COVERING AND PACKING PROBLEMS ON GRAPHS AND HYPERGRAPHS

BY

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DISSERTATION

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Abstract

In this thesis we consider several extremal problems for graphs and hypergraphs: packing, domination, and coloring. Graph packing problems have many applications to areas such as scheduling and partitioning. We consider a generalized version of the packing problem for hypergraphs. There are many instances where one may wish to cover the vertices or edges of a graph. A dominating set may be thought of as a covering of the vertex set of a graph by stars. Similarly a proper coloring may be thought of as a covering of the vertex set of a graph by independent sets. We consider special cases of domination and coloring on graphs.

Two n-vertex hypergraphs G and H *pack* if there is a bijection $f\colon V(G) \to V(H)$ such that for every edge $e \in E(G)$, the set $\{f(v)\colon v \in e\}$ is not an edge in H. Sauer and Spencer showed that any two n-vertex graphs G and H with $|E(G)| + |E(H)| < \frac{3n-2}{2}$ pack. Bollobás and Eldridge proved that, with 7 exceptions, if graphs G and H contain no spanning star and $|E(G)| + |E(H)| \le 2n - 3$, then G and H pack. In Chapter 2 we generalize the Bollobás – Eldridge result to hypergraphs containing no edges of size 0, 1, $n-1$, or n. As a corollary we get a hypergraph version of the Sauer – Spencer result.

In 1996 Reed proved that for every n-vertex graph G with minimum degree 3 the domination number $\gamma(G)$ is at most $\frac{3n}{8}$. While this result is sharp for cubic graphs with no connectivity restriction, better upper bounds exist for connected cubic graphs. In Chapter 3, improving an upper bound of Kostochka and Stodolsky, we show that for $n > 8$ the domination number of every n-vertex connected cubic graph is at most $\lfloor \frac{5n}{14} \rfloor$. This bound is sharp for $8 < n \le 18$ and nears the best known lower bound of $\frac{7n}{20}$.

An *acyclic coloring* is a proper coloring with the additional property that the union of any two color classes induces a forest. In Chapter 4 we show that every graph with maximum degree at most 5 has an acyclic 7-coloring. We also show that every graph with maximum degree at most r has an acyclic $(1 + \lfloor \frac{(r+1)^2}{4} \rfloor)$-coloring.

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Chapter 1

Introduction

Many mathematical and real-world problems have natural graph-theoretic models. In this thesis we will discuss several extremal problems on graphs and hypergraphs. Although we approach the subject primarily from a theoretical viewpoint, many of these problems have applications to real-world problems.

In Chapter 2 we discuss packing problems on hypergraphs. Problems such as laying out circuits, building networks, scheduling, and partitioning may be thought of in terms of packing appropriate graphs or hypergraphs. Covering problems arise very naturally. Given a set of train stations (vertices) we may consider two stations adjacent if the distance between them is at most k. Consider the problem of guaranteeing that a station with greater amenities be located at a reasonable distance from any given station, but minimizing the number of such costly upgraded stations. This is a covering problem which can be solved by considering the domination number of the resulting graph. In Chapter 3 we study the domination number of a specific class of graphs, namely 3-regular or cubic graphs.

Coloring problems are well studied and appear in many varieties. While domination may be considered a covering of a graph by stars, the problem of properly coloring the vertices of a graph may be thought of as covering a graph by independent sets. In Chapter 4 we consider a further restriction of the standard chromatic number on graphs with bounded degree.

Section 1.1 gives some of the basic definitions used. Sections 1.2 – 1.4 describe the results appearing in this thesis.

The results of Chapter 2 are in preparation [13], the results of Chapter 3 have been accepted and will appear in Ars Mathematica Contemporanea [18], and the results of Chapter 4 have been published [17].

1.1 Basic Definitions

In this section we review some of the basic definitions, terms, and concepts used in this thesis. In most cases we will follow the notation given in *Introduction to Graph Theory* by West [33].

A *graph* G consists of two sets: a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where each element of $E(G)$ consists of exactly two members of $V(G)$. We call the vertices contained in an edge its *endpoints*. We specify an edge with endpoints u and v as uv. We say that two vertices u and v are *adjacent* or *neighbors* if uv is an edge in $E(G)$. The *degree* of a vertex v is the number of vertices adjacent to it. We generally denote the degree of a vertex v as $d_G(v)$, or as $d(v)$ when the graph is understood. We let $\Delta(G)$ denote the maximum degree of G and $\delta(G)$ denote the minimum degree of G. A graph is *regular* if every vertex has the same degree. We say that a graph is *r-regular* if every vertex has degree r. We may say that a graph is *cubic* in the special case where it is 3-regular.

The *neighborhood* of a vertex v, denoted $N_G(v)$, is the set of all vertices adjacent to v; note that $d_G(v) = |N_G(v)|$. The *closed neighborhood* of a vertex v, denoted $N_G[v]$, is $N_G(v) \cup v$. The neighborhood of a set $X \subseteq V(G)$, denoted $N_G(X)$, is $\left(\bigcup_{v \in X} N_G(v) \right) - X$. The closed neighborhood of X, denoted $N_G[X]$, is $N_G(X) \cup X$.

A graph H is a *subgraph* of a graph G if there exists an injection $f\colon V(H) \to V(G)$ such that for every edge $uv \in E(H)$, $f(e)f(v) \in E(G)$. Such a graph H is an *induced subgraph* if it has the additional property that if $uv \notin E(H)$, then $f(e)f(v) \notin E(G)$. If $S \subseteq V(G)$, then the *subgraph of G induced by S*, denoted $G[S]$, is the graph obtained from G by deleting all vertices not in S and all edges incident to vertices not in S.

If G is a graph and $F \subseteq E(G)$, then $G - F$ is the subgraph of G with the vertex set $V(G)$ and the edge set $E(G) - F$. When F consists of a single edge e, we write $G - e$ instead of $G - \{e\}$. If $X \subseteq V(G)$, then $G - X$ denotes the subgraph of G induced by the vertices in $V(G) - X$. Again when X consists of a single vertex v we write $G - v$ instead of $G - \{v\}$.

A *path* is a graph whose vertices may be ordered so that two vertices are adjacent if and only if they are consecutive in the list. The *endpoints* of a path are the vertices having degree 1. The remaining vertices are *internal vertices*. The *length* of a path is the number of edges contained in the path. The (unlabeled) path with n vertices is denoted P_n. A *cycle* is a graph whose vertices may be placed in a cycle so that two vertices are adjacent if and only if they are consecutive in the cycle. The (unlabeled) cycle with n

vertices is denoted C_n. A cycle is *even* if it has an even number of vertices and *odd* if it has an odd number of vertices. A graph is *acyclic* if it contains no cycles. We call an acyclic graph a *forest*. An n-vertex graph is called *hamiltonian* if it contains a copy of C_n as a subgraph.

Given a graph G with vertices u and v, a *u,v-path* is a path with endpoints u and v. We say that G is *connected* if for any two vertices $u, v \in V(G)$, there exists a u, v-path in G. A graph that is not connected is *disconnected*. The *components* of G are the maximal connected subgraphs. The *distance* between vertices u and v in G, denoted $d_G(u, v)$ or simply $d(u, v)$, is the length of the shortest u, v-path in G (if such a path exists).

A *tree* is a connected forest or, equivalently, an acyclic connected graph. A *leaf* in a tree is a vertex of degree 1. A *star* is an n-vertex tree with a vertex of degree $n - 1$. A *clique* is a set of pairwise adjacent vertices. The *complete graph* is the n-vertex graph whose vertices are pairwise adjacent. The (unlabeled) complete graph with n vertices is denoted K_n. A graph is *bipartite* if its vertices can be partitioned into two sets X and Y such that each of X and Y induces a subgraph containing no edges. We denote the (unlabeled) bipartite graph where $|X| = m$, $|Y| = n$, and all of X is adjacent to all of Y by $K_{m,n}$. A set $S \subseteq V(G)$ that induces no edges is an *independent set*. A *matching* in a graph G is a set of edges with no shared endpoints. A *perfect matching* is a matching in which every vertex of G is an endpoint of some edge in the matching.

A *proper coloring* of a graph G is an assignment of labels to the vertices so that adjacent vertices receive different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors in a proper coloring of G. The *color classes* in a proper coloring of G are the sets of like colored vertices.

A *hypergraph* is a generalization of a graph where edges are not required to have size 2. We may call the edges of a hypergraph with size 2 *graph edges* and the edges with size greater than 2 *hyperedges*.

1.2 Hypergraph Packing

Two n-vertex graphs G and H are said to *pack* if there exists a bijection $f\colon G \to H$ such that every edge of G is mapped to a non-edge of H. An important equivalent statement is that G and H pack if and only if H is a subgraph of the complement of G.

Graph packing has been well studied, and many of the results can be found in survey papers by Yap [35] and Wozniak [34]. If the total number of edges in two graphs G and H is small, a natural assumption is that G and H are more likely to pack. Sauer and Spencer [31] showed that this intuition is, in fact, correct. They proved that if $|E(G)| + |E(H)| < \frac{3n-2}{2}$, then G and H pack. To see that this result is sharp, we let G be a spanning star and define H as follows: If n is even we let $H = \frac{n}{2}K_2$ and if n is odd, we let $H = P_3 + \frac{n-3}{2}K_2$. In proving this result Sauer and Spencer also showed that if $|E(G)||E(H)| \leq \binom{n}{2}$, then G and H pack.

Bollobás and Eldridge [7] realized that the most important feature of the above example was the vertex of degree $n-1$ in G. They proved that with 7 exceptions, if G and H are n-vertex graphs with maximum degree at most $n-2$ and at most $2n-3$ total edges, then G and H pack.

Similar questions can be asked for hypergraphs. As in the graph case, two n-vertex hypergraphs G and H pack if and only if there exists a bijection from G to H that maps every edge of G to a non-edge of H. Pilśniak and Woźniak [29] proved that if an n-vertex hypergraph G has at most $n/2$ edges and $V(G)$ is not an edge in G, then G packs with itself. They also asked whether such a hypergraph G packs with every n-vertex hypergraph H satisfying the same conditions. Recently, Naroski [26] proved the stronger result that if the total number of edges in G and H is at most n and neither contains the edge of size n, then G and H pack. Naroski also extended the second result of Sauer and Spencer by proving that if G and H have no edges of size less than k or greater than $n-k$ and $|E(G)||E(H)| \leq \binom{n}{k}$, then G and H pack.

We say that a *universal vertex* in a hypergraph G is a vertex contained in a 2-edge with every other vertex of G. We will then prove the following hypergraph generalization of Bollobás and Eldridge's result:

Theorem 1.2.1. *Let G and H be n-vertex hypergraphs with $|E(G)| + |E(H)| \leq 2n-3$ containing no 1-edges and no edges of size at least $(n-1)$. With 14 exceptions, G and H do not pack if and only if one of G or H has a universal vertex and every vertex of the other hypergraph is incident to a graph edge, or G and H or one of G or H has $n-1$ edges of size $n-2$ not containing a given vertex v, and for every vertex x of the other hypergraph some edge of size $n-2$ does not contain x.*

As a corollary we get the following hypergraph generalization of the main result of Sauer and Spencer:

Corollary 1.2.1. *Let G and H be n-vertex hypergraphs containing no 1-edges and no edges of size at least $n-1$. If $|E(G)| + |E(H)| < \frac{3n-2}{2}$, then G and H pack.*

These results are based on joint work with P. Hamburger and A. V. Kostochka [13].

1.3 Domination in Cubic Graphs

A set D of vertices in a graph G *dominates* itself and its neighbors at distance 1. If a set D dominates all vertices of G, then it is a *dominating set* in G. The *domination number*, $\gamma(G)$, of a graph G is the minimum size of a dominating set in G.

We may think of domination problems as covering problems. A number of covering problems can be reduced to the problem of finding the domination number of an appropriate graph. Recreational problems such as dominating the spaces of a $n \times n$ grid with a specific chess piece as well as practical problems such as minimizing the number of higher-level nodes in a computer network may easily be modeled as domination problems.

Naturally, graphs G with high minimum degree have small domination number. Ore [27] proved that $\gamma(G) \leq n/2$ for every n-vertex graph without isolated vertices (i.e., with $\delta(G) \geq 1$). Blank [6] and independently McCuaig and Shepherd [24] proved that $\gamma(G) \leq 2n/5$ for every n-vertex graph with $\delta(G) \geq 2$ when $n \geq 8$. Reed [30] proved that $\gamma(G) \leq 3n/8$ for every n-vertex graph with $\delta(G) \geq 3$. Each of these bounds is sharp. Reed [30] conjectured that the domination number of each connected 3-regular n-vertex graph is at most $\lceil n/3 \rceil$. Kostochka and Stodolsky [19] disproved this conjecture. They gave a sequence $\{G_k\}_{k=1}^{\infty}$ of connected cubic graphs such that for every k, $|V(G_k)| = 46k$ and $\gamma(G_k) \geq 16k$. This gives $\frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{8}{23} = \frac{1}{3} + \frac{1}{69}$.

Kelmans [15] gave a sequence $\{G_k\}_{k=1}^{\infty}$ of cubic 2-connected graphs such that for every k, $|V(G_k)| = 60k$ and $\gamma(G_k) \geq 21k$. This implies $\frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{60}$, which is currently the best lower bound. In particular, for infinitely many n there exists an n-vertex cubic graph G with

$$\gamma(G) \geq \left(\frac{1}{3} + \frac{1}{60}\right) n.$$

Kelmans also found a 54-vertex connected cubic graph L with $\gamma(L) = 19 = \left(\frac{1}{3} + \frac{1}{54}\right) |V(L)|$.

Improving Reed's upper bound of $3n/8$, Kostochka and Stodolsky [20] proved that for connected cubic n-vertex graphs G with $n > 8$,

$$\gamma(G) \leq \frac{4n}{11} = \left(\frac{1}{3} + \frac{1}{33}\right) n.$$

A large portion of this thesis will be devoted to strengthening this upper bound. We will prove the following theorem:

Theorem 1.3.1. *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \leq \frac{5n}{14} = \left(\frac{1}{3} + \frac{1}{42}\right) n.$$

The bound $\lfloor \frac{5n}{14} \rfloor$ is sharp for $8 < n \leq 18$. For example, a 3-connected cubic 14-vertex hamiltonian graph G with $\gamma(G) = 5$ is presented in [10].

Our proofs exploit the ideas and techniques of Reed's seminal paper [30] and of [20]. We modify and elaborate the technique of [20] substantially.

These results are based on joint work with A. V. Kostochka [17].

1.4 Acyclic Coloring

A proper coloring of a graph G is *acyclic* if the union of any two color classes induces a forest. The *acyclic chromatic number*, $a(G)$, is the smallest integer k such that G is acyclically k-colorable.

We may think of the traditional vertex coloring problem as a type of covering problem. In particular, we seek to cover the vertices of a graph by some number of independent sets. Under this model we are allowing a vertex to cover only itself. The chromatic number is then the minimum number of independent sets needed to cover the vertices of a graph. If we add the additional constraint that any two independent sets cannot induce a cycle we then get the acyclic chromatic number.

The notion of acyclic coloring was introduced in 1973 by Grünbaum [12] and turned out to be interesting and closely connected to a number of other ideas in graph coloring.

Grünbaum proved that every planar graph has an acyclic 9-coloring and conjectured that, in general, 5 colors suffice. Mitchem [25], Albertson and Berman [2], and Kostochka [21] improved this result by proving that every planar graph is acyclically 8, 7, and 6-colorable, respectively. Borodin [8] showed that every planar graph is acyclically 5-colorable, thereby proving Grünbaum's conjecture.

Grünbaum also studied $a(r)$, which is the maximum value of the acyclic chromatic number over all graphs G with maximum degree at most r. He conjectured that always $a(r) = r + 1$ and proved this for $r \leq 3$. In 1979, Burstein [9] proved the conjecture for $r = 4$; this result was also proved independently by Kostochka [16]. It was also proved in [16] that for $k \geq 3$, the problem of deciding whether a graph is acyclically k-colorable is NP-complete. It turned out that for large r, Grünbaum's conjecture is incorrect in a strong sense. Albertson and Berman mentioned in [1] that Erdős proved that $a(r) = \Omega(r^{4/3-\epsilon})$ and conjectured that $a(r) = o(r^2)$. Alon, McDiarmid and Reed [4] sharpened Erdős' lower bound to $a(r) \geq c\, r^{4/3}/(\log r)^{1/3}$ and proved that $a(r) \leq 50\, r^{4/3}$. While we now have a reasonable understanding of the order of the magnitude of $a(r)$ for large r, the problem of estimating $a(r)$ for small r is less well understood and has received recent attention.

Fertin and Raspaud [11] showed among other results that $a(5) \leq 9$ and gave a linear-time algorithm that acyclically 9-colors any graph with maximum degree 5. Furthermore, for $r \geq 3$, they gave a fast algorithm that uses at most $r(r-1)/2$ colors to acyclically color any graph with maximum degree r. For large r this is much worse than the upper bound of Alon, McDiarmid, and Reed, but for $r < 1000$, it is better. Hocquard and Montassier [14] showed that every 5-connected graph G with $\Delta(G) = 5$ has an acyclic 8-coloring. Kothapalli, Varagani, Venkaiah, and Yadav [23] showed that $a(5) \leq 8$. Kothapalli, Satish, and Venkaiah [22] proved that every graph with maximum degree r is acyclically colorable with at most $1 + r(3r+4)/8$ colors. This is better than the bound $r(r-1)/2$ in [11] for $r \geq 8$. In this thesis we will prove the following theorem:

Theorem 1.4.1. *Every graph with maximum degree* 5 *has an acyclic* 7*-coloring, i.e.,* $a(5) \leq 7$.

The proof yields a linear-time algorithm to provides an acyclic coloring for any graph with maximum degree 5 using at most 7 colors. We also show that for $r \geq 6$, there exists a linear-time algorithm giving an acyclic coloring of any graph with maximum degree r using at most $1 + \lfloor \frac{(r+1)^2}{4} \rfloor$ colors. This is better than the bounds in [11] and [22] cited above for every $r \geq 6$ and better than the bounds in [4] for $r \leq 2825$.

These results are based on joint work with A. V. Kostochka [18].

Chapter 2

Hypergraph Packing

2.1 Introduction

Recall that a hypergraph is a pair (V, E) where V is a finite set (elements of V are called vertices) and E is a family of nonempty subsets of V (members of E are called edges). An important instance of combinatorial packing problems is that of *(hyper)graph packing.* Two n-vertex hypergraphs G and H *pack*, if there is a bijection $f\colon V(G) \to V(H)$ such that for every edge $e \in E(G)$, the set $\{f(v) : v \in e\}$ is not an edge in H. For graphs, this means that G is a subgraph of the complement $\overline{H}$ of H, or, equivalently, H is a subgraph of the complement $\overline{G}$ of G. A *universal vertex* in a hypergraph G is a vertex v which is contained in a 2-edge (graph edge) with every other vertex in G.

Many important results on extremal graph packing problems were obtained in the seventies. At this time, fundamental papers by Bollobás and Eldridge [7] and Sauer and Spencer [31] appeared. In particular, Sauer and Spencer [31] proved the following.

Theorem 2.1.1. [31] *Let G and H be n-vertex graphs with $|E(G)| + |E(H)| < \frac{3n-2}{2}$. Then G and H pack.*

The examples showing that Sauer and Spencer's result is sharp rely upon the existence of a universal vertex. Bollobás and Eldridge [7] obtained the following refinement of Theorem 2.1.1.

Theorem 2.1.2. [7] *Let G and H be n-vertex graphs with $|E(G)| + |E(H)| \le 2n - 3$. If neither of G and H has an universal vertex, and the pair $\{G, H\}$ is not one of the seven pairs in Figure 2.1, then G and H pack.*

Corollary 1 in [7] gives that Theorem 2.1.2 can be restated as follows:

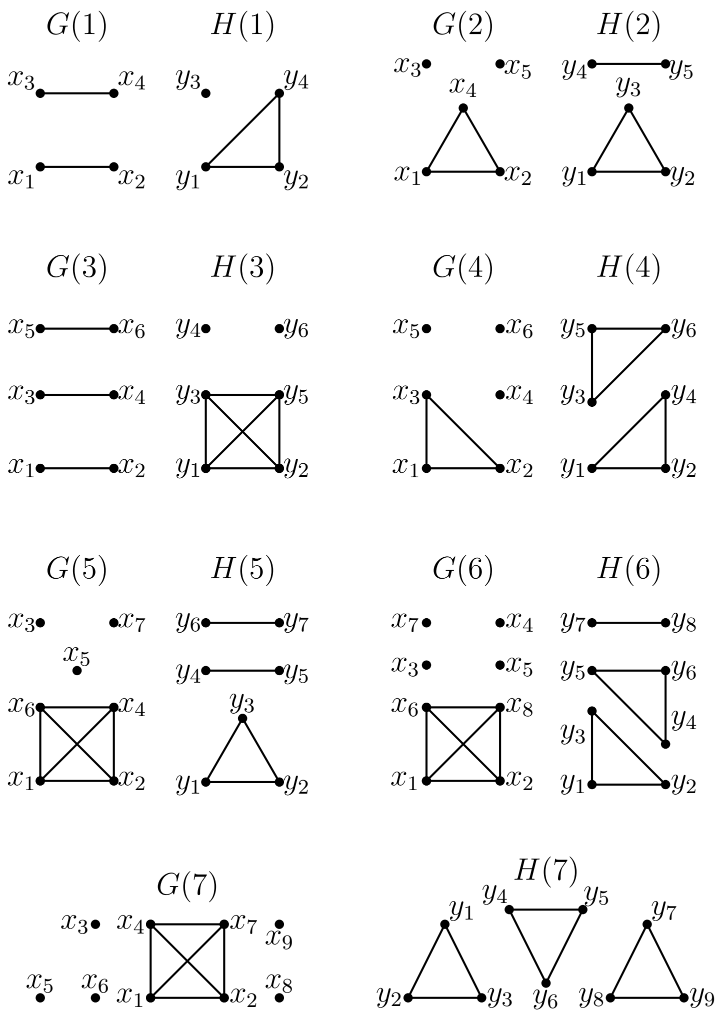


Figure 2.1: Bad pairs in Theorem 2.1.2.

Theorem 2.1.3. [7] *Let G and H be n-vertex graphs with $|E(G)|+|E(H)| \leq 2n-3$. Then G and H do not pack if and only if either $\{G, H\}$ is one of the seven pairs in Fig. 2.1, or one of G and H has an universal vertex and the other has no isolated vertices.*

If G and H are n-vertex non-uniform hypergraphs, packing may be more complicated. In general we will use *i-edge* to denote edges of size i in a hypergraph. We will sometimes distinguish edges of size 2 by calling them *graph edges*, and edges of size at least 3 by calling them *hyperedges*.

Edges of size 0, 1, $n-1$ or n make harder for hypergraphs to pack. For example, if $V(G)$ is an edge in G and $V(H)$ is an edge in H, then G and H do not pack. Similarly, if $\emptyset$ is an edge in both G and H, then G and H do not pack. Also if the total number of 1-edges or the total number of $(n-1)$-edges in G and H is at least $n+1$, then G and H again do not pack. These examples indicate that edges of size i and $n-i$ behave similarly. Indeed, a bijection $f : V(G) \to V(H)$ maps edge $e \in E(G)$ onto edge $g \in E(H)$ if and only if it maps set $V(G) - e$ onto $V(H) - g$. This motivates the notion of the *orthogonal hypergraph*: For a hypergraph F, the *orthogonal hypergraph* $F^{\perp}$ has the same set of vertices as F and $E(F^{\perp}) := \{V(F) - e : e \in E(F)\}$. By definition, two n-vertex hypergraphs G and H pack if and only if $G^{\perp}$ and $H^{\perp}$ pack.

Pilśniak and Woźniak [29] proved that if an n-vertex hypergraph G has at most $n/2$ edges and $V(G)$ is not an edge in G, then G packs with itself. They also asked whether such G packs with any n-vertex hypergraph H satisfying the same conditions. Recently, Naroski [26] proved the following stronger result.

Theorem 2.1.4. *Let G and H be n-vertex hypergraphs with no n-edges. If $|E(G)| + |E(H)| \leq n$, then G and H pack.*

By the above examples, the bound of n in Theorem 2.1.4 is sharp. We will prove a refinement of this theorem to hypergraphs with no 1-, $(n-1)$-, and n-edges. This refinement also generalizes and extends to hypergraphs Theorem 2.1.3.

We define a *bad pair* of hypergraphs to be either one of the pairs $(G(i)$,$H(i))$ in Fig. 2.1, or one of the pairs $(G(i)^{\perp}$,$H(i)^{\perp})$.

Our main result is the following:

Theorem 2.1.5. *Let G and H be n-vertex hypergraphs with $|E(G)| + |E(H)| \leq 2n-3$ containing no 0-, 1-, $(n-1)$-, and n-edges. Let $|E(G)| \leq |E(H)|$. Then G and H do not pack if and only if either*

(i) (G, H) or (H, G) is a bad pair, or

(ii) H has a universal vertex and every vertex of G is incident to a graph edge, or

(iii) $H^{\perp}$ has a universal vertex and every vertex of $G^{\perp}$ is incident to a graph edge.

Since each of the graphs in Fig. 2.1 has at most 9 vertices, for $n \geq 10$ the theorem says that ... *G and H do not pack if and only if either H has a universal vertex and every vertex of G is incident to a graph edge or $H^{\perp}$ has a universal vertex and every vertex of $G^{\perp}$ is incident to a graph edge.* Note that the theorem is sharp even for graphs: for infinitely many n there are n-vertex graphs G_n and H_n such that $|E(G)|+|E(H)| = 2n-2$, neither of G_n and H_n has a universal vertex, and G_n and H_n do not pack (see, e.g., [7, 32]).

In the same way Theorem 2.1.3 yields Theorem 2.1.1, Theorem 2.1.5 yields the following extension of Theorem 2.1.1 to hypergraphs.

Corollary 2.1.1. *Let G and H be n-vertex hypergraphs with $|E(G)| + |E(H)| < n - 1 + \lceil n/2 \rceil$ containing no 0-, 1-, $(n-1)$-, and n-edges. Then G and H pack.*

To prove Theorem 2.1.5, we consider a counter-example (G, H) with the fewest vertices. In the next section we set up the proof and derive simple properties of (G, H). In Section 3 we prove two more advanced properties of (G, H). In the last section we deliver the proof of Theorem 2.1.5.

2.2 Preliminaries

Consider a counterexample (G, H) to Theorem 2.1.5 with the least number of vertices n. This means that $|E(G)| + |E(H)| \leq 2n - 3$, $|E(G)| \leq |E(H)|$, neither (G, H) nor (H, G) is a bad pair, G and H do not pack, and if H (respectively, $H^{\perp}$) has a universal vertex, then G (respectively, $G^{\perp}$) has a vertex not incident with graph edges. If at least one of G, H, $G^{\perp}$ and $H^{\perp}$ is an ordinary graph, then the statement holds by Theorem 2.1.3. So we will assume that

$$\text{each of } G,\ H,\ G^{\perp} \text{ and } H^{\perp} \text{ has at least one hyperedge.} \tag{2.1}$$

Naroski [26] used the following hypergraph operation: For an n-vertex hypergraph F, the hypergraph $\widetilde{F}$ is obtained from F by replacing each edge $e \in E(F)$ of size at least $(n+1)/2$ with $V(F) - e$ and deleting multiple edges if they occur. This operation has the following useful property.

Lemma 2.2.1 ([26]). *Let F_1 and F_2 be n-vertex hypergraphs with no edge with size less than k and no edge with size greater than $n-k$. Then*
(a) $|E(\widetilde{F_1})| \le |E(F_1)|$ and $|E(\widetilde{F_2})| \le |E(F_2)|$,
(b) both $\widetilde{F_1}$ and $\widetilde{F_2}$ have no edges of size less than k and no edges of size greater than $\lfloor \frac{n}{2} \rfloor$, and
(c) if $\widetilde{F_1}$ and $\widetilde{F_2}$ pack, then F_1 and F_2 pack.

Lemma 2.2.2. *If $\widetilde{H}$ has a universal vertex and every vertex of $\widetilde{G}$ is incident to a graph edge, then G and H pack.*

Proof. Let S be the set of 2-edges of $\widetilde{G}$ and $\widetilde{H}$ that are 2-edges in G and H. Let S' be the set of 2-edges of $\widetilde{G}$ and $\widetilde{H}$ whose complementary $(n-2)$-edges exist in G and H. Suppose that $\widetilde{H}$ contains a universal vertex v. Then $\widetilde{G}$ contains at most $n-2$ edges and hence some vertex of $\widetilde{G}$ is contained in at most one 2-edge. We consider two cases.

Case 1: All 2-edges in $\widetilde{H}$ that contain v are contained in S (respectively, S'). By the symmetry between H and $H^{\perp}$, we may assume that they all are in S. Then under the conditions of the theorem, some vertex $w \in V(\widetilde{G})$ is not contained in any edge in S. We let H' be the hypergraph obtained from H by deleting v, and all 2-edges containing v, and replacing each hyperedge $e \in E(H)$ that contains v by $e-v$. We let G' be the hypergraph obtained from G by deleting w and replacing each edge $e \in E(G)$ containing w by the edge $e-w$. Then since $|E(G')| + |E(H')| \le 2n-3-(n-1) = n-2$, Theorem 2.1.4 yields that G' and H' pack. We extend this packing to a packing of G and H by mapping v to w.

Case 2: Vertex v is contained in a 2-edge of $\widetilde{H}$ that is not in S and in a 2-edge of $\widetilde{H}$ that is not in S'. Let w_1 be a vertex of $\widetilde{G}$ which is contained in exactly one 2-edge (if no such vertex exists, then some vertex w of $\widetilde{G}$ is not incident to 2-edges at all, and we proceed as in Case 1 (deleting all 2-edges of $\widetilde{H}$ incident with v)). Let w_1w_2 be the 2-edge in $\widetilde{G}$ containing w_1. By symmetry, we may assume that $w_1w_2 \in S$. Let vv' be an edge of $\widetilde{H}$ which is not in S. We let H'' be the hypergraph obtained from $H^{\perp}$ by first deleting v, v', and all 2-edges containing v and then removing v and v' from each edge e that contains any of them. We let G'' be the hypergraph obtained from $G^{\perp}$ by first deleting w_1, w_2, and the edge w_1w_2 and then truncating all edges containing either of w_1 and w_2. Then since $|E(G')| + |E(H')| \le 2n-3-(n-1)-1 = n-3$, Theorem 2.1.4 yields that G'' and H'' pack. We extend this packing to a packing of G and H by mapping v to w_1 and v' to w_2. □

In view of Lemmas 2.2.1 and 2.2.2, we will assume that G and H have no edges of size greater than $\frac{n}{2}$. We will study properties of the pair (G, H) and finally come to a contradiction.

Throughout the proof, for $i \in \{2, \dots, \lfloor \frac{n}{2} \rfloor\}$, G_i (respectively, H_i) denotes the subgraph of G (respectively, of H) formed by all of its edges of size i, and $d_i(v, G)$ (respectively, $d_i(v, H)$) denotes the degree of vertex v in G_i (respectively, in H_i). In particular, G_2 and H_2 are formed by graph edges in G and H, respectively. Then we let $l_i := |E(G_i)|$ and $m_i := |E(H_i)|$. Also, for brevity, let $m := \sum_{i=1}^{n} m_i$, $l := \sum_{i=1}^{n} l_i$, $\overline{m} = m - m_1 - m_2$ and $\overline{l} = l - l_1 - l_2$. In other words, $\overline{l}$ is the number of *hyperedges* in G, and $\overline{m}$ is the number of hyperedges in H. Recall that by the choice of G,

$$l \leq n - 2. \tag{2.2}$$

For n-vertex hypergraphs F_1 and F_2, let $x(F_1, F_2)$ denote the number of bijections from $V(F_1)$ onto $V(F_2)$ that are not packings. Since we have chosen G and H that do not pack,

$$x(G, H) = n!. \tag{2.3}$$

A nice observation of Naroski is:

Lemma 2.2.3 ([26])**.**

$$x(G, H) \leq 2(n-2)!\ m_2 l_2 + 3!(n-3)!\ \overline{m}\overline{l}. \tag{2.4}$$

Proof. For edges $e \in G$ and $f \in H$, let X_{ef} be the set of bijections in X that map the edge e onto the edge f. Then

$$\begin{aligned} x(G, H) = &\left| \bigcup_{e \in E(G), f \in E(H)} X_{ef} \right| \leq \sum_{e,f} |X_{ef}| = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{e,f: |e|=|f|=i} |X_{ef}| \\ &= \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{e,f: |e|=|f|=i} i!(n-i)! = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} m_i l_i i!(n-i)! \\ &\leq 2(n-2)! m_2 l_2 + 3!(n-3)! \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} m_i l_i \leq 2(n-2)! m_2 l_2 + 3!(n-3)! \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} m_i \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor} l_i \\ &= 2(n-2)! m_2 l_2 + 3!(n-3)!\ \overline{m}\overline{l}. \end{aligned}$$

□

Lemma 2.2.4. $n \geq 8$.

Proof. If $n \leq 5$, then $\lfloor \frac{n}{2} \rfloor \leq 2$, and G and H are graphs, a contradiction to (2.1). Suppose now that $n = 7$. By (2.4), $x(G, H) \leq 2 \cdot 5! m_2 l_2 + (3!)(4!)\overline{m}\overline{l}$. By (2.1), $\overline{m} \geq 1$ and $\overline{l} \geq 1$. And the maximum of the expression $2 \cdot 5! m_2 l_2 + (3!)(4!)\overline{m}\overline{l}$ under the conditions that $m_2 + l_2 + \overline{m} + \overline{l} \leq 11$, $\overline{m} \geq 1$ and $\overline{l} \geq 1$ is attained at $l_2 = 4$, $m_2 = 5$, $\overline{m} = \overline{l} = 1$ and is equal to

$$2 \cdot 5! \cdot 4 \cdot 5 + (3!)(4!) = 4800 + 144 < 5040 = 7!,$$

a contradiction to (2.3).

Finally, suppose that $n = 6$. Similarly to the case for $n = 7$, $x(G, H) \leq 2 \cdot 4! m_2 l_2 + (3!)^2 \overline{m}\overline{l}$, $\overline{m} \geq 1$ and $\overline{l} \geq 1$. Since $2 \cdot 4! \geq (3!)^2$, for nonnegative integers m_2, l_2 and positive integers $\overline{m}, \overline{l}$, the maximum of the expression $2 \cdot 4! m_2 l_2 + (3!)^2 \overline{m}\overline{l}$ under the condition that $m_2 + l_2 + \overline{m} + \overline{l} \leq 9$ is exactly 6! and is attained only if $m_2 = l_2 = 0$, $\overline{l} = 4$ and $\overline{m} = 5$. So, G and H are 3-uniform hypergraphs with 4 and 5 edges, respectively.

Now we show that even in this extremal case $x(G, H) < 6!$. In the proof of Lemma 2.2.3, for every pair of edges $e \in G$ and $f \in H$, we considered the cardinality of the set of bijections X_{ef} from $V(G)$ onto $V(H)$ that map the edge e onto the edge f and estimated $\Sigma := \sum_{e \in E(G)} \sum_{f \in E(H)} |X_{ef}|$. We will show that some bijection $F : V(G) \to V(H)$ maps at least two edges of G onto two edges of H, thus this bijection counts at least twice in Σ. For this, it is enough to (and we will) find edges $e_1, e_2 \in E(G)$ and $f_1, f_2 \in E(H)$ such that $|e_1 \cap e_2| = |f_1 \cap f_2|$, since in this case we can map e_1 onto f_1 and e_2 onto f_2.

If G has two disjoint edges e and e', then any third edge of G shares one vertex with one of e and e' and two vertices with the other. So, we may assume that any two edges in G intersect. Similarly, we may assume that any two edges in H intersect.

Now we show that

H has a pair of edges with intersection size 1 and a pair with intersection size 2. (2.5)

If the intersection of each two distinct edges in H contains exactly one vertex, then each vertex belongs to at most two edges, which yields $|E(H)| \leq 2 \cdot 6/3 = 4$, a contradiction to $\overline{m} = 5$. Finally, suppose that $|f_1 \cap f_2| = 2$ for all distinct $f_1, f_2 \in E(H)$. If two vertices in H, say v_1 and v_2, are in the intersection of at least three edges, then every other edge also must contain both v_1 and v_2. Since $n = 6$ and $\overline{m} = 5$, this is impossible. Hence we may assume that each pair of vertices is the intersection of at most two edges. Given the edges $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$, every other edge must contain v_3, v_4, and one of v_1 or v_2. Hence each edge of H is contained in $\{v_1, v_2, v_3, v_4\}$. Thus H has at most 4 edges, a contradiction. This proves (2.5). Hence the lemma holds. $\square$

Lemma 2.2.5. $m_2 l_2 > \frac{(n-2)^2}{2}$.

Proof. Suppose that $m_2 l_2 = C \le \frac{(n-2)^2}{2}$. It suffices to show that $x(G,H) < n!$. So, by Lemmas 2.2.3 and 2.2.4, it is enough to show that for $n \ge 8$ and any nonnegative integers m_2, l_2 and positive integers $\overline{m}, \overline{l}$ such that $m_2 + l_2 + \overline{m} + \overline{l} \le 2n - 3$, the expression $Y := 2(n-2)!\ m_2 l_2 + 3!(n-3)!\ \overline{m}\overline{l}$ is less than $n!$. Since $C \le \frac{(n-2)^2}{2}$, $m_2 + l_2 \ge 2\sqrt{C}$. Therefore, $\overline{m} + \overline{l} \le 2n - 3 - 2\sqrt{C}$ and so $\overline{m}\overline{l} \le (n - 1.5 - \sqrt{C})^2$. It follows that

$$\begin{aligned} Y &\le 2!\ (n-3)!\Big((n-2)C + 3(n-1.5-\sqrt{C})^2 \Big) \\ &= 2!\ (n-3)!\Big((n+1)C + 3(n-1.5)^2 - 6(n-1.5)\sqrt{C} \Big). \end{aligned}$$

The second derivative w.r.t. C of the last expression is positive, and so it is enough to check $C = 0$ and $C = \frac{(n-2)^2}{2}$. If $C = 0$, then $Y \le 2!\ (n-3)!3(n-1.5)^2$, which is less than $n!$ for $n \ge 8$. Similarly, if $C = \frac{(n-2)^2}{2}$ and $n \ge 8$, then

$$\begin{aligned} \frac{Y}{n!} &< \frac{2(n-2)!\frac{(n-2)^2}{2} + 3!(n-3)!(n - \frac{n-2}{\sqrt{2}})^2}{n!} \\ &= \frac{(n-2)^3 + 6(n - \frac{n-2}{\sqrt{2}})^2}{n(n-1)(n-2)} \\ &= \frac{n^3 - 6n^2 + 12n - 8 + 6n^2 - 6n(n-2)\sqrt{2} + 3(n-2)^2}{n(n-1)(n-2)} \\ &= \frac{n^3 - 6n(n-2)\sqrt{2} + 3n^2 + 4}{n(n-1)(n-2)} \\ &< 1, \end{aligned}$$

a contradiction to (2.3). $\square$

Corollary 2.2.1. $m_2 > n/2$.

