ON P-OPERATOR SPACES AND THEIR APPLICATIONS

BY

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DISSERTATION

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Abstract

There have been a lot of research done on the relationship between locally compact groups and algebras associated with them. For example, Johnson proved that a locally compact group G is amenable if and only if the convolution algebra $L_1(G)$ is amenable as a Banach algebra, and Ruan showed that G is amenable if and only if the Fourier algebra $A(G)$ of G is operator amenable. Motivated by Ruan's work, we want to study G through tools from p-operator spaces. We first introduce the p-operator space and various p-operator space tensor products. We then study p-operator space approximation property and p-operator space completely bounded approximation property which are related to p-operator space injective tensor product. We then apply these properties to the study of the pseudofunction algebra $PF_p(G)$, the pseudomeasure algebra $PM_p(G)$, and the Figà-Talamanca-Herz Algebra $A_p(G)$. Especially we show that if G is discrete, the most of approximation properties for the reduced group C*-algebra $C^*_\lambda(G)$, the group von Neumann algebra $VN(G)$, and the Fourier algebra $A(G)$ (related to amenability, weak amenability, and approximation property of G) have natural p-analogues for $PF_p(G)$, $PM_p(G)$, and $A_p(G)$. With help of Herz's work, we also study the stability of these properties. Finally we discuss the properties C_p, C'_p, and C''_p which are natural p-analogues of properties C, C', and C''.

To my family

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List of Abbreviations

p-cb	p-completely bounded
p-AP	p-approximation property
p-OAP	p-operator space approximation property
p-CBAP	p-operator space completely bounded approximation property

List of Symbols

$\mathbb{C}$	the field of complex numbers
$\mathbb{N}$	the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$.
$X \cong Y$	X is isometrically isomorphic to Y
$X \overset{pcb}{\cong} Y$	X is p-completely isometrically isomorphic to Y
$\mathcal{CB}_p(X, Y)$	the space of p-completely bounded operators between X and Y
$\mathcal{CB}_p^{\sigma}(X', Y')$	the space of weak*-continous p-completely bounded operators between X' and Y'
$\mathbb{M}_{n,r}(X)$	the linear space of all $n \times r$ matrices with entries in X
$\mathbb{M}_n(X)$	the linear space of all $n \times n$ matrices with entries in X
$\mathbb{M}_{n,r}$	$= \mathbb{M}_{n,r}(\mathbb{C})$
$\mathbb{M}_n$	$= \mathbb{M}_n(\mathbb{C})$
$\mathbb{M}_\infty(X)$	the linear space of all infinite matrices with entries in X
$\mathbb{M}_\infty$	$= \mathbb{M}_\infty(\mathbb{C})$
$\mathbb{M}_\infty^{\mathrm{fin}}(X)$	the linear space of all infinite matrices with only finitely many nonzero entries from X
$M_{n,r}(X)$	the normed linear space obtained by specifying a norm on $\mathbb{M}_{n,r}(X)$
$M_n(X)$	$= M_{n,n}(X)$
$M_{n,r}$	$= M_{n,r}(\mathbb{C})$
M_n	$= M_n(\mathbb{C})$
$M_\infty(X)$	the space of all $x \in \mathbb{M}_\infty(X)$ with $\|x\| < \infty$
M_∞	$= M_\infty(\mathbb{C})$
$\mathcal{K}_\infty(X)$	the closure of $\mathbb{M}_\infty^{\mathrm{fin}}(X)$ in $\mathbb{M}_\infty(X)$
$\otimes_p$	p-tensor product
$\oplus_p$	p-direct sum
E^n	n-fold p-direct sum $E \oplus_p \cdots \oplus_p E$ of an SQ_p space E.
$\ell_p^n(X)$	$= \ell_p^n \otimes_p X$
$\|\cdot\|_{pcb}$	p-completely bounded norm

$X \otimes Y$	the algebraic tensor product of X and Y
$X \overset{\wedge_p}{\otimes} Y$	p-operator space projective tensor product of X and Y
$X \overset{\vee_p}{\otimes} Y$	p-operator space injective tensor product of X and Y
$X \overset{h_p}{\otimes} Y$	p-operator space Haagerup tensor product of X and Y
$X \overset{\pi}{\otimes} Y$	Banach space projective tensor product of X and Y
$X \overset{\epsilon}{\otimes} Y$	Banach space injective tensor product of X and Y
$\langle \cdot, \cdot \rangle$	scalar pairing, $\langle [v_{ij}], [w_{ij}] \rangle = \sum_{i,j} \langle v_{ij}, w_{ij} \rangle$
$\langle\langle \cdot, \cdot \rangle\rangle$	matrix pairing, $\langle\langle [v_{ij}], [w_{kl}] \rangle\rangle = [\langle v_{ij}, w_{kl} \rangle]_{(i,k),(j,l)}$
$\mathcal{K}(L_p(\mu))$	the space of compact operators on $L_p(\mu)$
$\mathcal{N}(L_p(\mu))$	the space of nuclear operators on $L_p(\mu)$
$\mathcal{N}_n$	$= \mathcal{N}(\ell_p^n)$
$\mathcal{N}_\infty$	$= \mathcal{N}(\ell_p)$
$\\| \cdot \\|_p$	ℓ_p-norm of a vector (or a matrix): $\\|\{v_i\}\\| = (\sum_i \|v_i\|^p)^{1/p}$, $\\|[a_{ij}]\\| = \left(\sum_{i,j} \|a_{ij}\|^p\right)^{1/p}$
$\\| \cdot \\|_{1,n}$	a norm on $\mathbb{M}_n(V)$ given by $\\|v\\|_{1,n} = \inf\{\\|\alpha\\|_{p'} \\|w\\| \\|\beta\\|_p : r \in \mathbb{N}, v = \alpha w \beta, \alpha \in \mathbb{M}_{n,r}, \beta \in \mathbb{M}_{r,n}, w \in M_r(V)\}$
$\mathcal{N}_n(V)$	the normed space $(\mathbb{M}_n(V), \\| \cdot \\|_{1,n})$
ϵ_{ij}	standard basis for $\mathbb{M}_n$
$\mathrm{Ball}(X)$	the closed unit ball of a Banach space X. X_1 is also used.
X_1	the closed unit ball of a Banach space X. $\mathrm{Ball}(X)$ is also used.
$\lambda_p(s)$	the left regular representation, $(\lambda_p(s)f)(t) = f(s^{-1}t)$ for $f \in L_p(G)$ and $s \in G$.
$\rho_p(s)$	the right regular representation, $(\rho_p(s)f)(t) = f(ts)\Delta(s)^{1/p}$ for $f \in L_p(G)$ and $s \in G$.
$C(G)$	the space of all continuous functions on G
$C_b(G)$	the space of all bounded continuous functions on G
$C_{00}(G)$	the space of compactly supported functions on G
$C_0(G)$	the norm closure of $C_{00}(G)$ in $C(G)$, the space of continuous functions on G which vanish at infinity
$A_p(G)$	the Figà-Talamanca-Herz Algebra of G
$A_{p,c}(G)$	functions in $A_p(G)$ with compact support
$A_p(K)$	functions that are restrictions to K of functions in $A_p(G)$.
$PF_p(G)$	the p-pseudofunction algebra of G, that is, the norm closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$
$PM_p(G)$	the p-pseudomeasure algebra of G, that is, the weak*-closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$ = the weak*-closure of $\mathrm{span}\{\lambda_p(s) : s \in G\}$ in $\mathcal{B}(L_p(G))$

$\xi \star \eta$	the convolution of ξ and η; $\xi \star \eta(x) = \int_G \xi(y)\eta(y^{-1}x)dy$
$\mathbb{1}_S$	the characteristic function of S
$\mathbf{1}_K$	the constant function on K, that is, $\mathbf{1}_K(x) = 1$ for all $x \in K$
$\|S\|$	the (Haar) measure of S.
${}_x\varphi$	the left translation of a function φ by x; ${}_x\varphi(t) = \varphi(xt)$.

Chapter 1

Introduction to p-Operator Spaces

We introduce and study basics on p-operator spaces, which can be regarded as p-generalization of operator spaces. We define p-operator spaces and p-completely bounded maps, and give some examples.

1.1 Preliminaries

Throughout this writing, we always assume $1 < p < \infty$ unless stated otherwise. Given p, its conjugate exponent is denoted by p' so that $1/p + 1/p' = 1$. Every Banach space is over $\mathbb{C}$, the field of complex numbers. If X and Y are normed linear spaces, $\mathcal{B}(X, Y)$ will denote the normed linear space of the bounded linear operators from X into Y and we will use $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. For a normed linear space X, the dual space X' of X is the Banach space defined by $X' = \mathcal{B}(X, \mathbb{C})$, that is, the space of all continuous linear functionals on X. For a measure space (Ω, Σ, μ), if there is no risk of confusion, we will simply write $L_p(\mu)$ for $L_p(\Omega, \Sigma, \mu)$.

1.2 SQ_p Spaces

Definition 1.2.1. A Banach space E is called an L_p *space* if it is isometrically isomorphic to some $L_p(\mu)$. A Banach space E is called an SQ_p (denoted $E \in SQ_p$) *space*[1] if it is isometrically isomorphic to a quotient of a subspace of an L_p space.

Remark 1.2.2. $E \in SQ_p$ if and only if E is isometrically isomorphic to a subspace of quotient of an L_p space. Indeed, if $X \subseteq Y \subseteq L_p(\mu)$ for some measure μ, then $Y/X \subseteq L_p(\mu)/X$. Conversely, if $\pi_X : L_p(\mu) \to L_p(\mu)/X$ denotes the canonical quotient map and if $W \subseteq L_p(\mu)/X$, then one can show that $\pi_X^{-1}(W)/X$ is isometrically isomorphic to W via the map $\Phi : \pi_X^{-1}(W)/X \to W$ defined by $\Phi(f + X) = \pi_X(f)$, $f \in \pi_X^{-1}(W)$.

To discuss properties of SQ_p spaces, we need some facts on Banach spaces. We include a lemma for convenience.

[1] In the literature, e.g. [Run05], QSL_p *space* is also used.

Lemma 1.2.3. *Let $X \subseteq Y \subseteq Z$ be Banach spaces and let*

$$X^\perp = \{\varphi \in Z' : \varphi|_X \equiv 0\} \quad \text{and} \quad Y^\perp = \{\varphi \in Z' : \varphi|_Y \equiv 0\}.$$

Then there is an isometric isomorphism

$$(Y/X)' \cong X^\perp / Y^\perp.$$

Proof. First note that the function $\Phi : (Y/X)' \to \{\varphi \in Y' : \varphi|_X \equiv 0\}$ defined by

$$\Phi(f)(y) = f(\hat{y}), \quad f \in Y/X, \quad y \in Y,$$

where $\hat{y} := y + X \in Y/X$, is an isometric isomorphism onto its image. Note also that the function $\Psi : Y' \to Z'/Y^\perp$ defined by

$$\Psi(\varphi) = \bar{\varphi} + Y^\perp, \qquad \varphi \in Y'$$

is also a well-defined isometric isomorphism onto its image, where $\bar{\varphi}$ is a Hahn-Banach extension of φ to Z'. The result follows since the image of $(Y/X)'$ under $\Psi \circ \Phi$ is $X^\perp/Y^\perp$. $\square$

Remark 1.2.4. Now we can state some properties of SQ_p spaces.

1. Every SQ_p space is reflexive by [Meg98, Theorem 1.11.16, Corollary 1.11.18].

2. By 1 above and Lemma 1.2.3, E is an SQ_p space if and only if E' is an $SQ_{p'}$ space.

3. A calculation similar to that in Remark 1.2.2 shows that SQ_p space is closed under taking subspaces and quotients.

4. Since every subspace or quotient of a Hilbert space is again a Hilbert space, it follows that $SQ_2 = \{\text{Hilbert spaces}\}$.

5. If E and F are SQ_p spaces, then so is $E \oplus_p F$, where $\oplus_p$ is the *p-direct sum* defined on $E \oplus F$ by $\|e \oplus f\| = (\|e\|^p + \|f\|^p)^{1/p}$ for $e \in E$ and $f \in F$. Similarly, we can define the *n-fold p-direct sum* $E_1 \oplus_p \cdots \oplus_p E_n$ of SQ_p spaces $E_1, \ldots, E_n$. In particular, if $E_1 = \cdots = E_n = E$, and there is no risk of confusion, then we will use the notation E^n for $E \oplus_p \cdots \oplus_p E$.

6. If $p = 1$ or $p = \infty$, then $SQ_p = \{\text{Banach spaces}\}$. To see this, it suffices to check that $\Phi : \ell_1(X_1) \to X$ (resp. $\Psi : X \to \ell_\infty(X_1'))$ given by $\{a_x\} \mapsto \sum_x a_x x$ (respectively, $x \mapsto [\Psi(x)(f) = f(x), f \in \ell_\infty(X_1')]$) is

a quotient map (respectively, an isometry). If X is a separable Banach space, then X can be regarded as a quotient of ℓ_1 [Mor01, Theorem 2.19] or a subspace of ℓ_∞. To see this, first note that X_1' is weak*-metrizable [Meg98, Theorem 2.6.23] and therefore weak*-separable by Alaoglu's theorem. Let $\{f_n\}_{n=1}^\infty$ be a weak*-dense subset of X_1' and define a map from X to ℓ_∞ by $x \mapsto \{f_n(x)\}_{n=1}^\infty$: this gives an isometry.

7. By [Her71, Corollary 2], if $p \leq q \leq 2$ or $2 \leq q \leq p$, then an L_q space is an SQ_p space. Therefore, if $p \leq q \leq 2$ or $2 \leq q \leq p$, then every SQ_q space is an SQ_p space.

8. For another characterization of SQ_p spaces, see [Kwa72].

1.3 p-Operator Spaces

Let (Ω, Σ, μ) be a measure space and let X be a Banach space. Let $L_p(\mu, X)$ be the space of Bochner p-integrable functions from Ω to X.[2] We define a norm on the algebraic tensor product $L_p(\mu) \otimes X$ by embedding $L_p(\mu) \otimes X$ into $L_p(\mu, X)$ in the natural way, that is, $f \otimes x \mapsto f(\cdot)x$ for $f \in L_p(\mu)$ and $x \in X$. Let $L_p(\mu) \otimes_p X$ denote the completion of $L_p(\mu) \otimes X$ in $L_p(\mu, X)$ with respect to the norm in $L_p(\mu, X)$. It follows easily that $L_p(\mu) \otimes_p X$ is isometrically isomorphic to $L_p(\mu, X)$.[3] If $X = L_p(\Omega', \Sigma', \mu')$, then we have the isometric isomorphism

$$L_p(\mu) \otimes_p L_p(\mu') \cong L_p(\mu \times \mu'),$$

where $\mu \times \mu$ denotes the product measure on $\Sigma \times \Sigma'$ [DF93, §7.2]. In particular, if I and J are index sets, then we have

$$\ell_p(I) \otimes_p \ell_p(J) = \ell_p(I \times J).$$

If $L_p(\mu) = \ell_p^n$, $\mathbb{C}^n$ equipped with ℓ_p-norm, then we will also use notation $\ell_p^n(X)$ for $\ell_p^n \otimes_p X = X^n$.

Now we are ready to define the main subject of the thesis.

Definition 1.3.1. A Banach space X is called a *concrete p-operator space* if X is a closed subspace of $\mathcal{B}(E)$ for some $E \in SQ_p$.

Let $\mathbb{M}_n(X)$ denote the linear space of all $n \times n$ matrices with entries in X. For a concrete p-operator space $X \subseteq \mathcal{B}(E)$ and for each $n \in \mathbb{N}$, define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X)$ by identifying $\mathbb{M}_n(X)$ as a subspace of $\mathcal{B}(\ell_p^n \otimes_p E) = \mathcal{B}(\ell_p^n(E))$, and let $M_n(X)$ denote the corresponding normed space. The norms $\|\cdot\|_n$ then satisfy

[2] See [DF93, Appendix B12] for details.
[3] See [DF93, §7.2] for details.

$\mathcal{D}_\infty$ for $u \in M_n(X)$ and $v \in M_m(X)$, we have $\|u \oplus v\|_{M_{n+m}(X)} = \max\{\|u\|_n, \|v\|_m\}$.

$\mathcal{M}_p$ for $u \in \mathbb{M}_m(X)$, $\alpha \in \mathbb{M}_{n,m}$, and $\beta \in \mathbb{M}_{m,n}$, we have $\|\alpha u \beta\|_n \leq \|\alpha\|\|u\|_m\|\beta\|$, where $\|\alpha\|$ is the norm of α as a member of $\mathcal{B}(\ell_p^m, \ell_p^n)$, and similarly for β.

Remark 1.3.2. When $p = 2$, these are Ruan's axioms and 2-operator spaces are simply operator spaces because the SQ_2 spaces are exactly the same as Hilbert spaces.

As in operator spaces, we can also define abstract p-operator spaces.

Definition 1.3.3. An *abstract p-operator space* is a Banach space X together with a sequence of norms $\|\cdot\|_n$ defined on $\mathbb{M}_n(X)$ satisfying the conditions $\mathcal{D}_\infty$ and $\mathcal{M}_p$ above.

Thanks to the following theorem by Le Merdy, we do not distinguish between concrete p-operator spaces and abstract p-operator spaces and we will merely speak of p-operator spaces.

Theorem 1.3.4. *[LeM96, Theorem 4.1] An abstract p-operator space X can be isometrically embedded in $\mathcal{B}(E)$ for some $E \in SQ_p$ in such a way that the canonical norms on $\mathbb{M}_n(X)$ arising from this embedding agree with the given norms.*

Example 1.3.5.

1. Suppose E and F are SQ_p spaces and let $L = E \oplus_p F$, then the mapping

$$x \mapsto \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$$

   is an isometric embedding of $\mathcal{B}(E, F)$ into $\mathcal{B}(L)$ and using this we can view $\mathcal{B}(E, F)$ as a p-operator space. Note that $M_n(\mathcal{B}(E, F))$ is isometrically isomorphic to $\mathcal{B}(\ell_p^n(E), \ell_p^n(F))$.

2. The identification $L_p(\mu) = \mathcal{B}(\mathbb{C}, L_p(\mu)) \subseteq \mathcal{B}(\mathbb{C} \oplus_p L_p(\mu))$ gives a p-operator space structure on $L_p(\mu)$ called the *column p-operator space structure* of $L_p(\mu)$, which we denote by $L_p^c(\mu)$. Similarly, we denote by $L_{p'}^r(\mu)$ the p-operator space structure on $L_{p'}(\mu)$ which is called the *row p-operator space structure* of $L_{p'}(\mu)$ and defined by the identification $L_{p'}(\mu) = \mathcal{B}(L_p(\mu), \mathbb{C})$. In general, we can define E^c and $(E')^r$ for any $E \in SQ_p$.

3. Unless stated otherwise, we give $\mathbb{C}$ the obvious p-operator space structure, that is, $M_n(\mathbb{C}) = \mathcal{B}(\ell_p^n)$.[4]

[4]We will also use M_n to denote $\mathcal{B}(\ell_p^n)$.

1.4 p-Completely Bounded Maps

Note that a linear map $u : X \to Y$ between p-operator spaces X and Y induces a map $u_n : M_n(X) \to M_n(Y)$ by applying u entrywise.

Definition 1.4.1. We say that u is *p-completely bounded* (p-cb) if $\|u\|_{pcb} := \sup_n \|u_n\| < \infty$. Similarly, we define the notions of *p-completely contractive*, *p-completely isometric*, and *p-completely quotient*. We write $\mathcal{CB}_p(X, Y)$ for the space of all p-completely bounded maps from X into Y and $\mathcal{CB}_p(X)$ for the space $\mathcal{CB}_p(X, X)$.

Before proceeding, we discuss subspaces and quotients of a p-operator spaces as in [Daw10]. If Y is a subspace of a p-operator space X, then inclusions $\mathbb{M}_n(Y) \subseteq \mathbb{M}_n(X)$ and the corresponding norms determine a p-operator space matrix norm on Y. This determines the *p-operator subspace structure*.

Example 1.4.2. Let $\mathcal{K}(L_p(\mu)) \subseteq \mathcal{B}(L_p(\mu))$ denote the space of compact operators on $L_p(\mu)$, then $\mathcal{K}(L_p(\mu))$ has a p-operator space structure inherited from that of $\mathcal{B}(L_p(\mu))$. Using the fact that a compact operator on $L_p(\mu)$ can be approximated by finite rank operators [Rya02, Example 4.5 and Corollary 4.13][5], it is easily shown that $M_n(\mathcal{K}(L_p(\mu)))$ can be identified with $\mathcal{K}(L_p(\mu)^n) = \mathcal{K}(L_p(\mu) \oplus_p \cdots \oplus_p L_p(\mu))$.

Given a closed subspace Y of a p-operator space X, we use the identification $\mathbb{M}_n(X/Y) = \mathbb{M}_n(X)/\mathbb{M}_n(Y)$ to define a norm on $\mathbb{M}_n(X/Y)$, and it is easy to check that X/Y becomes a p-operator space and that the quotient map $\pi : X \to X/Y$ is a p-completely quotient map. This determines the *p-operator quotient structure*.

For a p-operator space X and for each $n \in \mathbb{N}$, we can give $M_n(X)$ a natural p-operator space structure using the identification $M_r(M_n(X)) = M_{rn}(X)$. We also want to turn the mapping space $\mathcal{CB}_p(X, Y)$ between two p-operator spaces X and Y into a p-operator space: let us define a norm on $\mathbb{M}_n(\mathcal{CB}_p(X, Y))$ by identifying this space with $\mathcal{CB}_p(X, M_n(Y))$. Using Le Merdy's theorem, one can show that $\mathcal{CB}_p(X, Y)$ itself is a p-operator space. In particular, the *p-operator dual space* of X is defined to be $\mathcal{CB}_p(X, \mathbb{C})$. The next lemma by Daws shows that we may identify the Banach dual space X' of X with the p-operator dual space $\mathcal{CB}_p(X, \mathbb{C})$ of X.

Lemma 1.4.3. *[Daw10, Lemma 4.2] Let X be a p-operator space, and let $\varphi \in X'$, the Banach dual of X. Then φ is p-completely bounded as a map to $\mathbb{C}$. Moreover, $\|\varphi\|_{pcb} = \|\varphi\|$.*

Remark 1.4.4. If $\varphi = [\varphi_{ij}] \in M_n(X') = M_n(\mathcal{CB}_p(X, \mathbb{C})) = \mathcal{CB}_p(X, M_n)$ for some p-operator space X, then

$$\|\varphi\| = \sup\{\|\langle\langle \varphi, x \rangle\rangle\| : m \in \mathbb{N},\ x \in M_m(X),\ \|x\| \le 1\},$$

[5] That is, $L_p(\mu)$ has the Approximation Property.

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the matrix paring as in [ER00, §1.1].

We can slightly generalize Lemma 1.4.3 as in the following proposition.

Proposition 1.4.5. *Let X be a p-operator space and $\varphi : X \to M_n = \mathcal{B}(\ell_p^n)$ be bounded and linear. Then φ is p-completely bounded and $\|\varphi\|_{pcb} \leq n\|\varphi_n\|$.*[6]

Proof. Fix $m \geq n$. We need to show that $\|\varphi_m\| \leq n\|\varphi_n\|$. Let $x = [x_{ij}] \in M_m(X)$. Choose $\tilde{\xi} \in \ell_p^{mn}$ and $\tilde{\eta} \in \ell_{p'}^{mn}$ such that $\|\tilde{\xi}\| = \|\tilde{\eta}\| = 1$ and consider $\langle \tilde{\eta}, \varphi_m(x)\tilde{\xi}\rangle$. One can find $mn \times mn$ permutation matrices P and Q such that

$$\Phi := Q\varphi_m(x)P = \begin{bmatrix} [\varphi_{1,1}(x_{ij})] & \cdots & [\varphi_{1,n}(x_{ij})] \\ \vdots & \ddots & \vdots \\ [\varphi_{n,1}(x_{ij})] & \cdots & [\varphi_{n,n}(x_{ij})] \end{bmatrix}.$$

Letting $\xi = P^{-1}\tilde{\xi}$ and $\eta = (Q^T)^{-1}\tilde{\eta}$, we have $\|\xi\| = \|\eta\| = 1$ and $\langle \tilde{\eta}, \varphi_m(x)\tilde{\xi}\rangle = \langle \eta, \Phi\xi\rangle$. Write

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}, \quad \xi_k = \begin{bmatrix} \xi_{k,1} \\ \vdots \\ \xi_{k,m} \end{bmatrix} \in \ell_p^m, \quad \eta_l = \begin{bmatrix} \eta_{l,1} \\ \vdots \\ \eta_{l,m} \end{bmatrix} \in \ell_{p'}^m, \quad 1 \leq k, l \leq n,$$

and for each $1 \leq k, l \leq n$, put $u_k = \frac{1}{\|\xi_k\|}\xi_k$ and $v_l = \frac{1}{\|\eta_l\|}\eta_l$. Finally letting

$$\alpha = \begin{bmatrix} \|\xi_1\| \\ \vdots \\ \|\xi_n\| \end{bmatrix}, \quad \beta = \begin{bmatrix} \|\eta_1\| \\ \vdots \\ \|\eta_n\| \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix}_{mn \times n}, \quad \text{and} \quad V = \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{bmatrix}_{mn \times n},$$

we obtain $\langle \eta, \Phi\xi\rangle = \langle V\beta, \Phi U\alpha\rangle = \langle \beta, V^T\Phi U\alpha\rangle$ with U, V contractive and $\|\alpha\|_{\ell_p^n} = \|\beta\|_{\ell_{p'}^n} = 1$. Thus the result will follow once we show that $\|V^T\Phi U\|_{\mathcal{B}(\ell_p^n)} \leq n\|\varphi_n\|\|x\|$. Note that

$$\begin{aligned} V^T\Phi U &= \begin{bmatrix} v_1^T & & \\ & \ddots & \\ & & v_n^T \end{bmatrix}_{n \times mn} \begin{bmatrix} & & \\ & [\varphi(x_{ij})_{k,l}]_{1\leq i,j\leq m} & \\ & & \end{bmatrix}_{mn \times mn} \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix}_{mn \times n} \\ &= \begin{bmatrix} & & \\ & v_k^T[\varphi(x_{ij})_{k,l}]_{1\leq i,j\leq m}u_l & \\ & & \end{bmatrix}_{n \times n} \end{aligned}$$

[6]When $p = 2$, we have $\|\varphi\|_{cb} = \|\varphi_n\|$. See [ER00, Proposition 2.2.2].

$$= \begin{bmatrix} & & \\ & \varphi(v_k^T x u_l)_{k,l} & \\ & & \end{bmatrix}_{n\times n}.$$

Denote $\varphi(v_k^T x u_l) \in M_n$ by y^{kl}, then the last matrix above is nothing but

$$\begin{bmatrix} (y^{11})_{1,1} & \cdots & (y^{1n})_{1,n} \\ \vdots & \ddots & \vdots \\ (y^{n1})_{n,1} & \cdots & (y^{nn})_{n,n} \end{bmatrix},$$

which can be computed as

$$R[y^{kl}]_{n^2\times n^2} C = R\varphi_n \left(V^T \begin{bmatrix} x & \cdots & x \\ \vdots & \ddots & \vdots \\ x & \cdots & x \end{bmatrix}_{mn\times mn} U \right) C,$$

where R is an $n \times n^2$ matrix whose only nonzero entry is 1 at positions $(1,1), (2, n+2), (3, 2n+3), \cdots$, and at (n, n^2) and $C = R^T$. Since

$$\begin{bmatrix} x & x & \cdots & x \\ x & x & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x \end{bmatrix} = \begin{bmatrix} x & & & \\ & x & & \\ & & \ddots & \\ & & & x \end{bmatrix} + \begin{bmatrix} & x & & \\ & & x & \\ & & & \ddots \\ x & & & \end{bmatrix} + \cdots + \begin{bmatrix} & & & x \\ x & & & \\ & \ddots & & \\ & & & x \end{bmatrix}, \tag{1.1}$$

we get

$$\begin{aligned} \|V^T \Phi U\|_{\mathcal{B}(\ell_p^n)} &\leq \|R\| \cdot \|\varphi_n\| \cdot \|V^T\| \cdot \left\| \begin{bmatrix} x & \cdots & x \\ \vdots & \ddots & \vdots \\ x & \cdots & x \end{bmatrix} \right\| \cdot \|U\| \cdot \|C\| \\ (\|R\|, \|V^T\|, \|U\|, \|C\| \leq 1) &\leq \|\varphi_n\| \cdot \left\| \begin{bmatrix} x & \cdots & x \\ \vdots & \ddots & \vdots \\ x & \cdots & x \end{bmatrix} \right\| \\ (\text{by } (1.1)) &\leq n\|\varphi_n\|\|x\|, \end{aligned}$$

and this completes the proof. □

1.5 p-Operator Spaces on L_p Space

When $E = L_p(\mu)$ and if $X \subseteq \mathcal{B}(E) = \mathcal{B}(L_p(\mu))$, then we say that X is a *p-operator space on L_p space.* These p-operator spaces are often easier to work with as we will see soon. Let $\kappa_X : X \to X''$ denote the canonical inclusion from a Banach space X into its second dual. Contrary to operator spaces, κ_X is *not* always p-completely isometric. Thanks to the following theorem by Daws, however, we can easily characterize those p-operator spaces with the property that the canonical inclusion is p-completely isometric.

Proposition 1.5.1. *[Daw10, Proposition 4.4] Let X be a p-operator space. Then κ_X is a p-complete contraction. Moreover, κ_X is a p-complete isometry if and only if $X \subseteq \mathcal{B}(L_p(\mu))$ p-completely isometrically for some measure μ.*[7]

We say that a p-operator space X is *reflexive* if the canonical isometric inclusion $\kappa_X : X \to X''$ is a p-completely isometric isomorphism from X onto X''.

Lemma 1.5.2. *A p-operator space X is reflexive if and only if X is reflexive as a Banach space and there is a measure μ such that $X \subseteq \mathcal{B}(L_p(\mu))$. In particular, for any measure μ, $L_p^c(\mu)$ and $L_{p'}^r(\mu)$ are reflexive.*

Proof. ($\Longrightarrow$) If X is reflexive as a p-operator space, then clearly X is reflexive as a Banach space. By [Daw10, Proposition 4.4], if κ_X is p-completely isometric, then there exists a measure μ such that $X \subseteq \mathcal{B}(L_p(\mu))$.
($\Longleftarrow$) Since $X \subseteq \mathcal{B}(L_p(\mu))$, by [Daw10, Proposition 4.4], for each $n \in \mathbb{N}$, $(\kappa_X)_n$ is isometric. That $(\kappa_X)_n : M_n(L_p^c(\mu)) \to M_n((L_p^c(\mu))'')$ is surjective follows from the fact that $\kappa_X : X \to X''$ is surjective. ☐

As pointed out in [Daw10, §4.1], Le Merdy gives an example of (even finite dimensional) p-operator spaces X such that κ_X is not a p-complete isometry, and this is a first significant problem we confront with when extending results from operator spaces.[8] At the same time, Proposition 1.5.1 shows that p-operator spaces on L_p space are easier to work with.

Lemma 1.5.3 below exhibits another problem we face when extending results from operator spaces to p-operator spaces.[9]

Lemma 1.5.3. *[Daw10, Lemmas 4.5 and 4.6] Let X and Y be p-operator spaces. If $u \in \mathcal{CB}_p(X, Y)$, then the adjoint mapping u' belongs to $\in \mathcal{CB}_p(Y', X')$ with $\|u'\|_{pcb} \leq \|u\|_{pcb}$. If u is a p-complete quotient map, then u' is a p-complete isometry.*

Remark 1.5.4. We do not know whether we always have $\|u'\|_{pcb} = \|u\|_{pcb}$. We cannot simply apply the same argument as in operator spaces theory because we lack p-analogue of Roger Smith's lemma [ER00,

[7] That is, κ_X is a p-complete isometry if and only if X is a p-operator space on L_p space.
[8] For any operator space X, the canonical inclusion is always completely isometric [ER00, Proposition 3.2.1].
[9] For operator spaces X, Y and for $u \in \mathcal{CB}(X, Y)$, we always have $\|u\|_{cb} = \|u'\|_{cb}$ [ER00, Proposition 3.2.2].

Lemma 2.2.1]. Similarly, when u is a p-complete isometry, we do *not* know whether u' is a p-complete quotient because a p-analogue of Arveson-Wittstock-Hahn-Banach theorem is not available yet at this moment.[10]

However, with an additional condition that Y is a p-operator space on L_p space, we do get the equality in Lemma 1.5.3 as explained in the following.

Proposition 1.5.5. *Let X and Y be p-operator spaces with $Y \subseteq \mathcal{B}(L_p(\mu))$ for some measure μ. If $u \in \mathcal{CB}_p(X,Y)$, then the adjoint $\|u'\|_{pcb} = \|u\|_{pcb}$.*

Proof.

$$\begin{aligned}
\|u\|_{pcb} &= \sup\left\{\|[u(x_{ij})]\|_{M_n(Y)} : n \in \mathbb{N}, [x_{ij}] \in M_n(X), \|[x_{ij}]\| \leq 1\right\} \\
&= \sup\left\{\|[u(x_{ij})]\|_{M_n(Y'')} : n \in \mathbb{N}, [x_{ij}] \in M_n(X), \|[x_{ij}]\| \leq 1\right\} \\
&= \sup\left\{\|[u(x_{ij})]\|_{\mathcal{CB}_p(Y',M_n)} : n \in \mathbb{N}, [x_{ij}] \in M_n(X), \|[x_{ij}]\| \leq 1\right\} \\
&= \sup\{\|\langle\langle u(x_{ij}), \varphi_{kl}\rangle\rangle\| : n, m \in \mathbb{N}, [x_{ij}] \in M_n(X), \|[x_{ij}]\| \leq 1, [\varphi_{kl}] \in M_m(Y'), \|[\varphi_{kl}]\| \leq 1\} \\
&= \sup\{\|\langle\langle x_{ij}, u'(\varphi_{kl})\rangle\rangle\| : n, m \in \mathbb{N}, [x_{ij}] \in M_n(X), \|[x_{ij}]\| \leq 1, [\varphi_{kl}] \in M_m(Y'), \|[\varphi_{kl}]\| \leq 1\} \\
\text{(Remark1.4.4)} \quad &= \sup\left\{\|u'(\varphi_{kl})\|_{M_n(X')} : m \in \mathbb{N}, [\varphi_{kl}] \in M_m(Y'), \|[\varphi_{kl}]\| \leq 1\right\} \\
&= \|u'\|_{pcb},
\end{aligned}$$

where the second equality comes from Proposition 1.5.1. □

We close this section with the following proposition.

Proposition 1.5.6. *Let $X \subseteq \mathcal{B}(L_p(\mu))$ and $Y \subseteq \mathcal{B}(L_p(\nu))$ be p-operator spaces. Then the adjoint mapping*

$$\Phi : \mathcal{CB}_p(X,Y) \to \mathcal{CB}_p(Y',X'), \qquad T \mapsto T'$$

is a p-completely isometric isomorphism from $\mathcal{CB}_p(X,Y)$ onto $\mathcal{CB}_p(Y',X')$ if and only if either $X = \{0\}$ or Y is reflexive.

Proof. ($\Longrightarrow$) Suppose $X \neq \{0\}$. By Lemma 1.5.2, to show that Y is reflexive, it suffices to show that Y is reflexive as a Banach space. Let $\varphi \in Y''$. Let $x \neq 0$ be a vector in X, then there exists $x' \in X'$ such that $x'(x) = 1$. Define $S : Y' \to X'$ by $S(f) = \varphi(f)x'$, then $S \in \mathcal{CB}_p(Y',X')$ and hence $S = T'$ for some $T \in \mathcal{CB}_p(X,Y)$. Since $\varphi(f) = \langle S(f), x\rangle = \langle f, Tx\rangle$ for all $f \in Y'$, we see that the canonical inclusion κ_Y is

[10]Having a positive answer to this question is equivalent to p-analogue of Arveson-Wittstock-Hahn-Banach Theorem because

$$(u')_n : M_n(Y') = \mathcal{CB}_p(Y, M_n) \to \mathcal{CB}_p(X, M_n) = M_n(X')$$

is given by the restriction.

onto, that is, Y is reflexive as a Banach space.

($\Longleftarrow$) If $X = \{0\}$, then the result is trivial. Suppose that Y is reflexive and fix $n \in \mathbb{N}$. Let $S = [S_{ij}] \in M_n(\mathcal{CB}_p(Y', X')) = \mathcal{CB}_p(Y', M_n(X')) = \mathcal{CB}_p(Y', \mathcal{CB}_p(X, M_n))$. Since $X \subseteq \mathcal{B}(L_p(\mu))$, for each i, j, we can define $T_{ij} = S'_{ij}|_X \in \mathcal{CB}_p(X, Y)$. Now for any $f \in Y'$ and for any $x \in X$,

$$\begin{aligned} S(f)(x) &= [\langle S_{ij}(f), \kappa_X(x)\rangle] \\ &= [\langle f, T_{ij}(x)\rangle] \\ &= [\langle T'_{ij}(f), x\rangle]. \end{aligned}$$

This shows that $S = T'$, where $T = [T_{ij}] \in M_n(\mathcal{CB}_p(X, Y))$ and hence the adjoint mapping is onto. □

1.6 Examples of p-Completely Isometric Isomorphisms

In this section, we will give specific examples of p-operator spaces and identify some of them via p-completely isometric isomorphisms. Let $\mathcal{N}(L_p(\mu))$ denote the Banach space of all nuclear operators on $L_p(\mu)$. Since $L_p(\mu)$ has the Approximation Property [Rya02, Example 4.5], we have

$$\mathcal{N}(L_p(\mu)) = L_{p'}(\mu) \overset{\pi}{\otimes} L_p(\mu), \tag{1.2}$$

where $\overset{\pi}{\otimes}$ denotes the Banach space projective tensor product [Rya02, Chapter 2]. Since $L_p(\mu)$ is reflexive (hence has the Radon-Nikodým property, [Rya02, Corollary 5.45]), we have the isometric isomorphisms

$$\mathcal{K}(L_p(\mu))' = \mathcal{N}(L_p(\mu)) \quad \text{and} \quad \mathcal{N}(L_p(\mu))' = \mathcal{B}(L_p(\mu)).^{11} \tag{1.3}$$

Giving $\mathcal{K}(L_p(\mu))$ a p-operator subspace structure in $\mathcal{B}(L_p(\mu))$ and giving $\mathcal{N}(L_p(\mu))$ a *p-operator space structure by duality*, that is, by regarding $\mathcal{N}(L_p(\mu))$ as a p-operator subspace of $\mathcal{B}(L_p(\mu))'$[12], we can regard $\mathcal{K}(L_p(\mu))$ and $\mathcal{N}(L_p(\mu))$ as p-operator spaces. With this structure, we can say more about the second isometry in (1.3) as in the following lemma.

Proposition 1.6.1. *[Daw10, Lemma 5.1] With the dual p-operator space structure on $\mathcal{N}(L_p(\mu))$, we have $\mathcal{N}(L_p(\mu))' = \mathcal{B}(L_p(\mu))$ p-completely isometrically.*

As an application of Proposition 1.6.1, let us take a closer look at the representation of a dual p-operator

[11] See Corollary 4.8, Corollay 4.13, and Theorem 5.33 in [Rya02]. See also [Rya02, §2.2].

[12] This p-operator space structure on $\mathcal{N}(L_p(\mu))$ will be called the *dual p-operator space structure* on $\mathcal{N}(L_p(\mu))$.

space. Since X' is again a p-operator space for a p-operator space X, by Theorem 1.3.4, there exists $E \in SQ_p$ such that $X' \subseteq \mathcal{B}(E)$ p-completely isometrically. Daws showed that in fact we can choose E to be $\ell_p(I)$ for some index set I [Daw10, Theorem 4.3]. We can still improve Daws' result as in the following.

Proposition 1.6.2. *Let V be a p-operator space. Then there exists an index set I such that V' can be identified with a weak*-closed subspace of $\mathcal{B}(\ell_p(I))$ and the restriction map*

$$\pi : \omega \in \mathcal{N}(\ell_p(I)) \to \omega|_{V'} \in V$$

is p-completely quotient. Therefore, we have the p-complete isometry $V = \mathcal{N}(\ell_p(I))/V'_\perp$, where $V'_\perp$ is the pre-annihilator of V' in $\mathcal{N}(\ell_p(I))$.

