EXTREMAL GRAPH THEORY: RAMSEY-TURÁN NUMBERS, CHROMATIC THRESHOLDS, AND MINORS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

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Abstract

This dissertation investigates several questions in extremal graph theory and the theory of graph minors. It consists of three independent parts; the first two parts focus on questions motivated by Turán's Theorem and the third part investigates a problem related to Hadwiger's Conjecture.

Let H be a graph, t an integer, and $f(n)$ a function. Erdős, Hajnal, Sós, and Szemerédi defined the t-Ramsey-Turán number of H, $\mathbf{RT}_t(n, H, f(n))$, to be the maximum number of edges in an n-vertex, H-free graph with K_t-independence number less than $f(n)$, where the *K_t-independence number* of a graph G is the maximum number of vertices in a K_t-free induced graph of G. In the first part of this thesis, we study the Ramsey-Turán numbers for several graphs and hypergraphs, proving two conjectures of Erdős, Hajnal, Simonovits, Sós, and Szemerédi. In joint work with József Balogh, our first main theorem is to provide the first lower bounds of order $\Omega(n^2)$ on $\mathbf{RT}_t(n, K_{t+2}, o(n))$. Our second main theorem is to prove lower bounds on $\mathbf{RT}(n, \mathrm{TK}^s(r), o(n))$, where $\mathrm{TK}^s(r)$ is the r-uniform hypergraph formed from K_s by adding $r-2$ new vertices to every edge.

Let $\mathcal{F}$ be a family of r-uniform hypergraphs. Introduced by Erdős and Simonovits, the chromatic threshold of $\mathcal{F}$ is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs with minimum degree at least $c\binom{n}{r-1}$ has bounded chromatic number. The problem of chromatic thresholds of graphs has been well studied, but there have been no previous results about the chromatic thresholds of r-uniform hypergraphs for $r \geq 3$. Our main result in this part of the thesis, in joint work with József Balogh, Jane Butterfield, Ping Hu, and Dhruv Mubayi, is to prove a structural theorem about hypergraphs with bounded chromatic number. Corollaries of this result show that the chromatic threshold of the family of F-free hypergraphs is 0 for several hypergraphs F, including a hypergraph generalization of cycles.

A graph H is a *minor* of a graph G if starting with G, one can obtain H by a sequence of vertex deletions, edge deletions, and edge contractions. Hadwiger's famous conjecture from 1943 states that every t-chromatic graph G has K_t as a minor. Hadwiger's Conjecture implies the following weaker conjecture: every graph G has $K_{\lceil n/\alpha(G)\rceil}$ as a minor, where $\alpha(G)$ is the independence number of G. The main theorem in the last part of this thesis, in joint work with József Balogh and Hehui Wu, is to prove that every graph has $K_{n/(2\alpha(G)-\Theta(\log \alpha(G)))}$ as a minor.

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Chapter 1

Introduction

1.1 Overview

Extremal graph theory studies the maximum or minimum values obtained by graph parameters over familes of graphs. One of the foundational results of extremal graph theory is Turán's Theorem [66], which states that the unique n-vertex, K_r-free graph G with the most edges is the complete $(r-1)$-partite graph with part sizes as equal as possible, denoted $T_{r-1}(n)$ and called the Turán graph. This thesis focuses on two problems motived heuristically by asking about the "stability" or "robustness" of Turán's Theorem. If we impose some other condition on G in addition to being K_r-free, how radically does G differ from $T_{r-1}(n)$?

In 1970, Erdős and Sós [23] introduced a problem motivated by noting that the Turán graph has huge independent sets. What happens if we forbid G to have large independent sets? Let $f(n)$ be a function and define $\mathbf{RT}(n, H, f(n))$ to be the maximum number of edges in an n-vertex, H-free graph with independence number less than $f(n)$. Having made this definition, we notice that in fact it is also related to Ramsey's Theorem [50]. Ramsey's Theorem [50] states that for every integer d, there exists an integer n_0 such that for all $n > n_0$, any two coloring of the edges of K_n contains a monochromatic copy of K_d. An independent set in G is a clique in the complement of G, so $\mathbf{RT}(n, H, f(n))$ is also the maximum number of red edges used in a red/blue coloring of $E(K_n)$ that avoids a copy of H in red and a copy of $K_{f(n)}$ in blue. For this reason, the quantity $\mathbf{RT}(n, H, f(n))$ is called the *Ramsey-Turán number of* H. Heuristically, we expect that the problem of determining $\mathbf{RT}(n, H, f(n))$ will be similar to Ramsey's Theorem [50] when $f(n)$ is small and similar to Turán's Theorem [66] when $f(n)$ is large.

The main result in Chapter 2, in joint work with József Balogh, is the solution to two problems of Erdős, Hajnal, Simonovits, Sós, and Szemerédi [20]. Let $\mathbf{RT}_t(n, H, f(n))$ be the maximum number of edges in an n-vertex, H-free graph G where for every set S of $\lceil f(n) \rceil$ vertices of G, $\omega(G[S]) \geq t$.

Theorem 1. *For $t \geq 2$ and $2 \leq \ell \leq t$, let $u = \left\lceil \frac{t}{2} \right\rceil$. Then*

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}_t(n, K_{t+\ell}, \epsilon n)}{n^2} \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2}.$$

A different question about the robustness of Turán's Theorem [66] is the following: for triangle-free graphs, what happens if we forbid G to be bipartite? Consider the following simple construction:

let G be $K_{\lceil \frac{n}{2} \rceil - 2, \lfloor \frac{n}{2} \rfloor}$ with parts X and Y. Divide Y into Y_1, Y_2 with $|Y_1| = \left\lceil \frac{|Y|}{2} \right\rceil$ and $|Y_2| = \left\lfloor \frac{|Y|}{2} \right\rfloor$, and add two vertices a and b such that ab is an edge, a is joined to Y_1, and b is joined to Y_2. This graph is triangle-free but not bipartite because it contains C_5, and the number of edges is about $n^2/4 - n/2$. From this, we can see that restricting G to be triangle-free and non-bipartite we still have roughly the same number of edges as $T_2(n)$. But there is one big difference between G and $T_2(n)$: the minimum degree has been cut in half. These observations motivated Erdős and Simonovits [24] to ask the following question: "If G is non-bipartite, what bound on $\delta(G)$ forces G to contain a triangle?" This question is extended to all graphs by replacing the triangle by a general graph H and replacing non-bipartite with chromatic number at least k, leading to the following definition. Let $\mathcal{F}$ be a family of graphs. The chromatic threshold of $\mathcal{F}$ is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of graphs with minimum degree at least cn has bounded chromatic number. Using this definition, the question of Erdős and Simonovits [24] can be rephrased as determining the chromatic threshold of the family of triangle-free graphs.

In joint work with József Balogh, Jane Butterfield, Ping Hu, and Dhruv Mubayi, the main result in Chapter 3 is to determine the chromatic threshold for a large family of hypergraphs. An r-uniform hypergraph H is *near r-partite* if there exists a partition $V_1 \cup \cdots \cup V_r$ of $V(H)$ such that all edges of H either cross the partition (have one vertex in each V_i) or are contained entirely in V_1, and in addition $H[V_1]$ is a partial matching.

Theorem 2. *Let H be an r-uniform, near r-partite hypergraph. In addition, assume that H does not contain three distinct vertices x, y, and z and three distinct edges E_1, E_2, and E_3 such that $x, y \in E_1$, $x, z \in E_2$, and $y, z \in E_3$. Then the chromatic threshold of the family of H-free graphs is zero.*

For a subfamily of the hypergraphs considered in Theorem 2, we determine the exact extremal hypergraph. An r-uniform hypergraph H is *stable with respect to $T_r(n)$* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if G is an H-free r-uniform hypergraph with at least $(1-\delta)\,|T_r(n)|$ edges then there is a partition of $V(G)$ into $U_1, \ldots, U_r$ such that all but at most ϵn^r edges of G have exactly one vertex in each part.

Theorem 3. *Let H be an r-uniform, near r-partite hypergraph with partition $V_1, \ldots, V_r$. Assume that $H[V_1]$ contains only one edge E, E has $r-2$ vertices of degree one, and H is stable with respect to $T_r(n)$. Then there exists some n_0 such that if $n > n_0$, the unique n-vertex, r-uniform, H-free hypergraph with the most edges is $T_r(n)$.*

Switching gears, Chapter 4 looks at an extremal problem involving graph minors: when G has independence number k, what is the maximum integer t such that K_t is a minor of G? This is related to Hadwiger's Conjecture [33] from 1943, which states that every t-chromatic graph G has K_t as a minor. Bollobás, Catlin, and Erdős [12] described Hadwiger's Conjecture as "one of the deepest unsolved problems in graph theory." Since in a t-coloring of G every color class is an independent set, it follows that $\alpha(G)$ is at least $|V(G)|/t$. Thus Hadwiger's Conjecture implies that if G is an n-vertex graph with independence number k, then $K_{\lceil n/k \rceil}$ is a minor G.

In joint work with József Balogh and Hehui Wu, Chapter 4 proves the following lower bound on the maximum t such that K_t is a minor of G.

Theorem 4. *Let G be an n-vertex graph. Then $K_{\lceil n/r \rceil}$ is a minor of G, where*

$$r = 2\alpha(G) - \lceil \log_\tau(\tau\alpha(G)/2) \rceil \text{ and } \tau = \frac{2\sqrt{2}}{\sqrt{2}-1} \approx 6.83.$$

1.2 Basic definitions

A *hypergraph* H is a pair of sets $(V(H), E(H))$ where $V(H)$ is any finite set and $E(H)$ is a family of subsets of $V(H)$. Elements of $V(H)$ are the *vertices* of H and elements of $E(H)$ are *hyperedges* or just *edges* of H. For a positive integer r, a hypergraph H is *r-uniform* if every element in $E(H)$ has size r. A *graph* is a 2-uniform hypergraph. When H is a hypergraph, the *shadow graph* of H is the graph G with $V(G) = V(H)$ and $xy \in E(G)$ if and only if there exists some edge E of H with $x, y \in E$.

For a hypergraph H, we say that vertices a and b are *adjacent* (or *neighbors*) if there is some edge of G containing both a and b. For $a \in V(H)$, the *neighborhood of a*, denoted $N_H(a)$ or $N(a)$, is the collection of subsets $B \subseteq V(H)$ where $B \cup \{a\} \in E(H)$. For graphs, elements in $N(a)$ are vertex sets of size 1, so we identify $N(a)$ with a subset of $V(H)$. The *degree of a in H*, written $d_H(a)$ or $d(a)$, is the size of $N_H(a)$. The *minimum degree* of H is

$$\delta(H) = \min \{d_H(a) : a \in V(H)\}.$$

The *maximum degree* of H is

$$\Delta(H) = \max \{d_H(a) : a \in V(H)\}.$$

A hypergraph is *regular* if $\delta(H) = \Delta(H)$.

A hypergraph H is a *subhypergraph* (or in the case of graphs, a *subgraph*) of a hypergraph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, H is an *induced subhypergraph* (for graphs, an *induced subgraph*) of G if H is a subhypergraph of G and for every $A \subseteq V(H)$, if $A \in E(G)$ then $A \in E(H)$. Let $X \subseteq V(H)$. The *induced subhypergraph* (for graphs, the *induced subgraph*) of H with vertex set X, denoted $H[X]$, is the hypergraph with vertex set X and edge set $\{A : A \subseteq X \text{ and } A \in E(H)\}$. A subhypergraph H of G is *spanning* if $V(H) = V(G)$. A vertex a is *connected* to a vertex b if $a = b$ or if there exists a list $E_1, \ldots, E_t$ of edges such that $a \in E_1$, $b \in E_t$, and $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq t-1$. A *component* of a hypergraph is an induced subhypergraph $H[A]$ such that for every $a, b \in A$, a is connected to b, and for every $a \in A$ and $b \notin A$, a is not connected to b. A hypergraph is *connected* if it consists of only one component.

For a subset A of $V(H)$, the set A is *independent* in H if there is no edge E of H with $E \subseteq A$. A subset A is *strongly independent* in H if there is no edge E of H with $|E \cap A| \geq 2$. The *independence number* (or the *stability number*) of H, denoted $\alpha(H)$, is the maximum number of vertices in an

independent set, and the *strong independence number* of H is the maximum number of vertices in a strong independent set. For graphs G, a *clique* is a subset A of $V(G)$ where $G[A]$ has $\binom{|A|}{2}$ edges (in other words, for every $a, b \in A$, $ab \in E(G)$.) The *clique number* of G, denoted $\omega(G)$, is the maximum number of vertices in a clique. Let G be a graph and define the *K_t-independence number* to be

$$\alpha_t(G) = \max\{|S| : S \subseteq V(G), \omega(G[S]) < t\}.$$

Let k be an integer and H a hypergraph. A map $f : V(H) \to \{1, \ldots, k\}$ is a *proper coloring* if for every edge E in H, there exists vertices $a, b \in E$ such that $f(a) \neq f(b)$. A hypergraph H is *k-colorable* if there exists a proper coloring $f : V(H) \to \{1, \ldots, k\}$. The chromatic number of H, denoted $\chi(H)$, is the minimum k such that H is k-colorable. For an integer s, a hypergraph H is *s-partite* if there exists a partition $V_1 \cup \cdots \cup V_s = V(H)$ such that V_i is strongly independent for all $1 \leq i \leq s$. A 2-partite graph is a *bipartite graph*.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. For two vertex sets $T, S \subseteq V(G)$, we say T *touches* S if $T \cap S \neq \emptyset$ or there is an edge $xy \in E(G)$ with $x \in T$ and $y \in S$. The *cycle of length ℓ*, denoted C_ℓ, is the graph with vertex set $\{v_1, \ldots, v_\ell\}$ and edges $v_i v_{i+1}$ for $1 \leq i \leq \ell - 1$ and in addition the edge $v_\ell v_1$. The *path on ℓ vertices*, denoted P_ℓ, is the graph with vertex set $\{v_1, \ldots, v_\ell\}$ and edges $v_i v_{i+1}$ for $1 \leq i \leq \ell - 1$. The *length* of P_ℓ is $\ell - 1$. A graph G is *chordal* if G has no induced cycle of length at least 4. A vertex is *simplicial* if its neighborhood is a clique. A *simplicial elimination ordering* is an order $v_n, \ldots, v_1$ of the vertices of G in which vertices can be deleted so that each vertex v_i is a simplicial vertex of the graph induced by $\{v_1, \ldots, v_i\}$. Dirac [18] proved that a graph is chordal if and only if it has a simplicial elimination ordering, and Berge [9] observed that by greedily coloring the vertices of a simplicial elimination ordering one obtains an $\omega(G)$-coloring of G, proving that chordal graphs are perfect. A *tree* is a connected graph with no cycle as a subgraph.

The *complete n-vertex, r-uniform hypergraph*, denoted $K_n^{(r)}$ (and for graphs denoted K_n), is the unique n-vertex, r-uniform hypergraph with $\binom{n}{r}$ edges. The *complete s-partite, r-uniform hypergraph with part sizes $t_1, \ldots, t_s$*, denoted $K_{t_1,\ldots,t_s}^{(r)}$ and for graphs $K_{t_1,\ldots,t_s}$, is the r-uniform hypergraph whose vertex set is $V_1 \cup \cdots \cup V_s$, where $|V_i| = t_i$ for all $1 \leq i \leq s$, and whose edges consist of all r-sets with at most one vertex in each V_i for $1 \leq i \leq s$. The numbers $t_1, \ldots, t_s$ are the *part sizes* of H. The hypergraph $T_{r,s}(n)$ is the unique n-vertex, r-uniform, s-partite hypergraph with part sizes as equal as possible. Let $t_{r,s}(n)$ denote the number of edges in $T_{r,s}(n)$.

Let $\mathrm{TK}^r(s)$ be the r-uniform hypergraph obtained from the complete graph K_s by enlarging each edge with $r - 2$ new vertices. The *core vertices* of $\mathrm{TK}^r(s)$ are the s vertices of degree larger than one. For $s > r$, let $\mathcal{TK}^r(s)$ be the family of r-uniform hypergraphs such that there exists a set S of s vertices where each pair of vertices from S are contained together in some edge. The set S is the set of *core vertices* of the hypergraph. For $s \leq r$, let $\mathcal{TK}^r(s)$ be the family of r-uniform hypergraphs such that there exists a set S of s vertices where for each pair of vertices $x \neq y \in S$, there exists an edge E with $E \cap S = \{x, y\}$ (the definition is different when $s \leq r$ so that a hypergraph which is just a single edge is not in $\mathcal{TK}^r(s)$). It is obvious that $\mathrm{TK}^r(s) \in \mathcal{TK}^r(s)$.

A hypergraph G is *H-free* if H is not a subhypergraph of G. For an integer n and a r-uniform hypergraph H, let $\mathrm{ex}(n, H)$ denote the maximum number of edges in an n-vertex, r-uniform, H-free hypergraph. For an r-uniform hypergraph H, let

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{r}},$$

the *Turán density* of H. Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The *chromatic threshold* of $\mathcal{F}$ is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number. Let H be an r-uniform hypergraph and $f(n)$ a function. The *Ramsey-Turán number of H*, $\mathbf{RT}(n, H, f(n))$, is the maximum number of edges in an n-vertex, r-uniform, H-free hypergraph with independence number at most $f(n)$. We say that an r-uniform hypergraph H is *stable* with respect to $T_r(n)$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if G is an H-free r-uniform hypergraph with at least $(1-\delta)t_r(n)$ edges then there is a partition of $V(G)$ into $U_1, \ldots, U_r$ such that all but at most ϵn^r edges of G have exactly one vertex in each part.

For a graph G and an edge $ab \in E(G)$, the graph formed by *contracting the edge ab* is the graph with vertex set $V(G) - \{b\}$ and edges $\{xy : \{x, y\} \cap \{a, b\} = \emptyset\}$ together with the edges $\{ax : ax \in E(G) \text{ or } bx \in E(G)\}$. A graph H is a *minor* of G if H can be formed by starting with the graph G and preforming a sequence of vertex deletions, edge deletions, and edge contractions. Equivalently, H is a minor of G if there exists vertex subsets $X_1, \ldots, X_h$ of G such that H has vertices $\{v_1, \ldots, v_h\}$, $X_i \cap X_j = \emptyset$ for $i \neq j$, and for every edge $v_iv_j \in E(H)$, the set X_i touches the set X_j. A *complete minor of G* is such a collection $X_1, \ldots, X_n$ when $H = K_n$. The size of such a complete minor is n. The *Hadwiger number* of G, denoted $h(G)$, is the maximum size of a complete minor of G. We say that G has an *odd complete minor of order at least ℓ* if there are ℓ disjoint trees in G such that every two of them are joined by an edge, and in addition, all the vertices of the trees are two-colored in such a way that edges within trees are bichromatic and edges between trees are monochromatic.

Let f and g be two non-negative real-valued functions defined on the set $\mathbb{N}$ of natural numbers. In addition, assume $g(n) > 0$ for all $n \in \mathbb{N}$. We write

- $f = O(g)$ if there exists a $c > 0$ and an $n_0 \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n > n_0$,
- $f = \Omega(g)$ if there exists a $c > 0$ and an $n_0 \in \mathbb{N}$ such that $f(n) \geq cg(n)$ for all $n > n_0$,
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$,
- $f = o(g)$ if the ratio $\frac{f(n)}{g(n)}$ tends to 0 as n tends to infinity.

For $a, b \in \mathbb{R}$, the *interval with endpoints a and b*, denoted $[a, b]$, is the set $\{a \leq x \leq b : x \in \mathbb{R}\}$. A *box* in $\mathbb{R}^d$ is a set of the form $B = \Pi_{i=1}^{d}[a_i, b_i]$. The *volume* of a box B, denoted $vol(B)$, is $\Pi_i(b_i - a_i)$.

For any subset $A \subseteq \mathbb{R}^d$, the *outer measure* of A is

$$\lambda^*(A) = \inf \left\{ \sum_{B \in \mathcal{B}} vol(B) : \mathcal{B} \text{ is a countable collection of boxes whose union covers } A \right\}.$$

A is *measurable* (or *Lebesgue measurable*) if for every $S \subseteq \mathbb{R}^d$, $\lambda^*(S) = \lambda^*(A \cap S) + \lambda^*(S - A)$. The *Lebesgue measure* is defined by $\lambda(A) = \lambda^*(A)$ for any Lebesuge measurable set. A *Borel measure* on $\mathbb{R}^d$ is a function ν from subsets of $\mathbb{R}^d$ to the real line, where ν is defined on all boxes in $\mathbb{R}^d$, $\nu(\emptyset) = 0$, $\nu(A) \geq 0$ for all A, and $\nu(\cup_i A_i) = \sum_i \nu(A_i)$ for all pairwise disjoint $A_1, \ldots, A_t$. The *k-dimensional unit sphere*, denoted $\mathbb{S}^k$, is the subset of $\mathbb{R}^{k+1}$ consisting of all points with Eucliden distance exactly one from the origin. The *surface area of* $\mathbb{S}^k$ is the Lebesgue measure of $\mathbb{S}^k$, which is $(k+1)c_{k+1}$, where

$$c_d = \frac{\pi^{d+2}}{\Gamma(d/2+1)},$$

where $\Gamma(x)$ is the *gamma function*, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

A *hyperplane* in $\mathbb{R}^d$ is a set $\left\{ \vec{x} \in \mathbb{R}^d : \vec{x} \cdot \vec{a} = b \right\}$ for some vector $\vec{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$. A *halfspace* is a subset of $\mathbb{R}^d$ of the form $\left\{ \vec{x} \in \mathbb{R}^d : \vec{x} \cdot \vec{a} \geq b \right\}$, where $\vec{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$.

Chapter 2

Ramsey-Turán Numbers

2.1 Introduction

Recall from Section 1.2 that if H is an r-uniform hypergraph and $f(n)$ a function, then the **Ramsey-Turán number of** H, $\mathbf{RT}(n, H, f(n))$, is the maximum number of edges in an n-vertex, r-uniform, H-free hypergraph with independence number at most $f(n)$.

In 1970, Erdős and Sós [23] initiated the study of Ramsey-Turán numbers of graphs when they investigated if large independent sets are required to produce K_{2s+1}-free graphs with a large number of edges. One of the main problems in Ramsey-Turán theory is to determine the threshold function for $\mathcal{H}$ (see [58] for a survey). The **threshold function** for $\mathcal{H}$ is a function $t(n)$ such that $\mathbf{RT}(n, \mathcal{H}, t(n)) = \Omega(n^r)$ and if $f(n) = o(t(n))$ then $\mathbf{RT}(n, \mathcal{H}, f(n)) = o(n^r)$. Define

$$\gamma(\mathcal{H}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}(n, \mathcal{H}, \epsilon n)}{n^r},$$

and define $\mathbf{RT}(n, \mathcal{H}, o(n)) = \gamma(\mathcal{H}) n^r + o(n^r)$. An easy diagonalization argument shows that $t(n) = n$ is a threshold function for $\mathcal{H}$ if and only if $\mathrm{ex}(n, \mathcal{H}) = \Omega(n^r)$ and $\gamma(\mathcal{H}) = 0$. Very few threshold functions are known exactly; instead we study the easier problem of deciding whether $t(n) = n$ is a threshold function or not.

Erdős, Hajnal, Sós, and Szemerédi [21, p. 80] proposed a problem about an extension of the Ramsey-Turán number. Recall from Section 1.2 that the K_t-independence number of a graph G, $\alpha_t(G)$, is the maximum number of vertices in a K_t-free induced subgraph of G. Define $\mathbf{RT}_t(n, H, f(n))$ to be the maximum number of edges in an H-free graph G on n vertices with $\alpha_t(G) \leq f(n)$ and define

$$\mathbf{RT}_t(n, H, o(n)) = n^2 \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n^2} \mathbf{RT}_t(n, H, \epsilon n) + o(n^2). \tag{2.1}$$

For $t = 2$, it is easy to show that the limit in (2.1) exists; for $t \geq 3$, the fact that these limits exist is not obvious, it was one of the main results in [20].

The material presented in this chapter is joint work with József Balogh. It was submitted to the Israel Journal of Mathematics under the title *On the Ramsey-Turán Numbers of Graphs and Hypergraphs*, see [7]

For complete graphs of odd order, Erdős and Sós [23] proved that

$$\mathbf{RT}(n, K_{2s+1}, o(n)) = \frac{1}{2}\left(1 - \frac{1}{s}\right) n^2 + o(n^2).$$

The first celebrated result in Ramsey-Turán theory was that $\mathbf{RT}(n, K_4, o(n)) = \frac{1}{8}n^2 + o(n^2)$. In one of the first applications of the Regularity Lemma to graph theory, Szemerédi [60] proved that $\mathbf{RT}(n, K_4, o(n)) \leq \frac{1}{8}n^2 + o(n^2)$ in 1972. Four years later, Bollobás and Erdős [13] provided a surprising geometric construction which proved that Szemerédi's upper bound was tight. It was not until 1983 that Erdős, Hajnal, Sós, and Szemerédi [21] were able to extended this result to all complete graphs of even order. They proved that

$$\mathbf{RT}(n, K_{2s}, o(n)) = \frac{1}{2}\frac{6s-10}{6s-4}n^2 + o(n^2).$$

They also proved that $\mathbf{RT}(n, H, o(n)) \leq \mathbf{RT}(n, K_s, o(n))$ where s is the minimum integer such that $V(H)$ can be partitioned into $\lceil s/2 \rceil$ sets $V_1, \ldots, V_{\lceil s/2 \rceil}$ such that $V_1, \ldots, V_{\lfloor s/2 \rfloor}$ span forests and if s is odd $V_{\lceil s/2 \rceil}$ spans an independent set. If $s \geq 5$, then an extension of Rödl's Graph [53] can be used to show that $\mathbf{RT}(n, H, o(n)) \geq \mathbf{RT}(n, K_s, o(n))$. For $s = 4$ no general lower bounds are known, even the simplest case of $H = K_{2,2,2}$ is still open. The exact threshold function for K_s for $s \geq 4$ is also still unknown, but Sudakov [59] showed, using the so called "dependent random choice method," that $\mathbf{RT}(n, K_4, n2^{-\theta\sqrt{\log n}}) = o(n^2)$, where $\theta = \theta(n)$ is any function going to infinity arbitrarily slowly. Note that $n2^{-\theta\sqrt{\log n}}/n^{1-\delta} \to \infty$ as $n \to \infty$ for any fixed δ.

No results about $\mathbf{RT}_t(n, H, o(n))$ for $t \geq 3$ were known until a paper of Erdős, Hajnal, Simonovits, Sós, and Szemerédi [20] in 1994. Their main results were that the limit in (2.1) exists when H is a complete graph, $\mathbf{RT}_t(n, K_s, o(n)) \leq \frac{1}{2}\left(1 - \frac{t}{s-1}\right) n^2$, and this is sharp for all $s \equiv 1 \pmod{t}$. Note that when $t = 2$, this is Erdős and Sós's [23] result. They also had results for some special cases: if $\ell = 1, 2, 3, 4$ or 5 and $\ell \leq t + 1$, then $\mathbf{RT}_t(n, K_{t+\ell}, o(n)) \leq \frac{\ell-1}{4t}n^2$. In their paper, Erdős et al. [20] provided a construction which they claimed proved that $\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2$. Unfortunately, the proof that the construction has small independence number relies on a theorem of Bollobás [11] which has been retracted as incorrect [10]. Therefore, no $\Omega(n^2)$ lower bounds were known for any $\mathbf{RT}_t(n, K_s, o(n))$ with $s \leq 2t$. Erdős et al. [20] posed several open problems.

Problem 5. ([20, Problem 2.12]) Find the minimum ℓ such that $\mathbf{RT}_t(n, K_{t+\ell}, o(n)) = \Omega(n^2)$.

In [21], Erdős et al. stated that to solve Problem 6 below "an analogue of the Bollobás-Erdős graph would be needed which we think will be extremely hard to find."

Problem 6. ([20, p. 306], [21, p. 80], and [58, Problem 17]) Determine if $\mathbf{RT}_3(n, K_5, o(n)) = o(n^2)$.

Problem 7. ([20, p. 306], [21, p. 80], and [58, Problem 19]) Erdős et al. [20] proved that

- $\mathbf{RT}_3(n, K_5, o(n)) \leq \frac{1}{12}n^2 + o(n^2)$,
- $\mathbf{RT}_3(n, K_6, o(n)) \leq \frac{1}{6}n^2 + o(n^2)$,

- $\mathbf{RT}_3(n, K_8, o(n)) \leq \frac{3}{11}n^2 + o(n^2)$,
- $\mathbf{RT}_3(n, K_9, o(n)) \leq \frac{3}{10}n^2 + o(n^2)$.

Are any of these bounds tight?

2.2 Results

The main result of this chapter is to solve Problems 5 and 6 by providing the first lower bounds of order $\Omega(n^2)$ for $\mathbf{RT}_t(n, K_s, o(n))$ when $t + 2 \leq s \leq 2t$.

Theorem 8. *For $t \geq 2$ and $2 \leq \ell \leq t$, let $u = \lceil \frac{t}{2} \rceil$. Then*

$$\mathbf{RT}_t(n, K_{t+\ell}, o(n)) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2} n^2.$$

For comparison, $\mathbf{RT}_t(n, K_{t+1}, o(n)) = o(n^2)$. This lower bound can be easily extended to all $\mathbf{RT}_t(n, K_{qt+\ell}, o(n))$ with $q \geq 1$ and $2 \leq \ell \leq t$. Let G be the extremal graph from Theorem 8 and let T be a complete $(q-1)$-partite graph with almost equal part sizes. In each part of T, insert a K_{t+1}-free graph with small K_t-independence number. (Such a graph exists by the Erdős-Rogers Theorem [22].) Lastly, completely join G and T. Any copy of $K_{qt+\ell}$ which appears in this graph can have at most t vertices in each part of T. This forces G to contain $t+\ell$ vertices of the copy of $K_{qt+\ell}$, which is a contradiction. This graph also has a small K_t-independence number. By optimizing the relative size of G and T, we obtain the following corollary.

Corollary 9. *For $t, q \geq 2$ and $2 \leq \ell \leq t$, let $u = \lceil \frac{t}{2} \rceil$. For any $0 < a < 1$,*

$$\mathbf{RT}_t(n, K_{qt+\ell}, o(n)) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2}(an)^2 + \binom{q-1}{2}\left(\frac{n-an}{q-1}\right)^2 + (n-an)an.$$

In order to compare with the upper bounds stated in Problem 7, we compute the above bound for $t = 3$ and small values of ℓ and q.

- $\mathbf{RT}_3(n, K_5, o(n)) \geq \frac{1}{64}n^2$.
- $\mathbf{RT}_3(n, K_6, o(n)) \geq \frac{1}{48}n^2$.
- $\mathbf{RT}_3(n, K_8, o(n)) \geq 0.253968n^2$ $(a \approx 0.508)$.
- $\mathbf{RT}_3(n, K_9, o(n)) \geq 0.255319n^2$ $(a \approx 0.511)$.

Note that in [20] it was proved that $\mathbf{RT}_3(n, K_7, o(n)) = \frac{1}{4}n^2 + o(n^2)$.

Theorem 8 is a corollary of a result about hypergraphs. Recall that $\mathrm{TK}^r(s)$ is the r-uniform hypergraph obtained from the complete graph K_s by enlarging each edge with $r-2$ new vertices. The **core vertices** of $\mathrm{TK}^r(s)$ are the s vertices of degree larger than one. For $s > r$, let $\mathcal{TK}^r(s)$

be the family of r-uniform hypergraphs such that there exists a set S of s vertices where each pair of vertices from S are contained together in some hyperedge. The set S is called the set of **core vertices** of the hypergraph.

Mubayi [45] showed that $\mathrm{ex}(n, \mathcal{TK}^r(s)) = |T_{r,s-1}(n)|$ and $\mathrm{ex}(n, \mathrm{TK}^r(s)) = (1+o(1))\,|T_{r,s-1}(n)|$ for $s > r$. Recently, Pikhurko [47] has shown that for large n, $\mathrm{ex}(n, \mathrm{TK}^r(s)) = |T_{r,s-1}(n)|$ and that $T_{r,s-1}(n)$ is the unique extremal example. Since the extremal hypergraphs have large independent sets, it is interesting to study the behavior of $\mathbf{RT}(n, \mathrm{TK}^r(s), f(n))$. One simple observation is the following.

Proposition 10. *For $r \geq 2$,*

- $\mathbf{RT}(n, \mathrm{TK}^r(r+1), o(n)) = o(n^r)$,
- $\mathbf{RT}(n, \mathcal{TK}^r(2r-1), o(n)) = o(n^r)$.

Proof. We only prove the three uniform case; the proof can be easily extended to r-uniform.

Let $\mathcal{H}$ be a 3-uniform, n-vertex hypergraph with independence number at most ϵn^3 and at least $9\epsilon n^3 + 144n^2$ edges. For simplicity, assume 3 divides n and let $\mathcal{H}'$ be a 3-partite subhypergraph of $\mathcal{H}$ with equal part sizes. We can always find such a hypergraph $\mathcal{H}'$ with at least $\frac{1}{9}$ of the edges of $\mathcal{H}$. Recall that for a pair of vertices x and y, the **codegree** is the number of vertices z such that $\{x, y, z\}$ is an edge. For each pair x, y of vertices in different parts, delete all edges containing x and y if the codegree is at most 16. We delete at most $16n^2$ hyperedges. Thus we have a 3-partite hypergraph $\mathcal{H}'$ with at least ϵn^3 edges and the codegree of any pair of vertices from different parts is zero or at least 16.

Since $\mathcal{H}'$ has at least ϵn^3 edges, the maximum codegree is at least ϵn. So let x, y be a pair of vertices with codegree at least ϵn, and let Z be the set of vertices z such that $\{x, y, z\}$ is an edge. Since the independence number of $\mathcal{H}$ is at most ϵn, there exists a hyperedge E of $\mathcal{H}$ contained in Z. The vertices in E together with x, y form a hypergraph in $\mathcal{TK}^3(5)$. (In the r-uniform case, the edge E together with $r-1$ vertices will form a copy of $\mathcal{TK}^r(2r-1)$.) Thus any 3-uniform, n-vertex, $\mathcal{TK}^3(5)$-free hypergraph with independence number at most ϵn can have at most $9\epsilon n^3 + 144n^2$ edges.

To find a copy of $\mathrm{TK}^3(4)$, let z_1 and z_2 be two vertices from E. The core vertices in a copy of $\mathrm{TK}^3(4)$ are x, y, z_1, and z_2. z_1 and z_2 are contained together in the edge E, and since x and z_1 are contained toegether in a hyperedge of $\mathcal{H}'$, the codegree of x and z_1 is at least 16. Thus we can find an edge of $\mathcal{H}$ containing x and z_1 where the third vertex avoids all previously used vertices. Similarly we can find edges containing x, z_2 and x, y and y, z_i where the third vertex has not yet been used. Thus we find a copy of $\mathrm{TK}^3(4)$ in $\mathcal{H}$. (In the r-uniform case, take as core vertices two vertices from E together with $r-1$ other vertices to find a copy of $\mathrm{TK}^r(r+1)$.) □

Using our construction, we prove the following lower bounds.

