} \geq \sum_{(i,j)\in E} b_{ij} y_{ij}^{kl,w} + C x_n^{kl,w} + t_v^{kl} - t_w^{kl} \quad \forall (k,l) \in E \\
&\boldsymbol{t}, \boldsymbol{t}_r, \boldsymbol{t}_w \geq \mathbf{0}.
\end{aligned}$$

where $\boldsymbol{t} = (t^{kl})_{(k,l)\in E}$, $\boldsymbol{t}_v = (t_v^{kl})_{(k,l)\in E}$ and $\boldsymbol{t}_w = (t_w^{kl})_{(k,l)\in E}$.

Second set:

$$x_j - x_i + y_{ij} - z_{ij} \epsilon_{ij} \geq t_{ij} \quad \forall (i,j) \in E$$

becomes

$$\begin{aligned}
&x_i^0 - x_j^0 - y_{ij}^0 + t_{ij} + \Omega \|\boldsymbol{\gamma}_{ij}\|_2 && \\
&\quad + \sum_{(k,l)\in E} (\gamma_{v,ij}^{kl} \bar{v}_{kl} + \gamma_{v,ij}^{kl} \bar{w}_{kl}) \leq 0 && \forall (i,j) \in E \\
&\gamma_{ij}^{kl} \geq x_i^{kl,v} - x_j^{kl,v} - y_{ij}^{kl,v} + \varepsilon_{ij} \delta_{ij}^{kl} - \gamma_{v,ij}^{kl} + \gamma_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E, \\
&\gamma_{ij}^{kl} \geq x_i^{kl,w} - x_j^{kl,w} - y_{ij}^{kl,w} - \varepsilon_{ij} \delta_{ij}^{kl} + \gamma_{v,ij}^{kl} - \gamma_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E \\
&\boldsymbol{\gamma}_{ij}, \boldsymbol{\gamma}_{v,ij}, \boldsymbol{\gamma}_{w,ij} \geq \mathbf{0} && \forall (i,j) \in E,
\end{aligned}$$

where $\boldsymbol{\gamma}_{ij} = (\gamma_{ij}^{kl})_{(k,l)\in E}$, $\boldsymbol{\gamma}_{v,ij} = (\gamma_{v,ij}^{kl})_{(k,l)\in E}$, $\boldsymbol{\gamma}_{w,ij} = (\gamma_{w,ij}^{kl})_{(k,l)\in E}$, and $\delta_{ij}^{kl} = 1$ if $(i,j) = (k,l)$ and 0 otherwise.

Third set:

$$y_{ij} \geq 0 \quad \forall (i,j) \in E$$

becomes

$$
\begin{aligned}
&-y_{ij}^0 + \Omega\|\boldsymbol{\alpha}_{ij}\|_2 + \sum_{(k,l)\in E} (\alpha_{v,ij}^{kl}\bar{v}_{kl} + \alpha_{w,ij}^{kl}\bar{w}_{kl}) \le 0 && \forall (i,j) \in E \\
&\alpha_{ij}^{kl} \ge -y_{ij}^{kl,v} - \alpha_{v,ij}^{kl} + \alpha_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E \\
&\alpha_{ij}^{kl} \ge -y_{ij}^{kl,w} + \alpha_{v,ij}^{kl} - \alpha_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E \\
&\boldsymbol{\alpha}_{ij}, \boldsymbol{\alpha}_{v,ij}, \boldsymbol{\alpha}_{w,ij} \ge \mathbf{0} && \forall (i,j) \in E,
\end{aligned}
$$

where $\boldsymbol{\alpha}_{ij} = (\alpha_{ij}^{kl})_{(k,l)\in E}$, $\boldsymbol{\alpha}_{v,ij} = (\alpha_{v,ij}^{kl})_{(k,l)\in E}$ and $\boldsymbol{\alpha}_{w,ij} = (\alpha_{w,ij}^{kl})_{(k,l)\in E}$.

Fourth set:

$$t_{ij} + z_{ij}\epsilon_{ij} - y_{ij} \ge M_{ij} \quad \forall (i,j) \in E$$

becomes

$$
\begin{aligned}
&y_{ij}^0 + \Omega\|\boldsymbol{\beta}_{ij}\|_2 + \sum_{(k,l)\in E} (\beta_{v,ij}^{kl}\bar{v}_{kl} + \beta_{w,ij}^{kl}\bar{w}_{kl}) \\
&\qquad \le t_{ij} - M_{ij} && \forall (i,j) \in E \\
&\beta_{ij}^{kl} \ge y_{ij}^{kl,v} - \varepsilon_{ij}\delta_{ij}^{kl} - \beta_{v,ij}^{kl} + \beta_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E \\
&\beta_{ij}^{kl} \ge y_{ij}^{kl,w} + \varepsilon_{ij}\delta_{ij}^{kl} + \beta_{v,ij}^{kl} - \beta_{w,ij}^{kl} && \forall (i,j) \in E, (k,l) \in E \\
&\boldsymbol{\beta}_{ij}, \boldsymbol{\beta}_{v,ij}, \boldsymbol{\beta}_{w,ij} \ge \mathbf{0} && \forall (i,j) \in E,
\end{aligned}
$$

where $\boldsymbol{\beta}_{ij} = (\beta_{ij}^{kl})_{(k,l)\in E}$, $\boldsymbol{\beta}_{v,ij} = (\beta_{v,ij}^{kl})_{(k,l)\in E}$ and $\boldsymbol{\beta}_{w,ij} = (\beta_{w,ij}^{kl})_{(k,l)\in E}$.

Fifth set:

$$x_n \le D$$

becomes

$$
\begin{aligned}
&x_n^0 + \Omega\|\boldsymbol{\psi}\|_2 + \sum_{(k,l)\in E} (\psi_v^{kl}\bar{v}_{kl} + \psi_s^{kl}\bar{w}_{kl}) \le D \\
&\psi^{kl} \ge x_n^{kl,v} - \psi_v^{kl} + \psi_w^{kl} && \forall (k,l) \in E \\
&\psi^{kl} \ge -x_n^{kl,w} + \psi_v^{kl} - \psi_w^{kl} && \forall (k,l) \in E \\
&\boldsymbol{\psi}, \boldsymbol{\psi}_v, \boldsymbol{\psi}_w \ge \mathbf{0},
\end{aligned}
$$

where $\boldsymbol{\psi} = (\psi^{kl})_{(k,l)\in E}$,$\boldsymbol{\psi}_v = (\psi_v^{kl})_{(k,l)\in E}$ and $\boldsymbol{\psi}_w = (\psi_w^{kl})_{(k,l)\in E}$.

Putting all the above together, we end up with a second order conic program, for which we use CPLEX version 10 to solve. For our computational experiment, we create a fictitious project with the activity network in the

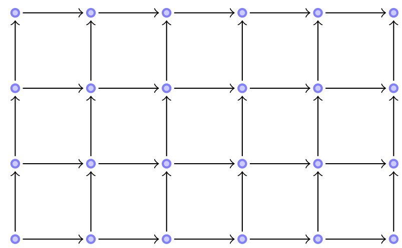

Figure 2.1: Project management grid with $H = 3$ and $W = 5$

form of a H by W grid (see Figure 2.1). There are a total of $(H+1)\times(W+1)$ nodes, with the first node at the bottom left corner and the the last node at the upper right corner. Each arc on the graph either points upwards or to the right.

In an instance of the uncertain project management problem, the related parameters t_{ij}, M_{ij}, b_{ij} are generated randomly. Specifically, on horizontal arcs, t_{ij} is generated from $U[6, 10]$ (the uniform distribution in $[6, 10]$), the minimum duration time M_{ij} is generated from $U[1, 5]$ and the resource unit cost b_{ij} is generated from $U[1, 10]$. On the vertical arcs, t_{ij} is generated from $U[4, 6]$, M_{ij} is generated from $U[1, 3]$ and b_{ij} is generated from $U[1, 5]$. Let $\epsilon_{ij} = t_{ij} - M_{ij}$. We also fix $C = 0.3$ and let $\bar{v}_{ij} = \rho_+$ and $\bar{w}_{ij} = \rho_-$ for some positive constants ρ_+ and ρ_- (the exact values of ρ_+ and ρ_- will be specified later). It is clear that ρ_+ and ρ_- measure the asymmetry of the primitive uncertainties.

We use two difference criteria to compare the performance of the EAARC and the AARC. In the first criterion, we measure the improvement of the optimal objective value of the EAARC relative to the AARC. In the second criterion, we compare the simulated average costs incurred using decision rules (2.17) derived from the EAARC and the AARC. Here's a precise description of how this is done:

- For an instance of the uncertain project management problem, we solve the EAARC and AARC respectively to derive decision rules (2.17).

- Generate 100 samples of $(z_{ij})_{(i,j)\in E}$ from $U[-\rho_-, \rho_+]$ for the instance of the uncertain project management problem.

- For each sample, compute the cost of the project management problem when the decisions x_i and y_{ij} are determined by the decision rules

(2.17) derived from solving the EAARC and the AARC. The average costs of the EAARC and the AARC are then defined as the average of the corresponding costs of all samples. Since we assume that only the primitive uncertainty $\boldsymbol{z}$ is observable, in the implementation of the EAARC decision rule, we let $v_{kl} = \max(z_{kl}, 0)$ and $w_{kl} = \max(-z_{kl}, 0)$.

- Compute the percentage of improvement:

$$\text{Percentage of Improvement} = \\ \frac{(\text{Average Cost of EAARC}) - (\text{Averaged Cost of AARC})}{(\text{Average Cost of AARC})} \times 100\%.$$

We now illustrate the impacts of the due date, the asymmetry property of the primitive uncertainties, problem size, information set, and level of robustness on the performance of the EAARC and the AARC.

Experiment 2.5.1 (Algorithm Improvement vs. due date constraint)**.** *We use the* 3×4 *grid network. Let* $\Omega = 3.0$*,* $\rho_+ = 1$ *and* $\rho_- = 0.7$*. We also use the complete information set, i.e., for each event* i*, the information set* I_i *consists of the realization of all past primitive uncertainties. We pick a range of due dates* D *between* 24 *and* 90*. The percentage of improvement vs due-date relation is shown in Figure 2.2.*

Our experiment indicates that there is a lower bound l *on the due date (*$l = 24$ *in our example), below which the due-date constraint would become so tight that both the EAARC and the AARC become infeasible. On the other hand, when the due-date goes above an upper bound* u *(*$u = 85$ *in our example), the time constraint becomes so loose that no project needs to be shortened, and therefore, the EAARC and the AARC yield the same cost.*

From Figure 2.2, we observe that the EAARC outperforms the AARC under both criteria (simulated average or optimal value). We also observe that the largest percentage of improvement always appears somewhere in the middle of l *and* u*. While for due-dates near* l *or* u*, the costs derived from the EAARC and the AARC are close. The explanation is as follows: when the due-date is too loose or too tight, the problem becomes somewhat simplified, i.e. all project must be shortened (in the tight case), or no project needs to*

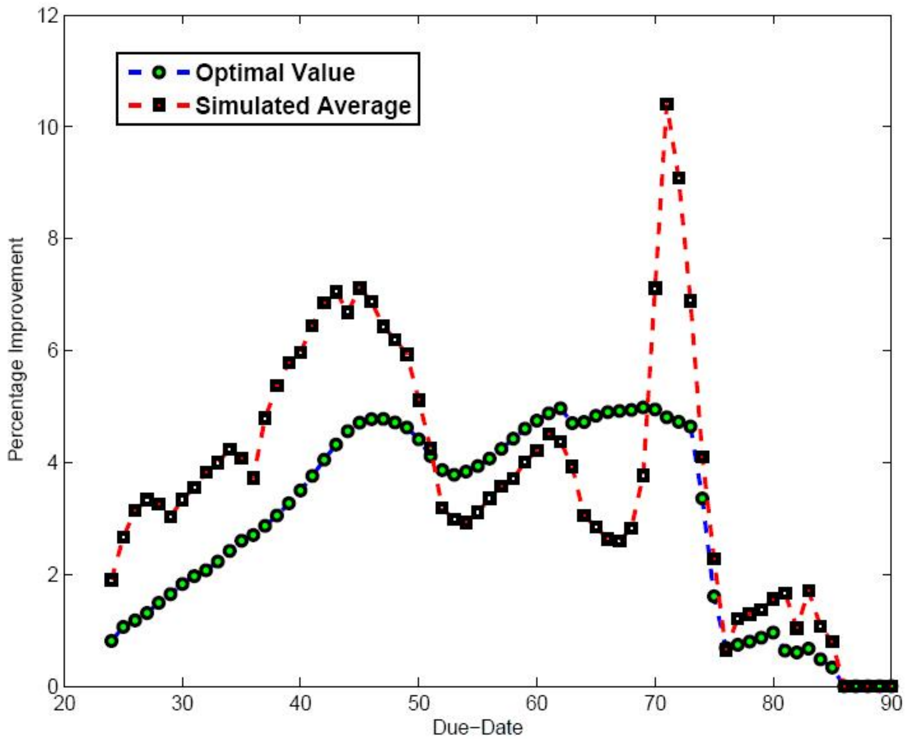

Figure 2.2: Algorithm improvement vs. due-date

ρ_-	1	0.9	0.8	0.7	0.6	0.5
Optimal	0.0%	2.3%	3.7%	4.5%	4.8%	4.9%
Simulated	0.0%	2.1%	2.9%	3.4%	3.5%	3.5%
ρ_-	0.4	0.3	0.2	0.1	0	
Optimal	4.6%	3.9%	2.9%	0.5%	0%	
Simulated	3.3%	2.9%	2.3%	1.4%	0%	

Table 2.1: Algorithm Improvement vs Asymmetric Uncertainty

be shortened (in the loose case). In either case, the EAARC does not have a big advantage over the AARC.

Around the mid-point of l and u, however, it is not immediately clear which project to shorten and how much to shorten. This adds more variability to the problem, which would in turn demand additional flexibility in the response function. The seemingly erratic shape in the middle portion suggests that the percentage of improvement is sensitive to due date. Especially we observe that there is a big jump at a due date around 71 *in the graph. One possible interpretation is that when the due date becomes rather loose, the optimal costs of both EAARC and AARC decrease rapidly and thus their ratio as well as the percentage of improvement becomes unstable. This is in fact a common observation throughout our experiments. We also observe that not surprisingly the percentage of improvement under the two criteria demonstrates a certain degree of correlation. The percentage of improvement under the simulated average cost criterion, however, appears to be more volatile than the other.*

Experiment 2.5.2 (Algorithm Improvement vs. Asymmetric Uncertainty Set)**.** *As described before, our primitive uncertainty set can model asymmetric uncertainties by adjusting ρ_+ and ρ_-. We are interested in comparing the performances of the EAARC and the AARC under different levels of asymmetry. Without loss of generality, we fix ρ_+ to be* 1 *and let ρ_- change. We set $D = 60$ and still use the 3×4 grid network with Ω set to* 3.0*. The computational results are shown in Table 2.1.*

Surprisingly, when $\rho_- = 1$ (completely symmetric uncertainty set), the EAARC and the AARC always give the same cost in our experiment. On the other end, when $\rho_- = 0$, which means the uncertainty set lies completely in the positive orthant, and therefore EAARC reduces to AARC (and gives 0% *improvement). When ρ_- takes value in the middle range, under both criteria,*

Size	3×4	4×4	4×5
Optimal	1.3%	2.4%	3.8%
Simulated	2.6%	3.5%	5.1%

Table 2.2: Algorithm Improvement vs Problem Size

the percentage of improvement gets higher. And peaks at around $\rho_- = 0.5$.

Experiment 2.5.3 (Algorithm Improvement vs. Problem Size)**.** *We now evaluate the algorithm improvement with different problem size. To do this, three grid networks are selected with size 2×3, 3×3, 3×4, respectively. The due dates are set to be $25, 30, 35$, respectively. Again, let $\Omega = 3.0$, $\rho_+ = 1$ and $\rho_- = 0.7$.*

The percentage of improvements are listed in Table 2.2. From this table, it is clear that the improvements in both the optimal objective values and the simulation averages of using the EAARC grow when the problem size grows. Interestingly, the improvement of the simulated average cost outperforms that of the optimal objective value.

Experiment 2.5.4 (Algorithm Improvement vs. Information Set)**.** *This experiment is carried out on a 3×4 grid network with $\Omega = 3.0$, $D = 60$, $\rho_+ = 1$ and $\rho_- = 0.7$. In the experiment, we compare the performance of the EAARC and the AARC using the complete information set, in which all the past information is available, and the partial information set, in which information too distant away in the past is "lost".*

To be more precise, we define the degree of information availability L as follows: for arc (i, j) to be in the information set for node k, activity (i, j) must complete before event k and there is a path from event i to event k using no more than L arcs. In our experiment, we vary L from 0 (information become lost immediately, e.g., no information available) to 7 (for 3×4 grid this means no information are lost).

Results are listed in Table 2.3 ("INF" stands for infeasible).

As we can easily observe, the percentage of improvement doesn't change much when information set shrinks. The explanation is as follows: the decision on a node depends heavily on the most "recent" information. Even though we're shrinking the information set, the most recent ones are still kept. Therefore, the performance doesn't change much. When $L = 0$, there

D	0	1	2	3	4	5	6	7
Optimal	INF	3.1%	3.2%	3.2%	3.2%	3.2%	3.2%	3.2%
Simulated	INF	4.4%	4.6%	4.6%	4.6%	4.6%	4.6%	4.7%

Table 2.3: Algorithm Improvement vs Information Set

Ω	2.0	2.5	3.0	3.5	4.0
Optimal	13.6%	8.1%	5.0%	3.2%	1.7%
Simulated	10.4%	6.3%	4.0%	4.5%	3.3%

Table 2.4: Algorithm Improvement vs Level of Robustness

are essentially no information available, and both the EAARC and the AARC become infeasible easily.

To further justify our explanation, we've also tried information set that includes all past information except the most recent ones. Both algorithms become infeasible frequently under this information set. This further confirms the intuition that for our project management problem, decision in each stage relies mostly on recent information.

Experiment 2.5.5 (Algorithm Improvement vs. Level of Robustness(Ω))**.** *This experiment is conducted on the* 3×4 *network with* $D = 60$*,* $\rho_+ = 1$ *and* $\rho_- = 0.7$*. We vary the values for* Ω *to adjust the level of robustness, and report the results in Table 2.4.*

Clearly, the EAARC outperforms the AARC by larger percentages when Ω *is small. When* Ω *grows large, we essentially put more value on robustness: both the EAARC and the AARC need to attain feasibility for a larger portion of primitive uncertainties of the problem. In this situation, the flexibility of the EAARC is confined, therefore the percentage of improvement decreases as* Ω *grows.*

In summary, the EAARC improves upon the AARC and its improvement depends on the tightness of the due-date, the asymmetric property of the uncertainty set, the information set, the size of the problem and the level of robustness. Specifically, our experiment demonstrates that the EAARC brings significant advantage over the AARC for due-dates that are not too tight or too loose, information sets which include the most recent history, larger problem size, and less stringent robustness. However, it is less clear how this improvement depends on the level of asymmetry of the uncertainty sets.

2.6 Conclusion

In this chapter, we propose the extended affinely adjustable robust counterpart to modeling and solving a class of multi-stage uncertain linear programs with fixed recourse. Our approach ends up with well structured conic programming formulations, which are tractable and scalable to multi-stage problems and allows for large scale implementation. We demonstrate both theoretically and computationally that the splitting based extended affinely adjustable robust counterpart may significantly improve upon the affinely adjustable robust counterpart.

