ANALYSIS IN THE HEISENBERG GROUP: WEAK S-JOHN DOMAINS AND THE DIMENSIONS OF GRAPHS OF HÖLDER FUNCTIONS

BY

JOHN MICHAEL MAKI

DISSERTATION

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Doctoral Committee:

Professor Jang-Mei G. Wu, Chair
Associate Professor Jeremy T. Tyson, Director of Research
Professor John P. D'Angelo
Assistant Professor Sergiy A. Merenkov

In this thesis, we provide connections between analytic properties in Euclidean $\mathbb{R}^n$ and analytic properties in sub-Riemannian Carnot groups. We introduce weak s-John domains, in analogy with weak John domains, and we prove that weak s-John is equivalent to a localized version. This is applied in showing that a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^3$ will be a weak s-John domain in the first Heisenberg group. This result is sharp, giving a precise value of s that depends only on α. We follow upon this by showing that a weak s-John domain in a general Carnot group will be a (q,p)-Poincaré domain for certain p and q that depend only on s and the homogeneous dimension of the Carnot group. The final result gives, in a general Carnot group, an upper bound on the lower box dimension of the graph of an Euclidean Hölder function, with application to the dimension of a Sobolev graph.

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CHAPTER 1

Introduction

1.1 Main results

Carnot groups, which are described in detail in Chapter 2, are sub-Riemannian spaces that provide a rich setting for analysis. One by-product of their unusual geometry, however, is that direct calculation is often more difficult than in a Riemannian space. One strategy for circumventing this problem is to bootstrap our way from properties in Euclidean $\mathbb{R}^n$ to sub-Riemannian properties.

Monti and Morbidelli used this idea, proving that bounded domains with Euclidean $C^{1,1}$ boundary in two-step Carnot groups are "non-tangentially accessible" in the intrinsic Carnot geometry [49, Theorem 3.2]. In our first theorem, we establish a similar statement that yields weak s-John domains in the Heisenberg group $\mathbb{H}^1$. Weak s-John domains are defined in analogy with weak John domains [32, Section 9.1]:

Definition 1.1.1. Let Ω be a bounded domain in a metric space X, and let $s \geq 1$. We say Ω is a *weak s-John domain* if there exists a constant $\lambda > 0$ and a point $x_0 \in \Omega$ such that, for every point $x \in \Omega$, there exists a continuous curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda\, d(\gamma(t), x)^s \tag{1.1.1}$$

for every $t \in [0, 1]$.

We make this definition easier to verify by establishing one particularly useful tool (Corollary 3.2.13), which shows that, in a broad class of spaces, it is sufficient to check that (1.1.1) holds for points arbitrarily close to the boundary. This result also leads to a similar statement for John domains (Corollary 3.2.15), although we are not certain the John domain result is new.

Theorem 1.1.2. *Let Ω be a bounded $C^{1,\alpha}$ domain in Euclidean $\mathbb{R}^3$, $0 < \alpha \leq 1$. Then Ω is a weak s-John domain in $\mathbb{H}^1$ for $s \geq 2/(\alpha+1)$.*

This theorem is sharp: there exist $C^{1,\alpha}$ domains in $\mathbb{R}^3$, for any $\alpha \in (0,1)$, which fail to be weak s-John domains in $\mathbb{H}^1$ for all $s < 2/(\alpha+1)$ (see Example 3.4.1).

It is natural that (weak) s-John domains would arise in a sub-Riemannian setting. The sub-Riemannian metric does not scale linearly in all directions, which can lead to smooth curves (in Euclidean geometry) exhibiting cusp-like behavior within the Carnot-Carathéodory geometry. As s-John domains permit some types of cusps, they can appear as a natural consequence of Euclidean regularity hypotheses.

To demonstrate the usefulness of the first theorem, we establish a Poincaré inequality with weak s-John as a hypothesis:

Theorem 1.1.3. *Let $\mathbb{G}$ be a Carnot group with homogeneous dimension Q, and let $\Omega \subset \mathbb{G}$ be a weak s-John domain, for some $s \in (1, Q/(Q-1))$. Then Ω is a (q,p)-Poincaré domain for each p and q satisfying $1 \leq p < \frac{Q}{Q-(Q-1)s}$ and $p \leq q < \frac{Qp}{Q-p(Q-(Q-1)s)}$.*

The definition of a (q,p)-Poincaré inequality can be found at the beginning of Section 4.1. In an expository paper [34], Heinonen highlights Poincaré inequalities as critical tools for analysis on non-smooth spaces. These inequalities serve a role similar to the Fundamental Theorem of Calculus, providing information about a function through knowledge of its derivative.

Theorem 1.1.3 mirrors a result by Hajłasz and Koskela for s-John domains in $\mathbb{R}^n$ [31, Corollary 5]. In Chapter 4, we offer an alternate proof of their result in $\mathbb{R}^n$ (sans borderline exponents), using an approach adapted from [40]. Further adaptations enable us to prove a similar result for general Carnot groups.

In our final result, we bootstrap from Euclidean smoothness of a function to the sub-Riemannian (lower box) dimension of its graph set, contributing to a program of similar questions. Consider a function $F\colon \mathbb{R}^2 \to \mathbb{R}$. It is known that, if F is Euclidean C^1, then its graph has Hausdorff dimension *exactly* three when considered as a subset of the Heisenberg group $\mathbb{H}^1$ (a consequence of Pansu's isoperimetric inequality [51]). It was shown in [4] that if F is Lipschitz, then its graph will have Hausdorff dimension *at least* three in $\mathbb{H}^1$. Yet, there are examples of functions of bounded variation whose graphs have dimension two [4, 7]. It is natural to ask how the dimensions of graphs interpolate between these results, particularly as there is a class of functions, Sobolev functions, whose regularity lies between Lipschitz and BV.

Our approach to this question is to extend a result established in $\mathbb{R}^n$: Cuzick [22] proved estimates for the lower box dimensions of graphs of Euclidean Hölder functions in $\mathbb{R}^n$. We address the same question in Carnot-Carathéodory geometry and obtain the following result:

Corollary 1.1.4. *Let $F : \Omega \to \mathbb{R}$ be an Euclidean α-Hölder function, defined on a bounded set Ω in $\mathbb{R}^2$. Let $\mathrm{Gr}\, F = \{(x, F(x)) \mid x \in \Omega\}$ in $\mathbb{R}^3$ be the graph of F. Then*

$$\text{CC-}\underline{\dim}_B \, \mathrm{Gr}\, F \leq 4 - \alpha,$$

where we have viewed $\mathrm{Gr}\, F$ as a subset of the Heisenberg group $\mathbb{H}^1$ equipped with its Carnot-Carathéodory metric.

An upper bound on the dimensions of Hölder graphs yields an upper bound on the graphs of supercritical Sobolev functions. Recent work by Magnani [43] in this setting has provided a sharp lower bound of three for the dimension of graphs of Sobolev functions. Both our upper bound and Magnani's lower bound are independent of which coordinates are used for the domain and which is used for the range. While we currently lack examples to demonstrate sharpness of our upper bound, the program has still been substantially advanced.

1.2 Historical background

In 1961, Fritz John [35] studied conditions under which bi-Lipschitz mappings are good approximations of isometries. The appropriate setting — now called a John domain — has proven useful in many other capacities, and similarly-conceived domains have also become established as fundamental.

We will use the term "geometric domains" to refer to these various classes of domains, as their definitions only rely on basic elements of geometry: lengths of curves and distance to the boundary. The importance of geometric domains comes from their ease of use, their hefty analytic implications, and their natural appearance in diverse branches of mathematics. Such domains allow us to remove specific smoothness hypotheses on the boundary, extending analysis into non-smooth settings.

John domains have appeared in many fields. There are numerous examples in quasiconformal mapping theory; for example, if the target of a quasiconformal mapping is a John domain, Heinonen [33] showed quantitatively that the mapping is also quasisymmetric and that the domain is linearly locally connected. In the study of complex dynamics, Carleson, Jones, and Yoccoz [18] characterized when the basin of attraction at infinity and the bounded Fatou components of complex polynomials of degree at least two are John domains.

John domains are particularly important in partial differential equations (PDE). John domains support Sobolev-Poincaré inequalities, essential tools for establishing regularity of solutions to PDE. Demonstrating this aspect of John domains was the work of many hands: Boman [12] introduced Boman domains (i.e., domains which satisfy the Boman chain condition); Bojarski [11] showed that Boman domains support a Sobolev-Poincaré inequality with optimal exponents; Buckley, Koskela, and Lu [15] showed that "Boman equals John" under rather general circumstances. The utility of making this last connection comes about because the John condition is typically much simpler to check than the Boman chain condition, and we receive a robust analytic tool for this light work.

The essential aspect of the definition of any geometric domain is understanding how the domain is shaped at its boundary. For a John domain, the defining condition requires escaping from the boundary toward the interior of the domain with a certain amount of elbow-room. More specifically, fix a central point in the domain as the endpoint of the escape. Then, from every other point in the domain, there needs to be an escape curve to this central point such that a "twisted cone" around the curve stays inside the set. (The escape curve acts as the central axis of the cone, so that the cone twists along with the curve. The tip of the cone is at the starting point of the curve.) The opening angles of the cones must be uniformly bounded away from zero.

This twisted cone requirement may not seem like much control, but, in the work mentioned above, each ball in a Boman chain supports a Sobolev-Poincaré inequality, and this local condition is stitched into a global condition along the chain of overlapping balls. A chain of such balls can access any point in a John domain since the chain can descend into each twisted cone. So, a minor constraint on the geometry of the boundary — the John condition — is just enough to make this work.

In Chapter 4 of this thesis, we operate in a philosophically similar manner, employing local Poincaré inequalities to guarantee certain global Poincaré inequalities hold on weak s-John domains. The mediator in our case is a Whitney decomposition, and the weak s-John condition provides crucial quantitative information about the decomposition.

A generalization of John domains are s-John domains, for $s \geq 1$ (see Definition

2.3.5). (A 1-John domain is the same as a John domain.) These domains allow a more degenerate access to the boundary, as we replace the twisted cones of John domains with "twisted power cusps." Domains with outward-pointing cusps are not John but, depending on the severity of the cusp, can be s-John domains for some s. A simple example in $\mathbb{R}^2$ is given by the bounded set $\Omega = B(0,1) \cap \{(x,y) \mid y > |x|^{1/s}\}$. This set demonstrates exactly how severe the cusp can be for an s-John domain. (It fails to be s'-John for any $s' < s$.)

A consequence of weakening the definition of John domains is a loss of some implications; for example, while Martio and Vuorinen [46] showed that the dimension of the boundary of a John domain in $\mathbb{R}^n$ is bounded away from n, Nieminen [50] demonstrated that the boundary of an s-John domain in $\mathbb{R}^n$ (for $s > 1$) can have dimension n. (In fact, he showed the stronger statement: the boundary of an s-John domain in $\mathbb{R}^n$ can have positive Lebesgue n-measure.)

In the two decades since their inception, s-John domains have not received much attention. Nevertheless, there is some profit to be obtained from them, as the results in Chapter 4 demonstrate.

Sub-Riemannian spaces (also called Carnot-Carathéodory spaces) embody the idea of constrained motion, and are the setting for our work. At each point, the allowed directions of motion are constrained to a subspace of the tangent space. The natural (Carnot-Carathéodory) distance between points is given by the shortest curve joining them that always follows allowed directions, if such a curve exists. Geodesic curves have great value in applications as they describe the most efficient path from one position to another. In applications to robotic control theory, the space describing the robot's possible configurations is a sub-Riemannian space, and geodesics result in faster/cheaper/more energy-efficient changes of state. The sub-Riemannian model applies equally well to more abstract applied settings: the precession of a Foucault pendulum is understandable as a geodesic in the parameter space of its Hamiltonian, a consequence of “Berry phase” [48]. (For an introduction to this area, the text by Montgomery [48] is an excellent starting point. The existence of “singular” geodesics in sub-Riemannian spaces is one surprise awaiting the interested reader.)

Carnot groups are a fruitful class of sub-Riemannian spaces. A Carnot group can be identified pointwise with $\mathbb{R}^n$ (for some n), but, being sub-Riemannian, it can look very different from the inside: the restrictions on allowed movement distort distances and dimensions away from Euclidean expectations. (In the special case of no restrictions on movement, the Carnot group is abelian and coincides with Euclidean $\mathbb{R}^n$.) Further, it has been shown that non-abelian Carnot groups are not bi-Lipschitzly embeddable — even locally — into any Euclidean $\mathbb{R}^n$, Hilbert space, or many other Banach spaces. (This follows from works by Pansu [52], Semmes [53], Cheeger [19], and Cheeger-Kleiner [20].) In other words, the distortion in these spaces makes them locally quite different from many familiar spaces.

This fact alone motivates some interest in these spaces. For example, in the context of general metric measure spaces, there is a program to classify spaces into bi-Lipschitz equivalence classes. As such, a non-abelian Carnot group lies in a different equivalence class than does Euclidean $\mathbb{R}^n$ and thus exemplifies a distinct geometry.

Recently, it was shown by LeDonne [42] that a sub-Riemannian space of topological dimension n embeds in $\mathbb{R}^{n+1}$ via a path isometry. This result shows that perhaps sub-Riemannian spaces and $\mathbb{R}^n$ are not immeasurably different after all. This direct analytic connection between sub-Riemannian spaces and $\mathbb{R}^n$ opens a promising direction for proving new results in sub-Riemannian spaces.

There are other good reasons to study Carnot groups. Mitchell [47] showed metric tangent cones of a general, regular sub-Riemannian space are isometric to Carnot groups. (In sub-Riemannian analysis, the tangent cone is the replacement for the concept of tangent space.) In hyperbolic geometry, Carnot groups were used in the proof of Mostow’s Rigidity Theorem; also, the boundary of complex hyperbolic space can be identified with the Heisenberg group, a relatively simple Carnot group. In PDE, Hörmander's condition, which is satisfied by Carnot groups, leads to existence of solutions of nonhomogeneous hypoelliptic equations; this overlap has led to some interplay between the topics.

While Carnot groups are attractive locations for new developments, they are difficult settings for calculations. The Carnot-Carathéodory metric mentioned above is defined somewhat abstractly, and precise formulas are not forthcoming in general. (We often abandon this natural metric in favor of a more explicit "equivalent" metric. However, even these metrics are unwieldy, due to the complicated nature of the group multiplication and the desire that any equivalent metric we use be invariant under left-multiplication.) Each non-abelian Carnot group has a unique arrangement of non-commutativity among its directions, and this variation complicates establishing specific analytic properties for a broad class of Carnot groups by direct computation. To partially avoid this mess, one strategy is to work with two pairs of glasses: one pair sees the space as Euclidean $\mathbb{R}^n$ and the other sees the Carnot structure. We have great practice in establishing analytic properties in Euclidean $\mathbb{R}^n$, so we exploit this by creating implications of the form: "Euclidean hypothesis" implies "sub-Riemannian conclusion."

Examples of this strategy abound. In [8], Balogh, Tyson, and Warhurst showed that sharp bounds on the Hausdorff and box dimensions of subset in Carnot groups can be stated based on their respective dimensions as subsets in Euclidean $\mathbb{R}^n$. Arcozzi and Ferrari [2] showed that certain Euclidean smoothnesses of surfaces in $\mathbb{R}^3$ guarantee related levels of regularity (in both Carnot and Euclidean terms) for the functions describing the Carnot-Carathéodory distance to the surfaces. As discussed in Section 1.1, the main results in this dissertation also pursue this strategy.

1.3 Organization

Chapter 2 provides the necessary background material, which can be broadly broken into three subjects: analysis on general metric spaces, geometric domains, and Carnot groups. We establish conventions of notation in Section 2.1. Section 2.2 has a few selected definitions from analysis on general metric spaces that will be needed. Section 2.3 contains the definitions, examples, and important theorems for geometric domains. Section 2.4 gives an introduction to Carnot groups and provides specific information about the Heisenberg group.

In Chapter 3, we show "$C^{1,\alpha}$ smoothness implies weak s-John" and various ancillary results. After placing our result in context in Section 3.1, we use Section 3.2 to recharacterize the weak s-John condition, making it easier to verify. In Section 3.3, we consider a motivating example set in a model domain, introduce some technical lemmas, and then prove the main theorem. Sections 3.4 and 3.5 contain noteworthy examples: the first shows that the main theorem is sharp, and the second shows that weak s-John and s-John differ in a setting where weak John and John do not.

Chapter 4 establishes an implication from weak s-John to Poincaré inequalities in Carnot groups. Such an implication was previously shown in $\mathbb{R}^n$ by Hajłasz and Koskela. Section 4.1 provides necessary background information and recalls the definition of a Whitney decomposition in $\mathbb{R}^n$. Section 4.2 offers a new proof of Hajłasz and Koskela's result in $\mathbb{R}^n$. In Section 4.3, we extend this new approach to prove the implication in general Carnot groups, utilizing a generalization of Whitney decomposition appropriate to more general spaces.

Chapter 5 tells the story of dimensions of graphs in $\mathbb{H}^1$. In Section 5.1, we discuss the current state of knowledge for this general question, including our result, which provides an upper bound on the dimensions of graphs of Euclidean Hölder functions. Section 5.2 looks at the background related to my result and summarizes my result as applied to $\mathbb{H}^1$. After some preliminary definitions in Section 5.3, we prove theorems giving upper bounds on the lower box dimension of graphs of Euclidean Hölder functions in general two-step Carnot groups.

CHAPTER 2

Background

2.1 Notations and conventions

2.1.1 Notations

We will use the following notations in this dissertation. Let (X, d) be a metric space, and let $x \in X$. We represent the *open ball* of radius r centered at x as $B(x, r)$ and a similar *closed ball* as $D(x, r)$. We will write the ϵ-neighborhood of a set K as

$$N_\epsilon(K) := \{x \in X \mid d(x, k) < \epsilon \text{ for some } k \in K\}.$$

For a set Ω, we write its *diameter* as $\operatorname{diam} \Omega$ and its *cardinality* as $\operatorname{card} \Omega$.

The distance between two sets A and B is denoted by

$$\operatorname{dist}(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}.$$

Similarly, the distance between a set A and a point x is written as

$$\operatorname{dist}(A, x) := \inf\{d(a, x) \mid a \in A\}.$$

Several different metrics on $\mathbb{R}^n$ will be used. The standard Euclidean distance between two points x and y will be denoted $d(x, y)$. Other metrics on $\mathbb{R}^n$ will be indicated via a subscript (e.g., $d_K(x, y)$). We will also use subscripts on the notations for balls, distance, and diameter (e.g., $B_K(x,r)$, $\operatorname{dist}_K(A,B)$, $\operatorname{diam}_K \Omega$) when a different metric is being used in their definitions.

For a curve $\gamma(t)$, the notation $\gamma|_{[a,b]}$ denotes the subcurve of γ formed as t ranges from a to b. For points x and y in the trace of γ, the subcurve of γ from x to y will be written $\gamma_{x,y}$. (If γ should encounter x or y multiple times, then $\gamma_{x,y}$ indicates the subcurve from the first encounter with x to the final encounter with y.) The length of a rectifiable arc γ is denoted by $l(\gamma)$. Recall that the length of a rectifiable curve γ with domain $[a,b]$ is given by

$$l(\gamma) = \sup\left\{ \sum_i d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; a = t_1 < \cdots < t_n = b \text{ is a finite partition} \right\}.$$

The class $C^{1,1}(\Omega, \mathbb{R})$ is the set of all differentiable functions from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}$ whose derivative is Lipschitz continuous. The class $C^{1,\alpha}(\Omega, \mathbb{R})$ is the set of all differentiable functions from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}$ whose derivative is Hölder continuous with exponent α.

A $C^{1,\alpha}$ domain is a domain whose boundary can be described locally as the graph of a $C^{1,\alpha}$ function. (A more formal statement is given in Definition 3.1.1.)

2.1.2 Conventions

For a curve γ, we will make no distinction between the curve and its trace, denoting both as simply γ.

In general, we will use C to indicate a constant, and we will reuse this letter from one instance to the next, even if the value of the constant has changed. In circumstances where the dependence of a constant on certain data is being established, we will use notation to indicate this dependence: for example, "$C = C(k, \alpha)$" would indicate the dependence of C on k and α. To help trace these dependencies during the course of a proof, we may employ subscripts, primes, or tildes as needed.

Finally, we will use the term *domain* to indicate an open, connected set.

2.2 Basic definitions from analysis on metric spaces

Let X be a metric measure space with metric d and measure μ, denoted (X, d, μ).

Definition 2.2.1. We say that μ is a *doubling measure* if there exists a constant $C > 0$ such that, for every $x \in X$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

holds for all $r > 0$. We call (X, d, μ) a *doubling space* if μ is a doubling measure.

Definition 2.2.2. We call (X, d, μ) an *Ahlfors Q-regular space* if μ is a Borel regular measure and there exists a constant $C > 0$ such that

$$\frac{r^Q}{C} \leq \mu(B(x, r)) \leq Cr^Q$$

holds for every $x \in X$ and $0 < r < \operatorname{diam} X$.

Definition 2.2.3. The *box dimension* (or *Minkowski dimension*) of a bounded set Ω is given by

$$\dim_B \Omega = \lim_{\delta \to 0} \frac{\log M_\delta(\Omega)}{-\log \delta}$$

where $M_\delta(\Omega)$ is the number of boxes of diameter δ required to cover Ω, if the limit exists. The *lower box dimension*, on the other hand, is always defined:

$$\underline{\dim}_B \Omega = \liminf_{\delta \to 0} \frac{\log M_\delta(\Omega)}{-\log \delta}.$$

Remark 2.2.4. The preceding definition is flexible: the boxes of diameter δ may be replaced by other sets of diameter δ, such as balls. See, for example, Falconer [24, p. 43] for several equivalent definitions.

Definition 2.2.5. The *Hausdorff s-measure* $\mathcal{H}^s$ of a set $\Omega \subset X$ is given by

$$\mathcal{H}^s(\Omega) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} \mu(U_i)^s \;\middle|\; \{U_i\} \text{ is a cover of } \Omega, \operatorname{diam} U_i < \delta \text{ for each } i \right\}.$$

Since the infimum increases as δ decreases, this limit either exists or is infinite.

Definition 2.2.6. The *Hausdorff dimension* of $\Omega \subset X$ is given by

$$\dim_H \Omega = \inf\{s \geq 0 \mid \mathcal{H}^s(\Omega) = 0\}.$$

Remark 2.2.7. For a bounded set Ω, we always have the inequality: $\dim_H \Omega \leq \underline{\dim}_B \Omega$.

2.3 Basic definitions and properties of geometric domains

The geometry of the boundary of a domain has implications for analytic questions such as existence of solutions to partial differential equations on a given domain, extension domains, compactness of function spaces, integrability of functions, and so on. As such questions have been considered in various contexts, a hierarchy of domains has arisen, classified by certain geometric behaviors of the boundary. One rendition of this hierarchy is:

$$\text{Lipschitz} \subset \text{NTA} \subset \text{uniform} \subset \text{John} \subset s\text{-John}.$$

Variations of these types (uniformly John, weak s-John, etc.) could be placed in this sequence, but this listing is sufficient to visualize the progression.

2.3.1 John domains

The type of domain that came to be called a John domain was introduced by Fritz John in 1961, in the context of approximations of bi-Lipschitz mappings [35]. The term itself was first used in a paper by Martio and Sarvas [45].

Definition 2.3.1. Let Ω be a bounded domain in a metric space X. We say Ω is a *John domain* if there exists a constant $\lambda > 0$ and a point $x_0 \in \Omega$ (the *John center*) such that, for every point $x \in \Omega$, there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda\, l\big(\gamma|_{[0,t]}\big) \tag{2.3.1}$$

for all $t \in [0, 1]$.

Remarks 2.3.2. 1. The condition (2.3.1) is often referred to as a "twisted cone condition." The twisted cone is formed by taking the union of the balls $B\big(\gamma(t), \lambda\, l\big(\gamma|_{[0,t]}\big)\big)$ over all $t \in [0, 1]$. The condition amounts to the assertion that this twisted cone lies in Ω. This recasting allows us to verify the condition by checking that each of these balls individually lies in Ω.

2. The John constant λ places a uniform lower limit on how narrow the opening angle of the twisted cone can be.

3. The specific choice of the John center x_0 is not important. If Ω is a John domain with center x_0 and if $\tilde{x}_0 \in \Omega$, then Ω is also a John domain with center $\tilde{x}_0$. The John constant λ generally will depend upon the choice of center point and will become worse (i.e., smaller) as the center point is taken closer to the boundary. (An analogous result for weak s-John domains is given by Proposition 3.2.1.)

4. The requirement that γ be rectifiable can be problematic – a metric space may have few (or no) rectifiable curves. Also, rectifiability is not preserved under

quasiconformal mappings. To bypass such issues, alternate definitions have been introduced which avoid the issue of rectifiability. Typically, the new definitions have been shown to be equivalent to the original one in many circumstances.

(a) Martio and Sarvas showed [45, Lemma 2.7] that a bounded domain $\Omega \subset \mathbb{R}^n$ is John if and only if there exists a constant $\lambda > 0$ such that, for any $x \in D$, there exists an arc γ in Ω from x to x_0 which satisfies

$$\gamma_{x,y} \subset B\left(y, \frac{1}{\lambda}\, d(y, \Omega^c)\right)$$

for all $y \in \gamma$. Capogna and Tang verified this equivalent definition applies in the setting of the Heisenberg group [17, Proposition 2.6].

(b) Hajłasz and Koskela introduced the following definition [32, Chapter 9], which also utilizes distance instead of length.

Definition 2.3.3. A bounded domain Ω is a *weak John domain* if there exists a constant λ and a point $x_0 \in \Omega$ such that, for every point $x \in \Omega$, there exists a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda\, d(x, \gamma(t)) \tag{2.3.2}$$

for all $t \in [0,1]$.

In [32, Theorem 9.6], Hajłasz and Koskela showed that weak John is equivalent to John under rather general circumstances.

(c) Another noteworthy equivalent class of domains are Boman domains, i.e., domains which satisfy a Boman chain condition. The advantages of the Boman chain condition are twofold: it is stated in terms of properties that only require a metric, and it lends itself to constructive approaches.

In [15, Theorem 3.1] it was shown that John and Boman domains coincide (at least) when the domains are proper and bounded and satisfy a "strong geodesic condition."

5. It is worth noting, in passing, that one of the alternative definitions for John allows us to define a John property for unbounded domains [41]. The idea is to abandon using a John center point and instead require any two points be joined by two twisted cones which avoid the boundary and meet at the midpoint of the John curve.

2.3.2 s-John domains

This type of domain is a generalization of John domains; in particular, some outward-pointing cusps are allowed. The concept was introduced in 1990 by Smith and Stegenga [54].

Remark 2.3.4. In a 1994 paper, Buckley and Koskela [14] refer to these as "John-α domains"; this nomenclature was likely influenced by the general use of the term "C-John domain" when the John constant needed to be explicitly referenced. Their choice did not become widespread, so one should check the definitions in a paper to identify which domain is really under consideration.

Definition 2.3.5. Let Ω be a bounded domain in a metric space X, and let $s \geq 1$. We say Ω is an *s-John domain* if there exists a constant $\lambda > 0$ and a point $x_0 \in \Omega$ such that, for every point $x \in \Omega$, there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda \, l\left(\gamma|_{[0,t]}\right)^s \tag{2.3.3}$$

for every $t \in [0,1]$.

Remarks 2.3.6. 1. A 1-John domain is a John domain, as the definitions coincide when $s = 1$.

2. The s-John condition (2.3.3) is often referred to a "twisted cusp condition."

As before, we may consider alternate definitions for s-John that are well-behaved in general spaces. We introduce the following class of domains, naming it in analogy with the weak John domains described above.

Definition 2.3.7. Let Ω be a bounded domain in a metric space X, and let $s \geq 1$. We say Ω is a *weak s-John domain* if there exists a constant $\lambda > 0$ and a point $x_0 \in \Omega$ such that, for every point $x \in \Omega$, there exists a continuous curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda\, d(\gamma(t), x)^s \tag{2.3.4}$$

for every $t \in [0,1]$.

On occasion, we will use a scale-invariant version of this definition, where the effect on the constant λ of rescaling the space X is explicitly separated from λ. The only change required for a scale-invariant definition is to replace (2.3.4) by:

$$\operatorname{dist}(\gamma(t), \Omega^c) \geq \lambda(\operatorname{diam}\Omega)^{1-s}\, d(\gamma(t), x)^s \tag{2.3.5}$$

for every $t \in [0,1]$.

Remarks 2.3.8. 1. In Proposition 3.2.1, we prove that, as with John domains, the particular choice of center point for a weak s-John domain is unimportant.

2. In Section 3.5, we show with an example that s-John and weak s-John are not equivalent in a setting where John and weak John are equivalent.

