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NEUTRONICS ANALYSIS FOR CYLINDRICAL ASSEMBLY USING  
GREEN'S FUNCTIONS

BY

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THESIS

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# ABSTRACT

## NEUTRONICS ANALYSIS FOR CYLINDRICAL ASSEMBLY USING GREEN'S FUNCTIONS

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When designing a reactor a preliminary design is done in order to obtain a rough estimate of various reactor properties. These properties include the neutron flux, criticality condition, or distribution of material in the assembly. It is possible to obtain an analytic solution for the neutron flux for a reactor represented in cylindrical coordinates using a Green's function or Green's function matrix method for both one group and two group neutron diffusion. The analytic results for a two-dimensional cylindrical system are simplified by assuming a cosine flux shape in the horizontal direction. This assumption makes the flux in the horizontal direction a neutron sink term in the radial direction. From the neutron flux, expressions for the criticality condition and fuel distribution cross section can be obtained by specifying the form factor, which is equivalent to specifying the power shape. A specific form factor, such as a constant power shape or parabolic power shape allow for a comparison between the results obtained by using a one group diffusion model against a two group diffusion model.

*To my little brothers, keep on working hard.*

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# CHAPTER 1

## INTRODUCTION

### 1.1 Motivation

The following is an extension of work done by Axford[1][2] in neutronics analysis. It has been demonstrated that by using either a Green's function or a Green's function matrix it is possible in such a way to solve for the criticality condition and the fuel absorption rate by specifying the form factor also known as the flux shape. Specifying the flux shape is equivalent to giving the power shape. Using this method, one obtains an analytic solution to quantities of interest that are necessary in preliminary design calculations.

This method is useful in that the resulting second order inhomogeneous linked differential equations may be solved analytically in terms of Green's functions. The Green's functions may be obtained in various ways, such as integral transforms, closed form representations, or a method of bilinear representations.

In work done by Axford, this method is constructed for a generalized neutron diffusion. The goal of this study is to perform a neutronics analysis for a thermal cylindrical system for one and two group neutron diffusion.

### 1.2 General Description of Work

Starting from either the one or two group generalized neutron diffusion equations, assumptions are made that are typical of preliminary reactor design in order to simplify the models. These include various assumptions, such as the flux being independent of azimuthal angle or certain coefficients being constant throughout the assembly.

One of the important simplifications made is that the flux in the horizontal

direction is transformed into a neutron sink term in the radial direction. This is done by assuming a cosine flux shape in the horizontal direction. Once simplified, the equations are transformed in such a way that the end result is that the flux shape may be specified to obtain the criticality condition and fuel absorption cross section throughout the core.

The equations are solved using either a Green's function or a Green's function matrix. The Green's functions are obtained by either using a Hankel transform or by construction of a closed form representation of Green's functions. General equations for an arbitrary power shape are obtained for the criticality condition and fuel absorption cross section.

Finally, a sample reactor is constructed in which the power shape is specified. The power shapes that are used are a flat power shape and a parabolic type power shape.

### 1.3 General Overview of Chapters

This thesis is organized in the following way.

In Chapter 2 we start from the generalized one group neutron diffusion equation and obtain a simplified model after certain assumptions used in preliminary reactor design are used. The neutron flux is then solved for in terms of a Green's function. The corresponding Green's function is obtained using a Hankel transform and a closed method representation. Following this expressions for the criticality condition and fuel distribution are obtained for a generalized power shape.

In Chapter 3 we start from the generalized two group neutron diffusion equations and perform the same analysis as was done in Chapter 2. The difference in this chapter is that a Green's function matrix must be employed in order to solve for the quantities of interest. The Green's functions are solved using an infinite Hankel transform.

Chapter 4 presents a model reactor in which the power shape is then specified. Plots are obtained for both one and two group diffusion in which the power shape is flat or of a parabolic type.

Finally Chapter 5 discusses directions for further research and provides a conclusion for this thesis.

# CHAPTER 2

## DEVELOPMENT OF ONE GROUP EQUATIONS FOR NEUTRON ANALYSIS FOR SPECIFIED POWER SHAPE

### 2.1 Introduction

In this section, neutron analysis for a specified power shape is carried out for a cylindrical system. The system and parameters specifying the dimensions are presented below.

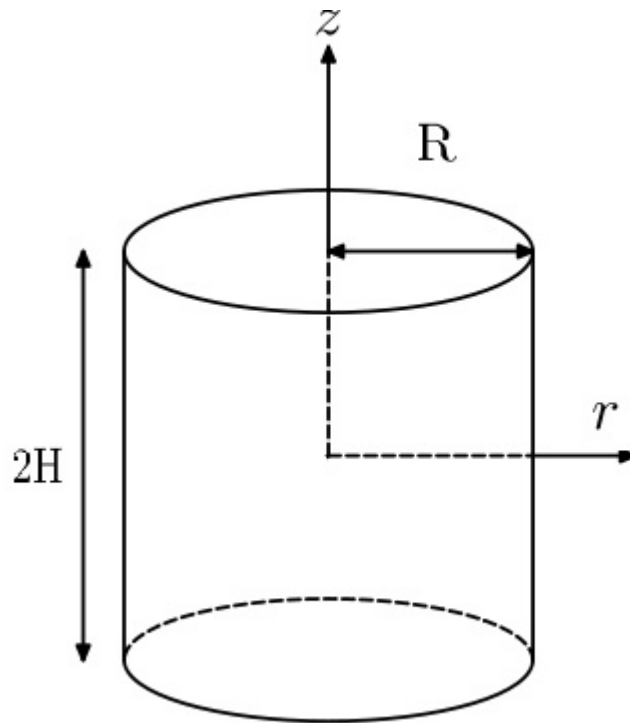


Figure 2.1: Geometric representation of core

Equations for the fuel distribution and the critically condition,  $k_{eff}$ , for a specified power shape will be developed starting from the general neutron diffusion equation in a cylindrical system. These equations will be formulated

by means of a Green's function.

Having obtained expressions for the fuel distribution and  $k_{eff}$ , in Chapter 4, the power shape is then specified and analytic results are obtained for the case in which the power shape is constant and for the case in which the power shape is of an arbitrary parabolic type.

The following section will assume basic knowledge of neutron diffusion. An introduction into the development of the governing equations for neutron diffusion can be found in either text from Lamarsh or by Duderstadt and Hamilton.[3][4]

## 2.2 Derivation of Governing Equations

We will start with the general one-group neutron diffusion equation:

$$\nabla_{\mathbf{r}} \cdot (D(\mathbf{r}) \nabla_{\mathbf{r}} \phi(\mathbf{r})) - \Sigma_a(\mathbf{r}) \phi(\mathbf{r}) + S(\mathbf{r}) = 0 \quad (2.1)$$

Where  $\phi(\mathbf{r})$  is the scalar flux,  $D(\mathbf{r})$  is the diffusion coefficient,  $\Sigma_a(\mathbf{r})$  is the absorption cross section, and  $S(\mathbf{r})$  is an arbitrary source term.

The following assumptions are made in order to simplify the problem. First the diffusion coefficient,  $D(\mathbf{r})$ , is assumed to be independent of position. This assumption is justified in the case of a homogeneous or uniform mixture of material.

To determine  $S(\mathbf{r})$ , we let  $\Sigma_f$  to be the absorption cross section of the fuel, and let  $\nu$  be the average number of neutrons released per fission. This gives us a source term of  $\nu \Sigma_f \phi(\mathbf{r})$ . To complete the derivation of the source term, we multiply this term by the multiplication factor,  $\frac{1}{k_{eff}}$ . This is done in order to perform a criticality analysis. It is now assumed no external source term is used. Thus, the source term is given by:

$$S(\mathbf{r}) = \frac{\nu \Sigma_f}{k_{eff}} \phi(\mathbf{r}) \quad (2.2)$$

For the problem we are trying to solve we will use a two-dimensional cylindrical coordinate system. This assumes that the neutron scalar flux is independent of the azimuthal angle. Thus the governing equation for the problem

becomes:

$$D\nabla^2\phi(r, z) - \Sigma_a\phi(r, z) + \frac{\nu\Sigma_f}{k_{eff}}\phi(r, z) = 0 \quad (2.3)$$

As a second order differential equation, two boundary conditions are required for a well posed problem. The first condition is that the flux,  $\phi$  remains finite at  $R = 0$ . The other is a Dirichlet boundary condition in which the scalar flux,  $\phi$  goes to 0 at some extrapolated distance from the surfaces of the core. This boundary condition is more of mathematical construct that allows simplification of the problem, rather than an accurate capture of the physics of the problem. The extrapolation distance can be found by methods from neutron transport theory.[1] In the case where the extrapolation distance is much smaller than the radius of the system, one can simplify the problem and assume that  $\phi$  goes to 0 at  $R$ . A Neumann boundary condition at the surface in which the neutron current density into the core is 0 is also a possibility and indeed captures the physics of the problem more precisely. However, in order to simplify the problem mathematically a Dirichlet boundary condition is used. The justifications for this can be found in Duderstadt and Hamilton.[4]

Expanding the Laplacian operator that appears in 2.3 and dividing by  $D$  gives us the following:

$$\frac{\partial^2\phi(r, z)}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi(r, z)}{\partial r}\right) - \frac{\Sigma_a}{D}\phi(r, z) + \frac{\nu\Sigma_f}{k_{eff}D}\phi(r, z) = 0 \quad (2.4)$$

In order to further simplify the problem and reduce it to a one-dimensional problem an assumption on the flux in the vertical direction will be used. We will assume that the flux can be split into a product of the radial and vertical fluxes. Furthermore, in the horizontal direction a chopped cosine flux distribution will be used.[5] The flux is thus represented by:

$$\phi(r, z) = \phi(r)\cos\left(\frac{\pi z}{H + 2\delta}\right) \quad (2.5)$$

In the above equation, the assumption that the horizontal flux goes to 0 at an extrapolated distance of  $H + \delta$ , in which  $\delta$  is the reflector diffusion length, is used. The use of the flux in this form allows the problem to be transformed from a two-dimensional problem to a one-dimensional problem. The horizontal term will then be captured as a neutron sink in the equation

for the flux.

Calculating derivatives and cancelling out,  $\cos\left(\frac{\pi z}{H+2\delta}\right)$ , which appears in every term gives us the following:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi(r)}{dr} \right) - \left( \frac{\pi}{H+2\delta} \right)^2 \phi(r) - \frac{\Sigma_a}{D} \phi + \frac{\nu \Sigma_f}{D k_{eff}} \phi(r) = 0 \quad (2.6)$$

Here, it is seen that the horizontal flux is captured as a neutron sink in the  $\left(\frac{\pi}{H+2\delta}\right)^2 \phi(r)$  term.

The neutron absorption cross section is now split up into the absorption cross sections for the moderator and fuel. The moderator absorption cross section is assumed to be constant, while the fuel absorption cross section is allowed to vary with position. Thus the absorption cross section becomes:

$$\Sigma_a = \Sigma_a^M + \Sigma_a^F(r) \quad (2.7)$$

With this 2.6 becomes:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi(r)}{dr} \right) - \alpha^2 \phi(r) - \frac{\Sigma_a^F(r)}{D} \phi(r) + \frac{\nu \Sigma_f}{D k_{eff}} \phi(r) = 0 \quad (2.8)$$

Where  $\alpha^2$  is defined as:

$$\alpha^2 = \left[ \left( \frac{\pi}{H+2\delta} \right)^2 + \frac{\Sigma_a^M}{D} \right] \quad (2.9)$$

A further simplification that can be made is rewriting the last term in terms of the fuel regeneration factor. This can be done by combining the last two terms in 2.8 and factoring out  $\Sigma_a^F$ . This is shown below:

$$\begin{aligned} \frac{-\Sigma_a^F(r)}{D} \phi(r) + \frac{\nu \Sigma_f}{k_{eff} D} \phi(r) &= \frac{1}{D} \left( \frac{\nu \Sigma_f}{\Sigma_a^F} \frac{1}{k_{eff}} - 1 \right) \Sigma_a^F \phi(r) \\ &= \frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) \Sigma_a^F(r) \phi(r) \end{aligned} \quad (2.10)$$

Where  $\eta$  is the fuel regeneration factor defined by  $\eta = \frac{\nu \Sigma_f}{\Sigma_a^F}$ . The final step is to define the fuel absorption rate as  $A(r) = \Sigma_a^F(r) \phi(r)$ . This is done because as has been mentioned the fuel absorption rate is proportional to the power density. We finally have the equation in a form suitable for analysis. This is

shown below:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi(r)}{dr} \right) - \alpha^2 \phi(r) + \frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) A(r) = 0 \quad (2.11)$$

This is an inhomogeneous differential equation that will be solved using the Green's Function Method. The corresponding Green's Function for 2.11 is:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) - \alpha^2 K(r|r') + \frac{\delta(r - r')}{2\pi r D} = 0 \quad (2.12)$$

Where  $r'$  refers to the source point and  $r$  is the field point.