Theorem 11. *Let $r \geq 3$ and let $u = \lceil \frac{r}{2} \rceil$.*

(i) $\mathbf{RT}(n, \mathrm{TK}^r(r+2), o(n)) \geq 2^{-\binom{ur}{2} + r\binom{u}{2}} \left(\frac{n}{r}\right)^r$.

(ii) $\mathbf{RT}(n, \mathcal{TK}^r(2r), o(n)) \geq 2^{-\binom{ur}{2} + r\binom{u}{2}} \left(\frac{n}{r}\right)^r$.

Note that unlike the extremal case where for large n it is the case that $\mathrm{ex}(n, \mathrm{TK}^r(s)) = \mathrm{ex}(n, \mathcal{TK}^r(s))$, the Ramsey-Turán numbers for $\mathrm{TK}^r(s)$ and $\mathcal{TK}^r(s)$ are different. Let $\mathcal{F}^r(s)$ be the subfamily of $\mathcal{TK}^r(s)$ containing those hypergraphs where each edge contains exactly two core vertices. We suspect that $\mathcal{F}^r(s)$ will behave like $\mathrm{TK}^r(s)$. That is, we make the following conjecture:

Conjecture 12. $\mathbf{RT}(n, \mathcal{F}^r(s), o(n)) = \mathbf{RT}(n, \mathrm{TK}^r(s), o(n))$.

Theorems 8 and 11 are corollaries of the following theorem, which is our main result.

Theorem 13. *For any integer $r \geq 2$, any small $\alpha, \beta > 0$, and any integer N, there exists an $m \geq N$ and constants c_1, c_2 depending only on r such that there exists an rm-vertex, r-uniform hypergraph $\mathcal{G}$ with vertex partition $W_1, \ldots, W_r$ with $|W_i| = m$ and the following properties:*

- *$\mathcal{G}$ does not have as a subhypergraph any hypergraph in $\mathcal{TK}^r(4)$ arranged so that W_i contains two core vertices and W_j contains two core vertices for some $i \neq j$.*
- *For every i, $\mathcal{G}[W_i]$ does not have as a subhypergraph any connected hypergraph $\mathcal{F}$ with $|V(\mathcal{F})| \leq r^3$ and $|V(\mathcal{F})| < r + (r-1)(|\mathcal{F}| - 1)$.*
- *The independence number of $\mathcal{G}$ is at most $c_1 \beta m$.*
- *$|\mathcal{G}| \geq 2^{r\binom{u}{2} - \binom{ru}{2}} m^r - c_2 \alpha m^r$, where $u = \lceil r/2 \rceil$.*

Similar to Sudakov's [59] result for K_4, we show that if we force the independence number of a hypergraph with n vertices to be at most $n2^{-\theta(\log n)^{2/3}}$, then a $\mathrm{TK}^3(6)$-free hypergraph has $o(n^3)$ edges. The proof extends to all $\mathrm{TK}^r(2r)$ for $r \geq 3$ with independence number at most $n2^{-\theta(\log n)^{(r-1)/r}}$. We omit the proof which is similar to the proof of Theorem 14.

Theorem 14. *If $\theta(n)$ is any function going to infinity arbitrarily slowly, then* $\mathbf{RT}(n, \mathrm{TK}^3(6), f(n)) = o(n^3)$, *where $f(n) = n2^{-\theta(n)(\log n)^{2/3}}$.*

The remainder of this chapter is organized as follows. In Section 2.3 we state several properties of the k-dimensional unit sphere which will be used during the construction. In Section 2.4 we state two earlier constructions by Bollobás and Erdős [13] and Rödl [53]. In Section 2.5 we state our construction and prove several results about it, and in Section 2.6 we show how the construction in Section 2.5 can be modified to prove Theorems 8, 11, and 13. Lastly, in Section 2.7 we prove Theorem 14.

2.3 Properties of the unit sphere

Let μ be the Lebesgue measure on the k-dimensional unit sphere $\mathbb{S}^k \subseteq \mathbb{R}^{k+1}$ normalized so that $\mu(\mathbb{S}^k) = 1$. For $A \subseteq \mathbb{S}^k$, define $\mathrm{diam}(A) = \sup\{d(x,y) : x, y \in A\}$ where $d(x,y)$ is the Euclidean distance in $\mathbb{R}^{k+1}$. For $A, B \subseteq \mathbb{S}^k$, define $d_{\max}(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$.

A **spherical cap** is the intersection of the unit sphere $\mathbb{S}^k$ with a halfspace. The **center** of a spherical cap is the point in the spherical cap at maximum distance from H, where H is the

hyperplane bounding the halfspace. The **height** of a spherical cap is the minimum distance between the center and H and the **diameter** of a spherical cap is the diameter of the sphere formed by the intersection of the spherical cap with H. Note that if a is the maximum distance between the center and a point of the spherical cap and h is the height, then $2h = a^2$.

Given any $\alpha, \beta > 0$, it is possible to select $\epsilon > 0$ small enough and then k large enough so that Properties (P1), (P2), and (P3) are satisfied.

(P1) Let C be a spherical cap in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$ (this means all points of the spherical cap are within distance $\sqrt{2} - \epsilon/\sqrt{k}$ of the center). Then $\mu(C) \geq \frac{1}{2} - \alpha$.

(P2) Let $C_1, \ldots, C_t$ be spherical caps in $\mathbb{S}^k$ with height h, where $2h = \left(\sqrt{2} - \epsilon/\sqrt{k}\right)^2$. Let z_i be the center of C_i. Assume for all $1 \leq i \neq j \leq t$ that $d(z_i, z_j) \leq \sqrt{2}$. Then $\mu(C_1 \cap \ldots \cap C_t) \geq \frac{1}{2^t} - t\alpha$.

(P3) Let C be a spherical cap with diameter $2 - \epsilon/(2\sqrt{k})$. Then $\mu(C) \leq \beta$.

(P4) For any $0 < \gamma < \frac{1}{4}$, it is impossible to have $p_1, p_2, q_1, q_2 \in \mathbb{S}^k$ such that $d(p_1, p_2) \geq 2 - \gamma$, $d(q_1, q_2) \geq 2 - \gamma$, and $d(p_i, q_j) \leq \sqrt{2} - \gamma$ for all $1 \leq i, j \leq 2$.

(P5) Let $A \subseteq \mathbb{S}^k$ and let C be a spherical cap of the same measure. Then $\operatorname{diam}(A) \geq \operatorname{diam}(C)$.

(P6) Let $A, B \subseteq \mathbb{S}^k$ with equal measure and let C be a cap of the same measure. Then $d_{\max}(A, B) \geq \operatorname{diam}(C)$.

Properties (P1) and (P2) follow directly from the formula for the measure of a spherical cap, Properties (P3), (P5), and (P6) are all folklore results that are easy corollaries of the isoperimetric inequality on the sphere [40], and Property (P4) can be proved by examining distances and using the triangle inequality.

Proof of Property (P1). Let z be the center of C, and let C' be the cap of the sphere centered at z with height 1 such that $\mu(C') = \frac{1}{2}$. Let $A = C' \setminus C$. We need to prove that we can choose ϵ small enough so that $\mu(A) \leq \alpha$.

Recall that the surface area of the k-dimensional sphere of radius r in $\mathbb{R}^{k+1}$ is $(k+1)c_{k+1}r^k$, where $c_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$. If we slice A into $(k-1)$-dimensional spheres along the z-direction with x the distance from the origin, then $0 \leq x \leq \epsilon/\sqrt{k}$ covers all of A and each ball has radius $r = \sqrt{1-x^2}$ and surface area $kc_k r^{k-1}$. Therefore

$$\begin{aligned}
\mu(A) = \frac{\operatorname{Area}(A)}{(k+1)c_{k+1}} &\leq \frac{1}{(k+1)c_{k+1}} \int_0^{\epsilon/\sqrt{k}} kc_k \left(1-x^2\right)^{(k-1)/2} \mathrm{d}x \\
&\leq \frac{kc_k}{(k+1)c_{k+1}} \int_0^{\epsilon/\sqrt{k}} \left(1-x^2\right)^{(k-1)/2} \mathrm{d}x \\
&\leq \frac{c_k}{c_{k+1}} \int_0^{\epsilon/\sqrt{k}} \mathrm{d}x \\
&\leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{k+1}{2}+1)}{\Gamma(\frac{k}{2}+1)} \frac{\epsilon}{\sqrt{k}}.
\end{aligned}$$

Since $\Gamma(x+\frac{1}{2}) \approx \sqrt{x}\Gamma(x)$ we know that $\Gamma(\frac{k}{2}+\frac{3}{2}) \leq \sqrt{k}\Gamma(\frac{k}{2}+1)$. Thus $\mu(A) \leq \frac{\epsilon}{\sqrt{\pi}}$ so we can choose ϵ small enough so that $\mu(A) < \alpha$ for any dimension k. □

Proof of Property (P2). For each i, let C_i' be the cap C_i extended to height 1 so $\mu(C_i') = \frac{1}{2}$. Property (P1) shows that we can choose ϵ small enough so that for any dimension k, we have $\mu(C_i' \setminus C_i) < \alpha$ for all i. Since $d(z_1, z_2) \leq \sqrt{2}$, the smallest intersection between C_1' and C_2' is when C_2' cuts C_1' in half in which case $\mu(C_1' \cap C_2') = \frac{1}{4}$. This implies that $\mu(C_1' \cap C_2') \geq \frac{1}{4}$. Similarly, for $2 \leq i \leq t$ we have C_i' in the worst case cuts each of $C_1', C_2', \ldots, C_{i-1}'$ in half so that $\mu\big(\big(C_1' \cap \ldots \cap C_{i-1}'\big) \cap C_i'\big) \geq \frac{1}{2}\mu(C_1' \cap \ldots \cap C_{i-1}')$. Thus $\mu(C_1' \cap \ldots \cap C_t') \geq 2^{-t}$. Since $\mu(C_i' \setminus C_i) < \alpha$ for each i, $\mu(C_1 \cap \ldots \cap C_t) \geq 2^{-t} - t\alpha$. □

For Property (P3), we need the following theorem (called concentration in measure or the spherical isoperimetric inequality).

Theorem 15. *If $A \subseteq \mathbb{S}^k$, then $\mu(A_\delta) \geq 1 - \frac{2}{\mu(A)}e^{-k\delta^2/16}$, where $A_\delta = \{x : d(x, A) < \delta\}$.*

Proof of Property (P3). If C is a cap with diameter $2 - \epsilon/(2\sqrt{k})$ and $\delta = 1 - h$ where h is the height of C, then $\delta^2 + 1 = \left(1 - \epsilon/(2\sqrt{k})\right)^2$ so that $\delta^2 \approx \epsilon/\sqrt{k}$. Let A be a cap of the sphere with $\mu(A) = \frac{1}{2}$ where A_δ is disjoint from C (this is possible since $\delta = 1 - h$). By Theorem 15,

$$\mu(C) \leq 1 - \mu(A_\delta) \leq 4e^{-k\delta^2/16} \leq c_1 e^{-c_2\epsilon\sqrt{k}}.$$

For any fixed ϵ, we can make k large enough so that the measure of C is less than β. □

Proof of Property (P4). We will show that $(p_1 + p_2 - q_1 - q_2)^2 < 0$ which is a contradiction. Let $\delta = 2 - \gamma$ and $\eta = \sqrt{2} - \gamma$. We have

$$2p_1p_2 = (p_1^2 + p_2^2) - (p_1 - p_2)^2 < 2 - \delta^2 \tag{2.2}$$

since $p_1^2 = p_2^2 = 1$ and $(p_1 - p_2)^2 = d(p_1, p_2)^2 > \delta^2$. Similarly we have $2q_1q_2 < 2 - \delta^2$. Next,

$$2\sum_{i=1}^{2}\sum_{j=1}^{2} p_iq_j = \sum_{i=1}^{2}\sum_{j=1}^{2}(p_i^2 + q_j^2) - \sum_{i=1}^{2}\sum_{j=1}^{2}(p_i - q_j)^2 > 4(2 - \eta^2), \tag{2.3}$$

since $p_i^2 = q_j^2 = 1$ and $d(p_i, q_j) < \eta$. Combining (2.2) with (2.3) gives

$$\begin{aligned}
(p_1 + p_2 - q_1 - q_1)^2 &= p_1^2 + p_2^2 + q_1^2 + q_2^2 + p_1p_2 + q_1q_2 - 2\sum_{i=1}^{2}\sum_{j=1}^{2} p_iq_j \\
&< 4 + 4 - 2\delta^2 - 8 + 4\eta^2 \\
&= 4\left(\sqrt{2} - \gamma\right)^2 - 2\left(2 - \gamma\right)^2 \\
&= 4\left(2 - 2\sqrt{2}\gamma + \gamma^2\right) - 2\left(4 - 4\gamma + \gamma^2\right) \\
&= 8(1 - \sqrt{2})\gamma + 2\gamma^2.
\end{aligned}$$

This is negative since $\gamma < \frac{1}{4}$. □

The proof of Property (P6) closely follows the ideas in the proof in [11], even though [11] has been retracted as incorrect [10]. Before the proof, we state some definitions. Let A be a subset of $\mathbb{S}^k$ and $z \in \mathbb{S}^k$. The idea of the proof of Property (P6) is to define a compression operator $\gamma_z(A)$ (also called the *two-point symmetrization*) which moves points of A towards z. This will gradually transform A closer to a spherical cap centered at z. The key property is that when we fix z and apply γ_z to both A and B, it will not decrease $d_{\max}(A, B)$. For $x \in \mathbb{S}^k$, let $x^+(z) = x^+ = x - \langle x, z\rangle\, z + |\langle x, z\rangle|\, z$ and $x^-(z) = x^- = x - \langle x, z\rangle\, z - |\langle x, z\rangle|\, z$, where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^{k+1}$. The two points x^+ and x^- are reflections of each other across the hyperplane through the origin with the normal vector z. Thus $d(x^+, y^+) = d(x^-, y^-)$, $d(x^+, y^-) = d(x^-, y^+)$, and $d(x^-, y^+) \geq d(x^+, y^+)$. The compression operator γ_z of A is defined as follows.

$$\begin{aligned} &\text{If } A \cap \{x^+, x^-\} = \{x^-\} \quad &&\text{then} \quad \gamma_z(A) \cap \{x^+, x^-\} = \{x^+\}. \\ &\text{If } A \cap \{x^+, x^-\} \neq \{x^-\} \quad &&\text{then} \quad \gamma_z(A) \cap \{x^+, x^-\} = A \cap \{x^+, x^-\}. \end{aligned}$$

In other words, if we have x^- and not x^+ then we remove x^- and add x^+. Note, if A is measurable then so is $\gamma_z(A)$ and $\mu(\gamma_z(A)) = \mu(A)$.

Lemma 16. *If $A, B \subseteq \mathbb{S}^k$ and $z \in \mathbb{S}^k$, then $d_{max}(\gamma_z(A), \gamma_z(B)) \leq d_{max}(A, B)$.*

Proof. Assume $d_{\max}(\gamma_z(A), \gamma_z(B)) > d$ so there exist $a \in \gamma_z(A)$ and $b \in \gamma_z(B)$ with $d(a, b) \geq d$. It suffices to show $d_{\max}(A, B) \geq d$. Note that if $a \notin A$ then we must have had a^- in A and the compression removed a^- and added $a^+ = a$. There are six cases:

- If $a \in A$ and $b \in B$ then $d_{\max}(A, B) \geq d(a, b) \geq d$.
- If $a \notin A$ and $b \notin B$ then $a^+ = a$, $b^+ = b$, $a^- \in A$ and $b^- \in B$. Therefore, $d_{\max}(A, B) \geq d(a^-, b^-) = d(a^+, b^+) = d(a, b) \geq d$.
- If $a \notin A$ and $b \in B$ and $b^+ = b$, then $a^- \in A$ and $a^+ = a$ so $d_{\max}(A, B) \geq d(a^-, b^+) \geq d(a^+, b^+) = d(a, b) \geq d$.
- If $a \notin A$ and $b \in B$ and $b^- = b$, then $a^- \in A$ and $a^+ = a$. Also, $b = b^- \in \gamma_z(B) \cap B$ forces $b^+ \in B$. Therefore, $\operatorname{diam}(A, B) \geq d(a^-, b^+) = d(a^+, b^-) = d(a, b) \geq d$.
- The cases when $a \in A$ and $b \notin B$ are similar to the above two cases. □

Let H be the metric space of closed non-empty subsets of $\mathbb{S}^k$ with the Hausdorff metric

$$d_H(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\} = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}.$$

Let $H' = H \times H$ with the product topology and the max norm $d_{H'}((A, B), (C, D)) = \max\{d_H(A, C), d_H(B, D)\}$.

Lemma 17. *Let ν be a Borel measure on $\mathbb{S}^k$ and let $(A_1, B_1), (A_2, B_2), \ldots \in H'$, $(A_s, B_s) \to (A, B)$. Then $\nu(A) + \nu(B) \geq \lim_{s\to\infty} \nu(A_s) + \nu(B_s)$.*

Proof. Given $\epsilon > 0$, let $\delta > 0$ be such that $\nu(A_\delta) < \nu(A) + \epsilon/2$ and $\nu(B_\delta) < \nu(B) + \epsilon/2$, where $A_\delta = \{x \in \mathbb{S}^k : d(x, A) < \delta\}$. Choose s large enough so that $d_{H'}((A_s, B_s), (A, B)) \leq \delta$ so $A_s \subseteq A_\delta$ and $B_s \subseteq B_\delta$. Thus $\nu(A_s) + \nu(B_s) < \nu(A) + \nu(B) + \epsilon$. □

Lemma 18. *Let $(A_1, B_1), (A_2, B_2), \ldots \in H'$, $(A_s, B_s) \to (A, B)$. Then $d_{max}(A, B) = \lim_{s\to\infty} d_{max}(A_s, B_s)$.*

Proof. Let $\delta > 0$ and $X_1, X_2, Y_1, Y_2 \subseteq \mathbb{S}^k$ and suppose $d_H(X_1, X_2) < \delta$ and $d_H(Y_1, Y_2) < \delta$ where $X_1, X_2, Y_1, Y_2 \in H$. Take any $x_1 \in X_1$ and $y_1 \in Y_1$. Then there exists $x_2 \in X_2$ and $y_2 \in Y_2$ so that $d(x_1, x_2) \leq \delta$ and $d(y_1, y_2) \leq \delta$. Clearly $d(x_2, y_2) \geq d(x_1, y_1) - 2\delta$. Therefore, $d_{\max}(X_2, Y_2) \geq d_{\max}(X_1, Y_1) - 2\delta$. Symmetry shows $d_{\max}(X_1, Y_1) \geq d_{\max}(X_2, Y_2) - 2\delta$.

Hence, given $\epsilon > 0$ we can choose s large enough so $d_{H'}((A_s, B_s), (A, B)) \leq \epsilon/2$, which implies $d_H(A_s, A) \leq \epsilon/2$ and $d_H(B_s, B) \leq \epsilon/2$. Therefore, $|d_{\max}(A_s, B_s) - d_{\max}(A, B)| \leq \epsilon$. □

Proof of Property (P6). Let $A, B \subseteq \mathbb{S}^k$ with $\mu(A) = \mu(B) = m$ and let C be a cap of measure m. If $m = 0$ the result is trivial. Since $d_{\max}(A, B) = d_{\max}(\bar{A}, \bar{B})$ we can assume A and B are closed sets. Define $\gamma'_z : H' \to H'$ by $\gamma'_z(A, B) = (\gamma_z(A), \gamma_z(B))$. Let K be the minimal closed subset of H' containing (A, B) and closed under γ'_z for every $z \in \mathbb{S}^k$. For every pair (M, N) in K we have $\mu(M) \geq m$ and $\mu(N) \geq m$. By Lemmas 16 and 18, for all $(M, N) \in K$ we have $d_{\max}(M, N) \leq d_{\max}(A, B)$.

For a Borel subset M of $\mathbb{S}^k$, define $\nu(M) = \mu(M \cap C)$. Then ν is a Borel measure on $\mathbb{S}^k$ so by Lemma 17 the supremum of $\nu(M) + \nu(N)$ on K is attained on some $(M, N) \in K$. We just need to show that both M and N contain the cap C, since then $d_{\max}(A, B) \geq d_{\max}(M, N) \geq \operatorname{diam}(C)$.

Suppose M does not contain the cap C. Then there is a non-empty cap D such that $D \subseteq C \setminus M$. Let x be the center of D and h the height of D. Since $\mu(M) \geq \mu(C)$ we know $\mu(M \setminus C) > 0$. Thus there is a nonempty cap E of height at most h with $E \cap C = \emptyset$ and $\mu(M \cap E) > 0$. Let y be the center of E. We can assume the height of E is equal to h by reducing the height of D. Let $z = (x - y)/d(x, y)$. Then $\gamma_z(E) = D, \gamma_z(C) = C$ and $\gamma_z(C \setminus D) = C \setminus D$. Therefore,

$$\begin{aligned}\mu(\gamma_z(M) \cap C) &= \mu(\gamma_z(M) \cap (C \setminus D)) + \mu(\gamma_z(M) \cap D) \\ &\geq \mu(M \cap (C \setminus D)) + \mu(M \cap E) = \mu(M \cap C) + \mu(M \cap E) > \mu(M \cap C).\end{aligned}$$

Also, since $\gamma_z(C) = C$ we know $\mu(\gamma_z(N) \cap C) \geq \mu(N \cap C)$. Therefore,

$$\begin{aligned}\nu(\gamma_z(M)) + \nu(\gamma_z(N)) &= \mu(\gamma_z(M) \cap C) + \mu(\gamma_z(N) \cap C) \\ &> \mu(M \cap C) + \mu(N \cap C) \\ &= \nu(M) + \nu(N),\end{aligned}$$

but this contradicts the choice of (M, N) since $(\gamma_z(M), \gamma_z(N)) \in K$. The case when M contains the cap C but N does not contain C is symmetric. □

2.4 Former constructions

In this section, we describe two previous constructions; our construction will use ideas from both.

The Bollobás-Erdős Graph. [13] In order to prove that $\mathbf{RT}(n, K_4, o(n)) \geq \frac{n^2}{8}$, we need to construct, for every $\alpha, \beta > 0$, a K_4-free graph G with n vertices, independence number at most βn, and at least $\frac{n^2}{8}(1-\alpha)$ edges. Given $\alpha, \beta \geq 0$, take ϵ small enough and k large enough so that Properties (P1) and (P3) hold. Divide the k-dimensional unit sphere $\mathbb{S}^k$ into $n/2$ domains having equal measure and diameter at most $\frac{\epsilon}{10\sqrt{k}}$. Choose a point from each set and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domain of the sphere. Take as vertex set of G the disjoint union of two sets V_1 and V_2 both isomorphic to P. For $x, y \in V_i$ we make xy an edge of G if $d(x, y) \geq 2 - \epsilon/\sqrt{k}$. For $x \in V_1, y \in V_2$ we make xy an edge of G if $d(x, y) \leq \sqrt{2} - \epsilon/\sqrt{k}$. Then Property (P1) shows every vertex in V_1 has at least $\frac{1}{2}|V_2|(1-\alpha)$ neighbors in V_2 so the total number of edges is at least $\frac{1}{8}n^2(1-\alpha)$. If I is a set in V_1 with $|I| \geq \beta |V_1| = \beta\frac{n}{2}$, then $\mu(\phi(I)) = |I|/|P| \geq \beta$. Let C be a spherical cap of measure $\mu(\phi(I))$. Properties (P3) and (P5) show that $2 - \epsilon/(2\sqrt{k}) \leq \operatorname{diam}(C) \leq \operatorname{diam}(\phi(I))$. For $p \in I$, each $\phi(p)$ has diameter at most $\epsilon/(10\sqrt{k})$ so we can find two points $p_1, p_2 \in I$ with $d(p_1, p_2) \geq 2 - \epsilon/\sqrt{k}$, showing that I is not independent. Finally, Property (P4) shows this graph has no K_4 as a subgraph since any K_4 must take two vertices from V_1 and two vertices from V_2 (the graph spanned by V_i is triangle-free). To summarize, we have constructed a K_4-free graph G on n vertices with independence number at most βn and at least $\frac{1}{8}n^2(1-\alpha)$ edges. Since this construction holds for any $\alpha, \beta > 0$, we have proved that $\mathbf{RT}(n, K_4, o(n)) \geq \frac{n^2}{8}$.

The Rödl Graph. [53] A reasonable lower bound on $\mathbf{RT}(n, K_{2,2,2}, o(n))$ is still an open problem (we do not even know if it is $\Omega(n^2)$ or not). Erdős suggested that perhaps some modified version of the Bollobás-Erdős graph could be used to show it is $\Omega(n^2)$. In this direction, Rödl showed how to modify the Bollobás-Erdős graph to exclude both K_4 and $K_{3,3,3}$, proving that $\mathbf{RT}(n, \{K_4, K_{3,3,3}\}, o(n)) \geq \frac{n^2}{8}$. The Rödl Graph is formed by blowing up the Bollobás-Erdős Graph so each vertex is blown up into an independent set of size t and then randomly delete edges from inside each V_i (see Theorem 24). By randomly deleting edges inside each V_i, we can destroy (almost) all short cycles while not changing the density between V_1 and V_2. Since the original graph does not contain K_4, blowing up the graph will not produce any K_4s. Also, after removing all short cycles, any graph which is not the union of a bipartite graph with a forest, such as $K_{3,3,3}$, will not be a subgraph.

2.5 Construction

Erdős et al. [20] conjectured (see [20, Conjecture 2.9] and [58, Conjecture 18]) that the asymptotically extremal graphs for $\mathbf{RT}_t(n, K_s, o(n))$ with $s = tq + \ell$ $(1 \leq \ell \leq t)$ have the following structure. Partition n vertices into $q+1$ classes $V_0, \ldots, V_q$. For each pair $\{i, j\} \neq \{0, 1\}$ we almost completely join V_i to V_j and between V_0 and V_1 we place a graph with density $(\ell - 1)/t + o(1)$. Lastly, inside each V_i we insert $o(n^2)$ edges. By optimizing the sizes of the V_is, the number of edges in this graph

will be approximately

$$\left(1 - \frac{2t-\ell+1}{p(2t-\ell+1)-\ell+1}\right)\binom{n}{2}.$$

Erdős et al. [20] suggested that some modified version of the Bollobás-Erdős graph should be used between V_0 and V_1, but did not know how to reduce the density of the Bollobás-Erdős graph while still maintaining some useful properties. Our construction is a modified version of the Bollobás-Erdős graph where we are able to reduce the density to roughly 2^{-t^2}. Unfortunately this is too low to match the conjecture but still enough to give a $\Omega(n^2)$ lower bound on $\mathbf{RT}_t(n, K_s, o(n))$.

Our construction depends on four parameters: two integers r and z and two small constants $\alpha, \beta > 0$. Fix an integer $r \geq 3$. Given $\alpha, \beta > 0$, fix ϵ and k so that Properties (P1), (P3), and (P2) hold. Define $\theta = \epsilon/\sqrt{k}$ and $u = \lceil \frac{r}{2} \rceil$.

For sufficiently large integers z, divide the k-dimensional unit sphere $\mathbb{S}^k$ into z domains having equal measure and diameter at most $\theta/4$. Choose a point from each set and let P be the set of these points. Let $\phi : P \to \mathcal{P}(\mathbb{S}^k)$ map points of P to the corresponding domain of the sphere. The vertices of our hypergraph will be r copies of u-tuples of points from P. Define

$$V = \left\{(p_1, \ldots, p_u) : p_i \in P \text{ and } d(p_i, p_j) \leq \sqrt{2} - \theta \text{ for all } i \neq j\right\}.$$

We will denote vertices in V as $\vec{v}$ and $\left\langle v^{(1)}, \ldots, v^{(u)} \right\rangle$ as the coordinates of $\vec{v}$. Let $V_1, \ldots, V_r$ be distinct sets isomorphic to V. Let $\mathcal{H} = \mathcal{H}(r, z, \alpha, \beta)$ be the hypergraph with vertex set $V_1 \dot\cup \ldots \dot\cup V_r$ and the following hyperedges. For each $1 \leq i \leq r$, make $E = \{\vec{v}_1, \ldots, \vec{v}_r\} \subseteq V_i$ a hyperedge if $|E| = r$ and for every pair $\vec{v}_\ell, \vec{v}_m$ (with $\ell \neq m$), there exists a coordinate $1 \leq j \leq u$ such that $d(\vec{v}_\ell^{(j)}, \vec{v}_m^{(j)}) \geq 2 - \theta$. For cross-hyperedges, make $\{\vec{v_1}, \ldots, \vec{v_r}\} \subseteq V(\mathcal{H})$ a hyperedge if $\vec{v}_1 \in V_1, \ldots, \vec{v}_r \in V_r$ and $d(v_i^{(j)}, v_\ell^{(m)}) \leq \sqrt{2} - \theta$ for all $1 \leq i, \ell \leq r$ and all $1 \leq j, m \leq u$.

We first claim some properties of $\mathcal{H}$.

Lemma 19. *$\mathcal{H}$ does not contain any hypergraph in $\mathcal{TK}^r(4)$ arranged so that V_i contains two core vertices and V_j contains two core vertices with some $i \neq j$.*

Proof. Assume without loss of generality that there exists $\vec{v}_1, \vec{v}_2 \in V_1$ and $\vec{v}_3, \vec{v}_4 \in V_2$ which are all core vertices. We will find four points violating Property (P4). By the definition of hyperedges inside V_1, there is some coordinate i such that $d(v_1^{(i)}, v_2^{(i)}) \geq 2 - \theta$. Similarly, there is some coordinate j such that $d(v_3^{(j)}, v_4^{(j)}) \geq 2 - \theta$. By the definition of cross-hyperedges, we know that all cross distances are at most $\sqrt{2} - \theta$. We therefore obtain four points $v_1^{(i)}, v_2^{(i)}, v_3^{(j)}, v_4^{(j)}$ which contradict Property (P4). □

Lemma 20. *Let $A_1, A_2 \subseteq P$ with $|A_1| = |A_2| \geq 2\beta z$ and let $t = |A_1|/2$. Then there exists t distinct points $p_1, \ldots, p_t \in A_1$ and t distinct points $q_1, \ldots, q_t \in A_2$ such that $d(p_i, q_i) \geq 2 - \theta$.*

Proof. Let G be the auxiliary bipartite graph on vertex set $A_1 \dot\cup A_2$ where $p \in A_1$ and $q \in A_2$ are adjacent if $d(p, q) \geq 2 - \theta$. We would like to find a matching of size at least t in G. Let M be a maximum matching in G, and assume $|E(M)| < t$. Let $G' = G - V(M)$ with $A_1' = A_1 - V(M)$ and

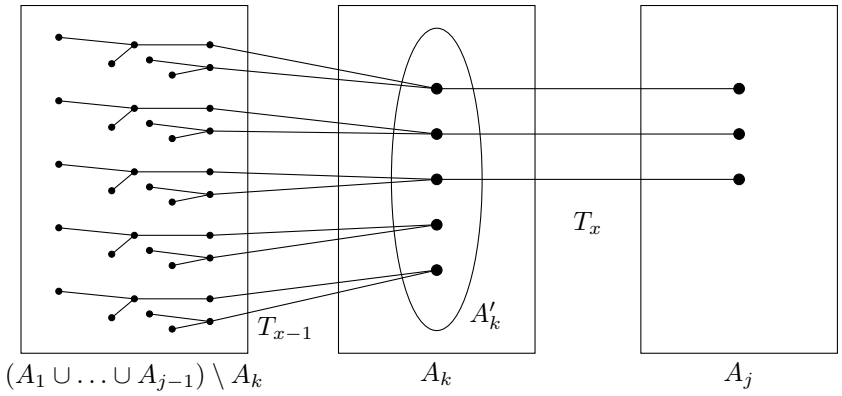


Figure 2.1: Embedding T during Lemma 21.

$A'_2 = A_2 - V(M)$. We will show that G' does not span an independent set, contradicting that M is a maximum matching.

Since $|E(M)| < t$, $|A'_i| > t \geq \beta z$. Let $B_i = \phi(A'_i)$ so that $\mu(B_i) = |A'_i|/z \geq \beta$. Let C be a spherical cap of measure β so $\mu(B_i) \geq \mu(C)$. Properties (P3) and (P6) show that $2 - \theta/2 \leq \text{diam}(C) \leq d_{\max}(B_1, B_2)$. Since each $\phi(p)$ has diameter at most $\theta/4$, we must have $p \in A'_1$ and $q \in A'_2$ with $d(p,q) \geq 2 - \theta$. In other words, pq is an edge of G' which contradicts that M was a maximum matching. □

Lemma 21. *If $A_1, \ldots, A_r \subseteq P$ with $|A_i| \geq 2^r \beta z$ and T is a tree on vertex set $[r]$, then there exists $p_1 \in A_1, \ldots, p_r \in A_r$ such that if $ij \in E(T)$ then $d(p_i, p_j) \geq 2 - \theta$.*

Proof. Assume $|A_i| = 2^r \beta z$. Let G be the auxiliary graph on vertex set $A_1 \dot\cup \ldots \dot\cup A_r$ where $p \in A_i$ and $q \in A_j$ are adjacent if $i \neq j$ and $d(p,q) \geq 2 - \theta$. We would like to find an embedding of T into G such that $i \in V(T)$ is embedded into A_i.

Let $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{r-1} = T$ be subtrees of T where T_x is formed by deleting a leaf of T_{x+1}. We prove by induction on x that we can find $2^{r-x}\beta z$ vertex disjoint embeddings of T_x into G such that $i \in V(T_x)$ is embedded into A_i. Since T_1 is just a single edge, Lemma 20 shows that we can find $|A_i|/2 = 2^{r-1}\beta z$ vertex disjoint embeddings of T_1.

Assume $x \geq 2$. By induction, we can find at least $2^{r-x+1}\beta z$ vertex disjoint embeddings of T_{x-1} into G. Let $j \in V(T_x)$ be the leaf of T_x deleted to form T_{x-1} and let k be the neighbor of j in T_x. Let A'_k be the set of vertices in A_k used by the embeddings of T_{x-1}, so that $|A'_k| \geq 2^{r-x+1}\beta z$. We now apply Lemma 20 to A'_k and A_j to find a matching between A'_k and A_j using at least $|A'_k|/2 \geq 2^{r-x}\beta z$ edges. Since the vertices of this matching are distinct, at least $2^{r-x}\beta z$ of the embeddings of T_{x-1} extend to embeddings of T_x. □

Lemma 22. *For all i, $\alpha(\mathcal{H}[V_i]) \leq r^u 2^{u+r} \beta z^u$.*

Proof. Without loss of generality, fix any set $X \subseteq V_1$ with $|X| = r^u 2^{u+r} \beta z^u$. Let $T_1, \ldots, T_u$ be trees such that $V(T_i) = [r]$ and $\cup T_i$ is the complete graph on vertex set $[r]$. This is possible because

$u = \lceil \frac{r}{2} \rceil$. Note that if $\vec{v}_1, \ldots, \vec{v}_r \in X$ such that $d(v_i^{(j)}, v_i^{(\ell)}) \geq 2-\theta$ when $j\ell \in E(T_i)$, then $\{\vec{v}_1, \ldots, \vec{v}_r\}$ forms a hyperedge inside X. We will find these vertices by repeatedly applying Lemma 21.