Proof. Suppose that $m_2 \le n/2$. By Lemma 2.2.5, $l_2 m_2 > \frac{(n-2)^2}{2}$. Therefore

$$l_2 > \frac{(n-2)^2}{2} \cdot \frac{2}{n} > n - 4.$$

Also, by (2.2) and (2.1), $l_2 \leq n-3$. So, $l_2 = n-3$, and thus $l = n-2$ and $m \leq n-1$. Hence by Lemma 2.2.3, for $n \geq 8$

$$\begin{aligned}
x(G,H) &\leq 2(n-2)!\; m_2(n-3) + 3!(n-3)!\; (m-m_2)\cdot 1 \\
&\leq 2\cdot(n-3)!\Big((n-2)(n-3)m_2 + 3(n-1-m_2)\Big) \\
&\leq 2\cdot(n-3)!\Big((n-2)(n-3)\frac{n}{2} + 3(0.5n-1)\Big) \\
&= (n-2)!\Big((n-3)n+3\Big) \\
&< n!,
\end{aligned}$$

a contradiction to (2.3). □

2.3 Two more lemmas

We need some definitions.

Definition. *For a hypergraph F without 1-edges and $A \subset V(F)$, the hypergraph $F-A$ has vertex set $V(F)-A$ and $E(F-A) := \{e-A : e \in E(F) \text{ and } |e-A| \geq 2\}$, where multiple edges are replaced with a single edge.*

An edge e of G *belongs to* a component C of G_2 if strictly more than $|e|/2$ vertices of e are in $V(C)$. By definition, each e belongs to at most one component of G_2. A component C of G_2 is *clean* if no hyperedge belongs to C. A *clean tree-component* of G is a clean component of G_2 which is a tree. In particular, each single-vertex component of G_2 is a clean tree-component. By definition, for each component C of G_2, at least $|V(C)|-1$ graph edges belong to C. Moreover,

$$\text{if exactly } |V(C)|-1 \text{ edges belong to } C\text{, then } C \text{ is a clean tree-component.} \tag{2.6}$$

Since $l_2 \leq n-3$, G_2 has at least 3 tree-components. Since $l \leq n-2$, by (2.6), at least two components of G_2 are clean tree-components. Since each non-clean component has at least two vertices,

$$\text{the smallest clean tree-component of } G_2 \text{ has at most } \max\{\tfrac{n}{3}, \tfrac{n-2}{2}\} = \tfrac{n-2}{2} \text{ vertices.} \tag{2.7}$$

Lemma 2.3.1. *Among the smallest clean tree-components of* G_2*, there exists a component* T *such that* $G - T$ *does not have a universal vertex.*

Proof. Let T be the vertex set of a smallest clean tree-component of G_2 and let $|V(T)| = t$.

Case 1: $|E(G)| \leq n-3$. Since $G - T$ is an $n-t$ vertex hypergraph containing only $n-t-2$ edges, $G-T$ cannot have a universal vertex.

Case 2: $|E(G)| = n-2$. Assume that $G-T$ contains a universal vertex, say w. Since $G-T$ has at most $n-t-1$ edges, each edge in $G-T$ is a graph edge connecting w with some other vertex. In particular, every hyperedge in G has all but 2 of its vertices in T. Hence for each hyperedge e in G, the edge $e-T$ connects an isolated vertex of G_2 to w. Since G_2 contains at least 3 components, we get that G_2 contains at least one isolated vertex. Then since any isolated vertex is a clean tree-component, $t = 1$.

Assume that G_2 contains k isolated vertices $v_1, v_2, \ldots, v_k$. Each of these vertices then forms a smallest clean tree-component. If $G - v_i$ does not contain a universal vertex for some $i \leq k$, we are done. Hence we may assume that $G - v_i$ contains a universal vertex w_i for each $i \leq k$. It follows that every edge of G has size at most 3 and contains w_i for every i. In particular, G_2 has at most one non-singleton component. Since $l_2 \leq l-1 \leq n-3$, G_2 has at least 3 components. Hence $k \geq 2$. Furthermore, each of the $v_i's$ is contained in each 3-edge, hence $k \leq 3$. If $k = 3$, then we have exactly one 3-edge $v_1v_2v_3$ in G. But then one the vertices of this edge is w_i for some i and hence is incident with $n-3$ graph edges. Since $n \geq 8$, vertex of degree $n-2$ is not isolated. So, $k = 2$.

Since G contains a 3-edge, we have an edge v_1v_2w where w is necessarily the universal vertex in $G - v_1$ and in $G - v_2$. Thus v_1v_2w is the only 3-edge in G, and so wu is an edge of G_2 for every $u \in V(G) - v_1 - v_2 - w$.

Case 2.1: H_2 contains an isolated vertex y. Since $m = n-1$ and $n \geq 8$, there exist vertices y_1 and y_2 such that $\{y, y_1, y_2\}$ is not a 3-edge in H. Then we may map w to y, v_1 to y_1 and v_2 to y_2, and the rest of $V(G)$ arbitrarily to the rest of $V(H)$ to get a packing of G and H, a contradiction to their choice.

Case 2.2: H_2 has no isolated vertices. Since $|E(H_2)| \leq n-2$, H_2 necessarily contains a vertex y of degree 1. Suppose $yy_1 \in E(H_2)$. Since H contains at most $n-1-n/2$ 3-edges, there exists some $y_2 \in V(H)$ which is not in a 3-edge with y and y_1. Then we may pack G and H as in Case 2.1. $\square$

Lemma 2.3.2. *Let* $t \leq (n-2)/2$*. Let* T *be a* t*-vertex clean tree in* G_2 *and let* $S \subset V(H)$

with $|S| = t$ be such that S intersects at least $t+1$ graph edges. If $G[T]$ and $H[S]$ pack, then either $G' := G - T$ or $H' := H - S$ has a universal vertex.

Proof. Assume that the lemma does not hold. Since the (graph) edges of T and the graph edges in H incident with S do not correspond to any edge in G' and H', we have

$$|E(G')| + |E(H')| \le |E(G)| + |E(H)| - (t-1) - (t+1) \le 2(n-t) - 3. \tag{2.8}$$

We claim that if G' and H' pack, then so do G and H. Indeed suppose that σ' is a packing of G' onto H' and σ'' is a packing of $G[T]$ onto $H[S]$. We will check that $\sigma' \cup \sigma''$ is a packing of G onto H. Suppose the contrary: that an edge A of G is mapped onto edge B of H. If $A \subset T$, this is impossible, since σ'' is a packing of $G[T]$ onto $H[S]$. So, suppose $A' := A \cap V(G') \neq \emptyset$ and $B' := B \cap V(H') \neq \emptyset$. Since T is a clean component of G_2, $|A'| \ge 2$. So, $|B'|$ is also at least 2. Then, by the definition of $G-T$ and $H-S$, A' is an edge of G' and B' is an edge of H'. Hence σ' does not send A' to B', a contradiction to the choice of A and B. Thus since G and H do not pack, neither do G' and H'. So by (2.8) and the minimality of n, either (G', H') is a bad pair or the lemma holds. Hence we may assume that (G', H') is a bad pair.

Let $k = n - t$. Note that for each bad pair $(G(i), H(i))$ in Fig. 1, the total number of edges in $G(i)$ and $H(i)$ is $2|V(G(i))| - 3 = 2|V(H(i))| - 3$. Hence $|E(H)| - |E(H-S)| = t+1$ and S covers exactly $t+1$ graph edges. Then

$$|E(G(i))| + |E(H(i))| = 2k - 3 \text{ and } |V(G)| = |V(H)| \le 2k - 2. \tag{2.9}$$

By the definition of bad pairs, either all edges in G' and H' are graph edges or all of them are $(k-2)$-edges. In the latter case, H has only $t + 1 \le n/2$ graph edges, a contradiction to Corollary 2.2.1. Thus, we may assume that $\{G', H'\} = \{G(i), H(i)\}$ in Fig. 1 for some $i \in \{1, \dots, 8\}$.

Case 1: $\bar{l} + \overline{m} \ge 2k - 3$. Then $l_2 + m_2 \le (2n-3) - (2k-3) = 2n - 2k$, and hence $l_2 m_2 \le (n-k)^2$. Since $4 \le k \le 9$ and $k \ge (n+2)/2$, we get

$$l_2 m_2 \le (n-k)^2 \le \left(\frac{n-2}{2}\right)^2 < \frac{(n-2)^2}{2},$$

a contradiction to Lemma 2.2.5.

Since we proved that $\bar{l} + \overline{m} < 2k - 3$ at least one edge of G' or H' is a graph edge in G or H. Furthermore, since T was a clean component, all the hyperedges of G become graph edges of G'. Let e_G be some such edge of G'. If none of the edges of H' was obtained from a hyperedge of H, then it is enough to pack $G' - e_G$ with H', which is possible by Theorem 2.1.3. So, there are $e \in E(G')$ and $f \in E(H')$ such that one of them is a graph edge and the other is a hyperedge in (G, H).

Case 2: (G', H') is one of the unordered pairs $\{G(1), H(1)\}$, $\{G(3), H(3)\}$, $\{G(4), H(4)\}$, $\{G(7), H(7)\}$. By symmetry, we may assume that $e = x_1x_2$ and $f = y_1y_2$. In all cases, we define mapping $\phi(x_j) = y_j$ for $j = 1, \dots, k$. This mapping together with the packing of $G[T]$ with $H[S]$ yields a packing of G with H, a contradiction.

Case 3: (G', H') is one of the unordered pairs $\{G(2), H(2)\}$, $\{G(5), H(5)\}$, $\{G(6), H(6)\}$. By symmetry, we may assume that $e = x_1x_2$ and either $f = y_1y_2$ or $f = y_{k-1}y_k$. If $f = y_1y_2$, then we let $\phi(x_j) = y_j$ for $j = 1, \dots, k$, and if $f = y_{k-1}y_k$, then we let $\phi(x_j) = y_{k+1-j}$ for $j = 1, \dots, k$.

□

Remark. Practically the same proof will verify the lemma with the roles of G and H switched, that is, with T being be a t-vertex clean tree in H_2 and S being a subset of $V(G)$ with $|S| = t$ such that S intersects at least $t + 1$ graph edges in G. The only difference is that if all edges of G' and H' are $(k-2)$-edges, then H has only $t - 1 \leq n/2$ graph edges (those that are the graph edges of T), and we get the same contradiction to Corollary 2.2.1.

2.4 Proof of Theorem 2.1.5

By Lemma 2.3.1, there is a smallest clean tree-component T of G_2 such that

$$G - T \text{ does not contain a universal vertex.} \tag{2.10}$$

We let $t = |V(T)|$.

Case 1: $t = 1$. Let $V(T) = \{u\}$. By Corollary 2.2.1, $\Delta(H_2) \geq 2$. Let $w \in V(H)$ with $d_2(w, H) = \Delta(H_2)$. Let $G' = G - u$ and let $H' = H - w$. By Lemma 2.3.2 and (2.10), H' contains a universal vertex, say w'.

Let $y = \Delta(H_2)$. Since H contains at least $n - 2$ edges forming the star in H' plus y graph edges incident to w, we get that $l + (n-2) + y \leq l + m \leq 2n - 3$. Since $l_2 \leq l - 1$, we get $l_2 + y \leq n - 2$. By Lemma 2.2.5, $m_2 > \frac{(n-2)^2}{2l_2}$. Also, w' is contained in at least

$n-2-y$ 3-edges, hence

$$(l_2+1)+\frac{(n-2)^2}{2l_2}+(n-2-y)<l+m\leq 2n-3,$$

which gives that $l_2-y+\frac{(n-2)^2}{2l_2}<n-2$. Adding these expressions gives

$$(l_2+y)+(l_2-y+\frac{(n-2)^2}{2l_2})<2(n-2)$$

or $l_2+\frac{(n-2)^2}{4l_2}<n-2$. This can be rewritten as $(2l_2-(n-2))^2<0$ which is false. This contradiction finishes Case 1, so below we assume that $t>1$.

Case 2: $t=2$. Let $V(T)=\{v_1,v_2\}$. If H contains a vertex w with $d_2(w,H)>n/2$, let w' be a non-neighbor of w in H_2. Then $G'=G-v_1-v_2$, and $H'=H-w-w'$ are $(n-2)$-vertex graphs with $|E(G')|+|E(H')|<\frac{3(n-2)-2}{2}$, so G' and H' pack by the minimality of n (we simply apply Corollary 2.1.1). Mapping v_1 to w and v_2 to w' will complete the packing of G with H. So, $\Delta(H_2)\leq n/2$.

Case 2.1: $\Delta(H_2)\geq 3$. Given non-adjacent vertices w_1 and w_2 in H_2 with $d_2(w_1,H)=\Delta(H_2)$, we let $G'=G-v_1-v_2$ and $H'=H-w_1-w_2$. By Lemma 2.3.2 and (2.10), H' contains a universal vertex.

Let $y=\Delta(H_2)\leq n/2$. Then $l+(n-3)+y\leq l+m\leq 2n-3$. Since H' contains a universal vertex, $m-m_2\geq n-3-y$, so $l+m_2+(n-3-y)\leq l+m\leq 2n-3$. Adding these gives $2(2n-3)\geq 2l+m_2+2(n-3)$, or

$$2n\geq 2l+m_2. \tag{2.11}$$

By Lemma 2.2.5, $l_2>\frac{(n-2)^2}{2m_2}$. So if $l-l_2\geq 2$ or $m-m_2\geq n-1-y$, then $2n>4+m_2+\frac{(n-2)^2}{m_2}$. And since $m_2+\frac{n-2)^2}{m_2}\geq 2(n-2)$, we get $2n>2n$, a contradiction. Hence we may assume that $l-l_2=1$ and that $m-m_2\leq n-2-y$. Furthermore, if $l_2m_2\leq\frac{(n-1)^2}{2}$, Lemma 2.2.3 gives

$$\begin{aligned}x(G,H)&\leq 2(n-2)!\,\frac{(n-1)^2}{2}+3!(n-3)!\,1(n-2-y)\\&\leq 2(n-2)!\,\frac{(n-1)^2}{2}+3!(n-3)!\,1(n-5)\\&=(n-1)!\left[(n-1)+\frac{6(n-5)}{(n-1)(n-2)}\right]\\&<n!\text{ (since }n\geq 8),\end{aligned}$$

a contradiction to (2.3). Thus $l_2 m_2 > \frac{(n-1)^2}{2}$ which gives $l = 1 + l_2 > 1 + \frac{(n-1)^2}{2m_2}$. Applying this to (2.11) we obtain $2n > 2 + m_2 + \frac{(n-1)^2}{m_2} \geq 2 + 2(n-1) = 2n$, a contradiction.

Case 2.2: $\Delta(H_2) \leq 2$. By Corollary 2.2.1, $\Delta(H_2) \geq 2$. Thus $\Delta(H_2) = 2$. Let w_1 be a vertex with $d_2(w_1, H) = 2$. If there exists some w_2 in H with $w_1 w_2 \notin E(H)$ and $d_2(w_2, H) \geq 1$, then we proceed as in Case 2.1. Hence we may assume that every vertex in H_2 that is not adjacent to w_1 is an isolated vertex. We then have that $m_2 \leq 3$, and $m_2 l_2 \leq 3(n-3)$. Lemma 2.2.5 then gives that $3(n-3) > (n-2)^2/2$ or $(n-5)^2 < 3$, a contradiction to $n \geq 8$.

Case 3: $t \geq 3$ and H_2 has an isolated vertex w. Let y be a leaf of T and let x be the neighbor of y in G_2. Let $G' = G - x$ and let $H' = H - w$. Since $t \geq 3$, $d_2(x, G) \geq 2$ and hence $|E(G')| \leq n - 4$. Therefore, $|E(G')| + |E(H')| \leq 2(n-1) - 3$, and G' does not have a universal vertex. Thus by the remark to Lemma 2.3.2, H' has a universal vertex, say w'. Let $G'' = G' - y$ and let $H'' = H' - w'$. Since w' was universal in H',

$$\begin{aligned}
|E(G'')| + |E(H'')| &= |E(G')| + |E(H')| - (n-2) \\
&\leq 2(n-1) - 3 - (n-2) \\
&= n - 3 \\
&< \frac{3(n-2) - 2}{2}.
\end{aligned}$$

So by the minimality of n and Corollary 2.1.1, G'' and H'' pack. We may then extend the packing of G'' and H'' to a packing of G and H by mapping x to w and y to w'. This finishes Case 3.

If n_1 vertices of G are in clean tree-components, then $l \geq \frac{n_1(t-1)}{t} + (n - n_1)$. Moreover, if $n = n_1$, then (since G has a hyperedge) $l \geq 1 + \frac{n_1(t-1)}{t} \geq 2 + \frac{(n-2)(t-1)}{t}$. Since $n - n_1 \neq 1$, we conclude that $l \geq n - \lfloor \frac{n-2}{t} \rfloor$. So

$$m \leq 2n - 3 - l \leq n - 3 + \lfloor \frac{n-2}{t} \rfloor. \tag{2.12}$$

We consider two cases depending on the maximum degree of H_2.

Case 4: $t \geq 3$ and $\Delta(H_2) \geq \lfloor \frac{n-2}{t} \rfloor$. Let w_1 be a vertex of maximum degree in H_2. Let v_1 be a leaf in T and choose $v_2, v_3, \ldots, v_t$ in T so that for each i with $2 \leq i \leq t$, the set $\{v_1, v_2, \ldots, v_i\}$ induce a tree in G_2 with v_i as a leaf with neighbor $v_{(i-1)'}$. We map v_1 to w_1 and proceed by induction to pack $V(T)$ into $V(H)$ so that for every $i = 1, \ldots, t$, the image, W_i, of $\{v_1, v_2, \ldots, v_i\}$ is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges. Assume that $v_1, v_2, \ldots, v_i$ have been mapped in this way to $w_1, w_2, \ldots, w_i$, so that

$W_i = \{w_1, w_2, \ldots, w_i\}$. In particular, W_i is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges in H.

Case 4.1: W_i is incident to at least $\lfloor \frac{n-2}{t} \rfloor + i$ graph edges. It suffices to map v_{i+1} to a vertex w_{i+1} in $V(H)$ such that for each $j \leq i$, $w_j \neq w_{i+1}$ and $w_j w_{i+1}$ is not an edge. Since v_{i+1} is adjacent only to $v_{i'}$ in $\{v_1, v_2, \ldots, v_i\}$, if $i + d_2(w_{i'}, H - W_i) < n$, then we can choose as w_{i+1} any vertex in $V(H) - W_i$ not adjacent to $w_{i'}$ in H_2. Hence we may assume that $d_2(w_{i'}, H - W_i) \geq n - i$. Since G_2 contains no isolated vertices, by the choice of G and H, $\Delta(H_2) \leq n-2$, so $i \neq 1$. Since v_1 is a leaf in T and $i \geq 2$, $i' \neq 1$. So, by the choice of w_1,

$$m_2 \geq d_2(w_{i'}, H - W_i) + d_2(w_1, H - w_{i'}) \geq 2d_2(w_{i'}, H - W_i) \geq 2(n-i).$$

Also, $i \leq t-1$. Hence $m \geq 1 + m_2 \geq 1 + 2(n-i) \geq 2n - 2t + 3$. So, by (2.12), $2n - 2t + 3 \leq n - 3 + \frac{n-2}{t}$. This gives $0 \leq 2t^2 - (n+6)t + (n-2)$, but for $2 \leq t \leq \frac{n-2}{2}$, this expression is at most -6.

Case 4.2: W_i is incident to exactly $\lfloor \frac{n-2}{t} \rfloor + i - 1$ graph edges. If there exists some $w_{i+1} \in V(H) - W_i$ not adjacent to W_i in H_2, then we can map v_{i+1} onto this w_{i+1}. Hence we may assume that $i + \lfloor \frac{n-2}{t} \rfloor + i - 1 \geq n$. This yields $0 \leq 2t^2 - (n+3)t + (n-2)$, but for $2 \leq t \leq \frac{n-2}{2}$, this expression is at most -3.

So, we can pack T into H in such a way that at least $\lfloor \frac{n-2}{t} \rfloor + t - 1$ graph edges of H are covered. Let $G' = G - v_1 - v_2 - \ldots - v_t$ and $H' = H - w_1 - w_2 - \ldots - w_t$. Since by (2.7), $\lfloor \frac{n-2}{t} \rfloor \geq 2$, Lemma 2.3.2 and (2.10) yield that H' has a universal vertex. But

$$|E(H')| \leq n - 3 + \lfloor \frac{n-2}{t} \rfloor - \lfloor \frac{n-2}{t} \rfloor - t + 1 = n - t - 2,$$

a contradiction.

Case 5: $t \geq 3$ and $\Delta(H_2) \leq \lfloor \frac{n-2}{t} \rfloor - 1$. By Corollary 2.2.1, $\Delta(H_2) \geq 2$. Hence $2 \leq \lfloor \frac{n-2}{t} \rfloor - 1$, which yields $t \leq (n-2)/3$. Define $v_1, v_2, \ldots, v_t$ as in Case 4. We map v_1 to a vertex w_1 of maximum degree in H_2. Since $\Delta(H_2) \geq 2$, we may proceed as in Case 4, to get a packing of T into H which covers at least $\Delta(H_2) + t - 1 \geq t + 1$ graph edges in H. Again by Lemma 2.3.2 and (2.10), H' has a universal vertex, say z. Then z is contained in at least $n - t - 1 - \Delta(H_2)$ hyperedges in H. Hence $m - m_2 \geq n - t - \lfloor \frac{n-2}{t} \rfloor \geq n - t - \frac{n-2}{t}$. We also have that $m - m_2 \leq 2n - 3 - (l_2 + m_2) - (l - l_2)$. These inequalities together give

$$(l_2 + m_2) + (l - l_2) \leq n - 3 + t + \frac{n-2}{t}. \tag{2.13}$$

By Lemma 2.2.5, $l_2+m_2>\sqrt{2}(n-2)$.

We consider two cases.

Case 5.1: $l-l_2\geq 2$. Then by (2.13) and Lemma 2.2.5 we have $\sqrt{2}(n-2)+2<n-3+t+\frac{n-2}{t}$. As $n-3+t+\frac{n-2}{t}$ achieves its maximum for extremal values of t, we need only to check the inequality for $t=3$ and $t=\frac{n-2}{3}$. For $t=3$ we get $\sqrt{2}(n-2)<(4/3)(n-2)$ and for $t=\frac{n-2}{3}$ we get $\sqrt{2}<4/3$; both inequalities are false.

Case 5.2: $l-l_2=1$. By (2.13), we have $l_2+m_2\leq n-2+t+\frac{n-2}{t}$. For fixed n, the expression $n-2+t+\frac{n-2}{t}$ achieves its maximum at extremal values of t. So, we check $t=3$ and $t=\frac{n-2}{3}$. In either case,

$$l_2+m_2\leq \frac{4(n-2)}{3}+1. \tag{2.14}$$

Since $l-l_2=1$ and $l+m\leq 2n-3$, by Lemma 2.2.3, the number $x(G,H)$ of "bad" bijections from $V(G)$ onto $V(H)$ satisfies

$$x(G,H)\leq m_2l_22(n-2)!+3!(n-3)!(m-m_2)\leq m_2l_22(n-2)!+3!(n-3)!(2n-3-l_2-1-m_2).$$

So, denoting $y:=(l_2+m_2)/2$, we have

$$x(G,H)\leq h(y):=y^22\cdot(n-2)!+3!(n-3)!(2n-4-2y).$$

Since $y\geq m_2/2>n/4\geq 2$, we have $h'(y)=4\cdot(n-2)!y-3!(n-3)!2=4\cdot(n-3)!((n-2)y-3)>0$. Thus by (2.14),

$$\begin{aligned}\frac{x(G,H)}{n!}&\leq\frac{h(2(n-2)/3+1/2)}{n!}\\&=\frac{|X|}{n!}\\&\leq\frac{1}{n!}\left[2(n-2)!(\frac{2}{3}(n-2)+\frac{1}{2})^2+3!(n-3)!\frac{2n-7}{3}\right]\\&=\frac{16n^3-72n^2+177n-302}{18n(n-1)(n-2)}.\end{aligned}$$

As this is less than 1 for $n\geq 8$, $x(G,H)<n!$, a contradiction to (2.3). $\square$

Chapter 3

Domination in Cubic Graphs

3.1 Introduction

Recall that the domination number, $\gamma(G)$, of a graph G is the minimum size of a dominating set in G.

Given an n-vertex graph G with no restrictions, the domination number can be as large as n when G consists only of isolated vertices. Forbidding isolated vertices or equivalently requiring that G have minimum degree at least one gives that every vertex cover is a dominating set. It is natural to conclude that graphs with higher minimum degree should in general have a smaller domination number. Arnautov [5] and Payan [28] independently gave the following bound on the domination number in terms of the minimum degree:

Theorem 3.1.1. ([5], [28]) *Every n-vertex graph G with minimum degree k satisfies*

$$\gamma(G) \leq n\left(\frac{1+\ln(k+1)}{k+1}\right).$$

Since every vertex in a k-regular graph can dominate at most $k+1$ vertices giving a domination number of at least $\frac{n}{k+1}$ this bound is relatively strong. For large k, Alon [3] proved the following:

Theorem 3.1.2. [3] *For all sufficiently large k and for infinitely many n there exist k-regular n-vertex graphs G with*

$$\gamma(G) \geq (n+o(1))\frac{\ln k}{k}.$$

As the two above bounds are asymptotically equal for large k, interest turned to

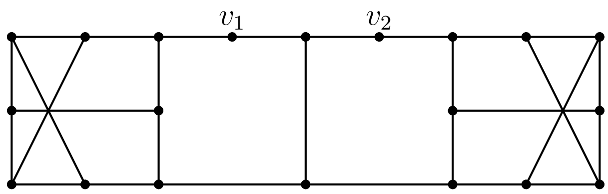


Figure 3.1

establishing bounds on the domination number of graphs with small minimum degree. Ore [27] proved that $\gamma(G) \le n/2$ for every n-vertex graph with $\delta(G) \ge 1$. Blank [6] proved that $\gamma(G) \le 2n/5$ for every n-vertex graph with $\delta(G) \ge 2$ if $n \ge 8$. Reed [30] proved that $\gamma(G) \le 3n/8$ for every n-vertex graph with $\delta(G) \ge 3$. All these bounds are sharp. Reed [30] conjectured that the domination number of each connected 3-regular (cubic) n-vertex graph is at most $\lceil n/3 \rceil$. Kostochka and Stodolsky [19] disproved this conjecture. They proved:

Theorem 3.1.3. [19] *There is a sequence $\{G_k\}_{k=1}^{\infty}$ of cubic connected graphs such that for every k, $|V(G_k)| = 46k$ and $\gamma(G_k) \ge 16k$, and thus $\frac{\gamma(G_k)}{|V(G_k)|} \ge \frac{8}{23} = \frac{1}{3} + \frac{1}{69}$.*

The current best lower bounds come from an example of Kelmans [15] which gives the following:

Theorem 3.1.4. [15] *There is a sequence $\{G_k\}_{k=1}^{\infty}$ of cubic 2-connected graphs such that for every k, $|V(G_k)| = 60k$ and $\gamma(G_k) \ge 21k$, and thus $\frac{\gamma(G_k)}{|V(G_k)|} \ge \frac{1}{3} + \frac{1}{60}$.*

These graphs are created by replacing each edges in a cycle by a copy of the graph in Figure 3.1 where the endpoints of the original edge are replaced by v_1 and v_2.

Kostochka and Stodolsky [20] improved Reed's upper bound of $3n/8$ for connected cubic graphs to the following:

Theorem 3.1.5. [20] *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \le \frac{4n}{11} = \left(\frac{1}{3} + \frac{1}{33}\right) n.$$

The main result of this chapter is the following improvement:

Theorem 3.1.6. *Let $n > 8$. If G is a connected cubic n-vertex graph, then*

$$\gamma(G) \le \frac{5n}{14} = \left(\frac{1}{3} + \frac{1}{42}\right) n.$$

The bound $\lfloor \frac{5n}{14} \rfloor$ is sharp for $8 < n \leq 18$. One 3-connected cubic 14-vertex hamiltonian graph G with $\gamma(G) = 5$ is presented in [10]. There are four such nonisomorphic graphs.

Our proofs exploit the ideas and techniques of Reed's seminal paper [30] and of [20]. We modify and elaborate the technique of [20] substantially. In the next section, we describe the setup of the Reed's paper [30] with some small changes and the procedure of constructing a dominating set. In the same section we state the basic lemmas that we will prove later. In Section 3.3, we describe a discharging that proves the bound modulo basic lemmas. In the next three sections we prove the basic lemmas.

This chapter is based on joint work with A. V. Kostochka.

3.2 The setup

We elaborate and extend the proof in [20]. A *vdp-cover* of a graph G is a covering of $V(G)$ by vertex- disjoint paths. The *order*, $|P|$, of a path P is the number of its vertices. When describing a specific path with vertices $v_1, v_2, \ldots, v_k$ where v_i is adjacent to v_j if and only if $|i-j| = 1$ we will write $(v_1 v_2 \ldots v_k)$. For $i \in \{0,1,2\}$, a path P is an *i-path*, if $|P| \equiv i \pmod 3$. If P is a path, $x \in V(P)$ and $P - x$ consists of an i-path and a j-path, then x is called an *(i,j)-vertex of P*.

Let G be a connected cubic graph and S be a vdp-cover of G. An endpoint x of a path $P \in S$ is an *out-endpoint* if x has a neighbor outside of P. An endpoint x of a 2-path $P \in S$ is a *$(2,2)$-endpoint* if x is not an out-endpoint and is adjacent to a $(2,2)$-vertex of P. By S_i we denote the set of i-paths in S.

A vdp-cover S of G is *optimal* if

(R1) $2|S_1| + |S_2|$ is minimized;

(R2) Subject to (R1), $|S_2|$ is minimized;

(R3) Subject to (R1) and (R2), $\sum_{P \in S_0} |P|$ is minimized;

(R4) Subject to (R1)–(R3), $\sum_{P \in S_1} |P|$ is minimized;

(R5) Subject to (R1)–(R4), the total number of out-endpoints of all paths in S is maximized;

(R6) Subject to (R1)–(R5), the total number of $(2,2)$-endpoints of all 2-paths in S is maximized.

It turns out that optimal vdp-covers possess several useful properties. The next lemma is Lemma 1 in [20].

Lemma 3.2.1. *Suppose that an out-endpoint x of a 1-path or a 2-path P_i in an optimal vdp-cover S is adjacent to a vertex $y \in P_j$, where $j \neq i$. Let $P_j = P_j' y P_j''$. Then*

(B1) P_j is not a 1-path;
(B2) If P_j is a 0-path, then both P_j' and P_j'' are 1-paths;
(B3) If P_j is a 2-path, then both P_j' and P_j'' are 2-paths;
(B4) If P_j is a 2-path and z is the common endpoint of P_j and P_j', then each neighbor of z on P_j'' should be a $(2,2)$-vertex.

Properties (B3), (R1), (R2) and (R3) yield the following fact.

Lemma 3.2.2. *If a path $(v_1,\dots,v_5)$ in an optimal vdp-cover S has chord v_1v_4 (see Fig. 1a), or chord v_1v_5, then none of its vertices is adjacent to an end vertex of another path in S.*

We also will use the following result.

Theorem 3.2.1. [10] *If G is a hamiltonian cubic $(3k+1)$-vertex graph, then $\gamma(G) \le k$.*

A path P in a vdp-cover S is a *special path of type 1* (respectively, *of type 2*), if P has 35 vertices (respectively, 38 vertices) and none of the hamiltonian paths on $V(P)$ has an out-endpoint or a $(2,2)$-endpoint. A *special vertex* in a special path P is a vertex at distance 17 in P from some of its end. By definition, each special path of type 1 has exactly one special vertex (its center), and each special path of type 2 has two special vertices (at distance 3 from each other). A special path P in a vdp-cover S will be called *very special* if there exists a path P_1 in S whose end-vertex is adjacent to the special vertices of at least two special paths one of which is P. The other special paths in the definition of a very special path are, by definition, also very special.

Now we essentially repeat construction in [20] of a dominating set with some modifications. Let S be an optimal vdp-cover.

(C1) If a 1-path $P \in S$ has no dominating set of size at most $(|P|-1)/3$, but has an out-endpoint, choose a vertex $y \notin V(P)$ which is a neighbor of an out-endpoint $x(P)$ of P. Call this $y \notin V(P)$ *an acceptor for P*. If $x(P)$ or the other endvertex of P has an outneighbor that is not a special vertex of a special path, then let the acceptor of P be not the special vertex of a special path. Furthermore, if there is a choice between special paths of type 1 and type 2, then we choose the acceptor in a path of type 2. In particular, if $|P| = 4$ and $G[V(P)]$ is a 4-cycle, then we choose, if possible, an outneighbor of $V(P)$ that is not a special vertex of a special path.

(C2) Say that a path $P \in S$ with $|P| = 5$ forms a *δ-subgraph*, if for some hamiltonian path on P, the center vertex, x' is adjacent to an endpoint of the path (see Fig 1.b) and the other end of P has an outneighbor. For each P forming a δ-subgraph, choose an

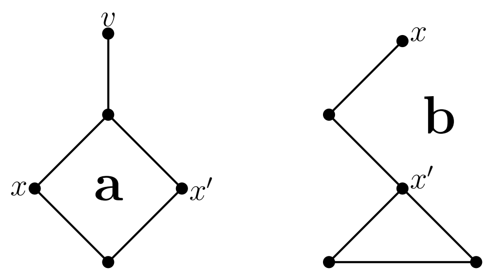


Figure 3.2

outneighbor of x and call it an *acceptor for* P. If $G[V(P)]$ is the 5-cycle, then choose as *acceptors* the outneighbors of two adjacent vertices of $G[V(P)]$. If $G[V(P)]$ is $K_{2,3}$, then choose as *acceptors* the outneighbors of two vertices of degree two in $G[V(P)]$. In all cases, if there is a choice, we try to minimize the number of acceptors that are special vertices of special paths.

(C3) Let $P \in S$ be a 2-path not described in (C2). If either P has two out-endpoints, or $|P| \leq 11$ and P has one out-endpoint, then for each of the out-endpoints of P, choose a neighbor outside P and designate it as an acceptor corresponding to that endpoint. If possible, choose the acceptors that are not special vertices of special paths.

Call a path *accepting* if at least one of its vertices was designated as an acceptor.

(C4) Construct a family $A \subseteq S$ of 2-paths as follows. Initially, let A be the set of accepting 2-paths in S. While there is any out-endpoint x of a path in A for which we have not already chosen an acceptor (because the path has only one out-endpoint), choose a neighbor y of x in $G - P$ and designate it as an acceptor for x. Moreover, if we can choose an acceptor that is not a special vertex of a special path, we do not choose a special vertex. If we have choice between special vertices of special paths of type 1 and type 2, then we choose the vertex in a special path of type 2. If y is on a previously non-accepting 2-path P', then add P' to A. Continue this process until there is an acceptor for every out-endpoint in A. In addition, for each $(2,2)$-endpoint x of each path P in A, designate a $(2,2)$-vertex y adjacent to x as an in-acceptor for x.

(C5) When we finish the procedure above, we look at special paths again. If a special vertex y of a special path $P \in S$ was designated as the acceptor for a path P_1 with an endvertex x_1 adjacent to y and some other vertex of P also is an acceptor, then we leave the situation as it is. If y is the only acceptor in P and x_1 has an outneighbor y' in a path that has other acceptors, then we redesignate the y' as the acceptor for x_1 (and P_1). Moreover, if P_1 is a path with 4 vertices, and $G[P_1]$ is a 4-cycle, then we choose y as an acceptor only if each other outneighbor of this 4-cycle also is a special vertex of a special path and no other vertices on all these paths are acceptors. If P_1 is a path with 5 vertices, and $G[P_1]$ is a 5-cycle or $K_{2,3}$, then we also, if possible switch to an acceptor in a path that contains another acceptor.

Each accepting 2-path $P \in S$ can be written in the form $P_1P_2P_3$, where P_1 and P_3 are both 1-paths containing no acceptors (including in-acceptors) and are maximal with this property. By (B3), the second and the penultimate vertices of P_2 are acceptors. The paths P_1 and P_3 are called *tips of* P, and P_2 is the *central path of* P. Now a dominating set D is defined as follows.

(C6) For each 0-path $P \in S$, every $(1,1)$-vertex of P is included in D.

(C7) For each accepting 2-path $P \in S$, every $(2,2)$-vertex of P that is in the central path of P is included in D.

(C8) Let $P \in S$ be a 1-path. If $G[P]$ has a dominating set D' with $|D'| \leq \lfloor |P|/3 \rfloor$, then we include D' into D. If no such set exists and P has an out-endpoint, then P has an out-endpoint, say $x(P)$, adjacent to the acceptor of P. In this case, choose some $\lfloor |P|/3 \rfloor$ vertices that dominate all vertices of P except for $x(P)$, and include these $\lfloor |P|/3 \rfloor$ vertices in D.

(C9) For each non-accepting 2-path in S on 5 vertices that forms a δ-subgraph, include vertex x' from the definition of δ-subgraphs into D. If $G[V(P)]$ is $K_{2,3}$, then include into D the vertex of degree two in $G[V(P)]$ that is not adjacent to the acceptors of P. If $G[V(P)] = C_5$, then include into D the vertex not adjacent to the two vertices adjacent with the acceptors of P.