Our extended affinely adjustable robust counterpart is rather flexible. However, a significant challenge is how to choose an appropriate extended affinely decision rule. Specifically, the following question is of great interest: for a given constant $\rho \geq 1$, can we construct a tractable EAARC such that

$$X_{EAARC} \subseteq X_0 \subseteq \rho X_{EAARC}?$$

Chapter 3

Preservation of Quasi-K-Concavity and its Application to Joint Inventory-Pricing Models with Concave Ordering Costs

3.1 Introduction

The concept of quasi-K-concavity was introduced in Porteus [49] to prove the optimality of a generalized (s, S) policy for inventory systems with concave ordering costs. To apply this concept to characterize optimal inventory policies, one relies heavily on some preservation properties under certain optimization operations. In this chapter, we provide a new preservation property of quasi-K-concavity, which says that under mild technical conditions, $\max_d[\alpha(d) + \beta(y - d)]$ is quasi-K-concave if the one-dimensional functions $\alpha(\cdot)$ and $\beta(\cdot)$ are concave and quasi-K-concave respectively.

The preservation property plays a critical role in analyzing joint inventory-pricing models with concave ordering costs. Specifically, consider a firm managing an inventory system with concave ordering cost, which may arise when the firm replenishes from a single supplier providing incremental quantity discount or multiple suppliers with different fixed costs and variable costs. Demand is random and depends on the selling price. Unsatisfied demand in each period is fully backlogged. At the beginning of each period, the firm makes pricing and inventory replenishment decisions simultaneously so as to maximize the total expected discounted profit over a finite planning horizon.

For such a model, we show by employing the preservation property of quasi-K-concavity that when demand is a deterministic function of the selling price plus a random perturbation with a positive Pólya or uniform distribution, the value functions belong to the class of quasi-K-concave functions and therefore a generalized (s, S, p) policy is optimal. Under such a policy, inventory is managed based on a generalized (s, S) policy. That is, there is a sequence of reorder points s_i and order-up-to levels S_i (both are increasing in i) such that if the starting inventory level is lower than the reorder point s_i but higher than s_{i+1}, the firm places an order to raise its inventory level to S_i. The optimal price is set according to the inventory level after replenishment. For the special case with two suppliers, one with only variable cost while the other with both fixed and variable costs, we prove that the generalized (s, S, p) policy is still optimal when the additive random component in the demand function has a strongly unimodal density.

Our model falls within the growing research stream on inventory and pricing coordination. Recently, significant progress has been made on analyzing integrated inventory and pricing models with fixed ordering cost and stochastic demand for both backlog (see Chen and Simchi-Levi [26, 27]; Huh and Janakiraman [43]) and lost sales (see Chen et al. [22]; Huh and Janakiraman [43]; Song et al. [54]) cases. For a recent review of this literature, readers are referred to Chen et al. [28]. However, we are not aware of any paper analyzing inventory and pricing models with concave ordering cost, which may be partly due to the technical complexity involved.

Our paper is closely related to classical stochastic inventory models with general concave ordering costs analyzed in Porteus [50, 49], who introduced the concept of quasi-K-concavity to prove the optimality of generalized (s, S) inventory policies when demand is a positive Pólya or uniform random variable. Recently, Fox et al. [38] analyze a special case of Porteus's model with two suppliers, one with only variable cost while the other with both fixed and variable costs. Using the concepts of K-concavity introduced in Scarf [52] and quasi-concavity (equivalently, quasi-0-concavity), they prove that the generalized (s, S) policy (indeed, a bit simplified policy) is optimal when demand has a strongly unimodal density. Our results and analysis, building upon the new preservation property of quasi-K-concavity as well as preservation properties of K-concavity and quasi-concavity, extend those in Porteus [50, 49] and Fox et al. [38] to include pricing decision.

The rest of this chapter is organized as follows. In Section 3.2, we present our major technical results, which are then applied to characterize the optimal policy for our inventory and pricing model with concave ordering cost in Section 3.3.

3.2 Main Technical Results

In this section, we present our preservation property of quasi-K-concavity. Quasi-K-concavity was introduced by Porteus [50] to prove the optimality of a generalized (s, S) policy for inventory systems with general concave ordering costs. By definition, a one-dimensional function f is quasi-K-concave if for any $x_1 \le x_2$ and $\lambda \in [0, 1]$, $f((1-\lambda)x_1 + \lambda x_2) \ge \min\{f(x_1), f(x_2) - K\}$. For brevity, readers are referred to Porteus [49] for properties of this class of functions.

Among all quasi-K-concave functions, we mainly consider one class called *quasi-K-concave function with changeover a*. A function f is quasi-K-concave with changeover a if it is increasing on $(-\infty, a]$ and non-K-increasing on $[a, \infty)$ (non-K-increasing means that for $x_1 < x_2$, $f(x_1) \ge f(x_2) - K$). An important property for this class of functions is that the quasi-K-concavity is preserved under integral convolution with respect to a positive Pólya or a positive uniform random variable. Positive Pólya (also called one-sided Pólya) distribution includes, among others, all finite convolutions of exponentially distributed random variables. Thus, as a special case, Erlang distribution is positive Pólya. Though the positive Pólya distribution appears to be restrictive, Cox [29] notes that for any given μ and $\sigma^2 \in [\mu^2/n, \mu^2]$ for some natural number n, a random variable with mean μ and variance σ^2 can be generated through a convolution of n exponential random variables. We refer to Porteus [50] for more details on this class of random variables and its relationship with quasi-K-concave functions.

We now present our major result of this section, which says that quasi-K-concavity can be preserved under a maximization operation. Let $\alpha(\cdot)$ and $\beta(\cdot)$ be one-dimensional continuous functions defined in a bounded interval $\mathcal{D} = [\underline{d}, \bar{d}]$ and in the real line respectively. Define a new function

$$\Gamma(y) = \max_{d \in \mathcal{D}} [\alpha(d) + \beta(y - d)]. \tag{3.1}$$

Lemma 3.2.1. *If $\alpha(\cdot)$ is a differentiable concave function and $\beta(\cdot)$ is a continuously differentiable quasi-K-concave function with some finite changeover ξ^0. Then the function $\Gamma(\cdot)$ defined in problem* (3.1) *is quasi-K-concave with a finite changeover no less than ξ^0.*

Since the proof is quite involved and long, it is provided in Appendix A.

Here we only briefly sketch the main idea of the proof. First, we show that $y-d(y)$ is non-increasing in y, where $d(y)$ is the smallest maximizer for problem (3.1). Second, we show that $d(y_0)$ is a maximizer of function $\alpha(\cdot)$, where y_0 is the largest point such that $\Gamma(\cdot)$ is nondecreasing in $(-\infty, y_0]$. Third, we show that y_0 is no less than the largest changeover of $\beta(\cdot)$. Finally, we use the results from the previous steps to prove that $\Gamma(\cdot)$ is non-K-increasing for $y \geq y_0$ and thus is quasi-K-concave with y_0 as its changeover.

In the next section, we will use Lemma 3.2.1 to analyze an inventory and pricing model with concave ordering cost. For a special case of the model involving two suppliers, one with only variable cost while the other with both fixed and variable costs, we can prove that a generalized (s, S, p) policy is optimal under a bit relaxed conditions using similar preservation properties of quasi-concavity (equivalently, quasi-0-concavity) and K-concavity.

Lemma 3.2.2. *Let $\alpha(\cdot)$ and $\beta(\cdot)$ be two continuous functions and $\Gamma(\cdot)$ be defined in problem* (3.1)*. We have the following results:*

(a) if $\alpha(\cdot)$ and $\beta(\cdot)$ are both quasi-concave, $\Gamma(\cdot)$ is also quasi-concave;

(b) if $\alpha(\cdot)$ is concave and $\beta(\cdot)$ is K-concave, $\Gamma(\cdot)$ is also K-concave.

Note that different from Lemma 3.2.1, part (a) of the above result only requires the quasi-concavity of α. We also comment that K-concavity was first introduced by Scarf [52] to show that an (s, S) policy is optimal for stochastic inventory models with fixed ordering costs. Chen and Simchi-Levi [26] implicitly use Theorem 3.2.2 part (b) to prove the optimality of (s, S, p) policy for an inventory and pricing problem with fixed ordering cost and additive demand.

3.3 Applications: Optimality of Generalized (s, S, p) Policy

In this section, we show how to apply our preservation properties in the previous section to analyze joint inventory and pricing models with concave ordering costs. It is worthwhile mentioning that a similar approach can be used to analyze another important application, namely inventory models incorporating sales effort/promotion decisions and concave ordering costs.

3.3.1 The Model

Consider a single product periodic-review inventory system in which a firm needs to replenish its inventory and set the selling price simultaneously at the beginning of each period over a finite planning horizon with length T. Customer demand is random but depends on the price. Unsatisfied demand is fully backlogged. The firm faces a concave piecewise linear ordering cost, which can be viewed as ordering from different (say M) suppliers with different fixed and variable ordering costs. Delivery leadtimes from all suppliers are assumed to be zero, as is common in the literature of joint inventory and pricing optimization. Ordering from supplier i incurs a fixed cost K_i and a unit cost c_i. Without loss of generality, assume that $c_1 > c_2 > \cdots > c_M \geq 0$, and $0 \leq K_1 < K_2 < \cdots < K_M$.

The remaining inventory at the end of each period t incurs a unit holding cost h_t while the unsatisfied demand incurs a unit backlog cost b_t. We use $L_t(x) = h_t \max\{x, 0\} + b_t \max\{-x, 0\}$ to denote the inventory holding and customer backlog cost given the ending inventory x. Let γ be the discount factor, $0 \leq \gamma \leq 1$. Similar to Porteus [50] (Assumption B1 in Chapter 9.4), we assume that $(c_1 - \gamma c_M) \leq b_t$, which implies that it is more cost effective to fill an order now from a more expensive supplier than delaying it until the next period using a cheaper supplier (in terms of only variable cost). The selling price of the product in period t is $p_t \in [\underline{p}_t, \overline{p}_t]$, and the demand has the following additive form:

$$D_t(p_t, \epsilon_t) = D_t(p_t) + \epsilon_t,$$

in which ϵ_t is a continuous random variable with cdf $F_t(\cdot)$ and mean μ_t. We also make the following assumption on the function $D_t(p_t)$.

Assumption 3.3.1. *For all $t = 1, 2, \cdots, T$, $D_t(p)$ has an inverse $D_t^{-1}(d)$, which is continuous and strictly decreasing. Furthermore, the expected revenue*

$$R_t(d) \triangleq (d + \mu_t) D_t^{-1}(d)$$

is differentiable and concave in d.

Assumption 1 implies that there is a one-to-one correspondence between the selling price $p_t \in [\underline{p}_t, \bar{p}_t]$ and $d_t = D_t(p_t) \in \mathcal{D}_t \equiv [\underline{d}_t, \bar{d}_t]$, where $\underline{d}_t = D_t(\bar{p}_t)$

and $\bar{d}_t = D_t(\underline{p}_t)$. Therefore, in what follows, to facilitate the analysis, we will use d instead of p as the decision variable. The concavity requirement for $R_t(d)$ is standard in the literature. Demand functions that satisfy this requirement include, among others, linear demand $D_t(p) = a - bp$ and exponential demand $D_t(p) = ae^{-bp}$.

We seek an optimal ordering and pricing policy for the firm so as to maximize its total expected discounted profit over the entire planning horizon. Let $v_t(x)$ be the optimal total expected discounted profit from period t to T. Note that $v_t(x)$ is a maximization over possible ordering from all M available suppliers:

$$v_t(x) = \max_{1 \le i \le M} \Big\{ -c_i(y-x) + \sup_{y \ge x} [\hat{H}_t(y) - K_i \delta(y-x)] \Big\}$$

where $\delta(q) = 1$ if $q > 0$ and 0 otherwise, $\hat{H}_t$ is given as

$$\hat{H}_t(y) \quad = \quad \max_{d \in \mathcal{D}_t} \Big\{ R_t(d) + \mathbb{E}[\hat{G}_t(y - d - \epsilon_t)] \Big\},$$

and $\hat{G}_t(x)$, including inventory holding and customer backlog cost as well as the discounted profit from next period, is given by

$$\hat{G}_t(x) \quad = \quad -L_t(x) + \gamma v_{t+1}(x).$$

To facilitate our analysis in the sequel, we define $G_{it}(x) = \hat{G}_t(x) - c_i x$, $H_{it}(y) = \hat{H}_t(y) - c_i y$ and $\hat{R}_{it}(d) = R_t(d) - c_i d$. With these definitions, we easily rewrite the above equations as:

$$\begin{aligned} v_t(x) &= \max_{1 \le i \le M} \Big\{ c_i x + \sup_{y \ge x} [H_{it}(y) - K_i \delta(y-x)] \Big\} && (3.2) \\ H_{it}(y) &= \max_{d \in \mathcal{D}_t} \Big\{ \hat{R}_{it}(d) + \mathbb{E}[G_{it}(y - d - \epsilon_t)] - c_i \mu_t \Big\} && (3.3) \\ G_{it}(x) &= -c_i x - L_t(x) + \gamma v_{t+1}(x). && (3.4) \end{aligned}$$

We assume without loss of generality that $v_{T+1}(\cdot) = 0$. Note that $\lim_{|x| \to \infty} G_{it}(x) =$

$\lim_{|x|\to\infty} H_{it}(y) = \lim_{|x|\to\infty} v_t(x) = -\infty$ as $\lim_{|x|\to\infty} L_t(x) = +\infty$ and $b_t \geq c_1 - \gamma c_M$.

We end this section with the definition of *generalized* (s, S) *policy.*

Definition 3.3.1. *A policy π is called generalized (s, S) if there exists an m and a sequence of parameters*

$$s_m \leq s_{m-1} \leq \cdots \leq s_1 \leq S_1 \leq S_2 \leq \cdots \leq S_m,$$

such that, given starting inventory level x, the optimal order-up-to level $\pi(x)$ is given by S_m if $x < s_m$, S_i if $s_{i+1} \leq x < s_i$ for $i = 1, 2, \cdots, m-1$, and x otherwise.

3.3.2 Analysis

In this section, we analyze the optimization problem (3.2)-(3.4) and characterize the optimal policies.

Let V^* denote the set of continuous functions $v : \mathbb{R} \to \mathbb{R}$ such that $-c_M x + v(x)$ is non-decreasing on $(-\infty, 0]$ and that $-c_i x + v(x)$ is non-K_i-increasing for each i on $\mathbb{R}$. The following result provides a characterization of the optimal policy of problem (3.2)-(3.4).

Theorem 3.3.1. *If $v_{t+1} \in V^*$ and ϵ_t is a positive Pólya or a positive uniform random variable, then*

(a) H_{it} is quasi-K_i-concave with changeover at some $a_{it} \geq 0$ for each i;

(b) There exists a generalized (s, S, p) policy that is optimal in period t;

(c) $v_t \in V^*$.

Thus, for our joint inventory and pricing problem (3.2)-(3.4)*, a generalized (s, S, p) policy is optimal.*

Proof. For part (a), we first rewrite

$$G_{it}(y) = -[(c_i - \gamma c_M)y + L_t(y)] + \gamma[v_{t+1}(y) - c_M y].$$

The property of $v_{t+1}(y)$, together with the assumptions $(c_1 - \gamma c_M) \leq b_t$ and $c_i < c_1$, implies that each term in $G_{it}(y)$ is increasing in $(-\infty, 0]$ and thus $G_{it}(y)$ is increasing in $(-\infty, 0]$. Moreover, for $y > 0$, we rewrite $G_{it}(y)$ as

$$G_{it}(y) = -[(1-\gamma)c_i y + L_t(y)[+\gamma[-c_i y + v_{t+1}(y)].$$

Because $-[(1-\gamma)c_i y + L_t(y)[$ is decreasing and hence non-$(1-\gamma)K_i$-increasing and $\gamma[-c_i y + v_{t+1}(y)]$ is non-γK_i-increasing as $v_{t+1} \in V^*$, $G_{it}(y)$ is non-K_i-increasing for $y > 0$. Thus, $G_{it}(y)$ is quasi-K_i-concave with changeover 0. Since ϵ_t has positive Pólya distribution, $\mathbb{E}[G_{it}(y - d - \epsilon_t)]$ is quasi-K_i-concave in y with a positive changeover. $\lim_{|x|\to\infty} G_{it}(x) = -\infty$ implies that this changeover is finite. In addition, since ϵ_t is a continuous random variable, $\mathbb{E}[G_{it}(y - d - \epsilon_t)]$ is continuously differentiable. Thus, by Lemma 3.2.1, $H_{it}(y)$ is quasi-K_i-concave with a changeover at some $a_{it} \geq 0$, and part (a) is proven.

For part (b), since $H_{it}(y)$ is quasi-K_i-concave, it is optimal to replenish inventory following a generalized (s, S) policy, which follows directly from Lemma 9.13 in Porteus [50]. Moreover, there exists an optimal $d^*_{it}(y)$, such that

$$d^*_{it}(y) = \underset{d \in [\underline{d}_t, \overline{d}_t]}{\operatorname{argmax}} \{\hat{R}_{it}(d) + \mathbb{E}[G_{it}(y - d - \epsilon_t)]\}.$$

Note that the optimal $d^*_{it}(y)$ is set based on the resulting inventory level y after the replenishment decision, and we can find the optimal price p^* through $D_t^{-1}(d^*_{it}(y)) = p^*$ given i is the supplier being ordered from.

For the proof of part (c), readers are referred to Porteus [49] pp. 147-148 for detailed steps. $\square$

We now focus on a special case of the model presented above. Specifically, we assume that there are only two suppliers: supplier H and supplier L, where supplier H charges a variable cost c_1 per unit but no fixed cost $(K_1 = 0)$ while supplier L charges a variable cost c_2 $(c_1 > c_2)$ per unit plus a fixed cost $K_2 = K > 0$. Such a cost structure is commonly seen in the practice of a dual sourcing strategy as discussed in Fox et al. [38]. Similar to the general model, we assume $b_t \geq c_1 - \gamma c_2$.

Given this cost structure, Fox et al. [38] proved for a corresponding inventory model without pricing decisions the optimality of generalized (s, S) type policies when demand has *strongly unimodal densities.* The class of strongly unimodal density functions is a broader class of random variables and includes many commonly used probability distributions such as normal, uniform, gamma distribution with shape parameter $p \geq 1$. A salient property of strongly unimodal density functions is the preservation of quasi-concavity, i.e., $\mathbb{E}[f(x-\epsilon)]$ is still quasi-concave if f is and ϵ has a strongly unimodal density (for more discussion on strongly unimodal density functions see Dharmadhikari and Joag-Dev [31]).

The result in Fox et al. [38] can be extended to our setting with pricing decisions. Specifically, the following result implies that the optimal inventory policy is a hybrid version of a base-stock policy plus an (s, S) policy.

Theorem 3.3.2. *For our joint inventory and pricing problem with two suppliers, under Assumption 3.3.1 with ϵ_t having a strongly unimodal density, there exist parameters s_t, S_t^L, S_t^H for period t such that the optimal order-up-to level y_t^* takes one of the two forms: If $S_t^H \leq s_t$, order from supplier L based on the following (s_t, S_t^L) policy,*

$$y_t^* = \begin{cases} S_t^L, & \text{if } x \leq s_t \\ x_t, & \text{if } x > s_t; \end{cases}$$

otherwise, follow an (s_t, S_t^H, S_t^L) mixed-ordering policy,

$$y_t^* = \begin{cases} S_t^L \ (\text{order from supplier } L) & \text{if } x \leq s_t \\ S_t^H \ (\text{order from supplier } H) & \text{if } s_t < x \leq S_t^H \\ x & \text{if } x > S_t^H \ . \end{cases}$$

Finally, set the optimal price $p^* = D_t^{-1}(d_t^*(y_t^*))$ based on the inventory level after replenishment.