## 2.3 Solution of Neutron Flux in Terms of Green's Function

In this section, the neutron flux will be solved for in terms of the neutron diffusion equation for the system and its corresponding Green's Function. These equations are repeated below:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi(r)}{dr} \right) - \alpha^2 \phi(r) + \frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) A(r) = 0 \quad (2.13)$$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) - \alpha^2 K(r|r') + \frac{\delta(r - r')}{2\pi r D} = 0 \quad (2.14)$$

We start by multiplying 2.13 by  $K(r|r')$  and 2.14 by  $\phi(r)$  to get the following:

$$K(r|r') \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \alpha^2 K(r|r') \phi(r) + \frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) K(r|r') A(r) = 0 \quad (2.15)$$

$$\phi(r) \frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) - \alpha^2 \phi(r) K(r|r') + \frac{\phi(r) \delta(r - r')}{2\pi r D} = 0 \quad (2.16)$$

We now subtract 2.16 from 2.15, integrate over the volume,  $\int r dr$  to get:

$$\int r dr K(r|r') \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \phi(r) \frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right)$$



$$+\frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) K(r|r') A(r) - \frac{\phi(r) \delta(r-r')}{2\pi r D} = 0 \quad (2.17)$$

Using integration by parts in 2.17 the following term is set equal to 0:

$$\int r dr K(r|r') \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \phi(r) \frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) = 0 \quad (2.18)$$

We are now left with the following:

$$\frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) \int K(r|r') A(r) r dr - \int r dr \frac{\phi(r) \delta(r-r')}{2\pi r D} = 0 \quad (2.19)$$

Using the sifting property of the Dirac delta function, we can solve for  $\phi(r)$ . After rearranging terms we come to the following:

$$\phi(r') = \left( \frac{\eta}{k_{eff}} - 1 \right) \int A(r) K(r|r') 2\pi r dr \quad (2.20)$$

Using the symmetry of the Green's Function, we can rewrite this in terms of the field point:

$$\phi(r) = \left( \frac{\eta}{k_{eff}} - 1 \right) \int A(r') K(r|r') 2\pi r' dr' \quad (2.21)$$

## 2.4 Derivation of Green's Function

In this section, an expression for the Green's Function will be found. It will be assumed that we have an infinite reflector in which the core moderator has the same properties as the reflector. Two ways of constructing the Green's Function will be used as a consistency check.

### 2.4.1 Solution via Closed Form Representation

The problem we are trying to solve is presented below:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) - \alpha^2 K(r|r') + \frac{\delta(r-r')}{2\pi r D} = 0 \quad 0 \leq r < \infty \quad (2.22)$$

With the following boundary conditions:

$$\begin{aligned}\lim_{r \rightarrow 0} K(r|r') &= \text{finite} \\ \lim_{r \rightarrow \infty} K(r|r') &= 0\end{aligned}\tag{2.23}$$

The solution will be done using a closed form representation method. The validity of this technique follows theorems proved Axford.[1] The proof will not be repeated here.

Using the method for a closed form representation, we will have two functions  $v(r)$  and  $u(r)$  that satisfy the following differential equations:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{du(r)}{dr} \right) - \alpha^2 u(r) = 0 \quad 0 \leq r \leq r' \tag{2.24}$$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dv(r)}{dr} \right) - \alpha^2 v(r) = 0 \quad r' \leq r \leq \infty \tag{2.25}$$

Using the above equations, the solution to the Green's Function is given by:

$$K(r|r') = \begin{cases} Au(r) & 0 \leq r \leq r' \\ Bv(r) & r' \leq r \leq \infty \end{cases} \tag{2.26}$$

Where the constants  $A$  and  $B$  are found by the internal boundary conditions at the source location. These are given by:

$$\begin{aligned}Au(r') &= Bv(r') \\ B \frac{dv(r')}{dr} - A \frac{du(r')}{dr} &= \frac{-1}{2\pi r D}\end{aligned}\tag{2.27}$$

These equations imply that the Green's function at the location of the source term is continuous, but that the derivative is discontinuous.

We will first solve 2.24. The solution to this differential equation after applying the boundary condition of finiteness at the center is thus given by:

$$u(r) = I_0(\alpha r) \tag{2.28}$$

Where  $I_0$  is the modified Bessel function of the first kind.

The solution to 2.25, after applying the boundary condition at infinity, is given by:

$$v(r) = K_0(\alpha r) \tag{2.29}$$

Where  $K_0$  is the modified Bessel function of the second kind.

Using the following identity for the Wronskian of modified Bessel functions[6]:

$$W[K_\nu(r), I_\nu(r)] = I_\nu(r)K_{\nu+1}(r) + I_{\nu+1}(r)K_\nu(r) = \frac{1}{z} \quad (2.30)$$

and solving the system of equations given by 2.27 gives us solutions for the constants  $A$  and  $B$ . These are:

$$A = \frac{1}{2\pi D} K_0(\alpha r') \quad (2.31)$$

and

$$B = \frac{1}{2\pi D} I_0(\alpha r') \quad (2.32)$$

so that the final result for the Green's Function is given by:

$$K(r|r') = \begin{cases} \frac{1}{2\pi D} K_0(\alpha r') I_0(\alpha r) & 0 \leq r \leq r' \\ \frac{1}{2\pi D} I_0(\alpha r') K_0(\alpha r) & r' \leq r \leq \infty \end{cases} \quad (2.33)$$

#### 2.4.2 Solution via Hankel Transform

The problem is repeated below:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dK(r|r')}{dr} \right) - \alpha^2 K(r|r') + \frac{\delta(r-r')}{2\pi r D} = 0 \quad 0 \leq r < \infty \quad (2.34)$$

This equation will be solved using a Hankel transform as a consistency check for the solution given by 2.33.

First we define the Hankel transform for  $K(r|r')$  by:

$$K(k|r') = \int_0^\infty J_0(kr) K(r|r') r dr \quad (2.35)$$

and the inverse transform as:

$$K(r|r') = \int_0^\infty J_0(kr) K(k|r') k dk \quad (2.36)$$

Taking the transform of 2.34 and solving for  $K(k|r')$  gives us the following

expression for the Green's function in the transform space:

$$K(k|r') = \frac{1}{2\pi D} \frac{J_0(r'k)}{k^2 + \alpha^2} \quad (2.37)$$

Finally taking the inverse transform of 3.38 gives us the following:

$$K(r|r') = \frac{1}{2\pi D} \int_0^\infty \frac{J_0(r'k)J_0(rk)}{k^2 + \alpha^2} k dk \quad (2.38)$$

Which when the integration is carried out gives us the following solution for the Green's function:

$$K(r|r') = \begin{cases} \frac{1}{2\pi D} K_0(\alpha r') I_0(\alpha r) & 0 \leq r \leq r' \\ \frac{1}{2\pi D} I_0(\alpha r') K_0(\alpha r) & r' \leq r \leq \infty \end{cases} \quad (2.39)$$

Both methods of solving the for the Green's function give identical results as shown in 2.33 and 2.39. Furthermore, the solution for the Green's function shows the symmetry property of the Green's function, as changing the field point and source point leaves the Green's function unchanged. Having obtained the Green's function, a neutron analysis can now be done for various power shapes.

## 2.5 Expressions for Criticality Factor and Fuel Distribution

Before finding expressions for  $k_{eff}$  and the fuel distribution for a flat power shape and parabolic power shape, the expressions will be solved for in terms of an arbitrary power shape denoted by  $F(r)$ . This is done to simplify the mathematics and to avoid repetition of certain steps when the analysis is carried out for specified power shapes.

We start with the equation for the flux as given by 2.21 in section 2.3. This is repeated below:

$$\phi(r) = \left( \frac{\eta}{k_{eff}} - 1 \right) \int A(r') K(r|r') 2\pi r' dr' \quad (2.40)$$

Where the fuel absorption rate is defined by  $A(r) = \Sigma_a^F(r)\phi(r)$ .

We multiply the above equation for the flux by  $\Sigma_a^F(r)$  to give us an equation

for the fuel absorption rate given by:

$$A(r) = \Sigma_a^F(r) \left( \frac{\eta}{k_{eff}} - 1 \right) \int A(r') K(r|r') 2\pi r' dr' \quad (2.41)$$

The fuel absorption rate is now defined by:

$$A(r) = A_0 F(r) \quad (2.42)$$

where  $F(r)$  is the relative absorption rate in the fuel, such that  $F(0) = 1.0$ . The fuel absorption rate is defined by,  $A_0 = \Sigma_a^F(0)\phi(0)$ . Defining the fuel absorption this way allows us to perform a neutron analysis based on the absorption rate of the fuel at the center of the core. Substituting this expression for  $A(r)$  into 2.41 and rewriting terms, we now write 2.41 as follows:

$$A_0 \left[ F(r) - \Sigma_a^F(r) \left( \frac{\eta}{k_{eff}} - 1 \right) \int F(r') K(r|r') 2\pi r' dr' \right] = 0 \quad (2.43)$$

In the case of a subcritical reactor  $A_0 = 0$ . The assumption is made that the reactor is critical. This is justified by the fact that criticality analysis is being performed on the reactor. In this case  $A_0 \neq 0$  and we can cancel out  $A_0$  out of the preceding equation. This allows us to solve for the relative absorption rate in the fuel in terms of the Green's function as follows:

$$F(r) = \Sigma_a^F(r) \left( \frac{\eta}{k_{eff}} - 1 \right) \int F(r') K(r|r') 2\pi r' dr' \quad (2.44)$$

From 2.44 we can solve for the two quantities of interest. First solving for the fuel distribution as a function of the relative absorption rate gives us the following:

$$\Sigma_a^F(r) = \frac{F(r)}{\left( \frac{\eta}{k_{eff}} - 1 \right) \int F(r') K(r|r') 2\pi r' dr'} \quad (2.45)$$

The final step is to substitute in the appropriate expression for the Green's function. The expression for the fuel absorption given by the fuel absorption

cross section given by equation 2.45 becomes:

$$\Sigma_a^F(r) = \frac{F(r)}{\frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) \left[ K_0(\alpha r) \int_0^r F(r') I_0(\alpha r') r' dr' + I_0(\alpha r) \int_r^R F(r') K_0(\alpha r') r' dr' \right]} \quad (2.46)$$

To calculate  $k_{eff}$ , 2.44 is evaluated at  $r = 0$  to obtain  $k_{eff}$  in terms of  $\Sigma_a^F(0)$ . The integral in 2.44 is evaluated from 0 to  $R$ , the radial dimensions of the core.  $F(r)$  is also evaluated at  $r = 0$ , which by definition of the relative absorption rate will give us  $F(0) = 1$ . This gives us the following expression:

$$k_{eff} = \frac{\Sigma_a^F(0) \eta \int_0^R F(r') K_0(\alpha r') 2\pi r' dr'}{1 + \int_0^R F(r') K_0(\alpha r') \Sigma_a^F(0) 2\pi r' dr'} \quad (2.47)$$

Substitution of the corresponding Green's function gives the following expression:

$$k_{eff} = \frac{\frac{\Sigma_a^F(0) \eta}{D} \int_0^R F(r') K_0(\alpha r') r' dr'}{1 + \frac{\Sigma_a^F(0)}{D} \int_0^R F(r') K_0(\alpha r') r' dr'} \quad (2.48)$$

## 2.6 Conclusion

Starting with the general one-group neutron diffusion equation, expressions for the fuel distribution and criticality condition were found. Using a chopped cosine allowed a two-dimensional cylindrical coordinate system problem to be reduced to a one dimension problem. The term representing the 2nd dimension was captured as a neutron sink term in the radial dimension. The problem was then solved by using a Green's function, which was found using a closed form representation and a Hankel transform as a consistency check.

In the following chapter a similar approach is carried out for a two-group neutron diffusion model to find  $k_{eff}$  and the fuel distribution.

# CHAPTER 3

## DEVELOPMENT OF TWO GROUP EQUATIONS FOR NEUTRON ANALYSIS FOR ARBITRARY POWER SHAPE

### 3.1 Introduction

This chapter is a continuation of the previous chapter. In the previous chapter neutron analysis was carried out for the one group neutron diffusion equation. In this chapter the same analysis is done for a two-group neutron diffusion equation.

The system's parameters will remain unchanged for this section. This chapter will follow that of the previous chapter in which starting from a the general two-group neutron diffusion equations, an equation will be found that takes the horizontal flux and turns it into a sink term in the radial direction.

Following this, expressions for the Green's functions will be found. This is done by using a Green's function matrix. Finally, expressions for the criticality condition,  $k_{eff}$ , and for the fuel distribution as a function of the radial direction are found.