(C10) For each other non-accepting 2-path $P \in S$ in which each of the ends is either an out-endpoint or a $(2,2)$-endpoint, include in D all $(2,2)$-vertices of P. Note that there are $\lfloor |P|/3 \rfloor$ of them and these $(2,2)$-vertices dominate all vertices of P except possibly for the out-endpoints of P. If a non-accepting 2-path $P \in S$ has exactly one out-endpoint x and $|P| \leq 11$, then include into D a smallest subset of $V(P)$ that dominates $V(P) - x$.

(C11) Let $P \in S$ be a 1-path, or a non-accepting 2-path with no out-endpoints, or a non-accepting 2-path with exactly one out-endpoint and $|P| \geq 14$. Choose a smallest dominating set in $G[V(P)]$ and include it in D. Note that in any case, this set has at most $\lceil |P|/3 \rceil$ vertices.

(C12) Let P_1 be a tip of an accepting 2-path $P \in S$ and x be the common end of P and P_1. If x is an out-endpoint or a $(2,2)$-endpoint, then include in D all $(2,2)$-vertices of P that are in P_1. There are $\lfloor |P_1|/3 \rfloor$ of them and these $(2,2)$-vertices dominate all vertices of P_1 except for x (which is dominated by a vertex already included in D by (C6) or (C7)). If x is neither an out-endpoint nor a $(2,2)$-endpoint, then include in D a smallest dominating set in the subgraph of G induced by P_1. Similarly to (C11), this set has at most $\lceil |P_1|/3 \rceil$ vertices.

(C13) An *exceptional path* is a non-accepting 2-path $P \in S$ such that
(i) both ends of P are out-endpoints and P does not form a δ-subgraph,
(ii) the acceptors of both ends are vertices of 2-paths $P' = P'_1P'_2P'_3$ and $P'' = P''_1P''_2P''_3$ with no outneighbors,
(iii) $|P'_1| \geq 16$, $|P'_3| \geq 16$, $|P''_1| \geq 16$, and $|P''_3| \geq 16$,
(iv) paths P' and P'' do not contain other acceptors, $|P'_2| = |P''_2| = 3$, and
(v) according to (C12), $|D \cap V(P')| = (|P'| + 4)/3$ and $|D \cap V(P'')| = (|P''| + 4)/3$.

The paths P' and P'' in the definition of an exceptional path P are called *dependants of* P.

For every exceptional path, we replace the $\lfloor |P|/3 \rfloor$ vertices of D in P (they dominated P apart from the endpoints) by a set of size $1 + \lfloor |P|/3 \rfloor$ dominating all vertices of P, but replace the $(|P'|+4)/3 + (|P''|+4)/3$ vertices of D in $P' \cup P''$ by $(|P'|+1)/3 + (|P''|+1)/3$ vertices dominating $V(P' \cup P'')$.

This finishes the definition of D.

By construction (see [30, P. 283]), the set D is dominating. We will prove that $|D| \leq 5|V(G)|/14$ if $|V(G)| > 8$ and G is connected. Note that a path P (or P_1) can contribute to D more than $|P|/3$ (or $|P_1|/3$) vertices only in cases (C11), (C12) or (C13). Thus the following lemmas will be helpful (and are extensions of Lemmas 2, 3, and 4 in [20]).

Lemma 3.2.3. *If a* 1*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths in* $G[V(P)]$ *has no out-endpoints, then either some* $(|P| - 1)/3$ *vertices dominate all vertices of* P *or* P *has at least* 28 *vertices.*

Lemma 3.2.4. *If a* 2*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths in* $G[V(P)]$ *has at most one out-endpoint, then either some* $(|P| - 2)/3$ *vertices dominate all vertices of* P *apart from an out-endpoint or* P *has at least* 14 *vertices.*

Lemma 3.2.5. *Let* $P_1 = (x_1, \ldots, x_k)$ *be a tip of an accepting* 2*-path* P *in an optimal vdp-cover. Let* $X(P_1)$ *be the set of the hamiltonian paths in* $G[V(P_1)]$ *one of whose ends is* x_k*. If none of the other ends of any path in* $X(P_1)$ *is an out-endpoint of* P *or a* $(2,2)$*-endpoint, then either some* $(k-1)/3$ *vertices dominate* $V(P_1)$*, or* $k \geq 16$*.*

In the next section, we will use discharging in order to prove our upper bound on $|D|$ provided that Lemmas 3.2.3, 3.2.4 and 3.2.5 hold. In the subsequent sections we prove these lemmas.

3.3 Discharging

Consider the following discharging. Initially, every vertex in D has charge 1 and every other vertex of G has charge 0, so the total sum of charges is $|D|$. We will change the charges of vertices in such a way that
(a) the sum of charges does not decrease, and
(b) the charge of every vertex becomes at most 5/14.

The properties (a) and (b) together imply that $|D| \leq 5|V(G)|/14$. We do the discharging in several steps and at every step check that the charge of each so far involved vertex is not greater than 5/14.

Step 1: For each 0-path P, every $(1,1)$-vertex of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex of each 0-path P has charge 1/3.

Step 2: For each accepting 2-path P, every $(2,2)$-vertex of P that is in the central path of P gives 1/3 of its charge to either of the two neighbors on P. After this step, each vertex in the central path of each accepting 2-path P has charge 1/3.

Step 3: If P is a 1-path and $D \cap V(P)$ dominates all vertices in P, then we distribute the charges of vertices in $D \cap V(P)$ evenly among vertices in P. If $|D \cap V(P)| \leq \lfloor |P|/3 \rfloor$, then each vertex of P will have charge less than 1/3. If $|D \cap V(P)| > \lfloor |P|/3 \rfloor$, then, by (C8) and (C11), P has no out-endpoints and $|D \cap V(P)| = (|P|+2)/3$. Furthermore, by Lemma 3.2.3, $|P| \geq 28$ and hence the charge of each vertex will be at most $\frac{1}{3} + \frac{2}{3|P|} \leq \frac{1}{3} + \frac{2}{3 \cdot 28} = \frac{5}{14}$.

Step 4: If P is a 1-path and $D \cap V(P)$ does not dominate all vertices in P, then by (C8) and (C11), P has an out-endpoint, say $x(P)$, adjacent to the acceptor of P. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the vertices in $V(P) - \{x(P)\}$. After this step, the vertex $x(P)$ has charge 0 and every other vertex of P has charge 1/3.

Step 5: Let P be a non-accepting and non-exceptional 2-path that does not form a δ-subgraph and in which each of the ends is either an out-endpoint or a $(2,2)$-endpoint. Distribute the charges of the $\lfloor |P|/3 \rfloor$ vertices of D in $V(P)$ evenly among the internal vertices of P. After this step, either of the ends of P has charge 0 and every other vertex of P has charge 1/3.

Step 6: For each 2-path P on 5 vertices forming a δ-subgraph, the only vertex x' of P in D gives $1/4$ to each of its neighbors. After this step, the out-endpoint x of P has charge 0 and every other vertex of P has charge $1/4$.

Step 7: Let P be a non-accepting 2-path with at most one out-endpoint that does not form a δ-subgraph. Since P has at most one out-endpoint, it is not exceptional. If $|V(P)| \geq 14$ or P has no out-endpoints, then similarly to Step 3, distribute the charges of the vertices in $D \cap V(P)$ evenly among the vertices of P. In this case, if $|V(P)| < 14$, then by Lemma 3.2.4, $|D \cap V(P)| < |V(P)|/3$, and each vertex of P will have charge less than $1/3$. If $|V(P)| \geq 14$, then

$$|D \cap V(P)| \leq (|V(P)|+1)/3 = (1+1/|V(P)|)|V(P)|/3 \leq (1+1/14)|V(P)|/3 = 5|V(P)|/14,$$

and, hence, each vertex of P has charge at most $5/14$. Suppose now that $|V(P)| \leq 11$ and P has exactly one out-endpoint $x(P)$. Distribute the charges of the vertices in $D \cap V(P)$ evenly among the vertices of $P - x(P)$. By (C10) and Lemma 3.2.4, $|D \cap V(P)| \leq (|V(P)| - 2)/3$, and so each vertex of $P - x(P)$ has the charge less than $1/3$, and $x(P)$ has charge 0.

Step 8: Let P be an exceptional path and P' and P'' be its dependants. By the definition of exceptional paths, P is non-accepting, and P' and P'' contain acceptors only for P. Distribute the charges of the vertices in $D \cap (V(P) \cup V(P') \cup V(P''))$ evenly among vertices in $V(P) \cup V(P') \cup V(P'')$. Recall that $|V(P) \cup V(P') \cup V(P'')| \geq 2+35+35 = 72$. By (C13),

$$|D \cap (V(P) \cup V(P') \cup V(P''))| = \frac{|V(P)| + |V(P')| + |V(P'')|}{3} + 1.$$

Hence, the charge of each vertex in $V(P) \cup V(P') \cup V(P'')$ is at most $1/3 + 1/72 = 25/72 < 5/14$.

Step 9: Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is either an out-endpoint or a $(2,2)$-endpoint of P. Distribute the charges of the $\lfloor |P_1|/3 \rfloor$ vertices of D in $V(P_1)$ evenly among the vertices of P_1 apart from $x(P_1)$. After this step, $x(P_1)$ has charge 0 and each other vertex of P_1 has charge $1/3$.

Step 10: Let P_1 be a tip of an accepting 2-path P such that the common end, $x(P_1)$, of P and P_1 is neither an out-endpoint nor a $(2,2)$-endpoint of P, and the central path of P has more than 3 vertices. Since the central path of P has more than 3 vertices, P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1 \dots x_k)$, $P_2 = (y_1 \dots y_m)$, and $P_3 = (z_1 \dots z_l)$, so that $P = (x_1 \dots x_k y_1 \dots y_m z_1 \dots z_l)$. Recall that, by definition, y_2 is an acceptor for an out-endpoint y' of a path or for $y' = z_l$ if z_l is a $(2,2)$-endpoint. Recall also that so far all out-endpoints and $(2,2)$-endpoints of non-exceptional paths had charges equal to 0. If $|V(P_1)| \geq 16$, then we distribute the charges of at most $(|V(P_1)| + 2)/3$ vertices of $D \cap V(P_1)$ as follows: each vertex of P_1 gets $5/14$, then we add $1/42$ to the charge of each of y_1, y_2 and y_3 and give $3/14$ to the vertex y' whose acceptor is y_2. The total charge that the vertices of $P_1 \cup \{y_1, y_2, y_3, y'\}$ get at this step is $5|P_1|/14 + 3/42 + 3/14$ which is at least $(|V(P_1)| + 2)/3$ when $|P_1| \geq 16$. Each of y_1, y_2 and y_3 had charge $1/3$ after Step 2 and for each of them the charge changed to $5/14$. Note that, since $m > 3$, the vertices y_1, y_2, y_3, and y' will not get any charge from the tip P_3.

If $|V(P_1)| < 16$, then since $x(P_1)$ is not an out-endpoint, by Lemma 3.2.5, $|D \cap V(P_1)| < |V(P_1)|/3$, and after distributing the charges of vertices of $D \cap V(P_1)$ evenly among vertices of P_1, each vertex of P_1 will have charge less than $1/3$.

Step 11: Let P be an accepting 2-path such that exactly one endpoint of P is an out-endpoint or a $(2,2)$-endpoint, and the central path of P has exactly 3 vertices. By definition, P is not a dependant of an exceptional path. Suppose that $P_1 = (x_1 \ldots x_k)$, $P_2 = (y_1 y_2 y_3)$, and $P_3 = (z_1 \ldots z_l)$, so that $P = (x_1 \ldots x_k y_1 y_2 y_3 z_1 \ldots z_l)$. We may assume that x_1 is neither an out-endpoint nor a $(2,2)$-endpoint of P. By definition, y_2 is an acceptor for an out-endpoint y' of a path P' or for $y' = z_l$ if z_l is a $(2,2)$-endpoint. Since z_l is either a $(2,2)$-endpoint or an out-endpoint of P, the charges of vertices in P_3 were defined at Step 9 (if the acceptor of z_l is on a 2-path, then the charge of z_l could be changed at Step 10 or Step 11). We define the charges of vertices in P_1, P_2 and the charge of y' exactly as at Step 10.

Step 12: Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a $(2,2)$-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| \leq (|V(P)| + 1)/3$. By Lemma 3.2.5, $|P| \geq 16$. Hence, after distributing the charges of vertices of $D \cap V(P)$ evenly among all vertices of P, each vertex of P will have charge at most

$$\frac{|V(P)| + 1}{3|V(P)|} = \frac{1}{3} + \frac{1}{3|V(P)|} \leq \frac{1}{3} + \frac{1}{48} < \frac{5}{14}.$$

Step 13: Let P be an accepting 2-path such that each of the endpoints of P is neither an out-endpoint nor a $(2,2)$-endpoint, the central path of P has exactly 3 vertices, and $|D \cap V(P)| > (|V(P)| + 1)/3$. If P is a dependant of an exceptional path, then we are done at Step 8. Suppose not. Let P_1, P_2, and P_3 be defined as at Step 11. Then $|D \cap V(P)| = (|V(P)| + 4)/3$ and this may happen only if $|D \cap V(P_1)| = (|P_1| + 2)/3$ and $|D \cap V(P_3)| = (|P_3| + 2)/3$. In this case, by Lemma 3.2.5, $k \geq 16$ and $l \geq 16$. If

$k+3+l > 38$, then $k+3+l \geq 41$ and $|D \cap V(P)| \leq \lceil k/3 \rceil + 1 + \lceil l/3 \rceil = (|V(P)|+4)/3$. Distributing the charge evenly among the vertices of $V(P) \cup \{y'\}$, where y' is the out-endpoint of another path P' whose acceptor is y_2, we obtain that the charge of each vertex in $V(P) \cup \{y'\}$ is at most

$$\frac{|V(P)|+4}{3(|V(P)|+1)} = \frac{1}{3} + \frac{3}{3(|V(P)|+1)} \leq \frac{1}{3} + \frac{1}{42} = \frac{5}{14}.$$

This is the only case so far that the end-vertex of a tip of a non-exceptional path gets charge greater than 3/14. Note that it happens only when each of the tips of P has at least 16 vertices, P has no out-endpoints or (2, 2)-endpoints, $|D \cap V(P)| = (|V(P)|+4)/3$, and P accepts only one vertex. Recall that the other possibility for an end-vertex $y*$ of a 1-path or of a tip of a 2-path to get a positive charge occurs only at Step 10 or 11. In such a case, the following conditions hold:

(r1) $y*$ receives at most 3/14 of charge;

(r2) the accepting vertex y is either the second or the penultimate vertex in the central path, say P_2^*, of some 2-path P^*;

(r3) if P_2^* has more than 3 vertices (Case 10), then the closest to y tip of P^* has at least 16 vertices and no out-endpoints;

(r4) if P_2^* has exactly 3 vertices (Case 11), then one of the tips of P^* has at least 16 vertices and no out-endpoints and the other tip has either an out-endpoint or a (2,2)-endpoint.

The only case we have not yet considered is that $|P_2| = 3$, $k, l \geq 16$ and $k+l+3 \leq 38$. In particular, this means that P is a special path. In this case, $|D \cap V(P)| = 13$, when P has type 1 and $|D \cap V(P)| = 14$, when P has type 2. In both cases, the only accepting vertex is a special vertex. In both cases, y' has the current charge 0. We give to y' and to every vertex of P charge 5/14, but $(35+1) \cdot 5/14 = 13 - 1/7$ and $(38+1) \cdot 5/14 = 14 - 1/14$; so we need to distribute either 1/7 (when P has type 1) or 1/14 (when P has type 2) among some other vertices. Consider the following cases for distributing this charge.

Case 1: Vertex y' is the out-endpoint of a 1-path P' of length at least 4. In this case, we add 1/42 to the charge of each of the vertices of $P' - y'$. At Step 3 or Step 4, each of these vertices got charge 1/3, so now each of them has charge 5/14. If P is a special path of type 2 or P' has at least 7 vertices, then we are done; so suppose that P has type 1 and $|P'| = 4$. Let $P' = (w_1 w_2 w_3 w_4)$, where $y' = w_1$. If w_1 has another outneighbor v apart from its acceptor, then by (C1) and (C5), v is a special vertex of a special path P'' of type 1, and this path is non-accepting. In this case, every vertex of P'' has charge 12/35, and after distributing evenly our surplus charge of 1/14 among vertices of P'', each of these vertices will have charge $12/35 + 1/(14 \cdot 35) < 5/14$. So, w_1 has no other outneighbors. By (C1), no vertex in P' dominates all the others. If w_1 has two neighbors in P' and no vertex in P' dominates all the others, then $G(P')$ is the 4-cycle (w_1, w_2, w_3, w_4). By (C5), the outneighbors of w_2, w_3 and w_4 are special vertices of special paths which are not accepting. So, we can distribute our surplus $1/14$ among these vertices, as above.

Case 2: Vertex y' is the out-endpoint of a tip of an accepting 2-path P'. Then P' can be written as $P'_1P'_2P'_3$, where P'_1 and P'_3 are the tips, and P'_2 is the center. Suppose that $P'_2 = (v_1v_2 \dots v_t)$. Note that by the definition of a center, v_2 is the acceptor for a vertex v' and the charge of v' (maybe received because of P'_3 at Step 10 or 11) is at most $3/14$. We give $1/7$ to v'.

Case 3: Vertex y' is the out-endpoint of a 2-path P' that forms a δ-subgraph. From (B3) we get that the center vertex is the only possible accepting vertex, but it has degree 3 in P'. Hence P' is non-accepting. We give $1/28$ to each of the remaining vertices of F. Since each of them got the charge $1/4$ at Step 6, now it will have $1/4 + 1/28 = 2/7$.

Case 4: Vertex y' is the out-endpoint of a non-accepting 2-path P' that does not form a δ-subgraph. Let $P' = (w_1 \dots w_s)$, where $y' = w_1$. Since P' is not accepting and we chose an acceptor for w_1, according Rules (C2)-(C5), either w_s also is an out-endpoint or $s \leq 11$. Suppose first that $s \leq 11$ and w_s has no outneighbors. Then $s \in \{5, 8, 11\}$ and on Step 7 each of the vertices of $P' - w_1$ got the charge $\frac{s-2}{3(s-1)}$. We distribute $1/7$ evenly among these vertices so that each of them will now have charge

$$\frac{s-2}{3(s-1)} + \frac{1}{7(s-1)} = \frac{7s-11}{21(s-1)} < \frac{1}{3}.$$

Suppose now that w_s is an out-endpoint. Since P' is not an exceptional path, the path P'' accepting w_s does not give charge to any vertex apart from w_s and by (r1) w_s has charge at most $3/14$. Adding the surplus to this vertex leaves it with charge $3/14 + 1/7 = 5/14$.

Case 5: The path P' containing y' has no other vertices. Since P is special, by (C1) this might happen only if P is very special and y' is adjacent to special vertices in special paths P_1 and P_2 that are non-accepting. We add $1/(14 \cdot 35)$ to the charge of each vertex in P_1 and P_2. This finishes the discharging.

Thus, what is left to prove Theorem 3.1.5 is to prove Lemmas 3.2.3, 3.2.4 and 3.2.5. We will do it in the next sections. In Section 3.4 we describe the approach we use and prove a number of auxiliary statements. Applying these statements, we prove Lemmas 3.2.4 and 3.2.5 in Section 3.5. Lemma 3.2.3 has the longest proof. It will be proved in Section 3.6.

3.4 Structure of proofs and technical statements

We will need some notation. Let G' be a subgraph of a graph G and $u, v \in V(G')$, $u \neq v$. Say that u is (G', v)-*distant* if G' contains a hamiltonian v, u-path. Sometimes, if it is clear which G' we have in mind, we will simply say that u is v-*distant*.

A v-*lasso* is a graph consisting of a cycle, say C, and a path connecting v with C. In this case, C is the *loop* of this v-lasso, and H is the remaining path which we will call the *handle*. If $v \in V(C)$, then C itself is a v-lasso. A v-lasso with k vertices, l of whose belong to the loop, will be sometimes called a (v, k, l)-*lasso*.

A typical structure used in the proofs of Lemmas 3.2.3, 3.2.4, and 3.2.5 will be as follows. We will consider a path $P_1 = (v_1 \dots v_k)$ and let $G_1 = G[V(P_1)]$. We will know that k is not large, for example, $k \leq 11$. For some reasons, we will know that v_1 has no neighbors outside of P_1 and, moreover, that no (G_1, v_k)-distant vertex has a neighbor outside of P_1. If k is $2 \pmod 3$, then we will want to prove that some $(k-2)/3$ vertices dominate $V(P_1) - v_k$. If k is $1 \pmod 3$, then we will want to prove that some $(k-1)/3$ vertices dominate $V(P_1)$. We will show that we do not need to consider the case of $k = 0 \pmod 3$. Thus, we need that some $\lfloor k/3 \rfloor$ vertices dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1. For example, if $P_1 = P = (v_1 \dots v_8)$ and v_8 is the only out-endpoint of P, then we will prove that some two vertices dominate $V(P_1) - v_8$. We will do this as follows.

Since v_1 has no neighbors outside of P_1, it has two neighbors, v_i and v_j distinct from v_2 on P_1. Path P_1 together with edge $v_1 v_i$ forms a v_k-lasso. Among all v_k-lassos on $V(P_1)$ choose a lasso L with the largest loop C. By renumbering vertices, we may assume that L consists of the cycle $C = (v_1 \dots v_r)$ and the path $(v_r \dots v_k)$. If r is divisible by 3, then the set $D = \{v_3, v_6, \dots, v_{3\lfloor k/3 \rfloor}\}$ dominates what we need. So, we will need to consider only $r \not\equiv 0 \pmod 3$. The problem of finding $\lfloor k/3 \rfloor$ vertices that dominate the first $3\lfloor k/3 \rfloor + 1$ vertices of P_1 reduces to the problem of finding $\lfloor r/3 \rfloor$ vertices that dominate $\{v_1, \dots, v_{3\lfloor r/3 \rfloor + 1}\}$, since the remaining $3(\lfloor k/3 \rfloor - \lfloor r/3 \rfloor)$ vertices of P_1 that we need to dominate are easily dominated by the vertices $v_{3(\lfloor r/3 \rfloor + 1)}, v_{3(\lfloor r/3 \rfloor + 2)}, \dots, v_{3(\lfloor k/3 \rfloor)}$.

Let $G' = G[V(C)]$. By the above condition on P_1, no (G', v_r)-distant vertex has a neighbor outside of P_1. By the maximality of $|C|$, no (G', v_r)-distant vertex has a neighbor in $V(P_1) - V(C)$. Thus, no (G', v_r)-distant vertex has a neighbor outside of C. In the rest of this section we will prove that under these conditions, some $\lfloor r/3 \rfloor$ vertices dominate $\{v_1, \dots, v_{3\lfloor r/3 \rfloor + 1}\}$ for $r = 4, 5, 7, 8, 10, 11, 13$ and 14. This will be heavily used later.

Lemma 3.4.1. *Let G' be the subgraph of a cubic graph G induced by vertices v_1, v_2, v_3, and v_4. If v_1 has no neighbor outside of G', then v_1 dominates $V(G')$.*

Proof. This is because the only possible neighbors of v_1 are v_2, v_3, and v_4. □

Lemma 3.4.2. *Let G' be the subgraph of a cubic graph G induced by the vertices of a path $(v_1v_2v_3v_4v_5)$. If no (G', v_5)- distant vertex has a neighbor outside of $V(G')$, then some vertex dominates $V(G') - v_5$.*

Proof. If $v_1v_3 \in E(G)$, then v_3 dominates $V(G') - v_5$. Suppose that $v_1v_3 \notin E(G)$. Then $v_1v_4, v_1v_5 \in E(G)$. The paths $(v_3v_2v_1v_4v_5)$ and $(v_2v_3v_4v_1v_5)$ show that each of v_2 and v_3 can play the role of v_1 and thus by the above argument should be adjacent to v_5 if no vertex dominates $V(G') - v_5$. But v_5 cannot be adjacent to all of v_1, v_2, v_3, v_4. □

Lemma 3.4.3. *If a graph G' on $3k+1$ vertices has a hamiltonian path $P = (v_1 \ldots v_{3k+1})$ and an edge v_iv_{i+3j-1}, where i is not divisible by 3, then G' has a dominating set of size k.*

Proof. If $i = 3m+1$, then we let $D = \{v_2, v_5, \ldots, v_{3m-1}, v_{3m+3}, v_{3m+6}, \ldots, v_{3k}\}$. Note that then $v_{i+3j-1} \in D$. Thus every $v \in D$ dominates its neighbors on P, and v_{i+3j-1} also dominates v_i.

If $i = 3m+2$, then we let $D = \{v_2, v_5, \ldots, v_{3m+3j-1}, v_{3m+3j+3}, v_{3m+3j+6}, \ldots, v_{3k}\}$. In this case $v_i \in D$, every $v \in D$ dominates its neighbors on P, and $v_i = v_{3m+2}$ also dominates $v_{i+3j-1} = v_{3m+3j+1}$. □

An immediate corollary of this lemma is the following fact.

Lemma 3.4.4. *If a graph G' on $3k+1$ vertices has a hamiltonian cycle $(v_1 \ldots v_{3k+1})$ and an edge v_iv_j with $j-i+1$ divisible by 3, then G' has a dominating set of size k.*

Lemma 3.4.5. *Let graph G' on $3k+1$ vertices form a subdivision of K_4 with the set R of the 4 branching vertices. Then either G' has a dominating set of size k or the lengths* (mod 3) *of the paths between the vertices in R in this subdivision of G' are equivalent to those in one of the three graphs in Figure 3.3 (graphs D, E, and F).*

Proof. A *thread* in a graph is a path connecting two vertices of degree at least 3 whose all internal vertices have degree 2. Say that two subdivisions of K_4 are *equivalent* if

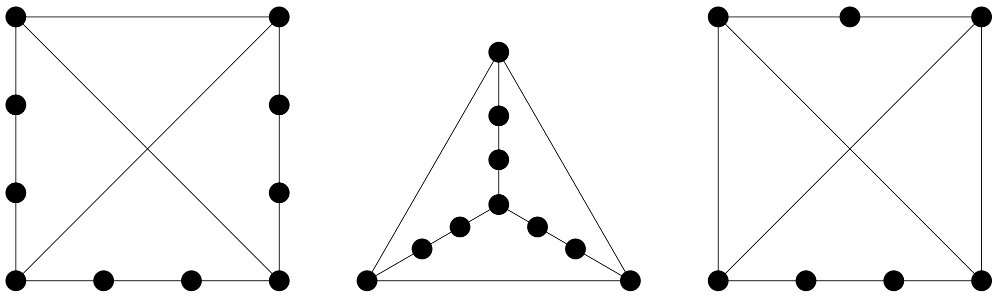


Figure 3.3: Graphs D, E, and F

the lengths of their threads are the same $\pmod 3$. Since every vertex of degree 2 dominates exactly three consecutive vertices in a thread, it is enough to prove the lemma for subdivisions of K_4 in which the length of each thread is in $\{1, 2, 3\}$. Since every edge subdivision in a graph adds one vertex and one edge, each K_4 subdivision with $3k + 1$ vertices has $3k + 3$ edges.

Case 1: Two threads of length 2 share an endvertex v. Then v dominates all but a path with $3k - 3$ vertices. Taking the natural dominating set in this path yields a dominating set of G' with size k.

Case 2: G' contains two vertex disjoint threads of length 2, but Case 1 does not hold. Since G' has $3k + 3$ edges, the other threads necessarily have the lengths $1, 1, 3$, and 3. This yields two possible graphs. The graphs and their dominating sets are shown as graphs G and H in Figure 3.4.

Case 3: Exactly one thread has length 2. The possible lengths of the remaining threads are $1, 1, 1, 1, 3$ or $1, 3, 3, 3, 3$. This yields four possible graphs, the bad case shown as F, and the three graphs shown with their dominating sets of size k are shown as graphs I, J, and K.

Case 4: All threads have the same length $\pmod 3$. This yields the graphs L and M each of which has a dominating set of size k.

Case 5: The lengths of the threads in our subdivision are $1, 1, 1, 3, 3, 3$. The three possible graphs with these thread lengths are graphs D, E in Figure 3.3, and graph N in Figure 3.4. □

Sometimes, it will be simpler to check that Case 1 of Lemma 3.4.5 holds. We state this case as a separate claim:

Lemma 3.4.6. *Suppose that a graph G' on $3k + 1$ vertices has a spanning subgraph G''*

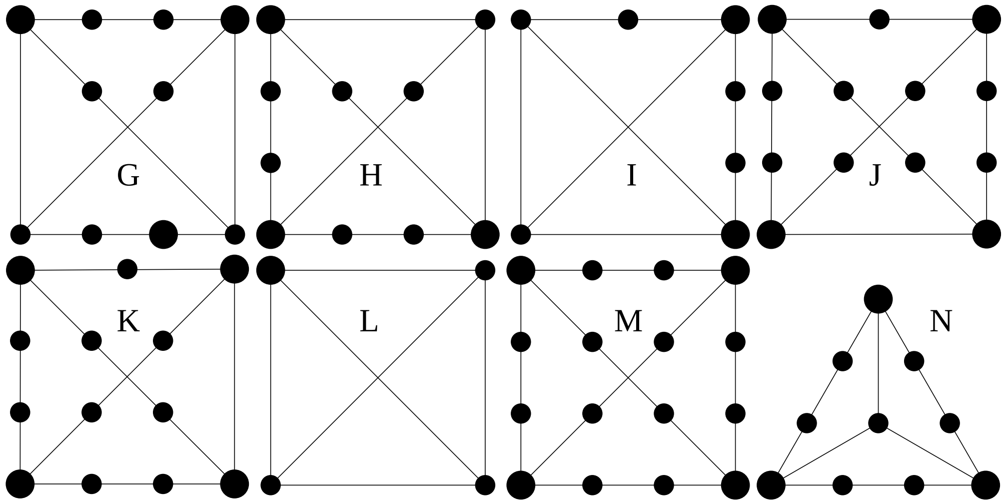


Figure 3.4: Graphs G, H, I, J, K, L, M, and N along with their dominating sets

consisting of 3 *internally disjoint paths* P_1, P_2 *and* P_3 *connecting some vertices* x *and* y. *Suppose that the distances on* P_1 *from an internal vertex* z *of* P_1 *to* x *and to* y *are* 2 (mod 3). *Then either* G' *has a dominating set of size* k, *or* z *has no third neighbor in* G', *or the third neighbor of* z *belongs to* P_1.

Lemma 3.4.7. *Let* G' *be the subgraph of a cubic graph* G *induced by vertices* $v_1, v_2, \ldots, v_7$. *If* G' *contains a hamiltonian cycle* $(v_1 v_2 \ldots v_7)$ *and* v_7 *has an outneighbor, then either some two vertices dominate* $V(G')$, *or there are two* (G', v_7)*-distant vertices such that each of them has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. For each $i = 1, \ldots, 7$, the *third neighbor* of v_i is the in-neighbor different from v_{i-1} and v_{i+1} (if it exists). Since both v_1 and v_6 are (G', v_7)-distant, under conditions of the lemma, at least one of them has no outneighbors. By symmetry, we may assume that v_1 has no outneighbors. By Lemma 3.4.4, the only possible third neighbors of v_1 are v_4 and v_5.

Case 1: $v_1 v_5 \in E(G')$. By Lemma 3.4.4, v_4 has no third neighbors in G'. Thus it has an outneighbor. But the path $(v_4 v_3 v_2 v_1 v_5 v_6 v_7)$ is hamiltonian in G'. So if the lemma does not hold, then v_6 has no outneighbors. Symmetrically to v_1, the possible third neighbors of v_6 are v_2 and v_3. If $v_6 v_3 \in E(G')$, then $\{v_1, v_3\}$ dominates $V(G')$. If $v_6 v_2 \in E(G')$, then symmetrically to v_4, v_3 must have an outneighbor, a contradiction to our assumptions.

Case 2: $v_1 v_4 \in E(G')$. If $\{v_1, v_6\}$ dominates $V(G')$, then we are done. Suppose not. Then $v_6 v_3 \notin E(G)$. Thus by Lemma 3.4.4, v_3 has an outneighbor. Since the path $(v_3 v_2 v_1 v_4 v_5 v_6 v_7)$ is hamiltonian in G', v_3 is v_7-distant. Hence if the lemma does not hold, then v_6 has the third neighbor in G'. By the symmetry with v_1, it should be v_2 or v_3. But we assumed that $v_6v_3 \notin E(G)$. Hence, $v_6v_2 \in E(G)$ and we have Case 1 again. $\square$

Lemma 3.4.8. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_8$. If G' contains a hamiltonian cycle $(v_1v_2 \ldots v_8)$ and v_8 has an outneighbor, then either some two vertices v_i and v_j dominate $V(G') - v_8$, or some (G', v_8)-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. In particular, this implies that v_1 and v_7 have third neighbors in G'. If $v_1v_7 \in E(G')$, then Lemma 3.4.7 yields our lemma. Let $v_1v_7 \notin E(G')$. By Lemma 3.4.3, $v_1v_6 \notin E(G')$ and $v_1v_3 \notin E(G')$. Hence, the only possible third neighbors for v_1 are v_4 and v_5, and by symmetry, the only possible third neighbors for v_7 are v_4 and v_3. If v_4 is not a neighbor of $\{v_1, v_7\}$, then $v_7v_3, v_1v_5 \in E(G')$ and hence $\{v_3, v_5\}$ dominates $V(G') - v_8$. Thus, (by symmetry) we may assume that $v_1v_4 \in E(G')$ and hence $v_7v_3 \in E(G')$.

The existence of the path $(v_6v_5v_4v_1v_2v_3v_7v_8)$ yields that v_6 has no outneighbors. The only possible third in-neighbor for v_6 is v_2. Then v_5 must have an outneighbor, but this contradicts the existence of the hamiltonian path $(v_5v_4v_3v_7v_6v_2v_1v_8)$. $\square$

Lemma 3.4.9. [20] *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{10}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{10})$, and that v_{10} has an outneighbor. Then either some three vertices dominate $V(G')$, or some (G', v_{10})-distant vertex has an outneighbor.*

Lemma 3.4.10. [20] *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{10}, v_{11}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{11})$, and that v_{11} has an outneighbor. Then either some three vertices dominate $V(G') - v_{11}$, or some (G', v_{11})-distant vertex has an outneighbor.*

Lemma 3.4.11. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{13}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{13})$ and v_{13} has an outneighbor. Then either some four vertices dominate $V(G')$, or some (G', v_{13})-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. By Lemma 3.4.4,

$$\text{no edge of the form } v_iv_{i+3j-1} \text{ is present in } G'. \tag{3.1}$$

Further, if the hamiltonian cycle is drawn as a planar graph, then any two crossing edges along with the hamiltonian cycle determine a K_4 subdivision on 13 vertices. Hence Lemma 3.4.5 may be applied whenever a potential edge crosses an edge already forced. Since v_1 is v_{13}-distant, it has a third neighbor in G'.

Case 1: $v_1v_4 \in E(G')$. The path $(v_{13}v_{12}...v_4v_1v_2v_3)$ forces v_3 to have its third neighbor in G'. Since any such neighbor forces an edge crossing v_1v_4, Lemma 3.4.5 restricts this neighbor to one of v_7 and v_{10}.

Case 1.1: $v_3v_7 \in E(G')$. Then Lemmas 3.4.4 and 3.4.5 forbid edges $v_{12}v_2$, $v_{12}v_5$, and $v_{12}v_{10}$. So, the third neighbor of v_{12} is one of v_6, v_8, and v_9. If $v_{12}v_6 \in E(G')$, then the set $\{v_1,v_6,v_7,v_{10}\}$ dominates G'. Suppose that $v_{12}v_8 \in E(G')$. Then the path $(v_{13}v_1v_2...v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (3.1) and by Lemma 3.4.5 with $R = \{v_1, v_2, v_4, v_9\}$ and $R = \{v_3, v_5, v_7, v_9\}$, we have $v_9v_6 \in E(G')$. So, the set $\{v_1, v_6, v_7, v_{11}\}$ dominates G'. Thus, $v_{12}v_9 \in E(G')$, and by symmetry, $v_{10}v_6 \in E(G')$. Then $\{v_1, v_6, v_7, v_{12}\}$ dominates G'.

Case 1.2: $v_3v_{10} \in E(G')$. Lemma 3.4.5 applied successively with $R \supset \{v_1, v_4, v_{12}\}$ and $R \supset \{v_3, v_{10}, v_{12}\}$ restricts the third neighbor of v_{12} to one of v_6 and v_9. In either case, the set $\{v_1, v_6, v_9, v_{10}\}$ dominates G'.

Hence $v_1v_4 \notin G'$, and symmetry gives $v_{12}v_9 \notin G'$.

Case 2: $v_1v_5 \in E(G')$. The path $(v_{13}v_{12}...v_5v_1v_2v_3v_4)$ forces v_4 to have its third neighbor in G'. By (3.1), this neighbor is one of v_7,v_8,v_{10}, or v_{11}.

Case 2.1: $v_4v_7 \in E(G')$. The path $(v_{13}v_{12} \dots v_7v_4v_3v_2v_1v_5v_6)$ forces v_6 to have its third neighbor in G'. By Lemma 3.4.5 with $R \supset \{v_6, v_4, v_7\}$ and $R \supset \{v_6, v_1, v_5\}$, this neighbor must be v_{10}. Then by Lemma 3.4.5 with $R \supset \{v_{12}, v_6, v_{10}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2 and hence the set $\{v_1, v_2, v_7, v_{10}\}$ dominates G'.

Case 2.2: $v_4v_8 \in E(G')$. By Lemma 3.4.5 with $R \supset \{v_{12}, v_4, v_8\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2. Then the path $(v_{13}v_1v_2v_{12}v_{11} \dots v_3)$ forces v_3 to have its third neighbor in G', but Lemma 3.4.5 with $R \supset \{v_3, v_1, v_5\}$ eliminates all possible neighbors of v_3.

Case 2.3: $v_4v_{10} \in E(G')$. By Lemma 3.4.5 with $R \supset \{v_{12}, v_4, v_{10}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} must be v_2. But then the set $\{v_1, v_2, v_7, v_{10}\}$ dominates G'.