The proof of the above result is almost parallel to the one in Fox et al. [38], who essentially show that both quasi-concavity and K-concavity can be preserved under dynamic programming recursions. Thus, rather than presenting the complete proof we will only sketch the key steps to prove the preservation of quasi-concavity and K-concavity under dynamic programming recursions (3.2)-(3.4) while highlighting the major differences with Fox et al. [38]. The main idea of the proof is to show by induction that $v_t(x)$

is K-concave and $G_{1t-1}(x)$ is quasi-concave in two steps. In the first step, one can prove that if $H_{1t}(y)$ is quasi-concave with nonnegative changeover and $H_{2t}(y)$ is K-concave, then the policy described in Theorem 3.3.2 is optimal, and in addition, $v_t(x)$ is K-concave and $G_{1t-1}(x)$ is quasi-concave with nonnegative changeover. This step can be proven by following an argument similar to the one in Fox et al. [38].

In the second step, we prove that if $v_{t+1}(x)$ is K-concave and $G_{1t}(x)$ is quasi-concave with nonnegative changeover, then $H_{1t}(y)$ is quasi-concave with nonnegative changeover and $H_{2t}(y)$ is K-concave. Observe that quasi-concavity is preserved under integral convolution with a strongly unimodal densities while K-concavity is preserved under integral convolution with general densities. Thus, to complete the proof of the second step, it suffices to use Lemma 3.2.2 to show that quasi-concavity and K-concavity are preserved under the optimization operation (3.1), which constitutes the major difference between our proof and the one in Fox et al. [38].

Chapter 4

Stochastic Inventory Model with Reference Price Effects

4.1 Introduction

Joint inventory-and-pricing models have enjoyed a rapid growth in the recent few decades. The driving force behind this development is quite clear: it is a good attempt that brings together the focus of traditionally separated disciplines. From an operations perspective which focuses on inventory control issues, bringing in pricing gives the decision maker an addition set of tools to achieve profit maximization. On the other hand, from an economics perspective which traditionally focuses on price and demand curve, inventory control is a problem that's often overlooked, but important enough that practitioners are forced to analyze.

A first attempt to joint inventory-and-pricing models dates back to Whitin [58], which analyzed a one-period model. Recently, significant progress has been made in this area. With fixed ordering cost and stochastic demand, Chen and Simchi-Levi [26][27] and Huh and Janakiraman [43] analyzed the backlogging case. Lost sales case was dealt with in Chen et al. [22] and Huh and Janakiraman [43]. For comprehensive reviews of this area see Chen and Simchi-Levi [28], Elmaghraby and Keskinocak [34], Federgruen and Heching [35] and Yano and Gilbert [59].

The demand function involved in most of the joint inventory-and-pricing models are dependent on price only through an instantaneous effect, that is, demand is assumed a function of the current price only. In recent years, numerous empirical studies in marketing science which analyze comsumers' choice behavior have revealed a more intricate relation between price and demand. This lead to the notion of *reference price effects.* For a good review of this area as well as an introduction to the conceptual framework, the readers are referred to Mazumdar et al. [46]. In order to understand what reference price effect means, consider a customer who enters a shop. If he has previous shopping experience for a product, then he is likely to form an internal judgement of a “fair price” for that product based on previous knowledge. This perceived “fair price” is termed *reference price*. The customer would then base his decision upon this reference price. When the observed price is below it, the customer would see it as a bargain and would buy it with a higher probability. On the other hand, when the observed price is above the reference point, the customer would see it as a loss and would be less inclined to make the purchase. In psychology, this reference-point-dependent behavior has been well explained using prospect theory in Kahneman and Tversky [44].

Since reference price cannot be directly observed through transaction records, its exact nature has been under debate. Some argue that it should simply reflect the memory for past prices; Others say that a reference price should be an expected future price, which incorporates a subjective view of a ”fair price” for the product. Again, for a comprehensive review of the conceptual framework, see Mazumdar et al. [46]. This review paper also summarized most statistical models that have been proposed for reference price. In our study, we stick to the commonly-used assumption that reference price is defined by an exponentially-weighted average of historical prices. Details of this will be made clear when we discuss our model in the next section.

For any practitioner who is responsible for making pricing decisions, being aware of the reference price effect is of course beneficial as it leads to a more accurate judgement on the demand he faces, and hence a more profitable decision. Suppose a retailing firm has some knowledge of the reference price effect among its customers, how would she dynamically set her price in order to make the maximum profit out of this effect? It turns out that there has already been some work along this direction. In Greenleaf [41], the author analyzed the impact of reference price effect on a single-period promotion. Specifically the author argued how the reference price effect would create a trade-off between additional short-term profits and a better long-term prospect. He then described how an optimal promotion strategy should be designed in order to reap the most profit out of this effect. Also under a discrete-time framework, Kopalle et al. [45] used dynamic programming to study the optimal pricing policy under asymmetric reference price effect. They further discussed a setting where brands compete in an oligopoly. Recently, using a more general demand function than previous works, Popescu and Wu [48] provided analytical answers to several questions of interest. Namely, they discussed conditions under which a constant pricing strategy would be optimal. They also proved that in certain cases, given an arbitrary starting reference price, the system would converge to the constant pricing strategy. Finally, In a continuous-time framework, Fibich et al. [36] studied the dynamic pricing problem under both symmetric and asymmetric demand function. Their work provided an elegant explicit solution to the optimal price process using optimal control. Extension to oligopolistic competition was also discussed for which the solution tool changed to dynamic games.

What distinguishes our paper from this literature is that pricing with reference price effect and inventory decision are all integrated in one model. This poses a significant challenge to analyzing the model and establishing structural results. We are currently aware of only a few studies along this direction: Urban [57] analyzed a one-period joint inventory-and-pricing model with both symmetric and asymmetric reference price effect, and provided numerical analysis which indicates that accounting for reference prices can have a substantial impact on the firm's profitability. In Gimpl-Heersink [39] the author mainly analyzed the model for one-period and two-periods cases. By explicitly calculating the profit-to-go function and its partial derivatives, the author proved that a base-stock list-price policy is optimal for both the two periods. They also provided a discussion on multi-period model however the assumptions are relatively restrictive: the commonly-used linear demand function - among other demand functions - fails to satisfy their assumption. Chen et al. [23] analyzed the problem from an algorithmic perspective. Namely they looked at an Economic-Lot-Sizing(ELS) problem with dynamic pricing and with reference price effect. They developed strong polynomial time algorithms for a few special cases of the prolem, and for the general case, they provided a heuristic with error bound estimations. And finally, we would like to mention the work by Ahn et al. [1]. They also studied an ELS model with dynamic pricing in which demand can be a function of current period price as well as past period prices. Although their demand model is not of the reference price-type per se, it is closely related. In fact, one of their special cases is almost identical to a special case in Chen et al. [23]. They proved structural results for their model and developed closed-form solutions or heuristics for various special cases.

In our paper, we try to analyze the finite horizon as well as infinite horizon version of the model, under various forms of demand function, reference price effect, and demand uncertainty. We will first provide a transformation technique that makes the new revenue function jointly concave under linear demand function, thus addressing the difficulty raised in Gimpl-Heersink [39] regarding linear demand functions. We will prove the optimality of a base-stock policy. We will then prove that reference price will converge to some steady state in the optimal trajectory. Finally we will give characterizations to the steady states. The conditions under which these results are obtained will be made clear in the sequel.

The rest of this chapter is organized as follows. In Section 4.2 we present the mathematical formulation of our model in finite horizon and discuss base-stock policy. In Section 4.3 we move to the infinite horizon model and prove the convergence results. Characterization of steady states are given in Section 4.4, various comparative statics as well as economic intuitions are also provided. Finally Section 4.5 concludes the paper and points to interesting topics for future research.

4.2 Our Model

We describe the model in a finite horizon setting. The corresponding infinite horizon model can be formulated in the same way. Consider a firm making inventory and pricing decisions over a planning horizon of length T. Periods are labeled backwards as $T, \cdots, 1$ where T corresponds to the first period in time. We also add an artificial period 0 to denote the end of the planning horizon. At the beginning of each period, an ordering decision is made together with a pricing decision. The order is received immediately and incurs a per unit cost c. Let $\mathcal{P} \equiv [\underline{p}, \bar{p}]$ be the interval of allowed prices, we also assume that reference price r lies in $\mathcal{P}$. Demand is stochastic and depends on the current period price p as well as the reference price r. Specifically, we have:

Assumption 4.2.1 (Stochastic Demand Function)**.** *Demand is given as:*

$$D_t(p_t, r_t, \epsilon) = d_t(p_t, r_t)\epsilon_m + \epsilon_a,$$

$d_t(p, r)$ is the expected demand function and is assumed to be non-increasing in p and nondecreasing in r. $\epsilon = (\epsilon_m, \epsilon_a)$ is random with $\mathbb{E}[\epsilon_m] = 1$ and $\mathbb{E}[\epsilon_a] = 0$. Furthermore, $D_t(p, r, \epsilon) \geq$ for any $p, r \in \mathcal{P}$ and any ϵ.

The following expected demand function $d_t(p, r)$ will be used in most of our analysis. A more general form will be given and used in Section 4.4.

Definition 4.2.1 (Expected Demand Function)**.** *Expected demand is given by $d_t(p, r) = b_t - a_t p + Q_t(r - p)$ and*

$$Q_t(x) = \eta^+(x)^+ + \eta^-(-x)^+,$$

where $x^+ = \max(0, x)$ and $\eta^+, \eta^- \geq 0$. The demand is said to be:

- *Loss-Averse(LA): if $\eta^+ < \eta^-$.*
- *Loss-Neutral(LN): if $\eta^+ = \eta^-$(also called Symmetric Reference Price Effect).*
- *Loss-Seeking(LS): if $\eta^+ > \eta^-$.*

The first term in the expected demand function (with a slight abuse of notation we call it $d_t(p)$) is the base demand function and $Q_t(r - p)$ is the reference price effect. We defined $Q_t(x)$ to be a two-piece linear function, with its only kink at $x = 0$. Note that $r - p$ corresponds to consumers' perceived loss/gain, with $r - p < 0$ being a loss. Among two cases of LA and LS, Loss-Averse(LA) demand is more favored from a psychological point-of-view using Prospect Theory (Kahneman and Tversky [44]). Specifically it predicts, as a general rule, that a perceived loss would stimulate more reaction from a human, compared to a perceived gain. When the two slopes are equal: $\eta^+ = \eta^- \triangleq \eta$, consumers have equal response to loss and gain. That is why this is called Loss-Neutral(LN) demand, or symmetric reference price effect. As pointed out in Fibich et al. [36], this "bears many similarities to the model of symmetric sticky-price effects, i.e., when market price does not adjust instantaneously to changes in quantities supplied." In our analysis we will try to establish our results using LN demand, and extend it to LA or LS demand whenever possible.

4.2.1 Model formulation

With the demand function, we can define expected revenue as $\hat{R}_t(p,r) = \mathbb{E}[(p-c)D_t(p,r,\epsilon)]$ which equals $(p-c)d_t(p,r)$ by Assumption (4.2.1). Reference price evolves according to the following dynamics(remember periods are labeled backwards):

$$r_{t-1} = (1-\alpha)p_t + \alpha r_t. \tag{4.1}$$

We make a comment here that applies to the Bellman Equation which we are about to describe. By default, price p_t is used as the independent (decision) variable in the Bellman Equation. However from (4.1), given r_t, there is a one-to-one correspondence between p_t and r_{t-1}. Therefore choosing p_t is equivalent to choosing r_{t-1}, in their respective feasible region of course. Thus, in some of our subsequent analysis we will use r_{t-1} instead of p_t as the decision variable.

We assume unsatisfied demand is backlogged and let $h(x)$ be the inventory holding and backlogging cost function. Denote by $\Pi_t(y,p,r)$ the single period expected profit, i.e.,

$$\Pi_t(y,p,r) = \hat{R}_t(p,r) - \mathbb{E}[h^\gamma(y - d_t(p,r,\epsilon))],$$

$\gamma \in (0,1)$ is the discount factor and $h^\gamma(x) = h(x) + (1-\gamma)cx$ is a transformed holding cost function. This transformation is a standard technique on stochastic inventory problems. For more information on this one can see, for example, Simchi-Levi et al. [53]. We further assume that $h^\gamma(x)$ is convex. We are now ready to describe the finite horizon problem formulation. Let $\hat{\phi}_t(x,r)$ be the profit-to-go function at the end of period t with inventory x and reference price r. For the end of the planning horizon let $\hat{\phi}_0(x,r) \equiv 0$. The Bellman equation for period t $(t = 1, \cdots, T)$ is:

$$\hat{\phi}_t(x,r) = \max_{y \geq x, p \in \mathcal{P}} \Pi_t(y,p,r) + \gamma \mathbb{E}\hat{\phi}_{t-1}[y - d(p,r,\epsilon), (1-\alpha)p + \alpha r], \tag{4.2}$$

4.2.2 Transformation technique for LN demand

As pointed out in Gimpl-Heersink [39], for loss-neutral(LN) demand function, $\hat{R}_t(p,r)$ is not jointly-concave in (p,r). This poses significant challenge for the commonly-used approach which requires joint-concavity to be preserved from period to period. As a result, the analysis in Gimpl-Heersink [39] for multi-period model carries on under rather restrictive assumptions which does not include LN demand. We also like to point out that for a two-period setting, Gimpl-Heersink [39] proved the joint-concavity of the function to be maximized in the Bellman Equation, even though the revenue function is not jointly-concave, and nor is the profit-to-go function from period 1. Unfortunately, their approach of calculating the profit-to-go functions explicitly does not extend to multi-period analysis. Here, we show that by making a carefully-chosen transformation on $\hat{R}_t(p, r)$, we can make it jointly-concave and supermodular. Specifically, we make the following transformation on the profit-to-go function:

$$\phi_t(x, r) = \hat{\phi}_t(x, r) - \lambda_t r^2.$$

Where λ_t are real numbers that are yet to be specified. By introducing ϕ_t, we can write the recursion in terms of this new function:

$$\begin{aligned}\phi_t(x, r) = &\max_{y \geq x, p \in \mathcal{P}} \hat{R}_t(p, r) - \lambda_t r^2 - \mathbb{E}H[y - d_t(p, r)\epsilon_m] \\ &+ \gamma \mathbb{E}\phi_{t-1}[y - d_t(p, r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r] + \gamma\lambda_{t-1}[(1-\alpha)p + \alpha r]^2. \quad (4.3)\end{aligned}$$

We combine a few terms and write this as:

$$\begin{aligned}\phi_t(x, r) = &\max_{y \geq x, p \in \mathcal{P}} R_t(p, r; \lambda) - \mathbb{E}H[y - d_t(p, r)\epsilon_m] \\ &+ \gamma \mathbb{E}\phi_{t-1}[y - d_t(p, r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r] \quad (4.4)\end{aligned}$$

where

$$R_t(p, r; \lambda) = \hat{R}_t(p, r) - \lambda_t r^2 + \gamma\lambda_{t-1}[(1-\alpha)p + \alpha r]^2. \quad (4.5)$$

The next theorem gives a condition under which there indeed exists λ_t that makes $\hat{R}_t(p, r; \lambda)$ jointly-concave and supermodular.

Lemma 4.2.1. *When demand parameters vary proportionally $a_t/a_{t-1} = \eta_t/\eta_{t-1} = k_t$ and the ratio $k_t \geq \max\left\{\gamma\alpha, \gamma\left(\frac{\alpha a+\eta}{a+\eta}\right)^2\right\}$, there exist $\lambda_t \geq 0$*

such that $R_t(p, r; \lambda)$ is jointly-concave and supermodular in (p, r).

Proof. See Appendix B.1. $\square$

Remark. The condition here is a sufficient condition and by no means necessary. The expression in max sign is between $(0, 1)$, hence we are requiring demand fluctuation to have limited downside movement. As an example, with $\gamma = 0.99,\ \alpha = 0.3,\ a = 10,\ \eta = 6$, this bound is $\max\{0.297, 0.313\} = 0.313$ which allows a lot of room for demand fluctuation. As a special case, when parameters are stationary: $a_t = a$, $b_t = b$ and $\eta_t = \eta$, the condition trivially holds.

We carry out our subsequent analysis under the assumption that this transformation is possible. Since the choice of λ_t will no longer be of interest, we suppress λ_t and write the transformed revenue function as $R_t(p, r)$.

4.2.3 Optimality of base-stock policy

For the finite horizon model, we now prove that a base-stock policy is optimal. For each t, the decision variables y and p attaining maximum in (4.4) are denoted by $y_t^*(x, r)$ and $p_t^*(x, r)$. Optimality of base-stock policy is given by the theorem below.

Theorem 4.2.1. *For the finite horizon model under LN demand, at any period t, the optimal profit-to-go function $\phi_t(x, r)$ is jointly concave in (x, r). Furthermore, the optimal inventory decision follows a base-stock policy. That is, there exists a base-stock function $y_t^0(r)$ and:*

$$y_t^*(x, r) = \begin{cases} y_t^0(r) & \textit{when } x \leq y_t^0(r) \\ x & \textit{when } x > y_t^0(r). \end{cases}$$

Proof. By Lemma 4.2.1 and the discussion following it, we can assume that our revenue function $R(p, r)$ is jointly-concave. We now prove the theorem by induction. For $t = 0$ since $\phi_0(x, r) = 0$, ϕ_0 is jointly concave in (x, r). Now assume $\phi_{t-1}(x, r)$ is jointly concave in (x, r). Since $d_t(p, r)$ is a linear function of p and r, $\phi_{t-1}[y - d_t(p, r)\epsilon_m - \epsilon_a, (1 - \alpha)p + \alpha r]$ is jointly concave in (y, p, r) for any (ϵ_m, ϵ_a). Passing that through expectation sign, $\mathbb{E}\phi_{t-1}[y - d_t(p, r)\epsilon_m - \epsilon_a, (1 - \alpha)p + \alpha r]$ is jointly concave in (y, p, r). Similarly we can prove joint concavity of $\mathbb{E}H[y - d_t(p, r)\epsilon_m]$. Lastly, $R_t(p, r)$ is assumed to be jointly concave in (p, r). Therefore the function in (4.4) before maximization is jointly concave in (x, y, p, r). And finally since maximization preserves joint-concavity, $\phi_t(x, r)$ is jointly concave in (x, r).

Suppose the maximization in (4.4) is taken over p first, then we get a function $F_t(y, r)$ which is joint-concave in (y, r). $F_t(y, r)$ is the maximized over $y \geq x$. For each r let $y_t^0(r)$ be the smallest maximizer of $F_t(y, r)$, optimality of the base-stock policy follows from the concavity of $F_t(y, r)$. $\square$

Again, to compare with previous results, Gimpl-Heersink [39] proved that a base-stock list-price is optimal for the two-period model. Here, under a multi-period setting, we were able to prove the optimality of a base-stock policy. Whether a list-price policy is optimal, in other words whether $p^*(x, r)$ is a nonincreasing function of x for any given r, is still unknown.