Let $0 \leq j < u$. Assume we have already selected $v_1^{(1)}, \ldots, v_1^{(j)}, v_2^{(1)}, \ldots, v_2^{(j)}, \ldots, v_r^{(1)}, \ldots, v_r^{(j)}$, that is coordinates 1 through j for all r vertices to be found. For each i, define the set of candidates to continue the future vertex $\vec{v}_i$ as

$$C_i^{(j)} = \left\{ \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, q_{j+1}, \ldots, q_u \right\rangle \in X : q_{j+1}, \ldots, q_u \in P \right\}.$$

Initially, $C_i^{(0)} = X$. Throughout the selection process we maintain that the size of $\left|C_i^{(j)}\right|$ is at least $r^{u-j}2^{u-j+r}\beta z^{u-j}$.

We now show how to select $v_1^{(j+1)}, \ldots, v_r^{(j+1)}$. For $1 \leq i \leq r$, call a tuple $(v_i^{(1)}, \ldots, v_i^{(j)}, p)$ bad if

$$\left| \left\{ \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, p, q_{j+2}, \ldots, q_u \right\rangle \in C_i^{(j)} : q_{j+2}, \ldots, q_u \in P \right\} \right| < r^{u-j-1}2^{u-j-1+r}\beta z^{u-j-1}.$$

Form D_i by deleting all vertices $\vec{w}$ from $C_i^{(j)}$ where the first $j+1$ coordinates of $\vec{w}$ form a bad tuple. Counting the number of vertices we delete, there are r choices for i, there are at most z choices for p, and there are at most $r^{u-j-1}2^{u-j-1+r}\beta z^{u-j-1}$ choices for the rest of the coordinates. Thus the number of vertices we delete is at most $r^{u-j}2^{u-j-1+r}\beta z^{u-j}$ so $|D_i| \geq r^{u-j}2^{u-j-1+r}\beta z^{u-j}$.

Now define

$$A_i = \left\{ p \in P : \exists q_{j+2}, \ldots, q_u \in P \text{ where } \left\langle v_i^{(1)}, \ldots, v_i^{(j)}, p, q_{j+2}, \ldots, q_u \right\rangle \in D_i \right\}.$$

If $|A_i| < 2^r\beta z$, then $|D_i| < 2^r\beta z^{u-j} \leq 2^{u-j+r}\beta z^{u-j}$ which is a contradiction. Now apply Lemma 21 to $A_1, \ldots, A_r$ and T_{j+1} to obtain $v_1^{(j+1)} \in A_1, \ldots, v_r^{(j+1)} \in A_r$. Since none of the tuples $(v_i^{(1)}, \ldots, v_i^{(j+1)})$ are bad,

$$\left|C_i^{(j+1)}\right| \geq r^{u-j-1}2^{u-j-1+r}\beta z^{u-j-1}$$

for every i. □

Lemma 23. *Let $\mathcal{E} = \{\{\vec{v}_1, \ldots, \vec{v}_r\} \in \mathcal{H} : \vec{v}_i \in V_i\}$. Then there exists a constant c depending only on r such that*

$$|V(\mathcal{H})| \leq r2^{-\binom{u}{2}}z^u$$

and

$$|\mathcal{E}| \geq 2^{-\binom{ru}{2}}z^{ru} - c\alpha z^{ru} \geq 2^{r\binom{u}{2}-\binom{ru}{2}}\left(\frac{|V(\mathcal{H})|}{r}\right)^r - c\alpha\left|V(\mathcal{H})\right|^r.$$

Proof. By Property (P2), each V_i has size at most $z\prod_{i=1}^{u-1}\left(\frac{z}{2^i} - i\alpha z\right)$ so the number of vertices is

at most $r2^{-\binom{u}{2}}z^u$. Using Property (P2) there are at least

$$z\prod_{i=1}^{ru-1}\left(\frac{z}{2^i}-i\alpha z\right)$$

choices of ru points on the sphere with pairwise distance at most $\sqrt{2}-\theta$. Each of these ru-sets of points form a cross-hyperedge. □

2.6 Proofs of Theorems 8, 11, and 13

We now turn our attention to proving Theorems 8, 11, and 13. Consider the construction $\mathcal{H}$ from Section 2.5 and assume $\mathrm{TK}^r(r+2)$ is a subhypergraph. Lemma 19 tells us it is impossible to have two core vertices in two different parts, so we must have three core vertices in some part. $\mathcal{H}$ itself may contain a copy of $\mathrm{TK}^r(3)$ inside one part, but by using an idea of Rödl [53] we are able to eliminate this possibility by blowing up the hypergraph $\mathcal{H}$. In [53], Rödl proved a variant of the following theorem for graphs and the special case when $\mathcal{F}$ is a cycle.

Theorem 24. *Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices. Let $0<\gamma<1$ and let ℓ be a positive integer. Then there exists a $t=t(\mathcal{H},\ell,\gamma,r)$ and an r-uniform hypergraph $\mathcal{G}$ with vertex set $V(\mathcal{H})\times[t]$ with the following properties.*

- *For all $\{a_1,\dots,a_r\}\in\mathcal{H}$ and all sets $U_i\subseteq\{a_i\}\times[t]$ with $|U_i|\geq\gamma t$ for each $1\leq i\leq r$, there exists at least one hyperedge of $\mathcal{G}$ with one vertex in each U_i.*
- *$\mathcal{G}$ does not contain as a subhypergraph any v-vertex hypergraph $\mathcal{F}$ with m edges where $v\leq\ell$ and $v+(1+\gamma-r)(m-1)<r$.*

Proof. Let $\mathcal{H}'$ be the t-blowup of $\mathcal{H}$. That is, $V(\mathcal{H}')=V(\mathcal{H})\times[t]$ and the hyperedges are $\{\{(a_1,i_1),(a_2,i_2),\dots,(a_r,i_r)\}:\{a_1,\dots,a_r\}\in\mathcal{H},1\leq i_1,\dots,i_r\leq t\}$. Let $\mathcal{H}''$ be a random subhypergraph of $\mathcal{H}'$ where each hyperedge is chosen independently with probability $p=t^{1+\gamma-r}$ (note that $r\geq 2$ and γ is small so that $p<1$). Let $\mathcal{F}$ be a v-vertex hypergraph with $|\mathcal{F}|=m$ and where $v+(1+\gamma-r)(m-1)<r$. The expected number of copies of $\mathcal{F}$ in $\mathcal{H}''$ is bounded by $c_1t^vp^m=o(pt^r)$ where c_1 is some constant depending only on $\mathcal{H}$ and ℓ. We now delete one hyperedge from each copy of $\mathcal{F}$ in $\mathcal{H}''$. There are at most 2^{ℓ^r} such hypergraphs $\mathcal{F}$ so we can make t sufficiently large so that we delete fewer than $\frac{\gamma^r}{2}pt^r$ hyperedges. $\mathcal{G}$ is the resulting graph which now satisfies the second property.

Now fix a hyperedge $E=\{a_1,\dots,a_r\}\in\mathcal{H}$ and $U_i\subseteq\{a_i\}\times[t]$ with $|U_i|=\gamma t$ for $1\leq i\leq r$. We now show that the probability that all blowups of the hyperedge E intersecting all V_i are deleted is exponentially small. Before deletion, the expected number of blowups of E where the copy of a_i appears in U_i for each i is $p(\gamma t)^r$. By Chernoff's Inequality, the probability there are at most $\frac{1}{2}p(\gamma t)^r$ such blowups of E is bounded by $e^{-c_2pt^r}$ where c_2 is some constant depending only on γ. Since we only delete $\frac{1}{2}p(\gamma t)^r$ hyperedges in total, the probability we delete all blowups of E where the copy of a_i appears in U_i for each i is at most $e^{-c_2pt^r}$.

We now use the union bound to bound the probability that there is some hyperedge $E = \{a_1, \ldots, a_r\} \in \mathcal{H}$ and some $U_i \subseteq \{a_i\} \times [t]$ with $|U_i| = \gamma t$ for $1 \leq i \leq r$ where we deleted all blowups of the edge E where the copy of a_i appears in U_i for each i. This probability is bounded by

$$|\mathcal{H}| \binom{t}{\gamma t}^r e^{-c_2 p t^r} \leq |\mathcal{H}| \left(\frac{e}{\gamma}\right)^{\gamma r t} e^{-c_2 p t^r} \leq |\mathcal{H}|\, e^{c_3 t} e^{-c_4 t^{1+\gamma}} = O(e^{-t^{1+\gamma}})$$

where c_3 and c_4 are constants depending only on γ and r. □

By combining the construction from Section 2.5 and the previous theorem, we are able to prove Theorem 13.

Proof of Theorem 13. Let $z = N$ and let $\mathcal{H} = \mathcal{H}(r, z, \alpha, \beta)$ be the hypergraph constructed in Section 2.5 and $V_1, \ldots, V_r$ the partition of the vertex set of $\mathcal{H}$. Let $\mathcal{E}_1$ be the set of cross-hyperedges, that is $\mathcal{E}_1 = \{\{\vec{v}_1, \ldots, \vec{v}_r\} \in \mathcal{H} : \vec{v}_i \in V_i\}$ and let $\mathcal{E}_2 = \mathcal{H} - \mathcal{E}_1$ so $\mathcal{E}_2$ is the set of hyperedges which are inside some V_i. Let $\gamma = \beta$ and $\ell = r^3$ and apply Theorem 24 to $\mathcal{E}_2$ to obtain $\mathcal{E}_2'$ where $V(\mathcal{E}_2') = V(\mathcal{H}) \times [t]$. Let $\mathcal{G}$ be $\mathcal{E}_2'$ together with all the hyperedges

$$\{\{(\vec{v}_1, a_1), \ldots, (\vec{v}_r, a_r)\} : \{\vec{v}_1, \ldots, \vec{v}_r\} \in \mathcal{E}_1, 1 \leq a_i \leq t\}.$$

Let $m = |V_i|\, t \approx 2^{-u(u-1)/2} z^u t$ so that $\mathcal{G}$ has rm vertices, and let $W_i = V_i \times [t]$. Since we blow up all cross hyperedges, Lemma 19 shows that $\mathcal{G}$ has no hypergraph in $\mathcal{TK}^r(4)$ arranged so that W_i has two core vertices and W_j has two core vertices. By Lemma 23, $|\mathcal{E}_1| \geq 2^{-\binom{ru}{2}} z^{ru} - c_2 \alpha z^{ru}$. Because we blow up all cross hyperedges, $\mathcal{G}$ has at least $2^{-\binom{ru}{2}} z^{ru} t^r - c_2 \alpha z^{ru} t^r$ hyperedges where c_2 is some constant depending only on r. Also, Theorem 24 shows that $\mathcal{G}[W_i]$ does not contain as a subhypergraph any hypergraph $\mathcal{F}$ with $|V(\mathcal{F})| \leq r^3 = \ell$ and $|V(\mathcal{F})| + (1-r)(|\mathcal{F}| - 1) < r$.

Let I be a vertex set in $\mathcal{G}[W_1]$ with $|I| = r^u 2^{u+r+1} \beta z^u t$. For $\vec{v} \in V_1$, call $\vec{v}$ **γ-bad** if there are fewer than γt indices $1 \leq i \leq t$ such that $(\vec{v}, i) \in I$. Form I' by deleting all pairs $(\vec{v}, i)$ from I where $\vec{v}$ is γ-bad. We deleted at most $z^u \gamma t = z^u \beta t$ so $|I'| \geq r^u 2^{u+r} \beta z^u t$. Define $A = \{\vec{v} \in V_1 : (\vec{v}, i) \in I' \text{ for some } 1 \leq i \leq t\}$. Then $|A| \geq r^u 2^{u+r} \beta z^u$ so by Lemma 22 we must have a hyperedge $\{\vec{v}_1, \ldots, \vec{v}_r\}$ contained in $\mathcal{H}[A] \subseteq \mathcal{E}_2$. Define $B_i = (\{\vec{v}_i\} \times [t]) \cap I'$. Since each $\vec{v}_i$ is not γ-bad we have $|B_i| \geq \gamma t$ for every $1 \leq i \leq r$. By Theorem 24 there exists a hyperedge of $\mathcal{E}_2'[W_1] \subseteq \mathcal{G}[W_1]$ with one vertex in each B_i, which is a hyperedge contained in I. This shows that the independence number of $\mathcal{G}[W_i]$ is at most $r^u 2^{u+r+1} \beta z^u t$ for each $1 \leq i \leq r$, which implies that $\mathcal{G}$ has independence number at most $r^{u+1} 2^{u+r+1} \beta z^u t \leq c_1 \beta m$, where c_1 is a constant depending only on r. □

Proof of Theorem 11 part (i). Let $\mathcal{G}$ be the construction from Theorem 13 and assume that $\mathrm{TK}^r(r+2)$ is a subhypergraph. Since we cannot have two core vertices in two different parts, the copy of $\mathrm{TK}^r(r+2)$ must have three core vertices in one part. Let $\mathcal{F} = \mathrm{TK}^r(3)$. Then $|\mathcal{F}| = 3$ and $|V(\mathcal{F})| = 3 + 3(r-2) = 3r - 3 < r + 2(r-1) = r + (r-1)(|\mathcal{F}| - 1) = 3r - 2$ which contradicts Theorem 13.

Let $n = |V(\mathcal{G})|$. From Theorem 13, we know that $n = rm$ and that

$$|\mathcal{G}| \geq 2^{r\binom{u}{2}-\binom{ru}{2}}m^r - c_1\alpha m^r = 2^{r\binom{u}{2}-\binom{ru}{2}}\left(\frac{n}{r}\right)^r - c_2\alpha n^r,$$

where c_1 and c_2 are constants depending only on r. Thus for any $\alpha > 0$, we know that

$$\begin{aligned}\lim_{\beta\to 0}\lim_{n\to\infty}\frac{\mathbf{RT}(n, \mathrm{TK}^r(r+2), \beta n)}{n^r} &\geq \lim_{\beta\to 0}\lim_{n\to\infty}\frac{2^{r\binom{u}{2}-\binom{ru}{2}}\left(\frac{n}{r}\right)^r - c_2\alpha n^r}{n^r}\\ &\geq 2^{r\binom{u}{2}-\binom{ru}{2}}\left(\frac{1}{r}\right)^r - c_2\alpha\end{aligned}$$

so $\mathbf{RT}(n, \mathrm{TK}^r(r+2), o(n)) \geq 2^{r\binom{u}{2}-\binom{ru}{2}}(n/r)^r - c_2\alpha n^r$ for any α. □

Proof of Theorem 11 part (ii). The proof is similar to the proof of Theorem 11 part (i). Any copy of a hypergraph in $\mathcal{TK}^r(2r)$ cannot have two core vertices in two different parts so it must have at least $r+1$ core vertices in a single part. To complete the proof, we just need to show that every minimal hypergraph $\mathcal{F}$ in $\mathcal{TK}^r(r+1)$ satisfies $|V(\mathcal{F})| \leq r + (r-1)(|\mathcal{F}| - 1)$. Let $v_1, \ldots, v_{r+1}$ be the core vertices of $\mathcal{F}$. For $1 \leq a < b \leq r+1$, since $\mathcal{F} \in \mathcal{TK}^r(r+1)$ there exists some hyperedge containing both $\vec{v}_a$ and $\vec{v}_b$. Let $E_{a,b}$ be a hyperedge of containing both $\vec{v}_a$ and $\vec{v}_b$ (if there is more than one such hyperedge, pick one arbitrarily.) Now consider the ordering

$$E_{1,2}, E_{1,3}, \ldots, E_{1,r+1}, E_{2,3}, \ldots, E_{2,r+1}, E_{3,4}, \ldots, E_{r,r+1}.$$

Since $\mathcal{F}$ is minimal, all hyperedges of $\mathcal{F}$ appear in the ordering somewhere. Now let $F_1, \ldots, F_m$ be a list of the hyperedges of $\mathcal{F}$ where for each hyperedge $D \in \mathcal{F}$, we keep the first copy of D in the ordering and remove all other copies. By the choice of ordering, each F_i must use at least one vertex from the previous hyperedges. Therefore, $|V(\mathcal{F})| \leq r + (r-1)(m-1)$. In fact, the last hyperedge must use at least two previous vertices so we can reduce the bound by one to $|V(\mathcal{F})| \leq r + (r-1)(m-1) - 1$. □

We will now show that Theorem 8 follows by looking at the shadow graph of the hypergraph from Theorem 13. If $\mathcal{H}$ is a hypergraph, the **shadow graph** of $\mathcal{H}$ is a graph G with $V(G) = V(\mathcal{H})$ and $xy \in E(G)$ if and only if there exists some hyperedge E of $\mathcal{H}$ with $x, y \in E$.

Proof of Theorem 8. Let $r = t$ and let $\mathcal{G}$ be the hypergraph constructed in Theorem 13 with parts $W_1, \ldots, W_r$. Let G be the shadow graph of $\mathcal{G}[W_1 \cup W_2 \cup \ldots \cup W_\ell]$, so we only take the shadow graph of the first ℓ parts. Then $\alpha_t(G)$ is small because any hyperedge inside $\mathcal{G}[W_i]$ turns into a copy of K_t in $G[W_i]$ for $1 \leq i \leq \ell$. Assume G contains $K_{t+\ell}$. It is not possible to have two of the vertices in W_i and two of the vertices in W_j with $i \neq j$ because then $\mathcal{G}$ would contain a $\mathcal{TK}^r(4)$ arranged so that two core vertices are in W_i and two core vertices are in W_j. Thus we can assume without loss of generality that $G[W_1]$ contains K_{t+1}. This implies that $\mathcal{G}[W_1]$ contains a hypergraph in $\mathcal{TK}^r(r+1)$ which was forbidden by the proof of Theorem 11 part (ii).

To compute the number of edges of G, we must use Property (P1). Edges between W_i and W_j are chosen by picking $2u$ points within distance $\sqrt{2}-\theta$ on the sphere and then blowing each vertex up into size t. Therefore, we have at least $t^2 z \prod_{i=1}^{2u-1} \left(\frac{z}{2^i} - i\alpha z\right)$ edges between W_i and W_j. Thus

$$|E(G)| \geq \binom{\ell}{2} 2^{-\binom{2u}{2}} z^{2u} t^2 - c_1 \alpha z^{2u} t^2, \tag{2.4}$$

where c_1 is some constant depending only on r. Each W_i has size at most $2^{-\binom{u}{2}} z^u t$ so G has at most $\ell 2^{-\binom{u}{2}} z^u t$ vertices. Thus

$$\frac{2^{\binom{u}{2}}}{\ell} |V(G)| \leq z^u t. \tag{2.5}$$

Combining (2.4) with (2.5), we obtain

$$\begin{aligned} |E(G)| &\geq \binom{\ell}{2} 2^{-\binom{2u}{2}} \left(\frac{2^{\binom{u}{2}}}{\ell} |V(G)|\right)^2 - c_2 \alpha |V(G)|^2 \\ &\geq \frac{1}{2} \frac{\ell(\ell-1)}{\ell^2} 2^{u(u-1)-u(2u-1)} |V(G)|^2 - c_2 \alpha |V(G)|^2 \\ &\geq \frac{1}{2} \left(1 - \frac{1}{\ell}\right) 2^{-u^2} |V(G)|^2 - c_2 \alpha |V(G)|^2 \end{aligned}$$

for some constant c_2 depending only on r. □

2.7 Lower bounds on Ramsey-Turán thresholds

The main tool to prove Theorem 14 is dependent random choice. Dependent random choice is a simple yet surprisingly powerful technique which has found applications in Extremal Graph Theory, Ramsey Theory, Additive Combinatorics, and Combinatorial Geometry. Early versions of this technique were proved and applied by several researchers, starting with Gowers, Kostochka, Rödl, and Sudakov. Gowers [31] used a variant of dependant random choice in an alternate proof of Szemerédi's Theorem [61] for four-term arithmetic progressions, Kostochka and Rödl [39] used dependent random choice to investigate bipartite Ramsey numbers, and Sudakov [59] used dependent random choice to prove $\mathbf{RT}(n, K_4, 2^{-\theta\sqrt{\log n}}) = o(n^2)$. Since then, many other applications of the dependent random choice method have been found (see [28] for a survey).

Lemma 25. *(Dependent Random Choice, Lemma 2.1 in [28]). Let a, d, m, n, r, t be positive integers. Let G be an n-vertex graph with average degree $d = 2|E(G)|/n$. If*

$$\frac{d^t}{n^{t-1}} - \binom{n}{r} \left(\frac{m}{n}\right)^t \geq a,$$

then G contains a subset U of at least a vertices such that every r vertices in U have at least m common neighbors.

Conlon, Fox, and Sudakov [16] extended Lemma 25 to hypergraphs in a paper where they investigated the Ramsey numbers of sparse hypergraphs. The **weight** $w(S)$ of a set S of hyperedges in a hypergraph is the number of vertices in the union of these edges.

Lemma 26. *(Hypergraph Dependent Random Choice, Lemma 1 in [16]). Suppose s, Δ are positive integers, $\epsilon, \beta > 0$, and G_r is an r-uniform, r-partite hypergraph with parts $V_1, \ldots, V_r$, each part having size N. Suppose G_r has at least ϵN^r edges. Then there exists an $(r-1)$-uniform, $(r-1)$-partite hypergraph G_{r-1} on the vertex set $V_2 \cup \ldots \cup V_r$ which has at least $\frac{\epsilon^s}{2} N^{r-1}$ edges and such that for each nonnegative integer $w \leq (r-1)\Delta$, there are at most $4r\Delta\epsilon^{-s}\beta^s w^{r\Delta} r^w N^w$ dangerous sets of edges of G_{r-1} with weight w, where a set S of edges of G_{r-1} is dangerous if $|S| \leq \Delta$ and the number of vertices $v \in V_1$ such that for every edge $e \in S$, $e + v \in G_r$ is less than βN.*

The main idea of the proof of Theorem 14 is to first apply Lemma 26 to obtain a graph G and then apply Lemma 25 to G. Lemma 25 guarantees a set U which is large enough so that we can find a hyperedge E_3 contained inside U. The vertices of E_3 then have a large number of common neighbors in G, enough common neighbors to find a hyperedge E_2 among the common neighbors. Then the hypergraph dependent random choice lemma shows that we can extend the edges of G spanned by $E_2 \cup E_3$ to hyperedges. We are thus able to find the following hypergraph. Let F be the 3-uniform hypergraph with vertices $\{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ and edges $\{x_1x_2x_3, y_1y_2y_3, z_1z_2z_3\} \cup \{x_iy_jz_k : 1 \leq i, j, k \leq 3\}$. Note that $F \in \mathcal{TK}^3(9)$. For a 3-uniform hypergraph, the **codegree** $d(x, y)$ of a pair of vertices x, y is the number of edges E with $x, y \in E$.

Theorem 27. *Let $\theta = \theta(n)$ be any function going to infinity arbitrarily slowly. Let $\beta = \beta(n) = 2^{-\theta(\log n)^{2/3}}$. There exists a constant b such that if $\mathcal{H}$ is an n-vertex, 3-uniform hypergraph with independence number at most $\frac{1}{3}\beta n$ and at least $bn^3 2^{-\theta^3/28} + 144n^2 = o(n^3)$ edges, then $\mathcal{H}$ contains F and $\mathrm{TK}^3(6)$.*

Proof. Let $N = \frac{n}{3}, \Delta = 9, w = 6, r = 3, c = 4r\Delta w^{r\Delta} r^w, s = \frac{w+1}{\theta}\sqrt[3]{\log n}$, and $\epsilon = 2^{-\theta^2\sqrt[3]{\log n}/4}$. Let $b = 9c$, so $\mathcal{H}$ has at least $9c\epsilon^{1/s}N^3 + 144n^2$ edges and independence number at most $\frac{1}{3}\beta n$.

For simplicity, assume 3 divides n and let $\mathcal{H}'$ be a 3-partite subhypergraph of $\mathcal{H}$ with equal part sizes. We can always find such a hypergraph $\mathcal{H}'$ with at least $\frac{1}{9}$ of the edges of $\mathcal{H}$. For each pair x, y of vertices in different parts, delete all edges containing both x and y if the codegree $d(x, y)$ is at most 16. We delete at most $16n^2$ hyperedges. Thus we have a 3-partite hypergraph $\mathcal{H}'$ with at least $c\epsilon^{1/s}N^3$ edges and the codegree of any pair of vertices from different parts is zero or at least 16. Let V_1, V_2, V_3 be the parts of $\mathcal{H}'$, each part having size N.

We now apply Lemma 26 to $\mathcal{H}'$ to obtain a graph G on $V_2 \cup V_3$ with at least $\frac{1}{2}\left(c\epsilon^{1/s}\right)^s N^2 \geq 2\epsilon N^2$ edges and at most

$$4r\Delta\left(c\epsilon^{1/s}\right)^{-s}\beta^s w^{r\Delta} r^w N^w \leq \epsilon^{-1}\beta^s N^6 \leq 2^{\theta^2\sqrt[3]{\log n}/4 - 7\log n} N^6$$

dangerous sets of edges of weight 6. When n is large, the number of dangerous sets is at most $1/2$ so we can assume G has no dangerous sets of weight 6.

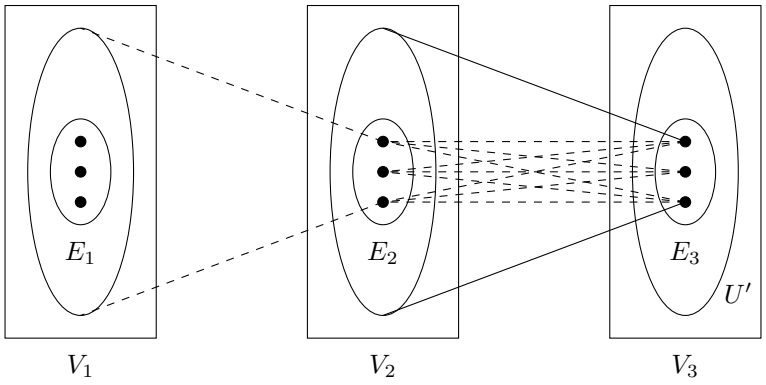


Figure 2.2: Embedding F in Theorem 27.

We now apply Lemma 25 to G with $t = \frac{4}{\theta}\sqrt[3]{\log n}$, $d = 4\epsilon N$, and $a = m = 2\beta N$. Let $n_1 = |V(G)| = 2N$. We check

$$\frac{d^t}{n_1^{t-1}} - \binom{n_1}{3}\left(\frac{m}{n_1}\right)^t \geq 4\epsilon^t N - 2N^3\beta^t \geq 2^{2-\theta(\log n)^{2/3}}N - 2^{1-4\log n}N^3$$
$$\geq 4\beta N - \frac{1}{2} \geq a = m.$$

Therefore we have a subset U of $V(G)$ with $|U| = m = 2\beta N$ such that every three vertices of U have at least βN common neighbors in G. Either V_2 or V_3 contains at least half of the vertices of U, so assume by symmetry that $U' = U \cap V_3$ has at least βN vertices.

U' contains a hyperedge E_3 of $\mathcal{H}$ since the size of U' is larger than the independence number of $\mathcal{H}$. The vertices of E_3 have at least βN common neighbors in G, so the common neighbors contain a hyperedge E_2. G is bipartite by Lemma 26 so $E_3 \subseteq V_3$ implies that $E_2 \subseteq V_2$. If we take S to be the 9 edges of G spanned by the vertices $E_2 \cup E_3$, then S has weight 6 so it is not dangerous. Therefore, we find at least βN vertices v in V_1 such that vxy is a hyperedge for all $x \in E_2$ and all $y \in E_3$. These βN vertices contain a hyperedge E_1 of $\mathcal{H}$.

Let $E_1 = \{x_1, x_2, x_3\}$, $E_2 = \{y_1, y_2, y_3\}$, and $E_3 = \{z_1, z_2, z_3\}$. These vertices form a copy of F within $\mathcal{H}$. We also find a copy of $\mathrm{TK}^3(6)$ with core vertices $x_1, x_2, y_1, y_2, z_1, z_2$. x_1, x_2 are contained together in the hyperedge $x_1x_2x_3$. Since x_i and y_j are contained together in at least one hyperedge of $\mathcal{H}'$, the codegree of x_i and y_j in $\mathcal{H}$ is at least 16. We can therefore find a distinct vertex in V_3 which is contained in a hyperedge together with x_i and y_j. The pairs x_i, z_j and y_i, z_j are handled similarly. □

2.8 Open problems

There are many open problems remaining in Ramsey-Turán theory.

- The exact value of $\mathbf{RT}_3(n, K_s, o(n))$ for small values of s are mostly still unknown. Erdős et

al. [20] proved that $\mathbf{RT}_3(n, K_s, o(n)) = \frac{1}{2}\left(1 - \frac{3}{s-1}\right) n^2 + o(n^2)$ when $s \equiv 1 \pmod 3$. Also, we know $\mathbf{RT}_3(n, K_4, o(n)) = o(n^2)$. The best bound for $s = 5$ is our lower bound of $\frac{1}{64}n^2$ and an upper bound of $\frac{1}{12}n^2$ by Erdős et al. [20]. For $s = 6$, the best lower bound is our lower bound of $\frac{1}{48}n^2$ and an upper bound of $\frac{1}{6}n^2$ by Erdős et al. [20]. For $\mathbf{RT}_3(n, K_8, o(n))$ and $\mathbf{RT}_3(n, K_9, o(n))$, Erdős et al. [20] proved upper bounds of $\frac{3}{11}n^2$ and $\frac{3}{10}n^2$ respectively and conjectured that the upper bounds are tight. The best lower bounds for K_8 and K_9 come from Corollary 9, which are $\mathbf{RT}_3(n, K_8, o(n)) \geq 0.253968n^2$ and $\mathbf{RT}_3(n, K_9, o(n)) \geq 0.255319n^2$.

- Erdős et al. [20, Conjecture 2.9] conjectured what the extremal graphs for $\mathbf{RT}_t(n, K_s, o(n))$ look like (see also [58, Conjecture 18]). According to that conjecture, $\mathbf{RT}_3(n, K_{11}, o(n)) = \frac{11}{32}n^2 + o(n^2)$ and $\mathbf{RT}_3(n, K_{14}, o(n)) = \frac{8}{21}n^2 + o(n^2)$ (see [58, Problem 25]). Our lower bounds from Corollary 9 are $\mathbf{RT}_3(n, K_{11}, o(n)) \geq 0.335106n^2$ (where $a \approx 0.34$) and $\mathbf{RT}_3(n, K_{14}, o(n)) \geq 0.376n^2$ (where $a \approx 0.256$.)

- In the area of the Ramsey-Turán theory, one of the major open problems is to prove a generalization of the Erdős-Stone Theorem [26] by proving that $\mathbf{RT}(n, H, o(n)) = \mathbf{RT}(n, K_s, o(n))$ where s is equal to some parameter depending only on H. Erdős et al. prove an upper bound using a parameter closely related to the arboricity. That is, they prove one can take s to be the minimum s such that $V(H)$ can be partitioned into $\lceil s/2 \rceil$ sets $V_1, \ldots, V_{\lceil s/2 \rceil}$ such that $V_1, \ldots, V_{\lfloor s/2 \rfloor}$ span forests and if s is odd $V_{\lceil s/2 \rceil}$ spans an independent set. For $s \geq 5$, an extension of Rödl's graph [53] can be used to prove that $\mathbf{RT}(n, H, o(n)) \geq \mathbf{RT}(n, K_s, o(n))$. What is the situation when $s = 4$? In several papers, Erdős mentioned the problem of solving the simplest open case when $H = K_{2,2,2}$, where one would like to have a lower bound of $\frac{1}{8}n^2 + o(n^2) = \mathbf{RT}(n, K_4, o(n))$. Even the question of determining if $\mathbf{RT}(n, K_{2,2,2}, o(n)) = \Omega(n^2)$ is still open (see [58, Problem 4], [21, p. 72], [59, Problem 1.3] among others).

- In their paper, Erdős et al. [20] gave a construction which they claim proved $\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2$. Unfortunately, the proof that the construction has small independence number relies on a theorem of Bollobás [11] which has been retracted as incorrect [10]. In [11], the following question was considered. For $A \subseteq \mathbb{S}^k$ and $t \geq 2$, define

$$d_t(A) = \sup\left\{\min_{i \neq j} d(x_i, x_j) : x_1, \ldots, x_t \in A\right\}.$$

  Is it true that if C is a spherical cap with $\mu(C) = \mu(A)$, then $d_t(A) \geq d_t(C)$? If this was true as claimed in [20], then $\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2$. In a private communication, Bollobás [10] provided the following counterexample. Take C to be a cap of the sphere in three dimensions with small but positive measure and let C' be another cap of the same measure which is far from C. Let $A = C \cup C'$. Then if $\mu(C)$ is small enough we can approximate C and C' by circles with radius r. Then $d_3(A) \approx 2r$ since we can take two points of C and one point of C'. But if D is a cap with the same measure as A then D has radius about $\sqrt{2}r$ so $d_t(D) \approx \sqrt{6}r > d_3(A)$. This counterexample can be extended to higher dimensions and more than three points, but only seems to work when C has small measure. Is it possible to

prove $d_t(A) \geq d_t(C)$ when we have some lower bound on the measure of C, perhaps a lower bound in terms of the dimension? Perhaps a slightly weaker statement like the following is true. Let $\delta = \delta_t$ be the diameter of the t-simplex ($\delta_t = \sqrt{\frac{2t}{t-1}}$). Let ϵ and $\eta > 0$ be fixed. Then for $k > k_0(t, \epsilon, \eta)$, if A is a measurable subset of $\mathbb{S}^k$ with $\mu(A) > \epsilon$, then there exist t points $x_1, \ldots, x_t \in A$ such that all $d(x_i, x_j) > \delta_t - \eta$. This slightly weaker statement is all that is needed to prove that $\mathbf{RT}_t(n, K_{2t}, o(n)) \geq \frac{1}{8}n^2$ (see [20, p. 315]).

Chapter 3

Chromatic Thresholds of Hypergraphs

3.1 Results

Definition. Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The **chromatic threshold** of $\mathcal{F}$ is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number.

The study of the chromatic thresholds of graphs was motivated by a question of Erdős and Simonovits [24]: "If G is non-bipartite, what bound on $\delta(G)$ forces G to contain a triangle?" This question was answered by Andrásfai, Erdős, and Sós [2], where they proved the following theorem.

Theorem 28. *If G is an n-vertex, triangle free graph with $\delta(G) > \frac{2n}{5}$, then G is bipartite.*

The blowup of C_5 shows the above theorem is sharp. The obvious next question is to ask what happens when the minimum degree drops below $2n/5$. This was answered by Häggkvist [34], where he proved the following theorem.

Theorem 29. *If G is an n-vertex, triangle free graph with $\delta(G) > \frac{3n}{8}$, then either G is bipartite or G is a subgraph of a blowup of C_5.*

Let H be the 8-vertex graph which is C_8 together with all four chords which connect vertices at distance four along the cycle. Then the blowup of H shows that the above theorem is sharp. This construction can be extended by taking longer cycles and adding more chords across the cycle; add three more vertices to the cycle and add one more chord incident to each vertex. This sequence of constructions has minimum degree approaching $1/3$. From the other side, Hajnal (see [24]) proved the following.