### 3.2 Derivation of Governing Equations

In following derivation of the governing equations for the system of interest, we start with the general two-group neutron diffusion equations. The two-groups are separated based on neutron energy levels. For the case of a thermal reactor, the fast energy group has neutrons that range from  $10MeV$  to  $.1 - .4ev$ . The slow energy group goes  $.1 - .4ev$  to  $0ev$ . The  $.1 - .4ev$  is chosen as the energy level to split the groups due to the fact that very few neutrons are found in this energy levels.

### 3.2.1 General Two-Group Equations

The equation for the fast group, denoted by group 1, is shown below:

$$\nabla_{\mathbf{r}} \cdot (D_1(\mathbf{r}) \nabla_{\mathbf{r}} \phi_1(\mathbf{r})) - \Sigma_{R1}(\mathbf{r}) \phi_1(\mathbf{r}) + S_1(\mathbf{r}) = 0 \quad (3.1)$$

In this equation  $\phi_1(\mathbf{r})$  represents the fast group scalar flux. The fast group diffusion coefficient is represented by  $D_1(\mathbf{r})$  and the fast group source term is given by  $S_1(\mathbf{r})$ . The term  $\Sigma_{R1}(\mathbf{r})$  gives the fast group removal cross section. This removal term represents the removal of neutrons due to absorption in the fast group and the loss of neutrons that occurs from the fast group due to down scattering into the slow group.

The equation for the slow group is denoted by group 2 and in the case of a thermal reactor this is also called the thermal group. This is given by:

$$\nabla_{\mathbf{r}} \cdot (D_2(\mathbf{r}) \nabla_{\mathbf{r}} \phi_2(\mathbf{r})) - \Sigma_{a2}(\mathbf{r}) \phi_2(\mathbf{r}) + \Sigma_{s(1 \rightarrow 2)}(\mathbf{r}) \phi_1(\mathbf{r}) + S_2(\mathbf{r}) = 0 \quad (3.2)$$

The slow group scalar flux is given by  $\phi_2(\mathbf{r})$ , the slow group diffusion coefficient by  $D_2(\mathbf{r})$  and the slow group source term by  $S_2(\mathbf{r})$ . In the above equation,  $\Sigma_{a2}(\mathbf{r})$  gives the absorption cross section of the slow group. The term  $\Sigma_{s(1 \rightarrow 2)}(\mathbf{r})$  gives the scattering transfer cross section. This represents the probability per unit path length that a neutron with entrance energy in the fast group is scattered and leaves the collision with exit energy in the slow group. This is for both elastic and inelastic scattering.

### 3.2.2 Source Terms

In this section the source terms are given and quantities of interest are explained.

We first start with the fast source term, this is given by:

$$S_1(\mathbf{r}) = S_1^{ext}(\mathbf{r}) + \frac{\chi_1}{k_{eff}} [\langle \nu \Sigma_f \rangle_1 \phi_1(\mathbf{r}) + \langle \nu \Sigma_f \rangle_2 \phi_2(\mathbf{r})] \quad (3.3)$$

The slow source term is given by:

$$S_2(\mathbf{r}) = S_2^{ext}(\mathbf{r}) + \frac{\chi_2}{k_{eff}} [\langle \nu \Sigma_f \rangle_1 \phi_1(\mathbf{r}) + \langle \nu \Sigma_f \rangle_2 \phi_2(\mathbf{r})] \quad (3.4)$$



In the above equations,  $\langle \nu \Sigma_f \rangle_1$  represents the average of the product  $\nu \Sigma_f$  over the entire fast group. Similarly,  $\langle \nu \Sigma_f \rangle_2$  represents the same average for the slow group.

The fission spectrum constant for either group is given by  $\chi_i$  where  $i = 1$  for the fast group and  $i = 2$  for the slow group. To understand what this quantity represents, consider the expression  $X(E)dE$  which represents the probability that a neutron born in fission has an energy between  $E$  and  $dE$ . If we integrate this quantity between the bounds for the given energy group, the integral results in the fission spectrum constant for that group. [4]

Since  $\chi_i$  represents a probability, this quantity is bound between 0 and 1. The value of the fission spectrum constant will depend on the type of reactor that is being analyzed. In the case of a fast reactor, the value for  $\chi_1$  and  $\chi_2$  will each be less than 1, and a constraint is given that  $\chi_1 + \chi_2 = 1$ . In the case of a thermal reactor,  $\chi_1 = 1.0$  and  $\chi_2 = 0$ . This says that all the neutrons are born in the fast group. Since the system we are analyzing is a thermal reactor, from here on out the values for the fission spectrum constants will be those of a thermal reactor.

In this analysis we will also assume that there are no external source terms for the fast group or slow group.

### 3.2.3 Further Assumptions

The following assumptions will also be made in order to simplify the analysis. The first that will be made is that the diffusion coefficients for both groups are given by that of the pure moderator. Furthermore, it is also assumed to be independent of position.

The removal of neutrons from the fast group is also assumed to be due to scattering off the moderator only. This is given by:

$$\Sigma_{R1}^M + \Sigma_{R1}^F = \Sigma_{R1}^M \quad (3.5)$$

A further assumption made in setting up the neutron diffusion equations is that there is a small amount of fast fission. This implies the following:

$$\langle \nu \Sigma_f \rangle_1 \phi_1(\mathbf{r}) + \langle \nu \Sigma_f \rangle_2 \phi_2(\mathbf{r}) = \langle \nu \Sigma_f \rangle_2 \phi_2(\mathbf{r}) \quad (3.6)$$

The last assumption made is that the scattering transfer cross section is due to scattering off the moderator only.

With all the assumptions, the neutron diffusion equation for the fast group becomes:

$$\nabla_{\mathbf{r}} \cdot (D_1 \nabla_{\mathbf{r}} \phi_1(\mathbf{r})) - \Sigma_{R1}^M(\mathbf{r}) \phi_1(\mathbf{r}) + \frac{1}{k_{eff}} < \nu \Sigma_f >_2 \phi_2(\mathbf{r}) = 0 \quad (3.7)$$

For the slow group, after expanding out the absorption term for the fuel and moderator becomes:

$$\nabla_{\mathbf{r}} \cdot (D_2 \nabla_{\mathbf{r}} \phi_2(\mathbf{r})) - \Sigma_{a2}^M(\mathbf{r}) \phi_1(\mathbf{r}) - \Sigma_{a2}^F(\mathbf{r}) \phi_1(\mathbf{r}) + \Sigma_{s(1 \rightarrow 2)}(\mathbf{r}) \phi_2(\mathbf{r}) = 0 \quad (3.8)$$

Expanding the Laplacian operator that appears in 3.7 and 3.8 in cylindrical coordinates gives us the following set of equations:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \phi_1(r, z) \right) + \frac{\partial^2}{\partial z^2} \phi_1(r, z) - \frac{\Sigma_{R1}^M(r, z)}{D_1^M} \phi_1(r, z) + \frac{1}{k_{eff}} \frac{\nu \Sigma_{f2}}{D_1^M} \phi_2(r, z) = 0 \quad (3.9)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \phi_2(r, z) \right) + \frac{\partial^2}{\partial z^2} \phi_2(r, z) - \frac{\Sigma_{a2}^M}{D_2^M} \phi_1(r, z) + \frac{\Sigma_{(1 \rightarrow 2)}}{D_2^M} \phi_1(r, z) \frac{\Sigma_{a2}^F}{D_2^M} \phi_1(r, z) = 0 \quad (3.10)$$

Again, as in the one group case, the equations for the neutron diffusion are second order differential equations. However, this time the equations are coupled. The boundary conditions remain the same as in the one-group case with the same justifications.

### 3.2.4 Chopped Cosine Substitution and Green's Function Matrix

Using the same idea as in the one-group case, the flux is assumed to be separable in the radial and vertical directions for both the slow group and fast group. In the horizontal direction a chopped cosine flux distribution is used. A further concept is needed that was not required in the one-group case. For any multi-group neutron diffusion equation, the neutron flux shape is the same for all energy levels. With this in mind, the flux is now given by:

$$\phi_1(r, z) = \phi_1(r) \cos \left( \frac{\pi z}{H + 2\delta} \right) \quad (3.11)$$

$$\phi_2(r, z) = \phi_2(r) \cos \left( \frac{\pi z}{H + 2\delta} \right) \quad (3.12)$$

In the above equations, just as the one-group case, it is assumed that the horizontal flux goes to 0 at a distance  $H + \delta$ , where  $\delta$  is the reflector diffusion length. The above substitutions for the flux allow the horizontal term to be transformed into an absorption term in the radial direction.

Calculating the derivatives and canceling out the  $\cos \left( \frac{\pi z}{H + 2\delta} \right)$  that appears in each terms gives us the following equation for the fast flux:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_1(r) \right) - \left( \frac{\pi}{H + 2\delta} \right)^2 \phi_1(r) - \frac{\Sigma_{R1}^M}{D_1^M} \phi_1(r) + \frac{1}{k_{eff}} \frac{\nu \Sigma_{f2}(r)}{D_1^M} \phi_2(r) = 0 \quad (3.13)$$

Through the use of the chopped cosine flux shape and the use of separable fluxes has allowed the flux in the horizontal direction to become an absorption term in the radial direction. Defining the absorption terms as:

$$\alpha_1^2 = \left[ \left( \frac{\pi}{H + 2\delta} \right)^2 + \frac{\Sigma_{R1}^M}{D_1^M} \right] \quad (3.14)$$

Allows the equation for the fast flux to be simplified to the following:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_1(r) \right) - \alpha_1^2 \phi_1(r) + \frac{1}{k_{eff}} \frac{\nu \Sigma_{f2}(r)}{D_1^M} \phi_2(r) = 0 \quad (3.15)$$

Following a similar procedure for the slow flux gives us the following equation:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_2(r) \right) - \alpha_2^2 \phi_2(r) + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} \phi_1(r) - \frac{\Sigma_{a2}^F}{D_2^M} \phi_2(r) = 0 \quad (3.16)$$

Where  $\alpha_2^2$  is defined as:

$$\alpha_2^2 = \left[ \left( \frac{\pi}{H + 2\delta} \right)^2 + \frac{\Sigma_{a2}^M}{D_2^M} \right] \quad (3.17)$$

The last step needed is to write the equations in terms of the fuel absorption rate. We define this quantity by  $A(r) = \Sigma_{a2}^F(r) \phi_2(r)$ . In the equation for the fast flux we also rewrite the last term in terms of the fuel regeneration factor which is defined by  $\eta = \frac{\nu \Sigma_{f2}(r)}{\Sigma_{a2}^F}$ . Using these final substitutions we have the

two-group neutron diffusion equations in a simplified form:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_1(r) \right) - \alpha_1^2 \phi_1(r) + \frac{\eta}{D_1^M k_{eff}} A(r) = 0 \quad (3.18)$$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_2(r) \right) - \alpha_2^2 \phi_2(r) + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} \phi_1(r) - \frac{A(r)}{D_2^M} = 0 \quad (3.19)$$

These coupled inhomogeneous differential equations will be solved using a Green's function matrix. To begin we define  $K_{ij}(r|r')$  as the flux in group  $i$  at point  $r$  due to a unit source in group  $j$  at point  $r'$ . With this in mind the corresponding Green's function matrix is:

$$\begin{bmatrix} K_{11}(r|r') & 0 \\ K_{21}(r|r') & K_{22}(r|r') \end{bmatrix} \quad (3.20)$$

The equation for the fast group in terms of the Green's function will be defined by:

$$\phi_1(r) = \frac{\eta}{k_{eff}} \int K_{11}(r|r') A(r') 2\pi r' dr' \quad (3.21)$$

For the slow group:

$$\phi_2(r) = \frac{\eta}{k_{eff}} \int K_{21}(r|r') A(r') 2\pi r' dr' - \int K_{22}(r|r') A(r') 2\pi r' dr' \quad (3.22)$$

### 3.3 Derivation of Greens Function Matrix Equations

In this section, using the definitions for the fluxes found in 3.21 and 3.50 the corresponding equations for the Green's functions will be found.

