A CHARACTERIZATION OF BI-LIPSCHITZ EMBEDDABLE METRIC SPACES IN TERMS OF LOCAL BI-LIPSCHITZ EMBEDDABILITY

BY

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DISSERTATION

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Abstract

We characterize uniformly perfect, complete, doubling metric spaces which embed bi-Lipschitzly into Euclidean space. Our result applies in particular to spaces of Grushin type equipped with Carnot-Carathéodory distance. Hence we obtain the first example of a sub-Riemannian manifold admitting such a bi-Lipschitz embedding. Our techniques involve a passage from local to global information, building on work of Christ and McShane. A new feature of our proof is the verification of the co-Lipschitz condition. This verification splits into a large scale case and a local case. These cases are distinguished by a relative distance map which is associated to a Whitey-type decomposition of an open subset Ω of the space. We prove that if the Whitney cubes embed uniformly bi-Lipschitzly into a fixed Euclidean space, and if the complement of Ω also embeds, then so does the full space.

To my family

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Chapter 1

Introduction

A map between two metric spaces is bi-Lipschitz if distances in the image and source should not exceed distances in the source and image respectively by more than a fixed, universal multiplicative constant. More precisely, a map f between metric spaces $(X,\, d_X)$ and $(Y,\, d_Y)$ is called bi-Lipschitz if there exists an $L \geq 1$ such that

$$\frac{1}{L}\, d_X(x,\, y) \leq d_Y(f(x),\, f(y)) \leq L\, d_X(x,\, y)$$

for all $x,\, y \in X$.

Bi-Lipschitz maps play a role in computer science as well as in many branches of mathematics. Solving the Sparsest cut problem approximately is important in the theory of approximation algorithms. The best known algorithm for this question is related to the Goemans-Linial conjecture. That is, every metric space (X, d) such that $(X, \sqrt{d})$ is isometric to a subset of l_2 can be bi-Lipschitz embedded into L^1 [11, 21]. Khot and Vishnoi [17] proved that the Goemans-Linial conjecture is not true. Indeed they constructed arbitrarily large metric spaces of negative type whose bi-Lipschitz constant for embeddings into L^1 tends to infinity. Recently, Cheeger and Kleiner [8] together with Lee and Naor [28] gave another counterexample to the Goemans-Linial conjecture which is based on classical analysis and a well understood metric space. They showed that the Heisenberg group admits a metric which is of negative type, yet does not admit a bi-Lipschitz embedding into L^1.

If two metric spaces are bi-Lipschitz equivalent then they have approximately the same behavior in terms of length, Hausdorff measure, topology and so on. Moreover, bi-Lipschitz maps are related to problems of differentiability by Rademacher's theorem. Lipschitz maps form the right substitute for smooth maps in the theory of analysis on metric spaces. We would like to know for which metric spaces the resulting analysis is genuinely new and for which ones the analysis can be seen as just classical analysis on a suitable subset of a Banach space. This leads to the question to characterize metric spaces that embed bi-Lipschitzly into classical Banach spaces. However, the characterization of metric spaces which are bi-Lipschitz equivalent to $\mathbb{R}^n$ or even of metric spaces which are bi-Lipschitzly embeddable into $\mathbb{R}^n$ remain difficult open problems in

Geometric Analysis.

Firstly, we state some known results concerning the question of which metric spaces are bi-Lipschitz equivalent to Euclidean space. It is well known that a metric space (X, d) is bi-Lipschitz equivalent to $\mathbb{R}$ if and only if it is of bounded turning and 1-Ahlfors regular. However, a bi-Lipschitz characterization of $\mathbb{R}^n$ is still unknown for $n \geq 2$. In the higher dimensional case, we might consider the condition of Linearly local connectivity (LLC) which is an analogue of the bounded turning condition. However, Ahlfors n-regularity and LLC are not sufficient conditions due to results of Semmes and Laakso. Indeed, Semmes [32] constructed a 3-regular set $E \subset \mathbb{R}^4$, which is the quasiconformal image of the hyperplane, such that (E, d) is not bi-Lipschitz equivalent to $\mathbb{R}^3$. Laakso [19] constructed a metric δ_ω on $\mathbb{R}^2$ deformed by a strong A_∞-weight ω such that $(\mathbb{R}^2, \delta_\omega)$ is Ahlfors 2-regular and satisfies the LLC condition, yet admits no bi-Lipschitz embedding into any Euclidean space. Therefore, in particular, it is not bi-Lipschitz equivalent to $\mathbb{R}^2$.

Bonk, Heinonen, and Saksman [4] showed that the bi-Lipschitz classification problem for Euclidean space is closely related to the quasiconformal Jacobian problem, asked by David and Semmes in 1990 [10]. That is, which locally integrable nonnegative functions ω can arise, up to a bounded multiplicative error, as Jacobian determinants $Jf(x) = \det(Df(x))$ of quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ where $n \geq 2$? Indeed, they showed that the two problems are equivalent in case $n = 2$ and the second question is stronger than the first one for any $n > 2$. However, it is still difficult to answer the quasiconformal Jacobian problem.

Now we come to the weaker question: which metric spaces embed bi-Lipschitzly into Euclidean space. We state progress on this problem, beginning with sufficient conditions. Assouad gave a partial answer: every snowflaked version of a doubling metric space embeds bi-Lipschitzly into some Euclidean space [1]. Even though the theorem of Assouad completely answers the question which metric spaces are quasisymmetrically embeddable into Euclidean space, this result does not guarantee bi-Lipschitz embeddability of the original metric space. In particular, the Heisenberg group, which is a doubling metric space, admits no bi-Lipschitz embedding into Euclidean space. Luosto [22] together with Luukkainen and Movahedi-Lankarani [23] gave a precise relationship between Assouad dimension and dimension of receiving Euclidean space for ultra metric spaces: an ultrametric space is bi-Lipschitzly embeddable into $\mathbb{R}^n$ if and only if its Assouad dimension is less than n.

Semmes [31] showed that $\mathbb{R}^n$ equipped with any metric δ_ω deformed by A_1-weight ω admits a bi-Lipschitz embedding into some $\mathbb{R}^N$. However, $(\mathbb{R}^n, \delta_\omega)$ may be not bi-Lipschitzly equivalent to $\mathbb{R}^n$. Bishop [3] constructed a Sierpinski carpet $E \subset \mathbb{R}^2$ and an A_1-weight ω which blows up on E. In this construction, he showed that w is not comparable to the Jacobian of any quasiconformal mapping.

We now turn to necessary conditions and counterexamples related to bi-Lipschitz embeddability. Pansu

[29] showed that a version of Rademacher's differentiation theorem holds for Lipschitz maps on Carnot groups: every Lipschitz map between Carnot groups is almost everywhere differentiable in some sense and its differential is a Lie group homomorphism. Semmes observed that Pansu's result implies that nonabelian Carnot groups admit no bi-Lipschitz embedding into Euclidean space (Theorem 7.1 in [32]). Cheeger proved a remarkable extension of Rademacher's theorem for doubling p-Poincaré spaces and gave a corresponding nonembedding theorem (see Section 10 and Theorem 14.3 in [5]).

By Cheeger's theorem, we can deduce nonembeddability of certain regular sub-Riemannian manifolds. However, his result does not apply to singular sub-Riemannian manifolds. This thesis is motivated by the question whether or not the Grushin plane embeds bi-Lipschitzly into Euclidean space. While the Grushin plane is one of the simplest singular sub-Riemannian manifold, the previous known nonembedding theorems do not apply. We will explain further in Chapter 4.

Now we come to the main result of this thesis. We will characterize uniformly perfect complete metric spaces which admit a bi-Lipschitz embedding in terms of uniform local bi-Lipschitz embeddability. Indeed, uniform perfectness and existence of a doubling measure yield existence of a type of Whitney decomposition. Furthermore, uniform local bi-Lipschitz embeddability of Christ cubes associated with such a decomposition implies global bi-Lipschitz embeddability.

Theorem 1.0.1. *A uniformly perfect complete metric space (X, d) admits a bi-Lipschitz embedding into some Euclidean space if and only if the following conditions hold:*

(1) it supports a doubling measure μ,

(2) there exists a closed subset Y of X which admits a bi-Lipschitz embedding into some $\mathbb{R}^{M_1}$,

(3) $\Omega = X \setminus Y$ admits uniformly Christ-local bi-Lipschitz embeddings into some $\mathbb{R}^{M_2}$.

The bi-Lipschitz constant and dimension of receiving Euclidean space depend on the data of the metric space X, the doubling constant of μ, M_1, M_2 and the bi-Lipschitz constants in conditions (2) *and* (3).

The structure of this thesis follows. In the second chapter, we shall see some basic propositions for Lipschitz maps, Lipschitz embedding, and extension theorems. We will review Michael Christ's construction of a system of dyadic cubes [9] in doubling metric spaces. We will next construct a Whitney-type decomposition which we call a Christ-Whitney decomposition (Lemma 2.4.2) for a uniformly perfect space supporting a doubling measure. We will also introduce some definitions and lemmas which set the stage for the main theorem.

In the following chapter, we shall characterize bi-Lipschitz embeddable metric spaces by proving Theorem 1.0.1. To this end, we first apply McShane's extension theorem to extend a Lipschitz map on Y to X.

We introduce the Whitney distance map d_{W} (Definition 2.4.12). It is the key tool for construction of a co-Lipschitz map. We break the Christ-Whitney decomposition into two parts using the Whitney distance map. After some basic preliminaries, we will construct a W-local co-Lipschitz and W-large scale co-Lipschitz map on these parts (Lemma 3.2.1 and Lemma 3.3.1).

In Chapter 4, we discuss our main application of Theorem 1.0.1 to the bi-Lipschitz embedding question for sub-Riemannian manifolds. We first recall the Rademacher-type theorems of Pansu and Cheeger. Then, we discuss their applications to the problem of bi-Lipschitz embedding. In contrast, as an application of Theorem 1.0.1 (Corollary 4.3.5) we will prove that spaces of Grushin type equipped with Carnot-Carathéodory distance embed bi-Lipschitzly into Euclidean space. These are the first examples of sub-Riemannian manifolds that embed bi-Lipschitzly into Euclidean space. In the last chapter, we will raise some remarks and questions arising from this thesis.

Chapter 2

Preliminaries

In Section 2.1 of this chapter we start with basic definitions and concepts. In Section 2.2 and Section 2.3 we state some well known Lipschitz embedding and extension theorems that will be used in our main theorem. In the final section of this chapter we introduce some definitions and lemmas. They will set the stage for Proposition 3.1.2 in Chapter 3.

2.1 Background on Metric Spaces

For a metric space $X = (X, d)$, we write $diam(A)$ (or $diam_d(A)$ in case we need to mention the metric) for the diameter of a set $A \subset X$, d_E for the Euclidean metric, and $dist(A, B)$ for the distance between nonempty sets $A,\ B \subset X$. We abbreviate $dist(A, x) = dist(A, \{x\})$ for a set $A \subset X$ and $x \in X$. We denote by $\overline{A}$ the closure of A and by $B(x, r)$ ($B^o(x, r)$) a closed ball (open ball) in X with radius r and center x. As customary, we let $C, c, \cdots$ denote finite positive constants. These constants may depend on auxiliary data a, b, etc ; we indicate this by writing $C(a, b)$ or $c(a, b)$. We also write $a \lesssim b$ if there is a constant C such that $a \leq C\, b$.

Definition 2.1.1. *A map $f : X \to Y$ is an embedding if it is a homeomorphism onto its image. An embedding f is L-bi-Lipschitz, $L \geq 1$, if*

$$\frac{1}{L}\, d_X(x, y) \leq d_Y(f(x), f(y)) \leq L\, d_X(x, y) \tag{2.1.1}$$

whenever $x, y \in X$. In the case $L = 1$, we call f an isometric embedding.

In other words, f and f^{-1} are L-Lipschitz. We say f is co-Lipschitz if f^{-1} is Lipschitz. We call any constant L satisfying equation (2.1.1) a bi-Lipschitz constant for f.

Definition 2.1.2. *An embedding $f : X \to Y$ is called quasisymmetric if there is a homeomorphism η :*

$[0, \infty) \to [0, \infty)$ *such that*

$$d_X(x, a) \leq t\, d_X(x,\, b) \quad \textit{implies} \quad d_Y(f(x),\, f(a)) \leq \eta(t)\, d_Y(f(x),\, f(b))$$

for all triples $a,\ b,\ x \in X$ *and for all* $t > 0$*.*

Obviously, every L-bi-Lipschitz embedding is quasisymmetric with $\eta(t) = L^2 t$. On the other hand, the identity map $(X,\, d_E) \to (X,\, {d_E}^{\epsilon})$ is quasisymmetric but not Lipschitz (Definition 2.2.1). Roughly speaking bi-Lipschitz embeddings distort both the shape and size of an object by a bounded amount, while quasisymmetry only preserves the approximate shape.

Properties of Lipschitz Maps. We denote by $LIP(X,\, Y)$ the collection of Lipschitz maps between metric spaces X and Y and by $LIP(f)$ the infimal Lipschitz constant.

Proposition 2.1.3. *Let* $(X,\, d)$ *be a metric space and* Y *be a normed vector space over* $\mathbb{R}$ *or* $\mathbb{C}$ *with metric endowed with* $d_Y(x,\, y) = \|x - y\|$*. Then,* $LIP(X,\, Y)$ *has a linear vector space structure. That is, for arbitrary* $f,\ g \in LIP(X,\, Y)$ *and* $\alpha \in \mathbb{R}$ *or* $\mathbb{C}$ *we have* $f + g \in LIP(X,\, Y)$ *and* $\alpha f \in LIP(X,\, Y)$*.*

We omit the proof.

Proposition 2.1.4. *If* $f,\ g \in LIP(X,\, \mathbb{R})$ *are bounded functions, then* $f \cdot g \in LIP(X,\, \mathbb{R})$*.*

Proof. It is obvious from the triangle inequality. $\square$

Proposition 2.1.5. *If* $f \in LIP(X,\, Y)$ *and* $g \in LIP(Y,\, Z)$*, then* $g \circ f \in LIP(X, Z)$ *and* $LIP(g \circ f) \leq LIP(g)LIP(f)$*.*

Proof. Suppose that f is L_1-Lipschitz and g is L_2-Lipschitz. Then,

$$d_Z(g \circ f(x),\, g \circ f(y)) \leq L_1\, d_Y(f(x),\, f(y)) \leq L_1\, L_2\, d_X(x,\, y)$$

for all $x,\, y \in X$. $\square$

The following proposition means that we may assume the domain of a bi-Lipschitz map to a complete metric space is closed.

