TRANSCHROMATIC GENERALIZED CHARACTER MAPS

BY

NATHANIEL J. STAPLETON

DISSERTATION

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Doctoral Committee:

Associate Professor Matthew Ando, Chair
Associate Professor Charles Rezk, Director of Research
Professor Randy McCarthy
Associate Professor Henry Schenck

Abstract

In [5], Hopkins, Kuhn, and Ravenel discovered a generalized character theory that proved useful in studying cohomology rings of the form $E^*(BG)$. In this paper we use the geometry of p-divisible groups to describe a sequence of "intermediate" character theories that retain more information about the cohomology theory E and yield the related result of [5] as a special case.

Dedicated to my Wife and my Parents

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Chapter 1

Introduction

In [5], Hopkins, Kuhn, and Ravenel discovered a generalized character theory that proved useful in studying cohomology rings of the form $E^*(BG)$. In this paper we describe a sequence of "intermediate" character theories that retain more information about the cohomology theory E and yield the related result of [5] as a special case. We begin with a brief summary of the work and then expound on this in much more detail.

Let E_n be Morava E-theory and G be a finite group. Hopkins, Kuhn, and Ravenel study the rings $E_n^*(BG)$ in terms of associated characters. They were inspired by Atiyah's theorem that

$$K^0(BG) \cong R(G)_I^\wedge$$

the K-theory of BG is isomorphic to the complex representation ring of G completed at the ideal of virtual representations of dimension 0. There is a natural map

$$R(G) \longrightarrow Cl(G, L)$$

taking a virtual representation to the sum of its characters in the class functions on G. As $R(G)$ can be studied via the associated character theory of the group, Hopkins, Kuhn, and Ravenel aimed to create a character theory for $E_n^*(BG)$. They created a cohomology theory built out of E_n^* that mimics the properties of $CL(G, L)$ and receives a map of equivariant cohomology theories from E_n.

$$E_n \longrightarrow L(E_n)$$

The cohomology theory that they construct is rational. The map they create therefore begins with a height n cohomology theory, E_n, and lands in a height 0 cohomology theory. It is thus transchromatic in nature, moving between cohomology theories of differing heights. In this paper we produce for every height t with $0 \leq t < n$ generalizations of their map such that the cohomology theory in the codomain has height t instead of 0 and their map is recovered when $t = 0$.

Let K be complex K-theory and let $R(G)$ be the complex representation ring of a finite group G. Consider a complex representation of G as a G-vector bundle over a point. Then there is a natural map $R(G) \to K^0(BG)$. This takes a virtual representation to a virtual vector bundle over BG by applying the Borel construction $EG \times_G -$. Work of Atiyah in the 50's and 60's shows that this map becomes an isomorphism after completing $R(G)$ with respect to the ideal of virtual bundles of dimension 0. [1]

Let L be the smallest characteristic zero field containing all roots of unity and let $Cl(G;L)$ be the ring of class functions on G taking values in L. A classical result in representation theory states that L is the smallest field such that the character map

$$\chi : R(G) \longrightarrow Cl(G,L)$$

taking a virtual representation to the sum of its characters induces an isomorphism $L \otimes R(G) \xrightarrow{\cong} Cl(G;L)$ for every finite G.

Hopkins, Kuhn, and Ravenel build, for each Morava E-theory, an equivariant cohomology theory that mimics the properties of $Cl(G,L)$ and is the receptacle for a map from Borel equivariant E_n. They begin by constructing a ring $L(E_n)^*$ out of E_n^*. We describe their construction. Let $\Lambda_k = (\mathbb{Z}/p^k)^n$, $\Lambda_k^* = \hom(\Lambda_k, S^1)$, and $\mathbb{G}_{E_n}$ be the formal group associated to E_n. The identity map $E_n^*(B\Lambda_k) \xrightarrow{id} E_n^*(B\Lambda_k)$ corresponds to a map $\Lambda_k^* \longrightarrow \mathbb{G}_{E_n}(E_n^*(B\Lambda_k))$. Localizing with respect to the nonzero image of this map gives a ring $S_k^{-1}E_n^*(B\Lambda_k)$ and then $L(E_n)^*$ is defined to be

$$L(E_n)^* = \operatorname*{colim}_k S_k^{-1} E_n^*(B\Lambda_k).$$

For X a G-space they define a G-space $\mathrm{Fix}(X) = \coprod_{\alpha \in \hom(\mathbb{Z}_p^n, G)} X^{\mathrm{im}\,\alpha}$ and their map takes the form

$$E_n^*(EG \times_G X) \xrightarrow{\Phi_G} L(E_n)^*(\mathrm{Fix}(X))^G.$$

The codomain of this map is closely related to the class functions on G taking values in $L(E_n)^0$. In fact, when X is a point the codomain reduces to precisely class functions on

$$\hom(\mathbb{Z}_p^n, G) = \{(g_1, \ldots, g_n) | g_i^{p^k} = e \text{ for some } k, [g_i, g_j] = e\}.$$