To start the process, we first introduce linear operators for the differential equations for the flux. These are defined below:

$$\hat{L}_1 \phi_1(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_1(r) \right) - \alpha_1^2 \phi_1(r) \quad (3.23)$$

$$\hat{L}_2 \phi_2(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \phi_2(r) \right) - \alpha_2^2 \phi_2(r) \quad (3.24)$$

Then in terms of the linear operators, the differential equations for  $\phi_1(r)$  and

$\phi_2(r)$  become

$$\hat{L}_1\phi_1(r) + \frac{\eta}{D_1^M k_{eff}} A(r) = 0 \quad (3.25)$$

$$\hat{L}_2\phi_2(r) + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} \phi_1(r) - \frac{A(r)}{D_2^M} = 0 \quad (3.26)$$

The first corresponding Green's function that will be found is for  $K_{11}(r|r')$ . The flux as defined in 3.21 is inserted into equation 3.25. This gives:

$$\frac{\eta}{k_{eff}} \hat{L}_1 \int K_{11}(r|r') A(r') 2\pi r' dr' + \frac{\eta}{D_1^M k_{eff}} A(r) = 0 \quad (3.27)$$

Switching integration and the operator and canceling  $\frac{\eta}{k_{eff}}$  gives us:

$$\int \hat{L}_1 K_{11}(r|r') A(r') 2\pi r' dr' + \frac{1}{D_1^M} A(r) = 0 \quad (3.28)$$

In order for the above equation to be satisfied, the integral must equal  $-\frac{1}{D_1^M} A(r)$ . In order for this to be satisfied, the operator acting on  $K_{11}(r|r')$  must equal  $\frac{\delta(r-r')}{2\pi D_1^M r'}$ . This leads to  $K_{11}(r|r')$  being found by:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} K_{11}(r|r') \right) - \alpha_1^2 K_{11}(r|r') + \frac{\delta(r-r')}{2\pi r D_1^M} = 0 \quad (3.29)$$

To find the remaining equations for the Green's function, we again substitute the equation for the fluxes given in 3.21 and 3.50 into 3.26. This gives us the following:

$$\begin{aligned} \int \hat{L}_2 K_{21}(r|r') \frac{\eta}{D_1^M k_{eff}} A(r) 2\pi r' dr' - \int \hat{L}_2 K_{22}(r|r') \frac{A(r)}{D_2^M} 2\pi r' dr' + \\ + \int \Sigma_{s(1 \rightarrow 2)} K_{11}(r|r') \frac{\eta}{D_1^M k_{eff}} A(r) 2\pi r' dr' - \frac{A(r)}{D_2^M} = 0 \end{aligned} \quad (3.30)$$

To find  $K_{22}(r|r')$  we require that the following equation to be satisfied:

$$\int \hat{L}_2 K_{22}(r|r') \frac{A(r)}{D_2^M} 2\pi r' dr' + \frac{A(r)}{D_2^M} = 0 \quad (3.31)$$

As in the case for  $K_{11}(r|r')$  we set the integral equal to  $-\frac{1}{D_2^M} A(r)$ . Using the

Dirac delta function as in for  $K_{11}(r|r')$  we get the following:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} K_{22}(r|r') \right) - \alpha_2^2 K_{22}(r|r') + \frac{\delta(r-r')}{2\pi r D_2^M} = 0 \quad (3.32)$$

Setting the terms left over terms equal to 0 gives us the following, after simplifying and switching  $\hat{L}_2$  with integration.

$$\int \left[ \hat{L}_2 K_{21}(r|r') + \Sigma_{s(1 \rightarrow 2)} K_{11}(r|r') \right] \frac{A(r)}{D_2^M} 2\pi r' dr' \quad (3.33)$$

The term inside the integral gives us the equation for  $K_{21}(r|r')$ :

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} K_{21}(r|r') \right) - \alpha_2^2 K_{21}(r|r') + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} K_{11}(r|r') = 0 \quad (3.34)$$

## 3.4 Derivation of Green's Function

In the following section, expressions for the Green's functions,  $K_{11}(r|r')$ ,  $K_{22}(r|r')$ , and  $K_{21}(r|r')$  will be found using Hankel transforms. The assumptions made in the derivation are that there is an infinite reflector in which the core moderator has the same properties as the reflector. This is the same technique that was used in the one-group case.

### 3.4.1 Solution via Hankel Transform

We will first solve the Green's function for  $K_{11}(r|r')$ . The equation is presented below:

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} K_{11}(r|r') \right) - \alpha_1^2 K_{11}(r|r') + \frac{\delta(r-r')}{2\pi r D_1^M} = 0 \quad (3.35)$$

The Hankel transform for  $K_{11}(r|r')$  is defined by:

$$K_{11}(k|r') = \int_0^\infty J_0(kr) K_{11}(r|r') r dr \quad (3.36)$$

and the inverse transform as:

$$K_{11}(r|r') = \int_0^\infty J_0(kr) K_{11}(k|r') k dk \quad (3.37)$$

Taking the transform of 3.35 and solving for  $K_{11}(k|r')$  gives us the following expression for the Green's function in the transform space:

$$K_{11}(k|r') = \frac{1}{2\pi D_1^M} \frac{J_0(r'k)}{k^2 + \alpha_1^2} \quad (3.38)$$

Finally taking the inverse transform of 3.38 and solving the resulting integration gives us the following solution for the Green's function:

$$K_{11}(r|r') = \begin{cases} \frac{1}{2\pi D_1^M} K_0(\alpha_1 r') I_0(\alpha_1 r) & 0 \leq r \leq r' \\ \frac{1}{2\pi D_1^M} I_0(\alpha_1 r') K_0(\alpha_1 r) & r' \leq r \leq \infty \end{cases} \quad (3.39)$$

The equation for  $K_{22}(r|r')$  is found using the same method by defining the Hankel transform and inverse transform for  $K_{22}(r|r')$  instead of for  $K_{11}(r|r')$ . The solution for  $K_{22}(r|r')$  is given below:

$$K_{22}(r|r') = \begin{cases} \frac{1}{2\pi D_2^M} K_0(\alpha_2 r') I_0(\alpha_2 r) & 0 \leq r \leq r' \\ \frac{1}{2\pi D_2^M} I_0(\alpha_2 r') K_0(\alpha_2 r) & r' \leq r \leq \infty \end{cases} \quad (3.40)$$

It remains to find the expression for  $K_{21}(r|r')$ . We define the Hankel transform and inverse transform as:

$$K_{21}(k|r') = \int_0^\infty J_0(kr) K_{21}(r|r') r dr \quad (3.41)$$

and:

$$K_{21}(r|r') = \int_0^\infty J_0(kr) K_{21}(k|r') k dk \quad (3.42)$$

The above definitions for the Hankel transform are applied to the following equation for  $K_{21}(r|r')$ :

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} K_{21}(r|r') \right) - \alpha_2^2 K_{21}(r|r') + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} K_{11}(r|r') = 0 \quad (3.43)$$

with the Hankel transform previously defined for  $K_{11}(r|r')$  to give the following:

$$(-1) (k^2 + \alpha_2^2) K_{21}(k|r') + \frac{\Sigma_{s(1 \rightarrow 2)}}{D_2^M} K_{11}(k|r') = 0 \quad (3.44)$$

Substitution of  $K_{11}(k|r')$  as found above and solving for  $K_{21}(r|r')$  gives:

$$K_{21}(k|r') = \frac{\Sigma_{s(1 \rightarrow 2)}}{2\pi D_1^M D_2^M} \frac{J_0(r'k)}{(k^2 + \alpha_2^2)(k^2 + \alpha_1^2)} \quad (3.45)$$

To find the inverse transform, partial fractions are used to simplify the resulting integral. The following change is made:

$$\frac{1}{(k^2 + \alpha_2^2)(k^2 + \alpha_1^2)} = \left( \frac{1}{k^2 + \alpha_1^2} - \frac{1}{k^2 + \alpha_2^2} \right) \left( \frac{1}{\alpha_2^2 - \alpha_1^2} \right) \quad (3.46)$$

With the above substitution the expression for  $K_{21}(r|r')$  becomes:

$$K_{21}(k|r') = \frac{1}{(\alpha_2^2 - \alpha_1^2)} \frac{\Sigma_{s(1 \rightarrow 2)}}{2\pi D_1^M D_2^M} J_0(r'k) \left( \frac{1}{k^2 + \alpha_1^2} - \frac{1}{k^2 + \alpha_2^2} \right) \quad (3.47)$$

This expression shows why partial fractions were used to expand the expression for  $K_{21}(k|r')$ . Through use of partial fractions the resulting transform will be two transforms that are the same as those found for  $K_{11}(r|r')$  and  $K_{22}(r|r')$  with the exception of the constants being different. This allows us to immediately write the expression for  $K_{21}(r|r')$  as:

$$K_{21}(r|r') = \frac{\Sigma_{s(1 \rightarrow 2)}}{2\pi D_1^M D_2^M} \frac{1}{(\alpha_2^2 - \alpha_1^2)} [K_0(\alpha_1 r') I_0(\alpha_1 r) - I_0(\alpha_2 r) K_0(\alpha_2 r')] \quad (3.48)$$

for  $0 \leq r \leq r'$ , and

$$K_{21}(r|r') = \frac{\Sigma_{s(1 \rightarrow 2)}}{2\pi D_1^M D_2^M} \frac{1}{(\alpha_2^2 - \alpha_1^2)} [I_0(\alpha_1 r') K_0(\alpha_1 r) - K_0(\alpha_2 r) I_0(\alpha_2 r')] \quad (3.49)$$

for  $r' \leq r \leq \infty$ .

### 3.5 Expression for Criticality Factor and Fuel Distribution

In this section expressions for  $k_{eff}$  and for the fuel distribution are found for an arbitrary power shape. As in the case for the one-group diffusion equation, the power shape is proportional to the relative fuel absorption rate which is denoted by  $F(r)$ . We start with the equation for the slow flux in



terms of Green's functions. This is given below:

$$\phi_2(r) = \frac{\eta}{k_{eff}} \int K_{21}(r|r')A(r')2\pi r' dr' - \int K_{22}(r|r')A(r')2\pi r' dr' \quad (3.50)$$

Where the fuel absorption rate has been defined as  $A(r) = \Sigma_{a2}^F(r)\phi_2(r)$ .

We multiply the above equation by  $\Sigma_{a2}^F(r)$  to get:

$$\begin{aligned} A(r) &= \Sigma_{a2}^F(r)\phi_2(r) = \\ &= \Sigma_{a2}^F(r)\frac{\eta}{k_{eff}} \int K_{21}(r|r')A(r')2\pi r' dr' - \Sigma_{a2}^F(r) \int K_{22}(r|r')A(r')2\pi r' dr' \end{aligned} \quad (3.51)$$

We now define the fuel absorption rate by:

$$A(r) = A_0 F(r) \quad (3.52)$$

Where  $F(r)$  is the relative absorption rate in the fuel with  $F(0) = 1.0$ .  $A_0$  will be given by  $A_0 = \Sigma_{a2}^F(0)\phi_2(0)$ . That is to say that  $A_0$  gives the fuel absorption rate at the center of the core. Making the above substitutions in equation 3.51 gives us the following:

$$A_0 \left[ F(r) - \Sigma_{a2}^F(r)\frac{\eta}{k_{eff}} \int K_{21}(r|r')F(r')2\pi r' dr' - \Sigma_{a2}^F(r) \int K_{22}(r|r')F(r')2\pi r' dr' \right] \quad (3.53)$$

In a subcritical reactor  $A_0 = 0$ . Since the goal is to perform criticality analysis, the assumption is made that the reactor is critical and that  $A_0 \neq 0$ . Canceling out  $A_0$  gives us the relative absorption rate in the fuel in terms of the Green's functions. This is presented below:

$$F(r) = \Sigma_{a2}^F(r)\frac{\eta}{k_{eff}} \int K_{21}(r|r')F(r')2\pi r' dr' - \Sigma_{a2}^F(r) \int K_{22}(r|r')F(r')2\pi r' dr' \quad (3.54)$$

The first quantity that will be solved for is the criticality condition given by  $k_{eff}$ . To do this we start with equation 3.54 and evaluate it at  $r = 0$ . This will give  $k_{eff}$  in terms of  $\Sigma_{a2}^F(0)$ . The integral in 3.54 is evaluated from 0 to  $R$ .  $F(r)$  is also evaluated at  $r = 0$ , which corresponds to  $F(0) = 1$  by definition. Solving 3.54 with the aforementioned conditions for  $k_{eff}$  results

in the following expression:

$$k_{eff} = \frac{\Sigma_{a2}^F(0)\eta \int_0^R F(r')K_{21}(0|r')2\pi r' dr'}{1 + \Sigma_{a2}^F(0) \int_0^R F(r')K_{22}(0|r')2\pi r' dr'} \quad (3.55)$$

Substituting the corresponding Green's function as found in section 3.4 into the above equation gives us the following expression for the criticality factor after simplifying:

$$k_{eff} = \frac{\frac{\Sigma_{a2}^F(0)\eta \Sigma_{s(1 \rightarrow 2)}}{D_1^M D_2^M (\alpha_2^2 - \alpha_1^2)} \int_0^R F(r') [K_0(\alpha_1 r') - K_0(\alpha_2 r')]}{1 + \frac{\Sigma_{a2}^F(0)}{D_2^M} \int_0^R F(r') K_0(\alpha_2 r') r' dr'} \quad (3.56)$$

This equation will be used in subsequent chapters to find the criticality condition for a parabolic and constant power shape.