Proof. We can apply a construction similar to that given in the proof of [ER00, Proposition 3.2.4] by constructing $\mathfrak{s}_n = M_n(V)_1$ and $\mathfrak{s} = \bigcup_{n=1}^{\infty} \mathfrak{s}_n$. We can obtain an index set I such that $\ell_p(I) = \bigoplus_{x \in \mathfrak{s}} \ell_p^{n(x)}$. The map

$$\Phi^\sigma : f \in V' \to \mathrm{diag}\,\{f_n(x) : x \in \mathfrak{s}_n, n \in \mathbb{N}\} \in \prod_{x \in \mathfrak{s}} \mathcal{B}(\ell_p^{n(x)}) \subseteq \mathcal{B}(\ell_p(I))$$

is a weak*-continuous p-completely isometric inclusion. In this case, we can identify V' with the weak*-closed subspace $\Phi^\sigma(V')$ in $\mathcal{B}(\ell_p(I))$ and V is equal to the quotient $\mathcal{N}(\ell_p(I))/V'_\perp$ via the restriction map

$$\pi : \omega \in \mathcal{N}(\ell_p(I)) \to \omega|_{V'} \in V.$$

Now for each $x = [x_{ij}] \in M_n(V)_1 = \mathfrak{s}_n$, we let $\iota_{n(x)}$ be the canonical inclusion of $\ell_p^{n(x)}$ into $\ell_p(I)$ and $P_{n(x)}$ be the contractive projection from $\ell_p(I)$ onto $\ell_p^{n(x)}$. Then the truncation map $\omega(y) = P_{n(x)} y \iota_{n(x)}$ on $\mathcal{B}(\ell_p(I))$ defines a contractive element ω in $M_n(\mathcal{N}(\ell_p(I)))$, which satisfies $\pi_n(\omega) = x$. This shows that π is actually a p-complete quotient map from $\mathcal{N}(\ell_p(I))$ onto V. The last part follows from Lemma 1.5.3. $\square$

The first isometric isomorphism in (1.3) also turns out to be p-completely isometric with the dual p-operator space structure on $\mathcal{N}(L_p(\mu))$. To prove this, we first need to study a norm structure on $\mathbb{M}_n(V)$ for a general p-operator space V.

Lemma 1.6.3. *Let $1 \leq p, p' \leq \infty$ with $1/p' + 1/p = 1$. Let $\lambda = \{\lambda_j\}_{1 \leq j \leq n}$ be a finite sequence in $\mathbb{C}$. Then*

$$\|\lambda\|_{\ell_p^n} \leq n^{|1/p - 1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}.$$

Proof. There is nothing to prove if $p = p' = 2$. It is trivial if $p = 1$. If $p > p'$, then $\|\lambda\|_{\ell_p^n} \leq \|\lambda\|_{\ell_{p'}^n} \leq n^{|1/p-1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}$ since $n^{|1/p-1/p'|} \geq 1$. Finally, assume $1 < p < p'$ and let $q = \frac{p'}{p} > 1$ and let q' be the conjugate exponent to q. By Hölder's inequality,

$$\|\lambda\|_{\ell_p^n}^p \leq \left(\sum_{j=1}^n |\lambda_j|^{pq}\right)^{1/q} \cdot n^{1/q'} = \left(\sum_{j=1}^n |\lambda_j|^{p'}\right)^{p/p'} \cdot n^{1-p/p'}$$

and hence $\|\lambda\|_{\ell_p^n} \leq n^{|1/p-1/p'|} \cdot \|\lambda\|_{\ell_{p'}^n}$. □

Lemma 1.6.4. *Let $\alpha = [\alpha_{ij}] \in \mathbb{M}_{n,r}$ and $\beta = [\beta_{kl}] \in \mathbb{M}_{r,n}$. Let $1 < p, p' < \infty$ with $1/p' + 1/p = 1$. Then we have*

$$\|\alpha\|_{\mathcal{B}(\ell_p^r,\ell_p^n)} \leq \|\alpha\|_{p'} \cdot n^{|1/p-1/p'|} \quad \textit{and} \quad \|\beta\|_{\mathcal{B}(\ell_p^n,\ell_p^r)} \leq \|\beta\|_p \cdot n^{|1/p-1/p'|},$$

where

$$\|\alpha\|_{p'} = \left(\sum_{i=1}^n \sum_{j=1}^r |\alpha_{ij}|^{p'}\right)^{1/p'} \quad \textit{and} \quad \|\beta\|_p = \left(\sum_{k=1}^r \sum_{l=1}^n |\beta_{kl}|^p\right)^{1/p}.$$

Proof. Suppose $\xi = \{\xi_j\}_{j=1}^r$ is a unit vector in ℓ_p^r. For each i, $1 \leq i \leq n$, let $\eta_i = \left|\sum_{j=1}^r \alpha_{ij}\xi_j\right|$, then by Hölder's inequality, $\eta_i \leq \left(\sum_{j=1}^r |\alpha_{ij}|^{p'}\right)^{1/p'}$ and by Lemma 1.6.3,

$$\left(\sum_{i=1}^n \eta_i^p\right)^{1/p} \leq n^{|1/p-1/p'|} \cdot \left(\sum_{i=1}^n \eta_i^{p'}\right)^{1/p'} \leq n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$$

and hence we get $\|\alpha\|_{\mathcal{B}(\ell_p^r,\ell_p^n)} \leq n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$. To prove the second inequality, let $\gamma : \ell_{p'}^r \to \ell_{p'}^n$ be the adjoint operator of β. Then by the argument above we have

$$\|\gamma\|_{\mathcal{B}(\ell_{p'}^r,\ell_{p'}^n)} \leq \|\gamma\|_p \cdot n^{|1/p-1/p'|}.$$

Since $\|\gamma\|_{\mathcal{B}(\ell_{p'}^r,\ell_{p'}^n)} = \|\beta\|_{\mathcal{B}(\ell_p^n,\ell_p^r)}$ and $\|\gamma\|_p = \|\beta\|_p$, we get the desired inequality. □

Let V be a p-operator space. Fix $n \in \mathbb{N}$ and define $\|\cdot\|_{1,n} : \mathbb{M}_n(V) \to [0,\infty)$ by

$$\|v\|_{1,n} = \inf\{\|\alpha\|_{p'}\|w\|\|\beta\|_p : r \in \mathbb{N}, \quad v = \alpha w \beta, \quad \alpha \in \mathbb{M}_{n,r}, \quad \beta \in \mathbb{M}_{r,n}, \quad w \in M_r(V)\}, \tag{1.4}$$

where $\|\cdot\|_{p'}$ and $\|\cdot\|_p$ as in Lemma 1.6.4.

Proposition 1.6.5. *Suppose that V is a p-operator space and $n \in \mathbb{N}$. Then $\|\cdot\|_{1,n}$ defines a norm on $\mathbb{M}_n(V)$.*

Proof. Suppose $v_1, v_2 \in \mathbb{M}_n(V)$. Let $\epsilon > 0$. For $i = 1, 2$, we can find α_i, β_i, and w_i such that $v_i = \alpha_i w_i \beta_i$

with $\|w_i\| \le 1$ and

$$\|\alpha_i\|_{p'} < (\|v_i\|_{1,n} + \epsilon)^{1/p'}, \quad \|\beta_i\|_p < (\|v_i\|_{1,n} + \epsilon)^{1/p}. \tag{1.5}$$

Let

$$\alpha = [\alpha_1 \ \alpha_2], \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} w_1 & \\ & w_2 \end{bmatrix},$$

then $\|\alpha\|_{p'}^{p'} = \|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}$, $\|\beta\|_p^p = \|\beta_1\|_p^p + \|\beta_2\|_p^p$, and $\|w\| \le 1$. Since $v_1 + v_2 = \alpha w \beta$, it follows that

$$\begin{aligned} \|v_1 + v_2\|_{1,n} &\le \|\alpha\|_{p'}\|\beta\|_p \\ \text{(Young's inequality)} \quad &\le \frac{\|\alpha\|_{p'}^{p'}}{p'} + \frac{\|\beta\|_p^p}{p} \\ &= \frac{\|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}}{p'} + \frac{\|\beta_1\|_p^p + \|\beta_2\|_p^p}{p} \\ \text{(by (1.5))} \quad &< \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon}{p'} + \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon}{p} \\ &= \|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we get $\|v_1 + v_2\|_{1,n} \le \|v_1\|_{1,n} + \|v_2\|_{1,n}$.

For any $c \in \mathbb{C}$, if $v = \alpha w \beta$, then we have $cv = \alpha(cw)\beta$ and hence $\|cv\|_{1,n} \le \|\alpha\|_{p'}|c|\|w\|\|\beta\|_p$. Taking the infimum, we get

$$\|cv\|_{1,n} \le |c|\|v\|_{1,n}. \tag{1.6}$$

Replacing c by $1/c$ and v by cv in (1.6) gives

$$|c|\|v\|_{1,n} \le \|cv\|_{1,n}, \tag{1.7}$$

so (1.6) together with (1.7) gives $\|cv\|_{1,n} = |c|\|v\|_{1,n}$.

Finally, suppose $\|v\|_{1,n} = 0$. To show that $v = 0$, it suffices to show that

$$\|v\| \le n^{2|1/p - 1/p'|} \cdot \|v\|_{1,n}. \tag{1.8}$$

Indeed, if $v = \alpha w \beta$ with $\alpha \in \mathbb{M}_{n,r}$, $\beta \in \mathbb{M}_{r,n}$, and $w \in M_r(v)$, then

$$\begin{aligned} \|v\| &\le \|\alpha\|\|w\|\|\beta\| \\ \text{(by Lemma 1.6.4)} \quad &\le \|\alpha\|_{p'} \cdot n^{|1/p - 1/p'|} \cdot \|w\| \cdot \|\beta\|_p \cdot n^{|1/p - 1/p'|} \end{aligned}$$

$$= \quad n^{2|1/p-1/p'|} \cdot \|\alpha\|_{p'} \cdot \|w\| \cdot \|\beta\|_p.$$

Taking the infimum, (1.8) follows. $\square$

Definition 1.6.6. For a p-operator space V, let $\mathcal{N}_n(V)$ denote the normed space $(\mathbb{M}_n(V), \|\cdot\|_{1,n})$.

We want to study the Banach dual of $\mathcal{N}_n(V)$. Let $f = [f_{ij}] \in M_n(V') = \mathcal{CB}_p(V, M_n)$. By Remark 1.4.4, we have

$$\|f\| = \sup\{\|\langle\langle f, \tilde{v}\rangle\rangle\| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leq 1\}.$$

Let $D^p_{n\times r}$ denote the closed unit ball of $\ell_p^{n\times r}$, then

$$\begin{aligned}
\|f\| &= \sup\{|\langle\langle\langle f, \tilde{v}\rangle\rangle\eta, \xi\ \rangle| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leq 1,\ \eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\} \\
&= \sup\left\{\left|\sum_{i,j,k,l} f_{ij}(\tilde{v}_{kl})\eta_{(j,l)}\xi_{(i,k)}\right| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leq 1,\ \eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\right\} \\
&= \sup\left\{\left|\sum_{i,j=1}^{n} \left\langle f_{ij}, \sum_{k,l=1}^{r} \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}\right\rangle\right| : r \in \mathbb{N},\ \tilde{v} = [\tilde{v}_{kl}] \in M_r(V),\ \|\tilde{v}\| \leq 1,\ \eta \in D^p_{n\times r},\ \xi \in D^{p'}_{n\times r}\right\}.
\end{aligned}$$

Note that $\sum_{k,l=1}^{r} \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}$ is the (i,j)-entry of the matrix product $\alpha\tilde{v}\beta$, where

$$\alpha = \begin{bmatrix} \xi_{(1,1)} & \cdots & \xi_{(1,r)} \\ \vdots & \ddots & \vdots \\ \xi_{(n,1)} & \cdots & \xi_{(n,r)} \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \eta_{(1,1)} & \cdots & \eta_{(n,1)} \\ \vdots & \ddots & \vdots \\ \beta_{(1,r)} & \cdots & \eta_{(n,r)} \end{bmatrix},$$

so

$$\begin{aligned}
\|f\| &= \sup\left\{\left|\sum_{i,j=1}^{n} \langle f_{ij}, (\alpha\tilde{v}\beta)_{ij}\rangle\right| : \|\tilde{v}\| \leq 1,\ \|\alpha\|_{p'} \leq 1,\ \|\beta\|_p \leq 1\right\} \\
&= \sup\{|\langle f, v\rangle| : v = \alpha\tilde{v}\beta,\ \|\tilde{v}\| \leq 1,\ \|\alpha\|_{p'} \leq 1,\ \|\beta\|_p \leq 1\} \\
&= \sup\{|\langle f, v\rangle| : \|v\|_{1,n} \leq 1\}. \qquad (1.9)
\end{aligned}$$

Lemma 1.6.7. *For a p-operator space V, the scalar pairing determines the isometric identification $\mathcal{N}_n(V)' = M_n(V')$.*

Proof. Define $\Phi : M_n(V') \to \mathcal{N}_n(V)'$ by $f \mapsto \langle f, \cdot\rangle$.

Φ is one-to-one: suppose $\Phi(f) = 0$, and let $v \in V$. For each i, j, consider an element $E_{ij}(v) \in \mathcal{N}_n(V)$ whose only nonzero element is v at (i,j)-position, then $\langle f, E_{ij}(v)\rangle = f_{ij}(v) = 0$ and hence $f_{ij} = 0$.

Φ is onto: Let $G \in \mathcal{N}_n(V)'$. For each i, j, define $E_{ij} : V \to \mathcal{N}_n(V)$ by $v \mapsto E_{ij}(v)$, where $E_{ij}(v)$ as as above. Let $f_{ij} = G \circ E_{ij} : V \to \mathbb{C}$ and let $f = [f_{ij}] \in M_n(V')$, then it follows that $\Phi(f) = G$.

Φ is isometric: this is immediate from (1.9). $\square$

Now we are ready to show that the first isometric isomorphism in (1.3) is also p-completely isometric with the dual p-operator space structure on $\mathcal{N}(L_p(\mu))$.

Proposition 1.6.8. *With the dual p-operator space structure on $\mathcal{N}(L_p(\mu))$, we have $\mathcal{K}(L_p(\mu))' = \mathcal{N}(L_p(\mu))$ p-completely isometrically.*

Proof. Fix $n \in \mathbb{N}$. We need to show that

$$M_n(\mathcal{N}(L_p(\mu))) = \mathcal{CB}_p(\mathcal{K}(L_p(\mu)), M_n)$$

isometrically. By the argument between Propositions 5.3 and 5.4 in [Daw10], we have an isometry

$$M_n(\mathcal{N}(L_p(\mu))) = \mathcal{N}_n(\mathcal{K}(L_p(\mu)))'.^{13}$$

Now the result follows from Lemma 1.6.7. $\square$

To explore more examples of p-completely isometric isomorphisms, we will make use of the following lemmas.

Lemma 1.6.9. *Let $\eta = [\eta_{ij}] \in M_{m,n}(L_p^c(\mu))$ and $\|\eta_{ij}\| < \epsilon$ for all i, j. Then $\|\eta\|_{M_{m,n}(L_p^c(\mu))} < m^{1/p} n^{1/p'} \epsilon$.*

Proof. Since $\|\eta\|_{M_{m,n}(L_p^c(\mu))} = \|\eta\|_{\mathcal{B}(\ell_p^n, \ell_p^m(L_p(\mu)))}$,

$$\|\eta\|_{M_{m,n}(L_p^c(\mu))} = \sup_{\substack{\lambda \in \ell_p^n \\ \|\lambda\| \le 1}} \left(\sum_{i=1}^m \left\| \sum_{j=1}^n \lambda_j \eta_{ij} \right\|^p \right)^{1/p}.$$

Here for $\|\lambda\| \le 1$, by Hölder's inequality,

$$\left\| \sum_{j=1}^n \lambda_j \eta_{ij} \right\| \le \sum_{j=1}^n |\lambda_j| \|\eta_{ij}\| \le \left(\sum_{j=1}^n \|\eta_{ij}\|^{p'} \right)^{1/p'} < n^{1/p'} \epsilon$$

and the result follows. $\square$

[13] In [Daw10], Daws used the notation $T_n(\mathcal{K}(L_p(\mu)))$ for $\mathcal{N}_n(\mathcal{K}(L_p(\mu)))$.

Lemma 1.6.10. *Let μ be a measure. Then for any $\xi = [\xi_{ij}] \in M_m(L_p^c(\mu))$ and for any $\epsilon > 0$, there exist a $k(m) \in \mathbb{N}$, a subspace F of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{k(m)}$, $\{f_1, \cdots, f_{k(m)}\} \subseteq F$, and matrices $\{\alpha^1, \cdots, \alpha^{k(m)}\} \subseteq M_m$ such that*

$$\|\xi - \tilde{\xi}\|_{M_m(L_p^c(\mu))} < \epsilon,$$

where

$$\tilde{\xi} = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t.$$

Moreover, we can have

$$\|\tilde{\xi}\| = \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\|_{M_{mk(m),m}}.$$

Proof. By standard properties of $L_p(\mu)$, there exist a $k(m) \in \mathbb{N}$, a subspace F of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{k(m)}$, and $\{\tilde{\xi}_{ij}\}_{i,j=1}^m \subseteq F$ such that $\|\xi_{ij} - \tilde{\xi}_{ij}\| < \frac{\epsilon}{m}$ for each i, j.[14] Let $\{f_1, \cdots, f_{k(m)}\} \subseteq F$ correspond to the canonical basis of $\ell_p^{k(m)}$ and write $\tilde{\xi}_{ij} = \sum_{t=1}^{k(m)} \alpha_{ij}^t f_t$, then we have

$$\tilde{\xi} := [\tilde{\xi}_{ij}] = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t,$$

where $\alpha^t = [\alpha_{ij}^t] \in M_m$. By Lemma 1.6.9, $\|\xi - \tilde{\xi}\|_{M_m(L_p^c(\mu))} < \epsilon$. Finally,

$$\begin{aligned}
\|\tilde{\xi}\| &= \left\| \left[\sum_{t=1}^{k(m)} \alpha_{ij}^t f_t \right] \right\|_{\mathcal{B}(\ell_p^m, \ell_p^m(L_p(\mu)))} \\
&= \sup_{\substack{\lambda \in \ell_p^m \\ \|\lambda\| \leq 1}} \left(\sum_{i=1}^m \left\| \sum_{j=1}^m \left(\sum_{t=1}^{k(m)} \alpha_{ij}^t \lambda_j f_t \right) \right\|^p \right)^{1/p} \\
&= \sup_{\substack{\lambda \in \ell_p^m \\ \|\lambda\| \leq 1}} \left(\sum_{i=1}^m \left\| \sum_{t=1}^{k(m)} \left(\sum_{j=1}^m \alpha_{ij}^t \lambda_j \right) f_t \right\|^p \right)^{1/p} \\
&= \sup_{\substack{\lambda \in \ell_p^m \\ \|\lambda\| \leq 1}} \left(\sum_{i=1}^m \sum_{t=1}^{k(m)} \left| \sum_{j=1}^m \alpha_{ij}^t \lambda_j \right|^p \right)^{1/p} \\
&= \sup_{\substack{\lambda \in \ell_p^m \\ \|\lambda\| \leq 1}} \left(\sum_{t=1}^{k(m)} \sum_{i=1}^m \left| \sum_{j=1}^m \alpha_{ij}^t \lambda_j \right|^p \right)^{1/p}
\end{aligned}$$

[14] See [LP68]. We will keep refer this property to the *rigid $\mathcal{L}_p$-structure of $L_p(\mu)$*.

$$= \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\|_{M_{mk(m),m}} .$$

$\square$

Proposition 1.6.11. *For any $L_p(\mu)$ and $L_p(\nu)$, there exists a natural p-completely isometric isomorphism*

$$\mathcal{CB}_p(L_p^c(\mu), L_p^c(\nu)) = \mathcal{B}(L_p(\mu), L_p(\nu)).$$

Proof. Fix $n \in \mathbb{N}$. For any matrix $T = [T_{kl}] \in M_n(\mathcal{B}(L_p(\mu), L_p(\nu))) = \mathcal{B}(\ell_p^n(L_p(\mu)), \ell_p^n(L_p(\nu)))$, the corresponding mapping $\tilde{T} \in M_n(\mathcal{CB}_p(L_p^c(\mu), L_p^c(\nu))) = \mathcal{CB}_p(L_p^c(\mu), M_n(L_p^c(\nu)))$ is defined by $\tilde{T}(\xi) = [T_{kl}(\xi)]$ for all $\xi \in L_p^c(\mu)$. We wish to show that $\|\tilde{T}\|_{pcb} = \|T\|$. Fix $\xi = [\xi_{ij}] \in M_m(L_p^c(\mu))$ and let $\epsilon > 0$. As in the proof of Lemma 1.6.10, there exist a $k(m) \in \mathbb{N}$, a subspace F of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{k(m)}$, and $\{\tilde{\xi}_{ij}\}_{i,j=1}^m \subseteq F$ such that

$$\|\xi_{ij} - \tilde{\xi}_{ij}\| < \min\left\{\frac{\epsilon}{nm\|T\|}, \frac{\epsilon}{m}\right\} \tag{1.10}$$

for each i, j. In particular,

$$\|\xi - \tilde{\xi}\| < \epsilon \tag{1.11}$$

by Lemma 1.6.9, where $\tilde{\xi} = [\tilde{\xi}_{ij}]$. Let $\{f_1, \cdots, f_{k(m)}\} \subseteq F$ correspond to the canonical basis of $\ell_p^{k(m)}$ and write $\tilde{\xi}_{ij} = \sum_{t=1}^{k(m)} \alpha_{ij}^t f_t$, then $\tilde{T}_m(\tilde{\xi}) = \sum_{t=1}^{k(m)} [T_{kl}(f_t)] \otimes \alpha^t$. Since $\|T_{kl}\| \leq \|T\|$ for each k and l, from (1.10) above and Lemma 1.6.9, we get

$$\|\tilde{T}_m(\xi) - \tilde{T}_m(\tilde{\xi})\| < \epsilon. \tag{1.12}$$

Again as in the proof of Lemma 1.6.10, there exist an $l(m) \in \mathbb{N}$, a subspace G of $L_p(\nu)$ which is isometrically isomorphic to $\ell_p^{l(m)}$, and $\{\eta_{kl}^t\}_{k,l,t} \subseteq G$ such that

$$\|T_{kl}(f_t) - \eta_{kl}^t\| < \min\left\{\frac{\epsilon}{nk(m)^{1/p'}}, \frac{\epsilon}{n\sum_{t=1}^{k(m)} \|\alpha^t\|}\right\} \tag{1.13}$$

for all k, l, and t. In particular, by Lemma 1.6.9,

$$\|[T_{kl}(f_t)] - [\eta_{kl}^t]\| < \frac{\epsilon}{\sum_{t=1}^{k(m)} \|\alpha^t\|} \quad \text{for each } t. \tag{1.14}$$

Let $\{g_1, \cdots, g_{l(m)}\} \subseteq G$ correspond to the canonical basis of $\ell_p^{l(m)}$ and write $\eta_{kl}^t = \sum_{s=1}^{l(m)} \beta_{kl}^{st} g_s$ so that for each t

$$[\eta_{kl}^t] = \sum_{s=1}^{l(m)} g_s \otimes \beta^{st}, \tag{1.15}$$

where $\beta^{st} = [\beta_{kl}^{st}] \in M_n$. Define $T_0 : \ell_p^n(F) \to \ell_p^n(G)$ by

$$\varphi := \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \overset{T_0}{\longmapsto} \begin{bmatrix} \sum_{t,l} \gamma_t^l \eta_{1l}^t \\ \vdots \\ \sum_{t,l} \gamma_t^l \eta_{nl}^t \end{bmatrix},$$

where $\varphi_l = \sum_{t=1}^{k(m)} \gamma_t^l f_t$. Note that

$$\begin{aligned} \|T_0(\varphi)\|^p &= \sum_{k=1}^n \left\| \sum_{t,l} \gamma_t^l \eta_{kl}^t \right\|^p \\ &= \sum_{k=1}^n \left\| \sum_{s,t,l} \gamma_t^l \beta_{kl}^{st} g_s \right\|^p \\ &= \sum_{k=1}^n \sum_{s=1}^{l(m)} \left| \sum_{t,l} \beta_{kl}^{st} \gamma_t^l \right|^p \\ &= \|\beta\gamma\|^p, \end{aligned}$$

where

$$\beta = \underbrace{\begin{bmatrix} \beta^{11} & \cdots & \beta^{1k(m)} \\ \vdots & \ddots & \vdots \\ \beta^{l(m)1} & \cdots & \beta^{l(m)k(m)} \end{bmatrix}}_{\in M_{nl(m),nk(m)}}, \quad \gamma_t = \underbrace{\begin{bmatrix} \gamma_t^1 \\ \vdots \\ \gamma_t^n \end{bmatrix}}_{\in M_{n,1}}, \quad \text{and} \quad \gamma = \underbrace{\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{k(m)} \end{bmatrix}}_{\in M_{nk(m),1}}.$$

Since $\|\varphi\| = \left(\sum_{t,l} |\gamma_t^l|^p\right)^{1/p} = \|\gamma\|$, it follows that

$$\|T_0\| = \|\beta\|. \tag{1.16}$$

Now for all $\varphi \in \ell_p^n(F)$, we obtain

$$\begin{aligned}
\|T(\varphi) - T_0(\varphi)\|^p &= \left\| \begin{bmatrix} \sum_{t,l} \gamma_t^l (T_{1l}(f_t) - \eta_{1l}^t) \\ \vdots \\ \sum_{t,l} \gamma_t^l (T_{nl}(f_t) - \eta_{nl}^t) \end{bmatrix} \right\|^p \\
&= \sum_{k=1}^n \left\| \sum_{t,l} \gamma_t^l (T_{kl}(f_t) - \eta_{kl}^t) \right\|^p \\
&\leq \sum_{k=1}^n \left(\sum_{t,l} |\gamma_t^l| \, \|T_{kl}(f_t) - \eta_{kl}^t\| \right)^p \\
\text{(Hölder's inequality)} \quad &\leq \sum_{k=1}^n \|\varphi\|^p \left(\sum_{t,l} \|T_{kl}(f_t) - \eta_{kl}^t\|^{p'} \right)^{p/p'} \\
\text{(by (1.13))} \quad &\leq \|\varphi\|^p \sum_{k=1}^n \left(nk(m) \frac{\epsilon^{p'}}{n^{p'} k(m)} \right)^{p/p'} \\
&= \epsilon^p \|\varphi\|^p
\end{aligned}$$

and hence

$$\|T_0\| \leq \|T|_{\ell_p^n(F)}\| + \|T|_{\ell_p^n(F)} - T_0\| \leq \|T\| + \epsilon. \tag{1.17}$$

Since

$$\begin{aligned}
\|\tilde{T}_m(\tilde{\xi}) - \textstyle\sum_{t=1}^{k(m)} [\eta_{kl}^t] \otimes \alpha^t\| &= \left\| \textstyle\sum_{t=1}^{k(m)} [T_{kl}(f_t)] \otimes \alpha^t - \sum_{t=1}^{k(m)} [\eta_{kl}^t] \otimes \alpha^t \right\| \\
&\leq \textstyle\sum_{t=1}^{k(m)} \|[T_{kl}(f_t)] - [\eta_{kl}^t]\| \|\alpha^t\| \qquad (1.18) \\
\text{(by (1.14))} \quad &\leq \epsilon,
\end{aligned}$$

we finally have

$$\begin{aligned}
\|\tilde{T}_m(\xi)\| &\leq \|\tilde{T}_m(\tilde{\xi})\| + \|\tilde{T}_m(\xi) - \tilde{T}_m(\tilde{\xi})\| \\
\text{(by (1.12))} \quad &\leq \left\| \sum_{t=1}^{k(m)} [\eta_{kl}^t] \otimes \alpha^t \right\| + \left\| \tilde{T}_m(\tilde{\xi}) - \sum_{t=1}^{k(m)} [\eta_{kl}^t] \otimes \alpha^t \right\| + \epsilon \\
\text{(by (1.18))} \quad &\leq \left\| \sum_{t=1}^{k(m)} [\eta_{kl}^t] \otimes \alpha^t \right\| + 2\epsilon \\
\text{(by (1.15))} \quad &= \left\| \sum_{t=1}^{k(m)} \sum_{s=1}^{l(m)} g_s \otimes \beta^{st} \otimes \alpha^t \right\| + 2\epsilon \\
&= \left\| \begin{bmatrix} \sum_{t=1}^{k(m)} \beta^{1t} \otimes \alpha^t \\ \vdots \\ \sum_{t=1}^{k(m)} \beta^{l(m)t} \otimes \alpha^t \end{bmatrix} \right\| + 2\epsilon \\
&\leq \left\| \begin{bmatrix} \beta^{11} & \cdots & \beta^{1k(m)} \\ \vdots & \ddots & \vdots \\ \beta^{l(m)1} & \cdots & \beta^{l(m)k(m)} \end{bmatrix} \right\| \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\| + 2\epsilon
\end{aligned}$$

$$\begin{aligned}\text{(by (1.16))} \quad &= \quad \|T_0\|\|\tilde{\xi}\| + 2\epsilon \\ \text{(by (1.17) and (1.11))} \quad &\leq \quad (\|T\| + \epsilon)(\|\xi\| + \epsilon) + 2\epsilon.\end{aligned}$$

This shows that $\|\tilde{T}\|_{pcb} \leq \|T\|$. For the reverse inequality, fix

$$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \ell_p^n(L_p(\mu)),$$

then there exist a $q(n) \in \mathbb{N}$, a subspace F of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{q(n)}$, $\{f_1, \cdots, f_{q(n)}\} \subseteq F$ corresponding to the canonical basis in $\ell_p^{q(n)}$, and a matrix $\delta = [\delta_{tl}] \in M_{q(n),n}$ such that $\|\xi_l - \tilde{\xi}_l\| < \min\left\{\frac{\epsilon}{n^{1/p}}, \frac{\epsilon}{n^{1/p}\|T\|}\right\}$ for all l, where $\tilde{\xi}_l = \sum_{t=1}^{q(n)} \delta_{tl} f_t$. Note that

$$\|\xi - \tilde{\xi}\| \leq \min\left\{\epsilon, \frac{\epsilon}{\|T\|}\right\}. \tag{1.19}$$

Put

$$\tilde{\xi} = \begin{bmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_n \end{bmatrix} \quad \text{and} \quad \delta_t = \begin{bmatrix} \delta_{t1} \\ \vdots \\ \delta_{tn} \end{bmatrix},$$

then by (1.19)

$$\left\|T(\xi) - T(\tilde{\xi})\right\| \leq \|T\|\|\xi - \tilde{\xi}\| < \epsilon \tag{1.20}$$

and

$$\begin{aligned} T(\tilde{\xi}) \quad &= \quad \begin{bmatrix} \sum_{l=1}^n T_{1l}(\tilde{\xi}_l) \\ \vdots \\ \sum_{l=1}^n T_{nl}(\tilde{\xi}_l) \end{bmatrix} \\ &= \quad \overbrace{\begin{bmatrix} T(f_1) & \cdots & T(f_{q(n)}) \end{bmatrix}}^{\in M_{n,nq(n)}} \overbrace{\begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{q(n)} \end{bmatrix}}^{\in M_{nq(n),1}} \end{aligned}$$

$$= \quad \tilde{T}_{1,q(n)}(\begin{bmatrix} f_1 & \cdots & f_{q(n)} \end{bmatrix}) \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{q(n)} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \|T(\tilde{\xi})\| &\leq \left\| \tilde{T}_{1,q(n)}(\begin{bmatrix} f_1 & \cdots & f_{q(n)} \end{bmatrix}) \right\| \left\| \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{q(n)} \end{bmatrix} \right\| \\ &\leq \|\tilde{T}\|_{pcb} \left\| \begin{bmatrix} f_1 & \cdots & f_{q(n)} \end{bmatrix} \right\|_{M_{1,q(n)}(L_p^c(\mu))} \|\tilde{\xi}\| \\ &= \|\tilde{T}\|_{pcb} \|\tilde{\xi}\| \end{aligned} \tag{1.21}$$

since $\{f_1, \cdots, f_{q(n)}\} \subseteq F$ corresponds to the canonical basis in $\ell_p^{q(n)}$ and hence

$$\left\| \begin{bmatrix} f_1 & \cdots & f_{q(n)} \end{bmatrix} \right\|_{M_{1,q(n)}(L_p^c(\mu))} = 1.$$

Finally we obtain from (1.19), (1.20), and (1.21)

$$\|T(\xi)\| \leq \|T(\tilde{\xi})\| + \left\| T(\xi) - T(\tilde{\xi}) \right\| \leq \|\tilde{T}\|_{pcb} \|\tilde{\xi}\| + \epsilon \leq \|\tilde{T}\|_{pcb} (\|\xi\| + \epsilon) + \epsilon,$$

which completes the proof. □

Corollary 1.6.12. *For any $L_p(\mu)$ and $L_p(\nu)$, we have the p-completely isometric isomorphisms*

$$(L_p^c(\mu))' = L_{p'}^r(\mu), \quad (L_{p'}^r(\mu))' = L_p^c(\mu), \quad \textit{and} \quad \mathcal{CB}_p(L_{p'}^r(\mu), L_{p'}^r(\nu)) = \mathcal{B}(L_p(\nu), L_p(\mu)).$$

Proof. By Proposition 1.6.11,

$$(L_p^c(\mu))' = \mathcal{CB}_p(L_p^c(\mu), \mathbb{C}) = \mathcal{B}(L_p(\mu), \mathbb{C}) = \mathcal{B}(L_{p'}(\mu)', \mathbb{C}) = L_{p'}^r(\mu).$$

By Lemma 1.5.2 and above calculation,

$$(L_{p'}^r(\mu))' = (L_p^c(\mu))'' = L_p^c(\mu).$$

Finally, by Proposition 1.5.6,

$$\mathcal{CB}_p(L^r_{p'}(\mu), L^r_{p'}(\nu)) = \mathcal{CB}_p((L^c_p(\mu))', (L^c_p(\nu))') = \mathcal{CB}_p(L^c_p(\nu), L^c_p(\mu)) = \mathcal{B}(L_p(\nu), L_p(\mu)).$$

□

We close this section with one more example. Let G be an index set. Then $\ell_\infty(G)$ has a natural p-operator space structure inherited from the embedding $\ell_\infty(G) \subseteq \mathcal{B}(\ell_p(G))$. We will always assume that $\ell_\infty(G)$ is given this p-operator space structure unless stated otherwise. Now give $\ell_1(G)$ the dual p-operator space structure, that is, $\ell_1(G)$ is given the p-operator space structure regarded as a subspace of $\ell_\infty(G)'$.

Proposition 1.6.13. *$\ell_1(G)' = \ell_\infty(G)$ p-completely isometrically.*

Proof. We claim that $\ell_\infty(G)$ is weak*-closed in $\mathcal{B}(\ell_p(G))$; once this is done, the result follows from [Daw10, Proposition 5.5]. To this end, suppose that $\{f_i\} \subseteq \ell_\infty(G)$ and that $f_i \to T$ in the weak*- topology. If $s, t \in G$ with $s \neq t$, then

$$T_{st} = \langle \delta^{p'}_s, T\delta^p_t \rangle = \lim_i f_i(s)\delta^p_t(s) = 0.$$

This shows that $T \subseteq \ell_\infty(G)$. □

Just like $p = 2$ case, we have the following characterization of the norm in $M_n(\ell_\infty(G))$.

Proposition 1.6.14. *If $[u_{ij}] \in M_n(\ell_\infty(G))$, then $\|[u_{ij}]\|_{M_n(\ell_\infty(G))} = \sup_{s\in G} \|[u_{ij}(s)]\|_{\mathcal{B}(\ell^n_p)}$.*

Proof. For all $s \in G$,

$$\|[u_{ij}(s)]\|_{\mathcal{B}(\ell^n_p)} = \sup\left\{\left|\sum_{i,j=1}^n a_i u_{ij}(s) b_j\right| : \sum_{i=1} |a_i|^{p'} \leq 1, \sum_{j=1} |b_j|^p \leq 1\right\}.$$

Since

$$\begin{aligned} & \sum_{i,j=1}^n a_i u_{ij}(s) b_j \\ = \;& \sum_{i,j=1}^n \sum_{x\in G} a_i \delta^{p'}_s(x) u_{ij}(x) b_j \delta^p_s(x) \\ = \;& \left\langle [a_i\delta^{p'}_s], [u_{ij}][b_j\delta^p_s] \right\rangle, \end{aligned}$$

we get

$$\left|\sum_{i,j=1}^n a_i u_{ij}(s) b_j\right| \leq \|[u_{ij}]\|_{M_n(\ell_\infty(G))}.$$

For the converse, suppose that $g = (g_i)_{i=1}^n \in \ell_{p'}^n \otimes_{p'} \ell_{p'}(G)$ with $\sum_{i,x} |g_i(x)|^{p'} \leq 1$. Similarly, suppose that $f = (f_j)_{j=1}^n \in \ell_p^n \otimes_p \ell_p(G)$ with $\sum_{j,x} |f_j(x)|^p \leq 1$. Then

$$\begin{aligned}
& \left| \sum_{i,j} \sum_x g_i(x) u_{ij}(x) f_j(x) \right| \\
\leq \quad & \sum_x \left| \sum_{i,j} g_i(x) u_{ij}(x) f_j(x) \right| \\
\leq \quad & \sum_x \|[u_{ij}(x)]\|_{\mathcal{B}(\ell_p^n)} \left(\sum_{i=1}^n |g_i(x)|^{p'} \right)^{1/p'} \left(\sum_{j=1}^n |f_j(x)|^p \right)^{1/p} \\
\leq \quad & \sup_{s \in G} \|[u_{ij}(s)]\|_{\mathcal{B}(\ell_p^n)}
\end{aligned}$$

and this completes the proof. $\square$

Chapter 2

Tensor Products of p-Operator Spaces

In this chapter, we study various tensor products on p-operator spaces. We mainly focus on the following three tensor products: p-projective tensor product, p-injective tensor product, and p-Haagerup tensor product.

2.1 p-Projective Tensor Product

The main source for this section is [Daw10].

Definition 2.1.1. Let X, Y be p-operator spaces. For $u \in \mathbb{M}_n(X \otimes Y)$, let

$$\|u\|_{\wedge_p} = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\},$$

where the infimum is taken over $r, s \in \mathbb{N}$, $\alpha \in M_{n,r\times s}$, $v \in M_r(X)$, $w \in M_s(Y)$, and $\beta \in M_{r\times s,n}$.

It was Daws who first defined and studied the p-projective tensor product. Note that $\|\cdot\|_{\wedge_p}$ gives the algebraic tensor product $X \otimes Y$ a p-operator space structure [Daw10, Proposition 4.8]. Furthermore, $\|\cdot\|_{\wedge_p}$ is the largest subcross p-operator space norm on $X \otimes Y$ in the sense that $\|x \otimes y\| \leq \|x\|_r\|y\|_s$ for all $x \in M_r(X)$ and all $y \in M_s(Y)$ [Daw10, Proposition 4.8]. The *p-operator space projective tensor product* is defined to be the completion of $X \otimes Y$ with respect to this norm and is denoted by $X \overset{\wedge_p}{\otimes} Y$.

Remark 2.1.2.

1. One can show that p-operator space projective tensor product is commutative, i.e., $X \overset{\wedge_p}{\otimes} Y = Y \overset{\wedge_p}{\otimes} X$ p-completely isometrically.

2. By universality of the Banach space projective tensor product $\overset{\pi}{\otimes}$ [BLM04, A.3.3], we have

   $$\|u\|_{\wedge_p} \leq \|u\|_\pi$$

   for all $u \in X \otimes Y$.

Let V, W and Z be p-operator spaces, and let $\psi : V \times W \to Z$ be a bilinear map. Define bilinear maps $\psi_{r,s;t,u}$ by

$$\psi_{r,s;t,u} : M_{r,s}(V) \times M_{t,u}(W) \to M_{r\times t, s\times u}(Z), \qquad (v, w) \mapsto (\psi(v_{i,j}, w_{k,l})),$$

and let $\psi_{r;s} = \psi_{r,r;s,s}$. Finally define

$$\|\psi\|_{jpcb} = \sup\{\|\psi_{r;s}\| : r, s \in \mathbb{N}\}.$$

We say that ψ is *jointly p-completely bounded* (respectively, *jointly p-completely contractive*) if $\|\psi\|_{jpcb} < \infty$ (respectively, $\|\psi\|_{jpcb} \leq 1$). The space of all jointly p-completely bounded maps from $V \times W$ to Z will be denoted by $\mathcal{CB}_p(V \times W, Z)$ and this space can be turned into a p-operator space in the same way as for $\mathcal{CB}_p(V, W)$. Here we collect some results on the p-projective tensor product for convenience.

Proposition 2.1.3. *[Daw10, Proposition 4.9] Let X, Y, and Z be p-operator spaces. Then we have natural p-completely isometric identifications*

$$\mathcal{CB}_p(X \overset{\wedge_p}{\otimes} Y, Z) = \mathcal{CB}_p(X \times Y, Z) = \mathcal{CB}_p(X, \mathcal{CB}_p(Y, Z)).$$

In particular,

$$(X \overset{\wedge_p}{\otimes} Y)' = \mathcal{CB}_p(X, Y').$$

For p-operator spaces X and Y, let $\mathcal{CB}_p^\sigma(X', Y')$ denote the space of all weak*-weak*-continuous p-completely bounded maps between X' and Y'.