Proposition 2.1.6. *Let* A *be a subset of an arbitrary metric space* $(X,\, d_X)$ *and let* $(Y,\, d_Y)$ *be a complete metric space. If* $f : A \to Y$ *is* L*-bi-Lipschitz, then there is a uniquely defined* L*-bi-Lipschitz map* $F : \overline{A} \to Y$ *such that* $F|_A = f$*.*

Proof. For any two points $x,\ y \in \overline{A}$, we have sequences $\{x_n\}$, $\{y_n\}$ such that $x_n \to x$ and $y_n \to y$. Whenever $\{x_n\}$ and $\{y_n\}$ are Cauchy, $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy because f is Lipschitz. Thus, we have $z,\ z' \in Y$ such that $z = \lim_{n\to\infty} f(x_n)$ and $z' = \lim_{n\to\infty} f(y_n)$. We now define $z = F(x)$ and $z' = F(y)$. We obtain a well defined extension F of f and we can write

$$L^{-1} \lim_{n\to\infty} d_X(x_n,\, y_n) \le d_Y(z,\, z') = \lim_{n\to\infty} d_Y(f(x_n),\, f(y_n)) \le L \lim_{n\to\infty} d_X(x_n,\, y_n).$$

Therefore, $L^{-1}\, d_X(x,\, y) \le d_Y(F(x),\, F(y)) \le L\, d_X(x,\, y)$. Uniqueness is obvious. □

Uniform perfectness, doubling metric measure space and p-Poincaré inequality. People have studied spaces whose infinitesimal data yields global information. Metric spaces with a doubling measure and a Poincaré inequality admit first-order differential calculus similar to that in Euclidean space. The doubling condition provides a kind of boundedness of the geometry of the space and the p-Poincaré inequality expresses global control of a function in terms of its derivative. For more details, see Theorem 4.2.2 in Chapter 4.

Definition 2.1.7. *A metric space $(X,\, d)$ is uniformly perfect if there exists a constant $A > 0$ such that for each $x \in X$ and $0 < r < diam X$ there is a point $y \in X$ which satisfies $A^{-1} r \le d(x, y) \le r$. We say that $(X,\, d)$ is A-uniformly perfect.*

Uniform perfectness implies nonexistence of separating annuli of large modulus and nonexistence of isolated points. Every connected metric space is uniformly perfect. This is useful since $\overline{B}(x,\, r) \setminus B(x,\, A^{-1} r)$ is nonempty for all $x \in X$ and $0 < r < diam X$ and then we can say $A^{-1} r \le diam B(x,\, r) \le 2r$.

Definition 2.1.8. *A Borel measure μ in a metric space is called doubling if balls have finite and positive measure for any nonempty ball and there is a constant $D \ge 1$ such that*

$$\mu(B(x, 2r)) \le D \mu(B(x, r)) \tag{2.1.2}$$

for all $x \in X$ and $r > 0$. We call D a doubling constant.

In this case $0 < \mu(B) < \infty$ for all balls B and

$$\mu(B(x, \lambda r)) \le D \lambda^s \mu(B(x, r)) \tag{2.1.3}$$

for all $x \in X$, $r > 0$ and $\lambda \ge 1$ with $s = \log_2(D)$.

Definition 2.1.9. *A metric space is called doubling if there is a constant C so that every set of diameter d in the space can be covered by at most C sets of diameter at most $d/2$.*

It is clear that subsets of doubling spaces are doubling with same doubling constant. The existence of a doubling measure on (X, d) implies that (X, d) is doubling. The converse of this statement is not true in general. Saksman [30] found a Jordan domain in $\mathbb{R}^2$ which does not carry a doubling measure. On the other hand, we have the following theorem due to Luukkainen-Saksman and Wu.

Theorem 2.1.10 (Luukkainen-Saksman [24], Wu [34])**.** *A complete metric space $(X,\, d)$ carries a doubling measure if and only if X is doubling.*

Proposition 2.1.11. *If X is a doubling metric space, then it has the following covering property: there exists a function $C : (0,\, \frac{1}{2}] \to (0,\, \infty)$ such that every set of diameter d can be covered by at most $C(\epsilon)$ sets of diameter at most $\epsilon\, d$. Moreover, the covering function $C(\epsilon)$ can be chosen to be the form*

$$C(\epsilon) = D\,\epsilon^{-\beta} \text{ for some } D \;\text{ and } \beta > 0. \tag{2.1.4}$$

The infimal β satisfying (2.1.4) *on a given doubling metric space $(X,\, d)$ is called the Assouad dimension of X.*

The following theorem asserts that the doubling property and uniform perfectness are invariant under bi-Lipschitz mappings. In fact, they are quasisymmetrically invariant quantatively [14].

Theorem 2.1.12. *Quasisymmetric images of doubling or uniformly perfect metric spaces are doubling or uniformly perfect respectively.*

Proof. Let $f : X \to Y$ be an η-quasisymmetric homeomorphism. Suppose $(X,\, d_X)$ is a doubling metric space. It suffices to show that every ball B of radius r' can be covered by at most C_2 of sets of diameter $\leq \frac{r}{2}$. Let $B = B(y,\, r)$ and let

$$L = \sup_{z \in B} |f^{-1}(y) - f^{-1}(z)|.$$

Then, we can cover $f^{-1}(B)$ by at most $C_1(\epsilon)$ sets of diameters at most $2\epsilon L$ for any $\epsilon \leq \frac{1}{2}$, where C_1 is a covering function of X. Let $A_1,\, A_2,\, \cdots,\, A_p$ be such sets, $p = p(\epsilon) \leq C(\epsilon)$. We may assume that $A_i \subset f^{-1}(B)$ for all $i = 1, \cdots, p$. Thus $f(A_i) \subset B$ and $f(A_1),\, \cdots,\, f(A_p)$ cover B. Therefore, by Proposition 10.8 [14], we have

$$\frac{diam\, f(A_i)}{diam\, B} \leq \eta\,(\frac{2\, diam\, A_i}{diam\, f^{-1}(B)}) \leq \eta(4\epsilon)$$

Thus, $diam\, f(A_i) \leq 2r\,\eta(4\epsilon)$. We complete the proof of quasisymmetric invariance of doubling condition by choosing $\epsilon = \epsilon(\eta)$ so small that $\eta(4\epsilon) \leq \frac{1}{4}$.

Now we assume that (Y, d_Y) is A'-uniformly perfect. It suffices to show that for any $x \in X$ and $r > 0$, there exists $A \geq 1$ so that the annulus $\overline{B}(x, r) \setminus B(x, A^{-1}r)$ is nonempty. We let

- $R_2 = \inf\{R > 0 \mid f(B(x, A^{-1}r)) \subset B(f(x), R)\}$.
- $R_1 = \sup\{R > 0 \mid B(f(x), R) \subset f(B(x, r))\}$.

Since f is η-quasisymmetric, we have $\dfrac{R_1}{R_2} \leq \eta(\dfrac{1}{A})$. We choose $A = A(\eta, A')$ so that $\eta(\dfrac{1}{A}) = \dfrac{1}{2A'}$. By A'-uniform perfectness of Y, the annulus $\overline{B}(f(x), R_1) \setminus B(f(x), R_2) \neq \emptyset$. Therefore, we have A such that $\overline{B}(x, r) \setminus B(x, A^{-1}r) \neq \emptyset$. This completes the proof of quasisymmetric invariance of uniform perfectness. □

Corollary 2.1.13. *Bi-Lipschitz images of doubling or uniformly perfect metric spaces are doubling or uniformly perfect respectively.*

Basic analysis involving only functions can be done in doubling metric measure spaces. However, the structure of a doubling metric measure space is not strong enough for calculus. For example, we consider the snowflaking of a doubling metric space, $X = ([0,1], \sqrt{d_E}, \mathcal{H}^2)$ where $\mathcal{H}^2$ is 2-dimensional Hausdorff measure. The function $f(x) = x$ is 1-Lipschitz on X, with

$$\frac{|f(x) - f(y)|}{\sqrt{|x - y|}} = \sqrt{|x - y|}.$$

If an appropriate notion of derivative were defined for Lipschitz maps on X, it would vanish for f and such infinitesimal information would imply that f is constant. Therefore, we need a requirement that must be consequence to any reasonable calculus. The following definition allows one to obtain bounds on a function using bounds on its derivatives and geometry of its domain of definition. Let (X, d, μ) be a metric space with a Borel measure, not necessarily doubling.

Definition 2.1.14. *Let (X, d) be a metric space. A Borel function $\rho : X \to [0, \infty]$ is said to be an upper gradient of a function $u : X \to \mathbb{R}$ if*

$$|u(b) - u(a)| \leq \int_\gamma \rho \, ds \tag{2.1.5}$$

whenever a, $b \in X$ and γ is a rectifiable curve in X with end points a and b.

As a trivial example, we have an upper gradient $\rho \equiv \infty$ of every function, and $\rho \equiv 0$ is an upper gradient of any function if X contains no rectifiable curves.

Definition 2.1.15. *We say that the space (X, d, μ) supports a p-Poincaré inequality, $1 \leq p < \infty$, if every* *pair $(u,\, \rho)$ of a continuous function u and its upper gradient ρ satisfies the inequality*

$$\fint_B |u - u_B|\, d\mu \leq C_p\, r \left(\fint_{\lambda B} \rho^p\, d\mu \right)^{\frac{1}{p}}, \tag{2.1.6}$$

on each ball B with radius r, where $\lambda \geq 1$, C_p are fixed constants. We denote

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu.$$

$\mathbb{R}^n$ supports a Poincaré inequality for any $p \geq 1$, known as the Sobolev-Poincaré inequality. In contrast, the above example $([0,\, 1], \sqrt{d_E}, \mathcal{H}^2)$ can not support a p-Poincaré inequality for any p because there are no rectifiable curves.

A doubling metric space supporting a p-Poincaré inequality gives a strong measurable differentiable structure. This allow us to have a differential for a Lipschitz map.

We emphasize that uniformly perfect metric spaces supporting a doubling measure will play an important role so as to have a type of Whitney decomposition, which we will discuss in the last section of this chapter.

2.2 Assouad Embedding Theorem

Characterizing metric spaces that admit a bi-Lipschitz embedding into Euclidean space is one of the celebrated open questions in Geometric analysis. Assouad provided a sufficient condition for bi-Lipschitz embeddability into Euclidean space. We start this section by introducing the definition of a snowflaked version of metric space.

Definition 2.2.1 (Snowflaking)**.** *If $(X,\, d)$ is a metric space, then its snowflaking is a metric space $(X,\, d^\epsilon)$, where $0 < \epsilon < 1$. We say that $(X,\, d^\epsilon)$ is a snowflaked version of $(X,\, d)$.*

Theorem 2.2.2 (Assouad [1])**.** *Each snowflaked version of a doubling metric space admits a bi-Lipschitz embedding into some Euclidean space. If $0 < \epsilon < 1$, then $(\mathbb{R},\, {d_E}^\epsilon)$ embeds bi-Lipschitzly into $\mathbb{R}^k$, where k is the smallest integer which is greater than $\frac{1}{\epsilon}$.*

The identity snowflaking $(X,\, d) \to (X,\, d^\epsilon)$ is t^ϵ-quasisymmetric and hence, each metric space is quasisymmetrically embedded in Euclidean space if and only if it is doubling by Theorem 2.2.2 and Theorem 2.1.12. However, Assouad's theorem does not answer whether or not the original metric space embeds bi-Lipschitzly. For example, whereas the snowflaking of the Heisenberg group endowed with Carnot-Carathéodory distance, $(\mathbb{H},\, {d_{cc}}^\epsilon)$, admits a bi-Lipschitz embedding into some Euclidean space, the Heisenberg group is not bi-Lipschitzly embeddable into any Euclidean space due to Pansu and Semmes ([32], [29]) or Cheeger's nonembedding theorem [5] that will discussed in Chapter 4. For more examples, we give references [20], [19], and [18].

We shall briefly sketch the proof of Assouad's embedding theorem. He builds a multiscale family of maps on scale 2^{-j} for each $j \in \mathbb{Z}$ and glues these maps together into an embedding using 2^{-j}-nets and a coloring map. A similar idea will appear in the proof of Theorem 3.1.2. In fact, we shall consider a Whitney-type decomposition instead of nets and use a coloring map to add dimensions of receiving Euclidean space.

Proof. A maximal ϵ-net in X is an ϵ-dense, ϵ-separated subset of X. (A set S is ϵ-separated if mutual distances between two points of S are at least ϵ ; S is ϵ-dense if every point of X is within distance ϵ from some point of S). Let N_0 be a maximal 1-net. Doubling condition yields there are M finitely many points in $N_0 \cap B(x,\, 12)$ for all $x \in X$. By a standard application of Zorn's lemma, we have a coloring map

$$K : N_0 \to \{1,\, 2,\, \cdots,\, M\}$$

so that $K(p_i) \neq K(p_j)$ if $d(p_i,\, p_j) < 12$.

We now write $\{e_j\}$ for the standard basis of $\mathbb{R}^M$ and define $\phi_0 : X \to \mathbb{R}^M$ by

$$\phi_0(x) = \sum_{p_i \in N_0} \max\{2 - d(x,\, p_i),\, 0\} e_{K(p_i)}.$$

Clearly, it is finite sum of 1-Lipschitz maps and hence $|\phi_0(x) - \phi_0(y)| \le C \min\{d(x,\, y),\, 1\}$ for all $x,\, y \in X$ and uniform constant C , while $2^{-1}8 \le d(x,\, y) \le 8$ implies $|\phi_0(x) - \phi_0(y)| \ge 1$. For each $j \in \mathbb{N}$, we have a finite number of points in a maximal 2^{-j}-net, N_j, intersecting $B(x,\, 2^{-j}12)$ for all $x \in X$ and we similarly obtain a map $\phi_j : X \to \mathbb{R}^M$ with the properties

(1) $|\phi_j(x) - \phi_j(y)| \ge 1$ if $2^{-j-1}8 < d(x,\, y) \le 2^{-j}8$,

(2) $|\phi_j(x) - \phi_j(y)| \le C \min\{2^j d(x,\, y),\, 1\}$ for all $x,\, y \in X$.

Next, consider $\mathbb{R}^{2N}$ with the standard basis $\{e_j\}$ cyclically extended to all $j \in \mathbb{Z}$. Then the following map

$$\Phi(x) := \sum_{j \in \mathbb{Z}} 2^{-\epsilon j} \phi_j(x) \otimes e_j \tag{2.2.1}$$

is the desired bi-Lipschitz map from $(X,\, d^\epsilon)$ to $\mathbb{R}^M \otimes \mathbb{R}^{2N}$, provided N is sufficiently large depending on the given data. To this end, here we may normalize the map ϕ_j so that $\phi_j(x_0) = 0$ for a fixed base point x_0.

We first note that the above series converges, because

$$|\Phi(x)| = |\Phi(x) - \Phi(x_0)| \leq C \sum_{j\in\mathbb{Z}} 2^{-\epsilon j} \min\{2^j d(x,\, y),\, 1\} < \infty.$$

We now fix $x,\, y \in X$ and choose $k \in \mathbb{Z}$ such that

$$2^{-k-1}8 < d(x,\, y) \leq 2^{-k}8$$

Then, we arrive at

$$\begin{aligned}
|\Phi(x) - \Phi(y)| &\leq \sum_{j\geq k+1} |\phi_j(x) - \phi_j(y)| + \sum_{j\leq k} |\phi_j(x) - \phi_j(y)| \\
&\leq C\,(2^{-\epsilon k} + 2^{k(1-\epsilon)} d(x,\, y)) \\
&\leq C\, d(x,\, y)^{\epsilon}.
\end{aligned}$$

Furthermore, provided N is large, we have

$$\begin{aligned}
|\Phi(x) - \Phi(y)| &\geq \Big| \sum_{-N+k<j\leq N+k} 2^{-\epsilon j}(\phi_j(x) - \phi_j(y)) \otimes e_j \Big| - \sum_{j>N+k} |\phi_j(x) - \phi_j(y)| - \sum_{j\leq -N+k} |\phi_j(x) - \phi_j(y)| \\
&\geq 2^{-\epsilon k}|\phi_k(x) - \phi_k(y)| - c\, 2^{-\epsilon(N+k)} - c\, 2^{-\epsilon(-N+k)} 2^{-N+k} d(x,\, y) \\
&\geq 2^{-\epsilon k}|\phi_k(x) - \phi_k(y)| - \frac{1}{2}\, 2^{-\epsilon k} \\
&\geq c\, 2^{-\epsilon k} \geq c\, d(x,\, y)^{\epsilon},
\end{aligned}$$

completing the proof. □

2.3 Lipschitz Extension Theorems

With some restrictions on X and Y, and for $A \subset X$, every Lipschitz function $f : A \to Y$ can be extended to a Lipschitz function $F : X \to Y$. In this section, we shall study three Lipschitz extension theorems of Kirszbraun-Valentine, McShane, and Whitney. The Kirszbraun-Valentine extension theorem is defined on any Hilbert space source and target and extension map preserves the Lipschitz constant. On the other hand, Whitney's extension theorem uses a Whitney decomposition on Euclidean space and an associated partition of unity to construct an explicit Lipschitz extension map to metric space with linear structure. McShane's Lipschitz extension map has no restriction on the source space. For further information, see [15].