A closely related problem is the monotonicity of the base-stock level $y_t^0(r)$ with respect to r. Gimpl-Heersink [39] proved for the one-period model under additive demand uncertainty that $y_t^0(r)$ is non-decreasing in r. We point out that this is simply a consequence of the one-period revenue function $R(d, r)$ (written in terms of variables (d, r)) is supermodular in (d, r). For one-period model we can set $\gamma = 0$ and furthermore under additive demand uncertainty, base-stock level $y^0(r)$ is simply the optimal demand $d^0(r)$ plus a fixed amount of safety stock. Hence monotonicity of $y^0(r)$ in the one-period model follows from the monotonicity of $d^0(r)$, and the latter is guaranteed by supermodularity of $R(d, r)$.

For multi-period models, monotonicity of the base-stock level becomes a nontrivial question. Technical difficulty arise from the fact that supermodularity of the profit-to-go function $\phi_t(x, r)$ is not preserved in the expression $\gamma \mathbb{E}\phi_{t-1}[y - d(p, r, \epsilon), (1 - \alpha)p + \alpha r]$. Interestingly, all our numerical results for the multi-period problem shows that the optimal base-stock $d_t^0(r)$ level is monotonically non-decreasing in r. This is even true under a significant amount of multiplicative uncertainty. As an example, Figure 4.2.3 shows the optimal base-stock level vs. reference price with the following parameters: $\gamma = 0.99,\ \alpha = 0.15,\ c = 0.2,\ h = 0.1,\ s = 0.15,\ b = 40,\ a = 20,\ \eta = 20,\ Var(\epsilon_a) = 5,\ Var(\epsilon_m) = 0.1$.

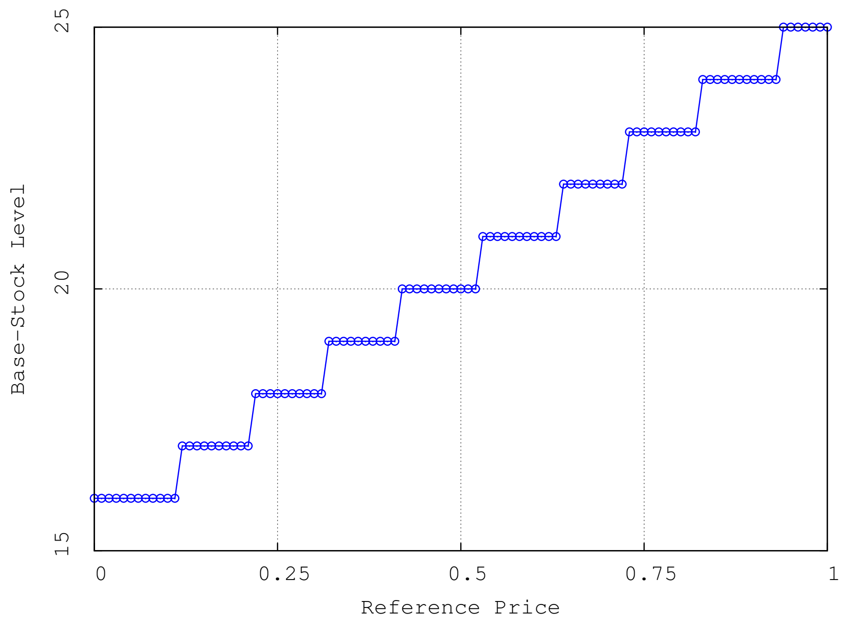

Figure 4.1: Base-stock level vs. reference price

4.3 Infinite Horizon Stochastic Model

We now turn to the infinite horizon version of our model. For infinite horizon, we are interested in the asymptotic property of the optimal trajectory, under some sample path. Namely we want to prove that the optimal trajectory converge to some stationary state. Furthermore, we want to find out properties that characterize these steady states. These two questions will be answered in this and the next section.

Since we are dealing with the infinite horizon problem now, subscript t is dropped and the profit-to-go function is now $\phi(x, r)$. We first prove a property of $\phi(x, r)$:

Theorem 4.3.1. $\phi(x, r)$ *satisfies:*

(a) $\phi(x, r)$ *is decreasing in* x *for fixed* r.

(b) If $\underline{p} \geq c$, *then* $\phi(x, r)$ *is increasing in* r *for fixed* x.

Proof. Part 1 of the theorem is quite obvious: right-hand-side of (4.4) is a maximization over $y \geq x$ and $r \in \mathcal{P}$. Increasing x shrinks the feasible set of that maximization, while leaving the function itself intact. Therefore $\phi(x, r)$

decreases in x for fixed r. As for part 2, remember the finite horizon Bellman Equation:

$$\phi_t(x,r) = \max_{y \geq x, p \in \mathcal{P}} \Pi(y,p,r) + \gamma \mathbb{E}\phi_{t-1}[y - D(p,r,\epsilon), (1-\alpha)p + \alpha r]$$

with $\phi_0(x,r) = 0$. We can prove that $\phi_t(x,r)$ converges to $\phi(x,r)$ point-wise. First observe that $\phi_0(x,r)$ is increasing in r for fixed x. Assume that $\phi_{t-1}(x,r)$ is increasing in r for fixed x. Let $r \geq c$ and (y,p) be optimal for the initial condition (x,r). For any $r' \geq r$ and close to r, we can pick $p' \geq p$ such that $D(p,r,\epsilon) = D(p',r',\epsilon)$ for any realization of ϵ. Therefore

$$\begin{aligned}
\phi_t(x,r') &\geq \Pi(y,p',r') + \gamma \mathbb{E}\phi_{t-1}[y - D(p',r',\epsilon), (1-\alpha)p' + \alpha r'] \\
&\geq \Pi(y,p,r) + \gamma \mathbb{E}\phi_{t-1}[y - D(p,r,\epsilon), (1-\alpha)p' + \alpha r'] \\
&\geq \Pi(y,p,r) + \gamma \mathbb{E}\phi_{t-1}[y - D(p,r,\epsilon), (1-\alpha)p + \alpha r] \\
&= \phi_t(x,r).
\end{aligned}$$

Since $\phi_t(x,r)$ converges to $\phi(x,r)$ point-wise, we have that $\phi(x,r)$ is increasing in r for fixed x. $\square$

Before moving on to the detailed analysis, we first give a brief roadmap. Compared to a classical joint inventory-and-pricing model, our Bellman equation has one more state variable r. This added dimension of state space brings significant challenge. Therefore we first propose a simplification of the problem, prove results for this simplified version, then demonstrate how we can use it as an auxiliary tool to establish results for our original problem.

Specifically we make the following simplification. Assume now that the retailer is allowed to return products back to the manufacturer and get a full refund. In mathematical terms, this is equivalent to allowing order-up-to level y to be less than initial inventory x. This allows us to divide and conquer the difficulties that our original model poses. Subsection (4.3.1) establishes structural results for this simplified model. Then Subsection (4.3.2) uses it to prove results for the original model.

4.3.1 When return is allowed

When return is allowed, the dynamic programming equation can be written as:

$$\phi(x,r) = \max_{y,p\in\mathcal{P}} R(p,r) - \mathbb{E}H[y - D(p,r)\epsilon_m] + \gamma\mathbb{E}\phi[y - D(p,r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r].$$

where $H(x) \triangleq \mathbb{E}h^\gamma(x-\epsilon_a)$ is convex because $h^\gamma(\cdot)$ is convex. It is easy to see that the right-hand-side(RHS) does not involve x at all. Therefore $\phi(x,r)$ is indeed only a function of r. This reduces the dimension of our dynamic programming equation to one, which is the main merit of this simplification. Note that $\mathbb{E}\phi(x-\epsilon_a, r) = \phi(r)$, the dynamic programming equation simplifies to:

$$\phi(r) = \max_{y,p\in\mathcal{P}} R(p,r) - \mathbb{E}H[y - d(p,r)\epsilon_m] + \gamma\phi[(1-\alpha)p + \alpha r]. \tag{4.6}$$

We now do a variable change: let $q = (1-\alpha)p + \alpha r$ be the new reference price if price is chosen to be p. Equivalently $p = \frac{q-\alpha r}{1-\alpha}$. The above formulation can be written in terms of q:

$$\begin{aligned}
\phi(r) &= \max_{y,q\in\mathcal{Q}_r} R\left(\frac{q-\alpha r}{1-\alpha}, r\right) - \mathbb{E}H\left[y - d\left(\frac{q-\alpha r}{1-\alpha}, r\right)\epsilon_m\right] + \gamma\phi(q) \qquad (4.7)\\
&\triangleq \max_{q\in\mathcal{Q}_r}[\tilde{R}(q,r) - (\min_y G(y,q,r)) + \gamma\phi(q)].
\end{aligned}$$

Where $\mathcal{Q}_r = (1-\alpha)\mathcal{P} + \alpha r$ and $\tilde{R}(\cdot)$ is the expected revenue function now in terms of (q,r). The above dynamic programming formulation can by optimized through some optimal response function $q^*(r)$. The next theorem presents monotonicity results of $q^*(r)$:

Lemma 4.3.1. *The response function $q^*(r)$ that maximizes (4.8) is nondecreasing in r.*

Proof. Our arguments will be based on properties of supermodular functions. One is referred to Topkis [56] for a general treatment of this topic.

For arbitrary $r, r' \in \mathcal{P}$, $r \le r'$ and $q \in \mathcal{Q}_r$, $q' \in \mathcal{Q}_{r'}$, since $\mathcal{Q}_r = (1-\alpha)\mathcal{P} +$

αr, it is not hard to prove that $\min(q, q') \in \mathcal{Q}_r$ and $\max(q, q') \in \mathcal{Q}_{r'}$, in other words, the set $\{(q, r) : q \in \mathcal{Q}_r, r \in \mathcal{P}\}$ is a lattice.

In order to prove that $q^*(r)$ is nondecreasing in r, by Theorem 2.8.2 in Topkis [56] we only need to prove the function $R(q, r) - \min_y G(y, q, r)$ is supermodular. We assumed that $\tilde{R}(p, r)$ is supermodular in (p, r), by Lemma 2 in Popescu and Wu [48] $\tilde{R}(q, r)$ should also be supermodular in (q, r). Also, by Lemma 2.6.1 in Topkis [56], sum of two supermodular functions is supermodular. What's left to prove is that the function $\min_y G(y, q, r)$ is submodular. This is, in its full form, the following function:

$$\min_y \mathbb{E}H\left[y - D\left(\frac{q - \alpha r}{1 - \alpha}, r\right)\epsilon_m\right].$$

First of all note that $H(\cdot)$ is convex. Therefore for any fixed ϵ_m, $H(y - d\epsilon_m)$ is jointly convex in y and d. Taking expectation over ϵ_m preserves convexity, that is: $\mathbb{E}H(y - d\epsilon_m)$ is convex. Minimizing this function with respect to y gives a convex function of d: $\min_y \mathbb{E}H(y - d\epsilon_m)$. Finally, replace d by the following expression:

$$d = a - \frac{a + \eta}{1 - \alpha}q + \left[\eta + \frac{\alpha(a + \eta)}{1 - \alpha}\right] r$$

This is a linear function in q and r with opposite signs. Therefore by Theorem 2.6.2 in Topkis [56] the resulting function is submodular in q and r.

This completes our argument that the one-period profit function is supermodular. And therefore $q^*(r)$ is nondecreasing in r. □

Lemma (4.3.1) leads directly to the following stability and convergence result:

Theorem 4.3.2. *In the case when return is allowed and under LN demand, starting from any state, with probability* 1 *the system will converge to a stationary price and a fixed base-stock level along an optimal trajectory. Furthermore, convergence to the stationary price is monotone.*

Proof. Theorem 4.3.2 is essentially a consequence of Lemma 4.3.1. It can be proved in much the same way as the proofs of Lemma 2 and Theorem 2 in Popescu and Wu [48]. We therefore omit the proof here. □

4.3.2 When return is not allowed

Now we come back to the case when return is not allowed, that is, the maximization over inventory is now taken for $y \geq x$. To simplify our subsequent discussion we introduce a few notations: Let I^* be the system when return is not allowed. And let I^0 be a system with the same parameters as I^* but allowing return. Let the profit-to-go functions in the two systems be $\phi^*(x,r), \phi^0(r)$, respectively (remember x is not a state variable for I^0). $\phi^0(r)$ should be the solution to the dynamic programming equation in (4.6) and $\phi^*(x,r)$ should be the solution to:

$$\begin{aligned}\phi^*(x,r) = \max_{y \geq x, p \in \mathcal{P}} & R(p,r) - \mathbb{E}H[y - d(p,r)\epsilon_m] \\ & + \gamma E[\phi^*(y - d(p,r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r] \end{aligned} \quad (4.8)$$

We first look at additive demand uncertainty, that is, $d(p,r,\epsilon) = d(p,r) + \epsilon_a$. For any state (x,r), denote the optimal decision for I^0 by $[y^0(r), p^0(r)]$ and the optimal decision for I^* by $[y^*(x,r), p^*(x,r)]$. The major obstacle to proving convergence in system I^* is that I^* cannot always mimic the behavior of I^0: when $y^0(r) < x$, $y^*(x,r)$ cannot be equal to $y^0(r)$ because of the constraint $y^*(x,r) \geq x$. This is due to a high level in initial inventory x so whenever this happens, we say the inventory level "blocks" the optimal solution $y^0(r)$. Because of this discrepancy, we cannot reduce I^* to a one-dimensional dynamic program like we did for I^0. However the following theorem guarantees the same type of path-wise convergence result for I^*:

Theorem 4.3.3. *Under LN demand with additive uncertainty, starting system I^* from any state and with probability* 1 *it will eventually converge to a steady state in which price remains stable and inventory is replenished up to some fixed base-stock level in each period.*

Proof. We prove by comparing the two systems I^0 and I^* and we first divide the state space $S = \{(x,r) \mid r \in \mathcal{P}\}$ into two complementary sets: $S_1 \triangleq \{(x,r) \in S \mid y^0(r) \geq x\}$, and $S_2 = S \setminus S_1$. See Figure 1 for a graphical representation. In words, if any state belongs to S_1, it means that inventory is low enough such that the optimal decision for I^0 is also feasible for I^*.

Our proof relies on proving the following three facts:

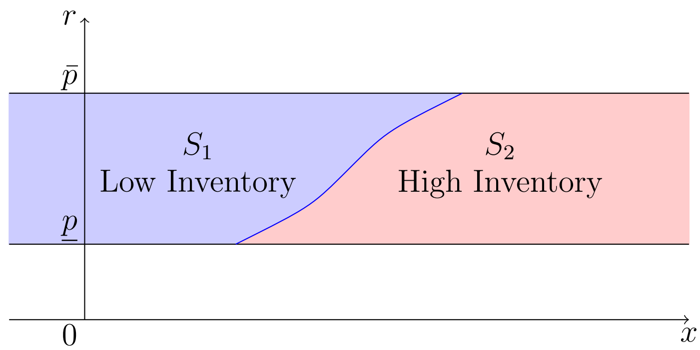

Figure 4.2: S_1 and S_2

(a) For $(x, r) \in S_1$, suppose I^* follows I^0's decisions $y^0(r)$ and $p^0(r)$, then the new state (x', r') is still in S_1, regardless of demand uncertainty ϵ_a.

(b) If $(x, r) \in S_1$, then it is indeed optimal for I^* to follow I^0's decisions: $y^*(x, r) = y^0(r)$, $p^*(x, r) = p^0(r)$.

(c) For $(x, r) \in S_2$, $y^*(x, r) = x$. In words, if the initial inventory is too high that it blocks the order quantity $y^0(r)$, then system I^* would choose to not make an order.

Intuitively, fact (c) guarantees that if the system starts at high inventory $(x, r) \in S_2$, then it will make no order and let its inventory drop down. After this transient phase of reducing inventory, the state will enter S_1. From then on, fact (a) and (b) jointly guarantees that I^* would follow I^0's strategies and the state will remain in S_1 from then on.

We first prove fact (a). For additive demand uncertainty: $D(p, r, \epsilon) = d(p, r) + \epsilon_a$, holding cost becomes $H[y - d(p, r)]$ and the maximization over y can be taken explicitly. Assume the largest minimizer of $H(\cdot)$ is x_{min} and the minimum is H^*, then $y^0(r) = \underset{y}{\operatorname{argmin}}\, H[y - d^0(r)] = x_{min} + d^0(r)$. where $d^0(r) = d[p^0(r), r]$ is the expected demand under optimal decisions. Suppose $(x, r) \in S_1$ and the optimal expected demands are d_1^0 for the current period and d_2^0 for the next period. Then after the current period inventory level would become $x_{min} + d_1^0 - d_1^0 - \epsilon_a$ while the next period requires an inventory level no more than $x_{min} + d_2^0$ so that the optimal order is not blocked. This leads to the inequality:

$$x_{min} + d_1^0 - d_1^0 - \epsilon_a \leq x_{min} + d_2^0.$$

But this inequality is equivalent to $d_2^0 + \epsilon_a \geq 0$ which always holds by Assumption (4.2.1). Therefore the state at the next period remains in S_1.

We now turn to fact (b), Note that $\phi^*(x,r) \leq \phi^0(r)$ since any feasible decision for system I^* is also feasible for I^0. Also, by fact 1, once the state enters S_1, if I^* mimics I^0's strategy it can continue doing so. This gives a feasible solution for I^* with a profit identical to the optimal profit in I^0. Therefore $\phi^*(x,r) = \phi^0(r)$ for $(x,r) \in S_1$ and it is indeed optimal for I^* to mimic I^0: $y^*(x,r) = y^0(r)$,$p^*(x,r) = p^0(r)$.

Finally we prove fact (c). We first introduce some more notations. Define $[\tilde{y}^*(x,r), \tilde{p}^*(x,r)]$ as follows:

$$\begin{aligned}
[\tilde{y}^*(x,r), \tilde{p}^*(x,r)] = \underset{y,p\in\mathcal{P}}{\operatorname{argmax}}\, & R(p,r) - \mathbb{E}H[y - d(p,r)\epsilon_m] \\
& + \gamma\mathbb{E}\phi^*[y - d(p,r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r] \qquad (4.9)
\end{aligned}$$

In case that there are multiple maximizers, choose the one with the smallest y. This is the same equation as (4.8) only relaxing the constraint $y \geq x$. Note the critical difference between this and (4.6): the profit-to-go function here is still $\phi^*(x,r)$ and in some sense we are only relaxing the constraint $y \geq x$ in the current period.

We want to prove that $\tilde{y}^*(x,r) \leq y^0(r)$. Assume the opposite is true, that is, $\tilde{y}^*(x,r) > y^0(r)$. To make our argument more concise, let $F^0(y,p;r)$ and $F^*(y,p;x,r)$ be the functions behind maximization sign in (4.6) and (4.9), respectively. Since $\phi^*(x,r) \leq \phi^0(r)$ for all (x,r), $F^*(y,p;x,r) \leq F^0(y,p;r)$ for all (y,p) and all (x,r). We now claim that the following inequalities hold:

$$\begin{aligned}
F^*[\tilde{y}^*(x,r), \tilde{p}^*(x,r)] &\leq F^0[\tilde{y}^*(x,r), \tilde{p}^*(x,r)] \\
&\leq F^0[y^0(r), p^0(r)] \\
&= F^*[y^0(r), p^0(r)].
\end{aligned}$$

The first inequality holds because $F^*(y,p;x,r) \leq F^0(y,p;r)$ for any (x,r) and any (y,p), as stated before. The second inequality holds because $[y^0(r), p^0(r)]$

maximizes $F^0(y, p; x, r)$. The equality on the last line requires a slightly more careful argument, which we now make. Remember that under additive demand, $y^0(r) = x_{\min} + d[p^0(r)]$. We can follow exactly the same argument when we were proving fact (a), and show that the new state would remain in S_1. More precisely, define $x' = y^0(r) - d[p^0(r)]\epsilon_m - \epsilon_a$ and $r' = (1-\alpha)p^0(r) + \alpha r$ then we can prove that $(x', r') \in S_1$ with probability 1. Therefore using fact (b) it is clear that $\phi^*(x', r') = \phi^0(r')$ for all the possible outcomes of x'. and hence $F^0(y^0(r), p^0(r)) = F^*(y^0(r), p^0(r))$.