2.3.3 Uniform domains

Definition 2.3.9. A domain Ω in a metric space is a *uniform domain* if there are constants α and β such that, for each pair of points x, y in Ω, there exists a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = y$, that satisfies the two conditions:

i. $l(\gamma) \leq \alpha\, d(x,y)$, and

ii. $\min\left\{l(\gamma|_{[0,t]}), l(\gamma|_{[t,1]})\right\} \leq \beta\, \mathrm{dist}(\gamma(t), \Omega^c)$

for all $t \in [0,1]$.

Remark 2.3.10. As was the case with John domains, one could choose to use any of the many equivalent definitions of uniform domain (see [44, 57]).

Martio and Sarvas introduced uniform domains in the same paper [45] in which they named John domains. This connection was natural, as uniform domains are similarly dependent upon purely metric conditions, and their conditions build upon the strictures imposed by the John domain definition.

These refinements of John domains also possess the features that make John domains fruitful: ease of verification and numerous, widespread analytic implications. Uniform domains are extension domains for different function classes: quasiconformal functions in $\mathbb{R}^2$ and locally Hölder functions [28], Sobolev functions [37], and functions of bounded mean oscillation [29, 36]; in the last case, uniform domains were shown to be precisely the extension domains for BMO functions. It should seem quite natural that geometric domains were put to this use: extending a function from a subset to the entire space requires being able to control and predict the behavior of the function near the boundary in order to preserve the function's regularity. Controlling the actual geometry of the boundary lets us control the behavior of the function.

Gehring [26, 27] has collected a few dozen implications of being a uniform domain. For example, uniform domains are linearly locally connected and satisfy a useful quasihyperbolic boundary condition, and the set of proper, simply connected uniform domains in $\mathbb{R}^2$ can be identified with the set of quasidisks. (In particular, interactions with complex analytic theory in $\mathbb{R}^2$ have made for a rich collection of results there.) More recently, Bonk, Heinonen, and Koskela [13] showed a deep connection between uniform domains and Gromov hyperbolic spaces.

2.4 Carnot groups

We now define Carnot groups. In Section 2.4.1, I will provide the background necessary to make sense of this definition. Basic analytic tools in Carnot groups are discussed in Section 2.4.2. Finally, we specifically consider the Heisenberg group in Section 2.4.3.

Definition 2.4.1. A *Carnot group* is a homogeneous, connected, simply connected Lie group with a nilpotent, graded Lie algebra.

Remark 2.4.2. Carnot groups are sometimes defined as having a *stratified* Lie algebra instead of a *graded* Lie algebra. These terms are used interchangeably in the literature.

2.4.1 Lie groups

We start by recalling the definition of a Lie group. The following definitions and basic results can be found in most standard texts on Lie groups.

Definition 2.4.3. A *Lie group* is a group with a smooth manifold structure such that the group operation and its associated inverse are C^∞-smooth operations.

I will limit my discussion to real matrix Lie groups, i.e., Lie groups which can be realized as a subgroup of $GL(n, \mathbb{R})$ for some n, since all Carnot groups are of this type. Here, the group operation is matrix multiplication, and the inverse operation is the matrix inverse.

Definition 2.4.4. The *Lie algebra* $\mathfrak{g}$ of a matrix Lie group $\mathbb{G}$ is a real vector space that is isomorphic to the tangent space at the group identity, along with an operation, the *Lie bracket* $[\cdot, \cdot]$:

$$[X, Y] := XY - YX$$

for all X, $Y \in \mathfrak{g}$.

Remark 2.4.5. The Lie bracket is bilinear, antisymmetric, and satisfies Jacobi's equation:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

We will develop the Carnot groups by building them from their Lie algebras. For this purpose, imagine that we are beginning with only a Lie algebra, i.e., an n-dimensional real vector space with basis $\{X_i\}$ and with values chosen for the n^2 Lie brackets $\{[X_i, X_j]\}$ such that the conditions in the above remark are satisfied.

Definition 2.4.6. The *exponential map* for matrix Lie groups, $\exp\colon \mathfrak{g} \to \mathbb{G}$, is

$$\exp(A) = I + \sum_{i=1}^{\infty} \frac{1}{i!} A^i.$$

The exponential map is the bridge from the Lie algebra to the Lie group. For a connected, simply connected Lie group, exp is a diffeomorphism onto the group. So, given a Lie algebra, we can generate the set of elements for the associated Lie group via the exponential map.

Now, given elements $\exp(X)$ and $\exp(Y)$ in the Lie group, we need to define the Lie group operation, $\exp(X) \cdot \exp(Y)$. The *Baker-Campbell-Hausdorff (BCH) formula* shows us how to infer this group operation from the Lie brackets:

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + R(X, Y)\right) \tag{2.4.1}$$

where $R(X, Y)$ is an infinite weighted sum of higher-order (i.e., iterated) brackets in X and Y.

This would be an awkward definition for the group operation if $R(X, Y)$ was an infinite sum. In the case of Carnot groups, the property of nilpotency saves the day.

Definition 2.4.7. A Lie algebra $\mathfrak{g}$ is *nilpotent* if there exists an integer k such that the k-iterated Lie brackets are zero for all choices of entries from $\mathfrak{g}$. A Lie group is *nilpotent* if its Lie algebra is nilpotent.

For Carnot groups, the infinite sum $R(X, Y)$ has only finitely many non-zero terms, which makes this a viable method of determining the group operation.

Carnot groups have additional structure on their Lie algebra, which will have significant implications for the analytic features on the spaces. Specifically, their Lie algebras are graded.

Definition 2.4.8. A Lie algebra $\mathfrak{g}$ is *graded* (or *stratified*) if it admits a vector space decomposition $\mathfrak{g} = v_1 \oplus ... \oplus v_k$ where $[v_1, v_j] = v_{j+1}$ for all $j \geq 1$, with $v_j = \{0\}$ for $j > k$. For such a graded Lie algebra, the *step* or *depth* of the grading is k.

Remark 2.4.9. Note that graded Lie algebras are necessarily nilpotent.

Definition 2.4.10. The operation of multiplication from the left in a Carnot group is called *left-translation*. We write L_g to denote left-translation by $g \in \mathbb{G}$. Hence, $L_g(h) := g \cdot h$, for every $h \in \mathbb{G}$.

Remark 2.4.11. The name "left-translation" is intended to be suggestive. In Euclidean $\mathbb{R}^n$ (a simple example of a Carnot group), left-multiplying by the element

$(x_1, \ldots, x_n)$ translates the space by the vector $\langle x_1, \ldots, x_n \rangle$. In any Carnot group, left-translation is the fundamental method for re-centering the space.

Definition 2.4.12. An operation ϕ which acts on subsets of a Lie group $\mathbb{G}$ is *left-invariant* if $\phi(H) = \phi\big(L_g(H)\big)$ for any element $g \in \mathbb{G}$ and any subset $H \subset \mathbb{G}$.

Remark 2.4.13. In a connected, simply connected Lie group, we may view the Lie algebra as the set of all left-invariant vector fields on the manifold.

We will explicitly demonstrate how this equivalent form is generated for a simple Carnot group, the Heisenberg group, in Section 2.4.3. (Essentially, we will use left-multiplication to carry a vector from the tangent space at the identity to vectors elsewhere in the space. Since the vector field was created using left-translation, it will be left-invariant.)

Definition 2.4.14. A Lie group $\mathbb{G}$ is *homogeneous* if left-translation is transitive on the group: that is, for every two elements x, $y \in \mathbb{G}$, there exists an element $g \in \mathbb{G}$ such that $L_g(x) = y$.

The effect of being homogeneous is, roughly speaking, that no point is distinguished. Every point is just a left-translation different from any other, which also means they all have a similarly structured tangent space, observed by left-translating the tangent space at the origin to each point in the space.

Left-invariance is a fundamentally useful tool in Carnot groups. If a point $g \in \mathbb{G}$ is under consideration, we may left-translate the space by g^{-1} and do our work at the origin. This manipulation would not be useful if our analytic results were disrupted by this action; indeed, we expressly choose our measures, metrics, norms, etc. to be left-invariant for this reason.

Example 2.4.15. Euclidean $\mathbb{R}^n$ is a one-step Carnot group. The Lie brackets in its Lie algebra are all zero, which makes the space abelian. The converse is also true: an abelian Carnot group is Euclidean $\mathbb{R}^n$ for some n. Consequently, most of our interest is in non-abelian Carnot groups.

One more basic operation on a graded Lie algebra is dilation, which we can push forward to an operation on the group via the exponential map.

Definition 2.4.16. The dilation $\tilde{\delta}_t \colon \mathfrak{g} \to \mathfrak{g}$, $t > 0$, is a Lie algebra automorphism such that $\tilde{\delta}_t(X) := t^i X$, where $X \in v_i$. In the Lie group, we have the related dilation $\delta_t \colon \mathbb{G} \to \mathbb{G}$, defined as $\delta_t := \exp \circ \tilde{\delta}_t \circ \exp^{-1}$.

Remark 2.4.17. Once you have defined the dilation on v_1 as mapping X to tX, which is natural, the definition on v_i, $i > 1$, is completely determined by dilation being a Lie algebra automorphism (such maps must interact in a specific manner with the Lie brackets, and the given definition follows).

2.4.2 Analysis on Carnot groups

With the underlying Lie structure of the Carnot groups now established, we can define analytic concepts for the space that interact well with its Lie structure. Consider a Carnot group $\mathbb{G}$ with Lie algebra $\mathfrak{g}$. As in the definition in the previous section, let $v_1 \oplus ... \oplus v_k$ be a stratification of the Lie algebra $\mathfrak{g}$. Here we will consider v_1, not as a subspace of $T_0(\mathbb{G})$, but as the collection of left-invariant vector fields on $\mathbb{G}$ created from the vectors in that subspace.

Notation 2.4.18. Let $p \in \mathbb{G}$. We will write $(v_1)_p$ to indicate the subspace of $T_p(\mathbb{G})$ spanned by the left-invariant vector fields in v_1 evaluated at p.

Definition 2.4.19. The *horizontal distribution* on $\mathbb{G}$ is the linear subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ given by

$$H\mathbb{G} = \{(p, V) \mid p \in \mathbb{G}, V \in (v_1)_p\}.$$

Definition 2.4.20. A curve $\gamma\colon [a,b] \to \mathbb{G}$ is *horizontal* if $\gamma'(t) \in (v_1)_{\gamma(t)}$ for all $t \in [a,b]$. In other words, γ is horizontal if it is tangent to the horizontal distribution everywhere.

Remark 2.4.21. In the above definition, it is sufficient that γ be piecewise C^1.

Fix a non-degenerate, positive definite inner product $\langle \cdot, \cdot \rangle_H$ defined on v_1.

Definition 2.4.22. Let $\gamma\colon [a,b] \to \mathbb{G}$ be an absolutely continuous, horizontal curve. The length of γ is given by

$$l(\gamma) = \int_a^b \Big(\big\langle \gamma'(t), \gamma'(t) \big\rangle_H \Big)^{1/2} \, dt.$$

Definition 2.4.23. A curve $\gamma\colon [a,b] \to \mathbb{G}$ is a *geodesic* in $\mathbb{G}$ if it is the shortest horizontal curve joining $\gamma(a)$ and $\gamma(b)$.

It is clear that we are distinguishing the first level of the stratification; the purpose is illuminated by the following definition and theorem.

Definition 2.4.24. Let M be a manifold, and let $K \subset TM$ be a distribution (i.e., a linear subbundle). Let $\{X_i\}$ be a local frame for K. Then K is *bracket-generating* if $\{X_i\}$ together with all iterated Lie brackets of elements from $\{X_i\}$ span TM.

Theorem 2.4.25 (Chow-Rashevsky)**.** *If K is a bracket-generating distribution on a connected manifold M, then any two points in M can be connected with a horizontal (i.e., tangent to K) curve.*

From the definitions, we can see that v_1 in our stratification is bracket-generating, and hence any two points in $\mathbb{G}$ can be connected by *some* horizontal curve. This fact leads us to define a natural distance in $\mathbb{G}$.

Definition 2.4.26. The *Carnot-Carathéodory metric* d_{CC} on $\mathbb{G}$ is

$$d_{CC}(p,q) := \inf\{ l(\gamma) \mid \gamma \text{ horizontal, joining } p \text{ to } q\}$$

for points $p,\, q \in \mathbb{G}$.

Remark 2.4.27. The metric d_{CC} respects dilations of $\mathbb{G}$:

$$d_{CC}\big(\delta_t(p), \delta_t(q)\big) = t\, d_{CC}(p, q)$$

for all $p, q \in \mathbb{G}$.

Remark 2.4.28. On Carnot groups, there is always a bi-invariant measure defined, the *Haar measure*. For a Carnot group $\mathbb{G}$ with topological dimension n, this measure is equivalent to Lebesgue n-measure. The Haar measure respects dilations, which leads to the following useful consequences:

- Carnot groups are Ahlfors Q-regular spaces, where

  $$Q = \sum_{i=1}^{k} i\, \dim v_i.$$

  The value Q is called the *homogeneous dimension* of the Carnot group.

- For any Carnot-Carathéodory ball $B_{CC}(x, r)$ in $\mathbb{G}$ and Haar measure μ, we have $\mu\big(B_{CC}(x, r)\big) = r^Q\, \mu\big(B_{CC}(0, 1)\big)$.

Definition 2.4.29. Fix a basis $\{X_1, ..., X_n\}$ for the Lie algebra $\mathfrak{g}$. We define *canonical coordinates of the first kind* in $\mathbb{G}$ as $(a_1, ..., a_n) \leftrightarrow \exp(a_1 X_1 + ... + a_n X_n)$.

2.4.3 The Heisenberg group

Now we will focus more precisely on the Heisenberg group, which will be the setting for the work in Chapter 3.

Definition 2.4.30. The n^{th} *Heisenberg group* $\mathbb{H}^n$ has topological dimension $2n + 1$, and its non-trivial Lie brackets are $[X_i, Y_i] = T$ for all $i \in 1, ..., n$ for the Lie algebra basis $\{X_1, ..., X_n, Y_1, ..., Y_n, T\}$.

Remark 2.4.31. We see that $\mathbb{H}^n$ has grading $v_1 = \text{span}\{X_1, ..., X_n, Y_1, ..., Y_n\}$ and $v_2 = \text{span}\{T\}$, hence all Heisenberg groups are two-step Carnot groups.

We will concern ourselves primarily with the first Heisenberg group, $\mathbb{H}^1$. An alternate way to define $\mathbb{H}^1$ is as the unique non-abelian Carnot group of topological dimension three. For topological dimensions one and two, no non-abelian Carnot groups exist, which makes $\mathbb{H}^1$ a common initial testing ground for exploring consequences of non-commutativity.

Let $\{X, Y, T\}$ be a basis for the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^1$. Note $[X, Y] = T$ is the only non-trivial Lie bracket, and $\mathfrak{h}$ has grading $v_1 = \text{span}\{X, Y\}$ and $v_2 = \text{span}\{T\}$. Consequently, dilations on $\mathfrak{h}$ by a factor of r will scale X and Y by r and T by r^2. This yields the dilation function in the group $\mathbb{H}^1$: $\delta_r(x, y, t) = (rx, ry, r^2t)$. We define an inner product $\langle \cdot, \cdot \rangle_H$ on v_1 in the obvious way:

$$\langle a_1X + b_1Y, a_2X + b_2Y\rangle_H := a_1a_2 + b_1b_2. \tag{2.4.2}$$

If we use canonical coordinates of the first kind on $\mathbb{H}^1$ (see Definition 2.4.29), we obtain the group operation:

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(xy' - yx')\right). \tag{2.4.3}$$

Remark 2.4.32. Left-translation L_g in $\mathbb{H}^1$ is an affine map. Let $g = (x, y, t)$ and $p = (x', y', t')$. Then the group operation given above gives:

$$L_g(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y/2 & x/2 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ t' \end{bmatrix} + \begin{bmatrix} x \\ y \\ t \end{bmatrix}.$$

Note that the t-direction is the only one that experiences a skew under this map.

Notation 2.4.33. There are two common alternate notations for points in $\mathbb{H}^1$.

1. The first alternate notation uses x_1, x_2, and x_3 in place of x, y, and t, respectively.

2. The second employs complex notation. Let the xy-plane be viewed as $\mathbb{C}$, and set $z = x + iy$; the value t remains a real number. Then points in $\mathbb{H}^1$ have the form (z, t), and the group operation (2.4.3) becomes:

$$(z,t)\cdot(z',t') = \left(z + z', t + t' - \frac{1}{2}\operatorname{Im}(z\cdot\bar{z}')\right).$$

Remark 2.4.34. As mentioned in Remark 2.4.13, the Lie algebra of a Carnot group is isomorphic to the tangent space at the identity, but is also isomorphic to the set of left-invariant vector fields in the group. We will now take the time to demonstrate this equivalence explicitly in $\mathbb{H}^1$. Note that the identity in the group is the origin, and a basis for the tangent space there would be: $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, and $T = \frac{\partial}{\partial t}$.

Consider X, and choose a point $g = (x, y, t) \in \mathbb{H}^1$. First, we represent the vector in a form which lies in the group. Let $\gamma\colon [-1, 1] \to \mathbb{H}^1$, $\gamma(s) = (s, 0, 0)$. Note that $\gamma(0) = 0$ and $\gamma'(0) = \frac{\partial}{\partial x}$.

We define the left-translation of X essentially by pulling back left-translation in the group:

$$(L_g)_*(X) := \left.\frac{d}{ds}L_g(\gamma(s))\right|_{s=0}.$$

With this straightforward calculation, we find $X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t}$. Similar calculations for Y and T yield their left-invariant vector fields:

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t},$$

$$Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t},$$

$$T = \frac{\partial}{\partial t}.$$

Remark 2.4.35. A consequence of the descriptions for X, Y, and $\langle \cdot, \cdot \rangle_H$ on $H\mathbb{H}^1$ is that the length of a horizontal curve is equal to the length of its orthogonal projection into the xy-plane.

Example 2.4.36. Geodesics in $\mathbb{H}^1$ are both known and easily described, which is unusual for Carnot groups. Consider a circle in $\mathbb{R}^2$ that passes through the origin, and let $c(t) = \big(x(t), y(t)\big)$ describe an arc of this circle. Let $z(t)$ be a solution of the ODE $z'(t) = \frac{1}{2}\big(x(t)y'(t) - x'(t)y(t)\big)$. Then the curve $\gamma(t) = \big(x(t), y(t), z(t)\big)$ is a geodesic in $\mathbb{H}^1$. The arbitrary choice of integration constant when finding $z(t)$ indicates that any geodesic can be shifted vertically and remain a geodesic (albeit between new endpoints).

In the limiting case where the radius of the circle is infinite, $c(t)$ is a line through the origin, and the related curve $\gamma(t)$ is again a geodesic.

For an insightful derivation of this example, see [48, Chapter 1].

Remark 2.4.37. In practice, the CC metric can be awkward to use. For example, the unit ball using this metric, $B_{CC}(0,1)$, is unusual to our Euclidean eyes: it is not convex and has two cusps. Explicitly describing the geodesics required is possible, but cumbersome.

Therefore, we often utilize two other metrics which are bi-Lipschitz equivalent to the CC metric, are left-invariant and respect dilations. Both metrics take the form $d(p,q) = \|p^{-1} \cdot q\|$, where the norm $\|\cdot\|$ is defined:

$\|(x,y,t)\|_K = \big((x^2+y^2)^2 + t^2\big)^{1/4}$ for the *Korányi* (or *gauge*) norm;

$\|(x,y,t)\|_M = \max\big\{(x^2+y^2)^{1/2}, |t|^{1/2}\big\}$ for the *max* norm.

In complex notation, the norms are

$$\|(z,t)\|_K = (|z|^4 + t^2)^{1/4} \quad \text{and} \quad \|(z,t)\|_M = \max\left\{|z|, |t|^{1/2}\right\}.$$

These "norms" are not norms, strictly speaking, since $\mathbb{H}^1$ is not a vector space. However, in the spirit of Bellaïche [9], we may view the Carnot group as similar to a vector space, using the group operation as "addition" and dilations as "scalar multiplication."

For reference, we give the explicit forms of these metrics, where $p = (x, y, t)$ and $q = (x', y', t')$:

$$d_K(p,q) = \left[\left((x'-x)^2 + (y'-y)^2\right)^2 + (t'-t)^2\right]^{1/4} \quad \text{(Korányi metric)} \tag{2.4.4}$$

$$d_M(p,q) = \max\left\{\left((x'-x)^2 + (y'-y)^2\right)^{1/2}, |t'-t|^{1/2}\right\} \quad \text{(max metric)} \tag{2.4.5}$$

(Note that $p^{-1} = (-x, -y, -t)$.)

Definition 2.4.38. A *homogeneous* norm is a continuous function $\|\cdot\| : \mathbb{G} \to \mathbb{R}$ that respects the dilation and satisfies $\|p\| = \|p^{-1}\|$.

Both of the norms given above are homogeneous; also, a norm defined using the CC metric, $\|p\|_{CC} := d_{CC}(0, p)$, is homogeneous.

Remark 2.4.39. Note that $(\mathbb{H}^1, d_{CC})$ is Ahlfors 4-regular, hence it has Hausdorff dimension four. The Haar measure on $\mathbb{H}^1$ is the Lebesgue 3-measure, but one may equivalently use the Hausdorff 4-measure.

Remark 2.4.40. Finally, let us enumerate the class of isometries on $\mathbb{H}^1$. The class can be decomposed into three basic types of maps:

- left-translations L_g, $g \in \mathbb{G}$;
- rotations about the t-axis by any angle; and

- a “reflection” which sends (x, y, t) to $(x, -y, -t)$.

(The last map is equivalent to rotating about the x axis by 180°.) All three types of maps are isometries, and compositions of these maps generate all possible isometries on $\mathbb{H}^1$, as noted in [3]. (This fact appears somewhat obscurely in [39] and more explicitly in [56, Section 10], but the idea may predate both references.)

CHAPTER 3

Euclidean $C^{1,\alpha}$ implies weak s-John in $\mathbb{H}^1$

3.1 Motivation and historical context

We begin with two important definitions.

Definition 3.1.1. A domain $\Omega \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ *domain* if, for every point x on the boundary of Ω, there exists a neighborhood V of x and a function $\Phi \in C^{1,\alpha}(V, \mathbb{R})$ such that $\Omega \cap V = \{y \in V \mid \Phi(y) > 0\}$. We say Ω is an *Euclidean* $C^{1,\alpha}$ *domain* if the regularity of Φ is established using the Euclidean metric in its domain and range.

Definition 3.1.2. (i) Let $V \subset \mathbb{R}^n$ be bounded. A function $\Phi \colon V \to \mathbb{R}$ is a $C^{1,\alpha}$ *function* if $\Phi \in C^1$ and the following norm is finite:

$$\|\Phi\|_{C^{1,\alpha}} := \sup_{x \in V} |\Phi(x)| + \sum_{i=1}^{n} \sup_{x \in V} \left| \frac{\partial \Phi}{\partial x_i}(x) \right| + \sum_{i=1}^{n} \sup_{\substack{x,y \in V \\ x \neq y}} \left\{ \frac{\left| \frac{\partial \Phi}{\partial x_i}(x) - \frac{\partial \Phi}{\partial x_i}(y) \right|}{d(x,y)^\alpha} \right\}.$$

(ii) The partial derivatives of a $C^{1,\alpha}$ function satisfy a Hölder condition with exponent α. In analogy with usual Hölder functions, we will associate a *Hölder constant* C_0 with a $C^{1,\alpha}$ function by using the final group of values in the norm:

$$C_0 := \sqrt{n} \max_{1 \leq i \leq n} \left\{ \sup_{\substack{x,y \in V \\ x \neq y}} \left\{ \frac{\left| \frac{\partial \Phi}{\partial x_i}(x) - \frac{\partial \Phi}{\partial x_i}(y) \right|}{d(x,y)^\alpha} \right\} \right\}. \tag{3.1.1}$$

Remark 3.1.3. The definition of $C^{1,\alpha}$ functions can be found, for example, in [23, Section 5.1]. My definition for the Hölder constant of such a function has been chosen for convenience.

Remark 3.1.4. Euclidean $C^{1,1}$ domains and $C^{1,1}$ functions can be defined in a similar fashion by taking $\alpha = 1$.

In a 1998 paper by Capogna and Garofalo [16], we find two theorems which conclude a domain is "non-tangentially accessible" (NTA) under assumptions about the Euclidean regularity of its boundary. (Recall that NTA domains are a type of geometric domain that is weaker than Lipschitz domains, but stronger than uniform and John domains.)

Theorem 3.1.5 (Capogna-Garofalo)**.** *Let $\mathbb{G}$ be a Carnot group of step 2. If $\Omega \subset \mathbb{G}$ is a bounded open set with Euclidean $C^{1,1}$ boundary having cylindrical symmetry in a neighborhood of every characteristic point, then Ω is NTA under the CC metric.*

Theorem 3.1.6 (Capogna-Garofalo)**.** *If $\Omega \subset \mathbb{R}^{2n+1}$ is a bounded, Euclidean $C^{1,1}$ domain, then Ω is an NTA domain in $(\mathbb{H}^n, d_{CC})$.*

The second theorem does not appear in the paper but can be constructed from the pieces therein; specifically, combining Propositions 9 and 10 with Proposition 4.2 from the 1995 paper by Capogna and Tang [17], the theorem follows.

One of the major results in a subsequent paper by Monti and Morbidelli [49, Theorem 3.2] is the following, which supersedes both of these:

Theorem 3.1.7 (Monti-Morbidelli)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Euclidean $C^{1,1}$ domain. If $(\mathbb{R}^n, d_{CC})$ is a Carnot group of step two, then Ω is an NTA domain in $(\mathbb{R}^n, d_{CC})$.*

They show that this result is sharp in two aspects. First, there exists a bounded domain $\Omega \subset \mathbb{R}^4$ with Euclidean C^∞ boundary which fails to be a John domain in a 3-step Carnot group on $\mathbb{R}^4$ (specifically, the "Engel group" in $\mathbb{R}^4$) [49, Example 5.3].

Second, for any $\alpha \in (0,1)$, there exists a bounded domain $\Omega \subset \mathbb{R}^3$ with Euclidean $C^{1,\alpha}$ boundary which fails to be a John domain in $\mathbb{H}^1$ [49, Example 5.1].

Taking particular notice of the second aspect, we see that not only are $C^{1,\alpha}$ domains potentially not NTA, but they can fail the much weaker requirements to be John domains. In Section 3.3 below, I extend Monti and Morbidelli's theorem into the class of $C^{1,\alpha}$ domains ($\alpha < 1$) using weak s-John domains (see Definition 2.3.7).

Specifically, we will prove the following:

Theorem 3.1.8. *Let Ω be a bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{R}^3$. Then Ω is a weak s-John domain in $\mathbb{H}^1$ for $s \geq 2/(\alpha+1)$.*

In Section 3.4, we demonstrate sharpness of this result with an example of a bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{R}^3$ which fails to be weak s-John for any $s < 2/(\alpha+1)$.

3.2 Results to simplify verification of a weak s-John domain

We begin by establishing the following result, which is analogous to a similar statement about John domains. We show that, if a domain is weak s-John, then any point in the domain may be used as the weak s-John center.

Proposition 3.2.1. *Let Ω be a bounded domain in a metric space (X,d). Assume Ω is a weak s-John domain with weak s-John center x_0 and weak s-John constant λ. Let $x_1 \in \Omega$, and let $\varepsilon \in (0,1)$ such that* $\operatorname{dist}(x_1, \partial\Omega) = \varepsilon \operatorname{diam}\Omega$. *Then Ω is a weak s-John domain with weak s-John center x_1 and weak s-John constant $\lambda' = \lambda'(\lambda, s, \varepsilon)$.*

Proof. Let $D = \operatorname{diam}\Omega$, and choose $x \in \Omega$. Let $\gamma_1 : [0,1] \to \Omega$ be a weak s-John curve from x_1 to x_0, and let $\gamma_2 : [0,1] \to \Omega$ be a weak s-John curve from x to x_0. We now define a curve from x to x_1 by concatenating the curves γ_2 and γ_1^{-1}:

$$\gamma(t) = \begin{cases} \gamma_2(2t), & \text{if } 0 \le t \le 1/2; \\ \gamma_1(2-2t), & \text{if } 1/2 < t \le 1. \end{cases}$$

We claim γ is a weak s-John curve with weak s-John constant $\lambda' = \min\left\{\lambda\left(\frac{\varepsilon}{2}\right)^s, \frac{\varepsilon}{2}\right\}$. Proving this claim will complete the proof. We proceed by considering γ in three parts.

Let $t \in [0, 1/2]$. Since γ_2 is a weak s-John curve and $\lambda' \le \lambda$, we get

$$\operatorname{dist}(z, \Omega^c) \ge \lambda D^{1-s} d(x,z)^s \ge \lambda' D^{1-s} d(x,z)^s$$

for every point $z = \gamma(t)$, $t \in [0, 1/2]$.

