COORDINATED PRICING AND INVENTORY MANAGEMENT

BY

PENG HU

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Industrial Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

Doctoral Committee:

Associate Professor Xin Chen, Chair
Professor Jong-Shi Pang
Assistant Professor Enlu Zhou
Associate Professor Nicholas Petruzzi
Associate Professor Yanfeng Ouyang

Abstract

This dissertation mainly focuses on coordinated pricing and inventory management problems, where the related background is provided in Chapter 1. Several periodic-review models are then discussed in Chapters 2, 3 4 and 5, respectively.

Chapter 2 analyzes a deterministic single-product model, where a price adjustment cost incurs if the current selling price is changed from the previous period. We develop exact algorithms for the problem under different conditions and find out that computation complexity varies significantly associated with the cost structure.

Chapter 3 develops a single-product model in which demand of a period depends not only on the current selling price but also on past prices through the so-called reference price. Strongly polynomial time algorithms are designed for the case without no fixed ordering cost, and a heuristic is proposed for the general case together with an error bound estimation. Moreover, our illustrates through numerical studies that incorporating reference price effect into coordinated pricing and inventory models can have a significant impact on firms' profits.

Chapter 4 discusses the stochastic version of the model in Chapter 3 when customers are loss averse. It extends the associated results developed in Gimpl-Heersink (2008) and proves that the reference price dependent base-stock policy is proved to be optimal under a certain conditions.

Instead of dealing with specific problems, Chapter 5 establishes the preservation of supermodularity in a class of optimization problems. This property and its extensions include several existing results in the literature as special cases, and provide powerful tools as we illustrate their applications to several operations problems: the stochastic two-product model with cross-price effects, the two-stage inventory control model, and the self-financing model.

To my parents and wife Juanjuan Huan, for their endless love.

Acknowledgments

I would like to thank the thesis committee members Prof. Xin Chen, Prof. Jong-Shi Pang, Prof. Enlu Zhou, Prof. Nicholas Petruzzi and Prof. Yanfeng Ouyan for their efforts and valuable suggestions.

First and foremost I would like to express tons of heartily thanks to my academic advisor Prof. Xin Chen, whose kindness and academic experience have been invaluable to me. He provides the financial support for me over the past five years which allows me to focus on studying, and encourages me with his expertise and patient guidance to develop an understanding of the research area from the very beginning – even before my arrival to University of Illinois at Urbana-Champaign in Fall 2006. More importantly, Prof. Chen's helps are never limited in research. Besides many suggestions for me to be a better communicator and to balance work and life, I will always remember his help in the hardest time I ever experienced that almost abandoned the studying. Without his unfailing support it is impossible to image I can successfully make the achievement.

It's my luck to have so many great and dedicated professors at such as Prof. Jiming Peng, Prof. Liming Feng, Prof. Carolyn Beck, Prof. Ramavarapu S. Sreenivas, etc. They designed so great courses that broaden my scope in many aspects and some of them also served on the committee of my qualifying exam. Among all professors, I want to especially thanks Prof. Uday Shanbhag, who gave me warmhearted advices and encouraged me a lot in my hardest time. Moreover, I appreciate for timely help from Prof. Angelia Nedich, Prof. Tolga Tezcan, and many other foremost experts in the research community at University of Illinois. The discussions on convex analysis with Prof. Nedich and on queuing systems with Prof. Tezcan greatly enhanced my understanding of these topics and contributed deeply to the thesis.

During the studying in the Department of Industrial and Enterprise Systems Engineering for five years, I was blessed with the staff including Michelle

Tipsword, Donna Eiskamp, Kevin Carmody, Laura Herriott, Lynnell Lacy, Randall Elkins and many others. Their dedicated work helped me to concentrate on studying and made my life much easier. I am also very fortunate to meet friends such as Ankur Kulkarni, Di Zhang, Yuhan Zhang, Limeng Pan, Seyed Mohammad Nourbakhsh, Rui Zhang, Shahzad Bhatti, Zisheng Chen and many others. I also want to thank my wife Juanjuan Huang for her incredible encouragement and inspiring supports.

Finally, I want to thank my parents for their endless love and numerous sacrifices. Though my father could not witness the graduation, I'll dedicate this thesis to him and hope his soul rest in peace.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Motivations

Thanks to recent advances in sophisticated information technologies such as enterprise resource planning systems and electronic tags, companies are now able to effectively gather information from customers and make dynamic pricing with a relatively small amount of effort. For example, it has been observed that Amazon.com, one of the leading e-commerce retailers, "obtain[s] all sorts of information ... at a minimum cost" to "adjust the prices of identical goods to correspond to a customer's willingness to pay (Weiss and Mehrotra, 2001)". Other examples include the airline industry, hotel management, car rental agencies and etc.

Capability of dynamic pricing has prompted a rethink over classical revenue management problems and inventory management problems. Due to the traditional organizational structures within a firm, typical decisions on the revenue management are made at first in order to maximize the total revenue. Such decisions include activities like establishing the price impact on demand, roughly estimating the sales target and etc. On the inventory management side decisions like replenishment are then made in order to minimize the associated cost, where demands are assumed exogenously determined from an operations perspective. The two kinds of problems both aim to match supply and demand, however, separately through marketing decisions and inventory decisions.

Because in practice demands are usually price sensitive, one can expect that incorporating the innovative dynamic pricing mechanism brings a certain flexibility to inventory management. By taking the advantage of coordinated dynamic pricing and inventory management, it is no surprise that companies can better match supply and demand hence improve their profit.

The academics in response developed and analyzed a variety of mathematical models that integrate the two kinds of decisions. Coordinated pricing and inventory models have enjoyed a rapid growth in the past decade. See, for example, Federgruen and Heching (1999); Petruzzi and Dada (1999); Chen and Simchi-Levi (2004a,b); Song et al. (2009); Huh and Janakiraman (2008) and etc. Significant progress has been made since the publication of the seminal paper by Whitin (1955), where an economic order quantity model and a newsvendor model both with price-dependent demand are analyzed. Detailed literature review will be presented for each model considered in later chapters. For a comprehensive review of this area, we refer to Chen and Simchi-Levi (2011) for an up-to-date survey. The reader can also consult some other excellent resources such as Eliashberg and Steinberg (1991); Yano and Gilbert (2003); Elmaghraby and Keskinocak (2003) and Chan et al. (2004).

This thesis belongs to the stream of research on dynamic pricing and inventory problems, where several periodic-review models are respectively discussed in Chapters 2 – 5. For these models, Figure 1.1 conceptually illustrates the typical decision process over the whole planning horizon. In all our the horizon is divided into finite number of periods and indexed by $1, 2, \cdots, T$. At the beginning of each period t, the system state x_t, e.g., the inventory level of a product, is observed, then pricing and replenishment decisions are simultaneously made. The pricing decision will influence demand D_t in this period or demands in later periods. In all our models, we assume zero lead time. That is, orders delivered instantaneously. During the period, demand, which depends on the current price and/or historical prices and maybe some uncertain components, arrives and satisfied by on-hand inventory. Revenue is obtained by the end of the period. We assume the replenishment decision z_t incurs a certain cost, unused inventory after satisfying demand is fully carried over to the next period incurring a inventory holding cost; moreover, under stochastic settings we assume unsatisfied demand is backlogged with a shortage backlogging cost and will be fulfilled by end of the planning horizon. At the beginning of the next period, the system state is updated to x_{t+1}. The same process will repeat in period $t = 1, \cdots, T$ sequentially, where the initial state x_1 is given as the system input. The objective is to maximize the total profit by appropriately deciding $\{p_1, \cdots, p_T\}$ and $\{z_1, \cdots, z_T\}$.

Incorporating two kinds of decisions will clearly lead to additional efforts. As has been evidenced in various empirical studies from the economics litera-

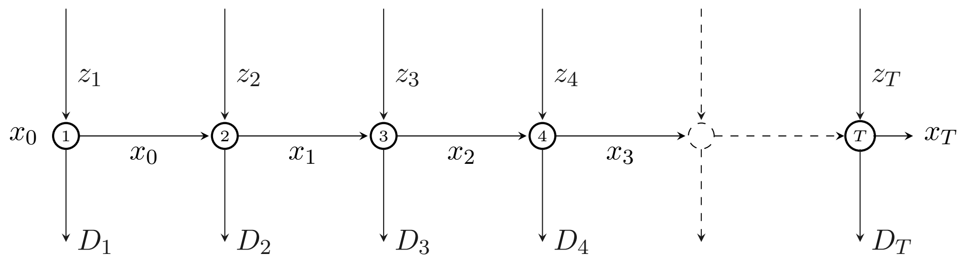

Figure 1.1: Coordinated pricing and inventory management

ture, price adjustment is not free and associated costs could be too significant to be ignored in many business settings. This observation motivates Chapter 2, where a deterministic single-product model with price adjustment costs is analyzed. This model takes into account the price adjustment cost, which basically says a cost is charged whenever the selling price at the current period is different from the one at the previous period, where the associated could depend on the price adjustment directions, i.e., either marked up or marked down. Such model includes several related models analyzed in the literature as special cases. For example, if the price adjustment cost is significantly high, then static pricing strategies should be applied, which reduces to the model analyzed by Kunreuther and Schrage (1973). In this chapter, we develop exact algorithms for solving the problem under different conditions, and discuss the corresponding computational complexity. Interestingly, computational complexity of our algorithms varies significantly with respect to the structure of price adjustment cost functions. Several numerical examples are also provided which show that dynamic pricing strategies may out perform static pricing strategies even when price adjustment cost accounts for a significant portion of the total profit.

Chapter 3 considers a deterministic single-product model with reference price effects. Notice that the major stream of revenue management literature and almost all existing inventory management models exclusively assume that the demand of a period only depends on the selling price at the current period. This assumption, appropriate to model impulse purchasing, is unreasonable when consumers may react to firms' pricing strategies. As demonstrated by plenty of empirical evidences, it would lead to improper managerial decisions if consumers' reaction to pricing strategies is not appropriately accounted for. An attempt of this chapter is to fill this gap by incorporating reference price effects into coordinated pricing and inventory management models. Specifically, a deterministic single-product model is developed, where demand in a period depends not only on the current selling price but also on past prices through the so-called reference price. Strongly polynomial time algorithms are designed for the case without no fixed ordering cost, and a heuristic is proposed for the general case together with an error bound estimation. Our numerical study illustrates that incorporating reference price effects can have a significant impact on firms' profits. In addition, sensitivity analysis is provided on profit with respect to three kinds of input parameters: the relative contribution of reference price effects to demands, the memory factor in the reference price model and the relative magnitude of loss and gain.

Chapter 4 discusses reference price effects model under stochastic settings when customers are loss averse. This chapter extends the associated results developed in Zhang (2011), where the author mainly focus on the loss neutral case. In this chapter, the reference price dependent base-stock policy is proved to be optimal under a certain conditions.

It turns out in many stochastic inventory management problems the preservation of a certain property under dynamic programming recursions plays a key role to characterize the structure of either the profit/cost-to-go functions or the optimal policies. Chapter 5 establishes a new preservation property of supermodularity under optimization operations. Compared to the well-known preservation result stated in Topkis (1998), our main result relaxes the lattice requirement on constraint sets. On the other hand we have to assume concavity of the objective function and impose a requirement on the dimension of the parameter vector. Despite the additional assumptions, our approach and its extensions provide powerful tools which can be used in many applications. Specifically we will discuss the following operations models in this chapter: the stochastic two-product model with cross-price effects, the two-stage inventory control model, and the self-financing model. In contrast to papers originally introduce these models, applying our approach gives significantly simpler proofs and provides additional insights.

1.2 Organization of the thesis

Chapter 2 analyzes a deterministic model with price adjustment costs. Section 2.1 introduces the background and reviews the literature. Section 2.2 presents the mathematical formulation of this problem and discuss some special cases. In Sections 2.3 and 2.4, exact algorithms together with corresponding computational complexity are provided to solve problem (2.2), where to illustrate the idea clearly, we first analyze the case without fixed ordering cost and focus solely on the pricing plan in Section 2.3, then move to the general case in Section 2.4 and deal with the interaction of ordering plan and pricing plan. We also consider a case in which selling prices are restricted to a predetermined finite set in Section 2.5.

Chapter 3 develops a deterministic model where demand of a period depends not only on the current selling price but also past observed prices through the memory-based reference price. It focuses on developing effective algorithms for solving the model. Section 3.2 presents its mathematical formulation. Section 3.3 analyzes the case without fixed ordering cost and develop strongly polynomial time algorithms to several special cases. The general model is dealt with in Section 3.4, followed by a numerical study in Section 3.5. Finally, the last section gives some suggestions for future research. To maintain a clear presentation, all technical proofs are presented in the appendix of this chapter.

Chapter 4 consider a stochastic version of the model studied in Chapter 3.The main problem is described in Section 4.1. Section 4.2 then proves that a reference price dependent base-stock policy is optimal when customers are loss averse or loss neutral case.

Chapter 5 studies a new preservation property of supermodularity under optimization operations when the constraint set may not be a lattice, where the background and related literature are introduced in Section 5.1. Section 5.2 presents the main result of this chapter, as well as some extensions and associated discussion. Section 5.3 considers three applications to illustrate the usage of the new preservation property. Section 5.4 summarizes this chapter and provides some future research problems. Most proofs of this chapter are provided in the appendix unless otherwise specified.

Finally, the last chapter concludes this thesis by pointing out some directions for future research.

Chapter 2

Costly Price Adjustment Model with Deterministic Demand

2.1 Background and literature review

Coordinated pricing and inventory management problem with deterministic demand has been extensively studied in many aspects. However, a predominate assumption made in the majority of existing literature is that price adjustment is costless. Yet, as evidenced by various empirical studies from the economics literature (e.g., Bergen et al., 2003), cost associated with price adjustments may be very significant to be ignored in many business settings. For instance, Levy et al. (1997) report that price adjustment may generate enormous costs for major retailer chains, and take up as much as 40% of the reported profits for some of these chains. There are usually two kinds of costs associated with price adjustment. One is called the *managerial cost* which corresponds to "the time and attention required of managers to gather the relevant information and to make and implement decisions"; and the other is the *physical cost* (or *menu cost*) which associates with physical activities such as manually changing shelf prices, "constructing new price lists, printing and distributing new list prices and monthly supplemental price sheets, and notifying suppliers(Zbaracki et al., 2004)". Both costs may be significant in retailing and other industries (see Levy et al., 1997; Slade, 1998; Aguirregabiria, 1999; Bergen et al., 2003; Kano, 2006).

The purpose of this chapter is to incorporate price adjustment costs into coordinated pricing and inventory models. Specifically, we develop and analyze a deterministic single-product periodic-review model over a finite planning horizon. On the supply side, the setting is similar to the classic economic lot sizing model. Namely, to satisfy demand for a finite horizon, replenishment is made at the beginning of each period incurring fixed and variable ordering costs and inventory is carried over from one period to the next incurring holding costs. On the demand side, a selling price is determined at the beginning of each period together with the replenishment decision, and demand of a period deterministically depends on the selling price in the current period. In contrast to the majority of the literature, we assume that a price adjustment cost is charged if the current selling price is changed from the previous period. The objective is to determine a coordinated ordering and pricing plan so as to maximize the total profit over the planning horizon.

This model includes several coordinated pricing and inventory models analyzed in the literature as special cases. For example, if price can be changed freely without incurring any adjustment cost, it reduces to the problem analyzed in Wagner and Whitin (1958a) and Thomas (1970). If the price adjustment cost is so high that prohibits any price change, then a constant price should be determined at the beginning of the planning horizon hence it reduces to the static pricing model analyzed in Kunreuther and Schrage (1973); Gilbert (2000) and van dan Heuvel and Wagelmans (2006). Great progress has been made recently on this class of models. For example, in deterministic settings, Deng and Yano (2006) and Geunes et al. (2006) extend the dynamic pricing model and Geunes et al. (2008) extend the static pricing model by incorporating ordering capacity constraints. However, almost all existing works in the literature predominately ignore the costs associated with price changes. As the only exceptions to our best knowledge, Aguirregabiria (1999) and Chen et al. (2008) incorporate these costs into their models in *stochastic* settings, where the former concentrates more on empirical studies and the latter focus on deriving structural properties of the optimal policies. Two other related papers are Netessine (2006), who recognizes the importance of the impact of price adjustment costs on pricing and inventory decisions and formulates a deterministic continuous-time model to optimize the timing of a fixed number of price changes, and Celik et al. (2009), who analyze a continuous-time stochastic revenue management problem with costly price changes. Celik et al. (2009) characterize the optimal pricing policies for settings with ample inventory and develop several heuristics based on fluid approximations. However, their model does not take into account inventory replenishment decisions and thus does not capture the intricate interaction of ordering and pricing.

Our work is the first to introduce price adjustment costs into coordinated pricing and inventory models in *deterministic* settings. Since our model takes into account both fixed ordering costs and price adjustment costs, it is much more involved to handle the interaction of ordering plans and pricing plans. Still, we manage to develop exact algorithms to solve this problem under different conditions. The main idea is to partition the planning horizon by price adjustment periods such that each member of the partition consists of consecutive periods with a constant price. The total profit is then appropriately allocated to these members. Each member corresponds to a static pricing problem and an equivalent longest path problem is finally constructed to obtain the optimal sequence of price adjustment periods.

The remainder of this chapter is organized as follows. In Section 2.2 we present the mathematical formulation of this problem and discuss some special cases. In Sections 2.3 and 2.4 we derive exact algorithms to solve problem (2.2) and present their related computational complexity. To illustrate the idea clearly, we first analyze the case without fixed ordering cost in Section 2.3, which allows us to focus solely on the pricing plan. The general case is handled in Section 2.4, where the interaction of ordering plans and pricing plans is taken into account. We also present an extension to our model introduced in Section 2.2, as well as a case in which selling prices are restricted to a predetermined finite set, in Section 2.5. We then present a numerical study in Section 2.6. Finally, we conclude the paper in the last section with some suggestions for further research.

2.2 Model and preliminaries

Consider a firm that makes replenishment and pricing decisions to satisfy a sequence of demands of a single product over a finite planning horizon with T periods. At the beginning of each period t, an ordering quantity z_t and a selling price p_t are determined simultaneously, where p_t belongs to some closed interval $[L, U]$. The replenishment incurs the cost

$$k_t\delta(z_t) + c_t z_t,$$

where $k_t, c_t \geq 0, \delta(0) = 0$ and $\delta(y) = 1$ whenever $y > 0$. That is, it consists of the fixed ordering cost k_t when $z_t > 0$, and a variable ordering cost $c_t z_t$. Since we focus on a deterministic model, without loss of generality, assume that orders are delivered instantaneously and no backlogging is allowed. Inventory left at period t, denoted by I_t, is carried over to the next period with the marginal holding cost h_t. Similar to Kunreuther and Schrage (1973) and van dan Heuvel and Wagelmans (2006), demand of period is modeled as a deterministic function of selling price p as

$$D_t(p) = a_t d(p) + b_t, \quad \forall p \in [L, U],\ t = 1, \dots, T$$

where a_t, b_t are positive coefficients, and $d(p)$, referred to as the *base demand function*, is a strictly decreasing function in term of p.

In contrast to most papers in the literature on coordinated pricing and inventory models, we assume that a cost $f(\tilde{P} - P)$ incurs when price is changed to $\tilde{P}$ from the value P in the previous period, where $f(0) = 0$ and for some non-negative U^+ and U^-,

$$f(\tilde{P} - P) = \begin{cases} U^+ & \text{if } P < \tilde{P}, \\ U^- & \text{if } P > \tilde{P}. \end{cases}$$

Note that all results in the following still hold when the price adjustment cost is time dependent. For simplicity, express the price adjustment cost as

$$f(\tilde{P} - P) = U^{\tilde{\alpha}}, \qquad \forall\, P \neq \tilde{P},$$

where $\tilde{\alpha}$, called the *price change indicator*, specifies the direction of price adjustment and satisfies the consistency condition

$$\tilde{\alpha} \in \{+1, -1\}, \quad \tilde{\alpha}(\tilde{P} - P) > 0. \tag{2.1}$$

That is, $\tilde{\alpha} = +1$ indicates markup and $\tilde{\alpha} = -1$ indicates markdown.

Similar price adjustment cost structures have been proposed and analyzed in the literature. For example, Aguirregabiria (1999) and Kano (2006) consider symmetric costs $U^+ = U^-$. Our model allows asymmetric price adjustment costs to reflect the fact that firms may take different actions in response to price markdown and price markup. For instance, a firm may advertise a price decrease. However, it is very unlikely for a firm to advertise a price increase.

The objective of the firm is to decide ordering quantities z_t and prices p_t

in all periods over the planning horizon so as to maximize the total profit without any backlogging. Mathematically, the firm faces the following coordinated pricing and inventory problem:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T} \{p_t D_t - f(p_t - p_{t-1}) - [k_t \delta(z_t) + c_t z_t + h_t I_t]\} && && \text{(2.2a)} \\
\text{subject to} \quad & D_t = a_t d(p_t) + b_t, && t = 1, 2, \cdots, T, && \text{(2.2b)} \\
& I_t = I_{t-1} + z_t - D_t, && t = 1, 2, \cdots, T, && \text{(2.2c)} \\
& I_t \geq 0, \quad z_t \geq 0, \quad p_t \in [L, U], && t = 1, 2, \cdots, T, && \text{(2.2d)}
\end{aligned}
$$

where $p_t D_t$, $f(p_t - p_{t-1})$, $k_t \delta(z_t) + c_t z_t$ and $h_t I_t$ in (2.2a) respectively represent the one-period revenue, the price adjustment cost, the ordering cost and the inventory holding cost in period t. Demand of period t depends on the selling price p_t through (2.2b). Constraint (2.2c) denotes the inventory balance equation, which together with $I_t \geq 0$ in (2.2d) ensures that no demand is backlogged. The feasible sets of the inventory level I_t, order quantity z_t and selling price p_t are given in (2.2d). Finally, we assume that $I_0 = 0$ and $p_0 = 0$, where $f(p_1 - p_0) = f(p_1)$ can be regarded as the cost of setting up the price p_1 in the first period.

Several important pricing and inventory models can be cast as special cases of the above problem. First, when $L = U$ in (2.2d), (2.2) reduces to the classical economic lot sizing problem, which is first, to our best knowledge, analyzed in Wagner and Whitin (1958b). The authors show that it can be solved in $O(T^2)$ by appropriately constructing an acyclic network and finding a shortest path in it. More efficient algorithms with a running time $O(T \log T)$ are proposed by Aggarwal and Park (1993); Federgruen and Tzur (1991) and Wagelmans et al. (1992), respectively. The so-called zero inventory ordering (ZIO for short) property, plays a key role in these works. It basically says that there exists an optimal ordering plan such that by following the plan, an order is placed precisely when the inventory level drops to zero. The ZIO property also implies that if t is a reorder period (i.e., $I_{t-1} = 0$), then the optimal ordering plan over periods $\{1, 2, \cdots, t-1\}$ can be determined independently of that over $\{t, t+1, \cdots, T\}$.

Second, if no price adjustment cost is incurred, i.e., $f(\Delta) = 0$ for any Δ, then (2.2) reduces to the coordinated inventory and dynamic pricing model studied by Wagner and Whitin (1958a) and Thomas (1970). Since price in each period can be independently decided, ZIO property still holds hence one can solve an equivalent longest path problem in an acyclic network in $O(T^2)$ time to determine the optimal ordering plan.

Finally, if the price adjustment cost is very high, i.e., $f(\Delta)$ is significantly large for any nonzero Δ, then (2.2) reduces to the static pricing model analyzed by Kunreuther and Schrage (1973). In this case, problem (2.2) becomes

$$\underset{p\in[L,U]}{\text{maximize}} \sum_{t=1}^{T} \{p[a_t d(p) + b_t] - C(d(p))\},$$

where the function $C(d)$ given below denotes the minimal total inventory-related cost with respect to the base demand d,

$$\begin{aligned} C(d) = \text{minimize} \quad & \sum_{t=1}^{T} [k_t\delta(z_t) + c_t z_t + h_t I_t] \\ \text{subject to} \quad & I_t = I_{t-1} + z_t - (a_t d + b_t), \qquad t = 1, 2, \cdots, T, \\ & I_0 = 0, \quad I_t \geq 0, \quad z_t \geq 0, \qquad t = 1, 2, \cdots, T. \end{aligned}$$

Kunreuther and Schrage (1973) show that $C(d)$ is concave and piecewise linear in d. Building upon a heuristic algorithm proposed in Kunreuther and Schrage (1973), van dan Heuvel and Wagelmans (2006) derive an exact algorithm to solve this problem. They prove that if $C(d)$ consists of S_T linear pieces, then all these linear pieces can be determined essentially by solving S_T economic lot sizing problems. Because it takes $O(T \log T)$ time to solve a T-period economic lot sizing problem, we make the following assumption throughout this chapter.

Assumption 2.1. *For a T-period coordinated inventory and static pricing model, the function $C(d)$ consists of $O(S_T)$ linear pieces and its expression can be determined in $O(S_T T \log T)$ time.*

We make two remarks on Assumption 2.1. First, if there is no speculative motive on holding inventories, i.e. $c_t + h_t \geq c_{t+1}$ for all $t < T$, then the T-period economic lot sizing problem can be solved in $O(T)$ time (see, e.g., Federgruen and Tzur, 1991). Therefore in this case we can remove the term $\log T$ from Assumption 2.1. Second, Gilbert (2000) proves that $S_T = O(T)$ when $b_t = 0$ and all cost parameters are time-independent. Finally, van dan

Heuvel and Wagelmans (2006) claim that $S_T = O(T^2)$ for the general case. However, as pointed out by van dan Heuvel (private communication), there is a flaw in their proof and it is not clear whether it can be fixed or not. We also impose the following assumption on the base demand function.

Assumption 2.2. *Given any constants A_1 and A_2, the function $pd(p) + A_1 d(p) + A_2 p$ in term of p has $O(1)$* local maximizers *in $[L, U]$, and it takes an $O(1)$ time to find all of these local maximizers.*

A weaker version of Assumption 2.2 is used by van dan Heuvel and Wagelmans (2006), who implicitly assume that a global maximizer of any function of the form $pd(p) + A_1 d(p) + A_2 p$ in $[L, U]$ can be found in an $O(1)$ time. As we will see later, it is not sufficient to consider only the global maximizer for our problem. Nevertheless, when the $d(p)$ is linear, $pd(p) + A_1 d(p) + A_2 p$ is concave hence both our assumption and van dan Heuvel and Wagelmans (2006)'s hold and are equivalent.

Before continuing on the analysis, we briefly introduce the basic idea to solve problem (2.2). Specifically, we will partition the planning horizon such that each member of the partition consists of a sequence of consecutive periods with a constant price. For convenience of the presentation, we provide an alternative yet equivalent representation of a price sequence. Specifically, for a given price sequence $\{p_1, p_2, \cdots, p_T\}$, we call $\{(s_n, \alpha_n, P_n) : 1 \le n \le N\}$ is the associated *pricing plan* where

1. $1 = s_1 < s_2 < \ldots < s_N < s_{N+1} = T + 1$ be the price adjustment periods such that $p_t = P_n$ for all $s_n \le t < s_{n+1}$ and $1 \le n \le N$;

2. α_n are the price change indicators at period s_n for all $1 \le n \le N$.

Note that the period $T + 1$ is introduced as an artificial price adjustment period for notation simplicity in latter discussion. Moreover, for any two triples (s, α, P), $(\tilde{s}, \tilde{\alpha}, \tilde{P})$ with s and $\tilde{s}$ being two consecutive price adjustment periods, the consistency condition (2.1) below holds,

$$1 \le s < \tilde{s} \le T + 1, \ \alpha, \tilde{\alpha} \in \{-1, +1\}, \tag{2.3}$$

Clearly there is a one-to-one correspondence between a price sequence and a pricing plan defined as above satisfying (2.1) and (2.3).

2.3 Zero fixed ordering cost case

We assume in this section that no fixed ordering cost is charged, i.e., $k_t = 0$ for all periods t. Define $c(s,t)$ as the marginal cost of satisfying the demand of period t by an order placed at period s, i.e.,

$$c(s,t) = c_s + h_s + \cdots . + h_{t-1}, \quad \forall 1 \le s \le t \le T+1.$$

It is straightforward to see that the optimal ordering periods is independent of pricing plans and can be recursively obtained by letting $\tau_1 = 1$ and

$$\tau_{m+1} = \min\big\{T+1, \min[t : \tau_m < t \le T, c_t \le c(\tau_m, t)]\big\}, \quad \forall m \ge 1.$$

That is, it is optimal to place an order at period t if the associated cost is no more than the cost of satisfying the demand of period t by earlier orders.

Figure 2.1 illustrates a typical (not optimal) ordering plan in this case, where orders are placed at period $\tau_1 = 1$, $\tau_2 = 4$ and $\tau_3 = 8$. Once the ordering plan is determined, by ZIO property the ordering quantities at periods 1, 4 and 8 should respectively covers demands from period 1 to period 3, from period 4 to period 7, and in period 8. Furthermore, the sequence of price $\{p_1, \ldots, p_8\}$ can be determined independently.

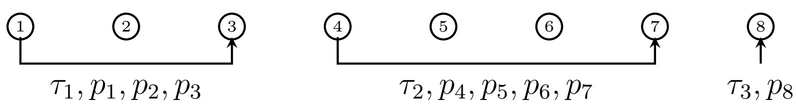

Figure 2.1: A typical coordinated pricing and ordering plan in zero fixed ordering cost case

What remains is to to decide the optimal pricing plan. Let $\bar{c}_t = c(\tau_m, t)$ for $\tau_m \le t < \tau_{m+1}$. Given a pricing plan $\{(s_n, \alpha_n, P_n) : 1 \le n \le N\}$, one can verify that the minimal total ordering and inventory holding cost is

$$\sum_{s_1 \le t < s_2} \bar{c}_t D_t(P_1) + \sum_{s_2 \le t < s_3} \bar{c}_t D_t(P_2) + \cdots + \sum_{s_N \le t < s_{N+1}} \bar{c}_t D_t(P_N),$$

the total price adjustment cost is $U^{\alpha_1} + U^{\alpha_2} + \cdots + U^{\alpha_N}$ and the total revenue is $\sum_{n=1}^{N} R(P_n; s_n, s_{n+1})$, where

$$R(P; s, \tilde{s}) = P[D_s(P) + D_{s+1}(P) + \cdots + D_{\tilde{s}-1}(P)]$$

denotes the accumulated revenue from periods s to $\tilde{s}-1$ when a constant price P is used in these periods. Therefore by defining

$$G(P; s, \alpha, \tilde{s}, \tilde{\alpha}) = R(P; s, \tilde{s}) - \sum_{s \le t < \tilde{s}} \bar{c}_t D_t(P) - U^{\alpha},$$

the associated total profit can be expressed as

$$G(P_1; s_1, \alpha_1, s_2, \alpha_2) + G(P_2; s_2, \alpha_2, s_3, \alpha_3) + \cdots + G(P_N; s_N, \alpha_N, s_{N+1}, \alpha_{N+1}),$$

where α_{N+1} could be either $+1$ or -1.

In the above discussion, we in fact partition the planning horizon into sub-planning horizons each of which consists of consecutive periods with a constant price. For the sub-planning horizon $\{s_n, \cdots, s_{n+1}-1\}$, the total profit is $G(P_n; s_n, \alpha_n, s_{n+1}, \alpha_{n+1})$. Observe that $P_{n-1} \neq P_n$ and $P_n \neq P_{n+1}$ imply that the sequence $\{(s_n, \alpha_n) : 1 \le n \le N\}$ remains unchanged if we slightly modify P_n. Therefore a necessary condition for an optimal pricing plan $\{(s_n, \alpha_n, P_n) : 1 \le n \le N\}$ is that for each n, P_n is a local maximizer of the function $G(P; s_n, \alpha_n, s_{n+1}, \alpha_{n+1})$ on $[L, U]$.

We are ready to convert problem (2.2) to a longest path problem on an acyclic network $(\mathcal{V}, \mathcal{E})$. Let $\mathcal{P}(s, \alpha, \tilde{s}, \tilde{\alpha})$ be the set of all local maximizers of $G(P; s, \alpha, \tilde{s}, \tilde{\alpha})$ on $[L, U]$. Define the node set $\mathcal{V}$ and the link set $\mathcal{E}$ as

$$\begin{aligned} \mathcal{V} &= \big\{\boldsymbol{v} = (P, s, \alpha, \tilde{s}, \tilde{\alpha}) : P \in \mathcal{P}(s, \alpha, \tilde{s}, \tilde{\alpha}) \text{ and (2.3) holds}\big\} \cup \{\boldsymbol{v}^0, \boldsymbol{v}^e\}, \\ \mathcal{E} &= \big\{ \langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle : \boldsymbol{v} = (P, s, \alpha, \tilde{s}, \tilde{\alpha}), \tilde{\boldsymbol{v}} = (\tilde{P}, \tilde{s}, \tilde{\alpha}, s', \alpha') \in \mathcal{V} \text{ and (2.1) holds}\big\}, \end{aligned}$$

where $\boldsymbol{v}^0 = (\underline{P}, 1, +1, 1, +1)$, $\boldsymbol{v}^e = (\underline{P}, T+1, -1, T+1, -1)$ for some $\underline{P} < L$ are artificial nodes serving as the origin and the destination of the longest path to be constructed. A typical node $\boldsymbol{v} \notin \{\boldsymbol{v}^0, \boldsymbol{v}^e\}$ specifies two consecutive price adjustment periods s and $\tilde{s}$ together with their associated price change indicators α and $\tilde{\alpha}$, and a constant price P, restricted to be a local maximizer of the profit function $G(p; s, \alpha, \tilde{s}, \tilde{\alpha})$ on $[L, U]$, used between periods s and $\tilde{s}-1$. There is a link between two nodes $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ only if the consistency condition (2.1) holds. Note that for the two artificial nodes $\boldsymbol{v}^0$ and $\boldsymbol{v}^e$, values of their price components do not correspond to any real price. Any $\underline{P}$ less than $\min[L, U]$ is sufficient to ensure that there is no incoming link to the origin $\boldsymbol{v}^0$ and any node $\boldsymbol{v}$ adjacent to $\boldsymbol{v}^0$ is of the form $\boldsymbol{v} = (P, 1, +1, s, \alpha)$.

Similarly, there is no outgoing link to the destination $\boldsymbol{v}^e$ and its adjacent nodes must have the form $\boldsymbol{v} = (P, s, \alpha, T+1, -1)$. By Assumption 2.2, each $\mathcal{P}(s, \alpha, \tilde{s}, \tilde{\alpha})$ has $O(1)$ elements hence the network $(\mathcal{V}, \mathcal{E})$ has $O(T^2)$ nodes and $O(T^3)$ links.

For each link $\langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle \in \mathcal{E}$, assign the length $\ell(\boldsymbol{v}, \tilde{\boldsymbol{v}}) = 0$ if $\boldsymbol{v} = \boldsymbol{v}^0$ and $G(P; s, \alpha, \tilde{s}, \tilde{\alpha})$ if $\boldsymbol{v} = (P, s, \alpha, \tilde{s}, \tilde{\alpha}) \neq \boldsymbol{v}^0$. To calculate all the link lengths, we need to find local maximizers of $G(P; s, \alpha, \tilde{s}, \tilde{\alpha})$ for all possible combinations $(s, \alpha, \tilde{s}, \tilde{\alpha})$. This amounts to solving $O(T^2)$ maximization problems, each of which has an objective function in the form $A_0 P d(P) + A_1 d(P) + A_2 P + A_3$ for some constants $A_l, l = 0, 1, 2, 3$ with $A_0 \geq 0$. From Assumption 2.2, all these link lengths can be computed in an $O(T^2)$ time.

Observe that any path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ in the acyclic network specifies a sequence $\{(s_n, \alpha_n, P_n) : 1 \leq n \leq N\}$. In addition, its length is equal to the associated total profit of the sequence. On the other hand, any optimal pricing plan can be represented as a sequence $\{(s_n, \alpha_n, P_n) : 1 \leq n \leq N\}$ with P_n being a local maximizer of the function $G(P; s_n, \alpha_n, s_{n+1}, \alpha_{n+1})$, and thus corresponds to a path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$. Therefore, determining an optimal pricing plan is equivalent to finding a longest path in the acyclic network $G = (\mathcal{V}, \mathcal{E})$, which can be solved in $O(|\mathcal{E}|)$ by applying well known algorithms from the network flow literature, where $|\mathcal{E}|$ denotes the number of elements in $\mathcal{E}$. In summary, we have the following results.

Theorem 2.1. *When $k_t = 0$ for all $t = 1, 2, \cdots, T$, solving problem (2.2) is equivalent to finding a longest path from node 1 to node $T+1$ in the acyclic network $(\mathcal{V}, \mathcal{E})$. Moreover,*

(a) this network contains $O(T^2)$ node and $O(T^3)$ links;

(b) it takes an $O(T^2)$ time to calculate all the link lengths in the network;

(c) a longest path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ can be determined in $O(T^3)$ time.

It is worth mentioning that the computational complexity can be reduced if the price adjustment costs are symmetric, i.e., $U^+ = U^- = U$. In this case, a given optimal pricing plan can be represented as a sequence $\{(s_n, P_n) : 1 \leq n \leq N\}$, where $\{s_n : 1 \leq n \leq N\}$ are the price adjustment periods and P_n is the price used from periods s_n to $s_{n+1} - 1$. In contrast to the asymmetric price adjustment cost case, here P_n is a global maximizer of the accumulated profit function (from periods s_n to $s_{n+1} - 1$)

$$G(P; s_n, s_{n+1}) = R(P; s_n, s_{n+1}) - \sum_{s_n \le t < s_{n+1}} \bar{c}_t D_t(P) - U.$$

This allows us to find an optimal pricing plan by solving a longest path problem yet in a different acyclic network network $(\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \{1, 2, \cdots, T+1\}, \qquad \mathcal{E} = \{\langle s, \tilde{s} \rangle : s, \tilde{s} \in \mathcal{V}, s < \tilde{s}\}.$$

The length of $\langle s, \tilde{s} \rangle \in \mathcal{E}$ is given by $\max_{P \in [L,U]} G(P; s_n, s_{n+1})$. Note that this network contains $O(T)$ nodes and $O(T^2)$ links, and it takes an $O(T^2)$ time to calculate all the link lengths and find a longest path from nodes 1 to $T+1$.

2.4 General case

In this section, we consider the general case with fixed ordering costs. Similar to Section 2.3, we partition the planning horizon such that each member of the partition consists of consecutive periods with a constant price and decompose the total profit over the planning horizon as the summation of the profit incurred over all members of the partition. Figure 2.2 shows a typical coordinated pricing and ordering plan in this case, where prices are adjusted in periods $s_2 = 2$ and $s_3 = 6$, and orders are placed in periods $\tau_1 = 1, \tau_2 = 4$ and $\tau_3 = 8$.

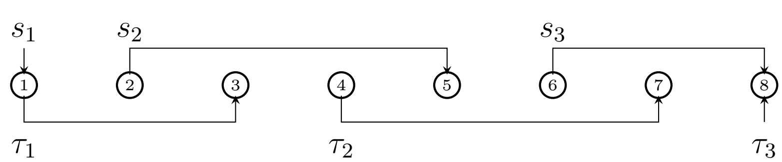

Figure 2.2: A typical coordinated pricing and ordering plan

However, the problem is significantly more complicated than the zero fixed ordering cost case as analyzed in the previous section. When there exist fixed ordering costs, the optimal ordering plan cannot be determined independent of the pricing plan due to the interaction of the two kinds of decisions. The key is to carefully take into account the ordering period associated to each price adjustment period. For this purpose, consider two consecutive price adjustment periods s and $\tilde{s}$. Let τ and $\tilde{\tau}$ be their corresponding ordering periods, that is, demands of periods s and $\tilde{s}$ are respectively satisfied by orders in periods τ and $\tilde{\tau}$. See Figure 2.3 for the illustration.

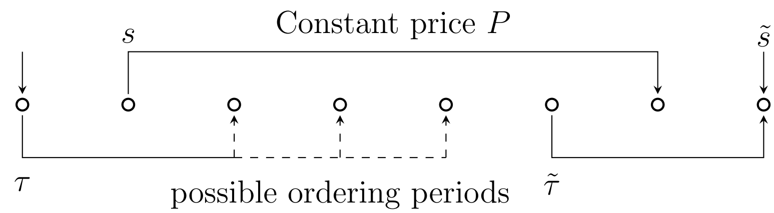

Figure 2.3: Price adjustment periods $s, \tilde{s}$ and associated ordering periods $\tau, \tilde{\tau}$

From the ZIO property, we have that

$$\text{either} \quad \tau = \tilde{\tau} \leq s < \tilde{s} \quad \text{or} \quad \tau \leq s < \tilde{\tau} \leq \tilde{s}, \tag{2.4}$$

where $\tau = \tilde{\tau}$ means that no replenishment is made between periods $\tau + 1$ and $\tilde{s} - 1$. In the case $\tau \leq s < \tilde{\tau} \leq \tilde{s}$, the ZIO property implies that the demands from periods τ to s are filled by the order at period τ whereas the demands from periods $\tilde{\tau}$ to $\tilde{s}$ are filled by the order at period $\tilde{\tau}$. Therefore the marginal ordering and holding cost to satisfy demand in period s is $c(\tau, s)$. Moreover, if no order is placed at period t for some $t \geq s$, then the marginal order and holding cost is $c(\tau, t)$. Observe that by the definition of $c(s, t)$,

$$c(\tau, t) = c_\tau + h_\tau + \cdots + h_s + \cdots + h_{t-1} = c(\tau, s) + h_s + \cdots + h_{t-1}.$$

It indicates that if introduce an artificial replenishment at period s and let the artificial marginal ordering cost be $c(\tau, s)$, then the marginal ordering and holding cost to satisfy demand in period t is equal to $c(\tau, t)$. Figure 2.4 illustrates the idea to introducing artificial replenishment at price adjustment periods.

The above observation plays the key role for solving the problem in general case. For any $(\tau, s, \tilde{\tau}, \tilde{s})$ satisfying (2.4), assume a constant price P is used between periods s and $\tilde{s} - 1$. Define $d = d(P)$ as the corresponding base demand. Let $C(d; \tau, s, \tilde{\tau}, \tilde{s})$ be the minimal ordering and inventory holding cost accumulated from period s up to period $\tilde{s} - 1$. There are two cases. If

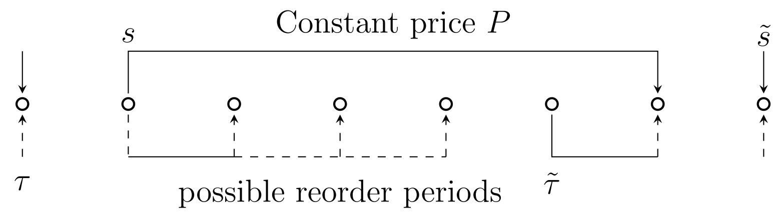

Figure 2.4: Introducing artificial replenishment at price adjustment periods

$\tau < \tilde{\tau}$ then we can express

$$\begin{aligned} C(d;\tau,s,\tilde{\tau},\tilde{s}) &= k_\tau(1-\delta(s-\tau)) + C^1(d;\tau,s,\tilde{\tau}) \\ &+ k_{\tilde{\tau}}\delta(\tilde{s}-\tilde{\tau}) + C^2(d;\tilde{\tau},\tilde{\tau},\tilde{s}), \end{aligned}$$

And when $\tau = \tilde{\tau}$, we have that

$$C(d;\tau,s,\tilde{\tau},\tilde{s}) = k_\tau(1-\delta(s-\tau)) + C^2(d;\tau,s,\tilde{s}),$$

where $C^1(d;\tau,s,\tilde{\tau})$ denotes the variable inventory costs accumulated from period s to $\tilde{\tau}-1$ provided demand in period s is satisfied by an order in period τ, and $C^2(d;\tau,s,\tilde{s}) = \sum_{t=s}^{\tilde{s}-1} c(\tau,t)(a_t d + b_t)$ denotes the variable inventory costs accumulated from periods s to $\tilde{s}-1$ by ordering at period τ. Observe that from the above expression, the fixed ordering cost $k_{\tilde{\tau}}$ is allocated to periods $\{s,\ldots,\tilde{s}-1\}$ only if $s \le \tilde{\tau} < \tilde{s}$ to avoid double counting.