Corollary 2.1.4. *Let X be a p-operator space on L_p space and Y be a p-operator space. Then we have*

$$\mathcal{CB}_p(X, Y') = \mathcal{CB}_p^\sigma(X'', Y') \tag{2.1}$$

p-completely isometrically.

Proof. Let $u \in \mathcal{CB}_p(X, Y')$. It is easy to verify that $\tilde{u} = \kappa'_Y \circ u'' : X'' \to Y'$ defines a weak*-weak*-continuous p-completely bounded extension of u with $\|\tilde{u}\|_{pcb} \leq \|u\|_{pcb}$. In fact, by Goldstine's Theorem, $\tilde{u}$ is a unique weak*-weak*-continuous extension of u. Since $X \subseteq X''$ p-completely isometrically, we also have $\|\tilde{u}\|_{pcb} \geq \|u\|_{pcb}$. If $T \in \mathcal{CB}_p^\sigma(X'', Y')$, then we must have $T = \widetilde{T|_X}$ and this shows that (2.1) holds isometrically. To show that (2.1) is a p-complete isometry, let us fix $n \in \mathbb{N}$. Then we have isometries

$$M_n(\mathcal{CB}_p(X, Y')) = \mathcal{CB}_p(X, M_n(Y')) = \mathcal{CB}_p(X, \mathcal{CB}_p(Y, M_n)) = \mathcal{CB}_p(X, (\mathcal{N}_n \overset{\wedge_p}{\otimes} Y)').$$[1]

Since we already showed that (2.1) holds isometrically, we have

$$\mathcal{CB}_p(X, (\mathcal{N}_n \overset{\wedge_p}{\otimes} Y)') = \mathcal{CB}_p(X'', (\mathcal{N}_n \overset{\wedge_p}{\otimes} Y)')$$

isometrically and it follows that

$$M_n(\mathcal{CB}_p(X, Y')) = \mathcal{CB}_p(X'', (\mathcal{N}_n \overset{\wedge_p}{\otimes} Y)') = \mathcal{CB}_p(X'', \mathcal{CB}_p(Y, M_n)) = \mathcal{CB}_p(X'', M_n(Y')) = M_n(\mathcal{CB}_p(X'', Y'))$$

isometrically. This completes the proof. $\square$

As in operator spaces, the p-operator space projective tensor product is projective in the following sense:

Proposition 2.1.5. *[Daw10, Proposition 4.10] Let $X, \tilde{X}, Y$, and $\tilde{Y}$ be p-operator spaces. If $u : X \to \tilde{X}$ and $v : Y \to \tilde{Y}$ are p-complete quotient maps, then $u \otimes v$ extends to a p-complete quotient map $u \otimes v : X \overset{\wedge_p}{\otimes} Y \to \tilde{X} \overset{\wedge_p}{\otimes} \tilde{Y}$.*

In operator spaces, the trace class operators on a Hilbert space H can be expressed in terms of operator space projective tensor product. To be more precise, we have the following natural isometries

$$(H^c)' \widehat{\otimes} H^c \cong H' \overset{\pi}{\otimes} H \cong \mathcal{T}(H),$$

where $\mathcal{T}(H)$ denotes the space of all trace class operators on H [ER00, Proposition 8.2.1]. We have the following p-analogue of this result.

Proposition 2.1.6. *For a measure μ, we have an isometric isomorphism $\mathcal{N}(L_p(\mu)) \cong L^r_{p'}(\mu) \overset{\wedge_p}{\otimes} L^c_p(\mu)$.*

Proof. Note that the adjoint of the canonical contraction[2]

$$\varphi : \mathcal{N}(L_p(\mu)) \overset{(1.2)}{=} L_{p'}(\mu) \overset{\pi}{\otimes} L_p(\mu) \to L^r_{p'}(\mu) \overset{\wedge_p}{\otimes} L^c_p(\mu)$$

is the natural mapping $\mathcal{CB}_p(L^c_p(\mu)) \to \mathcal{B}(L_p(\mu))$, which is an isometric surjection by Proposition 1.6.11. The

[1]Note that $M_n(Y') = \mathcal{CB}_p(Y, M_n)$ p-completely isometrically.
[2]See Remark 2.1.2.

result follows using the arguments in [ER00, §A.2].[3] $\square$

Since p-operator space projective tensor product resembles the Banach space projective tensor product in many ways, we can expect that the p-operator space $\mathcal{N}(L_p(\mu))$ to behave well with respect to the p-operator space projective tensor product. Our next goal is to make this clear. Let $\mathcal{N}_n = \mathcal{N}(\ell_p^n)$ so that $\mathcal{B}(\ell_p^n)' = \mathcal{K}(\ell_p^n)' = \mathcal{N}_n$ and $\mathcal{N}_n' = \mathcal{B}(\ell_p^n)$ p-completely isometrically. For a p-operator space V, we wish to study the Banach space structure of $\mathcal{N}_n \overset{\wedge_p}{\otimes} V$.

Proposition 2.1.7. *For any p-operator space V, we have a natural isometry*

$$\mathcal{N}_n(V) \cong \mathcal{N}_n \overset{\wedge_p}{\otimes} V,$$

where $\mathcal{N}_n(V)$ denotes the normed space in Definition 1.6.6.

Proof. By Proposition 2.1.3, we know that $(\mathcal{N}_n \overset{\wedge_p}{\otimes} V)'$ is isometrically isomorphic to $\mathcal{CB}_p(V, M_n) = M_n(V')$. Let's examine the duality between $\mathcal{N}_n \overset{\wedge_p}{\otimes} V$ and $M_n(V')$ more closely. Note that every element in $\mathcal{N}_n$, being a linear map from ℓ_p^n to ℓ_p^n, can be written as a linear combination of ϵ_{ij}, where $\{\epsilon_{ij}\}_{i,j=1}^n$ denotes a standard basis for $\mathbb{M}_n$. If $v = \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \in \mathcal{N}_n \overset{\wedge_p}{\otimes} V$ and $f = [f_{kl}] \in M_n(V')$, then

$$\begin{aligned}
\left\langle f, \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \right\rangle_{M_n(V'),\mathcal{N}_n \overset{\wedge_p}{\otimes} V} &= \sum_{i,j} \langle \epsilon_{ij}, [f_{kl}(v_{ij})]_{k,l=1}^n \rangle_{\mathcal{N}_n, M_n} \\
&= \sum_{i,j} f_{ij}(v_{ij}) \\
&= \langle f, v \rangle_{M_n(V'),\mathcal{N}_n(V)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|v\|_{\mathcal{N}_n \overset{\wedge_p}{\otimes} V} &= \sup \left\{ \left| \left\langle f, \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \right\rangle_{M_n(V'),\mathcal{N}_n \overset{\wedge_p}{\otimes} V} \right| : \|f\| \le 1,\ f \in M_n(V') \right\} \\
&= \sup \left\{ \left| \langle f, v \rangle_{M_n(V'),\mathcal{N}_n(V)} \right| : \|f\| \le 1,\ f \in M_n(V') \right\} \\
&= \|v\|_{\mathcal{N}_n(V)}.
\end{aligned}$$

$\square$

[3] That is, if $\psi : V \to W$ is a bounded linear map between two Banach spaces V and W, then ψ is an isometry if and only if ψ' is a quotient mapping. Under the same assumption, ψ is a quotient mapping if and only if ψ' is an isometry.

2.2 p-Haagerup Tensor Product

In [LeM96], Le Merdy considered the p-operator space Haagerup tensor product. Let X, Y be p-operator spaces. Given $v = [v_{ir}] \in M_{n,k}(X)$ and $w = [w_{rj}] \in M_{k,m}(Y)$, define $v \odot w = [\sum_{r=1}^{k} v_{ir} \otimes w_{rj}] \in \mathbb{M}_{n,m}(X \otimes Y)$.

Definition 2.2.1. Let X, Y be p-operator spaces. For $u \in \mathbb{M}_{n,m}(X \otimes Y)$, we define a norm

$$\|u\|_{h_p} = \inf\{\|v\|\|w\| : v \in M_{n,k}(X),\ w \in M_{k,m}(Y),\ u = v \odot w\}.$$

The *p-operator space Haagerup tensor product* is defined to be the completion of $(X \otimes Y, \|\cdot\|_{h_p})$.

Remark 2.2.2.

1. The p-Haagerup tensor product $\overset{h_p}{\otimes}$ is associative [LeM96, Remark 2.5].

2. $\|\cdot\|_{h_p}$ is a subcross norm, because for $v \in M_r(V)$ and $w \in M_s(W)$, $v \otimes w = (v \otimes I_s) \odot (I_r \otimes w)$. Hence it follows from [Daw10, Proposition 4.8] that $\|\cdot\|_{h_p} \leq \|\cdot\|_{\wedge_p}$ on the algebraic tensor product $V \otimes W$.

Just in operator spaces, sometimes $\overset{h_p}{\otimes}$ and $\overset{\wedge_p}{\otimes}$ produce the same norm. To give an example, we need a lemma.

Lemma 2.2.3. *Let V and W are p-operator spaces. Let $v = [v_{ij}] \in M_{n,m}(V)$ and $w = [w_{kl}] \in M_{m,n}(W)$ with $\|w_{kl}\| \leq \epsilon$ for all k, l, then $\|v \odot w\|_{\wedge_p} \leq \epsilon n^2 m \|v\|$.*

Proof. Observe that

$$v \odot w \quad = \quad \sum_{i,l=1}^{n} \sum_{k=1}^{m} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1_{i^{\text{th}}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} (v_{ik} \otimes w_{kl}) \ [0 \cdots 0 \quad 1_{l^{\text{th}}} \quad 0 \cdots 0]_{1\times n}.$$

Since $\|v_{ik}\| \leq \|v\|$ for all i, k, we get

$$\|v \odot w\|_{\wedge_p} \leq \sum_{i,l=1}^{n} \sum_{k=1}^{m} \|v\|\epsilon = n^2 m \|v\| \epsilon.$$

$\square$

Proposition 2.2.4. *Let V be a p-operator space and μ a measure. Then we have the p-complete isometries*

$$V \overset{h_p}{\otimes} L_p^c(\mu) = V \overset{\wedge_p}{\otimes} L_p^c(\mu), \qquad L_{p'}^r(\mu) \overset{h_p}{\otimes} V = L_{p'}^r(\mu) \overset{\wedge_p}{\otimes} V.$$

Proof. We prove only the first identification: the second one is similar. Suppose $u \in \mathbb{M}_n(V \otimes L_p^c(\mu))$ with $\|u\|_{h_p} < 1$. By Remark 2.2.2, it suffices to show that $\|u\|_{\wedge_p} < 1$. There exist an $m \in \mathbb{N}$, $v = [v_{ij}] \in M_{n,m}(V)$, and $\xi = [\xi_{kl}] \in M_{m,n}(L_p^c(\mu))$ such that $u = v \odot \xi$ with $\|v\|, \|\xi\| < 1$. Let $\epsilon > 0$. As in the proof of Lemma 1.6.10, there exist a $k(m) \in \mathbb{N}$, a subspace F of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{k(m)}$, and $\{\tilde{\xi}_{kl}\} \subseteq F$ such that

$$\|\xi_{kl} - \tilde{\xi}_{kl}\| < \min\left\{\frac{\epsilon}{2m^{1/p}n^{1/p'}}, \frac{\epsilon}{2n^2 m}\right\} \quad \text{for each } k, l. \tag{2.2}$$

Let $\{f_1, \cdots, f_{k(m)}\} \subseteq F$ correspond to the canonical basis of $\ell_p^{k(m)}$ and write $\tilde{\xi}_{kl} = \sum_{t=1}^{k(m)} \alpha_{kl}^t f_t$. If we let $\tilde{\xi} = [\tilde{\xi}_{kl}] \in M_{m,n}(L_p^c(\mu))$, then

$$\tilde{\xi} = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t = [f_1 \cdots f_{k(m)}] \odot \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix}$$

and by Lemma 1.6.9 and (2.2), we obtain $\|\xi - \tilde{\xi}\| < \frac{\epsilon}{2}$. In particular,

$$\left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\| = \|\tilde{\xi}\| < \|\xi\| + \frac{\epsilon}{2} < 1 + \frac{\epsilon}{2}.$$

If we let $\tilde{u} = [\tilde{u}_{il}] = v \odot \tilde{\xi} \in \mathbb{M}_n(V \otimes L_p^c(\mu))$, then

$$\begin{aligned} \tilde{u} &= \left[\sum_{j,t} \alpha_{j,l}^t v_{ij} \otimes f_t\right]_{(i,l)} \\ &= \left(v \otimes [f_1 \cdots f_{k(m)}]\right) \begin{bmatrix} \tilde{\alpha}^1 \\ \vdots \\ \tilde{\alpha}^m \end{bmatrix}, \end{aligned}$$

where

$$\tilde{\alpha}^k = \begin{bmatrix} \alpha_{k1}^1 & \cdots & \alpha_{kn}^1 \\ \vdots & \ddots & \vdots \\ \alpha_{k1}^{k(m)} & \cdots & \alpha_{kn}^{k(m)} \end{bmatrix} \in M_{k(m),n}$$

for each $k = 1, \cdots, m$. Since

$$\|[f_1 \cdots f_{k(m)}]\| = 1 \quad \text{and} \quad \left\| \begin{bmatrix} \tilde{\alpha}^1 \\ \vdots \\ \tilde{\alpha}^m \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\|,$$

we have

$$\|\tilde{u}\|_{\wedge_p} \leq \|v\| \|\tilde{\xi}\| < 1 + \frac{\epsilon}{2}.$$

Since $u - \tilde{u} = v \odot (\xi - \tilde{\xi})$, by Lemma 2.2.3 and (2.2), we have $\|u - \tilde{u}\|_{\wedge_p} < \frac{\epsilon}{2}$, so $\|u\|_{\wedge_p} \leq \|\tilde{u}\|_{\wedge_p} + \|u - \tilde{u}\|_{\wedge_p} < 1 + \epsilon$. Since ϵ is arbitrary, we are done. □

Corollary 2.2.5. *For a measure μ, we have an isometric isomorphism $\mathcal{N}(L_p(\mu)) \cong L_{p'}^r(\mu) \overset{h_p}{\otimes} L_p^c(\mu)$.*

Proof. This follows immediately from Propositions 2.1.6 and 2.2.4. □

Corollary 2.2.6. *Given a p-operator space V and measures μ and ν, we have the p-completely isometric isometry*

$$\mathcal{CB}_p(V, \mathcal{B}(L_p(\mu), L_p(\nu))) = (L_{p'}^r(\nu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_p^c(\mu))'.$$

Proof. Since p-Haagerup tensor product is associative and p-projective tensor product is commutative, by Proposition 2.2.4,

$$\begin{aligned} L_{p'}^r(\nu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_p^c(\mu) &= L_{p'}^r(\nu) \overset{\wedge_p}{\otimes} V \overset{\wedge_p}{\otimes} L_p^c(\mu) \\ &= V \overset{\wedge_p}{\otimes} L_p^c(\mu) \overset{\wedge_p}{\otimes} L_{p'}^r(\nu). \end{aligned}$$

Therefore, by Proposition 2.1.3, Corollary 1.6.12, and Proposition 1.6.11,

$$\begin{aligned} (L_{p'}^r(\nu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_p^c(\mu))' &= (V \overset{\wedge_p}{\otimes} L_p^c(\mu) \overset{\wedge_p}{\otimes} L_{p'}^r(\nu))' \\ &= \mathcal{CB}_p(V, (L_p^c(\mu) \overset{\wedge_p}{\otimes} L_{p'}^r(\nu))') \\ &= \mathcal{CB}_p(V, \mathcal{CB}_p(L_p^c(\mu), (L_{p'}^r(\nu))')) \\ &= \mathcal{CB}_p(V, \mathcal{CB}_p(L_p^c(\mu), L_p^c(\nu))) \end{aligned}$$

$$= \mathcal{CB}_p(V, \mathcal{B}(L_p(\mu), L_p(\nu))).$$

$\square$

If we replace L_p spaces by SQ_p spaces in Corollary 2.2.6, then we have a slightly weaker result.

Lemma 2.2.7. *If X and Y are SQ_p spaces, then we have the isometric isomorphism*

$$\mathcal{B}(X,Y) \cong ((Y')^r \overset{h_p}{\otimes} X^c)'.$$

Proof. Define

$$\Phi : \mathcal{B}(X,Y) \to ((Y')^r \overset{h_p}{\otimes} X^c)' \quad \text{by} \quad \Phi(u)(y' \otimes x) = \langle u(x), y' \rangle, \quad u \in \mathcal{B}(X,Y),\ x \in X,\ y' \in Y'$$

and

$$\Psi : ((Y')^r \overset{h_p}{\otimes} X^c)' \to \mathcal{B}(X,Y) \quad \text{by} \quad \langle \Psi(f)(x), y' \rangle = f(y' \otimes x), \quad f \in ((Y')^r \overset{h_p}{\otimes} X^c)',\ x \in X,\ y' \in Y'.$$ [4]

Then it is easy to show that Ψ and Φ are inverses of each other. For all $u \in \mathcal{B}(X,Y)$,

$$\begin{aligned} \|u\| &= \sup\{|\langle u(x), y'\rangle| : \|x\| \le 1, \|y'\| \le 1, x \in X, y' \in Y'\} \\ &= \sup\{|\Phi(u)(y' \otimes x)| : \|x\| \le 1, \|y'\| \le 1, x \in X, y' \in Y'\} \\ &\le \|\Phi(u)\|, \end{aligned}$$

where the last inequality comes from the fact that p-Haagerup tensor product is a subcross norm. Now suppose $z \in (Y')^r \otimes X^c$ with $\|z\|_{h_p} \le 1$. Let $\epsilon > 0$, then by definition of the p-Haagerup tensor product, there exist $y'_1, \cdots, y'_n \in Y'$ and $x_1, \cdots, x_n \in X$ such that $z = \sum_{i=1}^n y'_i \otimes x_i$ and

$$\|[y'_1 \cdots y'_n]\| \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\| = \left(\sum_{i=1}^n \|y'_i\|^{p'} \right)^{1/p'} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} < 1 + \epsilon.$$

[4]Recall that every SQ_p space is reflexive. See Remark 1.2.4.

By Hölder's inequality,

$$|\Phi(u)(z)| = \left|\sum_{i=1}^{n}\langle u(x_i), y_i'\rangle\right| \leq \|u\| \left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p} \left(\sum_{i=1}^{n}\|y_i'\|^{p'}\right)^{1/p'} < \|u\|(1+\epsilon)$$

and this shows that $\|\Phi(u)\| \leq \|u\|$. Therefore Φ is an isometric isomorphism. □

Remark 2.2.8. Lemma 2.2.7 is enough to show that the p-Haagerup tensor product is *not* injective for $p \neq 2$, in the sense of Remark 2.3.5.[5] Motivated by [LeM96, Remark 6.2], let E be a subspace of $L_p(\mu)$ which is not an L_p space.[6] Consider the inclusion mapping $J : (E')^r \overset{h_p}{\otimes} E^c \to (E')^r \overset{h_p}{\otimes} L_p^c(\mu)$. We claim that J is not even isometric. Indeed, if J were isometric, the adjoint $J' : \mathcal{B}(L_p(\mu), E) \to \mathcal{B}(E, E)$[7] would be a quotient map given by the restriction. In particular, id_E must extend to a map from $L_p(\mu)$ onto E and this would imply that E is a 1-complemented subspace of $L_p(\mu)$. This is equivalent to saying that E is an L_p space, which is a contradiction.

The fact that the p-Haagerup tensor product is not injective can be used to show that there is *no* p-analogue of polar decomposition. To make it precise, suppose $\beta \in \mathbb{M}_{r,n}$ with $r > n$. Regard β as an operator from ℓ_2^n to ℓ_2^r. If β has full rank, then we can always write $\beta = \tau\beta_0$, where $\tau \in \mathcal{B}(\ell_2^n, \ell_2^r)$ is an isometry and $\beta_0 \in \mathcal{B}(\ell_2^n, \ell_2^n)$ satisfies $\|\beta\| = \|\beta_0\|$.[8] It leads us to the following question.

Question 2.2.9. Do we have a similar decomposition if $p \neq 2$? That is to say, if $p \neq 2$, can we always write a full rank matrix $\beta \in \mathbb{M}_{r,n}$ as $\beta = \tau\beta_0$ where $\tau \in \mathcal{B}(\ell_p^n, \ell_p^r)$ is an isometry and $\beta_0 \in \mathcal{B}(\ell_p^n, \ell_p^n)$ satisfies $\|\beta\|_{\mathcal{B}(\ell_p^n, \ell_p^r)} = \|\beta_0\|_{\mathcal{B}(\ell_p^n, \ell_p^n)}$?

If the answer to Question 2.2.9 were yes, then the same argument as in [ER00, Lemma 9.2.3 and Proposition 9.2.5] could be used to show that p-Haagerup tensor product is injective. Since p-Haagerup tensor product is not injective, we conclude that the answer to Question 2.2.9 is *no*.

2.3 p-Injective Tensor Product

Definition 2.3.1. Let X, Y be p-operator spaces. Regarding the algebraic tensor product $X \otimes Y$ as a subspace of $\mathcal{CB}_p(X', Y)$, we define the *p-operator space injective tensor product* $X \overset{\vee_p}{\otimes} Y$ to be the completion of $X \otimes Y$ in $\mathcal{CB}_p(X', Y)$.

[5] The Haagerup tensor product for operator spaces is injective [ER00].

[6] ℓ_2^n can be regarded as a subspace of $L_p(\mathbb{C}^n, dx)$. See [DF93, Proposition 8.7]. Note that a Hilbert space cannot be isometric to any L_p space, $p \neq 2$, since L_p spaces do not have the parallelogram law unless $p = 2$.

[7] Note that E is reflexive, see Remark 1.2.4.

[8] One can take $\beta_0 = |\beta|$ using polar decomposition. See the proof of [ER00, Lemma 9.2.3].

Remark 2.3.2.

1. For each $u \in \mathbb{M}_n(X \otimes Y)$, we have $\|u\|_{\vee_p} \leq \|u\|_{h_p}$.[9] To see this, suppose $u = v \odot w$ with $v \in M_{n,r}(X)$ and $w \in M_{r,n}(Y)$. It follows that

$$\begin{aligned} \|u\|_{\vee_p} &= \sup\left\{ \left\| \left[\sum_{k=1}^{r} \varphi_{st}(v_{ik}) w_{kj} \right] \right\| : m \in \mathbb{N}, \varphi = [\varphi_{st}] \in M_m(X')_1 \right\} \\ &\leq \sup\left\{ \|\langle\langle \varphi, v \rangle\rangle\| \, \|w\| : m \in \mathbb{N}, \varphi = [\varphi_{st}] \in M_m(X')_1 \right\} \\ &\leq \|v\| \|w\|. \end{aligned}$$

   Taking infimum over v and w, we get the desired inequality.

2. By definition of the Banach space injective tensor product $\overset{\epsilon}{\otimes}$,[10] we have

$$\|u\|_\epsilon = \|u\|_{\mathcal{B}(X',Y)} \leq \|u\|_{\mathcal{CB}_p(X',Y)} = \|u\|_{\vee_p}$$

   for every $u \in X \otimes Y$.

3. If $Y \subseteq \mathcal{B}(L_p(\nu))$, then the p-operator space injective matrix norm $\|\cdot\|_{\vee_p}$ on $X \otimes Y$ satisfies

$$\|u\|_{\vee_p} = \sup\{ \|(\varphi \otimes \psi)_n(u)\| : m, k \in \mathbb{N}, \varphi \in M_m(X')_1, \psi \in M_k(Y')_1 \}$$

   for each matrix $u \in \mathbb{M}_n(X \otimes Y)$.

4. If $X \subseteq \mathcal{B}(L_p(\mu))$ as well, then $X \overset{\vee_p}{\otimes} Y = Y \overset{\vee_p}{\otimes} X$ p-completely isometrically.

Now we are ready to compare various tensor norms on $X \otimes Y$.

Proposition 2.3.3. *If X and Y are p-operator spaces, then the various tensor norms $X \otimes Y$ are ordered as follows:*

$$\|\cdot\|_\epsilon \leq \|\cdot\|_{\vee_p} \leq \|\cdot\|_{h_p} \leq \|\cdot\|_{\wedge_p} \leq \|\cdot\|_\pi.$$

Proof. Combine Remarks 2.1.2, 2.2.2, and 2.3.2. ☐

Proposition 2.3.4. *If $X \subseteq \mathcal{B}(L_p(\mu))$, then we have p-complete isometric isomorphisms*

$$M_n(X) = M_n \overset{\vee_p}{\otimes} X = \mathcal{CB}_p^\sigma(X', M_n).$$

[9] In particular, $\|u\|_{\vee_p}$ is a subcross norm.

[10] For Banach spaces E and F, the norm of $x \in E \otimes F$ is given by regarding x as a member of $\mathcal{B}(E', F)$ [Rya02, §3.1].

Proof. For each $m \in \mathbb{N}$, we have

$$M_m(M_n(X)) \subseteq M_m(M_n(X'')) = \mathcal{CB}_p(X', M_{mn}).$$

By definition, $M_m(M_n \overset{\vee_p}{\otimes} X) \subseteq M_m(\mathcal{CB}_p(X', M_n)) = \mathcal{CB}_p(X', M_{mn})$ and the first identification follows. For the second identification, simple calculation shows that $M_n(X) \subseteq \mathcal{CB}_p^\sigma(X', M_n)$. The other inclusion comes from applying [Con90, Theorem V.1.2] to each component of $T = [T_{ij}] \in \mathcal{CB}_p^\sigma(X', M_n)$. □

Remark 2.3.5. At this moment, we do not know whether the p-operator space injective tensor product is injective that is, if $u : X \to \tilde{X}$ and $v : Y \to \tilde{Y}$ are p-completely isometric injections, then we do not know whether $u \otimes v$ always extend to a p-completely isometric injection $u \otimes v : X \overset{\vee_p}{\otimes} Y \to \tilde{X} \overset{\vee_p}{\otimes} \tilde{Y}$. But if we assume that all the p-operator spaces under consideration are on L_p space, then we can show that $u \otimes v : X \overset{\vee_p}{\otimes} Y \to \tilde{X} \overset{\vee_p}{\otimes} \tilde{Y}$ is a p-complete isometry as in the following proposition. This fact supports that the terminology p-injective tensor product is still reasonable.

Proposition 2.3.6. *For $i = 1, 2$, suppose $X_i \subseteq Y_i \subseteq \mathcal{B}(L_p(\mu_i))$. Then*

$$X_1 \overset{\vee_p}{\otimes} X_2 \subseteq Y_1 \overset{\vee_p}{\otimes} Y_2$$

p-completely isometrically.

Proof. For $i = 1, 2$, let $\varphi_i : X_i \hookrightarrow Y_i$ denote the (p-completely isometric) inclusion. Since $\varphi_1 \otimes \varphi_2 = (\varphi_1 \otimes id_{Y_2}) \circ (id_{X_1} \otimes \varphi_2)$, by Remark 2.3.2 above, it suffices to show that

$$id_{X_1} \otimes \varphi_2 : X_1 \overset{\vee_p}{\otimes} X_2 \to X_1 \overset{\vee_p}{\otimes} Y_2$$

is p-completely isometric. Note that the following diagram commutes:

$$\begin{tikzcd}
X_1 \overset{\vee_p}{\otimes} X_2 \arrow[r, "id_{X_1} \otimes \varphi_2"] \arrow[d, hook] & X_1 \overset{\vee_p}{\otimes} Y_2 \arrow[d, hook] \\
\mathcal{CB}_p(X_1', X_2) \arrow[r, hook] & \mathcal{CB}_p(X_1', Y_2)
\end{tikzcd}$$

Since $X_1 \overset{\vee_p}{\otimes} X_2 \subseteq \mathcal{CB}_p(X_1', X_2)$, $X_1 \overset{\vee_p}{\otimes} Y_2 \subseteq \mathcal{CB}_p(X_1', Y_2)$, and $\mathcal{CB}_p(X_1', X_2) \subseteq \mathcal{CB}_p(X_1', Y_2)$ p-completely isometrically, we conclude that $id_{X_1} \otimes \varphi_2$ is p-completely isometric. □

Theorem 2.3.7. *If $X \subseteq \mathcal{B}(L_p(\nu))$, then there are p-complete isometries*

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} X \hookrightarrow \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu)),$$

$$\mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} X \hookrightarrow \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu)).$$

In particular, if $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$, then $V \overset{\vee_p}{\otimes} W \hookrightarrow \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu))$ p-completely isometrically.

Proof. By Proposition 1.6.8, we have a p-complete isometry $\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} X \hookrightarrow \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), X)$. By [Daw10, Proposition 5.3], we have the isometry $\mathcal{N}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}(L_p(\nu)) = \mathcal{N}(L_p(\mu) \otimes_p L_p(\nu))$ and hence we get

$$\mathcal{CB}_p(\mathcal{N}(L_p(\mu)), \mathcal{B}(L_p(\nu))) = (\mathcal{N}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}(L_p(\nu)))' = (\mathcal{N}(L_p(\mu) \otimes_p L_p(\nu)))' = \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu)) \tag{2.3}$$

isometrically. Therefore,

$$\begin{aligned}
M_n((\mathcal{N}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}(L_p(\nu)))') &= \mathcal{CB}_p(\mathcal{N}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}(L_p(\nu)), M_n) \\
&= \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), \mathcal{CB}_p(\mathcal{N}(L_p(\nu)), M_n)) \\
&= \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), M_n(\mathcal{B}(L_p(\nu)))) \\
&= \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), \mathcal{B}(\ell_p^n \otimes_p L_p(\nu))) \\
\text{(by (2.3))} &= \mathcal{B}(L_p(\mu) \otimes_p \ell_p^n \otimes_p L_p(\nu)) \\
&= M_n(\mathcal{B}(L_p(\mu) \otimes_p L_p(\nu)))
\end{aligned}$$

and this means that we have the p-complete isometry

$$(\mathcal{N}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}(L_p(\nu)))' \cong \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu)). \tag{2.4}$$

Therefore, (2.3) is in fact a p-complete isometry and this in particular shows that

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} X \subseteq \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), X) \subseteq \mathcal{CB}_p(\mathcal{N}(L_p(\mu)), \mathcal{B}(L_p(\nu))) = \mathcal{B}(L_p(\mu) \otimes_p L_p(\nu))$$

p-completely isometrically.

For the second identification, note that we have the following p-completely isometric inclusions

$$\mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} X \subseteq \mathcal{CB}_p(\mathcal{B}(L_p(\mu))', X) \subseteq \mathcal{CB}_p(\mathcal{B}(L_p(\mu))', \mathcal{B}(L_p(\nu))). \tag{2.5}$$

By Proposition 2.1.7 and [Daw10, Proposition 5.4], we get isometric isomorphisms

$$M_n(\mathcal{B}(L_p(\mu))') = \mathcal{CB}_p(\mathcal{B}(L_p(\mu)), M_n) = (\mathcal{B}(L_p(\mu)) \overset{\wedge_p}{\otimes} \mathcal{N}_n)' = (\mathcal{N}_n(\mathcal{B}(L_p(\mu))))' = (M_n(\mathcal{N}(L_p(\mu))))'' \tag{2.6}$$

and hence the closed unit ball of $M_n(\mathcal{N}(L_p(\mu)))$ is weak*-dense in the closed unit ball of $M_n(\mathcal{B}(L_p(\mu))')$. To be more precise, by Lemma 1.6.7, (2.6) means that for all $\psi = [\psi_{ij}] \in M_n(\mathcal{B}(L_p(\mu))')_1$, there is a net $\psi^\gamma = [\psi_{ij}^\gamma] \in M_n(\mathcal{N}(L_p(\mu)))_1$ such that for all $T = [T_{ij}] \in \mathcal{N}_n(\mathcal{B}(L_p(\mu))) = (M_n(\mathcal{N}(L_p(\mu))))'$,[11]

$$\sum_{i,j} \langle \psi_{ij}^\gamma, T_{ij} \rangle \to \sum_{i,j} \langle \psi_{ij}, T_{ij} \rangle. \tag{2.7}$$

Let $T_0 \in \mathcal{B}(L_p(\mu))$ and let $\delta > 0$. By considering $T = \epsilon_{ij} \otimes T_0 \in \mathbb{M}_n(\mathcal{B}(L_p(\mu)))$, (2.7) in particular yields that for each i, j, there is γ_{ij} such that

$$|\langle \psi_{ij}^\gamma - \psi_{ij}, T_0 \rangle| < \delta \quad \text{ for all } \gamma \succcurlyeq \gamma_{ij}. \tag{2.8}$$

Consider the following diagram

$$\begin{tikzcd}
M_n(\mathcal{N}(L_p(\mu))) \arrow[r] \arrow[d, equal] & M_n(\mathcal{B}(L_p(\mu))') \arrow[d, equal] \\
\mathcal{CB}_p^\sigma(\mathcal{B}(L_p(\mu)), M_n) \arrow[r, hook] & \mathcal{CB}_p(\mathcal{B}(L_p(\mu)), M_n),
\end{tikzcd}$$

where the first column comes from Proposition 2.3.4. (2.7) and (2.8) mean that $\psi^\gamma \in \mathcal{CB}_p^\sigma(\mathcal{B}(L_p(\mu)), M_n)_1$ converges to ψ in the point-norm topology because if $T_0 \in \mathcal{B}(L_p(\mu))$, then

$$\|\psi^\gamma(T_0) - \psi(T_0)\| \leq \sum_{i,j} |\langle \psi_{ij}^\gamma - \psi_{ij}, T_0 \rangle| \leq n^2 \delta$$

for γ large enough. Therefore, using the same argument as in [ER00, Proposition 8.1.2], we can replace $\mathcal{CB}_p(\mathcal{B}(L_p(\mu))', \mathcal{B}(L_p(\nu)))$ in (2.5) by $\mathcal{CB}_p(\mathcal{N}(L_p(\mu)), \mathcal{B}(L_p(\nu)))$ and the result follows from the previous case. The remaining part follows from Proposition 2.3.6. $\square$

[11] See Definition 1.6.6 for the definition of $\mathcal{N}_n(\mathcal{B}(L_p(\mu)))$.

Remark 2.3.8. In fact, in Theorem 2.3.7, one can show that $\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} X \hookrightarrow \mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} X$ using Proposition 2.3.6. We gave a different proof to exhibit that (2.4) is a p-complete isometry.

The following result provides another reason why the terminology p-injective tensor product is reasonable.

Theorem 2.3.9. *Let V and W be p-operator spaces on L_p spaces. Then there exist two index sets I and J such that we can identify V and W with p-operator subspaces of $\mathcal{B}(\ell_p(I))$ and $\mathcal{B}(\ell_p(J))$, respectively, and the canonical inclusion*

$$V \overset{\vee_p}{\otimes} W \hookrightarrow \mathcal{B}\left(\ell_p(I) \otimes_p \ell_p(J)\right)$$

is a p-completely isometric injection.

Proof. By assumption, $V \subseteq V''$ (respectively, $W \subseteq W''$) p-completely isometrically. By Proposition 1.6.2, there is an index set I (respectively, J) such that $V'' \subseteq \mathcal{B}(\ell_p(I))$ (respectively, $W'' \subseteq \mathcal{B}(\ell_p(J))$). We can conclude from Proposition 2.3.6 and Theorem 2.3.7 that the canonical inclusions

$$V \overset{\vee_p}{\otimes} W \hookrightarrow V'' \overset{\vee_p}{\otimes} W'' \hookrightarrow \mathcal{B}(\ell_p(I)) \overset{\vee_p}{\otimes} \mathcal{B}(\ell_p(I)) \hookrightarrow \mathcal{B}\left(\ell_p(I) \otimes_p \ell_p(J)\right)$$

are p-completely isometric injections.

□

Our next result is a p-operator space injective tensor product counterpart of Proposition 2.2.4.

Proposition 2.3.10. *Let μ and ν be measures and let $V \subseteq \mathcal{B}(L_p(\nu))$. Then we have the p-complete isometries*

$$L_p^c(\mu) \overset{h_p}{\otimes} V = L_p^c(\mu) \overset{\vee_p}{\otimes} V, \qquad V \overset{h_p}{\otimes} L_{p'}^r(\mu) = V \overset{\vee_p}{\otimes} L_{p'}^r(\mu).$$

Proof. Let us only give the sketch of the proof for the column space $L_p^c(\mu)$. The proof for the row space $L_{p'}^r(\mu)$ is similar. Let us first assume that $L_p(\mu) = \ell_p^n$. Then we have the p-completely isometric inclusion

$$(\ell_p^n)^c \overset{\vee_p}{\otimes} V = V \overset{\vee_p}{\otimes} (\ell_p^n)^c \hookrightarrow \mathcal{CB}_p(V', (\ell_p^n)^c) = M_{n,1}(V'').$$

It follows that every $v \in M_m((\ell_p^n)^c \overset{\vee_p}{\otimes} V) = M_{mn,m}(V)$ can be expressed by $v = I_{mn}v = I_{mn} \odot v$. Since we can regard I_{mn} as a contractive element in $M_{m,mn}((\ell_p^n)^c) = M_{mn}$, we get $\|v\|_{h_p} \leq \|v\|_{\vee_p}$. The general case follows by applying the rigid $\mathcal{L}_p$-structure of $L_p(\mu)$ [LP68]. □

Theorem 2.3.11. *Let V be a p-operator space on L_p space. For any measure μ, we have the p-complete isometries*

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} V = L_p^c(\mu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_{p'}^r(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_{p'}^r(\mu).$$

In particular, we have

$$\mathcal{K}(L_p(\mu)) = L_p^c(\mu) \overset{h_p}{\otimes} L_{p'}^r(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} L_{p'}^r(\mu).$$

Proof. If $L_p(\mu) = \ell_p^n$, we have $\mathcal{K}(\ell_p^n) = \mathcal{B}(\ell_p^n) = M_n$. In this case, we obtain

$$\mathcal{B}(\ell_p^n) \overset{\vee_p}{\otimes} V = M_n(V) = (\ell_p^n)^c \overset{h_p}{\otimes} V \overset{h_p}{\otimes} (\ell_{p'}^n)^r = (\ell_p^n)^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (\ell_{p'}^n)^r$$

by [LeM96, Proposition 6.3] and Proposition 2.3.10. For general case, we need to apply the rigid $\mathcal{L}_p$-structure of $L_p(\mu)$, i.e. we need to consider an increasing net $\{F_\alpha\}$ of finite dimensional subspaces in $L_p(\mu)$ such that each F_α is isometric to some $\ell_p^{d(\alpha)}$ and the union $\bigcup_\alpha F_\alpha$ is norm-dense in $L_p(\mu)$. For each α, we may identify the dual space F'_α with a subspace of $L_{p'}(\mu)$ and F'_α is isometric to $\ell_{p'}^{d(\alpha)}$. In this case, the p-operator spaces F_α^c, $(F'_\alpha)^r$, and $\mathcal{B}(F_\alpha)$ are 1-complemented subspaces of $L_p^c(\mu)$, $L_{p'}^r(\mu)$, and $\mathcal{K}(L_p(\mu))$, respectively. Then we obtain an increasing net of p-operator spaces

$$\mathcal{B}(F_\alpha) \overset{\vee_p}{\otimes} V = F_\alpha^c \overset{h_p}{\otimes} V \overset{h_p}{\otimes} (F'_\alpha)^r = F_\alpha^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (F'_\alpha)^r.$$

Since the unions $\bigcup_\alpha \mathcal{B}(F_\alpha)$, $\bigcup_\alpha F_\alpha^c \overset{h_p}{\otimes} V \overset{h_p}{\otimes} (F'_\alpha)^r$, and $\bigcup_\alpha F_\alpha^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (F'_\alpha)^r$ are norm-dense in $\mathcal{K}(L_p(\mu))$, $L_p^c(\mu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_{p'}^r(\mu)$, and $L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_{p'}^r(\mu)$, respectively, thanks to Proposition 2.3.6, we obtain the desired p-complete isometries

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} V = L_p^c(\mu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_{p'}^r(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_{p'}^r(\mu).$$

The second part is immediate by taking $V = \mathbb{C}$. $\square$

Now let us discuss the duality property between the p-operator space injective tensor product and p-operator space projective tensor product.[12]

Theorem 2.3.12. *Let V be a p-operator space on L_p space. For each $n \in \mathbb{N}$, we have the isometric isomorphism*

$$(M_n \overset{\vee_p}{\otimes} V)' = \mathcal{N}_n \overset{\wedge_p}{\otimes} V'.$$

[12]See also Lemma 5.1.1.