Suppose there exists $t \in (1/2, 1]$ such that $z := \gamma(t)$ satisfies $d(x_1, z) > \frac{1}{2}\operatorname{dist}(x_1, \partial\Omega) = \frac{\varepsilon D}{2}$. Then we verify the weak s-John condition:

$$\begin{aligned} \operatorname{dist}(z, \partial\Omega) &\ge \lambda D^{1-s}\, d(x_1, z)^s \quad \text{as } \gamma_1 \text{ is a weak } s\text{-John curve} \\ &> \left(\frac{\varepsilon}{2}\right)^s \lambda D \\ &\ge \left(\frac{\varepsilon}{2}\right)^s \lambda D^{1-s}\, d(x,z)^s \\ &\ge \lambda' D^{1-s}\, d(x,z)^s. \end{aligned}$$

Finally, let $t \in (1/2, 1]$ such that $z := \gamma(t)$ satisfies $d(x_1, z) \le \frac{1}{2}\operatorname{dist}(x_1, \partial\Omega) = \frac{\varepsilon D}{2}$. We again verify the weak s-John condition:

$$\operatorname{dist}(z, \partial\Omega) \ge \frac{\varepsilon D}{2} \ge \lambda' D \ge \lambda' D^{1-s}\, d(x,z)^s. \qquad \square$$

The focus of the remainder of this section is to localize the verification of being weak s-John to a neighborhood of the boundary. This localization will be necessary for the proof of Theorem 3.3.1 since $C^{1,\alpha}$ is expressed as a local condition, holding on neighborhoods of boundary points.

It is reasonable to presume that such a result could hold. After all, the essence of the condition for John and s-John domains is ensuring that we can "escape the boundary efficiently enough" as we journey to the center point. If points near the boundary can do it, one imagines that points farther into the interior should have no trouble. It is also reasonable that we might only need to verify that the weak s-John curve from a point near the boundary satisfies the weak s-John condition *for a short distance,* enough to get a bit away from the boundary. The condition usually becomes easier to satisfy the farther we get into the interior, so it seems that we should only be truly concerned about the initial progress of the curve.

We will now establish rigorously that it suffices simply to check points lying within δ of the boundary, for any fixed $\delta > 0$, and that we only need to verify the escape curves satisfy the weak s-John condition until they are a distance δ from the boundary. This is proven as Corollary 3.2.13, the culmination of several incremental advances.

Remark 3.2.2. This localization is not achieved without cost. A consequence of only checking points near the boundary is that the weak s-John constant produced is not optimal and gets worse the smaller δ is. However, we usually have no need for the constant, seeking mainly the classification of the domain as weak s-John or not. (For example, the implication in the next chapter, that weak s-John leads to certain Poincaré inequalities, depends quantitatively on s but not on the weak s-John constant.)

First, we must introduce a connectivity condition, which we will then work very hard to remove.

Notation 3.2.3. Recall that $N_\varepsilon(K)$ represents the (open) ε-neighborhood of a set K. Additionally, we define the following notations for this section of the thesis. Let Ω be a bounded domain in a metric space. We define:

- the ε-core of Ω as $A(\varepsilon) := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, and
- the δ-edge of Ω as $E(\delta) := \Omega \cap N_\delta(\partial\Omega)$.

Finally, recall the scale-invariant weak s-John condition, given by (2.3.5).

Lemma 3.2.4. *Let Ω be a bounded domain in a metric space (X, d). Assume there exists $\varepsilon \in (0, 1/2)$ such that $A := A(\varepsilon \operatorname{diam} \Omega)$ is a non-empty, pathwise-connected set. If there exists a curve γ from $x \in \Omega$ to $x_0 \in A$ satisfying (2.3.5) with weak s-John constant λ, then for any $\widetilde{x_0} \in A$ there exists a curve $\widetilde{\gamma}$ from x to $\widetilde{x_0}$ satisfying (2.3.5) with weak s-John constant $\widetilde{\lambda} = \widetilde{\lambda}(\lambda, \varepsilon)$.*

Proof. Let $\widetilde{\gamma}$ be the curve formed by concatenating γ and γ', where γ' is a path in A from x_0 to $\widetilde{x_0}$. As γ is a weak s-John curve, we have $d(z, \partial\Omega) \geq \lambda(\operatorname{diam} \Omega)^{1-s}\, d(x, z)^s$ for all $z \in \gamma$.

Now consider $z \in \gamma'$. We have $d(z, \partial\Omega) \geq \varepsilon \operatorname{diam} \Omega \geq \varepsilon(\operatorname{diam} \Omega)^{1-s}\, d(z, x)^s$. So, the weak s-John condition is satisfied here with constant ε.

Then $\widetilde{\lambda} = \min\{\lambda, \varepsilon\}$ is a valid weak s-John constant for $\widetilde{\gamma}$. □

Suppose we wish to show Ω is a weak s-John domain. Since we proved in Proposition 3.2.1 that the center point can be arbitrarily chosen, we may choose a prospective weak s-John center in the interior region A. It is clear from Lemma 3.2.4 that if it is possible to "escape" from points near the boundary into A with a uniform weak s-John constant, then we can extend each escape curve to the chosen center point in A with uniform control over the ultimate weak s-John constant, verifying the domain is weak s-John. We formally state this consequence in the following corollary.

Corollary 3.2.5. *Let Ω be a bounded domain in a metric space X. Assume there exists $\varepsilon \in (0, 1/2)$ such that $A := A(\varepsilon \operatorname{diam} \Omega)$ is a non-empty, pathwise-connected set. If there exists $\lambda > 0$ such that, for every point $x \in \Omega - A$, there is a curve from x to some point of A satisfying (2.3.5) with weak s-John constant at least λ, then Ω is a weak s-John domain.*

Remarks 3.2.6. 1. This corollary is the weak s-John version of a similar result regarding John domains [49, Proposition 2.4].

2. Note that, for different points in $\Omega - A$, the endpoints in A of their weak s-John curves need not be the same. (In the proof of Lemma 3.2.4, the weak s-John constant of the extension to $\widetilde{\gamma}$ did not depend on the first endpoint in A.)

3. It can occur that there exists an ε-core $A(\varepsilon)$ which is non-empty and path-connected, but such that $A(\varepsilon')$ is not path-connected for all $\varepsilon' < \varepsilon$.

4. Also, there exist John domains (hence also weak s-John) for which $A(\varepsilon \operatorname{diam} \Omega)$ is non-empty and not path-connected for all $\varepsilon \in (0, 1/2)$, which means that this requirement is sufficient but not necessary.

The last two remarks motivate the refinement presented in the next lemma.

Lemma 3.2.7. *Let Ω be a bounded domain in a metric space X. Assume there exist $\varepsilon \in (0, 1/2)$ and $\tilde{\varepsilon} \in (0, \varepsilon]$ such that $A := A(\varepsilon \operatorname{diam} \Omega)$ is a non-empty set with the property: every pair of points in A are connected by a path in $A(\tilde{\varepsilon} \operatorname{diam} \Omega)$. If there exists a curve γ from $x \in \Omega$ to $x_0 \in A$ satisfying (2.3.5) with weak s-John constant λ, then for any $\widetilde{x_0} \in A$ there exists a curve $\widetilde{\gamma}$ from x to $\widetilde{x_0}$ satisfying (2.3.5) with weak s-John constant $\widetilde{\lambda} = \widetilde{\lambda}(\lambda, \tilde{\varepsilon})$.*

Proof. The proof is as for Lemma 3.2.4, replacing all appearances of ε in the calculations with $\tilde{\varepsilon}$. □

We now get a refined corollary, as well.

Corollary 3.2.8. *Let Ω be a bounded domain in a metric space X. Assume there exists $\varepsilon \in (0, 1/2)$ and $\tilde{\varepsilon} \in (0, \varepsilon]$ such that $A := A(\varepsilon \operatorname{diam} \Omega)$ is a non-empty set with the property: every pair of points in A are connected by a path in $A(\tilde{\varepsilon} \operatorname{diam} \Omega)$. If there exists $\lambda > 0$ such that, for every point $x \in \Omega - A$, there is a curve from x to* *some point of A satisfying (2.3.5) with weak s-John constant at least λ, then Ω is a weak s-John domain.*

We have established a need for an ε-core which is connected by paths lying inside some (potentially) larger $\tilde{\varepsilon}$-core. We show in a fairly general setting that this requirement is satisfied by *any* core set.

Proposition 3.2.9. *Let Ω be a path-connected, bounded domain in a doubling, geodesic metric space (X, d). Let $\varepsilon > 0$, and let $A := A(\varepsilon)$. Then there exists $\tilde{\varepsilon} > 0$ such that the path-components of A are path-connected in $A(\tilde{\varepsilon})$.*

Proof. Let $\{P_\alpha\}_{\alpha \in I}$ be the path-components of A. First, we show that there are only finitely many components P_α that are (pairwise) not path-connected to each other inside $A(\varepsilon/2)$.

Let $x \in P_{\alpha_1}$ and $y \in P_{\alpha_2}$ be points in two such path-components of A. We immediately note that $B(x, \varepsilon) \subset \Omega$ and $B(y, \varepsilon) \subset \Omega$. Further, for every curve γ joining x to y in Ω, there must exist some point $z \in \gamma$ such that $\operatorname{dist}(z, \partial\Omega) < \varepsilon/2$ (else there would be a curve path-connecting P_{α_1} and P_{α_2} in $A(\varepsilon/2)$).

We briefly assume that $d(x, y) < \varepsilon/2$, and show why this cannot occur. If it did, then a geodesic from x to y would lie inside $B(x, \varepsilon/2)$, and hence it would entirely lie at least $\varepsilon/2$ from the boundary of Ω. This would contradict the previously established fact about the existence of $z \in \gamma$, and therefore cannot occur.

So, it is necessary that $d(x, y) \geq \varepsilon/2$. As X is a doubling space and Ω is bounded, a standard argument shows that there are only finitely many pairs of points in Ω that can be pairwise at least $\varepsilon/2$ distant from each other. Hence, there are only finitely many path-components of A that are still not path-connected in $A(\varepsilon/2)$.

To finish our proof, we note that, for this finite collection $\{P_{\alpha_i}\}_{i=1}^n$, the P_{α_i} are all path-connected in Ω, and hence there exists an exact value $\tilde{\varepsilon} > 0$ at which the first P_{α_i} becomes path-disconnected from the others in $A(2\tilde{\varepsilon})$. (The value $\tilde{\varepsilon}$ is distinct from zero since the collection $\{P_{\alpha_i}\}_i$ is finite.) So, in $A(\tilde{\varepsilon})$, the entire collection $\{P_\alpha\}_{\alpha\in I}$ is path-connected. $\square$

Remark 3.2.10. The conditions placed on the metric in Proposition 3.2.9 are satisfied, for example, by the Euclidean metric on $\mathbb{R}^n$ or the CC metric associated with any Carnot structure on $\mathbb{R}^n$. Also, any domain in these spaces will necessarily be path-connected, since these spaces are locally path-connected.

We can require weaker hypotheses on the metric. Close inspection of the proof of Proposition 3.2.9 reveals that the geodesic condition on the space can be weakened to a connectivity condition such as the following, from the monograph by Hajłasz and Koskela [32, Proposition 9.6]. (Recall that, given a ball $B(x,r)$, the ball δB denotes $B(x,\delta r)$.)

Definition 3.2.11. Let Ω be a bounded domain in a doubling metric space (X,d). We say Ω satisfies a *H-K connectivity condition* if there exists a constant $\delta \geq 1$ such that for every ball B with $\delta B \subset \Omega$, every two points x, $y \in B$ can be connected by a rectifiable curve which is

(i) contained in δB, and

(ii) of length less than or equal to $\delta\, d(x,y)$.

The purpose of the geodesic condition in the proof of Proposition 3.2.9 was to guarantee that if x and y are close to each other, then there is a curve joining them which is far from the boundary. We can adapt the proof as follows: for the set $B(x,\varepsilon/2)$, which is contained in Ω, the H-K connectivity condition says that any two points in $B(x,\varepsilon/2\delta)$ can be connected by a (rectifiable) curve contained in $B(x,\varepsilon/2)$. So, if $d(x,y) < \varepsilon/2\delta$, then we arrive at our contradiction, as before.

This weakening of the hypothesis actually offers a substantial improvement as we no longer need actual length metrics but can use metrics which are, for example, equivalent to a length metric, such as the Korányi metric (in place of the CC metric) in a Carnot group.

As a final observation, note that we actually only required part (i) of the H-K connectivity condition. Let me formally restate the preceding discussion, for reference.

Proposition 3.2.12. *Let Ω be a path-connected, bounded domain in a doubling metric space (X, d). Further, assume Ω satisfies part (i) of the H-K connectivity condition. Let $\varepsilon > 0$, and let $A := A(\varepsilon)$. Then there exists $\tilde{\varepsilon} > 0$ such that the path-components of A are path-connected in $A(\tilde{\varepsilon})$.*

Corollary 3.2.13. *Let Ω be a path-connected, bounded domain in a doubling metric space (X, d). Further, assume Ω satisfies part (i) of the H-K connectivity condition. Let $\varepsilon \in (0, 1/2)$. Then to conclude that Ω is a weak s-John domain, it suffices to show that there exists $\lambda > 0$ such that, for every point x in the edge set $E(\varepsilon \operatorname{diam} \Omega)$, there is a curve in Ω from x to some point of $A(\varepsilon \operatorname{diam} \Omega)$ satisfying (2.3.5) with weak s-John constant at least λ.*

Proof. We check that $E(\varepsilon \operatorname{diam} \Omega)$ is connected by weak s-John curves into the interior set $A(\varepsilon \operatorname{diam} \Omega)$. The path components of $A(\frac{\varepsilon}{2} \operatorname{diam} \Omega)$ are path-connected in $A(\tilde{\varepsilon})$, as in the conclusion of Proposition 3.2.12, which means we can apply Corollary 3.2.8 to conclude that Ω would be a weak s-John domain. $\square$

Remark 3.2.14. As noted in the discussion preceding Proposition 3.2.12, the complicated conditions on the space given in the hypotheses are satisfied, for example, by any doubling space with a metric equivalent to a length metric.

If we take $s = 1$, this corollary provides a localization for weak John domains. By [32, Proposition 9.6], if we amend the hypotheses of Corollary 3.2.13 to include the entire H-K connectivity condition, then we are in a setting where John domains and weak John domains are equivalent. This yields a result that localizes the verification of the John condition to a neighborhood of the boundary. While we are aware of a similar result for uniform domains [49, Proposition 2.5], this may be a new result for John domains, so we will state it here for reference.

Corollary 3.2.15. *Let Ω be a path-connected, bounded domain in a doubling metric space (X, d). Further, assume Ω satisfies the H-K connectivity condition. Let $\varepsilon \in (0, 1/2)$. Then to conclude that Ω is a John domain, it suffices to show that there exists $\lambda > 0$ such that, for every point x in the edge set $E(\varepsilon \operatorname{diam} \Omega)$, there is a curve in Ω from x to some point of $A(\varepsilon \operatorname{diam} \Omega)$ satisfying the John condition (2.3.1) with John constant at least λ.*

3.3 Main theorem

The proof of the following theorem is the goal of this section.

Theorem 3.3.1. *Let Ω be a bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{R}^3$. Then Ω is a weak s-John domain in $\mathbb{H}^1$ for $s \geq 2/(\alpha + 1)$.*

To clarify the structure of the proof, we first will introduce a motivating example which suggests our approach. Then we will discuss the setup for the proof and present a series of technical lemmas that are necessary to pursue this approach rigorously. Finally, we will give the proof itself.

3.3.1 Motivating example

Example 3.3.2. Let us consider a model $C^{1,\alpha}$ domain. Based on this model domain, we will understand why the Theorem 3.3.1 is reasonable and also what our fundamental strategy will be for its proof.

Let $\Omega \subset \mathbb{H}^1$ be a domain, given near the origin by the set $\{(z, t) \mid |z|^{1+\alpha} < t < 1\}$. (Recall the use of complex notation in the Heisenberg group, introduced in Notation 2.4.33.) This is the same domain as appears in [49, Example 5.1]; there, it was noted that this is a $C^{1,\alpha}$ domain.

Let $s = 2/(1+\alpha)$, and consider the point $p = (0, \varepsilon)$, $\varepsilon > 0$, on the t-axis. We will construct (the beginning of) a weak s-John curve from p. As we saw in Section 3.2, we are only concerned about the weak s-John curves escaping from points near the boundary, so we may assume that ε is small.

Why is this model domain — and this point p — a good test for the weak s-John condition? The boundary of the model domain is tangent to the horizontal distribution at the origin. This tangency makes it costly (in terms of distance) to move away from the boundary when starting near the origin. The cheapest directions to travel are horizontal directions, but those are ineffective ways to gain distance from the boundary where such tangency occurs. So, this setup gives a worst-case scenario for trying to satisfy the weak s-John condition in $\mathbb{H}^1$, meaning success here would strongly suggest that the domain is weak s-John.

To simplify our calculations, we will use the max metric (2.4.5) in $\mathbb{H}^1$.

Note that, by the symmetry of Ω, an optimal weak s-John curve from p towards the interior of Ω will follow the t-axis. (This curve maximizes the distance to the boundary over the set of points lying a fixed distance from p. Under the max metric, this set of points is a cylinder with top and bottom. The point on this cylinder furthest from the boundary lies above p on the t-axis.) Let the curve $\gamma(u) := (0, \varepsilon + u)$ be our candidate for a weak s-John curve leaving p.

Fix $u > 0$, u small; we will verify the weak s-John condition holds at $\gamma(u)$. Our major task is to determine the distance from $\gamma(u)$ to the boundary. Given $(z,t) \in \partial\Omega$ near the origin, the distance to $\gamma(u)$ is:

$$d_M\big((z,t), \gamma(u)\big) = \max\{|z|, |t - \varepsilon - u|^{1/2}\} = \max\left\{|t|^{\frac{1}{\alpha+1}}, |t - \varepsilon - u|^{1/2}\right\}.$$

To find the minimal distance to the boundary, we note that we may restrict t to the range $[0, \varepsilon + u]$. The first term in the max expression strictly (and continuously) increases in t and the second strictly (and continuously) decreases in t. Hence, the minimal value of the distance occurs when the two terms are equal, i.e., $|z| = |t - \varepsilon - u|^{1/2}$. Let (z_0, t_0) be a point on the boundary for which this equation holds. (Note, then, that $\mathrm{dist}_M\big(\gamma(u), \partial\Omega\big) = |z_0|$.) Combining this equation with the defining equation for the boundary, we get the relationship:

$$|z_0|^2 + |z_0|^{1+\alpha} = \varepsilon + u.$$

Trivially, we get $2|z_0|^{\alpha+1} \geq u$; hence, $\mathrm{dist}_M\big(\gamma(u), \partial\Omega\big) = |z_0| \geq Cu^{1/(\alpha+1)}$.

Since $d_M\big(p, \gamma(u)\big) = u^{1/2}$, we get

$$Cd_M\big(p, \gamma(u)\big)^s = Cu^{1/(1+\alpha)} \leq \mathrm{dist}_M\big(\gamma(u), \partial\Omega\big).$$

This shows that the curve γ is weak s-John for u small.

Remark 3.3.3. Let's elaborate on how this model domain underlies our approach to proving Theorem 3.3.1.

First, note that for any $C^{1,\alpha}$ domain which is tangent to the horizontal distribution at the origin and lies above the origin, there is a positive constant C_0 such that a modified form of the model domain, $\{(z,t) \mid C_0|z|^{1+\alpha} < t\}$, will (locally) fit inside of that domain. (This claim is proven below as Lemma 3.3.13.) This allows us to work within the more-easily-described model domain for our calculations.

Second, we can define a model domain to use in the cases where the tangent plane of the $C^{1,\alpha}$ domain is not horizontal (or vertical) at the origin. Let P denote the tangent plane. We vertically sum the previous model domain with P: if (z_0, t_0) lies on the boundary of the model domain, and (z_0, t_1) lies on P, then the point $(z_0, t_0 + t_1)$ is on the boundary of the new model domain. This new model domain then has a tangent plane at the origin coinciding with P. As before, by choosing C_0 appropriately, we can fit this model domain inside of a $C^{1,\alpha}$ domain with tangent plane P at the origin, and our calculations are performed inside of this skewed model domain.

In the case of a vertical tangent plane, some horizontal curve will efficiently carry us away from the boundary.

3.3.2 Technical lemmas

Throughout this section, let Ω be a bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{R}^3$.

The broad strategy of the proof is simple: pick a point "near" the boundary of Ω and build a "sufficiently long" weak s-John curve from it. By Corollary 3.2.13, we know this is enough, and the corollary gives us flexibility in choosing what "near" and "sufficiently long" will be. The majority of the work in the proof will be in verifying that the constructed curves are weak s-John curves in $\mathbb{H}^1$ and that the weak s-John constant for the curve doesn't depend on the particular point chosen.

The full setup for the proof is conceptually straightforward. First, we choose a point p in Ω "near" the boundary. Let q be a nearest boundary point to p, and let V be a neighborhood containing both p and q, with the boundary of Ω inside V described by an Euclidean $C^{1,\alpha}$ function. (Keep in mind that this neighborhood is the only place we have any specific functional description of the domain, so all of our work happens inside of this neighborhood.)

We then want to relocate our work near the origin in order to make our calculations easier. Specifically, we left-translate the space by q^{-1}. This maps q to the origin, p maps somewhere nearby, and V maps to a neighborhood of the origin. We use the $C^{1,\alpha}$ function describing the boundary of Ω inside V to (essentially) fit a model domain inside of Ω. Finally, we define a curve exiting from p and perform the necessary calculations to show it is a weak s-John curve, with weak s-John constant independent of the choice of p. (The curve chosen will depend on the model domain fitted inside Ω, which depends on the tangent plane at the origin. Four cases will have to be considered.)

There are a host of technical details to be careful about lurking in this setup.

- The model domain to be fitted inside of Ω depends on the tangent plane at the origin and the Hölder constant C_0 of the derivative of the $C^{1,\alpha}$ function that describes the boundary of Ω in the neighborhood being considered. We need to understand how C_0 will be affected by our choice of p and by the left-translation that moves our work to the origin.
  - Since the point p could be located anywhere in the domain, we need to have "$C^{1,\alpha}$ uniformity" in the domain. That is, when we describe the boundary of the domain near p using a $C^{1,\alpha}$ function, we must have a uniform value for the Hölder constant of the function's derivative. Lemma 3.3.4 shows this is possible for any $C^{1,\alpha}$ domain.
  - Recall, left-translation in a sub-Riemannian space applies a skew to the space. Having already described the boundary near p using a $C^{1,\alpha}$ function with known Hölder constant C_0, we must determine how the skew has changed the function. In fact, it is still $C^{1,\alpha}$, but the Hölder constant of its derivative can change by a bounded amount. (Since the change is bounded, we can simply change our uniform C_0 value to a new, uniform, worst-case value.) This is detailed in Lemma 3.3.5.
- The left-translation that moved q to the origin moved p *somewhere*, and we need to figure out where. The distance of p to the boundary, together with information about the tangent plane at the origin (which now lies on the relocated boundary), is sufficient to uniquely locate where p was moved to. Lemma 3.3.6 gives the new location.
- The neighborhood that contains p and q needs to be considered carefully. First, the definition for $C^{1,\alpha}$ domains gives the existence of a neighborhood about q on which a $C^{1,\alpha}$ function describes the domain, but it does not guarantee

that p would lie in this neighborhood. Further, to verify the weak s-John condition holds for a "sufficiently long" weak s-John curve exiting p, we need to have a description of the boundary on a large enough neighborhood around p. Lemma 3.3.9 guarantees that there is a minimum size of neighborhood available to us, which solves both issues. Choosing our values for "near" and "sufficiently long" small enough will guarantee that p lies in the neighborhood and that the curve's construction and verification take place entirely within these minimum-sized neighborhoods.

- In a final technical point, for our calculations, we will need an *explicit* function that describes the boundary of Ω, instead of an *implicit* one. In Definition 3.1.1 of a $C^{1,\alpha}$ domain, the boundary of the domain is described as the level set of a $C^{1,\alpha}$ function from $\mathbb{R}^3$ to $\mathbb{R}$. In order to write the boundary as the graph of a $C^{1,\alpha}$ function from $\mathbb{R}^2$ to $\mathbb{R}$, we need to apply the Implicit Function Theorem. Lemma 3.3.11 justifies the application of this step.

Lemma 3.3.4. *There exists a uniform constant C_0 such that, for any point q on the boundary of Ω, there is an Euclidean $C^{1,\alpha}$ function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ which describes Ω in a neighborhood of q and whose derivative has Hölder constant at most C_0.*

Proof. By the definition of an Euclidean $C^{1,\alpha}$ domain, at each point q on the boundary of Ω, there is a neighborhood V_q of q and an Euclidean $C^{1,\alpha}$ function $\Phi_q\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega \cap V_q = \{\Phi_q > 0\}$. The neighborhoods V_q form an open cover of $\partial\Omega$, which is a closed and bounded subset of $\mathbb{R}^3$, hence compact. Thus, there exists a finite subcover $\{V_{q_i}\}$ of $\partial\Omega$. Let C_0 be the largest Hölder constant associated with the derivatives of the functions Φ_{q_i}. $\square$

Lemma 3.3.5. *Let M be a bound for the (bounded) set Ω. Let $q \in \partial\Omega$, and let V be a neighborhood of q. Let $\Phi\colon V \to \mathbb{R}$ be an Euclidean $C^{1,\alpha}$ function such that Ω is described in V by $\{(x, y, t) \mid \Phi(x, y, t) > 0\}$. We left-translate the entire space by q^{-1},* *and define $\widetilde{\Phi}$ as the left-translated version of Φ:*

$$\widetilde{\Phi}(p) := \Phi(L_q(p)). \tag{3.3.1}$$

Then the function $\widetilde{\Phi}$ is also an Euclidean $C^{1,\alpha}$ function, and if the Hölder constant of the derivative of Φ is C_0, then the Hölder constant of $\widetilde{\Phi}$ is no worse than $C\,C_0$, where C is a constant that depends only on M.

Proof. Recall from Remark 2.4.32 that left-translation in $\mathbb{H}^1$ is an affine transformation:

$$L_q(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q_2/2 & q_1/2 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} =: Ap + q.$$

Note that we have the bound $|A_{ji}| \leq \max\{M/2, 1\}$. Also, since L_q is a smooth function, the composition $\Phi \circ L_q$ is still $C^{1,\alpha}$.

Before working with the Hölder constant for the derivative of $\widetilde{\Phi}$, it is useful to develop a bound. Let V' denote the domain of $\widetilde{\Phi}$. Let x and y be points in V'.

$$\begin{aligned} \left| \frac{\partial \widetilde{\Phi}}{\partial x_i}(x) - \frac{\partial \widetilde{\Phi}}{\partial x_i}(y) \right| &= \left| \sum_{j=1}^{3} \frac{\partial \Phi}{\partial x_j}(L_q(x)) \cdot A_{ji} - \sum_{j=1}^{3} \frac{\partial \Phi}{\partial x_j}(L_q(y)) \cdot A_{ji} \right| \\ &\leq \max\{M/2, 1\} \cdot \sum_{j=1}^{3} \left| \frac{\partial \Phi}{\partial x_j}(L_q(x)) - \frac{\partial \Phi}{\partial x_j}(L_q(y)) \right|. \end{aligned}$$

Note that, if $x \in V'$, then $L_q(x)$ lies in V. We now find an upper bound on the

Hölder constant for the derivative of $\widetilde{\Phi}$.

$$
\begin{aligned}
&\sqrt{3}\max_i \left\{ \sup_{\substack{x,y\in V' \\ x\neq y}} \frac{\left|\frac{\partial\widetilde{\Phi}}{\partial x_i}(x) - \frac{\partial\widetilde{\Phi}}{\partial x_i}(y)\right|}{d(x,y)^\alpha} \right\} \\
&\quad \leq \sqrt{3}\max_i \left\{ \sup_{\substack{x,y\in V' \\ x\neq y}} \frac{\max\{M/2,1\}\cdot\sum_{j=1}^3 \left|\frac{\partial\Phi}{\partial x_j}(L_q(x)) - \frac{\partial\Phi}{\partial x_j}(L_q(y))\right|}{d(x,y)^\alpha} \right\} \\
&\quad = 3\max\{M/2,1\}\cdot\sqrt{3}\max_j \left\{ \sup_{\substack{x,y\in V' \\ x\neq y}} \frac{\left|\frac{\partial\Phi}{\partial x_j}(L_q(x)) - \frac{\partial\Phi}{\partial x_j}(L_q(y))\right|}{d(x,y)^\alpha} \right\} \\
&\quad = 3\max\{M/2,1\}\, C_0.
\end{aligned}
$$

In the above, we used the fact that left-translation is an isometry, hence $d(x,y)^\alpha$ is the same as $d(L_q(x),L_q(y))^\alpha$ (which is what Φ would expect in the denominator for its Hölder constant). $\square$

In domains in $\mathbb{R}^n$ with smooth enough boundary, variational arguments show that the line joining a point and its nearest boundary point will be normal to the boundary. Available theorems do not extend to $C^{1,\alpha}$ smoothness — much less $C^{1,\alpha}$ domains in the setting of the Heisenberg group — so it becomes a reasonable concern whether the positional relationship between interior points and their nearest boundary points can be ascertained. In this technical lemma, we address this question, set in the specific circumstances that will be seen in the proof of Theorem 3.3.1.