It remains to decide the expression of $C^1(d;\tau,s,\tilde{\tau})$. Interestingly, this can be determined independent of periods out of $\{s,\ldots,\tilde{s}-1\}$ by solving the following economic lot sizing problem,

$$\begin{aligned} C^1(d;\tau,s,\tilde{\tau}) = \quad & \text{minimize} && [c(\tau,s)z_s + h_s I_s] + \sum_{s<t<\tilde{\tau}} [k_t\delta(z_t) + c_t z_t + h_t I_t] \\ & \text{subject to} && I_t = I_{t-1} + z_t - (a_t d + b_t), && s \le t < \tilde{s}, \\ & && I_{s-1} = 0, \quad I_t \ge 0, \quad z_t \ge 0, && s \le t < \tilde{s}, \end{aligned}$$

In this problem, the variable z_s denotes the quantity ordered at period τ and carried over to satisfy the demands at period s or even later periods. Inside the first pair of square brackets in the objective function, the term $c(\tau,s)z_s$ represents the variable ordering cost and the holding cost for inventory ordered at period τ and carried over to s, and the term $h_s I_s$ denotes the additional holding cost for inventory carried over to later periods. Although the order is placed at period τ, for cost accounting purpose it can be regarded as an order placed at period s by setting $I_{s-1} = 0$ and the variable ordering cost equal to $c(\tau, s)$.

Similar to Section 2.3, the accumulated profit between the two consecutive price adjustment periods s and $\tilde{s}$ taking into account their corresponding ordering periods τ and $\tilde{\tau}$ and price adjustment indicators α and $\tilde{\alpha}$ can be written as

$$G(P; \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}) = R(P; s, \tilde{s}) - C(d(P); \tau, s, \tilde{\tau}, \tilde{s}) - U^{\alpha}.$$

Using the same argument as in Section 2.3, we can prove that there is no loss of optimality to restrict P in $\mathcal{P}(\tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$, the set of local maximizers of the function $G(P; \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$ for $p \in [L, U]$. In addition, we can convert problem (2.2) to a longest path problem in some acyclic network $(\mathcal{V}, \mathcal{E})$, where the node set $\mathcal{V}$ and the link set $\mathcal{E}$ are defined by

$$\begin{aligned}
\mathcal{V} &= \big\{\boldsymbol{v} = (P, \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}) : P \in \mathcal{P}(\tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}),\ \text{(2.3) and (2.4) hold}\big\} \\
&\quad \cup \{\boldsymbol{v}^0 = (\underline{P}, 1, 1, +1, 1, 1, +1)\}, \\
&\quad \cup \{\boldsymbol{v}^e = (\underline{P}, T+1, T+1, -1, T+1, T+1, -1)\}, \\
\mathcal{E} &= \big\{\, \langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle : \boldsymbol{v} = (P, \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}) \in \mathcal{V}, \\
&\qquad\quad \tilde{\boldsymbol{v}} = (\tilde{P}, \tilde{\tau}, \tilde{s}, \tilde{\alpha}, \tau', s', \alpha') \in \mathcal{V},\ \text{(2.1) holds}\big\},
\end{aligned}$$

where $\underline{P} < L$ and $\boldsymbol{v}^0, \boldsymbol{v}^e$ respectively denote the origin and the destination of the longest path to be constructed. Like the construction in the previous section, there is no incoming link to node $\boldsymbol{v}^0$ and any node adjacent to $\boldsymbol{v}^0$ has the form $(P, 1, +, 1, \tau, s, \alpha)$ when (2.1) holds.

For each link $\langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle \in \mathcal{E}$, assign the length

$$\ell\,(\boldsymbol{v}, \tilde{\boldsymbol{v}}) = \begin{cases} 0, & \text{if } \boldsymbol{v} = \boldsymbol{v}^0, \\ G(P; \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}), & \text{if } \boldsymbol{v} = (P, \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}) \neq \boldsymbol{v}^0. \end{cases}$$

Following the same argument as in the previous section, we know that the longest path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ in the acyclic network corresponds to an optimal pricing plan to problem (2.2). The optimal ordering plan can be determined when the pricing plan is known by solving a classical economic lot sizing problem.

We now discuss the computational complexity of this algorithm. To obtain all the link lengths in the network, we need to know all $O(T^3)$ functions $C^i(d;\tau,s,t)$ for $i = 1, 2$. Clearly all $C^2(d;\tau,s,t)$ can be obtained in an $O(T^3)$ time. From Assumption 2.1, it takes an $O(S_T T^4 \log T)$ time in total to determine the expressions of all functions $C^1(d;\tau,s,t)$. Therefore, it takes an $O(S_T T^4 \log T)$ time to determine the expressions of all $G(P;\tau,s,\alpha,\tilde{\tau},\tilde{s},\tilde{\alpha})$. Because $C(d;\tau,s,\tilde{\tau},\tilde{s})$ has $O(S_T)$ linear pieces by Assumption 2.1, each $G(P;\tau,s,\alpha,\tilde{\tau},\tilde{s},\tilde{\alpha})$ consists of $O(S_T)$ pieces of the form $A_0Pd(P) + A_1d(P) + A_2P + A_3$ for some coefficients $A_l, l = 0, 1, 2, 3$ with $A_0 \geq 0$. By Assumption 2.2, the set $\mathcal{P}(\tau,s,\alpha,\tilde{\tau},\tilde{s},\tilde{\alpha})$ has at most $O(S_T)$ elements and can be determined in an $O(S_T)$ time. Therefore, the acyclic network $G = (\mathcal{V},\mathcal{E})$ has $O(S_T T^4)$ nodes and $O(S_T^2 T^6)$ links whose lengths can be constructed in an $O(S_T T^4 \log T)$ time.

In summary, we have the following theorem on problem (2.2).

Theorem 2.2. *Solving problem* (2.2) *is equivalent to finding a longest path from node* $\boldsymbol{v}^0$ *to node* $\boldsymbol{v}^e$ *in the acyclic network* $(\mathcal{V},\mathcal{E})$ *constructed as above. Moreover,*

(a) this network has $O(S_T T^4)$ *nodes and* $O(S_T^2 T^6)$ *links;*

(b) it takes an $O(S_T T^4 \log T)$ *time to construct the network;*

(c) a longest path from $\boldsymbol{v}^0$ *to* $\boldsymbol{v}^e$ *can be found in an* $O(S_T^2 T^6)$ *time.*

Again the computational complexity can be reduced for the symmetric price adjustment cost case with $U^+ = U^- = U$ by converting problem (2.2) to a longest path problem in a different acyclic network $G = (\mathcal{V},\mathcal{E})$. In this network, let

$$\begin{aligned}
\mathcal{V} &= \{\boldsymbol{v} = (\tau,s) : 2 \leq \tau \leq s \leq T\} \cup \{\boldsymbol{v}^0,\boldsymbol{v}^e\},\\
\mathcal{E} &= \big\{ \langle \boldsymbol{v},\tilde{\boldsymbol{v}}\rangle : \boldsymbol{v} = (\tau,s),\ \tilde{\boldsymbol{v}} = (\tilde{\tau},\tilde{s}) \text{ and (2.4) holds}\big\},
\end{aligned}$$

where $\boldsymbol{v}^0 = (1,1)$ and $\boldsymbol{v}^e = (T+1,T+1)$ are the origin and the destination of the longest path to be constructed. For any link $\langle \boldsymbol{v},\tilde{\boldsymbol{v}}\rangle$ with $\boldsymbol{v} = (\tau,s)$ and $\tilde{\boldsymbol{v}} = (\tilde{\tau},\tilde{s})$, assign a length $\ell(\boldsymbol{v},\tilde{\boldsymbol{v}}) = 0$ if $\boldsymbol{v} = \boldsymbol{v}^0$ and $\max_{P\in[L,U]} G(P;\tau,s,\tilde{\tau},\tilde{s})$ otherwise, where $G(P;\tau,s,\tilde{\tau},\tilde{s}) = R(P;s,\tilde{s}) - C(d(P);\tau,s,\tilde{\tau},\tilde{s}) - U$. The equivalence of the longest path problem in the acyclic network and problem (2.2) should be clear from the analysis of this section and the previous one. Note that we have $O(T^2)$ nodes and $O(T^4)$ links in the network $(\mathcal{V}, \mathcal{E})$ and it takes a total $O(S_T T^5 \log T)$ time to obtain all the link lengths.

2.5 Extensions and the finite price levels case

We introduce some extensions to the model discussed so far in this section. First, similar discussion can be made if the feasible set of selling price is not stationary, i.e., $p \in \mathcal{P}_t$ in period t, $t = 1, \cdots, T$. The only difference appears in the definition of $\mathcal{P}(\tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$, which becomes the set of local maximizers of $G(P; \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$ in $\mathcal{P}_s \cap \cdots \cap \mathcal{P}_{\tilde{s}-1}$.

Second, the demand function can be generalized to $D_t(p) = \beta_t + \alpha_t d(p)$, where $d(p)$, called the *price effect* (see Kunreuther and Schrage, 1973, for example), is some decreasing function in p. In this case, $G(P; \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}) = R(P; s, \tilde{s}) - C(d(P); \tau, s, \tilde{\tau}, \tilde{s}) - U^{\alpha}$, which consists of pieces of the form $A_1 p d(p) + A_2 d(p) + A_3 p + A_4$ for some coefficients $A_1, \cdots, A_4$. We can solve problem (2.2) in a similar way by considering all local maximizers of these functions. The computational complexity relies on the number of local maximizers and whether they can be found efficiently.

Finally, Celik et al. (2009) argue that the price adjustment cost may also depend on the inventory level I on hand of the form $cI + f(\Delta)$, where cI is the inventory-related cost. Our approach can be extended to the price adjustment cost function

$$f(\tilde{P}, P, I) = V_1(P; \tilde{\alpha}) + V_2(\tilde{P}; \tilde{\alpha}) + U(I; \tilde{\alpha}),$$

such that (2.1) holds for $P, \tilde{P}$ and $\tilde{\alpha}$, where $U(I; \alpha)$ and $V_i(p; \alpha), i = 1, 2$, can be general continuous functions in I and p, respectively. In this case, the total profit relies on the constant price P in periods $s, \cdots, \tilde{s} - 1$ through the following function

$$R(P; s, \tilde{s}) - C(-P; \tau, s, \tilde{\tau}, \tilde{s}) - \left[V_1(P; \tilde{\alpha}) + V_2(P; \alpha) + U(I_s; \alpha)\right],$$

where $I_s = \sum_{t=\tau}^{s-1} (b_t - a_t P)$ is the amount of inventory carried to period s. This problem can be dealt in a similar way.

So far we assumed that price can take all possible values within the interval $[L, U]$. As can be seen from the previous section, the complexity of our exact algorithm may be very high partly due to the possibly large number of local maximizers of the function G over $[L, U]$. A remedy to alleviate this high complexity is to restrict to a finite number of price levels in $[L, U]$. By doing this, we hope that a few price levels may capture a large portion of the total profit. Thus, in this section, we analyze problem (2.2) with the price admissible set $[L, U]$ replaced by a discrete set $\mathcal{P}$ with S predefined price levels. It is also worth mentioning that our approach here is consistent with many business practices in which candidates of price levels are determined a priori.

Problem (2.2) with the constraint $p \in \mathcal{P}$ can be handled using the same approach of Section 2.4. The only difference is that to construct the acyclic network, we can simply focus on all feasible price candidates in $\mathcal{P}$ rather than finding all local maximizers of the function G over $[L, U]$. Specifically, in the new acyclic network $(\mathcal{V}, \mathcal{E})$, let

$$
\begin{aligned}
\mathcal{V} &= \{(P, \tau, s, \alpha) : P \in \mathcal{P}, \tau \leq s\} \\
&\quad \cup \{\boldsymbol{v}^0 = (\underline{P}, 1, 1, +), \boldsymbol{v}^e = (\underline{P}, T+1, T+1, -)\}, \\
\mathcal{E} &= \{\langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle : \boldsymbol{v} = (P, \tau, s, \alpha), \tilde{\boldsymbol{v}} = (\tilde{P}, \tilde{\tau}, \tilde{s}, \tilde{\alpha}), \text{(2.1) and (2.4) hold}\},
\end{aligned}
$$

where $\underline{P} < \min \mathcal{P}$. This network contains $O(ST^2)$ nodes and $O(S^2T^4)$ links. Moreover, for each link $\langle \boldsymbol{v}, \tilde{\boldsymbol{v}} \rangle \in \mathcal{E}$, say $\boldsymbol{v} = (P, \tau, s, \alpha)$, $\tilde{\boldsymbol{v}} = (\tilde{P}, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$, assign the length

$$
\ell(\boldsymbol{v}, \tilde{\boldsymbol{v}}) = \begin{cases} 0, & \text{if } \boldsymbol{v} = \boldsymbol{v}^0 \\ G(P, \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha}), & \text{if } \boldsymbol{v} \neq \boldsymbol{v}^0. \end{cases},
$$

Observe that for a given P and a tuple $(\tau, s, \tilde{\tau})$, $C(d(P), \tau, s, \tilde{\tau}, \tilde{s})$ can be calculated by solving an economic lot sizing problem of the form $C(d(P), \tau, s, \tilde{\tau}, \tilde{\tau})$ with additional $O(T)$ addition operations for all $\tilde{s} \geq \tilde{\tau}$. Thus, for a given P, all values $C(d(P), \tau, s, \tilde{\tau}, \tilde{s})$ and $G(P, \tau, s, \alpha, \tilde{\tau}, \tilde{s}, \tilde{\alpha})$ can be derived in an $O(T^4 \log T)$; moreover, the length $\ell(\boldsymbol{v}, \tilde{\boldsymbol{v}})$ is independent of the value of $\tilde{P}$. Thus, it takes an $O(ST^4 \log T)$ time to obtain all link lengths in $(\mathcal{V}, \mathcal{E})$.

We now present a result parallel to Theorem 2.2.

Theorem 2.3. *Assume that the price admissible set $\mathcal{P}$ has S elements. Then solving problem (2.2) is equivalent to finding a longest path from node $\boldsymbol{v}^0$ to node $\boldsymbol{v}^e$ in the acyclic network $(\mathcal{V}, \mathcal{E})$ constructed as above. Moreover,*

(a) the network contains $O(ST^2)$ nodes and $O(S^2T^4)$ links;

(b) it takes an $O(ST^4 \log T)$ time to obtain all the link lengths;

(c) a longest path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ can be found in an $O(S^2T^4)$ time.

Table 2.1 summarizes the sizes of the constructed acyclic networks and the computational complexity of preparing all link lengths and finding longest paths under different settings in our model, where recall that S_T denotes the number of linear pieces of $C(d)$, the cost function corresponds to a T-period static pricing and inventory control problem. One can observe that the computational complexity heavily depends on the structure of cost functions.

Table 2.1: Computational complexity

	Nodes	Links	Obtain lengths	Total
$k_t = 0, U^+ = U^-$	$O(T)$	$O(T^2)$	$O(T^2)$	$O(T^2)$
$k_t = 0,$	$O(T^2)$	$O(T^3)$	$O(T^2)$	$O(T^3)$
	$O(T^2)$	$O(T^4)$	$O(S_T T^5 \log T)$	$O(S_T T^5 \log T)$
General case	$O(S_T T^6)$	$O(S_T T^4 \log T)$	$O(S_T T^4 \log T)$	$O(S_T^2 T^6)$
S price levels	$O(ST^2)$	$O(S^2T^4)$	$O(ST^4 \log T)$	$O(S^2T^4)$

2.6 Numerical Study

In this section we present an example typical in our numerical study to illustrate the effectiveness of dynamic pricing strategies. Consider a 12-period instance with $d(p) = 30 - p$ for $p \in [20, 30]$,

$$k_t = 150,\ c_t = 20, h_t = 5,\ V_t^{\pm} = 2,\ U_t^{\pm} = 15,$$

for $t = 1, 2, \ldots, 12$, and

$$\begin{aligned} \{a_1, a_2, \cdots, a_{12}\} &= \{5, 1, 3, 2, 2, 3, 5, 1, 10, 10, 5, 5\}, \\ \{b_1, b_2, \cdots, b_{12}\} &= \{4, 2, 2, 1, 13, 1, 1, 2, 2, 2, 13, 9\}. \end{aligned}$$

In addition to solving problem (2.2) with continuous prices, we also solve the problem with finite number of price levels and compare the two cases. Specifically, we consider eight price admissible sets with the numbers of price levels ranging from 3 to 10. These price admissible sets are derived by dividing the interval $[20, 30]$ equally and then rounding to an integers. That is, for $n = 3, \cdots, 10$,

$$\mathcal{P}_n = \left\{ 20 + \left\lfloor \tfrac{10k}{n-1} \right\rfloor : 0 \le k < n \right\},$$

where n represents the numbers of elements in $\mathcal{P}_n$ and $\lfloor x \rfloor$ denotes the largest integer number no more than x.

By applying the algorithms developed in previous sections, we obtained the optimal profits, pricing and ordering plans as reported in Table 2.2 for the continuous price case and for cases with predefined price admissible sets $\mathcal{P}_{10}, \cdots, \mathcal{P}_3$, respectively. The optimal profit and solution associated with the joint static pricing and inventory problem are also presented in the last column. Moreover, we report the ratio between the optimal profit of each different case with the optimal profit of the continuous price case in the third line of the table.

From Table 2.2, we make several observations. First, the dynamic pricing strategy significantly outperforms the static pricing strategy even when the price adjustment cost accounts for a considerable portion of the total profit. In fact, in the continuous price case, the price adjustment cost accounts for 38.4% of the total profit, while the static pricing strategy results in more than 19% profit off.

Second, the optimal profit of problem (2.2) with finite number of price levels depends on the values of price levels, and does not increase monotonically with respect to the number of price levels. For instance, in the example, when the price admissible set is $\mathcal{P}_5$, $\mathcal{P}_7$ or $\mathcal{P}_9$, the corresponding profit is 96.5% of the optimal profit of the continuous price case, while for $\mathcal{P}_8$ the corresponding profit is only 92%.

We tested many other examples and found that these observations are valid when the profit margin is relatively small. However, if the profit margin is high, solving problem (2.2) with four to five price levels derived by equally dividing the interval $\mathcal{P}$ provides very good approximation to problem (2.2) with continuous price levels.

Table 2.2: Numerical example

	General	$\mathcal{P}_{10}$	$\mathcal{P}_9$	$\mathcal{P}_8$	$\mathcal{P}_7$	$\mathcal{P}_6$	$\mathcal{P}_5$	$\mathcal{P}_4$	$\mathcal{P}_3$	Static
Profit	177.1	166	171	163	171	166	171	109	155	143.2
%	100	93.7	96.5	92.0	96.5	93.7	96.5	61.5	87.5	80.8
Prices										
$t=1$	25.4	26	25	26	25	26	25	27	25	26.1
$t=2$	29.8	30	30	30	30	30	30	30	30	26.1
$t=3$	29.8	30	30	30	30	30	30	30	30	26.1
$t=4$	29.8	30	30	30	30	30	30	30	30	26.1
$t=5$	29.8	30	30	30	30	30	30	27	30	26.1
$t=6$	29.8	26	30	26	30	26	30	27	30	26.1
$t=7$	29.8	26	30	26	30	26	30	27	30	26.1
$t=8$	29.8	26	30	26	30	26	30	27	30	26.1
$t=9$	25.4	26	25	26	25	26	25	27	25	26.1
$t=10$	25.4	26	25	26	25	26	25	27	25	26.1
$t=11$	25.4	26	25	26	25	26	25	27	25	26.1
$t=12$	28.2	28	28	29	28	28	28	27	25	26.1
Orders										
$t=1$	33.2	29	34	29	34	29	34	24	34	29.7
$t=2$	0	0	0	0	0	0	0	0	0	0
$t=3$	0	0	0	0	0	0	0	0	0	22.7
$t=4$	0	0	0	0	0	0	0	0	0	0
$t=5$	19.3	26	17	26	17	26	17	29	17	33.7
$t=6$	0	0	0	0	0	0	0	0	0	0
$t=7$	0	27	0	27	0	27	0	21	0	26.7
$t=8$	0	0	0	0	0	0	0	0	0	0
$t=9$	47.8	42	52	42	52	42	52	32	52	41.4
$t=10$	47.8	42	52	42	52	42	52	32	52	41.4
$t=11$	53.9	52	57	47	57	52	57	52	38	61.4
$t=12$	0	0	0	0	0	0	0	0	34	0

It is also interesting to see how price adjustment cost affects the number of prices in the optimal pricing plan for problem (2.2) with continuous price levels. To do this, we test our current instance with price adjustment cost scaled by a factor ε ranging from 0 to 3.6. As we can see from Table 2.3, the larger the scale factor ε, the less the number of price levels in the optimal pricing plan. This observation is consistent with our intuition. However, we also find examples in which the number of price levels in the optimal pricing plan does not decrease monotonically with the price adjustment cost.

Table 2.3: Number of price levels

#	10	9	8	7
ε	$[0, 0.03)$	$[0.03, 0.12)$	$[0.12, 0.15)$	$[0.15, 0.35)$
#	5	4	2	1
ε	$[0.35, 0.92)$	$[0.92, 1.50)$	$[1.50, 1.55)$	$[1.55, 3.60)$

2.7 Conclusion

In this chapter we present a coordinated pricing and inventory management model with deterministic demand and price adjustment cost. We develop exact time algorithms to solve the problem. Interestingly, depending on the structure of the ordering cost and the price adjustment cost, the computational complexity of our algorithms varies significantly (see Table 2.1). Employing these algorithms, we demonstrate through a typical example that dynamic pricing strategies can significantly outperform static pricing strategies even when the price adjustment cost accounts for a considerable portion of the total profit.

We plan to extend our model and its analysis along several directions. First, it is interesting to see whether the algorithms developed in this chapter can be improved. Observe that to construct the acyclic networks, we have to solve a variety of optimization subproblems with small differences. Thus, one direction is to identify and eliminate possible redundant computations in solving these subproblems. Moreover, solving such subproblem indeed corresponds to determining the optimal price for a certain joint static pricing and inventory model. As we remarked on Assumption 2.1, the associated computational complexity is not clear by now, which may constitute another direction.

Second, from our computational experiments, we observe that a few number of price levels, chosen appropriately, can capture a significant portion of the total profit. Thus, an interesting question is to solve the joint inventory and pricing problem which allows only at most N price levels for some given N. Unlike the model analyzed in Subsection 2.5, the N possible price levels are not given a priori and are decisions to be made in the problem.

Finally, it remains a challenge to incorporate ordering capacity constraints into our model. In this case, it is not clear how to modify the approach in Section 2.2 of breaking down the total profit to terms involving single constant prices, as the zero inventory ordering property does not hold anymore. Even if this could be done, it is likely that we have to solve the joint static pricing and inventory model with capacity constraints as a subroutine, which itself is challenging. To put this in perspective, we note that solving the joint static pricing and inventory model with general capacity constraints is NP-hard, while the algorithm developed in Geunes et al. (2008) takes an $O(T^9)$ time to solve the model with equal capacity constraints.

Chapter 3

Reference Price Effect Model with Deterministic Demand

3.1 Background and literature review

In this chapter, we develop a deterministic coordinated pricing and inventory model incorporating a class of well-studied consumer behavioral models in the marketing and economics literature: the memory-based reference price model. Similar to the model discussed in the previous section, on the supply side we consider a setting similar to the classic economic lot sizing model. On the demand side, a selling price is determined, which influences demand in the period, at the beginning of each period together with the replenishment decision. Moreover, the objective is to make ordering and pricing decisions so as to maximize the total profit over the planning horizon. What distinguishes our work from the literature is the incorporation of reference price effect and the demand depends not only on the selling price but also on the reference price. More specifically, in our model, demand is specified by a linear decreasing function of the current price plus a piecewise linear function of the difference between the current price and the reference price.

Reference price models have been studied in the marketing and economics literature over the past two decades. Such models argue that consumers develop price expectations from historical prices (referred to as reference prices) and use them to judge the current selling price of a product. That is, reference price is an internal anchor formed in consumers' minds as a result of experience based on information such as prices in observed periods (Kalyanaram and Little, 1994). Figure 3.1 illustrates the typical customers' behavior when taking into account the reference price effects. Although in most cases it can not be physically observed, researchers noticed that "comparison of the market price to . . . reference price Raman and Bass (2002)" influences consumers' evaluation on potential purchases before making their decision, especially "in a market with repeated interactions (Popescu and Wu, 2007)." The memory-based reference price model is not only validated by various empirical studies (Briesch et al., 1997; Hardie et al., 1993; Greenleaf, 1995, etc.) but also supported by the famous prospect theory (Tversky and Kahneman, 1991). Indeed, reference price models are now accepted as an empirical generalization in the marketing literature (see the review paper Mazumdar et al., 2005).

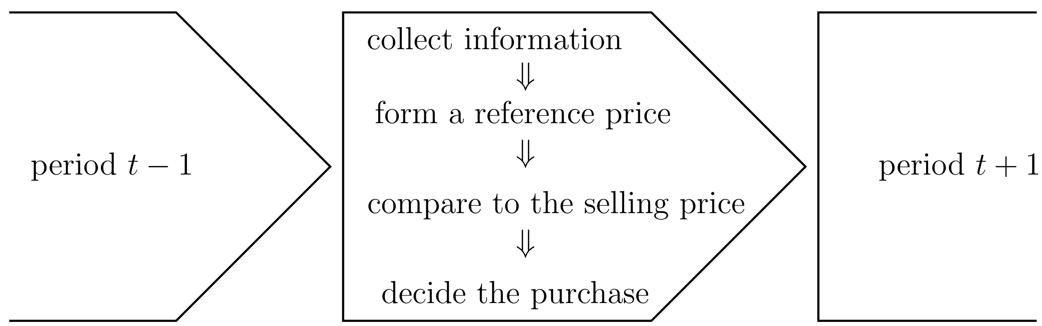

Figure 3.1: Reference price effects

Even though various reference price models are now well established in the marketing literature, their impact on pricing and inventory decisions is largely unexplored despite their profound effects in shaping consumer demand. Our work is among the limited initial attempts along this direction. Some related work from the operations literature includes Fibich et al. (2003), Popescu and Wu (2007), Urban (2008) and Gimpl-Heersink (2008). Among them, Fibich et al. (2003) derive a closed form solution for a deterministic and continuous time dynamic pricing problem in which demand rate is piecewise linear in the current price and the reference price. Focusing on a periodic review infinite horizon setting, Popescu and Wu (2007) extend Kopalle et al. (1996) by allowing more general demand functions (as functions of price and reference price). They illustrate that a stationary price would be optimal if consumers are more sensitive to losses than gains and provide a characterization of the optimal stationary price. Urban (2008) develops a single-period model with stochastic demand and derives its optimal solution. Finally, Gimpl-Heersink (2008) analyzes a stochastic periodic review finite horizon model in which demand is a function of the current price and the reference price with an additive random perturbation. Under rather restrictive assumptions, Gimpl-Heersink (2008) proves that a reference price dependent base-stock policy is optimal for a two-period setting with stationary loss-neutral demand model.

Unlike these aforementioned papers, this chapter focuses on developing effective algorithms for a periodic review finite horizon model with deterministic demand. Such a model is important as it explicitly models the interaction of operational and pricing decisions by capturing customers' response to dynamically adjusted prices in a tractable way through reference prices, and thus holds the promise to be incorporated into decision support systems. Indeed, it is arguably the simplest pricing and inventory model that captures customers' behavior towards dynamic pricing at an aggregated level. Nevertheless, the analysis is nontrivial even for cases with zero fixed ordering cost as we illustrate here. A closely related paper is Ahn et al. (2007), which develop algorithms for a periodic review finite horizon model in which demand also depends on past prices. Although a special case of Ahn et al. (2007) is almost identical to one of the special cases of our model, their demand functions are constructed based on a different mechanism and are totally different from the ones derived from reference price models. It is also appropriate to mention that none of the above papers incorporate fixed ordering costs in their models, despite the fact that such costs are ubiquitous in production planning settings.

Incorporating reference price effect into coordinated pricing and inventory models significantly complicates the analysis and algorithm design. Indeed, even without fixed ordering cost, our model is a maximization problem involving a piecewise quadratic objective function which may not be concave. Interestingly, in the case with zero fixed ordering cost, we identify certain technical conditions under which our problem can be solved in strongly polynomial time. For the general case, we propose an approximation heuristic that discretizes reference price and exploits the structure of the economic lot sizing problem. Lower and upper bounds to the optimal objective value of our model are also provided based on the heuristic. Our numerical study, employing the heuristic, illustrates that the more reference price effect contributes to demands, the higher the benefit of coordinated pricing and inventory models vs. the sequential decision making process.

The remainder of this chapter is organized as follows. In Section 3.2 the mathematical formulation of our model is presented. In Section 3.3, we analyze the model with zero fixed ordering cost and develop strongly polynomial time algorithms to several special cases. The general model is dealt with in Section 3.4, followed by a numerical study in Section 3.5. Finally, we conclude the paper in the last section with some suggestions for future research. To maintain a clear presentation, all technical proofs are presented in Appendix A.

3.2 Problem description

Consider a firm that has to make coordinated ordering and pricing decisions to satisfy a sequence of deterministic demands of a single product over a finite planning horizon with T periods. On the supply side the setting is exactly the same as the classic economic lot sizing model, i.e., at the beginning of period t, an ordering quantity y_t are determined and incurs the cost $k_t\delta(y_t)+c_ty_t$, where the order is assumed to be delivered immediately. Moreover, no shortage is allowed and the inventory left at the end of period t, denoted by I_t, is carried over to the next period with a unit inventory holding cost h_t. The objective of the firm is to decide its ordering quantities and prices so as to maximize its total profit over the planning horizon.

On the demand side a selling price p_t is decided simultaneously together with the replenishment decision in each period. Unlike most papers in the literature on pricing and inventory coordination, in our model demand at each period depends on not only the current selling price but also past observed prices. We adopt one of the well-studied consumer behavioral pricing models in the marketing literature, where the impact of past prices on the demand is captured by the *reference price effect*. This type of models argues that consumers develop price expectations, called the *reference prices*, based on past observed prices and use them to judge the purchase price of a product (see Mazumdar et al., 2005, for a review). Among many different reference price models, a memory-based model, is commonly used and empirically validated on scanner panel data for a variety of products (see, for example, Greenleaf, 1995). In this model, reference price is generated by exponentially weighting past prices. Specifically, starting with a given initial r_1, the reference price at period t, denoted by r_t, evolves as

$$r_{t+1} = \alpha r_t + (1-\alpha)p_t, \quad t = 1, 2, \cdots, T.$$

In the above evolution equations, $\alpha \in [0, 1)$ is called the memory factor or carryover constant (Kalyanaram and Little, 1994). Several papers, for example, Raman and Bass (2002); Krishnamurthi et al. (1992); Mayhew and Winer (1992), assume that $\alpha = 0$. In this case, the model reduces to $r_{t+1} = p_t$, which means the price of the previous period serves as the reference price. It captures the fact that "consumers... experience considerable difficulty in recalling accurately even the most recently encountered prices ... Thus, it is unlikely that consumers would retrieve from memory and use prices encountered much beyond the immediate past purchase occasion (Krishnamurthi et al., 1992)." Other values of α are also observed from empirical studies (see the related discussion in Section 3.5). Observe that as α increases, "consumers change their reference prices more slowly to incorporate new price information (Greenleaf, 1995)." When $\alpha = 1$, reference prices remain a constant over the whole planning horizon. Thus, we restrict $\alpha < 1$ to avoid the trivial case that past prices have no impact on demand.

Following Greenleaf (1995); Kopalle and Winer (1996) and Fibich et al. (2003), the demand at period t, with a given price p and a reference price r, is modeled as

$$b_t - a_t p + \eta(r - p),$$

where $b_t - a_t p$ is the base demand independent of reference prices, $\eta(r - p)$ is the additional demand induced by the reference price effect. The difference between reference price and selling price, i.e., $r - p$, in the above demand model is usually referred to as a perceived surcharge/discount (Popescu and Wu, 2007). If $r < p$, consumers perceive this as a loss, while if $r > p$, they perceive it as a gain. In this paper, we assume that η is a kinked function consisting of two linear pieces as

$$\eta(z) = \eta^+ \max\{z, 0\} + \eta^- \min\{z, 0\},$$

where non-negative coefficients η^+ and η^- represent the marginal reference price effect associated with gains and losses, respectively.

The kink of the function η indicates that consumers respond to losses and gains differently. Consumers are classified as loss averse, loss neutral and loss-seeking depending on whether $\eta^+ \leq \eta^-$, $\eta^+ = \eta^-$ or $\eta^+ \geq \eta^-$. See Figure 3.2 for the illustration. It is common in the marketing literature to

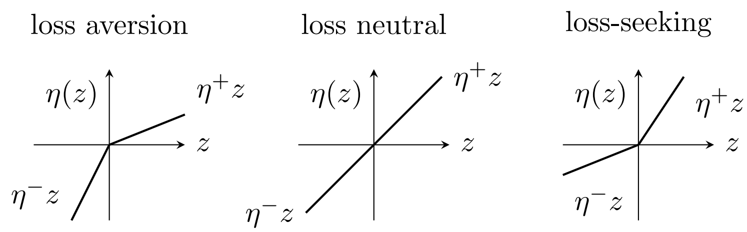

Figure 3.2: Reference price effects in demand

assume that consumers are more sensitive to losses than gains. Indeed, the loss averse assumption is consistent with the prospect theory (Tversky and Kahneman, 1991), which predicts that a perceived loss would stimulate more reaction from a human than a perceived gain, and has also been validated by several empirical studies (see, for example, Putler, 1992; Hardie et al., 1993). However, there is evidence that indicates consumers may be more sensitive to gains than losses in some situations (e.g., Greenleaf, 1995; Krishnamurthi et al., 1992). In this paper we consider the general case and make no assumption on the relative magnitudes of the two coefficients. It is also worth mentioning that our heuristic about to provided in Section 3.4 works for arbitrary demand models $d_t = d_t(p_t, r_t)$. See Remark 3.1 in that section.

We now present the mathematical model to maximize the firms' total profit over the planning horizon by simultaneously determining price and ordering quantity in each period:

$$V^* = \underset{p_t, y_t, r_t}{\text{maximize}} \quad \sum_{t=1}^{T} \{p_t d_t - [k_t \delta(y_t) + c_t y_t + h_t I_t]\} \tag{3.1a}$$

$$\text{subject to} \quad I_t = I_{t-1} + y_t - d_t, \qquad t = 1, \cdots, T, \tag{3.1b}$$

$$d_t = b_t - a_t p_t + \eta(r_t - p_t), \qquad t = 1, \cdots, T, \tag{3.1c}$$

$$r_{t+1} = \alpha r_t + (1-\alpha) p_t, \qquad t = 1, \cdots, T, \tag{3.1d}$$

$$I_t \geq 0, \quad y_t \geq 0, \quad p_t \in [L_t, U_t], \qquad t = 1, \cdots, T, \tag{3.1e}$$

where $I_0 = 0$ and the initial reference price r_1 is given as an input to the optimization model. In the above model, d_t defined by (3.1c) denotes the demand, and the term $p_t d_t$ in (3.1a) is the revenue at period t. The bracketed term in (3.1a) denotes inventory-related costs, including the fixed ordering cost $k_t \delta(y_t)$, the variable ordering cost $c_t y_t$ and the inventory holding cost $h_t I_t$. Constraint (3.1b) specifies the inventory balance equation (starting with zero inventory at the beginning of the planning horizon), which together with $y_t \geq 0$ in (3.1e) ensures that no demand is backlogged. The last set of constraints gives the feasible set for the decision variables. Throughout the paper, we assume that d_t is nonnegative for all feasible p_t and r_t.

Several coordinated pricing and inventory management models in literature can be cast as special cases of problem (3.1). For example, when $L_t = U_t$, i.e., prices are given, problem (3.1) reduces to the classical economic lot sizing model (Wagner and Whitin, 1958a). When $\eta(z) = 0$ or $\alpha = 1$, problem (3.1) reduces to the coordinated pricing and inventory model in which demand at each period relies on the current price only. Similar models have been studied by Wagner and Whitin (1958a); Thomas (1970), and etc. Note that when $k_t = 0$, $\eta^- = 0$ and $\alpha = 0$, our model is essentially identical to a special case analyzed by Ahn et al. (2007). In their general model, demand depends on prices of the current period and past periods. However, in general their demand model is totally different from the reference price model employed here. In addition, Ahn et al. (2007) do not take into account fixed ordering costs.

3.3 Zero fixed ordering cost case

In this section we focus on the case with no fixed ordering cost, i.e., $k_t = 0$, at any period. Similar to the discussion in Section 2.3, in this case the ordering plan can be determined independent of pricing plan, as well as the marginal ordering and inventory holding cost, denoted by $\bar{c}_t$, in each period t for $t = 1, \cdots, T$. Therefore the profit in period t with respect to given selling price p_t and reference price r_t can be expressed by

$$\pi_t(r_t, p_t) = (p_t - \bar{c}_t)[b_t - a_t p_t + \eta(r_t - p_t)].$$

Problem (3.1) then becomes

$$\begin{aligned} \underset{p_t, r_t : t \leq T}{\text{maximize}} \quad & \pi_1(r_1, p_1) + \pi_2(r_2, p_2) + \cdots + \pi_T(r_T, p_T) \\ \text{subject to} \quad & r_{t+1} = \alpha r_t + (1-\alpha) p_t, \quad p_t \in [L_t, U_t], \quad t = 1, \cdots, T. \end{aligned} \tag{3.2}$$

What remains is to determine the optimal pricing plan.

Observe that the per period profit function $\pi_t(r,p)$ is piecewise quadratic and not jointly concave in general, which poses a significant technical difficulty making the analysis of problem (3.2) rather challenging. In this section we will introduce conditions on the input parameters such that problem (3.2) can be solved in strongly polynomial time.

The following discussion depends on whether consumers are loss averse, loss neutral or loss seeking. We first present a useful property on the optimal pricing plan.

Proposition 3.1. *If $\bar{c}_t \le U_t$ in all periods, then there exists an optimal solution $\{p_1, \cdots, p_T\}$ to problem* (3.2) *such that $p_t \ge \bar{c}_t$ for all $t = 1, \cdots, T$.*

Proposition 3.1 basically says if the marginal ordering and holding costs do not exceed the upper bounds of feasible prices at all periods, then an optimal solution exists such that the firm can always make a profit in each period by selecting proper feasible prices. This assumption is quite reasonable and usually satisfied in practice.

Since the effect of past prices on period t's demand is summarized by the reference price r_t, it will be convenient to express profit in terms of reference prices. In particular, given the reference prices r_t, r_{t+1} at periods $t, t+1$, respectively, the price p_t and the profit, denoted by $\Pi_t(r_t, r_{t+1})$, at period t can be expressed as

$$p_t = \frac{r_{t+1} - \alpha r_t}{1-\alpha}, \quad \Pi_t(r_t, r_{t+1}) = \pi_t\left(r_t, \frac{r_{t+1} - \alpha r_t}{1-\alpha}\right).$$

Let $G_{t+1}(r_{t+1})$, $t \le T$, be the maximal accumulated profit up to period t when reference price r_{t+1} is specified at period $t+1$. That is,

$$\begin{aligned} G_{t+1}(r_{t+1}) = \quad & \underset{p_s, r_s : s \le t}{\text{maximize}} \quad \Pi_1(r_1, r_2) + \Pi_2(r_2, r_3) + \cdots + \Pi_t(r_t, r_{t+1}), \\ & \text{subject to} \quad \alpha r_s + (1-\alpha) p_s = r_{s+1}, \quad p_s \in [L_s, U_s], \quad s \le t. \end{aligned}$$

Apparently solving problem (3.2) amounts to maximizing $G_{T+1}(r)$. Thus, it suffices to determine the expression of $G_{T+1}(r)$, which can be inductively derived for $t = 2, \cdots, T$ through

$$\begin{aligned} G_{t+1}(r_{t+1}) = \quad & \underset{p_t, r_t}{\text{maximize}} \quad G_t(r_t) + \Pi_t(r_t, r_{t+1}), \\ & \text{subject to} \quad \alpha r_t + (1-\alpha) p_t = r_{t+1}, \quad p_t \in [L_t, U_t], \end{aligned} \tag{3.3}$$

where $G_2(r_2) = \Pi_1(r_1, r_2)$ and we specify $G_{t+1}(r_{t+1}) = -\infty$ if r_{t+1} leads to an empty feasible set in the above problem.

3.3.1 Loss averse and loss neutral cases

In this subsection we focus on cases in which consumers are loss neutral (i.e.,η is linear) or loss averse (i.e., η is concave). To simplify notation, we replace (r_t, r_{t+1}) by (x, r) in the definition of $\Pi_t(r_t, r_{t+1})$ and problem (3.3).

Since the per period profit function $\Pi_t(x, r)$ is not jointly concave in x and r, problem (3.3) may not be a concave maximization problem in general. To circumvent this difficulty, we first prove that a certain modified version of the per period profit function $\Pi_t(x, r)$ is jointly concave and supermodular on some feasible set under some technical condition. Such result is given in Proposition 3.2 below.

Proposition 3.2. *Let $\Omega = \{(x, r) : \alpha x + (1-\alpha)p = r, p \geq \bar{c}_t\}$ in the loss-averse case and $\Omega = \Re^2$ in the loss-neutral case. Then the function $\Pi_t(x, r) - A_t x^2 + B_t r^2$ is jointly concave and supermodular on Ω, where*

$$A_t = \frac{1}{2}\frac{2\alpha a_t + \alpha\eta^+ + (1-\alpha)\eta^-}{1-\alpha}, \quad B_t = \frac{1}{2}\frac{2a_t + \eta^+}{1-\alpha}.$$

For constants A_t given in Proposition 3.2, define $F_t(x) = G_t(x) + A_t x^2$ and rewrite problem (3.3) as

$$\begin{array}{lll} F_{t+1}(r) = & \underset{x}{\text{maximize}} & F_t(x) + (A_{t+1} - B_t)r^2 + \left[\Pi_t(x, r) - A_t x^2 + B_t r^2\right], \\ & \text{subject to} & \alpha x + (1-\alpha)p = r, \quad p \in [L_t, U_t]. \end{array}$$

One can expect that if $A_{t+1} \leq B_t$, or equivalently,

$$\alpha(2a_{t+1}) + (1-\alpha)(\eta^- - \eta^+) \leq 2a_t, \tag{3.4}$$

where we specify $a_{T+1} = 0$, then it is possible to inductively show the joint concavity of $F_t(x)$. Indeed we have the following proposition.