The inequalities show that $[y^0(r), p^0(r)]$ is another maximizer of F^* with $y^0(r) < \tilde{y}^*(x, r)$, which contradicts our assumption that $\tilde{y}^*$ is the smallest such maximizer. Therefore $\tilde{y}^*(x, r) \leq y^0(r)$ always holds.

Finally, remember that the inventory cost $H[y - d(p, r)\epsilon_m]$ is concave in y. From our proof in Theorem 4.2.1 we also know that $\phi^*[y - d(p, r)\epsilon_m - \epsilon_a, (1-\alpha)p + \alpha r]$ is concave in y. Therefore $F^*(y, p; x, r)$ is concave in y. For any state (x, r) with $x > y^0(r)$ and decision (y, p) with $y > x$, we know that $x > \tilde{y}^*(x, r)$. By concavity of F^* and that $\tilde{y}^*(x, r)$ is its maximizer, $F^*(y, p; x, r) \leq F^*(x, p; x, r)$ and therefore the optimal decision at (x, r) is to not make any order and let inventory remain at x. The proof is now complete. □

The proof above actually gives a good sketch of what the optimal reference price path would look like. When initial inventory is high, the system will go through a transient stage where it makes no replenishments. We do not know whether the reference price is monotone or not in this transient stage. But once the state (s, r) enters S_1 (and it always will by fact (c)), it will stay in S_1(fact (a)) and the reference price thereafter will converge to r^* monotonically.

We now extend this convergence result to LA demand, or the case where $\eta^+ \leq \eta^-$. Specifically, we show that when demand uncertainty is additive, the desired monotone convergence still holds. The technique we use here, which was also used both in Fibich et al. [36] and in Popescu and Wu [48], applies to a wide range of dynamic programs with kinked reward structure. Intuitively the argument goes as follows: With the kinked price shock function, under an optimal reference price trajectory, only one of those linear pieces would matter. This is because under an optimal pricing policy, the reference price monotonically converge to its optimal value, therefore the "price shock" $p_t - r_t$

is either always positive or always negative and only one side of the function $Q(r-p)$ would affect the outcome. This is the content of the next theorem:

Theorem 4.3.4. *Under demand with Loss Aversion defined in (4.2.1) and additive demand uncertainty, starting system I^* from any state and with probability* 1 *it will eventually converge to a steady state in which price remains stable and inventory is replenished up to some fixed base-stock level in each period.*

Proof. For system I^* denote its reference price effect by $Q^*(r-p) = \eta^+ \cdot (r-p)^+ + \eta^- \cdot (p-r)^+$. Because $\eta^+ \leq \eta^-$, we can write this as:

$$Q^*(r-p) = \min_{\eta \in [\eta^+, \eta^-]} \eta \cdot (r-p). \tag{4.10}$$

For any $\eta \in [\eta^+, \eta^-]$, we construct a system that is otherwise identical to I^* but has a loss-neutral reference price effect $Q^\eta(r-p) = \eta \cdot (r-p)$, we call it $I(\eta)$. From Equation (4.10) we know $Q^*(r-p) \leq Q^\eta(r-p)$ for any $\eta \in [\eta^+, \eta^-]$. Thus for any price trajectory in $\mathcal{P}$, system $I(\eta)$ generates more profit than I^* (remember we assumed that $\underline{p} \geq c$ so that profit margin is always non-negative). Let the profit-to-go function corresponding to $I(\eta)$ be $\phi(x, r; \eta)$. Then $\phi^*(x, r) \leq \phi(x, r; \eta)$ for any $\eta \in [\eta^+, \eta^-]$. The reason is that the two systems have exactly the same feasible set, while $I(\eta)$ always generate no less profit than I^*. From Theorem 4.3.3 we know that system $I(\eta^+)$ yields a steady state $r(\eta^+)$ while $I(\eta^-)$ yields a steady state $r(\eta^-)$.

Suppose system I^* starts from some $r < r(\eta^-)$, then one feasible trajectory for I^* is to mimic the optimal trajectory of $I(\eta^-)$. Since this trajectory is optimal for $I(\eta^-)$ and that $\phi^*(x, r) \leq \phi(x, r; \eta^-)$, this trajectory is indeed optimal for I^* as well. Hence reference price for I^* would be monotonically increase and converge to $r(\eta^-)$. The case where $r > r(\eta^+)$ is the same: reference price for I^* would monotonically decrease and converge to $r(\eta^+)$.

When $r(\eta^+) < r < r(\eta^-)$, it is not hard to show that there exists $\eta \in [\eta^+, \eta^-]$ such that r is the steady state for system $I(\eta)$. Similarly we have $\phi^*(x, r) \leq \phi(x, r; \eta)$ and since a stationary reference price at r is optimal for $I(\eta)$, it is optimal for I^* as well. $\square$

Remark. The proof of this theorem also give a good sketch of the optimal trajectory. In the loss-averse case, there will be an interval of steady states, each of those states corresponds to an unique $\eta \in [\eta^+, \eta^-]$. Starting from any reference price, in an optimal trajectory the system will converge to the nearest steady state and stay there.

We make a note that Theorem 4.3.3 and 4.3.4 do not cover multiplicative demand uncertainty. Specifically our proof of fact 1 and 3 in Theorem 4.3.3 requires additive uncertainty. Our numerical experiments do show that the same kind of convergence should hold for multiplicative demand as well. This is true even when $Var(\epsilon_m)$ gets very significant. As an example, Figure 4.3.2 shows the optimal trajectories of reference price (inventory is not plotted here) with the following parameters: $\gamma = 0.99,\ \alpha = 0.15,\ c = 0.2,\ h = 0.1,\ s = 0.15,\ b = 40,\ a = 20,\ \eta = 20,\ Var(\epsilon_a) = 5,\ Var(\epsilon_m) = 0.1$.

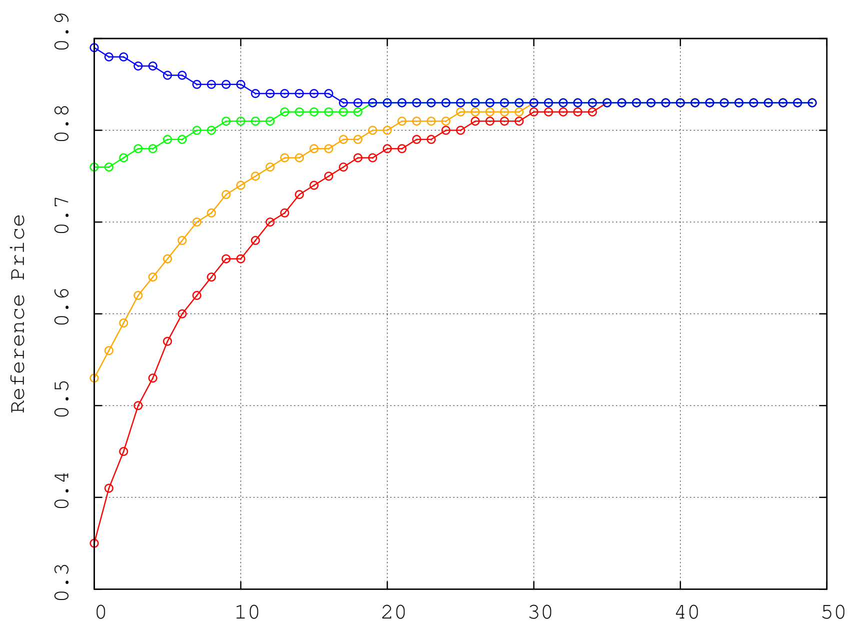

Figure 4.3: Reference price path under LN demand

This phenomenon suggests that for multiplicative demand uncertainty, there should also be some type of convergence at work. Although for multiplicative demand uncertainty, we cannot make the same arguments as we did for Theorem 4.3.3, and hence we cannot prove global convergence of the reference price path. Local convergence can indeed be guaranteed, and that is the content of the next theorem. Define S_1 in the same way as in the proof of Theorem 4.3.3 and we have:

Theorem 4.3.5. *Under LN demand with multiplicative uncertainty, there exists an open neighborhood $B(r^*)$ around the optimal steady state r^* such* *that, if system I^* starts from any state $(x, r) \in S_1$ with $r \in B(r^*)$, then with probability 1 it will eventually converge to the steady state r^* in which price remains stable and inventory is replenished up to some fixed base-stock level in each period.*

Proof. See Appendix B.2. □

Figure 4.3.2 gives an example of the optimal trajectories under LA demand. Remember that under LA demand there exists an interval of steady states instead of an unique one. Starting from any initial state and the system would converge to a state within that interval, and stay there. Of course under LN demand the interval of steady states shrinks to an unique point.

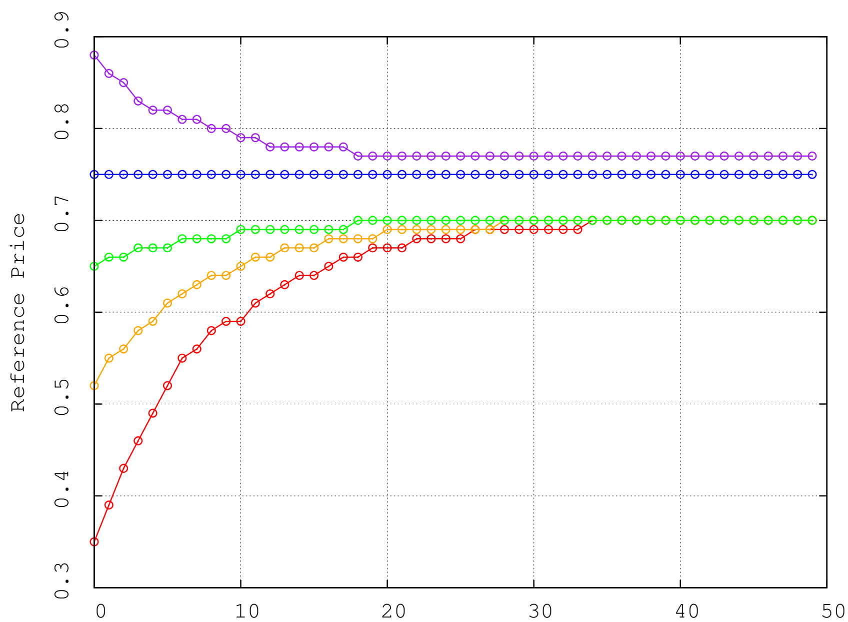

Figure 4.4: Reference price path under LA demand

The reason why there is not a similar convergence result for the loss-seeking case, is because an optimal trajectory can indeed cycle infinitely. Figure 4.3.2 provides some numerical examples of this phenomenon. Here we would like to point out that in a purely dynamic pricing setting, these properties were already observed in Popescu and Wu [48], alongside with various economic intuitions. What we have shown here is that these properties still hold when inventory cost is included explicitly in the model.

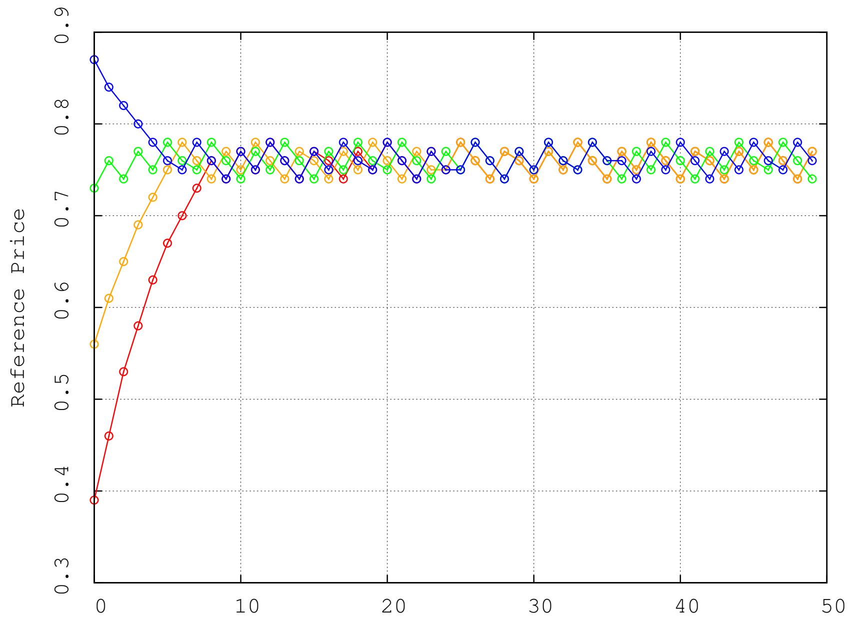

Figure 4.5: Reference price path under LS demand

4.4 Characterizing the Steady-State

In the previous section we've established the existence of steady states and proved global convergence properties. The next step is of course to characterize these steady states. Specifically, suppose r is an optimal steady-state reference price for the problem. we carry out a perturbation analysis, that is, we look at the value-to-go when price path stays at r, and then compare it to a perturbed path in which reference price takes a small perturbation around the steady-state, then returns back to it. The total discounted profit for the perturbed path should be no more than that of the steady-state path, and we therefore get the condition characterizing optimality of steady-state price.

In this section, we use a more general demand function, so that it becomes easier to compare with previous results obtained in Popescu and Wu [48].

Assumption 4.4.1. *Expected demand is now given by:*

$$d(p, r) = d(p) + Q(r - p, r) \tag{4.11}$$

where $Q(r - p, r)$ denotes the reference price effect in demand.

The corresponding revenue function can be written as:

$$R(p,r) = (p-c)d(p) + (p-c)Q(r-p,r) \triangleq R(p) + R^r(p,r). \qquad (4.12)$$

We assume that these functions are all differentiable.

4.4.1 Loss neutral demand

When demand is loss-neutral. The function $Q(x,r)$ should have equal left- and right-derivative. That is: $\lim_{\delta\to 0} \frac{Q(\delta,r)}{\delta} = \eta(r) > 0$. Consider two reference price paths: one stays constant at r while the other is perturbed: $r, r-(1-\alpha)\delta, r, r, r, \cdots$. The price path corresponding to this perturbation is $r-\delta, r+\alpha\delta, r, r, \cdots$. We first introduce a few notations: let $\Psi(d,y) = \mathbb{E}H(y-\epsilon_m d)$ be the expected holding cost, and let $\psi(d) \triangleq \min_y \Psi(d,y)$. We replacing the holding cost term in Equation (4.6) by our new notations:

$$\phi(r) = \max_p R(p,r) - \psi[d(p,r)] + \gamma\phi[(1-\alpha)p + \alpha r]. \qquad (4.13)$$

The total value without perturbation is:

$$V_0 = R(r,r) - \psi[d(r,r)] + \gamma[R(r,r) - \psi[d(r,r)]] + \gamma^2\phi(r).$$

Total value for the perturbed problem is:

$$\begin{aligned} V_\delta = R(r-\delta,r) - \psi[d(r-\delta,r)] + \gamma[R[r+\alpha\delta, r-(1-\alpha)\delta] \\ - \psi[d(r+\alpha\delta, r-(1-\alpha)\delta)]] + \gamma^2\phi(r). \end{aligned}$$

By optimality $V_0 \geq V_\delta$ should hold. When δ is small, we take the first-order Taylor expansion on each of the terms above, cancel out the principle components on both sides, and arrive at an inequality characterizing optimality:

$$
\begin{aligned}
V_\delta = R(r,r) - R_p(r,r)\delta - \psi[d(r,r)] + \psi'[d(r,r)]d_p(r,r)\delta + \gamma R(r,r) \\
- R_r(r,r)(1-\alpha)\gamma\delta + R_p(r,r)\alpha\gamma\delta - \psi[d(r,r)]\gamma \\
+ \psi'[d(r,r)]\gamma d_r(r,r)(1-\alpha)\delta - \psi'[d(r,r)]\gamma d_p(r,r)\alpha\delta + \gamma^2\phi(r) \\
\leq R(r,r) - \psi[d(r,r)] + \gamma R(r,r) - \gamma\psi[d(r,r)] + \gamma^2\phi(r) = V_0.
\end{aligned}
$$

We can simplify the above inequality and use the fact that δ can be both positive or negative to get the following condition: (suppressing function arguments(r,r))

$$
-R_p + \psi' d_p - (1-\alpha)\gamma R_r + \alpha\gamma R_p + \gamma\psi'(1-\alpha)d_r - \gamma\psi' d_p \alpha = 0.
$$

This further simplifies into:

$$
(1-\alpha\gamma)R'(r) + (\gamma-1)(r-c)\eta(r) - \psi'[d(r)]\cdot[(1-\alpha\gamma)d'(r) - (1-\gamma)\eta(r)] = 0. \tag{4.14}
$$

To compare with the result in Popescu and Wu [48], note that if we ignore the holding cost part $\psi(\cdot)$ and let $R(p,r)$ take the form in (4.12), then the inequality reduces to:

$$
(1-\alpha\gamma)R'(r) = (1-\gamma)(r-c)\eta(r). \tag{4.15}
$$

Not surprisingly this coincides with the result in Popescu and Wu [48]. Since the additional term in (4.14) corresponds to inventory costs, we can compare the solutions of (4.14) and (4.15) and find out the effect of adding inventory considerations. Namely we want to know whether inventory cost would raise or lower the steady state price. To answer this question, we make use of the following property of the function $\psi(d)$:

Lemma 4.4.1. $\psi'(d) \geq 0$ *for any* d.

Proof. Remember $\psi(d) = \min_y \Psi(y,d)$, by the envelope theorem:

$$
\psi'(d) = \left.\frac{\partial \Psi}{\partial d}\right|_{(y^*(d),d))}
$$

Let $f(\cdot)$ be the pdf of ϵ_m then:

$$\frac{\partial \Psi}{\partial d} = -\int_0^{\infty} H'(y-ud)uf(u)du$$

Since $y^*(d) = \arg\min_y \Psi(y,d)$, we have $\left.\frac{\partial \Psi}{\partial y}\right|_{(y^*(d),d)} = 0$. That is:

$$\int_0^{\infty} H'[y^*(d)-ud]f(u)du = 0.$$

$H(\cdot)$ is convex and continuously differentiable. Therefore there should exist a changeover point u_0 for which:

$$H'(u) \begin{cases} \geq 0 & \text{when } u \leq u_0 \\ \leq 0 & \text{when } u \geq u_0. \end{cases}$$

Therefore,

$$\begin{aligned} \psi'(d) &= -\int_0^{\infty} H'[y^*(d)-ud]uf(u)du \\ &= -\int_0^{\infty} H'[y^*(d)-ud]uf(u)du + u_0\int_0^{\infty} H'[y^*(d)-ud]f(u)du \\ &= -\int_0^{u_0} H'[y^*(d)-ud](u-u_0)f(u)du \\ &\quad -\int_{u_0}^{\infty} H'[y^*(d)-ud](u-u_0)f(u)du. \end{aligned}$$

Both the two terms are non-negative by our choice of u_0. Hence $\psi'(d) \geq 0$. □

Remark. This theorem states that larger expected demand d always leads to higher inventory cost. This is largely due to the multiplicative demand uncertainty. When d increases, variance of the stochastic demand $D = d\epsilon_m + \epsilon_a$ also increases. And intuitively speaking, increased demand uncertainty leads to increased inventory cost.