We now solve 3.54 for the fuel distribution as a function of the relative absorption rate. This gives the following expression:

$$\Sigma_{a2}^F(r) = \frac{F(r)}{\frac{\eta}{k_{eff}} \int F(r') K_{21}(r|r') 2\pi r' dr' - \int F(r') K_{22}(r|r') 2\pi r dr'} \quad (3.57)$$

Substitution of the appropriate Green's function gives the following:

$$\Sigma_{a2}^F(r) = \frac{F(r)}{\frac{\eta}{k_{eff}} \frac{1}{D_1 D_2} \frac{\Sigma_{s(1 \rightarrow 2)}}{\alpha_2^2 - \alpha_1^2} BI_{21}(r) - \frac{1}{D_2} BI_{22}(r)} \quad (3.58)$$

where  $BI_{12}(r)$  is a function containing the Bessel function integrals that come from  $K_{21}(r|r')$  and is defined by:

$$\begin{aligned} BI_{21}(r) = & K_0(\alpha_1 r) \int_0^r I_0(\alpha_1 r') F(r') r' dr' - K_0(\alpha_2 r) \int_0^r I_0(\alpha_2 r') F(r') r' dr' + \\ & + I_0(\alpha_1 r) \int_r^R K_0(\alpha_1 r') F(r') r' dr' - I_0(\alpha_2 r) \int_r^R K_0(\alpha_2 r') F(r') r' dr' \end{aligned} \quad (3.59)$$

and  $BI_{22}(r)$  is a function that results from the Bessel function integrals that come from  $K_{22}(r|r')$  and is defined below:

$$BI_{22}(r) = K_0(\alpha_2 r) \int_0^r I_0(\alpha_2 r') F(r') r' dr' + I_0(\alpha_2 r) \int_r^R K_0(\alpha_2 r') F(r') r' dr' \quad (3.60)$$

### 3.6 Conclusion

In this chapter expressions for the criticality condition and fuel distribution as functions of the relative absorption rate have been found starting from the general two-group neutron diffusion equations. The same procedure was carried out as in the one-group neutron diffusion case with the exception that a Green's function matrix was used for the two-group case. Using Hankel transforms the appropriate Green's function was found. The two-group case was important to find because it represents the minimum number of groups necessary to capture the physics of the problem.

In the remainder of this thesis the expressions for  $k_{eff}$  and the fuel distribution that were found for the one group and two group case will be used to get specific expressions for a flat power shape and a parabolic power shapes.

# CHAPTER 4

## NEUTRON ANALYSIS FOR SPECIFIED POWER SHAPES

### 4.1 Introduction

In this chapter we specify the power shape by specifying the relative absorption rate,  $F(r)$  in the expressions for  $k_{eff}$  and the fuel absorption rate. The analysis will first be done for a constant power shape,  $F(r) = 1$  for both groups and for a parabolic power shape, again for both groups. In each case, a model graphite reactor will be used with varying reactor parameters, such as the radius and height.

Various integrals involving the modified Bessels function appear throughout the chapter. The evaluation of these integrals appears in the Appendix.

### 4.2 Flat Power Shape for One-Group

In this section, we specify the relative absorption rate that corresponds to a flat power shape. After obtaining equations for the criticality condition and fuel distribution, a model reactor is specified and results are plotted.

#### 4.2.1 Specifying Flat Power Shape

We start by repeating the equations for the fuel distribution,  $\Sigma_a^F(R)$ , and the criticality condition,  $k_{eff}$ , as were solved in Chapter 2. These are repeated below:

$$\Sigma_a^F(r) = \frac{F(r)}{\frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) \left[ K_0(\alpha r) \int_0^r F(r') I_0(\alpha r') r' dr' + I_0(\alpha r) \int_r^R F(r') K_0(\alpha r') r' dr' \right]} \quad (4.1)$$

$$k_{eff} = \frac{\frac{\Sigma_a^F(0)\eta}{D} \int_0^R F(r') K_0(\alpha r') r' dr'}{1 + \frac{\Sigma_a^F(0)}{D} \int_0^R F(r') K_0(\alpha r') r' dr'} \quad (4.2)$$

A flat power shape will correspond to the relative absorption rate being equal to 1.0. This means:  $F(r) = 1.0$ . Substitution of this into the above equations gives the following after evaluation of the integrals and simplifying:

$$\Sigma_a^F(r) = \frac{1}{\frac{1}{D\alpha^2} \left( \frac{\eta}{k_{eff}} - 1 \right) [1 - \alpha R I_0(\alpha r) K_1(\alpha R)]} \quad (4.3)$$

$$k_{eff} = \frac{\frac{\Sigma_a^F(0)\eta}{D\alpha^2} [1 - \alpha R K_1(\alpha R)]}{1 + \frac{\Sigma_a^F(0)}{D\alpha^2} [1 - \alpha R K_1(\alpha R)]} \quad (4.4)$$

It is of interest to note that  $k_{eff}$  is independent of position and that only depends on  $\alpha$ , the reactor radius, and the fuel distribution at the center of the core. This is as expected as the criticality condition typically depends only on the physical dimensions and characteristics of the reactor.

The approach that will be taken to solve for the fuel distribution will be to specify  $k_{eff}$  in the above equation and solve for the fuel distribution at the center of the core. Doing so gives us the following equation:

$$\Sigma_a^F(0) = \frac{k_{eff} D \alpha^2}{[1 - \alpha R K_1(\alpha R)] [\eta - k_{eff}]} \quad (4.5)$$

We will graph the fuel absorption relative to that at the center of the core. To do this we divide  $\Sigma_a^F(r)$  by  $\Sigma_a^F(0)$ . Doing this yields the following:

$$\frac{\Sigma_a^F(r)}{\Sigma_a^F(0)} = \frac{1 - \alpha R K_1 \alpha R}{1 - \alpha R I_0(\alpha r) K_1(\alpha R)} \quad (4.6)$$

#### 4.2.2 Model Reactor for Flat Power

In this section we will model a graphite moderated reactor. The physical values of interest are shown in Table 4.2.2.

The extrapolation length will be calculated by taking the diffusion coefficient and multiplying it by 2.13 for this model reactor. This comes from simplifications from transport theory for a free surface.

We will assume we are working with a  $U^{235}$  fueled reactor. With this assumption, for the average number of fission neutrons emitted per neutron

Table 4.1: One Group Neutron Diffusion Parameters for Graphite at  $20^{\circ}C^a$

Property	Value
$D$	$0.84cm$
$\Sigma_a^M$	$2.4 \times 10^{-4} \frac{1}{cm}$

<sup>a</sup>Values from Introduction to Nuclear Reactor Theory for highly-purified graphite.[?]

absorbed,  $\eta$ , we use the value of 2.0651. It should be noted that this is for  $20^{\circ}C$  assuming a pure Maxwellian distribution in temperature.

We will first model a reactor that is typical for power production. We set the height to  $360cm$  and diameter to  $360cm$  as well. This is slightly under 12 feet. Using these values, we get a value for  $\alpha$  of  $0.0241 \frac{1}{cm}$ .

The approach we will take to find the fuel distribution is to find the value of  $\Sigma_a^F(0)$  that corresponds to a just critical reactor. We will set  $k_{eff} = 1$ . Using this we solve arrive at a value for  $\Sigma_a^F(0)$  of  $.000474 \frac{1}{cm}$ . Using this value we can now plot the fuel distribution with respect to the fuel distribution at the center as a function of position. This is shown below:

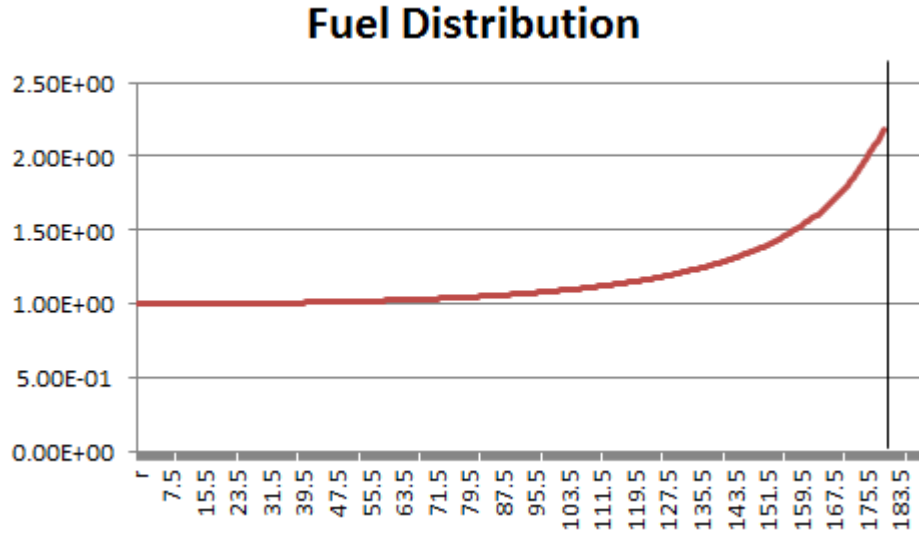


Figure 4.1: Fuel Distribution due to Flat Power Shape for One Group

This graph shows the expected physical result. The fuel absorption at the boundary is 2.18 times greater than that of the center. In order to flatten out the power shape there is an associated cost at the boundary, where the neutron leakage is increased.

### 4.3 Flat Power Shape for Two Group

In this section, the analysis produced in 4.2 is repeated for the two-group neutron diffusion equations.

#### 4.3.1 Specifying Flat Power Shape

The equation for the critical condition,  $k_{eff}$  and the fuel distribution,  $\Sigma_{a2}^F(r)$  are repeated below. These were solved in Chapter 3.

$$k_{eff} = \frac{\frac{\Sigma_{a2}^F(0)\eta\Sigma_{s(1\rightarrow2)}}{D_1^M D_2^M (\alpha_2^2 - \alpha_1^2)} \int_0^R F(r') [K_0(\alpha_1 r') - K_0(\alpha_2 r')] dr'}{1 + \frac{\Sigma_{a2}^F(0)}{D_2^M} \int_0^R F(r') K_0(\alpha_2 r') r' dr'} \quad (4.7)$$

$$\Sigma_{a2}^F(r) = \frac{F(r)}{\frac{\eta}{k_{eff}} \frac{1}{D_1 D_2} \frac{\Sigma_{s(1\rightarrow2)}}{\alpha_2^2 - \alpha_1^2} BI_{21}(r) - \frac{1}{D_2} BI_{22}(r)} \quad (4.8)$$

where  $BI_{12}(r)$  and  $BI_{22}(r)$  are defined by:

$$BI_{21}(r) = K_0(\alpha_1 r) \int_0^r I_0(\alpha_1 r') F(r') r' dr' - K_0(\alpha_2 r) \int_0^r I_0(\alpha_2 r') F(r') r' dr' + I_0(\alpha_1 r) \int_r^R K_0(\alpha_1 r') F(r') r' dr' - I_0(\alpha_2 r) \int_r^R K_0(\alpha_2 r') F(r') r' dr' \quad (4.9)$$

$$BI_{22}(r) = K_0(\alpha_2 r) \int_0^r I_0(\alpha_2 r') F(r') r' dr' + I_0(\alpha_2 r) \int_r^R K_0(\alpha_2 r') F(r') r' dr' \quad (4.10)$$

A flat power shape will again correspond to the relative absorption rate being equal to 1.0. Therefore, Substituting  $F(r) = 1$  into the equations for  $k_{eff}$  and  $\Sigma_{a2}^F(r)$  give the following expressions after simplifying.

$$k_{eff} = \frac{\frac{\Sigma_{a2}^F(0)\eta\Sigma_{s(1\rightarrow2)}}{D_1 D_2 (\alpha_2^2 - \alpha_1^2)} \left[ \left( \frac{1}{\alpha_1^2} - \frac{\alpha_1 R}{\alpha_1^2} K_1(\alpha_1 R) \right) - \left( \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} K_1(\alpha_2 R) \right) \right]}{1 + \frac{\Sigma_{a2}^F(0)}{D_2} \left[ \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} K_1(\alpha_2 R) \right]} \quad (4.11)$$

$$\Sigma_{a2}^F(r) = \frac{1}{\frac{\eta\Sigma_{s(1\rightarrow2)}}{k_{eff} D_1 D_2 (\alpha_2^2 - \alpha_1^2)} BI_{21}(r) - \frac{1}{D_2} BI_{22}(r)} \quad (4.12)$$

Table 4.2: Two Group Neutron Diffusion Parameters for Graphite at  $20^\circ C^a$

Property	Value
$D_1^M$	$1.016cm$
$\Sigma_{R1}^M$	$2.76 \times 10^{-3} \frac{1}{cm}$
$\Sigma_{s(1 \rightarrow 2)}^M$	$2.76 \times 10^{-3} \frac{1}{cm}$
$D_2^M$	$.84cm$
$\Sigma_{a2}^M$	$2.4 \times 10^{-4} \frac{1}{cm}$

<sup>a</sup>Values from Introduction to Nuclear Reactor Theory for highly-purified graphite.[?]