As in the case of the representation theorem there is an isomorphism

$$L(E_n)^0 \otimes_{E_n^0} E_n^0(EG \times_G X) \xrightarrow{\cong} L(E_n)^0(\mathrm{Fix}(X))^G.$$

The construction of $L(E_n)^0$ may seem ad hoc, but in fact it satisfies an important universal property: it is the smallest ring extension of E_n^0 such that $\mathbb{G}_{E_n}$, when pulled back over $L(E_n)^0$, is isomorphic to the constant group scheme $(\mathbb{Q}_p/\mathbb{Z}_p)^n$.

$$\begin{array}{ccc} \mathbb{Q}_p/\mathbb{Z}_p^n & \longrightarrow & \mathbb{G}_{E_n} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L(E_n)^0) & \longrightarrow & \mathrm{Spec}(E_n^0) \end{array}$$

This result can be rephrased in the language of p-divisible groups. Let R be a ring. A p-divisible group over R of height n is an inductive system (G_v, i_v) such that

1. G_v is a finite free commutative group scheme over R of order p^{vn}.

2. For each v, there is an exact sequence

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1}$$

   where i_v is the natural inclusion and p^v is multiplication by p^v in G_{v+1}.

Associated to every formal group, $\mathbb{G}$, over R is a p-divisible group

$$\mathbb{G} \rightsquigarrow \mathbb{G}[p] \xrightarrow{i_1} \mathbb{G}[p^2] \xrightarrow{i_2} \dots.$$

the ind-group scheme built out of the p^k-torsion for varying k. The only constant p-divisible groups are products of $\mathbb{Q}_p/\mathbb{Z}_p$. The ring that Hopkins, Kuhn, and Ravenel construct is the smallest extension of E_n such that $\mathbb{G}_{E_n}$ pulls back to a constant p-divisible group.

For $\mathbb{G}_{E_n}$, we have $\mathcal{O}_{\mathbb{G}_{E_n}[p^k]} \cong E_n^0(B\mathbb{Z}/p^k) = \pi_0 F(B\mathbb{Z}/p^k, E_n)$, the homotopy groups of the function spectrum. The pullback of $\mathbb{G}_{E_n}[p^k]$ constructed by Hopkins, Kuhn, and Ravenel in [5] factors through $\pi_0 L_{K(0)}(F(B\mathbb{Z}/p^k, E_n))$ the rationalization of the function spectrum. Spec of this Hopf algebra is the p^k-torsion of an ind-etale p-divisible group. Rezk noted that there are higher analogues of this: Fix an integer t such that $0 \leq t < n$. Then Spec of $\pi_0(L_{K(t)}F(B\mathbb{Z}/p^k, E_n))$ gives the p^k-torsion of a p-divisible group, $\mathbb{G}$, over $L_{K(t)}E_n^*$.

Associated to $\mathbb{G}$ is a short exact sequence of p-divisible groups

$$0 \longrightarrow \mathbb{G}_0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{et} \longrightarrow 0$$

where $\mathbb{G}_0$ is the formal group associated to $L_{K(t)}E_n$ and $\mathbb{G}_{et}$ is an ind-etale p-divisible group. The height of $\mathbb{G}$ is the height of $\mathbb{G}_0$ plus the height of $\mathbb{G}_{et}$.

These facts suggest that there may be results similar to those of [5] over a ring for which the pulled back p-divisible group actually has a formal component but for which the etale part has been made constant. The main theorem of this paper is that this is so.

Theorem. For each $0 \leq t < n$ there exists a ring extension of E_n^0, C_t, such that the pullback

$$\begin{array}{ccccc} \mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t} & \longrightarrow & \mathbb{G} & \longrightarrow & \mathbb{G}_{E_n} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(C_t) & \longrightarrow & \mathrm{Spec}(L_{K(t)}E_n^0) & \longrightarrow & \mathrm{Spec}\, E_n^0 \end{array}$$

is the sum of a height t formal group by a constant height $n-t$ p-divisible group. C_t is flat over E_n^0 and can be used to make a height t cohomology theory. Let $G_p = \hom(\mathbb{Z}_p^{n-t}, G)$ and $\mathrm{Fix}(X) = \coprod_{\alpha \in G_p} X^{\mathrm{im}\,\alpha}$ then for all finite G there is a map of equivariant theories

$$E_n^*(EG \times_G X) \longrightarrow C_t^*(EG \times_G \mathrm{Fix}(X))$$

so that when the domain is tensored with C_t the map becomes an isomorphism of equivariant cohomology theories.