Theorem 4.4.1. *Under LN demand given in Definition 4.2.1, adding inventory cost always lead to higher steady-state reference price.*

Proof. Remember that $R(r) = (r-c)d(r)$ therefore $R'(r) = d(r)+(r-c)d'(r)$. Using this and we can transform (4.14) into:

$$d(r) = (r - c - \psi'[d(r)]) \left[\frac{1-\gamma}{1-\alpha\gamma} \eta(r) - d'(r) \right]. \tag{4.16}$$

With LN demand, $\eta(r) \equiv \eta$ and $d(r) = b - ar$. The above equation further simplifies into:

$$b - ar = (r - c) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right] - \psi'(b - ar) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right]. \tag{4.17}$$

Let r_0 be the solution to this equation when the term $\psi'[b - ar]$ does not exist. This corresponds to the steady-state reference price when there are no inventory cost. If we put back inventory cost in Equation (4.17) and suppose the new solution is r. By Lemma 4.4.1 we know $\psi'[b - ar] \geq 0$. If $r < r_0$, then

$$\begin{aligned}
b - ar &> b - ar_0 \\
&= (r_0 - c) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right] \\
&> (r - c) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right] \\
&\geq (r - c) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right] - \psi'(b - ar) \left[\frac{1-\gamma}{1-\alpha\gamma} + a \right].
\end{aligned}$$

Which contradicts with Equation (4.17). Therefore in order to balance the equation r has to be no less than r_0. $\square$

The economic intuition behind this is very clear: When inventory cost is brought into the picture, total cost is raised and thus profit margin is lowered. Therefore in order to maintain the original profitability, price has to be raised.

4.4.2 Loss-averse and loss-seeking demand

When reference price effect function $Q(x, r)$ has a kink at $x = 0$, namely let $\lim_{\delta \to 0^+} \frac{Q(\delta, r)}{\delta} = \eta^+(r)$ and $\lim_{\delta \to 0^-} \frac{Q(\delta, r)}{\delta} = \eta^-(r)$ where $\eta^+ \neq \eta^-$. Again, we apply the perturbation analysis done in the previous section. Note that since the left and right derivative of $Q(x, r)$ are not equal, the sign of the perturbation matters and by perturbing the stationary reference price r in both directions, we get the following pair of inequalities:

$$(1-\alpha\gamma)R'(r) - (r-c)(\eta^+(r) - \gamma\eta^-(r)) \\ - \psi'[d(r)][(1-\gamma\alpha)d'(r) - \eta^+(r) + \gamma\eta^-(r)] \geq 0.$$

$$(1-\alpha\gamma)R'(r) - (r-c)(\eta^-(r) - \gamma\eta^+(r)) \\ - \psi'[d(r)][(1-\gamma\alpha)d'(r) - \eta^-(r) + \gamma\eta^+(r)] \leq 0.$$

We can write the pair in a more concise way:

$$\eta^+(r) - \gamma\eta^-(r) \leq \frac{(1-\alpha\gamma)[R'(r) - \psi'[d(r)]d'(r)]}{r - c - \psi'[d'(r)]} \leq \eta^-(r) - \gamma\eta^+(r). \quad (4.18)$$

From this it's clear that there exists a stationary reference price r only when $\eta^+(r) \leq \eta^-(r)$, which corresponds to the case when consumers are loss-averse. Furthermore, when reference price effect is kinked linear, η^+ and η^- are not dependent on r, and there is an easy parameterization that characterizes all the stationary reference prices. Since this follows directly from the inequality in (4.18), we state it as a proposition and omit the proof:

Proposition 4.4.1. *When $\eta^+ \leq \eta^-$, the set of all stationary reference prices is given by $\{r(\eta) \mid \eta \in [\eta^+, \eta^-]\}$ where $r(\eta)$ solves:*

$$\frac{(1-\alpha\gamma)[R'(r) - \psi'[d(r)]d'(r)]}{r - c - \psi'[d'(r)]} = (1-\gamma)\eta.$$

The proposition states that any steady state reference price for the loss-averse model is also the steady state reference price for some loss-neutral model with $\eta \in [\eta^+, \eta^-]$. Hence by Theorem (4.4.1), adding inventory costs still lead to higher steady states, which matches with our discussion in the previous section. In the loss-averse case, steady states form an interval with each point in the interval corresponding to some $\eta \in [\eta^+, \eta^-]$. Starting the system from any reference price and it will converge to the nearst state in that interval.

4.4.3 Strategic vs. myopic decision making

Reference price effect dictates that the decision makers have to be forward-looking when planning their inventory and pricing policies. This is why we adopted the dynamic programming formulation (4.13) to analyze the problem. Failure to take into account future consequences of current decisions(myopic decision making) would of course lead to suboptimal decisions. We would also like to know whether myopic decision making leads to higher or lower price and reference price.

A forward-looking(strategic) planner would carry out the maximization (4.13) in each period, while a myopic planner would simply ignore the last term $\gamma\phi[(1-\alpha)p+\alpha r]$. By Theorem (4.3.1) $\phi(r)$ is increasing in r, therefore the strategic planner who takes into account this term, would choose a price that's higher than what the myopic planner would choose. As for the steady state, we have the following proposition:

Proposition 4.4.2. *Under LN demand, reference price under a myopic decision maker would also reach a steady state r^M. r^M is given by the following condition:*

$$d(r) = (r - c - \psi'[b - ar])[\eta + a]. \tag{4.19}$$

Furthermore, $r \geq r^M$, that is, strategic decision making always leads to higher steady state reference price compared to myopic decision making.

Proof. Note that by setting γ in the dynamic programming equation (4.13) to 0, we end up with a myopic policy. Therefore convergence to a steady state r^M is guaranteed for the myopic planner.

This equation is very similar to Equation (4.16). Since $\gamma < 1$ and $\alpha < 1$, we know $\frac{1-\gamma}{1-\alpha\gamma} < 1$ Using an argument that's similar to the proof of Theorem 4.4.1, we can prove that $r \geq r^M$. $\square$

Remark. The intuition to this is very clear: under myopic decision making, the planner cares only about current period profit and he would reduce price to boost current period sales. This leads to lower steady-state reference price and is detrimental to the firm's long-term profit.

4.4.4 The effect of neglecting reference price effect

We are also interested in a closely related question: What are the consequences when the decision maker is not aware of the reference price effect. In other words the decision maker believes that demand function is $d(p)$ while in reality it's $d(p) + Q(r - p, r)$. Note the distinction between this case and the previous strategic vs. myopic case: a decision maker who ignores reference price effect would still make forward-looking decisions to maximize his firm's total discounted profit. The following proposition answers our question:

Proposition 4.4.3. *Under LN demand and when the reference price effect is neglected, a system would converge to its steady state r^N characterized by the following equation:*

$$d(p) = -[p - c - \psi'[d(p)]]d'(p). \tag{4.20}$$

Furthermore, $r \leq r^N$.

Proof. In this case, the decision maker's problem simplifies to the following one-period maximization problem:

$$\max_{p,r}(p - c)d(p) - \mathbb{E}H[y - d(p)\epsilon_m].$$

Equation (4.20) is exactly the first order optimality condition of this maximization problem. Equation (4.20) is also very similar to Equation (4.19), with p replacing r. So again, using an argument that's similar to the proof of Theorem 4.4.1, we can prove that $r \leq r^N$. $\square$

Remark. This fact also makes intuitive sense: with reference price effect, a higher price hurts demand not only through the base demand function, but also through reference price effect. A decision maker who is aware of this additional effect should of course be more wary of raising his price, Compared to a decision maker who only cares about the base demand function.

4.5 Conclusion

Our paper studies a joint inventory and pricing model under reference price effect. This provides new insights into how inventory decision interacts with pricing decision under the presence of reference price effect. The major difficulty in this integrated approach is that reference price effect links pricing decision in difference periods together. This increases the dimension of the dynamic program. It further links with inventory replenishment decisions in each period. Despite the difficulty, we were able to analyze both the finite horizon and infinite horizon model, and establish a number of structural results.

For the finite horizon model with LN demand, we proved that a base-stock policy is always optimal, regardless of the demand uncertainty being additive or multiplicative.

For infinite horizon model, we first analyzed a simplified model in which return of inventory is allowed. This allows us to reduce the dimension of our dynamic program, and establish convergence results: Namely we proved in Lemma 4.3.1 that in an optimal trajectory, reference price in the next period q^* is always a nondecreasing function in the current reference price r. We then proved the main convergence result (Theorem 4.3.2) which says that in an optimal trajectory, the state would converge to a steady state with probability 1. In this process, we made use of a jointly-concave and supermodular property on the revenue function. Since LN demand function does not satisfy this assumption, we introduced a transformation technique to cope with this issue.

We then went back to the case where return is not allowed. Specifically, under additive demand uncertainty, we were able to prove the same convergence result (Theorem 4.3.3). This result is extended from loss-neutral demand to loss-averse demand. For multiplicative demand uncertainty, unfortunately the same kind of convergence cannot be proved theoretically. We showed through numerical examples that that should be the case.

We analyzed the steady state that the system should converge to. This includes a characterization of the steady state for LN and LA demand. We answered some questions of central interest in this integrated model: Under LN demand and linear reference price effect, adding inventory considerations leads to higher steady-state reference price (Theorem 4.4.1); Neglecting reference price effect leads to higher steady-state reference price. Furthermore, forward-looking (strategic) decision making leads to higher steady-state reference price, compared to myopic decision making.

This chapter should only be taken as an initial attempt to inventory and pricing models with reference price effect. Several future tasks are specifically desirable. This includes giving a theoretical proof for convergence in the multiplicative demand case. Also, it would be very interesting to look at other types of inventory models. For example in a deterministic and continuous setting one can look at the EOQ (Economic Order Quantity) model,

Chapter 5

Stochastic Reference Price Effect: A Stochastic Optimal Control Perspective

5.1 Introduction

In this chapter, we continue to build on the model with reference price effect discussed in the previous chapter. The focus here however, is a purely economic question: how do one dynamically set the best price to maximize his profit? The operational question of setting the best inventory level will not be modeled here. Several seminal papers, already reviewed in the previous chapter, studied this dynamic pricing problem under reference price effect. In particular, in a continuous-time framework, Fibich et al. [36] studied the dynamic pricing problem under both symmetric and asymmetric demand function. Their work provided an elegant explicit solution to the optimal price process using optimal control. Extension to oligopolistic competition was also discussed for which the solution tool changed to dynamic games. Our study, which is also adopting a continuous-time model, has been inspired by their work.

The following argument marks our departure from previous work in this area: Since the notion of reference price is a subjective construction, which cannot be directly measured from transaction records, we argue that the dynamics of reference price may be subject to noise. Even if two identical customers are presented with the exact same scenario (in terms of historical prices), it is not reasonable to assume that they will arrive at the exact same reference price. Indeed, many previous empirical studies in this area have been focusing on proposing different models or factors for the formation of reference price. One is referred to Mazumdar et al. [46] for a review of these models. the goal in these studies has always been to explain a larger portion of the variance in the statistical model. In our opinion, it may be benificial to look at the residual variance and try to explain its nature: whether it is due to pure noise or whether it is due to some additional factors not taken into account. The possibility of there being a pure noise is of course very significant, again because of the subjective nature of reference price.

Therefore, it is more realistic to model the reference price process as a stochastic process rather than a deterministic one. We adopt a continuous-time model similar to the one presented in Fibich et al. [36] and try to understand the role that a stochastic reference price would play. The rest of this chapter will proceed as follows: In section 5.2 we propose a model that allows this random effect in the formation of reference price. We then analyze the optimal pricing problem in section 5.3. We obtain explicit solutions to the optimal steady-state price using stochastic optimal control. In section 5.4 we compare our results to that of Fibich et al. [36] and make some sense of how big a difference it makes by introducing stochastic reference price. In section 5.5 we extend our analysis to the case of oligopolistic competition with identical retailers. Finally, conclusions are given in section 5.6.

5.2 Model

We discuss the formation of reference price first. In [36], the authors considered reference price to be an exponentially-weighted average of historical prices. With a smoothing factor α, the reference price is given by:

$$r(t) = e^{-\alpha t}\left[r_0 + \alpha \int_0^t e^{\alpha s} p(s) ds\right], \quad t \geq 0 \tag{5.1}$$

where r_0 is the initial reference price and $p(\cdot)$ is the price process. This explains the work "exponentially-weighted average" because reference price is a weighted average of historical prices with the more recent ones weighted more heavily. In differential form this relationship can be written more concisely as:

$$\begin{cases} dr = \alpha[p(t) - r(t)]dt \\ r(0) = r_0 \end{cases} \tag{5.2}$$

The intuition behind this differential form is quite clear: reference price starts at an initial value r_0, and at a constant rate α, it would drift to close the gap $p(t) - r(t)$. The resulting $r(t)$ is a deterministic process. Which means, with a given initial value r_0 and a given price process $p(t)$, the reference price at any given time is a fixed value for the entire consumer population.

In reality however, since reference price is such a subjective construct, it is natural to assume that the dynamics of it be subject to noise. When consumers are forming their reference price, the exact same scenarios might lead to different reference price paths. Specifically, we extend the above reference price dynamics using a *stochastic differential equation*(SDE):

$$dr(t) = \alpha[p(t) - r(t)]dt + \sigma\sqrt{r(t)}dW(t). \tag{5.3}$$

Here $W(t)$ denotes a standard Wiener process and reference price $r(t)$ is now a stochastic process. At any given time it yields a probability distribution over all possible prices. For a good reference on the topic of SDE see [47]. This formulation has been inspired by the CIR model of interest rates in [30]. The special feature is the $\sqrt{r(t)}$ term which makes variance of the process smaller as $r(t)$ itself gets smaller. Specifically one can prove that the probability of r going negative is always zero.

Given the above dynamics, we introduce our optimal control problem. The demand rate function is given as:

$$D(r, p) = b - ap - \eta(p - r). \tag{5.4}$$

The first part of this represents a normal linear demand function and the second part is the reference price effect. $\eta > 0$ controls the magnitude of this effect. When $p(t) < r(t)$, consumers perceive the deal as a bargain and demand would rise. On the contrary when $p(t) > r(t)$ demand would fall. Given the demand rate, revenue would accumulate at the following rate:

$$F(r, p) = (p - c)D(r, p) = (p - c)[b - ap - \eta(p - r)] \tag{5.5}$$

where c is the unit production cost. Given an initial condition $r(0) = r_0$, our goal is to maximize the total discounted profit over a finite horizon of T(γ is the discount factor):

$$\Pi[p] = \mathbb{E}\left[\int_0^T e^{-\gamma t} F[r(t), p(t)]dt\right]. \tag{5.6}$$

5.3 Analysis

We adopt a dynamic programming approach. Let $V(r,t)$ be the optimal "profit-to-go" function at reference price r and time t. In order to better serve our purpose, we relax some of the mathematical rigor and derive the equation in a more intuitive way. Consider an infinitesimal increment of time dt. If one is to take this small step into the future, and wants to take the best possible step by carefully picking his control p, then $V(r,t)$ should satisfy the following equation:

$$V(r,t) = \max_p \mathbb{E}\left[F(r,p)dt + e^{-\gamma t}V(r+dr, t+dt)\right]. \tag{5.7}$$

Using Taylor series expansion and suppress the arguments (r,t), the last term above can be written as:

$$\begin{aligned} e^{-\gamma dt}V(r+dr,t+dt) &= V - \gamma V dt + V_t dt + V_r dr + \frac{1}{2}V_{rr} dr \cdot dr \\ &= V - \gamma V dt + V_t dt + V_r[\alpha(p-r)dt + \sigma\sqrt{r}dW] \\ &\quad + \frac{\sigma^2}{2} r V_{rr} dt. \end{aligned}$$

When passing this through the expectation sign, drift term $V_r \sigma dW$ vanishes since Wiener process has mean zero. Plugging this back into (5.7) and we have:

$$V(r,t) = \max_p \left[F(r,p) + V - \gamma V + V_t + \alpha(p-r)V_r + \frac{\sigma^2}{2} r V_{rr}\right] dt. \tag{5.8}$$

Cancel out the term $V(r,t)$ on both sides of the equation, and drop dt, we reach the Bellman equation:

$$\max_p F(r,p) - \gamma V + V_t + \alpha(p-r)V_r + \frac{\sigma^2}{2} r V_{rr} = 0. \tag{5.9}$$

Similar to previous literature, our interests is not in calculating the value function in its full form. Instead, we want to answer the following question: when time horizon is long enough, will the optimal price process converge to a steady state? In other words, would a fixed price be sufficient in the long run? The following theorem answers this question:

Theorem 5.3.1. *The Bellman equation (5.9) yields a solution. And when* *time horizon approaches infinity, the optimal price process converges to the following steady-state price:*

$$p_S^* = p_D^* + \frac{\sigma^2}{2a(\gamma+\alpha)+\gamma\eta}\left[\frac{a+\eta}{\alpha}\left(\frac{\gamma}{2}-\frac{\Delta}{2}\right)+\frac{2a+\eta}{2}\right] \tag{5.10}$$

where Δ is a constant given by:

$$\Delta = \sqrt{\gamma^2 + 2\alpha\frac{2a(\gamma+\alpha)+\gamma\eta}{\eta+a}}$$

and p_D^* is the optimal price in the deterministic problem:

$$p_D^* = \frac{(\gamma+\alpha)(b+ac)+\gamma\eta c}{2a(\gamma+\alpha)+\gamma\eta}.$$

The proof will be provided in Appendix C.

Here we want to point out p_D^* is the same as $p_{optimal}^{ss}$ given in [36]. Obviously when $\sigma = 0$, our model reduces to a deterministic one and our solution agrees with the one in [36]. Furthermore, it is easy to check that the coefficient associated with σ^2 is always positive, regardless of the parameters γ, α, η and a. Therefore when the stochastic reference price is taken into account, the optimal price is consistently higher. We want to point out that this conclusion cannot be made by only looking at economic intuitions, as it is entirely unclear, without careful analysis, how a stochastic reference price would change optimal price. Mathematically, one can argue that this has to do with the convexity of $V(r,t)$ in r. But from (5.9) one can see that even this is not a sufficient argument because there are two(potentially offsetting) terms involving p: The linear term $\alpha V_r p$ which is clearly non-decreasing in p as $V_r \geq 0$; and the demand rate $F(r,p)$ which can be either increasing or decreasing in p. Therefore the conclusion here is indeed a bit surprising: Regardless of the parameters we use in the model, it is always beneficial to have a slightly higher price to deal with stochastic reference price.

5.4 The Effect of Stochastic Reference Price

In the previous section we characterized the effect that optimal price goes up once stochastic reference price effect is taken into account. In this section we try to get an idea of the magnitude of this effect. in other words we want to know how big the change in optimal price is, given typical model parameters. To answer this, we have computed a table of the magnitude of this effect for various parameters. There are six parameters in total, they are: α(smoothing factor), γ(discount factor), b(constant coefficient in demand function), a(coefficient of p in demand function), η(coefficient of $r - p$ in demand function) and c(per-unit cost). We fix c at 0.2 and b at 1.0. Since a and η only matter through their magnitude relative to b, we choose different levels for a/b and η/b. various levels of γ and α are also chosen.