Proposition 3.3. *Suppose the inequality (3.4) holds for each $t \leq T$. If either (a) η is linear or (b) η is concave and $L_t \geq \bar{c}_t$ in each period t, then each function G_t is concave. Moreover, it consists of $O(t)$ quadratic pieces and can be obtained in an $O(t^2)$ time.*

The technical condition (3.4) holds under several plausible settings including (a) $\alpha = 0$ (i.e., the price in the previous period serving as the reference price) and $\eta^- - \eta^+ \le 2a_t$; (b) $a_{t+1} \le a_t$ and $\eta^- - \eta^+ \le 2a_t$. Note that several empirical studies (see Greenleaf, 1995, for example) support the hypothesis that demand is more sensitive to selling price than to reference price, under which the inequality $\eta^- - \eta^+ \le 2a_t$ apparently holds.

The proof of Proposition 3.3 is nontrivial. In fact in the loss-averse case, the reference price effects asymmetrically depend on whether the current price or the reference price is higher. Such problem can not be easily solved. As claimed in Fibich et al. (2003), the asymmetric reference price effects lead to "non-smooth optimization problems" and the explicit solutions are difficult to obtained by "using standard optimization method". By observing that "[t]he discrete formulation is . . . cumbersome for obtaining explicit solutions", Fibich et al. (2003) turns to a continuous-time formulation of this problem.

In the following we present an outline of the proof to illustrate its basic idea and refer to the appendix for the full proof. Observe that the per period profit function consists of two quadratic pieces, i.e.,

$$\Pi_t(x, r) = \begin{cases} \Pi_t^+(x, r), & \text{if } x \ge r, \\ \Pi_t^-(x, r), & \text{if } x \le r, \end{cases}$$

where Π_t^+ and Π_t^- are quadratic functions given by

$$\Pi_t^{\pm}(x, r) = \left(\frac{r - \alpha x}{1 - \alpha} - \bar{c}_t \right) \left(b_t - a_t \frac{r - \alpha x}{1 - \alpha} + \eta^{\pm} \frac{x - r}{1 - \alpha} \right).$$

In addition, we say $G_t(x)$ consists of N quadratic pieces if there exists $x^1 < \cdots < x^N$, called the breakpoints of G_t, such that

$$G_t(x) = G_t^n(x), \quad \forall x^{n-1} < x \le x^n, \quad 1 \le n \le N,$$

where $x^0 = -\infty$ and $G_t^n(x)$ are quadratic functions. The basic idea is to inductively prove that if G_t has N breakpoints, then G_{t+1} has at most $N + 5$ quadratic pieces, where N possible breakpoints associate with the breakpoints of G_t, 2 of them associate with the endpoints of the feasible set $[L_t, U_t]$ of p, and the remaining 3 possible breakpoints associate with the two-piece quadratic function Π_t.

More specifically We will show that there exists some sequence of points $\{R_m : m \leq M\}$ such that $M \leq N+5$ and exact one of the following cases holds when $r_{t+1} \in (R_{m-1}, R_m]$ for each m:

$$G_{t+1}(r_{t+1}) = [G_t^{i_m}(r_t) + \Pi_t^{j_m}(r_t, r_{t+1}) : \alpha r_t + (1-\alpha)U_t = r_{t+1}], \tag{3.5a}$$

$$G_{t+1}(r_{t+1}) = [G_t^{i_m}(r_t) + \Pi_t^{j_m}(r_t, r_{t+1}) : \alpha r_t + (1-\alpha)L_t = r_{t+1}], \tag{3.5b}$$

$$G_{t+1}(r_{t+1}) = \underset{r_t}{\text{maximize}}\ [G_t^{i_m}(r_t) + \Pi_t^{j_m}(r_t, r_{t+1})], \tag{3.5c}$$

where $R_0 = -\infty$, $i_m \leq N$ and $j_m \in \{+, -\}$ can be determined from the sequence $\{R_m : m \leq M\}$. That is, according to the range of r_{t+1}, it leads no loss of optimality to specify the quadratic pieces of Π_t, G_t and determine whether the constraint $p \in [L_t, U_t]$ is active or not in problem (3.3).

Proof of the above statement is divided into four steps:

Step 1: There exists some R such that it leads no loss of optimality to consider one quadratic piece of $\Pi_t(r_t, r_{t+1})$ in problem (3.3) depending on whether $r_{t+1} \in (-\infty, L_t] \cup [R, U_t)$ or not.

Step 2: There exist two numbers R_L, R_U such that an optimal solution to problem (3.3) satisfies $p_t = L_t$ when $r_{t+1} < R_L$ and $p_t = U_t$ when $r_{t+1} > R_U$; moreover, if $r_{t+1} \in [R_L, R_U]$, then the constraint $p \in [L_t, U_t]$ can be removed from problem (3.3) without loss of any optimality.

Step 3: There exists a non-decreasing sequence $\{r^n : 1 \leq n \leq N\}$ such that it leads no loss of optimality to specify $G_t = G_t^n$ in (3.3) when $r_{t+1} \in (r^{n-1}, r^n]$ for each n, where $r^0 = \infty$.

Step 4: Let $\{R_m : m \leq M\}$ be a sorting of L_t, U_t, R, R_L, R_U and $\{r^n : n \leq N\}$. By combining results in the previous steps, we derive the desirable expression (3.5) of G_{t+1}.

After verifying expression (3.5), we then show G_{t+1} is piecewise quadratic with breakpoints $\{R_m : m \leq M\}$. In fact, $G_{t+1}(r)$ is clearly quadratic on $(R_{m-1}, R_m]$ in cases (3.5a) and (3.5b). For case (3.5c), because the objective function is quadratic in term of r_t for any given r_{t+1}, its optimal solution is linear in r_{t+1}. Hence $G_{t+1}(r_{t+1})$ is also quadratic on $(R_{m-1}, R_m]$ in this case.

From Proposition 3.1, there is no loss of optimality to assume $L_t \geq \bar{c}_t$ when $\bar{c}_t \leq U_t$ at all periods. From Proposition 3.3, we can determine the expression of $G_{T+1}(r)$ in $O(T^2)$ time when $L_t \geq \bar{c}_t$. Since we can maximize a function consisting of $O(t)$ quadratic pieces in $O(t)$ time, our main result in this subsection follows.

Theorem 3.1. *Suppose* (3.4) *holds. Problem* (3.2) *can be solved in* $O(T^2)$ *time if either (a) consumers are loss neutral; or (b) consumers are loss averse and* $\bar{c}_t \leq U_t$ *for all* $t \leq T$.

3.3.2 The Loss-seeking Case

When consumers are loss-seeking (in this case, the function η is convex), functions G_t may not be concave anymore. Hence it would be challenging to obtain results similar to Proposition 3.3. In this subsection, we will focus on a special case, in which we assume the price of the previous period serves as the reference price ($r_{t+1} = p_t$) and consumers are insensitive to price decreases ($\eta^- = 0$). Interestingly, this case is similar to the model 2-($K = 1, Q = \infty$) in Ahn et al. (2007).

We now develop a strongly polynomial time algorithm for this special case under certain conditions. First observe that if a price markup incurs at some period τ, i.e., $p_{\tau-1} < p_\tau$, then the prices before period τ have no impact on prices in later periods and they can be independently determined. Specifically, for a given price sequence $\{p_1, \cdots, p_T\}$, let $1 = \tau_1 < \cdots < \tau_N < \tau_{N+1} = T + 1$ be all markup periods, i.e., $p_{t-1} < p_t$ if $t = \tau_n$ for some $1 < n \leq N$ and $p_{t-1} \geq p_t$ otherwise. Here period 1 and the artificial period $T + 1$ are counted as price markup periods for convenience. For each pair of consecutive price markup periods $(\tau, \tilde{\tau}) = (\tau_n, \tau_{n+1})$ for some n, one can verify that profit accumulated from period τ to period $\tilde{\tau} - 1$ is

$$\Pi_\tau(p_\tau, p_\tau) + \sum_{\tau < t < \tilde{\tau}} \Pi_t(p_{t-1}, p_t),$$

which is independent of prices specified before period τ and after period $\tilde{\tau} - 1$.

This observation allows us to partition the planning horizon by the price markup periods, determine prices between each consecutive price markup periods independently, and then find the optimal markup period sequence with maximal total profit. For this purpose, define

$$\begin{aligned} \ell(\tau, \tilde{\tau}) = \quad & \underset{p_t : \tau \le t < \tilde{\tau}}{\text{maximize}} \quad \Pi_\tau(p_\tau, p_\tau) + \sum_{\tau < t < \tilde{\tau}} \Pi_t(p_{t-1}, p_t) \\ & \text{subject to} \quad p_\tau \ge p_{\tau+1} \ge \cdots \ge p_{\tilde{\tau}-1}, \\ & \qquad\qquad\quad p_t \in [L_t, U_t], \quad \tau \le t < \tilde{\tau}. \end{aligned} \tag{3.6}$$

Note that if τ and $\tilde{\tau}$ turn out to be two consecutive price markup periods in an optimal solution to problem (3.2), then $\ell(\tau, \tilde{\tau})$ is exactly the maximal profit accumulated from period τ to period $\tilde{\tau} - 1$. From this observation, we can construct an acyclic network $(\mathcal{V}, \mathcal{E})$ where the node set and link set are respectively defined by

$$\mathcal{V} = \{1, 2, \ldots, T+1\}, \quad \mathcal{E} = \{(\tau, \tilde{\tau}) : 1 \le \tau < \tilde{\tau} \le T+1\}.$$

Moreover, let $\ell(\tau, \tilde{\tau})$ be the length of a link $(\tau, \tilde{\tau})$ in $\mathcal{E}$.

To calculate link lengths $\ell(\tau, \tilde{\tau})$, let $G_{\tau,\tau+1}(p) = \Pi_\tau(p, p)$ and recursively define $G_{\tau,t}$ for $t = \tau+1, \cdots, T$ through the optimization problem:

$$\begin{aligned} G_{\tau,t+1}(p_{t+1}) = \quad & \underset{p_t}{\text{maximize}} \quad G_{\tau,t}(p_t) + \Pi_t(p_t, p_{t+1}), \\ & \text{subject to} \quad p_t \ge p_{t+1}, \quad p_t \in [L_t, U_t], \end{aligned} \tag{3.7}$$

where we set $G_{\tau,t+1}(p_{t+1}) = -\infty$ when the feasible set of the above problem is empty. Observe that the function $G_{\tau,t}(p_t)$ can be interpreted as the maximal accumulated profit from period τ to period $t-1$ when the price at period $t-1$ is set at p_t (or equivalently the reference price of period t is set at p_t) and only price markdown is allowed. The length $\ell(\tau, \tilde{\tau})$ can be computed by maximizing the function $G_{\tau,\tilde{\tau}}(p_{\tilde{\tau}})$ over $p_{\tilde{\tau}}$.

We illustrate in the following theorem that an optimal solution to problem (3.2) can be derived by finding a longest path in the acyclic network and present the computational complexity.

Theorem 3.2. *Suppose $\alpha = 0$ and $0 = \eta^- \le \eta^+ \le 2a_t$ in each period. If either $\bar{c}_t \le U_t$ or $U_t = U$ holds for each $t = 1, \cdots, T$, then solving problem* (3.2) *is equivalent to finding a longest path from the origin* 1 *to the destination $T+1$ in the acyclic network $(\mathcal{V}, \mathcal{E})$, which contains $O(T)$ nodes and $O(T^2)$ links. Moreover, it takes an $O(T^3)$ time to construct the network and another*

$O(T^2)$ time to find a longest path.

Similar idea of converting (3.2) to a longest path problem has been used in Ahn et al. (2007) for the so-called 2-($K = 1, Q = \infty$) model, where subproblems similar to (3.6) are also derived to obtain link lengths of the acyclic network. We point out two important differences with their paper. First, subproblems in their model are automatically concave maximization problems, while for our model, certain technical conditions on the input parameters are required. More importantly, we calculate all link lengths by inductively obtaining functions $G_{\tau,t}(p_t)$ through (3.7) for all $1 \leq \tau < t \leq T+1$ in strongly polynomial time, while Ahn et al. (2007) suggest interior point methods whose running time is polynomial but not strongly polynomial in general. The construction of our strongly polynomial time algorithm follows from a similar idea as in the proof of Proposition 3.3.

In Theorem 3.2 we require that either $\bar{c}_t \leq U_t$ for all t or $U_t = U$ for all t. Without such assumption, the longest path in $(\mathcal{V}, \mathcal{E})$ may fail to associate with an optimal solution for problem (3.2). To see this, consider the 2-period example with parameters $(\eta^-, \eta^+) = (0, 1)$ and

$$(L_1, U_1, a_1, b_1, \bar{c}_1) = (0, 6, 1, 6, 0), \quad (L_2, U_2, a_2, b_2, \bar{c}_2) = (0, 1, 1, 1, 3),$$

where $\bar{c}_2 > U_2$ and $U_1 > U_2$. Calculation shows that $\ell(1, 2) = 9$, $\ell(2, 3) = 0$ and $\ell(1, 3) = 6$ in this acyclic network. Therefore, the longest path is $\{123\}$, which corresponds to the solution $(p_1, p_2) = (3, 1)$. However, the actual total profit associated with this solution is

$$p_1(6 - p_1) + (p_2 - 3)\big[1 - p_2 + (p_1 - p_2)\big] = 5,$$

which is strictly less than $\ell(1, 3) = 6$. In fact, node 2 appears in the longest path $\{123\}$ even though it is not a price markup period in the corresponding solution, and the algorithm fails to find the optimal solution, which is $(p_1, p_2) = (2, 1)$, to problem (3.2). Interestingly, we show in Appendix A.5 that an optimal solution can be derived by finding a longest path in an *extended* acyclic network again in $O(T^3)$ even when the assumption either $\bar{c}_t \leq U_t$ for all t or $U_t = U$ for all t is relaxed.

3.4 General case

In this section we consider the general problem with fixed ordering costs and general reference price effect functions. Apparently this general case is significantly more difficult than the special case with zero fixed cost presented in the previous section. In this section, rather than solving problem (3.1) to optimality, we propose a heuristic by discretizing the reference prices. In addition, we derive lower and upper bounds to the optimal objective value of problem (3.1) from the heuristic. To simplify our presentation, we assume a uniform feasible set $[L, U]$ for prices at all periods and $r_1 \in [L, U]$. With this assumption, we observe from (3.1d) that $r_t \in [L, U]$ for all $t \geq 2$.

We restrict reference prices to a predetermined finite set $\mathcal{R}_\varepsilon \subset [L, U]$ in our heuristic. To achieve a reasonable accuracy, the set $\mathcal{R}_\varepsilon$ is chosen as below with some positive scalar ε,

$$\max\{\min\{|r - r_\varepsilon| : \; r_\varepsilon \in \mathcal{R}_\varepsilon\} : r \in [L, U]\} \leq \tfrac{1}{2}\varepsilon,$$

That is, the distance between sets $\mathcal{R}_\varepsilon$ and any point in $[L, U]$ is no more than $\frac{1}{2}\varepsilon$. As a simple instance, we can set $\mathcal{R}_\varepsilon = \{L + (n-1)\varepsilon : n \leq S_\varepsilon\}$, where S_ε, the number of elements in $\mathcal{R}_\varepsilon$, is given as the integer part of $\varepsilon^{-1}(U - L)$.

Given the finite set $\mathcal{R}_\varepsilon$, we consider the following problem

$$V^\varepsilon = \underset{y_t, p_t, r_t}{\text{maximize}} \; \sum_{t=1}^{T} \{p_t d_t - [k_t \delta(y_t) + c_t y_t + h_t I_t]\} \tag{3.8a}$$

$$\begin{aligned} \text{subject to} \quad & \text{(3.1b) and (3.1c) hold} \\ & \left| r_{t+1} - \alpha r_t - (1-\alpha) p_t \right| \leq \varepsilon, \;\; t = 1, \cdots, T, \qquad \text{(3.8b)} \\ & I_t \geq 0, \;\; y_t \geq 0, \;\; r_t \in \mathcal{R}_\varepsilon, \;\; p_t \in [L, U], \;\; t = 1, \cdots, T. \end{aligned}$$

In the above problem, reference prices are restricted in $\mathcal{R}_\varepsilon$. In addition, the reference price evolution equation (3.1d) is approximated by (3.8b). Clearly, ε controls the accuracy of the approximation and smaller ε leads to better approximation to problem (3.1). We will construct lower and upper bounds for problem (3.1) based on problem (3.8).

Another lower bound to problem (3.1) is given by the following problem

$$
\begin{aligned}
V_0^{\varepsilon} = \quad & \underset{y_t, p_t, r_t}{\text{maximize}} \quad \sum_{t=1}^{T} \{p_t d_t - [k_t \delta(y_t) + c_t y_t + h_t I_t]\}, && (3.9)\\
& \text{subject to} \quad (3.1b), (3.1c) \text{ and } (3.1d) \text{ hold},\\
& \qquad I_t \geq 0, \quad y_t \geq 0, \quad r_t \in \mathcal{R}_{\varepsilon}, \quad p_t \in [L, U], \qquad t = 1, \cdots, T,
\end{aligned}
$$

where reference price evolves as the same as in problem (3.1) but are restricted to $\mathcal{R}_{\varepsilon}$.

The advantage of restricting the reference price to a finite set is its tractability. Indeed, we now show that both problems (3.8) and (3.9) can be solved in polynomial time in terms of T and the number of elements in $\mathcal{R}_{\varepsilon}$.

We start with problem (3.8). The idea is to partition the planning horizon according to ordering periods while taking into account the reference price levels at these periods. Specifically, let τ and $\tilde{\tau}$ with $\tau < \tilde{\tau}$ be two consecutive ordering periods. Once the reference price levels r and $\tilde{r}$ at periods τ and $\tilde{\tau}$ are specified, $p_\tau, \cdots, p_{\tilde{\tau}-1}$ can be optimized independent of prices in other periods. The optimal cumulative profit from periods τ to $\tilde{\tau} - 1$, denoted by $\ell(\tau, r, \tilde{\tau}, \tilde{r})$, can be calculated from the optimization problem

$$
\begin{aligned}
& \underset{p_t, r_t : \tau \leq t \leq \tilde{\tau}}{\text{maximize}} \quad -k_\tau + \sum_{t=\tau}^{\tilde{\tau}-1} [p_t - c(\tau, t)] [b_t - a_t p_t + \eta(r_t - p_t)] && (3.10)\\
& \text{subject to} \quad |r_{t+1} - \alpha r_t - (1-\alpha) p_t| \leq \varepsilon, \qquad \tau \leq t < \tilde{\tau},\\
& \qquad r_\tau = r, \; r_{\tilde{\tau}} = \tilde{r}, \; r_t \in \mathcal{R}_{\varepsilon}, \; p_t \in [L, U], \quad \tau \leq t < \tilde{\tau}.
\end{aligned}
$$

We now show that problem (3.8) can be solved by identifying a longest path in an acyclic network. Specifically, construct an acyclic network $(\mathcal{V}, \mathcal{E})$ with the node set $\mathcal{V}$ and link set $\mathcal{E}$ defined by

$$
\begin{aligned}
\mathcal{V} &= \{(\tau, r) : 2 \leq \tau \leq T+1, r \in \mathcal{R}_{\varepsilon}\} \cup \{\boldsymbol{v}^0, \boldsymbol{v}^e\},\\
\mathcal{E} &= \{\langle (\tau, r), (\tilde{\tau}, \tilde{r}) \rangle : \tau < \tilde{\tau}, (\tau, r), (\tilde{\tau}, \tilde{r}) \in \mathcal{V}\},
\end{aligned}
$$

where $\boldsymbol{v}^0 = (1, r_1)$ and the artificial node $\boldsymbol{v}^e = (T+1, *)$ are the origin and the destination respectively in the network. Here the symbol "*" denotes an arbitrary value since no reference price is specified at period $T+1$. The length of the link $\langle (\tau, r), (\tilde{\tau}, \tilde{r}) \rangle \in \mathcal{E}$ is given by $\ell(\tau, r, \tilde{\tau}, \tilde{r})$, where in the case

$(\tilde{\tau}, \tilde{r}) = \boldsymbol{v}^e$, it is obtained by simply removing the constraint $r_{T+1} = \tilde{r}$ from problem (3.10). It is straightforward to verify that the longest path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ corresponds to an optimal solution for problem (3.8), and its length corresponds to the maximal profit that can be achieved.

It remains to calculate all link lengths. Interestingly, for a given (τ, r), all the link lengths $\ell(\tau, r, \tilde{\tau}, \tilde{r})$ for $\tilde{\tau} > \tau$ and $\tilde{r} \in \mathcal{R}_\varepsilon$ can be obtained by solving another longest path problem in a different acyclic network. To see this, define acyclic networks $(\mathcal{V}^{\tau,r}, \mathcal{E}^{\tau,r})$ by

$$\begin{aligned}
\mathcal{V}^{\tau,r} &= \big\{(\tau, r)\big\} \cup \big\{(\bar{\tau}, \bar{r}) : \bar{\tau} < \tilde{\tau}, \bar{r} \in \mathcal{R}_\varepsilon\big\}, \\
\mathcal{E}^{\tau,r} &= \big\{\, \langle (t, \bar{r}), (t+1, \tilde{r}) \rangle : (t, \bar{r}), (t+1, \tilde{r}) \in \mathcal{V}^{\tau,r} \big\}.
\end{aligned}$$

Let the length of the link $\langle (t, \bar{r}), (t+1, \tilde{r}) \rangle$ be the optimal value of the following optimization problem

$$\begin{aligned}
\underset{p_t}{\text{maximize}} \quad & [p_t - c(\tau, t)]\,[b_t - a_t p_t + \eta(\bar{r} - p_t)] \\
\text{subject to} \quad & |\tilde{r} - \alpha \bar{r} - (1-\alpha) p_t| \le \varepsilon, \;\; p_t \in [L, U],
\end{aligned} \tag{3.11}$$

where we assume the optimal value is $-\infty$ when the feasible set is empty. Observe that in this acyclic network, the length of a longest path from node (τ, r) to node $(\tilde{\tau}, \tilde{r})$ is exactly $\ell(\tau, r, \tilde{\tau}, \tilde{r}) - k_\tau$. Moreover, problem (3.11) can be solved in an $O(1)$ time because its objective function consists of two concave quadratic pieces and its feasible set is either an interval or empty.

Since the acyclic network $(\mathcal{V}^{\tau,r}, \mathcal{E}^{\tau,r})$ contains $O(TS_\varepsilon^2)$ links, it takes an $O(TS_\varepsilon^2)$ time to find a longest path from (τ, r) to all other nodes in the network. Because there are $O(TS_\varepsilon)$ feasible pairs (τ, r), we know all link lengths in the network $(\mathcal{V}, \mathcal{E})$ can be calculated in an $O(T^2 S_\varepsilon^3)$ time.

Similar idea can be applied to problem (3.9) with the same networks $(\mathcal{V}, \mathcal{E})$ and $(\mathcal{V}^{\tau,r}, \mathcal{E}^{\tau,r})$ but different link lengths. In particular, when computing the length of link $\langle (t, \bar{r}), (t+1, \tilde{r}) \rangle$, we replace problem (3.11) by

$$\begin{aligned}
\underset{p_t}{\text{maximize}} \quad & [p_t - c(\tau, t)]\,[b_t - a_t p_t + \eta(\bar{r} - p_t)] \\
\text{subject to} \quad & \tilde{r} = \alpha \bar{r} + (1-\alpha) p_t, \quad p_t \in [L, U],
\end{aligned}$$

which is still solvable in an $O(1)$ time. Thus, all link lengths in the network $(\mathcal{V}, \mathcal{E})$ can be constructed in $O(T^2 S_\varepsilon^3)$ time as well.

In summary, solving problem (3.8) or problem (3.9) is equivalent to finding a longest path in an acyclic network $(\mathcal{V},\mathcal{E})$, which consists of $O(TS_\varepsilon)$ nodes, $O(T^2S_\varepsilon^2)$ links and can be constructed in $O(T^2S_\varepsilon^3)$ time. Hence we end up with the following result.

Theorem 3.3. *(3.8) and (3.9) can be solved in an $O(T^2S_\varepsilon^3)$ time.*

If no fixed ordering cost incurs at any period, then the optimal ordering plan can be determined independent of the demand as discussed in Section 3.3. In this case, it is unnecessary to decide the ordering plan by finding a longest path in $(\mathcal{V},\mathcal{E})$. Instead, it suffices to determine the price sequence by finding a longest path in $(\mathcal{V}^{1,r_1},\mathcal{E}^{1,r_1})$ from node $(1,r_1)$ to node $(T+1,*)$. The only modification is to replace $c(\tau,t)$ by $\bar{c}_t$ when we compute all link lengths in (3.10), where $\bar{c}_t$ denotes the marginal ordering and inventory holding cost associated with the optimal ordering plan.

Corollary 3.1. *If $k_t = 0$ for all t, problem (3.8) and problem (3.9) can be solved in $O(TS_\varepsilon^2)$ time.*

Finally, we provide the lower and upper bounds for V^*, the optimal objective value of problem (3.1), based on V^ε, V_0^ε and the problem parameters. Note that the bounds depend on the magnitude of η^+, η^-, L, U, c_t and h_t but not the fixed cost k_t.

Theorem 3.4. $\max\{V^\varepsilon - C_T^-\varepsilon, V_0^\varepsilon\} \le V^* \le V^\varepsilon + C_T\varepsilon$*, where*

$$\begin{aligned} C_T &= \tfrac{T}{2}\max\{\eta^+,\eta^-\}\max_{\tau\le t\le T+1}\max\left\{|U-c(\tau,t)|,\left|L-c(\tau,t)\right|\right\},\\ C_T^- &= 2\min\{(1-\alpha)^{-1},T\}C_T \end{aligned}$$

Remark 3.1. *The heuristic actually works for arbitrary demand models $d_t = d_t(p_t,r_t)$ besides the one given by (3.1c). In fact, discussions are almost the same by properly replacing the objective functions of optimization problems in this section. Depending on the efficiency to solve the one dimensional optimization problem (3.11), the computational complexity to solve (3.8) may be different. Similarly, depending on the specific expression of $d_t(p_t,r_t)$, the bounds for V^* may also differ slightly.*

3.5 Numerical study

In this section we conduct a numerical study to see how the firm's profit varies with parameters in problem (3.1) and the performance of the heuristic given in Section 3.4.

Consider a 10-period instance (i.e. $T = 10$) with the initial reference price $r_1 = 10$, the variable ordering cost $c_t = 4$, the holding cost $h_t = 1$, the fixed ordering cost $k_t = 15$ (the cost parameters are based on our experience with a large retailing company), the lower bound $L = 5$ and the upper bound $U = 15$. In addition, demand functions and reference price effect functions are given as follows:

$$\begin{aligned} d_t(p, r) &= (1 - \beta)(b - a_t p) + \beta\eta(r - p), \\ \eta(z) &= (1 - \lambda)\max\{z, 0\} + \lambda \min\{z, 0\}, \end{aligned}$$

where $0 \leq \beta, \lambda \leq 1, b = 20$ and

$$\{a_1, a_2, \cdots, a_{10}\} = \{2, 2, 1.5, 1.5, 1.5, 1.5, 1, 1, 1, 1\}.$$

In the above demand model, parameter β controls the relative contribution of reference price effect. Observe that when $\beta = 0$, demand at a period only depends on the price of the current period and reference prices do not play a role at all.

In the above reference price effect function, parameter λ controls the relative magnitude of loss and gain. Specifically, the ratio $\frac{\lambda}{1-\lambda}$ corresponds to η^-/η^+ in model (3.1). Note that when $\lambda > 0.5$, consumers exhibit loss aversion; when $\lambda = 0.5$, consumers are loss neutral; and when $\lambda < 0.5$, consumers exhibit loss-seeking behavior. Different values of λ have been reported from empirical studies. For instance, Greenleaf (1995) identify $\frac{\lambda}{1-\lambda} = \frac{1}{5.4}$ or $\lambda \approx 0.16$ in a study involving a national brand of peanut butter, Hardie et al. (1993) find that $\frac{\lambda}{1-\lambda} = 1.457$ or $\lambda \approx 0.65$ in a refrigerated orange juice purchase investigation, and Putler (1992) observes $\frac{\lambda}{1-\lambda} = 2.4$ or $\lambda \approx 0.7$ by studying egg sales.

We are interested in understanding how the parameters α, β and λ affect the firm's profit. Thus, in our study, we consider different combinations of the three parameters with $\alpha \in \{0.05n : 0 \leq n \leq 19\}, \beta \in \{0.1m : 1 \leq m \leq 4\}$

and $\lambda \in \{0, 0.16, 0.65, 0.7, 1\}$, which amounts to a total of 400 cases. Since reference price effect is usually dominated by direct price effect, we restrict $\beta \leq 0.5$. In the candidate set for λ, we include the values obtained by Greenleaf (1995); Hardie et al. (1993) and Putler (1992), as well as two special cases $\lambda = 1$ and $\lambda = 0$, which represent the gain-insensitive case and loss-insensitive case respectively.

We apply the heuristic developed in the previous section to derive an approximation to problem (3.1). We first decide the number of reference price candidates needed to achieve an acceptable accuracy level, which is measured by the ratio $(V_+ - V_-)/V$, where $V = V^\varepsilon$, $V_+ = V + C_T\varepsilon$ and $V_- = V_0^\varepsilon$ are the upper and lower bounds to the optimal objective value of problem (3.1) respectively. (In our computation, V_0^ε is always above $V - C_T^-\varepsilon$, another lower bound derived in Theorem 3.4.) For all the 400 instances, we set $\mathcal{R}_\varepsilon = \{5 + 0.1n : 0 \leq n \leq 100\}$, i.e., $\varepsilon = 0.1$ and 101 reference price candidates are included. The accuracy ratio is reported in Figure 3.3 for all possible combinations of α and (β, λ). From the figure, it appears that the accuracy deteriorates as α and/or β increase. However, only in one instance $(\beta, \lambda, \alpha) = (0.4, 1, 0.95)$, the ratio slightly exceeds 6%, and in most cases it is less than 3%. This implies that for our problem instance, 101 reference price levels provide reasonable accuracy.

To see how the accuracy ratio changes with respect to the number of reference price levels, we consider a typical instance with parameters $(\beta, \lambda, \alpha) = (0.4, 0.65, 0.9)$. Figure 3.4 illustrates how the profit $V = V^\varepsilon$ derived from the heuristic, as well as the upper bound $V_+ = V^\varepsilon + C_T\varepsilon$ and lower bound $V_- = V_0^\varepsilon$, vary as S_ε increases, where $S_\varepsilon \in \{1 + 10n : 1 \leq n \leq 25\}$.

From Figure 3.4, we see that V_+ and V_- roughly approximate V monotonically as S_ε increases. The estimated error $V_+ - V_-$ decreases rapidly as more candidates of reference price levels are included at first and then s-tays relatively flat when the number of reference price levels exceeds 101. Thus, in our remaining study, we restrict the reference price levels to set $\mathcal{R}_\varepsilon = \{5 + 0.1n : 0 \leq n \leq 100\}$.

Interestingly, our numerical study shows that for this instance, the optimal ordering plan is independent of the number of reference price levels. We also observe that optimal ordering plans remain quite stable for many additional problem instances we tested. This observation thus motivates a new heuristic. In the new heuristic, we first solve problem (3.8) with a small S_ε to get

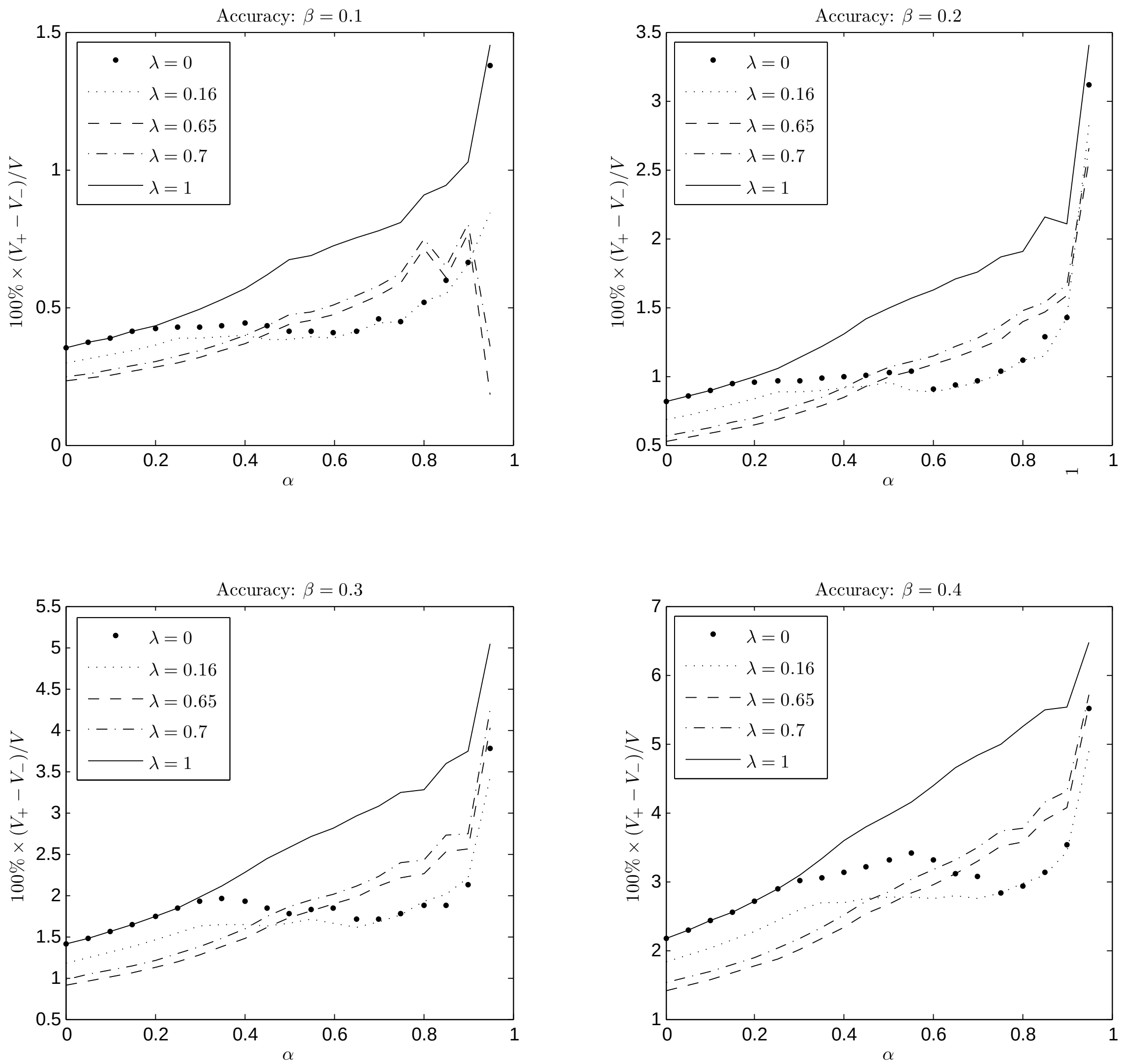

Figure 3.3: Accuracy with respect to input parameters α, β and γ

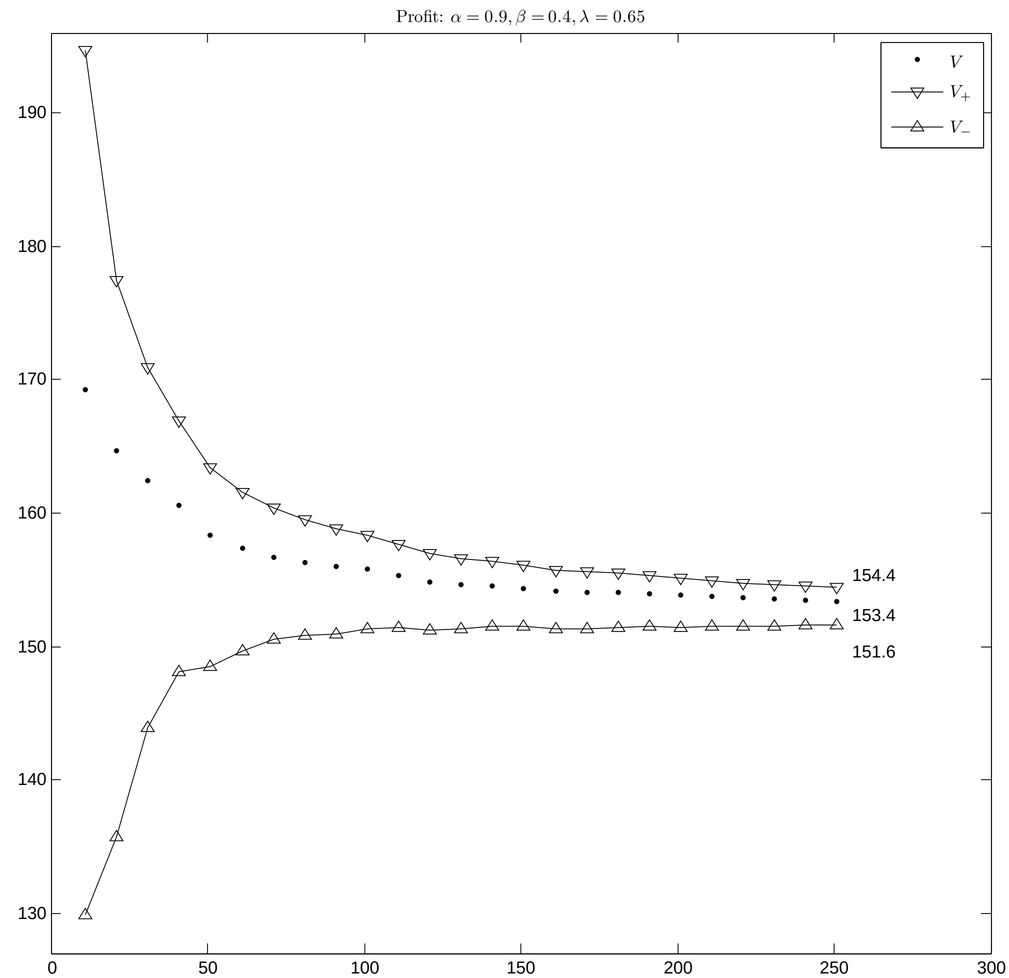

Figure 3.4: Profit with respect to the number of reference price levels

an ordering plan. We then fix the ordering plan and solve problem (3.8) using larger S_ε to get higher accuracy. Once the ordering plan is fixed, the complexity of solving problem (3.8) is only $O(TS_\varepsilon^2)$ instead of $O(T^2S_\varepsilon^3)$ for the general problem as stated in Corollary 3.1. The approach developed in Section 3.3 can also be used to reduce the computational complexity if the conditions are satisfied.

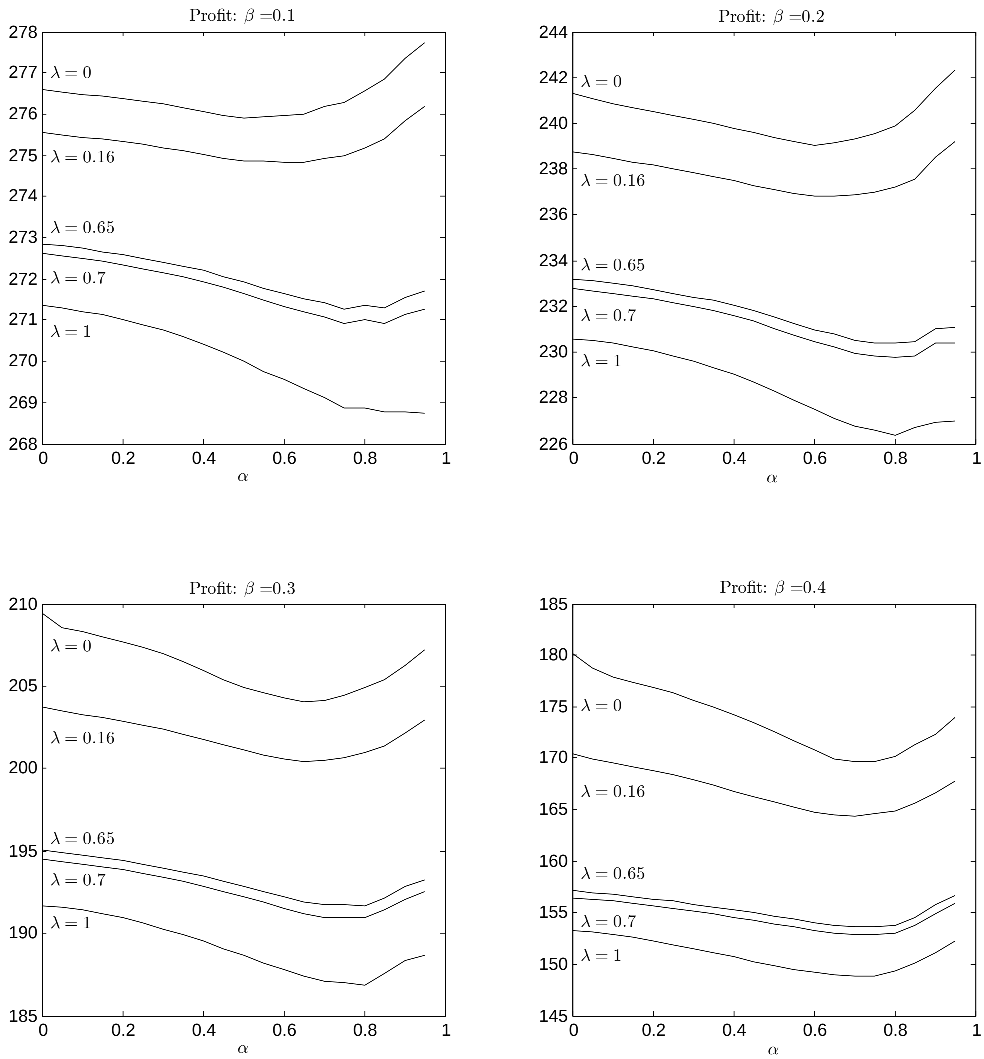

Figure 3.5: Profit with respect to all combinations of α, β and λ

We now illustrate how profit changes with respect to the parameters β, λ and α. The profits V calculated by our heuristic are reported in Figure 3.5 for all possible combinations of β and (α, λ). Several observations can be made from the figure.

1. For any combination (β, λ), profit does not depend on α monotonically. Roughly speaking, it decreases first and then increases in α. It also appears that profit is more sensitive to α when α approximates to 1.

2. For any fixed α and β, profit decreases as λ increases. This is not surprising as larger λ implies lower demand when the prices and reference prices are kept unchanged.

3. For any fixed α and λ, profit decreases as β increases. This observation follows from the fact that current price effect dominates reference price effect.

4. Profit is more sensitive to β compared with other factors. For example, profit is larger than 268 when $\beta = 0.1$ and drops to no more than 180 when $\beta = 0.4$. By contrast, profit varies less than 30 for each fixed β.

Throughout this chapter we assume that pricing and ordering decisions are simultaneously determined. However, in practice there are still significant barriers toward the integration of pricing and inventory decisions due to traditional organizational structures within a firm, even though great benefits of integration have been well demonstrated in various settings (see, for instance, Chen et al., 2004; Deng and Yano, 2006; Federgruen and Heching, 1999). In the following, we provide further evidence to demonstrate the potential of coordinated vs. sequential decision making when reference price effect is incorporated. We refer to Gimpl-Heersink (2008) for a similar study with reference price effect in stochastic settings but zero fixed ordering cost.