This map is intimately related to the algebraic geometry of the situation. In fact, when $X = *$ and $G = \mathbb{Z}/p^k$ this map recovers the global sections of the map on p^k-torsion $\mathbb{G}_0[p^k] \oplus (\mathbb{Z}/p^k)^{n-t} \longrightarrow \mathbb{G}_{E_n}[p^k]$. The map of Hopkins, Kuhn, and Ravenel is recovered when $t = 0$.

The thesis contains two chapters. In the first chapter we work with the algebraic geometry of p-divisible groups and in the second chapter we construct the transchromatic generalized character maps and study their basic properties.

The first chapter is split into two sections. We begin by proving that $\mathbb{G}$ is the middle term of a short exact sequence of p-divisible groups and studying the etale quotient in the exact sequence. Then we move on to constructing the ring extension of E_n^0 over which $\mathbb{G}$ splits as a sum of its formal part and a constant p-divisible group.

The second chapter contains three sections. The transchromatic generalized character maps can be split into two parts, a topological part and an algebraic part. In the first section we describe the topological part in terms of transport categories and work out some examples of the map for particular spaces. In the second section we describe the algebraic part of the character map and put the topological and algebraic parts together. In the third section we prove that the transchromatic character maps induce an isomorphism when the source is tensored up to C_t.

Chapter 2

Preliminaries

2.1 Conventions

Within this paper all rings are commutative with unit and all graded rings are graded commutative.

By a *cohomology theory* we mean a generalized cohomology theory on the category of finite spaces (spaces equivalent to a finite CW-complex), $\mathrm{Top}^{\mathrm{f}}$. That is a functor

$$(\mathrm{Top}^{\mathrm{f}})^{op} \longrightarrow \mathrm{AB}_*$$

from finite spaces to graded abelian groups that satisfies all of the Eilenberg-Steenrod axioms except for the dimension axiom. We choose finite spaces as our source category because it allows for flat extension of cohomology theories. By an equivariant cohomology theory we will always mean a Borel-equivariant theory.

For G an abelian group, let G^* be the dual group $\hom(G, S^1)$.

For any L-algebra R and ideal $I \subseteq L$, by R/I we mean $R/(I \cdot R)$ the quotient of R by the extension of I to R.

We always use the symbol $\otimes$ without a subscript although one is needed. Context provides sufficient information to work out what it ought to be.

2.2 Commutative Algebra

There are several basic theorems from commutative algebra that are important in the following chapters. Let R be a ring, I an ideal of R and M a finitely generated R-module. On several occasions a basis for M/IM as an R/I-module needs to be lifted to a basis for M as an R-module. The main result we need in this direction is a corollary of Nakayama's Lemma:

Proposition 2.2.1. [9] Let R, I, and M be as above. If $M = IM$ then there exists $r \in R$ such that $rM = 0$ and $r \equiv 1 \mod I$. If in addition I is contained in the Jacobson radical of R then $M = 0$.

Corollary 2.2.2. [9] Let R be a ring, I an ideal contained in the Jacobson radical of R and M an R-module. Suppose that $N \subseteq M$ is a submodule such that M/N is a finite R-module. Then $M = N + IM$ implies that $M = N$.

Corollary 2.2.3. Let R and I be as in the previous corollary. Let M be a finite R-module such that M/IM is a free R/I-module. Then every lift of a minimal basis of M/IM as an R/I-module to M is a minimal basis of M as an R-module.

Proof. Let $u_1, \ldots, u_n$ be a basis for M/IM and $m_1, \ldots, m_n$ a lifting of the basis to M. $M = \sum Rm_i + IM$, application of the previous corollary implies that $M = \sum Rm_i$ and the minimality follows from the minimality of the basis in the quotient. $\square$

2.3 Algebraic Geometry

Let R be a commutative ring. For the purposes of this paper a *p-divisible group* over R of height h is an inductive system $(G_v, i_v) = G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} \ldots$ such that

(i) G_v is a finite free commutative group scheme over R of order p^{vh}

(ii) For each v, there is an exact sequence

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1}.$$

For more information about p-divisible groups see [3] and [10].

Chapter 3

Transchromatic Geometry

Let $0 \leq t < n$ and fix a prime p. In this chapter we study the p-divisible group obtained from $\mathbb{G}_{E_n}$ by base change to $\pi_0 L_{K(t)} E_n$. In the first section we prove that it sits inside an exact sequence of p-divisible groups

$$0 \longrightarrow \mathbb{G}_0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{et} \longrightarrow 0$$

where the first group is formal and the last is ind-etale. In the second section we construct the ring extension of $\pi_0 L_{K(t)} E_n$ over which the p-divisible group splits as a sum of a height t formal group and a constant height $n-t$ ind-etale p-divisible group.