For the results, note that the optimal price in (5.10) is linear in σ^2, this coefficient would naturally become our criteria. Furthermore, it makes sense to divide this coefficient by p_D^* as it measures the relative magnitude of this effect on the optimal price. This ratio, which we call *relative price change*, will be quoted in percentage points, to signify its meaning of "percentage change in optimal price, per unit of variance σ^2'.

Two levels of discount factor γ are used: 0.01 and 0.05, their results are listed in Table 5.1 and Table 5.2, respectively.

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	10%	4%	2%	42%	17%	10%	81%	33%	20%
$a/b = 0.5$	4%	1%	1%	17%	7%	4%	37%	14%	9%
$a/b = 0.8$	2%	1%	0%	10%	4%	2%	21%	08%	5%

Table 5.1: Relative price change with discount factor $\gamma = 0.01$

	$\eta/b = 0.2$			$\eta/b = 0.5$			$\eta/b = 0.8$		
α	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
$a/b = 0.2$	6%	3%	2%	20%	12%	8%	34%	23%	16%
$a/b = 0.5$	2%	1%	1%	9%	5%	3%	18%	11%	7%
$a/b = 0.8$	1%	1%	0%	5%	3%	2%	11%	6%	4%

Table 5.2: Relative price change with discount factor $\gamma = 0.05$

A few observations are immediate from these tables. First of all, the relative change in optimal price can be very significant in many scenarios, keep in mind that these are "per unit of σ^2" figures, so the actual relative changes in optimal price are these numbers multiplied by σ^2. When η/b is large, reference price effect becomes the more dominating factor in demand function, in which case the relative price change understandably becomes large. On the other hand when a/b is large, relative price change becomes less significant. Furthermore, relative price change decreases when α gets larger, or when reference price adjusts to new price faster. This also makes intuitive sense, because as α gets larger, the merit of having a reference price model shrinks. In the limiting case reference price adjusts to current price instantaneously, and there would not be a reference price effect any more. Finally, relative price change decreases when γ gets larger, or when future profit is discounted more. This is somewhat less intuitive, however it is still helpful to think about the limiting case. When γ gets arbitrarily large, future profit is discounted so much so that essentially we would be dealing with a single-period problem. In which case the relative price change would of course become 0 as reference price effect itself would vanish.

5.5 Oligopolistic Competition

The results in the last section can be readily extended to the case of competing firms. Specifically, assume we have N retailers aiming at the same group of consumers. Furthermore, assume that they have production costs that are identical. Note that when there is only a single firm, setting price p is equivalent to setting production quantity q. When there are multiple firms, this is no longer the case. Price p is now determined by the total production quantity Q of all the firms through the demand function. That is, $Q = \sum_n q_n$, and $Q = a - \delta p - \gamma(p - r)$.

Let $V_n(r)$ be the value-to-go function for firm n, the Bellman equation in this case can be written as:

$$\alpha V_n(r) = \max_{q_n} \left\{ (p - c)q_n + \frac{dV_n(r)}{dr}\beta(p - r) + \frac{\sigma^2 r}{2}\frac{d^2V_n(r)}{dr^2} \right\}. \tag{5.11}$$

The maximum is attained by:

$$\hat{q}_n = a + \gamma r - \hat{Q} - (\delta + \gamma)c - \beta\frac{dV_n(r)}{dr}.$$

Where $\hat{Q} = \sum_n \hat{q}_n$. The firms are identical, and that allows us to sum all the equations and use $V(r) = \sum_n V_n(r)$. The PDE we obtain is:

$$\alpha V = \frac{N}{(\delta+\gamma)(N+1)^2}[a+\gamma r-(\delta+\gamma)c+\beta V_r]\cdot\left[a+\gamma r-(\delta+\gamma)c-\frac{\beta}{N}V_r\right]$$
$$+\frac{\beta V_r}{(\delta+\gamma)(N+1)}\cdot[a+(\delta+\gamma)Nc+\beta V_r-(\delta+\gamma)(N+1)r+\gamma r]+\frac{N\sigma^2 r}{2}V_{rr}. \tag{5.12}$$

The solution of (5.12) is given by:

$$V(r) = Ar^2 + Br + C$$

where A is the same as in Fibich et al. [36]:

$$A = (\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{8\beta^2 N} - \frac{\gamma}{2\beta}\pi\sqrt{\left((\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{8\beta^2 N}-\frac{\gamma}{2\beta}\right)^2-\frac{\gamma^2}{4\beta^2}}.$$

And B is given by:

$$B = \big(2[a-(\delta+\gamma)c][A\beta(N-1)+N\gamma]+2A\beta[a+(\delta+\gamma)NC](N+1) + NA(\delta+\gamma)(N+1)^2\sigma^2\big)/\big((\delta+\gamma)(N+1)^2(\alpha+\beta)-2\beta N(2A\beta+\gamma)\big).$$

5.6 Conclusion

This chapter studies a dynamic pricing problem under stochastic reference price effect. This stochasticity is adopted because it is more reasonable to assume that the dynamics of reference price is subject to noise. A stochastic differential equation for the reference price evolution was proposed. The corresponding optimal pricing model was analyzed using stochastic optimal control theory. By solving the HJB equations we were able to provide an explicit solution for the optimal steady-state price.

We then compared numerically our optimal price to the one in [36] to show that the relative change in the optimal price can be very significant when a stochastic reference price is assumed. Specifically, this relative change in price becomes larger if: (1) reference price effect itself becomes larger, (2) reference price adapts to new price at a higher speed, and (3) future profit is discounted more.

One interesting observation is that when noise is introduced in reference price, the resulting optimal price is always higher, regardless of model parameters. The managerial implication of this fact is clear: if a decision maker is adopting a reference price model to set his optimal price, and if he believes that the noisy dynamics of reference price is causing his model-implied optimal price to become suboptimal, then it is always beneficial for him to adjust his price upwards.

As a potential direction of extention, it would be interesting to see an analysis of the oligopolistic competition model with non-identical retailers. The technique presented in this thesis would no longer work as the Bellman equation becomes a system of equations.

It is important to point out that while reference price in our model is subject to noise, the consumer group is still homogeneous in the sense that they share the same parameters a, η, α, σ, etc. As a potential future direction of research, it would be interesting to look at a model that analyzes the heterogeneity of consumer groups explicitly. The meaning of heterogeneity is two folds here: consumers may differ in their reaction to a given reference price and a price, in other words they may have different utility functions. On the other hand, consumers may differ in their underlying reference price dynamics. We believe that a model that incorporates heterogeneity might lead to some interesting insights.

In the first case the interesting question is, given a demographic profile of consumers' utility functions, what would the aggregate-level demand function look like? Indeed, this question has already been raised in the literature. Bell and Lattin [5] argues that by simply using a loss-neutral reference price effect (so that demand has a linear term in $r - p$) but allowing the coefficient in front of $r - p$ to be heterogeneous among consumers, the aggregate-level reference price effect might demonstrate a loss-averse shape. Thus the commonly-adopted loss-averse reference price effect might, instead of being implied by consumer loss-aversion via Prospect Theory, simply be a consequence of heterogeneity in consumers. Arora et al. [2] also studied the reference price effect in which consumers differ in their risk preference. They also managed to derive important implications from this heterogeneity effect which makes them suggest that in may be worthwhile for a researcher in this field or a decision maker to develop a demographic profile of the risk preferences of consumers. In the second case, it would be interesting to look at - once again - a dynamic programming problem. When price discrimination is not allowed, a retailer would have to make his pricing decisions taking into account all his different consumer groups.

Chapter 6

Future Research

Chapter 2 extended the class of affine response functions and made them more versatile. There are two natural questions to ask. First, what is the fundamental structure on the problem that makes these types of extensions possible? Nemirovski et al. [7] provides an insightful answer to this question: when the convex hull of the extended uncertainty set is still in some tractable form, then the overall robust optimization problem is still efficiently solvable. The question now becomes this: which classes of nonlinear transformation can be used to form the extended uncertainty set, so that tractability is retained. The second question - which might be even harder to answer than the first - is, for which problems are affine decision rules guaranteed to be optimal? In other words, what kind of problem structure would render an extended decision rule worthless? To the best of our knowledge, this question has only been partially answered by Bertsimas et al. [16], which looks a one-dimensional multi-stage robust formulation, and proves the optimality of response functions that are affine in terms of disturbances.

There is another question that appears more specific to the EAARC that we proposed. Namely how to choose an appropriate extended affinely decision rule. Specifically, for a given constant $\rho \geq 1$, can we construct a tractable EAARC such that

$$X_{EAARC} \subseteq X_0 \subseteq \rho X_{EAARC}?$$

For the joint inventory-and-pricing problem with general concave cost discussed in chapter 3, Some further questions that can be asked include: what happens when inventory replenishments have (deterministic or stochastic) lead time? What happens with a lost-sales model instead of a backlogging model? These questions - we believe - would further add a significant challenge in terms of technical difficulty.

For the joint inventory-and-pricing problem with reference price effects studied in chapter 4, there are two interesting unanswered questions. Firstly, we proved that a base-stock policy is optimal for the model, but what about a list-price policy? Namely is the optimal price a decreasing function of current reference price level? Secondly, it will be interesting to see whether our convergence results would still hold under multiplicative demand uncertainty. Currently we still don't know the answer to that this question, although numerical results show that this is very likely the case.

Finally, chapter 5 represents only an initial attempt to use a more realistic reference price model that incorporates heterogeneity and randomness. The interesting question to ask is, among previous results for dynamic pricing models under reference price effect, which ones will cease to hold under these more realistic models? Which ones will be preserved under these models? We believe the answer to this question is greatly valuable to offering more reliable managerial insights for practitioners who are dealing with reference price effect.

Appendix A

Proofs for Chapter 3

A.1 Proof of Lemma 3.2.1 on page 42

To prove Lemma 3.2.1, we need the following result.

Proposition A.1.1. *Let $\alpha(\cdot)$ be a concave function in a bounded interval $\mathcal{D} = [\underline{d}, \bar{d}]$ and $\beta(\cdot)$ be a continuous function. There exists a $d(y)$ maximizing $\alpha(d) + \beta(y-d)$ for $d \in \mathcal{D}$ such that $y - d(y)$ is an increasing function of y.*

To prove the above result, one can first replace d by a new variable $\tilde{d} = y - d$. Since $\alpha(\cdot)$ is concave and $\beta(\cdot)$ is a function of a single variable, the function $\alpha(y - \tilde{d}) + \beta(\tilde{d})$ is supermodular in $(y, \tilde{d})$. Thus, there exists a $\tilde{d}(y)$ maximizing $\alpha(y - \tilde{d}) + \beta(\tilde{d})$ such that $\tilde{d}(y)$ is increasing in y (note that $\tilde{d}(y)$ can be chosen as either the largest optimal solution for all y or the smallest optimal solution for all y). Then the above lemma holds for $d(y) = y - \tilde{d}(y)$.

Now we move on to prove Lemma 3.2.1. Define $d(y) = \min\{d : d \in \arg\max_{d\in\mathcal{D}}[\alpha(d) + \beta(y-d)]\}$. By Proposition A.1.1, $y - d(y)$ is increasing. Define

$$y_0 = \sup\{y : \Gamma(y) \text{ is non-decreasing on } (-\infty, y]\}.$$

We claim that $d(y_0) \in \{\text{argmax}_{d\in\mathcal{D}}\, \alpha(d)\}$. In the sequel, we will first prove the lemma under this claim. The proof for the claim itself will be provided after that.

Note that if $\Gamma(\cdot)$ is indeed quasi-K-concave, y_0 defined above would be its largest changeover point. Therefore, to prove the lemma, it is sufficient to show that $\Gamma(y)$ is non-K-increasing for $y \geq y_0$ or for $y_0 \leq y_2 \leq y_1$:

$$\Gamma(y_2) \geq \Gamma(y_1) - K.$$

Let $\xi^0 > 0$ be the largest changeover of $\beta(y)$. One should note that, if $\xi^0 = \infty$, then $y_0 = \infty$ and the lemma is clearly true. So in the following proof, ξ^0 is assumed to be finite.

We first show the right-continuity of $y - d(y)$ at $y = y_0$. Since $y - d(y)$ is increasing in y, $\lim_{y \to y_0^+} y - d(y)$ always exists (superscript "+" means taking the right limit) and it is sufficient to show that it equals $y_0 - d(y_0)$. Assume $\lim_{y \to y_0^+} y - d(y) = y_0 - \tilde{d}$ for some $\tilde{d} \geq 0$. Then by continuity of $\Gamma(\cdot)$,

$$\Gamma(y_0) = \alpha(\tilde{d}) + \beta(y_0 - \tilde{d})$$

and so $\tilde{d} \in \{\text{argmax}_d[\alpha(d) + \beta(y_0 - d)]\}$. Furthermore, by the monotonicity of $y - d(y)$ we have $y_0 - \tilde{d} = \lim_{y \to y_0^+} y - d(y) \geq y_0 - d(y_0)$. Hence, $\tilde{d} \leq d(y_0)$. As $d(y_0)$ is assumed to be the smallest maximizer of $[\alpha(d) + \beta(y_0 - d)]$, $\tilde{d} = d(y_0)$ and $\lim_{y \to y_0^+} y - d(y) = y_0 - d(y_0)$.

We next show by contradiction that $y_0 - d(y_0) \geq \xi^0$. Suppose $y_0 - d(y_0) < \xi^0$, by right-continuity of $y - d(y)$ at y_0 there exists a number $\eta > 0$ such that for any $y \leq y'$ in the interval$[y_0, y_0 + \eta]$, $y_0 - d(y_0) \leq y - d(y) \leq y' - d(y') \leq \xi^0$. We can show that $\Gamma(y)$ is non-decreasing in this interval $[y_0, y_0 + \eta]$ with $\eta > 0$:

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y)) + \beta(y' - d(y)) \\
&\geq \alpha(d(y)) + \beta(y - d(y)) \\
&= \Gamma(y),
\end{aligned}$$

where the first inequality follows from the fact that $d(y')$ is optimal for $\Gamma(y')$; the second inequality holds because $\beta(\cdot)$ is increasing on $(-\infty, \xi^0]$. This contradicts with the definition of y_0. Therefore, $y_0 - d(y_0) \geq \xi^0$.

Now we focus our attention on $\xi^0 \leq y_2 - d(y_2) \leq y_1 - d(y_1)$, we verify the lemma by discussing several different cases.

If $d(y_2) \geq d(y_1)$, then $y_2 - d(y_1) \geq y_2 - d(y_2) \geq \xi^0$ and therefore

$$
\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_1)) + \beta(y_2 - d(y_1)) \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}
$$

where the first inequality follows from the optimality of $d(y_2)$ and the second one from the non-K-increasing of $\beta(y)$ for $y \geq \xi^0$.

If $d(y_2) < d(y_1)$, then we have the following two different cases:

Case I: $d(y_0) \geq d(y_2)$. In this case, obviously $y_2 - d(y_0) \leq y_2 - d(y_2)$.

$$
\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_0)) + \beta(y_2 - d(y_0)) \\
&\geq \alpha(d(y_0)) + \beta(y_1 - d(y_1)) - K \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}
$$

where the second inequality follows from $\xi^0 \leq y_0 - d(y_0) \leq y_2 - d(y_0) \leq y_2 - d(y_2) \leq y_1 - d(y_1)$ and the last one from the optimality of $d(y_0)$ for $\alpha(d)$ that we claimed.

Case II: $d(y_2) > d(y_0)$.

$$
\begin{aligned}
\Gamma(y_2) &= \alpha(d(y_2)) + \beta(y_2 - d(y_2)) \\
&\geq \alpha(d(y_2)) + \beta(y_1 - d(y_1)) - K \\
&\geq \alpha(d(y_1)) + \beta(y_1 - d(y_1)) - K \\
&= \Gamma(y_1) - K,
\end{aligned}
$$

where the first inequality follows from the non-K-increasing of $\beta(y)$ and the second one follows from the concavity of $\alpha(d)$ and that $d(y_0)$ is its maximizer.

The above cases cover all possibilities and we have proved the lemma under the claim that $d(y_0)$ is a maximizer of $\alpha(d)$. We now turn to prove the claim itself. Observe that $d(y_0)$ can either lie in the interior of $\mathcal{D} = [\underline{d}, \overline{d}]$ or on its boundary. We distinguish between these two cases. If $d(y_0)$ is an interior point of $\mathcal{D}$, then from the first order optimality condition,

$$\alpha'(d(y_0)) = \beta'(y_0 - d(y_0)).$$

If $\alpha'(d(y_0)) > 0$, then $\beta'(y_0 - d(y_0)) > 0$. Since $\beta(\cdot)$ is continuously differentiable, $\beta'(x) > 0$ for x in a small neighborhood of $y_0 - d(y_0)$. As $\lim_{y \to y_0^+} d(y) = d(y_0)$, one can show that there exists a small neighborhood $\mathcal{U}$ of y_0 such that for any $y', y \in \mathcal{U}$ with $y' > y > y_0$, $\beta(y' - d(y)) > \beta(y - d(y))$. Then

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y)) + \beta(y' - d(y)) \\
&> \alpha(d(y)) + \beta(y - d(y)) \\
&= \Gamma(y).
\end{aligned}$$

This contradicts with the definition of y_0.

If $\alpha'(d(y_0)) < 0$, then $\beta'(y_0 - d(y_0)) < 0$. There exists some $y' < y_0$ that is sufficiently close to y_0 such that $\beta(y' - d(y_0)) > \beta(y_0 - d(y_0))$. Then

$$\begin{aligned}
\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\
&\geq \alpha(d(y_0))) + \beta(y' - d(y_0)) \\
&> \alpha(d(y_0)) + \beta(y_0 - d(y_0)) \\
&= \Gamma(y_0),
\end{aligned}$$

which also contradicts with the definition of y_0. Therefore, $\alpha'(d(y_0)) = 0$ and $d(y_0)$ is an interior maximizer for $\alpha(\cdot)$.

We next consider the case where $d(y_0)$ is on the boundary of $\mathcal{D}$. Consider first $d(y_0) = \underline{d}$. From the first order optimality condition, we have that

$$\alpha'(\underline{d}) - \beta'(y_0 - \underline{d}) \leq 0.$$

We need to show that $\underline{d}$ is a maximizer of $\alpha(d)$ in $\mathcal{D}$. Suppose this is not true, then as $\alpha(d)$ is differentiable and concave, $\alpha'(\underline{d}) > 0$ and therefore

$\beta'(y_0 - \underline{d}) > 0$. By an argument similar to the one used in the previous two paragraphs, we can show that for $y' \geq y > y_0$ with y' sufficiently close to y_0,

$$\begin{aligned}\Gamma(y') &= \alpha(d(y')) + \beta(y' - d(y')) \\ &\geq \alpha(d(y)) + \beta(y' - d(y)) \\ &\geq \alpha(d(y)) + \beta(y - d(y)) \\ &= \Gamma(y),\end{aligned}$$

where the last inequality follows from the fact that $\beta'(y_0 - \underline{d}) > 0$ and the continuity of $\beta(\cdot)$. This contradicts with the definition of $\Gamma(y_0)$. Therefore, $\underline{d}$ is a maximizer of $\alpha(d)$. The case that $d(y_0) = \overline{d}$ can be similarly proven. Thus we have proved our claim that $d(y_0) \in \{\arg\min_{d\in\mathcal{D}} \alpha(d)\}$. This concludes the proof of Lemma 3.2.1.

A.2 Proof of Lemma 3.2.2 on page 43

Since part (b) has been implicitly proven and used in Chen and Simchi-Levi [26], we focus on part (a).