Lemma 3.3.6. *Let $\mathcal{O} = (0,0,0) \in \partial\Omega$. Assume that the tangent plane at $\mathcal{O}$ has normal vector $\langle A,0,C\rangle$, where $A \geq 0$ and $C \geq 0$, and that this vector is an inward-pointing normal for Ω. Assume $p := (a,b,c)$ is a point in Ω that has $\mathcal{O}$ as (one of) its closest boundary point(s) when distances are measured with the Korányi metric.*

(i) If $A = 0$, then $p = (0,0,c)$ for some $c > 0$.

(ii) If $C = 0$, then $p = (a,0,0)$ for some $a > 0$.

(iii) If A and C are both non-zero, let $\nu = \frac{A}{C}$. Then $p = (a(\delta), b(\delta), c(\delta))$ for some $\delta \in (0, \infty)$, where

$$a(\delta) = \frac{2\nu\delta^{1/2}}{\delta + 1}, \quad b(\delta) = \frac{2\nu\delta}{\delta + 1}, \quad c(\delta) = \frac{16\nu^2\delta^{3/2}}{\delta + 1}.$$

Remark 3.3.7. The points $(a(\delta), b(\delta), c(\delta))$ lie on a spiral path, but one which is unrelated to Heisenberg geodesics. The projection of the path into the x_1x_2-plane is an open semicircle with radius ν, centered at $(0, \nu)$ and lying in the first quadrant. When δ is small and positive, we are near the origin. As δ goes to infinity, we approach $(0, 2\nu)$ in the x_1x_2-plane, and x_3 goes to infinity. It is worth noting, for reference, its distance to the origin:

$$d_K\big((a(\delta), b(\delta), c(\delta)), \mathcal{O}\big) = 2\nu \left(\frac{\delta}{\delta + 1}\right)^{1/2} (1 + 16\delta)^{1/4}. \tag{3.3.2}$$

Remark 3.3.8. There is no guarantee of the existence of a point $p \in \Omega$ for a given distance and tangent plane at $\mathcal{O}$. However, if such a point is presumed to exist, then the distance from p to the origin, in conjunction with the tangent plane at $\mathcal{O}$, uniquely specifies the location of p.

Proof of Lemma 3.3.6. The main strategy here is to employ a variational argument: since we assume that $d_K(p, \mathcal{O})$ is minimal among points in the boundary, we can generate equations and inequalities which constrain the possible location of p. (This is, essentially, the Euler-Lagrange method.)

To perform the calculations, we need to describe the boundary near the origin as the *graph* of a $C^{1,\alpha}$ function, which requires that we use the Implicit Function Theorem. The easiest way to do this is to separate the proof into two parts: $C > 0$ and $C = 0$.

Part I. $C > 0$.

By the Implicit Function Theorem, the domain Ω is described in a neighborhood of the origin by $\{x_3 > \phi(x_1, x_2)\}$, where $\phi(x_1, x_2)$ is a $C^{1,\alpha}$ function. Since $\mathcal{O} \in \partial\Omega$, then $\phi(0,0) = 0$. Further, the normal of the tangent plane at $\mathcal{O}$ implies that $\phi_{x_1}(0,0) = -\nu$, $\nu \geq 0$, and $\phi_{x_2}(0,0) = 0$.

We explicitly separate out the linear behavior of ϕ by defining $\psi(x_1, x_2)$:

$$\phi(x_1, x_2) = -\nu x_1 + \psi(x_1, x_2)$$

for all (x_1, x_2) in the domain of ϕ. Note that $\psi(0,0) = 0$ and $\nabla\psi(0,0) = 0$.

Case (i): $A = 0$ (i.e., $\nu = 0$).

1. Our first equation comes from variation of the x_1-coordinate. Let $q = (\varepsilon, 0, \psi(\varepsilon, 0)) \in \partial\Omega$. By minimality, the derivative of the distance function $d_K(p, q)$ with respect to ε is zero when $\varepsilon = 0$. This yields the constraint:

$$4a(a^2 + b^2) + bc = 0. \tag{3.3.3}$$

2. Our second equation comes from variation of the x_2-coordinate. Let $q = (0, \varepsilon, \psi(0, \varepsilon)) \in \partial\Omega$. Again, the derivative of the distance function $d_K(p, q)$ with respect to ε is zero when $\varepsilon = 0$. This yields the constraint:

$$4b(a^2 + b^2) - ac = 0. \tag{3.3.4}$$

3. Let $q = (0, 0, -\varepsilon)$, with $\varepsilon > 0$. This point is in Ω^c for ε small enough. As $\mathrm{dist}_K(p, \partial\Omega) = \mathrm{dist}_K(p, \Omega^c)$, we get the inequality $d_K(p, q) - d_K(p, \mathcal{O}) \geq 0$. It is equivalent that $d_K(p, q)^4 - d_K(p, \mathcal{O})^4 \geq 0$, which yields the inequality:

$$\begin{aligned} d_K(p,q)^4 - d_K(p,\mathcal{O})^4 &= (a^2 + b^2)^2 + (c + \varepsilon)^2 - (a^2 + b^2)^2 - c^2 \\ &= 2\varepsilon c + o(\varepsilon). \end{aligned}$$

As we are working in a neighborhood of $\mathcal{O}$, we may take ε small enough that the $O(\varepsilon)$ terms are dominant. For ε small and positive, we get the constraint:

$$c \geq 0. \tag{3.3.5}$$

We now consider what the constraints imply. Squaring and adding equations (3.3.3) and (3.3.4), we get:

$$0 = \big(4a(a^2+b^2)+bc\big)^2 + \big(4b(a^2+b^2)-ac\big)^2 = (a^2+b^2)\big(16(a^2+b^2)+c^2\big).$$

Clearly, we get $a = b = 0$. Thus, for Case (i), the point p has the form $(0, 0, c)$, where $c > 0$.

Case (iii): $A > 0$ (i.e., $\nu > 0$).

We seek a similar approach in this case. Since $\nu > 0$ here, it will appear in the description for q in the x_1-variation. It does not appear in the other variations, so the constraints (3.3.4) and (3.3.5) will also apply in this case.

We perform the variation of the x_1-coordinate. Let $q = \big(\varepsilon, 0, -\nu\varepsilon + \psi(\varepsilon, 0)\big) \in \partial\Omega$. By minimality, the derivative of the distance function $d_K(p, q)$ with respect to ε is zero when $\varepsilon = 0$. This yields the constraint:

$$4a(a^2+b^2) + bc = 2\nu c. \tag{3.3.6}$$

Now we consider the consequences. First, it is straightforward to verify, using the constraints and the fact that $\nu > 0$, that a, b, and c must either all be zero or all be non-zero. As $(0, 0, 0) \notin \Omega$, the coordinates of p must all be non-zero. The constraints also imply that the coordinates must all be positive.

Finally, we derive the coordinates of p as functions of δ. Proper rearranging of the equations (3.3.6) and (3.3.4) yields $a^2 + b^2 = 2\nu b$, which shows that a and b are constrained to the open semicircle with center $(0, \nu)$ and radius ν, lying in the first quadrant.

The given expressions for $a(\delta)$ and $b(\delta)$ can be verified to satisfy this constraint. Further, $b(\delta)$ is a strictly increasing function of δ, going from zero at $\delta = 0$ and limiting to 2ν as δ goes to infinity. Hence, this particular parametrization of the open semicircle covers the entire open semicircle.

Substituting $a(\delta)$ and $b(\delta)$ into (3.3.4) and solving will produce the desired $c(\delta)$.

From this method of proof, it should be clear that the given parametrization is not unique, nor does the proof guarantee the existence of p for given values of $d_K(p, \mathcal{O})$ and ν. If such a p exists, however, it must be located as described.

Part II. $C = 0$.

By the Implicit Function Theorem, the domain Ω is described in a neighborhood of the origin by $\{x_1 > \phi(x_2, x_3)\}$, where $\phi(x_2, x_3)$ is a $C^{1,\alpha}$ function. Since $\mathcal{O} \in \partial\Omega$, then $\phi(0, 0) = 0$. Further, the normal of the tangent plane at $\mathcal{O}$ implies that $\nabla\phi(0, 0) = 0$.

Case (ii): $C = 0$.

1. Our first equation comes from variation of the x_2-coordinate. Let $q = \big(\phi(\varepsilon, 0), \varepsilon, 0\big) \in \partial\Omega$. By minimality, the derivative of the distance function $d_K(p, q)$ with respect to ε is zero when $\varepsilon = 0$. This yields the constraint:

$$4b(a^2 + b^2) = ac. \tag{3.3.7}$$

2. Our second equation comes from variation of the x_3-coordinate. Let $q = \big(\psi(0, \varepsilon), 0, \varepsilon\big) \in \partial\Omega$. Again, the derivative of the distance function $d_K(p, q)$ with respect to ε is zero when $\varepsilon = 0$. This yields the constraint:

$$c = 0. \tag{3.3.8}$$

3. Let $q = (-\varepsilon, 0, 0)$, where $\varepsilon > 0$. Thus, q is in Ω^c, and, as we saw earlier, we can proceed using the inequality $d_K(p,q)^4 - d_K(p,\mathcal{O})^4 \geq 0$. This yields the inequality:

$$4a(a^2 + b^2) + bc \geq 0. \tag{3.3.9}$$

These constraints imply that $b = c = 0$ and $a \geq 0$. As $(0,0,0)$ is not a point in Ω, we can conclude in Case (ii) that p has the form $(a,0,0)$ where $a > 0$. $\square$

Lemma 3.3.9. *Let q be a boundary point of Ω. Then there exists a positive constant κ such that the left-translated domain $L_{q^{-1}}\Omega$ is given by $\{p \mid \Phi(p) > 0\}$ inside the ball $B_K(\mathcal{O},\kappa)$, where Φ is an Euclidean $C^{1,\alpha}$ function.*

Remark 3.3.10. The point here is, since the left-translated domain is still $C^{1,\alpha}$, there will be a $C^{1,\alpha}$ function which describes it *on some neighborhood* of the origin. The lemma guarantees a minimum size for this neighborhood, which will give us the necessary room for building a weak s-John curve later on.

Proof. For every boundary point x, there is a neighborhood V_x on which an Euclidean $C^{1,\alpha}$ function Φ_x describes Ω inside V_x. These sets V_x form an open cover of the compact set $\partial\Omega$, so, by Lebesgue's Number Lemma, there is a value $\kappa > 0$ such that $B_K(q,\kappa)$ is entirely contained in some V_{x_0}. Consequently, we treat this $B_K(q,\kappa)$ as our neighborhood for q, and Ω is described in $B_K(q,\kappa)$ by the function Φ_{x_0}.

Let Φ be the left-translated version of Φ_{x_0}, as in Lemma 3.3.5: that is, $\Phi(p) := \Phi_{x_0}(L_q(p))$. When we left-translate the space by q^{-1}, the $C^{1,\alpha}$ function Φ then describes $L_{q^{-1}}\Omega$ inside $B_K(\mathcal{O},\kappa)$. $\square$

Lemma 3.3.11. *Let $\mathcal{O} = (0,0,0) \in \partial\Omega$. Assume that the tangent plane at $\mathcal{O}$ has normal vector $\langle A, 0, C\rangle$, where $A \geq 0$ and $C \geq 0$, and that this vector is an inward-pointing normal for Ω. Then there exists a positive constant κ such that:*

(i) If $A \leq C$, we may write $\partial\Omega \cap B(\mathcal{O},\kappa)$ in the form $x_3 = \phi(x_1,x_2)$, where $\phi \in C^{1,\alpha}(\mathbb{R}^2,\mathbb{R})$.

(ii) If $A > C$, we may write $\partial\Omega \cap B(\mathcal{O}, \kappa)$ in the form $x_1 = \phi(x_2, x_3)$, where $\phi \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R})$.

Proof. Let Φ be an Euclidean $C^{1,\alpha}$ function which describes Ω inside $B_K(\mathcal{O}, \kappa)$, as given by Lemma 3.3.9. If needed, we reduce κ so that $\kappa < 1/(2C_0)^{1/\alpha}$. Also, we may renormalize Φ such that $|\nabla\Phi(\mathcal{O})| \geq 2$.

Case (i): $A \leq C$.

The goal is to apply the Implicit Function Theorem on Φ to produce ϕ; note that ϕ would inherit the necessary regularity from Φ.

To guarantee that the IFT may be used, we must show that the third component of $\nabla\Phi$ is non-zero in $B_K(\mathcal{O}, \kappa)$. We argue by contradiction: assume $\nabla\Phi(x) = \langle A', B', 0\rangle$ for some $x \in B_K(\mathcal{O}, \kappa)$. The Hölder condition on $\frac{\partial\Phi}{\partial x_3}$ gives:

$$C = \left|\frac{\partial\Phi}{\partial x_3}(x) - \frac{\partial\Phi}{\partial x_3}(\mathcal{O})\right| \leq C_0\, d(x, \mathcal{O})^\alpha. \tag{3.3.10}$$

As $|\nabla\Phi(\mathcal{O})| = \sqrt{A^2 + C^2} \geq 2$ and $C \geq A \geq 0$, we know that $C \geq 1$. So, the above inequality shows $1 \leq C_0\, d(x, \mathcal{O})^\alpha$. For κ less than $1/(2C_0)^{1/\alpha}$, such a point x will not lie inside of $B_K(\mathcal{O}, \kappa)$, which is our contradiction. (Note also that this condition on κ guarantees that $\frac{\partial\Phi}{\partial x_3}$ is bounded away from zero in $B_K(\mathcal{O}, \kappa)$, a fact which is useful in verifying the regularity of ϕ.)

The second case is shown similarly. $\square$

Remark 3.3.12. In both cases above, the Hölder constant C_0 of the function ϕ is related to the Hölder constant $\widetilde{C_0}$ of Φ:

$$C_0 \leq 4\sqrt{2}(1 + \widetilde{C_0}) \max_i \left\{ \sup_{x \in B(\mathcal{O}, \kappa)} \left|\frac{\partial\Phi}{\partial x_i}(x)\right| \right\}.$$

In a manner similar to the proof of Lemma 3.3.4, we can show that there is a uniform bound on the $C^{1,\alpha}$ norm of Φ. If M is this bound, then we have a uniform bound on the Hölder constant for ϕ:

$$C_0 \leq 4\sqrt{2}M(1+\widetilde{C_0}).$$

For the final technical lemma, we have arrived at the setup referred to in Remark 3.3.3. Specifically, we have an Euclidean $C^{1,\alpha}$ domain described near the origin by $\{(x,y,t) \mid \psi(x,y) < t\}$ with $\nabla\psi(0,0) = \langle 0,0\rangle$. Geometrically, this lemma says that the model domain $\{(z,t) \mid C_0|z|^{1+\alpha} < t\}$ will lie inside of this one, shown by comparing the t-coordinates of their boundaries, for any choice of x and y. Computationally, this lemma lets us convert "$C^{1,\alpha}$ regularity" into a very usable form.

Lemma 3.3.13. *Let $\mathcal{O} := (0,0)$ and let $B := B(\mathcal{O}, r) \subset \mathbb{R}^2$. Let $\psi \colon B \to \mathbb{R}$ be a $C^{1,\alpha}$ function such that $\psi(\mathcal{O}) = 0$ and $\nabla\psi(\mathcal{O}) = \langle 0,0\rangle$. Then there exists a constant $C_0 > 0$ such that*

$$|\psi(x_1,x_2)| \leq C_0(x_1^2+x_2^2)^{(\alpha+1)/2}. \tag{3.3.11}$$

Proof. Let $x \in B$. Recall the definitions for a $C^{1,\alpha}$ function and its associated Holder constant C_0 (see Definition 3.1.2). Since $\psi \in C^{1,\alpha}$,

$$\left|\frac{\partial\psi}{\partial x_i}(x)\right| = \left|\frac{\partial\psi}{\partial x_i}(x) - \frac{\partial\psi}{\partial x_i}(\mathcal{O})\right| \leq \frac{1}{\sqrt{2}}C_0\, d(x,\mathcal{O})^\alpha,$$

for $i = 1,2$.

Let γ be a geodesic from $\mathcal{O}$ to x in B. We integrate $\nabla\psi$ along γ:

$$\begin{aligned}
|\psi(x_1,x_2)| = |\psi(x_1,x_2) - \psi(\mathcal{O})| &= \left|\int_\gamma \nabla\psi\, ds\right| \\
&\leq \max_{y\in\gamma} |\nabla\psi(y)| \cdot l(\gamma) \\
&\leq \sqrt{2}\max_{i=1,2}\left\{\max_{y\in\gamma}\left|\frac{\partial\psi}{\partial x_i}(y)\right|\right\} \cdot d(x,\mathcal{O}) \\
&\leq C_0\, d(x,\mathcal{O})^{\alpha+1} \\
&= C_0\,(x_1^2+x_2^2)^{(\alpha+1)/2}.
\end{aligned}$$

□

3.3.3 Proof of main theorem

Proof of Theorem 3.3.1. Let $s = 2/(\alpha+1)$. It is sufficient to demonstrate Ω is a weak s-John domain in $\mathbb{H}^1$, as this implies Ω is a weak s'-John domain for all $s' > s$.

Let $D := \operatorname{diam}_K \Omega$, and let $\varepsilon > 0$ such that $\varepsilon D << 1$. We apply Corollary 3.2.13: it suffices to choose an arbitrary point within a distance εD of the boundary, construct a curve from that point toward the interior of Ω, and verify that the curve satisfies the weak s-John condition (2.3.5) until it has reached a distance of εD from the boundary. (Note that $(\mathbb{H}^1, d_K)$ is a doubling metric space, with the Korányi metric d_K equivalent to the length metric d_{CC}; by Remark 3.2.14, we may apply Corollary 3.2.13 here.)

In Corollary 3.2.13, ε may be chosen as small as desired, and this freedom is needed in order to adapt to the particular features of Ω. Ultimately, our choice for ε will depend on α, s, $\operatorname{diam}\Omega$, λ, and the Hölder constant C_0 of the domain. We will see in (3.3.39) what choice of ε is required.

Let $\tilde{p} \in \Omega$ be chosen such that $\operatorname{dist}_K(\tilde{p}, \partial\Omega) < \varepsilon D$, and let $q \in \partial\Omega$ be a nearest boundary point to $\tilde{p}$. We now left-multiply the space by q^{-1} to relocate our work near the origin. This will succeed in making our subsequent calculations easier.

Let $\mathcal{O} := (0,0,0)$. By Lemmas 3.3.4 and 3.3.5, there is an Euclidean $C^{1,\alpha}$ function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ which describes the left-translated domain $L_{q^{-1}}\Omega$ near $\mathcal{O}$ and whose derivative has Hölder constant (at most) C_0. By Lemma 3.3.9, the neighborhood on which Φ describes Ω is (at least) a ball $B_K(\mathcal{O}, \kappa)$, where κ is some positive constant.

Using isometries (see Remark 2.4.40), we rotate (and possibly reflect) the domain $L_{q^{-1}}\Omega$ until the inward-pointing normal for the tangent plane at $\mathcal{O}$ is $\langle A, 0, C\rangle$, where A and C are non-negative. Call this domain Ω'. By Lemma 3.3.11, if we take κ such that $\kappa < 1/(2C_0)^{1/\alpha}$, then:

- If $A \leq C$, we may write $\Omega' \cap B_K(\mathcal{O}, \kappa)$ in the form $\{x_3 > \phi(x_1, x_2)\}$, where $\phi \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R})$.

- If $A > C$, we may write $\Omega' \cap B_K(\mathcal{O}, \kappa)$ in the form $\{x_1 > \phi(x_2, x_3)\}$, where $\phi \in C^{1,\alpha}(\mathbb{R}^2, \mathbb{R})$.

The original point $\tilde{p}$ has been relocated as a consequence of these gyrations. Call its new location p. We apply Lemma 3.3.6 to see where p may actually lie. From here on, we consider four cases, distinguished by the tilt of the tangent plane at $\mathcal{O}$.

Case 1: $A = 0$ (a horizontal tangent plane).
According to Lemma 3.3.6, the point p is given by $(0, 0, c)$, where c is some positive constant. Note that the domain $\Omega' \cap B_K(\mathcal{O}, \kappa)$ is given by $\{x \in B_K(\mathcal{O}, \kappa) \mid x_3 > \phi(x_1, x_2)\}$.

We will employ the curve $\gamma(t) = (0, 0, c + t)$, $t \geq 0$, as the weak s-John curve.

The scale-invariant weak s-John condition requires

$$\operatorname{dist}_K(\gamma(t), \partial\Omega) \geq \lambda D^{1-s}\, d_K(\gamma(t), p)^s \tag{3.3.12}$$

for $t > 0$, for some $\lambda > 0$ independent of $\tilde{p}$. We only need to check this condition until the left-hand side is at least εD; to assure this, it is sufficient to verify the right-hand side is greater than εD.

Now we recast the weak s-John condition given above, noting that the point $\gamma(t)$ being a certain distance from the boundary is equivalent to having a ball of that radius, centered at $\gamma(t)$, that lies wholly inside the domain. We begin by reformulating such a Heisenberg ball:

$$\begin{aligned} B_K\big(\gamma(t), \lambda D^{1-s}\, d_K(\gamma(t), p)^s\big) &= L_{\gamma(t)}\Big(B\big(\mathcal{O}, \lambda D^{1-s}\, d_K(\gamma(t), p)^s\big)\Big) \\ &= \big\{\gamma(t) \cdot h \;\big|\; h \in \mathbb{H}^1, |h|_K < \lambda D^{1-s} d_K(\gamma(t), p)^s\big\}. \end{aligned} \tag{3.3.13}$$

So, satisfying the weak s-John condition is equivalent to having $\gamma(t) \cdot h \in \Omega'$ for all $h \in \mathbb{H}^1$ such that $|h|_K < \lambda D^{1-s}\, d_K(\gamma(t), p)^s$.

Let subscripts on a point denote which coordinates of the point are being referenced. We can rewrite $\gamma(t) \cdot h \in \Omega'$ as

$$(\gamma(t) \cdot h)_3 > \psi\big((\gamma(t) \cdot h)_{1,2}\big). \tag{3.3.14}$$

We use the $C^{1,\alpha}$ regularity of ϕ, writing C_0 to indicate the uniform Hölder constant for ϕ (see Remark 3.3.12). Applying Lemma 3.3.13 to (3.3.14 and rearranging the result, we find it suffices to show

$$C_0\big[(\gamma(t) \cdot h)_1^2 + (\gamma(t) \cdot h)_2^2\big]^{(\alpha+1)/2} < (\gamma(t) \cdot h)_3 \tag{3.3.15}$$

holds for all appropriate h and $t \geq 0$, until we are far enough from the boundary.

With our chosen curve $\gamma(t)$, we get $\gamma(t) \cdot h = (h_1, h_2, c + t + h_3)$, which makes the condition to be verified

$$C_0\big[h_1^2 + h_2^2\big]^{(\alpha+1)/2} - h_3 < c + t. \tag{3.3.16}$$

We moved the term h_3 to the left-hand side because it may be negative, which would work against our efforts to maintain the inequality. (We will do this maneuver in later cases also, each time we encounter a term which depends on h.)

To help verify this and later inequalities, we will employ some standard inequalities and a few observations:

- Jensen's inequality: for $a \geq 0$, $b \geq 0$, and $s \geq 1$, we have $(a+b)^s \leq 2^{s-1}(a^s + b^s)$.
- For $a \geq 0$, $b \geq 0$, and $0 < \alpha \leq 1$, we have $(a + b)^\alpha \leq a^\alpha + b^\alpha$.
- Young's inequality: for $a \geq 0$, $b \geq 0$, $1 < p < \infty$, $1 < q < \infty$, and $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$.
- $\pm h_1 \leq |h_1| \leq |h|_K$.

- $\pm h_2 \leq |h_2| \leq |h|_K$.
- $\pm h_3 \leq |h_3| \leq |h|_K^2$.

Further, we have a bound on $|h|_K$:

$$|h|_K < \lambda D^{1-s} t^{s/2}.$$

This bound depended on γ, so it will change in each case.

Additionally, note that $|h|_K$ is less than εD. (Once it's bigger than that, the curve γ is sufficiently far from the boundary and our work is done.) As $\varepsilon D << 1$, we can always assume that $|h|_K$ is smaller than one.

With these tools, we can now dispatch this case, the easiest one, in a single bite. We assume that ε is small enough that $0 \leq t \leq 1$, an assumption we will justify at the very end of our proof. (Determining this is a valid assumption will require knowing how small λ needs to be chosen, and we will only have that answer after seeing all of the constraints placed on it by each case.)

$$\begin{aligned} C_0(h_1^2 + h_2^2)^{(\alpha+1)/2} - h_3 &\leq 2C_0 |h|_K^{\alpha+1} + |h|_K^2 \\ &\leq 2C_0(\lambda D^{1-s})^{\alpha+1} t^{s(\alpha+1)/2} + (\lambda D^{1-s})^2 t^s \\ &< c + t, \end{aligned}$$

where the last inequality holds if the exponents $s(\alpha+1)/2$ and s are at least one (which is true) and if $\lambda \leq 2^{-1/2} D^{s-1}$ and

$$\lambda < D^{s-1} \left(\frac{1}{4C_0} \right)^{1/(\alpha+1)}. \tag{3.3.17}$$

Note that (3.3.17) supersedes the requirement on λ immediately preceding it. As new restrictions appear on λ, we will keep track of which is the most constraining.

We now observe two issues that recur and one that doesn't. First, the need for

$s(\alpha+1)/2$ to be at least one, from which derives the condition in the hypotheses of the theorem, only appears this one time. In the other three cases, the situation with the exponent is never as tight as it is here. (As discussed in Example 3.3.2, this is not unexpected.) We'll find the later verifications become more complicated, but that is mainly a consequence of the general difficulty of doing calculations in a Carnot group.

Second, note that c played no part in the ultimate verification of the inequality. Since c can be arbitrarily small (and positive), the burden must be carried by the other term.

Finally, the extra factors of 2 which crept into the constraints on λ arise because there were two terms, both of which needed to be bounded by a single t. So, each term was individually bounded by $\frac{1}{2}t$. In future cases, we will see such small integers appearing in a predictable fashion. They are all small and finite, so their involvement in the upper bounds on λ will not be a cause for concern.

Case 2: $C = 0$ (a vertical tangent plane).

The setup is quite similar, so we will primarily note the differences. The point p is located at $(a, 0, 0)$, where $a > 0$, and the domain $\Omega' \cap B(\mathcal{O}, \kappa)$ is given by $\{x \in B(\mathcal{O}, \kappa) \mid x_1 > \phi(x_2, x_3)\}$. We will employ the weak s-John curve $\gamma(t) = (a + t, 0, 0)$, $t \geq 0$.

The recharacterization given by (3.3.13) still applies. The condition $\gamma(t) \cdot h \in \Omega'$ becomes

$$(\gamma(t) \cdot h)_1 > \psi\big((\gamma(t) \cdot h)_{2,3}\big) \tag{3.3.18}$$

Again employing Lemma 3.3.13 and rearranging the result, we find it suffices to show

$$C_0\big[(\gamma(t) \cdot h)_2^2 + (\gamma(t) \cdot h)_3^2\big]^{(\alpha+1)/2} < (\gamma(t) \cdot h)_1 \tag{3.3.19}$$

holds for all appropriate h and $t \geq 0$, until we are far enough from the boundary.

For this choice of γ, we get $\gamma(t) \cdot h = (a + t + h_1, h_2, h_3 + \frac{1}{2}(a + t)h_2)$, which makes the condition to be verified

$$C_0\left[h_2^2+\left(h_3+\tfrac{1}{2}(a+t)h_2\right)^2\right]^{(\alpha+1)/2}-h_1<a+t. \tag{3.3.20}$$

The bound on $|h|_K$ in this case is $|h|_K<\lambda D^{1-s}t^s$.