Theorem 2.3.1 (Kirszbraun-Valentine)**.** *Let X and Y be Hilbert spaces and let A be a subset of X. Then any L-Lipschitz map $f : A \to Y$ extends to an L-Lipschitz map $F : X \to Y$ so that $F|_A = f$.*

The proof of Kirszbraun-Valentine's theorem uses the following intersection property: for any $\{x_i\}_{i\in I} \subset X$, $\{y_i\}_{i\in I} \subset Y$ and $\{r_i\}$, we assume that $d(y_i, y_j) \le d(x_i, x_j)$. Then $\bigcap_{i\in I} \overline{B}(x_i, r_i) \neq \emptyset$ implies that $\bigcap_{i\in I} \overline{B}(y_i, r_i) \neq \emptyset$. We omit the proof.

Next extension theorem is useful because the source metric space can be arbitrary.

Theorem 2.3.2 (McShane)**.** *Let X be an arbitrary metric space. If $A \subset X$ and $f : A \to \mathbb{R}$ is L-Lipschitz, then there exists an L-Lipschitz function $F : X \to \mathbb{R}$ which extends f. i.e. $F|_A = f$.*

Proof. We define $f_a(x) = f(a) + L\, d(x, a)$ for $a \in Y$. Then f_a is L-Lipschitz, $f_a \ge f(a)$ and $f_a(a) = f(a)$. Thus, $F(x) = \inf_{a\in Y} f_a(x)$ is the required L-Lipschitz extension of f. □

Corollary 2.3.3. *Let $f : A \to \mathbb{R}^M$ where $A \subset X$, be an L-Lipschitz function. Then, there exists an $\sqrt{M}L$-Lipschitz function $F : X \to \mathbb{R}^M$ such that $F|_A = f$.*

Proof. This corollary follows immediately by applying Theorem 2.3.2 to coordinate functions of f. □

We remark that Theorem 2.3.1 is sharper than Theorem 2.3.2 and Theorem 2.3.6 in the sense that the Lipschitz constant for a Lipschitz extension map does not increase.

In Euclidean space $\mathbb{R}^n$, a system of dyadic cubes is the collection D of cubes consisting of all (closed) cubes Q in $\mathbb{R}^n$ that have sides parallel to the coordinate axes, side length 2^k and vertices in the set $2^k\mathbb{Z}^n$, where $k \in \mathbb{Z}$. We divide D into generations, each consisting of essentially disjoint cubes with side length 2^k for a fixed k. Then, we can decompose any open subset of $\mathbb{R}^n$ into a disjoint union of cubes whose diameters are approximately proportional to their distances from its complement.

Definition 2.3.4 (Whitney decomposition)**.** *Let Y be a closed subset of $\mathbb{R}^n$. Then its complement Ω is the union of a sequence of cubes Q, whose interiors are mutually disjoint and whose diameters are approximately proportional to their distances from the closed set Y. More precisely,*

(1) $\Omega = \cup_{Q\in \mathrm{W}_\Omega} Q$,

(2) The interiors of any two cubes are mutually disjoint,

(3) $c_1\, dist(Q, Y) \le diam(Q) \le c_2\, dist(Q, Y)$.

The constants c_1 and c_2 are independent of Q.

Lemma 2.3.5. *We have a Lipschitz partition of unity associated to the Whitney decomposition* W_Ω*. More precisely, we have a collection* $\{\varphi_Q\}_{Q\in \mathrm{W}_\Omega}$ *satisfying the following:*

(1) $0 \le \varphi_Q \le 1$*,*

(2) $\varphi_Q|_Q \ge \frac{1}{C_1} > 0$ *and* $\varphi_Q|_{X\setminus \lambda Q} = 0$*,*

(3) φ_Q *is Lipschitz with constant* $C_2/diam(Q)$*,*

(4) For every $p \in \Omega$*, we have* $\varphi_Q(p) \neq 0$ *for at most* C_3 *elements in* W_Ω*,*

(5) $\sum_{Q\in \mathrm{W}_\Omega} \varphi_Q = 1$*.*

Here C_1*,* C_2*, and* C_3 *denote uniformly fixed constants depending on* n *while independent of the choice of element in* W_Ω *and* λ *is a universal fixed constant where* $1 < \lambda < \frac{5}{4}$ *which is independent of* n *and the choice of element in* W_Ω*. We denote* λQ *by a* λ *dilated Whitney cube with same center of* Q*.*

Proof. We define

$$\psi_Q(x) = \max\big\{0,\ \ 1 - \frac{dist(x,Q)}{dist(Q, X\setminus \lambda Q)}\big\}.$$

Then,

- $0 \le \psi_Q \le 1$, $\psi_Q|_{X\setminus\lambda Q} = 0$ and $\psi_Q|_Q = 1$.
- ψ_Q is Lipschitz with constant $\dfrac{c}{diam(Q)}$.

Now we define

$$\psi := \sum_{Q\in \mathrm{W}_\Omega} \psi_Q.$$

We note that $\psi \ge 1$ everywhere and ψ is locally finite because each point of Ω is contained in at most $c_2(n)$ of the cubes λQ (see Proposition 3 in Chapter VI [33]). Then, $\psi|_{\lambda Q}$ is $\dfrac{c\,c_2}{diam(Q)}$-Lipschitz.

Next, when we set

$$\varphi_Q = \frac{\psi_Q}{\psi},$$

then, $0 \le \varphi_Q \le 1$, $\varphi_Q|_{X\setminus\lambda Q} = 0$ and $\sum_{Q\in \mathrm{W}_\Omega} \varphi_Q = 1$. We further claim that φ_Q is $\dfrac{2c\,c_2}{diam(Q)}$-Lipschitz. To this end, let us first consider $x,\ y \in \lambda Q$. Then,

$$\begin{aligned}
|\varphi_Q(x) - \varphi_Q(y)| &= \frac{|\psi_Q(x)\psi(y) - \psi_Q(y)\psi(x)|}{\psi(x)\psi(y)} \\
&\leq |\psi_Q(x)\psi(y) - \psi_Q(y)\psi(x)| \\
&\leq |\psi_Q(x)\psi(y) - \psi_Q(y)\psi(y)| + |\psi_Q(y)\psi(y) - \psi_Q(y)\psi(x)| \\
&\leq \sup_{\lambda Q} |\psi| \frac{c}{diam(Q)}\, d(x,\, y) + \frac{c\, c_2}{diam(Q)}\, d(x,\, y) \\
&\leq \frac{2c\, c_2}{diam(Q)}\, d(x, y)
\end{aligned}$$

Next, let $x \in \lambda Q$ and $y \in X \setminus \lambda Q$. Then, we observe that

$$|\varphi_Q(x) - \varphi_Q(y)| \leq |\varphi_Q(x)| \leq |\psi_Q(x)| \leq \frac{c}{diam(Q)}\, d(x,\, y),$$

which proves the claim. □

Theorem 2.3.6 (Whitney [33])**.** *Let Y be a closed subset of $\mathbb{R}^n$ and f be an L-Lipschitz function from $(Y,\, d)$ into $\mathbb{R}^m$ for some m. Suppose that W_Ω is a Whitney decomposition of $\Omega = X \setminus Y$ and $\{\varphi_Q\}$ is an associated Lipschitz partition of unity as in Lemma 2.3.5. Then,*

$$g(p) = \begin{cases} \sum_{Q \in \mathrm{W}_\Omega} f(z_Q)\varphi_Q(p), & for\ p \in \Omega\ and\ z_Q \in Y\ \ such\ that\ dist(Y,\, Q) = dist(z_Q,\, Q); \\ f(p), & for\ p \in Y. \end{cases} \tag{2.3.1}$$

is a $L'(L,\, n)$-Lipschitz extension of f to $\mathbb{R}^n$.

Proof. Suppose that $p \in \Omega$ and $q \in Y$. Then

$$\begin{aligned}
|g(p) - g(q)| &= |\sum_{Q \in \mathrm{W}_\Omega} f(z_Q)\varphi_Q(p) - f(q)| \\
&\leq \sum_{Q \in \mathrm{W}_\Omega, q \in \lambda Q} |f(z_Q) - f(q)||\varphi_Q(p)| \\
&\leq c\, c_2\, L\, d(p,\, q),
\end{aligned}$$

where $c_2(n)$ is the number of Q such that $q \in \lambda Q$. The last inequality comes from $d(q,\, z_Q) \leq c\, d(p,\, q)$.

When $p,\, q \in Y$, we have two cases. If there exists $z \in Y$ such that $d(p,\, z) + d(q,\, z) \leq 100\, d(p,\, q)$, then

$$\begin{aligned}|g(p) - g(q)| &\leq |g(p) - g(z)| + |g(q) - g(z)| \\ &\leq c\, c_2\, L\, \{d(p,\, z) + d(q,\, z)\} \\ &\leq 100\, c\, c_2\, L\, d(p,\, q).\end{aligned}$$

Otherwise, for all $z \in Y$, $\min\{d(p,\, z), d(q,\, z)\} \geq 25\, d(p,\, q)$. We consider γ a line segment p to q. Then,

$$\begin{aligned}|g(p) - g(q)| &\leq \int_\gamma \|\nabla g\|\, d\gamma \\ &\leq C(n,\, L)\, d(p,\, q).\end{aligned}$$

We can get the above inequality from the following:

$$\begin{aligned}\frac{\partial}{\partial x_i} g(p) &= \sum_{Q \in \mathrm{W}_\Omega} f(z_Q) \frac{\partial}{\partial x_i} \varphi_Q(p) \\ &= \sum_{Q \in \mathrm{W}_\Omega,\, p \in \lambda Q} (f(z_Q) - f(p)) \frac{\partial}{\partial x_i} \varphi_Q(p) \\ &\leq \sum_{Q \in \mathrm{W}_\Omega,\, p \in \lambda Q} L\, d(z_Q,\, p) \frac{\partial}{\partial x_i} \varphi_Q(p) \\ &\lesssim c_2\, L\end{aligned}$$

because $\sum \varphi_Q(p) = 1$, $d(z_Q,\, p) \lesssim diamQ$ and $\varphi_Q(p) \lesssim \dfrac{1}{diamQ}$ and hence, the theorem is completed. $\square$

Although the Whitney Lipschitz extension map is defined explicitly, the map is somewhat complicated and the source metric space is restricted to Euclidean space. We will use McShane's extension map for an arbitrary metric space.

2.4 Preliminaries for the Main Theorem

In this section, we shall show that a uniformly perfect metric space equipped with a doubling measure allows us to have a Christ-Whitney decomposition on $X \setminus Y$. We also see some definitions and lemmas related to this decomposition.

As Euclidean space has a system of dyadic cubes, every doubling metric measure space also has a system of Christ cubes akin to classical dyadic cubes. The following Proposition 2.4.1 may be transparent if we think of Q_α^k as being essentially a cube of diameter roughly δ^k with center z_α^k. When $Q_\beta^{k+1} \subset Q_\alpha^k$, we say that Q_β^{k+1} is a child of Q_α^k and Q_α^k is a parent of Q_β^{k+1}.

Proposition 2.4.1 (Christ [9])**.** *Let (X, d, μ) be a doubling metric measure space. Then, there exists a collection of open subsets $\{Q_\alpha^k \subset X \mid k \in \mathbb{Z} \ , \ \alpha \in I_k\}$ where I_k is some index set depending on k, and constants $\delta \in (0,\, 1)$, $a_0 \in (0,\, 1)$, $\eta > 0$ and $C_1, \ c < \infty$ such that*

(1) $\mu(X \setminus \cup_{\alpha \in I_k} Q_\alpha^k) = 0$, for all $k \in \mathbb{Z}$.

(2) For any α, β, k, and l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \bigcap Q_\alpha^k = \emptyset$.

(3) Each Q_α^k has exactly one parent and at least one child for all $k \in \mathbb{Z}$.

(4) For each (α, k), there exists $z_\alpha^k \in X$ such that $B(z_\alpha^k,\, a_0\delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k,\, C_1\delta^k)$.

Proof. We will sketch the construction of $\{Q_\alpha^k\}$. Let $\delta \in (0,\, 1)$ be a small positive number and for each $k \in \mathbb{Z}$, fix a maximal collection of points $z_\alpha^k \in X$ satisfying

$$d(z_\alpha^k,\, z_\beta^k) \ \geq \ \delta^k \text{ , for all } \alpha \neq \beta. \tag{2.4.1}$$

By maximality, for each k, for each $x \in X$, there exists a α such that $d(x,\, z_\alpha^k) < \delta^k$. We now have a partial ordering $\leq$ on the set of all ordered pairs $(k,\, \alpha)$, which satisfies,

(1) $(k,\, \alpha) \leq (l,\, \beta)$ implies $l \leq k$.

(2) For each $(k,\, \alpha)$ and $l \leq k$ there is a unique β such that $(k,\, \alpha) \leq (l,\, \beta)$.

(3) If $(k,\, \alpha) \leq (k-1,\, \beta)$ then $d(z_\alpha^k,\, z_\beta^{k-1}) < \delta^{k-1}$.

(4) If $d(z_\alpha^k,\, z_\beta^{k-1}) < \frac{1}{2}\delta^{k-1}$ then $(k,\, \alpha) \leq (k-1,\, \beta)$

The above partial ordering is constructed as follows: For each $(k,\, \alpha)$ there exists at least one β for which $d(z_\alpha^k,\, d_\beta^{k-1}) < \delta^{k-1}$ and there exists at most one β for which $d(z_\alpha^k,\, z_\beta^{k-1}) < \frac{1}{2}\,\delta^{k-1}$ by the maximality. We check whether there exists β such that $d(z_\alpha^k,\, z_\beta^{k-1}) < \frac{1}{2}\delta^{k-1}$. If so, we give a partial order $(k,\, \alpha) < (k-1,\, \beta)$ and also $(k,\, \alpha)$ is not related to any other $(k-1,\, r)$. If there is no such β, then select any β for which $d(z_\alpha^k,\, z_\beta^{k-1}) < \delta^{k-1}$ and we give a partial order $(k,\, \alpha) < (k-1,\, \beta)$ and is not related to an other $(k-1,\, r)$.