Let $d(y) \in \arg\max_{d\in\mathcal{D}}[\alpha(d) + \beta(y-d)]$ (its existence is guaranteed by the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$) and d^* be a maximizer of $\alpha(\cdot)$ in $\mathcal{D}$. First note that if $\beta(\cdot)$ does not have a finite maximizer, then it must be monotone. In this case, we can show that $\Gamma(\cdot)$ is also monotone. We only prove the case in which $\beta(\cdot)$ is increasing (the case in which $\beta(\cdot)$ is decreasing can be proven similarly). Let $y_1 \leq y_2$. Then

$$\begin{aligned}\Gamma(y_1) = \alpha(d(y_1)) + \beta(y_1 - d(y_1)) &\leq \alpha(d(y_1)) + \beta(y_2 - d(y_1)) \\ &\leq \alpha(d(y_2)) + \beta(y_2 - d(y_2)) = \Gamma(y_2),\end{aligned}$$

where the first inequality holds since $\beta(\cdot)$ is increasing and the remaining equalities and inequality follow from the definition of $d(y)$.

We now assume that $\beta(\cdot)$ has a finite maximizer, denoted as x^*. It is not hard to show that $y^* = x^* + d^*$ is a maximizer of the function $\Gamma(\cdot)$. We can show that $\Gamma(\cdot)$ is increasing in $(-\infty, y^*]$ and decreasing in $[y^*, \infty)$. Let

$y < y^*$ and $\eta = y^* - y$. To this end, we prove that $\Gamma(\cdot)$ is increasing in a neighborhood of y. Since $y < y^*$, either $d^* - d(y)$ or $y^* - d^* - (y - d(y))$ must be no less than $\eta/2$. We focus on the case with $d^* - d(y) \geq \eta/2$ (Note that the other case is symmetric). In this case, for any $y' \in [y, y + \eta/2]$, let $d = d(y) + y' - y$. Then $d(y) \leq d \leq d^*$, $y' - d = y - d(y)$, and therefore

$$\Gamma(y) = \alpha(d(y)) + \beta(y - d(y)) \leq \alpha(d) + \beta(y' - d) \leq \Gamma(y'),$$

where the first inequality holds since $\alpha(d)$ is increasing for $d \leq d^*$. We next prove that $\Gamma(y') \leq \Gamma(y)$ for any $y' \in [y - \eta/2, y]$. For a given $y' \in [y - \eta/2, y]$, if $y' - d(y') \leq y^* - d^*$, then

$$\Gamma(y') = \alpha(d(y')) + \beta(y' - d(y')) \leq \alpha(d(y')) + \beta(y - d(y')) \leq \Gamma(y),$$

where the first inequality holds since $\beta(\cdot)$ is increasing in $(-\infty, y^* - d^*]$; if $y' - d(y') \geq y^* - d^*$, denoting $d = d(y') + (y - y')$, we have that $d(y') \leq d \leq d^*$, $y - d = y' - d(y')$ and

$$\Gamma(y') = \alpha(d(y')) + \beta(y' - d(y')) \leq \alpha(d) + \beta(y - d) \leq \Gamma(y),$$

where the first inequality holds since $\alpha(d)$ is increasing for $d \leq d^*$. Thus, $\Gamma(\cdot)$ is increasing in $(-\infty, y^*]$. Similarly, we can prove that $\Gamma(\cdot)$ is decreasing in $[y^*, \infty)$.

Appendix B

Proofs for Chapter 4

B.1 Proof of Lemma 4.2.1 on page 56

Proof. $R_t(p, r; \lambda)$ is a quadratic function in (p, r). By choosing the appropriate λ_t, we want to make its Hessian negative semidefinite and also make $\frac{\partial^2 R}{\partial p \partial r} \geq 0$. The Hessian of R is:

$$\begin{bmatrix} -2(a_t + \eta_t) + 2\gamma\lambda_{t-1}(1-\alpha)^2 & \eta_t + 2\gamma\lambda_{t-1}(1-\alpha)\alpha \\ \eta_t + 2\gamma\lambda_{t-1}(1-\alpha)\alpha & -2\lambda_t + 2\gamma\lambda_{t-1}\alpha^2 \end{bmatrix}.$$

Negative semi-definiteness requires the following inequalities:

$$\lambda_t - \gamma\lambda_{t-1}\alpha^2 \geq 0 \tag{B.1}$$

$$\lambda_t \leq \frac{a_t + \eta_t}{\gamma(1-\alpha)^2} \tag{B.2}$$

$$\begin{aligned} &\left[-2(a_t + \eta_t) + 2\gamma\lambda_{t-1}(1-\alpha)^2\right]\left[-2\lambda_t + 2\gamma\lambda_{t-1}\alpha^2\right] \\ &\qquad - \left[\eta_t + 2\gamma\lambda_{t-1}(1-\alpha)\alpha\right]^2 \geq 0. \end{aligned} \tag{B.3}$$

Supermodularity requires

$$\eta_t + 2\gamma\lambda_{t-1}(1-\alpha)\alpha \geq 0. \tag{B.4}$$

By our assumption in this lemma, demand fluctuates proportionally, therefore we can write a_t and η_t both as multiples of an "underlying demand rate" ρ_t: $a_t = a\rho_t$ and $\eta_t = \eta\rho_t$. It is therefore not hard to conjecture that the appropriate λ_t should also be some multiple of the underlying demand rate. In fact, we looking for λ_t having the following form: $\lambda_t = \lambda\rho_{t+1}$. Using this new set of notations, and after some simple calculations, equations (B.1)-(B.4) can be rewritten as:

$$\lambda \leq \frac{a+\eta}{\gamma(1-\alpha)^2}. \tag{B.5}$$

$$\lambda\rho_{t+1} - \gamma\alpha^2\lambda\rho_t \geq 0. \tag{B.6}$$

$$\begin{aligned} &-(a+\eta)\lambda\rho_t\rho_{t+1} + \gamma\alpha^2(a+\eta)\lambda\rho_t^2 + \gamma(1-\alpha)^2\lambda^2\rho_t\rho_{t+1} \\ &\qquad + \eta^2\rho_t^2/4 + \gamma\alpha(1-\alpha)\eta\lambda\rho_t^2 \leq 0. \end{aligned} \tag{B.7}$$

$$\lambda \geq -\frac{\eta}{2\gamma\alpha(1-\alpha)}. \tag{B.8}$$

In (B.7), divide both sides by ρ_t^2 and remember $k_t = \rho_{t+1}/\rho_t$, this inequality further simplifies into a quadratic inequality in λ:

$$\left[\gamma(1-\alpha)^2 k_t\right]\lambda^2 - \left[(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)\eta\right]\lambda + \eta^2/4 \leq 0. \tag{B.9}$$

We now restrict ourselves in looking for $\lambda \geq 0$. (B.6) now becomes $k_t \geq \gamma\alpha^2$. This is always satisfied because of the lemma assumption $k_t \geq \gamma\alpha$ and that $\alpha < 1$. Equation (B.8) also becomes trivial because of the restriction $\lambda \geq 0$. In summary, we now search for $0 \leq \lambda \leq \frac{a+\eta}{\gamma(1-\alpha)^2}$ satisfying (B.9).

(B.9) is a quadratic inequality. We denote it by $A\lambda^2 + B\lambda + C \leq 0$ for short. The first thing to check is whether $\Delta = B^2 - 4AC \geq 0$. Some calculation would reveal that

$$\begin{aligned} \Delta &= \left[(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)\eta\right]^2 - \gamma\eta^2(1-\alpha)^2 k_t \\ &= (k_t - \gamma\alpha^2)\left[k_t(a+\eta)^2 - \gamma(\alpha a + \eta)^2\right]. \end{aligned}$$

This expression is always nonnegative, because by our assumption $k_t \geq \gamma\alpha^2$ and also $k_t \geq \frac{\gamma(\alpha a+\eta)^2}{(a+\eta)^2}$. We now pick $\lambda = \operatorname{argmin}(A\lambda^2 + B\lambda + C) = -\frac{B}{2A}$, that is:

$$\lambda = \frac{(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)\eta}{2\gamma(1-\alpha)^2 k_t}. \tag{B.10}$$

The only thing left to show is that $0 \leq \lambda \leq \frac{a+\eta}{\gamma(1-\alpha)^2}$. For the upper bound part:

$$\begin{aligned}
\lambda &= \frac{(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)\eta}{2\gamma(1-\alpha)^2 k_t} \\
&\leq \frac{(a+\eta)k_t}{2\gamma(1-\alpha)^2 k_t} \\
&= \frac{a+\eta}{2\gamma(1-\alpha)^2} \\
&\leq \frac{a+\eta}{\gamma(1-\alpha)^2}.
\end{aligned}$$

As for the lower bound part:

$$\begin{aligned}
\lambda &= \frac{(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)\eta}{2\gamma(1-\alpha)^2 k_t} \\
&\geq \frac{(a+\eta)(k_t - \gamma\alpha^2) - \gamma\alpha(1-\alpha)(a+\eta)}{2\gamma(1-\alpha)^2 k_t} \\
&= \frac{(k_t - \gamma\alpha)(a+\eta)}{2\gamma k_t(1-\alpha)^2} \\
&\geq 0.
\end{aligned}$$

The final inequality holds because of the assumption that $k_t \geq \gamma\alpha$. The proof is now complete. □

B.2 Proof of Theorem 4.3.5 on page 68

Proof. In Proof of Theorem 4.2.1 we have shown that the function behind maximization sign in (4.4) is jointly-concave in all its variables. Similarly the function behind maximization sign in (4.6), which we call $F^0(y, p; r)$, is also jointly-concave in (y, p, r). Furthermore, if we apply the linear transform $d = b - ap + \eta(r - p)$ and call the new function $\tilde{F}^0(y, d; r)$, then this function is still jointly-concave in (y, d, r). And therefore the maximizing $y^0(r)$ should be continuous in (r).

Assumption 4.2.1 assumed an uniform lower bound on the stochastic demand, namely $D = d\epsilon_m + \epsilon_a \geq \delta > 0$ for any valid expected demand d and any ϵ. By continuity of $y^0(r)$, there exists a neighborhood $B(r^*)$ such that $|y^0(r) - y^0(r')| \leq \delta$ for all $r, r' \in B(r^*)$.

Suppose that the system starts at a state $(x, r) \in S_1$ and $r \in B(r^*)$. We use the decision $y^0(r)$ and $p^0(r)$ and denote $d^0(r) = d[p^0(r), r]$, by Theorem 4.3.2 we know $r' = (1 - \alpha)p^0(r) + \alpha r \in B(r^*)$. The new inventory level $x' = y^0(r) - d^0(r)\epsilon_a - \epsilon_a \leq y^0(r) - \delta \leq y^0(r')$ which means in the next period, the optimal inventory decision would not be blocked by a high x'. Carrying this argument on in a similar fashion as the proof of fact 2 in Theorem 4.3.3, we can now claim that (x, r) would remain in S_1 under the policy $y^0(\cdot)$ and $p^0(\cdot)$. And finally since $\phi^*(x, r) \leq \phi^0(x, r)$ for all (x, r), it is indeed optimal for I^* to follow the policy $y^0(\cdot)$ and $p^0(\cdot)$.

The proof of local convergence is now complete. □

Appendix C

Proofs for Chapter 5

C.1 Proof of Theorem 5.3.1 on page 83

Substitute in the expression for $F(r,p)$, the first order optimality condition on p gives:

$$p = \frac{c}{2} + \frac{b+\eta r}{2(a+\eta)} + \frac{\alpha V_r}{2(a+\eta)}. \tag{C.1}$$

Plugging this back into the Bellman equation (5.9), we get:

$$\frac{\sigma^2}{2} r \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial t} - \gamma V + \frac{\alpha^2}{4(a+\eta)} \left(\frac{\partial V}{\partial r}\right)^2 + \left[\frac{\alpha c}{2} - \alpha r + \frac{\alpha(b+\eta r)}{2(a+\eta)}\right] \frac{\partial V}{\partial r} \\ - \frac{c(b+\eta r)}{2} + \frac{(b+\eta r)^2}{4(a+\eta)} + \frac{c^2(a+\eta)}{4} = 0.$$

Introducing a few new notations, this can be written concisely as:

$$\frac{\partial V}{\partial t} - \gamma V + Ar\frac{\partial^2 V}{\partial r^2} + B\left(\frac{\partial V}{\partial r}\right)^2 + P_1(r)\frac{\partial V}{\partial r} + P_2(r) = 0. \tag{C.2}$$

Where A and B are constants. $P_1(r)$, $P_2(r)$ are 1st-order and 2nd-order polynomials in r, respectively. Assume further that:

$$\begin{cases} P_1(r) &= p_{10} + p_{11} r \\ P_2(r) &= p_{20} + p_{21} r + p_{22} \end{cases}$$

Assume function $V(r,t)$ has the following form:

$$V(r,t) = Q(t)r^2 + R(t)r + M(t). \tag{C.3}$$

Then we get the following ordinary differential equation:

$$\frac{dQ}{dt} - \gamma Q + 4BQ^2 + 2p_{11}Q + p_{22} = 0. \tag{C.4}$$

$$\frac{dR}{dt} - \gamma R + 2AQ + 4BQR + 2p_{10}Q + p_{11}R + p_{21} = 0. \tag{C.5}$$

$$\frac{dM}{dt} - \gamma M + BR^2 + p_{10}R + p_{20} = 0. \tag{C.6}$$

The terminal condition $V(r,T) = 0 \; \forall r$ implies terminal conditions $Q(T) = R(T) = M(T) = 0$. Here's a list of the new notations:

$$\begin{aligned}
A &= \frac{\sigma^2}{2} \\
B &= \frac{\alpha^2}{4(a+\eta)} \\
p_{10} &= \frac{\alpha c}{2} + \frac{\alpha b}{2(a+\eta)} \\
p_{11} &= -\alpha + \frac{\alpha\eta}{2(a+\eta)} \\
p_{20} &= -\frac{bc}{2} + \frac{b^2}{4(a+\eta)} + \frac{c^2(a+\eta)}{4} \\
p_{21} &= -\frac{c\eta}{2} + \frac{b\eta}{2(a+\eta)} \\
p_{22} &= \frac{\eta^2}{4(a+\eta)}.
\end{aligned}$$

We will start by giving an explicit solution to the ODE (C.4). Rewrite it as:

$$\frac{dQ}{dt} = -4B(Q - Q_1)(Q - Q_2)$$

where $Q_1 < Q_2$ are the two distinct roots of the equation:

$$4BQ^2 - (\gamma - 2p_{11})Q + p_{22} = 0.$$

Namely:

$$\begin{aligned}
Q_1 &= \frac{\gamma - 2p_{11} - \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B}, \\
Q_2 &= \frac{\gamma - 2p_{11} + \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B}.
\end{aligned}$$

Therefore:

$$\frac{dQ}{(Q-Q_1)(Q-Q_2)} = -4Bdt$$

$$\frac{dQ}{Q_1-Q_2}\left[\frac{1}{Q-Q_1} - \frac{1}{Q-Q_2}\right] = -4Bdt$$

$$\ln\frac{Q-Q_1}{Q-Q_2} = -4B(Q_1-Q_2)t + C$$

$$\frac{Q-Q_1}{Q-Q_2} = D\cdot e^{-4B(Q_1-Q_2)t}. \tag{C.7}$$

Using the terminal condition of $Q(T) = 0$, we can determine the constant multiplier D:

$$D = \frac{Q_1}{Q_2}e^{4B(Q_1-Q_2)T}.$$

Plugging this back to (C.7) we get:

$$Q(t) = \frac{Q_1 e^{4B(Q_1-Q_2)T} - Q_1 e^{4B(Q_1-Q_2)t}}{Q_1/Q_2 e^{4B(Q_1-Q_2)T} - e^{4B(Q_1-Q_2)t}}. \tag{C.8}$$

Plugging this back into (C.5) and (C.6) it is not hard to solve for $R(t)$ and $M(t)$, and therefore we can get an explicit solution for $V(r,t)$ using (C.3). These steps can indeed be carried out. However, since this calculation is tedious, and we are only interested in the steady-state outcome, for simplicity we omit these steps and claim that we have an explicit $V(r,t)$ that is sufficiently differentiable on $[0,+\infty)\times[0,T]$.

Keep in mind that up till now we have solved our nonlinear second order partial differential equation (C.2) with terminal condition $V(r,T) = 0$. And by performing the minimization indicated in (5.9) we would be able to solve for the optimal control $p^*(t)$. We still need to prove that this $p^*(t)$ indeed solves the stochastic optimal control problem in steady state, with the optimal value function equal to $V(r,t)$. This is known as the verification step. Various forms of verification theorem exist in the optimal control literature, for example see Theorem 4.1 in chapter VI of Fleming et al. [37]. The typical condition for a verification theorem is that the value function $V(r,t)$ satisfies a "polynomial growth" criteria in terms of its state variable r. That is: there exist constants D and k, for which $|V(r,t)| \le D(1+|r|^k)$. Looking at our value function (C.3), which is itself a second-order polynomial in r, with t in a finite interval, the "polynomial growth" criteria is obviously satisfied with

$k = 2$ and D sufficiently large. Therefore by the verfication theorem, we have indeed obtained an optimal control to our problem.

In the sequel, let's turn our attention to the steady-state outcome. Looking at (C.8), if we hold t fixed, and let $T \to \infty$, then since $Q_1 < Q_2$, we would have $Q(t) \to Q_1 \triangleq Q$. Explicitly, we have:

$$Q = \frac{\gamma}{2\alpha^2}(a + \eta) + \frac{2a + \eta}{2\alpha} - \frac{a + \eta}{2\alpha^2}\Delta$$

where Δ is given by:

$$\Delta = \sqrt{\gamma^2 + 2\alpha\frac{2a(\gamma + \alpha) + \gamma\eta}{\eta + a}}.$$

Correspondingly, we have:

$$\begin{aligned} R &= \frac{2p_{10}Q + p_{21} + 2AQ}{\gamma - 4BQ - p_{11}} \\ &= \left[\frac{\gamma}{\alpha} + \frac{\sigma^2(a + \eta)}{\alpha^2} + \frac{c(a + \eta)}{\alpha}\right]\frac{\gamma - \Delta}{\gamma + \Delta} + \left[b + ca + \frac{\sigma^2(2a + \eta)}{2\alpha}\right]\frac{2}{b + \Delta}. \end{aligned}$$

Similarly we would be able to solve for M, however the explicit formula for M is not needed in the following analysis. With Q, R, M, the steady-state value function is now solely dependent on r: $Qr^2 + Rr + M$. To solve for the optimal steady-state price p^*, we look back to the first-order optimality condition (C.1). Note that in a steady-state, $V_r(r^*) = 2Qr^* + R$ and also $r^* = p^*$. This leads to:

$$p^*\left[1 - \frac{\eta}{2(a + \eta)} - \frac{\alpha Q}{a + \eta}\right] = \frac{c}{2} + \frac{b}{2(a + \eta)} + \frac{\alpha R}{2(a + \eta)}.$$

Therefore,

$$p_S^* = p_D^* + \frac{\sigma^2}{2a(\gamma + \alpha) + \gamma\eta}\left[\frac{a + \eta}{\alpha}\left(\frac{\gamma}{2} - \frac{\Delta}{2}\right) + \frac{2a + \eta}{2}\right].$$

Where p_D^* is the optimal price in the deterministic problem:

$$p_D^* = \frac{(\gamma + \alpha)(b + ac) + \gamma\eta c}{2a(\gamma + \alpha) + \gamma\eta}.$$

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