Theorem 30. *For every $t \geq 3$ and every $\epsilon > 0$, there exists a triangle-free graph G with $\delta(G) > (\frac{1}{3} - \epsilon)\,|V(G)|$ and $\chi(G) = t$.*

These results lead Erdős and Simonovits [24] to the following conjecture: if $\delta(G) > (1/3 + \epsilon)\,|V(G)|$ and G is triangle-free, then $\chi(G) < k_\epsilon$, where k_ϵ is a constant depending only on ϵ. Note that the conjecture is equivalent to the statement that the family of triangle-free graphs has chromatic threshold $1/3$. The conjecture was proven by Thomassen [63]. Subsequently, there have

The material presented in this chapter is joint work with József Balogh, Jane Butterfield, Ping Hu, and Dhruv Mubayi. It was submitted to Advances in Math under the title *On the Chromatic Thresholds of Hypergraphs*, see [5]

been three more proofs of the conjecture: one by Łuczak [42] using the Regularity Lemma, a result of Brandt and Thomassé [14] proving that one can take $k_\epsilon = 4$, and a recent proof by Łuczak and Thomassé [43] using the concept of Vapnik-Chervonenkis dimension (which is defined later in this paper).

For other graphs, Goddard and Lyle [30] proved that the chromatic threshold of the family of K_r-free graphs is $(2r-5)/(2r-3)$ while Thomassen [64] showed that the chromatic threshold of the family of C_{2k+1}-free graphs is zero for $k \geq 2$. Recently, Łuczak and Thomassé [43] gave another proof that the class of C_{2k+1}-free graphs has chromatic threshold zero for $k \geq 2$, as well as several other results about related families, such as Petersen-free graphs. The main result of Allen, Böttcher, Griffiths, Kohayakawa and Morris [1] is to determine the chromatic threshold of the family of H-free graphs for all H.

Definition. Let H be an r-uniform hypergraph. We say that H is **near r-partite** if there exists a partition $V_1 \cup \ldots \cup V_r$ of $V(H)$ such that all edges of H either cross the partition (have one vertex in each V_i) or are contained entirely in V_1, and in addition $H[V_1]$ is a partial matching. The edges in $H[V_1]$ are called the **special edges**. Say that H is **mono near r-partite** if $H[V_1]$ contains exactly one edge.

Our main theorem claims that for an infinite family of hypergraphs H the chromatic threshold of the family of H-free hypergraphs is 0. This is the first (non-trivial) family of hypergraphs whose chromatic threshold is determined.

Theorem 31. *Let H be an r-uniform, near r-partite hypergraph. If H is $\mathcal{TK}^r(3)$-free, then the chromatic threshold of the family of H-free hypergraphs is zero.*

The proof of Theorem 31 requires a slightly weaker condition on H, so we actually prove a statement slightly stronger than Theorem 31. The proof does not require that H is $\mathcal{TK}^r(3)$-free, just that any copy of a hypergraph in $\mathcal{TK}^r(3)$ in H has at most one core vertex in a special edge of H.

For a subfamily of the hypergraphs considered in Theorem 31 we determine the exact extremal hypergraph. For many hypergraphs H (for example the Fano plane), at first only asymptotic extremal results were proved and later the precise structure of extremal hypergraphs was determined. We prove that if a mono near r-partite hypergraph H has Turán density $r!/r^r$ and is stable with respect to $T_r(n)$, then its unique extremal hypergraph is the complete r-partite hypergraph. Similar phenomena occur for graphs; see Simonovits [57], where for critical graphs the Erdős-Stone Theorem [26] was sharpened.

Definition. Let H be an r-uniform hypergraph. We say that H is **critical** if

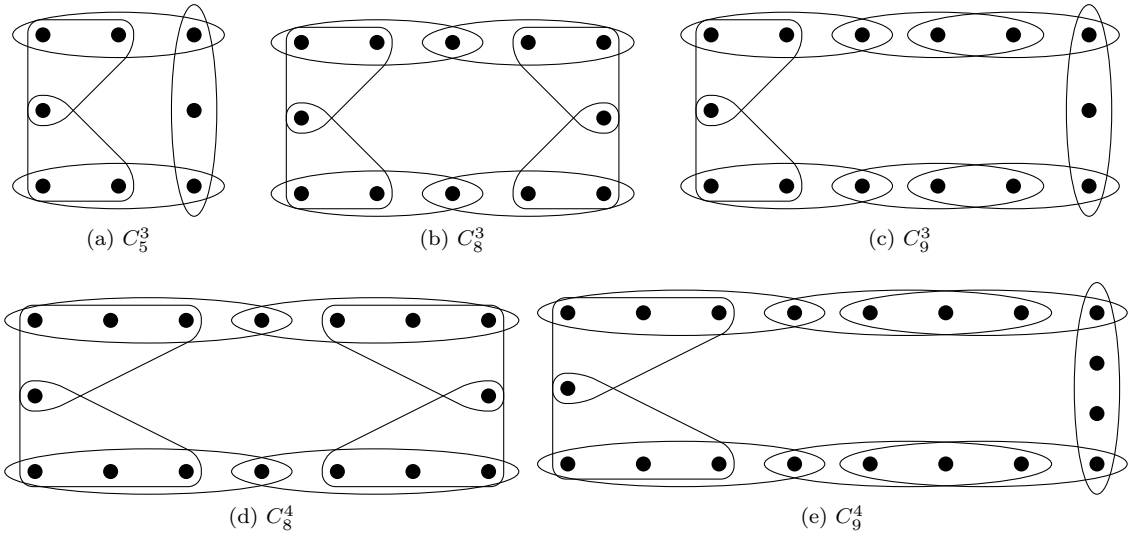
- H is mono near r-partite,
- the special edge of H has at least $r-2$ vertices of degree one,
- $\pi(H) = r!/r^r$,
- H is stable with respect to $T_r(n)$.

Theorem 32. *Let H be an r-uniform critical hypergraph. Then there exists some n_0 such that for $n > n_0$, $T_r(n)$ is the unique H-free hypergraph with the most edges.*

A particularly interesting critical family is one that generalizes cycles to hypergraphs.

Definition. Let C_m^r be the r-uniform hypergraph with m edges on n vertices $v_1, \ldots, v_n$ for which

1. the n vertices are arranged consecutively in a circle,
2. each edge contains r consecutive vertices,
3. if $m = 2k + 1$ for some integer $k > 0$ then $n = rk + (r - 1)$, and if $m = 2k$ then $n = rk$,
4. edges E_i and E_j share vertices if and only if $i \in \{j - 1, j + 1\}$ or $i = 1$ and $j = m$,
5. for $i \leq m - 1$, if i is odd then $|E_i \cap E_{i+1}| = 1$; if i is even then $|E_i \cap E_{i+1}| = r - 1$, and
6. if m is even then $|E_1 \cap E_m| = 1$; if m is odd then $|E_1 \cap E_m| = r - 1$.



(a) C_5^3 (b) C_8^3 (c) C_9^3

(d) C_8^4 (e) C_9^4

Figure 3.1: Hypergraph Cycles

We say that C_m^r is **odd** if m is odd, and **even** otherwise.

Lemma 33. *If m is odd then C_m^r is not r-partite.*

Proof. Suppose $m = 2k + 1$ for some integer K. Notice that edge E_{2k+1} consists of the vertices $v_{rk+1}, v_{rk+2}, \ldots, v_{rk+r-1}, v_1$. Suppose $f : V \to [r]$ is an r-coloring of the vertices of C_{2k+1}^r such that each color class induces a strongly independent set. Then vertices $v_1, v_{r+1}, v_{2k+1}, \ldots, v_{rk+1}$ must all have the same color. In particular, $f(v_1) = f(v_{rk+1})$, which is a contradiction because v_1 and v_{rk+1} are both in E_{2k+1}. □

It is easy to see that C_{2k+1}^r is mono near r-partite. A theorem of Keevash and Mubayi [38], combined with a theorem of Pikhurko [48], the supersaturation result of Erdős and Simonovits [25], and the hypergraph removal lemma of Gowers, Nagle, Rödl, and Skokan [32, 46, 54, 55, 62] prove that C_{2k+1}^3 and C_{2k+1}^4 are critical. It remains an open question whether C_{2k+1}^r is critical for $r \geq 5$.

Theorem 34. *The cycles C^3_{2k+1} and C^4_{2k+1} are critical for every $k \geq 2$.*

Theorems 31, 32, and 34 together with the simple observation that C^r_{2k+1} is near r-partite and $\mathcal{TK}^r(3)$-free for all $r, k \geq 2$ proves the following corollary, which extends the results in [64] and [43] that the chromatic threshold of the family of C_{2k+1}-free graphs is zero.

Corollary 35. *For $r = 3$ or $r = 4$, there exists some n_0 such that for $n > n_0$, the unique n-vertex, r-uniform, C^r_{2k+1}-free hypergraph with the largest number of edges is $T_r(n)$. For all $r, k \geq 2$, the chromatic threshold of the family of C^r_{2k+1}-free hypergraphs is zero.*

Note that Łuczak and Thomassé [43] proved Theorem 31 for graphs, and they conjectured that the family of H-free graphs has chromatic threshold zero if and only if H is near acyclic and triangle free. (A graph G is **near acyclic** if there exists an independent set S in G such that $G - S$ is a forest and every odd cycle has at least two vertices in S.) This conjecture is announced to have been verified in [1]. We pose a similar question for hypergraphs.

Problem 36. *Characterize the r-uniform hypergraphs H for which the chromatic threshold of the family of H-free hypergraphs has chromatic threshold zero.*

The rest of the chapter is organized as follows. First, in Section 3.2 we define and motivate fiber bundles and fiber bundle dimension, the main tools in the proof of Theorem 31. Next, in Section 3.3 we show the power of fiber bundle dimension by giving a relatively short proof of Theorem 31. We prove our key theorem about fiber bundle dimension, Theorem 37, in Section 3.4. In Section 3.5, we prove that C^r_{2k+1} is critical (Theorem 34), and then prove Theorem 32.

3.2 Fiber bundles and fiber bundle dimension

The proof of Theorem 31 is based on an idea pioneered by Łuczak and Thomassé [43] to color graphs, which itself was based on the Vapnik-Chervonenkis dimension. Let H be a hypergraph. A subset X of $V(H)$ is **shattered** by H if for every $Y \subseteq X$, there exists an $E \in H$ such that $E \cap X = Y$. Introduced in [56] and [67], the **Vapnik-Chervonenkis dimension** (or VC-dimension) is the maximum size of a vertex subset shattered by H.

Definition. A **fiber bundle** is a tuple (B, γ, F) such that B is a hypergraph, F is a finite set, and $\gamma : V(B) \to 2^{2^F}$. That is, γ maps vertices of B to collections of subsets of F, which we can think about as hypergraphs on vertex set F. The hypergraph B is called the **base hypergraph** of the bundle and F is the **fiber** of the bundle. For a vertex $b \in V(B)$, the hypergraph $\gamma(b)$ is called the **fiber over** b.

We should think about a fiber bundle as taking a base hypergraph and putting a hypergraph "on top" of each base vertex. There is one canonical example of a fiber bundle. Given a hypergraph B, define the **neighborhood bundle of** B to be the bundle (B, γ, F) where $F = V(B)$ and γ maps $b \in V(B)$ to $\{A \subseteq F : A \cup \{b\} \in E(B)\}$.

Why define and use the language of fiber bundles? We can consider that in some sense fiber bundles are a generalization of directed graphs to hypergraphs, where we think of $\gamma(x)$ as the "out-neighborhood" of x. In the neighborhood bundle, $\gamma(x)$ is related to the neighbors of x so we can consider the neighborhood bundle as some sort of directed analogue of the undirected hypergraph B, where each edge is directed "both ways". By thinking of the "out-neighborhood" of x as $\gamma(x)$ and not requiring any dependency between $\gamma(x)$ and $\gamma(y)$ for $x \neq y$, we have no dependency between the neighborhood of x and the neighborhood of y, which is one of the defining differences between directed and undirected graphs. Note that the definition of a fiber bundle differs from the usual definition of *directed hypergraph* used in the literature, which is the reason we use the term "fiber bundle" instead of "directed hypergraph."

A fiber bundle (B, γ, F) is (r_b, r_γ)**-uniform** if B is an r_b-uniform hypergraph and $\gamma(b)$ is an r_γ-uniform hypergraph for each $b \in V(B)$. Given $X \subseteq V(B)$, the **section of** X is the hypergraph with vertex set F and edges $\cap_{x \in X} \gamma(x)$. In other words, the section of X is the collection of subsets of F that appear in every fiber over x for $x \in X$. Motivated by a definition of Łuczak and Thomassé [43], we define the dimension of a fiber bundle. Let H be a hypergraph and define $\dim_H(B, \gamma, F)$ to be the maximum integer d such that there exist d disjoint edges $E_1, \ldots, E_d$ of B (i.e. a matching) such that for every $x_1 \in E_1, \ldots, x_d \in E_d$, the section of $\{x_1, \ldots, x_d\}$ contains a copy of H. Our definition of dimension will coincide with the definition of paired VC-dimension in [43] when (B, γ, F) is $(2, 1)$-uniform and $H = \{\{x\}\}$, the complete 1-uniform, 1-vertex hypergraph.

Let A be an r-uniform hypergraph. Our method of proving an upper bound on the chromatic threshold of the family of A-free hypergraphs, used in Theorem 31, is the following. Let G be an A-free r-uniform hypergraph with minimum degree at least $c\binom{|V(G)|}{r-1}$. We now need to show that G has bounded chromatic number, which we do in two steps. Let (G, γ, F) be the neighborhood bundle of G. First, we show that the dimension of (G, γ, F) is bounded by showing that if the dimension is large then we can find A as a subhypergraph. Given that $\dim_H(G, \gamma, F)$ is bounded, we use the following theorem to bound the chromatic number of G. In most applications, we will let H be an $(r-1)$-uniform, $(r-1)$-partite hypergraph.

Theorem 37. *Let $r_b \geq 2$, $r_\gamma \geq 1$, $d \in \mathbb{Z}^+$, $0 < \epsilon < 1$, and H be an r_γ-uniform hypergraph with zero Turán density. Then there exists constants $K_1 = K_1(r_b, r_\gamma, d, \epsilon, H)$ and $K_2 = K_2(r_b, r_\gamma, d, \epsilon, H)$ such that the following holds. Let (B, γ, F) be any (r_b, r_γ)-uniform fiber bundle where $dim_H(B, \gamma, F) < d$ and for all $b \in V(B)$,*

$$|\gamma(b)| \geq \epsilon \binom{|F|}{r_\gamma}.$$

If $|F| \geq K_1$, then $\chi(B) \leq K_2$.

The above theorem is sufficent for our purposes, but our proof of Theorem 37 proves something slightly stronger. The conclusion of the above theorem can be reworded to say that either F is small, the chromatic number of B is bounded, or $\dim_H(B, \gamma, F)$ is large, which means that we can find d hyperedges $E_1, \ldots, E_d$ such that every section of $x_1 \in E_1, \ldots, x_d \in E_d$ contains a copy of H. In fact, the proof shows that if F is large and the chromatic number of B is large, we can guarantee

not only one copy of H but at least $\Omega(|F|^h)$ copies of H in each section, where h is the number of vertices in H.

We conjecture a similar statement for all r_γ-uniform hypergraphs H, instead of just those hypergraphs with a Turán density of zero.

Conjecture 38. *Let $r_b \geq 2$, $r_\gamma \geq 1$, $d \in \mathbb{Z}^+$, $0 < \epsilon < 1$, and H be an r_γ-uniform hypergraph. Then there exists a constants $K_1 = K_1(r_b, r_\gamma, d, \epsilon, H)$ and $K_2 = K_2(r_b, r_\gamma, d, \epsilon, H)$ such that the following holds. Let (B, γ, F) be any (r_b, r_γ)-uniform fiber bundle where $dim_H(B, \gamma, F) < d$ and for all $b \in V(B)$,*

$$|\gamma(b)| \geq (\pi(H) + \epsilon) \binom{|F|}{r_\gamma}.$$

If $|F| \geq K_1$, then $\chi(B) \leq K_2$.

The motivation behind defining and using the language of fiber bundles rather than using the language of hypergraphs is that in the course of the proof of Theorem 37, we will modify B and γ and apply induction. As mentioned above, fiber bundles can be thought of as a directed version of a hypergraphs. When applying Theorem 37 in Section 3.3, we start with the neighborhood bundle, which carries no "extra" information beyond just the hypergraph B. But if we tried to prove Theorem 37 in the language of hypergraphs, we would run into trouble when we needed to modify γ. In the neighborhood bundle, γ is related to the neighborhood of a vertex and if we restricted ourselves to neighborhood bundles or just used the language of hypergraphs, modifying $\gamma(x)$ would imply that some $\gamma(y)$'s would change at the same time. The notion of a fiber bundle allows us to change the "out-neighborhood" of x independently of changing the "out-neighborhood" of $y \neq x$, and this power is critical in the proof of Theorem 37.

3.3 Chromatic threshold of near r-partite hypergraphs

In this section we show an application of Theorem 37 by proving Theorem 31. Fix $\epsilon > 0$ and let G be an n-vertex, r-uniform, H-free hypergraph with $\delta(G) \geq \epsilon\binom{n}{r-1}$. We would like to use Theorem 37 to bound the chromatic number, so we need to choose an appropriate bundle. We will not use the neighborhood bundle of G, but rather a closely related bundle. Once we have defined this bundle, we show it has bounded dimension by proving that if the dimension is large then we can find a copy of H in G.

As preparation, we need the following lemma, which tells us something about the structure of near r-partite graphs.

Lemma 39. *Let H be an r-uniform, near r-partite, $\mathcal{TK}^r(3)$-free hypergraph. Let $E_1, \ldots, E_k$ be the special edges of H. For $x \in V(H)$, let $\gamma(x) = \{E - x : x \in E \in H\}$. Let $N_1, \ldots, N_{r^k}$ be the r^k possible hypergraphs $\gamma(x_1) \cap \gamma(x_2) \cap \ldots \cap \gamma(x_k)$ where $x_1 \in E_1, \ldots, x_k \in E_k$. Then*

- *$V(N_i) \cap V(N_j) = \emptyset$ for $i \neq j$ (let $V(A) = \cup_{E \in A} E$).*

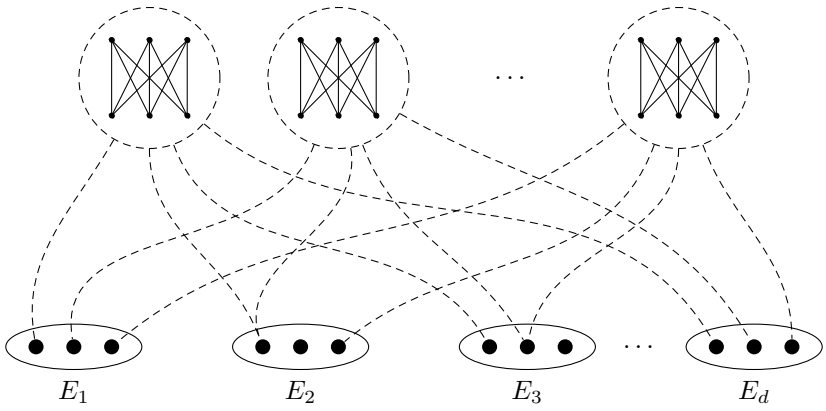


Figure 3.2: The structure guaranteed by dimension d.

- *N_i is $(r-1)$-partite for every i.*

Proof. Assume $x \in V(N_i) \cap V(N_j)$ and let E_t be a special edge such that N_i selects y from E_t and N_j selects z from E_t with $y \neq z$. Then x, y, z are the core vertices of some hypergraph in $\mathcal{TK}^r(3)$, a contradiction. Secondly, N_i is $(r-1)$-partite because $H \setminus E_1 \setminus \ldots \setminus E_k$ is r-partite by the definition of near r-partite. □

Proof of Theorem 31. Let H be an r-uniform, near r-partite, m-vertex, $\mathcal{TK}^r(3)$-free hypergraph, and let $\epsilon > 0$ be fixed. Let G be an n-vertex, H-free hypergraph with $\delta(G) \geq \epsilon\binom{n}{r-1}$. We need to show that the chromatic number of G is bounded by a constant depending only on ϵ and H.

First, choose an equitable partition $X_1, \ldots, X_r$ of $V(G)$ such that the sizes of $X_1, \ldots, X_r$ are as equal as possible and for every $x \in V(G)$ the number of edges containing x and one vertex from each X_i is at least $\frac{1}{2r^r}\epsilon\binom{n}{r-1}$. (This can be done by randomly choosing the partition $X_1, \ldots, X_r$.) We will show how to bound the chromatic number of $G[X_1]$; the same argument can be applied to bound the chromatic number of each $G[X_i]$ and thus the chromatic number of G.

Define the $(r, r-1)$-uniform fiber bundle (B, γ, F) as follows. Let $B = G[X_1]$, let $F = X_2 \cup \ldots \cup X_r$, and for $x \in X_1$ define

$$\gamma(x) = \{\{x_2, \ldots, x_r\} \subseteq F : x_2 \in X_2, \ldots, x_r \in X_r, \{x, x_2, \ldots, x_r\} \in G\}.$$

Then $\gamma(x)$ has size at least $\frac{1}{2r^r}\epsilon\binom{n}{r-1}$. Let L be the complete $(r-1)$-uniform, $(r-1)$-partite hypergraph on $(rm)^m$ vertices. Let $V_1, \ldots, V_r$ be the r-partition of $V(H)$ guaranteed by the definition of near r-partite and let d be the size of V_1. Using that the Turán density of a complete $(r-1)$-uniform $(r-1)$-partite hypergraph is zero, we apply Theorem 37 to show that there exists constants $K_1 = K_1(r, \epsilon, H)$ and $K_2 = K_2(r, \epsilon, H)$ such that one of the following holds: either $|F| \leq K_1$, $\chi(B) \leq K_2$, or $\dim_L(B, \gamma, F) \geq d$. Since $|F| = (1 - 1/r)\,|V(G)|$, if either of the first two possibilities occur then the chromatic number of $G[X_1]$ is bounded. It must therefore be the case that $\dim_L(B, \gamma, F) \geq d$.

We now show this implies that G contains a copy of H, which follows from Lemma 39. Since

$\dim_L(B, \gamma, F) \geq d$, there are d edges $E_1, \ldots, E_d$ such that for each $x_1 \in E_1, \ldots, x_d \in E_d$, we have that $\gamma(x_1) \cap \ldots \cap \gamma(x_d)$ contains a complete $(r-1)$-uniform, $(r-1)$-partite hypergraph on $(rm)^m$ vertices, see Figure 3.2. Since $m = |V(H)|$, from each $\gamma(x_1) \cap \ldots \cap \gamma(x_d)$ we can pick a copy of the complete $(r-1)$-uniform, $(r-1)$-partite hypergraph on m vertices so that all these copies are vertex disjoint. Assume $V_1 = A_1 \cup \ldots \cup A_\ell \cup \{a_{\ell+1}\} \cup \ldots \cup \{a_{\ell'}\}$, where $A_1, \ldots, A_\ell$ are the special edges of H. Using Lemma 39, we can embed a copy of H in G by mapping A_i to E_i for $1 \leq i \leq \ell$, mapping a_i to any vertex in E_i for $\ell + 1 \leq i \leq \ell'$, and mapping N_i to a subhypergraph of the corresponding complete $(r-1)$-uniform, $(r-1)$-partite hypergraph on m vertices. □

3.4 Coloring hypergraphs with bounded dimension

3.4.1 An insightful attempt at proving Theorem 37

The proof of Theorem 37, which appears in Sections 3.4.2 and 3.4.3, is complex, but it started as a simple idea built on three key ideas. In this section we attempt to motivate the essential ingredients behind the proof.

Consider the following proof strategy for Theorem 37: assume the chromatic number of B is large and give an algorithm that produces d edges which witness that $\dim_H(B, \gamma, F) \geq d$. The first key idea is to use the greedy algorithm, which selects the edges one by one while maintaining that all sections are large enough to force a copy of H. Initially, we can pick any edge E_1, since any section of $x \in E_1$ is the set $\gamma(x)$, and by assumption $\gamma(x)$ is large enough to force a copy of H. So where could the greedy algorithm get stuck? Assume the greedy algorithm selected $E_1, \ldots, E_i$ but cannot continue. That is, for every other edge E, there exists some section S of $x_1 \in E_1, \ldots, x_i \in E_i$ and there exists some $x \in E$ such that $S \cap \gamma(x)$ is too small to force a copy of H.

The second key idea in the proof is to assume that every edge of B has small overlap; that is, we assume that (B, γ, F) satisfies the condition that for every $x \neq y \in E \in B$, the number of edges in the hypergraph $\gamma(x) \cap \gamma(y)$ is small. With this assumption, the r^i sections $S_1, \ldots, S_{r^i}$ of $x_1 \in E_1, \ldots, x_i \in E_i$ are almost disjoint. Let $S_{r^i+1} = \binom{F}{r-1} \setminus S_1 \setminus \ldots \setminus S_{r^i}$ so that $S_1, \ldots, S_{r^i+1}$ almost form a partition. Recall that the greedy algorithm could not continue because for every edge E disjoint from $E_1, \ldots, E_i$, there existed some S_j and some $x \in E$ with $S_j \cap \gamma(x)$ small. Because $S_1, \ldots, S_{r^i+1}$ is almost a partition, this implies that there is some other $S_{j'}$ with $S_{j'} \cap \gamma(x)$ large. In other words, the greedy algorithm cannot continue if there exists sets $S_1, \ldots, S_{t+1}$ such that for every edge E, there exists some S_j and some $x \in E$ such that $S_j \cap \gamma(x)$ is large.

The third key idea is to use this in a density increment argument, similar to that used in the proof of the Regularity Lemma. If we have a partition P of (B, γ, F) (a partition will be formally defined later), we apply the greedy algorithm from the last two paragraphs in each part of P to refine the partition; the sets $S_1, \ldots, S_t$ are used to refine each part. If we define the density of a partition correctly, we can show that each time when we apply the greedy algorithm the density will increase by a constant amount and add only a constant number of new parts in the partition.

The last part of the proof is to reduce the full theorem to the case where (B, γ, F) satisfies the condition that for every $x \neq y \in E \in B$, $\gamma(x) \cap \gamma(y)$ is small.

3.4.2 A conditional proof of Theorem 37

Condition 1. Throughout this section, fix $r_b, r_\gamma \geq 1$, $d \in \mathbb{Z}^+$, $0 < \epsilon < \frac{1}{4} r_b^{-d}$, and H an r_γ-uniform hypergraph with zero Turán density. Also, let (B, γ, F) be an (r_b, r_γ)-uniform fiber bundle for which $\dim_H(B, \gamma, F) < d$ and if $b \in V(B)$, then $|\gamma(b)| \geq \epsilon \binom{|F|}{r_\gamma}$.

In this section, we will prove the following theorem.

Theorem 40. *Let (B, γ, F) be a fiber bundle satisfying Condition 1. Then there exists constants $0 < \lambda = \lambda(r_b, r_\gamma, d, \epsilon, H) < \epsilon$, $L_1 = L_1(r_b, r_\gamma, d, \epsilon, H)$, and $L_2 = L_2(r_b, r_\gamma, d, \epsilon, H)$ such that the following holds. Assume $|F| \geq L_1$ and for every $x \neq y \in E \in B$, $|\gamma(x) \cap \gamma(y)| \leq \lambda \binom{|F|}{r_\gamma}$. If $r_b = 1$, then $|B| \leq L_2$ and if $r_b \geq 2$ then $\chi(B) \leq L_2$.*

The only differences between this theorem and Theorem 37 are that we allow $r_b = 1$ and also assume a restriction on $\gamma(x) \cap \gamma(y)$ for $x \neq y \in E \in B$. We will remove the need for this assumption in the next section.
Define the following constants.

$$\alpha = \frac{1}{1000}\left(\frac{\epsilon}{4}\right)^{d+1}, \qquad \eta = \frac{1}{4}\epsilon^2\alpha, \qquad \beta = \alpha^{1/\eta}, \qquad L_2 = \left\lceil r_b d (r_b^d + 2)^{1/\eta} \right\rceil.$$

Next, pick L_1 large enough so that if $|F| \geq L_1$ and $S \subseteq \binom{F}{r_\gamma}$ with $|S| \geq 4^{-d}\epsilon^{d+1}\beta \binom{|F|}{r_\gamma}$, then S contains a copy of H. Lastly, pick $\lambda = \frac{\epsilon^{d+1}\beta}{d r_b^2 4^d}$.

Condition 2. If (B, γ, F) is a fiber bundle, then for every edge E in B and every $x \neq y \in E$, the section of x, y has at most $\lambda \binom{|F|}{r_\gamma}$ edges.

If (B, γ, F) is a fiber bundle, a **partition** P of (B, γ, F) is a family $P = \{(X_1, S_1), \ldots, (X_p, S_p)\}$ such that $X_1, \ldots, X_p$ is a partition of $V(B)$ and $S_1, \ldots, S_p$ is a partition of $\binom{F}{r_\gamma}$, where we allow $X_i = \emptyset$ or $S_i = \emptyset$. A partition Q is a refinement of a partition P if for each $(X, S) \in P$, there exist $(Y_1, T_1), \ldots, (Y_q, T_q) \in Q$ such that $X = \cup Y_i$ and $S = \cup T_i$. For $X \subseteq V(B)$ and $S \subseteq 2^F$, the **density of** (X, S) is

$$d(X, S) = \begin{cases} 1 & S = \emptyset \text{ or } X = \emptyset, \\ \min\left\{ \frac{|\gamma(x) \cap S|}{|S|} : x \in X \right\} & \text{otherwise,} \end{cases}$$

and define

$$d(P) = \min\left\{ d(X, S) : (X, S) \in P \right\}.$$

A partition P is a **partial coloring** if for every $(X, \emptyset) \in P$, we have that $B[X]$ independent. The **rank** of a partition P is the minimum of $|S|$ over all $(X, S) \in P$ with $S \neq \emptyset$.

Recall from the sketch in Section 3.4.1 that the general structure of our proof is to show how to refine a partial coloring P to a partial coloring Q where $d(Q) \geq \eta + d(P)$. This will imply that we can only refine the partition a constant number of times, at which point we will have a coloring with a bounded number of parts. The key lemma to facilitate this refinement is the following.

Lemma 41. *Let (B, γ, F) be a fiber bundle satisfying Conditions 1 and 2. Let $X \subseteq V(B)$ and $S \subseteq \binom{F}{r_\gamma}$ with $X \neq \emptyset$, $d(X, S) \geq \epsilon$, $|F| \geq L_1$, and $|S| \geq \beta\binom{|F|}{r_\gamma}$. Then there exists a partition $Y_1, \ldots, Y_n, Z$ of X and a partition $T_1, \ldots, T_n$ of S such that $n \leq r_b^d + 1$ and*

- $|T_i| \geq \alpha |S|$,
- $d(Y_i, T_i) \geq \min\{1, \eta + d(X, S)\}$,
- *$B[Z]$ is independent.*

This lemma has an easy corollary.

Corollary 42. *Let (B, γ, F) be a fiber bundle satisfying Conditions 1 and 2 and $|F| \geq L_1$. Let P be a partial coloring of (B, γ, F) where P has rank at least $\alpha^k\binom{|F|}{r_\gamma}$ with $k \leq \frac{1}{\eta}$. Then there exists a refinement Q of P such that*

- $|Q| \leq (r_b^d + 2)|P|$,
- *Q is also a partial coloring,*
- *the rank of Q is at least $\alpha^{k+1}\binom{|F|}{r_\gamma}$,*
- $d(Q) \geq \min\{1, \eta + d(P)\}$.

Proof. For each pair $(X, S) \in P$ with $X \neq \emptyset$ and $S \neq \emptyset$, apply Lemma 41. Since $k \leq \frac{1}{\eta}$, $|S| \geq \alpha^k\binom{|F|}{r_\gamma} \geq \alpha^{1/\eta}\binom{|F|}{r_\gamma} \geq \beta\binom{|F|}{r_\gamma}$. Lemma 41 produces $Y_1, \ldots, Y_n, Z$ and $T_1, \ldots, T_n$ with $n \leq r_b^d + 1$. We replace the pair (X, S) with the pairs $(Y_1, T_1), \ldots, (Y_n, T_n), (Z, \emptyset)$. The resulting partition satisfies all the required properties. □

We can now easily prove Theorem 40.

Proof of Theorem 40. By assumption, (B, γ, F) satisfies Conditions 1 and 2. Start with the partition $P = \left\{(V(B), \binom{F}{r_\gamma})\right\}$ and apply Corollary 42 repeatedly until the partition satisfies $d(P) = 1$. Since the value of $d(P)$ increases by η at each step, the partition is refined at most $1/\eta$ times, and so the resulting partition P has at most $(r_b^d + 2)^{1/\eta}$ parts. Consider a part $(X, S) \in P$. If $S = \emptyset$, then since P is a partial coloring $B[X]$ must be independent, so $\chi(B[X]) = 1$. If $S \neq \emptyset$, we know that $|S| \geq \beta\binom{|F|}{r_\gamma}$, which by the choice of β and L_1 forces a copy of H in S. Since $d(X, S) = 1$ we must have $S \subseteq \gamma(x)$ for every $x \in X$, so that a matching of size d in $B[X]$ witnesses that $\dim_H(B, \gamma, F) \geq d$. Therefore, the maximum size of a matching in $B[X]$ is $d - 1$. If $r_b = 1$, then $B[X]$ has at most $d - 1$ edges, so the total number of edges of B is $d(r_b^d + 2)^{1/\eta}$. If $r_b \geq 2$, then since the size of a maximal matching in $B[X]$ is $d - 1$, it is the case that $\chi(B[X]) \leq r_b(d - 1) + 1$. This implies that the chromatic number of B is at most $r_b d(r_b^d + 2)^{1/\eta}$. □

All that remains is to prove Lemma 41. Before proving this lemma, we make some definitions. If $E_1, \ldots, E_t \in B$ and $S \subseteq \binom{F}{r_\gamma}$, then the **minimum section density of $E_1, \ldots, E_t$ with respect to S** is

$$\delta(E_1, \ldots, E_t, S) = \min\left\{\frac{|\gamma(x_1) \cap \ldots \cap \gamma(x_t) \cap S|}{|S|} : x_1 \in E_1, \ldots, x_t \in E_t\right\}.$$

Notice that if $E_1, \ldots, E_d$ are disjoint, $\delta(E_1, \ldots, E_d, S) > 0$, S is a constant fraction of $\binom{F}{r_\gamma}$, and F is large, then $E_1, \ldots, E_d$ witness that $\dim_H(B, \gamma, F) \geq d$. Define constants $\psi_1, \ldots, \psi_d$ with $\psi_1 = 1$ by $\psi_{m+1} = \frac{1}{4}\epsilon\psi_m$ for $1 \leq m \leq d-1$.