This verification is not as easy as the first, but still quite manageable. We begin by bounding the left-hand side of our desired inequality:

$$\begin{aligned}
&C_0\left[h_2^2+\left(h_3+\tfrac{1}{2}(a+t)h_2\right)^2\right]^{(\alpha+1)/2}-h_1\\
&\qquad\le C_0|h|_K^{\alpha+1}+C_0\left|h_3+\tfrac{1}{2}(a+t)h_2\right|^{\alpha+1}+|h|_K\\
&\qquad\le C_0|h|_K^{\alpha+1}+C_02^\alpha|h|_K^{2\alpha+2}+\tfrac{1}{2}C_0(a+t)^{\alpha+1}|h|_K^{\alpha+1}+|h|_K\\
&\qquad\le 3C_02^\alpha|h|_K+\tfrac{1}{2}C_0(a+t)^{\alpha+1}|h|_K^{\alpha+1}.
\end{aligned}$$

We seek to bound each of the final terms by $\frac{1}{2}(a+t)$. The first term is readily bounded:

$$3C_02^\alpha|h|_K<3C_02^\alpha\lambda D^{1-s}t^s\le\frac{1}{2}t,$$

where the last inequality holds if

$$\lambda\le\frac{D^{s-1}}{3C_02^\alpha}. \tag{3.3.21}$$

This constraint supersedes (3.3.17) for certain values of α and C_0.

To bound the second term, let $M:=\max\{a,t\}$. Note that $M<1$, which follows from $t<1$ and

$$a=d_K(p,\mathcal{O})=\operatorname{dist}(\tilde{p},\partial\Omega)<\varepsilon D<<1.$$

We bound the second term:

$$\begin{aligned}\frac{1}{2}C_0(a+t)^{\alpha+1}|h|_K &< \frac{1}{2}C_0(2M)^{\alpha+1}(\lambda D^{1-s}M^s)\\ &\leq C_0 2^\alpha M^{\alpha+1+s}(\lambda D^{1-s})\\ &\leq \frac{1}{2}(a+t),\end{aligned}$$

where the final inequality holds if

$$\lambda \leq \frac{D^{s-1}}{C_0 2^{\alpha+1}}, \tag{3.3.22}$$

which is superseded by the constraint (3.3.21).

This concludes case (2). Notice that bounding the first term required $s \geq 1$; for the other bounds, the exponentsgave us a generous amount of room and led to no new constraints.

Case 3: $0 < A \leq C$ (a not-very-steep tangent plane).
Let $\nu = A/C$. The point $p = (a, b, c)$ is given by

$$a(\delta) = \frac{2\nu\delta^{1/2}}{\delta+1}, \quad b(\delta) = \frac{2\nu\delta}{\delta+1}, \quad c(\delta) = \frac{16\nu^2\delta^{3/2}}{\delta+1}.$$

The domain $\Omega' \cap B(\mathcal{O}, \kappa)$ is given by $\{x \in B(\mathcal{O}, \kappa) \mid x_3 > \phi(x_1, x_2)\}$.

Part of the reasoning behind employing four cases is to specify the linear behavior of ϕ at the origin. In the next two cases, it will be useful for us to separate out this term. To do this, we define the function ψ by extracting ϕ's linear behavior:

$$\phi(x_1, x_2) = -\nu x_1 + \psi(x_1, x_2).$$

As ϕ is $C^{1,\alpha}$, so is ψ; we also note that $\psi(0,0) = 0$ and $\nabla\psi(0,0) = \langle 0, 0\rangle$.

The recharacterization given by (3.3.13) still applies. The condition $\gamma(t) \cdot h \in \Omega'$

becomes

$$(\gamma(t) \cdot h)_3 > -\nu(\gamma(t) \cdot h)_1 + \psi((\gamma(t) \cdot h)_{1,2}).$$

Again employing Lemma 3.3.13 and rearranging the result, we find it suffices to show

$$C_0\big[(\gamma(t) \cdot h)_1^2 + (\gamma(t) \cdot h)_2^2\big]^{(\alpha+1)/2} < (\gamma(t) \cdot h)_3 + \nu(\gamma(t) \cdot h)_1 \tag{3.3.23}$$

holds for all appropriate h and $t \geq 0$, until we are far enough from the boundary.

The not-very-steep tangent plane at the origin means that, while this isn't a "worst-case" for escaping the boundary, it is a bad case. To manage the verification, we need to make further manipulations to the space before we begin. The point p lies above the boundary point $r = (a, b, \phi(a, b))$. We left-translate the space by r^{-1} (an isometry), which takes r to the origin and p to a point $p' = (0, 0, c-\phi(a, b))$ on the $+x_3$-axis. Next, we rotate the space about the x_3-axis (another isometry) until the inward-pointing normal of the tangent plane to the boundary at the origin has the form $\langle A', 0, C'\rangle$, with A' and C' both non-negative. After these operations, the boundary near the origin will be described by a $C^{1,\alpha}$ function $\widetilde{\phi}(x_1, x_2) = -\nu x_1 + \widetilde{\psi}(x_1, x_2)$, where now $\nu = A'/C'$. (The function $\widetilde{\psi}$ serves the same role as ψ did.)

Before continuing, let us address why the ratio A'/C' is well-defined. To understand this ratio, we develop values for A' and C' by relating the original normal $\langle A, 0, C\rangle$ to $\langle A', 0, C'\rangle$.

Recall that, by our preparatory steps, $\nabla\phi(0, 0) = \langle -\nu, 0\rangle = \langle -A/C, 0\rangle$. By $\tilde{p}$ being chosen within distance εD of the boundary, we know that $|p|_K = d_K(p, \mathcal{O}) < \varepsilon D$. Also, we are assuming that $\varepsilon D << 1$. Since $\phi \in C^{1,\alpha}$, we have:

$$|\nabla\phi(r) - \nabla\phi(0, 0)| \leq C_0(a^2 + b^2)^{\alpha/2} \leq C_0|p|_K^\alpha < C_0(\varepsilon D)^\alpha.$$

Let $\nabla\phi(r) = \langle M, N\rangle$. The preceding inequality implies $(M+\nu)^2 + N^2 \leq C_0^2(\varepsilon D)^{2\alpha}$. Thus, there exists values μ_1 and μ_2 (not necessarily positive) smaller than $C_0(\varepsilon D)^\alpha$ such that $M = -\nu + \mu_1 = (-A/C) + \mu_1$ and $N = \mu_2$. Hence, the tangent plane to the surface at r has inward-pointing normal vector $\langle (A/C) - \mu_1, -\mu_2, 1\rangle$, which we normalize to $\langle A - C\mu_1, -C\mu_2, C\rangle$.

Left-translating the space by r^{-1} sends this normal vector to the vector

$$\left\langle A - C\mu_1, -C\mu_2, C - \frac{b}{2}(A - C\mu_1) + \frac{a}{2}(-C\mu_2)\right\rangle$$

at the origin. Finally, the rotation sends this vector to the vector

$$\left\langle \sqrt{(A - C\mu_1)^2 + (C\mu_2)^2}, 0, C - \frac{b}{2}(A - C\mu_1) + \frac{a}{2}(-C\mu_2)\right\rangle.$$

Hence, the ratio A'/C' is given by:

$$\nu = \frac{\sqrt{(A - C\mu_1)^2 + (C\mu_2)^2}}{C - \frac{b}{2}(A - C\mu_1) + \frac{a}{2}(-C\mu_2)}.$$

We note that the numerator is bounded above by $|A - C\,C_0(\varepsilon D)^\alpha| + C\,C_0(\varepsilon D)^\alpha$. If $A \geq C\,C_0(\varepsilon D)^\alpha$, then the upper bound on the numerator is A. If $A < C\,C_0(\varepsilon D)^\alpha$, then we have the upper bound $2C\,C_0(\varepsilon D)^\alpha$.

For the denominator, we have the lower bound $C(1 - \frac{b}{2} - \frac{a}{2})$. The values of both a and b are bounded above by $|p|_K$, which is less than εD, giving us the lower bound $C(1 - \varepsilon D)$.

Thus, we arrive at

$$\frac{A'}{C'} \leq \frac{\max\{A, 2C\,C_0(\varepsilon D)^\alpha\}}{C(1 - \varepsilon D)}. \tag{3.3.24}$$

We now need to assume that ε is chosen small enough that $\varepsilon D < 1/C_0^{1/\alpha}$. (As mentioned earlier, we will see in (3.3.39) how small ε will need to be.) This, together with the fact that $A/C \leq 1$ in this case, shows that A'/C' is bounded above, hence

C' is not zero, and the ν associated with $\widetilde{\phi}$ and $\widetilde{\psi}$ is well-defined.

Continuing on, we wish to escape from p' with a weak s-John curve. This is shown by first escaping the origin with a curve γ that satisfies the weak s-John condition for $t > 0$. (At $t = 0$, it is on the boundary, so the condition fails there.) Specifically, we choose:

$$\gamma(t) = \begin{cases} (t, 0, 0), & \text{if } 0 < t \leq t_1; \\ (t_1, 0, t - t_1), & \text{if } t \geq t_1. \end{cases} \tag{3.3.25}$$

We set the choice of t_1 as $t_1 = (\nu/4C_0)^{1/\alpha}$. (This is not an optimal choice, but it is good enough.) Close inspection of (3.3.24) shows $\nu \leq 4$; hence, $t_1 \leq \nu$. In the case where $\nu = 0$, we have $t_1 = 0$, and γ would have only one segment.

When this curve is translated upward so that it escapes from p', the translation only increases the distance to the boundary for each point on the curve, since the boundary lies beneath the domain. The weak s-John condition is maintained, with the same weak s-John constant.

Verification on the first piece of γ: $t \in (0, t_1]$

Note $\gamma(t) \cdot h = (t + h_1, h_2, h_3 + th_2)$ for the first piece of γ. The inequality (3.3.23) becomes

$$C_0 \left[(t + h_1)^2 + (h_2)^2\right]^{(\alpha+1)/2} < (h_3 + th_2) + \nu(t + h_1),$$

which rearranges to the form

$$C_0 \left[(t + h_1)^2 + (h_2)^2\right]^{(\alpha+1)/2} - h_3 - th_2 - \nu h_1 < \nu t.$$

For the first piece of γ, the condition $|h|_K < \lambda D^{1-s} d(\gamma(t), \mathcal{O})^s$ gives $|h|_K < \lambda D^{1-s} t^s$. Also, $t < t_1$ yields $t^\alpha < \nu/(4C_0)$.

We now demonstrate the bound:

$$
\begin{aligned}
&C_0\left[(t+h_1)^2+(h_2)^2\right]^{(\alpha+1)/2}-h_3-th_2-\nu h_1\\
&\quad\le C_0\left[|t+h_1|^{\alpha+1}+|h_2|^{\alpha+1}\right]+|h|_K^2+t|h|_K+\nu|h|_K\\
&\quad\le C_0 2^\alpha t^{\alpha+1}+C_0 2^\alpha|h|_K^{\alpha+1}+C_0|h|_K^{\alpha+1}+|h|_K^2+t|h|_K+\nu|h|_K\\
&\quad\le C_0 2^\alpha t^{\alpha+1}+4C_0|h|_K^{\alpha+1}+t|h|_K+\nu|h|_K\\
&\quad< 2^{\alpha-2}\nu t+4C_0(\lambda D^{1-s}t^s)^{\alpha+1}+t(\lambda D^{1-s}t^s)+\nu(\lambda D^{1-s}t^s)\\
&\quad< 2^{\alpha-2}\nu t+(\lambda D^{1-s})^{\alpha+1}t^{2-\alpha}\nu+\lambda D^{1-s}t^{s+1-\alpha}\frac{\nu}{4C_0}+\lambda D^{1-s}t^s\nu\\
&\quad\le\left[2^{\alpha-2}+\lambda\left(\lambda^\alpha D^{(1-s)(\alpha+1)}+\frac{D^{1-s}}{4C_0}+D^{1-s}\right)\right]\nu t\\
&\quad\le \nu t,
\end{aligned}
$$

where the final inequality is satisfied if

$$\lambda\left(\lambda^\alpha D^{(1-s)(\alpha+1)}+\frac{D^{1-s}}{4C_0}+D^{1-s}\right)\le\frac{1}{2}.$$

This condition will hold, for example, if

$$\lambda\le\frac{1}{6}D^{s-1}. \tag{3.3.26}$$

Taking this as our constraint, note that it supercedes the previous constraints (3.3.17) and (3.3.21) for certain values of α and C_0.

Finally, note that the choice of t_1 was guided by the $2^{\alpha-2}$ term above; without λ to control that term, we needed to ensure that it was bounded away from one by an appropriate choice of t_1.

<u>Verification on the second piece of γ</u>: $t>t_1$

Note that, for this segment of γ,

$$\gamma(t) \cdot h = (t_1 + h_1, h_2, t - t_1 + h_3 + t_1 h_2).$$

Thus, (3.3.23) becomes

$$C_0 \left[(t_1 + h_1)^2 + h_2^2 \right]^{(\alpha+1)/2} \leq (t - t_1 + h_3 + t_1 h_2) + \nu(t_1 + h_1)$$

which rearranges to the form

$$C_0 \left[(t_1 + h_1)^2 + h_2^2 \right]^{(\alpha+1)/2} - h_3 - t_1 h_2 - \nu h_1 \leq (t - t_1) + \nu t_1.$$

We begin by bounding the left-hand side:

$$\begin{aligned} &C_0 \left[(t_1 + h_1)^2 + h_2^2 \right]^{(\alpha+1)/2} - h_3 - t_1 h_2 - \nu h_1 \\ &\quad \leq C_0 \left[(t_1 + h_1)^{\alpha+1} + |h_2|^{\alpha+1} \right] + |h|_K^2 + t_1 |h|_K + \nu |h|_K \\ &\quad \leq C_0 2^\alpha t_1^{\alpha+1} + C_0 2^\alpha |h|_K^{\alpha+1} + C_0 |h|_K^{\alpha+1} + |h|_K^2 + t_1 |h|_K + \nu |h|_K \\ &\quad \leq 2^{\alpha-2} \nu t_1 + 4 C_0 |h|_K^{\alpha+1} + 2\nu |h|_K \\ &\quad =: I_1 + I_2 + I_3. \end{aligned}$$

For the second part of γ, $|h|_K < \lambda D^{1-s} \, d(\gamma(t), \mathcal{O})^s$ leads to the following bound:

$$\begin{aligned} |h|_K &< \lambda D^{1-s} (t_1^4 + (t - t_1)^2)^{s/4} \\ &\leq \lambda D^{1-s} (t_1^s + (t - t_1)^{s/2}). \end{aligned}$$

We now address the three pieces (I_1, I_2, I_3) separately. For I_1, it is trivially true that $I_1 < (1/2)\nu t_1$. For I_2:

$$\begin{aligned} I_2 = 4C_0|h|_K^{\alpha+1} &< 4C_0\left(\lambda D^{1-s}\left[t_1^s + (t-t_1)^{s/2}\right]\right)^{\alpha+1} \\ &\leq C_0 2^{\alpha+2}(\lambda D^{1-s})^{\alpha+1}(t_1^2 + (t-t_1)) \\ &= 2^\alpha(\lambda D^{1-s})^{\alpha+1}\nu t_1^{2-\alpha} + 2^{\alpha+2}C_0(\lambda D^{1-s})^{\alpha+1}(t-t_1) \\ &\leq \frac{1}{4}\nu t_1 + \frac{1}{2}(t-t_1), \end{aligned}$$

where the last inequality holds if

$$\lambda \leq D^{s-1}\left(\frac{1}{C_0 2^{\alpha+3}}\right)^{1/(\alpha+1)}.$$

A simpler, slightly stricter constraint on λ would be

$$\lambda \leq \frac{D^{s-1}}{16C_0}, \tag{3.3.27}$$

which supercedes all previous conditions on λ.

$$\begin{aligned} I_3 = 2\nu|h|_K &< 2\nu\lambda D^{1-s}(t_1^s + (t-t_1)^{s/2}) \\ &\leq 2\lambda D^{1-s}\nu t_1^s + 2\lambda D^{1-s}\left(\frac{\alpha}{\alpha+1}\right)\nu^{(\alpha+1)/\alpha} + 2\lambda D^{1-s}\left(\frac{1}{\alpha+1}\right)(t-t_1) \\ &\leq 2\lambda D^{1-s}\nu t_1^s + 2\lambda D^{1-s}\left(\frac{\alpha}{\alpha+1}\right)(4C_0)^{1/\alpha}\nu t_1 + 2\lambda D^{1-s}\left(\frac{1}{\alpha+1}\right)(t-t_1) \\ &\leq \frac{1}{8}\nu t_1 + \frac{1}{8}\nu t_1 + \frac{1}{2}(t-t_1) \end{aligned}$$

where the last inequality holds if

$$\lambda \leq \frac{1}{8(4C_0)^{1/\alpha}}D^{s-1}. \tag{3.3.28}$$

This is a stronger condition than (3.3.27) and will be the ultimate requirement on λ.

Note that the bound on λ is independent of ε; this is important since the choice of ε will depend on λ.

This concludes the bounds on the second part of γ. We did not encounter any constraints on how large t could be, so we can continue the curve until we have achieved a distance of εD from the boundary. This distance will certainly have been achieved once $t = t_1 + (\varepsilon D^s/\lambda)^{2/s}$. Finally, as indicated earlier, the curve is now shifted up so that it originates from p' on the $+x_3$-axis, to produce the desired weak s-John curve.

Case 4: $A > C > 0$ (a steep tangent plane).

Let $\nu = A/C$, as before; the point $p = (a, b, c)$ is the same as in Case 3, as well. However, we are forced to approach this case differently than Case 3. Here, ν can be arbitrarily large. Not only would this prevent some of our previous calculations from working, but we cannot trust that we could perform the initial manipulation of the space needed to prepare for the calculations: the boundary point r (as defined in Case 3) may not lie within $B(\mathcal{O}, \kappa)$!

So, unlike in Case 3, we perform this case without any further manipulations of the space. The unwieldy description of p, given by Lemma 3.3.6, must be used explicitly. As a consequence, the calculations that follow are rather intricate.

The domain $\Omega' \cap B(0, \kappa)$ is given by $\{x \in B(0, \kappa) \mid x_1 > \phi(x_2, x_3)\}$. We employ the weak s-John curve $\gamma(t) = (a + t, b, c)$, $t \geq 0$. Again, we separate out the linear behavior of ϕ at the origin, thereby defining ψ:

$$\phi(x_2, x_3) = -\frac{1}{\nu}x_3 + \psi(x_2, x_3).$$

The recharacterization given by (3.3.13) still applies. The condition $\gamma(t) \cdot h \in \Omega'$ becomes

$$(\gamma(t) \cdot h)_1 > -\frac{1}{\nu}(\gamma(t) \cdot h)_3 + \psi\big((\gamma(t) \cdot h)_{2,3}\big). \tag{3.3.29}$$

Again employing Lemma 3.3.13 and rearranging the result, we find it suffices to show:

$$C_0\big[(\gamma(t)\cdot h)_2^2 + (\gamma(t)\cdot h)_3^2\big]^{(\alpha+1)/2} < (\gamma(t)\cdot h)_1 + \frac{1}{\nu}(\gamma(t)\cdot h)_3 \tag{3.3.30}$$

holds for all appropriate h and $t \geq 0$, until we are far enough from the boundary.

For the chosen curve γ, we get $\gamma(t)\cdot h = \big(a+t+h_1, b+h_2, c+h_3+\frac{1}{2}((a+t)h_2-bh_1)\big)$. We multiply both sides of the condition by ν to arrive at the inequality to be verified:

$$\begin{aligned} C_0\nu\left[(b+h_2)^2 + \big(c+h_3+\tfrac{1}{2}((a+t)h_2-bh_1)\big)^2\right]^{(\alpha+1)/2} & \\ - h_3 - \tfrac{1}{2}((a+t)h_2-bh_1) - \nu h_1 &< c + \nu(a+t). \end{aligned} \tag{3.3.31}$$

The bound on $|h|_K$ in this case is $|h|_K < \lambda D^{1-s}\big(t^4 + (\frac{1}{2}tb)^2\big)^{s/4}$.

The fact that $\nu \geq 1$ in this case has some interesting ramifications. As mentioned in a remark following Lemma 3.3.6, the distance from p to the origin is given by:

$$d_K(p,\mathcal{O}) = 2\nu\left(\frac{\delta}{\delta+1}\right)^{1/2}(1+16\delta)^{1/4}. \tag{3.3.32}$$

As the point $\tilde{p}$ was initially chosen within εD of the boundary, this equation leads to several useful inequalities. Assume that $\varepsilon < \frac{1}{D\sqrt{2}}$. We avoid the awkward term $(1+16\delta)^{1/4}$ in the following work by bounding it below by one.

I. $\nu b < \frac{1}{2}\varepsilon^2 D^2$, and hence also $b < 1$.

This inequality comes from rearranging the terms:

$$\varepsilon D > 2\nu\left(\frac{\delta}{\delta+1}\right)^{1/2} = \sqrt{2\nu}\left(\frac{2\nu\delta}{\delta+1}\right)^{1/2} = \sqrt{2\nu b}. \tag{3.3.33}$$

II. $\delta \leq 1$ and $\delta^{1/2} < \varepsilon D$.

$\frac{\varepsilon D}{2\nu} > \left(\frac{\delta}{\delta+1}\right)^{1/2}$ and $\nu \geq 1$ implies that $\frac{1}{2}\varepsilon D > \left(\frac{\delta}{\delta+1}\right)^{1/2}$. This inequality easily leads to $\delta \le 1$ (using our standing assumption on ε), and hence $\delta^{1/2} < \varepsilon D$.

III. $c < 4\varepsilon^2 D^2$.

Again, this inequality comes from rearranging the terms:

$$\varepsilon D > 2\nu \left(\frac{\delta}{\delta+1}\right)^{1/2} = \frac{1}{2\delta^{1/4}}\left(\frac{16\nu^2\delta^{3/2}}{\delta+1}\right)^{1/2} = \frac{\sqrt{c}}{2\delta^{1/4}}. \tag{3.3.34}$$

Hence, $c < 4\varepsilon^2 D^2 \sqrt{\delta} \le 4\varepsilon^2 D^2$.

IV. $|h|_K \le 1$.

We've noted this inequality before: it derives from $|h|_K \le \varepsilon D << 1$.

Recall what needs to be established:

$$C_0\nu\left[(b+h_2)^2 + \left(c+h_3+\tfrac{1}{2}((a+t)h_2 - bh_1)\right)^2\right]^{(\alpha+1)/2} \\ - h_3 - \tfrac{1}{2}((a+t)h_2 - bh_1) - \nu h_1 < c + \nu(a+t). \tag{3.3.35}$$

We begin by bounding the left-hand side:

$$\begin{aligned}
&C_0\nu\left[(b+h_2)^2 + \left(c+h_3+\tfrac{1}{2}((a+t)h_2 - bh_1)\right)^2\right]^{(\alpha+1)/2} - h_3 - \frac{1}{2}((a+t)h_2 - bh_1) - \nu h_1 \\
&\quad\le C_0\nu\left[(b+h_2)^{\alpha+1} + \left(c+h_3+\tfrac{1}{2}((a+t)h_2 - bh_1)\right)^{\alpha+1}\right] + |h|_K^2 + \frac{1}{2}(a+t)|h|_K \\
&\qquad + \left(\tfrac{1}{2}b - \nu\right) h_1 \\
&\quad\le C_0\nu\Big[2^\alpha b^{\alpha+1} + 2^\alpha |h|_K^{\alpha+1} + 2^{2\alpha} c^{\alpha+1} + 2^{2\alpha}|h|_K^{2\alpha+2} + 2^{2\alpha}\left(\tfrac{1}{2}(a+t)|h|_K\right)^{\alpha+1} \\
&\qquad + 2^{2\alpha}(\tfrac{1}{2}b|h|_K)^{\alpha+1}\Big] + |h|_K^2 + \tfrac{1}{4}(a+t)^2 + \tfrac{1}{4}|h|_K^2 + \tfrac{\nu}{\delta+1}|h|_K \\
&\quad\le C_0 2^\alpha \nu b^{\alpha+1} + C_0 2^\alpha \nu |h|_K^{\alpha+1} + C_0 2^{2\alpha}\nu c^{\alpha+1} + C_0 2^{2\alpha}\nu|h|_K^{2\alpha+2} + C_0 2^{2\alpha-1}\nu(a+t)^{2\alpha+2} \\
&\qquad + C_0 2^{2\alpha-1}\nu|h|_K^{2\alpha+2} + C_0 2^{2\alpha-1}\nu b^{2\alpha+2} + C_0 2^{2\alpha-1}\nu|h|_K^{2\alpha+2} + |h|_K^2 + \frac{1}{4}(a+t)^2 \\
&\qquad + \tfrac{1}{4}|h|_K^2 + \tfrac{\nu}{\delta+1}|h|_K \\
&\quad =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12}.
\end{aligned}$$

Before we start to despair over the number of terms, we cut our work down to manageable proportions.

- By tool I, we know that $I_1 \geq I_7$.
- Since $I_{12} \geq \frac{1}{2}|h|_K$ (by tool II and $\nu \geq 1$), we have $2I_{12} \geq I_9 + I_{11}$.
- By tool IV, $3I_2 \geq I_4 + I_6 + I_8$.
- By tools II and IV, $C_0 2^\alpha I_{12} \geq I_2$.

It remains to show $2I_1$, I_3, I_5, I_{10}, and $(2 + 3C_0 2^\alpha)I_{12}$ can be bounded explicitly.

Bounding $2I_1$

We employ the definition of b to bound $2I_1$ by $\frac{1}{6}\nu a$:

$$\begin{aligned} 2I_1 = C_0 2^{\alpha+1}\nu b^{\alpha+1} &= C_0 2^{\alpha+2} b^\alpha \nu^2 \frac{\delta}{\delta+1} \qquad \text{(changing one power of } b\text{)} \\ &= C_0 2^{\alpha+1}\delta^{1/2}\nu a b^\alpha \qquad \text{(into a power of } a\text{)} \\ &\leq C_0 2^{\alpha+1}\varepsilon D \nu a \qquad \text{(since } b \leq 1\text{)} \\ &\leq \frac{1}{6}\nu a, \end{aligned}$$

where the last inequality holds if

$$\varepsilon \leq \frac{1}{6C_0 2^{\alpha+1} D}. \tag{3.3.36}$$

Bounding I_3

We begin by using the definition of c:

$$
\begin{aligned}
I_3 = C_0 2^{2\alpha}\nu c^{\alpha+1} &= C_0 2^{5\alpha+\frac{5}{2}}\delta^{(\alpha+1)/2}(\delta+1)^{1/2}\left(\nu\frac{2\nu\delta}{\delta+1}\right)^{\alpha+\frac{1}{2}}\left(\nu\frac{2\nu\delta^{1/2}}{\delta+1}\right)\\
&\le C_0 2^{5\alpha+\frac{5}{2}}\delta^{(\alpha+1)/2}(\delta+1)^{1/2}\left(\frac{1}{2}\varepsilon^2 D^2\right)^{\alpha+\frac{1}{2}}\nu a \qquad \text{(using tool I)}\\
&\le C_0 2^{4\alpha+\frac{5}{2}}\left(\varepsilon D\right)^{2\alpha+1}\nu a \qquad \text{(since } \delta\le 1\text{)}\\
&\le \frac{1}{6}\nu a,
\end{aligned}
$$

where the last inequality holds if

$$
\varepsilon \le \frac{1}{(6C_0 2^{4\alpha+5/2})^{1/(2\alpha+1)}D}. \tag{3.3.37}
$$

<u>Bounding I_5</u>

Our desired bound for I_5 is

$$
I_5 = 2^{2\alpha-1}C_0\nu(a+t)^{2\alpha+2} \le \tfrac{1}{6}\nu(a+t)
$$

which holds if

$$
a+t \le \left(\frac{1}{6C_0 2^{2\alpha-1}}\right)^{1/(2\alpha+1)}. \tag{3.3.38}
$$

We will return to this condition after looking at the next two bounds.

<u>Bounding I_{10}</u>

Our desired bound for I_{10} is

$$
I_{10} = \frac{1}{4}(a+t)^2 \le \frac{1}{6}\nu(a+t)
$$

which holds if

$$
a+t \le \frac{2}{3}.
$$

This condition is superceded by (3.3.38).

<u>Bounding $(2+3C_0 2^\alpha)I_{12}$</u>

Recall that $|h|_K < \lambda D^{1-s}\big(t^s + \big(\frac{1}{2}bt\big)^{s/2}\big)$ in Case 4.

$$\begin{aligned} I_{12} = \frac{\nu}{\delta+1}|h|_K &< \nu|h|_K \\ &< \nu\lambda D^{1-s}\big(t^s + (\tfrac{1}{2}bt)^{s/2}\big) \\ &\le \nu\lambda D^{1-s}(\tfrac{3}{2}t^s + \tfrac{1}{2}b^s) \\ &\le \tfrac{3}{2}\lambda D^{1-s}\nu t + \tfrac{1}{2}\lambda D^{1-s}(b^{s-1}\delta^{1/2})\nu a \\ &\le \tfrac{3}{2}\lambda D^{1-s}\nu t + \tfrac{1}{2}\lambda D^{1-s}\nu a \\ &\le \tfrac{1}{6(2+3C_0 2^\alpha)}\nu t + \tfrac{1}{6(2+3C_0 2^\alpha)}\nu a, \end{aligned}$$

where the final inequality holds if

$$\lambda \le \frac{1}{9(2+3C_0 2^\alpha)} D^{s-1},$$

which is weaker than the previous constraint (3.3.28).