Proof. Let us first assume that u is a contractive linear functional contained in $(M_n \overset{\vee_p}{\otimes} \mathcal{B}(E))' = \mathcal{B}(E^n)'$ with $E = L_p(\mu)$ and $\|u\| = 1$. Since $\|u\|_{pcb} = \|u\| = 1$ (Lemma 1.4.3), u is actually a p-completely contractive linear functional from $\mathcal{B}(E^n)$ into $\mathbb{C}$. We can apply Pisier's representation theorem [Pis90, Theorem 2.1(c)] to obtain an L_p space, a unital p-completely contractive homomorphism $\tilde{\pi}\mathcal{B}(E^n) \to \mathcal{B}(L_p)$ and contractive vectors $\eta \in L_p$, $\xi \in L_{p'}$ such that

$$u([a_{ij}]) = \langle \xi, \tilde{\pi}([a_{ij}])\eta \rangle.$$

Now using the submatrix system $\{e_{ij} \otimes 1\}$ in $M_n \overset{\vee_p}{\otimes} \mathcal{B}(E) = \mathcal{B}(E^n)$, we can split the range space L_p into the following n-copies of ℓ_p-direct sum

$$L_p = \tilde{\pi}(e_{11} \otimes 1)L_p \oplus_p \cdots \oplus_p \tilde{\pi}(e_{nn} \otimes 1)L_p.$$

It is easy to see that $\tilde{\pi}(e_{11} \otimes 1)L_p$ is a closed and contractively complemented subspace of L_p and thus is an L_p space, which we denote by $L_p(\nu)$. All other spaces $\tilde{\pi}(e_{ii} \otimes 1)L_p (1 \le i \le n)$ are isometrically isomorphic to $L_p(\nu)$ via $\{\tilde{\pi}(e_{ij} \otimes 1)\}$. Therefore (up to an isometric isomorphism), we can obtain a unital p-completely contractive homomorphism $\pi : \mathcal{B}(E) \to \mathcal{B}(L_p(\nu))$ and contractive vectors $[\xi_i] \in L_{p'}(\nu)^n$ and $[\eta_j] \in L_p(\nu)^n$ such that

$$u([a_{ij}]) = \langle [\xi_i], (id_{M_n} \otimes \pi)([a_{ij}])[\eta_j] \rangle = \langle [\xi_i], [\pi(a_{ij})][\eta_j] \rangle = \sum_{i,j=1}^{n} \langle \xi_i, \pi(a_{ij})\eta_j \rangle.$$

Now let us assume that the vectors $\eta_1, \dots, \eta_n$ are contained in a finite dimensional subspace $F \subseteq L_p(\nu)$ such that F is isometric to ℓ_p^k. If we let $f_1, \dots, f_k$ correspond to the canonical basis of ℓ_p^k, we can express each η_j as $\eta_j = \sum_{v=1}^{k} \beta_{j,v} f_v$ with $\sum_{j,v} |\beta_{j,v}|^p \le 1$. Similarly, we may assume that the vectors $\xi_1, \dots, \xi_n$ are contained in a finite dimensional subspace $G \subseteq L_{p'}(\nu)$ such that G is isometric to $\ell_{p'}^l$. In this case, we can express ξ_i as $\xi_i = \sum_{w=1}^{l} \alpha_{i,w} g_w$ with $\sum_{i,w} |\alpha_{i,w}|^{p'} \le 1$. Let ι_F and ι_G be the embedding of F and G into $L_p(\nu)$ and $L_{p'}(\nu)$, respectively. Then the map

$$\Phi = [\Phi_{w,v}] : a \in \mathcal{B}(E) \to (\iota_G)'\pi(a)\iota_F \in (\iota_G)'\mathcal{B}(L_p(\nu))\iota_F \cong M_{l,k}$$

defines a contractive element in $M_{l,k}(\mathcal{B}(E)') = \mathcal{CB}_p(\mathcal{B}(E), M_{l,k})$ such that

$$u = \langle [\xi_i], (id_{M_n} \otimes \pi)[\eta_j] \rangle = [\alpha_{i,w}](id_{M_n} \otimes \Phi)[\beta_{j,v}].$$

This shows that u corresponds to a contractive element in $\mathcal{N}_n \overset{\wedge_p}{\otimes} \mathcal{B}(E)'$.

In general, the vectors $\{\eta_j\}_{j=1}^n$ (respectively, $\{\xi_i\}_{i=1}^n$) can be approximated by vectors in some sufficiently large finite dimensional subspaces $F \subseteq L_p(\nu)$ such that $F \cong \ell_p^k$ (respectively, $G \subseteq L_{p'}(\nu)$ such that $G \cong \ell_{p'}^l$). It follows that u is a limit of contractive elements in $\mathcal{N}_n \overset{\wedge_p}{\otimes} \mathcal{B}(E)'$ and thus u itself is contractive in $\mathcal{N}_n \overset{\wedge_p}{\otimes} \mathcal{B}(E)'$. Therefore we have the isometric isomorphism

$$(M_n \overset{\vee_p}{\otimes} \mathcal{B}(E))' = \mathcal{N}_n \overset{\wedge_p}{\otimes} \mathcal{B}(E)'.$$

Now for general p-operator space $V \subseteq \mathcal{B}(L_p(\mu))$, if u is a contractive linear functional in $(M_n \overset{\vee_p}{\otimes} V)'$, we may extend u to a contractive linear functional $\tilde{u} \in (M_n \overset{\vee_p}{\otimes} \mathcal{B}(L_p(\mu)))'$. From the above discussion, we can find a p-complete contraction $\Phi : \mathcal{B}(L_p(\mu)) \to M_{l,k}$ and contractive $[\alpha_{i,w}]$ and $[\beta_{j,v}]$ such that $\tilde{u} = [\alpha_{i,w}](id_{M_n} \otimes \Phi)[\beta_{j,v}]$ is contractive in $\mathcal{N}_n \overset{\wedge_p}{\otimes} \mathcal{B}(E)'$. Then $\Phi|V$ is a p-complete contraction from V into $M_{l,k}$ and thus $u = [\alpha_{i,w}](id_{M_n} \otimes \Phi|_V)[\beta_{j,v}]$ is contractive in $\mathcal{N}_n \overset{\wedge_p}{\otimes} V'$. This completes the proof. □

Corollary 2.3.13. *Let V and W be p-operator spaces on L_p space. Then the canonical inclusion*

$$V' \overset{\vee_p}{\otimes} W \hookrightarrow \mathcal{CB}_p(V, W)$$

is a p-completely isometric injection.

Proof. It is known from Proposition 2.3.4 and Theorem 2.3.12 that for each $n \in \mathbb{N}$, we have the isometric isomorphisms

$$M_n(V'') = \mathcal{CB}_p(V', M_n) = (\mathcal{N}_n \overset{\wedge_p}{\otimes} V')' = M_n(V)''.$$

Therefore we can replace V'' by V in the definition $V' \overset{\vee_p}{\otimes} W \hookrightarrow \mathcal{CB}_p(V'', W)$ of the p-operator space injective tensor product as in the proof of the second half of Theorem 2.3.7. □

2.4 Infinite Matrices

As in operator spaces, we can develop the theory of infinite matrices for p-operator spaces. Most of the ideas in this section comes from those of [ER00, Chapter 11]. For the convenience of the reader, we briefly introduce some p-operator space analogues. Suppose that V is a p-operator space. We denote by $\mathbb{M}_\infty(V)$ the linear space of all infinite matrices $[v_{ij}]$ with $v_{ij} \in V$. For $1 \leq r, s < \infty$, we identify matrix spaces $\mathbb{M}_{r,s}(V)$ and $\mathbb{M}_r(V)$ in the obvious manner, and we let v^r denote the truncation of $v \in \mathbb{M}_\infty(V)$ to $\mathbb{M}_r(V)$. If $r \leq s$, then

$$\|v^r\| = \|(I_r \oplus 0_{s-r})v^s(I_r \oplus 0_{s-r})\| \leq \|v^s\|.$$

If $v \in \mathbb{M}_\infty(V)$, then we define

$$\|v\| = \sup\{\|v^r\| : r \in \mathbb{N}\} = \lim_{r\to\infty} \|v^r\|,$$

and we define $M_\infty(V)$ to be the space of all $v \in \mathbb{M}_\infty(V)$ for which $\|v\| < \infty$.
For any $m \in \mathbb{N}$, we identify $\mathbb{M}_m(M_\infty(V))$ with $M_{m\times\infty}(V)$. Let $M_m(M_\infty(V))$ denote $\mathbb{M}_m(M_\infty(V))$ with the corresponding norm.

Proposition 2.4.1. *If V is a p-operator space, then $M_\infty(V)$ is a p-operator space.*

Proof. This follows exactly as for operator spaces, as in [ER00, Proposition 10.1.1]. □

We define $\mathcal{K}_\infty(V)$ to be the closure of $\mathbb{M}_\infty^{\text{fin}}(V)$ in $M_\infty(V)$, where $\mathbb{M}_\infty^{\text{fin}}(V)$ denotes the linear space of matrices with only finitely many nonzero entries. Suppose that $V \subseteq \mathcal{B}(L_p(\mu))$, then the column mappings in the diagram

$$\begin{array}{ccc} M_r(V) & \cong & M_r \overset{\vee_p}{\otimes} V \\ \big\downarrow & & \big\downarrow \\ \mathcal{K}_\infty(V) & & \mathcal{K}_\infty \overset{\vee_p}{\otimes} V \end{array}$$

are isometric, and in each case the union of their ranges is dense. It follows that we have the isometry

$$\mathcal{K}_\infty(V) \cong \mathcal{K}_\infty \overset{\vee_p}{\otimes} V.$$

One can also consider the spaces $\mathbb{M}_{n,\infty}(V)$ and $\mathbb{M}_{\infty,n}(V)$. If $v \in \mathbb{M}_{n,\infty}(V)$, then we interpret the truncation v^r as an element of $\mathbb{M}_r(V)$ for $r \le n$, an as an element of $\mathbb{M}_{n,r}(V)$ for $r \ge n$, and we use a similar convention for $\mathbb{M}_{\infty,n}(V)$. We define $M_{n,\infty}(V)$ and $M_{\infty,n}(V)$ just as for $M_\infty(V)$. In particular, we wish to study $M_{n,\infty}$ and $M_{\infty,n}$ more closely.

Lemma 2.4.2. *Let $\alpha = [\alpha_{ij}] \in \mathbb{M}_{n,\infty}$ and $\beta = [\beta_{kl}] \in \mathbb{M}_{\infty,n}$. Let $1 < p, p' < \infty$ with $1/p' + 1/p = 1$. Then we have*

$$\|\alpha\|_{\mathcal{B}(\ell_p,\ell_p^n)} \le \|\alpha\|_{p'} \cdot n^{|1/p-1/p'|} \quad \textit{and} \quad \|\beta\|_{\mathcal{B}(\ell_p^n,\ell_p)} \le \|\beta\|_p \cdot n^{|1/p-1/p'|},$$

where

$$\|\alpha\|_{p'} = \left(\sum_{i=1}^{n}\sum_{j=1}^{\infty} |\alpha_{ij}|^{p'}\right)^{1/p'} \quad \textit{and} \quad \|\beta\|_p = \left(\sum_{k=1}^{\infty}\sum_{l=1}^{n} |\beta_{kl}|^{p}\right)^{1/p}.$$

Proof. The proof is almost identical to that of Lemma 1.6.4. We insert the proof for convenience. Suppose $\xi = (\xi_j)$ is a unit vector in ℓ_p. For each i, $1 \le i \le n$, let $\eta_i = \left|\sum_{j=1}^\infty \alpha_{ij}\xi_j\right|$, then by Hölder's inequality,

$\eta_i \leq \left(\sum_{j=1}^{\infty} |\alpha_{ij}|^{p'}\right)^{1/p'}$ and by Lemma 1.6.3,

$$\left(\sum_{i=1}^{n} \eta_i^p\right)^{1/p} \leq n^{|1/p-1/p'|} \cdot \left(\sum_{i=1}^{n} \eta_i^{p'}\right)^{1/p'} \leq n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$$

and hence we get $\|\alpha\|_{\mathcal{B}(\ell_p,\ell_p^n)} \leq n^{|1/p-1/p'|} \cdot \|\alpha\|_{p'}$. To prove the second inequality, let $\gamma : \ell_{p'} \to \ell_{p'}^n$ be the adjoint operator of β. Then by the argument above we have

$$\|\gamma\|_{\mathcal{B}(\ell_{p'},\ell_{p'}^n)} \leq \|\gamma\|_p \cdot n^{|1/p-1/p'|}.$$

Since $\|\gamma\|_{\mathcal{B}(\ell_{p'},\ell_{p'}^n)} = \|\beta\|_{\mathcal{B}(\ell_p^n,\ell_p)}$ and $\|\gamma\|_p = \|\beta\|_p$, we get the desired inequality. □

For any matrices $\alpha \in M_{n,\infty}$ and $\beta \in M_{\infty,n}$, we have

$$\lim_{r\to\infty} \|\alpha - \alpha^r\| = \lim_{r\to\infty} \|\beta - \beta^r\| = 0. \tag{2.9}$$

To see the first equality of (2.9), by Lemma 2.4.2, it suffices to show that $\lim_{r\to\infty} \|\alpha - \alpha^r\|_{p'} = 0$, but this is a simple consequence of $\ell_{p'}$-convergence. Similar argument works for the second equality in (2.9), and we see that $M_{n,\infty} = K_{n,\infty}$ and $M_{\infty,n} = K_{\infty,n}$.

If $v \in M_\infty(V)$ and $\alpha \in \mathbb{M}_\infty^{\text{fin}}$, then we define $\alpha v \in \mathbb{M}_\infty(V)$ by

$$(\alpha v)_{i,j} = \sum_k \alpha_{i,k} v_{k,j}.$$

As in the 2-operator spaces, it turns out that indeed $\alpha v \in M_\infty(V)$. Given $\alpha \in \mathcal{K}_\infty$ and $r \leq s$,

$$\|\alpha^s v - \alpha^r v\| \leq \|\alpha^s - \alpha^r\| \|v\|,$$

and thus $\alpha^r v$ is Cauchy. We let

$$\alpha v = \lim_{r\to\infty} \alpha^r v. \tag{2.10}$$

Similarly, for $\beta \in \mathcal{K}_\infty$, let

$$v\beta = \lim_{r\to\infty} v\beta^r. \tag{2.11}$$

Lemma 2.4.3. *With operations in (2.10) and (2.11), $M_\infty(V)$ is $\mathcal{K}_\infty$-bimodule.*

Proof. We follow the argument in [ER00]. Let $\alpha, \beta \in \mathcal{K}_\infty$, and $v \in M_\infty(V)$. Firstly, we show that $(\alpha\beta)v = \alpha(\beta v)$. Let $E^r = I_r \oplus 0 \in \mathcal{K}_\infty$, then $\|\alpha(I - E^r)\| \to 0$ as $r \to \infty$ since this is evident for $\alpha \in M_{r_0}$, and the general case follows from the density argument. Therefore,

$$\|(\alpha\beta)^r - \alpha^r\beta^r\| = \|E^r\alpha(I - E^r)\beta E^r\| \to 0$$

and

$$(\alpha\beta)v = \lim_{r\to\infty}(\alpha^r\beta^r)v = \lim_{r\to\infty}\alpha^r(\beta^r v).$$

But we have

$$\begin{aligned}\|\alpha(\beta v) - \alpha^r(\beta^r v)\| &\leq \|\alpha(\beta - \beta^r)v\| + \|(\alpha - \alpha^r)\beta^r v\| \\ &\leq \|\alpha\|\|\beta - \beta^r\|\|v\| + \|\alpha - \alpha^r\|\|\beta\|\|v\| \to 0,\end{aligned}$$

and associativity follows. The same argument applies on the right side. Lastly, noting that $(\alpha^r v)\beta^r = \alpha^r(v\beta^r) = \alpha^r v\beta^r$, we show that

$$(\alpha v)\beta = \lim_{r\to\infty}\alpha^r v\beta^r = \alpha(v\beta). \tag{2.12}$$

Indeed,

$$\begin{aligned}\|(\alpha v)\beta - (\alpha^r v)\beta^r\| &\leq \|(\alpha v)\beta - (\alpha^r v)\beta\| + \|(\alpha^r v)\beta - (\alpha^r v)\beta^r\| \\ &= \|(\alpha v - \alpha^r v)\beta\| + \|\alpha^r v(\beta - \beta^r)\| \\ &\leq \|\alpha v - \alpha^r v\|\|\beta\| + \|\alpha^r v\|\|\beta - \beta^r\| \to 0\end{aligned}$$

as $r \to \infty$ and the first equality of (2.12) follows. Likewise, we can prove the second equality of (2.12) and this completes the proof. $\square$

The above argument applies, as well, to the space $M_{n,\infty}(V)$ and $M_{\infty,n}(V)$. If $\alpha \in M_{n,\infty} = K_{n,\infty}$, $v \in M_\infty(V)$, and $\beta \in M_{\infty,n} = K_{\infty,n}$, then we have a corresponding element

$$\alpha v\beta = \lim_{r\to\infty}\alpha^r v\beta^r \in M_n(V). \tag{2.13}$$

Following [ER00, Theorem 10.1.4], for any p-operator space V, we get p-completely isometric isomorphism

$$(\mathcal{N}_\infty \overset{\wedge_p}{\otimes} V)' \cong M_\infty(V'), \tag{2.14}$$

where $\mathcal{N}_\infty$ denotes the space of all nuclear operators on ℓ_p.

Now let us suppose that $v \in M_\infty(V)$, $w \in M_\infty(W)$, $\alpha \in M_{n,\infty^2}$, and $\beta \in M_{\infty^2,n}$. We have $v \otimes w \in$

$M_{\infty^2}(V \overset{\wedge_p}{\otimes} W)$ since

$$\|(v \otimes w)^{r\times s}\|_{\wedge_p} = \|v^r \otimes w^s\|_{\wedge_p} \le \|v^r\|\|w^s\| \le \|v\|\|w\|,$$

and thus from (2.13) we have a well-defined element $u = \alpha(v \otimes w)\beta \in M_n(V \overset{\wedge_p}{\otimes} W)$. Moreover, proceeding as for 2-operator spaces, we get

$$\|u\|_{\wedge_p} \le \|\alpha\|\|v\|\|w\|\|\beta\|. \tag{2.15}$$

Proposition 2.4.4. *Given p-operator spaces V and W, and $u \in M_n(V \overset{\wedge_p}{\otimes} W)$,*

$$\|u\|_{\wedge_p} = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v \otimes w)\beta\},$$

where the infimum is taken over all such representations with $v \in M_\infty(V), w \in M_\infty(W), \alpha \in M_{n,\infty^2}$, and $\beta \in M_{\infty^2,n}$. Furthermore, we may assume that $v \in \mathcal{K}_\infty(V)$ and $w \in \mathcal{K}_\infty(W)$.

Proof. By (2.15), it suffices to show that $\|u\|_{\wedge_p} \ge \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u = \alpha(v\otimes w)\beta\}$. If $u \in M_n(V \overset{\wedge_p}{\otimes} W)$ and $\|u\|_{\wedge_p} < 1$, then there exists a sequence $\{u_k\}$ of elements in $M_n(V \otimes W)$ such that

$$u = \sum_{k=1}^{\infty} u_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_k\|_{\wedge_p} < 1.$$

Let $0 < \epsilon < 1 - \sum_{k=1}^{\infty} \|u_k\|_{\wedge_p}$. For each $u_k \in M_n(V \otimes W)$ we can choose $v_k \in M_{p_k}(V), w_k \in M_{q_k}(W), \alpha_k \in M_{n,p_k\times q_k}$, and $\beta_k \in M_{p_k\times q_k,n}$ such that

$$u_k = \alpha_k(v_k \otimes w_k)\beta_k$$

and

$$\|\alpha_k\|\|v_k\|\|w_k\|\|\beta_k\| < \|u_k\|_{\wedge_p} + \frac{\epsilon}{2^k}.$$

Without loss of generality, we can suppose that $\|v_k\| = \|w_k\| = 1$ and

$$\|\alpha_k\|^{p'} = \|\beta_k\|^p < \|u_k\|_{\wedge_p} + \frac{\epsilon}{2^k}. \tag{2.16}$$

Let $q = \max\{p', p\}$. Since $\sum_k(\|u_k\|_{\wedge_p} + \frac{\epsilon}{2^k}) < 1$, we can find a sequence $\{c_k\}$ such that $c_k \ge 1$, $c_k \to \infty$, and yet we still have

$$\sum_{k=1}^{\infty} c_k^q(\|u_k\|_{\wedge_p} + \frac{\epsilon}{2^k}) < 1. \tag{2.17}$$

Let

$$v = \bigoplus_{k=1}^{\infty} c_k^{-1} v_k \quad \text{and} \quad w = \bigoplus_{k=1}^{\infty} c_k^{-1} w_k,$$

then $v \in \mathcal{K}_\infty(V)$ and $w \in \mathcal{K}_\infty(W)$ with $\|v\|, \|w\| \leq 1$. Let

$$\alpha = [c_1\alpha_1 \ 0_{12} \ 0_{13} \ \cdots \ 0_{21} \ c_2\alpha_2 \ \cdots \ 0_{23} \ \cdots],$$

where 0_{rs} denotes the n by $p_r \times q_s$ zero matrix. We claim that $\alpha \in M_{n,\infty^2}$ with $\|\alpha\| < 1$. Indeed, let $\xi = [\xi_{11} \ \xi_{12} \ \cdots \ \xi_{21} \ \xi_{22} \ \cdots \ ]^T$ be a unit column vector in ℓ_p, where ξ_{rs} is a $p_r \times q_s$-dimensional row vector. Then

$$\|\alpha\xi\| = \left\| \sum_k c_k\alpha_k\xi_{kk} \right\| \leq \sum_k \|c_k\alpha_k\xi_{kk}\| \leq \left(\sum_k \|c_k\alpha_k\|^{p'} \right)^{1/p'} \left(\sum_k \|\xi_{kk}\|^p \right)^{1/p} < 1$$

by (2.16) and (2.17). This shows that $\|\alpha\| < 1$. Let β be the transpose of

$$[c_1\beta_1^T \ 0_{12} \ 0_{13} \ \cdots \ 0_{21} \ c_2\beta_2^T \ \cdots \ 0_{23} \ \cdots],$$

Again, by (2.16) and (2.17), it follows easily that $\beta \in M_{\infty^2,n}$ with $\|\beta\| < 1$. Now the result follows since $u = \alpha(v \otimes w)\beta$. □

We can also represent elements in the p-operator spaces Haagerup tensor product in terms of infinite matrices. First of all, using an argument similar to that used right before Proposition 2.4.4, we get that if $v \in M_{n,\infty}(V)$, $x \in M_\infty(X)$, and $w \in M_{\infty,n}(W)$, then we have a well-defined element $u = v \odot x \odot w$ in $M_n(V \overset{h_p}{\otimes} X \overset{h_p}{\otimes} W)$ and moreover $\|u\|_{h_p} \leq \|v\|\|x\|\|w\|$.

Proposition 2.4.5. *Given p-operator spaces V, W, X and $u \in M_n(V \overset{h_p}{\otimes} X \overset{h_p}{\otimes} W)$,*

$$\|u\|_{h_p} = \inf\{\|v\|\|x\|\|w\| : u = v \odot x \odot w\},$$

where the infimum is taken over all such representations with $v \in M_{n,\infty}(V), x \in M_\infty(X)$, and $w \in M_{\infty,n}(W)$. Furthermore, we may assume that $x \in \mathcal{K}_\infty(X)$.

Proof. If $u \in M_n(V \overset{h_p}{\otimes} X \overset{h_p}{\otimes} W)$ and $\|u\|_{h_p} < 1$, then there exists a sequence $\{u_k\}$ of elements in $M_n(V \otimes_{h_p} X \otimes_{h_p} W)$ such that

$$u = \sum_{k=1}^{\infty} u_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_k\|_{h_p} < 1.$$

Let $0 < \epsilon < 1 - \sum_{k=1}^{\infty} \|u_k\|_{h_p}$. For each $u_k \in M_n(V \otimes_{h_p} X \otimes_{h_p} W)$ we can choose $v_k \in M_{n,p_k}(V), x_k \in$

$M_{p_k \times q_k}(X)$, and $w_k \in M_{q_k,n}$ such that

$$u_k = v_k \odot x_k \odot w_k$$

and

$$\|v_k\|\|x_k\|\|w_k\| < \|u_k\|_{h_p} + \frac{\epsilon}{2^k}.$$

Without loss of generality, we can suppose that $\|x_k\| = 1$ and

$$\|v_k\|^{p'} = \|w_k\|^p < \|u_k\|_{h_p} + \frac{\epsilon}{2^k}. \tag{2.18}$$

Let $q = \max\{p', p\}$. Since $\sum_k (\|u_k\|_{h_p} + \frac{\epsilon}{2^k}) < 1$, we can find a sequence $\{c_k\}$ such that $c_k \geq 1$, $c_k \to \infty$, and yet we still have

$$\sum_{k=1}^{\infty} c_k^q (\|u_k\|_{h_p} + \frac{\epsilon}{2^k}) < 1. \tag{2.19}$$

Let

$$x = \bigoplus_{k=1}^{\infty} c_k^{-2} x_k,$$

then $x \in \mathcal{K}_\infty(X)$ with $\|x\| \leq 1$. Let

$$v = [c_1 v_1 \ c_2 v_2 \ \cdots \quad] \quad \text{and} \quad w = \begin{bmatrix} c_1 w_1 \\ c_2 w_2 \\ \vdots \\ \\ \end{bmatrix},$$

then as in the proof of Proposition 2.4.4, we get $v \in M_{n,\infty}(V)$ and $w \in M_{\infty,n}(W)$ with $\|v\|, \|w\| \leq 1$. Finally, $u = v \odot x \odot w$ and this finishes the proof. $\square$

Chapter 3

Figà-Talamanca-Herz Algebras

3.1 Basics on Locally Compact Groups

Throughout this section G will denote a locally compact group. Whenever it makes sense, the convolution $\xi \star \eta$ of two functions ξ and η on G is defined to be

$$\xi \star \eta(x) = \int_G \xi(y)\eta(y^{-1}x)dy,$$

where dy means the left Haar measure.

Lemma 3.1.1. *Let K be a compact subset of G. Then there is a function $f \in C_{00}(G)$ such that $f|_K \equiv 1$.*

Proof. Let L be a compact neighborhood of the neutral element $e \in G$, then KL is also compact being the image of a compact set under a continuous function: $K \times L \ni (k,l) \mapsto kl$. Consider the function

$$f = \frac{1}{|L|} \mathbb{1}_{KL} \star \mathbb{1}_{L^{-1}}.$$

Since

$$f(x) = \frac{1}{|L|} \int_{KL} \mathbb{1}_{L^{-1}}(y^{-1}x)dy = \frac{1}{|L|} \int_{KL} \mathbb{1}_L(x^{-1}y)dy = \frac{|KL \cap xL|}{|L|},$$

for every $k \in K$, we get

$$f(k) = \frac{|KL \cap kL|}{|L|} = \frac{|xL|}{|L|} = 1.$$

$f \in C_0(G)$ because f is given by the convolution of functions with compact supports [HR79, (20.16)] and $f \in C_{00}(G)$ because f vanishes off the compact set KLL^{-1}. $\square$

3.2 p-Pseudofunction Algebras and p-Pseudomeasure Algebras

Let G be a locally compact group and let $1 < p < \infty$. For each $s \in G$, there exists an isometric isomorphism $\lambda_p(s)$ on $L_p(G)$, called the *left regular representation*, given by

$$\lambda_p(s)(\xi)(t) = \xi(s^{-1}t), \qquad \xi \in L_p(G), \quad s,t \in G.$$

It is well-known (for instance, [Run02]) that $\lambda_p : G \to \mathcal{B}(L_p(G))$ is strong operator continuous (equivalently, weak operator continuous).[1] Note that λ_p induces a representation $\lambda_p : L_1(G) \to \mathcal{B}(L_p(G))$ via integration. We let $PF_p(G)$ denote the *p-pseudofunction algebra*, which is defined to be the norm closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$ so that for $f \in L_1(G)$, the norm of $\lambda_p(f)$ is defined by

$$\|\lambda_p(f)\| = \sup\{\|f \star \xi\|_p : \xi \in L_p(G),\ \|\xi\|_p \leq 1\}. \tag{3.1}$$

Remark 3.2.1.

1. $PF_p(G)$ is indeed a Banach algebra, because $\lambda_p(f \star g) = \lambda_p(f)\lambda_p(g)$ for all $f, g \in L_1(G)$.

2. By Young's inequality for convolution
$$\|\lambda_p(f)\| \leq \|f\|_1 \tag{3.2}$$
for any $f \in L_1(G)$. In particular, when $p = 1$, $\|\lambda_1(f)\| = \|f\|_1$ for all $f \in L_1(G)$ since $L_1(G)$ has a contractive approximate identity.

3. G is amenable if and only if for each positive $f \in L_1(G)$ one has $\|\lambda_p(f)\| = \|f\|_1$ [Lep68].

4. $PF_p(G)$ has a contractive approximate identity arising from contractive approximate identity of $L_1(G)$. If G is discrete, then $PF_p(G)$ has a unit.

5. If $p = 2$, $PF_2(G)$ is nothing but the reduced group C*-algebra of G.

We let $PM_p(G)$ denote the *p-pseudomeasure algebra*, which is defined to be the weak operator topology closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$. By Kreĭn-Šmulian Theorem, $PM_p(G)$ is the same as the weak* closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$.[2]

[1] Similarly one can define the *right regular representation* ρ_p, given by $\rho_p(s)(\xi)(t) = \xi(ts)\Delta(s)^{1/p}$, $\xi \in L_p(G)$, and $s, t \in G$, where Δ denotes the modular function.

[2] Equivalently, the weak*-closure of $\text{span}\{\lambda_p(s) : s \in G\}$ in $\mathcal{B}(L_p(G))$. So $PM_p(G)$ is always unital.

3.3 Introduction to Figà-Talamanca-Herz Algebras

In this section we introduce a p-generalization of the Fourier algebra.

Definition 3.3.1. Let G be a locally compact group. Let $\Lambda_p : L_{p'}(G) \overset{\pi}{\otimes} L_p(G) \to C_0(G)$ be defined by

$$\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)(f) \rangle, \quad s \in G,\ f \in L_p(G),\ g \in L_{p'}(G).$$

The *Figà-Talamanca-Herz Algebra* $A_p(G)$ is defined to be the coimage of Λ_p.[3]

Remark 3.3.2. We collect some facts about $A_p(G)$.

1. $A_p(G)$ consists of those $f \in C_0(G)$ such that there are sequences $(\xi_n) \subseteq L_{p'}(G)$ and $(\eta_n) \subseteq L_p(G)$ with $\sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty$ and $f = \sum_{n=1}^{\infty} \xi_n \star \check{\eta}_n$. Note that

$$\|f\|_{A_p(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| : f = \sum_{n=1}^{\infty} \xi_n \star \check{\eta}_n \right\}.$$

   Λ_p maps into $C_0(G)$ by Theorem [HR79, (20.16)]. Moreover, $\|\cdot\|_{A_p(G)} \geq \|\cdot\|_\infty$ by Young's inequality for convolution [HR79, (20.18)]. In particular, the convergence in $\|\cdot\|_{A_p(G)}$ implies uniform (hence pointwise) convergence.

2. $A_p(G)$ is a commutative Banach algebra under pointwise operations [Her71].

3. G is amenable if and only if $A_p(G)$ has bounded (by 1) approximate identities for any p, $1 < p < \infty$ [Her73, Theorem 6].

4. By Lemma 3.1.1, for any compact subset K of G, there exists $\varphi \in A_p(G)$ with $\varphi \equiv 1$ on K. See also 9 below.

5. If G is amenable, given a compact subset K and $\epsilon > 0$, there exists $\varphi \in A_p(G)$ with $\varphi \equiv 1$ on K and $\|\varphi\|_{A_p(G)} \leq 1 + \epsilon$. See Theorem 4 and Section 0 in [Her73].

6. $A_p(G)$ is isometrically isomorphic to $A_{p'}(G)$ via $f \mapsto \check{f}$. If G is abelian, then $A_p(G) = A_{p'}(G)$.

7. The Banach space dual $A_p(G)'$ can be identified with $PM_p(G)$ [Run02].

8. Let $f \in L_1(G)$, then f defines an element Φ_f in $A_p(G)'$ by $\int f(x)\omega(x)dx$, $\omega \in A_p(G)$. The norm $\|\Phi_f\|_{A_p(G)'}$ equals $\|\lambda_p(f)\|$ introduced in (3.1) above [Cow79, §4].

9. $A_p(G)$ is a regular tauberian algebra in the sense of [Her73, Proposition 3].[4]

[3] Some authors swap p and p'. For example, Runde in [Run02] uses $A_{p'}(G)$ instead of $A_p(G)$ in our definition.
[4] A Banach algebra A is said to be a regular tauberian algebra of functions on G if the following three conditions hold:

10. $A_p(G)$ is dense in $C_0(G)$ in $\|\cdot\|_\infty$. To see this, it suffices to show that for any $f \in C_{00}(G)$ and for any $\epsilon > 0$, there exists $u \in A_p(G)$ such that $\|u - f\|_\infty < \epsilon$. Since $f \in C_{00}(G)$, f is (right) uniformly continuous [Fol95, Proposition (2.6)], so there is an open neighborhood U of the neutral element of G such that $\|f(\cdot y) - f(\cdot)\|_\infty < \epsilon$ for all $y \in U$. By [Fol95, Proposition (2.19)] and (inner) regularity of the Haar measure, we can replace U with a compact set K with $0 < |K| < \infty$. Define $u := \frac{1}{|K|} f \star \mathbb{1}_{K^{-1}}$, then $u \in A_p(G)$ and for all $t \in G$,

$$
\begin{aligned}
|u(t) - f(t)| &= \left| \frac{1}{|K|} \int_{s\in G} f(s)\mathbb{1}_{K^{-1}}(s^{-1}t)ds - f(t) \right| \\
&= \frac{1}{|K|} \left| \int_{s\in tK} (f(s) - f(t))\, ds \right| \\
(y = t^{-1}s) \quad &\leq \frac{1}{|K|} \int_{y\in K} |f(ty) - f(t)|\, dy \\
&< \epsilon.
\end{aligned}
$$

11. If $x_i \to e$ in G, then $\|_{x_i}u - u\|_{A_p(G)} \to 0$. This follows from the fact that $A_{p,c}(G) = A_p(G) \cap C_{00}(G)$ is dense in $A_p(G)$.[5]

12. $A_p(G)$ is closed under the left and right translations since for $\varphi = \xi \star \check{\eta}$, $\xi \in L_{p'}(G), \eta \in L_p(G)$, we get

$$
\varphi(s^{-1}x) = (\lambda_{p'}(s)\xi \star \check{\eta})(x) \quad \text{and} \quad \varphi(xs) = (\xi \star \rho_p(s)\check{\eta})(x).
$$

Since $PM_p(G) \subseteq \mathcal{B}(L_p(G))$ is a p-operator space, $A_p(G)$ has a dual p-operator space structure and $A_p(G)' = PM_p(G)$ p-completely isometrically [Daw10, Proposition 5.5]. Unless stated otherwise, we assume that $A_p(G)$ carries this p-operator space structure.

Definition 3.3.3. A linear map $T : A_p(G) \to A_p(G)$ is said to be a *multiplier* if $T(uv) = T(u)v$ for all $u, v \in A_p(G)$. The set of multipliers of $A_p(G)$ is denoted by $MA_p(G)$.

Remark 3.3.4. If $T \in MA_p(G)$, then T is necessarily bounded. To see this, suppose $u_n \to u$ and $Tu_n \to v$ in $A_p(G)$. Fix $x \in G$ and take $h \in A_p(G)$ such that $h(x) = 1$. Then

$$
v(x) = v(x)h(x) = \lim_n Tu_n(x)h(x) = \lim_n T(u_n h)(x) = \lim_n u_n(x)Th(x) = u(x)Th(x) = Tu(x)h(x) = Tu(x)
$$

(a) Given a compact set $K \subseteq G$ and a closed subset F disjoint from K, there exists $u \in A$ such that $u \equiv 1$ on K and $u \equiv 0$ on F.

(b) The elements of compact support are dense in A.

(c) If M is a continuous multiplicative linear functional on A whose support (in the sense of [Her73, §3]) is a single point $\{x\} \subseteq G$, then $M = \delta_x$, i.e., $\langle u, M\rangle = u(x)$ for all $u \in A$.

[5] See Remark 3.3.2.9.

and the result follows by the Closed Graph Theorem.

To each $T \in MA_p(G)$, we associate a function $h_T : G \to \mathbb{C}$ in the following way: for $x \in G$, take $u \in A_p(G)$ with $u(x) = 1$ and define $h_T(x) = (Tu)(x)$. Note that h_T is well-defined since if $v \in A_p(G)$ is another function satisfying $v(x) = 1$, then

$$(Tu)(x) = (Tu)(x)v(x) = T(uv)(x) = T(vu)(x) = (Tv)(x)u(x) = (Tv)(x).$$

It is not difficult to show that the mapping $T \mapsto h_T$ is injective and this gives an alternative definition of multipliers.

Definition 3.3.5. A complex-valued function u on G is said to be a *multiplier* for $A_p(G)$ if the linear map $m_u(v) = uv$ maps $A_p(G)$ to $A_p(G)$.

Definition 3.3.6. For $u \in MA_p(G)$, let $M_u : PM_p(G) \to PM_p(G)$ denote the weak*-continuous linear map defined by $M_u = m_u'$ and $\bar{M}_u$ denote the restriction of M_u to $PF_p(G)$. $M_{cb}A_p(G)$ is defined to be the space of p-completely bounded multipliers, endowed with the norm $\|u\|_{M_{cb}A_p(G)} = \|M_u\|_{pcb}$.

Remark 3.3.7.

1. If $u \in MA_p(G)$, then u is necessarily in $C_b(G)$. To see this, suppose $g_i \to g$ in G. Let K be a compact neighborhood of the neutral element e of G and define $\psi \in A_p(G)$ by $\psi(s) = \langle \mathbb{1}_K, \lambda_p(s)\mathbb{1}_{g^{-1}K}\rangle$, then $\psi(g) = |K| > 0$. Since $u\psi \in A_p(G) \subseteq C_0(G)$ (Remark 3.3.2), $u(g_i)\psi(g_i) \to u(g)\psi(g) = u(g)|K|$. Since $\psi(g_i) \to \psi(g) = |K|$ (See Remark 3.3.2), we conclude that $u(g_i) \to u(g)$. For boundedness, one can rely on Theorem 3.3.8 below.

2. If $u \in MA_p(G)$, then m_u is a bounded linear map on $A_p(G)$. This follows from the closed graph theorem and the fact that $\psi_n \to \psi$ in $A_p(G)$ implies $\psi_n \to \psi$ pointwise.

We have some useful characterizations of these spaces.

Theorem 3.3.8. *[Daw10, Lemma 8.2, Theorem 8.3]*

1. *$u \in MA_p(G)$ if and only if there exists a bounded, weak*-continuous operator $M : PM_p(G) \to PM_p(G)$ such that $M(\lambda_p(s)) = u(s)\lambda_p(s)$.*

2. *$u \in M_{cb}A_p(G)$ if and only if there exist $E \in SQ_p$ and bounded continuous maps $\alpha : G \to E$ and $\beta : G \to E'$ such that $u(ts^{-1}) = \langle \beta(t), \alpha(s)\rangle$ for $s, t \in G$.*

Remark 3.3.9.