Let $a_0 \in (0,\, 1)$ be a small constant and we define

$$Q_\alpha^k = \cup_{(l,\, \beta) \leq (k,\, \alpha)} B^o(z_\beta^l,\, a_0\delta^l) \tag{2.4.2}$$

which is countable union of open balls with radius $a_0\delta^l$ and center z_β^l where $(l,\,\beta) \le (k,\,\alpha)$. Clearly, Q_α^k is open and contains $B^o(z_\alpha^k,\, a_0\delta^k)$. For $(l,\,\beta) \le (k,\,\alpha)$, there exists a chain $(k,\,\alpha) = (k,\,r_0) \ge (k+1,\,r_1) \ge (k+2,\,r_2)\cdots \ge (l,\beta)$. Then we can observe the following from triangle inequality

$$\begin{aligned}
d(z_\alpha^k,\, z_\beta^l) &\le d(z_\alpha^k,\, z_{r_1}^{k+1}) + d(z_{r_1}^{k+1},\, z_\beta^l) \le \delta^k + d(z_{r_1}^{k+1},\, z_\beta^l) \\
&\le \delta^k + d(z_{r_1}^{k+1},\, z_{r_2}^{k+2}) + d(z_{r_2}^{k+2},\, z_\beta^l) \le \delta^k + \delta^{k+1} + d(z_{r_2}^{k+2},\, z_\beta^l) \\
&\le \delta^k + \delta^{k+1} + \delta^{k+2} + \cdots \\
&= \frac{1}{1-\delta}\delta^k
\end{aligned}$$

and hence we have (4) with $C_1 = 2a_0 + \dfrac{1}{1-\delta}$.

From Lemma 15 in [9], if $Q_\alpha^k \cap Q_\beta^k \neq \emptyset$, then $\alpha = \beta$. For if $l \ge k$ and $Q_\alpha^k \cap Q_\beta^l \neq \emptyset$, choose γ so that $(l,\,\beta) \le (k,\,\gamma)$, whence $Q_\beta^l \subset Q_\gamma^k$. Then, $Q_\gamma^k \cap Q_\alpha^k \neq \emptyset$, so $\gamma = \alpha$ which complete (2).

For fixed k, we let $E = \cup_{\alpha \in I_k} Q_\alpha^k$. Given any $x \in X$ and any n, there exists z_α^n such that $d(z_\alpha^n,\, x) \le \delta^n$. If $n \ge k$, then $B^o(z_\alpha^n,\, a_0\delta^n) \subset B^o(x,\, (1+a_0)\delta^n)$, which we call B^o. Then, $\mu(B^o(z_\alpha^n,\, a_0\delta^n)) \ge c\,\mu(B^o)$ where $c \in (0,1]$ follows from doubling condition (see (2.1.3)). In other words,

$$\frac{\mu(E \cap B^o)}{\mu(B^o)} \ge c > 0.$$

Letting $n \to \infty$, we have

$$\limsup_{r \to 0} \frac{\mu(E \cap B^o(x,\, r))}{\mu(B^o(x,\, r))} \ge c > 0 \text{ for all } \ x \in X. \tag{2.4.3}$$

By Lebesgue's differentiation theorem, E has full measure in X as desired. This finishes the proof of (1). $\square$

We now introduce a Whitney-type decomposition on an open subset of a uniformly perfect metric space supporting a doubling measure. As open subset of Euclidean space has a Whitney decomposition from a system of dyadic cubes, we have a type of Whitney decomposition from a system of Christ cubes. We call it a Christ-Whitney decomposition. This decomposition has a comparability condition (see (4) Lemma 2.4.1) in addition to all conditions of a Whitney decomposition. This comparability condition together with doubling condition will play an important role in the proof of Lemma 2.4.14, which yields existence of a coloring map in Lemma 2.4.15.

Lemma 2.4.2. *Suppose that (X,d,μ) is a A-uniformly perfect metric space supporting a doubling metric measure, Y is a closed subset of X, and $\Omega = X \setminus Y$. Then Ω has a Christ-Whitney decomposition M_Ω satisfying the following properties:*

(1) $\mu(\Omega \setminus \cup_{Q \in \mathrm{M}_\Omega} Q) = 0$.

(2) $diam(Q) \leq dist(Q,\, Y) \leq \dfrac{4C_1\, A}{\delta}\, diam(Q)$.

(3) $Q \cap Q^{'} = \emptyset$.

(4) For any $Q \in \mathrm{M}_\Omega$*, there exists* $x \in \Omega$ *such that* $B(x,\, a_0\delta^k) \subset Q \subset B(x,\, C_1\delta^k)$ *for some* k*.*

The constants δ*,* a_0 *and* C_1 *are deduced from Proposition 2.4.1.*

Remark 2.4.3. *We say that* Q *is* $(C_1,\, a_0)$*-quasiball if the fourth condition holds. From now on, we will call a ball* $B(x,\, C_1\delta^k)$ *containing* Q *a* C_1*-quasiball of* Q *and denote it by* $\widetilde{B_Q}$*. We observe that* $diam(\widetilde{B_Q})$ *is comparable to* δ^k *by uniform perfectness of* X*.*

Proof. Since $\Omega = X \setminus Y$ is a doubling metric measure space, we have a family of subsets

$$\{Q_\alpha^k \subset \Omega \mid k \in \mathbb{Z}\,,\ \alpha \in I_k\}$$

for fixed constants δ and C_1 so that $\mu(\Omega \setminus \cup_{\alpha \in I_k} Q_\alpha^k) = 0$ from Proposition 2.4.1. We now consider layers, defined by $\Omega_k = \{x \mid c'\delta^k < dist(x,\, Y) \leq c'\delta^{k-1}\}$, where c' is a positive constant we shall fix momentarily. Obviously, $\Omega = \cup_{k=-\infty}^{\infty} \Omega_k$.

We now make an initial choice of Q's, and denote the resulting collection by M_0. Our choice is made as follows. We consider Q's chosen from $\mathcal{A}^k = \{Q_\alpha^k \mid \alpha \in I_k\}$ for each $k \in \mathbb{Z}$, (each such Q is of size approximately δ^k), and include a Q in M_0 if it intersects Ω_k. In other words,

$$\mathrm{M}_0 = \cup_k \{Q \in \mathcal{A}^k \mid Q \cap \Omega_k \neq \emptyset\}.$$

We then have $\mu(\Omega \setminus \cup_{Q \in \mathrm{M}_0} Q) = 0$. For an appropriate choice of c',

$$diam(Q) \leq dist(Q,\, Y) \leq \frac{4C_1\, A}{\delta}\, diam(Q). \tag{2.4.4}$$

Let us prove (2.4.4) first. Suppose $Q \in \mathcal{A}^k$, then $\dfrac{1}{A}\delta^k \leq diam(Q) \leq 2C_1\delta^k$ because of uniform perfectness. Since $Q \in \mathrm{M}_0$, there exists $x \in Q \cap \Omega_k$. Thus, $dist(Q, Y) \leq dist(x, Y) \leq c'\delta^{k-1} \leq \dfrac{c'A}{\delta}\, diam Q \leq \dfrac{4C_1\, A}{\delta}\, diam(Q)$ and $dist(Q,\, Y) \geq dist(x,\, Y) - diam(Q) \geq c'\delta^k - 2C_1\,\delta^k = 2C_1\,\delta^k \geq diam(Q)$. If we choose $c' = 4C_1$, we get the equation (2.4.4).

Notice that the collection M_0 has all required properties, except that Q's in it are not necessarily disjoint. To finish the proof of the lemma we need to refine our choice leading to M_0, eliminating Q's which were really unnecessary. We require the following observation. Suppose $Q \in \mathcal{A}^k$ and $Q' \in \mathcal{A}^{k'}$. If Q and Q' are not disjoint, then one of two must be contained in the other. Start now with any $Q \in \mathrm{M}_0$, and consider the unique maximal parent in M_0 which contains it. We let M_Ω denote the collection of maximal Q's in M_0. The last property comes straightforward from Proposition 2.4.1 and Lemma 2.4.2 is therefore proved. $\square$

We now define new concepts Q^* and Q^{**} corresponding to Q and a dilated Whitney cube λQ respectively in the classical Whitney decomposition.

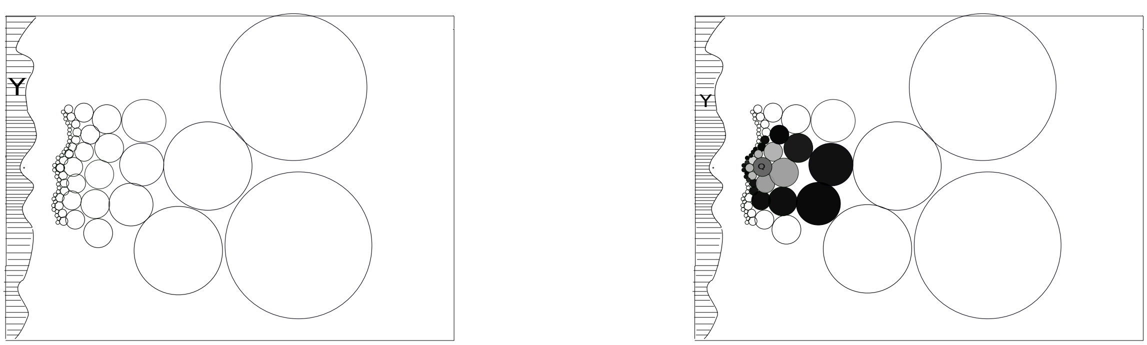
Definition 2.4.4. *For any fixed $Q \in \mathrm{M}_\Omega$, we denote Q^* by a collection of all $R \in \mathrm{M}_\Omega$ whose distance from Q does not exceed minimum diameters of R and Q by a fixed constant ϵ. We denote Q^{**} by a collection of all $S \in \mathrm{M}_\Omega$ whose distance from $R \in Q^*$ does not exceed minimum diameters of R and S by a fixed constant ϵ. Here ϵ is a fixed number such that $0 < \epsilon < 1$. In other words,*

(1) $Q^* = \cup\{R \in \mathrm{M}_\Omega \mid dist(Q, R) < \epsilon \min\{diam(Q), diam(R)\}\ \}$.

(2) $Q^{**} = \cup\{S \in \mathrm{M}_\Omega \mid dist(S, R) < \epsilon \min\{diam(S), diam(R)\}$ *for some* $R \in Q^*\ \}$.

Remark 2.4.5. *Q^* could have no other Christ-cubes except Q. For example, we can consider the Cantor set. In the rest of Chapter 2, we can choose any ϵ. However, in practice, we will restrict ϵ to a universal fixed number in $(0, 1)$ since we will consider condition of uniformly Christ-local bi-Lipschitz embeddings (Definition 2.4.11).*

Remark 2.4.6. *Figure 2.1, Figure 3.1 and Figure 3.2 illustrate an idea how our construction goes. Of course, actual shapes will depend on a metric space.*



(a) Christ-Whitney decomposition M_Ω (b) Definition of Q^* and Q^{**}

Figure 2.1: The gray balls are elements of Q^* and gray and black balls are elements of Q^{**}

We next see some propositions related to Q^* and Q^{**}.

Proposition 2.4.7. *For any fixed $Q \in \mathrm{M}_\Omega$, suppose $R \in Q^*$. Then,*

$$[\frac{4C_1\,A}{\delta}+1+\epsilon]^{-1}\,diam(R) \leq diam(Q) \leq [\frac{4C_1\,A}{\delta}+1+\epsilon]\,diam(R)$$

Proof. We suppose that $diam(R) \geq diam(Q)$. Then, we arrive at

$$\begin{aligned} diam(R) &\leq dist(R,\,Y) \\ &\leq diam(Q) + dist(Q,\,Y) + dist(R,\,Q) \\ &\leq [\frac{4C_1\,A}{\delta}+1+\epsilon]\,diam(Q) \end{aligned}$$

and the symmetrical implication proves the proposition. □

Proposition 2.4.8. *Let $(X,\,d)$ be a uniformly perfect metric space supporting a doubling measure μ. and let M_Ω be a Christ-Whitney decomposition as in Lemma 2.4.2.*

(1) Suppose $Q \in \mathrm{M}_\Omega$. Then there are at most N Christ cubes in M_Ω in Q^{**}.

(2) Any point in M_Ω is contained in at most N of Q^{**}.

The number N is independent of Q. It depends on the doubling constant of μ, ϵ and the data of X.

Proof. For any $R \in Q^{**}$, we have comparability between $diam(Q)$ and $diam(R)$ from Proposition 2.4.7. Therefore, $diam(Q^{**})$ is comparable to $diam(Q)$. Doubling condition yields that there are at most finite number of R's and hence there are at most $N(\mu,\,C_1,\,A,\,\delta,\,\epsilon)$ Christ cubes in Q^{**}.

Let p be a point in M_Ω and write $p \in R$. We now observe that for any $Q \in R^{**}$, we have $R \in Q^{**}$. We have $p \in Q^{**}$ for all $Q \in R^{**}$ and hence p is contained in at most N of Q^{**} from (1) of Proposition 2.4.8. □

We now consider a family of Lipschitz cutoff functions $\{\varphi_Q\}$. We will use these functions to construct a W-local co-Lipschitz map by composing with uniformly Christ-local bi-Lipschitz embeddings (see Lemma 3.3.1).

Lemma 2.4.9. *There exist functions $\varphi_Q : X \to \mathbb{R}$ where $Q \in \mathrm{M}_\Omega$ with the following properties:*

(1) $0 \leq \varphi_Q \leq 1$,

(2) $\varphi_Q|_{Q^*} = 1$,

(3) $\varphi_Q|_{X\setminus Q^{**}} = 0$,

(4) φ_Q is Lipschitz with constant $\dfrac{C}{diam(Q)}$,

(5) For all $p \in \Omega$, we have $\varphi_Q(p) \neq 0$ for at most N cubes $Q \in \mathrm{M}_\Omega$.

Here, C and N denote uniformly fixed constants independent of the choice of element $Q \in \mathrm{M}_\Omega$. They depend on the data of X, ϵ, and the doubling constant of μ quantatively.