For the proof of Lemma 41, recall the outline from Section 3.4.1. We will greedily select edges $E_1, \ldots, E_i$ as long as we can maintain that (using our new notation) $\delta(E_1, \ldots, E_i, S)$ is large enough to force a copy of H. Since $\dim_H(B, \gamma, F) < d$, the greedy algorithm must terminate before choosing d edges. Once the greedy algorithm terminates, we will let the sets $T_1, \ldots, T_{n-1}$ (the sets that we must find to prove Lemma 41) be all sections of $x_1 \in E_1, \ldots, x_i \in E_i$. If we let Y_i be the set of vertices y such that $|\gamma(y) \cap T_i| / |T_i|$ is at least $d(X, S) + \eta$, we will satisfy almost all of the requirements in Lemma 41. We need only prove that if Z are the vertices not in any Y_i then $B[Z]$ is independent. We do this by showing that if $B[Z]$ contained an edge E we could have continued the greedy algorithm by selecting E. In order for this to be true, we need the greedy algorithm to require a weaker lower bound on $\delta(E_1, \ldots, E_{i+1}, S)$ than the algorithm required on $\delta(E_1, \ldots, E_i, S)$. The constants ψ_i are used to define the greedy algorithm: the greedy algorithm will select edges while maintaining $\delta(E_1, \ldots, E_i, S) \geq \epsilon\psi_i$. Since $\psi_{i+1}/\psi_i = \frac{1}{4}\epsilon$, we lose a fraction of ϵ in each step. The careful choice of L_1 guarantees that even after losing a fraction of ϵ for the d steps per refinement over a maximum of $1/\eta$ refinements we still have enough edges to force a copy of H.

Proof of Lemma 41. Start by greedily selecting disjoint edges $E_1, \ldots, E_i$ of $B[X]$ such that $\delta(E_1, \ldots, E_i, S) \geq \epsilon\psi_i$. Since for every $x \in V(B)$

$$\frac{|\gamma(x) \cap S|}{|S|} \geq d(X, S) \geq \epsilon\psi_1,$$

the greedy algorithm can start with any edge E_1 in $B[X]$. Assume the greedy algorithm has selected $E_1, \ldots, E_m$ with $\delta(E_1, \ldots, E_m, S) \geq \epsilon\psi_m$ but for every other edge E in $B[X]$ disjoint from $E_1, \ldots, E_m$, we have $\delta(E_1, \ldots, E_m, E, S) < \epsilon\psi_{m+1}$.

First, we prove that $\dim_H(B, \gamma, F) \geq m$. Since $d(E_1, \ldots, E_m, S) \geq \epsilon\psi_m \geq \epsilon\psi_d$, we have that every section of $x_1 \in E_1, \ldots, x_m \in E_m$ has size at least $\epsilon\psi_d |S| \geq \epsilon\psi_d\beta\binom{|F|}{r_\gamma}$. By the choice of L_1, the section of $x_1, \ldots, x_m$ contains a copy of H, and so $m \leq d$. We make the following definitions.

- Let $R_1, \ldots, R_t$ be all r_b^m sections of $v_1 \in E_1, \ldots, v_m \in E_m$ intersected with S. If some $I \subseteq F$ appears in more than one R_i, remove it from all but the least indexed R_i.
- For $1 \leq i \leq t$, start with $T_i = R_i$ and remove elements from T_i (recall that elements of T_i are subsets of F) until $|T_i|$ is smaller than $2\epsilon |S|$. (If R_i is already smaller than $2\epsilon |S|$, nothing needs to be removed.)
- Let $T_{t+1} = S \setminus T_1 \setminus \ldots \setminus T_t$.
- For $1 \leq i \leq t+1$, define

$$Y_i = \left\{ x \in X : \frac{|\gamma(x) \cap T_i|}{|T_i|} \geq \min\{1, \eta + d(X, S)\} \right\}.$$

If some x appears in more than one Y_i, remove it from all but the least indexed Y_i.

- Let $Z = X \setminus Y_1 \setminus \ldots \setminus Y_{t+1}$.

By the definition of Y_i, $d(Y_i, T_i) \geq \min\{1, \eta + d(X, S)\}$. Therefore, to finish the proof we need to check that $|T_i| \geq \alpha |S|$ and $B[Z]$ is independent.

Claim: $|T_i| \geq \alpha |S|$ for all $1 \leq i \leq t+1$.

Proof. Since $\delta(E_1, \ldots, E_m, S) \geq \epsilon\psi_m$, before removing anything from R_i, each R_i has size at least $\epsilon\psi_m |S|$. Consider some $I \in R_i \cap R_j$ for some $j \neq i$. Since $j \neq i$, there must be some E_k such that R_i selects $\gamma(x)$ and R_j selects $\gamma(y)$ for $x \neq y \in E_k$. Thus $I \in \gamma(x) \cap \gamma(y)$, which has size at most $\lambda\binom{|F|}{r_\gamma}$ by Condition 2. In other words, every element removed from R_i is contained in $\gamma(x) \cap \gamma(y)$ for some $x \neq y \in E_k$. There are at most $mr_b(r_b - 1)$ choices of $x \neq y \in E_k$, so the maximum number of elements removed from R_i is $mr_b^2\lambda\binom{|F|}{r_\gamma}$. Since $m \leq d$ and $|S| \geq \beta\binom{|F|}{r_\gamma}$, we remove at most $dr_b^2\lambda\beta^{-1}|S|$ elements from R_i. By the choice of constants, $dr_b^2\lambda\beta^{-1} \leq \frac{1}{2}\epsilon\psi_d \leq \frac{1}{2}\epsilon\psi_m$, so

$$|R_i| \geq \frac{1}{2}\epsilon\psi_m |S|. \tag{3.1}$$

Since $\psi_m \leq 1$ and we remove elements from R_i to form T_i only if $|R_i| \geq 2\epsilon |S|$, equation (3.1) implies

$$|T_i| \geq \frac{1}{2}\epsilon\psi_m |S|, \tag{3.2}$$

which is at least $\alpha |S|$ by the choice of α. Now consider the size of T_{t+1}. Since each T_i with $i \leq t$ has size at most $2\epsilon |S|$ and we assumed that $\epsilon < \frac{1}{4}t^{-1}$ in Condition 1, the set T_{t+1} has at least $\frac{1}{2}|S| \geq \alpha |S|$ elements. □

Claim: $B[Z]$ is independent.

Proof. Assume E is an edge in $B[Z]$. We would like to show that there exists some $x \in E$ and some T_j such that

$$\frac{|\gamma(x) \cap T_j|}{|T_j|} \geq \min\{1, \eta + d(X, S)\}, \tag{3.3}$$

since this would show that $x \in Y_j$, contradicting that $x \in Z$. Assume E intersects some E_i for some $1 \leq i \leq m$, with $x \in E \cap E_i$. Since $x \in E_i$ there are many sections R_j that select x, since the sections R_j were formed by choosing one vertex from each of $E_1, \ldots, E_d$. Fix some such section R_j that selects x, in which case $R_j \subseteq \gamma(x)$. Then $T_j \subseteq R_j \subseteq \gamma(x)$ and $|\gamma(x) \cap T_j| / |T_j| = 1$ so (3.3) is satisfied.

Now assume E is disjoint from $E_1, \ldots, E_m$. Since the greedy algorithm could not continue, $\delta(E_1, \ldots, E_m, E, S) < \epsilon\psi_{m+1}$, which implies that there exists some $v_1 \in E_1, \ldots, v_m \in E_m, x \in E$ such that

$$|\gamma(v_1) \cap \ldots \cap \gamma(v_m) \cap \gamma(x) \cap S| < \epsilon\psi_{m+1} |S|.$$

By the definition of T_i, there exists some T_i such that $T_i \subseteq \gamma(v_1) \cap \ldots \cap \gamma(v_m) \cap S$. Therefore,

$$|\gamma(x) \cap T_i| < \epsilon\psi_{m+1} |S| \leq \frac{2\psi_{m+1}}{\psi_m} |T_i|,$$

where the last inequality uses (3.2). Assume that for every $j \neq i$, (3.3) fails. Then

$$|\gamma(x) \cap S| = |\gamma(x) \cap T_i| + \sum_{j \neq i} |\gamma(x) \cap T_j| \leq \frac{2\psi_{m+1}}{\psi_m} |T_i| + \sum_{j \neq i} (\eta + d(X,S)) |T_j|.$$

Dividing through by $|\gamma(x) \cap S|$ we obtain

$$1 \leq \frac{2\psi_{m+1}}{\psi_m} \frac{|T_i|}{|S|} \frac{|S|}{|\gamma(x) \cap S|} + (\eta + d(X,S)) \left(1 - \frac{|T_i|}{|S|}\right) \frac{|S|}{|\gamma(x) \cap S|}.$$

Because $|S| / |\gamma(x) \cap S| \leq \frac{1}{d(X,S)} \leq \frac{1}{\epsilon}$,

$$1 \leq \frac{2\psi_{m+1}}{\psi_m \epsilon} \frac{|T_i|}{|S|} + \left(\frac{\eta}{\epsilon} + 1\right) \left(1 - \frac{|T_i|}{|S|}\right). \tag{3.4}$$

Let $w = |T_i| / |S|$. The right hand side of the above inequality is a weighted average of $\frac{2\psi_{m+1}}{\psi_m \epsilon}$ and $(1 + \frac{\eta}{\epsilon})$:

$$\frac{2\psi_{m+1}}{\psi_m \epsilon} w + \left(1 + \frac{\eta}{\epsilon}\right)(1 - w).$$

Since $2\psi_{m+1}/(\psi_m \epsilon) = \frac{1}{2} < 1 + \frac{\eta}{\epsilon}$, this will be maximized when w is as small as possible. By (3.2), $w \geq \alpha$, so

$$\frac{2\psi_{m+1}}{\psi_m \epsilon} \alpha + \left(1 + \frac{\eta}{\epsilon}\right)(1 - \alpha) < \frac{1}{2}\alpha + 1 + \frac{\eta}{\epsilon} - \alpha \leq 1 + \frac{\eta}{\epsilon} - \frac{1}{2}\alpha < 1.$$

This implies that for any $w \geq \alpha$, the inequality in (3.4) is false. This contradiction shows that there must be some $j \neq i$ such that $|\gamma(x) \cap T_j| / |T_j|$ is at least $\eta + d(X,S)$, which contradicts that E is contained in $B[Z]$. □

Thus $B[Z]$ is independent and the proof is complete. □

3.4.3 Fiber bundles with large overlap

In the previous section, we proved Theorem 37 restricted to fiber bundles that satisfy Condition 2. To prove Theorem 37, we will divide the edges of B into two pieces. Let B' be the subset of edges of B that satisfy Condition 2; that is for every $x \neq y \in E \in B'$, $\gamma(x) \cap \gamma(y)$ has density at most λ. We apply Theorem 40 to (B', γ, F) to bound the chromatic number of B'. For the remaining edges, we will merge x and y into a new vertex z if $\gamma(x) \cap \gamma(y)$ has density at least λ (we define $\gamma(z)$ to be $\gamma(x) \cap \gamma(y)$). Let (M, ψ, F) be the fiber bundle after merging all such vertices. Since all edges

of $B - B'$ have some such pair x, y, all edges of M will have size at most $r_b - 1$. Then we apply induction on r_b to bound the chromatic number of M. To be able to apply induction, we need to verify that $\dim_H(M, \psi, F) \leq \dim_H(B, \gamma, F)$ and that there is a lower bound on the density of $\gamma(m)$ for $m \in V(M)$. The definition of $\gamma(z) = \gamma(x) \cap \gamma(y)$ satisfies both of these requirements. First, $\dim_H(M, \psi, F) \leq \dim_H(B, \gamma, F)$ because any copy of H in $\gamma(z)$ will be in both $\gamma(x)$ and $\gamma(y)$. Also, there is a lower bound on the density of $\gamma(z)$ because we only merge if $\gamma(x) \cap \gamma(y)$ has density at least λ. The magic in this proof is that Condition 2, the extra requirement needed in the previous section, fits exactly with the requirements to be able to apply induction after merging.

For technical reasons, our induction statement needs to be slightly stronger than Theorem 37; we no longer assume B is a uniform hypergraph. Instead, we allow the edges of B to have size between one and r_b. This is because after merging, all we know is that the edges have size between one and $r_b - 1$. This is also why we need to allow 1-uniform hypergraphs in Theorem 40.

Theorem 43. *Let $r_b, r_\gamma \geq 1$, $d \in \mathbb{Z}^+$, $0 < \epsilon < 1$, and H be an r_γ-uniform hypergraph with zero Turán density. Then there exists constants $K_1 = K_1(r_b, r_\gamma, d, \epsilon, H)$ and $K_2 = K_2(r_b, r_\gamma, d, \epsilon, H)$ such that the following holds. Let (B, γ, F) be any fiber bundle where the edges of B have size between 1 and r_b, $dim_H(B, \gamma, F) < d$, and for all $b \in V(B)$, $\gamma(b)$ is an r_γ-uniform hypergraph and $|\gamma(b)| \geq \epsilon\binom{|F|}{r_\gamma}$. Let A be the set of edges of size 1 in B. If $|F| \geq K_1$, then $|A| \leq K_2$ and $\chi(B - A) \leq K_2$.*

Proof. The proof is by induction on r_b. When $r_b = 1$, we can directly apply Theorem 40, since Condition 2 is trivially satisfied. So assume $r_b \geq 2$, and let $\lambda = \lambda(r_b, r_\gamma, d, \epsilon, H)$ be the constant from Theorem 40. Define a subhypergraph B' of B as follows:

$$B' = \left\{ E \in B : |E| = r_b \text{ and } \forall x \neq y \in E \quad |\gamma(x) \cap \gamma(y)| \leq \lambda \binom{|F|}{r_\gamma} \right\}.$$

Then (B', γ, F) is an (r_b, r_γ)-uniform fiber bundle to which we can apply Theorem 40 to bound the chromatic number of B'. To complete the proof, we will bound the chromatic number of $B - B'$.

Initially, let $B_0 = B - B'$ and $\gamma_0 = \gamma$. At stage i, assume we have defined (B_i, γ_i, F). Let E be some edge of B_i with $|E| = r_b$, where there exists $x \neq y \in E$ be such that $|\gamma_i(x) \cap \gamma_i(y)| \geq \lambda\binom{|F|}{r_\gamma}$. We form $(B_{i+1}, \gamma_{i+1}, F)$ by merging the vertices x and y. More precisely, let z be a new vertex and define

$$\begin{aligned}
V(B_{i+1}) &= V(B_i) - x - y + z, \\
E(B_{i+1}) &= \{E \in B_i : E \cap \{x, y\} = \emptyset\} \cup \{E - x - y + z : E \in B_i, E \cap \{x, y\} \neq \emptyset\}, \\
\gamma_{i+1}(w) &= \begin{cases} \gamma_i(w) & w \neq z, \\ \gamma_i(x) \cap \gamma_i(y) & w = z. \end{cases}
\end{aligned}$$

We do this until every edge of B_i has size at most $r_b - 1$; say this occurs at step s.

Through this modification, the dimension cannot increase. Consider step i, when we merge the vertices x and y in (B_i, γ_i, F) to form $(B_{i+1}, \gamma_{i+1}, F)$. Let S be any section in (B_i, γ_i, F) that selects

x or y, and let S' be a section in $(B_{i+1}, \gamma_{i+1}, F)$ that is identical to S except that it selects z instead of x or y. If S' contains a copy of H, then this copy of H is in $\gamma(z)$, which implies it is in both $\gamma(x)$ and $\gamma(y)$. Therefore, H is in S.

For each vertex z we add, we have $|\gamma(z)| \geq \lambda\binom{|F|}{r_\gamma}$. Therefore, we can apply induction on r_b to (B_s, γ_s, F) using $\epsilon = \lambda$ to bound the chromatic number of B_s. We now consider "un-merging" the vertices of B_s to obtain a coloring of B_0. Consider un-merging $z \in V(B_i)$ to obtain $x, y \in V(G_{i-1})$. If z was contained in an edge of size one, we color x and y using two new colors. Otherwise, we color x and y the same color as z. After the un-merging, we will have a proper coloring of $B_0 = B - B'$. By induction, there is a bounded number of edges of size one, so we will use a bounded number of new colors.

The colorings on B_0 and B' can be combined to obtain a coloring of B using a bounded number of colors. Let K_1 (the minimum size of F) be the maximum between the required size from Theorem 40 and the size required by induction. Lastly, any edge of size one in B will also appear in B_s, and by induction B_s has a bounded number of edges of size one. □

3.5 Extremal results for critical hypergraphs

In this section, we prove Theorems 32 and 34. First, it is easy to see that C^r_{2k+1} is mono near r-partite; the edge E_{2k+1} in C^r_{2k+1} will be the special edge. Also, E_{2k+1} has $r-2$ vertices, $v_{rk+2}, \ldots, v_{rk+r-1}$, that have degree one. Thus to complete the proof of Theorem 34 we need only prove that C^3_{2k+1} and C^4_{2k+1} are stable with respect to $T_3(n)$ and $T_4(n)$. One tool we will use is the hypergraph removal lemma of Gowers, Nagle, Rödl, and Skokan [32, 46, 54, 55, 62].

Theorem 44. *For every integer $r \geq 2$, $\epsilon > 0$, and r-uniform hypergraph H, there exists a $\delta > 0$ such that any r-uniform hypergraph with at most $\delta n^{|V(H)|}$ copies of H can be made H-free by removing at most ϵn^r edges.*

The second tool we will use is supersaturation, proved by Erdős and Simonovits [25]. There are several equivalent formulations of supersaturation, the one we will use is the following.

Theorem 45. *[25, Corollary 2] Let $K^r_{t_1,\ldots,t_r}$ be the complete r-uniform, r-partite hypergraph with part sizes $t_1, \ldots, t_r$. Let $t = \sum t_i$. For every $\epsilon > 0$, there exists a $\delta = \delta(r, t, \epsilon)$ such that any r-uniform hypergraph with at least ϵn^r edges contains at least δn^t copies of $K^r_{t_1,\ldots,t_r}$.*

For any hypergraph H, let $H(t)$ denote the hypergraph obtained from H by blowing up each vertex into an independent set of size t. An easy extension of supersaturation is the following (see Theorem 2.2 in the survey by Keevash [37]).

Corollary 46. *For every $r, t \geq 2$, $\epsilon > 0$, and r-uniform hypergraph H, there exists an n_0 such that if $n \geq n_0$ and G is an n-vertex, r-uniform hypergraph which contains at least $\epsilon n^{|V(H)|}$ copies of H, then G contains a copy of $H(t)$.*

Next, we will need stability results for F_5 and the book $B_{4,2}$, proved by Keevash and Mubayi [38] and Pikhurko [48] respectively. Let the **book** $B_{r,m}$ be the r-uniform hypergraph with vertices

$x_1, \ldots, x_{r-1}, y_1, \ldots, y_r$ and hyperedges $\{x_1, \ldots, x_{r-1}, y_i\}$ for $1 \leq i \leq m$ and $\{y_1, \ldots, y_r\}$. Note that $F_5 = B_{3,2}$.

Theorem 47. *[38] F_5 is stable with respect to $T_3(n)$.*

Theorem 48. *[48] $B_{4,2}$ is stable with respect to $T_4(n)$.*

The last piece of the proof of Theorem 34 is the following lemma.

Lemma 49. *If H is an r-uniform hypergraph that is stable with respect to $T_r(n)$, and F is a subhypergraph of $H(t)$ for some t, then F is also stable with respect to $T_r(n)$.*

Proof. If F is contained in $T_r(n)$, then for large n there is no r-uniform, F-free hypergraph with at least $(1-\delta)t_r(n)$ edges, so F is vacuously stable with respect to $T_r(n)$. So assume F is not a subhypergraph of $T_r(n)$. Let h denote the number of vertices in H and let $\epsilon > 0$ be fixed. We need to show how to define δ such that if G is a F-free hypergraph with at least $t_r(n) - \delta n^r$ edges, it differs from $T_r(n)$ in at most ϵn^r edges.

Since H is stable with respect to $T_r(n)$, there exists an $\alpha \leq \epsilon/2$ such that if G' has at least $t_r(n) - 2\alpha n^r$ edges and contains no copy of H, then G' differs from $T_r(n)$ in at most $\epsilon n^r/2$ edges. By Theorem 44, there exists $\beta = \beta(\alpha)$ such that if there are at most βn^h copies of H in G then by deleting at most αn^r edges of G we can remove all copies of H. Lastly, choose $\delta \ll \beta$.

Now, fix some G that contains no copy of F and has at least $t_r(n) - \delta n^r$ edges. Because G contains no copy of F it contains no copy of $H(t)$. Therefore, by Corollary 46 there are at most βn^h copies of H in G. By Theorem 44, we may therefore delete αn^r edges in order to find a subhypergraph G' of G that contains no copy of H. Notice that G' has at least $t_r(n) - (\delta + \alpha)n^r$ edges, and $(\delta + \alpha) < 2\alpha$, so G' differs from $T_r(n)$ in at most $\epsilon n^r/2$ edges. Therefore, G differs from $T_r(n)$ in at most $(\alpha + \epsilon/2)n^r$ edges, and $\alpha + \epsilon/2 < \epsilon$. □

It is easy to see that C^r_{2k+1} is a subhypergraph of $B_{r,2}(k)$. Thus Theorem 47 combined with Lemma 49 shows that C^3_{2k+1} is stable with respect to $T_3(n)$ and similarly Theorem 48 combined with Lemma 49 shows that C^4_{2k+1} is stable with respect to $T_4(n)$, which completes the proof of Theorem 34. Frankl and Füredi [29] determined $\pi(B_{r,2})$ for $r = 5, 6$, and the values come from blowups of small designs. In particular, $\pi(B_{r,2}) \neq t_r(n)$ for $r = 5, 6$. This means a different method must be used when attempting to prove that C^r_{2k+1} is critical for $r \geq 5$.

Proof of Theorem 32. Let H be a critical n-vertex, r-uniform hypergraph. Suppose H has h vertices and assume that E is the special edge of H. Suppose G is an H-free, r-uniform, n-vertex hypergraph with $|G| \geq t_r(n)$. We would like to show that $G = T_r(n)$. Partition the vertices of G into parts $X_1, \ldots, X_r$ such that the number of edges with one vertex in each X_i is maximized. Let $\epsilon_1 = (2r)^{-h}$, let $\epsilon_2 = \epsilon_1/8r^3$, let $\delta = \delta(r, h, \epsilon_2)$ from Theorem 45, and let $\epsilon < 2^{-2r}\epsilon_1\epsilon_2\delta$. Organize r-sets of vertices into the following sets.

- Let M be the set of r-sets with one vertex in each of $X_1, \ldots, X_r$ that are not edges of G (the missing cross-edges).

- Let B be the collection of edges of G that have at least two vertices in some X_i (the bad edges).
- Let $G' = G - B + M$, so that G' is a complete r-partite hypergraph.
- Let $B_i = \{W \in B : |W \cap X_i| \geq 2\}$.

Since $B = \cup_i B_i$, there is some B_i which has size at least $\frac{1}{r}|B|$. Assume without loss of generality that $|B_1| \geq \frac{1}{r}|B|$. For $a \in X_1$, make the following definitions.

- $B_a = \{W \in B_1 : a \in W\}$.
- Let $C_{a,i}$ be the edges in B_a which have exactly two vertices in B_1 and exactly one vertex in each X_j with $j \geq 1$ and $j \neq i$.
- Let $D_a = B_a \setminus C_{a,2} \setminus \cdots \setminus C_{a,r}$.

First, $|B| < \epsilon n^r$ because G is stable with respect to $T_r(n)$. Also, since $|G| \geq t_r(n)$, the number of r-sets in M is at most the number of edges in B, so $|M| \leq |B| < \epsilon n^r$.

In the rest of the proof, we will assume that B is non-empty and then count the r-sets in M in several different ways. Our counting will imply that $|M| \geq \epsilon n^r$, and this contradiction will force $B = \emptyset$ and so $G = T_r(n)$. We will count r-sets in M by counting embeddings of $H - E$ into G' that also map E to some element of B. Since G is H-free, each embedding must use at least one edge in M. Let Φ be the collection of embeddings $\phi : V(H) \to V(G')$ of $H - E$ into G', by which we mean that ϕ is an injection and for all $F \in H$, $\phi(F) = \{\phi(x) : x \in F\} \in G'$. We say that $\phi \in \Phi$ is **W-special** if $\phi(E) = W$ and **a-avoiding** if $a \in V(G)$ and some degree one vertex in E is mapped to a. If $W \in B$ and ϕ is W-special, then ϕ must use at least one edge of M. Call one of these edges the **missing edge of** ϕ.

Claim 1: For $\phi \in \Phi$ and $v \in V(H)$, there are at least $\frac{1}{2r}n$ embeddings $\phi' \in \Phi$ where $\phi(x) = \phi'(x)$ for $x \neq v$ and $\phi(v) \neq \phi'(v)$.

Proof. This follows easily because G' is a complete r-partite hypergraph for which each class has size about n/r, and $\phi(v)$ can be replaced by any unused vertex in the X_i that contains $\phi(v)$. □

Fix some $W \in B$, and consider when there exists a W-special embedding of $H - E$. Since $W \in B_i$ for some i, let $w_1 \neq w_2 \in W \cap X_i$. Then there exists an embedding of $H - E$ where w_1 and w_2 are used for the non degree one vertices in the special edge of H. Since the other vertices in the special edge have degree zero in $H - E$, the vertices in the special edge can then be embedded to W. Thus for any $W \in B$, by Claim 1 there are at least $\epsilon_1 n^{h-r}$ W-special embeddings of $H - E$, since we can vary any vertex of H not in W. The situation with a-avoiding is more complicated. If $W \in C_{a,i}$, then the only choice of w_1 and w_2 that we are guranteed to have are the two vertices in $W \cap X_1$, one of which is a. Thus in a W-special embedding, the only way we can guarantee an embedding is by mapping a non-degree one vertex to a. Therefore, only when $W \in D_a$ can we guarantee that there exists at least $\epsilon_1 n^{h-r}$ W-special, a-avoiding embeddings of $H - E$.

Claim 2: For every $a \in X_1$, $|D_a| \leq \epsilon_2 n^{r-1}$.

Proof. Assume there exists some $a \in X_1$ with $|D_a| \geq \epsilon_2 n^{r-1}$. We count a-avoiding, W-special embeddings of $H - E$ into G' where $W \in D_a$. For each $W \in D_a$, we argued above that there are at least $\epsilon_1 n^{h-r}$ embeddings. Since $|D_a| \geq \epsilon_2 n^{r-1}$, the number of a-avoiding embeddings which are W-special for some $W \in D_a$ is at least $\epsilon_1 \epsilon_2 n^{r-1} \cdot n^{h-r} = \epsilon_1 \epsilon_2 n^{h-1}$.

Fix some $L \in M$. We want to count the number of a-avoiding embeddings which are W-special for some $W \in D_a$ and have missing edge L. An upper bound on the number of such embeddings will be the number of choices for W times the number of choices for the $h - |W \cup L|$ vertices of H mapped outside $W \cup L$. Since all these embeddings are a-avoiding, L cannot contain a. For each $0 \leq \ell \leq r$, there exists at least $\binom{r}{\ell}$ choices for the intersection between L and W, at most $n^{r-\ell-1}$ choices of $W \in D_a$ with $|W \cap L| = \ell$ (here it is crucial that $a \in W$ and $a \notin L$), and at most $n^{h-2r+\ell}$ choices for the vertices of H not in $W \cup L$. Thus each $L \in M$ can kill at most $2^{-r} n^{h-r-1}$ embeddings. Since there are at least $\epsilon_1 \epsilon_2 n^{h-1}$ embeddings, M must have size at least $2^{-r} \epsilon_1 \epsilon_2 n^r$, contradicting the choice of ϵ. □

Claim 3: For every $a \in X_1$ and every $2 \leq i \leq r$, $|C_{a,i}| \leq \epsilon_2 n^{r-1}$.

Proof. Assume there exists some a and i with $|C_{a,i}| \geq \epsilon_2 n^{r-1}$. The proof is similar to the proof of Claim 2, except now we cannot count a-avoiding embeddings. In the previous claim, we used the a-avoiding property to imply that the missing edge does not contain a. In this proof, we will instead guarantee that the missing edge cannot contain a by only counting embeddings which map all neighbors of $\phi^{-1}(a)$ into G.

Let v be one of the non degree one vertices in the special edge of H, and define $H_v = \{F \in H : v \in F, F \neq E\}$, that is all edges of H containing v which are not the special edge. Let $Z_a = \{F \in G \setminus B : a \in F\}$, that is all cross-edges of G which contain a. We now count embeddings $\phi \in \Phi$ which are W-special for some $W \in C_{a,i}$, map v to a, and all edges of H_v are mapped to edges in Z_a. For these embeddings, since edges in H_v are mapped to edges in $Z_a \subseteq G$, the missing edge cannot contain a.

First, $|Z_a| \geq |C_{a,i}|$, because otherwise we could move a to X_i and increase the number of edges across the partition and we chose the partition $X_1, \ldots, X_r$ to maximize the number of cross-edges. Let $H' = \{F - v : F \in H_v\}$ and $Z' = \{F - a : F \in Z_a\}$. Then H' and Z' are $(r-1)$-uniform, $(r-1)$-partite hypergraphs, and Z' has at least $|C_{a,i}| \geq \epsilon_2 n^{r-1}$ edges. Let $t = |V(H')|$. Then Theorem 45 shows that Z' contains at least δn^t copes of H', so there are at least $\epsilon_2 n^{r-1} \cdot \delta n^t \cdot \epsilon_1 n^{h-r-t} = \epsilon_1 \epsilon_2 \delta n^{h-1}$ embeddings of $H - E$ which are W-special for some $W \in C_{a,i}$, map v to a, and the edges in H_v are embedded into Z_a.

Now fix $L \in M$, and consider how many of these embeddings have L as their the missing edge. The computation is almost the same as in the previous claim. For each ℓ_1, ℓ_2, there are $\binom{r}{\ell_1}$ choices for $L \cap W$, there are $\binom{r}{\ell_2}$ choices for $L \cap \phi(H_v)$, there are $n^{r-1-\ell_1}$ choices for W (here we use that L does not contain a), $n^{t-\ell_2}$ choices for $\phi(H_v)$, and $n^{h-2r-t+\ell_1+\ell_2}$ choices for the other vertices of H. Thus each L can kill at most $2^{2r} n^{h-r-1}$ embeddings. Since there are at least $\epsilon_1 \epsilon_2 \delta n^{h-1}$ embeddings, M must have size at least $2^{-2r} \epsilon_1 \epsilon_2 \delta n^r$, contradicting the choice of ϵ. □

Claims 2 and 3 imply that $|B_a| < 2r\epsilon_2 n^{r-1}$ for each a. Define

$$A = \left\{a \in X_1 : d_M(a) \geq 2r^2\epsilon_2 n^{r-1}\right\}.$$

As in the proofs of the previous two claims, we would like to count embeddings of $H - E$ to obtain a lower bound on $|M|$. Once again, the main difficulty is controlling how the missing edge can intersect W. If there were some W with $W \cap A = \emptyset$, there will be few missing edges intersecting this W, which is how we will overcome this difficulty in this part of the proof.

Claim 4: There exists some $W \in B_1$ with $W \cap A = \emptyset$.

Proof. Assume that every $W \in B_1$ contains an element of A. Then $\sum_{a\in A} |B_a| \geq |B_1|$. Since $|B_a| < 2r\epsilon_2 n^{r-1}$ for every a, we have the following contradiction.

$$2r\epsilon_2 n^{r-1} |A| > \sum_{a\in A} |B_a| \geq |B_1| \geq \frac{1}{r} |B| \geq \frac{1}{r} |M| \geq \frac{1}{r} \sum_{a\in A} d_M(a) \geq \frac{2r^2\epsilon_2}{r} n^{r-1} |A|.$$

□

We now finish the proof by counting the W-special embeddings whose missing edge does not intersect W. There are at least $\epsilon_1 n^{h-r}$ embeddings which are W-special by Claim 1. If at least half of these have missing edge intersecting W, then W would contain a vertex in A. Thus there are at least $\frac{\epsilon_1}{2} n^{h-r}$ W-special embeddings where the missing edge does not intersect W. Each $L \in M$ can kill at most n^{h-2r} such embeddings, so M has at least $\frac{\epsilon_1}{2} n^r$ elements, contradicting the choice of ϵ. □

Chapter 4

Complete Minors, Independent Sets, and Chordal Graphs

4.1 Introduction

For a graph G and an edge $ab \in E(G)$, the graph formed by **contracting the edge** ab is the graph with vertex set $V(G) - \{b\}$ and edges $\{xy : \{x, y\} \cap \{a, b\} = \emptyset\}$ together with the edges $\{ax : ax \in E(G) \text{ or } bx \in E(G)\}$. A graph H is a **minor** of G if H can be formed by starting with the graph G and preforming a sequence of vertex deletions, edge deletions, and edge contractions.

Hadwiger's Conjecture [33] from 1943 states the following (see [65] for a survey):

Conjecture. *For every k-chromatic graph G, K_k is a minor of G.*

Hadwiger's Conjecture for $k = 4$ was proved by Dirac [17], the case $k = 5$ was shown equivalent to the Four Color Theorem [3, 4, 51] by Wagner [68] and the case $k = 6$ was shown equivalent to the Four Color Theorem by Robertson et al. [52]. Hadwiger's Conjecture for $k \geq 7$ remains open. Let $h(G)$ denote the Hadwiger number, the size of the largest complete minor of G. Since $\alpha(G)\chi(G) \geq |V(G)|$, Hadwiger's Conjecture implies the following conjecture, which was observed in [19], [44], and [70].

Conjecture. *For every graph G, $\alpha(G)h(G) \geq |V(G)|$.*

This conjecture seems weaker than Hadwiger's Conjecture, however Plummer, Stiebitz, and Toft [49] showed that for graphs with $\alpha(G) = 2$, the two conjectures are equivalent. In 1981, Duchet and Meyniel [19] showed that $(2\alpha(G) - 1)h(G) \geq |V(G)|$. No general improvement on this theorem has been made for the case $\alpha(G) = 2$. Seymour asked for any improvement on this result for $\alpha(G) = 2$, conjecturing that there exists an $\epsilon > 0$ such that if $\alpha(G) = 2$, then G has a complete minor of size $(1/3 + \epsilon)n$. Recently, Kawarabayashi, Plummer, and Toft [35] showed that $(4\alpha(G) - 3)h(G) \geq 2|V(G)|$ when $\alpha(G) \geq 3$ and Kawarabayashi and Song [36] improved this to $(2\alpha(G) - 2)h(G) \geq |V(G)|$ when $\alpha(G) \geq 3$. Wood [69] proved $(2\alpha(G) - 1)(2h(G) - 5) \geq 2|V(G)| - 5$ for all graphs G. The main result of this chapter is to improve the bound for graphs with $\alpha(G) \geq 14$.

The material presented in this chapter is joint work with József Balogh and Hehui Wu. It was accepted for publication in Discussiones Mathematicae Graph Theory under the title *Complete Minors, Independent Sets, and Chordal Graphs*, see [8]

Theorem 50. *Let G be an n-vertex graph. Then $K_{\lceil n/r \rceil}$ is a minor of G, where*

$$r = 2\alpha(G) - \lceil \log_\tau(\tau\alpha(G)/2) \rceil \text{ and } \tau = \frac{2\sqrt{2}}{\sqrt{2}-1} \approx 6.83.$$

Using a more careful analysis, the bound for $\alpha(G) = 5$ can be improved as well.