1. Theorem 3.3.8.*1* shows that the range of $\bar{M}_u$ is contained in $PF_p(G)$.

2. Theorem 3.3.8.*2* shows that the left and right translations are isometries in $M_{cb}A_p(G)$.

3. Since $M_{cb}A_p(G) \subseteq \mathcal{CB}_p(A_p(G))$, $M_{cb}A_p(G)$ has a natural induced p-operator space structure. In fact, if $u = [u_{ij}] \in M_n(M_{cb}A_p(G))$, then

$$\begin{aligned}
& \|u\|_{M_n(M_{cb}A_p(G))} \\
= \quad & \|u\|_{M_n(\mathcal{CB}_p(A_p(G)))} \\
= \quad & \|u\|_{\mathcal{CB}_p(A_p(G),M_n(A_p(G)))} \\
= \quad & \sup\left\{\|[u_{ij}\varphi_{kl}]\|_{M_{nm}(A_p(G))} : m \in \mathbb{N}, \quad \varphi = [\varphi_{kl}] \in M_m(A_p(G))_1\right\} \\
= \quad & \sup\left\{\|[u_{ij}\varphi_{kl}]\|_{M_{nm}((PM_p(G))')} : m \in \mathbb{N}, \quad \varphi = [\varphi_{kl}] \in M_m(A_p(G))_1\right\} \\
= \quad & \sup\left\{\|[u_{ij}\varphi_{kl}]\|_{\mathcal{CB}_p(PM_p(G),M_{nm})} : m \in \mathbb{N}, \quad \varphi = [\varphi_{kl}] \in M_m(A_p(G))_1\right\} \\
= \quad & \sup\left\{\|\langle\langle u_{ij}\varphi_{kl}, T_{rs}\rangle\rangle\|_{M_{nmt}} : m,t \in \mathbb{N}, \quad \varphi = [\varphi_{kl}] \in M_m(A_p(G))_1, \quad T = [T_{rs}] \in M_t(PM_p(G))_1\right\} \\
= \quad & \sup\left\{\|\langle\langle \varphi_{kl}, M_{u_{ij}}T_{rs}\rangle\rangle\|_{M_{nmt}} : m,t \in \mathbb{N}, \quad \varphi = [\varphi_{kl}] \in M_m(A_p(G))_1, \quad T = [T_{rs}] \in M_t(PM_p(G))_1\right\} \\
= \quad & \sup\left\{\|M_{u_{ij}}T_{rs}\|_{M_{nt}(PM_p(G))} : t \in \mathbb{N}, \quad T = [T_{rs}] \in M_t(PM_p(G))_1\right\} \\
= \quad & \|M_{u_{ij}}\|_{\mathcal{CB}_p(PM_p(G),M_n(PM_p(G)))} \\
= \quad & \|M_{u_{ij}}\|_{M_n(\mathcal{CB}_p(PM_p(G)))}.
\end{aligned}$$

4. If G is a discrete group, the inclusion mapping $\iota : M_{cb}A_p(G) \to \ell_\infty(G)$ is p-completely contractive.[6] Indeed, it is easy to show that ι is contractive (See Remark 3.3.7). To prove that ι is p-completely contractive, fix $n \in \mathbb{N}$ and let $[u_{ij}] \in M_n(M_{cb}A_p(G))$. Then for each $s \in G$, we have

$$\begin{aligned}
& \|[u_{ij}(s)]\|_{\mathcal{B}(\ell_p^n)} \\
= \quad & \sup\left\{\left|\sum_{i,j=1}^{n} a_i u_{ij}(s) b_j\right| : \sum_{i=1} |a_i|^{p'} \le 1, \sum_{j=1} |b_j|^p \le 1\right\} \\
= \quad & \sup\left\{|v(s)| : v = [a_i]^T[u_{ij}][b_j], \sum_{i=1} |a_i|^{p'} \le 1, \sum_{j=1} |b_j|^p \le 1\right\} \\
\le \quad & \sup\left\{\|v\|_{\ell_\infty(G)} : v = [a_i]^T[u_{ij}][b_j], \sum_{i=1} |a_i|^{p'} \le 1, \sum_{j=1} |b_j|^p \le 1\right\} \\
\le \quad & \sup\left\{\|v\|_{M_{cb}A_p(G)} : v = [a_i]^T[u_{ij}][b_j], \sum_{i=1} |a_i|^{p'} \le 1, \sum_{j=1} |b_j|^p \le 1\right\}
\end{aligned}$$

[6]See the paragraph right before Proposition 1.6.13 for the discussion of the p-operator space structure of $\ell_\infty(G)$.

$$\leq \quad \|[u_{ij}]\|_{M_n(M_{cb}A_p(G))}.$$

By Proposition 1.6.14, taking supremum over $s \in G$, we get the desired result.

We close this section with an important observation on $A_p(G)$ given by Miao. We record it for convenience.

Lemma 3.3.10. *Let $\varphi \in MA_p(G)$ and $a \in A_{p,c}(G)$.*[7] *Then the map $x \mapsto ({}_x\varphi)a$ from G to $A_p(G)$ is continuous.*[8]

Proof. If x is in a neighborhood V of the neutral element e of G such that $\bar{V}$ is compact, then by Remark 3.3.2.9, there is $u \in A_p(G)$ such that $u(t) = 1$ for all $t \in \bar{V} \cdot \mathrm{supp}(a)$. Hence for all $x \in V$, $({}_x\varphi)a = ({}_x(\varphi u))a$ and $\varphi a = (\varphi u)a$. Thus for all $x \in V$, we have by Remark 3.3.2.11

$$\|({}_x\varphi)a - \varphi a\|_{A_p(G)} = \|({}_x(\varphi u))a - (\varphi u)a\|_{A_p(G)} \leq \|a\|_{A_p(G)}\|{}_x(\varphi u) - \varphi u\|_{A_p(G)} \to 0$$

as $x \to e$. $\square$

Corollary 3.3.11. *Let $\varphi \in MA_p(G)$ and $a \in A_{p,c}(G)$. Let V be a neighborhood of e. If $f_V \in L_1(G)$ with $\|f_V\|_1 = 1$ and $f_V(t) = 0$ for $t \notin V$, then*

$$\lim_{V \to e} \int_G \langle ({}_x\varphi)a, T\rangle f_V(x)dx = \langle \varphi a, T\rangle$$

for all $T \in PM_p(G)$.

Proof. It is immediate since

$$\left|\int_G \langle ({}_x\varphi)a, T\rangle f_V(x)dx - \langle \varphi a, T\rangle\right| = \left|\int_G \langle ({}_x\varphi)a - \varphi a, T\rangle f_V(x)dx\right| \leq \|({}_x\varphi)a - \varphi a\|_{A_p(G)}\|T\|.$$

$\square$

Corollary 3.3.12. *Let $\varphi \in MA_p(G)$ and $a \in A_{p,c}(G)$. For any $f \in L_1(G)$, there is an element $\eta \in A_p(G)$ such that $\eta = \int_G ({}_{x^{-1}}\varphi)af(x)dx$ and*

$$\langle T, \eta\rangle = \int_G \langle ({}_{x^{-1}}\varphi)a, T\rangle f(x)dx$$

[7] $A_{p,c}(G) = A_p(G) \cap C_{00}(G)$ is dense in $A_p(G)$. See Remark 3.3.2.9.

[8] Note that ${}_x\varphi \in MA_p(G)$. Indeed, if $u \in A_p(G)$, then ${}_x\varphi \cdot u = {}_x (\varphi \cdot {}_{x^{-1}} u) \in A_p(G)$ by Remark 3.3.2.12.

for all $T \in PM_p(G)$. Moreover, $\eta = (f \star \varphi)a$ and hence

$$\langle T, (f \star \varphi)a \rangle = \int_G \langle ({}_{x^{-1}}\varphi)a, T \rangle f(x)dx$$

for all $T \in PM_p(G)$.

Proof. This follows from [Ped89, page.76], because $x \mapsto ({}_{x^{-1}}\varphi)a$ is a bounded continous function and $f(x)dx$ is a bounded Radon measure. Second part follows from the calculation

$$(f \star \varphi)a(t) = \int_G f(x)\varphi(x^{-1}t)a(t)dx = \int_G ({}_{x^{-1}}\varphi(t))a(t)f(x)dx.$$

$\square$

Corollary 3.3.13. *Let $\varphi \in M_{cb}A_p(G)$ and $v \in L_1(G)$ with $\|v\|_1 = 1$. Then $v \star \varphi \in M_{cb}A_p(G)$ and $\|v \star \varphi\|_{M_{cb}A_p(G)} \leq \|\varphi\|_{M_{cb}A_p(G)}$.*

Proof. Let $n \in \mathbb{N}$ and let $T = [T_{ij}] \in M_n(PM_p(G))$. Suppose that $f = \{f_j\}_{j=1}^n \in L_p(G) \otimes_p \ell_p^n$ and $g = \{g_i\}_{i=1}^n \in L_{p'}(G) \otimes_{p'} \ell_{p'}^n$ have compact supports. Then

$$\begin{aligned}
|\langle g, (M_{v\star\varphi})_n(T)(f)\rangle| &= \left| \sum_{i,j=1}^n \langle g_i, M_{v\star\varphi}(T_{ij})(f_j)\rangle \right| \\
&= \left| \sum_{i,j=1}^n \left\langle g_i \star \check{f}_j, M_{v\star\varphi}(T_{ij})\right\rangle \right| \\
&= \left| \sum_{i,j=1}^n \left\langle (g_i \star \check{f}_j)(v \star \varphi), T_{ij}\right\rangle \right| \\
\text{(Corollary 3.3.12)} \quad &= \left| \sum_{i,j=1}^n \int_G \left\langle (g_i \star \check{f}_j)({}_{x^{-1}}\varphi), T_{ij}\right\rangle v(x)dx \right| \\
&= \left| \sum_{i,j=1}^n \int_G \left\langle g_i, M_{{}_{x^{-1}}\varphi}(T_{ij})(f_j)\right\rangle v(x)dx \right| \\
&= \left| \int_G \left\langle g, (M_{{}_{x^{-1}}\varphi})_n(T)(f)\right\rangle v(x)dx \right| \\
&\leq \|g\|_{p'} \|{}_{x^{-1}}\varphi\|_{M_{cb}A_p(G)} \|T\| \|f\|_p \\
\text{(Remark 3.3.9.2)} \quad &= \|g\|_{p'} \|\varphi\|_{M_{cb}A_p(G)} \|T\| \|f\|_p
\end{aligned}$$

This completes the proof. $\square$

3.4 Other Related Spaces and Some Open Questions

In this section, we collect some results from [Run05].

Definition 3.4.1. Let G be a locally compact group. A *representation* of G is a pair (π, E) where E is a Banach space and π is a group homomorphism from G into the invertible isometries on E which is continuous with respect to the given topology on G and the strong operator topology on $\mathcal{B}(E)$. We denote by $\mathrm{Rep}_p(G)$ the collection of all representations of G on a SQ_p space. A *coefficient function* of (π, E) is a function $f : G \to \mathbb{C}$ of the form

$$f(x) = \langle \pi(x)\xi, \varphi \rangle \quad (x \in G),$$

where $\xi \in E$ and $\varphi \in E'$. Finally, we let

$$B_p(G) = \{f : G \to \mathbb{C} : f \text{ is a coefficient of some } (\pi, E) \in \mathrm{Rep}_p(G)\}.$$

Remark 3.4.2. Let G be a locally compact group.

1. Any representation (π, E) of G induces a representation of the group algebra $L_1(G)$ on E via integration, that is,

$$\pi(f) := \int_G f(x)\pi(x)dx \quad (f \in L_1(G)).$$

2. A representation (π, E) is called *cyclic* if there is $\xi \in E$ such that $\pi(L_1(G))\xi$ is dense in E. We let

$$\mathrm{Cyc}_p(G) := \{(\pi, E) : (\pi, E) \text{ is cyclic}\}$$

and for $f \in B_p(G)$, we define $\|f\|_{B_p(G)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\|\|\varphi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in \mathrm{Cyc}_p(G)$ with $\xi_n \in E_n$ and $\varphi_n \in E_n'$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|\|\varphi_n\| < \infty \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \varphi_n \rangle \quad (x \in G).$$

Then $\|\cdot\|_{B_p(G)}$ defines a norm on $B_p(G)$.

Runde proved the following result.

Theorem 3.4.3. *Let G be a locally compact group and let $B_p(G)$ be equipped with $\|\cdot\|_{B_p(G)}$. Then*

1. *$B_p(G)$ is a commutative Banach algebra.*

2. *There are contractive inclusions $A_p(G) \subseteq B_p(G) \subseteq MA_p(G)$.*

3. For $2 \leq q \leq p$ or $p \leq q \leq 2$, the inclusion $B_q(G) \subseteq B_p(G)$ is a contraction.

Proof. Theorem 4.7 and Corollary 5.3 in [Run05]. □

Theorem 3.4.4. *Let G be a locally compact group, then $PF_p(G)'$ embeds contractively into $B_p(G)$.*

Proof. Theorem 6.6 in [Run05]. □

Let $f \in L_1(G)$, then f defines a bounded linear functional Φ_f on $A_p(G) \subseteq C_0(G)$ via integration: for $u \in A_p(G)$, $\Phi_f(u) = \int_G f(s)u(s)ds$.

Lemma 3.4.5. *With notations as above, $\|\Phi_f\| = \|\lambda_p(f)\|$ for all $f \in L_1(G)$.*

Proof. This is proved in [Cow79]. We include the proof for convenience. By (3.1),

$$\|\lambda_p(f)\| = \sup\{|\langle \xi, f \star \eta\rangle| : \xi \in L_{p'}(G),\ \eta \in L_p(G),\ \|\xi\|, \|\eta\| \leq 1\}.$$

Since

$$\begin{aligned} \langle \xi, f \star \eta\rangle &= \int_{s\in G} \xi(s) \int_{t\in G} f(t)\eta(t^{-1}s)dtds \\ &= \int_{t\in G} f(t) \int_{s\in G} \xi(s)\eta(t^{-1}s)dsdt \\ &= \Phi_f(\xi \star \check{\eta}), \end{aligned}$$

it follows that $\|\lambda_p(f)\| \leq \|\Phi_f\|$. For the other direction, let $\epsilon > 0$ and let $u \in A_p(G)$ with $\|u\|_{A_p(G)} = 1$ such that $|\Phi_f(u)| \geq \|\Phi_f\| - \epsilon$. Let $u = \sum_{n=1}^\infty \xi_n \star \check{\eta}_n$ with $\sum_{n=1}^\infty \|\xi_n\|_{p'}\|\eta_n\|_p < 1 + \epsilon$, then

$$\begin{aligned} |\Phi_f(u)| &= \left|\sum_{n=1}^\infty \langle \xi_n, f \star \eta_n\rangle\right| \\ &\leq \sum_{n=1}^\infty \|\lambda_p(f)\|\|\xi_n\|_{p'}\|\eta_n\|_p \\ &< (1+\epsilon)\|\lambda_p(f)\| \end{aligned}$$

and hence $\|\lambda_p(f)\| > \frac{\|\Phi_f\|-\epsilon}{1+\epsilon}$. Letting $\epsilon \to 0$, we get the desired inequality. □

Remark 3.4.6. By Remark 3.3.7, Theorems 3.4.3, 3.4.4, and Lemma 3.4.5, we have contractive inclusions

$$A_p(G) \subseteq PF_p(G)' \subseteq B_p(G) \subseteq M_{cb}A_p(G) \subseteq MA_p(G) \subseteq C_b(G).$$

Runde showed that some of these sets agree when G is amenable.

Theorem 3.4.7. *[Run05, Theorem 6.7] Let G be an amenable locally compact group, then $PF_p(G)'$, $B_p(G)$, and $MA_p(G)$ are equal with identical norms.*

By Remark 3.4.6, if G is amenable then $PF_p(G)' = B_p(G) = M_{cb}A_p(G) = MA_p(G)$. In fact, a certain equality among these sets implies the amenability of G as in the following remark.

Remark 3.4.8. For a locally compact group G,

1. $PF_p(G)' = MA_p(G)$ if and only if G is amenable [Cow79].
2. $B_2(G) = MA_2(G)$ if and only if G is amenable. [Neb82] for discrete case, [Los84] for general case.
3. If G is discrete, $B_2(G) = M_{cb}A_2(G)$ if and only if G is amenable [Boz85].

Remark 3.4.8 leads us to the following questions.[9]

Question 3.4.9. For a locally compact group G (see Theorem 3.4.7 and the explanation right after that),

1. Do we have G is amenable if (and only if) $PF_p(G)' = M_{cb}A_p(G)$?
2. Do we have G is amenable if (and only if) $PF_p(G)' = B_p(G)$?
3. Do we have G is amenable if (and only if) $B_p(G) = MA_p(G)$?
4. Do we have G is amenable if (and only if) $B_p(G) = M_{cb}A_p(G)$?

3.5 Amenability and Multiplier Algebras

In this section, G will always denote a locally compact group. Let K be a compact subset of G. Let $A_p(K)$ denote the space of restrictions to K of functions in $A_p(G)$ equipped with the norm defined by

$$\|w\|_{A_p(K)} = \inf\{\|u\|_{A_p(G)} : w = u|_K,\ u \in A_p(G)\}.$$

In other words, we identify $A_p(K)$ with the quotient space $A_p(G)/\mathcal{M}_K$, where $\mathcal{M}_K = \{u \in A_p(G) : u|_K \equiv 0\}$.

In [Cow79], Cowling gives a useful characterization of $PF_p(G)'$.

Theorem 3.5.1. *[Cow79, Theorem 4] $w \in L_\infty(G)$ belongs to $PF_p(G)'$ and has norm at most C if and only if, for any compact subset K of G,*

$$w|_K \in A_p(K) \quad \textit{and} \quad \|w|_K\|_{A_p(K)} \leq C.$$

[9] We have positive answers for the first two in Question 3.4.9. See Theorem 3.5.3 below.

We recall the proof of the following theorem originally proved by Cowling [Cow79].

Theorem 3.5.2. *For a locally compact group G, $PF_p(G)' = MA_p(G)$ isometrically if and only if G is amenable.*

Proof. ($\Longrightarrow$) First of all, by Remark 3.3.2, for each compact K, the constant function $\mathbf{1}_K \in A_p(K)$. If we assume that $\sup\{\|\mathbf{1}_K\|_{A_p(K)} : K \text{ compact}\} < \infty$, then we would be able to find a net $\{u_K \in A_p(G) : K \text{ compact}\}$ such that $u_K \equiv 1$ on K for each K and $C := \sup\{\|u_K\|_{A_p(G)}\} < \infty$. Now let $\varphi \in A_p(G)$ and $\epsilon > 0$. Consider $\varphi_0 \in A_{p,c}(G)$ such that $\|\varphi - \varphi_0\| < \epsilon$. For every K containing the support of φ_0, we get $\varphi_0 u_K = \varphi_0$ and it follows that

$$\|\varphi u_K - \varphi\| = \|(\varphi - \varphi_0)u_K + \varphi_0 u_K - u\| \leq (C+1)\epsilon.$$

This shows that $\{u_K \in A_p(G) : K \text{ compact}\}$ is a bounded approximate identity for $A_p(G)$ and hence G must be amenable. Therefore, if G were not amenable, then we must have $\sup\{\|\mathbf{1}_K\|_{A_p(K)} : K \text{ compact}\} = \infty$. By Theorem 3.5.1, the constant function $\mathbf{1}_G$ is not in $PF_p(G)'$ but $\mathbf{1}_G$ is always in $MA_p(G)$.
($\Longleftarrow$) From Theorems 3.4.3 and 3.4.4, we see that $PF_p(G)' \subseteq MA_p(G)$ contractively. Suppose $w \in MA_p(G)$ and let $\epsilon > 0$. If G is amenable, then by Remark 3.3.2, for every compact K there exists u_K such that $u_K \equiv 1$ on K with $\|u_K\|_{A_p(G)} \leq 1 + \epsilon$. Now $w|_K = (wu_K)|_K \in A_p(K)$ and hence $\|w|_K\|_{A_p(K)} \leq \|w\|_{MA_p(G)}(1+\epsilon)$. Since ϵ is arbitrary, it follows that $\|w|_K\|_{A_p(K)} \leq \|w\|_{MA_p(G)}$ and by Theorem 3.5.1, $w \in PF_p(G)'$. Finally, to prove that $\|w\|_{MA_p(G)} \leq \|w\|_{PF_p(G)'}$, it suffices to show that $\|w\varphi_0\|_{A_p(G)} \leq \|w\|_{PF_p(G)'}\|\varphi_0\|_{A_p(G)}$ for any compactly supported φ_0. Let $\epsilon > 0$ and K be the support of φ_0. By definition of $A_p(K)$, one can find $u \in A_p(G)$ such that $u|_K = w|_K$ and $\|u\|_{A_p(G)} \leq \|w|_K\|_{A_p(K)} + \epsilon$. Now by Theorem 3.5.1,

$$\|w\varphi_0\|_{A_p(G)} = \|u\varphi_0\|_{A_p(G)} \leq (\|w|_K\|_{A_p(K)} + \epsilon)\|\varphi_0\|_{A_p(G)} \leq (\|w\|_{PF_p(G)'} + \epsilon)\|\varphi_0\|_{A_p(G)}$$

and the result follows. ☐

We can now answer the first and the second questions in Question 3.4.9 positively.

Theorem 3.5.3. *For a locally compact group G, $PF_p(G)' = M_{cb}A_p(G)$ (respectively, $PF_p(G)' = B_p(G)$) isometrically if and only if G is amenable.*

Proof. ($\Longleftarrow$) This direction comes from Theorem 3.5.2.
($\Longrightarrow$) The proof is almost identical to that of Theorem 3.5.2. The only point to note is that $\mathbf{1}_G$ is always in $M_{cb}A_p(G)$ (respectively, $= B_p(G)$) as well. ☐

3.6 $Q_{pcb}(G)$ as the Predual of $M_{cb}A_p(G)$

In this section, we will show that $M_{cb}A_p(G)$ is a dual space, which is essential to study p-operator space approximation property.[10] To begin with, let $f \in L_1(G)$. For each $\varphi \in M_{cb}A_p(G) \subseteq C_b(G)$, the integration $\int_G f(t)\varphi(t)dt$ defines a bounded linear functional on $M_{cb}A_p(G)$ and hence we can embed $L_1(G)$ into $(M_{cb}A_p(G))'$. Let $\|\cdot\|_{Q,p}$ denote the norm on $L_1(G)$ inherited from this structure and let $Q_{pcb}(G)$ denote the norm closure of $(L_1(G), \|\cdot\|_{Q,p})$ in $(M_{cb}A_p(G))'$.

Remark 3.6.1. Let $f \in L_1(G)$. For each $\psi \in MA_p(G) \subseteq C_b(G)$, the integration $\int_G f(t)\psi(t)dt$ defines a bounded linear functional on $MA_p(G)$ and hence we can embed $L_1(G)$ into $(MA_p(G))'$. Let $Q_p(G)$ denote the norm closure of $L_1(G)$ with respect to the norm of $(MA_p(G))'$, then we have $Q_p(G)' = MA_p(G)$ [Mia04, Theorem 3.2].

Proposition 3.6.2 (Miao). *Let G be a locally compact group, then we have an isometric isomorphism $Q_{pcb}(G)' = M_{cb}A_p(G)$.*

Proof. We give a proof for convenience. First of all, it is clear that each $\varphi \in M_{cb}A_p(G)$ is in $Q_{pcb}(G)'$ with $\|\varphi\|_{Q_{pcb}(G)'} \leq \|\varphi\|_{M_{cb}A_p(G)}$. Coversely, let $m \in Q_{pcb}(G)'$ and $\|m\|_{Q_{pcb}(G)'} = 1$. Since $Q_{pcb}(G)$ is a closed subspace of $(M_{cb}A_p(G))'$, by the Hahn-Banach Theorem, we can extend m to a linear functional $\tilde{m}$ on $(M_{cb}A_p(G))'$ with $\|\tilde{m}\|_{(M_{cb}A_p(G))''} = 1$. By Goldstine's theorem, there is a net $\{m_\alpha\}$ in $M_{cb}A_p(G)$ such that $\|m_\alpha\|_{M_{cb}A_p(G)} \leq 1$ and $m_\alpha \to \tilde{m}$ in the $\sigma((M_{cb}A_p(G))'', (M_{cb}A_p(G))')$-topology. In particular, $\langle m_\alpha, f\rangle \to \langle m, f\rangle$ for any $f \in Q_{pcb}(G)$.

Let $f \in L_1(G)$. Since $M_{cb}A_p(G) \subseteq MA_p(G)$ with $\|\cdot\|_{MA_p(G)} \leq \|\cdot\|_{M_{cb}A_p(G)}$ on $M_{cb}A_p(G)$, we have $\|f\|_{Q_{pcb}(G)} \leq \|f\|_{Q_p(G)}$ and hence $m \in Q_p(G)' = MA_p(G)$ by Remark 3.6.1. We need to show that $\mathfrak{M} = m' : PM_p(G) \to PM_p(G)$ is p-completely contractive. So let $n \in \mathbb{N}$ and let $T = [T_{ij}] \in M_n(PM_p(G))$. Suppose that $f = \{f_i\}_{i=1}^n \in L_p(G) \otimes_p \ell_p^n$ and $g = \{g_j\}_{j=1}^n \in L_{p'}(G) \otimes_{p'} \ell_{p'}^n$ have compact supports. For each neighborhood V of the neutral element e of G, choose an f_V in $A_p(G)$ such that $f_V(t) = 0$ for $t \notin V$ and $\|f_V\|_1 = 1$. Then we have

$$\begin{aligned}
|\langle g, \mathfrak{M}_n(T)(f)\rangle| &= \left|\sum_{i,j=1}^n \langle g_i, \mathfrak{M}(T_{ij})(f_j)\rangle\right| \\
&= \left|\sum_{i,j=1}^n \left\langle g_i \star \check{f}_j, \mathfrak{M}(T_{ij})\right\rangle\right| \\
&= \left|\sum_{i,j=1}^n \left\langle (g_i \star \check{f}_j)m, T_{ij}\right\rangle\right|
\end{aligned}$$

[10] See Section 4.2.

$$
\begin{aligned}
\text{(Corollary 3.3.11)} \quad &= \left| \lim_V \sum_{i,j=1}^{n} \int_G \left\langle (g_i \star \check{f}_j)(_{x^{-1}}m), T_{ij} \right\rangle f_V(x) dx \right| \\
\text{(Corollary 3.3.12)} \quad &= \left| \lim_V \sum_{i,j=1}^{n} \left\langle (g_i \star \check{f}_j), M_{f_V \star m}(T_{ij}) \right\rangle \right|
\end{aligned}
$$

Here note that $\left\langle (g_i \star \check{f}_j), M_{f_V \star m}(T_{ij}) \right\rangle = \left\langle \omega_{T_{ij}, g_i \star \check{f}_j, f_V}, m \right\rangle$, where $\omega_{T_{ij}, g_i \star \check{f}_j, f_V} \in L_1(G)$ [Mia09, Lemma 3.1]. Therefore

$$
\begin{aligned}
\left| \lim_V \sum_{i,j=1}^{n} \left\langle (g_i \star \check{f}_j), M_{f_V \star m} T_{ij} \right\rangle \right| &= \left| \lim_V \sum_{i,j=1}^{n} \left\langle \omega_{T_{ij}, g_i \star \check{f}_j, f_V}, m \right\rangle \right| \\
&= \left| \lim_V \lim_\alpha \sum_{i,j=1}^{n} \left\langle \omega_{T_{ij}, g_i \star \check{f}_j, f_V}, m_\alpha \right\rangle \right| \\
&= \left| \lim_V \lim_\alpha \sum_{i,j=1}^{n} \langle g_i, M_{f_V \star m_\alpha}(T_{ij})(f_j) \rangle \right| \\
&= \left| \lim_V \lim_\alpha \langle g, (M_{f_V \star m_\alpha})_n(T)(f) \rangle \right| \\
&= \lim_V \lim_\alpha \|g\|_{p'} \|f_V \star m_\alpha\|_{M_{cb}A_p(G)} \|T\| \|f\|_p \\
\text{(Corollary 3.3.13)} \quad &\leq \|g\|_{p'} \|m_\alpha\|_{M_{cb}A_p(G)} \|T\| \|f\|_p \\
&\leq \|g\|_{p'} \|T\| \|f\|_p
\end{aligned}
$$

and it follows that $\mathfrak{M}$ is p-completely contractive. □

3.7 Description of $Q_{pcb}(G)$ for a Discrete Group G

For future applications, we want to characterize $Q_{pcb}(G)$ for a *discrete* group G. We begin with a lemma.

Lemma 3.7.1. *Let μ and ν be measures.*

1. *If $A \subseteq \mathcal{B}(L_p(\mu))$ is a p-operator space and $T : A \to A$ is a p-completely bounded map, then there exists a unique bounded map $\tilde{T} : \mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A \to \mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A$ such that*

$$\tilde{T}(b \otimes a) = b \otimes T(a), \quad b \in \mathcal{B}(L_p(\nu)), \quad a \in A,$$

   with $\|\tilde{T}\| \leq \|T\|_{pcb}$.

2. *If M is a weak*-closed subalgebra of $\mathcal{B}(L_p(\mu))$, and $T : M \to M$ is a weak* continuous p-completely*

bounded map, then there exists a unique weak-continuous bounded map* $\tilde{T} : \mathcal{B}(L_p(\nu))\bar{\otimes}M \to \mathcal{B}(L_p(\nu))\bar{\otimes}M$ *such that*

$$\tilde{T}(d \otimes c) = d \otimes T(c), \quad d \in \mathcal{B}(L_p(\nu)), \quad c \in M,$$

where $\mathcal{B}(L_p(\nu))\bar{\otimes}M$ *denotes the weak*-closure of the algebraic tensor product* $\mathcal{B}(L_p(\nu))\otimes M$ *in* $\mathcal{B}(L_p(\nu\times\mu))$*. Moreover, we have* $\|\tilde{T}\| \le \|T\|_{pcb}$*.*

Remark 3.7.2.

1. Once Lemma 3.7.1 is established, we can justify (and will use) the notation $\tilde{T} = id_{L_p(\nu)} \otimes T$.
2. In fact, $\tilde{T}$ is p-completely bounded with $\|\tilde{T}\|_{pcb} \le \|T\|_{pcb}$. See Lemma 3.7.3 below.

Proof of Lemma 3.7.1. 1. We use the rigid $\mathcal{L}_p$-structure of $L_p(\nu)$. Let $\{E_\gamma\}$ be a net of finite-dimensional subspaces of $L_p(\nu)$, directed by inclusion, such that $\bigcup_\gamma E_\gamma$ is dense in $L_p(\nu)$ and each E_γ is isometric to $\ell_p^{m(\gamma)}$ with $m(\gamma) = \dim E_\gamma$ [LP68]. Let P_γ denote the norm 1 projection from $L_p(\nu)$ onto E_γ, so that we identify $P_\gamma\mathcal{B}(L_p(\nu))P_\gamma$ with $M_{m(\gamma)}$. Define an operator T_0 on $\mathcal{B}(L_p(\nu))\otimes_{\text{alg}} A$ by $T_0(\sum b_i\otimes a_i) = \sum b_i\otimes Ta_i$, then for every $\sum b_i \otimes a_i \in \mathcal{B}(L_p(\nu)) \otimes_{\text{alg}} A$, we have

$$\begin{aligned}
& \left\|(P_\gamma \otimes id_{L_p(\mu)})T_0\left(\sum b_i \otimes a_i\right)(P_\gamma \otimes id_{L_p(\mu)})\right\| \\
= \ & \left\|\sum P_\gamma b_i P_\gamma \otimes Ta_i\right\| \\
\le \ & \|T_{m(\gamma)}\| \left\|\sum P_\gamma b_i P_\gamma \otimes a_i\right\| \\
\le \ & \|T\|_{pcb} \left\|(P_\gamma \otimes id_{L_p(\mu)})\left(\sum b_i \otimes a_i\right)(P_\gamma \otimes id_{L_p(\mu)})\right\| \\
\le \ & \|T\|_{pcb} \left\|\sum b_i \otimes a_i\right\|.
\end{aligned}$$

Since (P_γ) converges strongly to $id_{L_p(\nu)}$, it follows that

$$\left\|T_0\left(\sum b_i \otimes a_i\right)\right\| \le \|T\|_{pcb}\left\|\sum b_i \otimes a_i\right\|$$

and T_0 has a bounded extension $\tilde{T}$ to $\mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A$.

2. This is [Daw10, Theorem 6.4]. □

Lemma 3.7.3. *Let* T *be as in Lemma 3.7.1. Then* $id_{L_p(\nu)} \otimes T$ *(See Remark 3.7.2 for definition) is* p*-completely bounded with* $\|id_{L_p(\nu)} \otimes T\|_{pcb} \le \|T\|_{pcb}$*.*

Proof. Fix $n \in \mathbb{N}$ and consider $(id_{L_p(\nu)} \otimes T)_n : M_n(\mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A) \to M_n(\mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A)$. Since $M_n(\mathcal{B}(L_p(\nu)) \overset{\vee_p}{\otimes} A) \cong \mathcal{B}(L_p(\nu)^n) \overset{\vee_p}{\otimes} A$, we can identify $(id_{L_p(\nu)} \otimes T)_n$ with $id_{L_p(\nu)^n} \otimes T$. The result follows by Lemma 3.7.1 *2*. $\square$

Proposition 3.7.4. *Let G be a discrete group. Then*

$$Q_{pcb}(G) = \{\omega_{a,\varphi} : a \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G), \quad \varphi \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'\}.$$

Proof. $\supseteq$: Since $\|\omega_{a,\varphi}\| \leq \|a\|\|\varphi\|$ and $\operatorname{span}\{\lambda_p(s) : s \in G\}$ is norm dense in $PF_p(G)$, it suffices to show that $\omega_{a,\varphi} \in Q_{pcb}(G)$ for $a = b \otimes \lambda_s$ with $b \in \mathcal{B}(\ell_p)$. Indeed, for $u \in M_{cb}A_p(G)$, we get

$$\begin{aligned} \omega_{b\otimes\lambda_s,\varphi}(u) &= \langle (id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_u)(b \otimes \lambda_s), \varphi\rangle \\ &= \langle b \otimes u(s)\lambda_s, \varphi\rangle = u(s)\langle b \otimes \lambda_s, \varphi\rangle \\ &= \textstyle\sum_{t\in G} u(t)g(t), \end{aligned}$$

where $g(\cdot) = \langle b \otimes \lambda_\cdot, \varphi\rangle\delta_s(\cdot)$ is a function in $\ell_1(G)$. By definition of $Q_{pcb}(G)$, this shows that $\omega_{b\otimes\lambda_s,\varphi} \in Q_{pcb}(G)$.

$\subseteq$: We follow an idea similar to that given in [HK94]. Let

$$S = \{\omega_{a,\varphi} : a \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \quad \varphi \in ((\mathcal{B}(\ell_p)\bar{\otimes} PM_p(G))')_1\}.$$

Then by the argument above, S is contained in the closed unit ball of $Q_{pcb}(G)$. It is also easy to check that S is balanced. We claim that S is convex. To this end, first note that we can identify $\mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G)$ with $M_2(\mathcal{B}(\ell_p)) \overset{\vee_p}{\otimes} PF_p(G) = M_2(\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))$. By Theorem 2.3.12,

$$(M_2(\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)))' = \mathcal{N}_2 \overset{\wedge_p}{\otimes} (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$$

isometrically, where $\mathcal{N}_2$ denotes the space of nuclear operators on ℓ_p^2, and hence we can also identify $(\mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$ with $\mathcal{N}_2 \overset{\wedge_p}{\otimes} (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$. In particular, the duality is given in such a way that if $b = [b_{ij}] \in \mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi = [\varphi_{ij}] \in (\mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$, then

$$\langle b, \varphi\rangle = \sum_{i,j=1}^{2} \langle b_{ij}, \varphi_{ij}\rangle.$$

If T is a p-completely bounded operator on $PF_p(G)$, then for any $a = [a_{ij}] \in \mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G)$, we have

$$id_{\mathcal{B}(\ell_p\oplus_p\ell_p)} \otimes T(a) = [id_{\mathcal{B}(\ell_p)} \otimes T(a_{ij})]$$

(See Lemma 3.7.1 for the definition of $id_{\mathcal{B}(\ell_p \oplus_p \ell_p)} \otimes T$ and $id_{\mathcal{B}(\ell_p)} \otimes T$). Now let ω_{a_1,φ_1} and ω_{a_2,φ_2} be elements of S, and suppose $0 \leq \lambda \leq 1$. Let

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in \mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G), \quad \varphi = \begin{bmatrix} \lambda\varphi_1 & 0 \\ 0 & (1-\lambda)\varphi_2 \end{bmatrix} \in (\mathcal{B}(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G))'.$$

Then $\|a\| = \max\{\|a_1\|, \|a_2\|\} \leq 1$ and $\|\varphi\| \leq \lambda\|\varphi_1\| + (1-\lambda)\|\varphi_2\| \leq 1$. Now for any p-completely bounded operator T on $PF_p(G)$,

$$\begin{aligned} \omega_{a,\varphi}(T) &= \langle id_{\mathcal{B}(\ell_p \oplus_p \ell_p)} \otimes T(a), \varphi\rangle \\ &= \langle id_{\mathcal{B}(\ell_p)} \otimes T(a_1), \lambda\varphi_1\rangle + \langle id_{\mathcal{B}(\ell_p)} \otimes T(a_2), (1-\lambda)\varphi_2\rangle \\ &= (\lambda\omega_{a_1,\varphi_1} + (1-\lambda)\omega_{a_2,\varphi_2})(T). \end{aligned}$$

Moreover, since ℓ_p and $\ell_p \oplus_p \ell_p$ are isometrically isomorphic, $\omega_{a,\varphi} = \omega_{b,\psi}$ for some $b \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1$ and some $\psi \in ((\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))')_1$. This shows that S is convex.

Now we claim that S is norm dense in the closed unit ball of $Q_{pcb}(G)$. Suppose this is not the case and $\omega \in Q_{pcb}(G)_1$ is not in the closure of S. Using the geometric Hahn-Banach theorem and the fact that S is balanced, we can find $u \in M_{cb}A_p(G)$ such that

$$|\langle \omega_{a,\varphi}, u\rangle| \leq 1 < \langle \omega, u\rangle, \quad \forall \omega_{a,\varphi} \in S.$$

However, this implies

$$\begin{aligned} \|u\|_{pcb} &\leq \sup\{\|id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_u(a)\| : a \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1\} \\ &= \sup\{|\langle id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_u(a), \varphi\rangle| : a \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \varphi \in ((\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))')_1\} \\ &= \sup\{|\omega_{a,\varphi}(u)| : a \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \varphi \in ((\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))')_1\} \\ &\leq 1. \end{aligned}$$

But then this would imply $|\langle \omega, u\rangle| \leq 1$, a contradiction. This shows that S is norm dense in the closed unit ball of $Q_{pcb}(G)$.

Let $\ell_p^{(\infty)}$ denote the p-direct sum of a countably infinite number of copies of ℓ_p. Since ℓ_p is isometrically isomorphic to $\ell_p^{(\infty)}$, to complete the proof it suffices to show that every $\omega \in Q_{pcb}(G)$ is of the form $\omega = \omega_{a,\varphi}$ for some $a \in \mathcal{B}(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$ and some $\varphi \in (\mathcal{B}(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G))'$. Without loss of generality, we can assume that $\omega \in Q_{pcb}(G)_1$. Then there is an $\omega_1 \in S$ such that $\|\omega - \omega_1\| < \frac{1}{2}$. Since $2(\omega - \omega_1) \in Q_{pcb}(G)_1$, there is an $\omega_2 \in S$ such that $\|\omega - \omega_1 - \frac{1}{2}\omega_2\| < \frac{1}{2^2}$. Continuing in this fashion, we find a sequence $\{\omega_n\}$ in S such that

$$\left\|\omega - \sum_{i=1}^{n} \frac{1}{2^{i-1}}\omega_i\right\| < \frac{1}{2^n}.$$

Thus $\omega = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\omega_i$. Since $\omega_i \in S$, there are sequences $b_i \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1$ and $\psi_i \in ((\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))')_1$ such that $\omega_i = \omega_{b_i,\psi_i}$. Let $\alpha_i = (2^{-i+1})^{1/2}$, let $a_i = \alpha_i b_i$, and let $\varphi_i = \alpha_i \psi_i$. Let $a \in M_\infty(\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)) = \mathcal{B}(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$ be the diagonal matrix with diagonal entries $a_1, a_2, \cdots$. Since $\|a_i\| \to 0$, a in fact lies in $K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$. Moreover, since $\sum_{i=1}^{\infty} \|\varphi_i\| < \infty$, we can define $\varphi_0 \in (K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G))' = \mathcal{N}_\infty \overset{\wedge_p}{\otimes} (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$ by

$$\varphi_0([a_{ij}]) = \sum_{i=1}^{\infty} \langle a_{ii}, \varphi_i \rangle, \quad [a_{ij}] \in K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G).$$

Extend φ_0 to $\varphi \in (\mathcal{B}(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G))'$ using Hahn-Banach theorem, then it follows that $\omega = \omega_{a,\varphi}$ and this completes the proof. $\square$

Remark 3.7.5. The same argument as above actually works to show that

$$Q_{pcb}(G) = \{\omega_{a,\varphi} : a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G), \quad \varphi \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'\}.$$

Proposition 3.7.6. *Let G be a discrete group. Then*

$$Q_{pcb}(G) = \{\omega_{a,\varphi} : a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G), \quad \varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))\},$$

where $\mathcal{N}(\ell_p \otimes_p \ell_p(G))$ denotes the space of nuclear operators on $\ell_p \otimes_p \ell_p(G)$.

Proof. $\supseteq$: This direction is obtained by the same argument as in Proposition 3.7.4.