Where  $BI_{21}(r)$  and  $BI_{22}(r)$  have been simplified to the following:

$$BI_{21}(r) = \left( \frac{1}{\alpha_1^2} - \frac{\alpha_1 R}{\alpha_1^2} I_0(\alpha_1 r) K_1(\alpha_1 R) \right) - \left( \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} I_0(\alpha_2 r) K_1(\alpha_2 R) \right) \quad (4.13)$$

$$BI_{22}(r) = \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} I_0(\alpha_2 r) K_1(\alpha_2 R) \quad (4.14)$$

The same approach will be taken that was taken in the one-group case. We will specify that  $k_{eff} = 1$ . Therefore, the above expression must be solved for  $\Sigma_{a2}^F(0)$ . Doing so gives the following below.

$$\Sigma_{a2}^F(0) = \frac{k_{eff} D_2}{\frac{\Sigma_{s(1 \rightarrow 2)} \eta}{D_1(\alpha_2^2 - \alpha_1^2)} B - k_{eff} C} \quad (4.15)$$

Where  $B$  and  $C$  are equal to the following.

$$B = \left( \frac{1}{\alpha_1^2} - \frac{\alpha_1 R}{\alpha_1^2} K_1(\alpha_1 R) \right) - \left( \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} K_1(\alpha_2 R) \right) \quad (4.16)$$

$$C = \frac{1}{\alpha_2^2} - \frac{\alpha_2 R}{\alpha_2^2} K_1(\alpha_2 R) \quad (4.17)$$

### 4.3.2 Model Reactor For Flat Shape Two Group

We will repeat the analysis that was done for the one-group case with the same graphite moderated reactor. The physical values of interest are shown in Table 4.3.2.

Again we will assume that we are working with a  $U^{235}$  fueled reactor. The value for  $\eta$  will be 2.0651 at  $20^\circ C$  in a pure Maxwellian distribution in temperature. The same dimensions will be used as in the one-group case.



This is a height of  $360\text{cm}$  and diameter of  $360$ .

Using the values given, we get  $\alpha_1 = .054\frac{1}{\text{cm}}$  and  $\alpha_2 = .0241\frac{1}{\text{cm}}$ . Setting  $k_{eff} = 1$  for a just critical reactor gives  $\Sigma_{a2}^F(0) = .000596\frac{1}{\text{cm}}$ . This value is about 25

We can now plot the fuel distribution with respect to the fuel distribution at the center as a function of position. This is shown below:

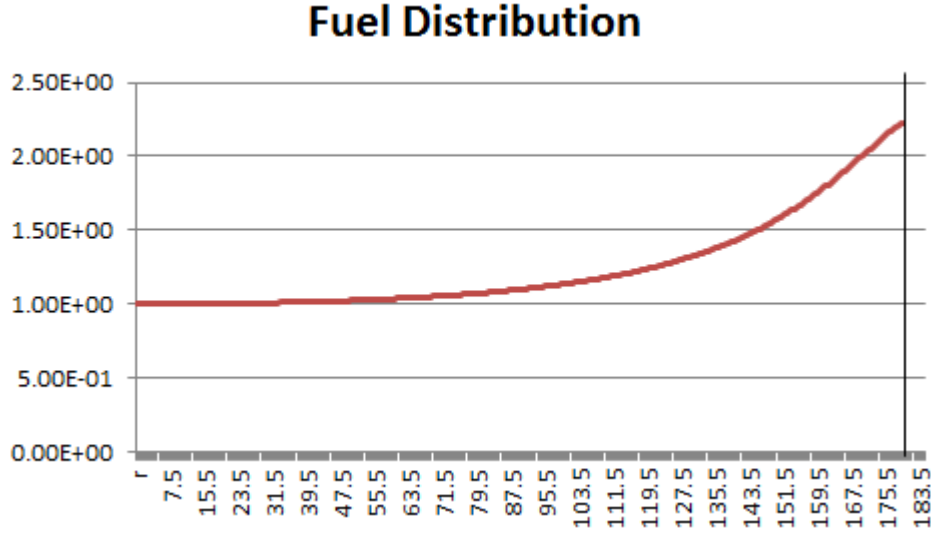


Figure 4.2: Fuel Distribution due to Flat Power Shape for Two Group

The two-group diffusion equations capture the same physical affect that the one-group diffusion equations captured. The fuel absorption at the boundary has increased to 2.23 times that of the center. This is slightly greater than the one-group case in which this was only 2.18 times. However, the overall behavior is the same and flattening the power shape increased the neutron leakage at the boundary.

#### 4.4 Parabolic Power Shape for One-Group

In this section, the power specified power shape is of the parabolic type. Equations for the criticality condition and the fuel distributions are obtained. Following this a model reactor is analyzed.

#### 4.4.1 Specifying the Parabolic Power Shape

The equations for the fuel distribution and criticality were found in Chapter 2. These are equations 4.1 and 4.2, which were shown earlier in this chapter. Key steps found in the evaluation of the integrals can be found in the appendix.

The parabolic power shape will correspond to the relative absorption rate also being parabolic. We will set the relative absorption rate to:

$$F(r) = 1 - (1 - \epsilon) \left( \frac{r}{R} \right)^2 \quad (4.18)$$

Here  $\epsilon$  is between 0 and 1 and controls the curvature of the parabola.  $\epsilon = 1$  will return the flat power shape, while  $\epsilon = 0$  will give the maximum curvature for this shape.

Substituting this into the equation for criticality, equation 4.2 gives us the following equation:

$$k_{eff} = \frac{\frac{\Sigma_a^F(0)\eta}{D} A}{1 + \frac{\Sigma_a^F(0)}{D} A} \quad (4.19)$$

Where  $A$  is a constant that is obtained by evaluating the integrals containing Bessel's functions and is equal to the following.

$$\begin{aligned} & \frac{1}{\alpha^2} [1 - (\alpha R) K_1(\alpha R)] + \\ & + \frac{(1 - \epsilon)}{R^2 \alpha^4} [2(\alpha R)^2 K_0(\alpha R) + [(\alpha R)^3 + 4(\alpha R)] K_1(\alpha R) - 4] \end{aligned} \quad (4.20)$$

Solving for the fuel distribution at the center in terms of  $k_{eff}$  gives us:

$$\Sigma_a^F(0) = \frac{k_{eff} D}{A(\eta - k_{eff})} \quad (4.21)$$

The fuel distribution will be given by the following after substitution of  $F(r)$  as specified above.

$$\Sigma_a^F(0) = \frac{1 - (1 - \epsilon) \left( \frac{r}{R} \right)^2}{\frac{1}{D} \left( \frac{\eta}{k_{eff}} - 1 \right) G(r)} \quad (4.22)$$

where  $G(r)$  is a function containing all the Bessel functions and is given by:

$$G(r) = \frac{1}{\alpha^2} [1 - (\alpha R)I_0(\alpha r)K_1(\alpha R)] - \frac{(1 - \epsilon)}{R^2\alpha^4} [(\alpha r)^2 + 4] + \frac{I_0(\alpha r)(1 - \epsilon)}{R^2\alpha^4} [2(\alpha R)^2 K_0(\alpha R) + (\alpha R)^3 K_1(\alpha R) + 4(\alpha R)K_1(\alpha R)] \quad (4.23)$$

It is of interest to note that for  $\epsilon = 1$ , the same equation for the fuel distribution for the case of  $F(r) = 1$  is obtained.

#### 4.4.2 Model Reactor for Parabolic Shape One Group

We will continue the theme and model a graphite moderated reactor. The physical values of the graphite to be used are given in Table 4.2.2. The same  $U^{235}$  fueled reactor will be used. This gives us  $\eta = 2.0651$  at  $20^\circ C$ . The same dimensions will be used as in the previous two cases. A height of  $360cm$  and diameter of  $360cm$ .

Letting the reactor be just critical, that is  $k_{eff} = 1$  and solving for the fuel distribution at the center gives a value of  $\Sigma_a^F(0) = .000533 \frac{1}{cm}$ . Using this we can now use the equation derived for the fuel distribution and plot the fuel distribution with respect to the fuel distribution at the center of the core as a function of position. Various fuel distributions are shown below with varying values for  $\epsilon$

Figure 4.7 for  $\epsilon = .7$  shows a similar behavior to the case for a flat power shape, where there is a fuel leakage towards the boundary. This is because for this value of  $\epsilon$  the curvature of the parabola is very low and the power shape is close to that of a straight line. However, note that the value of the fuel absorption at the boundary has decreased slightly compared to the flat power shape. As  $\epsilon$  is decreased the behavior of the fuel absorption changes. At  $\epsilon = .29$ , the fuel absorption cross section no longer peaks at the boundary, but right before it as is seen in Figure 4.8 As  $\epsilon$  is further decreased to a value of  $.1$ , the fuel leakage at the boundary starts to disappear where only about half the fuel absorption cross section is needed from the center value. For the limiting case of  $\epsilon = 0$ , there is no fuel leakage at the boundary and the absorption cross section goes to 0 at the boundary.

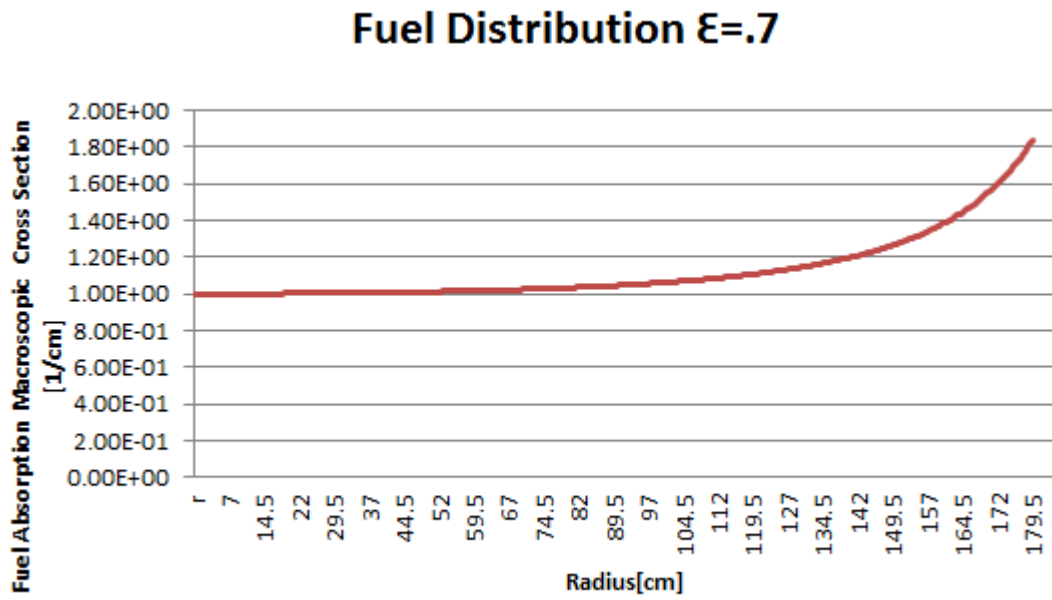


Figure 4.3: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .7$ )

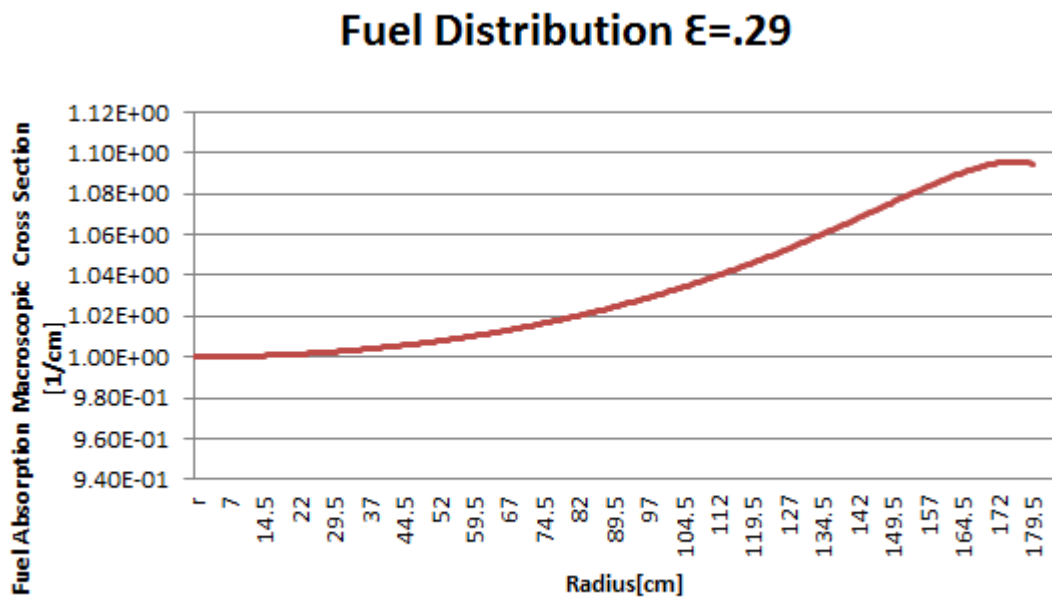


Figure 4.4: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .29$ )