Proof. We define

$$\varphi_Q(x) = \min\{1, \ \frac{dist(x, X \setminus Q^{**})}{dist(Q^*, \, X \setminus Q^{**})} \}.$$

Then, (1), (2) and (3) are obvious and (5) follows from Proposition 2.4.8. To check (4), consider

$$|\varphi_Q(p) - \varphi_Q(q)| \leq \frac{d(p, \, q)}{dist(Q^*, \, X \setminus Q^{**})}.$$

Thus, it suffices to show that

$$dist(Q^*, \, X \setminus Q^{**}) \geq c \, diam(Q)$$

To this end, let x be a point in Q^*. We write $x \in R$ for some $R \in Q^*$ and choose $y \in S \in X \setminus Q^{**}$. Then,

$$\begin{aligned} d(x, \, y) &\geq dist(R, \, S) \\ &\geq \epsilon \min\{diam(R), \, diam(S)\} \\ &\geq C(L_1, \, A, \, \delta, \, \epsilon) \, diam(Q). \end{aligned}$$

The last inequality is deduced from the comparability between $diam(R)$ and $diam(Q)$ in case $diam(S) \geq diam(R)$. Otherwise, $diam(R) \geq diam(S)$, we divide into two cases, either

$$(1) \; diam(R) \geq diam(S) \geq \frac{1}{2[\frac{4C_1 \, A}{\delta} + 1]} \, diam(R) \quad \text{or} \quad (2) \; diam(S) < \frac{1}{2[\frac{4C_1 \, A}{\delta} + 1]} \, diam(R).$$

In the first case, we have obviously comparability between $diam(S)$ and $diam(R)$. In the second case, we use the comparability condition of a Christ-Whitney decomposition. Then,

$$\begin{aligned} dist(R, \, S) &\geq dist(R, \, Y) - dist(S, Y) - diam(S) \\ &\geq diam(R) - [\frac{4C_1 \, A}{\delta} + 1] \, diam(S) \\ &\geq \frac{1}{2} \, diam(R) \\ &\geq C(L_1, \, A, \, \delta) \, diam(Q). \end{aligned}$$

Therefore, the proof of (4) is completed. $\square$

Remark 2.4.10. *We use the fact that* $\varphi_Q = 1$ *on* Q^* *and* $\varphi_Q = 0$ *off* Q^{**} *so that* $\widetilde{h}_Q = h_Q \cdot \varphi_Q$ *in Subsection* 3.3 *is bi-Lipschitz on* Q^* *and supported on* Q^{**}*. This is needed in the proof of Lemma 3.3.1, see case* (3)*.*

Definition 2.4.11. *Let* (X, d, μ) *be a uniformly perfect metric space supporting a doubling measure and let* Y *be a closed subset of* X*. We say that* $\Omega = X \setminus Y$ *admits uniformly Christ-local bi-Lipschitz embeddings if there exist bi-Lipschitz embeddings of each* Q^{**} *into a fixed Euclidean space with uniform bi-Lipschitz constant.*

The following relative distance map plays a key role to construct a co-Lipschitz map from a metric space into Euclidean space in Chapter 3. We will break M_Ω into two parts and construct co-Lipschitz maps on these parts (Definition 3.1.3) by using the Whitney distance map.

Definition 2.4.12. *The Whitney distance map* d_{W} *on* $\mathrm{M}_\Omega \times \mathrm{M}_\Omega$ *is defined by*

$$d_{\mathrm{W}}(Q, R) = \frac{dist(Q,\, R)}{\min(diam(Q), diam(R))}.$$

Remark 2.4.13. *The Whitney distance map* d_{W} *is not a metric. In fact, if* $\overline{Q} \cap \overline{R} \neq \emptyset$*, then* $d_{\mathrm{W}}(Q,\, R) = 0$*. We observe that* $d_{\mathrm{W}}(Q,\, R) \leq d_{\mathrm{W}}(Q,\, S) + d_{\mathrm{W}}(S,\, R) + 1$ *if* $diam(S) \leq \min\{diam(Q),\, diam(R)\}$*. Throughout this thesis, we will use the terminology Whitney distance ball of radius* ρ *for the set of all elements in* M_Ω *such that Whitney distance to a fixed center cube in* M_Ω *is less than* ρ*. We write* $B_{\mathrm{W}}(Q,\, \rho)$ *for the Whitney distance ball of radius* ρ *with center* Q*.*

The next lemma allows us to construct a coloring map that gives different colors to Christ cubes within a given Whitney distance ball. Indeed, this coloring map permits additional dimension of receiving Euclidean space.

Lemma 2.4.14. *Each Whitney distance ball of radius* ρ *contains a finite number of elements of the Christ-Whitney decomposition* M_Ω*. The number depends on the doubling constant of* μ *and* ρ*.*

Proof. We fix a Christ cube $Q \in \mathrm{M}_\Omega$ and we require to count the number of $R \in \mathrm{M}_\Omega$ such that $d_{\mathrm{W}}(Q, R) < \rho$. We have two cases either (1) $diam(Q) < diam(R)$ or (2) $diam(R) \leq diam(Q)$.

Suppose $diam(Q) < diam(R)$. Then, we have

$$dist(R,\, Y) - dist(Q,\, Y) < dist(Q,\, R) + diam(Q) < (\rho + 1)\, diam(Q).$$

Since $dist(Q,\, Y) \leq \dfrac{4C_1\, A}{\delta}\, diam(Q)$, we have an upper bound for $diam(R)$ in terms of $diam(Q)$. That is, $diam(R) < (\rho + 1 + \dfrac{4C_1\, A}{\delta})\, diam(Q)$.

Similarly, $diam(R)$ has a lower bound in terms of the size of Q in the case of $diam(R) \leq diam(Q)$:

$$diam(R) \geq (\rho + 1 + \frac{4C_1\, A}{\delta})^{-1}\, diam(Q).$$

Therefore, the number of $R \in \mathrm{M}_\Omega$ in $B_{\mathrm{W}}(Q,\, \rho)$ is the sum of the cardinality of the following sets:

$$\{R \in \mathrm{M}_\Omega \mid diam(Q) < diam(R) < (\rho + 1 + \frac{4C_1\, A}{\delta})\, diam(Q) \text{ and } dist(Q, R) < \rho\, diam(Q)\} \tag{2.4.5}$$

and

$$\{R \in \mathrm{M}_\Omega \mid (\rho + 1 + \frac{4C_1\, A}{\delta})^{-1}\, diam(Q) < diam(R) \leq diam(Q) \text{ and } dist(Q, R) < \rho\, diam(R)\} \tag{2.4.6}$$

Now we suppose that p and q are centers of C_1- quasiballs $\widetilde{B_Q}$ and $\widetilde{B_R}$ which have approximately sizes of Q and R. If R is in either the set (2.4.5) or the set (2.4.6), then we find that

$$d(p,\, q) \leq diam(Q) + dist(Q,\, R) + diam(R) < (2\rho + 1 + \frac{4C_1\, A}{\delta})\, diam(Q). \tag{2.4.7}$$

Thus, the number of $R \in \mathrm{M}_\Omega$ in $B_{\mathrm{W}}(Q,\, \rho)$ is at most twice of the number of centers q satisfying (2.4.7). In other words, we can count the number of R's in (2.4.5) and (2.4.6) by counting the number of centers of C_1-quasiballs $\widetilde{B_R}$. By the doubling condition, the ball centered at p with radius $(2\rho + 1 + \frac{4C_1\, A}{\delta})\, diam(Q)$ can be covered by finite number of C_1-quasiballs centered at such q. Finally, the comparability of the size of R and that of the ball centered at q concludes Lemma 2.4.14. □

We write the number of Christ cubes within Whitney distance ball of radius ρ as $m = m(\rho,\, D)$ in terms of ρ and the doubling constant D of μ.

Lemma 2.4.15. *There exists a coloring map*

$$K : \mathrm{M}_\Omega \longrightarrow \{1, 2, 3, \ldots, M\} \text{ for some } M \geq m(m-1)$$

such that any two boxes within Whitney distance ball of radius ρ have different colors. In other words, if R', R'' have $d_{\mathrm{W}}(R', R'') < \rho$, then $K(R') \neq K(R'')$.

Proof. We apply Zorn's lemma. Let us consider the partially ordered set $(\mathcal{P}, \leqslant)$ where $\mathcal{P}$ is the collection of maps k defined from $\mathcal{S} \subset \mathrm{M}_\Omega$ to $\{1, 2, \ldots, M\}$ so that $K(R) \neq K(R')$ for all $R,\, R' \in \mathcal{S}$ whose Whitney distance $< \rho$. The inequality $(k, \mathcal{S}) \leqslant (k', \mathcal{S}')$ means k' is a extension of k ($\mathcal{S} \subset \mathcal{S}' \in \mathcal{P}$ and $k'|_{\mathcal{S}} = k|_{\mathcal{S}}$).

By Zorn's lemma, there exists a maximal element $\widehat{k}$. If the domain of $\widehat{k}$ is M_Ω, then we can set $K = \widehat{k}$. Otherwise, take $Q' \in \mathrm{M}_\Omega \setminus \mathrm{domain}(\widehat{k})$. We now want to give a color to Q'. The color of Q' should differ from any color already assigned to any R where $d_{\mathrm{W}}(Q', R) < \rho$ and also differ from any color already assigned to any S where $d_{\mathrm{W}}(S, R) < \rho$ and $d_{\mathrm{W}}(Q', R) < \rho$. We observe that the number of such R is at most $m - 1$ and the number of S for given R is at most m. Thus, the total number of colors seen is at most $m(m-1)$. Since $M \geq m(m-1)$, it contradicts maximality of $\widehat{k}$. $\square$

Chapter 3

Main Theorem

Now we are ready to state the main theorem. It asserts that in a uniformly perfect complete metric space supporting a doubling measure, the local information of uniformly Christ-local bi-Lipschitz embeddability (Definition 2.4.11) can be turned into global information of bi-Lipschitz embeddability.

3.1 Main Theorem and Outline of Proof

Theorem 3.1.1. *A uniformly perfect complete metric space (X, d) admits a bi-Lipschitz embedding into some Euclidean space if and only if the following conditions hold:*

(1) it supports a doubling measure μ,

(2) there exists a closed subset Y of X which admits a bi-Lipschitz embedding into some $\mathbb{R}^{M_1}$,

(3) $\Omega = X \setminus Y$ admits uniformly Christ-local bi-Lipschitz embeddings into some $\mathbb{R}^{M_2}$.

The bi-Lipschitz constant and dimension of receiving Euclidean space depend on the data of the metric space X, the doubling constant of μ, M_1, M_2, and the bi-Lipschitz constants in conditions (2) and (3).

Outline of Proof

Suppose that we have a L-bi-Lipschitz embedding f from (X, d) into $\mathbb{R}^n$ for some n. Euclidean space is a doubling metric space and the doubling condition is bi-Lipschitz invariant. Hence, (X, d) is a complete doubling metric space. Thus, there exists a doubling measure μ from Theorem 2.1.10. The second condition is trivial, setting $Y = X$. The third condition is trivial since $\Omega = \emptyset$.

The content of the theorem is the other implication: a uniformly perfect complete space satisfying (1), (2), and (3) embeds bi-Lipschitzly in some $\mathbb{R}^n$ for some n. We will use Proposition 3.1.2 to complete the main theorem. Since the full measure set $\mathrm{M}_\Omega \cup Y$ is dense in X and the constructed map in Proposition 3.1.2 is uniformly continuous, the main theorem follows immediately. Therefore, we will focus on proving Proposition 3.1.2 in section 3.2, Section 3.3, and Section 3.4.

Proposition 3.1.2. *Let* (X, d, μ) *be a* A*-uniformly perfect, complete, doubling metric space and let* Y *be a closed subset of* X*. Then, the full measure set* $\mathrm{M}_\Omega \cup Y$ *admits a bi-Lipschitz embedding into some Euclidean space if the followings are satisfied:*

(1) Y *admits a bi-Lipschitz embedding into some* $\mathbb{R}^{M_1}$*,*

(2) $\Omega = X \setminus Y$ *admits uniformly Christ-local bi-Lipschitz embeddings into some* $\mathbb{R}^{M_2}$*.*

The bi-Lipschitz constant and dimension of receiving Euclidean space depend on the data of metric space X*, the doubling constant,* M_1*,* M_2*, and the bi-Lipschitz constants in conditions* (1) *and* (2)*.*

We briefly outline the proof of Proposition 3.1.2. We first extend a (bi)-Lipschitz map f on Y to a global Lipschitz map g on X, using McShane's extension theorem (see Theorem 2.3.2 and Corollary 2.3.3). We then suppose that f is a L_1-bi-Lipschitz embedding from Y into $\mathbb{R}^{M_1}$. From McShane's theorem, we have a $\sqrt{M_1}L_1$-Lipschitz extension map

$$g : X \longrightarrow \mathbb{R}^{M_1} \text{ such that } g|_Y = f.$$

From now on we fix such L_1 and M_1 is chosen sufficiently large relative to other data C_1, A, and δ. The precise choice of M_1 will be made so that Equation (3.2.1) make sense.

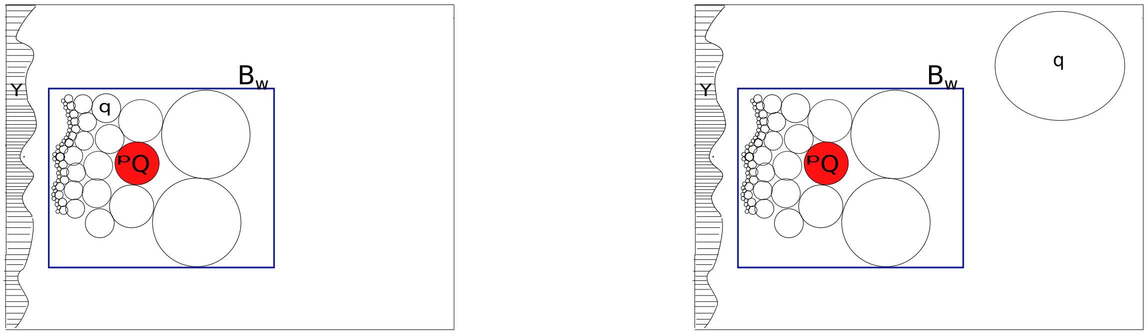
In general, the map g is not globally co-Lipschitz on a full measure set M_Ω of Ω. Therefore, we next shall construct a co-Lipschitz map using a local and large scale argument in the sense of Whitney distance on a Christ-Whitney decomposition (see Definition 2.4.12 and Definition 3.1.3).

Definition 3.1.3. *Let* Q *be any fixed cube in* M_Ω*. We say* $f : \mathrm{M}_\Omega \to \mathbb{R}^n$ *is* W*-local co-Lipschitz if it is co-Lipschitz for any two points* $p \in Q$*,* $q \in R$ *where* R *is in* $B_{\mathrm{W}}(Q, 16M_1{L_1}^2)$*. We say* f *is* W*-large scale co-Lipschitz if it is co-Lipschitz for any two points* $p \in Q$ *and* $q \in R$ *where* R *is not in* $B_{\mathrm{W}}(Q, 16M_1{L_1}^2)$*.*

In Section 3.2, we will construct a W-large scale co-Lipschitz map and global Lipschitz map on M_Ω. To this end, we will break the complement of arbitrary Whitney distance ball of radius $16M_1{L_1}^2$ into two parts using relative distance in terms of distance between two cubes and maximum diameter of them. We shall see that McShane's extension map g and distance map from Y, $dist(\cdot, Y)$, which are global Lipschitz maps, are W-large scale co-Lipschitz on these two parts respectively.

In Section 3.3, we will construct a W-local co-Lipschitz map on M_Ω via putting together all local patches of bi-Lipschitz embeddings. We will assign different colors to elements in a Christ-Whitney decomposition within arbitrary Whitney distance ball of radius $16M_1{L_1}^2$.

Finally, in Section 3.4, we will construct a global bi-Lipschitz embedding on the full measure set $\mathrm{M}_\Omega \cup Y$ of X completing the proof of Lemma 3.1.2.



(a) W-local co-Lipschitz (b) W-large scale co-Lipschitz

Figure 3.1: Let the square be the Whitney distance ball of radius $16M_1{L_1}^2$ centered at Q. W-local co-Lipschitz means $|f(p) - f(q)| \gtrsim d(p,\, q)$ for any $p \in Q$ and $q \in R$ where $d_{\mathrm{W}}(Q,\, R) < 16M_1{L_1}^2$. W-large scale co-Lipschitz means $|f(p) - f(q)| \gtrsim d(p,\, q)$ for $p \in Q$ and $q \in R$ with $d_{\mathrm{W}}(Q,\, R) \geq 16M_1{L_1}^2$.