Now we go back to (3.3.38). Our concerns are that a might be too large (as an initial condition!) or that, in escaping to the desired distance from the boundary, t would need to get too large.

To address these concerns, we first note that $a < \varepsilon D$, by the choice of $\tilde{p}$. Also, the distance of $\gamma(t)$ to the boundary is bounded below by $\lambda D^{1-s}t^s$, so we are unconcerned about all t such that $\lambda D^{1-s}t^s \ge \varepsilon D$, as we would be done already. That is, we are only concerned for $t < (\frac{\varepsilon D^s}{\lambda})^{1/s}$. Putting these together, we get our answers:

$$\begin{aligned} a + t &\le \varepsilon D + (\frac{\varepsilon D^s}{\lambda})^{1/s} \\ &= \varepsilon^{1/s} D(1 + 1/\lambda^{1/s}) \\ &\le \left(\frac{1}{6C_0 2^{2\alpha-1}}\right)^{1/(2\alpha+1)}, \end{aligned}$$

where the last inequality holds if

$$\varepsilon \leq \left[\frac{1}{D(1+1/\lambda^{1/s})} \left(\frac{1}{6C_0 2^{2\alpha-1}} \right)^{1/(2\alpha+1)} \right]^s . \tag{3.3.39}$$

This completes case (4).

As promised, we saw that the required upper bound on λ, given by (3.3.28), does not depend on ε, and the bounds on ε are explicit enough that we can be assured that it may be chosen appropriately. □

3.4 Sharpness example for main theorem

In this example, we consider a bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{H}^1$, where α can be chosen as any value in $(0, 1)$. We will verify that the domain fails to be weak s-John for all $s < 2/(\alpha + 1)$. This would show Theorem 3.3.1 is sharp.

Example 3.4.1. Let $\Omega \subset \mathbb{H}^1$ be a bounded, Euclidean $C^{1,\alpha}$ domain, given near the origin by the set $\{(z, t) \mid |z|^{\alpha+1} < t\}$. This is the same domain considered in the motivating example, Example 3.3.2. We assume that Ω is a weak s-John domain, for some $s < 2/(\alpha+1)$, and seek a contradiction. Let λ be the weak s-John constant for Ω.

Consider the point $p = (0, \varepsilon)$, $\varepsilon > 0$, on the t-axis. We may assume that ε is small. We saw in Example 3.3.2 that the (locally) optimal weak s-John curve exiting from p will travel directly up the t-axis; let us parametrize this curve as $\gamma(u) = (0, \varepsilon + u)$, $u \geq 0$.

As before, to simplify our calculations, we will use the max metric (2.4.5) in $\mathbb{H}^1$.

Consider the point on the curve $\gamma(\varepsilon) = (0, 2\varepsilon)$. We need to find the inequality that arises from the weak s-John condition applied at this point.The distance from

$(0, 2\varepsilon)$ to a boundary point (z, t) near the origin is:

$$d_M\big((z,t),(0,2\varepsilon)\big) = \max\left\{|z|, |t-2\varepsilon|^{1/2}\right\} = \max\left\{|t|^{\frac{1}{\alpha+1}}, |t-2\varepsilon|^{1/2}\right\}.$$

To find the minimal distance to the boundary, note that we may restrict t to the range $[0, 2\varepsilon]$.

So, consider both terms in the max expression as t ranges from 0 to 2ε. The first term strictly (and continuously) increases and the second strictly (and continuously) decreases. Hence, the minimal value of the max occurs when the two terms are equal, i.e., $|z| = |t-2\varepsilon|^{1/2}$. Let (z_0, t_0) be a point on the boundary for which this equation holds. (Note, then, that $\mathrm{dist}_M\big((0,2\varepsilon), \partial\Omega\big) = |z_0|$.) Combining this equation with the defining equation for the boundary, we get the relationship:

$$|z_0|^2 + |z_0|^{\alpha+1} = 2\varepsilon.$$

Trivially, we get $|z_0|^{\alpha+1} \leq 2\varepsilon$; hence, $\mathrm{dist}_M\big((0,2\varepsilon), \partial\Omega\big) = |z_0| \leq (2\varepsilon)^{1/(\alpha+1)}$.

From the weak s-John condition on γ at $(0, 2\varepsilon)$, we have the inequality:

$$\lambda(\mathrm{diam}_M\,\Omega)^{1-s}\, d_M\big((0,\varepsilon),(0,2\varepsilon)\big)^s \leq \mathrm{dist}_M\big((0,2\varepsilon),\partial\Omega\big) \leq (2\varepsilon)^{1/(\alpha+1)}.$$

The max-metric distance from $(0, \varepsilon)$ to $(0, 2\varepsilon)$ is easily calculated to be $\varepsilon^{1/2}$, so we make this substitution and rearrange the previous inequality:

$$2^{-1/(\alpha+1)}(\mathrm{diam}_M\,\Omega)^{1-s}\lambda \leq \varepsilon^{\frac{1}{\alpha+1}-\frac{s}{2}}.$$

For $s < 2/(\alpha+1)$, the exponent of ε here is positive. Hence, for ε small enough, this inequality fails to hold, and the optimal weak s-John curve exiting $(0, \varepsilon)$ (for this small ε) will fail to satisfy the weak s-John condition (with λ as its constant). This is our contradiction.

3.5 Weak s-John is not s-John in $\mathbb{H}^1$

The motivation for this example, a weak s-John domain which is not also s-John, comes from a result by Hajłasz and Koskela [32, Proposition 9.6].

Theorem 3.5.1 (Hajłasz-Koskela)**.** *Let X be a metric space which is doubling on $\Omega \subset X$. Assume that Ω has the following local connectivity property: there exists a constant $\delta \geq 1$ such that for every ball B with $\delta B \subset \Omega$, every two points x,$y \in B$ can be connected by a rectifiable curve contained in δB and of length less than or equal to $\delta d(x, y)$. Then Ω is a John domain if and only if Ω is a weak John domain.*

The conditions on the metric space are fairly easy to satisfy. One space in which this theorem applies is $(\mathbb{H}^1, d_{CC})$; this fact is notable because $(\mathbb{H}^1, d_{CC})$ is the setting for my counterexample in the s-John case ($s > 1$).

Example 3.5.2. Fix $\alpha \in (0, 1)$. We let Ω be as it was in Section 3.4; specifically, recall that Ω coincides with $\{(z, t) \mid |z|^{1+\alpha} < t\}$ in a neighborhood of the origin. This bounded, Euclidean $C^{1,\alpha}$ domain in $\mathbb{H}^1$ is weak s-John for any $s \geq 2/(1 + \alpha)$, by Theorem 3.3.1. Let $s = 2/(1 + \alpha)$; we will show that Ω is not s-John, thereby demonstrating that weak s-John and s-John domains do not coincide in $\mathbb{H}^1$.

We showed explicitly in Section 3.4 that this domain is $C^{1,\alpha}$ near the origin, and we assume that it is fully a $C^{1,\alpha}$ domain. We do not need explicitly to define the domain elsewhere, as the demonstration of the failure to be s-John will occur near the origin.

We argue by contradiction. Assume that Ω is s-John with s-John constant λ and s-John center $x_0 = (z_0, t_0)$, with $t_0 > 0$.

Let $0 < \varepsilon < \delta < t_0$. Then there are integers k_0 and k_1 such that

$$2^{-k_0-1} \leq \varepsilon < 2^{-k_0} \quad \text{and} \quad 2^{-k_1} \leq \delta < 2^{-k_1+1}.$$

Let $\widetilde{\gamma}$ be an s-John curve joining $(0, \varepsilon)$ to x_0. Hence, by the continuity of $\widetilde{\gamma}$, there is at least one point where $\widetilde{\gamma}$ intersects the plane $\{t = \delta\}$. Let the first such intersection be at the point $(z, \delta) \in \Omega$. Let γ denote the subcurve of $\widetilde{\gamma}$ from $(0, \varepsilon)$ to (z, δ). We will show, for small enough ε and δ, the s-John condition will fail to hold, providing our contradiction. For such an approach, we need estimates for the length of γ and for the distance from (z, δ) to the boundary of Ω.

To estimate the length of γ, we will use the constructive approach employed by Balogh and Monti in [5, pp. 101-102].

A Heisenberg box is given by $\mathrm{Box}(x, r) := \{xy \mid \|y\|_M \leq r\}$, where $\|\cdot\|_M$ is the max norm. The construction is to cover (most of) the set $\{(z, t) \mid 2^{-k-1} \leq t \leq 2^{-k}\} \cap \Omega$ with Heisenberg boxes with disjoint interiors. The boxes will be very short and stacked vertically, and they will extend (in the z direction) well past the boundary of Ω. We will bound the length of γ from below by bounding its length inside each box.

A standard piece of Carnot group machinery is the Box-Ball Theorem, stated here as in [5]:

Theorem 3.5.3 (Box-Ball Theorem)**.** *There exists a constant* $\lambda \in (0, 1)$ *such that* $\|y^{-1}x\|_M \leq d_{CC}(x, y) \leq \lambda^{-1}\|y^{-1}x\|_M$ *for all* $x, y \in \mathbb{H}^1$.

Let $\phi(t) = t^{1/(\alpha+1)}$. Then Ω near the origin is given by $\{(z, t) \mid |z| < \phi(t)\}$. For $k \in \mathbb{Z}$, let $p_k = (0, 2^{-k})$ and $r_k = 4\phi(2^{-k})$.

Consider the line segment $[p_{k+1}, p_k]$ on the vertical axis. This interval has change in its t-coordinate, Δt, equal to 2^{-k-1}. A Heisenberg box of radius r_k covers a t-interval with $\Delta t = 2r_k^2$. Hence, the number of vertically-stacked Heisenberg boxes with radius r_k needed to cover this segment is at least $\lfloor N_k \rfloor$, where N_k is given by:

$$N_k = \frac{2^{-k-1}}{2r_k^2}.$$

We call these boxes $\mathrm{Box}_{kj} := \mathrm{Box}((0, 2^{-k-1} + (2j-1)r_k^2), r_k)$, where $j = 1, \ldots, \lfloor N_k \rfloor$.

To verify that $N_k \geq 1$ (and hence there *are* such boxes), we simplify the right-hand side:

$$\frac{2^{-k-1}}{2r_k^2} = 2^{-6-k+2k/(\alpha+1)}.$$

Thus, N_k is at least one if the exponent above is positive, that is, if $k(\frac{2}{\alpha+1} - 1) > 6$. As we ultimately will be achieving our result for any ε and δ small enough, we can assume that we will only be using boxes with k as large as needed to satisfy this inequality.

We denote the upper point of intersection of $\partial \operatorname{Box}_{kj}$ with the t-axis as p_{kj}^+ and the lower point of intersection as p_{kj}^-. Let γ_{kj} denote the intersection of γ with Box_{kj}. Finally, we denote a point of intersection of γ with the top of $\partial \operatorname{Box}_{kj}$ as q_{kj}^+ and with the bottom of $\partial \operatorname{Box}_{kj}$ as q_{kj}^-.

To bound the length of γ_{kj}, we first use the triangle inequality:

$$l(\gamma_{kj}) \geq d_{CC}(q_{kj}^+, q_{kj}^-) \geq d_{CC}(p_{kj}^+, p_{kj}^-) - d_{CC}(p_{kj}^+, q_{kj}^+) - d_{CC}(p_{kj}^-, q_{kj}^-). \tag{3.5.1}$$

By the Box-Ball Theorem, $d_{CC}(p_{kj}^+, p_{kj}^-) \geq \|(p_{kj}^+)^{-1} p_{kj}^-\|_M \geq r_k = 4\phi(2^{-k})$. For the other two distances on the right side of (3.5.1), note that the "p" point lies on the t-axis and the "q" point has the same t-coordinate, so the CC-geodesic is a straight line with length equal to its usual Euclidean length. Also, the z-coordinate of each "q" point has norm bounded by $\phi(2^{-k})$, since the points must lie inside of Ω. Hence, $d_{CC}(p_{kj}^+, q_{kj}^+) \leq \phi(2^{-k})$ and $d_{CC}(p_{kj}^-, q_{kj}^-) \leq \phi(2^{-k})$. Hence, we arrive at the simplified bound:

$$l(\gamma_{kj}) \geq 2\phi(2^{-k}),$$

for all k and $j = 1, \ldots, \lfloor N_k \rfloor$.

Let γ_k be the intersection of γ with $\bigcup_{j=1}^{\lfloor N_k \rfloor} \operatorname{Box}_{kj}$. We bound the length of γ_k from below:

$$l(\gamma_k) \geq \lfloor N_k \rfloor l(\gamma_{kj}) \geq \frac{N_k}{2} l(\gamma_{kj}) \geq N_k \phi(2^{-k}) = \frac{2^{-k-1}\phi(2^{-k})}{2r_k^2} = \frac{2^{-k-6}}{\phi(2^{-k})}.$$

Now, for the lower bound on $l(\gamma)$, we note that we cannot say how many (if any) of the boxes $\{\mathrm{Box}_{k_0 j}\}_{j=1}^{\lfloor N_{k_0} \rfloor}$ or $\{\mathrm{Box}_{(k_1+1)j}\}_{j=1}^{\lfloor N_{k_1+1} \rfloor}$ are intersected by γ. (These are the boxes covering where $(0, \varepsilon)$ and (z, δ) lie.) So, we will only count through the boxes lying between these. Let $t_k = 2^{-k}$.

$$\begin{aligned} l(\gamma) &\geq \sum_{k=k_1}^{k_0-1} l(\gamma_k) \geq \sum_{k=k_1}^{k_0-1} \frac{2^{-k-6}}{\phi(2^{-k})} = \frac{1}{2^6} \sum_{k=k_1}^{k_0-1} \frac{t_{k-1} - t_k}{\phi(t_k)} \\ &\geq \frac{1}{2^6} \int_{2^{-k_0+1}}^{2^{-k_1}} t^{-1/(\alpha+1)}\, dt = \frac{\alpha+1}{2^6 \alpha} \left[2^{-k_1 \alpha/(\alpha+1)} - 2^{(-k_0+1)\alpha/(\alpha+1)} \right]. \end{aligned}$$

For $\varepsilon < 2^{-(2\alpha+1)/\alpha}\delta$, the difference in brackets above is at least one half the first term in the brackets:

$$\begin{aligned} l(\gamma) &\geq \frac{\alpha+1}{2^7 \alpha} 2^{-k_1 \alpha/(\alpha+1)} \\ &> \frac{\alpha+1}{2^7 \alpha} \left(\frac{\delta}{2} \right)^{\alpha/(\alpha+1)} \\ &> \frac{\alpha+1}{2^8 \alpha} \delta^{\alpha/(\alpha+1)}. \end{aligned}$$

Now, we bound from above the distance from (z, δ) to the boundary:

$$\mathrm{dist}_{CC}((z, \delta), \partial\Omega) \leq d_{CC}((z, \delta), p) \leq \phi(\delta) = \delta^{1/(\alpha+1)}, \tag{3.5.2}$$

where the point $p = (\tilde{z}, \delta)$ lies on the boundary and is found by moving radially outward from the point (z, δ). (This radial motion away from the t-axis is a CC-geodesic.)

We now consider the s-John condition, along with our two bounds:

$$\lambda\left(\frac{\alpha+1}{2^8\alpha}\delta^{\alpha/(\alpha+1)}\right)^s \le \lambda(l(\gamma))^s \le \operatorname{dist}((z,\delta),\partial\Omega) \le \delta^{1/(\alpha+1)}. \tag{3.5.3}$$

Recalling that we chose $s = 2/(\alpha+1)$, we can simplify to the inequality:

$$\lambda\left(\frac{\alpha+1}{2^7\alpha}\right)^s \le \delta^{(1-\alpha)/(1+\alpha)^2}.$$

The exponent of δ is positive, and we may apply these calculations for ε and δ as small as we wish (provided that we maintain the relationship between ε and δ asserted above). This limiting of ε to zero forces $\lambda = 0$, which is a contradiction. Hence, Ω is not an s-John domain.

CHAPTER 4

Weak s-John implies Poincare in $\mathbb{R}^n$ and in Carnot groups

4.1 Preliminaries

First, we recall two definitions.

Definition 4.1.1. Let $\mathbb{G}$ be a Carnot group with stratified Lie algebra $\mathfrak{g} = v_1 \oplus \cdots \oplus v_k$. Let $X_1, \ldots, X_m$ be a basis for v_1. Let $\Omega \subset \mathbb{G}$ and $u \in C^\infty(\Omega, \mathbb{R})$. Then the *horizontal gradient* of u, $\nabla_0 u$, is given by the vector field

$$\nabla_0 u = \sum_{i=1}^{m} (X_i u) X_i.$$

Definition 4.1.2. Let $\mathbb{G}$ be a Carnot group with Haar measure μ, and let $\Omega \subset \mathbb{G}$ be a domain with finite μ-measure. Let $1 \leq p \leq q < \infty$. We say Ω is a (q,p)-*Poincaré domain* if there exists $C > 0$ such that

$$\left(\int_\Omega |u - u_\Omega|^q \, d\mu \right)^{1/q} \leq C \left(\int_\Omega |\nabla_0 u|^p \, d\mu \right)^{1/p} \tag{4.1.1}$$

for all $u \in C^\infty(\Omega, \mathbb{R})$, where $u_\Omega = \mu(\Omega)^{-1} \int_\Omega u \, d\mu$ is the average of u on Ω.

Remark 4.1.3. Recall that we fixed an inner product on the horizontal subspace v_1 (just before Definition 2.4.22); hence, the norm on $\nabla_0 u$ above is given by

$$|\nabla_0 u|^2 = \langle \nabla_0 u, \nabla_0 u \rangle_H = (X_1 u)^2 + \cdots + (X_m u)^2.$$

Remark 4.1.4. A few words on nomenclature: We may generically refer to a domain which satisfies the above for some p and q as a Poincaré domain. Some instead use the term Sobolev-Poincaré domain in this manner, but this usage conflicts with another purpose for that term: in the literature the term Sobolev-Poincaré domain sometimes refers to a domain that satisfies the above with $q = \frac{Qp}{Q-p}$, the optimal exponent possible, where Q is the (homogeneous) dimension of the space.

In [31], Hajłasz and Koskela showed that an s-John domain Ω in $\mathbb{R}^n$ will necessarily be a (q,p)-Poincaré domain for certain p and q (see Theorem 4.2.12 for the precise statement). We first reproduce this result using a different approach, and then we will generalize our approach to demonstrate a similar result in Carnot groups.

Our approach closely follows that in [40], where Koskela, Onninen, and Tyson showed that domains satisfying a specific quasihyperbolic boundary condition are (q,p)-Poincaré domains, quantitatively. We begin with a different kind of geometric condition on the boundary but similarly arrive at such an analytic consequence. An important mediator between the geometry and the Poincaré inequality is a Whitney decomposition of the domain.

Definition 4.1.5. Let Ω be a bounded domain in $\mathbb{R}^n$. A *Whitney decomposition* W of Ω is a collection of closed cubes with edges parallel to the coordinate axes with the following properties:

i. $Q \subset \Omega$, for all $Q \in W$.

ii. The interiors of the cubes are disjoint.

iii. The diameter of each cube is $2^{-i} \operatorname{diam} \Omega$, for some $i \in \mathbb{Z}$. We denote by W_k the collection of all cubes in W with diameter $2^{-k} \operatorname{diam} \Omega$.

iv. $\Omega = \cup_{Q \in W} Q$.

v. $\operatorname{diam} Q \leq \operatorname{dist}(Q, \partial\Omega) \leq 4 \operatorname{diam} Q$ for all $Q \in W$.

A proof of the existence of such a decomposition can be found, for example, in Stein's book [55, pp. 167-168].

The approach stitches together local Poincaré inequalities – valid on individual Whitney cubes – into a global Poincaré inequality. We use our control on the geometry of the boundary to produce constraints on chains of Whitney cubes which allow us ultimately to achieve the stitching.

4.2 A new proof showing s-John implies Poincaré in $\mathbb{R}^n$

Throughout this section, let $\Omega \subset \mathbb{R}^n$ be an s-John domain for some $s \geq 1$, and let W be a Whitney decomposition of Ω.

Remark 4.2.1. The $s = 1$ case is the John case; John domains in $\mathbb{R}^n$ satisfy (q, p)-Poincaré inequalities for all p and q such that $1 \leq p < n$ and $p \leq q \leq \frac{np}{n-p}$. The upper limit $q = \frac{np}{n-p}$ is the optimal exponent possible for $\Omega \subset \mathbb{R}^n$. The original sources for this work are Gol′dshteĭn and Reshetnyak [30] on the Russian side of the Iron Curtain and, five years later, Bojarski [11] in the West.

Lemma 4.2.2. *Let $s > 1$, and let γ be a weak s-John curve in Ω from x to x_0 with weak s-John constant λ. Then*

$$\operatorname{card}\{Q \in W_1 \cup \cdots \cup W_k \mid Q \cap \gamma \neq \emptyset\} \leq C 2^{nk(s-1)/s}$$

where $C = C(n, s, \lambda)$.

Proof. Recall, for $Q \in W_i$, we have $\operatorname{diam} Q = 2^{-i} \operatorname{diam} \Omega$ and that

$$\operatorname{dist}(Q, \partial\Omega) \leq 4(2^{-i} \operatorname{diam} \Omega).$$

If $z \in Q \cap \gamma$, $Q \in W_i$, it follows that

$$\operatorname{dist}(z, \partial\Omega) \leq 5(2^{-i} \operatorname{diam} \Omega).$$

Combining this inequality with the (scale-invariant) weak s-John condition for γ, we get

$$\begin{aligned} d(x,z) &\leq \left(\frac{\operatorname{dist}(z, \partial\Omega)}{\lambda(\operatorname{diam} \Omega)^{1-s}} \right)^{1/s} \\ &\leq \left(\frac{5}{\lambda} \right)^{1/s} 2^{-i/s} \operatorname{diam} \Omega. \end{aligned} \tag{4.2.1}$$

Consequently,

$$\bigcup \{ z \mid z \in \gamma \text{ and } z \text{ lies in some } Q \in W_i \} \subset D\left(x, \left(\frac{5}{\lambda} \right)^{1/s} 2^{-i/s} \operatorname{diam} \Omega \right). \tag{4.2.2}$$

(Recall that $D(x, r)$ denotes the closed ball centered at x with radius r.)

Suppose $Q \cap \gamma \neq \emptyset$. Then there is some $z \in Q \cap \gamma$; by (4.2.2), such a point z lies within a known distance of x. By adding $\operatorname{diam} Q$ to the radius used in (4.2.2), we obtain a ball centered at x which contains Q as well. Recall that all Q in W_i have diameter $2^{-i} \operatorname{diam} \Omega$:

$$\begin{aligned} \bigcup \{ Q \mid Q \in W_i, Q \cap \gamma \neq \emptyset \} &\subset D\left(x, \left(\frac{5}{\lambda} \right)^{1/s} 2^{-i/s} \operatorname{diam} \Omega + 2^{-i} \operatorname{diam} \Omega \right) \\ &= D\left(x, 2^{-i} \operatorname{diam} \Omega \left[\left(\frac{5}{\lambda} \right)^{1/s} 2^{i(s-1)/s} + 1 \right] \right) \\ &\subset D\big(x, 2^{-i} \operatorname{diam} \Omega \left[C 2^{i(s-1)/s} \right] \big), \end{aligned} \tag{4.2.3}$$

where $C = C(s, \lambda)$.

We address the cardinality of $\{ Q \mid Q \in W_i, Q \cap \gamma \neq \emptyset \}$ using volume considerations:

$$\sum_{\substack{Q\in W_i\\ Q\cap\gamma\neq\emptyset}} |Q| = \operatorname{card}\{Q \mid Q\in W_i, Q\cap\gamma\neq\emptyset\}\cdot n^{-n/2}(2^{-i}\operatorname{diam}\Omega)^n$$
$$= \left|\bigcup\{Q \mid Q\in W_i, Q\cap\gamma\neq\emptyset\}\right|$$
$$\leq C\left(2^{-i}\operatorname{diam}\Omega\right)^n 2^{ni(s-1)/s},$$

where $C = C(n, s, \lambda)$. The second equality above follows by the disjointness of the interiors of the Q's, and the inequality is from the containment shown in (4.2.3) and monotonicity of the measure.

Hence, we get

$$\operatorname{card}\{Q\in W_i \mid Q\cap\gamma\neq\emptyset\} \leq C2^{ni(s-1)/s}, \tag{4.2.4}$$

where $C = C(n, s, \lambda)$.

Now, we can put together the upper bounds for $1 \leq i \leq k$.

$$\operatorname{card}\{Q\in W_1\cup\cdots\cup W_k \mid Q\cap\gamma\neq\emptyset\} \leq \sum_{i=1}^{k} C2^{ni(s-1)/s}$$
$$\leq C2^{nk(s-1)/s}$$

where $C = C(n, s, \lambda)$, as desired. □

Remark 4.2.3. For the case where $s = 1$, the above lemma holds with a resulting bound that is linear in k:

$$\operatorname{card}\{Q\in W_1\cup\cdots\cup W_k \mid Q\cap\gamma\neq\emptyset\} \leq Ck,$$

with $C = C(n, \lambda)$, which essentially coincides with the statement in [40].

We now introduce two definitions based on similar definitions in [40]. The first specifies paths within the Whitney decomposition which play the role of weak s-John chains; the second gives a related concept.

Definition 4.2.4. Fix $\widetilde{Q} \in W$, and let x be the center of $\widetilde{Q}$. Let $x_0 \in \Omega$, and fix a weak s-John curve γ from x to x_0 in Ω. We define the *path* $P(\widetilde{Q}, \gamma)$ in W as $\{Q \in W \mid Q \cap \gamma \neq \emptyset\}$.

Remark 4.2.5. For a point x in a weak s-John domain Ω, there is typically not a unique weak s-John curve from x to the center point x_0. However, it will not be important which of these weak s-John curves we choose in the above definition; what is needed is that our choice of curve remains fixed going forward. As such, we will henceforth suppress the dependence on γ in the notation: $P(\widetilde{Q}) := P(\widetilde{Q}, \gamma)$.

Definition 4.2.6. For each cube $\widetilde{Q} \in W$, fix a choice of weak s-John curve from its center point $x_{\widetilde{Q}}$ to x_0. (These choices determine the paths $P(\widetilde{Q})$ for all $\widetilde{Q} \in W$.) For a cube $Q \in W$, we define the *shadow* of Q as $S(Q) := \{\widetilde{Q} \in W \mid Q \in P(\widetilde{Q})\}$.

Remarks 4.2.7.

1. The name "shadow" is intended to be suggestive. Suppose a light source is placed at x_0, and the light travels backward along the weak s-John curves that were chosen. If Q is opaque, it casts a shadow backward along any paths that pass through it. (Specifically, we define its shadow to be comprised of the cubes that originate these obscured paths.)

2. There is a reciprocal relationship between paths and shadows that will be exploited below. First, we observe that $Q \in P(\widetilde{Q})$ if and only if $\widetilde{Q} \in S(Q)$. This idea leads later to more delicate interchanges; see, for example, the transition from (4.2.9) to (4.2.10).

3. As before, the specific choices of weak s-John curves do not play an important role beyond generating the paths $P(\widetilde{Q})$, so we suppress (in the notation) the dependence of the shadow on the choices of curves.

Now we see what quantitative control the weak s-John condition impresses on paths and shadows.