We first describe the sequential decision making process that is commonly seen in practice. In the process, the marketing department first makes pricing decision so as to maximize total revenue specified by the following problem.

$$\begin{aligned} R_0^s = \quad & \underset{p_t, r_t : t \leq T}{\text{maximize}} && p_1 d_1 + p_2 d_2 + \cdots + p_T d_T \\ & \text{subject to} && d_t = b_t - a_t p_t + \eta(r_t - p_t), && t = 1, \cdots, T, \\ & && r_{t+1} = \alpha r_t + (1-\alpha) p_t, \quad p_t \in [L, U], && t = 1, \cdots, T. \end{aligned}$$

Once the optimal prices and consequently the associated demand sequence $d_1^s, d_2^s, \cdots, d_T^s$ become available, the production/distribution department then takes the pricing decision from the marketing department as given and minimizes its inventory related cost given by the following optimization problem.

$$
\begin{aligned}
C_0^s = \quad & \underset{y_t : t \le T}{\text{minimize}} && \sum_{t=1}^{T} [k_t \delta(y_t) + c_t y_t + h_t I_t] \\
& \text{subject to} && I_t = I_{t-1} + y_t - d_t^s, \qquad t = 1, \cdots, T, \\
& && I_0 = 0, \;\; I_t \ge 0, \;\; y_t \ge 0, \qquad t = 1, \cdots, T.
\end{aligned}
$$

We apply the algorithms developed in Section 3.3 to compute the total revenue R_0^s if the technical conditions are satisfied; otherwise we turn to Section 3.4 to approximate R_0^s. The total cost C_0^s is simply given by the classical economic lot sizing problem and can be solved efficiently.

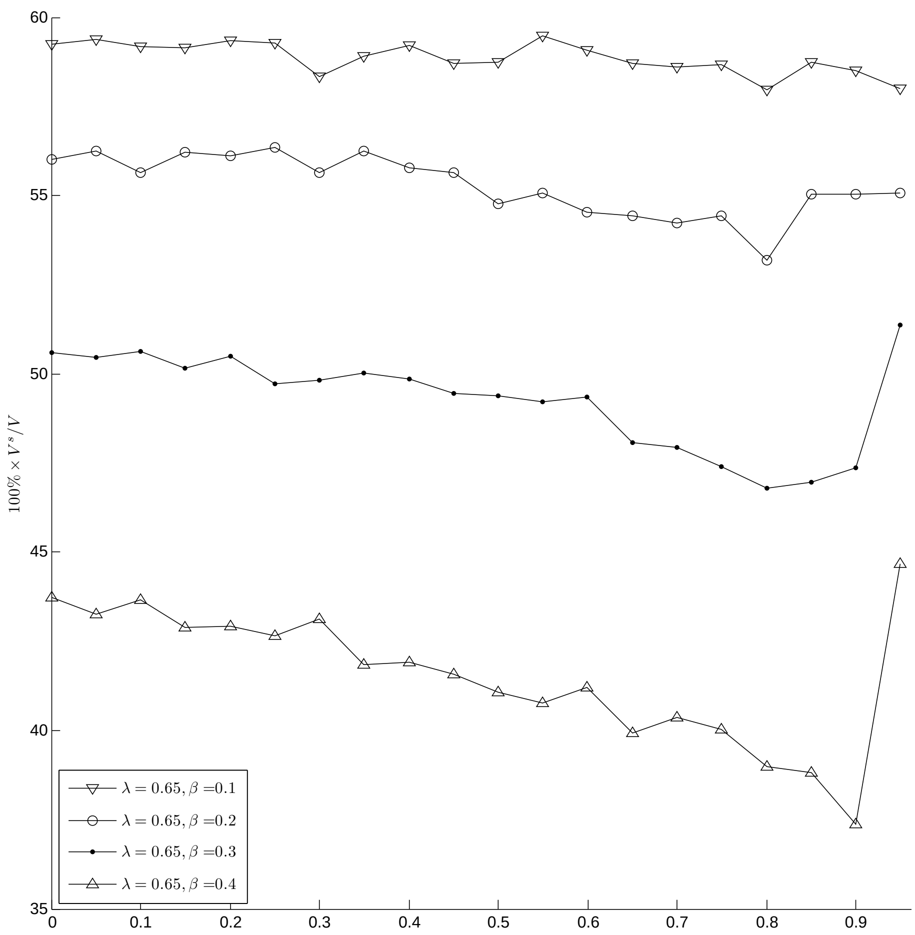

Figure 3.6: Sequential versus coordinated optimization

Figure 3.6 reports the ratio of V^s/V (in the y-axis), where V^s is the (estimated) total profit $R_0^s - C_0^s$ and V is the estimated profit in the coordinated optimization problem (3.8). In the figure, we select four combinations of (λ, β), namely, $\{(0.65, 0.1), (0.65, 0.2), (0.65, 0.3), (0.65, 0.4)\}$, and consider these ratios with respect to α in the x-axis. As can be seen from the figure, all ratios V^s/V are less than 60%, which implies that more than 40% profit could be potentially gained by coordinating pricing and inventory management decisions. Figure 3.6 also demonstrates that for fixed λ and α, the ratio of V_s/V decreases as β becomes larger. It means that more profit improvement can be obtained from pricing and inventory coordination when reference price effect becomes more important. For other combinations of (λ, β), we observe similar trends with respect to α. Note that due to the accuracy of our heuristic, the ratios vary greatly when α is close to 1.

3.6 Conclusion

In this chapter we propose and analyze an coordinated pricing and inventory model of a single product incorporating reference price effect in a deterministic setting. In this model, demand depends on both current selling price and reference price, where the latter evolves according to an exponentially smoothing process of past prices. When there is no fixed ordering cost, we develop strongly polynomial time algorithms under certain technical conditions. For the general setting with fixed ordering cost and kinked reference price effect functions, we develop efficient heuristic.

Our numerical study illustrates that incorporating reference price effect into pricing and inventory models have a great potential in improving firms' profit. Specifically, we demonstrate that the more reference price effect contributing to demand, the larger the profit can be gained through pricing and inventory integration. Indeed, in our study, the profit gains are more than 40% in all instances and 50% in most instances.

There are several interesting research questions. First, further improvement of the current algorithms and heuristic is important. Efficient algorithms become pertinent as we incorporate these models into decision support systems which usually involve many products.

Secondly, allowing capacity constraint may significantly complicate algorithm design when reference price effect is present. Indeed, it is an open question whether our model with capacity constraint and zero fixed ordering cost is NP-hard or not.

Thirdly, it is possible to extend our model and results to more general demand functions. We choose to focus on piecewise linear demand functions simply because they are relatively simple for implementation and have been empirically validated in a variety of settings. Nevertheless, as mention in Remark 3.1, the heuristic provided in Section 3.4 works for arbitrary demand models, where the computational complexity and bounds of the estimated errors could be different.

Finally, we would like to extend our model to settings with multi-products demand. Such models are not only quite relevant in practice but also challenging. Indeed, in settings with multiple products, it is not clear how dynamic pricing strategies affect the aggregated demand of a category of products and more empirical studies are necessary.

Chapter 4

Reference Price Effect Model with Stochastic Demand

4.1 Problem description

This chapter discusses the stochastic version of the reference price model in Chapter 3 when customers are loss-neutral or loss averse. Specifically, consider a periodic-review stochastic coordinated pricing and inventory management problem over a planning horizon with T period, where pricing and ordering decisions are simultaneously made at the beginning of each period. Demand of a period is *stochastic* and determined by the current selling price p_t and the reference price r_t, where p_t belongs to a closed interval $[L, U]$. And as the same settings in Chapter 3, reference prices evolve according to the memory-based model for some $0 \leq \alpha < 1$,

$$r_{t+1} = \alpha r_t + (1 - \alpha) p_t, \quad \forall 0 \leq t < T.$$

In addition, we assume that r_1, predetermined at the beginning of the planning horizon, belongs to the same feasible set of selling prices implying that $r_t \in [L, U]$ for all t.

Similar to the deterministic version studied in Chapter 3, given selling price p and reference price r at period t, we assume the expected demand d is

$$d = b_t - a_t p + \eta(r - p),$$

where $b_t, a_t \geq 0$, and for some $\eta^+, \eta^- \geq 0$,

$$\eta(z) = \eta^+ \max\{z, 0\} + \eta^- \min\{z, 0\}.$$

Recall that customers are called loss-averse if $\eta^- > \eta^+$, loss-seeking if $\eta^- < \eta^+$ and loss-neutral if $\eta^- = \eta^+$. In this chapter we will focus on the loss-averse or loss-neutral case, i.e., $\eta^+ \le \eta^-$.

Given the expected demand d, the associated realized demand, denoted by $D(d, \varepsilon_t)$, is

$$D(d, \varepsilon_t) = d\varepsilon_{m,t} + \varepsilon_{a,t},$$

where random variables $\varepsilon_t = (\varepsilon_{m,t}, \varepsilon_{a,t})$ are independent across the time index and satisfy that $\varepsilon_{m,t} > 0$, $\mathbb{E}[\varepsilon_{m,t}] = 1$ and $\mathbb{E}[\varepsilon_{a,t}] = 0$. Moreover, we assume $D(d, \varepsilon_t) \ge 0$ for all feasible p and r with probability 1. This model is known as the additive model when $\varepsilon_{m,t} = 1$, and it is known as the multiplicative model when $\varepsilon_{a,t} = 0$. Similar to existing literature on joint inventory and pricing optimization, such as Chen and Simchi-Levi (2011), some of our results depend on whether demand follows the additive or the multiplicative model.

Suppose that in each period, the order is received immediately with a per unit cost c_t. Because of uncertainties, realized customer demand may be larger than or smaller than the firm's inventory level in the period. We assume unsatisfied demand is backlogged and will be finally fulfilled with selling price in the period when it incurs, and unused inventory is carried over to the next period. Denote by $h_t(x)$ the inventory holding (if $x \ge 0$) and backlogging (if $x \le 0$) cost associated with the inventory level x. Assume that

$$h_t(x) = \begin{cases} h_t^-(-x) & \text{if } x \le 0, \\ h_t^+(x) & \text{if } x \ge 0, \end{cases},$$

where both $h_t^-(x)$ and $h_t^+(x)$ are increasing convex functions such that $h_t^-(x) = h_t^+(x) = 0$ for all $x \le 0$. In addition, suppose $c_t d \le c_{t+1} d + h_t^-(d)$ and $(L - c_{t+1})d - h_t^-(d)$ is increasing in d when $d \ge 0$. The two assumptions are commonly used in literature, where the inequality implies that there is no incentive to backlog demand, and the monotonicity assumption means that more profit can be obtained by selling more.

The objective is to find an ordering and pricing policy so as to maximize the total expected profit over the planning horizon. Let $v_{T+1}^0(x, r) = 0$ and $v_t^0(x, r)$ be the profit-to-go function at the end of period t associated with the inventory level x and reference price r. We can write down the dynamic programming recursion for $v_t^0(x,r)$, $t=1,\cdots,T$, as

$$
\begin{aligned}
v_t^0(x,r) = \quad & \underset{p,d,\tilde{r},y}{\text{maximize}} && [pd - c_t(y-x)] - \mathbb{E}h_t(y-\varepsilon_{m,t}d-\varepsilon_{a,t}) \\
& && +\mathbb{E}v_{t+1}^0(y-\varepsilon_{m,t}d-\varepsilon_{a,t},\tilde{r}) \\
& \text{subject to} && d = b_t - a_t p + \eta(r-p) \\
& && \tilde{r} = \alpha r + (1-\alpha)p, \quad y \geq x, \ L \leq p \leq U.
\end{aligned}
$$

where y denotes the order-up-to inventory level in the current period.

To our best knowledge, there are only a few papers that integrate stochastic inventory models with reference price effects. Urban (2008) analyzes a single-period joint inventory-and-pricing model with both symmetric and asymmetric reference price effect, and provide numerical analysis which indicates that accounting for reference prices has a substantial impact on the firm's profitability. Gimpl-Heersink (2008) analyzes a special case of the demand model with only additive uncertainty. Recognizing that the single-period expected revenue is not jointly concave as a function of the selling price and the reference price, they prove that a base-stock policy is optimal for one-period and two-period cases.

In this chapter, we extend Gimpl-Heersink (2008)'s results by analytically proving the optimality of the base-stock policy in a model with any number of periods for the general demand model when customers are either loss-neutral or loss-averse. There are two significant technical challenges in our model. The first one is that the single-period expected revenue fails to be jointly concave as a function of the selling price and the reference price. This challenge can be conquered by using the same transformation technique as the previous chapter such that a modified revenue function is jointly concave. The second one is the feasible set of the main problem is not convex due to the constraint $d = b_t - a_t p + \eta(r-p)$ for a general concave $\eta(z)$. Notice that Guler et al. (2010) consider a similar dynamic program as ours and face the same challenge. However, to circumvent such difficulty, they in fact replace the constraint by a much simpler one. It leads to a formulation which is not equivalent to the original problem. Under some reasonable conditions we manage to prove that it loses no optimality to properly modify the feasible set such that our main problem becomes a concave maximization problem. This allows us to prove that a reference price dependent base-stock policy is optimal for a general concave $\eta(z)$. When customers are loss-neutral and demand models involve only an additive random perturbation, we further prove that a higher reference price leads to a higher optimal base-stock level. Moreover, in the infinite horizon setting, we focus on the model when customers are loss-neutral. We prove that under some conditions on system parameters the reference price converges to some steady state in the optimal trajectory. Characterizations to the steady states are also provided.

The rest of this chapter is organized as follows. In Section 4.2, we consider the finite horizon problem and characterize the structure of the optimal policy. In Section 4.3, we analyze the infinite horizon counterpart. Finally, Section 4.4 concludes the paper and points to interesting topics for future research.

4.2 Finite horizon model

In this section, we focus on the finite horizon model when $\eta^+ \leq \eta^-$ in the demand model, that is, customers are either loss-neutral or loss-averse. Notice that $\eta(z)$ is a concave function in this case.

Similar to Chapter 3, at first we reformulate the main problem such that the objective function is jointly concave. Specifically, for each $1 \leq t \leq T$, define

$$v_t(x, r) = v_t^0(x, r) - A_t r^2 - c_t x,$$

where the constant A_t is given in Proposition 3.2. Let the expected demand d, relative difference $z = r - p$, order-up-to inventory level y and the reference price in the next period $\tilde{r}$ be decision variables. Then the selling price $p = \tilde{r} - \alpha z$, the current reference price $r = \tilde{r} + (1-\alpha)z$ and the original problem can be expressed by

$$v_t(x, r) = \underset{d,y,z,\tilde{r}}{\text{maximize}} \ \{\pi_t(z, \tilde{r}) + w_t(y, d) + \mathbb{E}[v_{t+1}(y - \varepsilon_{m,t} d - \varepsilon_{a,t}, \tilde{r})]\} \tag{4.1a}$$

$$\text{subject to } d = b_t - a_t(\tilde{r} - \alpha z) + \eta(z), \tag{4.1b}$$

$$\tilde{r} + (1-\alpha)z = r, \tag{4.1c}$$

$$y \geq x, \quad L \leq \tilde{r} - \alpha z \leq U, , \tag{4.1d}$$

where $\pi_t(z, \tilde{r})$ and $-w_t(y, d)$ defined below can be interpreted as the transformed one-period expected revenue and expected cost, respectively:

$$\begin{aligned}
\pi_t(z, \tilde{r}) &= [(\tilde{r} - \alpha z) - L)][b_t - a_t(\tilde{r} - \alpha z) + \eta(z)] \\
&- A_t[\tilde{r} + (1-\alpha)z]^2 + A_{t+1}\tilde{r}^2, \\
w_t(y, d) &= (L - c_{t+1})d - (c_t - c_{t+1})y - \mathbb{E}[h_t(y - \varepsilon_{m,t}d - \varepsilon_{a,t})].
\end{aligned}$$

By Proposition 3.2, we know that π_t is jointly concave and supermodular when (3.4) holds. In addition, the following proposition characterizes the function $w_t(y, d)$.

Proposition 4.1. *$w_t(y, d)$ is jointly concave, supermodular and increasing when $y \leq 0$ and $d \geq 0$.*

Proof. It is straightforward to see the concavity and supermodularity of $w_t(y, d)$ from the convexity of h_t. We only need to verify its monotonicity.

Since that h_t is convex, $h_t(x) - h_t(x - z)$ is increasing in x for any fixed non-negative z. To see the monotonicity in y when $y \leq 0$, for any $\delta \geq 0$, from $D(d, \varepsilon_t) - y \geq 0$ we know that

$$\begin{aligned}
& w_t(y, d) - w_t(y - \delta, d) + (c_t - c_{t+1})\delta \\
=\ & -\mathbb{E}[h_t(y - D(d, \varepsilon_t)) - h_t(y - \delta - D(d, \varepsilon_t)] \\
=\ & -\mathbb{E}[h_t^-(D(d, \varepsilon_t) - y) - h_t^-(D(d, \varepsilon_t) - y + \delta] \\
\geq\ & -[h_t^-(0) - h_t^-(\delta)] \geq 0,
\end{aligned}$$

where the last inequality holds by $c_t d \leq c_{t+1}d + h_t^-(d)$ for all $d \geq 0$.

To see the monotonicity in d, denote $\tilde{x} = \min\{y - D(d, \varepsilon_t), 0\}$. Then for any $\delta \geq 0$, by the monotonicity of $(L - c_{t+1})d - h_t^-(d)$ on $d \geq 0$,

$$\begin{aligned}
w_t(y, d + \delta) - w_t(y, d) &= -(c_{t+1} - L)\delta - \mathbb{E}[h_t(y - d_t^\varepsilon - \delta\varepsilon_{m,t}) - h_t(y - d_t^\varepsilon)] \\
&\geq -\mathbb{E}[(c_{t+1} - L)\delta\varepsilon_{m,t} + h_t(\tilde{x} - \delta\varepsilon_{m,t}) - h_t(\tilde{x})] \\
&= \mathbb{E}[(c_{t+1} - L)\delta\varepsilon_{m,t} + h_t^-(\delta\varepsilon_{m,t} - \tilde{x}) - h_t^-(-\tilde{x})] \geq 0.
\end{aligned}$$

We then complete this proof. □

We are ready to present the main theorem in this section.

Theorem 4.1. *Suppose* (3.4) *holds. Then* $v_t(x,r)$ *is decreasing in* x *and jointly concave. Moreover, a non-negative and reference price dependent base-stock policy is optimal, i.e., there exists* $y_t(r) \geq 0$ *such that it is optimal to raise the inventory up to* $y_t(r)$ *if* $x \leq y_t(r)$*, and no order is placed otherwise.*

Proof. Since the objective function is independent on x, $v_t(x,r)$ is decreasing in x for each t. It remains to show its concavity, which is straightforward when $t = T+1$. The statement will be proved inductively.

Suppose $v_{t+1}(x,r)$ is jointly concave. Because that the objective function (4.1a) is increasing in y and d when $y \leq 0 \leq d$ by Proposition 4.1, it leads no loss of optimality to restrict $y \geq 0$ and replace the equality in (4.1b) by inequality, that is,

$$\begin{aligned} v_t(x,r) = \quad & \underset{y,z,\tilde{r}}{\text{maximize}} \quad && \pi_t(z,\tilde{r}) + w_t(y,d) + \mathbb{E}[v_{t+1}(y - \varepsilon_{m,t} d - \varepsilon_{a,t}, \tilde{r})] \\ & \text{subject to} && d \leq b_t - a_t(\tilde{r} - \alpha z) + \eta(z), \\ & && \tilde{r} + (1-\alpha)z = r, \\ & && L \leq \tilde{r} - \alpha z \leq U, \quad y \geq \max\{0, x\}. \end{aligned}$$

Because the feasible set is convex by the concavity of η, and the objective is concave by Proposition 4.1, it implies that v_t is concave by Proposition 2.3.9, Bertsekas et al. (2003). That a reference price-dependent base-stock ordering policy is optimal can be proved by routine techniques. $\square$

An interesting question is how the optimal base-stock level $y_t(r)$ depends on the reference price r. Gimpl-Heersink (2008) proves that in the one-period case the optimal base-stock level is nondecreasing in r for the loss-neutral additive demand model. All our numerical studies also suggest that it holds in the multi-period case for general demand models. This is even true under a significant amount of multiplicative uncertainty. As an example, Figure 4.1 shows the optimal base-stock level vs. reference price when $\gamma = 0.99, \alpha = 0.15, c_t = 0.2, h(x) = 0.1\max\{x,0\} - 0.15\min\{x,0\}, b_t = 40, a_t = 20, \eta_t = 20$ and the variances of $\varepsilon_{a,t}$ and $\varepsilon_{m,t}$ are respectively 5 and 0.1.

Now we prove that in the multi-period case the optimal base-stock level is indeed nondecreasing in r for the loss-neutral additive demand model. A common and standard approach to prove this type of results is to employ Theorem 2.8.1 in Topkis (1998), which states that if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are lattices,

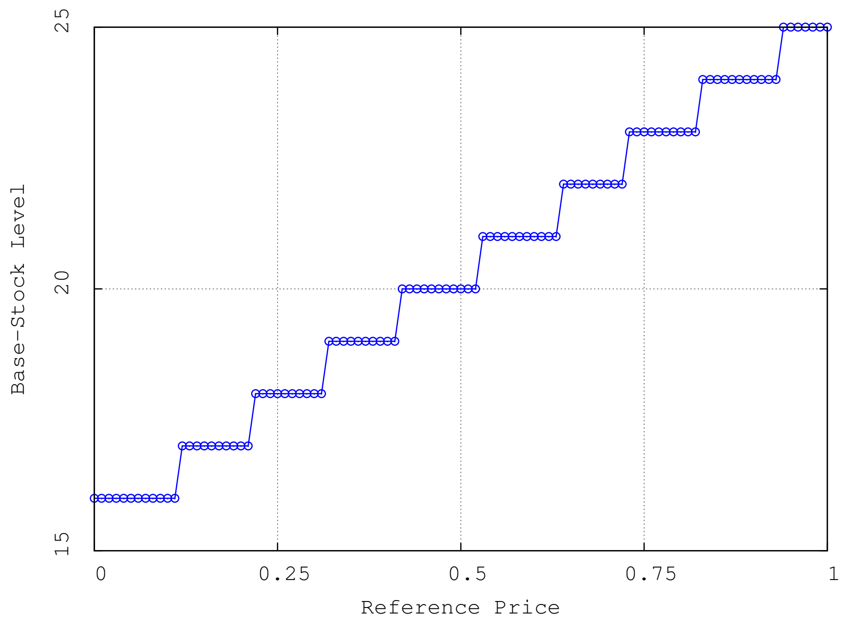

Figure 4.1: Base-Stock Level vs. Reference Price

$\boldsymbol{S}$ is a sublattice of $\boldsymbol{X} \times \boldsymbol{Y}$, and $g(\boldsymbol{x}, \boldsymbol{y})$ is supermodular in $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{S}$, then $f(\boldsymbol{x}) = \max_{\boldsymbol{y}}[g(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{S}]$ is supermodular on the set of $\boldsymbol{x}$ for which the maximization is well defined. The constraints in our dynamic programming model (4.1), however, do not seem to fit the needed sublattice condition. Nevertheless, Theorem 5.1, which will be introduced in the next chapter, provides a powerful tool for the analysis and leads to the following desirable results.

Theorem 4.2. *Suppose* (3.4) *holds. If the demand follows the loss-neutral and additive model, then in addition to the results stated in Theorem 4.1, we can select an optimal base-stock level* $y_t(r)$ *nondecreasing in reference price* r *for each* t*.*

Proof. We will show how to convert our problem into a format for which Theorem 5.1 can be applied.

Since that $\eta(z)$ is linear, say $\eta(z) = \eta z$, we can write the relative difference z given in (4.1b) as a linear function in terms of of d and $\tilde{r}$:

$$z = z_t(d, \tilde{r}) = \frac{d + a_t\tilde{r} - b_t}{\alpha a_t + \eta}.$$

Correspondingly, express the transformed one-period expected revenue in d

and $\tilde{r}$ as $\tilde{\pi}_t(d, \tilde{r}) = \pi_t(z_t(d, \tilde{r}), \tilde{r})$. That is,

$$\tilde{\pi}_t(d, \tilde{r}) = [\tilde{r} - \alpha z_t(d, \tilde{r}) - L]d + A_t[\tilde{r} + (1-\alpha)z_t(d, \tilde{r})]^2 - A_{t+1}\tilde{r}^2,$$

where $\tilde{\pi}_t(d, \tilde{r})$ is quadratic and concave by Proposition 3.2. In addition, calculation shows that the coefficient of the cross-term $d\tilde{r}$ is

$$\frac{\eta}{\alpha a_t + \eta} + 2(1-\alpha)A_t \frac{a + \eta}{(\alpha a_t + \eta)^2},$$

which is non-negative by the definition of A_t in Proposition 3.2. Therefore $\tilde{\pi}_t(d, \tilde{r})$ is also supermodular.

When demand is additive, $\varepsilon_{m,t} = 1$ for all t and (4.1) can be rewritten by

$$v_t(x, r) = \underset{y,q}{\text{maximize}} \ \{u_t(y, r) + (L - c_{t+1})y : y + q = x, q \leq 0\}, \tag{4.2a}$$

$$\begin{aligned} u_t(y, r) = \underset{d,\tilde{y},\tilde{r}}{\text{maximize}} \ & \tilde{\pi}_t(d, \tilde{r}) + \tilde{w}(\tilde{y}) + \mathbb{E}v_{t+1}(\tilde{y} - \varepsilon_{a,t}, \tilde{r}) \qquad (4.2b) \\ \text{subject to} \ & \tilde{y} + d = y, \\ & (a_t + \eta)\tilde{r} + (1-\alpha)d = (\alpha a_t + \eta)r + (1-\alpha)b, \\ & (a_t + \eta)L \leq b_t - d + \eta_t r \leq (a_t + \eta)U, \end{aligned}$$

where $-q$ and $\tilde{y}$ denote the ordering quantity and the expected inventory carried over to the next period, respectively, and the function $\tilde{w}_t$ defined below is concave by the concavity of h_t:

$$\tilde{w}_t(\tilde{y}) = -(L - c_{t+1})\tilde{y} - \mathbb{E}[h_t(\tilde{y} - \varepsilon_{a,t})].$$

Moreover, the last two constraints are obtained by substituting $z = z_t(d, \tilde{r})$ into (4.1c) and the second inequality in (4.1d).

Clearly (4.2) is a special case of the problem (5.1). If $v_{t+1}(y, r)$ is concave and supermodular, which can be verified straightforwardly when $t = T$, then so are $u_t(y, r)$ and $v_t(y, r)$ by Theorem 5.1. We then inductively prove that all u_t are supermodular. By Theorem 2.8.1, Topkis (1998), the optimal base-stock level $y_t(r)$ is nondecreasing in r. □

The intuition of the monotonicity of the optimal base-stock level is quite clear. In fact, in our demand model, higher reference price results in higher demand and thus it is reasonable to expect that the optimal base-stock level is also higher. Notice that in the special case without reference price effects (i.e, $\eta_t(z) = 0$), Federgruen and Heching (1999) show that under some conditions the list-price policy is optimal, i.e., the price is non-increasing in the inventory level of the firm. However, such policy may not be optimal in our problem. On one hand, when the inventory level is high, the firm should lower the price to increase the demand in order to clear inventory. On the other hand, the firm may raise the price to increase the reference prices and thus the demands in the future. This way, the firm cannot clear inventory in the current period, but can clear more inventory in future periods. Either effect can dominate depending on the demand function and customers sensitivity to the reference price effect.

4.3 Infinite horizon model

We next turn to the infinite horizon version of our model when customers are loss-neutral. We are interested in the asymptotic property of the optimal trajectory. To avoid tedious technicalities, we assume system parameters are stationary and focus the discounted profit case

$$\begin{aligned} v_t^0(x,r) = \quad & \underset{p,d,\tilde{r},y}{\text{maximize}} \quad && [pd - c(y-x)] - \mathbb{E}h(y - \varepsilon_m d - \varepsilon_a) \\ & && + \gamma \mathbb{E} v_{t+1}^0(y - \varepsilon_{m,t} d - \varepsilon_{a,t}, \tilde{r}) \\ & \text{subject to} && d = b - ap + \eta(r-p) \\ & && \tilde{r} = \alpha r + (1-\alpha)p, \quad y \geq x, \ L \leq p \leq U, \end{aligned}$$

where γ is the discount factor satisfying $0 < \gamma \leq 1$. Our goal is to study whether the optimal reference price path converges to some steady state, and to identify properties that characterize it.

It is worth noting that in loss-neutral case $\eta(z) = \eta z$, the expected demand d is linear in terms of the selling price p and the reference price r. Or equivalently we can express the selling price as

$$p = p(d,r) \triangleq \frac{b - d + \eta r}{a_t + \eta}.$$

In addition, when system parameters are stationary, (3.4) holds automatically and the constant A_t is too independent on t, i.e., we can denote $A_t = A$

for all t. Let

$$v_t(x,r) = v_t^0(x,r) - Ar^2 - cx.$$

Then the limit of v_t, denoted by $\varphi(x,r)$, is be determined by the Bellman equation as given in the following optimization problem:

$$\begin{aligned}
\varphi(x,r) = \quad \text{maximize} \quad & \pi(d,r) - \mathbb{E}h^\gamma(y - \varepsilon_m d - \varepsilon_a) \\
& + \mathbb{E}\left[\gamma\varphi(y - \varepsilon_m d - \varepsilon_a, q)\right], \qquad (4.3)\\
\text{subject to} \quad & q = \alpha r + (1-\alpha)p(d,r), \\
& L \le p(d,r) \le U, \ y \ge x,
\end{aligned}$$

where $h^\gamma(x) = h(x) + (1-\gamma)cx$ is the transformed holding and backlogging cost function, and

$$\pi(d,r) = [p(d,r) - c]d - Ar^2 + \gamma A[(1-\alpha)p(d,r) + \alpha r]^2$$

can be regarded as the modified single period expected revenue function. By Proposition 3.2, π is jointly concave and supermodular.

We call $\{(x_t, r_t) : t \ge 1\}$ a state path if (x_{t+1}, r_{t+1}) solves problem (4.3) when $(x,r) = (x_t, r_t)$ for any $t \ge 1$, and correspondingly $\{r_t : t \ge 1\}$ the reference price trajectory. Moreover, if $r_t = r^*$ implies that $r_{t+1} = r^*$, then we say r^* is a steady state.

Compared to most joint inventory-and-pricing models in the literature, our Bellman equation (4.3) has one more state variable r. This added dimension of state space brings significant challenge. Therefore, we first propose a simplification of the problem, prove results for this simplified version in Subsection 4.3.1, demonstrate how we can use it as an auxiliary tool to establish results for our original problem in Subsection 4.3.2, and then provide a characterization of the steady state in Subsection 4.3.3.

Before proceeding to the analysis, we introduce the following definitions.

$$\begin{aligned}
\omega(d) &= \underset{y}{\text{minimize}} \ \mathbb{E}[h^\gamma(y - \varepsilon_m d - \varepsilon_a)], \\
y(d) &= \arg\min_y \mathbb{E}[h^\gamma(y - \varepsilon_m d - \varepsilon_a]. \qquad (4.4)
\end{aligned}$$

The existence of $y(d)$ is ensured by the convexity of h^γ and the assumption $\lim_{|x|\to\infty} h^\gamma(x) = \infty$. If there are multiple optimal solutions, let $y(d)$ be the largest one. Lemma 4.1 below presents several properties of $\omega(d)$ and $y(d)$.

Lemma 4.1. *$\omega(d)$ is convex and $y(d)$ is increasing in d. In addition,*

(a) if the demand model is additive, then $y(d) = d + y_a$ and $\omega(d) = \omega_a$ for some y_a and ω_a;

(b) if the demand model is multiplicative, then $\omega(d)$ is increasing when $d \geq 0$; moreover, if $h^\gamma(x)$ is positive homogeneous with degree $\rho > 0$ (i.e., $h^\gamma(dx) = d^\rho h^\gamma(x)$ for any $d \geq 0$), then $\omega(d) = d^\rho \omega_m$ when $d \geq 0$, where $\omega_m \geq 0$ is the minimum of the function $\mathbb{E}h^\gamma(y - \varepsilon_m)$ over y.

Proof. The convexity of $\omega(d)$ follows from the convexity of $h^\gamma(x)$. By $\varepsilon_m \geq 0$, $\mathbb{E}[h^\gamma(y - \varepsilon_m d - \varepsilon_a)]$ is submodular in (y, d) hence $y(d)$ is increasing by Theorem 2.8.1, Topkis (1998).

(a) For the additive demand model, $\omega(d) = \min\limits_y \mathbb{E}[h^\gamma(y - d - \varepsilon_a)]$. Let y_a be the largest minimizer of the function $\mathbb{E}h^\gamma(y - \varepsilon_a)$ and $\omega_a = \min_y \mathbb{E}h^\gamma(y - \varepsilon_a)$. It is straightforward to show that $y(d) = d + y_a$ and $\omega(d) = \omega_a$.

(b) For the multiplicative demand model, $\omega(d) = \min_y \mathbb{E}[h^\gamma(y - \varepsilon_m d)] \geq \min_y [h^\gamma(y)] = 0$. Clearly $\omega(d)$ achieves its minimum at $d = 0$, which together with its convexity implies that $\omega(d)$ is increasing in d when $d \geq 0$. Furthermore, if $h^\gamma(x)$ is positive homogeneous with degree ρ, then

$$\omega(d) = d^\rho \min_y \mathbb{E}[h^\gamma(y/d - \varepsilon_m)] = d^\rho \min_y \mathbb{E}[h^\gamma(y - \varepsilon_m)] = d^\rho \omega_m.$$

We now complete the proof. □

4.3.1 When return is allowed

In this subsection, we study a simplified problem in which the retailer is allowed to return products back to the manufacturer and get a full refund. In this case the constraint $y \geq x$ disappears from the Bellman equation in (4.3). Thus, $\varphi(x, r)$ reduces to a function of r only, denoted by $\varphi^0(r)$, and it is given by the following optimality problem:

$$\begin{aligned} \varphi^0(r) = \quad & \underset{y,d}{\text{maximize}} \quad [R(d, r) - \mathbb{E}h^\gamma(y - D(d, \varepsilon)) + \gamma\varphi^0(q)], \\ & \text{subject to} \quad q = \alpha r + (1 - \alpha)p(d, r), \; L \leq p(d, r) \leq U \end{aligned}$$

Observe that allowing return reduces the dimension of the state space to one. In the following we denote by $(y^0(r), d^0(r))$ the optimal solution to the above problem. If there are multiple optimal solutions, we select the one with the largest $y^0(r)$.

It can be verified that the optimal order-up-to inventory level minimizes the expected single period holding and backlogging cost. Using equation (4.4), we have $y^0(r) = y(d^0(r))$, where $d = d^0(r)$ solves the following optimization problem

$$\begin{aligned} \varphi^0(r) = \quad & \underset{d}{\text{maximize}} \quad [R(d,r) - \omega(d) + \gamma\varphi^0(q)], \\ & \text{subject to} \quad q = \alpha r + (1-\alpha)p(d,r), \ L \leq p(d,r) \leq U. \end{aligned} \tag{4.5}$$

For technical convenience, we use q as the decision variable instead of d in further analysis. Thus, we denote by $\tilde{d}(q,r) = d\left(\frac{q-\alpha r}{1-\alpha}, r\right)$ and $\tilde{R}(q,r) = R(\tilde{d}(q,r), r)$ respectively the expected demand and transformed single period revenue in terms of q and r. Then, problem (4.5) can be reformulated as

$$\varphi^0(r) = \underset{q \in \mathcal{Q}(r)}{\text{maximize}} \left\{ \tilde{R}(q,r) - \omega(\tilde{d}(q,r)) + \gamma\varphi^0(q) \right\}, \tag{4.6}$$

where $\mathcal{Q}(r) = \{(1-\alpha)p + \alpha r : L \leq p \leq U\}$.

Before showing the uniqueness of steady state for the simplified model, we first present several structural results for a slightly generalized setting, which will be useful for our analysis.

Lemma 4.2. *Consider the following problem*

$$\psi(r) = \underset{q \in S_\varepsilon(r)}{\text{maximize}} \ \{\theta(q,r) + \gamma\psi(q)\}, \quad \forall L \leq r \leq U, \tag{4.7}$$

where for any $L \leq r \leq U$, $S_\varepsilon(r)$ is a subset of $[L,U]$ and the function θ is continuous on $S_\varepsilon = \{(q,r) : q \in S_\varepsilon(r), L \leq r \leq U\}$ assumed to be compact. Let $q(r)$ be the largest optimal solution for problem (4.7).

(a) If S_ε is a convex sublattice of $\Re^2$ and $\theta(q,r)$ is supermodular on S_ε, then $q(r)$ is nondecreasing and the reference price trajectory $\{r_t : t \geq 1\}$ of problem (4.7) *monotonically converges to a steady state.*

(b) If problem (4.7) *admits a steady state $r^* \in [L,U]$ then for sufficiently*

small positive δ,

$$\tfrac{1}{1-\gamma}\theta(r^*, r^*) = \underset{r\in[L,U]:|r-r^*|\le\delta}{\text{maximize}} \left\{\theta(r, r^*) + \tfrac{\gamma}{1-\gamma}\theta(r, r)\right\}.$$

Proof. Popescu and Wu (2007) consider a model similar to (4.6) without inventory related cost (i.e., $\omega(d) = 0$ and $c = 0$) and prove a result similar to part (a). Here we briefly mention the basic idea. Note that the monotonicity of $q(r)$ is an direct application of Theorem 2.8.1, Topkis (1998) since the objective function in problem (4.7) is supermodular and the set S_ε is a sublattice. It, together with the fact that r belong to the compact set $[L, U]$, implies that a reference price trajectory monotonically converges.

We now prove part (b), notice that $\{r^*, r, r, \cdots, r, \ldots\}$ gives a feasible and sub-optimal reference price trajectory whenever $r \in [L, U]$ and $|r - r^*|$ is sufficiently small. The corresponding total value $\theta(r, r^*) + \frac{\gamma}{1-\gamma}\theta(r, r)$ is no more than $\psi(r^*) = \frac{1}{1-\gamma}\theta(r^*, r^*)$, the value associated with the optimal reference price trajectory path $\{r^*, r^*, r^*, \cdots\}$. □

We now apply Lemma 4.2 to problem (4.6).

Theorem 4.3. *If return is allowed and demand is the loss-neutral, then reference price trajectory monotonically converges to a unique steady state r^*. Moreover, $d^* = b - ar^*$ is the unique optimal solution to the convex minimization problem*

$$\underset{d\in\mathcal{D}}{\text{minimize}} \ \left\{\omega(d) + \tfrac{1}{2}Bd^2 + Cd\right\}, \tag{4.8}$$

where $\mathcal{D} = \{b - ap : p \in [L, U]\}$, scalars $B = a^{-1} + \left(a + \eta\frac{1-\gamma}{1-\alpha\gamma}\right)^{-1} > 0$ and $C = c - a^{-1}b$.

Proof. Notice that in the loss-neutral case the expected demand $\tilde{d}(q, r)$ and the revenue $\tilde{R}(q, r) = R(\tilde{d}(q, r), r)$ have the expressions as follows:

$$\begin{aligned}
\tilde{d}(q, r) &= d\left(\tfrac{q-\alpha r}{1-\alpha}, r\right) = b - \tfrac{a+\eta}{1-\alpha}q + \tfrac{a\alpha+\eta}{1-\alpha}r, \\
\tilde{R}(q, r) &= \left(\tfrac{q-\alpha r}{1-\alpha} - c\right)\left(b - \tfrac{a+\eta}{1-\alpha}q + \tfrac{a\alpha+\eta}{1-\alpha}r\right) - \lambda r^2 + \gamma\lambda q^2,
\end{aligned}$$

Clearly $\tilde{R}(q, r)$ is supermodular and concave because $R(d, r)$ is jointly concave and $\tilde{d}(q, r)$ is linear. Moreover, because $\omega(d)$ is convex by Lemma 4.1 and

$\tilde{d}(q,r)$ is linear, decreasing in q and increasing in r, we know that $\omega(\tilde{d}(q,r))$ is submodular and convex in (q,r). Therefore the transformed expected single-period profit

$$\theta(q,r) = \tilde{R}(q,r) - \omega(\tilde{d}(q,r))$$

is concave and supermodular. The existence of a steady state immediately follows from Lemma 4.2(a).

To prove the necessary condition on a steady state, define

$$\Theta(q,r) = \theta(q,r) + \frac{\gamma}{1-\gamma}\theta(q,q).$$

By Lemma 4.2(b) we know that for a given steady state r^* and a sufficiently small $\delta > 0$,

$$\Theta(r^*, r^*) \geq \Theta(q, r^*), \quad \forall\, q \in [L, U] : |q - r^*| \leq \delta. \tag{4.9}$$

Observe that $\Theta(q,r)$ is jointly concave in (q,r). Let $\partial_q^+\Theta(q,r)$ and $\partial_q^-\Theta(q,r)$ be the right- and left-derivatives of $\Theta(q,r)$ at q for a given r, respectively. It is well-known from convex analysis (see, e.g., Rockafellar, 1970) that both $\partial_q^+\Theta(q,r)$ and $\partial_q^-\Theta(q,r)$ exist; moreover, the above inequality is equivalent to

$$\begin{cases} \partial_q^-\Theta(r^*, r^*) \geq 0, & \text{if } r^* > L, \\ \partial_q^+\Theta(r^*, r^*) \leq 0, & \text{if } r^* < U. \end{cases}$$

Recall that $d^* = b - ar^*$. Next we show that

$$\begin{cases} \partial_q^-\Theta(r^*, r^*) \geq 0 \text{ if and only if } \partial^+\omega(d^*) + Bd^* + C \geq 0, \\ \partial_q^+\Theta(r^*, r^*) \leq 0 \text{ if and only if } \partial^-\omega(d^*) + Bd^* + C \leq 0. \end{cases}$$

To see it, let $\partial^+\omega(d)$ and $\partial^-\omega(d)$ be the right- and left-derivatives of the convex function ω at d respectively. Since $\tilde{\omega}(q,r) = \omega(\tilde{d}(q,r))$, calculation shows that

$$\begin{aligned} \left[\tfrac{\partial}{\partial q}\tilde{R}(q,r)\right]\Big|_{q=r} &= \tfrac{1}{1-\alpha}(b-ar) - \tfrac{a+\eta}{1-\alpha}(r-c) + 2\lambda\gamma r, \\ \left[\partial_q^+\tilde{\omega}(q,r)\right]\Big|_{q=r} &= -\tfrac{a+\eta}{1-\alpha}\left[\partial^-\omega(d)\right]\Big|_{d=b-ar} = -\tfrac{a+\eta}{1-\alpha}\partial^-\omega(b-ar). \end{aligned}$$

The above equation implies that

$$\left[\partial_q^+\theta(q,r)\right]\Big|_{q=r} = \tfrac{1}{1-\alpha}(b-ar) + 2\lambda\gamma r - \tfrac{a+\eta}{1-\alpha}\left[(r-c) - \partial^-\omega(b-ar)\right].$$

In addition, since $\theta(q,q) = (q-c)(b-aq) - (1-\gamma)\lambda q^2 - \omega(b-aq)$,

$$\tfrac{\gamma}{1-\gamma}\left[\partial^+\theta(q,q)\right] = \tfrac{\gamma}{1-\gamma}(b-aq) - \tfrac{a\gamma}{1-\gamma}[(q-c) - \partial^-\omega(b-aq)] - 2\lambda\gamma q.$$

Thus,

$$\begin{aligned}\partial_q^+\Theta(r^*,r^*) &= \left(\tfrac{1}{1-\alpha} + \tfrac{\gamma}{1-\gamma}\right)(b-ar^*)\\ &- \left(\tfrac{1+\eta/a}{1-\alpha} + \tfrac{\gamma}{1-\gamma}\right)[a(r^*-c) - a\partial^-\omega(b-ar^*)].\end{aligned}$$

From the definitions of B and C we have that $\partial_q^+\Theta(r^*,r^*) \le 0$ if and only if $\partial^-\omega(d^*) + Bd^* + C \le 0$. Similarly $\partial_q^-\Theta(r^*,r^*) \ge 0$ if and only if $\partial^+\omega(d^*) + Bd^* + C \ge 0$. Therefore, (4.9) is equivalent to

$$\begin{cases}\partial^+\omega(d^*) + Bd^* + C \ge 0, & \text{if } d^* \text{ is not the upper boundary of } \mathcal{D},\\ \partial^-\omega(d^*) + Bd^* + C \le 0, & \text{if } d^* \text{ is not the lower boundary of } \mathcal{D}.\end{cases}$$

Since B is positive, d^* minimizes $\omega(d) + \frac{1}{2}Bd^2 + Cd$ over $\mathcal{D}$.