3.2 W-Large Scale Co-Lipschitz and Global Lipschitz Map on M_Ω

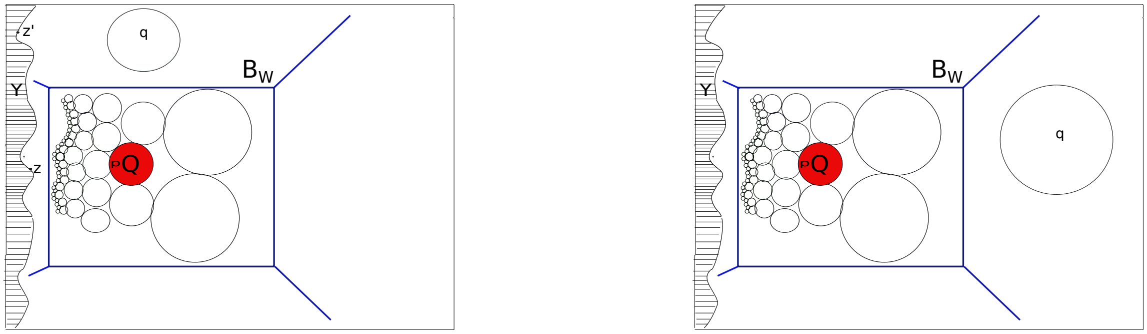
We construct a W-large scale co-Lipschitz and global Lipschitz map on a full measure set $\mathrm{M}_\Omega \subset \Omega$. Roughly speaking, McShane's extension map guarantees a W-large scale co-Lipschitz bound for points p, q in M_Ω whose distance is big enough with respect to maximum diameter of cubes containing them. Whenever $p \in Q$ and $q \in R$ with $dist(Q,\, R)$ exceeds maximum diameter of them by a fixed constant, we consider points z, z' in Y which give distances to p, q respectively. Then, $|g(p) - g(z)|$ and $|g(q) - g(z')|$ are approximately greater than maximum diameter and we can conclude co-Lipschitz from the triangle inequality. Furthermore, when the distance between two points is small enough with respect to maximum diameter of cubes containing them, $|d(p,\, Y) - d(q,\, Y)|$ is approximately greater than maximum diameter (see Figure 3.2).

Lemma 3.2.1. *Let Q be any fixed cube in M_Ω. For any two points $p \in Q$ and $q \in R$, where $d_{\mathrm{W}}(Q,\, R) \geq 16M_1{L_1}^2$, the McShane extension map g and $dist(\cdot,\, Y)$ guarantee W-large scale co-Lipschitz bounds. More precisely,*

(1) If $\dfrac{dist(Q,\, R)}{\max(diam(Q), diam(R))} \geq \dfrac{8M_1{L_1}^2}{1 + \frac{4\,C_1\,A}{\delta}}$, *then* $|g(p) - g(q)| \geq C(L_1,\, M_1)\, d(p,\, q)$.

(2) If $\dfrac{dist(Q,\, R)}{\max(diam(Q), diam(R))} \leq \dfrac{8M_1{L_1}^2}{1 + \frac{4\,C_1\,A}{\delta}}$, *then* $|dist(p,\, Y) - dist(q,\, Y)| \geq C(L_1,\, M_1)\, d(p,\, q)$.

Proof. We may assume that $diam(R) \geq diam(Q)$ without loss of generality. We choose z, $z' \in Y$ such that $dist(Y,\, Q) = dist(z,\, Q)$ and $dist(Y,\, R) = dist(z',\, R)$. We claim that $z \neq z'$. In fact, $d(z,\, z') \geq \frac{1}{2}\, d(p,\, q)$.



(a) $\frac{dist(Q,\, R)}{\max(diam(Q), diam(R))} \geq \frac{8M_1{L_1}^2}{1+\frac{4C_1A}{\delta}}$ (b) $\frac{dist(Q,\, R)}{\max(diam(Q), diam(R))} \leq \frac{8M_1{L_1}^2}{1+\frac{4C_1A}{\delta}}$

Figure 3.2: g and $dist(\cdot,\, Y)$ guarantee W-large scale co-Lipschitz bounds.

To conclude the claim, we suppose that $d(z,\, z') < \frac{1}{2}\, d(p,\, q)$. Then,

$$d(p,\, q) \leq d(p,\, z) + d(z,\, z') + d(z',\, q).$$

Thus, we have

$$\begin{aligned}
d(p,\, q) &\leq 2\,[d(p,\, z) + d(z',\, q)] \\
&\leq 2\,[dist(z,\, Q) + diam(Q) + dist(z',\, R) + diam(R)] \\
&\leq 2\,[dist(Y,\, Q) + diam(Q) + dist(Y,\, R) + diam(R)] \\
&\leq 2\,(\frac{4C_1A}{\delta} + 1)[diam(Q) + diam(R)] \\
&\leq 2\,(\frac{4C_1A}{\delta} + 1)(\frac{1+\frac{4C_1A}{\delta}}{8M_1{L_1}^2} + \frac{1}{16M_1{L_1}^2})\, dist(Q,\, R) \\
&\leq \frac{(1+\frac{4C_1A}{\delta})(3+\frac{4C_1A}{\delta})}{8M_1{L_1}^2}\, d(p,\, q).
\end{aligned}$$

This is a contradiction since provided M_1 is selected sufficiently large relative to C_1, A, and δ. Now,

$$\begin{aligned}
|g(p) - g(q)| &\geq |f(z) - f(z')| - |f(z) - g(p)| - |f(z') - g(q)| \\
&\geq \frac{1}{L_1}\, d(z,\, z') - C\, d(z,\, p) - C\, d(z',\, q).
\end{aligned}$$

where $C = \sqrt{M_1} L_1$ from McShane's theorem. We have

$$\begin{aligned} d(p, z) &\leq (\frac{4C_1\,A}{\delta} + 1)\, diam(Q) \leq \frac{(\frac{4C_1\,A}{\delta} + 1)}{16 M_1 {L_1}^2}\, dist(Q,\, R) \\ &\leq \frac{(\frac{4C_1\,A}{\delta} + 1)}{16 M_1 {L_1}^2}\, d(p,\, q). \end{aligned}$$

Similarly, we have

$$|g(q) - g(z')| \leq L_1\, d(z',\, q) \leq \frac{(\frac{4C_1\,A}{\delta} + 1)^2}{8 M_1 {L_1}^2}\, d(p,\, q).$$

In conclusion,

$$|g(p) - g(q)| \geq [\frac{1}{2L_1} - 2C \frac{(\frac{4C_1\,A}{\delta} + 1)^2}{8 M_1 {L_1}^2}]\, d(p,\, q) \tag{3.2.1}$$

$$\geq \frac{1}{2L_1}[1 - \frac{(\frac{4C_1\,A}{\delta} + 1)^2}{2\sqrt{M_1}}]\, d(p,\, q) \tag{3.2.2}$$

$$\geq \frac{1}{4L_1}\, d(p,\, q) \tag{3.2.3}$$

since we can choose M_1 sufficiently large. This complete the proof of the first case.

In second case, we have

$$16 M_1 {L_1}^2 diam(Q) \leq dist(Q,\, R) \leq \frac{8 M_1 {L_1}^2}{1 + \frac{4\,C_1\,A}{\delta}}\, diam(R).$$

Therefore, $2\,(1 + \dfrac{4\,C_1\,A}{\delta})\, diam(Q) \leq diam(R)$. We now have

$$\begin{aligned} |dist(p\,Y) - dist(q\,Y)| &\geq dist(q,\, Y) - dist(p,\, Y) \\ &\geq dist(R,\, Y) - dist(Q,\, Y) - diam(Q) \\ &\geq diam(R) - (1 + \frac{4\,C_1\,A}{\delta})\, diam(Q) \\ &\geq \frac{1}{2}\, diam(R) \end{aligned}$$

while $d(p,\, q) \leq diam(Q) + dist(Q,\, R) + diam(R) \lesssim diam(R)$. Thus, we proved the second case. □

3.3 W-Local Co-Lipschitz and Global Lipschitz Map on M_Ω

We next construct a W-local co-Lipschitz and global Lipschitz map on a full measure set $\mathrm{M}_\Omega \subset \Omega$ into some Euclidean space. In general, $M_1 + 1$, the dimension of the target space of $g(\cdot) \times dist(\cdot,\, Y)$ is not large enough to construct a co-Lipschitz map. Hence, we will use a coloring map that gives additional dimension of the Euclidean space (see Lemma 2.4.15).

Suppose that h_Q's are L_2-bi-Lipschitz embeddings of Q^{**} for each $Q \in \mathrm{M}_\Omega$ into $\mathbb{R}^{M_2}$ with uniformly determined L_2 and M_2. Now we consider everywhere defined map

$$\widetilde{h}_Q = h_Q \cdot \varphi_Q \ ;$$

it is bi-Lipschitz on Q^*, Lipschitz on M_Ω, and supported on Q^{**}. We recall that $\{\varphi_Q\}$ is a family of Lipschitz cutoff functions as in Lemma 2.4.9. Then, we may assume that for some c

$$\widetilde{h}_Q(Q^*) \subset B(0, c\, L_2 diam(Q)) \setminus B(0, \frac{1}{c\, L_2} diam(Q))$$

because we can postcompose with an isometric translation map of $\mathbb{R}^{M_2}$ if necessary. Next, we will put together all patches to make a W-local co-Lipschitz map by assigning different colors to each element in M_Ω. We will denote $\{e_1,\ e_2,\ \ldots,\ e_M\}$ by an orthonormal basis for $\mathbb{R}^M$.

Lemma 3.3.1. *The following map H from M_Ω into $(\mathbb{R}^{M_2})^M$ given by*

$$H(p) = \sum_{Q \in \mathrm{M}_\Omega} \widetilde{h}_Q(p) \otimes e_{K(Q)}, \tag{3.3.1}$$

is a global Lipschitz and W-local co-Lipschitz map. The (W-local) bi-Lipschitz constant depends on $L_1,\ L_2$ and M_1. That is,

$$|H(p) - H(q)| \geq C(L_1,\ L_2,\ M_1)\, d(p, q)$$

for any points p in any fixed Q and q in R where $d_{\mathrm{W}}(Q, R) < 16 M_1 {L_1}^2$.

Proof. Since $\widetilde{h}_Q$ is bi-Lipschitz on Q^* with the uniform bi-Lipschitz constant L_2, Lipschitz on M_Ω, and supported on Q^{**}, the map H is a N finite sum of Lipschitz maps on M_Ω from Proposition 2.4.8. Thus, it is Lipschitz on Ω. Now, we will show that H is a W-local co-Lipschitz map according to positions of two points p and q on M_Ω. There are three cases.

(1) If $p,\ q \in Q^*$, then $\widetilde{h}_Q$ is bi-Lipschitz on Q^* and Q is the element in M_Ω that shares the same color

at p and q. Therefore, we find that

$$\begin{aligned}|H(p)-H(q)| &\geq |\widetilde{h}_Q(p)-\widetilde{h}_Q(q)| \\ &= |h_Q(p)-h_Q(q)| \quad \text{since } \varphi_Q|_{Q^*}=1 \\ &\geq \frac{1}{L_2}\, d(p,\, q)\end{aligned}$$

since h_Q is L_2-bi-Lipschitz.

(2) If $p \in Q,\ q \notin Q^{**}$, then $\widetilde{h}_Q(q)=0$. Thus, we have

$$\begin{aligned}|H(p)-H(q)| \geq |\widetilde{h}_Q(p)-\widetilde{h}_Q(q)| &= |\widetilde{h}_Q(p)| \\ &\geq \frac{1}{c\, L_2}\, diam(Q).\end{aligned}$$

On the other hand, we observe that

$$\begin{aligned}d(p,\, q) &\leq diam(Q)+dist(Q,\, R)+diam(R) \\ &\leq diam(Q)+dist(Q,\, R)+dist(R,\, Y) \\ &\leq 2diam(Q)+2dist(Q,\, R)+dist(Q,\, Y).\end{aligned}$$

Since $dist(Q,\, Y) \leq \dfrac{4\, C_1\, A}{\delta}\, diam(Q)$ and $dist(Q,\, R) \leq 16 M_1 {L_1}^2 \min\{diam(Q),\, diam(R)\} \leq 16 M_1 {L_1}^2 diam(Q)$, we conclude

$$d(p,\, q) \lesssim diam(Q)$$

and so $|H(p)-H(q)| \gtrsim d(p,\, q)$ as desired.

(3) If $p \in Q,\ q \in Q^{**}$, then there is a $R \in Q^*$ so that $p,\ q \in R^*$ and $\widetilde{h}_R$ is bi-Lipschitz on R^*. Therefore, we conclude the following from the first case:

$$|H(p)-H(q)| \geq |\widetilde{h}_R(p)-\widetilde{h}_R(q)| \geq \frac{1}{L_2}\, d(p,\, q).$$

$\square$

3.4 Global Bi-Lipschitz Embedding on a Full Measure Set $\mathrm{M}_\Omega \cup Y$

Finally, we are ready to construct a global bi-Lipschitz embedding on a full measure set of X. We define the map F from $\mathrm{M}_\Omega \cup Y$ into $\mathbb{R}^{M_1} \times (R^{M_2})^M \times \mathbb{R}$ as following:

$$F(p) = \begin{cases} g(p) \times H(p) \times dist(p, Y), & for\; p \in \mathrm{M}_\Omega\,; \\ f(p) \times \{0\} \times \{0\}, & \text{for } p \in Y. \end{cases} \tag{3.4.1}$$

Then F is Lipschitz on a full measure set $\mathrm{M}_\Omega \subset \Omega$ because g and $dist(\cdot, Y)$ are Lipschitz on X and H is a finite sum of Lipschitz maps on M_Ω. Moreover, when we define $H(q) = 0$ for $q \in Y$, then for every $p \in \mathrm{M}_\Omega$ and any $q \in Y$, we arrive at

$$\begin{aligned} |H(p) - H(q)| = |H(p)| =& |\sum_{Q \in \mathrm{M}_\Omega} \widetilde{h}_Q(p) \otimes e_{K(Q)}| \\ &\leq N\, L_2\, diam(Q) \\ &\leq N\, L_2\, dist(Q, Y) \\ &\leq N\, L_2\, d(p, q) \end{aligned}$$

We have shown that F is co-Lipschitz on M_Ω by Lemma 3.2.1 and Lemma 3.3.1 and $F|_Y = f$ is co-Lipschitz. Finally, we have a bi-Lipschitz embedding F from a full measure set $\mathrm{M}_\Omega \cup Y$ of X into $\mathbb{R}^{M_1} \times (R^{M_2})^M \times \mathbb{R}$. The bi-Lipschitz constant depends on the data of metric space X, the doubling constant of μ, M_1, M_2, L_1 and L_2. Therefore, Proposition 3.1.2 is proved.

Chapter 4

Applications

In this chapter, we shall state theorems of Pansu and Cheeger which can be applied to get bi-Lipschitz nonembeddability of certain regular sub-Riemannian manifolds. In contrast, as a corollary of Theorem 3.1.1 we will prove bi-Lipschitz embeddability of spaces of Grushin type equipped with Carnot-Carathéodory distance. These are the first examples of sub-Riemannian manifolds which admit a bi-Lipschitz embedding.