Lemma 4.2.8. *For $\varepsilon > (s-1)/s$, there exists a constant $C = C(n, \varepsilon, s, \lambda, \operatorname{diam}\Omega)$ such that*

$$\sup_{\widetilde{Q}\in W} \sum_{Q\in P(\widetilde{Q})} |Q|^{\varepsilon} \le C.$$

Proof. Let $Q \in W$, and let $i \in \mathbb{Z}$ such that $Q \in W_i$. Then Q has side length $n^{-1/2}2^{-i} \operatorname{diam}\Omega$, making its Lebesgue measure

$$|Q| = n^{-n/2}2^{-in}(\operatorname{diam}\Omega)^n.$$

Let $\widetilde{Q} \in W$. We now sum the terms $|Q|^{\varepsilon}$ for all Q in $P(\widetilde{Q})$, noting we can utilize the i-dependence of $|Q|^{\varepsilon}$. To find the i-distribution of cubes in $P(\widetilde{Q})$, we apply the previous lemma.

$$\begin{aligned}\sum_{Q\in P(\widetilde{Q})} |Q|^{\varepsilon} &= \sum_{i=1}^{\infty} \sum_{Q\in W_i\cap P(\widetilde{Q})} |Q|^{\varepsilon} \\ &= \sum_{i=1}^{\infty} n^{-n\varepsilon/2}2^{-in\varepsilon}(\operatorname{diam}\Omega)^{n\varepsilon} \cdot \operatorname{card}\{Q \mid Q \in P(\widetilde{Q}) \cap W_i\} \qquad (4.2.5) \\ &\le \widetilde{C} n^{-n\varepsilon/2}(\operatorname{diam}\Omega)^{n\varepsilon} \sum_{i=1}^{\infty} 2^{-ni\left(\varepsilon - \frac{s-1}{s}\right)}.\end{aligned}$$

The final sum converges if and only if $\epsilon > (s-1)/s$. The upper bound depends (clearly) on n, ε, $\operatorname{diam}\Omega$, and s; by the presence of $\widetilde{C}$ from the previous lemma, the sum also depends on λ.

The upper bound does not depend on any data about $\widetilde{Q}$, so we obtain the same bound when the supremum is taken over all $\widetilde{Q} \in W$. □

Remark 4.2.9. If $s = 1$, we apply the bound from the remark following Lemma 4.2.2 when substituting into (4.2.5). In this way, we find that the final sum converges in this case if and only if $\varepsilon > 0$, which is again essentially the same as the result in [40].

Lemma 4.2.10. *There exists a constant $C = C(n, s, \lambda, \operatorname{diam}\Omega)$ such that* $\operatorname{diam} S(Q) \le$

$C(\operatorname{diam} Q)^{1/s}$. *Hence,* $|S(Q)| \leq C'|Q|^{1/s}$, *where* $C' = C'(C, n)$.

Proof. Let $Q \in W$ and $\widetilde{Q} \in S(Q)$. Hence, $Q \in P(\widetilde{Q})$, i.e., there exists a weak s-John curve γ from the center point x of $\widetilde{Q}$ to x_0 which passes through Q.

Let $z \in Q \cap \gamma$. Since $\operatorname{dist}(Q, \partial\Omega) \leq 4 \operatorname{diam} Q$, we have $\operatorname{dist}(z, \partial\Omega) \leq 5 \operatorname{diam} Q$. We combine this inequality with the weak s-John condition to get

$$\operatorname{dist}(x, Q) \leq d(x, z) \leq \left(\frac{5 \operatorname{diam} Q}{\lambda (\operatorname{diam} \Omega)^{1-s}} \right)^{1/s}.$$

To see how big $S(Q)$ could be in diameter, we start at the center of Q and stretch outwards. The above calculation shows that for any $\widetilde{Q}$ in $S(Q)$, its center point is within a certain distance of Q, and this distance does not depend upon any data about $\widetilde{Q}$. So, an upper bound on the reach of $S(Q)$ in any given direction from the center of Q is comprised of: half the diameter of Q, plus the upper bound on the distance from Q to the center of any cube in the shadow, plus half of the diameter of the biggest cube in the shadow (excluding Q, since we've already accommodated the distance to exit from it). The diameter of $S(Q)$ is then twice this amount:

$$\operatorname{diam} S(Q) \leq \operatorname{diam} Q + 2 \left(\frac{5 \operatorname{diam} Q}{\lambda (\operatorname{diam} \Omega)^{1-s}} \right)^{1/s} + \max_{\substack{\widetilde{Q} \in S(Q) \\ \widetilde{Q} \neq Q}} \operatorname{diam} \widetilde{Q}. \tag{4.2.6}$$

For $\widetilde{Q} \in S(Q)$, $\widetilde{Q} \neq Q$, the weak s-John curve γ associated with $\widetilde{Q}$ must travel (at least) to the edge of $\widetilde{Q}$ to reach Q. Starting from the center x of $\widetilde{Q}$, this journey requires at least $\frac{1}{2\sqrt{n}} \operatorname{diam} \widetilde{Q}$ length along γ in order to exit $\widetilde{Q}$. Hence, $d(x, z) \geq \frac{1}{2\sqrt{n}} \operatorname{diam} \widetilde{Q}$ for any $z \in \gamma \cap Q$. So,

$$\operatorname{diam} \widetilde{Q} \leq 2\sqrt{n}\, d(x, z) \leq 2\sqrt{n} \left(\frac{5 \operatorname{diam} Q}{\lambda (\operatorname{diam} \Omega)^{1-s}} \right)^{1/s} \tag{4.2.7}$$

for every $\widetilde{Q} \in S(Q)$, $\widetilde{Q} \neq Q$.

Using (4.2.7), (4.2.6) simplifies to

$$\operatorname{diam} S(Q) \leq 2(1+\sqrt{n}) \left(\frac{5 \operatorname{diam} Q}{\lambda (\operatorname{diam} \Omega)^{1-s}} \right)^{1/s} + \operatorname{diam} Q.$$

Let $i \in \mathbb{Z}$ such that $Q \in W_i$. Then

$$\begin{aligned} \operatorname{diam} S(Q) &\leq 2(1+\sqrt{n}) \left(\frac{5(2^{-i} \operatorname{diam} \Omega)}{\lambda (\operatorname{diam} \Omega)^{1-s}} \right)^{1/s} + 2^{-i} \operatorname{diam} \Omega \\ &= C 2^{-i/s} \operatorname{diam} \Omega + 2^{-i} \operatorname{diam} \Omega \\ &\leq C 2^{-i/s} \operatorname{diam} \Omega \\ &= C (\operatorname{diam} Q)^{1/s}, \end{aligned}$$

where $C = C(n, s, \lambda, \operatorname{diam} \Omega)$. $\square$

Remark 4.2.11. If $s = 1$, the previous lemma extends to this case without modification, concluding $\operatorname{diam} S(Q) \leq C \operatorname{diam} Q$.

The following theorem (a slight variation of Proposition 3.1 in [40]) brings the story together.

Theorem 4.2.12. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a weak s-John domain, for some $1 < s < n/(n-1)$, with $\operatorname{diam} \Omega = 1$. Then Ω is a (q,p)-Poincaré domain for each p and q satisfying $1 \leq p < \frac{n}{n-(n-1)s}$ and $p \leq q < \frac{np}{n-p(n-(n-1)s)}$.*

The proof of this theorem proceeds exactly as in [40]. That proof employs a technical lemma ([40][Lemma 3.8]); we provide the following substitute, which gives an identical conclusion as the original lemma.

Lemma 4.2.13. *Let Ω, p, and q be as in the statement of Theorem 4.2.12. Fix p and q. Then there exists a constant $C = C(n, p, q, s)$ such that*

$$\sum_{Q \in W} |S(Q) \cap E|^{p'} |Q|^{-p'/p^*} \leq C |E|^{p'/q'}$$

for any $E \subset \Omega$, where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.

Proof. First, we'll establish a preliminary bound.

$$\begin{aligned} |S(Q) \cap E|^{p'} &= |S(Q) \cap E|^{p'-1} |S(Q) \cap E| \\ &= |S(Q) \cap E|^{p'-1} \sum_{\widetilde{Q} \in S(Q)} |\widetilde{Q} \cap E| \qquad (4.2.8) \\ &\leq |E|^{\frac{p'}{p} - \frac{p'}{q}} |S(Q)|^{\frac{p'}{q}} \sum_{\widetilde{Q} \in S(Q)} |\widetilde{Q} \cap E|. \end{aligned}$$

Then

$$\sum_{Q \in W} |S(Q) \cap E|^{p'} |Q|^{-\frac{p'}{p^*}} \leq |E|^{p'-1-\frac{p'}{q}} \sum_{Q \in W} \left(\frac{|S(Q)|^{1/q}}{|Q|^{1/p^*}} \right)^{p'} \sum_{\widetilde{Q} \in S(Q)} |\widetilde{Q} \cap E| \qquad (4.2.9)$$

$$= |E|^{p'-1-\frac{p'}{q}} \sum_{\widetilde{Q} \in W} |\widetilde{Q} \cap E| \sum_{Q \in P(\widetilde{Q})} \left(\frac{|S(Q)|^{1/q}}{|Q|^{1/p^*}} \right)^{p'}. \qquad (4.2.10)$$

Now, to establish the desired bound, it is sufficient that

$$\sum_{Q \in P(\widetilde{Q})} \left(\frac{|S(Q)|^{1/q}}{|Q|^{1/p^*}} \right)^{p'} \leq C, \qquad (4.2.11)$$

for every $\widetilde{Q}$, as $\sum_{\widetilde{Q} \in W} C |\widetilde{Q} \cap E| = C|E|$, and $p' - 1 - \frac{p'}{q} + 1 = \frac{p'}{q'}$.

By Lemma 4.2.10, for $\operatorname{diam} \Omega = 1$ and $s > 1$ we have $\operatorname{diam} S(Q) \leq C(\operatorname{diam} Q)^{1/s}$. Hence, $|S(Q)| \leq C|Q|^{1/s}$, and so (4.2.11) is satisfied if

$$\sum_{Q \in P(\widetilde{Q})} |Q|^{(\frac{1}{qs} - \frac{1}{p^*})p'} \leq C, \qquad (4.2.12)$$

which happens via Lemma 4.2.8 as long as $(\frac{1}{qs} - \frac{1}{p^*})p' > \frac{s-1}{s}$.

Recall our assumption that $q < \frac{np}{n-p(n-(n-1)s)}$, and recall that $p' = \frac{p}{p-1}$ and $p^* = \frac{np}{n-p}$. Below, we substitute for p' and p^* and use the upper bound for q; the remainder is simple algebra.

$$\begin{aligned}\left(\frac{1}{qs}-\frac{1}{p^*}\right)p' &= \left(\frac{1}{qs}-\frac{n-p}{np}\right)\frac{p}{p-1}\\ &> \left(\frac{n-p(n-(n-1)s)}{nps}-\frac{n-p}{np}\right)\frac{p}{p-1}\\ &= \frac{s-1}{s}.\end{aligned}$$

□

Remark 4.2.14. In the $s = 1$ case, the critical convergence (4.2.12) becomes

$$\sum_{q\in P(\widetilde{Q})} |Q|^{(\frac{1}{q}-\frac{1}{p^*})p'} \le C, \tag{4.2.13}$$

which holds if $\varepsilon = \frac{1}{q} - \frac{1}{p^*} > 0$ by the remark following Lemma 4.2.8, i.e., if $q < \frac{np}{n-p}$. Hence, this approach does recover the result for John domains, excepting the borderline case $q = \frac{np}{n-p}$.

4.3 Weak s-John implies Poincaré in Carnot groups

We pursue a proof similar to the one in the previous section, with necessary adjustments. The first step in the generalization is to introduce a more general Whitney decomposition. This generalization is sometimes instead called a Whitney covering, which is more appropriate here as the targeted open set is not decomposed into disjoint pieces, but rather covered by sets in a highly prescribed manner.

General Whitney coverings appear in various places in the literature. An early example is by Coifman and Weiss [21]; more recent examples come from Vodop'yanov and Greshnov [58] and Björn, Björn, and Shanmugalingam [10]. As noted in the last paper, it can be more convenient simply to define a custom Whitney covering – and prove the basic results needed to make it useful – than to use an ill-suited one.

We, however, will borrow directly, as the Whitney covering described in [10] is basically what we need.

Let (X, d, μ) be a metric space with a doubling measure μ. (Doubling measures were defined in Definition 2.2.1.) Let F be a non-empty closed subset of X, fix a constant $R > 0$, and define $V := \{x \in X \mid 0 < \operatorname{dist}(x, F) \leq 16R\}$.

Let us adopt here the notational convention that $\lambda B = \lambda B(x, r) := B(x, \lambda r)$.

Theorem 4.3.1 (Björn-Björn-Shanmugalingam)**.** *There exists a countable family of balls*

$$W = \{B_{i,j} = B(x_{i,j}, r_i) \mid i \in \mathbb{N}, j \in J_i\}$$

such that for all $i \in \mathbb{N}$ and $j \in J_i$,

(i) $V \subset \cup_{B \in W} B \subset X - F$;

(ii) $r_i = 2^{-i} R$;

(iii) $8r_i < \operatorname{dist}(x_{i,j}, F) \leq 16r_i$;

(iv) the balls $\{\frac{1}{2}B \mid B \in W\}$ *are pairwise disjoint.*

Remark 4.3.2. If Ω is a bounded open set with non-empty boundary (as we will be using), then choosing $F = \Omega^c$ and $R \geq \frac{1}{16} \operatorname{diam} \Omega$ will yield a Whitney covering of Ω.

A further lemma from [10] will be useful for our work.

Lemma 4.3.3 (Björn-Björn-Shanmugalingam)**.** *Let $0 < \lambda < 8$. Then there exists a constant $M > 0$, depending only on λ and the doubling constant of μ, such that we have the following.*

(i) If $\lambda B_{i,j} \cap \lambda B_{k,l} \neq \emptyset$, *then* $r_i < (16 + \lambda) r_k / (8 - \lambda)$ *and hence*

$$i - \log_2 \left(\frac{16 + \lambda}{8 - \lambda} \right) < k < i + \log_2 \left(\frac{16 + \lambda}{8 - \lambda} \right).$$

(ii) If $B_{i,j} \in W$, then $\operatorname{card}\{B \in W \mid \lambda B_{i,j} \cap \lambda B \neq \emptyset\} \leq M$, *and hence for each* $x \in X$, $\sum_{B \in W} \chi_{\lambda B}(x) \leq M$.

Let $\mathbb{G}$ be a Carnot group with stratification $v_1 \oplus \cdots \oplus v_m$ of its Lie algebra. We recall the following, in order to establish notation which will be used throughout the remainder of this section. Let Q denote the homogeneous dimension of $\mathbb{G}$; recall $Q := \sum_{i=1}^{m} i \dim v_i$. Let μ denote a Haar (volume) measure on $\mathbb{G}$. Recall that $(\mathbb{G}, \mu)$ is Ahlfors Q-regular, indeed there exists a constant $K > 0$ such that $\mu(B(x,r)) = Kr^Q$ for every x; note that $K = \mu(B(0,1))$. Finally, observe that Ahlfors regularity for μ implies that μ is a doubling measure, so we may use the aforementioned Whitney covering.

Lemma 4.3.4. *Let $s > 1$, and let γ be a weak s-John curve in $\Omega \subset \mathbb{G}$ from x to x_0 with weak s-John constant λ. Then*

$$\operatorname{card}\{Q \in W_1 \cup \cdots \cup W_k \mid Q \cap \gamma \neq \emptyset\} \leq C 2^{Qk(s-1)/s},$$

where $C = C(Q, s, \lambda)$.

Proof. The proof here largely follows the proof of Lemma 4.2.2, which established this result in the $\mathbb{R}^n$ case. We note only the following adaptations:

- Let $R = \frac{1}{16} \operatorname{diam} \Omega$ in the definition of the Whitney covering of Ω. Then, for $B \in W_i$, we have $r_i = 2^{-i} R = 2^{-(i+4)} \operatorname{diam} \Omega$.

- If $z \in B_{i,j} \cap \gamma$, $B_{i,j} \in W_i$, it follows that

  $$\operatorname{dist}(z, \partial\Omega) \leq \operatorname{dist}(x_{i,j}, \partial\Omega) + d(x_{i,j}, z) \leq 16 r_i + r_i = 17 r_i.$$

  Consequently, we get

  $$d(x,z) \leq \left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam} \Omega \tag{4.3.1}$$

in place of (4.2.1). This inequality leads, by the same steps, to the containment:

$$\bigcup_{\substack{B\in W_i\\ B\cap\gamma\neq\emptyset}} B \subset D\big(x, 2^{-i}\operatorname{diam}\Omega\left[C2^{i(s-1)/s}\right]\big)\,, \tag{4.3.2}$$

where $C = C(s, \lambda)$.

- By our definition of Whitney covering, the set $\{\frac{1}{2}B \mid B \in W_i, B\cap\gamma \neq \emptyset\}$ is pairwise disjoint, and $\mu(\frac{1}{2}B) = K(\frac{1}{2}r_i)^Q$ for $B \in W_i$. We again bound the cardinality using volume considerations:

$$\begin{aligned}\sum_{\substack{B\in W_i\\ B\cap\gamma\neq\emptyset}} \mu\left(\frac{1}{2}B\right) &= \operatorname{card}\{B\in W_i \mid B\cap\gamma\neq\emptyset\}\cdot K\left(\frac{1}{2}r_i\right)^Q\\ &= \mu\left(\bigcup\left\{\frac{1}{2}B \,\middle|\, B\in W_i, B\cap\gamma\neq\emptyset\right\}\right)\\ &\leq CK\left(2^{-i}\operatorname{diam}\Omega\right)^Q 2^{Qi(s-1)/s},\end{aligned}$$

where $C = C(Q, s, \lambda)$. The second equality above follows by the disjointness of the $\frac{1}{2}B$'s, and the inequality is from the containment in (4.3.2) and monotonicity of the measure.

Hence, we get

$$\operatorname{card}\{B\in W_i \mid B\cap\gamma\neq\emptyset\} \leq C2^{Qi(s-1)/s},$$

where $C = C(Q, s, \lambda)$.

- The rest of the proof proceeds as before. □

Recall the definition of a (Whitney) path $P(B)$, introduced in the previous section.

Definition 4.3.5. Fix $\widetilde{B} \in W$, and let x be the center of $\widetilde{B}$. Let $x_0 \in \Omega$, and fix a weak s-John curve γ from x to x_0 in Ω. We define the *path* $P(\widetilde{B}, \gamma)$ (or simply $P(\widetilde{B})$) in W for this weak s-John curve as $\{B \in W \mid B\cap\gamma \neq \emptyset\}$.

Lemma 4.3.6. *Let $s > 1$. For $\varepsilon > (s-1)/s$, there exists a constant $C = C(Q, \varepsilon, s, \lambda, \operatorname{diam}\Omega, K)$ such that*

$$\sup_{\widetilde{B}\in W} \sum_{B\in P(\widetilde{B})} |B|^{\varepsilon} \leq C.$$

Proof. The proof proceeds essentially as for Lemma 4.2.8, with two slight modifications.

- Let $B \in W$, and let $i \in \mathbb{Z}$ such that $B \in W_i$. Then B has measure

$$\mu(B) = K r_i^Q = 2^{-Q(i+4)} K (\operatorname{diam}\Omega)^Q = C 2^{-Qi} (\operatorname{diam}\Omega)^Q.$$

- We apply Lemma 4.3.4 to bound the cardinality of the set in (4.2.5); Lemma 4.3.4 is the Carnot version of Lemma 4.2.2 and has an analogous result, so its effect on the calculations is the same.

- The rest follows as before. □

Recall the definition of a Whitney shadow $S(B)$, from the previous section.

Definition 4.3.7. For each ball $\widetilde{B}$ in W, fix a choice of weak s-John curve from its center point $x_{\widetilde{B}}$ to x_0. (These choices determine the paths $P(\widetilde{B})$ for all $\widetilde{B} \in W$.) For a ball $B_{i.j} \in W$, we define the *shadow* of $B_{i,j}$ as $S(B_{i,j}) := \{\widetilde{B} \in W \mid B_{i,j} \in P(\widetilde{B})\}$.

Lemma 4.3.8. *There exists a constant $C = C(s, \lambda, \operatorname{diam}\Omega, K, Q)$ such that $\mu(S(B)) \leq C\mu(B)^{1/s}$.*

Proof. Let $B := B_{i,j} \in W$ and $\widetilde{B} \in S(B)$. Hence, $B \in P(\widetilde{B})$, i.e., a weak s-John curve γ from the center point $\tilde{x}$ of $\widetilde{B}$ to x_0 was chosen which passes through B.

So, let $z \in B \cap \gamma$. Hence, $\operatorname{dist}(z, \partial\Omega) \leq \operatorname{dist}(x_{i,j}, \partial\Omega) + d(x_{i,j}, z) \leq 17r_i$. We combine this inequality with the weak s-John condition to get

$$d(\tilde{x}, z) \leq \left(\frac{17r_i}{\lambda(\operatorname{diam}\Omega)^{1-s}}\right)^{1/s} = \left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega. \tag{4.3.3}$$

Further, $\operatorname{dist}(\tilde{x}, B) \leq d(\tilde{x}, z)$, trivially, so we get

$$\operatorname{dist}(\tilde{x}, B) \leq \left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega.$$

To see how big $S(B)$ could be in diameter, we start at the center of B and stretch outwards. The above calculation shows that for any $\widetilde{B}$ in $S(B)$, its center point is within a certain distance of B, and this distance does not depend upon the particulars of $\widetilde{B}$. So, an upper bound on the reach of $S(B)$ in any given direction from the center of B is comprised of: the radius of B, plus the upper bound on the distance from B to the center of any cube in the shadow, plus the radius of the biggest cube in the shadow. The diameter of $S(B)$ is then twice this amount:

$$\begin{aligned}\operatorname{diam} S(Q) \leq 2r_i + 2\left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega \\ + 2\max\{r_k \mid \text{there exists } \widetilde{B} \in S(B) \cap W_k\}.\end{aligned} \tag{4.3.4}$$

By Lemma 4.3.3(i), for any $\widetilde{B} \in W_k \cap S(B)$ such that $B \cap \widetilde{B} \neq \emptyset$, we know $i-2 < k$ (using $\lambda = 1$ in the lemma).

For $\widetilde{B} \in W_j \cap S(B)$ such that $\widetilde{B} \cap B = \emptyset$, we must travel (at least) to the edge of $\widetilde{B}$ to reach B, so $d(\tilde{x}, z) \geq r_j$. Combining this inequality with (4.3.3), we bound r_j:

$$r_j \leq \left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega.$$

We can now get control of the max's value:

$$\begin{aligned}\max\{r_k \mid \text{there exists } \widetilde{B} \in S(B) \cap W_k\} &\leq \max\left\{\left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega, 2^{-i-2} \operatorname{diam}\Omega\right\} \\ &= \left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega.\end{aligned}$$

Now, (4.3.4) can be simplified and pushed to the end.

$$\begin{aligned}
\operatorname{diam} S(Q) &\leq 2r_i + 4\left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s} \operatorname{diam}\Omega \\
&= \left(\frac{1}{8}2^{-i} + 4\left(\frac{17}{16\lambda}\right)^{1/s} 2^{-i/s}\right) \operatorname{diam}\Omega \\
&= \left(\frac{1}{8} + 4\left(\frac{17}{16\lambda}\right)^{1/s}\right) 2^{-i/s} \operatorname{diam}\Omega \\
&= 16\left(\frac{1}{8} + 4\left(\frac{17}{16\lambda}\right)^{1/s}\right) (\operatorname{diam}\Omega)^{1-1/s}(2^{-i} \operatorname{diam}\Omega)^{1/s} \\
&= \widetilde{C} r_i^{1/s}.
\end{aligned}$$

Finally, $\mu(S(B)) \leq \mu(B(x_{i,j}, \widetilde{C} r_i^{1/s})) = K(\widetilde{C} r_i^{1/s})^Q = C(K r_i^Q)^{1/s} = C\mu(B)^{1/s}$. Note that C absorbs both $\widetilde{C}^Q$ and a factor of $K^{1-1/s}$. Hence, $C = C(s, \lambda, \operatorname{diam}\Omega, K, Q)$, as expected. $\square$

We finish, as we did before, by relying on a variation of Proposition 3.1 in [40].

Theorem 4.3.9. *Let $\Omega \subset \mathbb{G}$ be a weak s-John domain, for some $1 < s < Q/(Q-1)$, with $\operatorname{diam}_{CC}\Omega = 1$. Then Ω is a (q,p)-Poincaré domain for each p and q satisfying $1 \leq p < \frac{Q}{Q-(Q-1)s}$ and $p \leq q < \frac{Qp}{Q-p(Q-(Q-1)s)}$.*

Remark 4.3.10. The proof of this theorem proceeds essentially as in [40]. One encounters the following adaptations for the proof in the Carnot case.

- The recharacterization of what needs to be shown, taken from [31, Lemma 1 and Theorem 1(I)], was originally written for $\mathbb{R}^n$. However, the proofs require only obvious modifications (e.g., the gradient must be replaced by the horizontal gradient) to hold also in the Carnot case.

- In producing the bound $\frac{1}{C}\mu(A_g)^{p/q} \leq \int_\Omega |\nabla_0 u|^p d\mu$:

  – We require a (p^*, p)-Poincaré inequality to hold on balls from the Whitney covering. This result is known; see, for example, Garofalo and Nhieu [25, Theorem 1.5, together with Remark 1.2 and Theorem 1.15(I)].

– We use part (ii) of Lemma 4.3.3, which says that any point in Ω is contained in at most M of the Whitney covering balls, to make the inequality chain:

$$\sum \mu(A_g)^{p/q} \le \sum \mu(A \cap B)^{p/q} \le \sum \int_B |\nabla_0 u|^p d\mu \le M \int_\Omega |\nabla_0 u|^p d\mu, \tag{4.3.5}$$

where the sums are taken over all balls in the Whitney covering that intersect A_g and have $u_B \le 1/2$.

- In producing the bound $\frac{1}{C}\mu(A_b)^{p/q} \le \int_\Omega |\nabla_0 u|^p d\mu$:

  – We require a $(1,1)$-Poincaré inequality to hold on balls from the Whitney covering. This fact follows from the known result mentioned above.

  – The standard chaining argument referenced works for the general Whitney covering, with slightly worse constant.

  – We substitute the following technical lemma in place of ([40][Lemma 3.8]); the substitute arrives at an identical conclusion as the original lemma.

Lemma 4.3.11. *Let Ω, p, and q be as in the statement of Theorem 4.3.9. Fix p and q. Then there exists a constant $C = C(Q, p, q, s)$ such that*

$$\sum_{B \in W} |S(B) \cap E|^{p'} |B|^{-\frac{p'}{p^*}} \le C|E|^{\frac{p'}{q'}}$$

for any $E \subset \Omega$, where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p.

Proof. The proof proceeds exactly as for Lemma 4.2.13 with two slight differences.

- In (4.2.8), the equality is an inequality ("less than or equal to"), which does not affect the calculation.

- It is sufficient, as before, to verify that (4.2.11) holds, because

$$\sum_{\widetilde{B}\in W} C|\widetilde{B}\cap E| \le CM|E|.$$

We have invoked the second part of Lemma 4.3.3 to produce this inequality. $\square$

CHAPTER 5

Box dimension of Hölder graphs in Carnot groups

5.1 History and motivation

Measurements of distance (typically, as diameters of covering sets) play an intrinsic role in definitions of dimension taken from geometric measure theory. In Carnot groups, it is natural to use the Carnot-Carathéodory metric in these definitions; we will refer to such dimensions using a "CC" prefix to indicate a sub-Riemannian metric is being used. Using this metric leads to some unusual results: for example, an Euclidean line in a k-step Carnot group can have CC-Hausdorff dimension equal to any integer from 1 to k, depending on how the line is situated.

A fruitful approach to Carnot groups has been to establish implications of the form "Euclidean hypotheses yield a Carnot conclusion." For questions of dimension in Carnot groups, there have been two main types of Euclidean hypotheses. The first compares the Euclidean and CC dimensions of a general subset of a Carnot group. This question, known as Gromov's dimension comparison problem, has been explored in [4,6,7]; most recently, Balogh, Tyson, and Warhurst [8] gave sharp answers to this question in general Carnot groups, for both Hausdorff and box dimensions. The second type of Euclidean hypothesis connects the Euclidean smoothness of a function to the CC-dimension of its graph.

We focus on the Heisenberg group. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. If f is a C^1 smooth function, then its graph will have CC-Hausdorff dimension exactly equal to three in $\mathbb{H}^1$. (This follows from, for example, Pansu's isoperimetric inequality [51].) In fact, if f is only Lipschitz, then its graph still has CC-Hausdorff dimension exactly three in $\mathbb{H}^1$; [4, Proof of Theorem 6.7] shows the CC-Hausdorff dimension is at least three, and Corollary 5.2.5 below shows the CC-Hausdorff dimension is at most three (take $\alpha = 1$).

However, there is an example, the "Heisenberg square," which is the graph of a function f of bounded variation but has CC-Hausdorff dimension two [4]. This set demonstrates that there is a change of behavior as the regularity of the function is weakened. This example led us to explore the question of the dimensions of graphs of Sobolev functions, whose regularity lies between BV and Lipschitz.

Definition 5.1.1. The *Sobolev space* $W^{1,p}(\mathbb{R}^n, \mathbb{R})$, $p \geq 1$, is the set of all locally integrable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f \in L^p(\mathbb{R}^n)$ and the first weak partial derivatives of f are also in $L^p(\mathbb{R}^n)$.