$\subseteq$: Let

$$S = \{\omega_{a,\varphi} : a \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \quad \varphi \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1\}.$$

Then by the argument above, S is contained in the closed unit ball of $Q_{pcb}(G)$. It is easy to check that S is bounded. We claim that S is convex. Let ω_{a_1,φ_1} and ω_{a_2,φ_2} be elements of S, and suppose $0 \le \lambda \le 1$. Let

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in M_2(\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))$$

and

$$\varphi = \left[\begin{array}{cc} \lambda\varphi_1 & 0 \\ 0 & (1-\lambda)\varphi_2 \end{array}\right] \in \mathcal{N}_2 \overset{\wedge_p}{\otimes} T(\ell_p \otimes_p \ell_p(G)),$$

then $\varphi \in (M_2(\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)))' = \mathcal{N}_2 \overset{\wedge_p}{\otimes} (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$ with $\|a\| \leq 1$ and $\|\varphi\| \leq 1$. Now, after isometric identification

$$M_2(\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)) = K(\ell_p \oplus_p \ell_p) \overset{\vee_p}{\otimes} PF_p(G)$$

and

$$\mathcal{N}_2 \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p \otimes_p \ell_p(G)) = \mathcal{N}((\ell_p \otimes_p \ell_p(G)) \oplus_p (\ell_p \otimes_p \ell_p(G))) = \mathcal{N}(\ell_p \oplus_p \ell_p) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G)),$$

we have for any p-completely bounded operator T on $PF_p(G)$,

$$\begin{aligned} \omega_{a,\varphi}(T) &= \langle id_{K(\ell_p \oplus_p \ell_p)} \otimes T(a), \varphi\rangle \\ &= \langle id_{\mathcal{K}(\ell_p)} \otimes T(a_1), \lambda\varphi_1\rangle + \langle id_{\mathcal{K}(\ell_p)} \otimes T(a_2), (1-\lambda)\varphi_2\rangle \\ &= (\lambda\omega_{a_1,\varphi_1} + (1-\lambda)\omega_{a_2,\varphi_2})(T). \end{aligned}$$

Moreover, since ℓ_p and $\ell_p \oplus_p \ell_p$ are isometrically isomorphic, $\omega_{a,\varphi} = \omega_{b,\psi}$ for some $b \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1$ and some $\psi \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1$. This shows that S is convex.

Now we claim that S is norm dense in the closed unit ball of $Q_{pcb}(G)$. Suppose this is not the case and $\omega \in Q_{pcb}(G)_1$ is not in the closure of S. Using the geometric Hahn-Banach theorem and the fact that S is balanced, we can find $u \in M_{cb}A_p(G)$ such that

$$|\langle \omega_{a,\varphi}, u\rangle| \leq 1 < \langle \omega, u\rangle, \quad \forall \omega_{a,\varphi} \in S.$$

However, this implies

$$\begin{aligned} \|u\|_{pcb} &\leq \sup\{\|id_{\mathcal{K}(\ell_p)} \otimes \bar{M}_u(a)\| : a \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1\} \\ &= \sup\{|\langle id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_u(a), \varphi\rangle| : a \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \varphi \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1\} \\ &= \sup\{|\omega_{a,\varphi}(u)| : a \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1, \varphi \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1\} \\ &\leq 1. \end{aligned}$$

But then this would imply $|\langle \omega, u\rangle| \leq 1$, a contradiction. This shows that S is norm dense in the closed unit ball of $Q_{pcb}(G)$.

Let $\ell_p^{(\infty)}$ denote the p-direct sum of a countably infinite number of copies of ℓ_p. Since ℓ_p is isometrically isomorphic to $\ell_p^{(\infty)}$, to complete the proof it suffices to show that every $\omega \in Q_{pcb}(G)$ is of the form $\omega = \omega_{a,\varphi}$ for some $a \in K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$ and some $\varphi \in \mathcal{N}(\ell_p^{(\infty)}) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$. Without loss of generality, we can assume that $\omega \in Q_{pcb}(G)_1$. Then there is an $\omega_1 \in S$ such that $\|\omega - \omega_1\| < \frac{1}{2}$. Since $2(\omega - \omega_1) \in Q_{pcb}(G)_1$, there is an $\omega_2 \in S$ such that $\|\omega - \omega_1 - \frac{1}{2}\omega_2\| < \frac{1}{2^2}$. Continuing in this fashion, we find a sequence $\{\omega_n\}$ in S such that

$$\left\|\omega - \sum_{i=1}^{n} \frac{1}{2^{i-1}}\omega_i\right\| < \frac{1}{2^n}.$$

Thus $\omega = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\omega_i$. Since $\omega_i \in S$, there are sequences $b_i \in (\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))_1$ and $\psi_i \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1$ such that $\omega_i = \omega_{b_i,\psi_i}$. Let $\alpha_i = (2^{-i+1})^{1/2}$, let $a_i = \alpha_i b_i$, and let $\varphi_i = \alpha_i \psi_i$. Let $a \in M_\infty(\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)) = \mathcal{B}(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$ be the diagonal matrix with diagonal entries $a_1, a_2, \cdots$. Since $\|a_i\| \to 0$, a in fact lies in $K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G)$. Moreover, since $\sum_{i=1}^{\infty} \|\varphi_i\| < \infty$, we can define $\varphi \in \mathcal{N}_\infty \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p \otimes_p \ell_p(G)) = \mathcal{N}(\ell_p^{(\infty)}) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$ by

$$\varphi([a_{ij}]) = \sum_{i=1}^{\infty} \langle a_{ii}, \varphi_i \rangle, \quad [a_{ij}] \in K(\ell_p^{(\infty)}) \overset{\vee_p}{\otimes} PF_p(G).$$

Now it follows that $\omega = \omega_{a,\varphi}$ and this completes the proof. □

Remark 3.7.7. The same argument as above actually works to show that

$$Q_{pcb}(G) = \{\omega_{b,\varphi} : b \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G), \quad \varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))\}$$

for a discrete group G.

3.8 More on $Q_{pcb}(G)$: for General Locally Compact Group G

Theorem 3.8.1. $Q_{pcb}(G) \supseteq \left\{\omega_{a,\varphi,f} : a \in PM_p(G)\bar{\otimes}\mathcal{K}(\ell_p), \varphi \in A_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p), f \in A_{p,c}(G)\right\}$.

Proof. It suffices to show that if $a \in PM_p(G)$, $\varphi \in A_{p,c}(G)$, and $f \in A_{p,c}(G)$, then $\omega_{a,\varphi,f} \in Q_{pcb}(G)$, where $\omega_{a,\varphi,f}(u) = \langle M_{f\star u}(a), \varphi\rangle$. We will find a $g \in L_1(G)$ such that $\omega_{a,\varphi,f} = \int u(x)g(x)dx$ for all $u \in M_{cb}A_p(G)$. Let $K = (\operatorname{supp} f)^{-1} \operatorname{supp} \varphi$. It is easy to show that

$$([f \star u]\varphi)(x) = (f \star \mathbb{1}_K u)(x) = \int_G f_y(x)\varphi(x)(\mathbb{1}_K u)(y^{-1}) \quad \text{for all } x \in G,$$

where $f_y(\cdot) = f(\cdot y)$. Define a map $\Phi : G \to A_p(G)$ by $[\Phi(y)](x) = f_y(x)\varphi(x)$, and define a measure μ on G by $d\mu(y) = (\mathbb{1}_K u)(y^{-1})dy$, then $d\mu$ is a bounded Radon measure. Take $v = \int_G \Phi(y)d\mu(y)$ (Analysis Now p.70) in $A_p(G)$ so that

$$\langle b, v\rangle = \int_G \langle b, \Phi(y)\rangle d\mu(y) \quad \text{for all } b \in PM_p(G).$$

In particular, if $b = \lambda_p(x)$, then we get $v(x) = (f \star \mathbb{1}_K u)(x)$ and thus

$$\omega_{a,\varphi,f}(u) = \langle a, v\rangle = \int_G \langle a, \Phi(y)\rangle d\mu(y) = \int_G u(y^{-1})\langle a, \Phi(y)\rangle \mathbb{1}_K(y^{-1})dy = \int_G u(y)g(y)dy,$$

where $g(y) = \langle a, \Phi(y^{-1})\rangle \mathbb{1}_K(y)\Delta(y^{-1})$. Now it is easy to verify that $g \in L_1(G)$. $\square$

Chapter 4

Approximation Properties

4.1 p-Conditional Expectation

In this chapter, G always denotes a discrete group, unless stated otherwise. Let $A_{p,c}(G) = C_{00}(G) \cap A_p(G)$, where $C_{00}(G)$ denotes the set of compactly supported functions on G. Define

$$\Delta : A_{p,c}(G) \to A_{p,c}(G \times G), \qquad \Delta\varphi = \sum_{g \in G} \varphi(g)\delta_{(g,g)}.$$

Note that $\Delta\varphi(s,t) = \delta_s(t)\varphi(s)$.

Lemma 4.1.1. *The map Δ extends to a linear contraction from $A_p(G)$ into $A_p(G \times G)$.*

Proof. Let $\epsilon > 0$. Suppose $\varphi \in A_{p,c}(G)$, then we can express φ as $\varphi = \sum_n \xi_n \star \check{\eta}_n$ with $\xi_n \in \ell_{p'}(G)$, $\eta_n \in \ell_p(G)$, and $\sum_n \|\xi_n\|_{p'}\|\eta_n\|_p < \|\varphi\|_{A_p(G)} + \epsilon$. Define $\tilde{\xi}_n \in \ell_{p'}(G \times G)$ by $\tilde{\xi}_n(s,t) = \delta_s(t)\xi_n(s)$ and $\tilde{\eta}_n \in \ell_p(G \times G)$ by $\tilde{\eta}_n(s,t) = \delta_s(t)\eta_n(s)$. It is easy to show that $\|\tilde{\xi}_n\|_{p'} = \|\xi_n\|_{p'}$ and $\|\tilde{\eta}_n\|_p = \|\eta_n\|_p$. Moreover,

$$\begin{aligned}
& \textstyle\sum_n \tilde{\xi}_n \star \check{\tilde{\eta}}_n(s,t) \\
= \; & \textstyle\sum_n \left(\sum_{(g,h)} \tilde{\xi}_n(g,h)\tilde{\eta}_n(s^{-1}g, t^{-1}h)\right) \\
= \; & \textstyle\sum_n \sum_{(g,h)} \delta_g(h)\xi_n(g)\delta_{s^{-1}g}(t^{-1}h)\eta_n(s^{-1}g) \\
= \; & \textstyle\sum_n \sum_g \xi_n(g)\delta_s(t)\eta_n(s^{-1}g) \\
= \; & \delta_s(t)\varphi(s) \\
= \; & \Delta\varphi(s,t).
\end{aligned}$$

Therefore, we get

$$\|\Delta\varphi\|_{A_p(G \times G)} \leq \|\varphi\|_{A_p(G)} + \epsilon$$

and hence $\|\Delta\| \leq 1$. Since $A_{p,c}(G)$ is norm dense in $A_p(G)$ [Run02], we can extend Δ to a contraction from $A_p(G)$ into $A_p(G \times G)$. $\square$

Remark 4.1.2. In particular, the proof above shows that if $\xi \star \check{\eta} \in A_{p,c}(G)$ with $\xi \in \ell_{p'}(G)$ and $\eta \in \ell_p(G)$, then $\Delta(\xi \star \check{\eta}) = \tilde{\xi} \star \check{\tilde{\eta}}$. In fact, more is true as we have the following

Lemma 4.1.3. *With the same notations as above, if $\xi \in \ell_{p'}(G)$ and $\eta \in \ell_p(G)$, then $\Delta(\xi \star \check{\eta}) = \tilde{\xi} \star \check{\tilde{\eta}}$.*

Proof. Since $C_{00}(G)$ is dense in $\ell_{p'}(G)$, we can find a sequence $(f_n) \in C_{00}(G)$ such that $f_n \to \xi$ and $\|f_n\| \leq \|\xi\|_{p'}$ for all n. Similarly, there is a sequence $(g_n) \in C_{00}(G)$ such that $g_n \to \eta$ and $\|g_n\| \leq \|\eta\|_p$. It is easy to check that $f_n \star \check{g}_n \to \xi \star \check{\eta}$ in $A_p(G)$ and hence $\Delta(f_n \star \check{g}_n) \to \Delta(\xi \star \check{\eta})$ in $A_p(G \times G)$. However, since $f_n \star \check{g}_n \in C_{00}(G)$, from Remark 4.1.2, we get $\Delta(f_n \star \check{g}_n) = \tilde{f}_n \star \check{\tilde{g}}_n$, which converges to $\tilde{\xi} \star \check{\tilde{\eta}}$ in $A_p(G \times G)$. □

Define an isometry $\Gamma_p : \ell_p(G) \to \ell_p(G \times G)$ by $\delta_g \mapsto \delta_{(g,g)}$. Note that $\ell_{p'}(G) \overset{\pi}{\otimes} \ell_p(G) = \mathcal{N}(\ell_p(G))$, the space of nuclear operators on $\ell_p(G)$, then $\Gamma_{p',p} \triangleq \Gamma_{p'} \overset{\pi}{\otimes} \Gamma_p : \mathcal{N}(\ell_p(G)) \to \mathcal{N}(\ell_p(G \times G)) = \mathcal{N}(\ell_p(G)) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$ is a contraction. As in [Daw10], define a map $\Lambda_p : \mathcal{N}(\ell_p(G)) \to C_0(G)$ by

$$\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)(f) \rangle \quad (s \in G, f \in \ell_p(G), g \in \ell_{p'}(G)),$$

then it is easy to show that Λ_p induces a map from $A_p(G)$ to $A_p(G \times G)$, which coincides with Δ above.

Proposition 4.1.4. *Let $\rho : PM_p(G \times G) \to PM_p(G)$ denote the adjoint of Δ. Then ρ is p-completely contractive.*

Proof. Fix $n \in \mathbb{N}$. Let $[T_{ij}] \in M_n(PM_p(G \times G))$. To estimate the norm of $[\rho(T_{ij})] \in M_n(PM_p(G))$, choose $f_i \in \ell_{p'}(G), g_j \in \ell_p(G)$, $1 \leq i, j \leq n$. We have

$$\begin{aligned}
& \left\langle \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} & & \\ & \rho(T_{ij}) & \\ & & \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \right\rangle_{\ell^n_{p'} \overset{p'}{\otimes} \ell_{p'}(G),\ \ell^n_p \otimes_p \ell_p(G)} \\
&= \textstyle\sum_{i,j=1}^n \langle f_i, \rho(T_{ij}) g_j \rangle_{\ell_{p'}(G),\ \ell_p(G)} \\
&= \textstyle\sum_{i,j=1}^n \langle f_i \star \check{g}_j, \rho(T_{ij}) \rangle_{A_p(G),\ PM_p(G)} \\
(\Delta' = \rho) \quad &= \textstyle\sum_{i,j=1}^n \langle \Delta(f_i \star \check{g}_j), T_{ij} \rangle_{A_p(G \times G),\ PM_p(G \times G)} \\
(\text{Lemma 4.1.3}) \quad &= \textstyle\sum_{i,j=1}^n \langle \tilde{f}_i \star \check{\tilde{g}}_j, T_{ij} \rangle_{A_p(G \times G),\ PM_p(G \times G)} \\
&= \textstyle\sum_{i,j=1}^n \langle \tilde{f}_i, T_{ij}(\check{\tilde{g}}_j) \rangle_{\ell_{p'}(G \times G),\ \ell_p(G \times G)} \\
&= \left\langle \begin{bmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \begin{bmatrix} & & \\ & T_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_n \end{bmatrix} \right\rangle_{\ell^n_{p'} \overset{p'}{\otimes} \ell_{p'}(G \times G),\ \ell^n_p \otimes_p \ell_p(G \times G)}
\end{aligned}$$

Since $\|f_i\|_{p'} = \|\tilde{f}_i\|_{p'}$ and $\|g_j\|_p = \|\tilde{g}_j\|_p$, $1 \le i, j \le n$, we are done. $\square$

Lemma 4.1.5. *For all $s, t \in G$, $\rho(\lambda_{(s,t)}) = \delta_s(t)\lambda_s$.*

Proof. For all $g, s, t \in G$, by the duality between $A_p(G)$ (respectively, $A_p(G\times G)$) and $PM_p(G)$ (respectively, $PM_p(G \times G)$), we get

$$\langle \delta_g, \rho(\lambda_{(s,t)})\rangle = \langle \Delta\delta_g, \lambda_{(s,t)}\rangle = \langle \delta_{(g,g)}, \lambda_{(s,t)}\rangle = \delta_{(g,g)}(s,t) = \delta_g(s)\delta_s(t) = \delta_s(t)\langle \delta_g, \lambda_s\rangle = \langle \delta_g, \delta_s(t)\lambda_s\rangle$$

and the result follows because $A_{p,c}(G)$ is dense in $A_p(G)$. $\square$

As in [Daw10], define $W_p : \ell_p(G \times G) \to \ell_p(G \times G)$ by $(W_p\xi)(s,t) = \xi(s, st)$, then W_p is an isometric isomorphism on $\ell_p(G \times G)$ with the inverse $(W_p^{-1}\eta)(s,t) = \eta(s, s^{-1}t)$. It is straightforward to check that $(W_p^{-1})' = W_{p'}$. Define

$$\gamma : PM_p(G) \to \mathcal{B}(\ell_p(G \times G))(= \mathcal{B}(\ell_p(G) \otimes_p \ell_p(G))), \qquad T \mapsto W_p^{-1}(T \otimes I)W_p \quad (T \in PM_p(G)).$$

Before we proceed, we need a lemma about the Banach space projective tensor product.

Lemma 4.1.6. *Let $X \overset{\pi}{\otimes} Y$ denote the Banach space projective tensor product of Banach spaces X and Y. Let $1 < p', p < \infty$ with $1/p' + 1/p = 1$. Then for every $u \in X \overset{\pi}{\otimes} Y$, the norm $\pi(u)$ of u is given by*

$$\pi(u) = \inf\left\{\left(\sum_{n=1}^{\infty}\|x_n\|^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{\infty}\|y_n\|^{p}\right)^{\frac{1}{p}}\right\}, \tag{4.1}$$

where the infimum is taken over all expressions $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ with

$$\left(\sum_{n=1}^{\infty}\|x_n\|^{p'}\right)^{\frac{1}{p'}}\left(\sum_{n=1}^{\infty}\|y_n\|^{p}\right)^{\frac{1}{p}} < \infty.$$

Proof. Let $u \in X \overset{\pi}{\otimes} Y$ and $\epsilon > 0$. It is well known that

$$\pi(u) = \inf\left\{\sum_{n=1}^{\infty}\|x_n\|\|y_n\| : \sum_{n=1}^{\infty}\|x_n\|\|y_n\| < \infty, \quad u = \sum_{n=1}^{\infty} x_n \otimes y_n\right\}. \tag{4.2}$$

Therefore u can be written as $u = \sum_{n=1}^{\infty} \xi_n \otimes \eta_n$, $\xi_n(\neq 0) \in X$, $\eta_n(\neq 0) \in Y$ with $\sum_{n=1}^{\infty}\|\xi_n\|\|\eta_n\| < \pi(u)+\epsilon$. For each n, let $\lambda_n = \|\eta_n\|^{\frac{p}{p+p'}}\|\xi_n\|^{-\frac{p'}{p+p'}}$ so that

$$\|\lambda_n\xi_n\|^{p'} = \left\|\frac{1}{\lambda_n}\eta_n\right\|^{p} = \|\xi_n\|\|\eta_n\|.$$

Now let $x_n = \lambda_n \xi_n$ and $y_n = \frac{1}{\lambda_n}\eta_n$ then $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ and

$$\left(\sum_{n=1}^{\infty} \|x_n\|^{p'}\right)^{1/p'} \left(\sum_{n=1}^{\infty} \|y_n\|^{p}\right)^{1/p} = \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \pi(u) + \epsilon.$$

Since ϵ is arbitrary, this shows that the right hand side of (4.1) is less than or equal to $\pi(u)$. To show the reverse inequality, let's assume, for contradiction, that the right hand side of (4.1) is strictly less than $\pi(u)$. This means u can be written as $u = \sum_{n=1}^{\infty} x_n \otimes y_n$ with $(\sum_{n=1}^{\infty} \|x_n\|^{p'})^{1/p'}(\sum_{n=1}^{\infty} \|y_n\|^{p})^{1/p} < \pi(u)$. By Hölder's inequality, we would get $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \pi(u)$ but this is impossible because of (4.2). □

Lemma 4.1.7. *γ is weak*-weak* continuous.*

Proof. Suppose $T_\alpha, T \in PM_p(G)$ and $T_\alpha \to T$ in the weak* topology. For every $\sum_n f_n \otimes g_n \in \ell_{p'}(G) \overset{\pi}{\otimes} \ell_p(G)$ with $f_n \in \ell_{p'}(G), g_n \in \ell_p(G)$, and $\sum_n \|f_n\|\|g_n\| < \infty$, we get

$$\sum_n \langle f_n, T_\alpha g_n\rangle \to \sum_n \langle f_n, T g_n\rangle.$$

We claim that $T_\alpha \otimes I \to T \otimes I$ in the weak* topology in $\mathcal{B}(\ell_p(G \times G))$. By Lemma 4.1.6, every $u \in \ell_{p'}(G \times G) \overset{\pi}{\otimes} \ell_p(G \times G)$ can be expressed as $u = \sum_n \xi_n \otimes \eta_n$ with $\xi_n \in \ell_{p'}(G \times G), \eta_n \in \ell_p(G \times G)$, and $(\sum_n \|\xi_n\|^{p'})^{1/p'}(\sum_n \|\eta_n\|^{p})^{1/p} < \infty$. Since $\ell_{p'}(G \times G) = \ell_{p'}(G) \otimes_{p'} \ell_{p'}(G)$, each ξ_n can be written as $\xi_n = \sum_{s \in G} f_s^n \otimes \delta_s^{p'}$, where $f_s^n \in \ell_{p'}(G)$ with $\sum_{s \in G} \|f_s^n\|^{p'} = \|\xi_n\|^{p'}$. In particular, f_s^n is nonzero only for at most countably many $s \in G$. Similarly each η_n can be expressed as $\eta_n = \sum_{t \in G} g_t^n \otimes \delta_t^p$, where $g_t^n \in \ell_p(G)$ with $\sum_{t \in G} \|g_t^n\|^p = \|\eta_n\|^p$. Now

$$\sum_n \langle \xi_n, (T_\alpha \otimes I)\eta_n\rangle = \sum_n \sum_{s,t \in G} \langle f_s^n \otimes \delta_s^{p'}, T_\alpha g_t^n \otimes \delta_t^p\rangle = \sum_n \sum_{s \in G} \langle f_s^n, T_\alpha g_s^n\rangle.$$

Here the last term is a countable sum and

$$\begin{aligned} \textstyle\sum_n \sum_{s \in G} \|f_s^n\|\|g_s^n\| &\leq \left(\textstyle\sum_n \sum_{s \in G} \|f_s^n\|^{p'}\right)^{1/p'} \left(\textstyle\sum_n \sum_{s \in G} \|g_s^n\|^{p}\right)^{1/p} \\ &= \left(\textstyle\sum_n \|\xi_n\|^{p'}\right)^{1/p'} \left(\textstyle\sum_n \|\eta_n\|^{p}\right)^{1/p} < \infty \end{aligned}$$

and hence $\sum_n \sum_{s \in G} \langle f_s^n, T_\alpha g_s^n\rangle \to \sum_n \sum_{s \in G} \langle f_s^n, T g_s^n\rangle$. This shows that $T_\alpha \otimes I \to T \otimes I$ in the weak* topology in $\mathcal{B}(\ell_p(G \times G))$. Finally, for every $u = \sum_n \xi_n \otimes \eta_n$ with $\xi_n \in \ell_{p'}(G \times G), \eta_n \in \ell_p(G \times G)$, and

$(\sum_n \|\xi_n\|^{p'})^{1/p'}(\sum_n \|\eta_n\|^p)^{1/p} < \infty$, we obtain

$$\begin{aligned}\langle\gamma(T_\alpha), u\rangle &= \textstyle\sum_n \langle \xi_n, W_p^{-1}(T_\alpha \otimes I)W_p\eta_n\rangle \\ &= \textstyle\sum_n \langle W_{p'}\xi_n, (T_\alpha \otimes I)W_p\eta_n\rangle \\ &\quad \to \textstyle\sum_n \langle W_{p'}\xi_n, (T \otimes I)W_p\eta_n\rangle \\ &= \langle\gamma(T), u\rangle\end{aligned}$$

since $(W_p^{-1})' = W_{p'}$ and $W_{p'}, W_p$ are isometries and $T_\alpha \otimes I \to T \otimes I$ in the weak* topology. □

Remark 4.1.8. It is easy to show that $\gamma(\lambda_s) = \lambda_{(s,s)}$ and Lemma 4.1.7 shows that the range of γ is contained in $PM_p(G)\bar{\otimes}PM_p(G) = PM_p(G \times G)$.

Let us define the p-trace $tr_p : PM_p(G \times G) \to \mathbb{C}$ by $tr_p(T) = \langle T\delta^p_{(e,e)}, \delta^{p'}_{(e,e)}\rangle$.

Proposition 4.1.9. *Define $\mathcal{E} : PM_p(G \times G) \to PM_p(G \times G)$ by $\mathcal{E} = \gamma \circ \rho$. Then*

1. *The range of $\mathcal{E}$ is the weak* closure of* $\mathrm{span}\{\lambda^p_{(s,s)} : s \in G\}$
2. *$\mathcal{E}$ is weak*-weak* continuous;*
3. *$\mathcal{E}^2 = \mathcal{E}$;*
4. *$tr_p \circ \mathcal{E} = tr_p$;*
5. *$\mathcal{E}$ is unital and p-completely contractive.*

Proof. (1) is easy to verify. (2) follows from definition of ρ and Lemma 4.1.7. (3) and (4) are immediate from Lemma 4.1.5 and Remark 4.1.8. For (5), it is obvious that $\mathcal{E}$ is unital. By Proposition 4.1.4, it suffices to show that γ is p-completely contractive. Fix $n \in \mathbb{N}$ and let $[T_{ij}] \in M_n(PM_p(G))$. To compute the norm of $[\gamma(T_{ij})]$, let $\xi_i \in \ell_{p'}(G \times G)$, $\eta_j \in \ell_p(G \times G)$, $1 \le i, j \le n$. Since $\ell_{p'}(G \times G) = \ell_{p'}(G) \overset{p'}{\otimes} \ell_{p'}(G)$, each ξ_i can be written as $\xi_i = \sum_{s\in G} f^i_s \otimes \delta^{p'}_s$, where $f^i_s \in \ell_{p'}(G)$ with $\sum_{s\in G} \|f^i_s\|^{p'} = \|\xi_i\|^{p'}$. Likewise, each η_j can be written as $\eta_j = \sum_{t \in G} g_t^j \otimes \delta_t^p$, where $g_t^j \in \ell_p(G)$ with $\sum_{t \in G} \|g_t^j\|^p = \|\eta_j\|^p$. Since

$$\begin{aligned}
& \left\langle \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} & & \\ & \gamma(T_{ij}) & \\ & & \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G \times G),\ \ell_p^n \otimes_p \ell_p(G \times G)} \\
= & \left\langle \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} & & \\ & W_p^{-1}(T_{ij} \otimes I) W_p & \\ & & \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G \times G),\ \ell_p^n \otimes_p \ell_p(G \times G)} \\
((W_p^{-1})' = W_{p'}) \quad = & \left\langle \begin{bmatrix} W_{p'}\xi_1 \\ \vdots \\ W_{p'}\xi_n \end{bmatrix}, \begin{bmatrix} & & \\ & T_{ij} \otimes I & \\ & & \end{bmatrix} \begin{bmatrix} W_p\eta_1 \\ \vdots \\ W_p\eta_n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G \times G),\ \ell_p^n \otimes_p \ell_p(G \times G)}
\end{aligned}$$

and $W_{p'}$ (respectively, W_p) is an isometric isomorphism on $\ell_{p'}(G \times G)$ (respectively, $\ell_p(G \times G)$), for norm calculation of $[\gamma(T_{ij})]$, we can replace the last term above by

$$\begin{aligned}
& \left\langle \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \begin{bmatrix} & & \\ & T_{ij} \otimes I & \\ & & \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G \times G),\ \ell_p^n \otimes_p \ell_p(G \times G)} \\
= & \textstyle\sum_{i,j=1}^n \langle \xi_i, (T_{ij} \otimes I)\eta_j \rangle_{\ell_{p'}(G \times G),\ \ell_p(G \times G)} \\
= & \textstyle\sum_{i,j=1}^n \langle \sum_{s \in G} f_s^i \otimes \delta_s^{p'}, \sum_{t \in G} T_{ij} g_t^j \otimes \delta_t^p \rangle_{\ell_{p'}(G \times G),\ \ell_p(G \times G)} \\
= & \textstyle\sum_{i,j=1}^n \sum_{s \in G} \langle f_s^i, T_{ij} g_s^j \rangle_{\ell_{p'}(G),\ \ell_p(G)} \\
= & \textstyle\sum_{s \in G} \sum_{i,j=1}^n \langle f_s^i, T_{ij} g_s^j \rangle_{\ell_{p'}(G),\ \ell_p(G)} \\
= & \textstyle\sum_{s \in G} \left\langle \begin{bmatrix} f_s^1 \\ \vdots \\ f_s^n \end{bmatrix}, \begin{bmatrix} & & \\ & T_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} g_s^1 \\ \vdots \\ g_s^n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G),\ \ell_p^n \otimes_p \ell_p(G)} .
\end{aligned}$$

Here

$$
\begin{aligned}
& \left| \sum_{s\in G} \left\langle \begin{bmatrix} f_s^1 \\ \vdots \\ f_s^n \end{bmatrix}, \begin{bmatrix} & & \\ & T_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} g_s^1 \\ \vdots \\ g_s^n \end{bmatrix} \right\rangle_{\ell_{p'}^n \overset{p'}{\otimes} \ell_{p'}(G),\ \ell_p^n \otimes_p \ell_p(G)} \right| \\
\leq \quad & \|[T_{ij}]\| \cdot \sum_{s\in G} \left(\sum_{i=1}^n \|f_s^i\|^{p'}\right)^{1/p'} \left(\sum_{j=1}^n \|g_s^j\|^p\right)^{1/p} \\
\text{(Hölder's inequality)} \quad \leq \quad & \|[T_{ij}]\| \cdot \left(\sum_{s\in G}\sum_{i=1}^n \|f_s^i\|^{p'}\right)^{1/p'} \left(\sum_{s\in G}\sum_{j=1}^n \|g_s^j\|^p\right)^{1/p} \\
= \quad & \|[T_{ij}]\| \cdot \left(\sum_{i=1}^n \sum_{s\in G} \|f_s^i\|^{p'}\right)^{1/p'} \left(\sum_{j=1}^n \sum_{s\in G} \|g_s^j\|^p\right)^{1/p} \\
= \quad & \|[T_{ij}]\| \cdot \left(\sum_{i=1}^n \|\xi_i\|^{p'}\right)^{1/p'} \left(\sum_{j=1}^n \|\eta_j\|^p\right)^{1/p}.
\end{aligned}
$$

This completes the proof. □

As an application, we get a p-analogue of Lemma 2.5 in [Haa86].

Corollary 4.1.10. *Let G be a discrete group and let T be a weak*-continuous p-completely bounded map on $PM_p(G)$ or p-completely bounded map on $PF_p(G)$. Define $\varphi : G \to \mathbb{C}$ by*

$$\varphi(s) = \langle T(\lambda_s^p)\delta_e^p, \delta_s^{p'}\rangle.$$

Then

1. *$\varphi \in M_{cb}A_p(G)$ with $\|\varphi\|_{M_{cb}A_p(G)} \leq \|T\|_{pcb}$.*
2. *If T is of finite rank, then $\varphi \in \ell_p(G) \subseteq A_p(G)$.*

Proof. (1) First of all, suppose that T is a weak*-continuous p-completely bounded map on $PM_p(G)$. For simplicity of notation, let $M = PM_p(G)$. Define

$$S = \rho \circ (T \otimes id_{PM_p(G)}) \circ \gamma,$$

where $T \otimes id_{PM_p(G)}$ as in Lemma 3.7.1, then S is a weak*-continuous p-completely bounded map on $PM_p(G)$ with $\|S\|_{pcb} \leq \|T\|_{pcb}$. Suppose that $T(\lambda_s)\delta_e = \sum_{t\in G} c_{s,t}\delta_t$ with $c_{s,t} \in \mathbb{C}$ and $\sum_{t\in G} |c_{s,t}|^p < \infty$. We claim that $S(\lambda_s) = c_{s,s}\lambda_s$. To show this, let $\psi = \sum_{g\in G} a_g\delta_g \in A_{p,c}(G)$ with $a_g \in \mathbb{C}$. Since

$$\langle T(\lambda_s) \otimes \lambda_s, \delta_g \otimes \delta_g\rangle_{PM_p(G\times G), A_p(G\times G)} = \langle \delta_g^{p'} \otimes \delta_g^{p'}, \sum_{t\in G} c_{s,t}\delta_t^p \otimes \delta_s^p\rangle_{\ell_{p'}(G\times G), \ell_p(G\times G)},$$

we obtain

$$
\begin{aligned}
& && \langle S(\lambda_s), \psi\rangle \\
& &=\;& \langle(\rho\circ(T\otimes id_{PM_p(G)})\circ\gamma)(\lambda_s),\psi\rangle \\
&(\rho=\Delta') &=\;& \langle T(\lambda_s)\otimes\lambda_s, \Delta\psi\rangle \\
& &=\;& \textstyle\sum_{g\in G} a_g\langle T(\lambda_s)\otimes\lambda_s, \delta_g\otimes\delta_g\rangle \\
& &=\;& a_s c_{s,s} \\
& &=\;& \langle c_{s,s}\lambda_s, \psi\rangle
\end{aligned}
$$

and this proves the claim. On the other hand, by definition, $\varphi(s) = c_{s,s}$ and this shows that $S = M_\varphi$ and therefore $\|\varphi\|_{M_{cb}A_p(G)} = \|S\|_{pcb} \le \|T\|_{pcb}$.

Now assume that T is a p-completely bounded map on $PF_p(G)$. Define

$$S = \rho\circ(T\otimes id_{PF_p(G)})\circ\gamma|_{PF_p(G)},$$

where $T\otimes id_{PF_p(G)}$ as in Lemma 3.7.1, then S is a well-defined p-completely bounded map on $PF_p(G)$ the result follows using the same argument as above.

(2) We adapt the idea used in [Haa86]. Without loss of generality, we can assume that $T = f\otimes b$, where $f\in PM_p(G)'$ and $b\in PM_p(G)$. Then $T(\lambda_s) = f(\lambda_s)b$ and

$$\varphi(s) = \langle T(\lambda_s)\delta_e^p, \delta_s^{p'}\rangle = f(\lambda_s)\langle b\delta_e^p, \delta_s^{p'}\rangle.$$

This shows that $\varphi(s)$ is the s-component of $b\delta_e^p \in \ell_p(G)$. □

4.2 p-AP and p-OAP

Let G be a locally compact group. We say that G has the *p-approximation property (p-AP)* if there is a net $\{u_\alpha\}$ in $A_p(G)$ such that $u_\alpha \to 1$ in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$-topology. Let V be a p-operator space. We say that V has the *p-operator space approximation property (p-OAP)* if there is a net (T_α) of bounded finite rank maps on V such that for every $a\in\mathcal{K}(\ell_p)\overset{\vee_p}{\otimes} V$, $(id_{\mathcal{K}(\ell_p)}\otimes T_\alpha)(a)\to a$ in the norm topology (See Lemma 3.7.1 for the definition of $id_{\mathcal{K}(\ell_p)}\otimes T_\alpha$). V is said to have the *strong p-operator space approximation property (strong p-OAP)* if there is a net (T_α) of finite rank maps on V such that for every $a\in\mathcal{B}(\ell_p)\overset{\vee_p}{\otimes} V$, $(id_{\mathcal{B}(\ell_p)}\otimes T_\alpha)(a)\to a$ in the norm topology.

Remark 4.2.1. 2-AP is the same as the approximation property studied in [HK94]. Similarly, 2-OAP is the same as the operator space approximation property in [ER00].

Lemma 4.2.2. *Suppose that $T_\alpha, T \in \mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G)))$ and that $T_\alpha \to T$ in the stable point-norm topology. Then for any $L_p(\mu)$ and for any $c \in \mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$,*

$$id_{\mathcal{K}(L_p(\mu))} \otimes T_\alpha(c) \to id_{\mathcal{K}(L_p(\mu))} \otimes T(c)$$

in norm.

Proof. By Theorem 2.3.11, we have a p-complete isometry

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G) = L_p^c(\mu) \overset{h_p}{\otimes} PF_p(G) \overset{h_p}{\otimes} L_{p'}^r(\mu).$$

By Proposition 2.4.5, $c \in \mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$ can be written as $v \odot x \odot w$, where $v \in M_{1,\infty}(L_p^c(\mu))$, $x \in \mathcal{K}_\infty(PF_p(G))$, and $w \in M_{\infty,1}(L_{p'}^r(\mu))$ and hence

$$\begin{aligned} & \|id_{\mathcal{K}(L_p(\mu))} \otimes (T_\alpha - T)(c)\| \\ = \; & \|v \odot (id_{\mathcal{K}(\ell_p)} \otimes (T_\alpha - T))(x) \odot w\| \\ \leq \; & \|v\| \|(id_{\mathcal{K}(\ell_p)} \otimes (T_\alpha - T))(x)\| \|w\| \\ \to \; & 0. \end{aligned}$$

□

Lemma 4.2.3. *Suppose that $T_\alpha, T \in \mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G)))$ and that $T_\alpha \to T$ in the stable point-norm topology. Then $T_\alpha \to T$ in the $\sigma(\mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G))), PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G)))$ topology.*

Proof. By assumption, for any $a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$, $id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(a) \to id_{\mathcal{K}(\ell_p)} \otimes T(a)$ in norm. Let $u \in PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$, then by Proposition 2.4.4, we may assume that $u = \gamma(a \otimes f)\delta = \sum_{i,j,k,l} \gamma_{ik} a_{ij} \otimes f_{kl} \delta_{jl}$, where $\gamma \in M_{1,\infty^2}$, $\delta \in M_{\infty^2,1}$, $a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$, and $f \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} \mathcal{N}(\ell_p(G))$. The operators $\bar{\gamma} \in M_{1,\infty}$ and $\bar{\delta} \in M_{\infty,1}$ defined by

$$\bar{\gamma} = [\gamma^1 \; \gamma^2 \; \cdots \;], \quad \text{where} \quad \gamma^i = [\gamma_{i1} \; \gamma_{i2} \; \cdots \;]$$

and

$$\bar{\delta} = \begin{bmatrix} \delta^1 \\ \delta^2 \\ \vdots \\ \\ \end{bmatrix}, \quad \text{where} \quad \delta^l = \begin{bmatrix} \delta_{1l} \\ \delta_{2l} \\ \vdots \\ \\ \end{bmatrix}$$

have the same norm as γ and δ. Let $\gamma^n, \bar{\gamma}^n, a^n, f^n, \delta^n$, and $\bar{\delta}^n$ be the truncations of $\gamma, \bar{\gamma}, a, f, \delta$, and $\bar{\delta}$, respectively, then it follows easily that u is the norm limit of $\gamma^n(a^n \otimes f^n)\delta^n$ in $PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$ and that $\bar{\gamma} \odot f \odot \bar{\delta}$ is the norm limit of $\bar{\gamma}^n \odot f^n \odot \bar{\delta}^n$ in $\ell^r_{p'} \overset{h_p}{\otimes} \mathcal{N}(\ell_p(G)) \overset{h_p}{\otimes} \ell^c_p = \mathcal{N}_\infty \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$. Since

$$\begin{aligned}
\langle T_\alpha, u\rangle &= \lim_{n\to\infty} \sum_{i,j,k,l\leq n} \gamma_{ik}\langle T_\alpha(a_{ij}), f_{kl}\rangle\delta_{jl} \\
&= \lim_{n\to\infty} \langle (T_\alpha)_n(a^n), \bar{\gamma}^n \odot f^n \odot \bar{\delta}^n\rangle \\
&= \lim_{n\to\infty} \langle id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(a), \bar{\gamma}^n \odot f^n \odot \bar{\delta}^n\rangle \\
&= \langle id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(a), \bar{\gamma} \odot f \odot \bar{\delta}\rangle,
\end{aligned}$$

the result follows. $\square$

Lemma 4.2.4. *Suppose that $T_\alpha, T \in \mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G)))$. Then*

$$T_\alpha \to T \text{ in the } \sigma(\mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G))), PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))) \text{ topology}$$

if and only if for every $b \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))$,

$$\langle id_{\mathcal{B}(\ell_p)} \otimes T_\alpha(b), \varphi\rangle \to \langle id_{\mathcal{B}(\ell_p)} \otimes T(b), \varphi\rangle.$$

Proof. Let $u \in PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$, then by Proposition 2.4.4, we may assume that $u = \gamma(b \otimes f)\delta = \sum_{i,j,k,l} \gamma_{ik} b_{ij} \otimes f_{kl}\delta_{jl}$, where $\gamma \in M_{1,\infty^2}$, $\delta \in M_{\infty^2,1}$, $b \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$, and $f \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} \mathcal{N}(\ell_p(G))$. The operators $\bar{\gamma} \in M_{1,\infty}$ and $\bar{\delta} \in M_{\infty,1}$ defined by

$$\bar{\gamma} = [\gamma^1 \ \gamma^2 \ \cdots\ ], \quad \text{where} \quad \gamma^i = [\gamma_{i1} \ \gamma_{i2} \ \cdots\ ]$$

and

$$\bar{\delta} = \begin{bmatrix} \delta^1 \\ \delta^2 \\ \vdots \\ \ \end{bmatrix}, \quad \text{where} \quad \delta^l = \begin{bmatrix} \delta_{1l} \\ \delta_{2l} \\ \vdots \\ \ \end{bmatrix}$$

have the same norm as γ and δ. Let $\gamma^n, \bar{\gamma}^n, b^n, f^n, \delta^n$, and $\bar{\delta}^n$ be the truncations of $\gamma, \bar{\gamma}, b, f, \delta$, and $\bar{\delta}$, respectively, then it follows easily that u is the norm limit of $\gamma^n(b^n \otimes f^n)\delta^n$ in $PF_p(G) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$ and that $\bar{\gamma} \odot f \odot \bar{\delta}$ is the norm limit of $\bar{\gamma}^n \odot f^n \odot \bar{\delta}^n$ in $\ell^r_{p'} \overset{h_p}{\otimes} \mathcal{N}(\ell_p(G)) \overset{h_p}{\otimes} \ell^c_p = \mathcal{N}_\infty \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G)) = \mathcal{N}(\ell_p \otimes_p \ell_p(G))$.