Case 2.4: $v_4v_{11} \in E(G')$. By Lemma 3.4.5 with $R \supset \{v_{12}, v_4, v_{11}\}$ and $R \supset \{v_{12}, v_1, v_5\}$, the third neighbor of v_{12} is either v_2 or v_8. If $v_{12}v_2 \in E(G')$, then the path $(v_{13}v_{12}v_2v_1v_5v_6 \dots v_{11}v_4v_3)$ forces v_3 to have its third neighbor in G', but Lemma 3.4.5 with $R \supset \{v_3, v_1, v_5\}$ eliminates all possible neighbors of v_3. So, $v_{12}v_8 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6 \dots v_{10})$ forces v_{10} to have its third neighbor in G', but Lemma 3.4.5

with $R \supset \{v_{12}, v_8, v_{10}\}$ eliminates all possible neighbors of v_{10}.

Hence $v_1v_5 \notin G'$, and symmetry gives $v_{12}v_8 \notin G'$.

Case 3: $v_1v_7 \in E(G')$. Each allowable edge from v_{12} crosses v_1v_7, and Lemma 3.4.5 gives a dominating set of size 4.

Hence $v_1v_7 \notin G'$, and symmetry gives $v_{12}v_6 \notin G'$.

Case 4: $v_1v_{10} \in E(G')$. Each allowable edge from v_{12} crosses v_1v_{10}, and Lemma 3.4.5 gives a dominating set of size 4.

Hence $v_1v_{10} \notin G'$, and symmetry gives $v_{12}v_3 \notin G'$.

Case 5: $v_1v_8 \in E(G')$. The two possible third neighbors of v_{12} are v_2, and v_5.

Case 5.1: $v_{12}v_2 \in E(G')$. The path $(v_{13}v_1v_8v_7 \ldots v_2v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (3.1), this third neighbor is not in $\{v_4, v_7, v_{11}\}$. Then Lemma 3.4.5 with $R = \{v_1, v_9, v_8, v_i\}$ for $i \in \{4, 6\}$ forces $v_9v_5 \in E(G')$. Now the path $(v_{13}v_{12} \ldots v_8v_1v_2 \ldots v_7)$ forces v_7 to have its third neighbor in G'. This contradicts Lemma 3.4.6 with $x = v_5$, $y = v_9$ and $z = v_7$.

Case 5.2: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2 \ldots v_5v_{12}v_{11} \ldots v_6)$ forces v_6 to have its third neighbor in G'. Lemma 3.4.6 with $x = v_1$, $y = v_8$ and $z = v_6$ forces this neighbor to be one of v_2 and v_3. If $v_6v_3 \in E(G')$, then the set $\{v_1, v_5, v_6, v_{10}\}$ dominates G'. So, $v_6v_2 \in E(G')$. By the symmetry between v_6 and v_7, $v_7v_{11} \in E(G)$. The path $(v_{13}v_1v_2v_6v_7 \ldots v_{12}v_5v_4v_3)$ forces v_3 to have its third neighbor in G', a contradiction to Lemma 3.4.6 with $x = v_1$, $y = v_8$ and $z = v_3$.

Hence $v_1v_8 \notin G'$, and symmetry gives $v_{12}v_5 \notin G'$.

Case 6: $v_1v_{11} \in E(G')$, and $v_{12}v_2 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10} \ldots v_3)$ forces v_3 to have its third neighbor in G'. By (3.1), this neighbor is one of v_6,v_7,v_9, and v_{10}. Note that v_{10} is symmetric with v_3.

Case 6.1: $v_3v_6 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10} \ldots v_6v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. Lemma 3.4.5 with $R \supset \{v_6, v_3, v_5\}$ restricts this neighbor to v_9. By (3.1), $v_{10}v_8 \notin E(G)$. By Lemma 3.4.6 with $x = v_5$, $y = v_9$ and $z = v_7$, $v_{10}v_7 \notin E(G)$. So, $v_{10}v_4 \in E(G)$. So, the path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_6v_5v_9v_8v_7)$ forces v_7 to have its third neighbor in G', but no possible third neighbor remains.

Case 6.2: $v_3v_7 \in E(G')$. By symmetry, we may assume that v_{10} is adjacent to either v_4 or v_6. If $v_{10}v_4 \in E(G')$, then the path $(v_{13}v_1v_{11}v_{12}v_2v_3v_4v_{10}v_9 \ldots v_5)$ forces v_5 to have its third neighbor in G', a contradiction to Lemma 3.4.6 with $x = v_3$, $y = v_7$ and $z = v_5$. So, $v_{10}v_6 \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_7v_8v_9v_{10}v_6v_5v_4)$ forces v_4 to have its third neighbor in G'. By (3.1), this neighbor must be v_8, but Lemma 3.4.5 with $R = \{v_4, v_8, v_6, v_{10}\}$ gives a dominating set of size 4.

Case 6.3: $v_3v_9 \in E(G')$, and necessarily $v_{10}v_4 \in E(G')$. The path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_9v_8 \ldots v_5)$ forces v_5 to have its third neighbor in G', and (3.1) forces it to be v_8. Finally the path $(v_{13}v_{12}v_2v_1v_{11}v_{10}v_4v_3v_9v_8v_5v_6v_7)$ forces v_7 to have its third neighbor in G' which is impossible.
Case 6.4: $v_3v_{10} \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_{10}v_9 \ldots v_4)$ forces v_4 to have its third neighbor in G'. By (3.1), this neighbor is one of v_7 and v_8. Note that v_9 is symmetric with v_4. If $v_4v_7 \in E(G')$, then Lemma 3.4.5 with $R \supset \{v_4, v_7, v_9\}$ eliminates v_5 and v_6 as possible third neighbors of v_9. So, $v_4v_8 \in E(G')$, and by symmetry, $v_9v_5 \in E(G')$. The path $(v_{13}v_1v_{11}v_{12}v_2v_3v_{10}v_9v_5v_4v_8v_7v_6)$ forces v_6 to have its third neighbor in G' which is impossible. This proves the lemma. $\square$

Lemma 3.4.12. *Let G' be the subgraph of a cubic graph G induced by vertices $v_1, v_2, \ldots, v_{14}$. Suppose that G' contains a hamiltonian cycle $(v_1v_2 \ldots v_{14})$ and v_{14} has an outneighbor. Then either some four vertices dominate $V(G') - v_{14}$, or some (G', v_{14})-distant vertex has an outneighbor.*

Proof. Suppose that the lemma does not hold for some choice of G and G'. Then by Lemma 3.4.3, for every hamiltonian path $(u_1 \ldots u_{13})$ in $G' - v_{14}$,

$$\text{if } u_iu_{i+3j-1} \in E(G'), \text{ then } i = 0 \pmod 3. \tag{3.2}$$

By (3.2) for the path P=$(v_1v_2 \ldots v_{13})$, the only possible third neighbors of v_1 are v_4,v_5,v_7,v_8,v_{10},v_{11}, and v_{13}. Note that v_{13} is symmetric with v_1.
Case 1: $v_1v_4 \in E(G')$. The path $(v_{13}v_{12} \ldots v_4v_1v_2v_3)$ forces v_3 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_5,v_7,v_8,v_{10},v_{11}, and v_{13}.
Case 1.1: $v_3v_5 \in E(G')$. The path $(v_{13}v_{12} \ldots v_5v_3v_4v_1v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is either v_8, or v_{11}. In either case, the set $\{v_5$,v_8,v_{11},$v_{14}\}$ dominates G'.
Case 1.2: $v_3v_7 \in E(G')$. The third neighbor of v_{13} is amongst v_6,v_9, and v_{10}.
Case 1.2.1: $v_{13}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{13}v_{12} \ldots v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for P, this neighbor is among v_5,v_8,v_9,v_{11}, and v_{12}. If $v_2v_8 \in E(G')$, then the set $\{v_4, v_8, v_{10}, v_{13}\}$ dominates G'. If $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_4$,v_6,v_8,$v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_4$,v_6,v_9,$v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_2v_5 \in E(G')$. The path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8 \ldots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for P, this neighbor is either v_8, or v_9. If $v_{12}v_8 \in E(G')$, then

the path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8v_{12}v_{11}\dots v_9)$ forces v_9 to have its third neighbor in G'. Then (3.2) for this path disallows all possible third neighbors. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5v_2v_1v_4v_3v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. Thus $v_{10}v_8 \in E(G')$, and the set $\{v_2, v_3, v_{10}, v_{13}\}$ dominates G'.

Case 1.2.2: $v_{13}v_9 \in E(G')$. The path $P' = (v_1v_2\dots v_9v_{13}v_{12}\dots v_{10})$ forces v_{10} to have its third neighbor in G', and (3.2) for P' forces $v_{10}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{10}v_{11}v_{12}v_{13}v_9v_8v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. If $v_2v_5 \in E(G')$, then the path $(v_{13}v_9v_8v_7v_3v_2v_1v_4v_5v_6v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G', and hence $v_{12}v_8 \in E(G')$. Then the set $\{v_2, v_3, v_{10}, v_{12}\}$ dominates $G'-v_{14}$. If $v_2v_8 \in E(G')$ or $v_2v_{12} \in E(G')$, then the set $\{v_4, v_6, v_8, v_{12}\}$ dominates $G'-v_{14}$. Finally, if $v_2v_{11} \in E(G')$, then the set $\{v_4, v_6, v_9, v_{11}\}$ dominates $G' - v_{14}$.

Case 1.2.3: $v_{13}v_{10} \in E(G')$. The path $(v_1v_2\dots v_{10}v_{13}v_{12}v_{11})$ forces v_{11} to have its third neighbor in G'. By (3.2) for P and the symmetry between v_{11} and v_3, $v_{11}v_6 \in E(G')$. The path $(v_1v_4v_5v_6v_{11}v_{12}v_{13}v_{10}v_9v_8v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. If $v_2v_5 \in E(G')$, then the path $(v_1v_2v_5v_4v_3v_7v_6v_{11}v_{12}v_{13}v_{10}v_9v_8)$ forces v_8 to have the third neighbor in G', hence $v_8v_{12} \in E(G')$. Then the set $\{v_2, v_5, v_8, v_{10}\}$ dominates $G' - v_{14}$. If $v_2v_8 \in E(G')$, then the set $\{v_4, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 1.3: $v_3v_8 \in E(G')$. The path $(v_{13}v_{12}\dots v_8v_3v_2v_1v_4v_5\dots v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for this path this neighbor must be amongst v_{10}, v_{11}, and v_{13}.

Case 1.3.1: $v_7v_{10} \in E(G')$. Then (3.2) with the path $(v_{13}v_{12}v_{11}v_{10}v_7v_6v_5v_4v_1v_2v_3v_8v_9)$ forces $v_{13}v_6 \notin G'$. Hence by 3.2) for P $v_{13}v_9 \in E(G')$. Then the path $(v_{13}v_9v_8v_3v_2v_1v_4v_5v_6v_7v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Using (3.2) on this path forces $v_{12}v_6 \in E(G')$. Then the set $\{v_3, v_6, v_{10}, v_{14}\}$ dominates G'.

Case 1.3.2: $v_7v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_7v_6v_5v_4v_1v_2v_3v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. Then (3.2) for this path and P forces $v_{10}v_{13} \in E(G')$. This is then symmetric with Case 1.2.

Case 1.3.3: $v_7v_{13} \in E(G')$. The path $(v_{13}v_7v_6v_5v_4v_1v_2v_3v_8v_9\dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_2, v_5, v_6, and v_9. If $v_{12}v_2 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_2, v_5, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_6 \in E(G')$, then the set $\{v_3, v_6, v_{10}, v_{14}\}$ dominates G'. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_7v_6v_5v_4v_1v_2v_3v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (3.2) for P forces $v_{10}v_6 \in E(G')$. Then the set $\{v_3, v_6, v_{12}, v_{14}\}$ dominates G'.

Case 1.4: $v_3v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5\dots v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is amongst v_2, v_5, v_6, v_{11}, and v_{13}.
Case 1.4.1: $v_9v_2 \in E(G')$. The set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.
Case 1.4.2: $v_9v_5 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5v_9v_8v_7v_6)$ forces v_6 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_2, v_{11}, and v_{13}. If $v_6v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_6v_{11} \in E(G')$, then by (3.2) for P $v_{13}v_8 \notin E(G')$ and hence $v_{13}v_7 \in E(G')$. So, in this case $\{v_2, v_5, v_7, v_{11}\}$ dominates $G' - v_{14}$. Thus, $v_6v_{13} \in E(G')$. The path $(v_1v_2\dots v_6v_{13}v_{12}\dots v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for P, $v_7v_{11} \in E(G')$. Then the path $(v_1v_4v_5v_9v_8v_7v_6v_{13}v_{12}v_{11}v_{10}v_3v_2)$ forces v_2 to have the third neighbor in G', and (3.2) for this path yields $v_2v_{12} \in E(G')$. Thus the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$.
Case 1.4.3: $v_9v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_3v_2v_1v_4v_5v_6v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for P, this neighbor is one of v_{11}, and v_{13}. If $v_7v_{11} \in E(G')$, then by (3.2) for P, no vertex in G' can be adjacent to v_{13}. If $v_7v_{13} \in E(G')$, then the path $(v_1v_2\dots v_6v_9v_{10}\dots v_{13}v_7v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path, this neighbor is one of v_{11}, or v_{12}. If $v_8v_{11} \in E(G')$, then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. If $v_8v_{12} \in E(G')$, then the set $\{v_3, v_6, v_{12}, v_{14}\}$ dominates G'.
Case 1.4.4: $v_9v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_9v_{10}v_3v_2v_1v_4v_5\dots v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is one of v_2, and v_5. If $v_8v_2 \in E(G')$, then the path $(v_{13}v_{12}v_{11}v_9v_{10}v_3v_4v_1v_2v_8v_7v_6v_5)$ forces v_5 to have its third neighbor in G'. Then by (3.2) for this path (3.2) for P eliminates the remaining possible neighbors of v_5. So, $v_8v_5 \in E(G')$. The path
$(v_{13}v_{12}v_{11}v_9v_{10}v_3v_2v_1v_4v_5v_8v_7v_6)$ forces v_6 to have its third neighbor in G'. By (3.2) for this path this neighbor is one of v_2 and v_{13}. If $v_6v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_6v_{13} \in E(G')$, then the set $\{v_1, v_8, v_{10}, v_{13}\}$ dominates G'.
Case 1.4.5: $v_9v_{13} \in E(G')$. The path $(v_{13}v_9v_8\dots v_4v_1v_2v_3v_{10}v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is amongst v_2, v_5, and v_8. If $v_{12}v_2 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_1, v_7, v_{10}, v_{12}\}$ dominates G'. If $v_{12}v_8 \in E(G')$, then the set $\{v_1, v_6, v_{10}, v_{12}\}$ dominates G'.
Case 1.5: $v_3v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5\dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for P, this neighbor is amongst v_6, v_7, and v_{13}.
Case 1.5.1: $v_{10}v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for P, this neighbor is v_{13}. Now the path

$(v_1v_2\dots v_7v_{13}v_{12}\dots v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path, $v_8v_{12} \in E(G')$. Then the set $\{v_3, v_6, v_8, v_{14}\}$ dominates G'.
Case 1.5.2: $v_{10}v_7 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_3v_2v_1v_4v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path and for P, $v_8v_6 \in E(G')$. Then by (3.2) for P $v_{13}v_9 \in E(G')$, and the set $\{v_1, v_6, v_9, v_{11}\}$ dominates G'.
Case 1.5.3: $v_{10}v_{13} \in E(G')$. The path $(v_1v_4v_5\dots v_{10}v_{13}v_{12}v_{11}v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for P, this neighbor is amongst v_5, v_6, v_8, v_9, and v_{12}. If $v_2v_5 \in E(G')$ or $v_2v_8 \in E(G')$, then the set $\{v_5, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_2v_6 \in E(G')$ or $v_2v_9 \in E(G')$, then the set $\{v_4, v_6, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$.
Case 1.6: $v_3v_{13} \in E(G')$. The path $(v_1v_4v_5\dots v_{13}v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for this path and for the path $(v_2v_1v_4v_3v_{13}v_{12}\dots v_5)$ this neighbor is amongst v_5, v_8, and v_{11}.
Case 1.6.1: $v_2v_5 \in E(G')$. Identifying the vertices $v_{13}, v_{14}, v_1, v_2, v_3$, and v_4 as one vertex v gives a new graph G'' on 8 vertices. A hamiltonian path in G'' starting at v has a corresponding hamiltonian path in G' which starts at v_{14} by using either the path $(v_{14}v_{13}v_3v_2v_1v_5)$ or the path $(v_{14}v_1v_4v_5v_2v_3v_{13})$. A dominating set of $G'' - v$ not using v can be extended to a dominating set of $G' - v_{14}$ with size 2 greater by including the vertices v_3 and v_4. A dominating set of G'' which contains v can be extended to a dominating set of G' with size 2 greater by replacing v by the vertices v_2, v_5, and v_{13}. Hence Lemma 3.4.8 gives the desired result for G'' which extends to G'.
Case 1.6.2: $v_2v_8 \in E(G')$. The path $(v_{13}v_3v_2v_1v_4v_5\dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path and P, this neighbor is one of v_5 or v_9. Also the path $(v_1v_4v_5\dots v_8v_2v_3v_{13}v_{12}\dots v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path and P, this neighbor is one of v_5 or v_{12}. This then forces the edge $v_9v_{12} \in E(G')$. Next the paths $(v_1v_4v_5\dots v_8v_2v_3v_{13}v_{12}v_9v_{10}v_{11})$ and $(v_1v_2v_8v_7\dots v_3v_{13}v_{12}v_9v_{10}v_{11})$ force v_{11} to have its third neighbor in G', and these paths along with (3.2) force this edge to be to v_5. Finally the path $(v_{13}v_3v_2v_1v_4v_5\dots v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (3.2) on this path forces this edge to be to v_6. Then the set $\{v_2, v_4, v_6, v_{12}\}$ dominates $G' - v_{14}$.
Case 1.6.3: $v_2v_{11} \in E(G')$. The path $(v_1v_4v_5\dots v_{11}v_2v_3v_{13}v_{12})$ forces v_{12} to have its third neighbor in G'. The set $\{v_2, v_3, v_6, v_9\}$ is a dominating set if v_{12} is adjacent to either of v_6 or v_9. By (3.2) on the path P the third neighbor of v_{12} must be either v_5 or v_8.
Case 1.6.3.1: $v_{12}v_5 \in E(G')$. The path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4\dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) on P this vertex must be either v_6 or v_7. The inclusion of the

edge $v_{10}v_6$ gives the amended path $(v_1v_2v_{11}v_{12}v_{13}v_3 \ldots v_6v_{10} \ldots v_7)$ forcing $v_7v_9 \notin E(G')$. Then the path $(v_1v_2v_{11}v_{12}v_{13}v_3 \ldots v_6v_{10}v_9v_7v_8)$ gives the (G', v_{14})-distant vertex v_8 with an outneighbor. Hence the edge $v_{10}v_7 \in E(G')$. Then the path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4 \in v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. Hence $v_8v_6 \in E(G')$. Then the set $D = \{v_1, v_6, v_{10}, v_{13}\}$ dominates G'.

Case 1.6.3.2: $v_{12}v_8 \in E(G')$. The path $(v_1v_2v_{11}v_{12}v_{13}v_3v_4 \ldots v_{10})$ forces v_{10} to have its third neighbor in G'. Since $G' - \{v_{14}, v_9, v_{10}, v_{11}\}$ has the hamiltonian cycle $(v_1v_2v_3v_{13}v_{12}v_8v_7 \ldots v_4)$, v_{10} dominates all but a P_9 in $G' - v_{14}$ so $G' - v_{14}$ has a dominating set of size 4.

Case 2: $v_1v_5 \in E(G')$. The path $(v_{13}v_{12} \ldots v_5v_1v_2v_3v_4)$ forces v_4 to have its third neighbor in G'. By (3.2) for P, this neighbor is amongst v_7, v_8, v_{10}, v_{11}, and v_{13}.

Case 2.1: $v_4v_7 \in E(G')$. Then by the symmetry with v_1, the third neighbor of v_{13} is in $\{v_3, v_6, v_9\}$.

Case 2.1.1: $v_{13}v_3 \in E(G')$. The set $\{v_1, v_7, v_{10}, v_{13}\}$ dominates G'.

Case 2.1.2: $v_{13}v_6 \in E(G')$. The path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8 \ldots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is amongst v_3, v_8, and v_9. If $v_{12}v_3 \in E(G')$, then the set $\{v_1, v_7, v_9, v_{12}\}$ dominates G'. If $v_{12}v_8 \in E(G')$, then the path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (3.2) for this path, forces $v_9v_3 \in E(G')$. Then the set $\{v_1, v_7, v_9, v_{12}\}$ dominates G'. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5v_1v_2v_3v_4v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (3.2) for P forces $v_{10}v_3 \in E(G')$. Then the set $\{v_1, v_3, v_7, v_{12}\}$ dominates G'.

Case 2.1.3: $v_{13}v_9 \in E(G')$. The path $(v_1v_2 \ldots v_9v_{13}v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. If $v_{10}v_3 \in E(G')$, then the set $\{v_1, v_7, v_{10}, v_{12}\}$ dominates G'. So, $v_{10}v_6 \in E(G')$. The path $(v_1v_5v_6v_{10}v_{11}v_{12}v_{13}v_9v_8v_7v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for this path, this neighbor is one of v_8 and v_{12}. If $v_2v_8 \in E(G')$, then the set $\{v_2, v_4, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{12} \in E(G')$, then the set $\{v_2, v_4, v_9, v_{10}\}$ dominates $G' - v_{14}$.

Case 2.2: $v_4v_8 \in E(G')$. The path $(v_{13}v_{12} \ldots v_8v_4v_3v_2v_1v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is amongst v_{10}, v_{11}, and v_{12}.

Case 2.2.1: $v_7v_{10} \in E(G')$. The path $(v_{13}v_{12} \ldots v_{10}v_7v_6v_5v_1v_2v_3v_4v_8v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for P, $v_9v_{11} \notin E(G)$. If $v_9v_{12} \in E(G)$ or for some $i \in \{2, 3\}$, $v_9v_i \in E(G)$, then the set $\{v_i, v_5, v_7, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_9v_6 \in E(G)$. Now by (3.2) for P, only v_3 can be the third neighbor of v_{13}. Then the set $\{v_3, v_5, v_8, v_{11}\}$

dominates $G' - v_{14}$.
Case 2.2.2: $v_7v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_7v_6v_5v_1v_2v_3v_4v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for this path, this neighbor is one of v_3, or v_{13}. Symmetry with Case 1 forces $v_{10}v_3 \in E(G')$. The set $\{v_3, v_5, v_8, v_{12}\}$ dominates $G' - v_{14}$.
Case 2.2.3: $v_7v_{13} \in E(G')$. The path $(v_{13}v_7v_6v_5v_1v_2v_3v_4v_8v_9 \ldots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_2, v_3, v_6, and v_9. If $v_{12}v_2 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{10}\}$ dominates G'. So, either $v_{12}v_6 \in E(G')$ or $v_{12}v_9 \in E(G')$.
Case 2.2.3.1: $v_{12}v_6 \in E(G')$. The path $(v_1v_5v_6v_7v_{13}v_{12} \ldots v_8v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for the path P, this neighbor is one of v_9 and v_{11}. If $v_2v_9 \in E(G')$, then the set $\{v_2, v_4, v_7, v_{11}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_2, v_5, v_9, v_{13}\}$ dominates G'.
Case 2.2.3.2: $v_{12}v_9 \in E(G')$. The path $(v_{13}v_7v_6v_5v_1v_2v_3v_4v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have the third neighbor in G'. By (3.2) for P, this neighbor is one of v_3, or v_6. If $v_{10}v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{12}\}$ dominates G'. If $v_{10}v_6 \in E(G')$, then the set $\{v_1, v_4, v_6, v_{12}\}$ dominates G'.
Case 2.3: $v_4v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_3v_2v_1v_5v_6 \ldots v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_2, v_3, v_6, v_{11}, and v_{13}.
Case 2.3.1: $v_9v_2 \in E(G')$. The set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.
Case 2.3.2: $v_9v_3 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_5v_3v_9v_8 \ldots v_5v_1v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is one of v_6, or v_{11}. If $v_2v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. So, $v_2v_{11} \in E(G')$. The third neighbor of v_{13} is one of v_6, or v_7. If $v_{13}v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{13}\}$ dominates G'. If $v_{13}v_7 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{11}\}$ dominates G'.
Case 2.3.3: $v_9v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_4v_3v_2v_1v_5v_6v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for P, this neighbor is amongst v_3, v_{11}, and v_{13}. If $v_7v_3 \in E(G')$, then the set $\{v_1, v_3, v_9, v_{12}\}$ dominates G'. If $v_7v_{11} \in E(G')$, then the path $(v_{13}v_{12}v_{11}v_7v_6v_5v_1v_2v_3v_4v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. So (3.2) for this path and for P forces $v_8v_3 \in E(G')$. Then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. Thus, $v_7v_{13} \in E(G')$. The path $(v_1v_2 \ldots v_7v_{13}v_{12} \ldots v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_3, v_{11}, and v_{12}. If $v_8v_3 \in E(G')$, or $v_8v_{11} \in E(G')$, then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'. If $v_8v_{12} \in E(G')$, then the set $\{v_1, v_4, v_6, v_{12}\}$ dominates G'.
Case 2.3.4: $v_9v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_9v_{10}v_4v_3v_2v_1v_5v_6v_7v_8)$ forces v_8 to have

its third neighbor in G', and (3.2) for this path and for P, forces $v_8v_3 \in E(G')$. Then the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'.
Case 2.3.5: $v_9v_{13} \in E(G')$. The path $(v_1v_5v_6 \dots v_9v_{13}v_{12}v_{11}v_{10}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is one of v_6, or v_{11}. If $v_2v_6 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. If $v_2v_{11} \in E(G')$, then the set $\{v_2, v_4, v_7, v_{13}\}$ dominates G'.
Case 2.4: $v_4v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. Then (3.2) for this path and for P limits this neighbor to one of v_6, v_7, and v_{13}. By the symmetry between v_1 and v_{13}, $v_{10}v_{13} \notin E(G)$.
Case 2.4.1: $v_{10}v_6 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for P and this path, this neighbor is v_{13}. The path $(v_1v_2 \dots v_6v_{10}v_{11}v_{12}v_{13}v_7v_8v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path, this neighbor is one of v_3 and v_{12}. If $v_9v_3 \in E(G')$, then the set $\{v_1, v_3, v_7, v_{11}\}$ dominates G'. If $v_9v_{12} \in E(G')$, then the set $\{v_1, v_4, v_7, v_9\}$ dominates G'.
Case 2.4.2: $v_{10}v_7 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_4v_3v_2v_1v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G', and (3.2) for this path and for P, forces $v_8v_2 \in E(G')$. Then the set $\{v_2, v_5, v_{10}, v_{13}\}$ dominates G'.
Case 2.5: $v_4v_{13} \in E(G')$. The path $(v_1v_5v_6 \dots v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a P_9 (i.e., a path with 9 vertices) undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.
Case 3: $v_1v_7 \in E(G')$. The path $(v_{13}v_{12} \dots v_7v_1v_2 \dots v_6)$ forces v_6 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst $v_2, v_3, v_8, v_{10}, v_{11}$, and v_{13}.
Case 3.1: $v_6v_2 \in E(G')$. By the symmetry between v_1 and v_{13} and by (3.2) for P, the third neighbor of v_{13} is either v_4 or v_3. If $v_{13}v_4 \in E(G)$, then as in Case 2.5, the set $\{v_2, v_4, v_8, v_{11}\}$ dominates $G' - v_{14}$. So, $v_{13}v_3 \in E(G)$. The path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for P, this neighbor is amongst v_5, v_8, and v_9. If $v_{12}v_5 \in E(G')$, then the set $\{v_3, v_5, v_7, v_{10}\}$ dominates $G' - v_{14}$. If $v_{12}v_8 \in E(G')$, then the path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (3.2) for this path and P forces $v_9v_5 \in E(G')$. In this case, the set $\{v_3, v_5, v_7, v_{11}\}$ dominates $G' - v_{14}$. Thus, $v_{12}v_9 \in E(G')$. The path $(v_{13}v_3v_4v_5v_6v_2v_1v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have the third neighbor in G', and (3.2) for P forces $v_{10}v_4 \in E(G')$. Then $\{v_1, v_4, v_7, v_{12}\}$ dominates G'.
Case 3.2: $v_6v_3 \in E(G')$. By the symmetry between v_1 and v_{13} and by (3.2) for P, $v_{13}v_4 \in E(G')$. The path $(v_1v_7v_8 \dots v_{13}v_4v_5v_6v_3v_2)$ forces v_2 to have its third neighbor

in G'. As in Case 2.5, v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a path with 9 vertices undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 3.3: $v_6v_8 \in E(G')$. The path $(v_{13}v_{12} \dots v_8v_6v_7v_1v_2 \dots v_5)$ forces v_5 to have its third neighbor in G'. By (3.2) for this path and for P this neighbor is one of v_2 and v_{11}. If $v_5v_{11} \in E(G')$, then the set $\{v_3, v_8, v_{11}, v_{14}\}$ dominates G'. Thus, $v_5v_2 \in E(G')$. By the symmetry between v_1 and v_{13}, the third neighbor of v_{13} is either v_4 or v_3. If $v_{13}v_4 \in E(G)$, then as in Case 2.5, the set $\{v_2, v_8, v_{11}, v_{13}\}$ dominates G'. If $v_3v_{13} \in E(G')$, then the set $\{v_2, v_3, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 3.4: $v_6v_{10} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_6v_5 \dots v_1v_7v_8v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path and for P, and by the symmetry with Case 2, this neighbor is in $\{v_2, v_5, v_{11}\}$. If $v_9v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_9v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_9v_{11} \in E(G')$. Then v_{13} is adjacent to one of v_3 and v_4. If $v_{13}v_3 \in E(G')$, then the set $\{v_1, v_5, v_9, v_{13}\}$ dominates G'. If $v_{13}v_4 \in E(G')$, then the set $\{v_1, v_4, v_7, v_{11}\}$ dominates G'.

Case 3.5: $v_6v_{11} \in E(G')$. The path $(v_{13}v_{12}v_{11}v_6v_5 \dots v_1v_7v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for P and the symmetry with Case 2, this neighbor is one of v_3 and v_4. If $v_{10}v_3 \in E(G')$, then $v_{13}v_4 \in E(G')$, and the set $\{v_1, v_4, v_8, v_{11}\}$ dominates G'. So, $v_{10}v_4 \in E(G')$. Then $v_{13}v_3 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12}v_{11}v_{10}v_4v_5 \dots v_9)$ forces v_9 to have its third neighbor in G', and (3.2) for this path forces $v_9v_5 \in E(G')$. Then the set $\{v_1, v_3, v_9, v_{11}\}$ dominates G'.

Case 3.6: $v_6v_{13} \in E(G')$. The path $(v_{13}v_6v_5 \dots v_1v_7v_8 \dots v_{12})$ forces v_{12} to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is in $\{v_2, v_5, v_8, v_9\}$. If $v_{12}v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_{12}v_9 \in E(G')$, then the path $(v_{13}v_6v_5 \dots v_1v_7v_8v_9v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G', and (3.2) for this path and for P forces $v_{10}v_4 \in E(G')$. In this case, the set $\{v_1, v_4, v_7, v_{12}\}$ dominates G'. Thus, $v_{12}v_8 \in E(G')$. The path $(v_{13}v_6v_5 \dots v_1v_7v_8v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (3.2) for this path and for P, this neighbor is either v_2 or v_5. If $v_9v_2 \in E(G')$, then the set $\{v_4, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_9v_5 \in E(G')$, then the set $\{v_3, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$.

Case 4: $v_1v_8 \in E(G')$. The third neighbor of v_{13} is amongst v_3, v_4, and v_6.

Case 4.1: $v_{13}v_3 \in E(G')$. The set $\{v_3, v_5, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 4.2: $v_{13}v_4 \in E(G')$. The path $(v_{13}v_{12} \dots v_8v_1v_2 \dots v_7)$ forces v_7 to have its third

neighbor in G'. By (3.2) for this path, this neighbor is amongst v_3, v_{10}, and v_{11}.
Case 4.2.1: $v_7v_3 \in E(G')$. The path $(v_1v_8v_9 \dots v_{13}v_4v_5v_6v_7v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3 and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.
Case 4.2.2: $v_7v_{10} \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12} \dots v_5)$ forces v_5 to have its third neighbor in G'. By (3.2) for this path, this neighbor is in $\{v_3, v_9, v_{11}, v_{12}\}$. If $v_5v_3 \in E(G')$, then the set $\{v_1, v_5, v_{10}, v_{12}\}$ dominates G'. If $v_5v_9 \in E(G')$, then the set $\{v_2, v_5, v_7, v_{12}\}$ dominates $G' - v_{14}$. If $v_5v_{12} \in E(G')$, then the set $\{v_1, v_4, v_5, v_{10}\}$ dominates G'. Thus, $v_5v_{11} \in E(G')$. The path $(v_1v_8v_9v_{10}v_7v_6v_5v_{11}v_{12}v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3, and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.
Case 4.2.3: $v_7v_{11} \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12} \dots v_5)$ forces v_5 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_3, v_9, and v_{12}.
Case 4.2.3.1: $v_5v_3 \in E(G')$. The path $(v_1v_2v_3v_5v_4v_{13}v_{12} \dots v_6)$ forces v_6 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_9, v_{10}, and v_{12}. If $v_6v_9 \in E(G')$, then the set $\{v_3, v_9, v_{11}, v_{14}\}$ dominates G'. If $v_6v_{10} \in E(G')$, then the set $\{v_3, v_8, v_{10}, v_{13}\}$ dominates G'. If $v_6v_{12} \in E(G')$, then the set $\{v_1, v_4, v_6, v_{10}\}$ dominates G'.
Case 4.2.3.2: $v_5v_9 \in E(G')$. The set $\{v_1, v_4, v_5, v_{11}\}$ dominates G'.
Case 4.2.3.3: $v_5v_{12} \in E(G')$. The path $(v_1v_8v_9v_{10}v_{11}v_7v_6v_5v_{12}v_{13}v_4v_3v_2)$ forces v_2 to have its third neighbor in G'. Then v_2 dominates v_1, v_2, v_3, and one vertex of the cycle $(v_4v_5 \dots v_{13})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.
Case 4.3: $v_6v_{13} \in E(G')$. The set $\{v_3, v_6, v_8, v_{11}\}$ dominates $G' - v_{14}$.
Case 5: $v_1v_{10} \in E(G')$. Then v_{13} is adjacent to one of v_3 and v_4.
Case 5.1: $v_{13}v_3 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12} \dots v_4)$ forces v_4 to have its third neighbor in G'. By (3.2) for this path, this neighbor is amongst v_7, v_8, and v_{11}.
Case 5.1.1: $v_4v_7 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12} \dots v_7v_4v_5v_6)$ forces v_6 to have its third neighbor in G'. By (3.2) for this path, this neighbor is one of v_8 and v_{11}. If $v_6v_8 \in E(G')$, then the set $\{v_1, v_4, v_8, v_{12}\}$ dominates G'. So, $v_6v_{11} \in E(G')$. The path $(v_{13}v_3v_2v_1v_{10}v_9v_8v_7v_4v_5v_6v_{11}v_{12})$ forces v_{12} to have the third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13} and one vertex of the cycle $(v_1v_2 \dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.
Case 5.1.2: $v_4v_8 \in E(G')$. The path $(v_1v_2v_3v_{13}v_{12} \dots v_8v_4v_5v_6v_7)$ forces v_7 to have its third neighbor in G', and (3.2) for the path P forces $v_7v_{11} \in E(G')$. Now the path

$(v_{13}v_3v_2v_1v_{10}v_9v_8v_4v_5v_6v_7v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. So v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2 \dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.1.3: $v_4v_{11} \in E(G')$. The path $(v_{13}v_3v_2v_1v_{10}v_9 \dots v_4v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2 \dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.2: $v_{13}v_4 \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12} \dots v_5)$ forces v_5 to have its third neighbor in G'. By (3.2) for this path and for the path $(v_5v_6 \dots v_{10}v_1v_2v_3v_4v_{13}v_{12}v_{11})$, this neighbor is amongst v_8, v_9, and v_{11}. Note that a similar argument works for v_9.

Case 5.2.1: $v_5v_8 \in E(G')$. The path $(v_1v_2v_3v_4v_{13}v_{12} \dots v_8v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for the path P, this neighbor is one of v_3 and v_{11}. If $v_7v_3 \in E(G')$, then the set $\{v_3, v_5, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. So, $v_7v_{11} \in E(G')$. The path $(v_{13}v_4v_3v_2v_1v_{10}v_9v_8v_5v_6v_7v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2 \dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 5.2.2: $v_5v_9 \in E(G')$. The path $(v_{13}v_{12}v_{11}v_{10}v_1v_2 \dots v_5v_9v_8v_7v_6)$ forces v_6 to have its third neighbor in G', and (3.2) for this path and for the path $(v_6v_7v_8v_9v_5v_4v_{13}v_{12}v_{11}v_{10}v_1v_2v_3)$ forces $v_6v_3 \in E(G')$. Similarly, $v_8v_{11} \in E(G')$, and then the set $\{v_1, v_6, v_8, v_{13}\}$ dominates G'.

Case 5.2.3: $v_5v_{11} \in E(G')$. The path $(v_{13}v_4v_3v_2v_1v_{10}v_9 \dots v_5v_{11}v_{12})$ forces v_{12} to have its third neighbor in G'. Then v_{12} dominates v_{11}, v_{12}, v_{13}, and one vertex of the cycle $(v_1v_2 \dots v_{10})$ leaving only a P_9 undominated. Hence $G' - v_{14}$ can be dominated by 4 vertices.

Case 6: $v_1v_{11} \in E(G')$. By the symmetry between v_1 and v_{13}, $v_{13}v_3 \in E(G')$, and the set $\{v_3, v_5, v_8, v_{11}\}$ dominates $G' - v_{14}$.

Case 7: $v_1v_{13} \in E(G')$. As in the proof of Lemma 3.4.11, for the cycle $C = (v_1v_2 \dots v_{13})$ (3.1) holds. The path $(v_1v_{13}v_{12} \dots v_2)$ forces v_2 to have its third neighbor in G'. By (3.1), this neighbor is amongst $v_5, v_6, v_8, v_9, v_{11}$, and v_{12}. Note that a similar argument works for v_{12}.

Case 7.1: $v_2v_5 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_5v_2v_3v_4)$ forces v_4 to have the third neighbor in G'. By (3.2) for this path and (3.1) for C, this neighbor is one of v_8 and v_{11}.