Finally, problem (4.8) admits a unique minimizer over the compact feasible set $\mathcal{D}$ because its objective function strictly convex. Thus we conclude that the steady state is unique. □

The above result extends the existence and uniqueness of the steady state in Theorem 2, Popescu and Wu (2007) by incorporating inventory related costs $\omega(d)$ and c. Unlike their loss-neutral setting, the single period expected profit function here may be non-smooth at $r = r^*$. Yet, we still have the uniqueness of the steady state and provide a characterization of the stead state as the optimal solution of a minimization problem with a strictly convex objective function. The parametric analysis of the steady state will be presented in Proposition 4.2 .

It can also be shown that the optimal base-stock level $y^0(r)$ is nondecreasing in r. To see this, observe that in problem (4.5), if we write q as a function of d and r in the objective function, we can show that the objective function is supermodular in (d, r) and the set

$$\{(d, r) : L \leq p(d, r), r \leq U\}$$

a is a sublattice of $\Re^2$. Therefore, $d^0(r)$ is nondecreasing in r and $y^0(r) = y(d^0(r))$ is nondecreasing as well by Lemma 4.1. This, together with Theorem 4.3, implies that the optimal order-up-to levels also monotonically converge.

4.3.2 When return is not allowed

Now we move our attention back to the system in which return is not allowed. For exposition purpose, we denote I^* as the model when return is not allowed, and denote I^0 as the system with the same parameters as I^* when return is allowed.

Theorem 4.4 below shows that I^* and I^0 share the same type of convergence results on the reference price path for the additive demand model.

Theorem 4.4. *Consider the system I^* when demand is additive. There exists an optimal policy such that under the optimal policy, the reference price trajectory $\{r_t : t \geq 1\}$ converges to a steady state, which is unique and equal to r^*, the steady state of system I^0.*

Proof. Let $\{(x_t^0, r_t^0) : t \geq 1\}$ be the state path of I^0 under the optimal policy starting with some given initial state (x_1^0, r_1^0). Recall that at the beginning of period $t \geq 1$ when a typical state $(x_t^0, r_t^0) = (x, r)$ is observed, we respectively denoted by $d^0(r)$ and $y^0(r)$ the expected demand and inventory level after replenishment/return under the optimal policy for I^0.

For I^*, we claim the following policy, referred to as *policy* I^0, is optimal: given the state (x, r) at the beginning of any period, if $x > y^0(r)$, no order is placed and solve problem (4.3) by setting $y = x$ to get the reference price of the next period; if $x \leq y^0(r)$, place an order to raise the inventor level to $y^0(r)$ and determine the reference price of the next period, denoted by q, from $d^0(r) = \tilde{d}(q, r)$.

To see the optimality of policy I^0, observe that if $x > y^0(r)$, no order is placed. Since the expected demand is assumed to be always positive and $\mathbb{E}[\varepsilon_{a,t}] = 0$, eventually the inventory level will drop down to a point at which an order is necessary. Suppose at the beginning of some period $\tau \geq 1$ the observed state (x, r) satisfies that $x \leq y^0(r)$. We next prove that the state path $\{(x_t, r_t) : \tau \geq \tau\}$ in I^* under policy I^0 satisfies the condition $x_t \leq y^0(r_t)$ for all $t \geq \tau$. In fact, To see this, note that $y^0(r) = y_a + d^0(r)$ by Lemma 4.1(a). If $x_t \leq y^0(r_t)$, then

$$x_{t+1} = y^0(r_t) - [d^0(r_t) + \varepsilon_{a,t}] = y_a - \varepsilon_{a,t} \leq y_a + d^0(r^0_{t+1}) = y^0(r^0_{t+1}),$$

where the inequality follows from the nonnegativity of the realized demand. Thus, at each period $t \geq \tau$, I^* under policy I^0 gives exactly the same state path as I^0 with $(x^0_\tau, r^0_\tau) = (x, r)$ under its optimal policy and results the same realized discounted total profit. Since $\phi(x, r) \leq \phi^0(r)$ for any (x, r), policy I^0 is optimal for any initial state (x, r) with $x \leq y^0(r)$. For a given r, we can show that a stationary reference price dependent base-stock policy is optimal by extending Theorem 4.1 to its infinite horizon counterpart. Since it is optimal to order up to $y^0(r)$ when $x < y^0(r)$, $y^0(r)$ must be the optimal base-stock level and thus policy I^0 is actually optimal for I^*.

The above analysis indeed shows that I^* will simply mimic I^0 once an order is placed. Thus, the reference price trajectory $\{r_t : t \geq 1\}$ converges to the unique steady state of I^0, which is also a steady state of I^*. Observe that any steady state of I^* is also a steady state of I^0. This implies that the steady state of I^* is unique as well. □

We can prove a weaker result for the multiplicative demand model.

Theorem 4.5. *Consider the system I^* when demand is multiplicative. There exists an optimal policy such that under the optimal policy, the reference price trajectory $\{r_t : t \geq 1\}$ converges to the unique steady state r_* of system I^0 if we start with an initial state (x, r) with $x \leq y^0(r)$ and $r \leq r^*$.*

Proof. The logic of the proof is similar to the one for Theorem 4.4. The key difference is that in general, we cannot guarantee that the inventory levels x_t following policy I^0 will always be no more than $y^0(r_t)$ if we start with initial state (x, r) with $x \leq y^0(r)$. However, if $x \leq y^0(r)$ and $r \leq r^*$, then we can show that the state path $\{(x_t, r_t) : t \geq 1\}$ in system I^* is optimal under policy I^0. In fact, if $x_t \leq y^0(r_t)$ and $r_t \leq r^*$, then

$$x_{t+1} = y^0(r_t) - d^0(r_t)\varepsilon_{m,t} \leq y^0(r_t) \leq y^0(r_{t+1}),$$

where the first inequality holds by the nonnegativity of the realized demand and the second one holds since $r_t \leq r_{t+1}$ by Theorem 4.3 and $y^0(r)$ is non-decreasing. $\square$

Although we can analytically prove the convergence of reference price only when the initial state lies in a certain region in the multiplicative model, our numerical experiments show that the convergence result holds with any initial states. As an example, Figure 4.2 shows the optimal reference price trajectories of reference price (inventory is not plotted here) with $\gamma = 0.99, \alpha = 0.15, c = 0.2, h(x) = 0.1\max\{x, 0\} - 0.15\min\{x, 0\}, b = 40, a = 20, \eta = 20$ and the variances of ε_a and ε_m are 5 and 0.1, respectively.

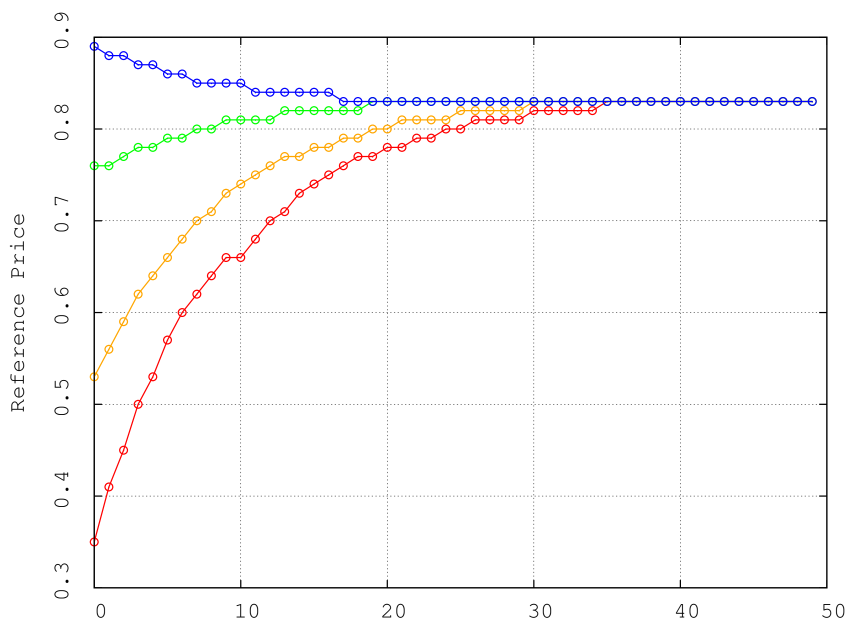

Figure 4.2: Reference price path under loss-neutral demand

4.3.3 Characterizing the steady-state

The previous subsection establishes some conditions for the optimality of reference-price dependent base-stock policy and global convergence properties of reference price trajectory. In this section we characterize the steady state r^*. In particular, we have the following proposition.

Proposition 4.2. *The steady state r_* is decreasing in α, η, a and increasing in γ, b, c.*

Proof. From Theorem 4.3, we have that $d^* = b - ar^*$ is the unique optimal solution to the problem

$$\underset{d \in \mathcal{D}}{\text{minimize}} \ \left[\omega(d) + \tfrac{1}{2}Bd^2 + (a^{-1}b - c)d\right],$$

where $B = \frac{1}{a} + \frac{1}{a+s}$ with $s = \eta\frac{1-\gamma}{1-\alpha\gamma}$. Clearly d^* and r^* depend on α, γ and η through s. Since the objective function in the minimization problem is supermodular in (d, B), by Theorem 2.8.1, Topkis (1998), d^* is decreasing in B. Since B is decreasing in s and $r^* = (b - d^*)/a$ is decreasing in d^*, we know that r^* is decreasing in s. In addition, because s is decreasing in γ and increasing in α and η, we then conclude the monotonicity of r^* in have that is decreasing in α, η and increasing in γ.

For the monotonicity in b and c, note that r^* solves the problem

$$\underset{r \in \mathcal{P}}{\text{minimize}} \ \left\{\omega(b - ar) + \tfrac{1}{2}B(b - ar)^2 + (ac - b)r\right\},$$

where $\omega(b - ar)$ and $(b - ar)^2$ are submodular in (r, b) by the convexity of $\omega(d)$ and d^2. One can easily verify that the objective function of the above minimization problem is submodular in (r, b, c). Therefore r^* is increasing in b and c by Theorem 2.8.1, Topkis (1998) again.

To see the monotonicity of r^* in a, observe from the above convex minimization problem that r^* is the projection of p^* onto the interval $[L, U]$, where $p = p^*$ is the unique solution satisfying the following condition:

$$\begin{aligned} & \partial^-\omega(b - ap) - \left(1 + \tfrac{a}{a+s}\right)p + \left(c + \tfrac{b}{a+s}\right) \\ \leq \ & 0 \\ \leq \ & \partial^+\omega(b - ap) - \left(1 + \tfrac{a}{a+s}\right)p + \left(c + \tfrac{b}{a+s}\right). \end{aligned}$$

Notice that by the convexity of ω, the first and the last terms in the above inequality are strictly decreasing in a and p for $a > 0$ and $p \geq 0$ (it is easy to show that $p^* \geq 0$). Since $p = p^*$ is the unique solution satisfying the above inequalities, p^* and hence r^* are decreasing in a. $\square$

The above proposition extends the results in Section 4.2, Popescu and Wu (2007) where there is no inventory-related cost to a model in which inventory is a consideration. Recall that α measures the contribution of historical prices to the reference price, η measures the reference price effect in demand, and

a measures the sensitivity to the current selling price. That the steady state is decreasing in a, α and η suggests that in the long run a lower price should be charged to the market in which consumers are more sensitive to the base price effect and to the reference price effect. Furthermore, the monotonicity in the discount factor γ, the market size b and the per unit ordering cost c suggests that a firm should charge a higher price in the long run if it cares more about profit in future, or when it has a larger market size or a higher production cost. Specifically, if the firm makes myopic decision, i.e., the firm uses the optimal solution for the integrated inventory and pricing model in which $\gamma = 0$, then it will underprice the product in the long run. The intuition is clear. Under myopic decision making, the inventory manager cares only about the current period profit and he/she would reduce price to boost current period sales. This however leads to lower reference price and is detrimental to the firm's long-term profit.

It is also interesting to study the impact of the demand uncertainties on the steady state. For this purpose, we use the convex order defined as follows (see, e.g., Shaked and Shanthikumar, 1994, for a comprehensive discussion).

Definition 4.1. *Given two random variables ζ_1 and ζ_2, we say ζ_1 is smaller than ζ_2 in the convex order if $\mathbb{E}h(\zeta_1) \leq \mathbb{E}h(\zeta_2)$ for all convex functions $h : \Re \mapsto \Re$.*

Intuitively, ζ_1 is smaller than ζ_2 in the convex order if ζ_2 is "more variable" than ζ_1. For example, when ζ_i follows the normal distribution with zero mean, ζ_1 is smaller than ζ_2 in convex order if and only if ζ_1 has the smaller variance than ζ_2 does.

Proposition 4.3 below basically states that the higher uncertainty in the market, the higher a firm should charge in the long run.

Proposition 4.3. *Consider the demand model $D(d, \varepsilon) = d\varepsilon_m + \varepsilon_a$.*

(a) In the additive demand case, the steady state is independent of ε_a.

(b) In the multiplicative demand case, if h^γ is positive homogeneous with degree $\rho > 0$, and ε_m is smaller than $\tilde{\varepsilon}_m$ in the convex order, then the corresponding steady states, respectively denoted by $r^*(\varepsilon_m)$ and $r^*(\tilde{\varepsilon}_m)$, satisfy that $r(\varepsilon_m) \leq r(\tilde{\varepsilon}_m)$.

Proof. Recall that the steady states depends on $(\varepsilon_m, \varepsilon_a)$ only through

$$\omega(d) = \underset{y}{\text{minimize}} \;\; \mathbb{E}h^{\gamma}\left(y - d\varepsilon_m - \varepsilon_a\right),$$

where the transformed transformed holding and backlogging cost function h^{γ} is convex as assumed.

(a) Since $\omega(d)$ is a constant by Lemma 4.1(a), the optimal solution $d^* = b - ar^*$ to problem (4.8) and hence the steady state r^* are independent on ε_a.

(b) By Lemma 4.1(b), $\omega(d) = d^{\rho}\omega_m$ when $d \geq 0$ for some $\omega_m \geq 0$. In this case, (4.8) reduces to

$$\underset{d \in \mathcal{D}}{\text{minimize}} \;\; \left[\omega_m d^{\rho} + \tfrac{1}{2}Bd^2 + Cd\right],$$

where all elements in $\mathcal{D} = \{b - ap : L \leq p \leq U\}$ are nonnegative.

Observe that the objective of the above problem is a supermodular function in (ω_m, d). Denote $d(\omega_m)$ as the optimal solution associated with any given ω_m. Then $d(\omega_m)$ is decreasing in ω_m by Theorem 2.8.1 in Topkis (1998). Finally, by the definition of convex order and the definition of ω_m in Lemma 4.1(b),

$$\omega_m = \min_y \mathbb{E}h^{\gamma}(y - \varepsilon_m) \leq \min_y h^{\gamma}(y - \tilde{\varepsilon}_m) = \tilde{\omega}_m.$$

Therefore $b - ar^*(\varepsilon_m) = d(\omega_m) \geq d(\tilde{\omega}_m) = b - ar^*(\tilde{\varepsilon}_m)$ implying that $r^*(\varepsilon_m) \leq r^*(\tilde{\varepsilon}_m)$.

□

4.4 Conclusion

This chapter studies a joint inventory and pricing model with reference price effect. This provides new insights into how inventory decision interacts with pricing decision under the presence of reference price effect. The major challenge is that the reference price effect links pricing decisions in difference periods, which further links with the inventory replenishment decisions in each period. This increases the dimension of the dynamic programming problem. Despite the difficulty, we were able to analyze both the finite horizon and infinite horizon model, and establish a number of structural results.

For the finite horizon model, we prove that a base-stock policy is always optimal under general settings on demand uncertainty when customers are either loss-neutral or loss-averse. For the infinite horizon model, we first analyzed a simplified model in which the return of inventory is allowed. This allows us to reduce the dimension of our dynamic program, and show that the reference price converges to a steady state eventually. We then move our attention back to the case where return is not allowed and proved similar convergence result. We also analyzed how the optimal steady reference price varies with model parameters.

Our work should only be taken as an initial attempt to inventory and pricing models with reference price effect. Several future tasks are specifically desirable. First, our discussion is based on the assumption that unsatisfied demand is fully backlogged. Whether similar results hold in the lost-sales case is still a challenge. Second, we also want to know if optimal order-up-to inventory level is monotone in given reference price or not when customers are loss-averse. Though our numerical studies suggest that the optimal base-stock level is monotonically nondecreasing in r for multi-period problems, it is still an open question whether we can prove the monotonicity theoretically. Finally, it would be interesting to see if the reference price trajectory converges for the infinite horizon model for more general demand models.

Chapter 5

Preservation of Supermodularity in 2-dimensional Parametric Optimization Problems

5.1 Background and preliminaries

In many dynamic programing problems, one is concerned whether a kind of property can be preserved hence structural results of optimal solutions can be derived. For example in Chapter 3 we recursively proved in Proposition 3.3 that all functions G_t defined by (3.3) are concave and consist of $O(t)$ quadratic pieces. As another example, the concept of supermodularity provides a convenient tool in deriving monotone comparative statics in many dynamic programing problems. One of the key preservation properties states that if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are lattices, $\boldsymbol{D}$ is a sublattice of $\boldsymbol{X} \times \boldsymbol{Y}$, and $g(\boldsymbol{x}, \boldsymbol{y})$ is supermodular in $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}$, then the function

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \; [g(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D}]$$

is supermodular on the set of $\boldsymbol{x}$ for which the maximization is well defined (see Topkis, 1998, Theorem 2.7.6). Under the above conditions, one can also show that the optimal solution set is increasing in $\boldsymbol{x}$. The above preservation property is powerful and widely used in many problems. However, to apply it, the set $\boldsymbol{D}$ is required to be a sublattice. Relaxing the lattice requirement has been proven a significant challenge. Indeed, without the lattice condition, the analysis becomes much more complicated even in some very simple settings in which supermodularity can be preserved.

The objective of this chapter is to establish a new preservation property of supermodularity under optimization operations when the constraint set may not be a lattice. Specifically, consider the following optimization problem parameterized by *two dimensional* vectors $\boldsymbol{x} \in \boldsymbol{S} = \{A\boldsymbol{y} : \boldsymbol{y} \in \boldsymbol{D}\}$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \; \big\{g(\boldsymbol{y}) : A\boldsymbol{y} = \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\big\}, \tag{5.1}$$

where A is a $2 \times n$ matrix, $\boldsymbol{D}$ is a closed convex sublattice of $\Re^n$ and g is an n-dimensional function defined on $\boldsymbol{D}$. Throughout of this chapter, we assume that the maximization is well defined whenever $\boldsymbol{x} \in \boldsymbol{S}$. The main result Theorem 5.1 shows that f is concave and supermodular on $\boldsymbol{S}$ if A is non-negative and g is concave and supermodular on $\boldsymbol{D}$.

The significance of the above result is that the constraint set

$$\{(\boldsymbol{x}, \boldsymbol{y}) : A\boldsymbol{y} = \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\}$$

is not a lattice in general and may not be mapped to become one by a variable transformation. Of course, by relaxing the lattice requirement, we have to assume concavity of the objective function and impose a requirement on the dimension of the parameter vector. In addition, in general the optimal solution set is not monotone in $\boldsymbol{x}$. Though it may appear restrictive, relaxing the above assumptions even slightly may render the preservation property invalid. More importantly, the property and its extensions include several existing results in the literature as special cases, and they prove quite powerful as we illustrate their applications to several operations models.

We notice that our results can be applied to many applications. For example, Gong and Chao (2011) consider the capacitated inventory systems with remanufacturing and characterize the optimal policies by showing some preservation property under the minimization operation in associated dynamic programing problems. By probably introducing additional variables we observe that their problems are in fact special cases of (5.1) and some of their results can be easily implied by Theorem 5.1. Another example is the production planning problem with emissions trading discussed by Gong and Zhou (2011), where their key technical results can be directly ensured from our results as we will show later in Section 5.2. In addition, when analyzing the stochastic coordinated pricing and inventory model with reference price effects in the previous chapter, we already adopted the main result in this chapter to prove Theorem 4.2 when customers are loss-neutral and the demand model involves only an additive random perturbation.

In this chapter we focus on three other applications. The first application is a two-product coordinated pricing and inventory control problem with cross-price effects over a finite planing horizon. In this model, the retailer observes the initial inventories of two products at the beginning of each period, and then simultaneously decide their prices and the ordering quantities. The demand of each product during a period is stochastic and depends not only on its own price but also the price of the other product. The objective is to maximize the total expected profit over the planning horizon assuming zero lead time and backlogging of unfilled demand.

In the second application, we consider a two-stage coordinated dynamic pricing and inventory control problem over a finite planning horizon. In the model, the firm observes the initial raw material inventory level and the finished product inventory level at the beginning of each period, and then decides the amount of raw material to be purchased, the amount of product produced from the raw material, and the selling price of product. Demand of the product is stochastic and depends on its price. There is no lead time for delivery and unused inventory is carried over to the next period. The objective is to maximize the total profit over the whole horizon.

In the third application, we consider a self-financing retailer who sells a single product over a finite planning horizon with its operational decisions limited by its cash flow. At the beginning of each period, the retailer observes the initial inventory level of the product and its available capital on hand, and then decides the amount of product to be ordered such that the ordering costs do not exceed the available capital. The delivery is immediate, unused capital is deposited to the savings account. After demand during the period is realized, unused inventory is carried over to the next period and unsatisfied demand is lost. The retailer obtains its profit by either depositing the unused capital or selling the product. The objective is to maximize the total profit over the planning horizon.

The first and second applications fall into the literature on coordinated pricing and inventory models. Papers directly related to the first application include Zhu and Thonemann (2009), Song and Xue (2007) and Ceryan et al. (2009), who analyze models with substitutable products and develop structural properties of the optimal policies. The second and third applications are respectively extensions of Yang (2004) and Chao et al. (2008). Compared with their papers, our approach based on results developed in this chapter is significantly simpler and provides additional insights to these applications. For instance, for the first application, Zhu and Thonemann (2009), Song and Xue (2007) and Ceryan et al. (2009) prove the submodularity of the profit-to-functions by analyzing the first-order optimality condition (the KKT condition) of the optimization problems resulted from the dynamic programming recursion. Their proofs are lengthy and unfortunately not very insightful. They also require smoothness assumptions on objective functions and can only deal with simple feasible sets. In fact, for tractability, all these three papers ignore the lower and upper bound constraints on prices when they analyze the KKT conditions, even though such constraints are indispensable in particular for linear demand models. Our approach allows us to treat coordinated pricing and inventory models with complementary products and substitutable products in a unified framework and derives new structural results. Yang (2004) analyzes a model related to our second application without pricing decisions. Again, his approach is also based on complicated analysis on the first-order optimality condition of the optimization problems resulted from the dynamic programming recursion.

Before continuing on the discussion, we introduce the notations and basic concepts used in this chapter. Sets are expressed by boldface capital letters (e.g., $\boldsymbol{D}$ and $\boldsymbol{S}$), matrices by regular capital letters (e.g., A and B), vectors by boldface lowercase letters (e.g., $\boldsymbol{x}$ and $\boldsymbol{y}$) and real numbers by regular lowercase letters. We also write $A = [a_{i,j}]_{m,n}$ or $\boldsymbol{x} = [x_i]_n$ sometimes to emphasize entries of A or components of $\boldsymbol{x}$, where subscripts outside the bracket indicate the size of A or the dimension of $\boldsymbol{x}$. All vectors are column vectors, and $\mathbf{0}, \boldsymbol{e}$ are the vectors with all components $0, 1$, respectively.

Given any $m \times n$ matrix A and subset $\boldsymbol{S}$ of $\Re^n$, denote by $A \geq 0$ if all its entries are non-negative, $|A|$ the determinant of A when $m = n$, and $A(\boldsymbol{S}) = \{A\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{S}\} \subset \Re^m$. Given vectors $\boldsymbol{x} = [x_i]_n$ and $\boldsymbol{y} = [y_i]_n$, denote by $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for all i, and

$$\boldsymbol{x} \vee \boldsymbol{y} = [\max(x_i, y_i)]_n, \quad \boldsymbol{x} \wedge \boldsymbol{y} = [\min\{x_i, y_i\}]_n.$$

$\boldsymbol{S}$ is called a *convex set* if

$$\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S} \text{ and } 0 \leq \lambda \leq 1 \Longrightarrow \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \boldsymbol{S}.$$

$\boldsymbol{S}$ is a *sublattice* (of $\Re^n$) if

$$\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S} \Longrightarrow \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \vee \boldsymbol{y} \in \boldsymbol{S}.$$

For example, the box set $[\boldsymbol{l}, \boldsymbol{u}] = \{\boldsymbol{x} : \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ forms a convex sublattice of $\Re^n$, where some components of vectors $\boldsymbol{l}, \boldsymbol{u}$ could be $-\infty, +\infty$, respectively (if $u_i = +\infty$, for example, $x_i \leq u_i$ is to be understood as $x_i < +\infty$).

Given a function f defined on a subset $\boldsymbol{S}$ of $\Re^n$ (in case $\boldsymbol{S}$ is not specified, we implicitly assume $\boldsymbol{S} = \Re^n$), we say f is *increasing* if

$$\boldsymbol{x} \leq \boldsymbol{y} \in \boldsymbol{S} \Longrightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{y}).$$

f is *concave* if $\boldsymbol{S}$ is convex and for all $0 \leq \lambda \leq 1$ and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$,

$$\lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \leq f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}).$$

And f is *supermodular* if $\boldsymbol{S}$ forms a sublattice and for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$,

$$f(\boldsymbol{x}) + f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) + f(\boldsymbol{x} \vee \boldsymbol{y}).$$

In addition, we say f is *decreasing*, *convex* or *submodular* if $-f$ is increasing, concave or supermodular. f is *monotone* if it is either increasing or decreasing, *bimonotone* if it is a bivariate function increasing in one variable and decreasing in the other, a *valuation* if it is submodular and supermodular. When referring to a convex function, we assume it is well-behaved (i.e., closed, proper and lower semi-continuous) and may take the value $+\infty$. For details on these concepts, we refer to Rockafellar (1970), Topkis (1998) and Simchi-Levi et al. (2005).

This chapter is organized as follows. In Section 5.2, we present our main result, its special cases and extensions. In addition, several examples are also given to demonstrate the applicability and limitation of our results on a few examples. In Section 5.3, we apply the main result to the three aforementioned applications. Section 5.4 summarizes this chapter and provides some future research problems. Throughout this chapter, many proofs are provided in Appendix B unless otherwise specified.

5.2 Main results

In this section, we first show in Theorem 5.1 that concavity and supermodularity can be preserved in problem (5.1) if $A \geq 0$. We then present a preservation property of components-wise concavity and supermodularity in Proposition 5.1 on a special case of problem (5.1) and an extension of Theorem 5.1 by replacing the constraint $A\boldsymbol{y} = \boldsymbol{x}$ with $A\boldsymbol{y} = B\boldsymbol{x}$ for some matrix B with two columns in Proposition 5.2. From Corollary 5.2 to Corollary 5.3 we discuss another special case of problem (5.1) and show several preservation properties. Finally, the applicability and limitation of our results are demonstrated on several examples including linear programs and quadratic programs. We point out that several results in the literature can be directly derived from ours as we go along.

Theorem 5.1. *Assume that A is a non-negative $2 \times n$ matrix in problem* (5.1). *If $\boldsymbol{D}$ is a closed convex sublattice, then so is $\boldsymbol{S}$; if g is concave and supermodular on $\boldsymbol{D}$, then so is f on $\boldsymbol{S}$.*

Proof. It is straightforward to see $\boldsymbol{S} = A(\boldsymbol{D})$ is closed and convex. Concavity of f on $\boldsymbol{S}$ follows from Theorem 5.4, Rockafellar (1970). It remains to prove that $\boldsymbol{S}$ is a sublattice and f is supermodular on $\boldsymbol{S}$, i.e., $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}, \boldsymbol{x} \vee \tilde{\boldsymbol{x}} \in \boldsymbol{S}$ and $f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) \leq f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}})$ for any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \boldsymbol{S}$.

Let $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ be the optimal solutions associated with $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ in problem (5.1), respectively. Because $\boldsymbol{D}$ is a sublattice, $\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}, \boldsymbol{y} \vee \tilde{\boldsymbol{y}} \in \boldsymbol{D}$ and $\boldsymbol{a} = A(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}), \boldsymbol{b} = A(\boldsymbol{y} \vee \tilde{\boldsymbol{y}}) \in \boldsymbol{S}$. Since $A \geq 0$, $\boldsymbol{a} \leq \boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ hence $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ belongs to the convex hull of $\{\boldsymbol{a}, \boldsymbol{x}, \tilde{\boldsymbol{x}}\}$ (see Figure 5.1 for the illustration). We know from the convexity of $\boldsymbol{S}$ that $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}} \in \boldsymbol{S}$. Similarly, we also have $\boldsymbol{x} \vee \tilde{\boldsymbol{x}} \in \boldsymbol{S}$.

Denote $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}} = \lambda \boldsymbol{a} + \mu \boldsymbol{x} + \nu \tilde{\boldsymbol{x}}$ for some $0 \leq \mu, \nu, \lambda \leq \lambda + \mu + \nu = 1$. Then $\boldsymbol{x} \vee \tilde{\boldsymbol{x}} = \lambda \boldsymbol{b} + \mu \tilde{\boldsymbol{x}} + \nu \boldsymbol{x}$ by $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{x} \wedge \boldsymbol{y} + \boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{x} + \tilde{\boldsymbol{x}}$. The concavity of f implies that

$$\lambda[f(\boldsymbol{a}) + f(\boldsymbol{b})] + (1 - \lambda)[f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}})] \leq f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}}).$$

In addition, the definition of f and the supermodularity of g lead to

$$f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) = g(\boldsymbol{y}) + g(\tilde{\boldsymbol{y}}) \leq g(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) + g(\boldsymbol{y} \vee \tilde{\boldsymbol{y}}) \leq f(\boldsymbol{a}) + f(\boldsymbol{b}).$$

By combining the above two inequalities we conclude the desirable inequality

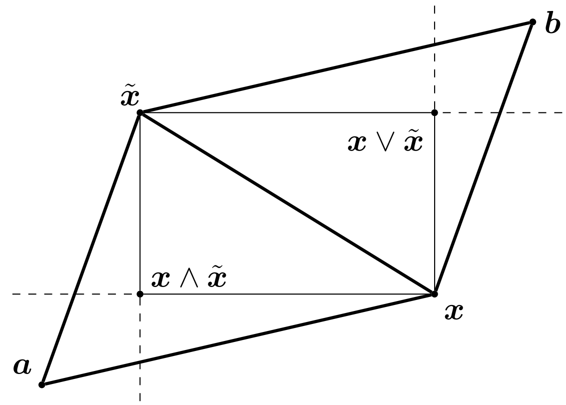

Figure 5.1: Construction of $\boldsymbol{a}$ and $\boldsymbol{b}$ in the proof of Theorem 5.1

$f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) \leq f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}})$. Therefore f is supermodular on $\boldsymbol{S}$. $\square$

Remark 5.1. *Theorem 5.1 still holds when the equality constraints $A\boldsymbol{y} = \boldsymbol{x}$ in* (5.1) *are replaced by inequality constraints. Indeed, it suffices to add non-negative slack or surplus variables to the inequality constraints and apply Theorem 5.1 in the current format to establish the same result.*

Remark 5.2. *The statement of Theorem 5.1 remains valid for some discrete cases. Specifically, suppose all entries of A are integers, $\boldsymbol{D} = [\boldsymbol{l}, \boldsymbol{u}] \cap \mathcal{Z}^n$, where $\mathcal{Z}$ denotes the set of all integers and $\boldsymbol{l}, \boldsymbol{u} \in \mathcal{Z}^n$. We can show that if g is supermodular on $\boldsymbol{D}$ and* integrally concave *(see Section 3.4, Murota, 2003) then so is f on $\boldsymbol{S}$. The proof is almost identical except that we now deal with the concave extensions of g and f instead.*

Note that in the proof of Theorem 5.1, we only need the supermodularity of g and the concavity of f (not the concavity of g) to ensure the supermodularity of f. The concavity of g does provide a sufficient condition for the concavity of f though. One may ask whether the concavity of g can be replaced by component-wise concavity such that supermodularity can still be preserved. Though the answer is negative in general as we illustrate later in this section on an unconstrained quadratic program, the concavity of g can be weakened for a special case of problem (5.1). The key is to observe that in the proof of Theorem 5.1, we construct $\boldsymbol{a}$ such that $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ can be expressed as a convex combination of $\boldsymbol{a}, \boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$, and then apply the concavity of f on $\boldsymbol{S}$. If one can guarantee that $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ and $\boldsymbol{a}$ lie on the same vertical line, then the concavity of $f(x_1, x_2)$ in x_2 is sufficient to complete the proof. This observation motivates the proposition below.

Proposition 5.1. *Consider the optimization problem parameterized by* $[t, x] \in \boldsymbol{S} \subset \Re^2$*:*

$$f(t, x) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{g(t, \boldsymbol{y}) : at + \boldsymbol{b}'\boldsymbol{y} = x, (t, \boldsymbol{y}) \in \boldsymbol{D}\},$$

where $\boldsymbol{D} \subset \Re^n$*,* $a \geq 0$*,* $\boldsymbol{b} \geq \boldsymbol{0}$ *and*

$$\boldsymbol{S} = \{(t, at + \boldsymbol{b}'\boldsymbol{y}) : (t, \boldsymbol{y}) \in \boldsymbol{D}\}.$$

Denote $\boldsymbol{D}(t) = \{\boldsymbol{y} : [t, \boldsymbol{y}] \in \boldsymbol{D}\}$ *and* $\boldsymbol{S}(t) = \{x : [t, x] \in \boldsymbol{S}\}$*. If* $\boldsymbol{D}$ *is a sublattice and* $\boldsymbol{D}(t)$ *is convex for all* t*, then so is* $\boldsymbol{S}$*. Moreover, if* $g(t, \boldsymbol{y})$ *is supermodular on* $\boldsymbol{D}$ *and concave in* $\boldsymbol{y}$ *on* $\boldsymbol{D}(t)$ *for all* t*, then* $f(t, x)$ *is supermodular on* $\boldsymbol{S}$ *and concave in* t *on* $\boldsymbol{S}(t)$ *for all* t*.*

When $n = 2$, $A \geq 0$ and $|A| > 0$, Theorem 5.1 implies that if g on $\boldsymbol{D}$ is concave and supermodular, then so is $g(P\boldsymbol{x})$ on $A(\boldsymbol{D})$, where $P = A^{-1}$. Notice that the matrix P has non-negative diagonal entries and non-positive off-diagonal entries (any matrix with this property will be referred to as an L_0-matrix thereafter). A stronger result can be obtained from Proposition 5.1, which provides a sufficient condition such that supermodularity is preserved under linear variable transformations.

Corollary 5.1. *Given any* 2×2 L_0*-matrix* P *and function* g *defined on a subset* $\boldsymbol{D}$ *of* $\Re^2$*, if* $\boldsymbol{D}$ *is a closed convex sublattice then so is the set* $\boldsymbol{S} = \{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{D}\}$*; moreover, if* g *on* $\boldsymbol{D}$ *is component-wise concave and supermodular then so is the function* $g(P\boldsymbol{x})$ *in term of* $\boldsymbol{x} \in \boldsymbol{S}$*.*

The following proposition presents an extension of problem (5.1).

Proposition 5.2. *Given some* $m \times n$ *matrix* A *and* $m \times 2$ *matrix* B *such that* $B'A \geq 0$ *and* $B'B$ *is an* L_0*-matrix, closed convex sublattice* $\boldsymbol{D}$ *of* $\Re^n$*, and concave and supermodular function* g *on* $\boldsymbol{D}$*, define*

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \big\{g(\boldsymbol{y}) : A\boldsymbol{y} = B\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}\big\},$$

for all $\boldsymbol{x} \in \boldsymbol{S} = \{\boldsymbol{x} :$ *there exists* $\boldsymbol{y} \in \boldsymbol{D}$ *such that* $A\boldsymbol{y} = B\boldsymbol{x}\}$*. Then* $\boldsymbol{S}$ *is a closed convex sublattice, and* f *is concave and supermodular on* $\boldsymbol{S}$*.*

Next we consider a special case of problem (5.1) below. Given convex sublattices $\boldsymbol{S}_n$ of $\Re^2$ and real-valued functions f_n defined on $\boldsymbol{S}_n$ for $n = 1, 2,$

let $\boldsymbol{S}$ be the *Minkowski sum* of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, i.e.,

$$\boldsymbol{S} = \{\boldsymbol{x}_1 + \boldsymbol{x}_2 : \boldsymbol{x}_1 \in \boldsymbol{S}_1,\ \boldsymbol{x}_2 \in \boldsymbol{S}_2\}.$$

Furthermore, for any $\boldsymbol{x} \in \boldsymbol{S}$ define

$$f(\boldsymbol{x}) = \underset{\boldsymbol{x}_1 \in \boldsymbol{S}_1, \boldsymbol{x}_2 \in \boldsymbol{S}_2}{\text{maximize}} \left\{ f_1(\boldsymbol{x}_1) + f_2(\boldsymbol{x}_2) : \boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{x} \right\}. \tag{5.2}$$

Note that when $\boldsymbol{S}_n = \Re^2$ for both $n = 1, 2$, $-f$ is called the *infimal convolution* of $-f_1$ and $-f_2$ (see Rockafellar, 1970, Section 5).

The following result is an immediate corollary of Theorem 5.1.

Corollary 5.2. *Suppose P is a non-singular 2×2 matrix. For problem (5.2), if all $P^{-1}(\boldsymbol{S}_n)$ are convex sublattices of $\Re^2$, and $f_n(P\boldsymbol{x})$ are concave and supermodular on $P^{-1}(\boldsymbol{S}_n)$, then $P^{-1}(\boldsymbol{S})$ forms a convex sublattice of $\Re^2$, and $f(P\boldsymbol{x})$ is concave and supermodular on $P^{-1}(\boldsymbol{S})$.*

We omit the detailed proof of Corollary 5.2. Observe that if P is the identity matrix, then Corollary 5.2 states the preservation of concavity and supermodularity in problem (5.2). In general, we may have some flexibility of choosing the matrix P depending on applications. Three interesting instances of P are listed below:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad J_1 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

where all of them are projections, i.e., $J(J\boldsymbol{x}) = J_1(J_1\boldsymbol{x}) = J_2(J_2\boldsymbol{x}) = \boldsymbol{x}$.

The linear transformation J maps a vector $[x_1, x_2]$ to $[x_1, -x_2]$. Geometrically, $J(\boldsymbol{S})$ is the reflection of the set $\boldsymbol{S}$ at the horizontal axis. Interestingly the transformation shows a simple but useful relation between two dimensional submodular functions and supermodular functions, whose proof follows directly from the definitions of supermodularity and submodularity and thus is omitted.

Lemma 5.1. *When both $\boldsymbol{S}$ and $J(\boldsymbol{S})$ are sublattices of $\Re^2$, $f(\boldsymbol{x})$ is supermodular on $\boldsymbol{S}$ if and only if $f(J\boldsymbol{x})$ is submodular on $J(\boldsymbol{S})$.*

This above lemma allows us to convert a statement on supermodularity to the related statement on submodularity. For example, we know from

Corollary 5.1 that if g is component-wise concave and submodular on $\Re^2$ then so is the function $g(B\boldsymbol{x})$ in $\boldsymbol{x}$ for any 2×2 non-negative matrix B.

The other two transformations J_1 and J_2 map a vector $[x_1, x_2]$ to $[x_1 - x_2, -x_2]$ and $[-x_1, x_2 - x_1]$, respectively. We will use them in the following to analyze a closely related concept, $L^\natural$*-concavity*, which finds applications in inventory models (see, for instance, Zipkin, 2008).

Definition 5.1. *A function f is $L^\natural$-concave on $\boldsymbol{S} \subset \Re^n$ if the set $\boldsymbol{S}_+ = \{(\boldsymbol{x}, \xi) : \boldsymbol{x} - \xi\boldsymbol{e} \in \boldsymbol{S}\} \subset \Re^{n+1}$ forms a sublattice and $f(\boldsymbol{x} - \xi\boldsymbol{e})$ is supermodular on $\boldsymbol{S}_+$. And f on $\boldsymbol{S}$ is $L^\natural$-convex if and only if $-f$ is $L^\natural$-concave on $\boldsymbol{S}$.*

when $\boldsymbol{S} = \Re^n$, the above definition is consistent with the one given in Murota (2003). In the *two dimensional* space one can verify that if $\boldsymbol{S}$ can be expressed by

$$\{[x_1, x_2] : l_1 \leq x_1 \leq u_1,\ l_2 \leq x_2 \leq u_2,\ l_0 \leq x_1 - x_2 \leq u_0\}, \tag{5.3}$$

then the corresponding $\boldsymbol{S}_+$ given below is a clearly convex sublattice:

$$\{[x_1, x_2, x_3] : l_1 \leq x_1 - x_3 \leq u_1,\ l_2 \leq x_2 - x_3 \leq u_2,\ l_0 \leq x_1 - x_2 \leq u_0\}.$$

As pointed out by Murota (2003), an $L^\natural$-concave function f is also concave and supermodular. Moreover, its Hessian matrix $\nabla^2 f(\boldsymbol{x})$, provided the existence, has non-positive diagonal entries and non-negative off-diagonal entries, and possesses the diagonal dominance property, i.e., the summation of entries in each row is non-positive. It should be mentioned that depending on applications, one may also assume $\xi \leq 0$ or $\xi \geq 0$ in the definition of $L^\natural$-concavity. For example, Zipkin (2008) uses $\{\xi : \xi \leq 0\}$ as the domain of ξ when he applies it to inventory models with lost sales.

The following lemma characterizes $L^\natural$-concavity through supermodularity for two dimensional functions.