4.1 Sub-Riemannian Manifolds

Definition 4.1.1. *[26] A sub-Riemannian manifold is a triple (M, H, g), where M is a connected manifold, H, called the horizontal distribution, is a subbundle of tangent bundle TM and g is a metric on the horizontal distribution. A horizontal curve is a continuous, almost everywhere differentiable curve, which is tangent to the horizontal distribution. The length of a horizontal curve $\gamma : [0,1] \to M$ is defined via the Riemannian metric on H. That is,*

$$l(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t),\, \gamma'(t))}\, dt$$

The Carnot-Carathéodory distance between two points p and q is induced by length of horizontal curves:

$$d_{cc}(p,\, q) = \inf\{l(\gamma) \mid \gamma\ :\ [0,1]\ \to M \ \text{ such that } \gamma(0) = p,\ \gamma(1) = q\}$$

We abbreviate the Carnot-Carathéodory distance by *cc*-distance. The distance is infinite if there is no such curve joining p to q.

Definition 4.1.2 (Hörmander condition)**.** *Let M be a connected manifold and let $H \subset TM$ be a distribution. The distribution H satisfies the Hörmander condition if, for each $x \in M$, the following holds. There is a local basis $X_1,\ X_2,\ \cdots, X_m$ of sections of H such that iterated brackets $[X_i,\, X_j]$, $[[X_i,\, X_j],\, X_k]$, etc. span the tangent space T_xM.*

We sometimes say that a collection of vector fields defining H satisfies the Hörmander condition.

Theorem 4.1.3 (Chow, Rashevsky [26]). *Suppose M is connected manifold and a distribution $H \subset TM$ satisfies the Hörmander condition. Then, any two points of M can be joined by a finite length of horizontal curve.*

To avoid complications, we will from now on restrict attention to the case of vector fields on Euclidean domains. This is the context of our main result in this chapter.

Let $\Omega \subset \mathbb{R}^N$ be a domain and let $X_1, X_2, \cdots, X_m$ be a system of vector fields verifying the Hörmander condition. Let $L^1 = L^1(X_1, X_2, \cdots, X_m)$ be the set of linear combinations with smooth coefficients of the vector fields $X_1, X_2, \cdots, X_m$. We define recursively

$$L^s = L^{s-1} + [L^1, L^{s-1}],$$

so that L^s is generated by the vector fields

$$X_I = [X_{i_1}, [X_{i_2}, \cdots, [X_{i_{l-1}}, X_{i_l}], \cdots]]$$

with $1 \leq l \leq s$. We denote $L^s(x)$ by the subspace of $T_x\Omega$ spanned by values at x of the brackets of length $\leq s$ of vector fields. Theorem 4.1.3 states that for each $x \in \Omega$, there is a smallest integer $r = r(x)$ such that $L^r(x) = T_x\Omega$. For each $x \in \Omega$, there is an increasing sequence of vector subspaces,

$$0 = L^0(x) \subset L^1(x) \subset \cdots \subset L^s(x) \subset \cdots \subset L^{r(x)}(x) = T_x\Omega.$$

Definition 4.1.4. *The step or depth of the distribution at x is the first integer r such that $L^{r(x)}(x) = T_x\Omega$. We say that x is a regular point if the integer $n_s(y) = dim L^s(y)$ $(s = 1, 2, \cdots)$ is constant for y in some neighborhood of x. Otherwise we say that x is a singular point.*

Carnot groups are particular examples of regular sub-Riemannian manifolds. They provide infinitesimal models for sub-Riemannian manifolds and all the fundamental results of sub-Riemannian geometry are easy to prove and understand in the case of Carnot groups.

Definition 4.1.5. *A Carnot (or stratified nilpotent) group is a simply connected group N with a distinguished vector space V_1 such that Lie algebra of the group has the direct sum decomposition:*

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_m, \tag{4.1.1}$$

where $V = V_1$ and $[V_i, V_j] = V_{i+j}$, and $V_s = 0$ for $s > r$. The number m is the step of the group. The number

$Q = \sum_{i=1}^{m} i\, dimV_i$ *is called the homogenous dimension of the group. The Carnot groups admit dilations* δ_λ *which are mappings such that*

$$d_{cc}\,(\delta_\lambda x,\, \delta_\lambda y) = \lambda\, d_{cc}(x,\, y) \;\; \textit{for all} \;\; x,\, y \in N. \tag{4.1.2}$$

Since the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra. For more information of Carnot-Carathéodory geometry, see [12]. Here we give examples of sub-Riemannian manifolds.

Example 4.1.6. $\mathbb{R}^n$ *with addition is the only commutative Carnot group.*

Example 4.1.7. *The Heisenberg group* $\mathbb{H}$ *is* $\mathbb{R}^3$ *with horizontal distribution spanned by two vectors*

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \qquad \textit{and} \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

It is the first non trivial example of step 2 Carnot group and it has dilations $\delta_\lambda(x,\, y,\, z) = (\lambda x,\, \lambda y,\, \lambda^2 z)$.

Example 4.1.8. *The Grushin plane* $\mathbb{G}$ *is* $\mathbb{R}^2$ *with horizontal distribution spanned by*

$$X_1 = \frac{\partial}{\partial x} \quad \textit{and} \quad X_2 = x\,\frac{\partial}{\partial y}.$$

The points on the line $x = 0$ *are singular, while the other points in the plane are regular. It has dilations* $\delta_\lambda(x,\, y) = (\lambda x,\, \lambda^2 y)$.

Nagel, Stein and Wainger studied the geometry of Carnot-Carathéodory spaces and showed that space with a system of vector fields satisfying the Hörmander condition is locally doubling measure space with respect to Lebesgue measure in the following sense. It also satisfies locally Poincaré inequality due to Jerison.

Theorem 4.1.9 (Nagel-Stein-Wainger [27])**.** *Let* $X_1,\ X_2,\ \cdots,\ X_m$ *be a system of vector fields satisfying the Hörmander condition and let* d_{cc} *be the associated Carnot-Carathéodory metric. Then for every compact* $K \subset \Omega$, *there are* $r_0 > 0$ *and* $C \geq 1$ *such that*

$$\mu(B(x,\, 2r)) \leq C\mu(B(x,\, r))$$

for Lebesgue measure μ *whenever* $x \in K$ *and* $r \leq r_0$.

Theorem 4.1.10 (Jerison [16])**.** *Let* $X_1,\ X_2,\ \cdots,\ X_m$ *be a system of vector fields satisfying the Hörmander condition in* Ω. *Then for every compact set* $K \subset \Omega$ *there are constants* $C > 1$ *and* $r_0 > 0$ *such that for*

$u \in Lip(B)$

$$\fint_B |u - u_B|\, dx \le C\, r \fint_{2B} |Xu|\, dx$$

whenever B is a ball centered at K with radius $r \le r_0$.

For further information about analysis on Carnot-Carathéodory spaces, see [13]. We remark the following.

Remark 4.1.11. *The Grushin plane with Carnot-Carathéodory distance is a globally doubling measure space and satisfies globally Poincaré inequality with respect to Lebesgue measure .*

The Grushin plane has dilations, $d_{cc}(\delta_\lambda p,\, \delta_\lambda q) = \lambda\, d_{cc}(p,\, q)$ for all $p,\, q \in \mathbb{G}$ because X_1 and X_2 are homogenous of degree one with respect to the dilations. We fix a compact K which contains a neighborhood of the origin and $r_0 > 0$ so that the above two theorems are true. For any $p \in M$ and any $r > 0$, we choose $\lambda > 0$ so that $\delta_\lambda(B(p,\, 2r)) = B(\delta_\lambda(p),\, 2\lambda\, r)$ is contained in K and $\lambda\, r \le r_0$. Then the doubling condition holds for $\delta_\lambda(B(p,\, r)) = B(\delta_\lambda(p),\, \lambda r)$ and $\delta_\lambda(B(p,\, 2r)) = B(\delta_\lambda(p),\, 2\lambda r)$. Since $\mu(\delta_\lambda(E)) = \lambda^3 \mu(E)$ for any set $E \subset \mathbb{G}$ we conclude the doubling condition for $B(p,\, r)$. A similar argument applies to the Poincaré inequality.

4.2 Bi-Lipschitz Nonembedding Theorems

In Euclidean space, Rademacher's theorem states that a Lipschitz function is differentiable almost everywhere and the derivative is linear. We shall state theorems of Pansu and Cheeger which are analogues of Rademacher's theorem in some sense. These theorems can be applied to get nonembeddability of some metric spaces into Euclidean space. The first proof of the nonembeddability result of the Carnot groups is based on a differentiability result due to Pansu. His theorem takes into account the algebraic structure in Carnot groups, which appear as tangent space from Mitchell's theorem [25].

Theorem 4.2.1 (Pansu [29])**.** *Let $(M,\, \bullet)$ and $(N,\, \star)$ be Carnot groups. Every Lipschitz mapping f between open sets in M and N is differentiable almost everywhere. Moreover, the differential*

$$df_y(x) = \lim_{t \to 0} \delta_{t^{-1}} [f(y)^{-1} \star f(y \bullet \delta_t(x))]$$

is a Lie group homomorphism almost everywhere.

Semmes [32] observed that Theorem 4.2.1 implies that nonabelian Carnot groups M can not be embedded bi-Lipschitzly in Euclidean space. If M had a bi-Lipschitz embedding f into some Euclidean space $\mathbb{R}^n$, then f must be differentiable in the sense of Pansu and its differential should be an isomorphism. This gives a contradiction because it has nontrivial kernel. Hence M cannot be bi-Lipschitz embeddable. In particular, the Heisenberg group does not admit a bi-Lipschitz embedding into Euclidean space.

Rademacher's theorem states that infinitesimal behavior of any Lipschitz functions on $\mathbb{R}^n$ is approximated at almost every point by some linear function; that is, a linear combination of the coordinate functions. Cheeger proved a remarkable extension of Rademacher's theorem in doubling metric measure spaces supporting a p-Poincaré inequality. He constructed coordinate charts that span the differentials of Lipschitz functions. Moreover, his work gives a way to get nonembeddability results by using a purely geometric and analytic method.

Theorem 4.2.2 (Cheeger [5]). *If (X, d, μ) is a doubling metric measure space supporting a p-Poincaré inequality for some $p \geq 1$, then (X, d, μ) has a strong measurable differentiable structure, i.e. a countable collection of coordinate patches $\{(X_\alpha, \pi_\alpha)\}$ that satisfy the following conditions:*

(1) Each X_α is a measurable subset of X with positive measure and the union of the X_α's has full measure in X.

(2) Each π_α is a $N(\alpha)$-tuple of Lipschitz functions, for some $N(\alpha) \in \mathbb{N}$, where $N(\alpha)$ is bounded from above independently of α.

(3) Given a Lipschitz function $f : X \longrightarrow \mathbb{R}$, there exists an L^∞ function $df^\alpha : X_\alpha \longrightarrow \mathbb{R}^{N(\alpha)}$ so that

$$\limsup_{y \to x} \frac{|f(y) - f(x) - df^\alpha(x) \cdot (\pi_\alpha(y) - \pi_\alpha(x))|}{d(x, y)} = 0 \quad \textit{for } \mu - a.e\ x \in X_\alpha.$$

Cheeger also provided a uniform statement that covers many of the known nonembedding results.

Theorem 4.2.3 (Cheeger). *If a doubling p-Poincaré space X admits a bi-Lipschitz embedding into some finite dimensional Euclidean space, then at almost every point $x \in X_\alpha$, the tangent cone of X at x is bi-Lipschitz equivalent to $\mathbb{R}^{N(\alpha)}$.*

We can deduce from Cheeger's theorem the known nonembedding results both for the Carnot groups and for Laakso spaces. Cheeger and Kleiner generalized the almost everywhere differentiability for Lipschitz maps on PI space to any Banach space V with Radon-Nikodým property (RNP). That is, every Lipschitz map $f : \mathbb{R} \to V$ is differentiable almost everywhere. Moreover, a bi-Lipschitz nonembedding theorem holds whenever the target has RNP ([6], [7]).

We now check nonembeddability of the Heisenberg group $\mathbb{H}$ by applying Cheeger's nonembedding theorem. The Heisenberg group has a strong measurable differentiable structure with a single coordinate patch

$(\mathbb{H}, \pi_1, \pi_2)$, where $\pi_1(x, y, t) = x$ and $\pi_2(x, y, t) = y$. If we assume that the Heisenberg group admits a bi-Lipschitz embedding into some Euclidean space, then every tangent cone at almost every point in $\mathbb{H}$ must be bi-Lipschitz equivalent to $\mathbb{R}^2$. Since the Hausdorff dimension of $\mathbb{H}$ is not equal to 2, we conclude bi-Lipschitz nonembeddability.

In contrast to the Heisenberg group, Cheeger's nonembedding theorem does not answer whether or not the Grushin plane locally embeds into some Euclidean space. The Grushin plane $\mathbb{G}$ with Lebesgue measure is a doubling metric measure space supporting p-Poincaré inequality for any $p \geq 1$ (see Remark 4.1.11). Let K be any compact subset of $\mathbb{G}$ and $\mathbb{A}$ be set of singular points, y-axis. It has a Cheeger's coordinate patch $(K \setminus \mathbb{A}, \pi_1, \pi_2)$, where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Since every tangent cone to $K \setminus \mathbb{A}$ is bi-Lipschitz equivalent to $\mathbb{R}^2$, we cannot conclude non-embeddability of the Grushin plane, unlike the case of the Heisenberg group. Indeed, we prove in the next section that the Grushin plane admits a bi-Lipschitz embedding into some Euclidean space.

4.3 Spaces of Grushin Type

In this section, we will prove that spaces of Grushin type endowed with Carnot-Carathéodory distance embed bi-Lipschitzly into some Euclidean space. To do so, we will check the conditions in Theorem 3.1.1. Throughout this section, points in $\mathbb{R}^n \times \mathbb{R}^l$ are denoted by $p = (x,\, y)$, where $x = (x_1,\, x_2,\, \cdots,\, x_n) \in \mathbb{R}^n$ and $y = (y_1,\, y_2,\, \cdots,\, y_l) \in \mathbb{R}^l$. We let $\alpha = (\alpha_1,\, \alpha_2,\, \cdots,\, \alpha_n)$ be an n-tuple of non-negative integers with length $|\alpha| = \sum_{i=1}^{n} \alpha_i$. If $x = (x_1,\, x_2,\, \cdots,\, x_n) \in \mathbb{R}^n$, we put $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 4.3.1. *The space of Grushin type is $\mathbb{R}^n \times \mathbb{R}^l$ for $n,\, l \in \mathbb{N}$ with horizontal distribution spanned by X_i and Y_j for $i = 1,\ 2,\ ,\cdots,\ n$ and $j = 1,\ 2,\ ,\cdots,\ l$*

$$X_i = \frac{\partial}{\partial x_i} \quad \textit{and} \quad Y_j = x^{\alpha^j} \frac{\partial}{\partial y_j}$$

where for each j, α^j is an n-tuple of non-negative integers α_i^j and $|\alpha^j| = k$. We denote the space of Grushin type by $\mathbb{GS}$.