Case 7.1.1: $v_4v_8 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_8v_4v_3v_2v_5v_6v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for this path and (3.1) for C, this neighbor is one of v_3 and v_{11}. If $v_7v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. So, $v_7v_{11} \in E(G')$.

The path $(v_1v_{13}v_{12}v_{11}v_7v_6v_5v_2v_3v_4v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for this path and (3.1) for C, this neighbor is one of v_3 and v_6. If $v_{10}v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. If $v_{10}v_6 \in E(G')$, then the set $\{v_1, v_4, v_{10}, v_{11}\}$ dominates G'.

Case 7.1.2: $v_4v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for this path and (3.1) for C, this neighbor is amongst v_3, v_6, and v_7. If $v_{10}v_3 \in E(G')$, then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'. If $v_{10}v_7 \in E(G')$, then the path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. So (3.2) for this path and (3.1) for C force $v_8v_{12} \in E(G')$. Then the set $\{v_2, v_5, v_{10}, v_{12}\}$ dominates $G' - v_{14}$. Thus, $v_{10}v_6 \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_4v_3v_2v_5v_6v_{10}v_9v_8v_7)$ forces v_7 to have its third neighbor in G'. So (3.2) for this path and (3.1) for C force $v_7v_3 \in E(G')$. Then the set $\{v_5, v_7, v_{10}, v_{13}\}$ dominates G'.

Case 7.2: $v_2v_6 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_6v_2v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. By (3.1) for C, this neighbor is amongst v_8, v_9, v_{11}, and v_{12}.

Case 7.2.1: $v_5v_8 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_8v_5v_4v_3v_2v_6v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for this path, and the path $(v_7v_6v_2v_1v_{13}v_{12} \dots v_8v_5v_4v_3)$, this neighbor is one of v_3 and v_{11}.

If $v_7v_3 \in E(G')$, then the set $\{v_3, v_5, v_{10}, v_{13}\}$ dominates G'. Thus, $v_7v_{11} \in E(G')$. Then the path

$(v_1v_{13}v_{12}v_{11}v_7v_6v_2v_3v_4v_5v_8v_9v_{10})$ forces v_{10} to have its third neighbor in G'. So, (3.2) for this path and (3.1) for C force $v_{10}v_4 \in E(G')$. Then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.2.2: $v_5v_9 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_9v_5v_4v_3v_2v_6v_7v_8)$ forces v_8 to have its third neighbor in G'. By (3.2) for this path and (3.1) for C, this neighbor is either v_{11} or v_{12}. If $v_8v_{12} \in E(G')$, then the path $(v_{13}v_1v_2v_6v_7v_8v_{12}v_{11}v_{10}v_9v_5v_4v_3)$ forces v_3 to have its third neighbor in G', and (3.2) for this path forces $v_3v_{10} \in E(G')$. In this case, the set $\{v_1, v_5, v_8, v_{10}\}$ dominates G'. So, $v_8v_{11} \in E(G')$. By (3.1) for C, we need $v_{12}v_3 \in E(G')$. Then the path $(v_1v_{13}v_{12}v_{11}v_8v_7v_6v_2v_3v_4v_5v_9v_{10})$ forces v_{10} to have its third neighbor in G'. If $v_{10}v_4 \in E(G')$, then the set $\{v_2, v_4, v_8, v_{12}\}$ dominates $G' - v_{14}$. If $v_{10}v_7 \in E(G')$, then the set $\{v_1, v_5, v_7, v_{12}\}$ dominates G'.

Case 7.2.3: $v_5v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_6v_7 \dots v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.2) for the path $(v_{10}v_9 \dots v_6v_2v_1v_{13}v_{12}v_{11}v_5v_4v_3)$ this neighbor is either v_3 or v_7. If $v_{10}v_3 \in E(G')$, then the set $\{v_3, v_5, v_8, v_{13}\}$ dominates G'. So, $v_{10}v_7 \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_6v_7v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. Then (3.1) for C and (3.2) for the path $(v_8v_9v_{10}v_7v_6v_2v_1v_{13}v_{12}v_{11}v_5v_4v_3)$

eliminate all possible third neighbors of v_8.

Case 7.2.4: $v_5v_{12} \in E(G')$. The path $P' = (v_{13}v_1v_2v_6v_7 \dots v_{12}v_5v_4v_3)$ forces v_3 to have its third neighbor in G', and (3.2) for P' and (3.1) for C force $v_3v_9 \in E(G')$. Now path $(v_1v_{13}v_{12}v_5v_4v_3v_2v_6v_7 \dots v_{11})$ forces v_{11} to have its third neighbor in G'. By (3.2) for the path $(v_{10}v_{11}v_{12}v_{13}v_1v_2v_3v_9v_8 \dots v_4)$, $v_{11}v_8 \in E(G')$. Hence, the set $\{v_3, v_6, v_{11}, v_{14}\}$ dominates G'.

Case 7.3: $v_2v_8 \in E(G')$. By the symmetry between v_2 and v_{12} and by (3.1) for C, v_{12} is adjacent to one of v_3, v_5 and v_6.

Case 7.3.1: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2 \dots v_5v_{12}v_{11} \dots v_6)$ forces v_6 to have its third neighbor in G'. By (3.1) for C, this neighbor is amongst v_3, v_9, and v_{10}. The case $v_6v_{10} \in E(G)$ contradicts Lemma 3.4.6 with $x = v_6$, $y = v_{10}$ and $z = v_8$.

Case 7.3.1.1: $v_6v_3 \in E(G')$. The path $(v_{13}v_1v_2v_8v_9 \dots v_{12}v_5v_4v_3v_6v_7)$ forces v_7 to have its third neighbor in G'. By (3.1) for C, this neighbor is amongst v_4, v_{10}, and v_{11}. If $v_7v_4 \in E(G')$, then the set $\{v_2, v_7, v_9, v_{12}\}$ dominates $G' - v_{14}$. If $v_7v_{10} \in E(G')$, then the set $\{v_2, v_5, v_{10}, v_{14}\}$ dominates G'. Thus, $v_7v_{11} \in E(G')$. The path $(v_{13}v_1v_2v_8v_7v_6v_3v_4v_5v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G', and (3.1) for C eliminates all possible third neighbors of v_9.

Case 7.3.1.2: $v_6v_9 \in E(G')$. The path $(v_1v_{13}v_{12}v_5v_4v_3v_2v_8v_7v_6v_9v_{10}v_{11})$ forces v_{11} to have its third neighbor in G', and (3.2) for this path and (3.1) for C force $v_{11}v_7 \in E(G')$. Then the set $\{v_1, v_4, v_9, v_{11}\}$ dominates G'.

Case 7.3.2: $v_{12}v_6 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_8v_2v_3 \dots v_7)$ forces v_7 to have its third neighbor in G'. By (3.2) for this path and the symmetric path, this neighbor is one of v_3 and v_{11}. W.l.o.g. assume that $v_7v_3 \in E(G')$. Then the path $(v_1v_{13}v_{12} \dots v_8v_2v_3v_7v_6v_5v_4)$ forces v_4 to have its third neighbor in G', and (3.2) for this path and (3.1) for C force $v_4v_{11} \in E(G')$. So, the set $\{v_1, v_4, v_6, v_9\}$ dominates G'.

Case 7.3.3: $v_{12}v_3 \in E(G')$. The path $(v_{13}v_1v_2v_8v_7 \dots v_3v_{12}v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By (3.1) for C, this neighbor is in $\{v_5, v_6\}$. The path $(v_{13}v_1v_2v_8v_9 \dots v_{12}v_5v_4v_3v_6v_7)$ forces v_7 to have its third neighbor in G'. If $v_5v_9 \in E(G)$, this contradicts Lemma 3.4.6 with $x = v_5$, $y = v_9$ and $z = v_7$. Thus $v_6v_9 \in E(G)$. Then the path $(v_1v_{13}v_{12} \dots v_9v_6v_7v_8v_2v_3v_4v_5)$ forces v_5 to have its third neighbor in G'. By (3.1) for C, it is v_{11}. Now the path $(v_1v_{13}v_{12}v_{11}v_5v_4v_3v_2v_8v_7v_6v_9v_{10})$ forces v_{10} to have its third neighbor in G'. This contradicts Lemma 3.4.6 with $x = v_8$, $y = v_2$ and $z = v_{10}$.

Case 7.4: $v_2v_9 \in E(G')$. The path $(v_1v_{13}v_{12} \dots v_9v_2v_3 \dots v_8)$ forces v_8 to have its third neighbor in G'; so by Lemma 3.4.6 with $x = v_6$, $y = v_{10}$ and $z = v_8$,

$$v_6 v_{10} \notin E(G). \tag{3.3}$$

By the symmetry between v_2 and v_{12} and by (3.1) for C, v_{12} is adjacent to either v_3 or v_5.

Case 7.4.1: $v_{12}v_3 \in E(G')$. The path $(v_{13}v_1v_2v_9v_8 \dots v_3v_{12}v_{11}v_{10})$ forces v_{10} to have its third neighbor in G'. By (3.1) for C, this neighbor is in $\{v_4, v_6, v_7\}$. By (3.3), it is in $\{v_4, v_7\}$, and the set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.4.2: $v_{12}v_5 \in E(G')$. The path $(v_{13}v_1v_2 \dots v_5v_{12}v_{11} \dots v_6)$ forces v_6 to have its third neighbor in G'. By (3.1) for C and (3.3), this neighbor is v_3. Symmetrically, $v_8v_{11} \in E(G')$. The path $(v_1v_{13}v_{12}v_{11}v_8v_7 \dots v_2v_9v_{10})$ forces v_{10} to have its third neighbor in G', and similarly v_4 has its third neighbor in G'. Thus $v_4v_{10} \in E(G')$, and the set $\{v_2, v_4, v_7, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.5: $v_2v_{11} \in E(G')$ and symmetrically $v_{12}v_3 \in E(G')$. Let G'' be obtained from $G' - \{v_1, v_2, v_{13}, v_{12}\}$ by identifying v_3 and v_{11} into a new vertex v^*. Graph G'' with 8-cycle $C'' = (v_4v_5 \dots v_{10}v^*)$ satisfies the conditions of Lemma 3.4.8. So by this lemma, either (a) some v^*-distant vertex $x \in G''$ has an outneighbor in G, or (b) a set $\{y, z\}$ of two vertices dominates $G'' - v^*$. Suppose (a) holds. By symmetry, we may assume that a hamiltonian path P in G'' from v^* to x starts from the edge v^*v_4. Then adding to $P - v^*$ the path $v_{14}v_{13}v_1v_2v_{11}v_{12}v_3v_4$ we produce a hamiltonian in G' path from v_{14} to the vertex x having an outneighbor, a contradiction. Thus (b) holds. Since v^* has only two neighbors in $G'' - v^*$, $v^* \notin \{y, z\}$. Hence the set $\{y, z, v_2, v_{12}\}$ dominates $G' - v_{14}$.

Case 7.6: $v_2v_{12} \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11} \dots v_3)$ forces v_3 to have its third neighbor in G'. By (3.1) for C, this neighbor is amongst v_6, v_7, v_9, and v_{10}.

Case 7.6.1: $v_3v_6 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11} \dots v_6v_3v_4v_5)$ forces v_5 to have its third neighbor in G', and (3.2) for this path and (3.1) for C force $v_5v_9 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}v_{10}v_9v_5v_4v_3v_6v_7v_8)$ forces v_8 to have its third neighbor in G', and (3.2) for this path and (3.1) for C force $v_8v_4 \in E(G')$, a contradiction to Lemma 3.4.6 with $x = v_4$, $y = v_8$ and $z = v_6$.

Case 7.6.2: $v_3v_7 \in E(G')$. Symmetry forces v_{11} to be adjacent to one of v_4 and v_5. By Lemma 3.4.6 with $x = v_3$, $y = v_7$ and $z = v_5$, $v_{11}v_5 \notin E(G')$. Thus, $v_{11}v_4 \in E(G')$. The path $(v_1v_{13}v_{12}v_2v_3v_7v_6v_5v_4v_{11}v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G', and (3.1) for C forces $v_8v_5 \in E(G')$. Then the set $\{v_1, v_7, v_8, v_{11}\}$ dominates G'.

Case 7.6.3: $v_3v_9 \in E(G')$. Then v_{11} is adjacent to one of v_4 and v_5. Both cases are

forbidden by Lemma 3.4.6 with $x = v_9$, $y = v_3$ and $z = v_{11}$.
Case 7.6.4: $v_3v_{10} \in E(G')$. This forces $v_{11}v_4 \in E(G')$. Then the path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9 \dots v_5)$ forces v_5 to have its third neighbor in G'. By (3.1) for C, this neighbor is one of v_8 and v_9. If $v_5v_8 \in E(G')$, then $v_9v_6 \in E(G')$. Now the path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9v_6v_5v_8v_7)$ forces v_7 to have its third neighbor in G', but no possible neighbor exists. Thus, $v_5v_9 \in E(G')$. The path $(v_{13}v_1v_2v_{12}v_{11}v_4v_3v_{10}v_9v_5v_6v_7v_8)$ forces v_8 to have its third neighbor in G', but (3.1) for C eliminates all possible neighbors of v_8. □

3.5 Proofs of Lemmas 3.2.4 and 3.2.5

For convenience, we restate Lemma 3.2.4 here.

Lemma 3.2.4 *If a* 2*-path* P *in an optimal vdp-cover is such that each of the hamiltonian paths in* $G[V(P)]$ *has at most one out-endpoint, then either some* $(|P| - 2)/3$ *vertices dominate all vertices of* P *apart from an out-endpoint or* P *has at least* 14 *vertices.*

Proof. If a 2-path $P = (v_1v_2 \dots v_k)$ has at most 11 vertices, then $k \in \{2, 5, 8, 11\}$. If $k = 2$, then clearly both vertices of P are out-endpoints. The case $k = 5$ was considered in Reed's paper [30], and the case $k = 8$ is proved in [20]. Hence we may assume that $k = 11$. If one of v_1 and v_{11} is an out-endpoint, then we may assume that it is v_{11}. Consider a v_{11}-lasso on $V(P)$ with a largest loop. As described in Section 3.4, we may assume that this loop is the cycle $C = (v_1 \dots v_r)$. Let $G' = G[V(P)]$ and $G'' = G[V(C)]$.
Case 1: Vertex v_{11} is an out-endpoint of P. By Lemma 3.4.3, if $r \in \{3, 6, 9\}$, then there exists a dominating set of $G' - v_{11}$ of size 3. If $r = 11$, then by Lemma 3.4.10, some three vertices dominate $V(P) - v_{11}$. Consider the remaining cases.
Case 1.1: $r = 10$. Since v_{11} is an out-endpoint of P, it has at most two neighbors in $V(G'')$ (one of which is v_{10}), and we are done by Lemma 3.4.9.
Case 1.2: $r = 8$. By Lemma 3.4.8 either there exists a dominating set of $G'' - v_8$ of size two, and this set together with v_9 dominates $V(P) - v_{11}$, or a (G'', v_8)-distant vertex is adjacent to a vertex in $\{v_9, v_{10}, v_{11}\}$, a contradiction to the maximality of r.
Case 1.3: $r = 7$. By Lemma 3.4.7, either there exists a dominating set of G'' with size two, or a (G'', v_7)-distant vertex is adjacent to a vertex in $\{v_8, v_9, v_{10}, v_{11}\}$, a contradiction to the maximality of r.
Case 1.4: $r \leq 5$. Since $d_{G''}(v_1) = 3$ and by Lemma 3.4.3 $v_1v_3 \notin E(G)$, $r = 5$ and $v_1v_4, v_1v_5 \in E(G'')$. Then the path $P_1 = (v_2v_3v_4v_1v_5v_6 ... v_{11})$ shows that v_2 is (G', v_{11})-

distant. Hence, v_2 has a neighbor in G' distinct from v_1 and v_3. This neighbor is not in $\{v_4, v_5\}$, since $v_1v_4, v_1v_5 \in E(G)$. This contradicts the maximality of r.

Case 2: P has no out-endpoints. We consider a lasso on G' with the largest loop. Since a cubic graph must have an even number of vertices, some vertex of G' must have an outneighbor. In particular, some vertex in G' is not the end of a hamiltonian path in G'. This then gives that $r \neq 11$. Consider the remaining cases.

Case 2.1: $r = 10$. Since G' has no out-endpoints, v_{11} has all three of its neighbors in G'. Viewing G' as the 10-cycle C together with the extra vertex v_{11}, we conclude that each vertex v_i adjacent along C to a neighbor of v_{11} is the end of a hamiltonian path on G' connecting v_i with v_{11}. It follows that

$$\text{each } v_i \text{ adjacent along } C \text{ to a neighbor of } v_{11} \text{ has no outneighbors.} \tag{3.4}$$

If two neighbors of v_{11} are adjacent along C, then G' is hamiltonian contradicting the maximality of r. If the shortest distance along C between two neighbors of v_{11} is at least 3, then we may assume that $v_{11}v_3 \in E(G')$ and $v_{11}v_7 \in E(G')$. Then by (3.4), only v_5 has an outneighbor. Then any choice of neighbors for v_4 gives a hamiltonian path starting at v_5. Hence every vertex of G' is the end of a hamiltonian path in G' which is a contradiction. Thus, the shortest distance along C between two neighbors of v_{11} is 2. We may assume that $v_{11}v_2 \in E(G')$.

Case 2.1.1: $v_8v_{11} \in E(G')$. By (3.4), v_1 has its third neighbor in G'. Each of the edges v_1v_3, v_1v_7, or v_1v_9 then forces a hamiltonian cycle in G'. Hence this third neighbor is amongst v_4, v_5, and v_6. If $v_1v_5 \in E(G')$, then every vertex of G' is the end of some hamiltonian path, a contradiction. Hence v_1 is adjacent to one of v_4 or v_6. Symmetry forces v_9 to be adjacent to the other of these vertices, and again every vertex in G' is the end of some hamiltonian path. Hence $v_8v_{11} \notin E(G')$ and $v_4v_{11} \notin E(G')$ by symmetry.

Case 2.1.2: $v_7v_{11} \in E(G')$. Then adding an edge from v_1 to v_3, v_6, v_8, or v_9 gives the hamiltonian cycles $(v_1v_3v_4 \ldots v_{11}v_2)$, $(v_1v_6v_5 \ldots v_2v_{11}v_7v_8v_9v_{10})$, $(v_8v_9v_{10}v_{11}v_7v_6 \ldots v_1)$, and $(v_1v_{10}v_{11}v_2v_3 \ldots v_9)$ respectively. Thus v_1 must be adjacent to one of v_4 or v_5. However, if v_1 is adjacent to either v_4, or v_5, the other is the start of a hamiltonian path in G', so G' has an out-endpoint for some hamiltonian path which contradicts the assumption of Case 2. Hence v_{11} is not adjacent to v_7 or v_5.

Case 2.1.3: $v_6v_{11} \in E(G')$. Then by the symmetry between v_{11} and v_1, in order to avoid Cases 2.1.1 and 2.1.2, we need $v_6v_1 \in E(G')$. But v_6 cannot have 4 neighbors.

Case 2.2: $r = 9$. The maximality of r restricts the neighbors of v_{11} to v_3, v_4, v_5, or v_6. If

v_{11} is adjacent to either of v_3, or v_6, then the set $\{v_3, v_6, v_9\}$ dominates G'. Hence v_{11} is adjacent to both of v_4, and v_5. This then gives the lasso having the loop $(v_1 \dots v_4 v_{11} v_5 \dots v_9)$ which contradicts the maximality of r.

Case 2.3: $r = 8$. The maximality of r restricts the neighbors of v_{11} to v_4 and v_9. Then v_{10} has a third neighbor in G', but any possible neighbor contradicts the maximality of r.

Case 2.4: $r = 7$. The only possible neighbors of v_{11} not contradicting the maximality of r are v_8 and v_9. Then the path $(v_1 \dots v_9 v_{11} v_{10})$ is also hamiltonian in G', and similarly we have $v_{10}v_8 \in E(G)$. Then $d(v_8) > 3$, a contradiction.

Case 2.5: $r = 6$. Since G' has maximum degree 3, the lowest indexed neighbor of v_{11} is at least v_7. So, by Lemmas 3.4.1 and 3.4.2, a single vertex dominates $\{v_{11}, v_{10}, v_9, v_8\}$, and this vertex along with v_3 and v_6 gives a dominating set of G' with size 3. □

Case 2.6: $r \leq 5$. The highest indexed neighbor of v_1 is smaller than the lowest indexed neighbor of v_{11}. So, by Lemmas 3.4.1 and 3.4.2, a vertex dominates $\{v_1, v_2, v_3, v_4\}$, a vertex dominates $\{v_{11}, v_{10}, v_9, v_8\}$, and a v_6 dominates v_5 and v_7. □

For convenience, we also restate Lemma 3.2.5.

Lemma 3.2.5 *Let $P_1 = (x_1, \dots, x_k)$ be a tip of an accepting 2-path P in an optimal vdp-cover. Let $X(P_1)$ be the set of the hamiltonian paths in $G[V(P_1)]$ one of whose ends is x_k. If none of the other ends of any path in $X(P_1)$ is an out-endpoint of P or a $(2,2)$-endpoint, then either some $(k-1)/3$ vertices dominate $V(P_1)$, or $k \geq 16$.*

Proof. For $k \leq 7$, it was proved in [30][Fact 11], for $k = 10$ it was proved in [20][Lemma 14]. Both cases will also be clear from the proof for $k = 13$ below. So, suppose that a tip $P_1 = (v_1 v_2 \dots v_{13})$ of an accepting 2-path P has no out-endpoint and no $(2,2)$-endpoint. Let v_{14} be the second (i.e. the other than v_{12}) neighbor of v_{13} in the path P. Let G' be the subgraph of G induced by $V(P_1) + v_{14}$. Since our system of paths was chosen to maximize the number of out-endpoints and $(2,2)$-endpoints and taking into account (B4) of Lemma 3.2.1,

$$\text{no } (G', v_{14})\text{-distant vertex in } G' \text{ has an outneighbor (with respect to } V(G')). \quad (3.5)$$

We choose a (G', v_{14})-distant vertex in G' and an edge incident to this vertex so that to maximize the length of the loop of a v_{14}-lasso in G'. We renumber the vertices in G' so that this vertex is v_1 and this loop is $(v_1 v_2 \dots v_r)$. Then let G'' be the graph induced by the set $\{v_1, v_2, \ldots, v_r\}$. By the maximality of r and (3.5),

$$\text{no } (G'', v_r)\text{-distant vertex in } G'' \text{ has an outneighbor with respect to } G''. \tag{3.6}$$

If $r = 14$, then we are done by Lemma 3.4.12.

Let $r < 14$. Then v_1 has two neighbors in $G'' - v_2$. By Lemma 3.4.3,

$$v_1 v_{3j} \notin E(G) \text{ for } j = 1, 2, 3, 4, \tag{3.7}$$

and hence $r \notin \{3, 6, 9, 12\}$.

Case 1: $r = 13$. By Lemma 3.4.11, either some 4 vertices dominate $V(P_1)$ (in which case we are done), or some (G'', v_{13})-distant vertex v_j has an outneighbor with respect to G'', a contradiction to (3.6).

Case 2: $r \in \{10, 11\}$. By Lemma 3.4.10 (if $r = 11$) or Lemma 3.4.9 (if $r = 10$), either some 3 vertices dominate $v_1, v_2, \ldots, v_{10}$ (then this set along with v_{12} dominates $G' - v_{14}$), or some (G'', v_r)-distant vertex v_j has an outneighbor, a contradiction to (3.6).

Case 3: $r \in \{7, 8\}$. By Lemma 3.4.8 (if $r = 8$) or Lemma 3.4.7 (if $r = 7$), either some 2 vertices dominate $v_1, v_2, \ldots, v_7$ (then this set along with v_9 and v_12 dominates G'), or some (G'', v_r)-distant vertex v_j has an outneighbor, a contradiction to (3.6).

Case 4: $r \leq 5$. By (3.7) $r = 5$ and the three neighbors of v_1 are v_2, v_4, and v_5. Since there is the path $(v_3 v_2 v_1 v_4 v_5 \ldots v_{13})$, by (3.6), v_3 has no neighbors outside of G''. So by (3.7), $v_3 v_5 \in E(G)$, but v_5 already has 3 other neighbors. □

3.6 Proof of Lemma 3.2.3

Recall that Lemma 3.2.3 states that each 1-path P in an optimal vdp-cover S that does not have an out-endpoint and does not contain a dominating set of size at most $(|P| - 1)/3$, has at least 28 vertices. Fact 9 in [30] states that such a path must have at least 16 vertices. Lemma 2 in [20] extends this by proving that such a path has at least 22 vertices. Hence we need to prove that such path cannot have 25 vertices and cannot have 22 vertices. We will prove this in two big lemmas. But first we introduce the notion of (H, v)-distant vertices for $v \notin V(H)$. If H is a subgraph of G and $x \in V(G) - V(H)$, then a vertex $y \in V(H)$ is (H, x)-*distant*, if H contains a hamiltonian path connecting y with a neighbor of x.

Lemma 3.6.1. *If a* 1*-path* P *in an optimal vdp-cover* S *does not have an out-endpoint and does not contain a dominating set of size at most* $(|P|-1)/3$*, then* P *cannot have* 22 *vertices.*

Proof. Let $P = (v_1 v_2 \dots v_{22})$ be a counter-example to the lemma, and let $G' = G[V(P)]$. Consider a v_{22}-lasso on $V(P)$ with a largest loop $C = (v_1 \dots v_r)$. Let $H = G' - C$. If $r = 22$, then by the definition of P, no vertex of P has an outneighbor. So in this case G' is a cubic hamiltonian graph and by Theorem 3.2.1 is dominated by 7 vertices. Thus $r \le 21$. Also by Lemma 3.4.3, r is not divisible by 3. If $r \le 14$, then since each end of every hamiltonian path in G' has no outneighbors, Lemmas 6, 7 and 12–17 imply that for some i, some set D of i vertices dominates the set $\{v_1, \dots, v_{3i+1}\}$. Then the set $D \cup \{v_{3(i+1)}, v_{3(i+2)}, \dots, v_{21}\}$ dominates G' and has 7 vertices. Thus $r \in \{16, 17, 19, 20\}$.

Case 1: $r = 16$. By the maximality of r, for each (H, v_{16})-distant vertex of H, only v_7, v_8, and v_9 are possible neighbors on C. By Lemma 3.4.3, $v_{22}v_8 \notin E(G)$. So $N(v_{22}) - v_{21} \subset \{v_7, v_9, v_{18}, v_{19}\}$.
Case 1.1: $|N(v_{22}) \cap \{v_7, v_9\}| = 1$. By symmetry, we may assume that $v_{22}v_7 \in E(G')$.
Case 1.1.1: $v_{22}v_{18} \in E(G')$. Because of the path $(v_{16}v_{17}v_{18}v_{22}v_{21}v_{20}v_{19})$, vertex v_{19} is (H, v_{16})-distant. By Lemma 3.4.3 for P, v_{19} has only two neighbors in H. Since v_7 already has 3 neighbors, v_{19} is adjacent to v_9. If v_{17} has two neighbors in C, then since it is (H, v_7)-distant, the second (apart from v_{16}) neighbor in C should be v_{14}. On the other hand, since v_{17} is (H, v_9)-distant, this neighbor should be v_2, a contradiction. So v_{17} has two neighbors in H. If $v_{17}v_{20} \in E(G)$, then the path $(v_{16}v_{17}v_{20}v_{19}v_{18}v_{22}v_{21})$ shows that v_{21} is (H, v_{16})-distant. Hence the third neighbor of v_{21} is in $H \cup \{v_7, v_9\}$. But all vertices in this set already have degree 3. Thus $v_{17}v_{21} \in E(G)$. Then the path $(v_{16}v_{17}v_{21}v_{22}v_{18}v_{19}v_{20})$ shows that v_{20} is (H, v_{16})-distant, and all possible neighbors of v_{20} already have degree 3.
Case 1.1.2: $v_{22}v_{19} \in E(G')$. If $v_{17}v_{20} \in E(G)$, then the set $\{v_{17}, v_{22}, v_2, v_5, v_8, v_{11}, v_{14}\}$ dominates G'. If $v_{17}v_{21} \in E(G)$, then we have Case 1.1.1 with v_{17} in place of v_{22}. So $v_{17}v_{14} \in E(G)$. The path $(v_{17}v_{18}v_{19}v_{22}v_{21}v_{20})$ shows that v_{20} is (H, v_{16})-distant and (H, v_{14})-distant. So if its third neighbor is in C, then it should be v_9 because of v_{16} and v_5 because of v_{14}, a contradiction. So $v_{20}v_{18} \in E(G')$, a contradiction to Lemma 3.4.3 for the path $(v_{17}v_{18} \dots v_{22}v_7v_8 \dots v_{16}v_1 \dots v_6)$.
Case 1.2: $v_{22}v_{18} \in E(G')$ and $v_{22}v_{19} \in E(G')$. Because of the path
$H' = (v_{16}v_{17}v_{18}v_{22}v_{19}v_{20}v_{21})$, vertex v_{21} can play the role of v_{22}. By Lemma 3.4.3, $v_{21}v_{17} \notin E(G')$. So we have Case 1.1 for C and H'.
Case 1.3: $v_{22}v_9 \in E(G')$ and $v_{22}v_7 \in E(G')$. Consider G' as a lasso with the cycle $C' = (v_7v_8 \dots v_{22})$ and handle $H' = (v_{16}v_1v_2 \dots v_6)$. As above, only v_7 and v_9 can be the

neighbors of v_6 on C'. Since v_9 already has 3 neighbors, we are in Case 1.1 for C' and H', which is proved.

Case 2: $r = 17$. By the maximality of r and Lemma 3.4.3, only v_7, and v_{10} can be the neighbors on C of any (H, v_{17})-distant vertex. So as in Case 1, by Lemma 3.4.3, $N(v_{22}) - v_{21} \subset \{v_7, v_{10}, v_{18}, v_{19}\}$.

Case 2.1: Exactly one of v_7 and v_{10} is a neighbor of v_{22}. Again, we may assume that $v_{22}v_7 \in E(G')$. If $v_{22}v_{18} \in E(G)$, then v_{21} is (H, v_7)-distant and v_{19} is (H, v_{17})-distant. They are not adjacent by Lemma 3.4.3 for P, and so $v_{21}v_{14} \in E(G')$ and $v_{19}v_{10} \in E(G')$. Now the set $\{v_{19}, v_{21}, v_2, v_5, v_8, v_{12}, v_{16}\}$ dominates G'. Thus $v_{22}v_{19} \in E(G)$. Then v_{20} is (H, v_{17})-distant. If $v_{20}v_{18} \in E(G')$, then the set $\{v_{18}, v_{22}, v_2, v_5, v_9, v_{12}, v_{15}\}$ dominates G'. So, $v_{20}v_{10} \in E(G')$. If $v_{18}v_{21} \in E(G')$, then the set $\{v_{18}, v_2, v_5, v_7, v_{10}, v_{12}, v_{15}\}$ dominates G'. Otherwise, since v_{18} is (H, v_7)-distant it is adjacent to v_{14}, but since it also is (H, v_{10})-distant it is adjacent to v_3, a contradiction.

Case 2.2: $v_{22}v_{18} \in E(G')$ and $v_{22}v_{19} \in E(G')$. We just repeat the proof of Case 1.2.

Case 2.3: $v_{22}v_7 \in E(G')$ and $v_{22}v_{10} \in E(G')$. Then by symmetry v_{18} has its third neighbor, say v_i. Since Case 2.1 is proved, $i < 17$, and v_i is at distance 7 along C from both v_7, and v_{10}, an impossibility.

Case 3: $r = 19$. First note that if v_{21} has its third neighbor in G', then v_{21} dominates all but a P_{18}, which can be dominated by 6 vertices. Thus v_{21}'s third neighbor is outside of G'. Also if $v_{20}v_{22} \in E(G')$, then v_{20} dominates all but a P_{18}. Thus we may assume that each of v_{20} and v_{22} has two neighbors on C. Furthermore, each vertex in G' that is adjacent to a neighbor of v_{20} or v_{22} is an endpoint of a hamiltonian path in G', and hence has its third neighbor in G'.

Case 3.1: The neighbors of v_{20} and v_{22} on C do not alternate. Let d be the maximum of the distance between the neighbors of v_{20} on C and the distance between the neighbors of v_{22} on C. We can assume that v_{20} is adjacent to v_{19} and v_d on C. We can further assume that the neighbors of v_{22} on C are v_{d+a} and v_{d+a+c}. Let $b = 19 - d - a - c$ (See the left graph in Figure 3.5). By symmetry, we may assume that $a \le b$. Maximality of r forces the neighbors of v_{20} to be at least distance 4 apart on C from the neighbors of v_{22}, in particular, $b \ge a \ge 4$. It also forces $c, d \ge 2$. Lemma 3.4.3 for P and symmetry eliminate all cases where neighbors of v_{20} are distance $5, 8, 11$, or 14 apart on C from the neighbors of v_{22}. Summarizing, we have

$$b \ge a \ge 4, \quad a, b, a+c, b+c, a+d, b+d \notin \{5, 8, 11, 14\}, \text{ and } 2 \le c \le d. \tag{3.8}$$

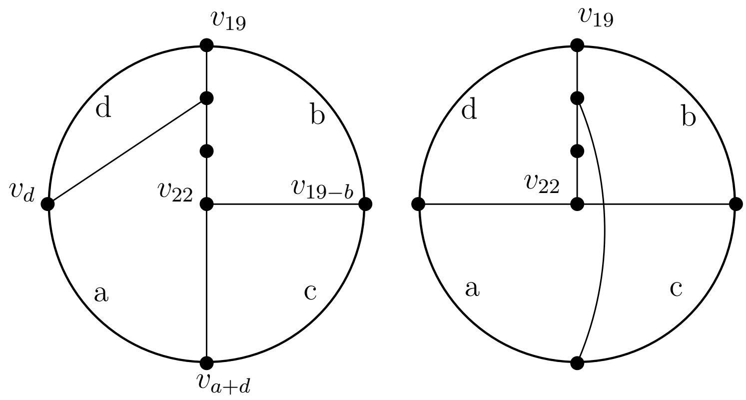


Figure 3.5

Case 3.1.1: $d > 9$. Then $c = 19 - (a + b + d) \leq 19 - (4 + 4 + 10) = 1$, a contradiction to (3.8).

Case 3.1.2: $d = 9$. Then $a + b = 19 - c - 9 = 10 - c \leq 8$. So by (3.8), $a = b = 4$. By Lemma 3.4.4, the third neighbor of v_{14} is one of $v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{17}$, and v_{18}. The path $(v_{14}v_{15} \ldots v_{22}v_{13}v_{12} \ldots v_1)$ along with Lemma 3.4.3 restricts this set of possible neighbors to $\{v_1, v_4, v_7, v_{10}, v_{17}, v_{18}\}$. Then Lemma 3.4.3 with the path $(v_{14}v_{13} \ldots v_9v_{20}v_{21}v_{22}v_{15}v_{16} \ldots v_{19}v_1v_2 \ldots v_8)$ restricts the set of possible neighbors of v_{14} to $\{v_{10}, v_{18}\}$. Since either of these edges forms a 4-arc in G' and since v_{16}, and v_{12} both have third neighbors in G', by Lemma 3.4.6, no good third neighbor exists for v_{14}.

Case 3.1.3: $d = 8$. If $a = 6$, then $a + d = 14$, a contradiction to (3.8). Similarly, $b \neq 6$. Hence $a = b = 4$. By Lemma 3.4.4, the third neighbor of v_{14} is one of $v_1, v_2, v_4, v_5, v_7, v_{10}, v_{11}, v_{17}$, and v_{18}. The path $(v_{13}v_{14} \ldots v_{22}v_{12}v_{11} \ldots v_1)$ along with Lemma 3.4.3 restricts this set of possible neighbors to $\{v_2, v_5, v_{11}, v_{17}, v_{18}\}$. The symmetry of the role of H with the role of the set $\{v_{18}, v_{17}, v_{16}\}$ eliminates v_{17} as a possible neighbor of v_{14}. If $v_{14}v_{18} \in E(G')$, then Lemma 3.4.6 with $x = v_{14}$, $y = v_{18}$ and $z = v_{16}$ yields that v_{16} has no third neighbor in G', a contradiction to the fact it is adjacent to a neighbor of v_{22}. Thus $v_{14}v_{18} \notin E(G')$. Now Lemma 3.4.6 with $x = v_{12}$, $y = v_{14}$ and $z = v_{19}$ eliminates all remaining potential neighbors of v_{14}.

Case 3.1.4: $d = 7$. If $a = 4$, then $a + d = 11$, a contradiction to (3.8). Similarly, $b \neq 4$.