A locally integrable function $g\colon \mathbb{R}^n \to \mathbb{R}$ is a *first weak partial derivative* of f with respect to x_i, $i \in \{1, \ldots, n\}$, if

$$\int_{\mathbb{R}^n} \phi g \, dx = - \int_{\mathbb{R}^n} f \frac{\partial}{\partial x_i} \phi \, dx$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$ (the space of smooth functions on $\mathbb{R}^n$ with compact support).

In $\mathbb{H}^1$, we will be working with functions from (Euclidean) $W^{1,p}(\mathbb{R}^2, \mathbb{R})$. Consideration of these functions can naturally split into three cases: $p > 2$, $p = 2$, and $1 \leq p < 2$. For each case, specific tools are available. Of particular benefit to our work, for $p > 2$, Sobolev functions are α-Hölder functions, where $\alpha = 1 - 2/p$. I provide an upper bound on the lower CC-box dimension for such Sobolev functions by producing an upper bound in the case of Hölder functions. Hence, CC-$\underline{\dim}_B \operatorname{Gr} f \leq 3 + 2/p$, for all $p > 2$. Note that this upper bound limits to 4 as p limits toward 2 from above.

For $1 \leq p \leq 2$, we still only have the trivial upper bound of 4 (the homogeneous dimension of $\mathbb{H}^1$).

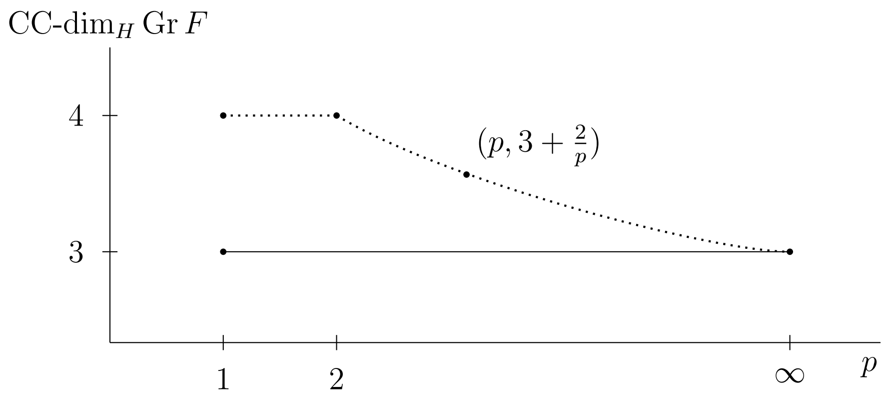

Figure 5.1: Upper and lower bounds on the CC-Hausdorff dimension in $\mathbb{H}^1$ of the graph of an Euclidean Sobolev function $F \in W^{1,p}(\mathbb{R}^2, \mathbb{R})$.

The lower bound on the CC-Hausdorff dimension of the graph of a Sobolev function is 3 for $1 \leq p < \infty$, as shown recently by Magnani [43, Theorem 2].

The lower bound is sharp; a Lipschitz function will be in $W^{1,p}$ for all p, so its graph realizes the lower bound of 3 on its CC-dimension. We do not currently have examples showing sharpness of the upper bound for Sobolev functions or for Hölder functions. We conjecture that an appropriately chosen, nowhere-differentiable function (e.g., a Weierstrass function) might serve this purpose.

5.2 Prior results and Heisenberg group results

Kahane [38] is generally credited with the first theorem giving upper bounds on the dimensions of graphs of α-Hölder functions $f : \mathbb{R} \to \mathbb{R}$, although it may have been a classical result prior to that [1, p. 193].

Theorem 5.2.1 (Kahane)**.** *Let $F\colon \mathbb{R} \to \mathbb{R}$ satisfy an α-Hölder condition on an interval $I_0 \subset \mathbb{R}$, where $0 < \alpha \leq 1$. The graph of F, $\operatorname{Gr} F := \{(t, F(t)) \mid t \in I_0\}$, satisfies:*

$$\dim_H \operatorname{Gr} F \leq 2 - \alpha.$$

A few years later Yoder [59] generalized the theorem to α-Hölder functions $f\colon \mathbb{R}^m \to$

$\mathbb{R}^n$, followed by a generalization by Cuzick [22] which utilized separate α-Hölder conditions for each coordinate function.

Definition 5.2.2. A function $F\colon \mathbb{R}^m \to \mathbb{R}^n$ satisfies a *Hölder condition of order* $\alpha = (\alpha_1, \ldots, \alpha_n)$ if the i^{th} coordinate function of F is α_i-Hölder, for $i = 1, \ldots, n$.

Theorem 5.2.3 (Cuzick)**.** *Let $F\colon \mathbb{R}^m \to \mathbb{R}^n$ satisfy a Hölder condition of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ on a cube $I_0 \subset \mathbb{R}^m$, where $0 < \alpha_1 \leq \cdots \leq \alpha_n \leq 1$. The graph of F, $\operatorname{Gr} F := \{(t, F(t)) \mid t \in I_0\}$, satisfies:*

$$\dim_H \operatorname{Gr} F \leq \min\left[\frac{m + \sum_{i=1}^n(\alpha_n - \alpha_i)}{\alpha_n}, m + \sum_{i=1}^n (1 - \alpha_i)\right].$$

Below are corollaries of the first two theorems in Section 5.4, applying the theorems to the first Heisenberg group, respectively. The second corollary is the one applicable to the dimension-of-graphs problem discussed in the previous section.

Corollary 5.2.4. *Consider the Heisenberg group $\mathbb{H}^1$. Let $I_1 = [a_1, b_1]$ be an interval on the x_1-axis in $\mathbb{H}^1$. Let $F\colon I_1 \to \mathbb{R}^2$ satisfy an Euclidean Hölder condition of order $\alpha = (\alpha_1, \alpha_2)$, $0 < \alpha_1 = \alpha_2 \leq 1$. Let $\operatorname{Gr} F = \{(t, F(t)) \mid t \in I_1\}$. Then*

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq \begin{cases} 4 - 2\alpha_1, & \textit{if } 0 < \alpha_1 \leq \frac{1}{2}; \\ 1 + \frac{1}{\alpha_1}, & \textit{if } \frac{1}{2} < \alpha_1 \leq 1. \end{cases} \tag{5.2.1}$$

Corollary 5.2.5. *Consider the Heisenberg group $\mathbb{H}^1$. Let $\Omega \subset \mathbb{R}^2$ be a bounded set, and let $F\colon \Omega \to \mathbb{R}$ be an Euclidean α-Hölder function, $0 < \alpha \leq 1$. Let $\operatorname{Gr} F = \{(t_1, t_2, F(t_1, t_2)) \mid (t_1, t_2) \in \Omega\}$. Then*

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq 4 - \alpha. \tag{5.2.2}$$

Remark 5.2.6. As shown in Theorem 5.4.6, the upper bounds in (5.2.1) and (5.2.2) are independent of the choice of which directions contain the domain and which contain the range. The lower bound produced by Magnani [43, Theorem 2] is similarly independent of this choice.

5.3 Preliminaries

Let $\mathbb{G}$ be a two-step Carnot group, and let the grading of its algebra $\mathfrak{g}$ be given by $\mathfrak{g} = v_1 \oplus v_2$.

Definition 5.3.1. The *growth vector* of the distribution is the 2-tuple (d_1, d_2), where $d_i = \dim v_i$.

Definition 5.3.2. The *weighting* $\{w_i\}_{i=1}^{d_1+d_2}$ associated with the growth vector is the assignment

$$(w_1, ..., w_{d_1+d_2}) := (\underbrace{1, ..., 1}_{d_1}, \underbrace{2, ..., 2}_{d_2}).$$

Let $\{V_i\}_{i=1}^{d_1+d_2}$ be a basis for the algebra such that $V_i \in v_j$ iff $w_i = j$. (Hence, $\{V_1, ..., V_{d_1}\}$ is a basis for v_1, and $\{V_{d_1+1}, ..., V_{d_1+d_2}\}$ is a basis for v_2.)

We describe $\mathbb{G}$ using canonical coordinates of the first kind (see Definition 2.4.29).

5.4 Theorems

Let m and n be positive integers such that $m + n = d_1 + d_2$. Let $F : \mathbb{R}^m \to \mathbb{R}^n$ satisfy a Hölder condition of order $\alpha = (\alpha_1, ..., \alpha_n)$, where $0 < \alpha_i \leq 1$ for every $i = 1, ..., n$. We view the graph of F, which is a subset of $\mathbb{R}^{m+n}$, as a subset of the Carnot group $\mathbb{G}$ and seek upper bounds on the lower box dimension of this subset under the Carnot-Carathéodory metric. Note that we trivially have the lower bound m.

We assume that the domain of F lies in some (Euclidean) m-dimensional coordinate plane. We do not require that the coordinates which parameterize this plane are "first-step" coordinates (i.e., initially generated by $\exp(tV_i)$, where $1 \leq i \leq d_1$). Clearly, if $m > d_1$, it would not even be possible. In particular, in Theorem 5.4.2, using higher step coordinates in the domain may not be avoidable.

In the following theorem, the domain of F is one-dimensional (i.e., $m = 1$), and we choose the domain to lie in a first-step coordinate interval.

Theorem 5.4.1. *Let $\mathbb{G}$ be a two-step Carnot group with topological dimension $1+n$, with a graded Lie algebra as indicated in the previous section. Let $I_1 = [a_1, b_1]$ be an interval on the x_1-axis in $\mathbb{G}$. Let $F\colon I_1 \to \mathbb{R}^n$ satisfy an Euclidean Hölder condition of order $\alpha = (\alpha_2, ..., \alpha_{n+1})$, where $0 < \alpha_i \leq 1$ for every $i = 2, ..., n+1$. Let $\operatorname{Gr} F = \{(t, F(t)) \mid t \in I_1\}$. Then*

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq \min_{0<\beta\leq 1} \left\{ \frac{1 + \sum_{i=2}^{n+1} (\beta w_i - \min\{\beta, \alpha_i\})}{\beta} \right\}.$$

Proof. We follow the approach employed by Cuzick [22]: for each coordinate in the range of F, the Hölder condition constrains the amount that the graph can vary. We require a certain amount of stacked, identically-sized boxes to cover this possible variation of F. Then we subdivide the domain of F; the Hölder condition yields a possible variation of F on each subinterval, which lets us cover the graph of F with stacks of smaller boxes. At each stage, we have a covering of the graph of F with smaller and smaller boxes, leading to an upper bound on the lower box dimension.

As we want to bound the CC-lower box dimension, we must cover the graph of F using stacks of Carnot boxes. Where the difficulty arises is in the awkward behavior of Carnot boxes. To separate the issues involved, we will cover the graph of F using Euclidean boxes and then cover those Euclidean boxes with stacks of Carnot boxes.

We begin by partitioning the interval I_1 into $2^t\lceil b_1 - a_1 \rceil$ equal pieces, $t \in \mathbb{Z}^+$.

Consider the consequent partitioning of $\mathrm{Gr}\, F$:

$$A_s := ([a_1 + s2^{-t}, a_1 + (s+1)2^{-t}] \times \mathbb{R}^n) \cap \mathrm{Gr}\, F,$$

where $0 \leq s \leq 2^t \lceil b_1 - a_1 \rceil - 1$. Since F_i is α_i-Hölder, we have

$$|F_i(x) - F_i(y)| \leq C|x - y|^{\alpha_i},$$

where x and y lie in $[a_1 + s2^{-t}, a_1 + (s+1)2^{-t}]$, for every $i = 2, \ldots, n+1$. Hence, there are choices for $a_2, ..., a_{n+1} \in \mathbb{R}$ such that A_s is contained within the Euclidean box:

$$E_s := [a_1 + s2^{-t}, a_1 + (s+1)2^{-t}] \times [a_2, a_2 + C2^{-t\alpha_2}] \times \cdots \times [a_{n+1}, a_{n+1} + C2^{-t\alpha_{n+1}}].$$

(I am using a single Hölder constant C here; even though the constants may differ, we may simply take the maximum one as a uniform choice without weakening our result.) Normally, the Euclidean box E_s would be covered by a stack of smaller, uniformly-sized Euclidean cubes; we will instead cover it with a stack of smaller, uniformly-sized Carnot boxes.

Let $\beta \in (0, 1]$. We select a Carnot box at the origin:

$$\mathrm{Box}(0, t) := \{x = (x_1, \ldots, x_{n+1}) \in \mathbb{G} \mid |x_i| \leq 2^{-t\beta w_i - 1}\}.$$

The parameter β ultimately allows an optimization to the most efficient covering. To get other Carnot boxes, we will left-translate this box; we denote the left-translated box $L_g(\mathrm{Box}(0, t))$ as $\mathrm{Box}(g, t)$.

An Euclidean cube optimizes the volume of a box for a given diameter; our choice of Carnot box similarly aims to make the Carnot-Carathéodory lengths of the sides the same to optimize our covering. As the lengths of the sides of $\mathrm{Box}(0, t)$ depend upon the step of the direction that the side resides in, our choice of Carnot box uses the weights w_i in the exponent to equalize the side lengths. Note that this Carnot box and all left-translated versions of it have diameter proportional to $2^{-t\beta}$.

Recall that left-translation is an affine map; in particular, it produces a skew in directions that are not in the first step. So, we will describe the construction of our covering in two parts: first-step directions and second-step directions.

Part I. First-step directions ($w_i = 1$)

Using the BCH formula (see (2.4.1)), we note that, for $w_i = 1$, left-translating an x_i-interval $[a_i, b_i]$ by an element g sends $[a_i, b_i]$ to $[a_i + g_i, b_i + g_i]$, where g_i is the i^{th} coordinate of g. Since there is no skew, the process of stacking left-translated boxes to cover E_s in that direction will be straightforward.

For E_s, the i^{th}-coordinate interval is $[a_i, a_i + C2^{-t\alpha_i}]$; for $\text{Box}(0, t)$, the width of its i^{th}-coordinate interval is $2^{-t\beta}$, which means we will stack

$$\frac{C2^{-t\alpha_i}}{2^{-t\beta}} = C2^{t(\beta-\alpha_i)} \text{ boxes in this coordinate direction.}$$

Note that if $C2^{t(\beta-\alpha_i)} \leq 1$, then the covering would only be one box deep in that direction. Also, covering the x_1 direction of E_s (which is only 2^{-t} thick) is always one box deep.

Part II. Second-step directions ($w_i = 2$)

Soon it will be useful to know that $\operatorname{Gr} F$ is bounded. This fact follows from F being continuous and the domain of F being compact.

There is a skew that applies to second-step directions when left-translating. This is described by the BCH formula:

$$(g \cdot x)_i = g_i + x_i + \frac{1}{2} \sum_{1 \leq p < q \leq d_1 + d_2} c_{pq}(g_p x_q - g_q x_p),$$

where $g \in \mathbb{G}$, $x \in \text{Box}(0, t)$, and c_{pq} is the coefficient of v_i when the Lie bracket $[v_p, v_q]$

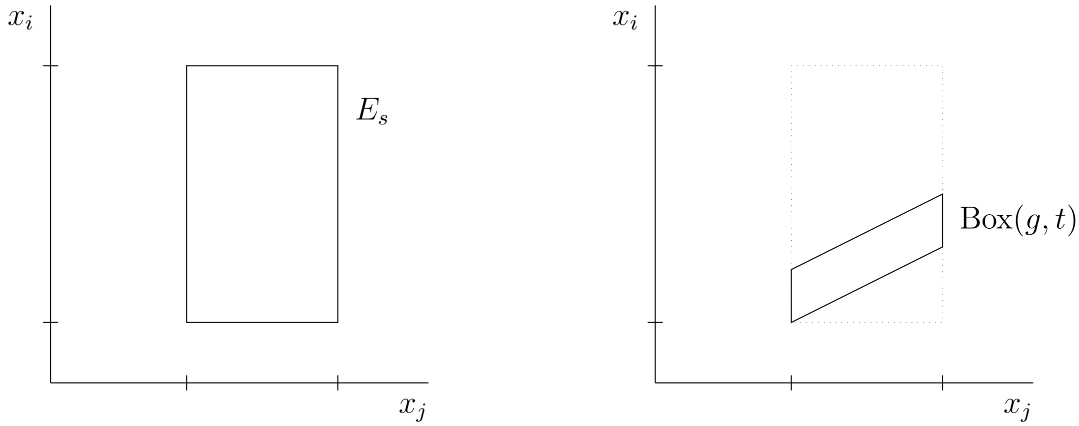

Figure 5.2: Comparing sections of E_s and $\mathrm{Box}(g,t)$ above a first-step direction x_j.

is taken. Note that, by the BCH formula, the only possible p and q for which $c_{pq} \neq 0$ are when both p and q correspond to first-step directions.

The skew in the i^{th} direction during left-translation is described by the summation, whose value we can bound. In $\mathrm{Box}(0,t)$, the values x_q and x_p can be uniformly bounded by $2^{-t\beta-1}$, as the values are first-step coordinates of some point x lying within the box at the origin. The values of g_i are bounded, as the point g lies in (or very near) E_s, which is bounded and intersects $\mathrm{Gr}\,F$, which is also bounded. We bound the summation term:

$$\left|\frac{1}{2}\sum_{1\leq p<q\leq d_1+d_2} c_{pq}(g_p x_q - g_q x_p)\right| \leq \frac{1}{2}\sum_{1\leq p<q\leq d_1+d_2} |c_{pq}|(|g_p|+|g_q|)2^{-t\beta-1} \leq C2^{-t\beta-1}$$

where C depends only on the bracket relations of $\mathbb{G}$ and the bound on $\mathrm{Gr}\,F$.

The circumstance that we now encounter is represented by Figure 5.2. If we fix g and x_i, the value of $(g\cdot x)_i$ is linear in the first-step coordinates. (This explains why the top and bottom of $\mathrm{Box}(g,t)$ are linear and parallel.) The figure is essentially a two-dimensional slice of E_s and $\mathrm{Box}(g,t)$, capturing sets along the x_i direction and some first-step direction. This slice is a simplification of the actual situation, but it embodies the essential elements: the Euclidean box E_s has sides parallel to the

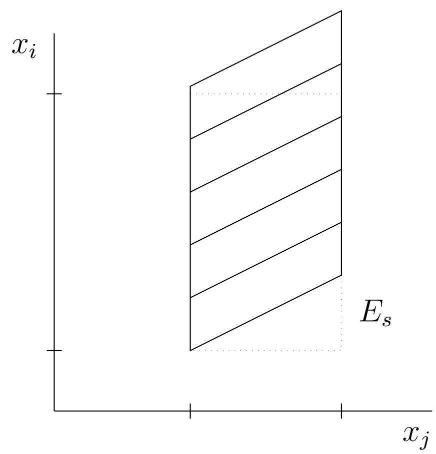

Figure 5.3: The upward stack of Carnot boxes leaves a part of E_s uncovered due to the skew.

axes, but the left-translation of $\mathrm{Box}(0,t)$ has skewed the Carnot box. As we stack the Carnot boxes, they will not neatly cover E_s. However, having bounded the extent of the skew, we can bound the number of Carnot boxes required.

Consider the bottom of the box in the i^{th} coordinate, i.e., the points $g \cdot x$ in $\mathrm{Box}(g,t)$ with $x_i = -2^{-2t\beta-1}$. The bottom ranges in its i^{th} coordinate at most from $g_i - 2^{-2t\beta-1} - C2^{-t\beta-1}$ to $g_i - 2^{-2t\beta-1} + C2^{-t\beta-1}$. A similar statement may be made about the top of the box.

We begin our covering as before, by dividing the i^{th} interval of E_s by the width of the i^{th} interval of $\mathrm{Box}(0,t)$. This operation gives:

$$\frac{C2^{-t\alpha_i}}{2^{-2t\beta}} = C2^{t(2\beta-\alpha_i)} \text{ subdivisions.}$$

To understand what occurs next, picture $\mathrm{Box}(g,t)$ such that the lowest part of its bottom is all that touches the i^{th} bottom of E_s, due to the skew of $\mathrm{Box}(g,t)$. From this starting position, stacking $C2^{t(2\beta-\alpha_i)}$ Carnot boxes upward will cover all of E_s in this direction except for the lower "triangle" (see Figure 5.3). Note that, with this particular covering, the top of E_s in this direction is covered, despite the skew.

To cover the rest of the E_s, we must stack additional Carnot boxes *under* the lowest Carnot box. By our previous calculation, the skew can add at most $C2^{-t}$ to the interval that must be covered. Covering this portion contributes a few more boxes:

$$\frac{C2^{-t\beta}}{2^{-2t\beta}} = C2^{t(2\beta-\beta)} \text{ subdivisions.}$$

Overall, then, there are $C(2^{t(2\beta-\beta)} + 2^{t(2\beta-\alpha_i)})$ subdivisions required in a step-2 direction. The dominant term here may be summarized as $C2^{t(2\beta-\min\{\beta,\alpha_i\})}$.

Recall that there are $2^t\lceil b_1 - a_1\rceil$ Euclidean boxes E_s, and we have covered each one using

$$\left(\prod_{i=2}^{d_1} \max\left\{1, C2^{t(\beta-\alpha_i)}\right\}\right)\left(\prod_{j=d_1+1}^{d_1+d_2} \max\{1, C2^{t(2\beta-\min\{\beta,\alpha_j\})}\}\right) \text{ Carnot boxes.}$$

The lower box dimension of $\operatorname{Gr} F$ is now bounded above:

$$\begin{aligned}
\text{CC-}\underline{\dim}_B \operatorname{Gr} F &= \liminf_{\delta\to 0} \frac{\log M_\delta(\operatorname{Gr} F)}{-\log\delta} \\
&\leq \lim_{t\to\infty} \frac{\log\left((2^t\lceil b_1-a_1\rceil)\left(\prod_{i=2}^{d_1}\max\{1, C2^{t(\beta-\alpha_i)}\}\right)\left(\prod_{j=d_1+1}^{d_1+d_2}\max\{1, C2^{t(2\beta-\min\{\beta,\alpha_j\})}\}\right)\right)}{-\log(2^{-t\beta})} \\
&= \lim_{t\to\infty} \frac{t\log 2 + \sum_{i=2}^{d_1}\max\{0, \log C + t(\beta-\alpha_i)\log 2\}}{t\beta\log 2} \\
&\quad + \lim_{t\to\infty} \frac{\sum_{j=d_1+1}^{d_1+d_2}\max\{0, \log C + t(2\beta-\min\{\beta,\alpha_j\})\log 2\}}{t\beta\log 2} \\
&= \frac{1+\sum_{i=2}^{d_1}(1-\alpha_i) + \sum_{j=d_1+1}^{d_1+d_2}(2\beta-\min\{\beta,\alpha_j\})}{\beta}.
\end{aligned}$$

As $w_i = 1$ for all first-step directions, we may condense the above result into the final formulation:

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq \frac{1+\sum_{i=2}^{n+1}(\beta w_i - \min\{\beta,\alpha_i\})}{\beta}.$$

To finish, we observe that this bound holds for any choice of β in $(0,1]$, so we may choose the minimum over such β of these upper bounds. □

Now we consider a second variation, where the range of F is 1-dimensional, thereby producing a full-fledged "Hölder surface." We choose the range to lie in a second-step direction here.

Theorem 5.4.2. *Let $\mathbb{G}$ be a two-step Carnot group with topological dimension $m+1$, with a graded Lie algebra as indicated in the previous section. Let $\Omega \subset \mathbb{R}^m$ be a bounded set, and let $F\colon \Omega \to \mathbb{R}$ be a Euclidean α-Hölder function, where $0 < \alpha \leq 1$. Let $\operatorname{Gr} F = \{(t_1, \dots, t_m, F(t_1, \dots, t_m)) \mid (t_1, \dots, t_m) \in \Omega\}$. Then*

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq d_1 + 2d_2 - \alpha.$$

Remark 5.4.3. Recall that the Hausdorff dimension of $\mathbb{G}$ is d_1+2d_2, so the conclusion above may be stated as $\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq \dim_H \mathbb{G} - \alpha$.

Proof. The proof proceeds as before, and we will only mention the necessary modifications.

- We begin by putting the domain Ω inside of an Euclidean box, $I_1 \times \cdots \times I_{m+1}$, where $I_i = [a_i, b_i]$ for each i.
- Each I_i is initially subdivided into $2^{-t}\lceil b_i - a_i \rceil$ equal pieces. For each first-step direction, each piece will be one box deep when we cover using the left-translated Carnot boxes. For second-step directions, it will require a depth of $2^{t(2\beta-1)}$ boxes, plus additional boxes to cover the skew.
- The interval in the $m+1^{\text{st}}$ direction for E_s is $C2^{-t\alpha}$ wide, from the Hölder condition.
- The skew of the left-translated Boxes in second-step directions still occurs as before, requiring an extra $C2^{t(2\beta-\beta}$ boxes in each direction to cover the skew, even though some of these directions are part of the domain now.

- There are $C(2^t)^{d_1+d_2-1}$ sets E_s, and each one is covered by $\max\{1, C2^{t\beta}\}^{d_2-1} \cdot \max\{1, C2^{t(2\beta-\min\{\beta,\alpha\})}\}$ Carnot boxes. These values yield:

$$\begin{aligned} \text{CC-}\underline{\dim}_B \operatorname{Gr} F &\leq \min_{0<\beta\leq 1} \left\{ \frac{d_1 + d_2 - 1 + \beta(d_2 - 1) \ + (2\beta - \min\{\beta, \alpha\})}{\beta} \right\} \\ &= d_1 + 2d_2 - \alpha. \end{aligned}$$ □

We now state the general theorem in the case of two-step Carnot groups. Its proof is largely an exercise in notation — the essential ideas have already been laid out in detail in the two proofs above — and we will omit it.

Theorem 5.4.4. *Let $\mathbb{G}$ be a two-step Carnot group with topological dimension $m+n$, with a graded Lie algebra as indicated in the previous section. Let $\Omega \subset \mathbb{R}^m$ be a bounded set, and let $F : \Omega \to \mathbb{R}^n$ satisfy an Euclidean Hölder condition of order $\alpha = (\alpha_1, ..., \alpha_n)$, where $0 < \alpha_i \leq 1$ for every $i = 1, ..., n$. Let $\operatorname{Gr} F$ denote the graph of F as a subset of $\mathbb{R}^{m+n}$. We allow the domain of F to consist of any combination of m-many first-step and second-step directions. Let R denote the set of indices for directions in the range of F, and let D_2 denote the number of second-step directions in the domain of F. Then*

$$\text{CC-}\underline{\dim}_B \operatorname{Gr} F \leq \min_{0<\beta\leq 1} \left\{ D_2 + \frac{m + \sum_{j\in R} \beta w_j - \sum_{i=1}^{n} \min\{\beta, \alpha_i\}}{\beta} \right\}. \tag{5.4.1}$$

Remark 5.4.5. The situation in higher-step Carnot groups is similar in many ways, and it can be approached in a similar manner. The most significant difference, conceptually, is that the guiding picture (Figure 5.2) has changed: when left-translating the Carnot box, the skew in a third-step direction is linear in second-step variables and *quadratic* in first-step variables. This dependence means the bottom and top, while still identical (except for a shift), are not planes. This doesn't greatly affect the calculations, however; the BCH formula still provides a bound for the amount of skew, and the dimension calculations proceed as before.

Finally, we can simplify Theorem 5.4.4 by a simple observation: swapping a second-step direction in the domain for a first-step direction in the range will have zero net effect on the upper bound in (5.4.1). The value of D_2 will decrease by one, but the sum $\sum_{j\in R}\beta w_j$ will increase by β. The result is that the upper bound is independent of the assignment of coordinates to domain and range.

Theorem 5.4.6. *Let $\mathbb{G}$ be a two-step Carnot group with topological dimension $m+n$, with a graded Lie algebra as indicated in the previous section. Let $\Omega \subset \mathbb{R}^m$ be a bounded set, and let $F : \Omega \to \mathbb{R}^n$ satisfy an Euclidean Hölder condition of order $\alpha = (\alpha_1, ..., \alpha_n)$, where $0 < \alpha_i \leq 1$ for every $i = 1, ..., n$. Let $\mathrm{Gr}\, F$ denote the graph of F as a subset of $\mathbb{R}^{m+n}$. Then*

$$\text{CC-}\underline{\dim}_B \,\mathrm{Gr}\, F \leq \min_{0<\beta\leq 1}\left\{d_1 + 2d_2 - m + \frac{m - \sum_{i=1}^n \min\{\beta, \alpha_i\}}{\beta}\right\}. \tag{5.4.2}$$

Remark 5.4.7. Again, recall that $\dim_H \mathbb{G} = d_1 + 2d_2$. Hence, we can write the upper bound (5.4.2) as

$$\text{CC-}\underline{\dim}_B \,\mathrm{Gr}\, F \leq \min_{0<\beta\leq 1}\left\{\dim_H \mathbb{G} - m + \frac{m - \sum_{i=1}^n \min\{\beta, \alpha_i\}}{\beta}\right\}.$$

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