Since

$$\begin{aligned}
\langle T_\alpha, u\rangle &= \lim_{n\to\infty} \sum_{i,j,k,l\le n} \gamma_{ik}\langle T_\alpha(b_{ij}), f_{kl}\rangle \delta_{jl} \\
&= \lim_{n\to\infty} \langle (T_\alpha)_n(b^n), \bar{\gamma}^n \odot f^n \odot \bar{\delta}^n\rangle \\
&= \lim_{n\to\infty} \langle id_{\mathcal{B}(\ell_p)} \otimes T_\alpha(b), \bar{\gamma}^n \odot f^n \odot \bar{\delta}^n\rangle \\
&= \langle id_{\mathcal{B}(\ell_p)} \otimes T_\alpha(b), \bar{\gamma} \odot f \odot \bar{\delta}\rangle,
\end{aligned}$$

we get the result. $\square$

Proposition 4.2.5. *Suppose that $T_\alpha, T \in \mathcal{CB}_p(PF_p(G), \mathcal{B}(\ell_p(G)))$ and that $T_\alpha \to T$ in the stable point-norm topology. Then for any $c \in \mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi \in \mathcal{N}(L_p(\mu) \otimes_p \ell_p(G))$, we have*

$$\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_\alpha(c), \varphi\rangle \to \langle id_{\mathcal{B}(L_p(\mu))} \otimes T(c), \varphi\rangle.$$

Proof. Suppose $c \in \mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi \in \mathcal{N}(L_p(\mu) \otimes_p \ell_p(G))$. By Proposition 2.4.4, we can write $\varphi = \gamma([f_{ij}] \otimes [g_{kl}])\delta = \sum_{i,j,k,l} \gamma_{ik} f_{ij} \otimes g_{kl}\delta_{jl}$, where $\gamma \in M_{1,\infty^2}$, $\delta \in M_{\infty^2,1}$, $f = [f_{ij}] \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} \mathcal{N}(L_p(\mu))$, and $g = [g_{kl}] \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} \mathcal{N}(\ell_p(G))$. Note that the map $\tilde{f} : \mathcal{B}(L_p(\mu)) \otimes PF_p(G) \to \mathcal{K}(\ell_p) \otimes PF_p(G)$ defined by

$$\tilde{f}(x \otimes y) = [f_{ij}(x)y]$$

extends to a map from $\mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$ to $\mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$. Now, using the same notation as in the proof of Lemma 4.2.3, we claim that

$$\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_\alpha(c), \varphi\rangle = \langle id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(\tilde{f}(c)), \bar{\gamma} \odot g \odot \bar{\delta}\rangle. \tag{4.3}$$

It suffices to consider $c = a \otimes b$, where $a \in \mathcal{B}(L_p(\mu))$ and $b \in PF_p(G)$. Then

$$\begin{aligned}
\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_\alpha(c), \varphi\rangle &= \lim_{n\to\infty} \sum_{i,j,k,l\le n} \langle a \otimes T_\alpha(b), \gamma_{ik} f_{ij} \otimes g_{kl}\delta_{jl}\rangle \\
&= \lim_{n\to\infty} \sum_{i,j,k,l\le n} \gamma_{ik}\langle f_{ij}, a\rangle \langle g_{kl}, T_\alpha(b)\rangle \delta_{jl} \\
&= \lim_{n\to\infty} \langle [f^n_{ij}(T_\alpha(a))] \otimes b, \bar{\gamma}^n \odot g^n \odot \bar{\delta}^n\rangle \\
&= \langle id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(\tilde{f}(a \otimes b)), \bar{\gamma} \odot g \odot \bar{\delta}\rangle,
\end{aligned}$$

and the claim is proved. Finally, the result follows since $id_{\mathcal{K}(\ell_p)} \otimes T_\alpha(\tilde{f}(c)) \to id_{\mathcal{K}(\ell_p)} \otimes T(\tilde{f}(c))$ in norm. $\square$

Lemma 4.2.6. *For any $x \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G) \overset{\vee_p}{\otimes} PF_p(G)$ and any $\varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))$, we have*

$$\langle id_{\mathcal{B}(\ell_p)} \otimes \rho(x), \varphi\rangle = \langle x, id_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p}(\varphi)\rangle,$$

where $id_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p} : \mathcal{N}(\ell_p) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G)) \to \mathcal{N}(\ell_p) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G)) \overset{\wedge_p}{\otimes} \mathcal{N}(\ell_p(G))$.

Proof. It is enough to verify this for $x = T \otimes \lambda_s \otimes \lambda_t$ and $\varphi = \psi \otimes \delta_w^{p'} \otimes \delta_z^p$, where $T \in \mathcal{B}(\ell_p)$, $\lambda_s, \lambda_t \in PF_p(G)$, $\psi \in \mathcal{N}(\ell_p)$, $\delta_w^{p'} \in \ell_{p'}(G)$, and $\delta_z^p \in \ell_p(G)$. Indeed, by Lemma 4.1.5,

$$\langle id_{\mathcal{B}(\ell_p)} \otimes \rho(x), \varphi\rangle = \delta_s(t)\langle T \otimes \lambda_s, \varphi\rangle = \delta_s(t)\delta_w(sz)\langle T, \psi\rangle.$$

On the other hand

$$\langle x, id_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p}(\varphi)\rangle = \langle x, \psi \otimes \delta_{w,w}^{p'} \otimes \delta_{z,z}^p\rangle = \langle T, \psi\rangle \delta_w(sz)\delta_w(tz),$$

and the result follows. ☐

Theorem 4.2.7. *If $PF_p(G)$ has the p-OAP, then G has the p-AP.*

Proof. Let (T_α) be a net of bounded finite rank maps on $PF_p(G)$ such that $(id_{\mathcal{K}(\ell_p)} \otimes T_\alpha)(a) \to a$ in the norm topology for every $a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G) = \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$. Define $u_\alpha : G \to \mathbb{C}$ by $u_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_e^p, \delta_s^{p'}\rangle$, then by Corollary 4.1.10, $u_\alpha \in \ell_p(G) \subseteq A_p(G)$ and moreover, it is easy to check that

$$\bar{M}_{u_\alpha} = \rho \circ (id_{PF_p(G)} \otimes T_\alpha) \circ \gamma|_{PF_p(G)}.$$

We claim that $u_\alpha \to 1$ in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ topology. Let $a \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))$ (See Remark 3.7.7). Since

$$\omega_{a,\varphi}(u_\alpha) = \langle id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{u_\alpha}(a), \varphi\rangle,$$

we need to show that

$$\langle id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{u_\alpha}(a), \varphi\rangle \to \langle a, \varphi\rangle. \tag{4.4}$$

Since

$$id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{u_\alpha} = (id_{\mathcal{B}(\ell_p)} \otimes \rho) \circ (id_{\mathcal{B}(\ell_p)} \otimes (id_{PF_p(G)} \otimes T_\alpha)) \circ (id_{\mathcal{B}(\ell_p)} \otimes \gamma|_{PF_p(G)}),$$

from Lemma 4.2.6, (4.4) becomes

$$\langle (id_{\mathcal{B}(\ell_p)} \otimes (id_{PF_p(G)} \otimes T_\alpha)) \circ (id_{\mathcal{B}(\ell_p)} \otimes \gamma|_{PF_p(G)})(a),\ id_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p}(\varphi)\rangle$$

and the result follows from Proposition 4.2.5. $\square$

We can summarize this section in the following

Corollary 4.2.8. *Let G be a discrete group. Then the following are equivalent*

1. *G has the p-AP;*
2. *there exists a net $\{\varphi_\alpha\} \subseteq A_{p,c}(G)$ such that $id_{\mathcal{K}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{K}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$;*
3. *$PF_p(G)$ has the p-OAP;*
4. *there exists a net $\{\varphi_\alpha\} \subseteq A_{p,c}(G)$ such that $id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$;*
5. *$PF_p(G)$ has the strong p-OAP;*

Proof. *3*$\Longrightarrow$ *1* was proved in Theorem 4.2.7. Since *4*$\Longrightarrow$ *5*, *4*$\Longrightarrow$ *2*, *2*$\Longrightarrow$ *3*, and *5*$\Longrightarrow$ *3* are clear, to complete the proof, we only need to show that *1*$\Longrightarrow$ *4*. Suppose that G has the p-AP, then there exists a net $\{\psi_\beta\} \subseteq A_p(G)$ such that $\psi_\beta \to 1$ with respect to $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$-topology. Moreover we can assume that $\{\psi_\beta\} \subseteq A_{p,c}(G)$. Then $\bar{M}_{\psi_\beta}$ is a net of finite rank maps on $PF_p(G)$ such that for every $a \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$ and $\varphi \in (\mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G))'$,

$$\langle id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\psi_\beta}(a), \varphi\rangle = \langle \omega_{a,\varphi}, \psi_\beta\rangle \to \langle \omega_{a,\varphi}, 1\rangle = \langle a, \varphi\rangle.$$

Therefore $id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\psi_\beta}(a) \to a$ in the weak topology. By a standard convexity argument, we can choose a net φ_α such that $id_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{B}(\ell_p) \overset{\vee_p}{\otimes} PF_p(G)$.
This completes the proof. $\square$

4.3 p-Weak Amenability and p-CBAP

In this section we study properties which are stronger that those in the previous section. A locally compact group G is said to be *p-weakly amenable* if $A_p(G)$ has an approximate identity that is bounded in the

$M_{cb}A_p(G)$ norm, i.e., if there exist a net $\{u_\alpha\}$ in $A_p(G)$ and a constant K such that $\|u_\alpha v - v\|_{A_p(G)} \to 0$ for all $v \in A_p(G)$ and such that $\|u_\alpha\|_{M_{cb}A_p(G)} < K$ for all α. A p-operator space V is said to have the *p-completely bounded approximation property (p-CBAP)* if there is a positive number K such that the identity map id_V on V can be approximated in the point-norm topology by a net $\{T_\alpha\}$ of finite rank p-completely bounded maps whose p-cb norms are bounded by K.

Remark 4.3.1. 1. 2-weak amenability is the same as weak amenability studied in Haagerup's unpublished paper [Haa86]. Similarly, 2-CBAP is the same as CBAP.

2. Following Theorem 11.3.3 in [ER00], one can show that if V has the p-CBAP, then V has the p-OAP.

The following is the main result in this section and gives a link between p-weak amenability and p-CBAP.

Theorem 4.3.2. *Let G be a discrete group. Then the following are equivalent*

1. *$A_p(G)$ has an approximate identity $\{\varphi_\alpha\}$ such that $\|\varphi_\alpha\|_{M_{cb}A_p(G)} < K$;*

2. *there exists a net $\{T_\alpha\}$ of finite rank maps on $PF_p(G)$ such that $\|T_\alpha\| < K$ and $\|T_\alpha(x) - x\| \to 0$ for all $x \in PF_p(G)$;*

3. *there exists a net $\{T_\alpha\}$ of weak*-continuous finite rank maps on $PM_p(G)$ such that $\|T_\alpha\| < K$ and $\langle T_\alpha(x) - x, \varphi\rangle \to 0$ for all $x \in PM_p(G)$.*

Proof. *1*$\Longrightarrow$ *3*: We may assume that $\varphi_\alpha \in A_{p,c}(G)$. Then $\{m_{\varphi_\alpha}\}$ are finite rank p-cb maps on $A_p(G)$ such that $\|m_{\varphi_\alpha}\|_{pcb} < K$ and $\|m_{\varphi_\alpha}(\omega) - \omega\|_{A_p(G)} \to 0$ for all $\omega \in A_p(G)$. Then $T_\alpha := m^*_{\varphi_\alpha}$ satisfy *3*.
1$\Longrightarrow$ *2*: First note that for each $s \in G$, $\|\delta_s\|_{A_p(G)} = 1$, see Remark 3.3.2. From *1*, we can conclude that $|\varphi_\alpha(s) - 1| = \|(\varphi_\alpha(s) - 1)\delta_s\|_{A_p(G)} = \|\varphi_\alpha\delta_s - \delta_s\|_{A_p(G)} \to 0$. Since $\bar{M}_{\varphi_\alpha}(\lambda_s) = \varphi_\alpha(s)\lambda_s$, $\|\bar{M}_{\varphi_\alpha}\|_{pcb} < K$ and

$$\|\bar{M}_{\varphi_\alpha}(\lambda_s) - \lambda_s\| = \|\varphi_\alpha(s)\lambda_s - \lambda_s\| = |\varphi_\alpha(s) - 1|\|\lambda_s\| \to 0.$$

Then we can obtain *2*.
2$\Longrightarrow$ *1*: Given $\{T_\alpha\}$, define $\varphi_\alpha : G \to \mathbb{C}$ by $\varphi(s) = \langle T(\lambda^p_s)\delta^p_e, \delta^{p'}_s\rangle$. By Corollary 4.1.10, $\varphi_\alpha \in A_p(G) \subseteq M_{cb}A_p(G)$ with $\|\varphi_\alpha\|_{M_{cb}A_p(G)} < K$. Since $\|T_\alpha(\lambda^p_s) - \lambda^p_s\| \to 0$, we can conclude that $|\varphi_\alpha - 1| \to 0$. Then for any $\psi \in A_{p,c}(G)$, we get $\|\varphi_\alpha\psi - \psi\|_{A_p(G)} \to 0$. This implies that we have this true for all $\psi \in A_p(G)$.
3$\Longrightarrow$ *1*: For each $s \in G$, it is easy to see that the linear functional

$$a \in PM_p(G) \mapsto \langle a\delta^p_e, \delta^{p'}_s\rangle$$

is contained in $A_p(G)$. Then

$$\varphi_\alpha(s) = \langle T_\alpha(\lambda_s^p)\delta_e^p, \delta_s^{p'}\rangle \to \langle \delta_s^p, \delta_s^{p'}\rangle = 1.$$

Now we can get the proof as in *2*$\Longrightarrow$ *1*. $\square$

4.4 Comparison with the Classical Case

We take a closer look at Theorem 3.3.8.

Theorem 4.4.1. *Let G be a discrete group and μ a measure. Let $\mathcal{B}(L_p(\mu))\bar{\otimes}PM_p(G)$ denote the weak* closure of $\mathcal{B}(L_p(\mu))\otimes_{\mathrm{alg}} PM_p(G)$ in $\mathcal{B}(L_p(\mu)\otimes_p \ell_p(G))$. Then the following properties of a function $\varphi : G \to \mathcal{B}(L_p(\mu))$ are equivalent:*

1. *there exists a weak*-continuous p-complete contraction $M_\varphi : PM_p(G) \to \mathcal{B}(L_p(\mu))\bar{\otimes}PM_p(G)$ such that*

$$M_u(\lambda_p(s)) = \varphi(s) \otimes \lambda_p(s);$$

2. *there exists a p-complete contraction $\bar{M}_\varphi : PF_p(G) \to \mathcal{B}(L_p(\mu)) \overset{\vee_p}{\otimes} PF_p(G)$ such that*

$$\bar{M}_u(\lambda_p(s)) = \varphi(s) \otimes \lambda_p(s);$$

3. *There exist an SQ_p space E and bounded maps $\alpha : G \to \mathcal{B}(L_p(\mu), E)$ and $\beta : G \to \mathcal{B}(E, L_p(\mu))$ such that $\sup_{t\in G}\|x(t)\| \le 1$, $\sup_{s\in G}\|y(s)\| \le 1$ and*

$$\forall s,t \in G, \quad \varphi(st^{-1}) = y(s)x(t).$$

Proof. *1*$\Longrightarrow$*2* is obvious.

2$\Longrightarrow$*3* First of all, note that $\lambda(\theta)\otimes\varphi(\theta)$ makes sense by Theorem 7.9 in [DF93]. By Theorem 4.1 in [Daw10], there are an SQ_p space K, a p-representation $\pi : PF_p(G) \to \mathcal{B}(K)$, and operators $U : \ell_p(G)\otimes_p L_p(\mu) \to K$, $V : K \to \ell_p(G)\otimes_p L_p(\mu)$ with $\|U\|, \|V\| \le 1$ such that

$$\forall \theta \in G, \quad \lambda(\theta)\otimes\varphi(\theta) = M_\varphi(\lambda(\theta)) = V\pi(\lambda(\theta))U.$$

For $t \in G$, define $x(t) \in \mathcal{B}(L_p(\mu), K)$ by

$$x(t)f = \pi(\lambda(t^{-1}))U(\delta_t^p \otimes f), \quad f \in L_p(\mu).$$

For $s \in G$, define $y(s) \in \mathcal{B}(K, L_p(\mu))$ by

$$y(s)k = R_s V \pi(\lambda(s))k, \quad k \in K,$$

where $R_s \in \mathcal{B}(\ell_p(G) \otimes_p L_p(\mu), L_p(\mu))$ defined by $R_s\left(\sum_{t\in G} \delta_t \otimes g_t\right) = g_s$. Note that $\sup_{t\in G} \|x(t)\| \leq 1$ and $\sup_{s\in G} \|y(s)\| \leq 1$. Now for all $f \in L_p(\mu)$ and for all $f' \in L_{p'}(\mu)$,

$$\begin{aligned}
& \langle y(s)x(t)f, f'\rangle \\
= \quad & \langle R_s V \pi(\lambda(s))\pi(\lambda(t^{-1}))U(\delta_t^p \otimes f), f'\rangle \\
= \quad & \langle R_s V \pi(\lambda(st^{-1}))U(\delta_t^p \otimes f), f'\rangle \\
= \quad & \langle R_s \lambda(st^{-1}) \otimes \varphi(st^{-1})(\delta_t^p \otimes f), f'\rangle \\
= \quad & \langle R_s(\delta_s^p \otimes \varphi(st^{-1})f), f'\rangle \\
= \quad & \langle \varphi(st^{-1})f, f'\rangle
\end{aligned}$$

and the result follows.

(b)$\Longrightarrow$(a) Suppose $K \subseteq L_p(\nu)/E$ for some measure ν and a subspace E of $L_p(\nu)$. Let $Q : L_p(\nu) \to L_p(\nu)/E$ denote the quotient mapping. By the argument in Section 7.3 in [DF93], for any $T \in \mathcal{B}(\ell_p(G))$, $T \otimes id_{L_p(\nu)} \in \mathcal{B}(\ell_p(G) \otimes_p L_p(\nu))$. By Proposition 7.4 in [DF93], $T \otimes id_{L_p(\nu)}$ induces a continuous mapping $T \otimes id_{L_p(\nu)/E} : \ell_p(G) \otimes_p L_p(\nu)/E \to \ell_p(G) \otimes_p L_p(\nu)/E$. Finally let $\pi : PF_p(G) \to \mathcal{B}(\ell_p(G) \otimes_p K)$ denote the operator defined by $\pi(T) = T \otimes id_K$. Define operators U and V by

$$U : \ell_p(G) \otimes_p L_p(\mu) \to \ell_p(G) \otimes_p K, \quad \delta_t^p \otimes f \mapsto \delta_t^p \otimes x(t)f$$

and

$$V : \ell_p(G) \otimes_p K \to \ell_p(G) \otimes_p L_p(\mu), \quad \delta_s^p \otimes k \mapsto \delta_s^p \otimes y(s)k,$$

then $\|U\|, \|V\| \leq 1$ and for all $f \in L_p(\mu)$ and for all $f \in L_{p'}(\mu)$, we get

$$\begin{aligned}
& \langle V\pi(\lambda(\theta))U(\delta_t^p \otimes f), \delta_s^{p'} \otimes f'\rangle \\
= \quad & \langle V\pi(\lambda(\theta))(\delta_t^p \otimes x(t)f), \delta_s^{p'} \otimes f'\rangle \\
= \quad & \langle V(\delta_{\theta t}^p \otimes x(t)f), \delta_s^{p'} \otimes f'\rangle \\
= \quad & \langle \delta_{\theta t}^p \otimes y(s)x(t)f, \delta_s^{p'} \otimes f'\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \langle \varphi(st^{-1})f, f' \rangle, & \text{if } \theta = st^{-1} \\ 0, & \text{otherwise} \end{cases} \\
&= \langle (\lambda(\theta) \otimes \varphi(\theta))(\delta_t^p \otimes f), \delta_s^{p'} \otimes f' \rangle
\end{aligned}
$$

hence $M_\varphi(\lambda(\theta)) = V\pi(\lambda(\theta))U$ and the result follows by the converse of Theorem 4.1 in [Daw10] once we can prove that the mapping $T \mapsto \pi(T)$ (hence $T \mapsto V\pi(T)U$) is p-completely contractive. To this end, let $[T_{ij}] \in M_n(PF_p(G))$ with $\|[T_{ij}]\| \le 1$. Write $T_{ij}\delta_t = \sum_{s \in G} a_{i,s}^{j,t} \delta_s$ and for each $1 \le j \le n$, let $\xi_j = \sum_{t \in G} \xi_t^j \delta_t \in \ell_p(G)$ with $\left(\sum_{j=1}^n \|\xi_j\|^p\right)^{1/p} = \left(\sum_{j,t} |\xi_t^j|^p\right)^{1/p} \le 1$. Then we get

$$
\begin{aligned}
1 &\ge \left\| \begin{bmatrix} & & \\ & T_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\|^p \\
&= \sum_i \left\| \sum_j T_{ij} \xi_j \right\|^p \\
&= \sum_{i,s} \left| \sum_{j,t} a_{i,s}^{j,t} \xi_t^j \right|^p .
\end{aligned} \tag{4.5}
$$

For each $1 \le j \le n$, let $k_j = \sum_{t \in G} \delta_t \otimes k_t^j \in \ell_p(G) \otimes_p K$ with $\left(\sum_{j=1}^n \|k_j\|^p\right)^{1/p} = \left(\sum_{j,t} \|k_t^j\|^p\right)^{1/p} \le 1$. Then we get

$$
\begin{aligned}
&\left\| \begin{bmatrix} & & \\ & \pi(T_{ij}) & \\ & & \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \right\|^p \\
&= \sum_i \left\| \sum_j \pi(T_{ij}) k_j \right\|^p \\
&= \sum_{i,s} \left\| \sum_{j,t} a_{i,s}^{j,t} k_t^j \right\|^p .
\end{aligned}
$$

By Theorem 3.2 in [LeM96] (originally due to [Kwa72]), the last term $\sum_{i,s} \left\| \sum_{j,t} a_{i,s}^{j,t} k_t^j \right\|^p$ is dominated by $\sum_{i,s} \left| \sum_{j,t} a_{i,s}^{j,t} \xi_t^j \right|^p$, which is in turn dominated by 1 by (4.5). This completes the proof. □

Remark 4.4.2. Suppose $2 \le q \le p$ or $p \le q \le 2$. By [Her71, Corollary 2] and Theorem 4.4.1, it follows that the identity mapping $J_{q,p} : M_{cb}A_q(G) \to M_{cb}A_p(G)$ is contractive. Taking the adjoint, we have that

$J'_{q,p} : (M_{cb}A_p(G))' \to (M_{cb}A_q(G))'$ is also contractive and that $\|f\|_{Q,p} \geq \|f\|_{Q,q}$ for all $f \in L_1(G)$ (See the discussion in the beginning of Section 3.6). Therefore $Q_{pcb}(G) \subseteq Q_{qcb}(G)$ contractively.

Proposition 4.4.3. *Suppose* $2 \leq q \leq p$ *or* $p \leq q \leq 2$. *If a discrete group* G *is* q*-weakly amenable, then it is also* p*-weakly amenable.*

Proof. Since G is q-weakly amenable, there exist an approximate identity $\{\omega_\alpha\} \in A_{q,c}(G)$ and $C > 0$ such that $\|\omega_\alpha\|_{M_{cb}A_q(G)} \leq C$ for all α. By Remark 4.4.2, we see that $\omega_\alpha \in M_{cb}A_p(G)$ with $\|\omega_\alpha\|_{M_{cb}A_p(G)} \leq C$ for each α. Define M_{ω_α} on $PF_p(G)$ by $M_{\omega_\alpha}(\lambda_p(s)) = \omega_\alpha(s)\lambda_p(s)$, then by Theorem 3.3.8, it follows that $\{M_{\omega_\alpha}\}$ are finite rank bounded maps on $PF_p(G)$ with $\|M_{\omega_\alpha}\| \leq C$ for all α. Since

$$|\omega_\alpha(s) - 1| = \|\omega_\alpha\delta_s - \delta_s\|_{A_q(G)} \to 0$$

for each $s \in G$, we conclude that $\|M_{\omega_\alpha}(x) - x\| \to 0$ for all $x \in PF_p(G)$. Now the result follows from Theorem 4.3.2. □

Proposition 4.4.4. *Suppose* $2 \leq q \leq p$ *or* $p \leq q \leq 2$. *If a discrete group* G *has the* q*-AP, then it also has the* p*-AP.*

Proof. If G has the q-AP, then there exists a net $\{u_\alpha\} \subseteq A_{q,c}(G)$ such that $\langle u_\alpha, \omega\rangle \to \langle 1, \omega\rangle$ for all $\omega \in Q_{qcb}(G)$. Since G is discrete, $\{u_\alpha\} \subseteq A_{p,c}(G)$ and by Remark 4.4.2, we have $\langle u_\alpha, \omega\rangle \to \langle 1, \omega\rangle$ for all $\omega \in Q_{pcb}(G)$. □

4.5 1-Nuclearity

Lemma 4.5.1. *Let* $x = [x_{ij}] \in \mathcal{B}(\ell_1^m)$. *Then*

$$\|x\|_{\mathcal{B}(\ell_1^m)} = \max_{1\leq j\leq m} \left\{ \sum_{i=1}^m |x_{ij}| \right\}.$$

Proof. Easy. □

Suppose G is a countable discrete group. Write $G = \bigcup_{n=1}^\infty F_n$ with $\{e\} \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$, each F_n finite, and $|F_n| = N_n$. For each $n \in \mathbb{N}$, define $P_n \in \mathcal{B}(\ell_1(G))$ by

$$P_n\left(\sum_{s\in G} a_s\delta_s\right) = \sum_{s\in F_n} a_s\delta_s,$$

then we can identify $P_n\mathcal{B}(\ell_1(G))P_n$ with $M_{N_n} = \mathcal{B}(\ell_1^{N_n})$.

$$\begin{tikzcd} & P_n\mathcal{B}(\ell_1(G))P_n \arrow[dr, "\psi_n"] & \\ \ell_1(G) \arrow[ur, "\varphi_n"] & & \ell_1(G) \end{tikzcd}$$

Define

$$\varphi_n : \ell_1(G) \to P_n\mathcal{B}(\ell_1(G))P_n, \qquad f \mapsto P_n\lambda(f)P_n$$

and

$$\psi_n : P_n\mathcal{B}(\ell_1(G))P_n \to \ell_1(G), \qquad [a_{u,v}]_{u,v\in F_n} \mapsto \sum_{u\in F_n} a_{u,e}\delta_u.$$

Claim 4.5.2. *For all $f \in \ell_1(G)$, $\|\psi_n \circ \varphi_n(f) - f\| \to 0$.*

Proof. For $u \in F_n$, we have $\langle \delta_u, \varphi_n(f)\delta_e\rangle = \langle \delta_u, P_n f\rangle = f(u)$ and this calculation shows that $\psi_n \circ \varphi_n(f) = \chi_{F_n} f$. □

Note that $\ell_1(G)$ can be identified with $\lambda(\ell_1(G)) \subseteq \mathcal{B}(\ell_1(G))$ as convolution operators and naturally equipped with the 1-operator space structure inherited from this inclusion. For notational convenience, we fix n and drop the subscript n.

Claim 4.5.3. *φ is 1-completely contractive.*

Proof. Clear. □

Claim 4.5.4. *ψ is 1-completely contractive.*

Proof. Fix $k \in \mathbb{N}$. Let $[T_{ij}]_{1\le i,j\le k} \in M_k(P_n\mathcal{B}(\ell_1(G))P_n)$ with $\|[T_{ij}]\| \le 1$. We need to show that $\|[\psi(T_{ij})]\| \le 1$. To this end, suppose $g_j \in \ell_1(G)$ with $\sum_{j=1}^k \|g_j\|_1 \le 1$, then

$$\begin{aligned} & \left\| \begin{bmatrix} & & \\ & \psi(T_{ij}) & \\ & & \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \right\| \\ \le & \sum_{i=1}^k \left\| \sum_{j=1}^k \psi(T_{ij}) \star g_j \right\| \\ \le & \sum_{i=1}^k \sum_{j=1}^k \|\psi(T_{ij})\|_1 \|g_j\|_1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \|g_j\|_1 \sum_{i=1}^{k} \|\psi(T_{ij})\|_1 \\
\text{(By Lemma 4.5.1)} \quad &\leq \sum_{j=1}^{k} \|g_j\|_1 \, \|[T_{ij}]\| \\
&\leq 1.
\end{aligned}
$$

This completes the proof. □

4.6 Stability of Approximation Properties

Lemma 4.6.1. *Let H be a closed subgroup of a locally compact group G. For any function u on G, let $u|_H$ denote the restriction of u to H.*

1. *If $u \in MA_p(G)$, then $u|_H \in MA_p(H)$ and*

$$\|u|_H\|_{MA_p(H)} \leq \|u\|_{MA_p(G)}.$$

2. *If $u \in M_{cb}A_p(G)$, then $u|_H \in M_{cb}A_p(H)$ and*

$$\|u|_H\|_{M_{cb}A_p(H)} \leq \|u\|_{M_{cb}A_p(G)}.$$

Proof. (a) By Theorem 1 in [Her71], the restriction of functions gives a quotient mapping from $A_p(G)$ to $A_p(H)$. Therefore, for $\varphi \in A_p(H)$, we get

$$\|\varphi\|_{A_p(H)} = \inf\{\|\tilde{\varphi}\|_{A_p(G)} : \tilde{\varphi} \in A_p(G),\ \tilde{\varphi}|_H = \varphi\}.$$

Suppose $u \in MA_p(G)$. For $\varphi \in A_p(H)$, there exists $\tilde{\varphi}$ with $\tilde{\varphi}|_H = \varphi$. Since $u|_H\varphi = (u\tilde{\varphi})|_H$ and $u\tilde{\varphi} \in A_p(G)$, again by Theorem 1 in [Her71] we get $u|_H\varphi \in A_p(H)$. This shows that $u|_H \in MA_p(H)$ and moreover,

$$
\begin{aligned}
\|u|_H\|_{MA_p(H)} &= \inf\{\|u|_H\varphi\|_{A_p(H)} : \varphi \in A_p(H),\ \|\varphi\|_{A_p(H)} \leq 1\} \\
&\leq \inf\{\|u\tilde{\varphi}\|_{A_p(G)} : \tilde{\varphi} \in A_p(G),\ \tilde{\varphi}|_H = \varphi,\ \|\varphi\|_{A_p(H)} \leq 1\} \\
&\leq \|u\|_{MA_p(G)}.
\end{aligned}
$$

(b) By Theorem 8.3 in [Daw10], there exists $E \in SQ_p$ and bounded continuous maps $\alpha : G \to E$ and $\beta : G \to E'$ such that $u(ts^{-1}) = \langle \beta(t), \alpha(s) \rangle$ for all $t, s \in G$. It follows that $u|_H(\tau\sigma^{-1}) = \langle \beta|_H(\tau), \alpha|_H(\sigma) \rangle$

for all $\tau, \sigma \in H$. This shows that $u|_H \in M_{cb}A_p(H)$ and that $\|u|_H\|_{M_{cb}A_p(H)} \le \|u\|_{M_{cb}A_p(G)}$. □

Theorem 4.6.2. *Let H be a closed subgroup of a locally compact group G. If G is p-weakly amenable, then H is also p-weakly amenable.*

Proof. Let $\{u_i\}$ be an approximate identity in $A_p(G)$ such that $\sup_i \|u_i\|_{M_{cb}A_p(G)} \le k$ for some k. Put $v_i = u_i|_H$. We claim that $\{v_i\}$ is an approximate identity in $A_p(H)$ such that $\sup_i \|u_i\|_{M_{cb}A_p(G)} \le k$. Indeed, for any $\varphi \in A_p(H)$, we can always find $\tilde{\varphi} \in A_p(G)$ such that $\tilde{\varphi}|_H = \varphi$ and

$$\|v_i\varphi - \varphi\|_{A_p(H)} = \|(u_i\tilde{\varphi} - \tilde{\varphi})|_H\|_{A_p(H)} \le \|u_i\tilde{\varphi} - \tilde{\varphi}\|_{A_p(G)} \to 0.$$

Since $\sup_i \|v_i\|_{M_{cb}A_p(G)} \le k$ by Lemma 4.6.1, this completes the proof. □

Theorem 4.6.3. *Let H be a subgroup of a discrete group G. If G has p-AP, then H also has p-AP.*

Proof. Suppose $\{u_i\}$ is a net in $A_p(G)$ such that $u_i \to 1_G$ in $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$-topology. Let $v_i = u_i|_H$. We claim that $v_i \to 1_H$ in $\sigma(M_{cb}A_p(H), Q_{pcb}(H))$-topology. Suppose $\xi \in Q_{pcb}(H)$, then $\xi = \lim_n f_n$ for some $f_n \in L_1(H)$ in $M_{cb}A_p(H)'$. Define $g_n \in L_1(G)$ by

$$g_n(x) = \begin{cases} f_n(x), & x \in H, \\ 0, & x \in G \setminus H. \end{cases}$$

Since

$$\begin{aligned} \|g_n - g_m\|_{M_{cb}A_p(G)'} &= \sup\left\{\left|\sum_{x\in G}(g_n(x) - g_m(x))\varphi(x)\right| : \varphi \in M_{cb}A_p(G),\ \|\varphi\|_{M_{cb}A_p(G)} \le 1\right\} \\ &= \sup\left\{\left|\sum_{x\in H}(f_n(x) - f_m(x))\varphi|_H(x)\right| : \varphi \in M_{cb}A_p(G),\ \|\varphi\|_{M_{cb}A_p(G)} \le 1\right\} \\ &= \|f_n - f_m\|_{M_{cb}A_p(H)'}, \end{aligned}$$

$\{g_n\}$ is Cauchy in $M_{cb}A_p(G)'$. Let $\eta = \lim_n g_n$ in $M_{cb}A_p(H)'$, then

$$\begin{aligned} \langle v_i, \xi\rangle &= \lim_n \sum_{x\in H} v_i(x)f_n(x) = \lim_n \sum_{x\in G} u_i(x)g_n(x) \\ &= \langle u_i, \eta\rangle \\ &\to \langle 1_G, \eta\rangle \\ &= \lim_n \sum_{x\in G} g_n(x) = \lim_n \sum_{x\in H} f_n(x) \end{aligned}$$

$$= \langle 1_H, \xi \rangle.$$

This completes the proof. $\square$

Remark 4.6.4. Even when $p = 2$, it is unlikely that G has the p-AP implies G/H has the p-AP. See [HK94].

Lemma 4.6.5. *Let G be a locally compact group, and suppose that H is a closed subgroup of G such that $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. For each $f, g \in C_{00}(G)$, let $\Phi_{f,g}$ denote the map defined on $M_{cb}A_p(H)$ by*

$$\Phi_{f,g}(u) = f \star udh \star \check{g}, \qquad u \in M_{cb}A_p(H),$$

where dh is a fixed left Haar measure on H. Then $\Phi_{f,g}$ is a bounded linear map from $M_{cb}A_p(H)$ into $M_{cb}A_p(G)$ that is $\sigma(M_{cb}A_p(H), Q_{pcb}(H))$-$\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ continuous.

Proof. Let $\pi_H : G \to G/H$ denote the canonical mapping onto the left cosets of H and write $\dot{x} = \pi_H(x)$. Let $T_H : C_{00}(G) \twoheadrightarrow C_{00}(G/H)$ be as in [RS00, Definition 3.3.9], that is,

$$T_H f(\dot{x}) = \int_H f(xh)dh, \qquad f \in C_{00}(G), \qquad \dot{x} \in G/H.$$

By [RS00, Propositions 8.1.1, 8.1.3, 8.1.4], there is a measure μ on G/H that is invariant under the natural action of G on G/H and satisfies the relation

$$\int_{G/H} T_H f(\dot{x}) d\mu(\dot{x}) = \int_{G/H} \int_H f(xh) dh d\mu(\dot{x}) = \int_G f(x)dx, \qquad f \in C_{00}(G), \tag{4.6}$$

where dx is a fixed Haar measure on G. Let $g \in C_{00}(G)$ and let $u \in M_{cb}A_p(H)$. Then

$$(udh \star \check{g})(y) = \int_H u(h) g(y^{-1}h) dh, \qquad \forall y \in G.$$

Using the fact that u is bounded and g is uniformly continuous, one can easily show that $udh \star \check{g}$ is continuous, and hence for each $x \in G$, the function $y \mapsto f(y)(udh \star \check{g})(y^{-1}x)$ is in $C_{00}(G)$. Now, by (4.6),

$$\begin{aligned}
\Phi_{f,g}(u)(x) &= \int_G f(y)(udh \star \check{g})(y^{-1}x)dy \\
&= \int_{G/H} \int_H f(yh)(udh \star \check{g})(h^{-1}y^{-1}x) dh d\mu(\dot{y}) \\
&= \int_{G/H} \int_H f(yh) \int_H u(k) g(x^{-1}yhk) dk dh d\mu(\dot{y}) \\
&= \int_{G/H} \int_H \int_H f(yh) u(k) g(x^{-1}yhk) dk dh d\mu(\dot{y}).
\end{aligned} \tag{4.7}$$

Let $\epsilon > 0$. Since $u \in M_{cb}A_p(H)$, by [Daw10, Theorem 8.3], there exist $E \in SQ_p$ and bounded maps $\alpha : H \to E$ and $\beta : H \to E'$ such that $u(s^{-1}t) = \langle \beta(s), \alpha(t)\rangle$ for all $t, s \in H$ and $\|\alpha\|_\infty \|\beta\|_\infty < \|u\|_{M_{cb}A_p(H)} + \epsilon$. Define $P : G \to E'$ (respectively, $Q : G \to E$) by

$$P(x) = \int_H f(xh)\beta(h)dh \qquad \left(\text{respectively, } Q(x) = \int_H g(xh)\alpha(h)\right), \qquad x \in G.$$

For any $x \in G$ and for any $\varphi \in E'' = E$, we have

$$|\langle P(x), \varphi\rangle| \leq \int_H |f(xh)|\,|\langle \beta(h), \varphi\rangle|\, dh \leq \|\beta\|_\infty \|\varphi\| \int_H |f(xh)|\, dh,$$

and hence

$$\|P(x)\| \leq \|\beta\|_\infty \int_H |f(xh)|\, dh, \qquad \forall x \in G. \tag{4.8}$$

Since

$$\|P(x) - P(y)\| \leq \|\beta\|_\infty \int_H |f(xh) - f(yh)|\, dh$$

and $f \in C_{00}(G)$, it follows that P is continuous. Similarly,

$$\|Q(x)\| \leq \|\alpha\|_\infty \int_H |g(xh)|\, dh, \qquad \forall x \in G, \tag{4.9}$$

$$\|Q(x) - Q(y)\| \leq \|\alpha\|_\infty \int_H |g(xh) - g(yh)|\, dh, \tag{4.10}$$

and in particular, Q is also continuous. Now by (4.7),

$$\begin{aligned}
\Phi_{f,g}(u)(x^{-1}y) &= \int_{G/H}\int_H\int_H f(zh)u(k)g(y^{-1}xzhk)dkdhd\mu(\dot{z}) \\
(k \mapsto h^{-1}k) &= \int_{G/H}\int_H\int_H f(zh)u(h^{-1}k)g(y^{-1}xzk)dkdhd\mu(\dot{z}) \\
(z \mapsto x^{-1}z) &= \int_{G/H}\int_H\int_H f(x^{-1}zh)u(h^{-1}k)g(y^{-1}zk)dkdhd\mu(\dot{z}) \\
&= \int_{G/H}\int_H\int_H f(x^{-1}zh)g(y^{-1}zk)\langle \beta(h), \alpha(k)\rangle dkdhd\mu(\dot{z}) \\
&= \int_{G/H} \langle P(x^{-1}z), Q(y^{-1}z)\rangle d\mu(\dot{z}),
\end{aligned}$$

where the G-invariance of μ is used in the third equality. Let ρ, ω be as in the proof of [HK94, Lemma 1.16].