Then $a, b \geq 6$, and $a + b + c + d \geq 6 + 6 + 2 + 7 = 21$, a contradiction.
Case 3.1.5: $d = 6$. Then $a + b \leq 19 - 2 - 6 = 11$.
Case 3.1.5.1: $a = b = 4$. Let v_i be the third neighbor of v_{14}. By Lemma 3.4.4, $i \in \{1, 2, 4, 5, 7, 8, 11, 17, 18\}$. The path $(v_{14}v_{13} \ldots v_6v_{20}v_{21}v_{22}v_{15}v_{16} \ldots v_{19}v_1v_2 \ldots v_5)$ with Lemma 3.4.3 restricts this set to $\{2, 5, 7, 8, 11, 18\}$. The path $(v_{11}v_{12} \ldots v_{22}v_{10}v_9 \ldots v_1)$ with Lemma 3.4.3 shrinks this set to $\{7, 11, 18\}$. If $i = 18$, then by Lemma 3.4.6 with $x = v_{14}$, $y = v_{16}$ and $z = v_{18}$, graph $G'' = G' - \{v_{20}, v_{21}, v_{22}\}$ has a dominating set of size 6. If $i = 7$, then the path $(v_8v_9 \ldots v_{14}v_7v_6 \ldots v_1v_{19}v_{18} \ldots v_{15}v_{22}v_{21}v_{20})$ forces the third neighbor of v_8 to be in G', which contradicts the fact that the role of H can be switched with $\{v_7, v_8, v_9\}$. Thus $i = 11$. Then the path $(v_{12}v_{13}v_{14}v_{11}v_{10} \ldots v_1v_{19}v_{18} \ldots v_{15}v_{22}v_{21}v_{20})$ forces v_{12} to have its third neighbor in G'. By Lemma 3.4.3 for this path and Lemma 3.4.4 for C, this neighbor is in $\{v_2, v_5, v_8, v_{18}\}$. For $j \in \{2, 5, 8\}$, Lemma 3.4.5 with $R = \{v_j, v_{10}, v_{12}, v_{19}\}$ eliminates v_j from the list. Thus $v_{12}v_{18} \in E(G')$. Then the hamiltonian cycle $(v_{11}v_{10} \ldots v_1v_{19}v_{20}v_{21}v_{22}v_{15}v_{16}v_{17}v_{18}v_{12}v_{13}v_{14})$ contradicts the maximality of r.
Case 3.1.5.2: $a = 4, b = 6$. Let v_i be the third neighbor of v_{11}. By Lemma 3.4.4, $i \in \{1, 2, 4, 5, 7, 8, 14, 15, 17, 18\}$. The path $(v_{11}v_{12} \ldots v_{22}v_{10}v_9 \ldots v_1)$ and Lemma 3.4.3 further reduces this set to $\{1, 4, 7, 14, 15, 17, 18\}$. The path $(v_{12}v_{11} \ldots v_6v_{20}v_{21}v_{22}v_{13}v_{14} \ldots v_{19}v_1v_2 \ldots v_5)$ and Lemma 3.4.3 eliminate 15 and 18 from this list. If $i \in \{1, 4, 14, 17\}$, then Lemma 3.4.5 with $R = \{v_6, v_{11}, v_{13}, v_i\}$ gives a dominating set of size 7. Thus $i = 7$. Then Lemma 3.4.6 with $x = v_7$, $y = v_{11}$ and $z = v_9$ gives a dominating set of size 7 in G'.
Case 3.1.5.3: $a = 4, b = 7$. Lemma 3.4.4 limits the third neighbor of v_{11} to one of v_1, v_2, v_4, v_5, v_7, $v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10} \ldots v_6v_{20}v_{21}v_{22}v_{12}v_{13} \ldots v_{19}v_1v_2 \ldots v_{15})$ and Lemma 3.4.3 limit this neighbor to one of $v_2, v_5, v_7, v_8, v_{15}$ and v_{18}. The path $(v_{11}v_{12} \ldots v_{22}v_{10}v_9 \ldots v_1)$ and Lemma 3.4.3 further limit this neighbor to one of v_7, v_{15} and v_{18}. Now Lemma 3.4.6 with $x = v_7$, $y = v_9$ and $z = v_{11}$ (respectively, with $x = v_{11}$, $y = v_{13}$ and $z = v_{15}$) yields a dominating set of size 7 if $v_{11}v_7 \in E(G)$ (respectively, if $v_{11}v_{15} \in E(G)$). So $v_{11}v_{18} \in E(G)$. Then the hamiltonian cycle $(v_{12}v_{13} \ldots v_{18}v_{11}v_{10} \ldots v_1v_{19}v_{20}v_{21}v_{22})$ contradicts the maximality of r.

Thus $a, b \geq 6$, and hence $a + b + c + d \geq 20$, a contradiction.
Case 3.1.6: $d = 5$. By (3.8), $a, b \neq 11 - d = 6$. If $a, b \geq 7$, then $a + b + c + 5 > 19$, a contradiction. So one of a and b, say a, is 4. Then by the maximality of d, $b \geq 5$, and so $b \in \{5, 6, 7, 8\}$. Hence by (3.8), $b = 7$. Then Lemma 3.4.4 limits the third neighbor of v_{11} to one of $v_1, v_2, v_4, v_7, v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10}, \ldots v_1v_{19}v_{20}v_{21}v_{22}v_{12}v_{13} \ldots v_{18})$ and Lemma 3.4.3 eliminate v_{14} and v_{17} from the

list of possible neighbors. The path $(v_{10}v_{11}\dots v_{22}v_9v_8\dots v_1)$ and Lemma 3.4.3 limit this neighbor to one of v_2, v_8, v_{15}, and v_{18}. By Lemma 3.4.6 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{15}v_{11} \notin E(G)$. Lemma 3.4.5 with $R = \{v_9, v_{11}, v_{19}, v_2\}$ and $R = \{v_9, v_{11}, v_{19}, v_8\}$ eliminates v_2, and v_8 as neighbors of v_{11}. Thus $v_{11}v_{18} \in E(G')$. Then the hamiltonian cycle $(v_{18}v_{17}\dots v_{12}v_{22}v_{21}v_{20}v_{19}v_1v_2\dots v_{11})$ contradicts the maximality of r.

Case 3.1.7: $d = 4$. By (3.8), $a, b \notin \{4, 5, 7, 8\}$. Then if $\max\{a, b\} \geq 9$, then $a+b+c+d \geq 6+9+2+4 = 21$, hence $a, b = 6$. By Lemma 3.4.4, the third neighbor of v_{11} is in $\{v_1, v_2, v_5, v_7, v_8, v_{14}, v_{15}, v_{17}, v_{18}\}$. The path $(v_{11}v_{12}\dots v_{22}v_{10}v_9\dots v_1)$ and Lemma 3.4.4 shrink this set to $\{v_1, v_7, v_{14}, v_{15}, v_{17}, v_{18}\}$. The path $(v_{12}v_{11}\dots v_1v_{19}v_{20}v_{21}v_{22}v_{13}v_{14}\dots v_{18})$ and Lemma 3.4.4 yield that this neighbor is in $\{v_1, v_7, v_{14}, v_{17}\}$. By Lemma 3.4.6 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_7v_{11} \notin E(G)$. If $v_{11}v_1 \in E(G')$, then the hamiltonian cycle $(v_1v_2\dots v_{10}v_{22}v_{21}\dots v_{11})$ contradicts the maximality of r. If $v_{11}v_{17} \in E(G')$, then the set $\{v_2, v_5, v_8, v_{11}, v_{15}, v_{19}, v_{22}\}$ dominates G'. Finally, if $v_{11}v_{14} \in E(G')$, then symmetry gives $v_{12}v_9 \in E(G')$, and hence the set $\{v_1, v_3, v_6, v_9, v_{14}, v_{17}, v_{21})\}$ dominates G'.

Case 3.1.8: $d = 3$. In this case, $2 \leq c \leq 3$. If $a = 4$, then $b \in \{19-4-3-3, 19-4-2-3\} = \{9, 10\}$. If $a = 6$, then similarly, $7 \leq b \leq 8$, but by (3.8), $b \neq 8$. Finally, if $a \geq 7$, then $b \leq 19-7-2-4 = 7$, and hence in this case $a = b = 7$.

Case 3.1.8.1: $a = 4, b = 9$. Lemma 3.4.4 limits the third neighbor of v_8 to one of $v_1, v_2, v_4, v_5, v_{11}$, $v_{12}, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_8v_9\dots v_{22}v_7v_6\dots v_1)$ and Lemma 3.4.3 limit this neighbor to one of v_1, v_4, $v_{11}, v_{12}, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_9v_8\dots v_3v_{20}v_{21}v_{22}v_{10}v_{11}\dots v_{19}v_1v_2)$ and Lemma 3.4.3 limit this neighbor to one of v_1, v_4, v_{11}, v_{14} or v_{17}. By Lemma 3.4.6 with $x = v_4$, $y = v_8$, and $z = v_6$, $v_8v_4 \notin E(G)$. For $i \in \{1, 11, 14, 17\}$, Lemma 3.4.6 with $x = v_8$, $y = v_i$, and $z = v_3$ eliminates v_i as a neighbor of v_8.

Case 3.1.8.2: $a = 4, b = 10$. By Lemma 3.4.5 with $R \supset \{v_{19}, v_2, v_7\}$, the third neighbor of v_2 is in $\{v_4, v_5, v_6\}$. Lemma 3.4.3 for P yields $v_2v_4 \notin E(G)$. By Lemma 3.4.6 with $x = v_2$, $y = v_6$, and $z = v_4$, $v_2v_6 \notin E(G)$. Thus $v_2v_5 \in E(G')$. Then the 21-cycle $(v_3v_4v_5v_2v_1v_{19}v_{18}\dots v_7v_{22}v_{21}v_{20})$ contradicts the maximality of r.

Case 3.1.8.3: $a = 6, b = 7$. Lemma 3.4.4 for C limits the third neighbor of v_{11} to one of v_1, v_2, v_4, v_5, v_7, $v_8, v_{14}, v_{15}, v_{17}$, and v_{18}. The path $(v_{11}v_{10}\dots v_3v_{20}v_{21}v_{22}v_{12}v_{13}\dots v_{19}v_1v_2)$ limits this neighbor to one of $v_2, v_4, v_5, v_7, v_8, v_{15}$, and v_{18}. The path $(v_{10}v_{11}\dots v_{22}v_9v_8\dots v_1)$ limits the neighbor to one of v_2, v_5, v_8, v_{15}, and v_{18}. For $i \in \{2, 5, 8\}$, Lemma 3.4.6 with $x = v_i$, $y = v_{11}$, and $z = v_{19}$ eliminates v_i as a neighbor of v_8. By Lemma 3.4.6 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{11}v_{15} \notin E(G)$. Thus $v_{11}v_{18} \in E(G')$. Then the Hamiltionian cycle $(v_{19}v_{20}v_{21}v_{22}v_{12}v_{13}\dots v_{18}v_{11}v_{10}\dots v_1)$ contradicts the maximality of r.

Case 3.1.8.4: $a = 7, b = 7$. Lemma 3.4.6 with $x = v_{12}$, $y = v_3$, and $z = v_1$ yields that the third neighbor of v_1 is in $\{v_{13}, v_{14}, \ldots, v_{18}\}$. Lemma 3.4.4 for C shortens the list to $\{v_{13}, v_{14}, v_{16}, v_{17}\}$. For $i = 14, 17$, Lemma 3.4.5 with $R = \{v_1, v_3, v_{19}, v_i\}$ gives $v_1v_i \notin E(G')$. If $v_1v_{13} \in E(G)$, then the cycle $(v_1v_2 \ldots v_{12}v_{22}v_{21}v_{20}v_{19}v_{18} \ldots v_{13})$ contradicts the maximality of r. So $v_1v_{16} \in E(G)$. By symmetry, $v_2v_6 \in E(G)$. Again by symmetry and by Lemma 3.4.4 for C, we may assume that the third neighbor of v_{11} is in $\{v_7, v_8\}$. Edge v_7v_{11} contradicts Lemma 3.4.6 with $x = v_{11}$, $y = v_7$, and $z = v_9$. So $v_8v_{11} \in E(G)$. Then the set $\{v_1, v_3, v_5, v_8, v_{14}, v_{18}, v_{22})\}$ dominates G'.

Case 3.1.9: $d = 2$. By (3.8), $a, b \notin \{5, 6, 8, 9, 11, 12\}$. Since $c = 2$, we have $d + c = 4$ and hence $a, b \notin \{4, 7, 10\}$. This proves the case.

Case 3.2: The neighbors of v_{20} and v_{22} alternate on C. Let $v_{22}v_d, v_{20}v_{d+a}, v_{22}v_{19-b} \in E(G')$. Define $c = 19 - a - b - d$ (see the graph on the right in Figure 3.5). By symmetry, we may assume that $d = \max\{a, b, c, d\}$. By Lemma 3.4.3, $5 \notin \{a, b, c, d\}$. By the maximality of r, $\min\{a, b, c, d\} \geq 4$, and so $d \leq 7$. Furthermore, if $d = 7$, then $a, b, c = 4$, and the set $\{v_2, v_5, v_9, v_{13}, v_{17}, v_{20}, v_{22}\}$ dominates G'. If $d = 6$, $a + b + c = 13$, a contradiction to $5 \notin \{a, b, c, d\}$. Finally, if $d = 4$, then $a + b + c + d = 16 < 19$.

Case 4: $r = 20$. Lemma 3.4.3 for the paths P and $\{v_{22}v_{21}v_{20}v_1v_2 \ldots v_{19}\}$ gives the possible neighbors of v_{22} as v_i, $i \in \{1, 4, 7, 10, 13, 16, 19\}$. By the maximality of r, $i \in \{4, 7, 10, 13, 16\}$. It follows that the distance on C from a neighbor of v_{22} to both neighbors of v_{21} is the same modulo 3 (and vice versa). Thus the distance on C between the two neighbors of $v \in \{v_{21}, v_{22}\}$ is 0 (mod 3). Since $r \equiv 2 \pmod 3$ and the neighbors of v_{21} and v_{22} are distance 1 (mod 3) apart on C, these neighbors cannot alternate. Let $v_{22}v_d, v_{22}v_{d+a}, v_{21}v_{20-b} \in E(G')$. Define $c = 20 - a - b - d$. We may assume that $d \leq c$ and $a \leq b$. Therefore,

$$d \leq c,\ a \leq b,\ d + a \leq 10, \quad c, d \equiv 1 \pmod 3, \text{ and } \quad a, b \equiv 0 \pmod 3. \tag{3.9}$$

Case 4.1: $d = 4$. By (3.9), $d + a \in \{7, 10\}$. Then v_5 has its third neighbor. By Lemma 3.4.3 for the paths $(v_5v_6 \ldots v_{20}v_1v_2v_3v_4v_{22}v_{21})$ and $(v_{d+a-1}v_{d+a-2} \ldots v_1v_{20}v_{19} \ldots v_{d+a}v_{22}v_{21})$, for either choice of $d + a$, the possible neighbors of v_5 are in $\{v_1, v_8, v_9, v_{12}, v_{15}, v_{18}\}$. If $v_5v_1 \in E(G')$, then the hamiltonian cycle $(v_5v_6 \ldots v_{22}v_4v_3v_2v_1)$ contradicts the maximality of r. By Lemma 3.4.6 with $x = v_{20}$, $y = v_{d+a}$, and $z = v_5$, the third neighbor of v_5 must be in the set $\{v_1, v_2, \ldots, v_{d+a-1}\}$. This contradicts the above statement when $d + a = 7$ and leaves v_8 and and v_9 as possible neighbors of v_5 when $d + a = 10$. In this case by (3.9), $c = 4$. Note that now v_5 is symmetric with v_9, v_{15}, and v_{19}.

Case 4.1.1: $v_5v_9 \in E(G)$. The path $(v_8v_7v_6v_5v_9v_{10} \dots v_4v_{22}v_{21})$ yields that v_8 has its third neighbor in G'. By Lemma 3.4.6 with $x = v_{20}$, $y = v_{10}$, and $z = v_8$, this neighbor is in $\{v_1, v_2, v_3\}$. By Lemma 3.4.3 for paths $(v_9v_8 \dots v_1v_{20}v_{19} \dots v_{10}v_{22}v_{21})$ and $(v_8v_7v_6v_5v_9v_{10} \dots v_{20} \dots v_4v_{22}v_{21})$ eliminates v_3 and v_2 from this list. So $v_8v_1 \in E(G)$. Then the cycle $(v_8v_7v_6v_5v_9v_{10} \dots v_{20}v_{21}v_{22}v_4v_3v_2v_1)$ contradicts the minimality of r.

Case 4.1.2: $v_5v_9 \notin E(G)$. Then $v_5v_8 \in E(G)$. Since v_5 is symmetric with v_9, v_{15}, and v_{19}, we conclude that $v_6v_9, v_{15}v_{18}, v_{19}v_{16} \in E(G)$. The path $(v_7v_6v_9v_5v_4 \dots v_1v_{20} \dots v_{10}v_{22}v_{21})$ yields that v_7 has its third neighbor, say v_i, in G'. If $i \in \{3, 12, 13, 17\}$, then for $j \in \{12, 13\}$ the set $\{v_3, v_5, v_{10}, v_j, v_{14}, v_{17}, v_{20}\}$ dominates G'. Since vertices v_1, v_2 and v_{11} with respect to v_8 are symmetric to v_{13}, v_{12} and v_3, respectively, no possible neighbors for v_8 left.

Case 4.2: $d = 7$. By (3.9), $d + a = 10$ and so $c = 7$. By Lemma 3.4.6 with $x = v_{17}$, $y = v_7$, and $z = v_{19}$, the third neighbor of v_{19} is in $\{v_1, v_2, \dots, v_6\}$. Lemma 3.4.3 for P and the 16-vertex path $(v_{19}v_{18}v_{17}v_{21}v_{22}v_{10}v_9 \dots v_1v_{20})$ reduces this list to $\{v_3, v_6\}$. If $v_{19}v_6 \in E(G)$, then the cycle $(v_{19}v_{18} \dots v_7v_{22}v_{21}v_{20}v_1 \dots v_6)$ contradicts the maximality of r. So $v_{19}v_3 \in E(G)$. Symmetrically, $v_8v_4 \in E(G)$. Now the hamiltonian cycle $(v_{19}v_{18} \dots v_8v_4v_5v_6v_7v_{22}v_{21}v_{20}v_1v_2v_3)$ contradicts the maximality of r. $\square$

Lemma 3.6.2. *If a* 1*-path* P *in an optimal vdp-cover* S *does not have an out-endpoint and does not contain a dominating set of size at most* $(|P| - 1)/3$*, then* $|P| \neq 25$.

Proof. Let $P = (v_1v_2 \dots v_{25})$ be a counter-example to the lemma, and let $G' = G[V(P)]$. Consider a v_{25}-lasso on $V(P)$ with the largest loop. Call the loop C, and the remaining handle H. We may assume that it is a $(v_{25}, 25, r)$-lasso. If $r = 25$, then no vertex of G' has an outneighbor, and hence $G = G'$. But a cubic graph cannot have 25 vertices. Thus $r \leq 24$. Also r is not divisible by 3 by Lemma 3.4.3. If $r \leq 17$, then by the maximality of r each neighbor of an (H, v_r)-distant vertex must lie in H. Thus again by the maximality of r, considering the largest lasso L in H with v_{r+1} as the endpoint of the handle, we know that the loop in L has at most 12 vertices. So, we may apply one of the Lemmas 3.4.1, 3.4.2, 3.4.7, 3.4.8, 3.4.9, and 3.4.10 to L. This then gives a contradiction to the maximality of the loop in L or a dominating set extendable to a dominating set of size 8 of G'. Thus $r \in \{19, 20, 22, 23\}$.

Case 1: $r = 19$. By the maximality of r, and Lemma 3.4.3, each (H, v_{19})-distant vertex of H has only v_7, v_9, v_{10}, and v_{12} as possible neighbors in C. Also by the maximality of r, if a vertex z of C is adjacent to the end of a handle H, then a vertex adjacent to z along C cannot have neighbors in H.

Case 1.1: Vertex v_{25} has two neighbors on C. By symmetry and the maximality of r, we have the following three cases:

Case 1.1.1: $v_{25}v_7 \in E(G')$ and $v_{25}v_9 \in E(G')$. By Lemma 3.4.3 for the paths $(v_{20}v_{21} \dots v_{25}v_7v_8 \dots v_{19}v_1v_2 \dots v_6)$ and $(v_{20}v_{21} \dots v_{25}v_9v_{10} \dots v_{19}v_1v_2 \dots v_8)$, the third neighbor of v_{20} is in $\{v_{16}, v_{23}, v_{24}\}$.

Case 1.1.1.1: $v_{20}v_{16} \in E(G')$. In this case, v_8 has its third neighbor in G', and by Lemma 3.4.3 for the paths $(v_8v_9 \dots v_{19}v_1v_2 \dots v_7v_{25}v_{24} \dots v_{20})$, $(v_8v_9 \dots v_{16}v_{20}v_{21} \dots v_{25}v_7v_6 \dots v_1v_{19}v_{18}v_{17})$, and $(v_8v_7 \dots v_1v_{19}v_{20} \dots v_{25}v_9v_{10} \dots v_{18})$, this neighbor is in $\{v_1, v_4, v_{12}, v_{15}\}$. If $v_8v_1 \in E(G')$ (which is symmetric with the case $v_8v_{15} \in E(G')$), then the cycle $(v_1v_2 \dots v_7v_{25}v_{24} \dots v_8)$ contradicts the maximality of r. If $v_8v_4 \in E(G')$ (which is symmetric with the case $v_8v_{12} \in E(G')$), then the cycle $(v_8v_9 \dots v_{25}v_7v_6v_5v_4)$ gives $r \geq 22$ contradicting the maximality of r.

Case 1.1.1.2: $v_{20}v_{23} \in E(G')$. The path $(v_8v_9 \dots v_{19}v_1v_2 \dots v_7v_{25}v_{24}v_{23}v_{20}v_{21}v_{22})$ forces v_{22} to have its third neighbor in G'. By Lemma 3.4.3 for this path, $(v_8v_7 \dots v_1v_{19}v_{18} \dots v_9v_{25}v_{24}v_{23}v_{20}v_{21}v_{22})$, and P, and by the maximality of r, $v_{22}v_{16} \in E(G')$. Then by Lemma 3.4.3 for the paths $(v_8v_9 \dots v_{19}v_1v_2 \dots v_7v_{25}v_{24} \dots v_{20})$, $(v_8v_9 \dots v_{16}v_{22}v_{21}v_{20}v_{23}v_{24}v_{25}v_7v_6 \dots v_1v_{19}v_{18}v_{17})$, and $(v_8v_7 \dots v_1v_{19}v_{20} \dots v_{25}v_9v_{10} \dots v_{18})$, the third neighbor of v_8 is in $\{v_1, v_4, v_{12}, v_{15}\}$. Just as in Case 1.1.1.1, each of these possibilities forces $r > 19$.

Case 1.1.1.3: $v_{20}v_{24} \in E(G')$. The path $(v_8v_7 \dots v_1v_{19}v_{18} \dots v_9v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. Since the path $(v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ covers H, Lemma 3.4.3 forces $v_{23}v_{16} \in E(G')$. Then just as in Case 1.1.1.1, we eliminate all neighbors of v_8.

Case 1.1.2: $v_{25}v_7 \in E(G')$ and $v_{25}v_{10} \in E(G')$. The maximality of r and Lemma 3.4.3 for the path $(v_{20}v_{21} \dots v_{25}v_7v_6 \dots v_1v_{19}v_{18} \dots v_8)$ force the third neighbor of v_{20} to be in $\{v_{17}, v_{23}, v_{24}\}$. Note that equivalent paths restrict the third neighbor of each (H, v_{25})-distant vertex to be in H or to be v_{17}. If an (H, v_{25})-distant vertex v_i is adjacent to v_{17}, then v_{18} has its third neighbor in G', and by Lemma 3.4.3 for the paths $(v_{18}v_{17} \dots v_{10}v_{25}v_{24} \dots v_{19}v_1v_2 \dots v_9)$ and $(v_{18}v_{19}v_1 \dots\dots v_7v_{25} \dots v_iv_{17}v_{16} \dots v_8)$, this neighbor is either in H or in $\{v_3, v_6, v_{11}, v_{14}\}$. In any case, as in Case 1.1.1, any such neighbor contradicts the maximality of r. If $v_{20}v_{23} \in$

$E(G')$, then v_{22} is (H, v_{25})-distant and hence its third neighbor is in H. In this case, $v_{22}v_{24} \in E(G)$, a contradiction to Lemma 3.4.3 for P. Thus, $v_{20}v_{24} \in E(G')$. Similarly v_{23} is an (H, v_{25})-distant vertex; hence its third neighbor is in H, but Lemma 3.4.3 for the path $(v_{23}v_{22}v_{21}v_{20}v_{24}v_{25}v_{10}v_9 \ldots v_1v_{19}v_{18} \ldots v_{11})$ eliminates all possible neighbors.

Case 1.1.3: $v_{25}v_7 \in E(G')$, and $v_{25}v_{12} \in E(G')$. In this case, no (H, v_{25})-distant vertex can have a neighbor in C other than v_{19}. Hence the third neighbor of v_{20} lies in H. By Lemma 3.4.3 for the path $(v_{20}v_{21} \ldots v_{25}v_7v_6 \ldots v_1v_{19}v_{18} \ldots v_8)$, this neighbor is one of v_{23} and v_{24}. If $v_{20}v_{23} \in E(G')$, then the path $(v_{22}v_{21}v_{20}v_{23}v_{24}v_{25}v_7v_6 \ldots v_1v_{19}v_{18} \ldots v_8)$ forces v_{22} to have its third neighbor in G'. But then Lemma 3.4.3 for P forces this neighbor to be in C, a contradiction. If $v_{20}v_{24} \in E(G')$, then the path $(v_8v_9 \ldots v_{19}v_1v_2 \ldots v_7v_{25}v_{24}v_{20}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. But then Lemma 3.4.3 for this path forces this neighbor to be in C, a contradiction.

Hence v_{25} (and by symmetry v_{20}) has at most one neighbor in C.

Case 1.2: Each of v_{20} and v_{25} has exactly one neighbor in C. Lemma 3.4.3 for P restricts the second neighbor of v_{25} in H to one of v_{21} and v_{22}. Similarly, the second neighbor of v_{20} in H is in $\{v_{23}, v_{24}\}$. If $v_{20}v_{23} \in E(G')$ and $v_{25}v_{21} \in E(G')$, then the path $(v_1v_2 \ldots v_{20}v_{23}v_{22}v_{21}v_{25}v_{24})$ forces v_{24} to have its third neighbor in G'. There is no room for this neighbor in H, so it is in C. Hence the set $\{v_{21}, v_{24}\}$ dominates all G' but a P_{18}. If $v_{20}v_{23} \in E(G')$, and $v_{25}v_{22} \in E(G')$, then the set $\{v_{20}, v_{25}\}$ dominates the set $\{v_{19}, v_{20},...,v_{25}\}$ leaving only a P_{18} undominated. If $v_{20}v_{24} \in E(G')$ and $v_{25}v_{21} \in E(G')$, then the path $(v_1v_2 \ldots v_{20}v_{24}v_{25}v_{21}v_{22}v_{23})$ forces v_{23} to have its third neighbor in G'. By Lemma 3.4.3 for P, this neighbor is in C. Hence the set $\{v_{21}, v_{23}\}$ dominates all but a P_{18}. Finally, suppose that $v_{20}v_{24} \in E(G')$ and $v_{25}v_{22} \in E(G')$. In our case, v_{25} has a neighbor v_i in C. Then the path $(v_{i+1}v_{i+2} \ldots v_{19}v_1v_2 \ldots v_iv_{25}v_{22}v_{23}v_{24}v_{20}v_{21})$ forces v_{21} to have its third neighbor in G'. Lemma 3.4.3 for the path $(v_{i+1}v_{i+2} \ldots v_{19}v_1v_2 \ldots v_iv_{25}v_{24} \ldots v_{20})$ forces this neighbor to be in C. Then the set $\{v_{21}, v_{24}\}$ dominates all but a P_{18}.

Case 1.3: Vertex v_{25} has no neighbors in C. By Lemma 3.4.3, $N(v_{25}) = \{v_{24}, v_{22}, v_{21}\}$. Then both, v_{23} and v_{24} are (H, v_{20})-distant, and hence at least one of them has a neighbor in C. Thus we have Case 1.2.

Case 2: $r = 20$. Let $\mathrm{dist}_C(x, y)$ denote the distance on C between the vertices x and y. Suppose that $i, j \geq 21$ and that v_i is an (H, v_j)-distant vertex. If $v_{i'}$ is a neighbor of v_i in C, and $v_{j'}$ is a neighbor of v_j in C, then the maximality of r and Lemma 3.4.3 imply that

$$\mathrm{dist}_C(v_{i'}, v_{j'}) \in \{7, 10\}. \tag{3.10}$$

Case 2.1: Some (H, v_{21})-distant vertex, say v_{25}, has two neighbors in C. We claim that

$$\text{an } (H, v_{25})\text{-distant vertex has a neighbor in } C \text{ distinct from } v_{20}. \tag{3.11}$$

Indeed, otherwise v_{21} has a neighbor in H distinct from v_{22}. It could be only v_{24}. Then v_{23} is (H, v_{25})-distant and cannot have 3 neighbors in H. This proves (3.11).

By (3.11) and (3.10), v_{25} cannot be adjacent to both of v_7, and v_{13}. So we may assume that $v_{25}v_7, v_{25}v_{10} \in E(G')$. By (3.10), a neighbor of $H - v_{25}$ in C distinct from v_{20} can be only v_{17}. Then the path $(v_{21}v_{22} \dots v_{25}v_7v_6 \dots v_1v_{20}v_{19} \dots v_8)$ forces v_8 to have the third neighbor in G'. By Lemma 3.4.6 with $x = v_{20}$, $y = v_{10}$, and $z = v_8$, this third neighbor is in $\{v_1, v_2, \dots, v_6\}$. By Lemma 3.4.3 for the paths $(v_9v_8 \dots v_1v_{20}v_{19} \dots v_{10}v_{25}v_{24} \dots v_{21})$ and $(v_8v_9 \dots v_{17}$ H $v_7v_6 \dots v_1v_{20}v_{19}v_{18})$, this third neighbor is either v_1 or v_4. If $v_8v_1 \in E(G')$, then the hamiltonian cycle $(v_1v_2 \dots v_7v_{25}v_{24} \dots v_8)$ contradicts the maximality of r. If $v_8v_4 \in E(G')$, then the cycle $(v_4v_5v_6v_7v_{25}v_{24} \dots v_8)$ forces $r \geq 22$.

Case 2.2: No (H, v_{21})-distant vertex has two neighbors in C. We claim that

$$\text{an } (H, v_{21})\text{-distant vertex, say } v_{25}, \text{ has a neighbor in } C. \tag{3.12}$$

Indeed, otherwise v_{25} has 3 neighbors in H, which implies $v_{25}v_{22}, v_{25}v_{21} \in E(G)$. Then v_{24} is (H, v_{21})-distant and has no room in H for the third neighbor. This proves (3.12). Suppose that v_j is the neighbor of v_{25} in C. By Lemma 3.4.3 for P, the neighbor of v_{25} in $H - v_{24}$ is either v_{21} or v_{22}. Similarly, the neighbor of v_{21} in $H - v_{22}$ is either v_{24} or v_{25}.

Case 2.2.1: $v_{25}v_{22} \in E(G')$. Then $v_{25}v_{21} \notin E(G')$ and hence $v_{21}v_{24} \in E(G')$. The path $(v_{25}v_{22}v_{21}v_{24}v_{23})$ shows that v_{23} is (H, v_{25})-distant. Also, v_{23} is (H, v_{21})-distant. Since v_{23} cannot have the third neighbor in H, it has a neighbor, v_i, in C. Since $r = 20$, $\min\{\mathrm{dist}_C(v_i, v_{20}), \mathrm{dist}_C(v_j, v_{20}), \mathrm{dist}_C(v_i, v_j)\} \leq 6$, a contradiction to (3.10).

Case 2.2.2: $v_{25}v_{21} \in E(G')$. In this case, v_{22} is (H, v_{21})-distant and by Lemma 3.4.3 has no third neighbor in H. Therefore, v_{22} has a neighbor, v_h, in C. Similarly, v_{23} is (H, v_{22})-distant and hence has a neighbor, v_ℓ, in C and v_{24} is (H, v_{25})-distant and hence has a neighbor, v_q, in C. By (3.10) for v_h and v_{20}, and for v_{20} and v_j, $\mathrm{dist}_C(v_h, v_{20}) \equiv 1 \pmod 3$ and $\mathrm{dist}_C(v_{20}, v_j) \equiv 1 \pmod 3$. Vertices v_h, v_{20} and v_j partition C into three paths that we will call $P_{j,h}$, $P_{j,20}$, and $P_{20,h}$, where P_{i_1,i_2} connects v_{i_1} with v_{i_2} and does not contain v_{i_3} for distinct $i_1, i_2, i_3 \in \{j, h, 20\}$. Since $20 \equiv 2 \pmod 3$, the number of edges in $P_{j,h}$ is $0 \pmod 3$. If $v_\ell \notin V(P_{j,h})$, then $\mathrm{dist}_C(v_\ell, v_j) \equiv 1 \pmod 3$ and hence v_q

cannot have $\mathrm{dist}_C(v_\ell, v_q) \equiv 1 \pmod 3$ and $\mathrm{dist}_C(v_q, v_j) \equiv 1 \pmod 3$ at the same time. So, $v_\ell \in V(P_{j,h})$. But then by the maximality of r, $P_{j,h}$ has at least 7 edges. Since each of $P_{j,20}$ and $P_{20,h}$ also has at least 7 edges, this is impossible for the 20-cycle C.

Case 3: $r = 22$. If v_{24} has its third neighbor in G', then v_{24} dominates all G' but a P_{21} which can be dominated by 7 vertices. Thus v_{24}'s third neighbor is outside of G'. Also if $v_{23}v_{25} \in E(G')$, then v_{23} dominates all but a P_{21}. Thus we may assume that each of v_{23} and v_{25} has exactly two neighbors in C. These four neighbors of v_{23} and v_{25} partition C into four paths. Suppose that the lengths of these paths are a, b, c, and d.

Case 3.1: The two neighbors of v_{23} in C and the two neighbors of v_{25} in C alternate on C for *each* representation of G' as a lasso with $r = 22$. We may assume that $v_{25}v_d, v_{23}v_{d+a}, v_{25}v_{22-b} \in E(G')$, $c = 22 - a - b - d$ and that $d = \max\{a, b, c, d\}$. By the maximality of r, $\min\{a, b, c, d\} \geq 4$ and hence $d = \max\{a, b, c, d\} \leq 22 - a - b - c \leq 10$. So, by Lemma 3.4.3,

$$\text{each of } a, b, c, d \text{ is in } \{4, 6, 7, 9, 10\}. \tag{3.13}$$

Case 3.1.1: $d \geq 8$. If $d = 10$, then by (3.13), $a = b = c = 4$ and the set $\{v_2, v_5, v_8, v_{12}, v_{16}, v_{20}, v_{23}, v_{25}\}$ dominates G'. If $d = 9$, then $a + b + c = 13$, which contradicts (3.13). By (3.13), $d \neq 8$.

Case 3.1.2: $d = 7$. By (3.13), $\{a, b, c\} = \{4, 4, 7\}$. By symmetry, there are two subcases: either $(d, a, c, b) = (7, 4, 4, 7)$ or $(d, a, c, b) = (7, 4, 7, 4)$. If $(d, a, c, b) = (7, 4, 4, 7)$, then the set $\{v_2, v_5, v_9, v_{13}, v_{17}, v_{20}, v_{23}, v_{25}\}$ dominates G'. If $(d, a, c, b) = (7, 4, 7, 4)$, then the set $\{v_2, v_5, v_9, v_{13}, v_{16}, v_{20}, v_{23}, v_{25}\}$ dominates G'.

Case 3.1.3: $d \leq 6$. If $d \leq 5$ then by the maximality of d, we have $a + b + c + d \leq 20 < 22$. So, $d = 6$. By (3.13), $\{a, b, c\} = \{4, 6, 6\}$. So by symmetry we may assume that $(d, a, c, b) = (6, 6, 6, 4)$. Then the cycle $(v_1v_2 \dots v_{18}v_{25}v_{24}v_{23}v_{22})$ with the handle v_{19}, v_{20}, v_{21} is a new lasso L with $r = 22$. By our assumption, the neighbors of v_{21} and the neighbors of v_{20} also alternate along the cycle in L. Since $d = 6$, each such adjacent pair of such neighbors along the cycle in L must be at distance 4 or 6. Since only one such distance can be 4, v_{19} is adjacent to v_6, but v_6 already has 3 neighbors.

Case 3.2: There exists a representation of G' as a lasso with $r = 22$ such that the neighbors of v_{23} along C, and the neighbors of v_{25} along C do not alternate. We may assume that $v_{23}v_d, v_{25}v_{d+a}, v_{25}v_{22-b} \in E(G')$, and $c = 22 - a - b - d$. We may assume further that $d \geq c$ and $a \leq b$. By the maximality of r, $a, b \geq 4$ and $c, d \geq 2$. Similarly to (3.8) in the proof of Lemma 3.6.1, we have

$$b \geq a \geq 4, \quad a, b, a + c, b + c, a + d, b + d \notin \{5, 8, 11, 14, 17\}, \text{ and } 2 \leq c \leq d. \tag{3.14}$$

By (3.14), $d \leq 22 - 4 - 4 - 2 = 12$.

Case 3.2.1: $d = 12$. By (3.14), $a = b = 4$, and $c = 2$. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{17}v_{18} \dots v_{25}v_{16}v_{15} \dots v_1)$ and $(v_{17}v_{16} \dots v_{12}v_{23}v_{24}v_{25}v_{18}v_{19} \dots v_{22}v_1v_2 \dots v_{11})$, the third neighbor of v_{17} is either v_{13} or v_{21}. Assume by symmetry that $v_{17}v_{13} \in E(G')$. Since v_{15} has its third neighbor in G', by Lemma 3.4.6 with $x = v_{13}$, $y = v_{17}$, and $z = v_{15}$, $G'[C]$ has a dominating set of size 7 and hence G' has a dominating set of size 8.

Case 3.2.2: $d = 11$. By (3.14), $a = b = 4$, and $c = 3$. Then Lemma 3.4.4 for C and Lemma 3.4.3 for the path $(v_{16}v_{17} \dots v_{25}v_{15}v_{14} \dots v_1)$ forces the third neighbor of v_{17} to be amongst $v_2, v_5, v_8, v_{14}, v_{20}$, and v_{21}. If $v_{17}v_{21} \in E(G)$, then since v_{19} has its third neighbor in G', Lemma 3.4.6 with $x = v_{17}$, $y = v_{21}$, and $z = v_{19}$ yields a dominating set in $G'[C]$ of size 7. If $v_{17}v_i \in E(G')$ for $i \in \{2, 5, 8, 14\}$, then the set $\{v_{17}, v_{19}, v_{22}, v_{25}\}$ dominates 13 vertices and leaves only a collection of paths whose lengths are divisible by 3. So, in this case G' can be dominated by 8 vertices. If $v_{17}v_{20} \in E(G')$, then v_{20} dominates all but a P_{21} in G', and hence G' has a dominating set of size 8.

Case 3.2.3: $d = 10$. In this case, $a + b + c = 12$, and no combination of values for a, b, and c satisfies (3.14): if $a = 4$, then $b + c = 8$, a contradiction; otherwise $6 \leq a \leq b$, and $a + b + c \geq 14$.

Case 3.2.4: $d = 9$. By (3.14), $(a, b, c) \in \{(4, 4, 5), (4, 6, 3), (4, 7, 2)\}$.