Theorem 51. *Let G be an n-vertex graph with $\alpha(G) = 5$. Then $K_{5n/38}$ is a minor of G.*

Let $f : V(G) \to \mathbb{Q}^+$ be a weight function on $V(G)$. For $A \subseteq V(G)$, define $f(A) = \sum_{v \in A} f(v)$. Then the **weighted independence number of G relative to f** is

$$\alpha_f(G) = \max \{f(A) : A \text{ is an independent set in } G\}.$$

Recall from Section 1.2 that a graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G. We shall need the following result.

Theorem 52. *Let H be a perfect graph and f a weight function on $V(H)$. Then*

$$\omega(H) \geq \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil.$$

A graph G is **chordal** if G has no induced cycle of length at least 4. A **partial simplicial elimination ordering** is an ordered vertex set $U = (v_1, \ldots, v_k)$ such that v_i are distinct and for each $v_i v_j \notin E(G)$ with $i < j$ and $v_i, v_j \in U$ and each component C of $G - \{v_1, \ldots, v_j\}$, at most one of v_i or v_j touches C. The goal of our algorithm is to find a minor H of G such that H is a chordal graph, and then to devise a weight function on the vertices of H to which we apply Theorem 52. Most of the time, the weight of a vertex v in H is the number of vertices of G which are contracted to v. The algorithm builds the minor H by using two operations: extension and breaking. The key property is that at each step, the algorithm uses the operations to increase the number of vertices in a partial simplicial elimination ordering. Once all vertices are included in the partial simplicial elimination ordering, we have a simplicial elimination ordering, so that the algorithm has produced a chordal graph.

In Section 4.3.2, we provide an algorithm which yields an alternate proof of Kawarabayashi and Song's [36] result.

Theorem 53. *Let G be an n-vertex graph. Then $K_{\lceil r \rceil}$ is a minor of G, where*

$$r = \begin{cases} n & \alpha(G) = 1, \\ \frac{n}{3} & \alpha(G) = 2, \\ \frac{n}{2\alpha(G)-2} & \alpha(G) \geq 3. \end{cases}$$

The rest of the chapter is organized as follows: in Section 4.2, we discuss several recent improvements on the above results, in Section 4.3 we define the operations and define the algorithms, in Section 4.4 we prove some lemmas and theorems about the operations used during the algorithms,

and in Section 4.5 we analyze the algorithm. In Section 4.6, we specialize the algorithm to $\alpha(G) = 5$ and by changing the weight function we find a complete minor of size at least $5n/38$, which is slightly larger than the $n/8 = 5n/40$ produced by the general algorithm.

4.2 Recent results on this problem

Since completing the work presented in this chapter, there have been several improvements on this problem. These extensions use the following structural theorem of Chudnovsky and Fradkin [15]. A **claw** is the bipartite graph $K_{1,3}$.

Theorem 54. *If G has chromatic number k and does not contain a claw as an induced subgraph, then G has a complete minor of size at least $2k/3$.*

Using Theorem 54, Fox [27] proved the following.

Theorem 55. *Every graph G has a complete minor of size at least $\frac{|V(G)|}{(2-c)\alpha(G)}$ where $c \approx 0.017$.*

Fortunately, Fox's technique is different from the technique we use in this chapter. Balogh and Kostochka [6] exploit this by combining Lemma 64 from this chapter with ideas from Fox's proof to improve the bound even further.

Theorem 56. *Every graph G has a complete minor of size at least $\frac{|V(G)|}{(2-c)\alpha(G)}$ where $c \approx 1/19.2$.*

We say that G has an **odd complete minor of order at least** ℓ if there are ℓ disjoint trees in G such that every two of them are joined by an edge, and in addition, all the vertices of the trees are two-colored in such a way that edges within trees are bichromatic and edges between trees are monochromatic. Gerards and Seymour conjectured that if a graph has no odd complete minor of order ℓ, then it is $(\ell - 1)$-colorable. This is substantially stronger than Hadwiger's Conjecture. The algorithm used by Duchet and Meyniel to prove $(2\alpha(G) - 1)h(G) \geq |V(G)|$ produces odd complete minors. The algorithm in our alternate proof of Kawarabayashi and Song's [36] result in Section 4.3.2 can be shown to produce an odd complete minor, so every graph with $\alpha(G) \geq 3$ has an odd complete minor of size at least $n/(2\alpha(G) - 2)$. With a little more work, we can show that our algorithm in Section 4.3.3 not only produces a complete minor but actually produces an odd complete minor. Therefore every graph G has an odd complete minor of size at least $n/(2\alpha(G) - \log_\tau(\tau\alpha(G)/2))$. Theorems 55 and 56 do not produce odd complete minors, so Theorem 50 is the best bound when considering the size of the largest odd complete minor.

4.3 Definition of the algorithms

The algorithm first builds a family of disjoint vertex sets spanning connected graphs which partition $V(G)$ and a spanning subgraph of G. We start with the empty family and at each step apply an operation which either adds a new set to the family, adds vertices to an existing set in the family, or updates the spanning subgraph. To identify the spanning subgraph, we color the edges of G:

initially all edges are blue and during the algorithm we color some edges red. We denote the spanning subgraph induced on the blue edges by G_b. When we color some edges red, we make sure that each element in the family spans a connected graph in G_b. Once we have obtained a partition of $V(G)$, we define a graph H by starting with G_b and contracting each set of the partition to a single vertex. We need the spanning subgraph G_b because starting from G and contracting each set in the partition might not yield a chordal graph. Throughout this chapter, a subscript of G is implied on α and N.

4.3.1 Operations used in the algorithm

There are two operations that are carried out by the algorithm: extending and breaking. We are given a labeled (ordered) family of disjoint vertex sets $\mathcal{F}$ and a red/blue coloring of the edges of G. Let $U = V(G) - \cup_{T\in\mathcal{F}} T$, and let G_b be the spanning subgraph of blue edges. We define the following operations:

Extending T into X by k: Let $T \in \mathcal{F}$, let $X \subseteq U$ such that $G_b[X \cup T]$ is connected, there are no red edges between T and X, and let $k \in \mathbb{Z}^+$ such that $k \leq \alpha(X - N(T))$. The operation extends T into X by k by adding at most $2k$ vertices from X into T so that the new $G_b[T]$ is still connected and we increase $\alpha(T)$ by at least k. When extending T into X, the order of the sets in $\mathcal{F}$ is unchanged. In the extension we always follow the algorithm described in the proof of Lemma 64, which shows that such an extension is possible. Extending T into X by k is always **acceptable**.

Breaking X by k: Let k be a positive integer, and let $X \subseteq U$ such that X does not touch $U - X$ in G (i.e. X is a union of some components of $G[U]$).

Step (a): For any $T \in \mathcal{F}$ and any component D of $G[X]$ with $\alpha(D - N(T)) = \alpha(D)$, we color all edges between T and D red.

Step (b): If there exists a component D of $G[X]$ with independence number at least k, let I be an independent set in D with $|I| \geq k$ and let v be any vertex in I. Add $T = \{v\}$ to $\mathcal{F}$ and then extend T into $D - T$ by $k - 1$. Lemma 65 shows that $T, D - T, k$ satisfy the conditions in the extension. We then set $X := X - T$ and continue Step (b) until every component in $G[X]$ has independence number strictly less than k. The new sets produced are added last in the ordering of $\mathcal{F}$.

Definition. We say that breaking X by k is **acceptable** if both of the following conditions hold before we start breaking (before Step (a)):

- For all $T \in \mathcal{F}$ and every component D of $G[X]$ either the edges between T and D are already red, or $\alpha(D - N(T)) = \alpha(D)$ (the edges will become red in Step (a)), or $\alpha(D - N(T)) < k$.
- For every component D of $G[X]$, $\alpha(D) < 2k$.

In other words, an acceptable breaking means each set T in the original family and each component D of U will either have the edges between T and D colored red or touch with blue edges every set born during Step (b) in D, and the new sets will touch each other as well.

Definition. We say that $\mathcal{F}$ is **formed by acceptable operations** in G if $\mathcal{F}$ is formed by starting with the empty family and then performing any sequence of acceptable operations. When we extend T into X by k we say that the **amount** of the extension is k. For $T \in \mathcal{F}$, define $\text{ext}(T)$ to be one plus the total amount of extensions of T, which includes the extensions in the breaking when T was born and all other extensions of T.

In Theorem 57 we show that we obtain a chordal graph when we start with the graph G_b and contract each set of the partition.

Theorem 57. *Let $\mathcal{F}$ be a partition of $V(G)$ formed by acceptable operations in G, and let G_b be the spanning subgraph of blue edges. Let H be the graph obtained by starting from G_b and contracting each set of $\mathcal{F}$ to a single vertex. Then H is a chordal graph.*

Lemma 58. *Consider a family $\mathcal{F}$ formed by a sequence of operations in G. Then for every $T \in \mathcal{F}$, $|T| \leq 2\,\text{ext}(T) - 1$. Also, $\text{ext}(T) \leq \alpha(T)$.*

An acceptable breaking of X by k requires that for each component D of $G[X]$ and each $T \in \mathcal{F}$ we have $\alpha(D - N(T))$ in the correct range. The following lemma shows that we can control $\alpha(D - N(T))$ by using the extension operation.

Lemma 59. *Let T' be the set formed by extending T into X by k. Then $\alpha(X - N(T)) - k \geq \alpha(X - T' - N(T'))$. That is, extending T into X by k using the procedure in Lemma 64 reduces $\alpha(X - N(T))$ by at least k.*

4.3.2 The $2\alpha(G) - 2$ algorithm

Let $n = |V(G)|$. We are going to build a partition $\mathcal{F}$ of $V(G)$ using only a sequence of breaking operations. At any stage of the algorithm, let $U = V(G) - \cup_{T \in \mathcal{F}} T$.

Case $\alpha(G) = 1$: Note that this conclusion is obvious but we give a detailed argument to make the reader more familiar with the definitions. The algorithm is to break $V(G)$ by 1. This is an acceptable operation because before the breaking $\mathcal{F}$ is empty and every component of G has independence number 1. Breaking $V(G)$ by 1 does not color any edges red because the family before the breaking is empty, and so the breaking results in a family of singleton sets $\mathcal{F} = \{\{v\} : v \in V(G)\}$ with $G_b = G$. Theorem 57 shows G is chordal, and using the weight function $f(v) = 1$ we have that the total weight is n and $\alpha_f(G) = \alpha(G) = 1$. Thus Theorem 52 shows that $\omega(G) \geq n$.

Case $\alpha(G) = 2$: We first break $V(G)$ by 2. This is acceptable because before the breaking $\mathcal{F}$ is empty and every component of G has independence number at most 2. No edges are colored red, and so this breaking results in a family $\mathcal{F}$ of disjoint induced P_3s (P_3 is the unique connected graph on three vertices with independence number 2). This family $\mathcal{F}$ is maximal because the remaining vertices (the set U) induce a disjoint union of cliques. We next break U by 1. This is acceptable because each $T \in \mathcal{F}$ dominates U so $\alpha(U - N(T)) = 0$, and each component of $G[U]$ is a clique. Also, no edge is colored red because each P_3 in $\mathcal{F}$ dominates U. Thus the two breaking operations produce a partition of $V(G)$ into a maximal family of induced P_3s and singleton sets of the remaining

vertices, with all edges colored blue ($G_b = G$). We now contract each P_3 to form the graph H, and use the weight function $f(v) = 3$ for a vertex v obtained by contracting a P_3, and $f(v) = 1$ otherwise. Thus $f(v)$ records the number of vertices in the set in $\mathcal{F}$ that is contracted down to v, and the total weight $f(V(H)) = n$. Theorem 57 shows that H is a chordal graph. To compute $\alpha_f(H)$, take any independent set I in H. This independent set corresponds to a pairwise non-touching subfamily $\mathcal{I}$ of $\mathcal{F}$. Since no edges are colored red, $\mathcal{I}$ is pairwise non-touching in G. Then either $\mathcal{I}$ contains one P_3 and nothing else (in which case $f(I) = 3$) or at most two single vertices (in which case $f(I) = 2$). Thus $\alpha_f(H) \leq 3$ so Theorem 52 shows that $\omega(H) \geq \lceil n/3 \rceil$, that is we have a complete minor of G of size at least $\lceil n/3 \rceil$.

Case $\alpha(G) \geq 3$: Initially, $U = V(G)$ and $\mathcal{F} = \emptyset$.

- Step 1: Let C be any component of $G[U]$. If $\alpha(C)$ is 1 or 2, then we break C like in the above two cases. If $\alpha(C) \geq 3$, then we break C by $\alpha(C) - 1$.
- Step 2: We now update $U := U - \cup_{T \in \mathcal{F}} T$ and continue Step 1 with a new C until $U = \emptyset$.

First, all the breakings are acceptable. Consider a component C we are about to break in Step 1. Now consider any set T that has already been produced, say T was born when C' was broke. If C is not contained in C' then there are no edges between T and C so $\alpha(C - N(T)) = \alpha(C)$. If $C \subseteq C'$, then $\alpha(T) = \alpha(C') - 1$ so that $\alpha(C' - N(T)) \leq 1$ which implies that $\alpha(C - N(T)) \leq 1$. Thus breaking C by $\alpha(C) - 1$ is acceptable. Because $\alpha(C - N(T)) \leq 1$, the only possibility for edges to be colored red in Step 1 is when we choose a component C with $\alpha(C) = 1$. Thus for each $T \in \mathcal{F}$, we have $G[T] = G_b[T]$.

Now consider the graph H formed by starting with G_b and contracting each set of $\mathcal{F}$. Consider the weight function f on $V(H)$ where we assign to each vertex of H the size of the set of $\mathcal{F}$ which it came from. Thus the total weight of f on H is n. By Theorem 57 we know that H is a chordal graph.

Next, we show that $\alpha_f(H) \leq 2\alpha(G) - 2$. Consider any independent set I in H. This corresponds to a pairwise non-touching (in G_b) subfamily $\mathcal{I}$ of $\mathcal{F}$. By Lemma 58, $|T| \leq 2\,\mathrm{ext}(T) - 1$ so that we can bound the total weight of I as follows:

$$f(I) = \sum_{T \in \mathcal{I}} |T| \leq 2 \sum_{T \in \mathcal{I}} \mathrm{ext}(T) - |I|\,.$$

If $|I| = 1$ then the largest breaking we ever do is by $\alpha(G) - 1$ which produce sets with $\mathrm{ext}(T) \leq \alpha(G) - 1$ which have size at most $2\alpha(G) - 3$. So assume $|\mathcal{I}| \geq 2$.

Claim 60. *For any pairwise non-touching family $\mathcal{I}$ in G_b, $\sum_{T \in \mathcal{I}} \mathrm{ext}(T) \leq \alpha(G)$.*

Proof. Define $\mu(\mathcal{I})$ to be the total number of red edges between sets of $\mathcal{I}$. Assume we have a counterexample to Lemma 60 where $\mu(\mathcal{I})$ is minimized. In other words, a pairwise non-touching family $\mathcal{I}$ in G_b where $\sum_{T \in \mathcal{I}} \mathrm{ext}(T) > \alpha(G)$ and $\mu(\mathcal{I})$ is minimized. If $\mu(\mathcal{I}) = 0$, then $\mathcal{I}$ is a pairwise non-touching family in G so that $\sum_{T \in \mathcal{I}} \alpha(T) \leq \alpha(G)$. By Lemma 58, $\mathrm{ext}(T) \leq \alpha(T)$ so $\sum_{T \in \mathcal{I}} \mathrm{ext}(T) \leq \alpha(G)$.

Assume now that $\mu(\mathcal{I}) \geq 1$ and take some $T, R \in \mathcal{I}$ where there is a red edge between T and R. We will produce a subfamily $\mathcal{I}'$ spanning fewer red edges. For edges to be colored red one of T or R must be a single vertex. Assume $|R| = 1$, and let C be the component containing R (with $\alpha(C) = 1$) chosen in Step 1 which caused the edges between T and R to be colored red. Thus $\alpha(C - N(T)) = 1$ so there exists a vertex $v \in V(C) - N(T)$. Let $\mathcal{I}' = \mathcal{I} - R + \{v\}$. Note that since $v \in V(C)$ and $\alpha(C) = 1$, $\{v\} \in \mathcal{F}$. We now show that v does not touch any other set in $\mathcal{I}'$. Say that there exists an $S \in \mathcal{I} \cap \mathcal{I}'$ where S touches v in G. Since $v \in V(C) - N(T)$, we must have $T \neq S$. First assume $\text{ext}(S) = 1$, so that $S = \{s\}$ for some vertex s. Then since s touches v we must have $s \in V(C)$. Note that when singletons are born, their component must be a clique. But then s touches R using a blue edge, contradicting that $S \in \mathcal{I}$. So we can assume $\text{ext}(S) \geq 2$.

First assume T is indexed lower than S, and let C' be the component chosen in Step 1 when T was born. Then $\alpha(T) \geq \text{ext}(T)$ and $\text{ext}(T)$ is one plus the number of extensions during the breaking so $\text{ext}(T) = \alpha(C') - 1$. Since S touches v and T is connected to v by a path of length 2 using a vertex of C we have that S is contained inside C'. Since $\alpha(S) \geq \text{ext}(S) \geq 2$ we must have T touching S using blue edges, contradicting that both are in $\mathcal{I}$. Now assume that S is indexed lower than T, and let C' be the component chosen in Step 1 when S was born. Then $\alpha(S) \geq \text{ext}(S) = \alpha(C') - 1$ and since S touches v and T is connected to v by a path of length 2 using a vertex of C we have that T is contained inside C'. Since $\alpha(T) \geq \text{ext}(T) \geq 2$ we have that S touches T using blue edges, contradicting that both are in $\mathcal{I}$. Thus v does not touch any other set in $\mathcal{I}'$, so $\mathcal{I}'$ is pairwise non-touching in G_b and we have reduced the number of red edges. Also, $\sum_{T \in \mathcal{I}'} \text{ext}(T) = \sum_{T \in \mathcal{I}} \text{ext}(T) > \alpha(G)$ contradicting that $\mathcal{I}$ was a minimum counterexample. □

Using Claim 60, we can immediately complete the proof since then $f(I) \leq 2\alpha(G) - |I| \leq 2\alpha(G) - 2$. To summarize, we can find a complete minor of G of size $\lceil r \rceil$, where r is defined as in Theorem 53. □

4.3.3 The $2\alpha(G) - \log_\tau(\tau\alpha(G)/2))$ algorithm

Given a graph G, we use the operations of breaking and extending to produce a partition $\mathcal{F}$ of $V(G)$ and a spanning subgraph G_b of blue edges. When we start the algorithm, $\mathcal{F}$ will be the empty family and $G_b = G$. The improvement from $2\alpha(G) - 2$ to $2\alpha(G) - \log_\tau(\tau\alpha(G)/2)$ comes from breaking each component C by $\lceil (\alpha(C) + 1)/2 \rceil$ so we produce sets of size approximately $\alpha(C)$, and then we extend the sets of $\mathcal{F}$ before future breakings only if it would prevent the breaking from being acceptable.

Given a graph G, set $G_b = G$ so all edges are colored blue and set $\mathcal{F} = \emptyset$.

We pick C to be any component of $G[V(G) - \cup_{T \in \mathcal{F}} T]$. If $\alpha(C) = 1$, we break C by 1 which constitutes Step C. So assume $\alpha(C) \geq 2$, and run the following substeps inside C, which constitutes Step C.

- Substep 1: For each $T \in \mathcal{F}$ with $\alpha(C - N(T)) = \alpha(C)$, color all edges between T and C red. Then let $b = \left\lceil \frac{\alpha(C)+1}{2} \right\rceil$, and let $A = V(C)$. Partition $\mathcal{F}$ into three classes.
  - $\mathcal{H}_0 = \{T \in \mathcal{F} :$ all edges between T and C are colored red$\}$,

– $\mathcal{H}_1 = \{T \in \mathcal{F} - \mathcal{H}_0 : \alpha(C - N(T)) < \sqrt{2}(b-1)\}$,
– $\mathcal{H}_2 = \mathcal{F} - \mathcal{H}_0 - \mathcal{H}_1$.

- Substep 2: For any $T \in \mathcal{H}_1$ and any component D of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend T into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Substep 2 until no pair T, D satisfies $b \leq \alpha(D - N(T)) < \alpha(D)$. Note, for the first T selected during Substep 2 we will have $D = C$ so that T satisfies $b \leq \alpha(C - N(T)) < \alpha(C)$ and thus is extended.

If there exists some $T \in \mathcal{H}_1$ which was not extended during Substep 2 and some component D of $G[A]$ such that $b \leq \alpha(D - N(T)) = \alpha(D)$ then we do not continue to Substep 3, instead we finish Step C. Otherwise, continue to Substep 3.

- Substep 3: For any $T \in \mathcal{H}_2$ and any component D of $G[A]$ with $b \leq \alpha(D - N(T)) < \alpha(D)$, we extend T into $V(D)$ by $\alpha(D - N(T)) - b + 1$. We then update $A := A - T$ and continue Step 3 until no pair T, D satisfies $b \leq \alpha(D - N(T)) < \alpha(D)$.

- Substep 4: Break A by b.

If $\mathcal{F}$ is not yet a partition of $V(G)$, pick a new component C.

In Sections 3 and 4, we prove that using this algorithm we can find a complete minor of G of size $\lceil n/r \rceil$, where r is defined in Theorem 50.

4.4 Analysis of the operations

4.4.1 Proofs of Theorems 52 and 57

If $V(G) = \{v_1, \ldots, v_n\}$ and $H_1, \ldots, H_n$ are pairwise disjoint graphs, then the **composition** $G[H_1, \ldots, H_n]$ is the graph formed by the vertex disjoint union of $H_1, \ldots, H_n$ plus the edges xy where $x \in V(H_i), y \in V(H_j)$ and $v_i v_j \in E(G)$. In 1972, Lovász [41] proved that a composition of perfect graphs is perfect.

Proof of Theorem 52. First, we modify f by multiplying each weight by their common denominator so that $f : V(H) \to \mathbb{Z}^+$. Multiplying every weight by a constant does not change $f(V(H))/\alpha_f(H)$. For $v \in V(H)$, define $H_v = f(v)K_1$ to be an independent set of size $f(v)$. Then define $H' = H[H_{v_1}, \ldots, H_{v_n}]$ as a composition of H. Then $f(V(H)) = |V(H')|$, $\omega(H) = \omega(H')$, and $\alpha_f(H) = \alpha(H')$. (If I' is a maximal independent set in H', then either $H_v \subseteq I'$ or $H_v \cap I' = \emptyset$ because H_v is an independent set and every vertex in H_v has the same neighborhood.) Since H' is a perfect graph, we have

$$\omega(H) = \omega(H') = \chi(H') \geq \left\lceil \frac{|V(H')|}{\alpha(H')} \right\rceil = \left\lceil \frac{f(V(H))}{\alpha_f(H)} \right\rceil.$$

□

We say that $\mathcal{F} = \{T_1, \ldots, T_k\}$ is a **partial simplicial elimination ordering** in G if for every non-touching pair T_i, T_j with $i < j$ and for every component C of $G - T_1 - \ldots - T_j$, at most one of T_i or T_j touches C. This corresponds exactly to a partial simplicial elimination ordering in the graph obtained by contracting each set of $\mathcal{F}$. We first prove that using acceptable operations we get a partial simplicial elimination ordering in the blue subgraph.

Theorem 61. *Let $\mathcal{F}_0$ be a partial simplicial elimination ordering in G, and let $\mathcal{F}$ be any family formed by starting with $\mathcal{F}_0$ and performing any sequence of acceptable operations. Let G_b be the spanning subgraph of blue edges after the operations. Then $\mathcal{F}$ is a partial simplicial elimination ordering in G_b.*

Proof. Let G_b be the spanning subgraph of G of the blue edges at the end of all operations. We need to prove that after every acceptable operation we have a partial simplicial elimination ordering.

Lemma 62. *Let $\mathcal{F}$ be a partial simplicial elimination ordering in G_b. Let U be the set of vertices $V(G) - \cup_{T \in \mathcal{F}} T$, and let $X \subseteq U$ be a union of some components of $G[U]$. Let k be an integer such that breaking X by k is an acceptable operation. Then the family obtained by breaking X by k is a partial simplicial elimination ordering in G_b.*

Proof. Let $\mathcal{F} = \{T_1, \ldots, T_m\}$ be the original family, and let $R_1, \ldots, R_\ell$ be the sets produced when we broke X by k. We consider a non-touching pair in G_b, and show that the pair satisfies the condition for a partial simplicial elimination ordering. We only need to consider pairs which contain at least one R_j.

Let T_i, R_j be a non-touching pair in G_b. Let D be the component of $G[X]$ containing R_j (then D is also a component of $G[U]$). We first show that all edges between T_i and D are colored red.

First assume T_i does not touch R_j in G. Then $k = \alpha(R_j) \leq \alpha(D - N(T_i))$ so by the condition in the definition of acceptable breaking we have $\alpha(D - N(T_i)) = \alpha(D)$ or all edges between T_i and D are colored red, thus after the breaking all edges between T_i and D are red.

Now assume the edges between T_i and R_j are red. Then we either had all edges between T_i and D red before the breaking or we colored all edges between T_i and D red during the breaking. Thus we have all the edges between T_i and D colored in red.

Let C be any component of $G_b - T_1 - \ldots - T_m - R_1 - \ldots - R_j$. We want to show that at least one of T_i or R_j does not touch C using blue edges. C is either contained inside D or disjoint from D, because D is a component of $G[U] = G - T_1 - \ldots - T_m$ and C is a connected subgraph of $G[U]$. If C is disjoint from D then R_j does not touch C in G. If C is contained inside D, then T_i does not touch C using blue edges because all edges between T_i and C are red.

Now consider a non-touching pair R_i, R_j in G_b with $i < j$. Assume R_i and R_j are contained in the same component D of $G[X]$. Then since $2k > \alpha(D)$ we must have R_i touching R_j in G so touching in G_b (we only color edges red which have exactly one endpoint in an existing set). Thus we must have R_i and R_j in different components of $G[X]$. So let C be any component of $G_b - T_1 - \ldots - T_m - R_1 - \ldots - R_j$, so that $V(C) \subseteq U$. Then $C[V(C) \cap X]$ is contained inside some component of $G_b[X]$ so $C[V(C) \cap X]$ cannot touch both R_i and R_j in G_b. Since there are no edges between X and $U - X$ in G, if R_i has no edges to $C[V(C) \cap X]$ then R_i has no edges to C and

similarly if R_j has no edges to $C[V(C) \cap X]$. Thus at most one of R_i or R_j touches C using blue edges. □

Lemma 63. *Let $\mathcal{F}$ be a partial simplicial elimination ordering in G_b. Let $X \subseteq V(G) - \cup_{T \in \mathcal{F}} T$ where $G_b[X]$ is connected, and let T_i be an element of $\mathcal{F}$. Then the family obtained by extending T_i into X is still a partial simplicial elimination ordering in G_b.*

Proof. Let $\mathcal{F} = \{T_1, \ldots, T_m\}$ before the extension, and let T' be the set T_i plus the vertices added during the extension. Now consider a $T_j \in \mathcal{F}$ where T_j does not touch T' in G_b, let $\ell = \max\{i, j\}$ and consider any component C of $G - T_1 - \ldots - T_\ell - T'$. Let D be the component of $G - T_1 - \ldots - T_\ell$ which contains C. Because $\mathcal{F}$ is a partial simplicial elimination ordering, at least one of T_i or T_j does not touch D using blue edges. Since $G[X]$ is connected, X is either contained inside $V(D)$ or disjoint from $V(D)$. If X is not contained inside $V(D)$, then $D = C$ and at least one of T_i or T_j does not touch C using blue edges. Extension does not change this, so one of T' or T_j does not touch C using blue edges. If X is contained inside $V(D)$, then T_i touches D using blue edges (T_i touches the new vertices in T' and we only extend using blue edges) so T_j does not touch D using blue edges so does not touch C using blue edges. □

Clearly, Lemma 62 and Lemma 63 imply Theorem 61. □

Proof of Theorem 57. Assume that $\mathcal{F}$ is formed by acceptable operations. Initially, we have that $\mathcal{F}_0 = \emptyset$ which is trivially a partial simplicial elimination ordering. Then by Theorem 61, $\mathcal{F}$ is a partial simplicial elimination ordering in G_b. Let H be the graph obtained from G_b by contracting each $T_i \in \mathcal{F}$ into a single vertex $v_i \in V(H)$.

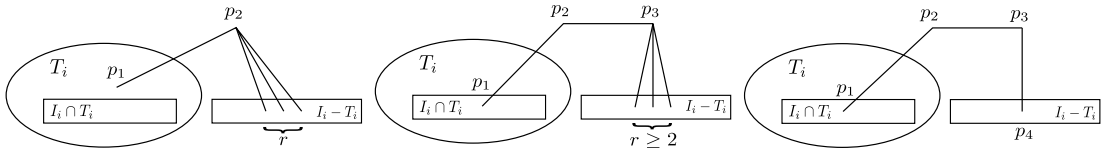
We show that H is chordal by giving a simplicial elimination ordering of H. We order the vertices of H according to the ordering of the sets of $\mathcal{F}$. For each $v_i \in V(H)$, define $B_i = N(v_i) \cap \{v_1, \ldots, v_i\}$. Assume we had $v_j, v_k \in B_i$ with $j < k < i$ where $v_j v_k \notin E(H)$. Let D be the component of $G_b - T_1 - \ldots - T_k$ which contains T_i. Then T_j does not touch T_k in G_b, so by the condition on partial simplicial elimination ordering one of T_j or T_k does not touch D in G_b. This contradicts that $v_j, v_k \in B_i$, so B_i spans a clique. This happens for each i, yielding that H is a chordal graph. □

4.4.2 Some properties of the operations

In this subsection, let G be any graph and G_b any spanning subgraph of G. Let $T, X \subseteq V(G)$ and k any integer with $T \cap X = \emptyset$, $G_b[T]$ connected, $G_b[X \cup T]$ connected, no red edges between T and X, and $k \leq \alpha(X - N(T))$. (These are the conditions when we extend T into X by k during the algorithm.)

Lemma 64. *It is possible to extend T into X by k such that $G_b[T]$ remains connected and $\alpha(T)$ increases by at least k and $|T|$ increases by at most $2k$.*

Proof. Let $T_0 = T$ so T_0 is the initial T. We use the following algorithm to produce $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_k$, where $\alpha(T_i) \geq \alpha(T_0) + i$ and $|T_i| \leq |T_0| + 2i$. (Note that we do not define T_i for every $i < k$.) Initially, let I_0 be any maximal independent set in $G[X - N(T_0)]$ with $|I_0| \geq k$.



(a) Case 1 (b) Case 2 (c) Case 3

Figure 4.1: Extensions of T_i.

Assume we have defined T_i and I_i with $i < k$. We now show how to define T_{i+r} and I_{i+r} for some $1 \leq r \leq k - i$.

Step 1. Choose P to be a shortest path in $G_b[X \cup T_0]$ between T_i and $I_i - T_i$. The length of P is at most three because I_i is a maximal independent set in $G_b[X - N(T_0)]$. The algorithm maintains that there are no edges between T_i and $I_i - T_i$ when $i < k$, so the length of P is at least two.

Step 2.

- **Case 1:** Consider when $P = (p_1, p_2, p_3)$ with $p_1 \in T_i$ and $p_3 \in I_i - T_i$. Then we add p_2 and

$$r = \min\{k - i, |N(p_2) \cap (I_i - T_i)|\} \geq 1$$

  vertices from $N(p_2) \cap (I_i - T_i)$ to T_i to form T_{i+r}. Let $I_{i+r} = I_i$. Thus $\alpha(T_{i+r}) = \alpha(T_i) + r$ and $|T_{i+r}| = |T_i| + 1 + r$.

- **Case 2:** Consider when $i \leq k-2$ and $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $|N(p_3) \cap (I_i - T_i)| \geq 2$. Here, we add p_2, p_3, and

$$r = \min\{k - i, |N(p_3) \cap (I_i - T_i)|\} \geq 2$$

  vertices of $N(p_3) \cap (I_i - T_i)$ to T_i to form T_{i+r}. Let $I_{i+r} \subset G[X - T_i - N(T_i)]$ be a maximal independent set containing I_i. Then $\alpha(T_{i+r}) \geq \alpha(T_i) + r$ and $|T_{i+r}| = |T_i| + 2 + r$. Since $r \geq 2$, the increase in the number of vertices is at most twice the increase of i.

- **Case 3:** Consider when $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $N(p_3) \cap I_i = \{p_4\}$. We set $I_{i+1} = I_i - \{p_4\} + \{p_3\}$ and then extend I_{i+1} to be a maximal independent set in $G[X - T_i - N(T_i)]$. Then I_{i+1} is still a maximal independent set of size at least k, and we can now add p_2 and p_3 to T_i to get T_{i+1}. This increases the number of vertices by two and the independence number by one.

- **Case 4:** Consider when $i = k - 1$ and $P = (p_1, p_2, p_3, p_4)$ with $p_1 \in T_i$ and $|N(p_3) \cap I_i| \geq 2$. Here, we add p_2 and p_3 to T_i to get T_k. Let $I_k = I_i - \{p_4\} + \{p_3\}$. Thus $\alpha(T_k) \geq \alpha(T_0) + k$ and $|T_k| \leq |T_i| + 2$.

Consider one step which did not produce T_k. If this step is Case 1, then we added the entire set $N(p_2) \cap I_i$ to T_i. In Case 2, we added the entire set $N(p_3) \cap I_i$ to T_i. In Case 3, we added the entire $N(p_2) \cap I_{i+1}$ to T_i. In Case 4, we always produce T_k. Note that we always maintain that there are

no edges between T_i and $I_i - T_i$ if $i < k$. If we ever added all vertices of I_i to T_i we would have increased $\alpha(T_i)$ to $\alpha(T_0) + k$ because $I_i \subseteq X - N(T_0)$ and $|I_i| \geq k$. We continue the algorithm if $i < k$ so we will eventually produce a T_k. □

Lemma 65. *The extension of $T = \{v\}$ into $D - T$ by $k - 1$ during Step (b) of a breaking operation satisfies all the conditions of the extension.*

Proof. Any edge colored red has at least one endpoint in a set of $\mathcal{F}$, so that $G_b[D] = G[D]$ where G_b is the spanning subgraph of blue edges at the time of extension. Since D is a component of $G[U]$, we have $G_b[D]$ connected. Also, since $T \subseteq V(D)$ we have no red edges between T and $D - T$ at the time of extension. Finally, since we chose $v \in I$ where I is an independent set of size at least k, we have $\alpha(D - N(v)) \geq |I - v| \geq k - 1$. □

Proof of Lemma 58. The extending operation does not produce new sets, so the only way to produce a new set is by breaking some set X by k. In Step (b) of the breaking, we initially have $|T| = 1$, and then we extend T by $k - 1$ which adds at most $2k - 2$ new vertices, so T has at most $2k - 1$ vertices. Since the independence number increased by at least $k - 1$, we have $\alpha(T) \geq k$. Extending T by k increases the number of its vertices by at most $2k$ and its independence number by at least k. Thus $|T| \leq 2\,\mathrm{ext}(T) - 1$ and $\alpha(T) \geq \mathrm{ext}(T)$. □

Proof of Lemma 59. Let $B = G[X - N(T)]$. Assume towards a contradiction that $\alpha(B) - k + 1 \leq \alpha(X - T' - N(T'))$, and let $I = I_k$ be the independent set used at the end of the proof of Lemma 64. Then $\alpha(B) \geq \alpha(B \cap T') + \alpha(B - T' - N(T'))$. Since $B - T' - N(T') = X - T' - N(T')$, we have $\alpha(B) \geq \alpha(B \cap T') + \alpha(B) - k + 1$, i.e. $\alpha(B \cap T') < k$.