Then

$$\begin{aligned}
\int_G \|Q(y^{-1}x)\|^p d\omega(x) &= \int_{G/H} \|Q(y^{-1}\rho(\dot{x}))\|^p d\mu(\dot{x}) \\
\text{(by (4.9))} \quad &\leq \int_{G/H} \left(\|\alpha\|_\infty \int_H \left|g(y^{-1}\rho(\dot{x})h)\right| dh\right)^p d\mu(\dot{x}) \\
(\pi_H(\rho(\dot{x})) = \dot{x}) \quad &= \int_{G/H} \left(\|\alpha\|_\infty \int_H \left|g(y^{-1}xh)\right| dh\right)^p d\mu(\dot{x}) \\
(G\text{-invariance of } \mu) \quad &= \int_{G/H} \left(\|\alpha\|_\infty \int_H |g(xh)|\, dh\right)^p d\mu(\dot{x}).
\end{aligned}$$

Put $L(g) = \int_{G/H} \left(\int_H |g(xh)|\, dh\right)^p d\mu(\dot{x})$ and $F = L_p(G, E, \omega) = L_p(G, \omega) \otimes_p E$, then F is an SQ_p space (See Theorem 3.1 and the remarks after it in [Run05]) and the function $B : G \to F$ defined by

$$B(x)(y) = Q(x^{-1}y), \qquad x, y \in G,$$

satisfies $\|B(x)\|^p \leq \|\alpha\|_\infty^p L(g)$ for all $x \in G$. Moreover, for any $y, z \in G$,

$$\begin{aligned}
\|B(y) - B(z)\|^p &= \int_G \|Q(y^{-1}x) - Q(z^{-1}x)\|^p d\omega(x) \\
&= \int_{G/H} \|Q(y^{-1}\rho(\dot{x})) - Q(z^{-1}\rho(\dot{x}))\|^p d\mu(\dot{x}) \\
\text{(by (4.10))} \quad &\leq \int_{G/H} \|\alpha\|_\infty^p \left(\int_H \left|g(y^{-1}\rho(\dot{x})h) - g(z^{-1}\rho(\dot{x})h)\right| dh\right)^p d\mu(\dot{x}) \\
&\leq \int_{G/H} \|\alpha\|_\infty^p \left(\int_H \left|g(y^{-1}xh) - g(z^{-1}xh)\right| dh\right)^p d\mu(\dot{x}) \\
&\leq \int_{G/H} \|\alpha\|_\infty^p 2^{p-1} \|T_H|g|\|_\infty^{p-1} \int_H \left|g(y^{-1}xh) - g(z^{-1}xh)\right| dh d\mu(\dot{x}) \\
&= \|\alpha\|_\infty^p 2^{p-1} \|T_H|g|\|_\infty^{p-1} \int_G \left|g(y^{-1}x) - g(z^{-1}x)\right| dx
\end{aligned}$$

and it follows that B is continuous. Similarly,

$$\int_G \|P(y^{-1}x)\|^{p'} d\omega(x) \quad \leq \quad \int_{G/H} \left(\|\beta\|_\infty \int_H |f(xh)|\, dh\right)^{p'} d\mu(\dot{x}).$$

Put $M(f) = \int_{G/H} \left(\int_H |g(xh)|\, dh\right)^p d\mu(\dot{x})$ and define $A : G \to L_{p'}(G, \omega) \otimes_{p'} E' \subseteq F'$ [DF93, Proposition 15.10] by $A(x)(y) = P(x^{-1}y)$, then $\|A(x)\|^{p'} \leq \|\beta\|_\infty^{p'} M(f)$ for all $x \in G$. Moreover, the same argument as above gives

$$\|A(y) - A(z)\|^{p'} \quad \leq \quad \|\beta\|_\infty^{p'} 2^{p'-1} \|T_H|g|\|_\infty^{p'-1} \int_G \left|f(y^{-1}x) - f(z^{-1}x)\right| dx$$

and hence A is also continuous. Now it follows that

$$\begin{aligned}
\langle A(x), B(y)\rangle &= \int_G \langle P(x^{-1}z), Q(y^{-1}z)\rangle d\omega(z) \\
&= \int_G \int_H \int_H f(x^{-1}zh)g(y^{-1}zk)\langle \beta(h), \alpha(k)\rangle dh dk d\omega(z) \\
&= \int_{G/H} \int_H \int_H f(x^{-1}\rho(\dot{z})h)g(y^{-1}\rho(\dot{z})k)\langle \beta(h), \alpha(k)\rangle dh dk d\mu(\dot{z}) \\
&= \int_{G/H} \int_H \int_H f(x^{-1}zh)g(y^{-1}zk)\langle \beta(h), \alpha(k)\rangle dh dk d\mu(\dot{z}) \\
&= \Phi_{f,g}(u)(x^{-1}y)
\end{aligned}$$

and this shows that $\Phi_{f,g}(u) \in M_{cb}A_p(G)$. Moreover, by taking the infimum over α and β, we get

$$\|\Phi_{f,g}\| \leq L(g)^{1/p} M(f)^{1/p'}.$$

It remains to show that $\Phi_{f,g}$ is $\sigma(M_{cb}A_p(H), Q_{pcb}(H))$ - $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ continuous. To this end, it suffices to show that $\Phi'_{f,g}$ maps $Q_{pcb}(G)$ into $Q_{pcb}(H)$. Since $L_1(G)$ is dense in $Q_{pcb}(G)$, it suffices to show that $\Phi'_{f,g}(\xi) \in Q_{pcb}(H)$ for every $\xi \in L_1(G)$. So let $\xi \in L_1(G)$ and define a function $\xi_{f,g}$ on H by

$$\xi_{f,g}(h) = \int_G f(x)(\xi \star g)(xh)dx, \qquad h \in H.$$

It is easy to check that $\xi_{f,g} \in L_1(H)$ and that

$$\langle u, \Phi'_{f,g}(\xi)\rangle \quad = \quad \langle \Phi_{f,g}(u), \xi\rangle = \int_H u(k)\xi_{f,g}(k)dk = \langle u, \xi_{f,g}\rangle,$$

from which it follows that $\Phi'_{f,g}(\xi) = \xi_{f,g} \in L_1(H) \subseteq Q_{pcb}(H)$. □

Theorem 4.6.6. *Let G be a locally compact group and H a closed normal subgroup of G. If H and G/H have the p-AP, then G has the p-AP.*

Proof. Let $\{u_i\} \subseteq A_{p,c}(H)$ be such that $u_i \to 1_H$ in the $\sigma(M_{cb}A_p(H), Q_{pcb}(H))$ topology. By Lemma 4.6.5 and [RS00, Proposition 3.3.17], for any $f, g \in C_{00}(G)$, $\Phi_{f,g}(u_i) \to \Phi_{f,g}(1_H)$ in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ topology. Since $\Phi_{f,g}(u_i) \in A_{p,c}(G)$ for all i, it suffices to show that 1_G is in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ closure of $\{\Phi_{f,g}(1_H) : f, g \in C_{00}(G)\}$. Let $f, g \in C_{00}(G)$ and put $\xi = T_H f$, $\eta = T_H g$. It follows from (4.7) that

$$\Phi_{f,g}(1_H)(x) \quad = \quad \int_{G/H} \int_H \int_H f(yh)g(x^{-1}yhk)dk dh d\mu(\dot{y})$$

$$
\begin{aligned}
&= \int_{G/H} \xi(\dot{y})\eta(\dot{x}^{-1}\dot{y})d\mu(\dot{y}) \\
&= \xi \star \check{\eta}(\dot{x}), \qquad \forall x \in G.
\end{aligned}
$$

Since T_H is a map from $C_{00}(G)$ onto $C_{00}(G/H)$ [RS00, Proposition 3.4.2], we obtain

$$\{\Phi_{f,g}(1_H) : f, g \in C_{00}(G)\} = \{(\xi \star \check{\eta}) \circ \pi_H : \xi, \eta \in C_{00}(G/H)\}. \tag{4.11}$$

Moreover, T_H extends to a map from $L_1(G)$ onto $L_1(G/H)$ [RS00, Proposition 3.4.5]. Define a map Ψ on $M_{cb}A_p(G/H)$ by $\Psi(u) = u \circ \pi_H$, $u \in M_{cb}A_p(G/H)$. Since $\Psi(u)(st^{-1}) = u(\dot{s}\dot{t}^{-1})$, we see that Ψ maps $M_{cb}A_p(G/H)$ contractively into $M_{cb}A_p(G)$. We claim that Ψ is $\sigma(M_{cb}A_p(G/H), Q_{pcb}(G/H))$ - $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ continuous. Let $\zeta \in L_1(G)$, then for any $u \in M_{cb}A_p(G/H)$, we get

$$
\begin{aligned}
\langle u, T_H\zeta\rangle &= \int_{G/H} u(\dot{x}) \int_H \zeta(xh)dhd\mu(\dot{x}) \\
&= \int_{G/H}\int_H u(\dot{x})\zeta(xh)dhd\mu(\dot{x}) \\
&= \int_{G/H}\int_H \Psi(u)(xh)\zeta(xh)dhd\mu(\dot{x}) \\
&= \langle \Psi(u), \zeta\rangle
\end{aligned}
$$

and using the same argument as in the proof of Lemma 4.6.5, we get the desired continuity of Ψ. Let R denote the linear span of $\{\xi \star \check{\eta} : \xi, \eta \in C_{00}(G/H)\}$, then R is dense in $A_p(G/H)$ in the $A_p(G/H)$ norm and hence $1_{G/H}$ is in the $\sigma(M_{cb}A_p(G/H), Q_{pcb}(G/H))$ closure of R. Now from the continuity of Ψ and (4.11), we conclude that 1_G is in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ closure of $\{\Phi_{f,g}(1_H) : f, g \in C_{00}(G)\}$ as desired. $\square$

Chapter 5

Conditions C_p, C_p', and C_p'' for p-Operator Spaces

5.1 Introduction

Conditions C, C', and C'' for operator spaces are studied in [ER00, Chapter 14]. To be precise, it is known that an operator space W is locally reflexive if and only if W satisfies condition C'' [ER00, Theorem 14.3.1]. It is also known that an operator space V is exact if and only if V satisfies condition C' [ER00, Theorem 14.4.1]. In this chapter, we define p-analogues of these definitions, which will be called conditions C_p, C_p', and C_p'', and show that a p-operator space satisfies condition C_p if and only if it satisfies both conditions C_p' and C_p'' .

Lemma 5.1.1. *Let V and W be p-operator spaces. Then the bilinear mapping*

$$\tilde{\Psi} : V' \times W' \to (V \overset{\vee_p}{\otimes} W)', \qquad (f, g) \mapsto f \otimes g$$

is jointly p-completely contractive and hence the canonical mapping $\Psi : V' \overset{\wedge_p}{\otimes} W' \to (V \overset{\vee_p}{\otimes} W)'$ is p-completely contractive.

Proof. The second half follows from Proposition 2.1.3, so it suffices to show that the bilinear map $\tilde{\Psi}$ is jointly p-completely contractive. Recall the definition

$$\tilde{\Psi}_{r;s} : M_r(V') \times M_s(W') \to M_{rs}((V \overset{\vee_p}{\otimes} W)'), \qquad ([f_{ij}], [g_{kl}]) \mapsto [f_{ij} \otimes g_{kl}], \qquad r, s \in \mathbb{N}.$$

Here we can identify $[f_{ij} \otimes g_{kl}]$ with $[f_{ij}] \otimes [g_{kl}] : V \overset{\vee_p}{\otimes} W \to M_{rs}$ and hence we get $\|[f_{ij} \otimes g_{kl}]\| \leq \|[f_{ij}]\| \cdot \|[g_{kl}]\|$. $\square$

Lemma 5.1.2. *Let $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$. Then $\|\cdot\|_{\vee_p}$ is a subcross matrix norm. In particular, for every $u \in M_n(V \otimes W)$, we have $\|u\|_{\vee_p} \leq \|u\|_{\wedge_p}$.*

Proof. Let $v = [v_{ij}] \in M_r(V)$ and $w = [w_{kl}] \in M_q(W)$, then by Remark 2.3.2,

$$\|v \otimes w\|_{\vee_p} = \sup \{\|(f \otimes g)_{rq}(v \otimes w)\| : f \in M_s(V')_1, g \in M_t(W')_1\}.$$

Note that $(f \otimes g)_{rq}(v \otimes w) = f_r(v) \otimes g_q(w)$ and hence

$$\|(f \otimes g)_{rq}(v \otimes w)\| = \|f_r(v)\| \cdot \|g_q(w)\| \leq \|v\| \cdot \|w\|.$$

The second half of the lemma follows immediately because $\| \cdot \|_{\wedge_p}$ is the largest subcross matrix norm. $\square$

5.2 Conditions C_p', C_p'', and C_p for p-Operator Spaces

Let $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$. Fix $\varphi \in (V \overset{\vee_p}{\otimes} W)'$. For $v_0 \in V$, we define a bounded linear functional ${}_{v_0}\varphi$ on W by

$${}_{v_0}\varphi(w) = \varphi(v_0 \otimes w), \qquad w \in W.$$

In general, when $v_0 = [v_{ij}] \in M_r(V)$ and $\varphi = [\varphi_{kl}] \in M_n((V \overset{\vee_p}{\otimes} W)')$, we define ${}_{v_0}\varphi = [{}_{v_{ij}}\varphi_{kl}] \in M_{rn}(W')$. Similarly, for $w_0 \in W$, we define $\varphi_{w_0} \in V'$ by

$$\varphi_{w_0}(v) = \varphi(v \otimes w_0), \qquad v \in V.$$

As in ${}_{v_0}\varphi$ above, we can extend the definition of φ_{w_0} for $w_0 \in M_r(W)$ and $\varphi \in M_n((V \overset{\vee_p}{\otimes} W)')$.

Define a linear map $\Phi^R_{V,W} : V \otimes W'' \to (V \overset{\vee_p}{\otimes} W)''$ by

$$\Phi^R_{V,W}(v \otimes w'')(\varphi) = \langle {}_v\varphi, w'' \rangle_{W',W''}, \qquad v \in V, \quad w'' \in W'', \quad \varphi \in (V \overset{\vee_p}{\otimes} W)'.$$

Similarly, define a linear map $\Phi^L_{V,W} : V'' \otimes W \to (V \overset{\vee_p}{\otimes} W)''$ by

$$\Phi^L_{V,W}(v'' \otimes w)(\varphi) = \langle \varphi_w, v'' \rangle_{V',V''}, \qquad v'' \in V'', \quad w \in W, \quad \varphi \in (V \overset{\vee_p}{\otimes} W)'.$$

Lemma 5.2.1. *The map $\Phi^R_{V,W}$ (respectively, $\Phi^L_{V,W}$) defined above extends to a p-completely contractive map $\Phi^R_{V,W} : V \overset{\wedge_p}{\otimes} W'' \to (V \overset{\vee_p}{\otimes} W)''$ (respectively, $\Phi^L_{V,W} : V'' \overset{\wedge_p}{\otimes} W \to (V \overset{\vee_p}{\otimes} W)''$).*

Proof. Consider the bilinear map $\Phi : V \times W'' \to (V \overset{\vee_p}{\otimes} W)''$ given by

$$(v, w'') \mapsto (\varphi \mapsto \langle {}_v\varphi, w'' \rangle_{W',W''}).$$

We get

$$\Phi_{r;s} : M_r(V) \times M_s(W'') \to M_{rs}((V \overset{\vee_p}{\otimes} W)''), \qquad ([v_{ij}], [w_{kl}'']) \mapsto [\Phi(v_{ij}, w_{kl}'')].$$

Following the notation in [ER00], we obtain

$$\|[\Phi(v_{ij}, w_{kl}'')]\| = \sup_n \left\{ \|\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle\| : \varphi \in M_n((V \overset{\vee_p}{\otimes} W)'), \|\varphi\| \le 1 \right\}.$$

Since $\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle = \langle\langle {}_v\varphi, w'' \rangle\rangle$, we have

$$\|\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle\| = \|\langle\langle {}_v\varphi, w'' \rangle\rangle\| \le \|{}_v\varphi\|_{M_{rn}(W')} \cdot \|w''\|_{M_s(W'')}$$

and the result follows because

$$\begin{aligned} \|{}_v\varphi\|_{M_{rn}(W')} &= \sup_m \{\|\langle\langle {}_v\varphi, w \rangle\rangle\|_{M_{rnm}} : w \in M_m(W), \|w\| \le 1\} \\ &= \sup_m \{\|\langle\langle \varphi, v \otimes w \rangle\rangle\|_{M_{rnm}} : w \in M_m(W), \|w\| \le 1\} \\ &\le \|\varphi\| \cdot \|v\| \\ &\le \|v\|. \end{aligned}$$

□

Let $\Psi : V' \overset{\wedge_p}{\otimes} W' \to (V \overset{\vee_p}{\otimes} W)'$ denote the canonical map, and consider the following commutative diagram

$$\begin{tikzcd} V \otimes W'' \arrow[r, "\Phi^R_{V,W}"] \arrow[d, equal] & (V \overset{\vee_p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge_p}{\otimes} W')' \arrow[d, equal] \\ \mathcal{CB}^{\sigma}_{p,F}(V', W'') \arrow[rr, hook, "\iota"] & & \mathcal{CB}_p(V', W'') \end{tikzcd}$$

where $\mathcal{CB}^{\sigma}_{p,F}(V', W'')$ denotes the space of all weak*-continuous p-completely bounded finite rank maps from V' to W'' and ι is the inclusion map. This commutative diagram shows that $\Phi^R_{V,W}$ is one-to-one, so one can equip $V \otimes W''$ with the p-operator space norm inherited from $(V \overset{\vee_p}{\otimes} W)''$, which will be denoted by, following the notation in [ER00], $V \otimes_{\vee_p} : W''$. We say that V satisfies *condition* C'_p (or V has *property* C'_p) if this induced norm coincides with the injective tensor product norm with every $W \subseteq \mathcal{B}(L_p(\nu))$.

Similarly, the following diagram

$$\begin{tikzcd}
V'' \otimes W \arrow[r, "\Phi^L_{V,W}"] \arrow[d, equal] & (V \overset{\vee_p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge_p}{\otimes} W')' \arrow[d, equal] \\
\mathcal{CB}^{\sigma}_{p,F}(W', V'') \arrow[rr, hook, "\iota"] & & \mathcal{CB}_p(W', V'')
\end{tikzcd}$$

is also commutative, $\Phi^L_{V,W}$ is one-to-one, and one can hence equip $V'' \otimes W$ with the p-operator space norm inherited from $(V \overset{\vee_p}{\otimes} W)''$, which will be denoted by $V'' : \otimes_{\vee_p} W$. We say that V satisfies *condition* C''_p (or V has *property* C''_p) if this induced norm coincides with the injective tensor product norm with every $W \subseteq \mathcal{B}(L_p(\nu))$.

In order to define *condition* C_p for p-operator spaces, we consider the following diagram

$$\begin{tikzcd}
 & (V \overset{\wedge_p}{\otimes} W'')'' \arrow[dr, "(\Phi^R_{V,W})''"] & & \\
V'' \otimes W'' \arrow[ur, "\Phi^L_{V,W''}"] \arrow[dr, "\Phi^R_{V'',W}"'] & & (V \overset{\vee_p}{\otimes} W)'''' \arrow[r, "P"] & (V \overset{\vee_p}{\otimes} W)'' \\
 & (V'' \overset{\wedge_p}{\otimes} W)'' \arrow[ur, "(\Phi^L_{V,W})''"'] & &
\end{tikzcd} ,$$

where P is the restriction mapping. Note that Lemma 5.2.1 was used here to consider $(\Phi^R_{V,W})''$ and $(\Phi^L_{V,W})''$.

For p-operator spaces $V \subseteq \mathcal{B}(L_p(\mu))$, $W \subseteq \mathcal{B}(L_p(\nu))$, we consider the following p-complete contraction:

$$(V \overset{\wedge_p}{\otimes} W)' \overset{pcb}{\cong} \mathcal{CB}_p(V, W') \xrightarrow{\text{adj}} \mathcal{CB}_p(W'', V') \overset{pcb}{\cong} (V \overset{\wedge_p}{\otimes} W'')'.$$

For $\varphi \in (V \overset{\wedge_p}{\otimes} W)'$, let $\varphi^{\wedge} \in (V \overset{\wedge_p}{\otimes} W'')'$ denote the image of φ under this map. Then we have

$$\varphi^{\wedge}(v \otimes w'') = \langle {}_v\varphi, w'' \rangle_{W', W''} = \Phi^R_{V,W}(v \otimes w'')(\varphi), \qquad v \in V, \quad w'' \in W''.$$

Moreover, $\varphi^{\wedge}$ is weak*-continuous in the second variable. Similarly, we also consider the p-complete contraction

$$(V \overset{\wedge_p}{\otimes} W)' \overset{pcb}{\cong} \mathcal{CB}_p(W, V') \xrightarrow{\text{adj}} \mathcal{CB}_p(V'', W') \overset{pcb}{\cong} (V'' \overset{\wedge_p}{\otimes} W)'$$

and define $^\wedge\varphi$, and then we get that

$$^\wedge\varphi(v'' \otimes w) = \langle \varphi_w, v'' \rangle_{V', V''} = \Phi^L_{V,W}(v'' \otimes w)(\varphi), \qquad v'' \in V'', \quad w \in W,$$

and that $^\wedge\varphi$ is weak*-continuous in the first variable.

Following the idea in [Han07], we have the next result.

Theorem 5.2.2. *Let $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$. Let α be a subcross matrix norm on $V \otimes W$ and denote by $V \otimes_\alpha W$ the resulting normed space. Then the following are equivalent.*

1. *There exists a separately weak*-continuous extension*

$$\Phi : V'' \otimes W'' \to (V \otimes_\alpha W)''$$

of the natural inclusion $\iota : V \otimes W \to (V \otimes_\alpha W)''$.

2. *The following diagram commutes*

$$\begin{tikzcd}
 & (V \overset{\wedge_p}{\otimes} W'')'' \arrow[dr, "(\Phi^R_{V,W})''"] & & \\
V'' \otimes W'' \arrow[ur, "\Phi^L_{V,W''}"] \arrow[dr, "\Phi^R_{V'',W}"'] & & (V \otimes_\alpha W)'''' \arrow[r, "P"] & (V \otimes_\alpha W)'' \\
 & (V'' \overset{\wedge_p}{\otimes} W)'' \arrow[ur, "(\Phi^L_{V,W})''"'] & &
\end{tikzcd} .$$

3. *For every $\varphi \in (V \otimes_\alpha W)'$, two functionals $(^\wedge\varphi)^\wedge$ and $^\wedge(\varphi^\wedge)$ coincide on $V'' \otimes W''$.*

4. *For every $\varphi \in (V \otimes_\alpha W)'$, $L_\varphi : V \to W'$ is weakly compact, where $\langle L_\varphi(v), w \rangle = \varphi(v \otimes w)$, $v \in V$, $w \in W$.*

Proof. *2* $\Longleftrightarrow$ *3*: Every $\varphi \in (V \otimes_\alpha W)'$ can be regarded as a bounded linear functional on $V \overset{\wedge_p}{\otimes} W$ because $\| \cdot \|_{\wedge_p}$ is the largest subcross matrix norm and both $^\wedge\varphi$ and $\varphi^\wedge$ extend φ. We have

$$\varphi^\wedge(v \otimes w'') = \langle {}_v\varphi, w'' \rangle_{W', W''} = \langle \Phi^R_{V,W}(v \otimes w'', \varphi) \rangle = \langle v \otimes w'', ((\Phi^R_{V,W})' \circ \kappa)(\varphi) \rangle,$$

where κ is the natural inclusion from $(V \overset{\wedge_p}{\otimes} W)'$ into $(V \overset{\wedge_p}{\otimes} W)'''$. This shows that $\varphi^\wedge = ((\Phi^R_{V,W})' \circ \kappa)(\varphi)$. Similarly, $^\wedge\varphi = ((\Phi^L_{V,W})' \circ \kappa)(\varphi)$. Let Φ_1 (respectively, Φ_2) be the composition of the upper chain (respectively, lower chain) in the diagram in *2*, i.e., $\Phi_1 = P\circ(\Phi^R_{V,W})''\circ\Phi^L_{V,W''}$ and $\Phi_2 = P\circ(\Phi^L_{V,W})''\circ\Phi^R_{V'',W}$, then we have

$$\begin{aligned}
\langle\Phi_1(v''\otimes w''),\varphi\rangle &= \langle P\circ(\Phi^R_{V,W})''\circ\Phi^L_{V,W''}(v''\otimes w''),\varphi\rangle_{(V\otimes_\alpha W)'',(V\otimes_\alpha W)^*} \\
&= \langle\Phi^L_{V,W''}(v''\otimes w''),((\Phi^R_{V,W})'\circ\kappa)(\varphi)\rangle_{(V\otimes_\alpha W)'',(V\otimes_\alpha W)^*} \\
&= \langle\Phi^L_{V,W''}(v''\otimes w''),\varphi^\wedge\rangle_{(V\otimes_\alpha W)'',(V\otimes_\alpha W)^*} \\
&= \langle(\varphi^\wedge)_{W''},v''\rangle_{V',V''} \\
&= {}^\wedge(\varphi^\wedge)(v''\otimes w'').
\end{aligned}$$

Similarly,

$$\Phi_2(v''\otimes w''),\varphi\rangle = ({}^\wedge\varphi)^\wedge(v''\otimes w''),$$

and thus $\Phi_1=\Phi_2 \Longleftrightarrow {}^\wedge(\varphi^\wedge) = ({}^\wedge\varphi)^\wedge$ on $V''\otimes W''$.

1 $\Longleftrightarrow$ *2*: According to calculations above, Φ_1 (respectively, Φ_2) is left (respectively, right) weak*-continuous. If $\Phi_1=\Phi_2$, then it ($\Phi_1=\Phi_2\triangleq\Phi$) is a separately weak*-continuous map which extends the natural inclusion $\iota: V\otimes W\to(V\otimes_\alpha W)''$. Conversely, suppose that Φ is the separately weak*-continuous extension of ι. For $v''\in V''$ (respectively, $w''\in W''$), let us take a net $\{v_i\}$ in V converging to v'' in the weak* topology (respectively, $\{w_j\}$ in W converging to w'' in the weak* topology). It follows that

$$\begin{aligned}
\langle\Phi_1(v''\otimes w''),\varphi\rangle &= {}^\wedge(\varphi^\wedge)(v''\otimes w'') = \lim_i {}^\wedge(\varphi^\wedge)(v_i\otimes w'') = \lim_i \varphi^\wedge(v_i\otimes w'') \\
&= \lim_i\lim_j \varphi^\wedge(v_i\otimes w_j) = \lim_i\lim_j\varphi(v_i\otimes w_j) = \lim_i\lim_j\langle\Phi(v_i\otimes w_j),\varphi\rangle \\
&= \langle\Phi(v''\otimes w''),\varphi\rangle \\
&= \lim_j\lim_i\langle\Phi(v_i\otimes w_j),\varphi\rangle = \lim_j\lim_i\varphi(v_i\otimes w_j) = \lim_j\lim_i {}^\wedge\varphi(v_i\otimes w_j) \\
&= \lim_j {}^\wedge\varphi(v''\otimes w_j) = \lim_j({}^\wedge\varphi)^\wedge(v''\otimes w_j) = ({}^\wedge\varphi)^\wedge(v''\otimes w'') \\
&= \langle\Phi_2(v''\otimes w''),\varphi\rangle,
\end{aligned}$$

so we conclude that $\Phi_1=\Phi=\Phi_2$.

3 $\Longleftrightarrow$ *4*: Let $\varphi\in(V\otimes_\alpha W)'$. Since $\langle L_{\varphi^\wedge}(v),w''\rangle_{W''',W''} = \varphi^\wedge(v\otimes w'') = \langle{}_v\varphi,w''\rangle_{W',W''} = \langle L_\varphi(v),w''\rangle_{W',W''} = \langle\iota_{W'}\circ L_\varphi(v),w''\rangle_{W''',W''}$, we get

$$L_{\varphi^\wedge} = \iota_{W'}\circ L_\varphi.$$

On the other hand, observe that $\langle L_{{}^\wedge\varphi}(v''),w\rangle_{W',W} = {}^\wedge\varphi(v''\otimes w) = \langle\varphi_w,v''\rangle_{V',V''}$ and that $\langle P_{W'}\circ$

$L_\varphi{}''(v''), w\rangle_{W',W} = \langle L_\varphi{}''(v'', \iota_W(w))\rangle_{W''',W''} = \langle v'', L'_\varphi \circ \iota_W(w)\rangle_{V'',V'}$. Since

$$\langle L'_\varphi \circ \iota_W(w), v\rangle_{V',V} = \langle \iota_W(w), L_\varphi(v)\rangle_{W'',W'} = \langle w, L_\varphi(v)\rangle_{W,W'} = \varphi(v \otimes w) = \varphi_w(v),$$

we obtain

$$L_{^\wedge\varphi} = P_{W'} \circ L_\varphi{}''.$$

Therefore, we get

$$L_{(^\wedge\varphi)^\wedge} = \iota_{W'} \circ L_{^\wedge\varphi} = \iota_{W'} \circ P_{W'} \circ L_\varphi{}''$$

and

$$L_{^\wedge(\varphi^\wedge)} = P_{W'''} \circ L_{\varphi^\wedge}{}'' = P_{W'''} \circ (\iota_{W'} \circ L_\varphi)'' = P_{W'''} \circ \iota_{W'}{}'' \circ L_\varphi{}'' = L_\varphi{}''.$$

This shows that

$$\begin{aligned} (^\wedge\varphi)^\wedge \quad &= \quad {}^\wedge(\varphi^\wedge) \text{ on } V'' \otimes W'' \\ &\iff \quad L_{(^\wedge\varphi)^\wedge} = L_{^\wedge(\varphi^\wedge)} \\ &\iff \quad L_\varphi{}''(V'') \subseteq W' \\ &\iff \quad L_\varphi \text{ is weakly compact.} \end{aligned}$$

(See [Meg98, Theorem 3.5.8] for the last equivalence) $\square$

Theorem 5.2.3. *Let $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$. For every $\varphi \in (V \overset{\vee_p}{\otimes} W)'$, L_φ is weakly compact.*

Proof. Without loss of generality, we may assume $\|\varphi\| (= \|\varphi\|_{pcb}) \leq 1$. Using Theorem 2.3.9, we have two index sets I and J and p-complete isometries

$$\Phi^V : V \hookrightarrow V'' \hookrightarrow \mathcal{B}(\ell_p(I)) \qquad \text{and} \qquad \Phi^W : W \hookrightarrow W'' \hookrightarrow \mathcal{B}(\ell_p(J)).$$

Consider the diagram below:

$$\begin{tikzcd} V \overset{\vee_p}{\otimes} W \arrow[rr, "\varphi"] \arrow[d, hook] & & \mathbb{C} \\ \mathcal{B}(\ell_p(I)) \overset{\vee_p}{\otimes} \mathcal{B}(\ell_p(J)) \arrow[d, hook] & & \\ \mathcal{B}(\ell_p(I \otimes J)) \arrow[uurr, dashed, "\tilde{\varphi}"'] & & \end{tikzcd}$$

By Hahn-Banach Theorem, φ extends to $\tilde{\varphi} : \mathcal{B}(\ell_p(I \otimes J)) \to \mathbb{C}$. Applying the same technique as in the proof of Theorem 2.3.12, we can find a measure space (Ω, Σ, θ) together with two vectors $\xi \in L_p(\theta)$, $\eta \in L_{p'}(\theta)$, and a unital p-completely contractive homomorphism $\pi : \mathcal{B}(\ell_p(I \otimes J)) \to \mathcal{B}(L_p(\theta))$ such that $\tilde{\varphi}(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle$. Define $T : \mathcal{B}(\ell_p(I)) \to \mathcal{B}(\ell_p(J))'$ by

$$\langle T(x), y\rangle = \tilde{\varphi}(x \otimes y), \qquad x \in \mathcal{B}(\ell_p(I)), \quad y \in \mathcal{B}(\ell_p(J)).$$

Then it is easy to check that the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "L_\varphi"] \arrow[d, hook, "\Phi^V"'] & W' \\ \mathcal{B}(\ell_p(I)) \arrow[r, "T"] & \mathcal{B}(\ell_p(J))' \arrow[u, "(\Phi^W)'"'] \end{tikzcd}$$

Define $R : \mathcal{B}(\ell_p(I)) \to L_p(\theta)$ and $S : \mathcal{B}(\ell_p(J)) \to L_{p'}(\theta)$ by

$$R(x) = \pi(x \otimes 1)\xi, \quad x \in \mathcal{B}(\ell_p(I)), \qquad \text{and} \qquad S(y) = (\pi(1 \otimes y))'\eta, \quad y \in \mathcal{B}(\ell_p(J)),$$

then the diagram

$$\begin{tikzcd} \mathcal{B}(\ell_p(I)) \arrow[rr, "T"] \arrow[dr, "R"'] & & \mathcal{B}(\ell_p(J))' \\ & L_p(\theta) \arrow[ur, "S'"'] & \end{tikzcd}$$

is commutative, because

$$\langle S'R(x), y\rangle = \langle R(x), S(y)\rangle = \langle \pi(x \otimes 1)\xi, (\pi(1 \otimes y))'\eta\rangle = \langle \pi(x \otimes y)\xi, \eta\rangle = \tilde{\varphi}(x \otimes y) = \langle T(x), y\rangle.$$

Combining these two commutative diagrams, we finally have $L_\varphi = (\Phi^W)'S'R\Phi^V$, that is, L_φ is factorized through a reflexive Banach space $L_p(\theta)$, so L_φ is a weakly compact operator [Meg98, Propositions 3.5.4 and 3.5.11]. $\square$

Corollary 5.2.4. *There exists a (necessarily unique) separately weak*-continuous extension*

$$\Phi : V'' \otimes W'' \to (V \overset{\vee_p}{\otimes} W)''$$

of the natural inclusion $\iota : V \otimes W \to (V \overset{\vee_p}{\otimes} W)''$.

Proof. Combine Theorem 5.2.2 and Theorem 5.2.3. Uniqueness follows from separate weak*-continuity. $\square$

Now we are ready to define condition C_p for p-operator spaces. Let Φ be as in Corollary 5.2.4. The following commutative diagram

$$\begin{tikzcd}
V'' \otimes W'' \arrow[r, "\Phi"] \arrow[d, equal] & (V \overset{\vee_p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge_p}{\otimes} W')' \arrow[d, equal] \\
\mathcal{CB}^{\sigma}_{p,F}(V', W'') \arrow[rr, hook, "\iota"] & & \mathcal{CB}_p(V', W'')
\end{tikzcd}$$

shows that Φ is injective. Thus we can equip $V'' \otimes W''$ with the p-operator space structure induced by Φ, which will be denoted by $V'' \!:\! \otimes_{\vee_p} \!:\! W''$. We say that $V \subseteq \mathcal{B}(L_p(\mu))$ satisfies *condition* C_p (or has *property* C_p) if the map Φ is isometric with respect to the injective tensor product norm with every $W \subseteq \mathcal{B}(L_p(\nu))$.

Proposition 5.2.5. *Suppose that $V \subseteq \mathcal{B}(L_p(\mu))$. Then V satisfies condition C_p if and only if V satisfies both condition C_p' and C_p''.*

Proof. Suppose that V satisfies condition C_p and $W \subseteq \mathcal{B}(L_p(\nu))$. Note that, even though we do not have a p-analogue of Arveson-Wittstock-Hahn-Banach theorem, we still have p-completely isometric embedding $V \otimes_{\vee_p} W'' \subseteq V'' \otimes_{\vee_p} W''$ and the bottom row in the following commutative diagram

$$\begin{tikzcd}
V \otimes_{\vee_p} \!:\! W'' \arrow[r] \arrow[d, hook] & V \otimes_{\vee_p} W'' \arrow[d, hook] \\
V'' \!:\! \otimes_{\vee_p} \!:\! W'' \arrow[r] & V'' \otimes_{\vee_p} W''
\end{tikzcd}$$

is isometric. Therefore the top row is also isometric and hence V satisfies condition C_p'. That V satisfies condition C_p'' can be proved using a similar argument.

On the other hand, if V satisfies condition C_p'', we get

$$V'' \otimes_{\vee_p} W'' = V'' : \otimes_{\vee_p} \!:\! W'' \hookrightarrow (V \otimes_{\vee_p} W'')''.$$

If V also satisfies condition C_p', then

$$V \otimes_{\vee_p} W'' = V \otimes_{\vee_p} \!:\! W'' \hookrightarrow (V \otimes_{\vee_p} W)'',$$

and hence we have isometric inclusion

$$V'' \otimes_{\vee_p} W'' \hookrightarrow (V \otimes_{\vee_p} W)''''.$$

Since $V'' \otimes_{\vee_p} W'' \subset (V \otimes_{\vee_p} W)''$ and $(V \otimes_{\vee_p} W)'' \hookrightarrow (V \otimes_{\vee_p} W)''''$ isometrically, the inclusion $V'' \otimes_{\vee_p} W'' \subseteq (V \otimes_{\vee_p} W)''$ must be isometric. $\square$

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Vita

EDUCATION

University of Illinois at Urbana-Champaign

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M.S. in Applied Mathematics — *February 2001*

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RESEARCH INTEREST

- **p-operator spaces and their applications**
  - Extending results on classical operator spaces to p-operator spaces
  - Property A_p and exactness of discrete groups
- **harmonic analysis on locally compact group**
  - Characterization of amenability of a locally compact group
  - Applications of theory of p-operator spaces, especially to the study of Figà-Talamanca-Herz algebras and related algebras/multipliers

PUBLICATION

- An, G., **Lee, J.-J.**, Ruan, Z.-J., On p-approximation properties for p-operator spaces (Journal of Functional Analysis 259 (2010) 933-974)

TALKS

- **Annual Wabash Miniconference** October 3, 2009
  - Indiana University Purdue University at Indianapolis
  - Title: p-operator spaces and approximation properties
- **Student seminar** Summer 2007 – on QWEP conjecture
- **Working seminar** Fall 2006 – on von Neumann algebras

TEACHING EXPERIENCE

- **Full Instructor** 9 semesters, University of Illinois at Urbana-Champaign
  - Class organization, exam making/grading with full responsibility
  - **Four** appearances on the 'List of Teachers Ranked as Excellent', **twice ranked within top 10%**
  - Introductory Matrix Theory (Sp '06, Fa '06, Sp '07, Su '07, Fa '08‡, Sp '09†, Fa '09†, Sp '10)
  - Elementary Linear Algebra (Su '09‡)
- **Discussion Session Leader** 4 semesters, University of Illinois at Urbana-Champaign
  - **Two** appearances on the 'List of Teachers Ranked as Excellent'
  - Calculus II (Sp '08†)
  - Calculus I (Fa '07)
  - Calculus of Several Variables (Fa '05†)
  - Algebra (Fa '04)
- One of the finalists for the Graduate Teaching Award, Department of Mathematics, University of Illinois at Urbana-Champaign (2010)

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HONORS AND AWARDS

- **Graduate College Scholarship** University of Illinois at Urbana-Champaign
  - 2003-2004 academic year and 2004-2005 academic year
  - Each came with full tuition waiver and $7,500 stipend
- **Research Experiences for Graduate Students** University of Illinois at Urbana-Champaign
  - Project on vector-valued integration under the direction of Professor Marius Junge

- Came with summer tuition waiver and \$3,000 stipend

- **Korea University Presidential Award** Korea University, Seoul, South Korea
  - For best undergraduate academic achievement in College of Science
  - Came with full tuition waiver for senior semester

- **Freshmen Scholarship** Korea University, Seoul, South Korea
  - For excellence in the national university entrance examination
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- **Research Assistantship** University of Illinois at Urbana-Champaign
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