Case 3.2.4.1: $a = b = 4$, and $c = 5$. Let v_i be the third neighbor of v_{17}. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$ and $(v_{17}v_{16} \dots v_9v_{23}v_{24}v_{25}v_{18}v_{19} \dots v_{22}v_1v_2 \dots v_8)$, $i \in \{10, 14, 21\}$. Since v_{19} has its third neighbor in G', if $v_{17}v_{21} \in E(G')$, then Lemma 3.4.6 with $x = v_{17}$, $y = v_{21}$, and $z = v_{19}$ yields a dominating set of $G'[C]$ of size 7. Suppose that $v_{17}v_{10} \in E(G')$. Since v_{12} has a common neighbor with v_{25}, it has a third neighbor v_j. By Lemma 3.4.6 with $x = v_{17}$, $y = v_{10}$, and $z = v_{12}$, $j \in \{14, 15, 16\}$. By Lemma 3.4.4 for C, $j \neq 14$. Then the cycle $(v_1v_2 \dots v_9v_{23}v_{24}v_{25}v_{13}v_{14} \dots v_jv_{12}v_{11}v_{10}v_{17}v_{18} \dots v_{22})$ contradicts the maximality of r. So $v_{17}v_{14} \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{13}v_{12} \dots v_1v_{22}v_{21} \dots v_{17}v_{14}v_{15}v_{16})$ forces v_{16} to have its third neighbor in G'. By Lemma 3.4.3 for this path and Lemma 3.4.4 for C, this third neighbor is in $\{v_1, v_4, v_7, v_{10}, v_{20}\}$. If the neighbor is in $\{v_1, v_4, v_7, v_{20}\}$, then the set $\{v_1, v_4, v_7, v_9, v_{11}, v_{14}, v_{20}, v_{25}\}$ dominates G'. Hence $v_{16}v_{10} \in E(G')$. Symmetry then forces $v_{15}v_{21} \in E(G')$, and the set $\{v_1, v_4, v_7, v_{10}, v_{13}, v_{18}, v_{21}, v_{23}\}$ dominates G'.

Case 3.2.4.2: $a = 4, b = 6, c = 3$. Then v_{14} has its third neighbor in G', and by Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{14}v_{15} \dots v_{25}v_{13}v_{12} \dots v_1)$ and $(v_{15}v_{14} \dots v_9v_{23}v_{24}v_{25}v_{16}v_{17} \dots v_{22}v_1v_2 \dots v_8)$, this neighbor is in $\{v_1, v_4, v_7, v_{10}, v_{17}, v_{20}\}$.

Since v_{12} has its third neighbor in G', Lemma 3.4.6 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ eliminates v_{10} as the third neighbor. Now for each of the remaining vertices v_i, Lemma 3.4.6 with $x = v_i$, $y = v_{14}$, and $z = v_9$ yields a dominating set of $G'[C]$ of size 7.

Case 3.2.4.3: $a = 4, b = 7, c = 2$. Then v_{14} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{14}v_{13}\dots v_9v_{23}v_{24}v_{25}v_{15}v_{16}\dots v_{22}v_1v_2\dots v_8)$ and $(v_{14}v_{15}\dots v_{25}v_{13}v_{12}\dots v_1)$, this neighbor is in $\{v_{10}, v_{18}, v_{21}\}$. Since both of v_{12}, and v_{16} have third neighbors in G', Lemma 3.4.6 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ and with $x = v_{14}$, $y = v_{18}$, and $z = v_{16}$ forces $v_{14}v_{21} \in E(G')$. This then forces the hamiltonian cycle $(v_{14}v_{13}\dots v_1v_{22}v_{23}v_{24}v_{25}v_{15}v_{16}\dots v_{21})$ contradicting the maximality of r.

Case 3.2.5: $d = 8$. By (3.14), $(a, b, c) \in \{(4, 4, 6), (4, 7, 3)\}$.

Case 3.2.5.1: $a = b = 4, c = 6$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths
$(v_{17}v_{16}\dots v_8v_{23}v_{24}v_{25}v_{18}v_{19}\dots v_{22}v_1v_2\dots v_7)$ and
$(v_9v_{10}\dots v_{18}v_{25}v_{24}v_{23}v_8v_7\dots v_1v_{22}v_{21}v_{20}v_{19})$, this neighbor is in $\{v_9, v_{10}, v_{16}, v_{17}\}$. Since v_{11} has its third neighbor in C, by Lemma 3.4.6 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$, $v_9v_{13} \notin E(G')$. If $v_{13}v_{10} \in E(G')$, then the set $\{v_2, v_5, v_8, v_{10}, v_{15}, v_{18}, v_{21}, v_{25}\}$ dominates G'. If $v_{13}v_{16} \in E(G')$, then symmetry gives $v_{17}v_{14} \in E(G')$. Thus Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{15}v_{14}v_{17}v_{16}v_{13}v_{12}\dots v_1v_{22}v_{21}\dots v_{18}v_{25}v_{24}v_{23})$ and $(v_{15}v_{16}v_{13}v_{14}v_{17}v_{18}\dots v_{22}v_1v_2\dots v_{12}v_{25}v_{24}v_{23})$ eliminates all possible neighbors of v_{15}. The last possibility is that $v_{13}v_{17} \in E(G')$. The path
$P' = (v_{23}v_{24}v_{25}v_{18}v_{19}\dots v_{22}v_1v_2\dots v_{13}v_{17}v_{16}v_{15}v_{14})$ forces v_{14} to have its third neighbor, say v_i, in G'. By Lemma 3.4.3 for the path $(v_{13}v_{14}\dots v_{25}v_{12}v_{11}\dots v_1)$ and for P', $i \in \{2, 5, 11, 20, 21\}$. By Lemma 3.4.6 with $x = v_{14}$, $y = v_i$, and $z = v_{22}$, $i \notin \{2, 5, 11\}$. So, $i \in \{20, 21\}$. If $i = 20$, then the set $\{v_3, v_6, v_9, v_{12}, v_{16}, v_{20}, v_{22}, v_{25}\}$ dominates G'. If $i = 21$, then the cycle $(v_{18}v_{19}v_{20}v_{21}v_{14}v_{15}v_{16}v_{17}v_{13}\ v_{12}\dots v_1v_{22}v_{23}v_{24}v_{25})$ contradicts the maximality of r.

Case 3.2.5.2: $a = 4, b = 7, c = 3$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. Lemma 3.4.4 for C and Lemma 3.4.3 for the paths
$(v_{14}v_{13}\dots v_8v_{23}v_{24}v_{25}v_{15}v_{16}\dots v_{22}v_1v_2\dots v_7)$,
and $(v_{13}v_{14}\dots v_{22}v_1v_2\dots v_8v_{23}v_{24}v_{25}v_{12}v_{11}v_{10}v_9)$ forces this neighbor to be in $\{v_3, v_6, v_9\}$. Since v_{11} has its third neighbor in G', by Lemma 3.4.6 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$, $v_9v_{13} \notin E(G')$. Finally, for $i = 3, 6$, Lemma 3.4.6 with $x = v_{13}$, $y = v_i$, and $z = v_8$ eliminates the remaining possible neighbors for v_{13}.

Case 3.2.6: $d = 7$. In this case, by (3.14), $a = b = 6$, and $c = 3$. So, v_{14} has its third neighbor in G'. Lemma 3.4.4 for C and Lemma 3.4.3 for the paths

$(v_{15}v_{14}\dots v_1v_{22}v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{21})$ and $(v_{14}v_{15}\dots v_{25}v_{13}v_{12}\dots v_1)$, forces this neighbor to be in $\{v_1, v_4, v_{10}, v_{17}, v_{20}\}$. Since v_{12} has its third neighbor in C, by Lemma 3.4.6 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$, $v_{14}v_{10} \notin E(G')$. Furthermore, for $i = 17, 20$, Lemma 3.4.6 with $x = v_{14}$, $y = v_i$, and $z = v_{22}$ yields that the possible neighbor of v_{14} is either v_1 or v_4. If $v_{14}v_1 \in E(G')$, then the hamiltonian cycle $(v_{25}v_{24}\dots v_{14}v_1v_2\dots v_{13})$ contradicts the maximality of r. If $v_{14}v_4 \in E(G')$, then by symmetry $v_{15}v_3 \in E(G)$ and the 23-cycle $(v_1v_2v_3v_{15}v_{14}v_4v_5\dots v_{13}v_{25}v_{16}v_{15}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.7: $d = 6$. By (3.14), $(a, b, c) \in \{(4, 6, 6), (4, 7, 5), (4, 9, 3), (4, 10, 2), (6, 6, 4), (6, 7, 3), (7, 7, 2)\}$.

Case 3.2.7.1: $a = 4, b = c = 6$. Since $v_{10}v_{25} \in E(G')$, v_{11} has its third neighbor in G'. Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{11}v_{12}\dots v_{25}v_{10}v_9v_8\dots v_1)$ and $(v_{15}v_{14}\dots v_1v_{22}v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{21})$, this neighbor is in $\{v_1, v_4, v_7, v_{14}, v_{15}, v_{17}, v_{20}\}$. Since v_9 has its third neighbor in G', by Lemma 3.4.6 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. Also for $i \in \{4, 1, 20, 17\}$, Lemma 3.4.6 with $x = v_{11}$, $y = v_i$, and $z = v_6$ eliminates v_i as a neighbor of v_{11}. Thus v_{11} is adjacent to either v_{14} or v_{15}.

Case 3.2.7.1.1: $v_{11}v_{15} \in E(G')$. Then v_{14} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for and the paths $(v_{15}v_{14}\dots v_6v_{23}v_{24}v_{25}v_{16}\dots v_{22}v_1v_2\dots v_5)$ and $(v_{14}v_{13}v_{12}v_{11}v_{15}v_{16}v_{17}\dots v_{25}v_{10}v_9\dots v_1)$, this neighbor is in $\{v_1, v_4, v_7, v_{17}, v_{20}\}$. By Lemma 3.4.6 with $x = v_6$, $y = v_{16}$, and $z = v_{14}$, v_{14} is not adjacent to v_i for $i \in \{1, 4, 17, 20\}$. Thus $v_{14}v_7 \in E(G')$, and the hamiltonian cycle $(v_1v_2\dots v_6v_{23}v_{24}v_{25}v_{10}v_9v_8v_7v_{14}v_{13}v_{12}v_{11}v_{15}v_{16}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.7.1.2: $v_{11}v_{14} \in E(G')$ and by symmetry $v_5v_2 \in E(G')$. Then v_{15} has a neighbor in G', and by Lemma 3.4.4 for C and Lemma 3.4.3 for the path $(v_{15}v_{14}\dots v_{10}v_{25}v_{24}v_{23}v_{22}v_1v_2\dots v_9)$, this neighbor is in $\{v_3, v_9, v_{12}, v_{18}, v_{19}, v_{21}\}$. Since v_{17} has its third neighbor in G', by Lemma 3.4.6 with $x = v_{15}$, $y = v_{19}$, and $z = v_{17}$, $v_{15}v_{19} \notin E(G')$. If $v_{15}v_3 \in E(G')$, then the 23-cycle $(v_1v_2v_3v_{15}v_{14}\dots v_6v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{22})$ contradicts the maximality of r. If $v_{15}v_9 \in E(G')$, then the 23-cycle $(v_1v_2\dots v_9v_{15}v_{14}\dots v_{10}v_{25}v_{16}v_{17}\dots v_{22})$ contradicts the maximality of r. If $v_{15}v_{12} \in E(G')$, then the path $(v_{23}v_{24}v_{25}v_{10}v_9\dots v_1v_{22}v_{21}\dots v_{15}v_{12}v_{11}v_{14}v_{13})$ forces v_{13} to have the third neighbor in G'. Then Lemma 3.4.3 for this path, C, and the path $(v_{23}v_{24}v_{25}v_{16}v_{17}\dots v_{22}v_1v_2\dots v_{11}v_{14}v_{15}v_{12}v_{13})$ eliminates all possible neighbors of v_{13}. If $v_{15}v_{21} \in E(G')$, then the cycle $(v_1v_2\dots v_{15}v_{21}v_{20}\dots v_{16}v_{25}v_{24}v_{23}v_{22})$ contradicts the maximality of r. Thus, $v_{15}v_{18} \in E(G')$ and, by symmetry, $v_1v_{20} \in E(G')$. Then the set $\{v_2, v_5, v_7, v_{10}, v_{13}, v_{16}, v_{20}, v_{23}\}$ dominates G'.

Case 3.2.7.2: $a = 4, b = 7, c = 5$. Since $v_6v_{23} \in E(G')$, v_5 has its third neighbor in G'.

By Lemma 3.4.3 for C and Lemma 3.4.6 with $x = v_{22}$, $y = v_{10}$, and $z = v_5$, this neighbor is in $\{v_1, v_2, v_8, v_9\}$. Since v_7 has its third neighbor in G', by Lemma 3.4.6 with $x = v_5$, $y = v_9$, and $z = v_7$, $v_5v_9 \notin E(G')$. If $v_8v_5 \in E(G')$, then v_8 dominates all but a P_{21}, hence v_5 is adjacent to either v_1 or v_2.

Case 3.2.7.2.1: $v_5v_1 \in E(G')$. Then v_2 has its third neighbor in G', and by Lemma 3.4.4 for C and Lemma 3.4.6 with $x = v_{22}$, $y = v_{10}$, and $z = v_2$, this neighbor is either v_8 or v_9. If $v_2v_8 \in E(G')$, then v_8 dominates all but a P_{21} and hence $v_2v_9 \in E(G')$. Then the hamiltonian cycle $(v_1v_{22}v_{21} \ldots v_{10}v_{25}v_{24}v_{23}v_6v_7v_8v_9v_2v_3v_4v_5)$ contradicts the maximality of r.

Case 3.2.7.2.2: $v_5v_2 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7 \ldots v_{22}v_1v_2v_5v_4v_3)$ forces v_3 to have its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for this path, this third neighbor is one of v_9, v_{12}, v_{18}, and v_{21}. If this neighbor is in $\{v_{12}, v_{18}, v_{21}\}$, then the set $\{v_2, v_3, v_7, v_{10}, v_{23}\}$ dominates all but a P_9 or but a P_3 and a P_6. In both cases, G' can be dominated by 8 vertices. Hence $v_3v_9 \in E(G')$. In this case, the hamiltonian cycle $(v_1v_2v_5v_4v_3v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \ldots v_{22})$ contradicts the maximality of r.

Case 3.2.7.3: $a = 4, b = 9, c = 3$. Since $v_{10}v_{25} \in E(G')$, v_{11} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths
$(v_{12}v_{11} \ldots v_6v_{23}v_{24}v_{25}v_{13}v_{14} \ldots v_{22}v_1v_2 \ldots v_5)$ and
$(v_7v_8 \ldots v_{13}v_{25}v_{24}v_{23}v_6v_5 \ldots v_1v_{22}v_{21} \ldots v_{14})$, this neighbor is either v_7 or v_8. Since v_9 has its third neighbor in G', by Lemma 3.4.6 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. Hence $v_{11}v_8 \in E(G')$, and so v_8 dominates all but a P_{21} in G'.

Case 3.2.7.4: $a = 4, b = 10, c = 2$. Since v_5 is a neighbor of v_d, it has its third neighbor, say v_i, in G'. By Lemma 3.4.3 for P, $i \in \{1, 2, 8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 3.4.6 with $x = v_{10}, y = v_{22}, z = v_5$, $i \in \{1, 2, 8, 9\}$. If $v_5v_9 \in E(G')$, the hamiltonian cycle $(v_1v_2 \ldots v_5v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \ldots v_{22})$ contradicts the maximality of r. If v_8 has its third neighbor in G' then v_8 dominates all but a P_{21} in G'. Hence $i \in \{1, 2\}$.

Case 3.2.7.4.1: $v_5v_1 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7 \ldots v_{22}v_1v_5v_4v_3v_2)$ forces v_2 to have its third neighbor, say v_j, in G'. By Lemma 3.4.3 for this path
$j \in \{8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 3.4.6 with $x = v_{10}$, $y = v_{22}$, $z = v_2$, $j \in \{8, 9\}$. Since v_8 does not have its third neighbor in G', $v_2v_9 \in E(G')$. Then the hamiltonian cycle $(v_1v_5v_4v_3v_2v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \ldots v_{22})$ contradicts the maximality of r.

Case 3.2.7.4.2: $v_5v_2 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_6v_7 \ldots v_{22}v_1v_2v_5v_4v_3)$ forces v_3 to have its third neighbor, say v_j, in G'. By Lemma 3.4.3 for this path and P, $j \in \{8, 9, 11, 14, 15, 17, 18, 20, 21\}$. By Lemma 3.4.6 with $x = v_{12}, y = v_{22}, z = v_j$, $j \in$

$\{8, 9, 11, 15, 18, 21\}$. Since v_8 does not have its third neighbor in G', $j \in \{9, 11, 15, 18, 21\}$. If $v_3v_9 \in E(G')$, the hamiltonian cycle $(v_1v_2v_5v_4v_3v_9v_8v_7v_6v_{23}v_{24}v_{25}v_{10}v_{11} \ldots v_{22})$ contradicts the maximality of r. If $v_3v_{11} \in E(G')$, the cycle $(v_1v_2v_5v_4v_3v_{11}v_{10} \ldots v_6v_{23}v_{24}v_{25}v_{12}v_{13} \ldots v_{22})$ contradicts the maximality of r. If $v_3v_{21} \in E(G')$, the 23-cycle $(v_1v_2v_5v_4v_3v_{21}v_{20} \ldots v_6v_{23}v_{22})$ contradicts the maximality of r. Hence $j \in \{15, 18\}$. Since v_7 is a neighbor of v_d, it has its third neighbor, say v_h, in G'. By Lemma 3.4.3 for P and the path $(v_7v_8 \ldots v_{12}v_{25}v_{24}v_{23}v_6v_5 \ldots v_{22}v_{21} \ldots v_{13})$, $h \in \{11, 13, 16, 19\}$. Since v_9 has its third neighbor in G', by Lemma 3.4.6 with $x = v_{11}$, $y = v_7$, and $z = v_9$, $v_{11}v_7 \notin E(G')$. If $v_{13}v_7 \in E(G')$ the hamiltonian path $(v_{12}v_{11} \ldots v_7v_{13}v_{14} \ldots v_{22}v_1 \ldots v_6v_{23}v_{24}v_{25})$ contradicts the maximality of r. So $h \in \{16, 19\}$. If $h = j + 1$, then the 23-cycle $(v_1v_2v_5v_4v_3v_jv_{j-1} \ldots v_{10}v_{25}v_{24}v_{23}v_6v_7v_hv_{h+1} \ldots v_{22})$ contradicts the maximality of r. If $j = 15$ and $h = 19$, the set $\{v_2, v_3, v_7, v_{10}, v_{13}, v_{17}, v_{21}, v_{24}\}$ dominates G'. Hence $j = 18$, $h = 16$, and the set $\{v_5, v_7, v_{10}, v_{14}, v_{18}, v_{20}, v_{22}, v_{25}\}$ dominates G'.

Case 3.2.7.5: $a = b = 6, c = 4$. Since v_{11} has its third neighbor in G', by Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{11}v_{10} \ldots v_6v_{23}v_{24}v_{25}v_{12}v_{13} \ldots v_{22}v_1v_2 \ldots v_5)$, and $(v_{15}v_{14} \ldots v_6v_{23}v_{24}v_{25}v_{16}v_{17} \ldots v_{22}v_1v_2 \ldots v_5)$ this neighbor is in $\{v_7, v_8, v_{15}\}$. Since v_{13} has its third neighbor in G', by Lemma 3.4.6 with $x = v_{11}$, $y = v_{15}$, and $z = v_{13}$, $v_{11}v_{15} \notin E(G')$.

Case 3.2.7.5.1: $v_{11}v_7 \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{12}v_{13} \ldots v_{22}v_1v_2 \ldots v_7v_{11}v_{10}v_9v_8)$ forces v_8 to have its third neighbor in G'. By Lemma 3.4.3 for this path and the paths $(v_7v_8 \ldots v_{16}v_{25}v_{24}v_{23}v_6v_5 \ldots v_1v_{22}v_{21} \ldots v_{17})$, $(v_{15}v_{14} \ldots v_6v_{23}v_{24}v_{25}v_{16}v_{17} \ldots v_{22}v_1v_2 \ldots v_5)$, and $(v_8v_9v_{10}v_{11}v_7v_6v_{23}v_{24}v_{25}v_{12}v_{13} \ldots v_{22}v_1v_2 \ldots v_5)$, this neighbor is v_{15}. Then the 23-cycle $(v_1v_2 \ldots v_7v_{11}v_{10}v_9v_8v_{15}v_{14}v_{13}v_{12}v_{25}v_{16}v_{17} \ldots v_{22})$ contradicts the maximality of r.

Case 3.2.7.5.2: $v_{11}v_8 \in E(G')$. The path $(v_{23}v_{24}v_{25}v_{12}v_{13} \ldots v_{22}v_1v_2 \ldots v_8v_{11}v_{10}v_9)$ forces v_9 to have its third neighbor in G'. By Lemma 3.4.3 for this path and Lemma 3.4.4 for C, this neighbor is in $S = \{v_2, v_5, v_{15}, v_{18}, v_{21}\}$. For each $v_i \in S$ except v_{15}, the set $\{v_2, v_5, v_6, v_{11}, v_{14}, v_{18}, v_{21}, v_{25}\}$ dominates G'. So, $v_9v_{15} \in E(G')$. Then the 23-cycle $(v_1v_2 \ldots v_8v_{11}v_{10}v_9v_{15}v_{14}v_{13}v_{12}v_{25}v_{16}v_{17} \ldots v_{22})$ contradicts the maximality of r.

Case 3.2.7.6: $a = 6, b = 7, c = 3$. Let v_i be the third neighbor of v_{14}. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{21}v_{20} \ldots v_{12}v_{25}v_{24}v_{23}v_{22}v_1v_2 \ldots v_{11})$, $(v_{14}v_{13} \ldots v_6v_{23}v_{24}v_{25}v_{15}v_{16} \ldots v_{22}v_1v_2 \ldots v_5)$, and $(v_{13}v_{14} \ldots v_{25}v_{12}v_{11} \ldots v_1)$, $i \in \{18, 21\}$. Since now v_{16} has its third neighbor in G', by Lemma 3.4.6 with $x = v_{14}$, $y = v_{18}$, and $z = v_{16}$, $v_{14}v_{18} \notin E(G')$. Hence $v_{14}v_{21} \in E(G')$ and the hamiltonian cycle $(v_1v_2 \ldots v_{14}v_{21}v_{20} \ldots v_{15}v_{25}v_{24}v_{23}v_{22})$ contradicts the maximality of r.

Case 3.2.7.7: $a = b = 7, c = 2$. Then v_{14} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths $(v_{14}v_{13}\dots v_6v_{23}v_{24}v_{25}v_{15}v_{16}\dots v_{22}v_1v_2\dots v_5)$, and $(v_{14}v_{15}\dots v_{25}v_{13}v_{12}\dots v_1)$, this neighbor is in $\{v_7, v_{10}, v_{18}, v_{21}\}$. By symmetry, we may assume that it is in $\{v_7, v_{10}\}$. Since v_{12} has its third neighbor in G', Lemma 3.4.6 with $x = v_{10}$, $y = v_{14}$, and $z = v_{12}$ eliminates v_{10} as a possible neighbor of v_{14}. Thus $v_{14}v_7 \in E(G')$, and the hamiltonian cycle $(v_1v_2\dots v_6v_{23}v_{24}v_{25}v_{13}v_{12}\dots v_7v_{14}v_{15}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.8: $d = 5$. By (3.14), $(a, b, c) \in \{(4, 10, 3), (7, 7, 3)\}$.

Case 3.2.8.1: $a = 4, b = 10, c = 3$. Then v_{10} has its third neighbor in G'. If this neighbor lies on the 19-cycle $(v_5v_4\dots v_1v_{22}v_{21}\dots v_{12}v_{25}v_{24}v_{23})$, the set $\{v_7, v_{10}\}$ dominates all but a P_{18}, hence this neighbor is in $\{v_6, v_7, v_8\}$. By Lemma 3.4.4 for C, v_8 cannot be this neighbor. Since v_8 has its third neighbor in C, if $v_{10}v_6 \in E(G')$, Lemma 3.4.6 with $x = v_{10}$, $y = v_6$, and $z = v_8$ gives a dominating set of $G'[C]$ of size 7. Hence $v_{10}v_7 \in E(G')$. Then the set $\{v_2, v_5, v_7, v_{12}, v_{15}, v_{18}, v_{21}, v_{25}\}$ dominates G'.

Case 3.2.8.2: $a = b = 7, c = 3$. Since $v_{25}v_{12} \in E(G')$, v_{13} has its third neighbor in G'. By Lemma 3.4.4 for C and Lemma 3.4.3 for the paths
$(v_{14}v_{13}\dots v_5v_{23}v_{24}v_{25}v_{15}v_{16}\dots v_{22}v_1v_2\dots v_4)$ and
$(v_{13}v_{14}\dots v_{22}v_1v_2\dots v_5v_{23}v_{24}v_{25}v_{12}v_{11}\dots v_6)$, this neighbor is in $\{v_3, v_6, v_9, v_{16}, v_{19}\}$.
Lemma 3.4.6 with $x = v_5$, $y = v_{15}$, and $z = v_3$ shrinks the list to $\{v_6, v_9\}$. Since v_{11} has its third neighbor in G', by Lemma 3.4.6 with $x = v_9$, $y = v_{13}$, and $z = v_{11}$ yields $v_9v_{13} \notin E(G')$. Hence $v_{13}v_6 \in E(G')$. Now the hamiltonian cycle
$(v_1v_2\dots v_5v_{23}v_{24}v_{25}v_{12}v_{11}\dots v_6v_{13}v_{14}\dots v_{22})$ contradicts the maximality of r.

Case 3.2.9: $d = 4$. By (3.14), $(a, b, c) = (6, 9, 3)$. Let v_i be the third neighbor of v_5. By Lemma 3.4.6 with $x = v_{22}$, $y = v_{10}$, and $z = v_5$, $i \le 9$. Then Lemma 3.4.3 for the path $(v_5v_6\dots v_{10}v_{25}v_{24}v_{23}v_4v_3v_2v_1v_{22}v_{21}\dots v_{11})$, and Lemma 3.4.4 for C further yield that $i \in \{1, 8, 9\}$. Since v_3 has its third neighbor in G', by Lemma 3.4.6 with $x = v_1$, $y = v_5$, and $z = v_3$, $v_5v_1 \notin E(G')$. If $v_5v_9 \in E(G')$, then by the same argument, v_8 is adjacent to one of v_1 and v_5. So $v_8v_1 \in E(G')$. Then the 23-cycle $(v_1v_2v_3v_4v_{23}v_{22}\dots v_9v_5v_6v_7v_8)$ contradicts the maximality of r. Thus, $v_5v_8 \in E(G')$. The path $(v_{25}v_{24}v_{23}v_4v_3v_2v_1v_{22}v_{21}\dots v_8v_5v_6v_7)$ forces v_7 to have its third neighbor, say v_i, in G'. By Lemma 3.4.3 for this path and Lemma 3.4.4 for C, $i \in \{1, 11, 14, 17, 20\}$. If $v_7v_1 \in E(G')$, then the 23-cycle $(v_1v_2v_3v_4v_{23}v_{22}\dots v_8v_5v_6v_7)$ contradicts the maximality of r. If $i \in \{11, 14, 17, 20\}$, then the set $\{v_2, v_5, v_{10}, v_{11}, v_{14}, v_{17}, v_{20}, v_{23}\}$ dominates G'.

Case 3.2.10: $d = 3$. Since $a \le b$ and $c \ge 2$, $a \le (22 - 3 - 2)/2 = 8.5$. So by (3.14), $a \in \{4, 6, 7\}$.

Case 3.2.10.1: $a \in \{4, 7\}$. Since v_2 shares a neighbor with v_{23}, it has its third neighbor, say v_i in G'. By Lemma 3.4.6 with $x = v_{22}$, $y = v_{a+d}$, and $z = v_2$, $4 \le i \le d + a - 1 \le 9$. By Lemma 3.4.3 for P, $i \ne 4, 7$. If $d + a - 2 \le i \le d + a - 1$, then the cycle $(v_{23}v_{24}v_{25}v_{d+a}v_{d+a+1} \ldots v_{22}v_1v_2v_iv_{i-1} \ldots v_3)$ contradicts the maximality of r. This means that $a = 7$ and $5 \le i \le 6$. The edge v_2v_5 contradicts Lemma 3.4.6 with $x = v_3$, $y = v_{10}$, and $z = v_5$. So $v_2v_6 \in E(G)$, a contradiction to Lemma 3.4.6 with $x = v_2$, $y = v_6$, and $z = v_4$.

Case 3.2.10.2: $a = 6$. Then by (3.14), $c = 3$. Since v_1 shares a neighbor with v_{23}, it has its third neighbor, say v_i in G'. By Lemma 3.4.6 with $x = v_{12}$, $y = v_3$, and $z = v_1$, $13 \le i \le 21$. By Lemma 3.4.3 for P, $i \ne 15, 18, 21$. If $13 \le i \le 14$, then the cycle $(v_{12}v_{11} \ldots v_1v_iv_{i+1} \ldots v_{25})$ contradicts the maximality of r. Lemma 3.4.6 with $x = v_{12}$, $y = v_{22}$, and $z = v_i$, shows that $i \ne 20, 17$. So $i \in \{16, 19\}$. The same lemma with $x = v_i$, $y = v_1$, and $z = v_{21}$, shows that the third neighbor of v_{21} is some v_j with $i + 1 \le j \le 19$. It follows that $i = 16$ and $17 \le j \le 19$. By Lemma 3.4.3 for P, $j \ne 19$. By the symmetry between v_1 and v_{11}, $v_{11}v_{18} \in E(G)$ and hence $j = 17$. But then symmetrically v_{13} also is adjacent to v_{17}, a contradiction.

Case 3.2.11: $d = 2$. Since $c \le d$, $c = 2$, and again no triple (a, b, c) satisfies (3.14).

Case 4: $r = 23$. Since v_{25} is the endpoint of a hamiltonian path in P, it has two neighbors in C. This forces v_{24} to be the endpoint of another hamiltonian path, and so v_{24} also has two neighbors in C. By the maximality of r, the distance on C between any neighbor of v_{24} and any neighbor of v_{25} is at least 3. Then Lemma 3.4.3 for P and the path $(v_{25}v_{24}v_{23}v_1v_2 \ldots v_{22})$ forces the neighbors of v_{25} in C to be in $\{v_4, v_7, v_{10}, v_{13}, v_{16}, v_{19}\}$. By symmetry, we conclude that

$$\text{the distance on } C \text{ between any neighbor of } v_{24} \text{ and any neighbor of } v_{25} \text{ is in } \{4, 7, 10\}. \tag{3.15}$$

In particular, since each of these values is 1 modulo 3, the neighbors of v_{24}, and v_{25} cannot alternate around C. So, we may assume that $v_{25}v_d, v_{25}v_{d+a}, v_{24}v_{23-b} \in E(G')$, and $c = 23 - a - b - d$. We may assume further that $d \le c$ and $a \le b$. In particular, $d + a \le 11$ and hence $d \in \{4, 7\}$. Furthermore, since a is divisible by 3, $d + a \le 10$. As a neighbor of v_d, v_{d+1} has its third neighbor, say v_i, in C. By Lemma 3.4.6 with $x = v_{23}$, $y = v_{d+a}$, and $z = v_{d+1}$, $i \le d + a - 1$. If $i = 1$, then the hamiltonian cycle $(v_dv_{d-1} \ldots v_1v_{d+1}v_{d+2} \ldots v_{25})$ contradicts the maximality of r. By Lemma 3.4.3 for the path $(v_{d+1}v_{d+2} \ldots v_{23}v_1v_2 \ldots v_dv_{25}v_{24})$, $i \notin \{2, 5, d + 3, d + 6\}$. By Lemma 3.4.3 for the path $(v_{d+a-1}v_{d+a-2} \ldots v_1v_{23}v_{22} \ldots v_{d+a}v_{25}v_{24})$, $i \ne 3, d-1$. Summarizing and remembering that $d \in \{4, 7\}$ and $d + a \leq 10$, we have

$$\text{if } d = 4, \text{ then } 8 \leq i \leq d + a - 1 \leq 9; \text{ if } d = 7, \text{ then } i = 4. \tag{3.16}$$

Case 4.1: $d = 4$. By above, $d + a \in \{7, 10\}$. So, by (3.16), $i \in \{8, 9\}$.
Case 4.1.1: $i = 9$. The path $(v_{24}v_{25}v_4v_3v_2v_1v_{23}v_{22}\ldots v_9v_5v_6v_7v_8)$ forces v_8 to have its third neighbor, say v_j, in G'. By Lemma 3.4.3 for this path, $j \neq 2, 6$. By Lemma 3.4.6 with $x = v_{23}$, $y = v_{10}$, and $z = v_8$, $j \leq 9$. Thus, $j \in \{1, 3\}$. If $j = 1$, then the hamiltonian cycle $(v_8v_7v_6v_5v_9v_{10}\ldots v_{25}v_4v_3v_2v_1)$ contradicts the maximality of r. Thus, $v_3v_8 \in E(G')$. Then the set $\{v_3, v_6, v_{10}, v_{11}, v_{14}, v_{17}, v_{20}, v_{23}\}$ dominates G'.
Case 4.1.2: $i = 8$. By (3.15), $c \in \{4, 7\}$. The path $P' = (v_7v_6v_5v_8v_9\ldots v_{25}v_4v_3v_2v_1)$ forces v_7 to have its third neighbor, say v_j, in G'.
Case 4.1.2.1: $c = 4$. By the symmetry between v_5 and v_9, $v_6v_9 \in E(G')$. By Lemma 3.4.3 for P' and the symmetric path $(v_7v_8v_9v_6v_5\ldots v_1v_{23}v_{22}\ldots v_{14}v_{24}v_{25}v_{10}v_{11}v_{12}v_{13})$, we have $j \in \{3, 11, 17, 20\}$. By symmetry, we may assume that either $j = 11$ or $j = 17$. Then the set $\{v_1, v_4, v_9, v_{11}, v_{14}, v_{17}, v_{18}, v_{21}\}$ dominates G'.
Case 4.1.2.2: $c = 7$. Recall that v_j is the third neighbor of v_7. By Lemma 3.4.3 for P, P', and the path $(v_7v_6v_5v_8v_9v_{10}v_{25}v_{24}v_{17}v_{18}\ldots v_{23}v_1v_2v_3v_4)$, we have $j \in \{1, 11, 14\}$. If $j = 1$, then the hamiltonian cycle $(v_1v_2v_3v_4v_{25}v_{24}\ldots v_8v_5v_6v_7)$ contradicts the maximality of r. If $j = 11$, then the 24-cycle
$(v_1v_2v_3v_4v_{25}v_{10}v_9v_8v_5v_6v_7v_{11}v_{12}\ldots v_{23})$ contradicts the maximality of r. Finally, if $j = 14$, then the set $\{v_2, v_7, v_8, v_{12}, v_{16}, v_{19}, v_{22}, v_{25}\}$ dominates G'.
Case 4.2: $d = 7$. By above, $d + a = 10$. By (3.16), $i = 4$. The path
$P' = (v_1v_2v_3v_4v_8v_9\ldots v_{25}v_7v_6v_5)$ forces v_5 to have its third neighbor, say v_j, in G'. By Lemma 3.4.6 with $x = v_{23}$, $y = v_7$, and $z = v_5$, $j \leq 6$. Thus, $j \in \{1, 2, 3\}$. Lemma 3.4.3 for P' and for the path $(v_9v_8\ldots v_1v_{23}v_{22}\ldots v_{10}v_{25}v_{24})$ yields $j \neq 2$ and $j \neq 3$, respectively. So, $v_5v_1 \in E(G')$. Now the cycle $(v_1v_2v_3v_4v_8v_9\ldots v_{25}v_7v_6v_5)$ contradicts the maximality of r. $\square$

Chapter 4

Acyclic Coloring

4.1 Introduction

Remember that a proper coloring of the vertices of a graph G is an assignment of colors to the vertices of the graph such that no two adjacent vertices receive the same color. A proper coloring of a graph G is acyclic if the union of any two color classes induces a forest. The acyclic chromatic number, $a(G)$, is the smallest integer k such that G is acyclically k-colorable. The notion of acyclic coloring was introduced in 1973 by Grünbaum [12] and turned out to be interesting and closely connected to a number of other notions in graph coloring. Several researchers felt the beauty of the subject and started working on problems and conjectures posed by Grünbaum.

In particular, Grünbaum studied $a(r)$ – the maximum value of the acyclic chromatic number over all graphs G with maximum degree at most r. He conjectured that always $a(r) = r + 1$ and proved this for $r \leq 3$. In 1979, Burstein [9] proved the conjecture for $r = 4$. This result was proved independently by Kostochka [16]. It was also proved in [16] that for $k \geq 3$, the problem of deciding whether a graph is acyclically k-colorable is NP-complete. It turned out that for large r, Grünbaum's conjecture is incorrect in a strong sense. Albertson and Berman mentioned in [1] that Erdős proved that $a(r) = \Omega(r^{4/3-\epsilon})$ and conjectured that $a(r) = o(r^2)$. Alon, McDiarmid and Reed [4] sharpened Erdős' lower bound to $a(r) \geq c\, r^{4/3}/(\log r)^{1/3}$ and proved that

$$a(r) \leq 50\, r^{4/3}. \tag{4.1}$$

This established almost the order of the magnitude of $a(r)$ for large r. Recently, the problem of estimating $a(r)$ for small r was considered again.

Fertin and Raspaud [11] showed among other results that $a(5) \leq 9$ and gave a linear-time algorithm that acycliclically 9-colors any graph with maximum degree 5. Furthermore, for $r \geq 3$, they gave a fast algorithm that uses at most $r(r-1)/2$ colors for acyclic coloring of any graph with maximum degree r. Of course, for large r this is much worse than the upper bound (4.1), but for $r < 1000$, it is better. Hocquard and Montassier [14] showed that every 5-connected graph G with $\Delta(G) = 5$ has an acyclic 8-coloring. Kothapalli, Varagani, Venkaiah, and Yadav [23] showed that $a(5) \leq 8$. Kothapalli, Satish, and Venkaiah [22] proved that every graph with maximum degree r is acyclically colorable with at most $1 + r(3r+4)/8$ colors. This is better than the bound $r(r-1)/2$ in [11] for $r \geq 8$. The main result of this chapter is

Theorem 4.1.1. *Every graph with maximum degree* 5 *has an acyclic* 7*-coloring, i.e.,* $a(5) \leq 7$.