But $|I \cap T'| = k$ because the algorithm added at least k vertices of $I_k = I$ to form $T_k = T'$. Since $I \subseteq V(B)$, we have $\alpha(B \cap T') \geq k$, a contradiction. □

4.5 The analysis of the $2\alpha(G) - \log_\tau(\tau\alpha(G)/2)$ algorithm

Definition. Let $f : \{0\} \cup \mathbb{R}^{\geq 1} \to \mathbb{R}, \tau \in \mathbb{R}$ satisfy the following properties:

P1: $f(0) = 0$,

P2: $f(4\sqrt{2}) \leq 1$,

P3: If $1 \leq x \in \mathbb{R}$, then $f(\tau x) \leq 1 + f(x)$,

P4: If $1 \leq x, y \in \mathbb{R}$, then $f(2\sqrt{2}x + 2\sqrt{2}y) \leq f(x) + f(y)$,

P5: If $0 \leq x \leq y \in \mathbb{Z}$, then $f(y) \leq f(x) + y - x$,

P6: If $1 \leq x \leq y \in \mathbb{Z}$ and $1 \leq r \in \mathbb{R}$, then $f(ry) \leq f(rx) + y - x$,

P7: f is non-decreasing so by property P4, if $x_1, \ldots, x_k \in \mathbb{R}$ with $x_i \geq 1$, then $f(\sum_i x_i) \leq \sum_i f(x_i)$,

P8: If $1 \leq x \in \mathbb{Z}$, then either $\lceil \sqrt{2}(x-1) \rceil \geq \frac{\tau}{\tau - \sqrt{2}} x$ or $f(2\sqrt{2}x) \leq f(\tau)$.

P9: If $2 \leq x \in \mathbb{Z}$, then $f(2\sqrt{2}x) \leq 1 + f(x - \lceil (x+1)/2 \rceil)$,

P10: If $2 \leq x \in \mathbb{Z}$, then $\sqrt{2}x \leq \tau \lceil (x-1)/2 \rceil$,

P11: If $2 \leq x \in \mathbb{Z}$, then $f(\sqrt{2}x) \leq 2 \lceil (x-1)/2 \rceil$.

We can pick $f(x) = \max\left\{\left\lceil \log_\tau(\tau x/(4\sqrt{2}))\right\rceil, 0\right\}$ for $x > 0$ and $f(0) = 0$, where $\tau = 2\sqrt{2}/(\sqrt{2}-1) \approx 6.83$.

The goal of this section is to prove the following which implies our main result:

Theorem 66. *The algorithm in Section 2.3 produces a complete minor of size* $\lceil n/(2\alpha(G) - f(2\sqrt{2}\alpha(G)))\rceil$.

To prove Theorem 66 we use Theorems 52 and 57, so we need to prove that the algorithm uses acceptable operations and give an upper bound for the weight of an independent set.

Notation. Let $\mathcal{F}$ be the partition after the algorithm terminates, and let G_b be the spanning subgraph of blue edges after the algorithm terminates. Let $\mathcal{F}_C$ be the family before Step C begins, and define $A_C = V(C) - \cup_{R\in\mathcal{F}_C} R$. If $\alpha(C) > 1$, define $\mathcal{F}_C^i$ to be the family right before Substep i of Step C, define $\mathcal{F}_C^5$ to be the family after all substeps of Step C are completed, and let $A_C^i = V(C) - \cup_{R\in\mathcal{F}_C^i} R$. For $T \in \mathcal{F}$ and $1 \leq i \leq 5$, define

$$T^{C,i} = \begin{cases} S & S \in \mathcal{F}_C^i \text{ and } S \subseteq T \text{ if there exists such an S,} \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, $T^{C,i}$ is the set in $\mathcal{F}_C^i$ that gets extended to T during the rest of the algorithm.

Define $\mathcal{H}_0(C)$, $\mathcal{H}_1(C)$, and $\mathcal{H}_2(C)$ to be the partition chosen in Substep 1 of Step C.

Lemma 67. *Let* $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$ *such that* $T^{C,1}$ *was extended during Substep 2 or Substep 3 of Step* C*. This extension satisfies all the conditions of an extension.*

Proof. Let A be the set of vertices not in a set of $\mathcal{F}$ right before the extension, D the component of $G[A]$ which $T^{C,1}$ is extended into and G_b' the spanning subgraph of blue edges at the time of the extension. Since $T^{C,1} \notin \mathcal{H}_0(C)$ there are no red edges between $T^{C,1}$ and C so no red edges between $T^{C,1}$ and D. Since each red edge at the time of extension has at least one endpoint in $\mathcal{F}_C$, we have $G[V(D)] = G_b'[V(D)]$. Since D is a component of $G[A]$, we have that $G_b'[V(D)]$ is connected. Since we are extending $T^{C,1}$ we must have $\alpha(D - N(T^{C,1})) < \alpha(D)$ implying there exist edges between $T^{C,1}$ and D. These edges must be blue so $G_b'[V(D) \cup T^{C,1}]$ is connected. Finally, we extend by $\alpha(D - N(T^{C,1})) - b + 1$ which is smaller than $\alpha(D - N(T^{C,1}))$ since $b \geq 1$. □

Lemma 68. $\mathcal{F}$ *is formed by acceptable operations.*

Proof. Consider Step C. If $\alpha(C) = 1$, we break C by 1. This breaking is acceptable because for each $T \in \mathcal{F}_C$ we either have $\alpha(C - N(T)) = 0 < 1$ or $\alpha(C - N(T)) = \alpha(C)$. So assume $\alpha(C) \geq 2$.

The coloring in Substep 1 can be viewed as breaking $V(C)$ by $\alpha(C)+1$. For $T \in \mathcal{F}$, in Step (a) of breaking $V(C)$ by $\alpha(C)+1$ we colors all edges red between T and D, where D is a component of C (i.e. $D = C$) with $\alpha(D - N(T)) = \alpha(D)$. Since we are breaking by $\alpha(C)+1$, Step (b) of the breaking does not produce any new sets. This breaking is acceptable since for each $T \in \mathcal{F}$, $\alpha(C - N(T)) \leq \alpha(C) < \alpha(C)+1$.

Extensions are always acceptable, so if Step C does not continue to Substep 3 after Substep 2 then Step C uses acceptable operations. So assume that Step C continues to Substep 3 and then consider the breaking in Substep 4. Since $b = \lceil(\alpha(C)+1)/2\rceil$, for any component D of $G[A_C^4]$ we have $\alpha(D) \leq \alpha(C) \leq 2b$. So consider some $T^{C,4} \in \mathcal{F}_C^4$ and some component D of $G[A_C^4]$. If $T^{C,1} \in \mathcal{H}_0(C)$, then $T^{C,1} = T^{C,4}$ and $\alpha(D - N(T^{C,4})) \leq \alpha(C - N(T^{C,4})) < b$ or all edges between $T^{C,4}$ and D are colored red.

If $T^{C,1} \in \mathcal{H}_1(C)$, then we considered extending $T^{C,1}$ in Substep 2. If we extended $T^{C,1}$ then by Lemma 59 we must have $\alpha(D-N(T^{C,4})) < b$. So assume $T^{C,1} = T^{C,4}$ and that $b \leq \alpha(D-N(T^{C,1}))$, and let D' be the component of $G[A_C^3]$ which contains D. Then $b \leq \alpha(D' - N(T^{C,1}))$ and since we continued to Substep 3 we must have $\alpha(D' - N(T^{C,1})) < \alpha(D')$. In this case we should have continued Substep 2 with the pair $T^{C,1}, D'$. Thus we must have $\alpha(D - N(T^{C,1})) < b$.

If $T^{C,1} \in \mathcal{H}_2(C)$ then we considered extending $T^{C,1}$ in Substep 3. If we extended $T^{C,1}$ then by Lemma 59 we must have $\alpha(D - N(T^{C,4})) < b$. So assume $T^{C,1} = T^{C,4}$. If $b \leq \alpha(D - N(T^{C,4})) < \alpha(D)$, then we should have continued Substep 3 with the pair $T^{C,1}, D$. Thus either $\alpha(D-N(T^{C,4})) < b$ or $\alpha(D - N(T^{C,4})) = \alpha(D)$, showing that the breaking in Substep 4 is acceptable. □

We now need to bound the maximum weight of an independent set. Define the set of **independent subfamilies of** $\mathcal{F}$ in G_b by

$$\mathrm{IND}_{G_b}(\mathcal{F}) = \{\mathcal{I} \subseteq \mathcal{F} : \mathcal{I} \text{ is a pairwise non-touching subfamily in } G_b\}.$$

Independent subfamilies of $\mathcal{F}$ correspond to independent sets in H.

Using the weight function which measures a set with its size, the total weight is $|V(G)|$. Then the total weight of $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$ is $\sum_{T\in\mathcal{I}} |T|$. Using Lemma 58, we know that the weight of $\mathcal{I}$ is at most $2\sum_{T\in\mathcal{I}} \mathrm{ext}(T) - |\mathcal{I}|$. We will give an upper bound of $2\alpha(G) - f(2\sqrt{2}\alpha(G))$ on the weight. To do this, we prove the following inequality

$$f(2\sqrt{2}\alpha(G)) \leq |\mathcal{I}| + 2\alpha(G) - 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T).$$

Note that when we analyzed the Section 2.2 algorithm, we showed that either $|\mathcal{I}|$ is at least 2 or that $\alpha(G) - \sum_{T\in\mathcal{I}} \mathrm{ext}(T)$ is at least 1. That is, we showed that in order for the total amount of extensions of sets in $\mathcal{I}$ to be $\alpha(G)$ we need more than one set.

Consider Figure 4.2, where the vertices of the tree are the steps run by the algorithm. Each step of the algorithm corresponds to a component, and the tree is the containment tree of these components. Let $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$, with $T, S \in \mathcal{I}$. Say that T is born in the step labeled C_1 in the figure, and is extended during the steps labeled C_2, C_3, and C_4. Assume that S is born in the step

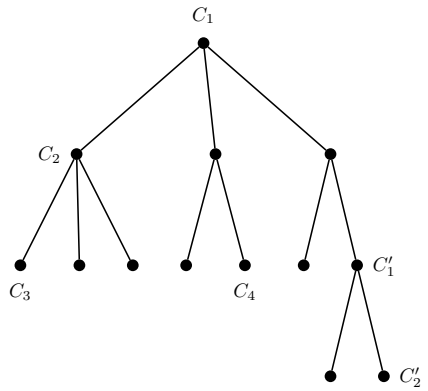


Figure 4.2: Containment tree of the steps run by the algorithm.

labeled C_1' and is extended in the step labeled C_2'. We would like to prove by induction on the steps of the algorithm that

$$|\{Q \in \mathcal{I} : Q \subseteq V(C)\}| + 2\alpha(C) - 2\sum\{\text{ext}(Q) : Q \in \mathcal{I}, Q \subseteq V(C)\}$$

is large. We are unable to prove this directly because when the induction reaches Step C_1 we must include $\text{ext}(T)$ into the sum for the first time because Step C_1 is the first step where T is completely contained inside the component for the step. Instead, we would like our inductive bound for Step C_2 to include the amount of extensions of T carried out in steps C_2 and C_3 which is only part of $\text{ext}(T)$, so that when we reach Step C_1 the inductive bounds for the smaller components contained inside C_1 already include most of the value $\text{ext}(T)$. So we define a notion of the gap between $\alpha(C)$ and $\sum\{\text{ext}(Q) : Q \in \mathcal{I}, Q \subseteq V(C)\}$ which allows us to include only the amount extensions of T into some subset of $V(C)$. Note that since $\mathcal{F}$ is a partial simplicial elimination ordering, we can have at most one set T which has part of its extensions inside C and part outside C. Define for any $T \in \mathcal{I}$

$$\begin{aligned}\text{ext}(C,T) &= \text{ the total amount of extensions of } T \text{ into } X \text{ where } X \subseteq V(C),\\ \text{gap}(C,\mathcal{I},T) &= \alpha(C - N(T^{C,1})) - \text{ext}(C,T) - \sum\{\text{ext}(Q) : Q \in \mathcal{I}, Q \neq T, Q \cap V(C) \neq \emptyset\}.\end{aligned}$$

Note that if $T \cap V(C) \neq \emptyset$ then for each Q in the sum we must have $Q \subseteq V(C)$ because $\mathcal{F}$ is a partial simplicial elimination ordering. In addition, define

$$\begin{aligned}\text{ext}(C,\emptyset) &= 0,\\ \text{gap}(C,\mathcal{I},\emptyset) &= \alpha(C) - \sum_{\substack{Q\in\mathcal{I}\\ Q\cap V(C)\neq\emptyset}} \text{ext}(Q).\end{aligned}$$

In the next lemma, we show that $|\mathcal{I}| + 2\,\text{gap}(C,\mathcal{I},T)$ is large by induction on the number of steps carried out by the algorithm. For comparison with the Section 2.2 algorithm, Claim 60 proves $0 \leq \text{gap}(G,\mathcal{I},\emptyset)$.

Lemma 69. *Consider any component C chosen as a Step during the algorithm. Let $\mathcal{I}_0 \in \mathrm{IND}_{G_b}(\mathcal{F})$, and let $\mathcal{I} = \{Q \in \mathcal{I}_0 : Q \cap V(C) \neq \emptyset\}$.*

(i) If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_1(C)$ then

$$f(\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T) - 1.$$

(ii) If $\alpha(C) > 1$ and there exists $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $T^{C,1} \in \mathcal{H}_2(C)$ then

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T) - 1.$$

(iii) Otherwise,

$$f(2\sqrt{2}\alpha(C)) \leq |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, \emptyset).$$

Using Lemma 69, we can prove Theorem 66.

Proof of Theorem 66. Let H be the graph formed from G_b by contracting each set of $\mathcal{F}$. Define g to be the weight function on $V(H)$ which assigns to each $v \in V(H)$ the size of the set of $\mathcal{F}$ which contracted to v. By Lemma 68 and Theorem 57 H is a perfect graph, so by Theorem 52 we just need to show that $\alpha_g(H) \leq 2\alpha(G) - f(2\sqrt{2}\alpha(G))$.

Let $C_1, \ldots, C_k$ be the components of G. Let I be any independent set in H, which corresponds to a subfamily $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. Then define $\mathcal{I}_i = \{T \in \mathcal{I} : T \cap V(C_i) \neq \emptyset\}$. Since $C_1, \ldots, C_k$ are components of G we have $\mathcal{I}_i = \{T \in \mathcal{I} : T \subseteq V(C_i)\}$.

For each $T \in \mathcal{I}_i$, we have $T \subseteq V(C_i)$ which implies $T^{C_i,1} = \emptyset$. Thus we apply the bound in case *(iii)* of Lemma 69 for each component C_i to obtain

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq \sum_{1 \leq i \leq k} |\mathcal{I}_i| + 2 \sum_{1 \leq i \leq k} \mathrm{gap}(C_i, \mathcal{I}_i, \emptyset). \tag{4.1}$$

Expanding the definition of $\mathrm{gap}(C_i, \mathcal{I}_i, \emptyset)$ in (4.1) gives,

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq |\mathcal{I}| + 2 \sum_{1 \leq i \leq k} \alpha(C_i) - 2 \sum_{1 \leq i \leq k} \sum_{T \in \mathcal{I}_i} \mathrm{ext}(T). \tag{4.2}$$

Since $\sum_i \alpha(C_i) = \alpha(G)$ and each $T \in \mathcal{I}$ appears in exactly one $\mathcal{I}_i$, (4.2) simplifies to

$$\sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)) \leq |\mathcal{I}| + 2\alpha(G) - 2 \sum_{T \in \mathcal{I}} \mathrm{ext}(T). \tag{4.3}$$

Using Lemma 58 to bound $|T|$ and rearranging (4.3), we have

$$g(I) = \sum_{T \in \mathcal{I}} |T| \leq 2 \sum_{T \in \mathcal{I}} \mathrm{ext}(T) - |\mathcal{I}| \leq 2\alpha(G) - \sum_{1 \leq i \leq k} f(2\sqrt{2}\alpha(C_i)).$$

By property P7,

$$g(I) \leq 2\alpha(G) - f(2\sqrt{2} \sum_{1 \leq i \leq k} \alpha(C_i)) = 2\alpha(G) - f(2\sqrt{2}\alpha(G)).$$

This says every independent set in H has weight at most $2\alpha(G) - f(2\sqrt{2}\alpha(G))$, we have $\alpha_g(H) \leq 2\alpha(G) - f(2\sqrt{2}\alpha(G))$. □

Before proving Lemma 69, we need some lemmas:

Lemma 70. *Let $T^{C,1} \in \mathcal{H}_1(C)$ and let R be a set born in Substep 4 of Step C. Then $T^{C,4}$ touches R with blue edges.*

Proof. Since $T^{C,1} \in \mathcal{H}_1(C)$, all edges between $T^{C,1}$ and C are colored blue at the start of Step C. We produced an R in Substep 4 so $\alpha(C) \geq 2$, so we consider extending $T^{C,1}$ in Substep 2. Since we continued to Substep 3 after Substep 2, we have for each component D of $G[A_C^3]$, $\alpha(D - N(T^{C,3})) < b$. Now consider the component D' of $G[A_C^4]$ which contains R. Then there exists a component D of $G[A_C^3]$ which contains D' so $\alpha(D' - N(T^{C,4})) \leq \alpha(D - N(T^{C,4})) < b$ so no edges incident to $T^{C,4}$ are colored red during Substep 4. Also, since $b \leq \alpha(R)$ by Lemma 58 we must have an edge of G between $T^{C,4}$ and R. This edge is colored blue at the end of the algorithm because no edges incident to $T^{C,4}$ are colored red during Substep 4 and all future edge colorings only color edges between an existing set in $\mathcal{F}$ and a vertex of A. □

Lemma 71. *Let $T^{C,1} \in \mathcal{H}_2(C)$, and assume that $T^{C,1}$ was extended in Substep 3 of Step C. Let R be a set born during Substep 4 of Step C. Then $T^{C,4}$ touches R with blue edges.*

Proof. Since $T^{C,1} \in \mathcal{H}_2(C)$ and we extended $T^{C,1}$, let A be the subset of vertices of C not yet in a set at the time we extend $T^{C,1}$, and let D be the component of $G[A]$ where $b \leq \alpha(D - N(T^{C,1})) < \alpha(D)$. By Lemma 59, $\alpha(D - T^{C,4} - N(T^{C,4})) < b$. Also, consider any other component D' of $G[A]$ besides D. Since $\alpha(C)/2 < b < \alpha(D)$, we must have $\alpha(D') < b$ since $\alpha(D) + \alpha(D') \leq \alpha(C)$. Thus for all components D' of $G[A - T^{C,4}]$ we have $\alpha(D' - N(T^{C,4})) < b$. Now R is connected so R must be contained inside some component D'' of $G[A_C^4]$ which is contained inside some component D' of $G[A - T^{C,4}]$. If $T^{C,4}$ does not touch R in G, then $\alpha(D - N(T^{C,4})) \geq \alpha(R) \geq b$ which gives a contradiction. If $T^{C,4}$ touches R with red edges, then we must have colored the edges between $T^{C,4}$ and D'' red during Substep 4 so $\alpha(D'' - N(T^{C,4})) = \alpha(D'') \geq \alpha(R) \geq b$, again giving a contradiction. □

Lemma 72. *Let $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. Then $\alpha(C - N(T^{C,1})) < \alpha(C)$.*

Proof. Assume $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. If $\alpha(C - N(T^{C,1})) = \alpha(C)$, then we color all edges between T and C red in Substep 1 of Step C. Since all edges are red, $T^{C,1} \in \mathcal{H}_0(C)$ which contradicts $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$. □

Proof of Lemma 69. The proof works by induction on $|V(C)|$, where Step C is a step carried out by the algorithm. Fix an $\mathcal{I}_0 \in \mathrm{IND}_{G_b}(\mathcal{F})$ and a component C chosen by the algorithm and consider

Step C. For the rest of this section, let $D_1, \ldots, D_k$ be the components of $G[A_C^5]$. We can then apply induction into each of the components D_i because at some future time in the algorithm D_i will be selected as a Step. Let $\mathcal{I} = \{T \in \mathcal{I}_0 : T \cap V(C) \neq \emptyset\}$ and $\mathcal{I}_i = \{T \in \mathcal{I} : T \cap V(D_i) \neq \emptyset\}$. We can apply induction into D_i with the independent subfamily $\mathcal{I}_i$.

If $\alpha(C) = 1$, we need to show $f(2\sqrt{2}) \leq |\mathcal{I}| + 2\operatorname{gap}(C, \mathcal{I}, \emptyset)$. If $\operatorname{gap}(C, \mathcal{I}, T) = 0$ then $|\mathcal{I}| = 1$. As f is non-decreasing, property P2 shows $f(2\sqrt{2}) \leq 1$.

Now assume $\alpha(C) > 1$ and consider the possibilities for $T \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$. We cannot have two sets $T, R \in \mathcal{I}$ with $T^{C,1} \neq \emptyset$ and $R^{C,1} \neq \emptyset$, because this would contradict that $\mathcal{F}$ forms a partial simplicial elimination ordering since T and R touch C with blue edges ($G_b[T]$ is connected and $T \cap V(C) \neq \emptyset$) but all edges between T and R are red. Also, we cannot have two sets $T, R \in \mathcal{I}$ which were born in Substep 4 of Step C because the sets born in Substep 4 are pairwise touching using blue edges. Thus define T to be the set in $\mathcal{I}$ with $T^{C,1} \neq \emptyset$ if it exists and otherwise define $T = \emptyset$, and define R to be the set in $\mathcal{I}$ which was born in Substep 4 of Step C, otherwise $R = \emptyset$. Note that $T \neq \emptyset$ implies that $T \in \mathcal{I}$ so that $T \cap V(C) \neq \emptyset$ which implies there are blue edges between T and C which implies $T^{C,1} \notin \mathcal{H}_0(C)$. Thus if $T \neq \emptyset$ we need to prove the inequality in either case *(i)* or *(ii)* of Lemma 69. If $T = \emptyset$ we need to prove the inequality in case *(iii)* of Lemma 69.

For each D_i, at most one of T or R can touch D_i using blue edges. (If both touch D_i using blue edges, then we contradict the partial simplicial elimination ordering.) Define Q_i to be T or R depending on which is contained in $\mathcal{I}_i$, and define $Q_i = \emptyset$ if neither is in $\mathcal{I}_i$. Define

$$\gamma_i = \begin{cases} 1 & Q_i^{D_i,1} \in \mathcal{H}_1(D_i), \\ \sqrt{2} & Q_i^{D_i,1} \in \mathcal{H}_2(D_i), \\ 2\sqrt{2} & Q_i = \emptyset. \end{cases}$$

Claim 73.

$$\sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) - \sum_{1 \leq i \leq k} 2\operatorname{gap}(D_i, \mathcal{I}_i, Q_i) \leq |\mathcal{I} - \{T, R\}|.$$

We actually use Claim 73 in the following form:

$$\sum_i f(\gamma_i \alpha(C - N(Q_i^{C,5}))) + 2\operatorname{gap}(C, \mathcal{I}, \emptyset) - 2\sum_i \operatorname{gap}(D_i, \mathcal{I}_i, \emptyset) \leq |\mathcal{I} - \{T, R\}| + 2\operatorname{gap}(C, \mathcal{I}, \emptyset). \tag{4.4}$$

Proof. Assume we have indexed the components so that for $1 \leq i \leq h_1$, $Q_i = T$ and for $h_1 < i \leq h_2$, $Q_i = R$ and for $h_2 < i \leq k$, $Q_i = \emptyset$. We consider Step D_i. Then $Q_i^{D_i,1}$ is the set of $\mathcal{I}_i$ which will touch D_i in blue and be considered in the statement of Lemma 69, and γ_i is the coefficient inside the function f. Thus for $1 \leq i \leq h_2$, we obtain

$$f(\gamma_i \alpha(D_i - N(Q_i^{D_i,1}))) \leq |\mathcal{I}_i| - 1 + 2\operatorname{gap}(D_i, \mathcal{I}_i, Q_i).$$

Note that $D_i \cap N(Q_i^{D_i,1}) = D_i \cap N(Q_i^{C,5})$ so we have

$$f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \leq |\mathcal{I}_i| - 1 + 2\,\mathrm{gap}(D_i, \mathcal{I}_i, Q_i).$$

Here, we can think of $|\mathcal{I}_i| - 1$ as counting the number of sets in $\mathcal{I}_i$ besides Q_i. For $h_2 < i \leq k$ we obtain

$$f(\gamma_i \alpha(D_i)) \leq |\mathcal{I}_i| + 2\,\mathrm{gap}(D_i, \mathcal{I}_i, \emptyset).$$

Again $|\mathcal{I}_i|$ is counting the sets of $\mathcal{I}_i$ besides Q_i. Thus

$$\sum_{i=1}^{h_2} (|\mathcal{I}_i| - 1) + \sum_{i=h_2+1}^{k} |\mathcal{I}_i| = |\mathcal{I} - \{T, R\}|\,.$$

Thus summing the inductive bounds over all i we obtain (for $h_2 < i \leq k$, $Q_i = \emptyset$ so that $\alpha(D_i - N(Q_i^{C,5})) = \alpha(D_i)$)

$$\sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))) \leq |\mathcal{I} - \{T, R\}| + \sum_{1 \leq i \leq k} 2\,\mathrm{gap}(D_i, \mathcal{I}_i, Q_i).$$

□

We finish the proof of Lemma 69 by showing that the inequality in Claim 73 simplifies in all cases to the inequalities in Lemma 69. For the simplification, we add $\mathrm{gap}(C, \mathcal{I}, T)$ to both sides of Claim 73 and then use lower bounds on $\mathrm{gap}(C, \mathcal{I}, T) - \sum_i \mathrm{gap}(D_i, \mathcal{I}_i, Q_i)$ and lower bounds on $\sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5})))$. Define

$$\begin{aligned}
\theta &= \mathrm{ext}(C, T) - \sum_{1 \leq i \leq k} \mathrm{ext}(D_i, T) = \text{ the amount of extensions of } T \text{ during Step } C,\\
\lambda &= \sum_{1 \leq i \leq k} \alpha(D_i - N(Q_i^{C,5})),\\
J &= \left\{ i : \alpha(D_i - N(Q_i^{C,5})) > 0 \right\}.
\end{aligned}$$

If $T = \emptyset$, we define $\theta = 0$.

We claim the following bounds.

Bound 1: If $T = R = \emptyset$, then $f(2\sqrt{2}\lambda) \leq \sum_{i \in J} f(\gamma_i \alpha(D_i - N(Q_i^{C,5})))$,

Bound 2: If $|J| \geq 2$, then $f(2\sqrt{2}\lambda) \leq \sum_{i \in J} f(\gamma_i \alpha(D_i - N(Q_i^{C,5})))$,

Bound 3: If $J = \{i\}$, then $f(\gamma_i \lambda) = \sum_{i \in J} f(\gamma_i \alpha(D_i - N(Q_i^{C,5})))$,

Bound 4: $f(\lambda) \leq \sum_i f(\gamma_i \alpha(D_i - N(Q_i^{C,5})))$,

Bound 5: If $R = \emptyset$, then $\mathrm{gap}(C, \mathcal{I}, T) - \sum_i \mathrm{gap}(D_i, \mathcal{I}_i, Q_i) = \alpha(C - N(T^{C,1})) - \theta - \lambda$,

Bound 6: If $R \neq \emptyset$, then $\mathrm{gap}(C, \mathcal{I}, T) - \sum_i \mathrm{gap}(D_i, \mathcal{I}_i, Q_i) = \alpha(C - N(T^{C,1})) - b - \lambda$.

We now justify these bounds. For Bound 1, $T = R = \emptyset$ implies that $Q_i = \emptyset$ and $\gamma_i = 2\sqrt{2}$ for all i so the inequality follows by property P7. Bound 2 follows from property P4 since $|J| \geq 2$. Bound 3 is an equality by definition of λ. For Bound 4, property P7 and $\gamma_i \geq 1$ imply

$$f(\lambda) \leq \sum_{1 \leq i \leq k} f(\alpha(D_i - N(Q_i^{C,5}))) \leq \sum_{1 \leq i \leq k} f(\gamma_i \alpha(D_i - N(Q_i^{C,5}))).$$

Now we justify Bound 5; assume $R = \emptyset$. First, using the definition of Q_i we have

$$\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq h_1} \text{ext}(D_i, T) + \sum_{h_1 < i \leq k} \text{ext}(D_i, \emptyset) = \sum_{1 \leq i \leq h_1} \text{ext}(D_i, T). \tag{4.5}$$

For $h_1 < i \leq k$, we have $Q_i = \emptyset$ implying that $T \cap V(D_i) = \emptyset$ which implies $\text{ext}(D_i, T) = 0$. Thus the equality in (4.5) expands to

$$\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq k} \text{ext}(D_i, T). \tag{4.6}$$

Then expanding the definition of gap,

$$\begin{aligned} \sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}_i, Q_i) &= \sum_{1 \leq i \leq k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) - \sum_{1 \leq i \leq k} \sum_{\substack{W \in \mathcal{I}_i \\ W \neq Q_i}} \text{ext}(W) \\ &= \lambda - \sum_{1 \leq i \leq k} \text{ext}(D_i, T) - \sum_{W \in \mathcal{I}, W \neq T} \text{ext}(W). \end{aligned} \tag{4.7}$$

Then expanding the definition of $\text{gap}(C, \mathcal{I}, T)$ and combining with the equality in (4.7) gives

$$\begin{aligned} \text{gap}(C, \mathcal{I}, T) - \sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}_i, Q_i) &= \alpha(C - N(T^{C,1})) - \text{ext}(T, C) - \sum_{\substack{W \in \mathcal{I} \\ W \neq T}} \text{ext}(W) \\ &\quad - \sum_{1 \leq i \leq k} \text{gap}(D_i, \mathcal{I}_i, Q_i) \\ &= \alpha(C - N(T^{C,1})) - \text{ext}(T, C) - \lambda + \sum_{1 \leq i \leq k} \text{ext}(D_i, T) \\ &= \alpha(C - N(T^{C,1})) - \lambda - \theta. \end{aligned}$$

This completes the proof of Bound 5.

Finally, consider Bound 6 and assume $R \neq \emptyset$. Using the definition of Q_i we have

$$\sum_{1 \leq i \leq k} \text{ext}(D_i, Q_i) = \sum_{1 \leq i \leq h_1} \text{ext}(D_i, T) + \sum_{h_1 < i \leq h_2} \text{ext}(D_i, R). \tag{4.8}$$

By Lemma 70 and Lemma 71 we did not extend T during Step C so that

$$\sum_{1\le i\le h_1} \operatorname{ext}(D_i, T) = \operatorname{ext}(T, C). \tag{4.9}$$

Also, if we have an index i with $Q_i \neq R$, this implies $R \cap V(D_i) = \emptyset$ so $\operatorname{ext}(D_i, R) = 0$. Combining (4.8) and (4.9) gives

$$\sum_{1\le i\le k} \operatorname{ext}(D_i, Q_i) = \operatorname{ext}(C, T) + \sum_{1\le i\le k} \operatorname{ext}(D_i, R).$$

Then expanding the definition of gap, we have

$$\begin{aligned}
\sum_{1\le i\le k} \operatorname{gap}(D_i, \mathcal{I}_i, Q_i) &= \sum_{1\le i\le k} \alpha(D_i - N(Q_i^{C,5})) - \sum_{1\le i\le k} \operatorname{ext}(D_i, Q_i) - \sum_{1\le i\le k} \sum_{\substack{W\in\mathcal{I}_i \\ W\neq Q_i}} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) - \sum_{1\le i\le k} \operatorname{ext}(D_i, R) - \sum_{W\in\mathcal{I}, W\neq T, W\neq R} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) + \operatorname{ext}(R) - \sum_{1\le i\le k} \operatorname{ext}(D_i, R) - \sum_{W\in\mathcal{I}, W\neq T} \operatorname{ext}(W) \\
&= \lambda - \operatorname{ext}(C, T) + b - \sum_{W\in\mathcal{I}, W\neq T} \operatorname{ext}(W).
\end{aligned} \tag{4.10}$$

The last inequality holds because $\operatorname{ext}(R) - \sum_i \operatorname{ext}(D_i, R)$ is one plus the number of extensions of R during Substep 4 of Step C which is b. Then expanding the definition of $\operatorname{gap}(C, \mathcal{I}, T)$ and combining with the equality in (4.10) gives

$$\begin{aligned}
\operatorname{gap}(C, \mathcal{I}, T) - \sum_{1\le i\le k} \operatorname{gap}(D_i, \mathcal{I}_i, Q_i) &= \alpha(C - N(T^{C,1})) - \operatorname{ext}(C, T) - \sum_{\substack{W\in\mathcal{I} \\ W\neq T}} \operatorname{ext}(W) \\
&\quad - \sum_{1\le i\le k} \operatorname{gap}(D_i, \mathcal{I}_i, Q_i) \\
&= \alpha(C - N(T^{C,1})) - \lambda - b.
\end{aligned}$$

This finishes the proof of all the bounds.

We now just need to show that in all the different cases, the inequality in Claim 73 simplifies to the inequalities in Lemma 69.

Case 1. $R = T = \emptyset$.

We apply Bounds 1 and 5 to simplify (4.4)

$$f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda \le |\mathcal{I}| + 2\operatorname{gap}(C, \mathcal{I}, \emptyset). \tag{4.11}$$

Since $\lambda \leq \alpha(C)$ we use property P6 with $r = 2\sqrt{2}$ to obtain

$$f(2\sqrt{2}\alpha(C)) \leq f(2\sqrt{2}\alpha(C)) + \alpha(C) - \lambda \leq f(2\sqrt{2}\lambda) + 2\alpha(C) - 2\lambda. \tag{4.12}$$

Combining (4.11) with (4.12) proves the inequality in case *(iii)* of Lemma 69.

Case 2. *$R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, and $|J| \geq 2$.*

We apply Bounds 2 and 5 to simplify (4.4):

$$f(2\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.13}$$

We have $\alpha(C - N(T^{C,1})) \geq \theta + \lambda$ since $\lambda = \sum_i \alpha(D_i - N(T^{C,5}))$ and $\alpha(T)$ increased by at least θ during Step C by adding vertices from $C - N(T^{C,1})$. Also, $\lambda \geq 1$ since $J \neq \emptyset$. Thus we can apply property P6 with $r = \sqrt{2}$ to get

$$\begin{aligned} f(2\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\theta - 2\lambda &\geq f(2\sqrt{2}\alpha(C - N(T^{C,1})) - 2\sqrt{2}\theta) \\ &\geq f(2\sqrt{2}(b-1)). \end{aligned} \tag{4.14}$$

The last inequality holds because $\theta \leq \alpha(C - N(T^{C,1})) - b + 1$ and f is non-decreasing. Now we combine (4.13) with (4.14) to obtain

$$f(2\sqrt{2}(b-1)) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.15}$$

Then by definition of b we have for $\alpha(C) \geq 2$

$$2\sqrt{2}(b-1) = 2\sqrt{2}\left\lceil \frac{\alpha(C) - 1}{2} \right\rceil \geq \sqrt{2}(\alpha(C) - 1). \tag{4.16}$$

Since we extended $T^{C,1}$ we have $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$ so by Lemma 72, $\alpha(C - N(T^{C,1})) \leq \alpha(C) - 1$. Then since f is non-decreasing, (4.15) and (4.16) simplify to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq f(\sqrt{2}(\alpha(C) - 1)) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

This proves the inequality in case *(i)* and *(ii)* of Lemma 69.