Remark 4.3.2. *When $n = l = 1$ and $|\alpha^1| = 1$, this is the Grushin plane described in Example 4.1.8. For more information about the Grushin plane, see [2].*

We can easily observe that k-th iterated Lie brackets generate the tangent space and $(k+1)$-th iterated

Lie brackets are zero. We next define dilations δ_λ on $\mathbb{R}^n \times \mathbb{R}^l$ by

$$\delta_\lambda(x,\, y) = (\lambda x,\, \lambda^{k+1} y) \tag{4.3.1}$$

whenever $p = (x,\, y) \in \mathbb{GS}$ and $\lambda > 0$. Then, X_i and Y_j for all i and j are homogeneous of degree one with respect to the dilations. Hence, the Carnot-Carathéodory distance satisfies

$$d_{cc}\,(\delta_\lambda(p,\, q)) = \lambda\, d_{cc}(p, q). \tag{4.3.2}$$

for all $p,\, q \in \mathbb{GS}$. The set of all singular points is

$$\mathbb{S} := \cup_{i=1}^n \{(x,\, y) \in \mathbb{R}^n \times \mathbb{R}^l \mid x_i = 0\}.$$

The metric on $\mathbb{GS}\backslash\mathbb{S}$ is the Riemannian metric ds^2 making X_i and Y_j where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, l$ into an orthonormal basis for the tangent space,

$$ds^2 = \sum_{i=1}^n dx_i^2 + \sum_{j=1}^l \frac{dy_j^2}{x^{2\alpha^j}}. \tag{4.3.3}$$

The metric can be extended across $\mathbb{S}$ as the Carnot-Carathéodory distance by means of the length elements ds^2, since the horizontal distribution satisfies the Hörmander condition.

For any horizontal curve $\gamma : [0, 1] \to \mathbb{GS}$, we write $\gamma(t) = (\mathrm{x}_1(t), \cdots, \mathrm{x}_n(t), \mathrm{y}_1(t), \cdots, \mathrm{y}_l(t))$ for a parametrized horizontal curve. Then, we have

$$\mathrm{length}(\gamma) = \int_0^1 \sqrt{\sum_{i=1}^n \mathrm{x}_i'(t)^2 + \sum_{j=1}^l \frac{\mathrm{y}_j'(t)^2}{\mathrm{x}(t)^{2\alpha^j}}}\, dt \tag{4.3.4}$$

where $\mathrm{x}(t)^{2\alpha^j} := \mathrm{x}_1(t)^{2\alpha_1{}^j}\mathrm{x}_2(t)^{2\alpha_2{}^j} \cdots \mathrm{x}_n(t)^{2\alpha_n{}^j}$.

The following proposition gives distance estimates for the cc-distance on $\mathbb{GS}$. We emphasize that $\mathbb{A}$ is proper subset of $\mathbb{S}$ and $\mu(\mathbb{S} \setminus \mathbb{A}) = 0$ with respect to Lebesgue measure.

Proposition 4.3.3. *Let $\mathbb{A}$ be $\{0\} \times \mathbb{R}^l$. The cc-distance on $\mathbb{A}$ is comparable to $\sqrt[k+1]{d_E}$. Now fix points $p = (x,\, y)$ and $q = (v,\, w)$ in $\mathbb{GS} \setminus \mathbb{A}$. We have the following distance estimates:*

$$c_1\Big\{\sum_{i=1}^n |x_i - v_i| + \sum_{j=1}^l \frac{|y_j - w_j|}{\sum_{i=1}^n \min(|x_i|, |v_i|)^k + \sum_{j=1}^l |y_j - w_j|^{\frac{k}{k+1}}}\Big\} \le d_{cc}(p, q) \le c_2\Big\{\sum_{i=1}^n |x_i - v_i| + \sum_{j=1}^l |y_j - w_j|^{\frac{1}{k+1}}\Big\} \tag{4.3.5}$$

Here c_1 and c_2 are constants independent of p and q.

Proof. The first estimation of the *cc*-distance on $\mathbb{A}$ is deduced from equation (4.3.1). The upper bound in equation (4.3.5) comes from the triangle inequality. We will use equation (4.3.4) to get the lower bound in equation (4.3.5). Let $\gamma(t) = (\mathrm{x}_1(t),\ \cdots,\ \mathrm{x}_n(t),\ \mathrm{y}_1(t),\ \cdots,\ \mathrm{y}_l(t))$ be a parametrized horizontal curve joining p to q where $t \in [0,1]$. Then, $(*)$ length$(\gamma) \geq |x_i - v_i|$ for all $i = 1,\ 2,\ \cdots,\ n$. If there exists K_i such that $|\mathrm{x}_i(t)| \leq K_i$ for all $t \in [0,1]$., then length$(\gamma) \geq K^{-k}|y_j - w_j|$ for all $j = 1,\ 2,\ ,\cdots,\ l$ where $K = \max_i\{K_i\}$. Otherwise, there exists $t_i \in [0,1]$ such that $|\mathrm{x}_i(t_i)| \geq K_i$ for all i and length$(\gamma) \geq$ length$(\widetilde{\gamma}) \geq \max\{|\mathrm{x}_i(t_i) - x_i|, |\mathrm{x}_i(t_i) - v_i|\} \geq K_i - \min\{|x_i|, |v_i|\}$ where $\widetilde{\gamma}$ is a subcurves of γ joining p to $(\mathrm{x}_1(t_i),\ \cdots,\ \mathrm{x}_n(t_i),\ \mathrm{y}_1(t_i),\ \cdots,\ \mathrm{y}_l(t_i))$ or q to $(\mathrm{x}_1(t_i),\ \cdots,\ \mathrm{x}_n(t_i),\ \mathrm{y}_1(t_i),\ \cdots,\ \mathrm{y}_l(t_i))$. Then , we have the following:

$$(**)\ \text{length}(\gamma) \geq \sup_{K_i > \max\{|x_i|,|v_i|\}} \min_{i,\ j}\{K_i - \min\{|x_i|, |v_i|\},\ K^{-k}|y_j - w_j|\}$$

For the sake of simplicity for computation, we denote $\min\{|x_i|, |v_i|\}$ by a_i and denote $|y_j - w_j|$ by b_j. When we choose K satisfying the following:

$$n\sum_i a_i + n\frac{\sum_j b_j}{\sum_i {a_i}^k + \sum_j {b_j}^{\frac{k}{k+1}}} \leq nK_i \leq nK \leq (1+n)nl^{\frac{1}{k}}(\sum_i {a_i}^k + \sum_j {b_j}^{\frac{k}{k+1}})^{\frac{1}{k}} \tag{4.3.6}$$

Then, we can see that

(1) $K_i - a_i \geq \dfrac{\sum_j b_j}{\sum_i {a_i}^k + \sum_j {b_j}^{\frac{k}{k+1}}}$,

(2) $\dfrac{b_j}{K^k} \geq \dfrac{\sum_j b_j}{\sum_i {a_i}^k + \sum_j {b_j}^{\frac{k}{k+1}}}$.

and, hence $(**)$ length$(\gamma) \geq \dfrac{|y_j - w_j|}{(1+n)^k l(\sum_i \min\{|x_i|, |v_i|\}^k + \sum_j |y_j - w_j|^{\frac{k}{k+1}})}$ for all j. We can compute the lower bound in equation (4.3.5) by averaging $(*)$ and $(**)$ for all i and j. $\square$

We next consider the lattice of points in $\mathbb{R}^n \times \mathbb{R}^l$ whose coordinates are integers. Then, this lattice determines a mesh $M_0 \times M_0$. For each $j \in \mathbb{Z}$, consider the submesh $M_j = 2^{-j}M_0 \times 2^{-j(k+1)}M_0$ which is set of cubes in $\mathbb{R}^n \times \mathbb{R}^l$ of sidelengths 2^{-j} and $2^{-j(k+1)}$ in $\mathbb{R}^n$ and in $\mathbb{R}^l$ respectively. From the above distance estimates, $\mathbb{GS} \setminus \mathbb{A}$ has a Whitney decomposition. We recall this in the following Proposition 4.3.4.

Proposition 4.3.4. *Let $\mathbb{A}$ be $\{0\} \times \mathbb{R}^l$. Then its complement $\Omega = \mathbb{GS} \setminus \mathbb{A}$ is the union of a sequence of cubes Q , whose interiors are mutually disjoint and whose diameters are approximately proportional to their distances from $\mathbb{A}$. More precisely,*

(1) $\Omega = \cup_{Q \in W_\Omega} Q$

(2) Any two cubes are mutually disjoint.

(3) $dist_{cc}(Q, \mathbb{A}) \leq diam_{cc}(Q) \leq 2c_2(n+l)dist_{cc}(Q, \mathbb{A})$.

The space of Grushin type is a locally doubling metric measure space with respect to Lebesgue measure (see Theorem 4.1.9). Because of self-similarity, it is a globally doubling metric measure space from similar argument as Remark 4.1.11. It is uniformly perfect. Since cc-distance on $\mathbb{A}$ is comparable to $\sqrt[k+1]{d_E}$, we apply Assouad's theorem. Then we have a L-bi-Lipschitz embedding f from $\mathbb{A}$ into $\mathbb{R}^m$ for some m and L. If we verify the condition of uniformly Christ-local bi-Lipschitz embeddings, then we can conclude the following corollary.

Corollary 4.3.5. *The space of Grushin type endowed with Carnot-Carathéodory distance admits a bi-Lipschitz embedding into some Euclidean space.*

It is enough to verify uniformly Christ-local bi-Lipschitz embeddings. In this case, Q^* is the set of all Whitney cubes which touch Q and Q^{**} is the set of all Whitney cubes which touch Q^* (see Definition 2.4.4).

Lemma 4.3.6. *The complement of* $\mathbb{A}$ *admits uniformly Christ-local bi-Lipschitz embeddings.*

Proof. We may assume that Q^{**} does not touch the set of singular points $\mathbb{S}$. If Q^{**} intersects $\mathbb{S}$, then $Q^{**} \cap \mathbb{S}$ is measure zero with respect to Lebesgue measure. Therefore, bi-Lipschitz embeddings on each $Q^{**} \setminus \mathbb{S}$ can be extended to Q^{**}. We observe that Q^{**} is a closed $(n+l)$-dimensional Riemannian manifold for each Q. For any two elements Q and Q' in W_Ω, we have $Q' = \Phi(Q)$ where Φ is composition of translation map ς with respect to $\{0\} \times \mathbb{R}^l$ and expansion map $\psi(x, y) = (2^{(j'-j)}x, 2^{(j'-j)(k+1)}y)$. Then, we have $diam(Q') = 2^{(j'-j)}diam(Q)$ from Proposition 4.3.4. Therefore, we can cover all Q^{**} by balls $B_1, B_2, \cdots, B_N$ of radius $diam(Q) > 0$ where N is independent of Q. For each i, there exist L-bi-Lipschitz diffeomorphisms for some L

$$\varphi_i : 5B_i \to \varphi_i(5B_i) \subset \mathbb{R}^{(n+l)}.$$

Without loss of generality, we may assume that $|\varphi_i(x)| \geq diam(Q)$ for all i and $x \in 5B_i$. let $u_i \in C_0^\infty(2B_i)$ be such that $0 \leq u_i \leq 1$ and $u_i|_{B_i} = 1$, and let $v_i \in C_0^\infty(5B_i)$ be such that $0 \leq v_i \leq 1$ and $v_i|_{4B_i} = 1$. Then, we define $\varphi : X \to \mathbb{R}^{(n+l)N} \times \mathbb{R}^{(n+l)N}$

$$\varphi(x) := (\varphi_1(x)u_1(x), \cdots, \varphi_N(x)u_N(x), \varphi_1(x)v_1(x), \cdots, \varphi_N(x)v_N(x))$$

Obviously φ is smooth, and hence it is Lipschitz with Lipschitz constant $2LN$. We will show that φ is co-Lipschitz. To this end, let us assume first that $d(x, y) > 3\,diam(Q)$. Then, there exists i such that

$u_i(x) = 1$ and $v_i(x) = 0$. Thus,

$$\begin{aligned}|\varphi(x) - \varphi(y)| &\geq |\varphi_i(x)u_i(x) - \varphi_i(y)u_i(y)| \\ &= |\varphi_i(x)| \\ &\geq diam(Q) \geq \frac{1}{C(C_1,\, A,\, \delta)}\, d(x,y)\end{aligned}$$

The last inequality arises from comparability of $diam(Q)$ and $diam(Q^{**})$ (see Proposition 2.4.7). On the other hand, if $d(x,y) \leq 3diam(Q)$, then there exists i such that $v_i(x) = 1 = v_i(y)$. Thus,

$$|\varphi(x) - \varphi(y)| \geq |\varphi_i(x) - \varphi_i(y)| \geq \frac{1}{L}\, d(x,\, y).$$

Therefore, we have uniformly local bi-Lipschitz embeddings on each Q^{**} into $\mathbb{R}^{2(n+l)N}$. The bi-Lipschitz constant and dimension of the target space are independent of Q. □

Chapter 5

Questions and Remarks

5.1 Questions and Remarks

So far we have given a characterization of Euclidean bi-Lipschitz embeddability of uniformly perfect metric spaces supporting a doubling measure. The hypothesis in Theorem 3.1.1 is based on a Christ-Whitney decomposition deduced from uniform perfectness and existence of a doubling measure. We emphasize that uniform perfectness is only used in Section 2.4 to show existence of a Christ-Whitney decomposition.

Question 5.1.1. *Can the condition of uniform perfectness be weakened?*

From Theorem 3.1.1, the dimension $M_1 + M\,M_2 + 1$ of the Euclidean space depends on the bi-Lipschitz constant L_1 and the doubling constant of μ. However, the number of colors M is not optimal. Thus, the following question naturally arises.

Question 5.1.2. *What is the minimal dimension of Euclidean space into which the metric space satisfying the conditions in Theorem 3.1.1 bi-Lipschitzly embeds?*

As an application of Theorem 3.1.1, we have considered the space of Grushin type with fixed length $|\alpha^j| = k$ for all $j = 1, 2, \cdots, n$. We now can consider the space of Grushin type with *extended* horizontal distribution on $\mathbb{R}^n \times \mathbb{R}^l$.

Definition 5.1.3. *The* extended space of Grushin type *is $\mathbb{R}^n \times \mathbb{R}^l$ for $n, l \in \mathbb{N}$ with horizontal distribution spanned by X_i and Y_j for $i = 1, 2, ,\cdots, n$ and $j = 1, 2, ,\cdots, l$*

$$X_i = \frac{\partial}{\partial x_i} \quad \textit{and} \quad Y_j = x^{\alpha^j} \frac{\partial}{\partial y_j}.$$

Remark 5.1.4. *We emphasize that the lengths $|\alpha_i^j|$ can be distinct in Definition 4.3.1.*

It seems that we can follow similar steps to prove embeddability. However, some of the technical details must be checked.

Conjecture 1. *The* extended space of Grushin type *equipped with Carnot-Carathéodory distance admits a bi-Lipschitz embedding into Euclidean space.*

In the case of spaces of Grushin type, horizontal distributions are good enough to have uniformly Christ-local embeddings. Therefore, the following problem naturally comes up.

Problem 5.1.5. *Find sufficient conditions on a higher dimensional horizontal distribution in a given sub-Riemannian manifold so as to guarantee the existence of uniformly Christ-local bi-Lipschitz embeddability.*

Even more generally, we meet the following problem:

Problem 5.1.6. *Characterize Christ-local bi-Lipschitz embeddability.*

If Problem 5.1.6 were solved, then we could characterize bi-Lipschitz embeddable metric spaces with geometric and analytic criteria. Therefore, we could determine which metric spaces admit a bi-Lipschitz embedding and we can classify metric spaces which are subsets of Euclidean space.

Problem 5.1.7. *Find other examples of sub-Riemannian manifolds that satisfy conditions in Theorem 3.1.1 and hence, embed bi-Lipschitzly into Euclidean space.*

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