We do not know whether $a(5)$ is 7 or 6, and do not have a strong opinion about it.

Our proof is different from that in [11, 14, 23] and heavily uses the ideas of Burstein [9]. He started from an uncolored graph G with maximum degree 4 and colored step by step more and more vertices (with some recolorings) so that each of the partial acyclic 5-colorings of G had additional good properties that enabled him to extend the coloring further. The proof yields a linear-time algorithm which gives an acyclic coloring using at most 7 colors of any graph with maximum degree 5. Using this approach we also show that for every fixed $r \geq 6$, there exists a linear-time algorithm giving an acyclic coloring of any graph with maximum degree r using at most $1 + \lfloor \frac{(r+1)^2}{4} \rfloor$ colors. This is better than the bounds in [11] and [22] cited above for every $r \geq 6$.

In the next section we introduce notation, prove two small lemmas and state the main lemma. In Section 4.3 we prove Theorem 4.1.1 modulo the main lemma. In Section 4.4 we derive linear-time algorithms for acyclic coloring of graphs with bounded maximum degree. In the last section we give the proof of the main lemma.

This chapter is based on joint work with A. V. Kostochka.

4.2 Preliminaries

Let G be a graph. A *partial coloring* of G is a coloring of some subset of the vertices of G. A *partial acyclic coloring* is then a proper partial coloring of G containing no bicolored cycles.

Given a partial coloring f of G, a vertex v is
(a) *rainbow* if all colored neighbors of v have distinct colors;
(b) *almost rainbow* if there is a color c such that exactly two neighbors of v are colored with c and all other colored neighbors of v have distinct colors;
(c) *admissible* if it is either rainbow or almost rainbow;
(d) *defective* if v is an uncolored almost rainbow vertex such that at least one of the two of its neighbors receiving the same color is admissible.

A partial acyclic coloring f of a graph G is *rainbow* if f is a partial acyclic coloring of G such that every uncolored vertex is rainbow.

A partial acyclic coloring f of a graph G is *admissible* if either f is rainbow or one vertex is defective and all other uncolored vertices are rainbow. In these terms, a coloring is rainbow if it is admissible and has no defective vertices. Note that both, rainbow and admissible colorings are partial acyclic colorings where additional restrictions are put only on uncolored vertices. The advantage of using admissible colorings is that they provide a stronger induction condition that places additional restrictions only on coloring of neighbors of uncolored vertices. So, the fewer uncolored vertices remaining, the weaker these additional restrictions.

All colorings in this section will be from the set $\{1, 2, \ldots, 7\}$.

Lemma 4.2.1. *Let v be a vertex of degree 4 in a graph G with $\Delta(G) \leq 5$. Let f be an admissible (respectively, rainbow) coloring in which v is colored with color c_1, each of the neighbors of v is colored, and exactly 3 colors appear on the neighbors of v. If at least one of the two neighbors of v receiving the same color and one of the other two neighbors of v each have a second (i.e., distinct from v) neighbor with color c_1, then we can recolor v and at most one of its neighbors so that the coloring remains admissible (respectively, rainbow). In particular, the new partial acyclic coloring has no new defective vertices. Moreover, if we need to recolor a vertex other than v, then we may choose a vertex with 5 colored neighbors and recolor it with a color incident to v in f.*

Proof. Let $N(v) = \{z_1, z_2, z_3, z_4\}$, $f(z_1) = f(z_2) = c_2$, $f(z_3) = c_3$, $f(z_4) = c_4$. Let z_2 and z_3 be the neighbors of v with colors c_2, and c_3 that are also adjacent to another vertex of color c_1. We may assume that z_2 is adjacent to a vertex of color c_5, since otherwise when we recolor v with c_5, no bicolored cycles appear and the coloring remains admissible (respectively, rainbow). Similarly, we may assume that z_2 is adjacent to vertices of colors c_6 and c_7. Then we may recolor z_2 with c_3 and repeat the above argument to get that z_3 also is adjacent to vertices with colors c_5, c_6, and c_7. In this case, we may change the original coloring by recoloring z_3 with c_2 and v with c_3. So, in this case only v and z_3 change colors. Note that either only v changes its color, or z_2 receives color c_3, or z_3 receives color c_2. $\square$

For partial colorings f and f' of a graph G, we say that f' is *larger than* f if it colors more vertices.

Lemma 4.2.2. *Let v be a vertex of degree 4 in a graph G with $\Delta(G) \leq 5$. Let f be a rainbow coloring in which v is colored with color c_1, the neighbors z_1, z_2, and z_3 of v receive the distinct colors c_2, c_3, and c_4, the neighbor z_4 of v is an uncolored rainbow vertex. Then either G has a rainbow coloring f_1 that colors the same vertices and differs from f only at v, or G has a rainbow coloring f' larger than f. Moreover, if the former does not hold, then z_4 has degree 5 and exactly one uncolored neighbor, say $z_{4,4}$, and we can choose the larger coloring f' so that all the following are true:*

1. *Every vertex colored in f is still colored.*
2. *Vertex z_4 is colored.*
3. *The only uncolored vertex apart from z_4 that may get colored is $z_{4,4}$, and it does only if it has neighbors of colors c_1, c_2, c_3, and c_4.*
4. *Apart from v, only one vertex w may change its color, and if it does, then (a) w is a neighbor of z_4, (b) w has four colored neighbors, (c) it changes a color in $\{c_5, c_6, c_7\}$ to another color in $\{c_5, c_6, c_7\}$, and (d) z_4 gets the former color of w. In particular, v is admissible in f'.*

Proof. Let v, z_1, z_2, z_3, and v_4 be as in the hypothesis. We may assume that z_4 is adjacent to a vertex $z_{4,1}$ of color c_5: otherwise, since v_4 is rainbow, when we recolor v with c_5, the new coloring will be rainbow. Similarly, we may assume that z_4 is adjacent to vertices $z_{4,2}$, and $z_{4,3}$ of colors c_6 and c_7. If z_4 has no other neighbors, then we can recolor v with c_5 and color z_4 with c_1. So, assume that z_4 has the fifth neighbor, $z_{4,4}$. If $z_{4,4}$ is colored, then $f(z_{4,4}) \in \{c_2, c_3, c_4\}$, since z_4 is rainbow. In this case, we let $f'(z_4) = c_1$ and $f'(v) = c_5$. So, we may assume that $z_{4,4}$ is not colored. If $z_{4,4}$ has no neighbor of color c_2, then coloring z_4 with c_2 leaves the coloring rainbow and makes it larger than f. Thus, we may assume that $z_{4,4}$ has a neighbor of color c_2 and similarly neighbors of colors c_3 and c_4. If $z_{4,4}$ has no neighbor of color c_1, then we let $f'(z_4) = c_1$ and $f'(v) = c_5$. So, let $z_{4,4}$ have such a neighbor.

If $z_{4,1}$ has no neighbor of color c_2, then by coloring z_4 with c_2 and $z_{4,4}$ with c_5, we get a rainbow coloring larger than f. So, we may assume (by symmetry) that $z_{4,1}$ has neighbors of colors c_2, c_3, c_4. If $z_{4,1}$ has no neighbor of color c_1, then we let $f'(z_4) = c_1$, $f'(z_{4,4}) = c_5$, and $f'(v) = c_6$. Finally, if $z_{4,1}$ also has a neighbor of color c_1, then we let $f'(z_{4,1}) = c_6$ and $f'(z_4) = c_5$. $\square$

The next lemma is our main lemma. We will use it in the next section and prove in Section 5.

Lemma 4.2.3. *Let f be an admissible partial coloring of a 5-regular graph G. Then G has a rainbow coloring f' that colors at least as many vertices as f.*

4.3 Proof of the Theorem

For convenience, we restate Theorem 4.1.1.

Theorem 4.1.1. Every graph with maximum degree 5 has an acyclic 7-coloring.

Proof. Let G be such a graph. If G is not 5-regular, form G' from two disjoint copies of G by adding for each $v \in V(G)$ of degree less than 5 an edge between the copies of v. Repeating this process at most five times gives a 5-regular graph G^* containing G as a subgraph. Since an acyclic 7-coloring of G^* yields an acyclic 7-coloring of its subgraph G, we may assume that G is 5-regular.

Let f be an admissible coloring of G from the set $\{1, 2, \ldots, 7\}$ with the most colored vertices. By Lemma 4.2.3, we may assume that f is rainbow.

Let H be the subgraph of G induced by the vertices left uncolored by f. Let x be a vertex of minimum degree in H. We consider several cases according to the degree $d_H(x)$.

Case 1: $d_H(x) = 0$. Since f is rainbow, any color in $\{1, 2, \ldots, 7\} - f(N_G(x))$ can be used to color x contradicting the maximality of f.

Case 2: $d_H(x) = 1$. Since f is rainbow, we may assume that x is adjacent to vertices of colors $1, 2, 3$, and 4. Let y be the uncolored neighbor of x. Since y is rainbow, coloring x with 5 gives either a rainbow coloring or an admissible coloring with the defective vertex y having the admissible neighbor x, a contradiction to the maximality of f.

Case 3: $d_H(x) = 2$. We may assume that x is adjacent to vertices with colors $1, 2, 3$, and two uncolored vertices y_1 and y_2. Since in our case y_1 is adjacent to at most 3 colored vertices, some color $c \in \{4, 5, 6, 7\}$ does not appear on the neighbors of y_1. Coloring x with c then yields either a rainbow coloring, or an admissible coloring with defective vertex y_2 and its admissible neighbor x, a contradiction to the maximality of f.

Case 4: $d_H(x) = 3$. We may assume that x is adjacent to vertices of colors 1 and 2. By the choice of x, each uncolored vertex of G has at most 2 colored neighbors. Since the three uncolored neighbors of x have at most 6 colored neighbors in total, some color $c \in \{3, 4, 5, 6, 7\}$ is present at most once among these 6 neighbors. Then coloring x with c again yields an admissible coloring, a contradiction to the maximality of f.

Case 5: $d_H(x) \geq 4$. Since each vertex of G has at most one colored neighbor, at most 5 colors are used in the second neighborhood of x. Hence x may be colored to give a rainbow coloring with more colored vertices.

We conclude that H is empty and that f is an acyclic 5-coloring of G. □

4.4 Algorithms

Theorem 4.4.1. *There exists a linear time algorithm for finding an acyclic* 7*-coloring of a graph with maximum degree* 5*.*

Proof. The proof of the Theorem 4.1.1, along with Lemmas 4.2.1–4.2.3 gives an algorithm. In order to control the efficiency of the algorithm we make the following modification: whenever the proof checks whether a vertex v is in a two-colored cycle, we check only for such a cycle of length at most 12, and if we do not find such a short cycle, then check whether two bicolored paths of length 6 leave v. This is enough, since the existence of such paths already makes the proofs of Theorem 4.1.1 and all the lemmas work. So, we need only to consider a bounded (at most 5^6) number of vertices around our vertex. It then suffices to compute the running time of this algorithm. Let n be the number of vertices in G. The process of creating a 5-regular graph takes $O(n)$ time since we apply this process at most 5 times, each time on at most $2^5 n$ vertices, each of degree at most 5. We may now assume that G is a 5-regular graph. We then create and maintain 6 databases D_j, $j = 0, 1, \ldots, 5$ (say doubly linked lists), each for the set of vertices with degree j in the current H. At the beginning, all vertices are in D_5, and it is possible to update the databases in a constant amount of time each time a vertex gains or loses a colored neighbor. Since there are at most $2^5 n$ possible searches for a vertex with the minimum number of uncolored neighbors, all the searches and updates will take $O(n)$ time. Note that the processes of Lemma 4.2.1 and Lemma 4.2.2 also take a constant amount of time to complete. Observe that each of the cases in Lemma 4.2.3 either finds a rainbow coloring, or finds an admissible coloring with more colored vertices, or reduces to a previous case in an amount of time bounded by a constant. Also when

Lemma 4.2.3 processes a defective vertex, it yields either a rainbow coloring, or a larger admissible coloring and the next defective vertex in a constant time. Finally, since we start from an uncolored graph and color each additional vertex in a constant time, the implied algorithm colors all vertices in $O(n)$ time. $\square$

For a partial coloring f of a graph G and a vertex $v \in V(G)$, we say that $u \in V(G)$ is *f-visible from v*, if either $vu \in E(G)$ or v and u have a common uncolored neighbor.

Theorem 4.4.2. *For every fixed r there exists a linear (in n) algorithm finding an acyclic coloring for any n-vertex graph G with maximum degree r using at most $1+\lfloor\frac{(1+r)^2}{4}\rfloor$ colors.*

Proof. We start from the partial coloring f_0 that has no colored vertices, and for $i = 1, \ldots, n$ at Step i obtain a rainbow partial acyclic coloring f_i from f_{i-1} by coloring one more vertex (without recoloring). The algorithm proceeds as follows: at Step i choose an uncolored vertex v_i with the most colored neighbors. Greedily color v_i with a color α_i in $C := \{1, \ldots, 1 + \lfloor\frac{(1+r)^2}{4}\rfloor\}$ that is distinct from the colors of all vertices f_{i-1}-visible from v_i. We claim that we always can find such α_i in C.

Suppose that at Step i, v_i has exactly k colored neighbors. Then it has at most $r-k$ uncolored neighbors, and each of these uncolored neighbors has at most k colored neighbors. So, the total number of vertices f_{i-1}-visible from v_i is at most

$$k + (r-k)k = k(r+1-k) \leq \lfloor\frac{(r+1)^2}{4}\rfloor = |C| - 1,$$

and we can find a suitable color α_i for v_i.

It now suffices to show that for each i, the coloring f_i is rainbow and acyclic. For f_0, this is obvious. Assume now that f_{i-1} is rainbow and acyclic. Since v_i is rainbow in f_{i-1}, coloring it with α_i does not create bicolored cycles. Thus, f_i is acyclic. Also since α_i is distinct from the colors of all vertices f_{i-1}-visible from v_i, f_i is rainbow.

For the runtime, note that at Step i the algorithm considers only v_i and vertices at distance at most 2 from v_i. As in the proof of Theorem 4.4.1, it is sufficient to maintain $r+1$ databases each containing all vertices with a given number of colored neighbors. This allows a constant time search for a vertex with the greatest number of colored neighbors. Moving a vertex as its number of colored neighbors changes takes a constant amount of time. Choosing and coloring v_i together with updating the databases then takes $O(r^2)$ time. Hence the running time of the algorithm is at most $c_r n$, where c_r depends on r. $\square$

4.5 Proof of Lemma 4.2.3

We will prove that under the conditions of the lemma, either its conclusion holds or there is an admissible coloring f'' larger than f. Since G is finite, repeating the argument eventually yields either an acyclic coloring of the whole G or a rainbow coloring. In both cases we do not have defective vertices.

Let H be the subgraph of G induced by the uncolored vertices. Let x be the sole defective vertex under f and let $y_1, y_2, \ldots, y_5$ be its neighbors. By the definition of a defective vertex, x has two neighbors of the same color. We will assume that $f(y_1) = f(y_2) = 1$ and that y_1 is admissible. When more then two neighbors of x are colored, we assume for $i = 3, 4, 5$ that if y_i is colored, then $f(y_i) = i - 1$. Also for $i = 1, \ldots, 5$, the four neighbors of y_i distinct from x will be denoted by $y_{i,1}, \ldots, y_{i,4}$ (some vertices will have more than one name, since they may be adjacent to more than one y_i). We consider several cases depending on $d_H(x)$.

Case 1: $d_H(x) = 0$. First we try to color x with colors $5, 6$, and 7. If this is not allowed, then for $j = 5, 6, 7$, G has a $1, j$-colored y_1, y_2-path. This forces that both of y_1 and y_2 have neighbors with colors $5, 6$, and 7, each of which is adjacent to another vertex of color 1. In particular, both y_1 and y_2 are admissible. For $i = 1, 2$ and $j = 1, 2, 3$, we suppose that $f(y_{i,j}) = j + 4$ and $y_{i,j}$ is adjacent to another vertex of color 1.

Case 1.1: For some $i \in \{1, 2\}$, $y_{i,4}$ is colored and $f(y_{i,4}) \notin \{5, 6, 7\}$. By symmetry, we may assume that $i = 1$ and $f(y_{1,4}) = 2$. Recolor y_1 with 3 and call the new admissible coloring f'. If we can now recolor y_2 so that the resulting coloring f'' is rainbow on $G - xy_2 - xy_1$ or the only defective vertex in f'' on $G - xy_2 - xy_1$ is $y_{2,4}$, then we do this recoloring and color x with 1. Since y_1 and y_2 have no neighbors of color 1 apart from x, we obtained an admissible coloring of G larger than f. If we cannot recolor y_2 to get such a coloring, then $y_{2,4}$ is colored with a color $c \in \{5, 6, 7\}$. Moreover, in this case by Lemma 4.2.1 applied to y_2 in coloring f' of $G - xy_2 - xy_1$, we can change the colors of only y_2 and some $y \in \{y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}\}$ to get an admissible coloring f_1 of $G - xy_2 - xy_1$. Moreover, by Lemma 4.2.1, $f_1(y) \in \{5, 6, 7\}$. Then by coloring x with 1 we obtain a rainbow coloring of G, as above.

Case 1.2: $y_{1,4}$ is not colored. By Lemma 4.2.2 for vertex y_1 in $G - xy_1$, either $G - xy_1$ has a rainbow coloring f' that differs from f only at y_1 (in which case by symmetry, we may assume that $f'(y_1) = 3$ and proceed further exactly as in Case 1.1), or $G - xy_1$ has a larger rainbow coloring f' satisfying statements 1)–4) of Lemma 4.2.2. In particular, by 4), none of y_2, y_3, y_4, y_5 changes its color and y_1 remains admissible. This finishes Case

1.2.

By the symmetry between y_1 and y_2, the remaining subcase is the following.

Case 1.3: $f(y_{1,4}) = 5$ and $f(y_{2,4}) = c \in \{5, 6, 7\}$. By Lemma 4.2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another admissible coloring f'. If $f'(y_1) \in \{5, 6, 7\}$, then f' is a rainbow coloring, as claimed. So, we may assume that $f'(y_1) = c_1 \in \{2, 3, 4\}$. If all the colors $5, 6, 7$ are present on neighbors of y_2, then again by Lemma 4.2.1 (applied now to y_2 in coloring f' of $G - xy_2$), G has an admissible coloring f'' that differs from f' only at y_2 and maybe at one neighbor of y_2. Then coloring x with 1 we get a rainbow coloring. So, some color in $\{5, 6, 7\}$ is not present in $f'(N(y_2))$. By Lemma 4.2.1, this may happen only if $y_{1,1}$ is a common neighbor of y_1 and y_2, and $c = f(y_{2,4}) \neq 5$. In particular, in this case, $y_{1,1}$ has neighbors of colors 1 (they are y_1 and y_2), $2, 3$, and 4. Since $c \neq 5$, we may assume that $c = 6$. By the symmetry between y_1 and y_2, we conclude that, in f, vertex $y_{2,2}$ also is a common neighbor of y_1 and y_2 and has neighbors of colors 1 (they are y_1 and y_2), $2, 3$, and 4. Returning to coloring f', we see that y_2 has no neighbors of color 5, and its neighbors $y_{1,1}$ (formerly of color 5) and $y_{2,2}$ (by the previous sentence) also have no neighbors of color 5. So, recoloring y_2 with 5 yields an admissible coloring of G. Now coloring x with 1 creates a larger rainbow coloring.

Case 2: $d_H(x) = 1$. We first try to color x with 4. If no bicolored cycle is formed, then either we have a rainbow coloring or an admissible coloring with defective vertex y_5 and an admissible neighbor x. Hence we may assume that coloring x with 4 creates a bicolored cycle. This then gives each of y_1 and y_2 a neighbor of color 4. A similar argument gives each of y_1 and y_2 a neighbor of color $5, 6$, and 7, i.e., both y_1 and y_2 are rainbow. Recoloring y_1 with color 2 allows us to repeat the argument at y_3. Then y_3 also has neighbors of each of the colors $4, 5, 6$, and 7. If y_5 has no neighbor of color 2, then recoloring (in the original coloring f) y_3 with 1, and coloring x with 2 yields a rainbow coloring. So, by the symmetry between colors $1, 2$, and 3, we may assume that for $i \in \{1, 2, 3\}$, $f(y_{5,i}) = i$. Since y_5 is rainbow, by the symmetry between colors $4, 5, 6$, and 7, we may assume that either $f(y_{5,4}) = 4$, or $y_{5,4}$ is not colored. In both cases, recolor (in the original coloring f) y_3 with 1, color x with 2 and y_5 with 5. We get an admissible coloring larger than f, where only $y_{5,4}$ may be defective.

Case 3: $d_H(x) = 3$. If one of the uncolored neighbors y_3, y_4, y_5 (say, y_3) of x has 4 colored neighbors, then we may color y_3 with some $c \notin f(N(y_3)) \cup \{1\}$ and thus create an admissible coloring larger than f. Hence we may assume that each of y_3, y_4, and y_5 has at most 3 colored neighbors.

Case 3.1: One of y_1 and y_2 has three neighbors of different colors such that each of these neighbors has another neighbor of color 1. Suppose for example that for $j = 1, 2, 3$, $f(y_{1,j}) = 1 + j$ and $y_{1,j}$ has another neighbor of color 1. If y_1 has a fourth color, say c, in its neighborhood, then we recolor y_1 with a color $c' \notin \{1, c, 5, 6, 7\}$ and get a rainbow coloring of G. Suppose now that color $c \in \{5, 6, 7\}$ appears twice on $N(y_1)$. Then by Lemma 4.2.1 applied to y_1 in $G - xy_1$, we can change the color of y_1 and at most one other vertex that is a neighbor of y_1 not adjacent to uncolored vertices to get another rainbow coloring of $G - xy_1$. Then this coloring will also be a rainbow coloring of G. Finally, suppose that y_1 has an uncolored neighbor $y_{1,4}$. Applying Lemma 4.2.2 to y_1 in $G - xy_1$ we either recolor only y_1 and get a rainbow coloring of G (finishing the case), or obtain a rainbow coloring f' of $G - xy_1$ larger than f satisfying the conclusions of the lemma. Since each of y_3, y_4 and y_5 has at least two neighbors left uncolored by f, none of them may play role of z_4 or $z_{4,4}$ in Lemma 4.2.2 when they get colored. Then f' is an admissible coloring of G where only x could be a defective vertex with admissible neighbor v. This proves Case 3.1.

Let T be the set of colors c such that more than one of the vertices y_3, y_4 and y_5 has a neighbor of color c. Since y_3, y_4 and y_5 have in total at most 9 colored neighbors, $|T| \leq 4$.

Case 3.2: $|T| \leq 3$. By symmetry, we may assume that $T \subseteq \{2, 3, 4\}$. If coloring x with $c \in \{5, 6, 7\}$ does not create a bicolored cycle, then it will yield an admissible coloring larger than f. So, we may assume that each of y_1 and y_2 has in its neighborhood vertices of colors $5, 6$, and 7, each of which is adjacent to another vertex of color 1. So, we have Case 3.1.

Case 3.3: $|T| = 4$. Let $T = \{2, 3, 4, 5\}$. As in Case 3.1, we may assume that each of y_1 and y_2 is adjacent to vertices of colors 6 and 7, each of which have another neighbor of color 1.

Let y_3 have exactly 3 colored neighbors labeled $y_{3,1}, y_{3,2}, y_{3,3}$ with colors $2, 3, 4$. Let $y_{3,4}$ be the uncolored neighbor of y_3. Then if $y_{3,4}$ has no neighbor of color 5, we may color y_3 with 5 to get a new admissible coloring. Hence $y_{3,4}$ is adjacent to a vertex of color 5. Similarly, $y_{3,4}$ has neighbors of color 6 and 7. By symmetry, we may assume that a vertex of color 2 is adjacent to at most one of y_4 and y_5.

Case 3.3.1: $y_{3,4}$ has no neighbor of color 1. We try to color y_3 with 1 and x with 2. If this does not produce a new admissible coloring, then one of y_1 or y_2, say y_1, has a neighbor of color 2 that is adjacent to another vertex of color 1. So, we again get Case 3.1.

Case 3.3.2: $y_{3,4}$ has a neighbor of color 1. If $y_{3,1}$ has no neighbor of color 1, then we again try to color y_3 with 1 and x with 2, but also color $y_{3,4}$ with 2. Then we simply repeat the argument of Case 3.3.1. So, suppose that $y_{3,1}$ has a neighbor of color 1. If $y_{3,1}$ has no neighbor of some color $\alpha \in \{5,6,7\}$, then we color $y_{3,4}$ with 2 and y_3 with α. Thus $y_{3,1}$ has neighbors of colors $1,5,6,7$. Then we recolor $y_{3,1}$ with 3 and color y_3 with 2.

Case 4: $d_H(x) = 2$. As at the beginning of Case 3, we conclude that each of the uncolored vertices y_4 and y_5 has at least one uncolored neighbor besides x.

Let B be the set of colors appearing in the neighborhoods of both, y_4 and y_5. By the previous paragraph, $|B| \leq 3$.

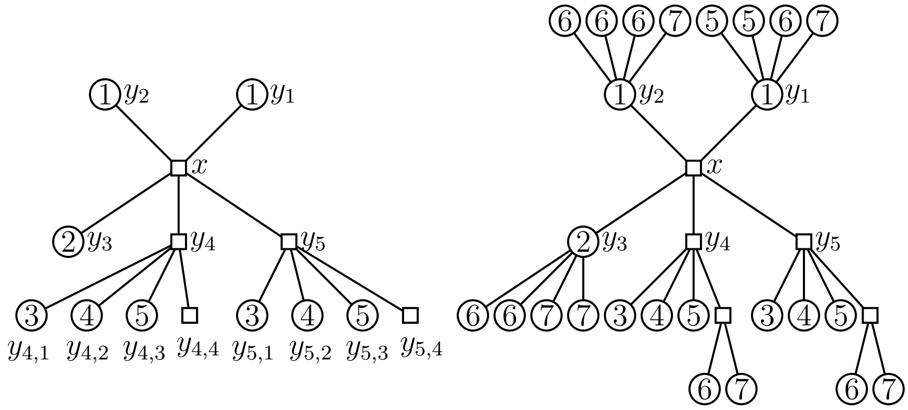
Case 4.1: $|B| \leq 1$. We may assume that $\{4,5,6,7\} \cap B = \emptyset$. Try to color x with 4. By the definition of B, either a two-colored cycle appears, or we get a new admissible coloring larger than f. Hence we may assume that coloring x with 4 creates a bicolored cycle. Since this cycle necessarily goes through y_1, y_1 is adjacent to a vertex with color 4. Similarly, y_1 is adjacent to vertices with colors $5,6$, and 7. Then recoloring y_1 with 3 yields a rainbow coloring of G.

Case 4.2: $|B| = 2$. If $1 \in B$ or $2 \in B$, then the argument of Case 4.1 holds. Assume that $B = \{3,4\}$. Similarly to Case 4.1, we may assume that for $i = 1,2$ and $j = 1,2,3$, y_i is adjacent to a vertex $y_{i,j}$ of color $j+4$ that is adjacent to another vertex of color 1 (in particular, y_1 and y_2 may have a common neighbor of color $j+4$).

If y_1 is rainbow, then uncoloring y_1 and coloring x with 7 gives Case 1 or Case 2. Thus we may assume that y_1 and (by symmetry) y_2 are not rainbow. So, we may assume that for $i = 1,2$, the fourth neighbor $y_{i,4}$ of y_i distinct from x has color $c_i \in \{5,6,7\}$. By symmetry, we may assume that $c_1 = 5$. Similarly to Case 1.3, by Lemma 4.2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another rainbow coloring f' of $G - xy_1$. If $f'(y_1) \in \{3,4,5,6,7\}$, then f' is a rainbow coloring of G, as claimed. So, we may assume that $f'(y_1) = 2$. Now practically repeating the argument of Case 1.3, we find a promised coloring.

Case 4.3: $|B| = 3$ (see Figure 4.1 on the left). If $2 \in B$, then we can repeat the argument of Case 4.2 for $B' = B - \{2\}$. Hence we may assume that $B \subseteq \{1,3,4,5,6,7\}$.

Case 4.3.0: $1 \in B$. Let $B = \{1,3,4\}$. Then some color in $\{5,6,7\}$, say 7, is not present on $N(y_4) \cup N(y_5)$. Again, we may assume that for $i = 1,2$ and $j = 1,2,3$, y_i is adjacent to a vertex $y_{i,j}$ of color $j+4$ that is adjacent to another vertex of color 1. If y_1 is rainbow, then we may uncolor y_1 and color x with 7 to get Case 1 or Case 2. Suppose now that y_1 and y_2 are not rainbow. By Lemma 4.2.1 applied to y_1 in $G - xy_1$, we can recolor y_1 and at most one other vertex (a neighbor of y_1) to obtain another admissible



Case 4.3. Case 4.3.4.3.

Figure 4.1

coloring f'. If $f'(y_1) \in \{3,4,5,6,7\}$, then f' is a rainbow coloring, as claimed. So, we may assume that $f'(y_1) = 2$. But then we can use the argument of Case 4.2 with the roles of y_3 and y_2 switched. This proves Case 4.3.0.

So, from now on, $B = \{3,4,5\}$. For $i = 4,5$ and $j = 1,2,3$, let $y_{i,j}$ be the neighbor of y_i of color $j+2$. We write *the* neighbor, since y_4 and y_5 are rainbow. As observed at the beginning of Case 4, y_4 and y_5 each have another uncolored neighbor, call them $y_{4,4}$ and $y_{5,4}$. In particular, y_4 and y_5 have no neighbors colored with 6 or 7. If x can be colored with either of 6 or 7 without creating a two-colored cycle, then we obtain a rainbow coloring. Hence we assume that for $i = 1,2$ and $j = 1,2$, $f(y_{i,j}) = j+5$ and $y_{i,j}$ has a neighbor of color 1 distinct from y_i.

Case 4.3.1: One of y_1 or y_2, say y_1, is rainbow. If $y_{4,4}$ has no neighbor of color $c \in \{6,7\}$, then we can color y_4 with c, a contradiction to the maximality of f. If $y_{4,4}$ has no neighbor of color $c' \in \{1,2\}$, then by uncoloring y_1 and coloring y_4 with c' and x with 6, we obtain an admissible coloring larger than f. So, $f(N(y_{4,4})) = \{1,2,6,7\}$. Then we may color $y_{4,4}$ with 3 and uncolor y_1 to get a new admissible coloring as large as f with one defective vertex y_4, for which Case 2 holds. This finishes Case 4.3.1.

So, below y_1 and y_2 are not rainbow and hence each of them is adjacent to at least three colored vertices.

Case 4.3.2: One of y_1 or y_2, say y_1, is adjacent to an uncolored vertex $y_{1,4} \neq x$. We may assume that $f(y_{1,1}) = f(y_{1,2}) = 6$ and $f(y_{1,3}) = 7$. First, we try to color x with 7 and y_1 with 3. Since the new coloring has at most one defective vertex, we may assume that a two-colored cycle is created. Hence each of $y_{1,1}$ and $y_{1,2}$ is adjacent to a vertex of color 3. The same argument gives these vertices neighbors of colors 4 and 5. Recall that one of $y_{1,1}$ and $y_{1,2}$, say $y_{1,1}$, has another neighbor of color 1. Then recoloring $y_{1,1}$ with 2 gives an admissible coloring in which y_1 is rainbow. Hence Case 4.3.1 applies to this new coloring.

So, from now on each of y_1 and y_2 has 4 colored neighbors. Since y_1 is admissible we may assume w.l.o.g. that y_1 is adjacent either to the colors $5, 6, 6, 7$ or the colors $5, 5, 6, 7$.

Case 4.3.3: y_1 has one neighbor of color 5 and three neighbors with colors 6 or 7. We may assume that $f(y_{1,1}) = 5$, $f(y_{1,2}) = f(y_{1,3}) = 6$, and $f(y_{1,4}) = 7$. If coloring y_1 with 3 or 4 yields an admissible coloring, then we are done; so we may assume that a two-colored cycle is formed in each case. It follows that each of $y_{1,2}$ and $y_{1,3}$ has neighbors colored with 3 and 4. By the symmetry between $y_{1,2}$ and $y_{1,3}$, we may assume that $y_{1,3}$ has a neighbor of color 1 other than y_1. If $y_{1,3}$ is almost rainbow, then we can uncolor it, recolor y_1 with 3, and color x with 7: this will give an admissible coloring with the same number of colored vertices as in f, and the only defective vertex $y_{1,3}$. Then either Case 1 or Case 2 applies to this new coloring. Hence we may assume that $y_{1,3}$ has two neighbors other than y_1 that receive the same color. Then since $y_{1,3}$ has no neighbor of color 2, y_1 may now be recolored with color 2 without creating a bicolored cycle. Repeating the above argument we derive that $y_{1,2}$ has neighbors of colors $2, 3$, and 4, and one of these colors appears twice on $N(y_{1,2}) - y_1$. By Lemma 4.2.1 applied to $y_{1,3}$ in the graph $G - y_{1,3}y_1$ for the original coloring, we can change its color and the color of at most one other vertex (that is a neighbor of $y_{1,3}$, all of whose neighbors are colored) to get an admissible coloring of $G - y_{1,3}y_1$. Since y_2 and y_3 are adjacent to the uncolored vertex x, their colors are not changed. If $y_{1,3}$ receives color 1, then we recolor y_1 with 3 and get a rainbow coloring of G. If $y_{1,3}$ receives a color other than 1, then we color x with 6 and again get a rainbow coloring of G.

Case 4.3.4: y_1 has two neighbors of color 5 (see Figure 4.1 on the right). We may assume that $f(y_{1,1}) = f(y_{1,2}) = 5$, $f(y_{1,3}) = 6$, and $f(y_{1,4}) = 7$. If y_1 can be recolored with either 3 or 4, this would give a rainbow coloring f'. Hence we assume that both of $y_{1,1}$ and $y_{1,2}$ are adjacent to vertices with colors 3 and 4.

Case 4.3.4.1: One of $y_{1,1}$ or $y_{1,2}$, say $y_{1,1}$, is rainbow. Then uncoloring $y_{1,1}$ and coloring y_1 with 3 and x with 7 yields either a rainbow coloring f' or a new admissible coloring (with the same number of colored vertices) with the defective vertex $y_{1,1}$ and admissible colored neighbor y_1. In the former case, we are done. In the latter, if one of the previous cases occurs, then we are done again. So, we may assume that Case 4.3.4 occurs. By the symmetry between colors 3 and 4, we may assume that apart from y_1, vertex $y_{1,1}$ has a neighbor of color 3, a neighbor of color 4, and two uncolored neighbors, say z_1 and z_2, each of whose has another uncolored neighbor and 3 colored neighbors. Moreover, the same 3 colors appear on the neighborhoods of z_1 and z_2, and since Case 4.3.4 holds, by the symmetry between colors 6 and 7, both of them are among these 3 colors. Then either coloring $y_{1,1}$ with 1 yields a rainbow coloring or coloring $y_{1,1}$ with 2 does.

Case 4.3.4.2: Each of $y_{1,1}$ and $y_{1,2}$ has a neighbor of color 2 that has another neighbor of color 5. Since $y_{1,1}$ is not rainbow, the fourth neighbor of $y_{1,1}$ has color $c \in \{2, 3, 4\}$. Since y_1 cannot be recolored with 3 or 4, some neighbor, say r, of $y_{1,1}$ of color c has another neighbor of color 5. If in the graph $G - y_1 y_{1,1}$, $y_{1,1}$ can be recolored with 1, then we may recolor y_1 with 3 and get a rainbow coloring of G. If $y_{1,1}$ can be recolored with either of 6 or 7, then we have Case 4.3.3. To disallow coloring $y_{1,1}$ with $1, 6$, and 7, r must be adjacent to vertices with each of these colors. By the symmetry between colors 3 and 4, we assume that $f(r) \neq 4$. If the neighbor r' of $y_{1,1}$ with $f(r') = 4$ has no neighbor of color $c' \in \{6, 7\}$, then we recolor r with 4 and $y_{1,1}$ with c' thus getting Case 4.3.3. If r' has no neighbor of color 1, then we recolor r with 4, $y_{1,1}$ with 1, and y_1 with 3 obtaining a rainbow coloring. Finally if $f(N(r') - y_{1,1}) = \{1, 5, 6, 7\}$, then we recolor r' with 3, $y_{1,1}$ with 4, and y_1 with 3.

The last subcase is:

Case 4.3.4.3: $y_{1,1}$ has no neighbor of color 2 that has another neighbor of color 5. Then recoloring y_1 with 2 creates another admissible coloring f'. We may then repeat our previous argument with y_3 playing the role of y_2 to conclude that y_3 has neighbors of color 6 and 7. If y_3 is admissible, then repeating the above argument we conclude that y_3 may be recolored with color 1 in the original coloring f. Then after this recoloring, by coloring x with 2 we get a rainbow coloring. Also, if y_2 is admissible in f, then we may recolor both of y_1 and y_2 with 2 and color x with 1 to get a rainbow coloring. Hence we may assume that all the neighbors of y_2 and y_3 apart from x are colored with 6 or 7. Recall that for $i = 4, 5$ and $j = 1, 2, 3$, $f(y_{i,j}) = j + 2$ and $y_{i,4}$ is uncolored. If for some $i \in \{4, 5\}$, $y_{i,4}$ has no neighbor of color $c \in \{6, 7\}$, then we can color y_i with c and get a better admissible coloring. Since none of y_1, y_2, or y_3 has a neighbor with color 3, if $y_{4,4}$ has no neighbor of color 1 or $y_{5,4}$ has no neighbor of color 2, then by coloring y_4 with 1, y_5 with 2 and x with 3 creates an admissible coloring with more colored vertices. By the symmetry between colors 1 and 2, each of $y_{4,4}$ and $y_{5,4}$ has neighbors of colors $1, 2, 6$, and 7.

If $y_{4,1}$ does not have a neighbor of color $c' \in \{1, 2, 6, 7\}$, then coloring $y_{4,4}$ with 3, y_4 with c' and x with 4 yields an admissible coloring. Otherwise, we recolor $y_{4,1}$ with 4 and color y_4 with 3. This proves the lemma. $\square$

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