Case 3. *$R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, $J = \{i\}$, and we continued to Substep 3.*

Then we use Bounds 3 and 5 to simplify (4.4):

$$f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.17}$$

Since $J \neq \emptyset$, we have $\lambda \geq 1$. Also, $\lambda + \theta \leq \alpha(C - N(T^{C,1}))$ since $\alpha(T)$ increased by at least θ during

Step C and $\lambda = \sum_i \alpha(C - N(T^{C,5}))$. Thus we can apply property P6 with $r = \gamma_i$ to obtain

$$\begin{aligned} f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - \theta - \lambda &\geq f(\gamma_i\alpha(C - N(T^{C,1})) - \gamma_i\theta) + \alpha(C - N(T^{C,1})) \\ &\geq f(\gamma_i(b-1)) + b - 1 - \lambda. \end{aligned} \tag{4.18}$$

The last inequality holds because $\theta \leq \alpha(C - N(T^{C,1})) - b + 1$ and f is non-decreasing. If $\lambda < b - 1$, then using properties P10 and P3 and $\gamma_i \geq 1$ we obtain

$$f(\gamma_i(b-1)) + b - 1 - \lambda \geq f(b-1) + 1 \geq f(\tau(b-1)) \geq f(\sqrt{2}\alpha(C)). \tag{4.19}$$

Since $\alpha(C - N(T^{C,1})) \leq \alpha(C)$, we combine (4.17), (4.18), and (4.19) to prove the inequality in case *(i)* and *(ii)* of Lemma 69.

Now assume $\lambda \geq b - 1$. Since $|J| = 1$ we have $\lambda = \alpha(D_i - N(Q_i^{C,5})) \leq \alpha(D_i)$ and since we ran Substep 4 we have $\alpha(D_i) \leq b - 1$ so $\lambda = b - 1$. We are also forced to have $\alpha(D_i - N(Q_i^{C,5})) = \alpha(D_i)$. Assume $Q_i \neq \emptyset$. Since $D_i \cap N(Q_i^{C,5}) = D_i \cap N(Q_i^{D_i,1})$ we will color all edges between $Q_i^{D_i,1}$ and D_i red in Substep 1 of Step D_i, which implies $Q_i^{C,5} \cap V(D_i) = \emptyset$. This contradicts that $Q_i \in \mathcal{I}_i$. Thus $Q_i = \emptyset$ and so by definition of γ_i, we have $\gamma_i = 2\sqrt{2}$. Then we combine (4.17) with (4.18) to obtain ($\lambda = b - 1$)

$$f(2\sqrt{2}(b-1)) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.20}$$

By the inequality in (4.16) we have $2\sqrt{2}(b-1) \geq \sqrt{2}(\alpha(C) - 1)$. Since f is non-decreasing, (4.20) simplifies to

$$f(\sqrt{2}(\alpha(C-1))) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Since $T^{C,1} \in \mathcal{H}_1(C) \cup \mathcal{H}_2(C)$ we have by Lemma 72 that $\alpha(C - N(T^{C,1})) < \alpha(C)$. Thus we have proved the inequality in case *(i)* and *(ii)* of Lemma 69.

Case 4. *$R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, $J = \{i\}$, and we did not continue to Substep 3 after Substep 2.*

In this case, we must have $T^{C,1} \in \mathcal{H}_1(C)$ because we did not run Substep 3. We apply Bounds 3 and 5 to simplify (4.4):

$$f(\gamma_i\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.21}$$

First assume that $\lambda < b - 1$. We use $\gamma_i \geq 1$ and that f is non-decreasing to simplify (4.21) to

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.22}$$

Because $\alpha(T)$ increased by at least θ during Step C, we have $\lambda + \theta \leq \alpha(C - N(T^{C,1}))$. We use

properties P6 ($J \neq \emptyset$ so $\lambda \geq 1$), P3, and P10 to obtain

$$\begin{aligned} f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta &\geq f(\alpha(C - N(T^{C,1})) - \theta) + \alpha(C - N(T^{C,1})) - \lambda - \theta \\ &\geq f(b-1) + b - 1 - \lambda \\ &\geq f(b-1) + 1 \\ &\geq f(\tau(b-1)) \\ &\geq f(\sqrt{2}\alpha(C)). \end{aligned} \tag{4.23}$$

Since $\alpha(C - N(T^{C,1})) \leq \alpha(C)$, combining (4.22) with (4.23) proves the inequality in both case *(i)* and *(ii)* of Lemma 69.

Now assume $\lambda \geq b - 1$ so that $\alpha(D_i - N(Q_i^{C,5})) \geq b - 1$. Since we did not continue to Substep 3 after Substep 2 we must have some component D_j and some $S^{C,1} \in \mathcal{H}_1(C)$ with $b \leq \alpha(D_j) = \alpha(D_j - N(S^{C,1})) \leq \sqrt{2}(b-1)$. If $i = j$ then we have $\alpha(D_i) \leq \sqrt{2}(b-1)$ and if $i \neq j$ then $\alpha(D_i) \leq \alpha(C) - \alpha(D_j) \leq \alpha(C) - b \leq \sqrt{2}(b-1)$. Thus

$$\alpha(D_i - N(Q_i^{C,5})) \geq b - 1 = \frac{1}{\sqrt{2}}\sqrt{2}(b-1) \geq \frac{\alpha(D_i)}{\sqrt{2}}. \tag{4.24}$$

Since

$$\alpha(D_i) \geq 2\left\lceil \frac{\alpha(D_i) - 1}{2} \right\rceil,$$

(4.24) implies that

$$\alpha(D_i - N(Q_i^{D_i,1})) \geq \frac{\alpha(D_i)}{\sqrt{2}} \geq \sqrt{2}\left(\left\lceil \frac{\alpha(D_i) + 1}{2} \right\rceil - 1 \right).$$

This shows that either $Q_i = \emptyset$ or $Q_i^{D_i,1} \in \mathcal{H}_0(D_i) \cup \mathcal{H}_2(D_i)$. If $Q_i \neq \emptyset$ and $Q_i^{D_i,1} \in \mathcal{H}_0(D_i)$ then we must have colored all edges between $Q_i^{D_i,1}$ and D_i red in Substep 1 of Step D_i which contradicts $Q_i \cap V(D_i) \neq \emptyset$. Thus either $Q_i = \emptyset$ so $\gamma_i = 2\sqrt{2}$ or $Q_i \neq \emptyset$ and $Q_i^{D_i,1} \in \mathcal{H}_2(D_i)$ so that $\gamma_i = \sqrt{2}$. Then (4.21) simplifies to

$$f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.25}$$

Because $\alpha(T)$ increased by at least θ during Step C, we have $\lambda + \theta \leq \alpha(C - N(T^{C,1}))$. Using property P6 ($J \neq \emptyset$ so $\lambda \geq 1$) we obtain

$$f(\sqrt{2}\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2\theta \geq f(\sqrt{2}\alpha(C - N(T^{C,1})) - \sqrt{2}\theta) \geq f(\sqrt{2}(b-1)). \tag{4.26}$$

Combining (4.25) with (4.26) gives

$$f(\sqrt{2}(b-1)) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Since $T^{C,1} \in \mathcal{H}_1(C)$ we have $\sqrt{2}(b-1) \geq \alpha(C - N(T^{C,1}))$ so we have proved the bound in case *(i)* of Lemma 69.

Case 5. *$R = \emptyset$, $T \neq \emptyset$, T was extended during Step C, and $J = \emptyset$.*

We apply Bounds 4 and 5 to (4.4) and then use $\lambda = 0$ and property P1 to obtain

$$2\alpha(C - N(T^{C,1})) - 2\theta \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Then $\theta \leq \alpha(C - N(T^{C,1})) - b + 1$ so

$$2(b-1) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.27}$$

By property P11, $f(\sqrt{2}\alpha(C)) \leq 2(b-1)$. Since $\alpha(C - N(T^{C,1})) \leq \alpha(C)$ we have that (4.27) simplifies to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

This proves the inequality in cases *(i)* and *(ii)* in Lemma 69.

Case 6. *$R = \emptyset$, $T \neq \emptyset$ and $T^{C,1} = T^{C,5}$.*

Since $T^{C,1} = T^{C,5}$, we have $\theta = 0$.

We apply Bounds 4 and 5 to simplify (4.4):

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.28}$$

Since $\lambda \leq \alpha(C - N(T^{C,1}))$, we use property P5 to obtain

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \geq f(\alpha(C - N(T^{C,1}))). \tag{4.29}$$

If $T^{C,1} \in \mathcal{H}_1(C)$, then (4.29) and (4.28) prove the inequality in case *(i)* of Lemma 69.

So assume $T^{C,1} \in \mathcal{H}_2(C)$. If $\lambda < \alpha(C - N(T^{C,1}))$ we can apply property P5 and P3 to obtain

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda \geq f(\alpha(C - N(T^{C,1}))) + 1 \geq f(\tau\alpha(C - N(T^{C,1}))). \tag{4.30}$$

Because $\tau \geq \sqrt{2}$, we can combine (4.30) with (4.28) to prove the inequality in case *(ii)* of Lemma 69.

So assume $\lambda = \alpha(C - N(T^{C,1}))$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we have $\lambda = \alpha(C - N(T^{C,1})) \geq \sqrt{2}(b-1) > \alpha(C)/2$. Then $|J| \geq 2$ since each component of $G[A_C^4]$ has independence number at most $\alpha(C)/2$ and $\lambda > \alpha(C)/2$. Using $|J| \geq 2$ we can apply Bounds 2 and 5 to Claim 73 to get

$$f(2\sqrt{2}\lambda) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Since $\lambda = \alpha(C - N(T^{C,1}))$, we have proved the bound in case *(ii)* of Lemma 69.

Case 7. *$R \neq \emptyset$ and $T \neq \emptyset$.*

First, $T^{C,1} \in \mathcal{H}_2(C)$ by Lemma 70 and $T^{C,4} = T^{C,1}$ by Lemma 71. Then we apply Bounds 4 and 6 to simplify (4.4):

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \leq |\mathcal{I}| - 2 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.31}$$

Because $b \leq \alpha(R^{C,5})$ and Q_i is T or R we have $\cup_i (D_i - N(Q_i^{C,5})) \bigcup R^{C,5} \subseteq C - N(T^{C,1})$ so that $\lambda + b \leq \alpha(C - N(T^{C,1}))$. (Note that $T^{C,1} \notin \mathcal{H}_0(C)$ so there are no red edges between $T^{C,1}$ and $R^{C,5}$.)

Assume $\alpha(C - N(T^{C,1})) = b$ so that $\lambda = 0$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we know $b = \alpha(C - N(T^{C,1})) \geq \sqrt{2}(b-1)$ so b is one or two so $\alpha(C - N(T^{C,1})) \leq \alpha(C) \leq 2b \leq 4$. Thus $f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq f(4\sqrt{2}) \leq 1$ by property P2. The left side of (4.31) is zero (using property P1) so adding one to both sides of (4.31) simplifies to

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq 1 \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T)$$

which is the inequality in case *(ii)* of Lemma 69.

We now assume $\alpha(C - N(T^{C,1})) > b$ and we use property P5 to obtain

$$f(\lambda) + 2\alpha(C - N(T^{C,1})) - 2\lambda - 2b \geq f(\alpha(C - N(T^{C,1})) - b). \tag{4.32}$$

Combining (4.31) with (4.32) and adding 1 to both sides we obtain

$$f(\alpha(C - N(T^{C,1})) - b) + 1 \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.33}$$

Using property P3 this transforms into

$$f(\tau\alpha(C - N(T^{C,1})) - \tau b) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T). \tag{4.34}$$

We now apply property P8 with $x = b$. Assume $f(2\sqrt{2}b) \leq f(\tau)$. Then $\alpha(C - N(T^{C,1})) - b \geq 1$ and $\alpha(C - N(T^{C,1})) \leq \alpha(C) \leq 2b$ imply that

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq f(2\sqrt{2}b) \leq f(\tau) \leq f(\tau\alpha(C - N(T^{C,1})) - \tau b). \tag{4.35}$$

Combining (4.34) and (4.35) we obtain

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T),$$

which is the inequality in case *(ii)* of Lemma 69.

Therefore, we can assume the other case of property P8 holds, namley that $\lceil \sqrt{2}(b-1) \rceil \geq \frac{\tau}{\tau - \sqrt{2}} b$. Since $T^{C,1} \in \mathcal{H}_2(C)$ we have $\alpha(C - N(T^{C,1})) \geq \sqrt{2}(b-1)$ so

$$\alpha(C - N(T^{C,1})) \geq \frac{\tau}{\tau - \sqrt{2}} b.$$

Manipulating this inequality, we find

$$\tau\alpha(C - N(T^{C,1})) - \tau b \geq \sqrt{2}\alpha(C - N(T^{C,1}))$$

so that

$$f(\tau\alpha(C - N(T^{C,1})) - \tau b) \geq f(\sqrt{2}\alpha(C - N(T^{C,1}))).$$

Combining this with (4.34) we obtain

$$f(\sqrt{2}\alpha(C - N(T^{C,1}))) \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, T),$$

which is the inequality in case *(ii)* of Lemma 69

Case 8. *$R \neq \emptyset$ and $T = \emptyset$.*

Using $T = \emptyset$ and $\theta = 0$, we apply Bounds 4 and 6 to simplify (4.4)

$$f(\lambda) + 2\alpha(C) - 2\lambda - 2b \leq |\mathcal{I}| - 1 + 2\,\mathrm{gap}(C, \mathcal{I}, \emptyset). \tag{4.36}$$

Since $\lambda = \sum_i \alpha(D_i - N(R^{C,5}))$ and $b \leq \alpha(R^{C,5})$ we have $\lambda + b \leq \alpha(C)$. We use property P5 to obtain

$$f(\alpha(C) - b) \leq f(\alpha(C) - b) + \alpha(C) - b - \lambda \leq f(\lambda) + 2\alpha(C) - 2\lambda - 2b. \tag{4.37}$$

We then combine (4.36) with (4.37) and add 1 to both sides to obtain

$$f(\alpha(C) - b) + 1 \leq |\mathcal{I}| + 2\,\mathrm{gap}(C, \mathcal{I}, T).$$

Then property P9 shows $f(2\sqrt{2}\alpha(C)) \leq f(\alpha(C) - b) + 1$ so we have proved the inequality in case *(iii)* of Lemma 69. □

4.6 The $\alpha(G) = 5$ algorithm

Let G be a graph with $\alpha(G) = 5$. At any stage of the algorithm, let U be the set of vertices of G not yet added to any set in $\mathcal{F}$. Initially, $U = |V(G)|$.

- Step 1: Let $\mathcal{F}$ be a maximal family of pairwise touching connected sets, with $\alpha(T) \leq 2$ and $|T| \leq 2\alpha(T) - 1$ for each $T \in \mathcal{F}$. We consider such a family with the maximum size, that is the maximum value $|\mathcal{F}|$. Set $U = V(G) - \cup_{T \in \mathcal{F}} T$.
- Step 2: For each $T \in \mathcal{F}$ with $|T| = 1$, we extend T into U by 1 if T touches U and T does not dominate U. We then repeat Step 2 until we have tried to extend every set.

There are now three cases. Case I is when $G[U]$ has no component with independence number 5 but has a component with independence number 4. Case II is when $G[U]$ has no component with

independence number 4 or 5. Case III is when $G[U]$ is connected and has independence number 5. We run different steps in the three cases.

Here are the steps in Case I:

- Step I.3: For any $T \in \mathcal{F}$ and any component C of $G[U]$ with $\alpha(C) = 4$ and $\alpha(C - N(T)) = 3$, we extend T into C by 1. We then update U and continue Step I.3 until no pair T, C satisfies the condition.
- Step I.4: Break U by 3.
- Step I.5: Break U by 2.
- Step I.6: Break U by 1.

Here are the steps in Case II:

- Step II.3: For any $T \in \mathcal{F}$ and any component C of $G[U]$ with $\alpha(C) = 3$ and $\alpha(C - N(T)) = 2$, we extend T into C by 1. We then update U and continue Step II.3 until no pair T, C satisfies the condition.
- Step II.4: Break U by 2.
- Step II.5: Break U by 1.

Here are the steps for Case III:

- Step III.3: Break U by 4.
- Step III.4: For any $T \in \mathcal{F}$ with $\alpha(T) = 2$ and any component C of $G[U]$ with $\alpha(C) = 3$ and $\alpha(C - N(T)) = 2$, we extend T into C by 1. We then update U and continue Step III.4 until no pair T, C satisfies the condition.
- Step III.5: Break U by 2.
- Step III.6: Break U by 1.

We claim that using this algorithm, we can find a complete minor of G of size $\frac{5n}{38}$.

We set up some notation for the sets in the family at different stages of the algorithm. Let $\mathcal{F}$ be the family at the end of the algorithm, and let G_b be the spanning subgraph of blue edges at the end of the algorithm. Let H be the graph obtained from G_b by contracting each set in $\mathcal{F}$. For each $T \in \mathcal{F}$ we use T to denote both the set in $V(G)$ and the vertex of H obtained by contracting T.

Let

$$\mathcal{F}(s, a) = \{T : T \text{ is a set in the final family}, T \text{ was first added during step s}, \text{ext}(T) = a\},$$

where $s \in \{1, I.3, \ldots, I.6, II.3, \ldots, II.5, III.3, \ldots, III.6\}$.

Define

$$\mathcal{F}(s) = \cup_a \mathcal{F}(s, a).$$

Note that we do not include the original sets which were added to the family in step s, but include the final configuration of the set which includes the original plus any extensions that were made. Define $\mathcal{F}_s$ to be the family right after Step s. Let U_s be the set of vertices not yet added into any set in $\mathcal{F}$ at the end of Step s. Set $n = |V(G)|$ and let λn be the size of the largest complete minor of G.

Claim 1. *$\mathcal{F}(1)$ is a pairwise touching family in G_b.*

Claim 2. *$\mathcal{F}$ is a partial simplicial elimination ordering in G_b, so H is a chordal graph.*

Proof. $\mathcal{F}(1)$ is pairwise touching so trivially is a partial simplicial elimination ordering. Then all breakings are acceptable so that by Theorem 61 $\mathcal{F}$ is a partial simplicial elimination ordering. Then we form a simplicial elimination ordering of the vertices of H, similarly to the proof of Theorem 57. □

Claim 3. *Let $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. Then $\sum_{T\in\mathcal{I}} \mathrm{ext}(T) \leq 5$.*

Proof. Can be checked by case analysis. □

Claim 4. *In the Case III algorithm (even if U_2 is not connected or U_2 has independence number less than 5), each $T \in \mathcal{F}(1,3)$ touches every set with extension number at least 2 using blue edges.*

Proof. Consider a $T \in \mathcal{F}(1,3)$. Then T touches every set in $\mathcal{F}(1)$ by Claim 1 and by Claim 3 every set in $\mathcal{F}(III.3)$. So we only need to show that T touches every set in $\mathcal{F}(III.5)$. Since $\mathrm{ext}(T) = 3$, we must have extended T in step III.4. By Lemma 59 we know T touches each set in $\mathcal{F}(III.5)$ using edges of G. It is impossible for these edges to be colored red because $\alpha(C - N(T))$ has been reduced to 1. □

Claim 5. *In the Case III algorithm (even if U_2 is not connected or U_2 has independence number less than 5), $|\mathcal{F}(1,2)| + |\mathcal{F}(1,3)| \leq (8\lambda - 1)n$.*

Proof. Let $f : V(H) \to \mathbb{Z}^+$ be defined by

$$f(T) = \begin{cases} |T| + 1 & T \in \mathcal{F}(1,2) \cup \mathcal{F}(1,3), \\ |T| & \text{otherwise.} \end{cases}.$$

Let $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. If $\mathcal{I}$ contains just a single set, the largest extension number of a set is 4 which has 7 vertices so we have $f(\mathcal{I}) \leq 8$. Assume $\mathcal{I}$ has at least two sets, and assume $\mathcal{I}$ does not contain any set in $\mathcal{F}(1,2) \cup \mathcal{F}(1,3)$ Then $f(\mathcal{I}) \leq 2\sum_{T\in\mathcal{I}} \mathrm{ext}(T)) - |\mathcal{I}| \leq 8$. Now assume $\mathcal{I}$ contains some sets in $\mathcal{F}(1,2) \cup \mathcal{F}(1,3)$. Since $\mathcal{F}(1,2) \cup \mathcal{F}(1,3) \subseteq \mathcal{F}(1)$ are all pairwise touching, $\mathcal{I}$ can contain at most one of these sets.

Assume $T \in \mathcal{I} \cap \mathcal{F}(1,2)$. Using Claim 3, there are two possibilities. One possibility is $\mathcal{I} = \{T, Q, R\}$ with $Q \in \mathcal{F}(III.5)$ and $R \in \mathcal{F}(III.6)$. For this $\mathcal{I}$, we have $f(\mathcal{I}) = |T| + 1 + |Q| + |R| \leq 8$. The other possibility is $\mathcal{I} = \{T, P, Q, R\}$ where $P, Q, R \in \mathcal{F}(III.6)$. For this $\mathcal{I}$, we have $f(\mathcal{I}) = |T| + 1 + 3 \leq 7$. Assume $T \in \mathcal{I} \cap \mathcal{F}(1,3)$. By Claim 4, the only possibility is $\mathcal{I} = \{T, R\}$ with $R \in \mathcal{F}(III.6)$. But for this $\mathcal{A}$, $f(\mathcal{I}) \leq |T| + 1 + |R| \leq 7$.

Thus $f(\mathcal{I}) \leq 8$, so by Theorem 52 we have

$$\lambda n \geq \frac{f(H)}{8} \geq \frac{n + |\mathcal{F}(1,2) \cup \mathcal{F}(1,3)|}{8}.$$

□

Claim 6. *In Cases I and II,* $|\mathcal{F}(1,2)| + |\mathcal{F}(1,3)| \leq (8\lambda - 1)n$.

Proof. Consider that instead of running the algorithm with Case I or II, we ran the Case III algorithm. Let $\mathcal{F}'$ be the family produced by the Case III algorithm. Then by Claim 5, $|\mathcal{F}'(1,2)| + |\mathcal{F}'(1,3)| \leq (8\lambda - 1)n$.

We have $\mathcal{F}(1) = \mathcal{F}'(1)$ and also have $\mathcal{F}(1,1) = \mathcal{F}'(1,1)$. No possible extension of sets in $\mathcal{F}(1,1)$ or $\mathcal{F}'(1,1)$ can happen in steps I.3, II.3, or III.4 because after Step 2, for every T in $\mathcal{F}(1,1)$ or $\mathcal{F}'(1,1)$ and each component C of U_2 either T dominates C or T does not touch C. □

Claim 7. *Assume that the algorithm selected Case I or Case II. Then let* $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. *Then* $\mathcal{I}$ *satisfies one of the following conditions.*

- $|\mathcal{I}| = 1$,
- $|\mathcal{I}| \geq 3$,
- $\mathcal{I} = \{T, R\}$ *with* $\mathrm{ext}(T) + \mathrm{ext}(R) < 5$,
- $\mathcal{I} = \{T, R\}$ *with* $T \in \mathcal{F}(1,2)$ *and* $R \in \mathcal{F}(I.4)$ *(in Case I),*
- $\mathcal{I} = \{T, R\}$ *with* $T \in \mathcal{F}(1,3)$ *and* $R \in \mathcal{F}(I.5)$ *(in Case I),*
- $\mathcal{I} = \{T, R\}$ *with* $T \in \mathcal{F}(1,3)$ *and* $R \in \mathcal{F}(II.4)$ *(in Case II).*

Proof. Say $\mathcal{I} = \{T, R\}$ with $\mathrm{ext}(T) + \mathrm{ext}(R) \geq 5$. We want to show that we must be in the last three options. Since $\mathrm{ext}(T) + \mathrm{ext}(R) \geq 5$ and the algorithms in Case I or II never produce a set with extension number 4, we must have $\mathrm{ext}(T) = 3$ and $\mathrm{ext}(R) = 2$ or $\mathrm{ext}(T) = 2$ and $\mathrm{ext}(R) = 3$. Since $\mathcal{F}(1)$ is pairwise touching in G_b, at most one of them can be in $\mathcal{F}(1)$. Now consider cases separately. In Case I, say $T \in \mathcal{F}(1,2)$. Then we must have $R \in \mathcal{F}(I.4)$ since $\mathrm{ext}(T) = 2$ and $\mathrm{ext}(R) = 3$. Consider $T \in \mathcal{F}(1,3)$. Then the only possibility of a set with extension number 2 for R is $\mathcal{F}(I.5)$. In Case II, the only place sets with extension number 3 are created is by extending a set in $\mathcal{F}(1)$. Thus $T \in \mathcal{F}(1,3)$ and the only possibility for R is $\mathcal{F}(II.4)$. □

Claim 8. *Assume that the algorithm selected Case I or II. Then* $\lambda \geq \frac{2}{15}$.

Proof. Use the following weight function $f : V(H) \to \mathbb{Q}^+$ by

$$f(T) = \begin{cases} |T| - 1 & T \in \mathcal{F}(1,3), \\ |T| & \text{otherwise.} \end{cases}$$

Using Claim 7, $\alpha_f(H) \leq 7$ so by Theorem 52 we have

$$n - |\mathcal{F}(1,3)| \leq 7\lambda n.$$

Combining with Claim 6 we have

$$n - (8\lambda n - n) \leq n - |\mathcal{F}(1,3)| \leq 7\lambda n$$

implying the claim. □

Claim 9. *Assume the algorithm selected Case III. Then every $T \in \mathcal{F}(1,1)$ dominates U_2 in G_b.*

Proof. Consider $T = \{x\} \in \mathcal{F}(1,1)$. In Step 2 we tried to extend T but failed. Since we are in Case III, U_2 is connected and $\alpha(U_2) = 5$. So because T was not extended in Step 2, we have T dominating U_2 or T does not touch U_2 in G. If T does not touch U_2 in G, we can form an independent set of size 6 by combining T with an independent set in U_2 of size 5. This contradicts $\alpha(G) = 5$, so that T dominates U_2 in G. Coloring edges red takes place during a breaking, but only if $\alpha(C - N(T)) = \alpha(C)$. Since T dominates U_2, we always have $\alpha(C - N(T)) = 0 < \alpha(C)$ so no edges incident to T are ever colored red. Thus T dominates U_2 in G_b. □

Claim 10. *Assume that the algorithm selected Case III. The possible subfamilies in $\mathrm{IND}_{G_b}(\mathcal{F})$ are a subset of one of the following cases:*

1. *One set from $\mathcal{F}(1,1)$,*
2. *One set from $\mathcal{F}(1,2)$, one set from $\mathcal{F}(III.5)$, and one set from $\mathcal{F}(III.6)$,*
3. *One set from $\mathcal{F}(1,2)$ and three sets from $\mathcal{F}(III.6)$,*
4. *One set from $\mathcal{F}(1,3)$ and two sets from $\mathcal{F}(III.6)$,*
5. *One set from $\mathcal{F}(III.3)$ and one set from $\mathcal{F}(III.6)$,*
6. *Two sets from $\mathcal{F}(III.5)$ and one set from $\mathcal{F}(III.6)$,*
7. *One set from $\mathcal{F}(III.5)$ and three sets from $\mathcal{F}(III.6)$,*
8. *Five sets from $\mathcal{F}(III.6)$.*

Proof. Let $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F})$. Assume $\mathcal{I} \cap \mathcal{F}(1) = \emptyset$. Then using Claim 3, conditions 5 - 8 list all possibilities. So assume $\mathcal{I} \cap \mathcal{F}(1) \neq \emptyset$. By Claim 1, we can include at most one set from $\mathcal{F}(1)$. Say $\mathcal{I} \cap \mathcal{F}(1) = \{T\}$. Consider $T \in \mathcal{I} \cap \mathcal{F}(1,1)$. Then by Claim 9, T dominates U_2 so $|\mathcal{I}| = 1$. Consider

$T \in \mathcal{I} \cap \mathcal{F}(1,2)$. Then conditions 2 and 3 cover the two possibilities. Consider $T \in \mathcal{I} \cap \mathcal{F}(1,3)$. Then T touches every set in $\mathcal{F}(III.5)$ by Claim 4. Thus the maximal non-touching family extending $\{T\}$ is adding two sets from $\mathcal{F}(III.6)$. □

Claim 11. *Assume that the algorithm selected Case III. Then* $\lambda \geq \frac{5}{38}$.

Proof. First,

$$n \leq |\mathcal{F}(1,1)| + 3\,|\mathcal{F}(1,2)| + 5\,|\mathcal{F}(1,3)| + 7\,|\mathcal{F}(III.3)| + 3\,|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|\,.$$

Now define $f : V(H) \to \mathbb{Z}^+$ to be:

$$f(T) = \begin{cases} 38 & T \in \mathcal{F}(1,1), \\ 21 & T \in \mathcal{F}(1,2), \\ 32 & T \in \mathcal{F}(1,3), \\ 35 & T \in \mathcal{F}(III.3), \\ 14 & T \in \mathcal{F}(III.5), \\ 3 & T \in \mathcal{F}(III.6). \end{cases}$$

Then considering Claim 10, we know $\alpha_f(H) \leq 38$. Thus using Theorem 52, we have

$$\begin{aligned} 38\lambda n \geq & 38\,|\mathcal{F}(1,1)| + 21\,|\mathcal{F}(1,2)| + 32\,|\mathcal{F}(1,3)| \\ & + 35\,|\mathcal{F}(III.3)| + 14\,|\mathcal{F}(III.5)| + 3\,|\mathcal{F}(III.6)| \end{aligned}$$

giving

$$38\lambda n \geq 5n + 33\,|\mathcal{F}(1,1)| + 6\,|\mathcal{F}(1,2)| + 7\,|\mathcal{F}(1,3)| - |\mathcal{F}(III.5)| - 2\,|\mathcal{F}(III.6)|\,.$$

Thus if we can show that

$$|\mathcal{F}(III.5)| + 2\,|\mathcal{F}(III.6)| \leq 6\,|\mathcal{F}(1,2)| + 7\,|\mathcal{F}(1,3)|$$

we will have $38\lambda n \geq 5n$ proving the claim.

Define $\mathcal{F}' = \mathcal{F}_2 \cup \mathcal{F}(III.6)$. Define L to be the graph formed by starting with $G_b[\cup_{T \in \mathcal{F}'} T]$ and contracting each set of $\mathcal{F}'$. Then $\mathcal{F}'$ is a partial simplicial elimination ordering, so that L is a chordal graph.

Define $f : V(L) \to \mathbb{Z}^+$ by

$$f(T) = \begin{cases} 5 & T \in \mathcal{F}_2 \text{ and } \operatorname{ext}(T) = 1, \\ 2 & T \in \mathcal{F}_2 \text{ and } \operatorname{ext}(T) = 2, \\ 1 & T \in \mathcal{F}(III.6). \end{cases}$$

Consider $\mathcal{I} \in \mathrm{IND}_L(\mathcal{F}')$. Say $T \in \mathcal{I} \cap \mathcal{F}_2$. If $\mathrm{ext}(T) = 1$ then we did not extend T in Step III.4 so $T \in \mathcal{F}(1,1)$ so by Claim 9, T dominates U_2 so $|\mathcal{I}| = 1$ so $f(\mathcal{I}) = 5$. If $\mathrm{ext}(T) = 2$ then we could have $\mathcal{I} = \{T, P, Q, R\}$ with $P, Q, R \in \mathcal{F}(III.6)$. Then $f(\mathcal{I}) = 5$. If $\mathcal{I} \cap \mathcal{F}_2 = \emptyset$ then $\mathcal{I}$ can have at most five sets from $\mathcal{F}(III.6)$ (by Lemma 3) so $f(\mathcal{I}) \leq 5$.

L is chordal and $\alpha_f(L) \leq 5$ so by Theorem 52, $\omega(L) \geq \frac{f(L)}{5}$. This clique in L is a pairwise touching subfamily of $\mathcal{F}'$ with size at least $\frac{f(L)}{5}$. This pairwise touching subfamily is a candidate for the choice of a family in Step 1. By the maximum choice in Step 1, we know

$$|\mathcal{F}(1)| \geq \frac{f(L)}{5} \geq \frac{2\,|\mathcal{F}(1)| + 3\,|\mathcal{F}(1,1)| + |\mathcal{F}(III.6)|}{5}.$$

Thus

$$3\,|\mathcal{F}(1,2)| + 3\,|\mathcal{F}(1,3)| \geq |\mathcal{F}(III.6)|\,. \tag{4.38}$$

Now define $\mathcal{F}' = \mathcal{F}(1,1) \cup \mathcal{F}(1,2) \cup \mathcal{F}(III.5) \cup \mathcal{F}(III.6)$. Define $L = G_b[\cup_{T \in \mathcal{F}'} T]$. Again, $\mathcal{F}'$ is a partial simplicial elimination ordering in $G_b[\cup_{T \in \mathcal{F}'} T]$ so that L is a chordal graph.

We then consider the weight function $f : V(L) \to \mathbb{Z}^+$

$$f(T) = \begin{cases} 5 & T \in \mathcal{F}(1,1), \\ 2 & T \in \mathcal{F}(1,2), \\ 2 & T \in \mathcal{F}(III.5), \\ 1 & T \in \mathcal{F}(III.6). \end{cases}$$

Then using Claim 10, we have $f(\mathcal{I}) \leq 5$ for each $\mathcal{I} \in \mathrm{IND}_{G_b}(\mathcal{F}')$. L is chordal and $\alpha_f(L) \leq 5$ so by Theorem 52 we have a clique in L of size $\frac{f(L)}{5}$. This clique corresponds to a pairwise touching subfamily of $\mathcal{F}'$ of size $\frac{f(L)}{5}$. Again, this subfamily is a possibility for the family in Step 1. By the maximum choice in Step 1,

$$|\mathcal{F}(1)| \geq \frac{5\,|\mathcal{F}(1,1)| + 2\,|\mathcal{F}(1,2)| + 2\,|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|}{5},$$

implying

$$3\,|\mathcal{F}(1,2)| + 5\,|\mathcal{F}(1,3)| \geq 2\,|\mathcal{F}(III.5)| + |\mathcal{F}(III.6)|\,. \tag{4.39}$$

Using 1.5 (4.38) + 0.5 (4.39) we obtain

$$6\,|\mathcal{F}(1,2)| + 7\,|\mathcal{F}(1,3)| \geq |\mathcal{F}(III.5)| + 2\,|\mathcal{F}(III.6)|\,.$$

□

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