Lemma 5.2. *Suppose the function f is defined on $\boldsymbol{S} \subset \Re^2$. If $\boldsymbol{S}$ is of the form* (5.3)*, then so are both $J_1(\boldsymbol{S})$ and $J_2(\boldsymbol{S})$. In addition, the four statements below are equivalent:*

(a) $f(\boldsymbol{x})$ is $L^\natural$-concave in $\boldsymbol{x}$ on its domain $\boldsymbol{S}$;

(b) $f(J_1\boldsymbol{x})$ is $L^\natural$-concave in $\boldsymbol{x}$ on its domain $J_1(\boldsymbol{S})$;

(c) $f(J_2\boldsymbol{x})$ *is* $L^\natural$*-concave in* $\boldsymbol{x}$ *on its domain* $J_2(\boldsymbol{S})$*;*

(d) $f(\boldsymbol{x}), f(J_1\boldsymbol{x})$ *and* $f(J_2\boldsymbol{x})$ *are respectively supermodular on their domains.*

By Lemmas 5.1, 5.2 and all above discussions, we have the following result on problem (5.2) from Corollary 5.2, where the proof is omitted.

Corollary 5.3. *Assume in problem* (5.2) *that all* $\boldsymbol{S}_n$ *are convex sublattices of* $\Re^2$*. Then* $\boldsymbol{S}$ *forms a convex sublattice of* $\Re^2$*. In addition,*

(a) if both f_n *are concave and supermodular on* $\boldsymbol{S}_n$*, then so is* f *on* $\boldsymbol{S}$*;*

(b) if $J(\boldsymbol{S}_n)$ *are sublattices of* $\Re^2$ *and both* f_n *are concave and submodular on* $\boldsymbol{S}_n$*, then* f *is concave and submodular on* $\boldsymbol{S}$*;*

(c) if $\boldsymbol{S}_n$ *is of the form* (5.3) *and* f_n *is* $L^\natural$*-concave on* $\boldsymbol{S}_n$ *for each* n*, then* f *is* $L^\natural$*-concave on* $\boldsymbol{S}$*.*

It should be mentioned that in Corollary 5.3(b) the condition on $J(\boldsymbol{S}_n)$ is indispensable. Actually it may fail if $J(\boldsymbol{S}_n)$ are not sublattices. For example, consider the problem below for all $x_1, x_2 \geq 0$:

$$\begin{aligned} f(x_1, x_2) = \quad & \text{maximize} \quad && y_1 \\ & \text{subject to} \quad && y_1 + z_1 = x_1,\ y_2 + z_2 = x_2, \\ & && 0 \leq y_1 \leq y_2,\ z_1, z_2 \geq 0, \end{aligned}$$

where it is straightforward to see that the set $\boldsymbol{S}_1 = \{[y_1, y_2] : 0 \leq y_1 \leq y_2\}$ forms a sublattice and the objective function is a valuation. Solving the above parameterized optimization problem gives us

$$f(x_1, x_2) = \min\{x_1, x_2\}, \quad \forall x_1, x_2 \geq 0,$$

which is supermodular as is consistent with Corollary 5.3(a). However, we cannot apply Corollary 5.3(b) because $J(\boldsymbol{S}_1)$ is not a sublattice. In fact, f is not submodular since

$$f(0,0) + f(1,1) = 0 + 1 > 0 + 0 = f(0,1) + f(1,0).$$

Next we demonstrate the applicability and limitation of our results on a few examples.

Example 5.1 (Linear Programs)**.** *Suppose $\boldsymbol{p}, \boldsymbol{u}$ are two given n-dimensional vectors and A is a $2 \times n$ matrix. Zipkin (2003) considers the linear programming problem*

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{\boldsymbol{p}'\boldsymbol{y} : A\boldsymbol{y} \le \boldsymbol{x}, \boldsymbol{0} \le \boldsymbol{y} \le \boldsymbol{u}\}.$$

Using intricate geometrical argument, he shows that $f(\boldsymbol{x})$ is supermodular in $\boldsymbol{x}$ over $\boldsymbol{x} \ge 0$ if A is non-negative. Interestingly, this result immediately follows form Remark 5.1 of Theorem 5.1.

Zipkin (2003) also proves that for arbitrary matrices A and C with proper sizes, the function defined below is supermodular over $\boldsymbol{x} \ge 0$ as long as the maximization above is well defined for all $\boldsymbol{x} \ge \boldsymbol{0}$,

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{\boldsymbol{p}'\boldsymbol{y} : A\boldsymbol{y} \le \boldsymbol{x}, C\boldsymbol{y} \le \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0}\}.$$

Unfortunately, our result does not cover this case. As we show later in Example 5.3, it does not work even for the case of quadratic objective. It is interesting to observe that $f(\boldsymbol{x})$ is not necessarily supermodular if $\boldsymbol{x}$ is not restricted in the non-negative orthant. Here is an example:

$$\boldsymbol{p} = [-1, 0, 0, -1], \qquad C = \boldsymbol{0}, \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -2 & -1 & 2 & 1 \end{bmatrix}.$$

Calculation shows

$$f(x_1, x_2) = \min\{0, x_1 + x_2, 2x_1 + x_2, 3x_1 + 2x_2\}.$$

One can verify the submodularity of f. However, f is not supermodular since

$$f(0,0) + f(1,-1) = 0 + 0 > -2 + 0 = f(0,-1) + f(1,0).$$

This example also indicates that without the condition $A \ge 0$, Theorem 5.1 may fail even if the objective function in problem (5.1) is linear.

Example 5.2 (quadratic programs I)**.** *Suppose P, Q are $n \times n$ symmetric matrices such that $P + Q$ is negative definite, and*

$$g(\boldsymbol{y}, \boldsymbol{z}) = \tfrac{1}{2}\boldsymbol{y}'P\boldsymbol{y} + \tfrac{1}{2}\boldsymbol{z}'Q\boldsymbol{z}.$$

For all $\boldsymbol{x} \in \Re^n$*, define*

$$f(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \ \{g(\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})\}.$$

One can easily verify that $f(\boldsymbol{x})$ *is a quadratic function. Moreover, denote* $\boldsymbol{y}(\boldsymbol{x})$ *as the optimal solution to the above problem for any given* $\boldsymbol{x}$*, and* $\nabla^2 f(\boldsymbol{x})$ *as the Hessian matrix of* $f(\boldsymbol{x})$*. Calculation shows that*

$$\boldsymbol{y}(\boldsymbol{x}) = (P+Q)^{-1}Q\boldsymbol{x}, \qquad \nabla^2 f(\boldsymbol{x}) = P(P+Q)^{-1}Q.$$

When $n = 2$*, we further have*

$$\nabla^2 f(\boldsymbol{x}) = P(P+Q)^{-1}Q = \tfrac{|Q|}{|P+Q|}P + \tfrac{|P|}{|P+Q|}Q.$$

There are some interesting observations on Example 5.2. First, it is a special case of problem (5.1) when $n = 2$. From the expression of $\nabla^2 f(\boldsymbol{x})$, we know that if g is supermodular then so is f, which is consistent with the statement of Theorem 5.1. This result does not seem to follow directly from Theorem 2.7.6, Topkis (1998) since the constraint set does not form a sublattice. One may simplify the example by eliminating $\boldsymbol{z}$ and the constraints. However, even when all entries of P are zero and Q has positive off-diagonal entries, we know that $g(\boldsymbol{y}, \boldsymbol{z})$ is supermodular in $(\boldsymbol{y}, \boldsymbol{z})$ but $g(\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y})$ is neither submodular nor supermodular in $(\boldsymbol{x}, \boldsymbol{y})$ in general.

Second, we can not weaken the concavity assumption on g in Theorem 5.1 to component-wise concavity. Consider Example 5.2 with matrices P and Q given below

$$P = \begin{bmatrix} -9 & 4 \\ 4 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 4 \\ 4 & -9 \end{bmatrix},$$

where g is component-wise concave and supermodular in this instance. On the other hand, calculation shows that the Hessian matrix of $f(\boldsymbol{x})$ is

$$\nabla^2 f(\boldsymbol{x}) = \frac{7}{18}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

It indicates that f is not supermodular.

Third, Theorem 5.1 does not hold in higher dimensional spaces, i.e., $n \geq 3$.

Consider Example 5.2 with matrices P and Q given below,

$$P = \begin{bmatrix} -11 & 8 & 0 \\ 8 & -16 & 5 \\ 0 & 5 & -10 \end{bmatrix}, \quad Q = \begin{bmatrix} -7 & 4 & 0 \\ 4 & -14 & 5 \\ 0 & 5 & -9 \end{bmatrix},$$

where g is $L^\natural$-concave, hence concave and supermodular, in the instance. On the other hand, the Hessian matrix of f is

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} -4.25 & 2.79 & -0.01 \\ 2.79 & -7.27 & 2.49 \\ -0.01 & 2.49 & -4.73 \end{bmatrix}.$$

Therefore f is neither supermodular nor submodular. Extending our results to higher dimensional spaces is interesting and challenging.

Finally, the optimal solution may not be monotone or may not have a clear monotonicity pattern even in cases in which we do have monotonicity. To see this, consider Example 5.2 with P, Q given below and their related optimal solutions $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$.

$$P = \begin{bmatrix} -6 & 5 \\ 5 & -6 \end{bmatrix}, \quad Q = \begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix}, \quad \boldsymbol{y}(\boldsymbol{x}) = \frac{1}{28}\begin{bmatrix} 21 & -7 \\ 13 & -3 \end{bmatrix}\boldsymbol{x};$$

$$P = \begin{bmatrix} -6 & 3 \\ 3 & -4 \end{bmatrix}, \quad Q = \begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix}, \quad \boldsymbol{y}(\boldsymbol{x}) = \frac{1}{65}\begin{bmatrix} 20 & 10 \\ -3 & 44 \end{bmatrix}\boldsymbol{x}.$$

In both instances, g are supermodular. However, in the first instance, for both $i = 1, 2$ $y_i(x_1, x_2)$ are increasing in x_1 but decreasing in x_2. In the second instance, $y_1(x_1, x_2)$ is increasing in both x_1 and x_2 but $y_2(x_1, x_2)$ is increasing in x_2 and decreasing in x_1.

Example 5.3 (quadratic programs II)**.** *Consider the problem*

$$\begin{aligned} f(x_1, x_2) = \quad & \text{maximize} \quad \tfrac{1}{2}(\boldsymbol{y} - \boldsymbol{e})' Q (\boldsymbol{y} - \boldsymbol{e}) \\ & \text{subject to} \quad \boldsymbol{\alpha}_1' \boldsymbol{y} \le x_1, \quad \boldsymbol{\alpha}_2' \boldsymbol{y} \le x_2, \quad \boldsymbol{y} \ge \boldsymbol{0}, \end{aligned}$$

for all $[x_1, x_2] \ge \boldsymbol{0}$*, where* $\boldsymbol{\alpha}_1 = [1, -\frac{1}{2}]$*,* $\boldsymbol{\alpha}_2 = [-\frac{1}{2}, 1]$ *and* $Q = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

Let $A = \begin{bmatrix} \boldsymbol{\alpha}_1' \\ \boldsymbol{\alpha}_2' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$*. Depending on whether each constraint* $\boldsymbol{\alpha}_i'\boldsymbol{y} \leq x_i$ *is active or not, we have*

$$f(x_1, x_2) = \begin{cases} 0, & \text{if } [x_1, x_2] \in \boldsymbol{S}_0, \\ -(x_1 - \frac{1}{2})^2, & \text{if } [x_1, x_2] \in \boldsymbol{S}_1, \\ -(x_2 - \frac{1}{2})^2, & \text{if } [x_1, x_2] \in \boldsymbol{S}_2, \\ \frac{1}{2}\left(A^{-1}\boldsymbol{x} - \boldsymbol{e}\right)' Q \left(A^{-1}\boldsymbol{x} - \boldsymbol{e}\right), & \text{if } \boldsymbol{x} = [x_1, x_2] \in \boldsymbol{S}_3, \end{cases}$$

where $\boldsymbol{S}_0 = \{\boldsymbol{x} : 2\boldsymbol{x} \geq \boldsymbol{e}\}$*,*

$$\boldsymbol{S}_i = \{[x_1, x_2] : 0 \leq 2x_i \leq 1, 3 \leq 2x_i + 4x_{3-i}\}, \quad \forall i = 1, 2$$

and $\boldsymbol{S}_3 = \{\boldsymbol{x} \geq \boldsymbol{0} : \boldsymbol{x} \notin \boldsymbol{S}_i, i = 0, 1, 2\}$*. That is, neither constraint is active when* $\boldsymbol{x} \in \boldsymbol{S}_0$*, only the constraint* $\boldsymbol{\alpha}_i'\boldsymbol{y} \leq x_i$ *is active when* $\boldsymbol{x} \in \boldsymbol{S}_i$ *for each* $i = 1, 2$*, and both constraints are active when* $\boldsymbol{x} \in \boldsymbol{S}_3$*.*

Calculation shows that $\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2) = -\frac{4}{3}$ *for any interior point* $\boldsymbol{x} \in \boldsymbol{S}_3$*. Hence, unlike the linear programming problems analyzed in Zipkin (2003),* f *is not supermodular over* $\boldsymbol{x} \geq \boldsymbol{0}$*. It also provides another instance to demonstrate that Theorem 5.1 may fail without the condition* $A \geq 0$*.*

To end this section, we apply Theorem 5.1 and its extensions to several applications in literature, where the corresponding analyses can be significantly simplied by our results.

Example 5.4. *The following optimization problem is presented in Chao et al. (2009) when they analyze a dynamic capacity expansion model,*

$$f_0(x_1, x_2) = \text{maximize } \{g_0(x_1, y_2) : x_2 \leq y_2 \leq x_1 + x_2\}.$$

Chao et al. (2009) prove that if $g_0(x_1, x_2)$ *is submodular and concave in* x_2*, then so is* $f_0(x_1, x_2)$*. Such result serves as the key technical tool in their analysis. They also comment that it is usually challenging to prove the preservation of submodularity under maximization.*

We now show that the above statement follows directly from our results. Define $g(\boldsymbol{x}) = g_0(J\boldsymbol{x})$ *and* $f(\boldsymbol{x}) = f_0(J\boldsymbol{x})$*. Then the problem can be rewritten by*

$$f(x_1, x_2) = \text{maximize } \left\{g(x_1, y_2) : y_1 + y_2 = x_2, 0 \leq y_1 \leq x_1\right\}.$$

Since the submodularity of g_0, f_0 is equivalent to the supermodularity of g, f, the same result as Chao et al. (2009) is ensured by Proposition 5.1.

Example 5.5. *Gong and Chao (2011) consider a periodic-review manufacturing system with product returns. Mathematically they face the problem*

$$\begin{aligned} v(x_1, x_2) = \quad \underset{y_1, y_2, z_1, z_2}{\text{minimize}} \quad & [rz_1 + pz_2 + h(y_1, y_2)] \\ \text{subject to} \quad & y_1 + z_1 = x_1, \; y_2 - z_2 = x_2, \\ & y_1 \geq 0, \; z_1 \geq 0, \; z_2 \geq 0, \\ & z_1 \leq k_r, z_2 \leq k_m, z_1 + z_2 \leq k, \end{aligned}$$

In their model, $v(x_1, x_2)$ is the cost-to-go function, where x_1 and x_2 denote the the inventory levels of returned products and total products (including both returned products and the serviceable products) at the beginning of a period, respectively. In the objective function, the term $rz_1 + pz_2$ denotes the costs associated with remanufactring z_1 units and manufacturing z_2 units, and $h(y_1, y_2)$ denotes the minimal expected total discounted cost to the end of the planning horizon when the inventory levels of returned products and total products respectively become y_1 and y_2 after manufacturing and remanufactring decisions in the period. In addition, constants k_r, k_m and k in the last set of constraints specify the capacities involved in remanufactring, manufacturing or total remanufactring/manufacturing operations, respectively.

By using $L^\natural$-convexity and lattice analysis, Gong and Chao (2011) characterize the optimal solution to the above problem under serval settings, e.g., $k_r = +\infty, k_m = -\infty$ and $k < +\infty$. When all k_r, k_m and k are finite, they claim that v is $L\natural$-convex if so is h. They do not provide the detailed proof as they state that the associated analysis is complicated. Interestingly, this claim can be directly derived by applying Corollary 5.3(c) to the following equivalent problem:

$$\begin{aligned} -v(x_1, x_2) = \quad \underset{y_1, y_2, z_1, z_2}{\text{maximize}} \quad & [-(rz_1 + pz_2) - h(y_1, y_2)] \\ \text{subject to} \quad & y_1 + y_2 = x_1, \; y_2 + z_2 = x_2, \\ & y_1 \geq 0, z_1 \geq 0, z_2 \leq 0, \\ & z_1 \leq k_r, -k_m \leq z_2, \; z_1 - z_2 \leq k. \end{aligned}$$

Example 5.6. *Gong and Zhou (2011) consider a production planning prob-*

lem with emissions trading, where their key technical result claims that for the problem

$$
\begin{aligned}
f(x_1, x_2) = \quad & \underset{u,v}{\text{minimize}} && g(x_1 + uv, x_2 - u), \\
& \text{subject to} && 0 \le u, \quad v_{\min} \le v \le v_{\max},
\end{aligned}
$$

if g is convex on $\Re^2$ and $0 < v_{\min} \le v_{\max}$, then the following preservation properties hold:

(a) If $g(y_1, y_2)$ is supermodular in (y_1, y_2) then so is $f(x_1, x_2)$ in (x_1, x_2);

(b) If $g(y_1, y_2 - y_1/v_{\min})$ is submodular in (y_1, y_2) then so is $f(x_1, x_2 - x_1/v_{\min})$ in (x_1, x_2);

(c) If $g(y_1 - v_{\max} y_2, y_2)$ is submodular in (y_1, y_2) then so is $f(x_1 - v_{\max} x_2, x_2)$ in (x_1, x_2).

The original proof of the above result in Gong and Zhou (2011) is much involved. For example, when prove the statement (a) by verifying that $f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) \le f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}}) + f(\boldsymbol{x} \wedge \tilde{\boldsymbol{x}})$ for any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \Re^2$, Gong and Zhou (2011) first assume $g(\boldsymbol{y}_\vee) = f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}})$ and $g(\boldsymbol{y}_\wedge) = f(\boldsymbol{x} \vee \tilde{\boldsymbol{x}})$ for some $\boldsymbol{y}_\vee$ and $\boldsymbol{y}_\wedge$, then construct two other vectors $\bar{\boldsymbol{y}}$ and $\hat{\boldsymbol{y}}$ such that

$$
f(\boldsymbol{x}) + f(\tilde{\boldsymbol{x}}) \le g(\bar{\boldsymbol{y}}) + g(\hat{\boldsymbol{y}}) \le g(\boldsymbol{y}_\vee) + g(\boldsymbol{y}_\wedge).
$$

Depending on the relative position of $\boldsymbol{y}_\vee$ and $\boldsymbol{y}_\wedge$, four cases together with six subcases are discussed. The other two statements are proved similarly.

From our main results we can prove these statements in a much easier way. To see it, introduce $y_1 = x_1 + uv$ and $y_2 = x_2 - u$ in the definition of f and properly change signs of variables. We have that

$$
\begin{aligned}
-f(x_1, x_2) = \quad & \underset{y_1, y_2, z_1, z_2}{\text{maximize}} && -g(y_1, y_2) \\
& \text{subject to} && y_1 + z_1 = x_1, \; y_2 + z_2 = x_2, \\
& && z_2 \ge 0, \quad v_{\max} z_2 \ge -z_1 \ge v_{\min} z_2
\end{aligned}
$$

Let $\boldsymbol{x} = [x_1, x_2]$, $\boldsymbol{y} = [y_1, y_2]$ and

$$
\boldsymbol{Z} = \{[z_1, z_2] : z_2 \ge 0, z_1 + v_{\min} z_2 \le 0 \le z_1 + v_{\max} z_2\}.
$$

Then for any nonsingular 2×2 matrix P,

$$-f(P\boldsymbol{x}) = \text{maximize } \left\{-g(P\boldsymbol{y}) : \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{x}, \boldsymbol{z} \in P^{-1}(\boldsymbol{Z})\right\}.$$

Notice that the above three statements indeed claim that $-f(P\boldsymbol{x})$ is supermodular in $\boldsymbol{x}$ when $-g(P\boldsymbol{y})$ is concave and supermodular in $\boldsymbol{y}$ for the following three instances of P:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1/v_{\min} & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -v_{\max} \\ 0 & 1 \end{bmatrix}.$$

It is straightforward to verify that the corresponding $P^{-1}(\boldsymbol{Z})$ are respectively

$$\begin{aligned}
&\{[z_1, z_2] : z_2 \leq 0,\ v_{\min} z_2 \geq z_1 \geq v_{\max} z_2\}, \\
&\{[z_1, z_2] : z_1 \leq v_{\min} z_2 \leq 0,\ (v_{\max} - v_{\min}) z_1 \leq v_{\min} v_{\max} z_2\}, \\
&\{[z_1, z_2] : z_2 \geq 0,\ 0 \leq z_1 \leq (v_{\max} - v_{\min}) z_2\},
\end{aligned}$$

where each $P^{-1}(\boldsymbol{Z})$ forms a convex sublattice of $\Re^2$. Then the key technical result of Gong and Zhou (2011) immediately follows from Corollary 5.3.

5.3 Applications

We have applied the results in the previous section to the reference price model in Chapter 4 (see Proposition 4.2). In this section we apply them several other periodic-review operational models. In all these models, we will deal with parameterized optimization problems with the following mathematical structure:

$$\begin{aligned}
v(x_1, x_2) = \quad & \underset{y_1, y_2}{\text{maximize}} && [f(z_1, z_2) + \bar{v}_+(y_1, y_2)] \\
& \text{subject to} && y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \\
&&& h_n(y_1, y_2) \leq 0,\ \forall\ 1 \leq n < M, \\
&&& h_n(z_1, z_2) \leq 0,\ \forall M \leq n < N, \\
&&& a_1 \leq y_1 \leq b_1, \quad a_2 \leq y_2 \leq b_2, \\
&&& l_1 \leq z_1 \leq u_1, \quad l_2 \leq z_2 \leq u_2,
\end{aligned}$$

where some components l_i, a_i could be $-\infty$ and b_i, u_i could be $+\infty$. Define vectors $\boldsymbol{l} = [l_1, l_2], \boldsymbol{u} = [u_1, u_2], \boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{b} = [b_1, b_2]$, and sets

$$\begin{aligned}\boldsymbol{Y} &= \{\boldsymbol{y} \in [\boldsymbol{a}, \boldsymbol{b}] : h_n(\boldsymbol{y}) \leq 0,\ \forall\, 1 \leq n < M\},\\ \boldsymbol{Z} &= \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h_n(\boldsymbol{z}) \leq 0,\ \forall\, M \leq n < N\},\end{aligned}$$

We can rewrite the above problem as

$$v(\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}}\ \left\{f(\boldsymbol{y} - \boldsymbol{x}) + \bar{v}_+(\boldsymbol{y}) : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y} - \boldsymbol{x} \in \boldsymbol{Z}\right\}, \tag{5.4}$$

where v is defined on the set $\boldsymbol{X} = \{\boldsymbol{y} - \boldsymbol{z} : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{z} \in \boldsymbol{Z}\}$.

Notice that in the stochastic coordinated pricing and inventory model with reference price effects analyzed in Section 4.2, when customers are loss-neutral and the demand uncertainty is additive, the corresponding problem (4.2) has the structure of the general problem (5.4).

We have the following theorem on problem (5.4).

Theorem 5.2. *In problem* (5.4)*,*

(a) if all h_n are convex and bimonotone, then $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{X}$ are convex sublattices in $\Re^2$. In addition, if $\bar{v}_+$ on $\boldsymbol{Y}$ and f on $\boldsymbol{Z}$ are concave and supermodular, then so is v on $\boldsymbol{X}$;

(b) if all h_n are convex and monotone, then $J(\boldsymbol{Y})$, $J(\boldsymbol{Z})$ and $J(\boldsymbol{X})$ are convex sublattices in $\Re^2$. In addition, if $\bar{v}_+(J\boldsymbol{y})$ on $J(\boldsymbol{Y})$ and $f(J\boldsymbol{z})$ on $J(\boldsymbol{Z})$ are concave and supermodular, and $\boldsymbol{X}$ also forms a sublattice, then v on $\boldsymbol{X}$ is concave and submodular.

Proof. Denote $\boldsymbol{Z}_- = \{\boldsymbol{z} \in \Re^2 : -\boldsymbol{z} \in \boldsymbol{Z}\}$. Observe that problem (5.4) can be converted to the format amicable to (5.2) as

$$v(\boldsymbol{x}) = \underset{\boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{z} \in \boldsymbol{Z}_-}{\text{maximize}}\ \left\{f(-\boldsymbol{z}) + \bar{v}_+(\boldsymbol{y}) : \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{x}\right\}.$$

Because given conditions on h_n ensure that $\boldsymbol{Y}$ and $\boldsymbol{Z}_-$ are convex sets (Theorem 4.6, Rockafellar, 1970) and sublattices (Example 2.2.7, Topkis, 1998), Part (a) holds by applying Theorem 5.1. Part (b) then follows from Lemma 5.1 and part (a). $\square$

If additional conditions are imposed on f and $\boldsymbol{Z}$ in problem 5.4, the optimal solution to problem (5.4) exhibits certain monotonicity properties. Note that in the following theorem we say $f(x_1, x_2)$ is separable if it can be expressed as $f_1(x_1) + f_2(x_2)$ for two univariate functions f_1 and f_2.

Theorem 5.3. *Suppose that $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$ and f is separable in problem* (5.4)*.*

(a) If $\bar{v}_+$ is supermodular, all h_n are continuous and bimonotone, and f is concave, then there exists an optimal solution $[y_1(x_1, x_2), y_2(x_1, x_2)]$ to problem (5.4) *such that $y_i(x_1, x_2)$ is increasing in both x_1 and x_2 for $i = 1, 2$.*

(b) If $\bar{v}_+$ is submodular, all h_n are continuous and monotone, and f is linear, then there exists an optimal solution $[y_1(x_1, x_2), y_2(x_1, x_2)]$ to problem (5.4) *such that $y_i(x_1, x_2)$ is increasing in x_i and decreasing in x_j for $i, j = 1, 2$ and $i \neq j$.*

The managerial interpretation and intuition of the above characterization on the optimal solution will become clear when we talk about the concrete applications. Notice that we introduce no concavity/convexity assumptions on $\bar{v}_+$ and h_n in Theorem 5.3. However, they will be required in all the following applications to inductively show the supermodularity/submodularity of profit-to-go functions. Moreover, with these concavity/convexity assumptions, if f_t is linear then more refined characterization of optimal solution $\boldsymbol{y}(\boldsymbol{x})$ is possible by partitioning the space of the parameter $\boldsymbol{x}$ into several regions, which is provided in Appendix B.6.

5.3.1 Coordinated pricing and inventory control with cross-price effects

Consider a retailer who decides the ordering quantities and prices of two products over a finite planning horizon with T periods. At the beginning of each period, the retailer observes the initial inventory levels x_i and then simultaneously decides the selling prices p_i and the order-up-to-levels y_i for products $i = 1, 2$. The demand of product i during a period is given by $d_i(p_1, p_2) + \varepsilon_i$, where ε_i is a random variable with expected value 0, $d_i(p_1, p_2)$ is the expected demand of product i depending on the prices of both products. Denote $\boldsymbol{x} = [x_1, x_2], \boldsymbol{y} = [y_1, y_2]$, $\boldsymbol{p} = [p_1, p_2]$, $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2]$ and $\boldsymbol{d}^{\varepsilon} = \boldsymbol{d}(\boldsymbol{p}) + \boldsymbol{\varepsilon}$. The demand function can be time dependent but we drop the time index for simplicity. We assume that random vectors are independent across time, there is no lead time for delivery, unsatisfied demand is backlogged and unused inventory is carried over to the next period.

As common in the literature, the expected demand $\boldsymbol{d}(\boldsymbol{p})$ is assumed to be linear as $\boldsymbol{d}(\boldsymbol{p}) = \boldsymbol{b} - A\boldsymbol{p}$ for some vector $\boldsymbol{b} \geq \boldsymbol{0}$ and the price sensitivity coefficient matrix $A = [a_{i,j}]_{2,2}$. Suppose $\boldsymbol{p} \in [\boldsymbol{l}, \boldsymbol{u}]$, where $\boldsymbol{l} \leq \boldsymbol{u}$ are lower and upper bounds on the prices such that $\boldsymbol{d}^{\varepsilon} \geq \boldsymbol{0}$ almost surely. For product i, coefficients $a_{i,i}$, $a_{i,j}$ respectively denote its own price sensitivity and the cross price sensitivity to the other product j ($j \neq i$). We assume that $a_{i,i} \geq 0$, that is, the demand of a product is decreasing in its own price. Depending on the nature of products, we focus on two cases: (a) the two products are *complements*, i.e., an increase in the price of one product will decrease the demanded amount of the other product, or equivalently $a_{i,j} \geq 0$; (b) the two products are *substitutes*, i.e., an increase in the price of one product will increase the demanded amount of the other product, or equivalently $a_{i,j} \leq 0$. In addition, we assume that the price change of one product has a stronger effect on its own demand than on the other product's demand, i.e., $a_{i,i} \geq |a_{i,j}|$. Note that A is positive semi-definite under these assumptions. For our purpose, we assume A is positive definite. In this case, there is a one-to-one correspondence between the expected demands and the prices.

It will be convenient to use the expected demands instead of prices as the decision variables. Denote the realized demand vector as $\boldsymbol{d}^{\varepsilon} = \boldsymbol{d} + \boldsymbol{\varepsilon}$ and the corresponding price vector as $\boldsymbol{p}(\boldsymbol{d}) = A^{-1}(\boldsymbol{b} - \boldsymbol{d})$. The expected one-period revenue is given by $r(\boldsymbol{d}) = \boldsymbol{d}'\boldsymbol{p}(\boldsymbol{d})$, which can be easily verified to be concave. Moreover, in the complementary product case, $r(\boldsymbol{d})$ is supermodular and $r(A\boldsymbol{d})$ is submodular; in the substitutable product case, $r(\boldsymbol{d})$ is submodular and $r(A\boldsymbol{d})$ is supermodular.

The ordering cost is proportional to the ordering quantity specified by $c(\boldsymbol{z}) = \boldsymbol{c}'\boldsymbol{z}$ for an ordering quantity vector $\boldsymbol{z} = [z_1, z_2]$. For an amount $\boldsymbol{x} = [x_1, x_2]$ of inventory carried over from one period to the next, an inventory holding and backorder cost $h(\boldsymbol{x}) = h_1(x_1) + h_2(x_2)$ is incurred, where $h_i(x_i)$, assumed to be convex, represents the inventory holding cost when $x_i > 0$ and the shortage penalty cost when $x_i < 0$. To avoid technicality, we assume that $\mathbb{E}[h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ is strictly convex, where $\mathbb{E}$ is the expectation operator corresponding to random variables $\boldsymbol{\varepsilon}$. The objective of the retailer is to find an ordering and pricing decision so as to maximize its expected total profit over the planning horizon. Let $v_t(\boldsymbol{x})$ be the profit-to-go function of period t starting with an inventory level $\boldsymbol{x}$. The dynamic program can be formulated as

$$\begin{aligned} v_t(\boldsymbol{x}) = \quad & \underset{\boldsymbol{y},\boldsymbol{d}}{\text{maximize}} \quad \{r(\boldsymbol{d}) - \boldsymbol{c}'\boldsymbol{z} + g_t(\boldsymbol{y} - \boldsymbol{d})\} \\ & \text{subject to} \quad \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z}, \quad \boldsymbol{0} \le \boldsymbol{z} \le \boldsymbol{k}, \quad \boldsymbol{l} \le A^{-1}(\boldsymbol{b} - \boldsymbol{d}) \le \boldsymbol{u}, \end{aligned}$$

where $g_t(\boldsymbol{x}) = \mathbb{E}[v_{t+1}(\boldsymbol{x} - \boldsymbol{\varepsilon}) - h(\boldsymbol{x} - \boldsymbol{\varepsilon})]$, the ordering quantity $\boldsymbol{z}$ is nonnegative and bounded above by $\boldsymbol{k}$ and without loss of generality, assume $v_{T+1}(\boldsymbol{x}) = \boldsymbol{c}'\boldsymbol{x}$. For any given nonsingular 2×2 matrix P, the above problem can be equivalently reformulated as

$$v_t(P\boldsymbol{x}) = \underset{\boldsymbol{y}}{\text{maximize}} \; \left\{ f_t(P\boldsymbol{y}) - \boldsymbol{c}'P\boldsymbol{y} \right\} + \boldsymbol{c}'P\boldsymbol{x} \tag{5.5a}$$

$$\text{subject to } \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z}, \quad \boldsymbol{0} \le P\boldsymbol{z} \le \boldsymbol{k}$$

$$f_t(P\boldsymbol{y}) = \underset{\boldsymbol{d}}{\text{maximize}} \; \left\{ r(P\boldsymbol{d}) + g_t(P\tilde{\boldsymbol{x}}) \right\}, \tag{5.5b}$$

$$\text{subject to } \boldsymbol{y} = \boldsymbol{d} + \tilde{\boldsymbol{x}}, \; \tilde{\boldsymbol{l}} \le A^{-1}P\boldsymbol{d} \le \tilde{\boldsymbol{u}},$$

where $\tilde{\boldsymbol{l}} = A^{-1}\boldsymbol{b} - \boldsymbol{u}$ and $\tilde{\boldsymbol{u}} = A^{-1}\boldsymbol{b} - \boldsymbol{l}$.

Because $r(\boldsymbol{d})$ and $\mathbb{E}[-h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ are strictly concave, problem (5.5a) has unique optimal solution, denoted by $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$, when P is the identity matrix. We have the following proposition. It is worth mentioning that Proposition 5.3 remains valid if all functions (e.g., r and h), system inputs (e.g., $\boldsymbol{l}$ and $\boldsymbol{u}$) except A, and the random variables $\boldsymbol{\varepsilon}$ are time-dependent.

Proposition 5.3. *In all periods, v_t and f_t are concave.*

(a) In the complementary product case, $v_t(\boldsymbol{x})$ and $f_t(\boldsymbol{y})$ are supermodular, $v_t(A\boldsymbol{x})$ and $f_t(A\boldsymbol{y})$ are submodular, and $y_i(x_1, x_2)$ is increasing in either x_1 or x_2.

(b) In the substitutable product case, $v_t(\boldsymbol{x})$ and $f_t(\boldsymbol{y})$ are submodular, $v_t(A\boldsymbol{x})$ and $f_t(A\boldsymbol{y})$ are supermodular, and $y_i(x_1, x_2)$ is increasing in x_i and decreasing in x_j for $i, j = 1, 2$ and $i \neq j$.

Proof. Let $\mathcal{L}$ be the collection of all 2×2 matrices $L = [\ell_{i,j}]_{2,2}$ such that $\ell_{i,1}\ell_{i,2} \le 0$ for $i = 1, 2$. Notice that if $P, A^{-1}P \in \mathcal{L}$, then sets $\{\boldsymbol{z} : \boldsymbol{0} \le P\boldsymbol{z} \le \boldsymbol{k}\}$ and $\{\boldsymbol{d} : \tilde{\boldsymbol{l}} \le A^{-1}P\boldsymbol{d} \le \tilde{\boldsymbol{u}}\}$ are sublattices (Example 2.2.7, Topkis, 1998); moreover, $-h(P\boldsymbol{x})$ is concave and supermodular in $\boldsymbol{x}$ by Lemma 2.6.2, Topkis (1998). We next verify these statements by selecting proper matrices P.

(a) Let P be the identity matrix and AJ in (5.5), respectively. It is straightforward to see $P, A^{-1}P \in \mathcal{L}$ and $r(P\boldsymbol{d})$ is concave and supermodular. Since v_{T+1} is linear as assumed, we can inductively prove that in all periods $v_t(P\boldsymbol{x})$ and $f_t(P\boldsymbol{y})$ are concave and supermodular by Theorem 5.2. That is, $v_t(\boldsymbol{x})$, $f_t(\boldsymbol{y})$, $v_t(AJ\boldsymbol{x})$ and $f_t(AJ\boldsymbol{y})$ are concave and supermodular. We then conclude the properties of v_t and f_t by Lemma 5.1. In addition, Theorem 5.3(a) implies that $y_i(x_1, x_2)$ is increasing in either x_1 or x_2.

(b) Let $P = J$ and A in (5.5), respectively. Similarly by Theorem 5.2 and Lemma 5.1, we can verify properties of v_t and f_t. In addition, Theorem 5.3(a) implies that $J\boldsymbol{y}(J\boldsymbol{x}) = [y_1(x_1, -x_2), -y_2(x_1, -x_2)]$ is increasing in $[x_1, x_2]$, that is, $y_i(x_1, x_2)$ is increasing in x_i and decreasing in x_j for $i \neq j$.

We now complete this proof. $\square$

It is not surprising that in the complementary product case, the optimal order-up-to-levels are increasing in the initial inventory levels of both products, and in the substitutable product case, the optimal order-up-to-level of a product is increasing in its own initial inventory level while decreasing in the other product's initial inventory level.

A simpler version of our model was analyzed by Zhu and Thonemann (2009), which deals with only the substitutable product case without the constraint $\boldsymbol{z} \leq \boldsymbol{k}$. Song and Xue (2007) consider a more general setting with more than two substitutable products and derive structural results of the optimal order-up-to levels similar to Zhu and Thonemann (2009) for the two product case. Ceryan et al. (2009) extend Zhu and Thonemann (2009) by introducing the constraint $\boldsymbol{z} \leq \boldsymbol{k}$ and an additional resource capacity constraint $z_1 + z_2 \leq k_0$. Notice that for Ceryan et al. (2009)'s model, we can characterize functions v_t, f_t and optimal solutions $\boldsymbol{y}(\boldsymbol{x})$ as the same as Proposition 5.3(b) by using the similar argument. It is appropriate to point out that all the structure results on the optimal inventory decision in these papers can be easily derived by our approach (we present in Figure B.3 in the appendix to illustrate the structure of $\boldsymbol{y}(\boldsymbol{x})$ for Ceryan et al. (2009)'s model).

Compared with these three papers, we deal with both the complementary product case and the substitutable product case in a unified framework. We are not aware of any paper which analyzes coordinated pricing and inventory models with complementary products. Moreover, in the substitutable product case, we develop theoretical results on $v_t(A\boldsymbol{x})$ that are not available in the literature. Even though these three papers present results on v_t almost identical to Proposition 5.3, our approach is significantly simpler. In fact, all these papers establish the submodularity of $v_t(\boldsymbol{x})$ recursively by analyzing the first-order optimality condition (the KKT condition) of problem (5.4). Their approaches can only handle simple feasible set and require some technical conditions on the objective functions (e.g., smoothness almost everywhere). And all three papers ignore the bound constraints $\boldsymbol{p} \in [\boldsymbol{l}, \boldsymbol{u}]$ on prices, though such constraints are imposed in their models. For example, Zhu and Thonemann (2009) discuss the range of optimal prices *after* deriving their structural results. Song and Xue (2007) mention that the price vector $\boldsymbol{p}$ belongs to some compact set in their introduction section but does not explicitly analyze it when proving the related theorem.

Remark 5.3. *In general, the optimal prices may not be monotone as illustrated in Zhu and Thonemann (2009). However, when the matrix A is symmetric, Ceryan et al. (2009) and Zhu and Thonemann (2009) prove that $\boldsymbol{p}(\boldsymbol{x})$ is decreasing in $\boldsymbol{x}$ (again by analyzing the KKT condition and ignoring the bound constraints on prices).*

Remark 5.4. *Proposition 5.3 could fail when demand follows the multiplicative model $\boldsymbol{d}^\varepsilon = \boldsymbol{b} - A^\varepsilon \boldsymbol{p}$, where entries of the price sensitivity coefficient matrix A^ε are random variables. To see it, consider a special case when $\boldsymbol{c} = \boldsymbol{0}, h(\boldsymbol{x}) = 0$ and $v_{t+1}(\boldsymbol{x}) = l_0(\boldsymbol{x}) - \boldsymbol{x}'B\boldsymbol{x}$ in* (5.5b) *for some linear term $l_0(\boldsymbol{x})$. Here the quadratic term $\boldsymbol{x}'B\boldsymbol{x}$ in v_{t+1} can be treated as a perturbation which is relatively small comparing to $l_0(\boldsymbol{x})$. Suppose $\mathbb{E}A^\varepsilon = A$ for some positive definite A and $\mathbb{E}(A^\varepsilon - A)B(A^\varepsilon - A) = Q$. Let $\boldsymbol{d} = \boldsymbol{b} - A\boldsymbol{p}$ be the* *decision variable. In this case problem* (5.5b) *becomes*

$$
\begin{aligned}
f_t(\boldsymbol{y}) = \quad & \underset{\boldsymbol{d}}{\text{maximize}} \quad -\boldsymbol{d}'(A^{-1} + A^{-1}QA^{-1})\boldsymbol{d} - (\boldsymbol{y} - \boldsymbol{d})'B(\boldsymbol{y} - \boldsymbol{d}) + l(\boldsymbol{y}, \boldsymbol{d}) \\
& \text{subject to} \quad \boldsymbol{l} \le A^{-1}(\boldsymbol{b} - \boldsymbol{d}) \le \boldsymbol{u},
\end{aligned}
$$

where $l(\boldsymbol{y}, \boldsymbol{d})$ *is some linear function in terms of* $\boldsymbol{y}$ *and* $\boldsymbol{d}$*. Let us consider the instance*

$$
A^{-1} = \begin{bmatrix} 0.54 & 0 \\ 0 & 0.98 \end{bmatrix}, \; B = \begin{bmatrix} 0.97 & 0.07 \\ 0.07 & 0.98 \end{bmatrix}, \; A^{\varepsilon} = A + \varepsilon \begin{bmatrix} 0.54 & -0.48 \\ -0.48 & 0.64 \end{bmatrix},
$$

where ε *is some random variable with the expected value* 0 *and variance* 1*. It is no hard to see the objective is quadratic and strictly concave. Moreover, by properly selecting* $\boldsymbol{l}$ *and* $\boldsymbol{u}$ *one can expect that the constraint is inactive when* $\boldsymbol{y}$ *belongs to some nonempty open subset of* $\Re^2$*. For these* $\boldsymbol{y}$ *the above problem reduces to a special case of Example 5.2. In this situation calculation shows that*

$$
\nabla^2 v_{t+1}(\boldsymbol{x}) = -2B = \begin{bmatrix} -1.94 & -0.14 \\ -0.14 & -1.96 \end{bmatrix}, \quad \nabla^2 f_t(\boldsymbol{y}) = \begin{bmatrix} -0.38 & 0.05 \\ 0.05 & -0.59 \end{bmatrix}.
$$

In this example, v_{t+1} *is submodular but* f_t *is not.*

5.3.2 Two-stage inventory control

Consider a two-stage coordinated dynamic pricing and inventory control problem with random supply and demand over a finite planning horizon. At the beginning of each period, the firm observes the initial raw material inventory level x_1^0 and the finished product inventory level x_2, and then decides the amount z_1^0 of raw material to be purchased (i.e., $z_1^0 \ge 0$) or sold (i.e., $z_1^0 \le 0$). Assume the is no lead time for delivery. With $x_1^0 + z_1^0$ amount of raw material on hand, the firm simultaneously determines the amount z_2 of raw material to be converted into finished product, and the selling price p of the finished product in the period. Suppose one unit of the finished product consumes one unit of the raw material and $0 \le z_2 \le x_1^0 + z_1^0$. Right after the production, the inventory level of raw material becomes $x_1^0 + z_1^0 - z_2$ and that of finished product becomes $x_2 + z_2$. By the end of this period, an additional amount ε_1^0 of raw material arrives and brings the raw material inventory level up to $x_1^0 + z_1^0 - z_2 + \varepsilon_1^0$, where ε_1^0 is a non-negative random variable. Moreover, an amount $d(p) - \varepsilon_2$ of demand for the finished product arrives and brings its inventory level down to $x_2 + z_2 - d(p) + \varepsilon_2$, where unsatisfied demand is backlogged, ε_2 is a random variable independent on ε_1^0 with expected value 0 and $d(p)$ denotes the expected demand given the price p. Assume that $d(p)$ is strictly deceasing in p, which implies that there is a one-to-one correspondence between expected demand and selling price. For convenience, we use $d = d(p)$ as the decision variable and denote the selling price by $p = p(d)$, where $d \in [l, u]$ for some $0 \leq l \leq u$ with $p(u) \geq 0$.