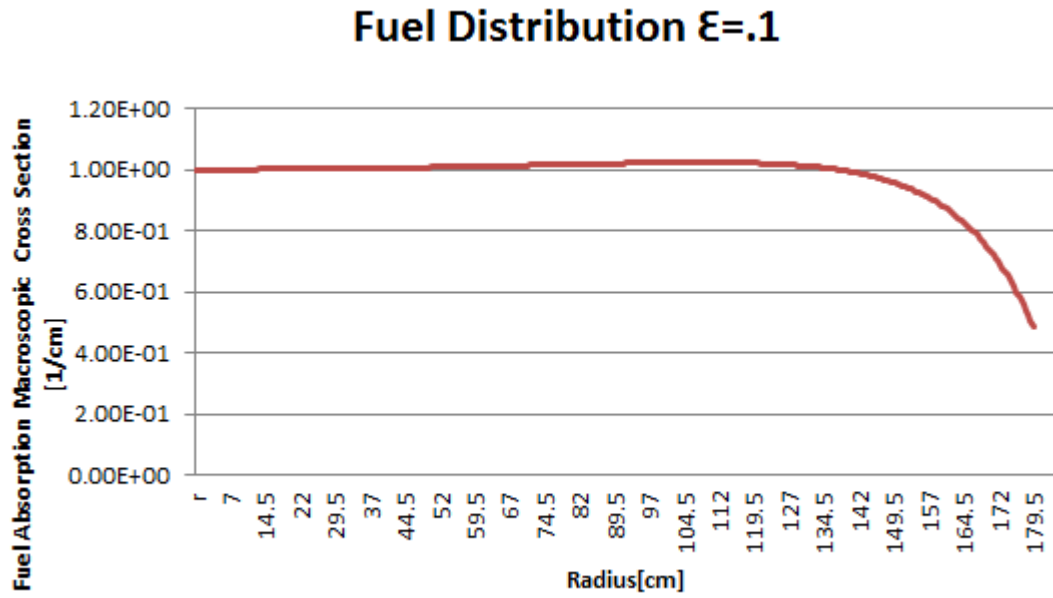


Figure 4.5: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .1$ )

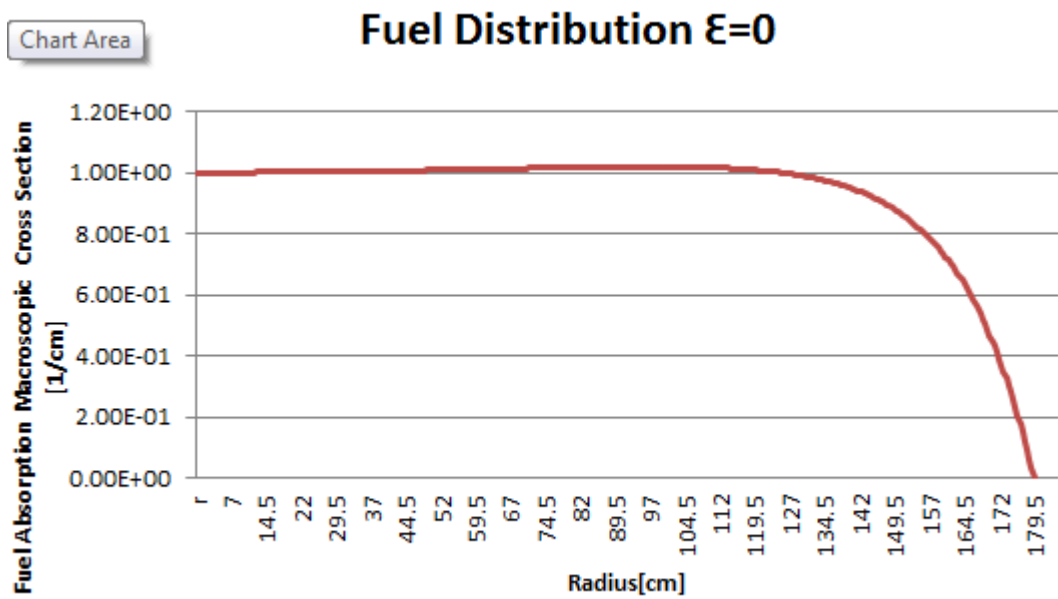


Figure 4.6: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = 0$ )

## 4.5 Parabolic Power Shape for Two Group

In this section, we repeat the previous analysis for two-group neutron diffusion.

### 4.5.1 Specifying Parabolic Power Shape

We start by working with equations 4.7 and 4.8 which specify the criticality condition and fuel distribution. The parabolic power shape will correspond to the same relative absorption rate as used in the one group case. This is repeated below:

$$F(r) = 1 - (1 - \epsilon) \left( \frac{r}{R} \right)^2 \quad (4.24)$$

Where  $\epsilon$  obtains a value between 0 and 1. Substitution of  $F(r)$  as specified above into equation 4.7 gives us the following criticality condition:

$$k_{eff} = \frac{\frac{\Sigma_{a2}^F(0)\eta\Sigma_{s(1\rightarrow2)}}{D_1^M D_2^M (\alpha_2^2 - \alpha_1^2)} B}{1 + \frac{\Sigma_{a2}^F(0)}{D_2^M} C} \quad (4.25)$$

Where  $B$  and  $C$  are defined below.

$$\begin{aligned} B = & \frac{1}{\alpha_1^2} [1 - \alpha_1 R K_1(\alpha_1 R)] - \frac{1}{\alpha_2^2} [1 - \alpha_2 R K_1(\alpha_2 R)] + \\ & + \frac{(1 - \epsilon)}{R^2 \alpha_1^4} [2(\alpha_1 R)^2 K_0(\alpha_1 R) + ((\alpha_1 R)^3 + 4(\alpha_1 R)) K_1(\alpha_1 R) - 4] - \\ & - \frac{(1 - \epsilon)}{R^2 \alpha_2^4} [2(\alpha_2 R)^2 K_0(\alpha_2 R) + ((\alpha_2 R)^3 + 4(\alpha_2 R)) K_1(\alpha_2 R) - 4] \end{aligned} \quad (4.26)$$

$$\begin{aligned} C = & \frac{1}{\alpha_2^2} [1 - \alpha_2 R K_1(\alpha_2 R)] + \\ & + \frac{(1 - \epsilon)}{R^2 \alpha_2^4} [2(\alpha_2 R)^2 K_0(\alpha_2 R) + ((\alpha_2 R)^3 + 4(\alpha_2 R)) K_1(\alpha_2 R) - 4] \end{aligned} \quad (4.27)$$

Since we will specify the criticality condition, we will solve for the fuel distribution at the center of the core. This gives us the following:

$$\Sigma_{a2}^F(0) = \frac{k_{eff} D_2^M}{\frac{\eta\Sigma_{s(1\rightarrow2)} B}{D_1^M (\alpha_2^2 - \alpha_1^2)} - k_{eff} C} \quad (4.28)$$

The fuel distribution will be given by the following equation.

$$\Sigma_{a2}^F(r) = \frac{1}{\frac{\eta \Sigma_{s(1 \rightarrow 2)}}{k_{eff} D_1 D_2 (\alpha_2^2 - \alpha_1^2)} B I_{21}(r) - \frac{1}{D_2} B I_{22}(r)} \quad (4.29)$$

Where  $B I_{21}(r)$  has been simplified to the following

$$\begin{aligned} B I_{21}(r) = & \frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^4} \frac{(1-\epsilon)}{R^2} (\alpha_1 r)^2 - \frac{4}{\alpha_1^4} \frac{(1-\epsilon)}{R^2} + \frac{1}{\alpha_2^4} \frac{(1-\epsilon)}{R^2} (\alpha_2 r)^2 + \\ & + \frac{4}{\alpha_2^4} \frac{(1-\epsilon)}{R^2} - \frac{1}{\alpha_1^2} (\alpha_1 R) I_0(\alpha_1 r) K_1(\alpha_1 R) + \frac{(1-\epsilon)}{\alpha_1^4 R^2} 2(\alpha_1 R)^2 I_0(\alpha_1 r) K_0(\alpha_1 R) + \\ & + \frac{(1-\epsilon)}{\alpha_1^4 R^2} (\alpha_1 R)^3 I_0(\alpha_1 r) K_1(\alpha_1 R) + \frac{(1-\epsilon)}{\alpha_1^4 R^2} 4(\alpha_1 R) I_0(\alpha_1 r) K_1(\alpha_1 R) + \\ & + \frac{1}{\alpha_2^2} (\alpha_2 R) I_0(\alpha_2 r) K_1(\alpha_2 R) - \frac{(1-\epsilon)}{\alpha_2^4 R^2} 2(\alpha_2 R)^2 I_0(\alpha_2 R) K_0(\alpha_2 R) - \\ & - \frac{(1-\epsilon)}{\alpha_2^4 R^2} (\alpha_2 R)^3 I_0(\alpha_2 r) K_1(\alpha_2 R) - \frac{(1-\epsilon)}{\alpha_2^4 R^2} 4(\alpha_2 R) I_0(\alpha_2 r) K_1(\alpha_2 R) \quad (4.30) \end{aligned}$$

and  $B I_{22}(r)$  is given by:

$$\begin{aligned} B I_{22}(r) = & \frac{1}{\alpha_2^2} - \frac{(1-\epsilon)}{R^2 \alpha_2^4} (\alpha_2 r)^2 - \frac{4}{\alpha_2^4} \frac{(1-\epsilon)}{R^2} - \frac{1}{\alpha_2^2} (\alpha_2 R) I_0(\alpha_2 r) K_1(\alpha_2 R) + \\ & + \frac{(1-\epsilon)}{R^2 \alpha_2^4} 2(\alpha_2 R)^2 I_0(\alpha_2 r) K_0(\alpha_2 R) + \frac{(1-\epsilon)}{R^2 \alpha_2^4} (\alpha_2 R)^3 I_0(\alpha_2 r) K_1(\alpha_2 R) + \\ & + \frac{(1-\epsilon)}{R^2 \alpha_2^4} 4(\alpha_2 R) I_0(\alpha_2 r) K_1(\alpha_2 R) \quad (4.31) \end{aligned}$$

It is worth mentioning that in the case that  $\epsilon = 1$  the same expressions for  $B I_{21}(r)$  and  $B I_{22}(r)$  are recovered that were found for the case of  $F(r) = 1$ .

#### 4.5.2 Model Reactor for Parabolic Power Shape Two Group

Following the theme of the chapter, the same Uranium fueled graphite moderated reactor will be used that has been used in the previous three examples. The physical values of interest for graphite are shown in Table 4.3.2. The same physical dimensions for the reactor will be kept. This gives us the following graphs of the fuel distribution with respect to the fuel distribution at the center of the core. Below are various plots for varying values of  $\epsilon$ .

In the above model reactor, the fuel distribution at the center is calculated

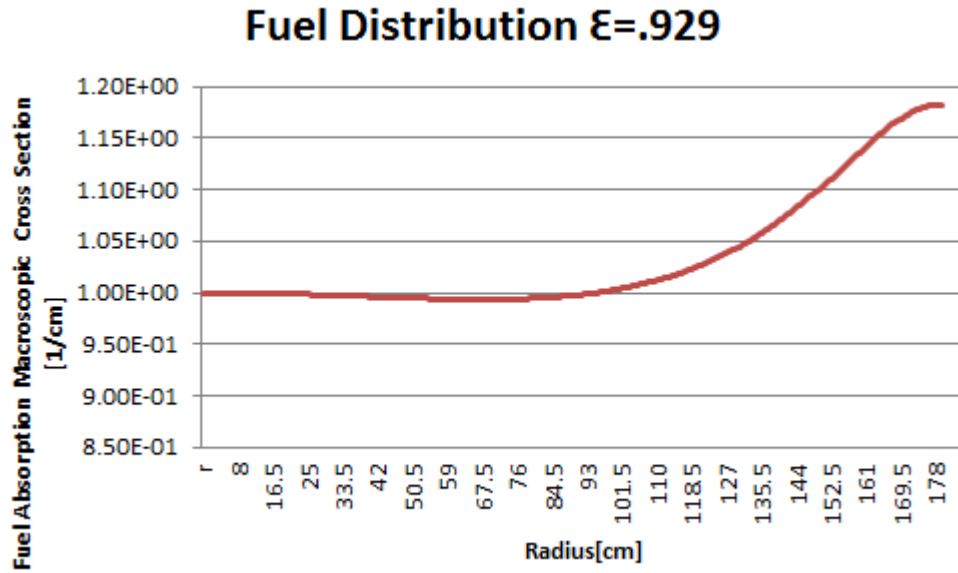


Figure 4.7: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .7$ )