Following the literature (e.g., Zipkin, 2000), we use *echelon inventory levels* $\boldsymbol{x} = [x_1, x_2]$ as system states, where $x_1 = x_1^0 + x_2$ is the total inventory of raw material and finished product. Suppose it incurs the costs $c_1(z_1^0)$, $c_2(z_2)$, $h_1(x_1^0)$ and $h_2(x_2)$ if z_1^0 units of raw material is purchased/sold, z_2 units of the finished product is produced, x_1^0 units of raw material inventory and x_2 units of product inventory are carried over to the next period, where $h_2(x_2)$ is to be understood as the shortage penalty cost if $x_2 < 0$. To avoid technicality, we assume that the expected one-period revenue $r(d) = dp(d)$ is strictly concave, and $c_1(z_1) + c_2(z_2)$ and $\mathbb{E}[h_1(x_1 - x_2 + \varepsilon_1^0) + h_2(x_2 + \varepsilon_2)]$ are strictly convex, where $\mathbb{E}$ denotes the expectation operator corresponding to ε_1^0 and ε_2. The firm's objective is to maximize the expected total profit over the T-period planning horizon.

Let $v_{T+1}(x_1, x_2) = 0$ and $v_t(x_1, x_2)$ be the profit-to-go functions with respect to the echelon inventory levels $[x_1, x_2]$ at the beginning of period $t = 1, \cdots, T$. We can formulate the problem as

$$\begin{aligned} v_t(x_1, x_2) = \quad & \underset{y_1, y_2, d}{\text{maximize}} \quad [r(d) - c_1(z_1) - c_2(z_2) + g_t(y_1 - d, y_2 - d)] \\ & \text{subject to} \quad y_1 = x_1 + z_1,\ y_2 = x_2 + z_2, \\ & \qquad\qquad\quad y_2 \leq y_1,\ z_2 \geq 0, \quad l \leq d \leq u, \end{aligned}$$

where $[y_1, y_2]$ denotes the echelon inventory levels right after the production, the constraint $y_1 \leq y_2$ indicates that the amount of finished product produced from raw material can not exceed the amount of on-hand raw material, and

$$\begin{aligned} g_t(x_1, x_2) &= \mathbb{E}[v_{t+1}(x_1 + \varepsilon_1^0 + \varepsilon_2, x_2 + \varepsilon_2)] \\ &- \mathbb{E}[h_1(x_1 - x_2 + \varepsilon_1^0) + h_2(x_2 + \varepsilon_2)]. \end{aligned}$$

Note that system inputs can be time-dependent which will not affect our later analysis.

The problem can be equivalently reformulated as

$$\begin{aligned} v_t(x_1, x_2) = &\underset{y_1, y_2}{\text{maximize}} \ \left[f_t(y_1, y_2) - c_1(z_1) - c_2(z_2)\right], &\quad (5.6\text{a}) \\ &\text{subject to} \ \ y_1 = x_1 + z_1, \ y_2 = x_2 + z_2, \\ &\qquad\qquad\quad y_2 \le y_1, \ z_2 \ge 0, \\ f_t(y_1, y_2) = &\,\text{maximize} \ \left[r(d) + g_t(y_1 - d, y_2 - d)\right], &\quad (5.6\text{b}) \\ &\text{subject to} \ \ d \in [l, u]. \end{aligned}$$

Since functions $r(d), -c_1(z_1), -c_2(z_2), -h_1(z_1^0)$ and $-h_2(z_2)$ are strictly concave, there exist unique optimal solutions $[y_1(x_1, x_2), y_2(x_1, x_2)]$ and $d(y_1, y_2)$ respectively to problems (5.6a) and (5.6b). Observe that the both problems are special cases of problem (5.4).

We have the following proposition from Theorems 5.2 and 5.3.

Proposition 5.4. *In all periods, v_t and f_t are $L^\natural$-concave, and $y_i(x_1, x_2)$ are increasing in x_1 and x_2 for $i = 1, 2$. Moreover, $d(y_1, y_2) \le d(y_1 + \delta, y_2 + \delta) \le d(y_1, y_2) + \delta$ for any $\delta \ge 0$.*

Proof. Suppose v_{t+1} is $L^\natural$-concave, which is true in the last period $t = T$. By

$$g_t(x_1 - \xi, x_2 - \xi) = \mathbb{E}[v_{t+1}(x_1 - \xi, x_2 - \xi) - h_1(x_1 - x_2) - h_2(x_2 - \xi)],$$

where h_1, h_2 are convex as assumed, one can easily verify the $L^\natural$-concavity of g_t. Because (5.6b) is a special case of (5.2), it is no hard to see from Corollary 5.3(a) that $f_t(\boldsymbol{y}), f_t(J_1\boldsymbol{y})$ and $f_t(J_2\boldsymbol{y})$ are supermodular. Therefore f_t is $L^\natural$-concave by Lemma 5.2, and so is v_t by Corollary 5.3(c).

The monotonicity of $y_i(x_1, x_2)$ follows from Theorem 5.3(a). To characterize $d(y_1, y_2)$, we need the results of Lemma 3 in Zipkin (2008), which claims that there exists $d_0(y_1, y_2)$ solving the unconstrained problem

$$\underset{d}{\text{maximize}} \ \left[r(d) + g_t(y_1 - d, y_2 - d)\right]$$

such that $d_0(y_1, y_2) \le d_0(y_1 + \delta, y_2 + \delta) \le d_0(y_1, y_2) + \delta$ for any $\delta \ge 0$. Because the objective function is concave, it follows that

$$d(y_1, y_2) = \max\{l, \min[d_0(y_1, y_2), u]\}.$$

Observe that $\max\{l, \min[d_0 + \delta, u]\} = \max\{l - \delta, \min[d_0, u - \delta]\} + \delta$. We then conclude the inequality on $d(y_1, y_2)$. $\square$

Remark 5.5. *Though Zipkin (2008) uses a slightly different definition of $L^\natural$-concavity by restricting $\xi \leq 0$, one can exactly follow his proof to see Lemma 3, Zipkin (2008) holds under our definition.*

The structure of $\boldsymbol{y}(\boldsymbol{x})$ is consistent with the intuition that higher initial inventory level leads to higher order-up-to-levels. Moreover, the two inequalities of $d(y_1, y_2)$ imply that lower price should be charged so as to reduce the inventory level of finished product; however, the reduction has bounded sensitivity. Furthermore, when c_1 and c_2 are linear, refined structure of $y(x_1, x_2)$ can derived as the problem 5.4. For simplicity, we omit the details here.

Yang (2004) considers a similar problem without pricing. He assumes that $c_1(z_1)$ is either strictly convex or linear, and $c_2(z_2)$ is linear. Different from our model in which the echelon inventory levels play the role of system states, he models the *minus cost-to-go function* v_t as below in the inventory levels of raw material and finished product.

$$\begin{aligned} v_t(x_1^0, x_2) = \quad & \underset{y_1, y_2}{\text{maximize}} \quad [-c_1(z_1^0 + z_2) - c_2(z_2) + \mathbb{E}g_t(y_1 + \varepsilon_1^0, y_2 - \varepsilon_2)] \\ & \text{subject to} \quad y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \quad y_1 \geq 0, \quad z_2 \geq 0, \end{aligned}$$

where $g_t(x_1, x_2) = -h_1(x_1^0) - h_2(x_2) + v_{t+1}(x_1, x_2)$. Yang (2004) then analyzes the related KKT conditions and inductively prove that all v_t are concave, supermodular and their Hessian matrices are diagonal dominant. Since that $L^\natural$-concavity implies concavity and the diagonal dominance property for smooth functions, our results immediately lead to the same concavity and diagonal dominance properties on v_t as Yang (2004). In addition, because $c_1(z_1^0 + z_2) + c_2(z_2)$ is supermodular in $[z_1^0, z_2]$, the supermodularity of v_t can also be obtained from Theorem 5.2.

5.3.3 Inventory control with self-financing

Consider a self-financing retailer who sells a single product over a finite planning horizon with the operational decisions limited by its cash flow. At the beginning of each period, the retailer observes the initial inventory level x_1 of the product and his/her capital level s on hand, and then places an order of size z_1 to raise the inventory level up to $y_1 = x_1 + z_1$. The order is received right away which incurs an ordering cost c per unit. We assume that the total ordering cost cz_1 can not exceed the available capital s. Unused capital $s - cz_1$ is deposited to a savings account and the earning is $r(s - cz_1)$ at the end of the period, where $r \geq 1$ and $r - 1$ is the interest rate. A demand d^ε arrives during the period. The retailer fills the demand from his/her available inventory with a unit price p and receives a revenue $p \min\{y_1, d^\varepsilon\}$ from sales. The revenue increases to $rp \min\{y_1, d^\varepsilon\}$ at the end of the period. Unused inventory is carried over to the next period and unsatisfied demand is lost, which incurs the inventory holding and shortage penalty cost $h(y_1 - d^\varepsilon)$. Assume that h is convex and $p \geq c$ (i.e., profit increases as the amount of sold product increases).

Define $x_2 = s + px_1$ as the current capital plus the revenue if all inventory on hand is sold out. It will be convenient to use $[x_1, x_2]$ as system states. Under this setting, the state $[\tilde{x}_1, \tilde{x}_2]$ in the next period satisfies $\tilde{x}_1 = (y_1 - d^\varepsilon)^+$ and $\tilde{x}_2 = r(y_2 - p\tilde{x}_1)$, where $y_2 = s - cz_1 + py_1 = x_2 + pz_1 - cz_1$.

Let $v_{T+1}(x_1, x_2) = 0$ and $v_t(x_1, x_2)$ be the profit-to-go functions in period $t = 1, \cdots, T$. The retailer's objective is to maximize the expected ending profit and faces the dynamic recursion

$$\begin{aligned} v_t(x_1, x_2) = \quad & \underset{y_1, y_2}{\text{maximize}} && \mathbb{E}\left[f_t\big(y_1 - d^\varepsilon, ry_2\big) - h(y_1 - d^\varepsilon)\right] \\ & \text{subject to} && y_1 = x_1 + z_1, \quad y_2 = x_2 + z_2, \\ &&& py_1 \leq y_2, \quad z_1 \geq 0, \quad z_2 = (p - c)z_1, \end{aligned} \tag{5.7}$$

where the expectation operator $\mathbb{E}$ associates with random variables d^ε, the constraint $py_1 \leq y_2$ corresponds to the cash flow limitation, and $f_t(x_1, x_2) = v_{t+1}(x_1^+, x_2 - px_1^+)$ with $x_1^+ = \max(x_1, 0)$. We assume that $\mathbb{E}[h(\boldsymbol{y} - \boldsymbol{\varepsilon})]$ is strictly convex to avoid technicality, which ensures the uniqueness of the optimal solution, denoted by $\boldsymbol{y}(\boldsymbol{x}) = [y_1(x_1, x_2), y_2(x_1, x_2)]$, to problem (5.7).

Apparently (5.7) is a special case of problem (5.4). We have the following results on problem (5.7).

Proposition 5.5. *All $v_t(x_1, x_2)$ are decreasing in x_1, increasing in x_2, jointly concave and supermodular. Moreover, the optimal solution $\boldsymbol{y}(\boldsymbol{x})$ is increasing in $\boldsymbol{x}$.*

Proof. Suppose these statements are true in period $t+1$, which are obvious in the last period $t = T$. From the definition of f_t, we know $f_t(x_1, x_2)$ is decreasing in x_1 and increasing in x_2. Then one can verify the monotonicity of v_t from the expression (5.7).

By Corollary 5.1, $v_{t+1}(x_1, x_2 - px_1)$ is concave and supermodular, which together with the monotonicity of v_{t+1} implies that $f_t(x_1, x_2)$ is concave and supermodular. Because $(p-c)z_1$ is increasing in z_1, properties on v_t and $\boldsymbol{y}(\boldsymbol{x})$ follow from Theorems 5.2 and 5.3. □

In Proposition 5.5, The monotonicity of $v_t(x_1, x_2)$ obeys the intuition that lower initial inventory level x_1 for higher initial total value $x_2 = s + px_1$ brings more flexibility for retailer's operations and hence leads to higher ending profit. Moreover, though omitted here, one can obtain some refined characterization of $\boldsymbol{y}(\boldsymbol{x})$ by similar arguments as problem (5.4).

A simpler version of the problem without the inventory holding and shortage penalty cost is analysed by Chao et al. (2008), where all parameters (including the cumulative distribution function of demand) are time-independent. The major difference between our model and the one in Chao et al. (2008) is the definition of system states. Chao et al. (2008) model the profit-to-go functions $v^0(x_1, s)$ in terms of the initial inventory level x_1 and capital s. Unlike our results, they only prove that $v^0(x_1, s)$ is jointly concave and increasing in s, and characterize the structure of optimal solution under some specific conditions.

5.4 Conclusion

In this chapter we study a class of two dimensional parameterized optimization problems, and establish the preservation of supermodularity together with concavity, where the constraint set may not be a lattice and may not be mapped to become one by a variable transformation. We also present several variations in Section 5.2 including the preservation of supermodularity together with the component-wise concavity, submodularity together with concavity and $L^\natural$-concavity.

Our results include several results in the literature as special cases. They significantly simplify the proofs of several operational models, some of which have not been treated rigorously, and shed new insights on these models. We believe our results can be applied in many other models.

Our results also bright up several interesting issues that need further research. First, as we comment in Example 5.2, our results can not be directly extended to higher dimensional space. A natural question is under what conditions the preservation of supermodularity in problem (5.1) holds when we have more than two parameters.

The second question is whether we can say anything about the structure of the optimal solution to problem (5.1). As we notice in Example 5.2, the optimal solution may fail to be monotone in general. It would be interesting to identify conditions under which the optimal solution is monotone.

Chapter 6

Future research

Though have been proposed over a half century, coordinated pricing and inventory management problems receive considerable attentions in the operations management community only in the past decade. Academics have recognized the importance of the coordination of different decisions which are previously made in a separate way. For example, in Chapter 3 where a deterministic model with reference price effect is developed, we pointed out through numerical examples that more than 40% profit could be potentially gained by coordinating pricing and inventory management decisions compared to a sequential decision making process. Recent years have witnessed phenomenal growth of successful deployments of pricing strategies. This thesis belongs to this stream of research on dynamic pricing and inventory problems and mainly focuses on the periodic-review models from Chapter 2 to Chapter 5. I would like to conclude this thesis by pointing out some potential directions for my furture research in addition to specific conclusions provided in previous chapters.

The first possible future research is to build and analyze more general coordinated pricing and inventory management problems. For example, one potential direction is to incorporate customer behavior into operational models. Consumer behavior has been extensively studied in the marketing and economics literature, however, its impact on pricing and inventory decisions is largely unexplored despite their profound effects in shaping consumer demand. A few works, including Chapter 3 and Chapter 4 in this thesis, have discussed several models with reference price effects, which assume that consumers will judge their purchase basing on historical selling prices. Nevertheless, this kind of work is quite limited, for example, to my best knowledge there is no research on multi-product or multi-echelon models with reference price effects so far. Furthermore, historical selling prices are not the only factor influences consumer behavior: other factors include brand loyalty, purchase frequency, promotion and so on. Incorporating customer behavior enriches the applicability of models and of course imposes enormous modeling and technical challenge. Because of the strong empirical and theoretical supports, it is definitely worth considering such extension of operations models, characterizing the associated structure of the optimal policies, and applying these results to decision support systems in practical problems.

Developing efficient algorithms for solving coordinated pricing and inventory models is another potential direction for the future research. The motivation is quite clear: efficient algorithms, either exact or heuristic, play important roles when connecting academic research to industrial practice. Well-designed algorithms should balance the accuracy and efficiency, and ensure robustness with respect to system inputs. For example, exact algorithms are given for the two deterministic models studied in Chapter 2 and Chapter 3, where computational complexity and sensitivity analysis of parameters are provided if possible. However, efficient algorithms for general stochastic, multi-product, multi-period models are poorly explored. Even in the deterministic single-product setting, incorporating pricing decisions significantly increases the computational complexity relative to pure inventory models. Because of the theoretical importance and practical relevance of stochastic models, it is critical to to develop efficient and effective computational approaches to solve them. A possible direction is to apply the so-called stochastic approximation algorithm. Such algorithm runs adaptively and is quite simple but useful. Moreover, it has been extensively applied in many research fields including operations research and management science.

Finally, advances in information technologies allow companies efficiently gather information from customers and make dynamic pricing to improve the profit. As the flip side of the coin, today customers are also able to dynamically respond to companies strategies correspondingly. For example, Groupon, a website that “features discounted gift certificates usable at local or national companies (wikipedia.org)”, plays the role of an agent standing for customers to argue with companies. Therefore it would be interesting to develop game models to capture the interaction of companies and consumers.

Appendix A

A.1 Proof of Proposition 3.1

For any feasible solution $P_{T+1} = \{p_1, \cdots, p_T\}$ to problem (3.2), let $p_t^* = \max\{p_t, \bar{c}_t\}$ at period t. The given condition in this proposition implies that $P_t = \{p_1, \cdots, p_{t-1}, p_t^*, \cdots, p_T^*\}$ is feasible to problem (3.2) for any $t = 1, \cdots, T$. We make two observations. First, the demand function $d_t(p, r)$ is non-negative, increasing in reference price r and decreasing in price p. Second, reference price at any period is independent of prices in later periods.

We next inductively prove that the profit corresponds to the price sequence

$$P_t = \{p_1, \cdots, p_{t-1}, p_t^*, \cdots, p_T^*\}$$

for $1 \leq t \leq T$ is higher than the one given by

$$P_{t+1} = \{p_1, \cdots, p_t, p_{t+1}^*, \cdots, p_T^*\}.$$

Observe that the two solutions correspond to the same accumulated profit before period t, as well as the same reference price r_t at period t. Profits associated with P_t and P_{t+1} are $(p_t^* - \bar{c}_t)d_t(p_t^*, r_t)$ and $(p_t - \bar{c}_t)d_t(p_t, r_t)$ at period t, respectively. If $p_t^* = p_t$, then both P_t and P_{t+1} correspond to the same profit at the period; otherwise $p_t^* = \bar{c}_t > p_t$, P_t corresponds to zero profit, which is higher than the negative profit associated with P_{t+1} at period t. Furthermore, for any period s with $s > t$, the demand under P_t is no less than the demand under P_{t+1} and in addition, $p_s^* - \bar{c}_s \geq 0$. It implies that the profit given by P_t is no less than that by P_{t+1} for all $s > t$.

A.2 Proof of Proposition 3.2

We introduce the following lemma, where its proof will be given later.

Lemma A.1. *$\varphi(y,z) = y\beta(z) - ay^2 - bz^2$ is jointly concave and supermodular when $y \geq 0$ if $a \geq 0$, $\beta(z)$ is concave, and there exists $\beta_0 \leq 2\sqrt{ab}$ such that $0 \leq \beta(z_1) - \beta(z_2) \leq \beta_0(z_1 - z_2)$ for all $z_1 \leq z_2$.*

By specifying $a = a_t + A_t - B_t, b = A_t - \alpha^2 B_t$ and $\beta_0 = \frac{1}{2}\eta^- + \alpha B_t - A_t$ in Lemma A.1, it immediately follows Proposition 3.2.

Next we prove Lemma A.1. Observe that $f(g_1(x_1), g_2(x_2))$ is supermodular if $f(x_1, x_2)$ is supermodular and both $g_1(x), g_2(x)$ are increasing. Specifically $y\beta(z)$ and thus $\varphi(y,z)$ are supermodular.

Given any function $f(y,z)$, $0 \leq \lambda \leq 1$ and points $(y_0, z_0), (y_1, z_1)$ with $y_0, y_1 \geq 0$, denote

$$[f((1-\lambda)y_0+\lambda y_1, (1-\lambda)z_0+\lambda z_1)-(1-\lambda)f(y_0,z_0)-\lambda f(y_1,z_1)] = (1-\lambda)\lambda\Delta(f).$$

It is straightforward to verify that $\Delta(-ay^2 - bz^2) = a(y_0 - y_1)^2 + b(z_0 - z_1)^2$. Moreover, because $y_0, y_1 \geq 0$ and $\beta(z)$ is concave, we have

$$\begin{aligned}(\mu y_0 + \lambda y_1)\beta(\mu z_0 + \lambda z_1) &\geq (\mu y_0 + \lambda y_1)[\mu\beta(z_0) + \lambda\beta(z_1)] \\ &= \mu y_0\beta(z_0) + \lambda y_1\beta(z_1) - \mu\lambda(y_0 - y_1)[\beta(z_0) - \beta(z_1)],\end{aligned}$$

which implies that $\Delta(y\beta(z)) \geq -(y_0 - y_1)[\beta(z_0) - \beta(z_1)]$. By conditions on β_0, we know that

$$\begin{aligned}\Delta(\varphi(y,z)) &\geq a(y_0 - y_1)^2 + b(z_0 - z_1)^2 - (y_0 - y_1)[\beta(z_0) - \beta(z_1)] \\ &\geq (2\sqrt{ab} - \beta_0^2)|(y_0 - y_1)(z_0 - z_1)| \geq 0,\end{aligned}$$

Therefore $\varphi(y,z)$ is concave when $y \geq 0$ by the definition.

A.3 Proof of Proposition 3.3

Let $F_t(x) = G_t(x) + A_t x^2$ and rewrite problem (3.3) as

$$\begin{aligned} F_{t+1}(r) = \quad & \underset{x}{\text{maximize}} && F_t(x) + (A_{t+1} - B_t)r^2 + \left[\Pi_t(x,r) - A_t x^2 + B_t r^2\right], \\ & \text{subject to} && \alpha x + (1-\alpha)p = r, \quad p \in [L_t, U_t], \end{aligned}$$

Recall the definition of F_t. Since we assumed $L_t \geq \bar{c}_t$ when η is concave, Proposition 3.2 ensures that $[\Pi_t(x,r) - A_t x^2 + B_t r^2]$ is concave. Note that $A_{t+1} \leq B_t$ by the condition (3.4), and the feasible set is convex in (x,r), Proposition 2.3.9 in Bertsekas et al. (2003) ensures that F_{t+1} is concave provided the concavity of F_t. Because $G_2(x) = \Pi_1(r_1, x)$, we know $F_2(x) = \Pi_1(r_1,x) + B_1 x^2 + (A_2 - B_1)x^2$ is concave in x by Proposition 3.2 again. We conclude that all F_t and G_t, $2 \leq t \leq T+1$, are concave.

We divide the remaining proof into several steps. Some useful observations on the function $\theta(x,r) = \partial_x^+[G_t(x) + \Pi_t(x,r)]$, the right derivative of the objective function of problem (3.3) at x for a fixed r, are made for further use.

1. $\theta(x,r)$ decreases in x when $r \geq \bar{c}_t$, and increases in r when $x \geq \bar{c}_t$.
2. If $\alpha > 0$, define $q(p,r) = [r - (1-\alpha)p]/\alpha$, then $\theta(q_r(p), r)$ increases in r when $p \geq \bar{c}_t$.
3. $\theta(r,r)$ decreases in r and $\theta(r,r) \leq 0$ if and only if $R \leq r$, where $R = \sup\{r : \theta(r,r) \geq 0\}$.

The first two follow from their expressions

$$\begin{aligned}
\theta(x,r) &= \partial^+ G_t(x) + \frac{\alpha}{1-\alpha}\left[\frac{2a_t(r-\alpha x)}{1-\alpha} - (b_t + a_t\bar{c}_t)\right] \\
&+ \frac{\eta^0}{1-\alpha}\left[\frac{(1+\alpha)r - 2\alpha x}{1-\alpha} - \bar{c}_t\right], \\
\theta(q_r(p), r) &= \partial^+ G_t(q_r(p)) + \frac{\alpha}{1-\alpha}\left[2a_t p - (b_t + a_t\bar{c}_t)\right] \\
&+ \frac{\eta^0}{1-\alpha}\left(2p - r - \bar{c}_t\right),
\end{aligned}$$

where $\eta^0 = \eta^+$ if $x \geq r$ and $q_r(p) \geq r$ (or $r \leq p$); otherwise $\eta^0 = \eta^-$. Moreover, we have

$$\begin{aligned}
\theta(r,r) &= \partial_x^+ G_t(r) + \frac{2\alpha a_t + \eta^+}{1-\alpha} r - \frac{\alpha(b_t + a_t\bar{c}_t) + \eta^+\bar{c}_t}{1-\alpha}, \\
&= \partial^+ F_t(r) - (\eta^- - \eta^+)r - \frac{\alpha(b_t + a_t\bar{c}_t) + \eta^+\bar{c}_t}{1-\alpha}.
\end{aligned}$$

Because F_t is concave and $\eta^+ \leq \eta^-$, it follows the last observation.

Step 1:

In this part we will claim that depending on r, it suffices to consider one quadratic piece of Π_t in problem (3.3). Specifically,

$$
\begin{aligned}
G_{t+1}(r) = \quad & \underset{x}{\text{maximize}} \quad G_t(x) + \Pi_t^j(x,r), \\
& \text{subject to} \quad \alpha x + (1-\alpha)p = r, \quad p \in [L_t, U_t],
\end{aligned}
$$

where $\Pi_t^j(x,r) = \Pi_t^-(x,r)$ if $r \in (-\infty, L_t] \cup [R, U_t)$ and $\Pi_t^j(x,r) = \Pi_t^+(x,r)$ otherwise. In the following We verify this statement in three cases.

1. For any $r < L_t$, a feasible x to problem (3.3) satisfies

   $$\alpha x \le r - (1-\alpha)L_t < r - (1-\alpha)r = \alpha r.$$

   Therefore $x < r$ and $\alpha > 0$ imply that $\Pi_t(x,r) = \Pi_t^-(x,r)$.

2. For any $r > U_t$, we have $\Pi_t(x,r) = \Pi_t^+(x,r)$ in problem (3.3) similarly.

3. For any $L_t \le r \le U_t$, note that r is feasible to problem (3.3) and $\theta(x,r)$ decreases in x by $r \ge L_t \ge \bar{c}_t$. There are two sub-cases:

   (a) if $R \le r$, then $\theta(r,r) \le 0$ hence $\theta(x,r) \le 0$ for any $x \ge r$, which implies $G_t(x) + \Pi_t(x,r)$ decreases in x when $x \ge r$. Therefore there exists an optimal solution $x^*(r)$ to problem (3.3) such that $x^*(r) \le r$. We then conclude that it leads no loss of optimality to let $\Pi_t(x,r) = \Pi_t^-(x,r)$ in problem (3.3).

   (b) if $R > r$, then we can let $\Pi_t(x,r) = \Pi_t^+(x,r)$ in problem (3.3) by using a similar argument.

Step 2:

We now distinguish whether the constraint $p \in [L_t, U_t]$ is active or not. Specifically, we will claim that

$$
G_{t+1}(r) = \begin{cases} [G_t(x) + \Pi_t(x,r) : \alpha x + (1-\alpha)U_t = r], & \text{if } r > R_U, \\ [G_t(x) + \Pi_t(x,r) : \alpha x + (1-\alpha)L_t = r], & \text{if } r < R_L, \end{cases}
$$

where $R_U = U_t$, $R_L = L_t$ if $\alpha = 0$; otherwise if $\alpha > 0$,

$$R_L = \sup\{r : \theta(q_r(L_t), r) \geq 0\}, \quad R_U = \inf\{r : \theta(q_r(U_t), r) \leq 0\}.$$

Moreover, if $r \in [R_L, R_U]$, the constraint $p \in [L_t, U_t]$ is inactive. That is,

$$\begin{aligned} G_{t+1}(r) = \quad & \underset{x}{\text{maximize}} \quad [G_t(x) + \Pi_t(x, r)] \\ & \text{subject to} \quad \alpha x + (1-\alpha)p = r. \end{aligned}$$

It is straightforward to verify the above statement when $\alpha = 0$. Note that $G_{t+1}(r) = -\infty$ when either $r > U_t$ or $r < L_t$ because such r leads to empty feasible set to problem (3.3) in this case.

When $\alpha > 0$, we can express the feasible set of problem (3.3) as an interval $[q_r(U_t), q_r(L_t)]$. In addition, $\theta(x, r)$ decreases in x on the interval by concavity of its objective function.

Different cases for $\alpha > 0$ are considered below.

1. For any $r < R_L$, we know $\theta(q_r(L_t), r) \geq 0$ because $\theta(q_r(L_t), r)$ is decreasing in r by $L_t \geq \bar{c}_t$. Moreover, from the monotonicity of $\theta(x, r)$ in x, we have

$$\theta(x, r) \geq \theta(q_r(L_t), r) \geq 0, \quad \forall x \leq q_r(L_t).$$

   Therefore $q_r(L_t)$ satisfying $\alpha q_r(L_t) + (1-\alpha)L_t = r$ solves problem (3.3).

2. For any $r > R_U$, the statement can be verified similarly.

3. For any $r \in [R_L, R_U]$, we have $\theta(q_r(L_t), r) < 0 < \theta(q_r(U_t), r)$. Hence an optimal solution $x^*(r)$ to problem (3.3) satisfies $q_r(U_t) < x^*(r) < q_r(L_t)$. That is, the boundary constraint $p \in [L_t, U_t]$ is inactive. Because of concavity of the objective function, there is no loss of optimality to remove such inactive constraint, which concludes the statement.

Step 3:

In this part we will claim that depending on r, it leads no loss of optimality to specify the quadratic piece of $G_t(x)$ in the problem (3.3). Specifically, there exists a non-decreasing sequence $\{r^n : n \leq N\}$ such that

$$\begin{aligned} G_{t+1}(r) = \quad & \underset{x}{\text{maximize}} \quad G_t^n(x) + \Pi_t(x,r), \\ & \text{subject to} \quad \alpha x + (1-\alpha)p = r, \quad p \in [L_t, U_t], \end{aligned}$$

for any $r \in (r^{n-1}, r^n]$, where $r_0 = -\infty$ and N is the number of breakpoints of G_t.

To verify the statement, define $x^*(r)$ as maximal optimal solution to problem (3.3), i.e., $x^*(r) = \max X^*(r)$ where

$$X^*(r) = \arg\max_x \{G_t(x) + \Pi_t(x,r) : \alpha x + (1-\alpha)p = r, p \in [L_t, U_t]\}.$$

Because the feasible set is increasing in r (see, e.g., Topkis, 1998, for the definition of increasing sets), and $\Pi_t(x,r)$ is supermodular by Proposition 3.2, we know $x^*(r)$ increases in r by Theorem 2.8.2 in Topkis (1998). Therefore there exists an increasing sequence $\{r^n : n \leq N\}$ such that

$$x^{n-1} < x^*(r) \leq x^n, \quad \forall r : r^{n-1} < r \leq r^n,$$

where $\{x^n : n \leq N\}$ is the breakpoint sequence of G_t. It implies that if $r^{n-1} < r \leq r^n$, then $x^{n-1} < x \leq x^n$ can be introduced as a redundant constraint to problem (3.3). Hence there is no loss of optimality to specify $G_t(x) = G_t^n(x)$ when $r^{n-1} < r \leq r^n$.

It remains to calculate $r^n, n \leq N$. Define $r_L^n = \alpha x^n + (1-\alpha)L_t$ and $r_U^n = \alpha x^n + (1-\alpha)U_t$ which are the breakpoints related to x^n of the following functions,

$$\begin{aligned} G_{t+1}^L(r) &= [G_t(x) + \Pi_t(x,r) : \alpha x + (1-\alpha)L_t = r] \\ G_{t+1}^U(r) &= [G_t(x) + \Pi_t(x,r) : \alpha x + (1-\alpha)U_t = r]. \end{aligned}$$

Moreover, for $j \in \{+,-\}$ define

$$r_j^n = \sup\left\{r : \partial_x^+ [G_t(x) + \Pi_t^j(x,r)]\Big|_{x=x^n} \leq 0\right\},$$

which are are the breakpoints related to x^n of the following functions, respectively,

$$\begin{aligned} G^+_{t+1}(r) &= \max_x [G_t(x) + \Pi^+_t(x,r)], \\ G^-_{t+1}(r) &= \max_x [G_t(x) + \Pi^-_t(x,r)]. \end{aligned}$$

By combining results from the first two steps, we know $r^n \in \{r^n_L, r^n_U, r^n_+, r^n_-\}$. The *consistence examination* can be applied to determine r^n from the four candidates. For example, if r^n_L belongs to an interval $(R_{m-1}, R_m]$ such that $G_{t+1}(r) = G^L_{t+1}(r)$, then it is consistent so that $r^n = r^n_L$. Otherwise, we do the same examination for r^n_U, r^n_+ and r^n_- sequentially. We summarize the consistence examination as below for easy reference.

1. If $\theta(x^n, r^n_L) > 0$, then $r^n = r^n_L$ and $G_{t+1}(r^n) = G^L_{t+1}(r^n)$.
2. If $\theta(x^n, r^n_U) < 0$, then $r^n = r^n_U$ and $G_{t+1}(r^n) = G^U_{t+1}(r^n)$.
3. If $r^n_+ \geq x^n$, then $r^n = r^n_+$ and $G_{t+1}(r^n) = G^+_{t+1}(r^n)$.
4. If $r^n_+ < x^n$, then $r^n = r^n_-$ and $G_{t+1}(r^n) = G^-_{t+1}(r^n)$.

The above consistence examination does not assign two different r^n and $\tilde{r}^n$ to the same x^n. Otherwise, the definition of $\{r^n\}$ implies that for either $r = r^n$ or $r = \tilde{r}^n$ we have the expression

$$\begin{aligned} G_t(x^n) + \Pi_t(x^n, r) = \quad & \underset{x}{\text{maximize}} && G_t(x) + \Pi_t(x,r) \\ & \text{subject to} && \alpha x + (1-\alpha)p = r, \quad p \in [L_t, U_t]. \end{aligned}$$

Without loss of generality, assume $r^n < \tilde{r}^n$. Since $x^*(r)$ is increasing, the above relation holds for all $r \in [r^n, \tilde{r}^n]$, which means that $\Pi_t(x^n, r)$ is a constant in the interval. It contradicts with the setting that $\Pi_t(x,r)$ is quadratic. Therefore each x^n corresponds to at most one r^n.

Step 4:

By all results so far, we can conclude the desirable expression (3.5). We next discuss the computational complexity to obtain G_{t+1} from G_t.

In **Step 1** we need to obtain the constant $R = \sup\{r : \theta(r,r) \geq 0\}$. Since G_t is concave and piecewise quadratic with N breakpoints, $\theta(r,r)$ is decreasing and consists of $N+1$ linear pieces. Therefore it takes $O(\log N)$ time to obtain R by a binary search algorithm. Similarly, we can also obtain R_L and R_U defined in **Step 2** in additional $O(\log N)$ time.

In **Step 3** we need to obtain the sequence $\{r_n : n \leq N\}$. Since both $\partial_x^+[G_t(x) + \Pi_t^+(x,r)]$ and $\partial_x^+[G_t(x) + \Pi_t^-(x,r)]$ are linear in r for any fixed x, it takes $O(1)$ time to calculate all four calculates of r_n. Therefore such sequence can be obtained in $O(N)$ time.

When determining the expression of G_{t+1} through (3.5), apparently each case of (3.5a), (3.5b) and (3.5c) can be solved in $O(1)$ time. In summary, it takes $O(N)$ time to obtain G_{t+1} from G_t.

Observe that $G_2(x) = \Pi_t(r_1, x)$. Hence each G_{t+1} is concave and consists of $O(t)$ quadratic pieces. In addition, we can obtain all G_{t+1} recursively for $t = 1, 2, \cdots, T$ in $O(T^2)$ time.

A.4 Proof of Theorem 3.2

We first verify the computational complexity to construct the network $(\mathcal{V}, \mathcal{E})$. Since $\Pi_t(x,p)$ is quadratic when $x \geq p$, it is straightforward to verify that $\Pi_t(x,p) - \frac{1}{2}\eta^+ x^2 + \frac{1}{2}\eta^+ p^2$ is jointly concave on $\{(x,p) : x \geq p\}$ when $\eta^+ \leq 2a_t$. For any $t > \tau$, define $F_{\tau,t}(p) = G_{\tau,t}(p) + \frac{1}{2}\eta^+ p^2$ and rewrite (3.7) as

$$\begin{aligned} F_{\tau,t+1}(p) = \quad & \underset{x}{\text{maximize}} \quad F_{\tau,t}(x) - \frac{1}{2}\eta^+ x^2 + \Pi_t(x,p) + \frac{1}{2}\eta^+ p^2, \\ & \text{subject to} \quad x \geq p, \quad p \in [L_t, U_t]. \end{aligned}$$

Similar to the proof of Proposition 3.3, we know both $F_{\tau,t}(p)$ and $G_{\tau,t}(p)$ are concave; moreover, $G_{\tau,t+1}(p)$ consists of $O(t-\tau)$ quadratic pieces and can be obtained in $O(t-\tau)$ time after $G_{\tau,t}(p)$ becomes available. Therefore, it takes $O(T^3)$ time to obtain all $G_{\tau,\tilde{\tau}}(p)$, and an additional $O(T^3)$ to obtain $\ell(\tau,\tilde{\tau})$ by maximizing $G_{\tau,\tilde{\tau}}(p)$ over p for all $1 \leq \tau < \tilde{\tau} \leq T+1$. Note that $G_{\tau,t}(p)$ is strictly concave when $\eta^+ > 0$, which implies that problem (3.6) yields a unique optimal solution.

We next show the equivalence of solving problem (3.2) and finding a longest path in $(\mathcal{V}, \mathcal{E})$. That is, we need to prove that the total profit incurred by an optimal price sequence is no more than the total length of some path in $(\mathcal{V}, \mathcal{E})$, and there exists a longest path in $(\mathcal{V}, \mathcal{E})$ with the total length no more than the profit associated with some feasible price sequence of problem (3.2).

On the one hand, given an optimal price sequence $\{p_1, \cdots, p_T\}$, let $1 = \tau_1 < \tau_2 < \cdots < \tau_{N+1} = T+1$ be its price markup periods. Then for any pair of consecutive price markup periods $(\tau, \tilde{\tau})$, we have $p_{\tau-1} \leq p_\tau, p_{\tilde{\tau}-1} \leq p_{\tilde{\tau}}$ and $p_t \geq p_{t+1}$ when $\tau \leq t < \tilde{\tau} - 1$. Clearly $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$ is feasible to problem (3.6), which implies the accumulated profit from period τ to $\tilde{\tau} - 1$ is no more than $\ell(\tau, \tilde{\tau})$. Therefore the total profit associated with the feasible price sequence $\{p_1, \cdots, p_T\}$ is no more than total length of the path $\{\tau_1 \tau_2 \cdots \tau_{N+1}\}$ with $\tau_1 = 1$ and $\tau_{N+1} = T + 1$.

On the other hand, suppose $\tau_1 \tau_2 \cdots \tau_N \tau_{N+1}$ with $1 = \tau_1 < \cdots < \tau_N < \tau_{N+1} = T + 1$ is a longest path in $(\mathcal{V}, \mathcal{E})$. Without loss of generality, we can assume $\ell(\tau_{n-1}, \tau_n) + \ell(\tau_n, \tau_{n+1}) > \ell(\tau_{n-1}, \tau_{n+1})$ for any $1 < n \leq N$; otherwise we can replace links (τ_{n-1}, τ_n) and (τ_n, τ_{n+1}) by a new link (τ_{n-1}, τ_{n+1}). Let $\{p_1, \cdots, p_T\}$ be the price sequence related to the longest path, i.e., $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$ solves problem (3.6) for any $(\tau, \tilde{\tau}) = (\tau_n, \tau_{n+1})$, $n = 1, \cdots, N$. If we can prove all τ_n are price markup periods, then this price sequence is feasible to problem (3.2) and its profit is the same as the length of the longest path.

Assume to the contrary that there exists a node τ on the longest path such that $p_{\tau-1} > p_\tau$. Let $\bar{\tau}, \tau, \tilde{\tau}$ with $\bar{\tau} < \tau < \tilde{\tau}$ be three consecutive nodes on the longest path. Since we assume $p_{\tau-1} > p_\tau$, $\{p_{\bar{\tau}}, \cdots, p_{\tilde{\tau}-1}\}$ is feasible to problem (3.6) with $(\tau, \tilde{\tau})$ replaced by $(\bar{\tau}, \tilde{\tau})$. It follows from $\Pi_\tau(r, p) = \Pi_\tau(p, p) + (p - \bar{c}_\tau)\eta(r - p)$ that

$$\begin{aligned}\ell(\bar{\tau}, \tilde{\tau}) &\geq \Pi_{\bar{\tau}}(p_{\bar{\tau}}, p_{\bar{\tau}}) + \sum_{t=\bar{\tau}+1}^{\tau-1} \Pi_t(p_{t-1}, p_t) + \Pi_\tau(p_{\tau-1}, p_\tau) + \sum_{t=\tau+1}^{\tilde{\tau}-1} \Pi_t(p_{t-1}, p_t) \\ &= \ell(\bar{\tau}, \tau) + \ell(\tau, \tilde{\tau}) + \eta^+\big[(p_\tau - \bar{c}_\tau)(p_{\tau-1} - p_\tau)\big]\end{aligned}$$

where the equality follows from the definition of $\{p_{\bar{\tau}}, \cdots, p_{\tau-1}\}$ and $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$. If $p_\tau \geq \bar{c}_\tau$, then $\ell(\bar{\tau}, \tilde{\tau}) \geq \ell(\bar{\tau}, \tau) + \ell(\tau, \tilde{\tau})$, which contradicts the assumption that $\ell(\bar{\tau}, \tau) + \ell(\tau, \tilde{\tau}) > \ell(\bar{\tau}, \tilde{\tau})$. It then follows that $p_{\tau-1} \leq p_\tau$ and τ is indeed a price markup period. Thus, it remains to prove $p_\tau \geq \bar{c}_\tau$.

In the case that $\bar{c}_t \leq U_t$ at all periods, Proposition 3.1 implies that it suffices to restrict our attention on $p_t \geq \bar{c}_t$ for all t. We now focus on the case when $U_t = U$ at all periods. In this case, assume that $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$

is the optimal solution to problem (3.6) with largest p_τ among all optimal solutions (it is well defined since all optimal solutions of problem (3.6) form a compact set). If $p_\tau < \bar{c}_\tau$, let s be the first period with nonnegative marginal profit after periods τ, i.e., $s = \min\{t : p_t \leq \bar{c}_t, \tau \leq t < \tilde{\tau}\}$, where we specify $s = \tilde{\tau}$ if $p_t < \bar{c}_t$ for all $\tau \leq t < \tilde{\tau}$. Since we assume $p_{\tau-1} > p_\tau$, we have that $p_t < p_{\tau-1} \leq U$ for $\tau \leq t < \tilde{\tau}$. Thus, for sufficient small $\varepsilon > 0$, $p_\tau + \varepsilon < p_{\tau-1}$ and $\{p_\tau + \varepsilon, \cdots, p_{s-1} + \varepsilon, p_s, \cdots, p_{\tilde{\tau}}\}$ is feasible to problem (3.6). Moreover, calculation shows that

$$\begin{aligned} & \Pi_\tau(p_{\tau-1}, p_\tau) + \sum_{t=\tau}^{s-1} \Pi_t(p_{t-1}, p_t) + \Pi_s(p_{s-1}, p_s) + \sum_{t=s+1}^{\tilde{\tau}-1} \Pi_t(p_{t-1}, p_t) \\ \leq \ & \Pi_\tau(p_{\tau-1}, p_\tau + \varepsilon) + \sum_{t=\tau}^{s-1} \Pi_t(p_{t-1} + \varepsilon, p_t + \varepsilon) \\ + \ & \Pi_s(p_{s-1} + \varepsilon, p_s) + \sum_{t=s+1}^{\tilde{\tau}-1} \Pi_t(p_{t-1}, p_t). \end{aligned}$$

This implies the price sequence $\{p_\tau + \varepsilon, \cdots, p_{s-1} + \varepsilon, p_s, \cdots, p_{\tilde{\tau}}\}$ gives a profit no less than that associated with $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$, which contradicts the assumption that $\{p_\tau, \cdots, p_{\tilde{\tau}-1}\}$ is the optimal solution with the largest p_τ. Therefore $p_\tau \geq \bar{c}_\tau$ and the proof is now complete.

A.5 Remarks on Theorem 3.2

In the case that neither $\bar{c}_t \leq U_t$ nor $U_t = U$ holds at some period, we are still able to solve problem (3.2) in $O(T^3)$ time. The key is to construct an expanded acyclic network and maintain the price consistency. That is, if some time period τ is included as a node in a feasible path, then it should represent a price markup period.

To take into account price consistency, we assume that $\eta^+ > 0$ for all t. This assumption ensures the uniqueness of the optimal solution to problem (3.6) (see proof of Theorem 3.2). Denote this solution by

$$\{\underline{p}(\tau, \tilde{\tau}), p_{\tau+1}, \cdots, p_{\tilde{\tau}-2}, \overline{p}(\tau, \tilde{\tau})\},$$

where the dependency on $(\tau, \tilde{\tau})$ for the first and the last elements is emphasized (other components also rely on $(\tau, \tilde{\tau})$). We now introduce the extended acyclic network $(\bar{\mathcal{V}}, \bar{\mathcal{E}})$

$$\begin{aligned}
\bar{\mathcal{V}} &= \big\{(\tau, \tilde{\tau}) : 1 \leq \tau < \tilde{\tau} \leq T+1\big\} \cup \big\{\boldsymbol{v}^0 = (1,1), \boldsymbol{v}^e = (T+1, T+1)\big\}, \\
\bar{\mathcal{E}} &= \big\{\, \langle(\bar{\tau}, \tau), (\tau, \tilde{\tau})\rangle : \overline{p}(\bar{\tau}, \tau) < \underline{p}(\tau, \tilde{\tau}), 1 \leq \bar{\tau} < \tau < \tilde{\tau} \leq T+1\big\} \\
&\quad \cup \big\{\, \big\langle \boldsymbol{v}^0, (1, \tau)\big\rangle : 2 \leq \tau \leq T+1\big\} \cup \big\{\, \langle(\tau, T+1), \boldsymbol{v}^e\rangle : 1 \leq \tau \leq T\big\},
\end{aligned}$$

where artificial nodes $\boldsymbol{v}^0, \boldsymbol{v}^e$ are the origin and the destination in the longest path problem to be constructed. Moreover, the length of any link $\langle(\bar{\tau}, \tau), (\tau, \tilde{\tau})\rangle$ containing no artificial node is given by $\ell(\bar{\tau}, \tau)$, and other links in $\bar{\mathcal{E}}$ are assigned with a zero length.

Our construction implies that a non-artificial node $(\tau, \tilde{\tau})$ represents consecutive periods starting from period τ and ending at period $\tilde{\tau} - 1$ with non-decreasing prices. In addition, two non-artificial nodes $(\bar{\tau}, \tau)$ and $(\tau', \tilde{\tau})$ are connected by a link in $\bar{\mathcal{E}}$ if and only if they share a common index which indicates a price markup period, that is, $\tau = \tau'$ and $\overline{p}(\bar{\tau}, \tau) < \underline{p}(\tau, \tilde{\tau})$. Clearly a path from $\boldsymbol{v}^0$ to $\boldsymbol{v}^e$ in the network $(\bar{\mathcal{V}}, \bar{E})$ corresponds to a feasible price sequence to problem (3.2), and an optimal solution to problem (3.2) corresponds to a path in the network.

The acyclic network contains $O(T^3)$ links, whose lengths can be constructed in $O(T^3)$ time. An additional $O(T^3)$ time allows us to find a longest path in $(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ from node $\boldsymbol{v}^0$ to node $\boldsymbol{v}^e$. Thus, problem (3.2) can be solved in $O(T^3)$ time if $r_{t+1} = p_t$, $\eta^- = 0$ and $0 < \eta^+ \leq 2a_t$ at all periods.

Observe in the loss aversion case, we assume that $\bar{c}_t \leq U_t$ for all t in Theorem 3.1. If this assumption is violated, a similar construction can be applied to the model if $r_{t+1} = p_t$, $\eta^+ = 0$ and $0 < \eta^- \leq 2a_t$ at all periods. In this case, problem (3.2) can also be solved in $O(T^3)$ time by converting it to a longest path problem in some acyclic network.

A.6 Proof of Theorem 3.4

We first prove $V^* \leq V^\varepsilon + C_T\varepsilon$. Suppose $\{p_t^* : t = 1, \cdots, T\}$ is the optimal price sequence to problem (3.1) and $\mathcal{T}^*$ is the associated optimal ordering plan. Then a feasible solution to problem (3.8) is constructed by keeping the price sequence and ordering plan unchanged (i.e. $p_t = p_t^*$ and $\mathcal{T} = \mathcal{T}^*$). In addition, let $r_t = \arg\min\{|r - r_t^*| : r \in \mathcal{R}_\varepsilon\}$ be the nearest element in $\mathcal{R}_\varepsilon$ to r_t^* for each $t \geq 2$, which implies $|r_t - r_t^*| \leq \frac{1}{2}\varepsilon$ and

$$|r_{t+1} - \alpha r_t - (1-\alpha)p_t^*| = |r_{t+1} - r_{t+1}^* + \alpha(r_t^* - r_t)| \leq \tfrac{\varepsilon}{2} + \tfrac{\alpha\varepsilon}{2} \leq \varepsilon,$$

Therefore such sequence $\{(p_t, r_t)\}$ is feasible to problem (3.8). Let V be the profit associated with the feasible solution $\{(p_t, r_t) : t = 1, \cdots, T\}$, and $\bar{c}_t$ be the marginal ordering and inventory holding costs given by the ordering plan $\mathcal{T}^*$. Because $|r_t - r_t^*| \leq \frac{\varepsilon}{2}$ and

$$|p_t - \bar{c}_t| \leq \max_{1 \leq \tau \leq t \leq T+1} \{|L_t - c(\tau, t)|, |U_t - c(\tau, t)|\},$$

we have that

$$|d_t^* - d_t| = |\eta(r_t^* - p_t^*) - \eta(r_t - p_t^*)| \leq \tfrac{1}{2} \max\{\eta^+, \eta^-\}\varepsilon.$$

which then implies that

$$|V - V^*| \leq \sum_{t=1}^{T} |p_t - \bar{c}_t||d_t - d_t^*| \leq C_T \varepsilon.$$

Hence it follows that $V^* \leq V + C_T\varepsilon \leq V^\varepsilon + C_T\varepsilon$.

Since $V_0^\varepsilon \leq V^*$ is trivial, it remains to show $V^\varepsilon \leq C_T^- \varepsilon + V^*$. Suppose $\{r_t^\varepsilon : t = 1, \cdots, T\}$ is the optimal reference price sequence to problem (3.8) and $\mathcal{T}^\varepsilon$ is the optimal ordering plan. Then a feasible solution to problem (3.1) is constructed by keeping the pricing and ordering plan unchanged (i.e. $p_t = p_t^\varepsilon$ and $\mathcal{T} = \mathcal{T}^\varepsilon$), and generating r_t through $r_{t+1} = \alpha r_t + (1-\alpha)p_t$. Let V be the profit associated with the feasible pricing sequence $\{(r_t, p_t) : t = 1, \cdots, T\}$ and the ordering plan $\mathcal{T}$ in problem (3.1). Since $p_t = p_t^\varepsilon$, $r_1^\varepsilon = r_1$ and

$$|r_{t+1}^\varepsilon - r_{t+1}| = |r_{t+1}^\varepsilon - \alpha r_t^\varepsilon - (1-\alpha)p_t^\varepsilon + \alpha(r_t^\varepsilon - r_t)| \leq \varepsilon + \alpha|r_t^\varepsilon - r_t|,$$

the following inequality can be proved recursively.

$$|r_t^\varepsilon - r_t| \leq \sum_{\tau=1}^{t-1} \alpha^\tau \varepsilon \leq \varepsilon \min\left\{(1-\alpha)^{-1}, (t-1)\right\},$$

which in turn implies $|d_t^* - d_t| \leq \max\{\eta^+, \eta^-\} \max\{(1-\alpha)^{-1}, t-1\}\varepsilon$. Thus, similar to the argument in the above paragraph, it follows that

$$V^{\varepsilon} \leq V + C_T^{-}\varepsilon \leq V^{*} + C_T^{-}\varepsilon.$$

In summary, we conclude the desirable inequalities.

Appendix B

B.1 Proof of Proposition 5.1

Apparently the concavity of g in term of $\boldsymbol{y}$ implies that $f(x_1, x_2)$ is concave in x_2. Rewrite the problem as below, which is a special case of (5.1) with $[1, \mathbf{0}]$ being the first row of the matrix A.

$$f(x_1, x_2) = \underset{(y_1, \boldsymbol{y}) \in \boldsymbol{D}}{\text{maximize}} \; \{g(y_1, \boldsymbol{y}) : y_1 = x_1, a_1 y_1 + \boldsymbol{\alpha}' \boldsymbol{y} = x_2\}.$$

For any $\boldsymbol{y}_+ = (y_1, \boldsymbol{y}), \tilde{\boldsymbol{y}}_+ = (\tilde{y}_1, \tilde{\boldsymbol{y}}) \in \boldsymbol{D}$, note that

$$(A\boldsymbol{y}_+) \wedge (A\tilde{\boldsymbol{y}}_+) = A(\boldsymbol{y}_+ \wedge \tilde{\boldsymbol{y}}_+).$$

Following the same proof of Theorem 5.1, we define $\boldsymbol{a}$ from any two $\boldsymbol{x}, \tilde{\boldsymbol{x}}$. It is easy to verify that $\boldsymbol{a} = [a_1, a_2]$ and $\boldsymbol{x} \wedge \tilde{\boldsymbol{x}} = [s_1, s_2]$ satisfy $a_1 = s_1$. Hence the concavity of $f(x_1, x_2)$ in x_2 completes the proof, too.

B.2 Proof of Corollary 5.1

The basic idea is to de composite P as $P = LU$ for some triangle matrices L and U, then sequentially discuss $g_1(\boldsymbol{x}) = g(L\boldsymbol{x})$ and $g_2(\boldsymbol{x}) = g_1(U\boldsymbol{x}) = g(P\boldsymbol{x})$. The statement is straightforward when both diagonal entries of P are zero. Without loss of generality we assume that the first diagonal entry of P is 1. Two cases are considered depending on the sign of $|P|$.

If $|P| \geq 0$ then we can express $P = LU$ as below:

$$P = \begin{bmatrix} 1 & -p \\ -\bar{p} & p_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\bar{p} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -p \\ 0 & |P| \end{bmatrix} = LU,$$

where $p, \bar{p}$ and p_2 are some non-negative real numbers. Denote $g_1(\boldsymbol{x}) = g(L\boldsymbol{x})$, i.e., $g_1(x_1, x_2) = g(x_1, x_2 - \bar{p}x_1)$. Apparently g_1 is component-wise concave. Moreover, we have

$$g_1(x_1, x_2) = \text{maximize } \{g(x_1, y) : \bar{p}x_1 + y = x_2, [x_1, y] \in \boldsymbol{D}\}.$$

Therefore g_1 is supermodular on $\{\boldsymbol{x} : L\boldsymbol{x} \in \boldsymbol{S}\}$ by Proposition 5.1. Following a similar argument, we can verify the component-wise concavity and supermodularity of $g_2(\boldsymbol{x}) = g_1(U\boldsymbol{x})$ on $\{\boldsymbol{x} : LU\boldsymbol{x} \in \boldsymbol{S}\}$, i.e., $g(P\boldsymbol{x})$ on $\{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{S}\}$.

If $|P| < 0$, consider $g(PJ_0\boldsymbol{x})$ for the linear transformation J_0 mapping a vector $[x_1, x_2]$ to $[x_2, x_1]$. Because $|PJ_0| = -|P| > 0$, and that $g(P\boldsymbol{x})$ on $\{\boldsymbol{x} : P\boldsymbol{x} \in \boldsymbol{S}\}$ is component-wise concave and supermodular if and only of so is $g(PJ_0\boldsymbol{x})$ on $\{\boldsymbol{x} : PJ_0\boldsymbol{x} \in \boldsymbol{S}\}$, we conclude this proof immediately.

B.3 Proof of Proposition 5.2

Recall that $B'B$ has non-negative diagonal entries and non-positive off-diagonal entries. If $B'B$ is singular, then some real numbers λ_1, λ_2, vector $\boldsymbol{v}$ satisfy that $\lambda_1\lambda_2 \leq 0$ and $B\boldsymbol{x} = (\lambda_1 x_1 + \lambda_2 x_2)\boldsymbol{v}$ for all $\boldsymbol{x} = [x_1, x_2]$. In this case f depends on $\boldsymbol{x}$ through $\lambda_1 x_1 + \lambda_2 x_2$ hence its supermodularity follows from its concavity. It leads no loss of generality to assume $B'B$ is non-singular.

For any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \boldsymbol{S}$, let $\boldsymbol{y}, \tilde{\boldsymbol{y}}$ be the corresponding optimal solutions. Since $\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}, \boldsymbol{y} \vee \tilde{\boldsymbol{y}} \in \boldsymbol{D}$, there exist $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{S}$ such that

$$A(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) = B\boldsymbol{a} \text{ and } A(\boldsymbol{y} \vee \tilde{\boldsymbol{y}}) = B\boldsymbol{b}.$$

By $B'A \geq 0$, it leads to

$$B'B\boldsymbol{a} = B'A(\boldsymbol{y} \wedge \tilde{\boldsymbol{y}}) \leq (B'A\boldsymbol{y}) \wedge (B'A\tilde{\boldsymbol{y}}) = (B'B\boldsymbol{x}) \wedge (B'B\tilde{\boldsymbol{x}}).$$

Note that the inverse of $B'B$ is non-negative. We know $\boldsymbol{a} \leq \boldsymbol{x} \wedge \tilde{\boldsymbol{x}}$ and in a similar way, $\boldsymbol{x} \vee \tilde{\boldsymbol{x}} \leq \boldsymbol{b}$. By the same remaining part of the proof of Theorem 5.1, f is concave and supermodular on $\boldsymbol{S}$.

B.4 Proof of Lemma 5.2

When $\boldsymbol{S}$ has the form (5.3), calculation shows that

$$\begin{aligned} J_1(\boldsymbol{S}) &= \{[x_1, x_2] : l_1 \le x_1 - x_2 \le u_1,\ l_2 \le -x_2 \le u_2,\ l_0 \le x_1 \le u_0\}, \\ J_2(\boldsymbol{S}) &= \{[x_1, x_2] : l_1 \le -x_1 \le u_1,\ l_2 \le x_2 - x_1 \le u_2,\ l_0 \le -x_2 \le u_0\}. \end{aligned}$$

Therefore both $J_1(\boldsymbol{S})$ and $J_2(\boldsymbol{S})$ have the form (5.3), too.

We next show (a) and (d) are equivalent. Let $\psi(x_1, x_2, \xi) = f(x_1 - \xi, x_2 - \xi)$ and $\boldsymbol{S}_+ = \{[x_1, x_2, x_3] : [x_1 - x_3, x_2 - x_3] \in \boldsymbol{S}\}$ be its domain. Observe that

$$f(J_1[x_1 - \xi, x_2 - \xi]) = \psi(x_1, \xi, x_2),\ \ f(J_2[x_1 - \xi, x_2 - \xi]) = \psi(\xi, x_2, x_1).$$

Therefore all the three functions in (d) are respectively supermodular on their domains if and only if ψ is supermodular in any two of its variables with the other one fixed on $\boldsymbol{S}_+$. On one hand from the definition we know that f is $L^\natural$-concave if and only if ψ is supermodular on $\boldsymbol{S}_+$. We then conclude the equivalence between (a) and (d) by Theorems 2.6.1 and 2.6.2, Topkis (1998).

By the equivalence between (a) and (d), $f(J_1\boldsymbol{x})$ is $L^\natural$-concave on $J_1(\boldsymbol{S})$ if and only if $f(J_1\boldsymbol{x}), f(J_1^2\boldsymbol{x}) = f(\boldsymbol{x})$ and $f(J_2 J_1\boldsymbol{x})$ are respectively supermodular on their domains. Notice that $J_2 J_1 = J_0 J_2$ where J_0 is the linear transformation mapping a vector $[x_1, x_2]$ to $[x_2, x_1]$, and that a function $g(x_1, x_2)$ is supermodular in $[x_1, x_2]$ if and only if so is the function $g(x_2, x_1)$ in $[x_1, x_2]$. Therefore $f(J_2 J_1\boldsymbol{x})$ is supermodular if and only if so is $f(J_2\boldsymbol{x})$. We then conclude the equivalence between (b) and (d). Similarly, (c) and (d) are equivalent, too.

B.5 Proof of Theorem 5.3

(a) Since f is separable, we can rewrite (5.4) as

$$\underset{\boldsymbol{y}=[y_1,y_2]}{\text{maximize}}\ \{[f_1(y_1 - x_1) + f_2(y_2 - x_2) + \bar{v}_+(\boldsymbol{y})] : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y} - \boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}]\},$$

where the objective, regarded as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, is supermodular (Lemma 2.6.2, Topkis, 1998) and the set $\{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{l} \le \boldsymbol{y} - \boldsymbol{x} \le \boldsymbol{u}\}$ forms a sublattice (Example 2.2.7, Topkis, 1998). The monotonicity of $\boldsymbol{y}(\boldsymbol{x})$

follows from Theorem 2.8.1, Topkis (1998).

(b) Note that $J\boldsymbol{y}(J\boldsymbol{x})$ solves the problem

$$\underset{\boldsymbol{y}}{\text{maximize}} \ \{f(J\boldsymbol{y}) + \bar{v}_+(J\boldsymbol{y}) : \boldsymbol{y} \in J(\boldsymbol{Y}(J\boldsymbol{x}))\},$$

where the objective is supermodular by Lemma 5.1, and $J(\boldsymbol{Y}(J\boldsymbol{x}))$ forms a sublattice by Example 2.2.7, Topkis (1998) and that all h_n are monotone. Therefore $J\boldsymbol{y}(J\boldsymbol{x})$ is increasing in $\boldsymbol{x}$ as we proved in part(a). The monotonicity of $\boldsymbol{y}(\boldsymbol{x})$ then follows.

B.6 Characterization of the optimal solution to problem (5.4) for linear f_t

Recall the definition of function $v(\boldsymbol{y})$ in the proof of Theorem 5.3, which is concave and supermodular (submodular) if v_{t+1} is concave and supermodular (submodular). Suppose $\boldsymbol{y}_0(\boldsymbol{x})$ maximizes $v(\boldsymbol{y})$ over $\boldsymbol{Y}$. Notice that $\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x})$ if $\boldsymbol{z}_0(\boldsymbol{x}) \in \boldsymbol{Z}$, where $\boldsymbol{z}_0(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) - \boldsymbol{x}$. Otherwise $\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) - \boldsymbol{x}$ belongs to the boundary of $\boldsymbol{Z}$. We only need to characterize $\boldsymbol{y}(\boldsymbol{x})$ for the latter case $\boldsymbol{z}_0(\boldsymbol{x}) \notin \boldsymbol{Z}$.

We start from $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$. When $\boldsymbol{z}(\boldsymbol{x}) = [z_1(x_1, x_2), z_2(x_1, x_2)]$ belongs to the boundary of $\boldsymbol{Z}$, there are four possible cases: $z_1(x_1, x_2) = l_1$, $z_2(x_1, x_2) = l_2$, $z_1(x_1, x_2) = u_1$ and $z_2(x_1, x_2) = u_2$. We focus on the first case; the others can be discussed by similar arguments. If $z_1(x_1, x_2) = l_1$ or equivalently $y_1(x_1, x_2) = x_1 + l_1$, $y_2(x_1, x_2)$ must solve the problem

$$\text{maximize} \ \{v(x_1 + l_1, y_2) : [x_1 + l_1, y_2] \in \boldsymbol{Y}, \ x_2 + l_2 \le y_2 \le x_2 + u_2\}.$$

Relax the constraint $x_2 + l_2 \le y_2 \le x_2 + u_2$ and denote $\bar{y}_2(x_1)$ as the related optimal solution. Because this is a concave maximization problem,

$$y_2(x_1, x_2) = \max\{x_2 + l_2, \min[\bar{y}_2(x_1), x_2 + u_2]\}.$$

Let $\gamma_1(x_1) = \bar{y}_2(x_1) - l_2$ and $\gamma_2(x_1) = \bar{y}_2(y_1) - u_2$, which are increasing (decreasing) functions by Theorem 5.3 if v is supermodular (submodular) and all h_n are bimonotone (monotone) in problem (5.4). Partition the state space of $\boldsymbol{x}$ by curves $x_2 = \gamma_1(x_1)$ and $x_2 = \gamma_2(x_1)$. Then it is optimal to let $y_2 = x_2 + l_2$ when $\boldsymbol{x}$ lies above the curve $x_2 = \gamma_1(x_1)$, $y_2 = x_2 + u_2$ when $\boldsymbol{x}$ lies below the curve $x_2 = \gamma_2(x_1)$, and $y_2 = \bar{y}_2(x_1)$ otherwise.

The structure of $\boldsymbol{y}(\boldsymbol{x})$ is conceptually illustrated in Figure B.1 and Figure B.2 when $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$, where the former corresponds to supermodular $\bar{v}_+$ and bimonotone h_n, and the latter corresponds to submodular $\bar{v}_+$ and monotone h_n. The space of $\boldsymbol{x}$ is partitioned into nine areas by four curves $x_2 = \gamma_k(x_1)$, $1 \le k \le 4$, where all functions γ_k are increasing (decreasing) if v_{t+1} is supermodular (submodular) and all h_n are bimonotone (monotone).

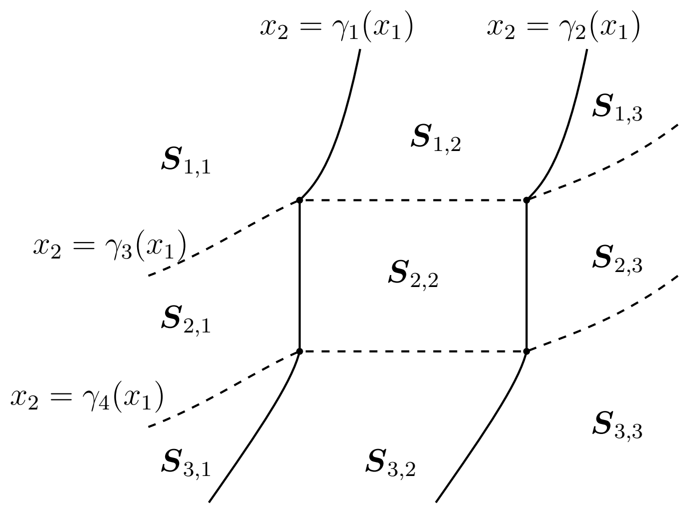

Figure B.1: Structure of $\boldsymbol{y}(\boldsymbol{x})$ for $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$: Supermodular case.

The structure of $\boldsymbol{y}(\boldsymbol{x})$ is described as below:

1. If $\boldsymbol{x} = [x_1, x_2]$ lies above the curve $x_2 = \gamma_3(x_1)$, then the constraint $y_2 - x_2 \ge l_2$ is active hence $y_2(x_1, x_2) = x_2 + l_2$. If $\boldsymbol{x}$ lies below the curve $x_2 = \gamma_4(x_1)$, then the constraint $y_2 - x_2 \le u_2$ is active hence $y_2(x_1, x_2) = x_2 + u_2$. If $\boldsymbol{x}$ lies between the two curves, then it leads no loss of optimality to remove constraints $l_2 \le y_2 - x_2 \le u_2$.

2. If $\boldsymbol{x} = [x_1, x_2]$ lies on the left side of the curve $x_2 = \gamma_1(x_1)$, then the constraint $y_1 - x_1 \le u_1$ is active hence $y_1(x_1, x_2) = x_1 + u_1$. If $\boldsymbol{x}$ lies on the right side of the curve $x_2 = \gamma_2(x_1)$, then the constraint $y_1 - x_1 \ge l_1$ is active hence $y_1(x_1, x_2) = x_1 + l_1$. If $\boldsymbol{x}$ lies between the two curves, then it leads no loss of optimality to remove constraints $l_1 \le y_1 - x_1 \le u_1$.

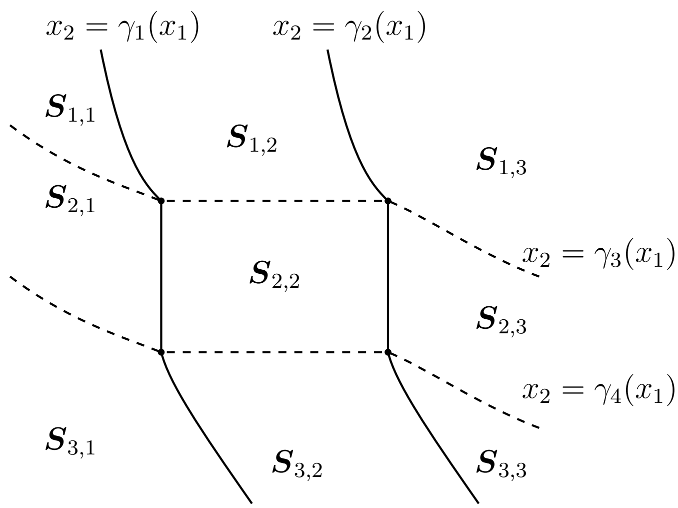

Figure B.2: Structure of $\boldsymbol{y}(\boldsymbol{x})$ for $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$: Submodular case.

We can characterize $\boldsymbol{y}(\boldsymbol{x})$ for $\boldsymbol{x}$ in each area. For example, if $\boldsymbol{x} \in \boldsymbol{S}_{1,2}$, i.e., $\boldsymbol{x}$ lies above $x_2 = \gamma_3(x_1)$ and between $x_2 = \gamma_1(x_1)$ and $x_2 = \gamma_2(x_1)$, then only the constraint $y_2 - x_2 \geq l_2$ is active. It is optimal to let $y_2(x_1, x_2) = x_2 + l_2$ and $y_1(x_1, x_2)$ maximizes $v(y_1, x_2 + l_2)$ over $\boldsymbol{Y}$. If $\boldsymbol{x} \in \boldsymbol{S}_{2,2}$, then it leads no loss of generality to remove the constraint $\boldsymbol{y} - \boldsymbol{x} \in [\boldsymbol{l}, \boldsymbol{u}]$. If $\boldsymbol{x} \in \boldsymbol{S}_{3,2}$, only the constraint $y_2 - x_2 \leq u_2$ is active therefore $y_2(x_1, x_2) = x_2 + u_2$ and $y_1(x_1, x_2)$ maximizes $v(y_1, x_2 + u_2)$ over $\boldsymbol{Y}$. Similar arguments can be made when $\boldsymbol{x}$ falls into other areas.

Next we consider $\boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h(\boldsymbol{z}) \leq 0\}$ for some convex h. Let $\bar{\boldsymbol{y}}(\boldsymbol{x})$ be the optimal solution associated with $\boldsymbol{Z} = [\boldsymbol{l}, \boldsymbol{u}]$, and $\bar{\boldsymbol{z}}(\boldsymbol{x}) = \bar{\boldsymbol{y}}(\boldsymbol{x}) - \boldsymbol{x}$. If $h(\bar{\boldsymbol{z}}(\boldsymbol{x})) \leq 0$, then $\boldsymbol{y}(\boldsymbol{x}) = \bar{\boldsymbol{y}}(\boldsymbol{x})$. Therefore we only needs to discuss these $\boldsymbol{x} \in \boldsymbol{\Omega} = \{\boldsymbol{x} : h(\bar{\boldsymbol{z}}(\boldsymbol{x})) \geq 0\}$.

Observe that $h(\boldsymbol{z}(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} \in \boldsymbol{\Omega}$. If h is bimonotone (monotone) then $h(z_1, z_2) = 0$ determines some increasing (decreasing) function $z_2 = \alpha(z_1)$. Let $\boldsymbol{a} = [l_0, \alpha(l_0)]$ and $\boldsymbol{b} = [u_0, \alpha(u_0)]$ be the intersection points of the curve $h(\boldsymbol{z}) = 0$ and the boundary of $[\boldsymbol{l}, \boldsymbol{u}]$. Then

$$\boldsymbol{y}(\boldsymbol{x}) = \arg\max \{v(\boldsymbol{y}) : \boldsymbol{y} \in \boldsymbol{Y}, \boldsymbol{y} - \boldsymbol{x} = [\xi, \alpha(\xi)],\ l_0 \leq \xi \leq u_0\}.$$

Recall that $\boldsymbol{y}_0(\boldsymbol{x})$ maximizes v over $\boldsymbol{y}$ and $\boldsymbol{z}_0(\boldsymbol{x}) = \boldsymbol{y}_0(\boldsymbol{x}) - \boldsymbol{x}$. Again, because it is a concave maximization problem, we can further partition the set $\boldsymbol{\Omega}$ into three parts depending on whether $\boldsymbol{z}_0(\boldsymbol{x}) < \xi$, $\xi \leq \boldsymbol{z}_0(\boldsymbol{x}) \leq \alpha(\xi)$

or $\boldsymbol{z}_0(\boldsymbol{x}) > \alpha(\xi)$.

Figure B.3 conceptually illustrates the structure of $\boldsymbol{y}(\boldsymbol{x})$ associated with $\boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h(\boldsymbol{z}) \leq 0\}$ for some linear $h(z_1, z_2) = z_1 + z_2 - k_0$. Comparing to Figure B.2, we further partition the space of $\boldsymbol{x}$ by adding one curve $x_2 = \bar{\gamma}(x_1)$ corresponding to the $h(\boldsymbol{z}) \leq 0$ such that the additional constraint is active if and only if $\boldsymbol{x} \in \boldsymbol{\Omega} = \{[x_1, x_2] : x_2 \leq \bar{\gamma}(x_1)\}$. $\boldsymbol{\Omega}$ is further partitioned into three parts $\boldsymbol{\Omega}_m, m = 1, 2, 3$, by two curves such that $\boldsymbol{z} = \boldsymbol{b}$ when $\boldsymbol{x} \in \boldsymbol{\Omega}_1$, the constraint $\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}]$ is inactive when $\boldsymbol{x} \in \boldsymbol{\Omega}_2$, and $\boldsymbol{z} = \boldsymbol{a}$ when $\boldsymbol{x} \in \Omega_3$. When $\boldsymbol{x} \notin \boldsymbol{\Omega}$, the characterization of $\boldsymbol{y}(\boldsymbol{x})$ is similar as $\bar{\boldsymbol{y}}(\boldsymbol{x})$.

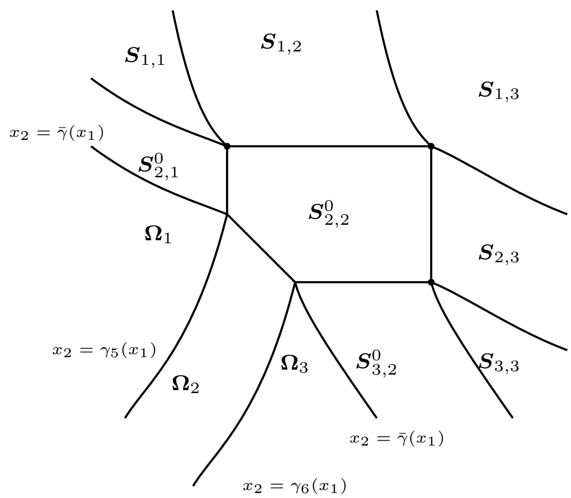

Figure B.3: Structure of $\boldsymbol{y}(\boldsymbol{x})$ for $\boldsymbol{Z} = \{\boldsymbol{z} \in [\boldsymbol{l}, \boldsymbol{u}] : h(\boldsymbol{z}) \leq 0\}$

When more constraints of the form $h_n(\boldsymbol{z}) \leq 0$ are involved in the expression of $\boldsymbol{Z}$, we can repeat the above discussions by adding constraints step by step, then characterize $\boldsymbol{y}(\boldsymbol{x})$ by further partitioning the space of $\boldsymbol{x}$.

*

References

Aggarwal, A., J. K. Park. 1993. Improved algorithms for economic lot size problems. *Oper. Res.* **41**(3) 549–571.

Aguirregabiria, V. 1999. The dynamics of markups and inventories in retailing firms. *Review of Economic Studies* **66** 275–308.

Ahn, H., M. Gumus, P. Kaminsky. 2007. Pricing and manufacturing decisions when demand is a function of prices in multiple periods. *Operations Research-Baltimore* **55**(6) 1039–1057.

Bergen, M., M. Ritson, S. Dutta, D. Levy, M. Zbaracki. 2003. Shattering the myth of costless price changes. *European Management Journal* **21** 663–669.

Bertsekas, D.P., A. Nedić, A.E. Ozdaglar. 2003. *Convex analysis and optimization*. Athena Scientific.

Briesch, R.A., L. Krishnamurthi, T. Mazumdar, S.P Raj. 1997. A comparative analysis of reference price models. *Journal of Consumer Research* **24**(2) 202–214.

Celik, S., A. Muharremoglu, S. Savin. 2009. Revenue management with costly price adjustments. *Operations research* **57**(5) 1206–1219.

Ceryan, O., O. Sahin, I. Duenyas. 2009. Managing demand and supply for multiple products through dynamic pricing and capacity flexibility. under submission .

Chan, L. M. A., Z. J. M. Shen, D. Simchi-Levi, J. Swann. 2004. Coordination of pricing and inventory decisions: A survey and classification. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era* 335–392.

Chao, X., H. Chen, S. Zheng. 2009. Dynamic capacity expansion for a service firm with capacity deterioration and supply uncertainty. *Operations Research* **57**(1) 82–93.

Chao, X., J. Chen, S. Wang. 2008. Dynamic inventory management with cash flow constraints. *Naval Research Logistics* **55**(8) 758–768.

Chen, H., O. Wu, D.D. Yao. 2004. Optimal pricing and replenishment in a single-product inventory system. Tech. rep., Working Paper.

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research* **52** 887–896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research* **29** 698–723.

Chen, X., D. Simchi-Levi. 2011. Pricing and inventory management. *Handbook of Pricing Management, eds. Ozer and Philips, to appear* .

Chen, X., S.X. Zhou, Y. Chen. 2008. Integration of inventory and pricing decisions with costly price adjustments. *Working Paper* .

Deng, S., C. A. Yano. 2006. Joint production and pricing decisions with setup costs and capacity constraints. *Manage. Sci.* **52** 741–756.

Eliashberg, J., R. Steinberg. 1991. Competitive strategies for two firms with asymmetric production cost structures. *Management Sci.* **37** 1452–1473.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Sci.* **49** 1287–1309.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Operations Research* **47**(3) 454–475.

Federgruen, A., M. Tzur. 1991. A simple forward algorithm to solve general dynamic lot sizing models with n periods in $O(n \log n)$ or $O(n)$ time. *Management Sci.* **37**(8) 909–925.

Fibich, G., A. Gavious, O. Lowengart. 2003. Explicit solutions of optimization models and differential games with nonsmooth (asymmetric) reference-price effects. *Operations Research* **51**(5) 721–734.

Geunes, J., Y. Merzifonluoglu, H. E. Romeij. 2008. Capacitated procurement planning with price-sensitive demand and general concave revenue functions. *Eur. J. Oper. Res.* to appear.

Geunes, J., H. E. Romeijn, K. Taaffe. 2006. Requirements planning with pricing and order selection flexibility. *Oper. Res.* **54** 394–401.

Gilbert, S. M. 2000. Coordination of pricing and multiple-period production across multiple constant priced goods. *Management Sci.* **46** 1602–1616.

Gimpl-Heersink, D. I. L. 2008. Joint pricing and inventory control under reference price effects. Ph.D. thesis, Vienna University of Economics and Business Administration.

Gong, X, X. Chao. 2011. Optimal control policy for capacitated inventory systems with remanufacturing. *working paper* .

Gong, X, S. X. Zhou. 2011. Optimal production planning with emissions trading. *submitted* .

Greenleaf, E.A. 1995. The impact of reference price effects on the profitability of price promotions. *Marketing Science* **14**(1) 82–82.

Guler, M., T. Bilgic, R. Gullu. 2010. Joint inventory and pricing decisions with reference effects. Tech. rep., Working paper.

Hardie, B.G.S., E.J. Johnson, P.S. Fader. 1993. Modeling loss aversion and reference dependence effects on brand choice. *Marketing Science* **12**(4) 378–394.

Huh, W. T., G. Janakiraman. 2008. (s, S) optimality in joint inventory-pricing control: An alternate approach. *Oper. Res.* **56** 783–790.

Kalyanaram, G., J.D.C. Little. 1994. An empirical analysis of latitude of price acceptance in consumer package goods. *Journal of Consumer Research* **21**(3) 408–418.

Kano, K. 2006. Menu costs, strategic interactions, and retail price movements. *Working* .

Kopalle, P.K., R.S. Winer. 1996. A dynamic model of reference price and expected quality. *Marketing Letters* **7**(1) 41–52.

Kopalle, Praveen K., Ambar G. Rao, Jo L. Assuno. 1996. Asymmetric reference price effects and dynamic pricing policies. *Marketing Science* **15**(1) 60–85.

Krishnamurthi, L., T. Mazumdar, S. P. Raj. 1992. Asymmetric response to price in consumer brand choice and purchase quantity decisions. *The Journal of Consumer Research* **19**(3) 387–400.

Kunreuther, H., L. Schrage. 1973. Joint pricing and inventory decisions for constant priced items. *Management Sci.* **19**(7) 732–738.

Levy, D., M. Bergen, S. Dutta, R. Venable. 1997. The magnitude of menu costs: Direct evidence from large u.s. supermarket chains*. *Quarterly Journal of Economics* **112** 791–825.

Mayhew, G. E., R. S. Winer. 1992. An empirical analysis of internal and external reference prices using scanner data. *The Journal of Consumer Research* **19**(1) 62–70.

Mazumdar, T., S. P. Raj, I. Sinha. 2005. Reference price research: Review and propositions. *Journal of Marketing* **69**(4) 84–102.

Murota, K. 2003. *Discrete convex analysis*. Society for Industrial and Applied Mathematics, Philadelphia.

Netessine, S. 2006. Dynamic pricing of inventory/capacity with infrequent price changes. *Eur. J. Oper. Res.* **174** 553–580.

Petruzzi, N.C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Oper. Res.* **47** 183–194.

Popescu, I., Y. Wu. 2007. Dynamic pricing strategies with reference effects. *Operations Research* **55**(3) 413–429.

Putler, D.S. 1992. Incorporating reference price effects into a theory of consumer choice. *Marketing Science* **11**(3) 287–309.

Raman, K., F.M. Bass. 2002. A general test of reference price theory in the presence of threshold effects. *Tijdschrift Voor Economie En Management* **47**(2) 205–226.

Rockafellar, R.T. 1970. *Convex analysis*. Princeton University Press.

Shaked, M., J.G. Shanthikumar. 1994. *Stochastic orders and their applications*. Academic Press.

Simchi-Levi, D., X. Chen, J. Bramel. 2005. *The logic of logistics: theory, algorithms and applications for logistics and supply chain management*. Springer Science.

Slade, M. E. 1998. Optimal pricing with costly adjustment: Evidence from retail-grocery prices. *The Review of Economic Studies* **65** 87–107.

Song, J.S., Z. Xue. 2007. Demand management and inventory control for substitutable products. Tech. rep., Working Paper.

Song, Y., S. Ray, T. Boyaci. 2009. Technical note—optimal dynamic joint inventory-pricing control for multiplicative demand with fixed order costs and lost sales. *Operations research* **57**(1) 245–250.

Thomas, J. 1970. Price-production decisions with deterministic demand. *Management Science* **16**(11) 747–750.

Topkis, D.M. 1998. *Supermodularity and complementarity*. Princeton University Press.

Tversky, A., D. Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics* **106**(4) 1039.

Urban, T.L. 2008. Coordinating pricing and inventory decisions under reference price effects. *International Journal of Manufacturing Technology and Management* **13**(1) 78–94.

van dan Heuvel, W., A. P. M. Wagelmans. 2006. A polynomial time algorithm for a deterministic joint pricing and inventory model. *Eur. J. Oper. Res.* **170**(2) 463–480.

Wagelmans, A. P. M., S. van Hoesel, A. Kolen. 1992. Economic lot sizing: An O($n \log n$) algorithm that runs in linear time in the wagner-whitin case. *Oper. Res.* **40** S145–S156.

Wagner, H. M., T. M. Whitin. 1958a. Dynamic version of the economic lot size model. *Management Sci.* **5** 89–96.

Wagner, H.M., T.M. Whitin. 1958b. Dynamic problems in the theory of the firm. *Naval Research Logistics Quarterly* **5**(1) 53–74.

Weiss, R.M., A.K. Mehrotra. 2001. Online dynamic pricing: Efficiency, equity and the future of e-commerce. *Virginia Journal of Law and Technology Association* **6** 11–17.

Whitin, T.M. 1955. Inventory control and price theory. *Management Science* 61–68.

Yang, J. 2004. Production control in the face of storable raw material, random supply, and an outside market. *Operations Research* **52**(2) 293–311.

Yano, C. A., S. M. Gilbert. 2003. *Coordinated Pricing and Production/Procurement Decisions: A Review. J. Eliashberg, A. Chakravarty, eds. Managing Business Interfaces: Marketing, Engineering, and Manufacturing Perspectives*. Kluwer, Norwell, MA.

Zbaracki, M.J., M. Ritson, D. Levy, S. Dutta, M. Bergen. 2004. Managerial and customer costs of price adjustment: direct evidence from industrial markets. *Review of Economics and Statistics* **86**(2) 514–533.

Zhang, Yuhan. 2011. Essays on robust optimization, integrated inventory and pricing, and reference price effect. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Zhu, K., U. W. Thonemann. 2009. Coordination of pricing and inventory control across products. *Naval Research Logistics* **56**(2) 175–190.

Zipkin, P. 2000. *Foundations of inventory management*, vol. 2. McGraw-Hill Boston, MA.

Zipkin, P. 2003. Some modularity properties of linear programs. *Working Paper* .

Zipkin, P. 2008. On the structure of lost-sales inventory models. *Operations Research* **56**(4) 937–944.