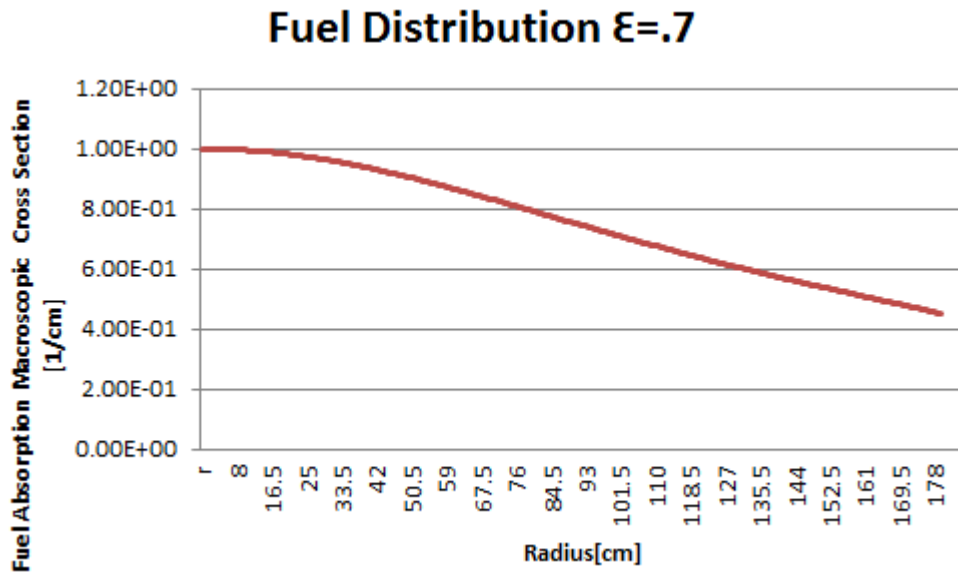


Figure 4.8: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .29$ )

to be  $.000671 \frac{1}{cm}$ . This is about one quarter greater than that of the one group case. There are also notable differences in the shape the fuel distribution



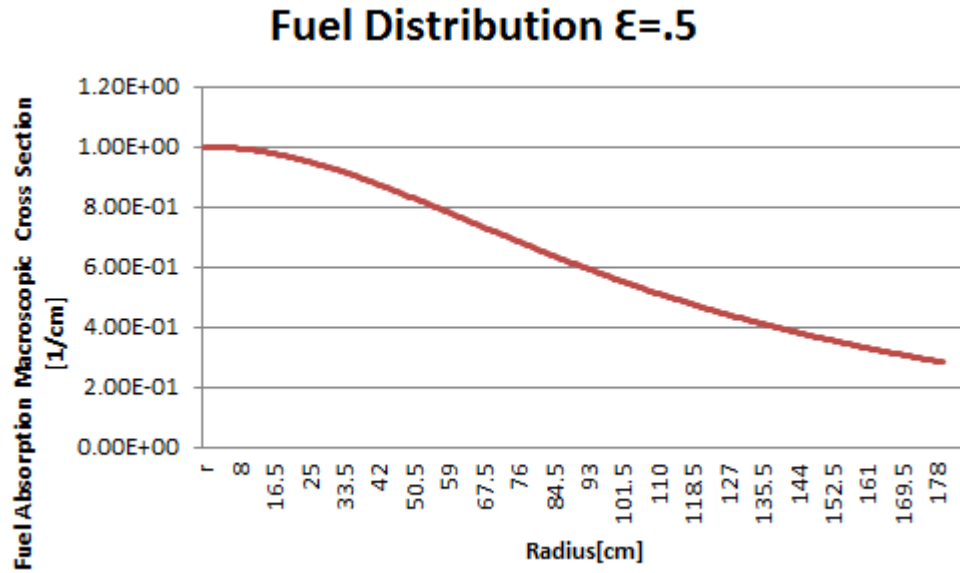


Figure 4.9: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = .1$ )

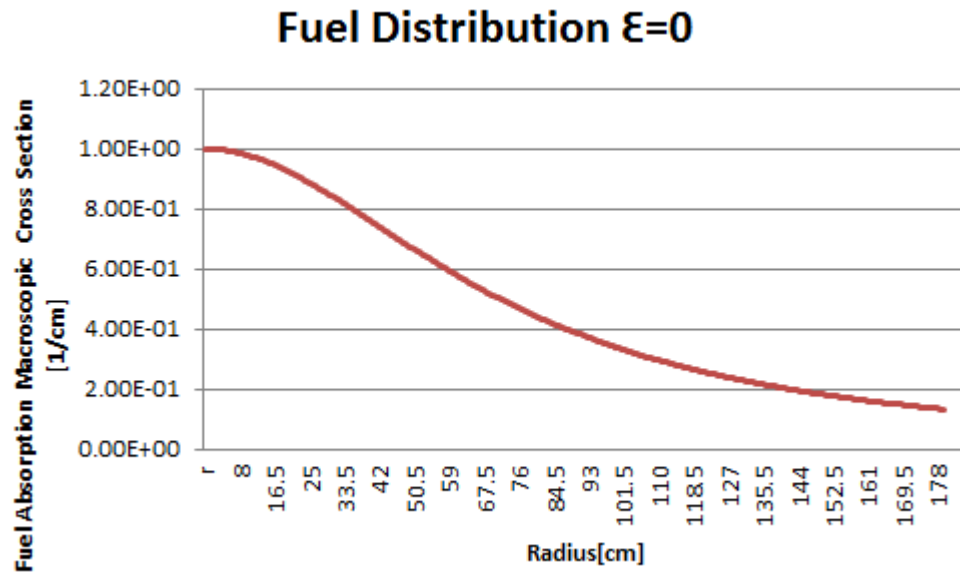


Figure 4.10: Fuel Distribution for One Group due to Parabolic Power Shape( $\epsilon = 0$ )

takes. In the one group case the fuel distribution stays constant until it approaches the edge and then it experiences a steep drop. In the two-group

case as the graphs above show, the fuel distribution goes through a gradual decline to get the desired power shape. Another difference is that in the one-group case the fuel distribution goes to 0 at the edge. This does not occur in the two group case. In this case, the fuel distribution approaches 0 but does not reach it. There is still fuel leakage at the boundary.

## 4.6 Conclusion

Specifying a specific power shape allowed us to obtain results in order for a comparison to be made between different group models. The graphs of the fuel absorption cross section show that in general the one group diffusion model underestimates the amount of fuel needed as compared to the two-group case. We were also able to obtain results in which it was seen that there is a cost in flattening the power shape as it leads to an increase in neutron leakage at the boundary.

## CHAPTER 5

# CONCLUSION AND DIRECTIONS FOR FURTHER RESEARCH

### 5.1 Conclusion

The proceeding study in specified power shape neutronics has shown the usefulness of the method of Green's functions. A cylindrical geometry thermal reactor was able to be studied and results for the fuel distribution function and the required fuel distribution at the center of the core needed for criticality were made.

In particular, analytic solutions were obtained in terms of modified Bessel functions. These expressions can be quickly solved for using programs that compute Bessel functions or using a table of Bessel function values.

One important simplification that was made along the way is that by specifying a chopped cosine power shape in the horizontal direction, the system was able to be transformed into a one dimension radial direction. The neutron flux term in the horizontal direction was captured as a lost term in the radial direction.

In the model reactor various results were obtained. One is that there is a cost in flattening a power shape as it leads to an increase in neutron leakage at the boundary. The results also show that in preliminary reactor design the one group diffusion model leads to an underestimate of the fuel absorption rate at the center of the core. The one group model underestimates this by roughly being 80

### 5.2 Directions for Further Research

Various assumptions were made in simplifying the model that does not represent physical reality. An example of this is assuming a homogeneous assem-

bly. An example of where this was used was assuming the diffusion length remains constant throughout the core.

Another assumption made is that the moderator and reflector properties were the same. This assumption was used in the construction of the Green's functions. A further study would assume that the reflector properties are different then the core and moderator properties. In this case the Green's functions would have to be found using a different method, but the same procedure could be used to perform a neutronics analysis.

Lastly, a further study would compare the analytic results obtained in this thesis to numerical methods such as Monte Carlo simulations.

# APPENDIX A

## APPENDIX

Throughout this work various integrals involving Bessel functions were used. The follow appendix list the integrals and some key steps that were used in the evaluation.

### A.1 Integrals involving $I_0(\alpha r)$

$$\int I_0(r)rdr = rI_1(r) \quad (\text{A.1})$$

$$\begin{aligned} \int I_0(r)r^3dr &= r^2rI_1(r) - 2 \int r^2I_1(r)dr = r^2rI_1(r) - 2r^2I_2(r) = \\ &= r^3I_1(r) - 2r^2 \left[ I_0(r) - \frac{2}{z}I_1(r) \right] = (r^3 + 4r)I_1(r) - 2r^2I_0(r) \end{aligned} \quad (\text{A.2})$$

Where integration by parts was used in the first step. The integral  $\int r^2I_1(r)dr$  was evaluated using the following property

$$\frac{d}{dr} [r^n I_n(r)] = r^n I_{n-1}(r) \quad (\text{A.3})$$

and the following property was used to simplify  $I_2(r)$ .

$$I_{n+1}(r) = I_{n-1}(r) - \frac{2n}{r}I_n(r) \quad (\text{A.4})$$

Integrating by substitution gives the desired result.

$$\int I_0(\alpha r)rdr = \frac{1}{\alpha^2} [(\alpha r)I_1(\alpha r)] \quad (\text{A.5})$$

$$\int I_0(\alpha r)r^3dr = \frac{1}{\alpha^4} [((\alpha r)^3 + 4(\alpha r)) I_1(\alpha r) - 2(\alpha r)^2 I_0(\alpha r)] \quad (\text{A.6})$$

$$\int_0^r I_0(\alpha r) r dr = \frac{1}{\alpha^2} [(\alpha r) I_1(\alpha r)] \quad (\text{A.7})$$

$$\int_0^r I_0(\alpha r) r^3 dr = \frac{1}{\alpha^4} [((\alpha r)^3 + 4(\alpha r)) I_1(\alpha r) - 2(\alpha r)^2 I_0(\alpha r)] \quad (\text{A.8})$$

The last two integrals have been obtained using the fact that  $I_n(0) = 0$  for all  $n$ .

## A.2 Integrals involving $K_0(\alpha r)$

$$\int K_0(r) r dr = -r K_1(r) \quad (\text{A.9})$$

$$\begin{aligned} \int K_0(r) r^3 dr &= -r^2 r K_1(r) + 2 \int r^2 K_1(r) dr = -r^3 K_1(r) - 2r^2 K_2(r) = \\ &= -r^3 K_1(r) - 2r^2 \left[ K_0(r) + \frac{2}{r} K_1(r) \right] = -2r^2 K_0(r) - [r^3 + 4r] K_1(r) \end{aligned} \quad (\text{A.10})$$

Where integration by parts was used in the first step. The integral  $\int r^2 K_1(r) dr$  was evaluated using the following property

$$\frac{d}{dr} [r^n K_n(r)] = -r^n K_{n-1}(r) \quad (\text{A.11})$$

and the following property was used to simplify  $K_2(r)$

$$K_{n+1}(r) = K_{n-1}(r) + \frac{2n}{r} K_n(r) \quad (\text{A.12})$$

Integration by substitution leads to the desired results.

$$\int K_0(\alpha r) r dr = \frac{-1}{\alpha^2} \alpha r K_1(\alpha r) \quad (\text{A.13})$$

$$\int K_0(\alpha r) r^3 dr = \frac{-1}{\alpha^4} [2(\alpha r)^2 K_0(\alpha r) + [(\alpha r)^3 + 4(\alpha r)] K_1(\alpha r)] \quad (\text{A.14})$$

$$\int_0^r K_0(\alpha r) r dr = \frac{1}{\alpha^2} [1 - \alpha r K_1(\alpha r)] \quad (\text{A.15})$$

Here the following property is used:

$$\lim_{r \rightarrow 0} r K_1(r) = 1 \quad (\text{A.16})$$

$$\int_0^r K_0(\alpha r) r^3 dr = \frac{-1}{\alpha^4} [2(\alpha r)^2 K_0(\alpha r) + [(\alpha r)^3 + 4(\alpha r)] K_1(\alpha r) - 4] \quad (\text{A.17})$$

Here the following property is used

$$\lim_{r \rightarrow 0} [2r^2 K_0(r) + (r^3 + 4r) K_1(r)] = 4 \quad (\text{A.18})$$

Both of the limits may be obtained by using a power series expansion around 0.

### A.3 Other Properties of Modified Bessel Functions

The following was used to simplify many resulting terms:

$$W[K_n(r), I_n(r)] = I_n(r) K_{n+1}(r) + I_{n+1}(r) K_n(r) = \frac{1}{r} \quad (\text{A.19})$$

Where  $W[.,.]$  is the Wronskian.

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Jose Rivera received his bachelors nuclear engineering from the University of Illinois Urbana Champaign in 2009. The same year he started the masters program at the University of Illinois where he continued to work with his undergraduate adviser, Professor Roy A. Axford. While a graduate student in nuclear, plasma, and radiological engineering, Jose took various mathematics classes which have led him to pursue his doctoral degree in the field of mathematics.