3.1 The Exact Sequence

This paper will be concerned with the Morava E-theories E_n and their localizations with respect to Morava $K(t)$-theory for $0 \leq t < n$: $L_{K(t)} E_n$. E_n is an even periodic height n theory and $L_{K(t)} E_n$ is an even periodic height t theory. Basic properties of these cohomology theories can be found in ([15], [6], [5], [13]) for instance. The coefficients of these theories are

$$E_n^0 \cong W(k)[\![u_1, \ldots, u_{n-1}]\!]$$
$$L_{K(t)} E_n^0 \cong W(k)[\![u_1, \ldots, u_{n-1}]\!][u_t^{-1}]^{\wedge}_{(p,\ldots,u_{t-1})}$$

The second isomorphism follows from [6]. Thus the ring $L_{K(t)} E_n^0$ is obtained from E_n^0 by inverting the element u_t and then completing with respect to the ideal $(p, u_1, \ldots, u_{t-1})$.

Let E be one of the cohomology theories above. Classically it is most common to study these cohomology theories in terms of the associated formal group $\mathbb{F}_E = \operatorname{Spf}(E^0(BS^1))$. However, in this paper we will be studying these cohomology theories in terms of their associated p-divisible group. First we fix a coordinate for the formal group $\mathcal{O}_{\mathbb{F}_E} \cong_x E^0[\![x]\!]$, this provides us with a formal group law $\mathbb{F}_E(x, y) \in E^0[\![x, y]\!]$. This coordinate can be used to understand the associated p-divisible group.

Let $\mathbb{G}_E[p^k] = \mathrm{Spec}(E^0(B\mathbb{Z}/p^k)) = \hom_{E^0-alg}(E^0(B\mathbb{Z}/p^k), -)$. As $B\mathbb{Z}/p^k$ is an H-space, $E^0(B\mathbb{Z}/p^k)$ is a Hopf algebra and $\mathbb{G}_E[p^k]$ is a commutative group scheme. It is a classical theorem ([5],[14]) that

Theorem 3.1.1. Given a generator $\beta^k \in (\mathbb{Z}/p^k)^* = \hom(\mathbb{Z}/p^k, S^1)$ there is an isomorphism $E^0(B\mathbb{Z}/p^k) \cong_{\beta^k} E^0[\![x]\!]/([p^k](x))$ where $[p^k](x)$ is the p^k-series for the formal group law associated to E.

The dual is needed because $\mathbb{Z}/p^k \longrightarrow S^1$ induces $E^0(BS^1) \longrightarrow E^0(B\mathbb{Z}/p^k)$ and allows us to use the coordinate for the formal group in order to understand the codomain. Now the Weierstrass preparation theorem implies that

Proposition 3.1.2. ([5]) If the height of E is n then $E^0[\![x]\!]/([p^k](x))$ is a free E^0-module with basis $\{1, x, \dots x^{p^{kn}-1}\}$.

Thus we see that $\mathbb{G}_E[p^k]$ is a finite free group scheme of order p^{kn}. We now have the group schemes that we would like to use to form a p-divisible group. We must define the maps that make them into a p-divisible group.

For each k fix a generator $\beta^k \in (\mathbb{Z}/p^k)^*$. Define $i_k : \mathbb{Z}/p^k \longrightarrow \mathbb{Z}/p^{k+1}$ to be the unique map such that $\beta^{k+1} \circ i_k = \beta^k$. Then, with the coordinate,

$$i_k^* = E^0(Bi_k) : E[\![x]\!]/([p^{k+1}](x)) \longrightarrow E[\![x]\!]/([p^k](x)) : x \mapsto x.$$

Spec of this map is the inclusion $i_k : \mathbb{G}_E[p^k] \longrightarrow \mathbb{G}_E[p^{k+1}]$ and makes the inductive sequence $\mathbb{G}_E[p] \xrightarrow{i_1} \mathbb{G}_E[p^2] \xrightarrow{i_2} \dots$ a p-divisible group.

Before continuing we establish some notation. Let $L = L_{K(t)}E_n^0$ (remember that this depends on t) and $m_L = (p, u_1, \dots u_{t-1})$. Note that m_L is not necessarily a maximal ideal. For a scheme X over $\mathrm{Spec}(R)$ and a ring map $R \longrightarrow S$ let

$$S \otimes X = \mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} X.$$

Given a p-divisible group $\mathbb{G}_E$ over E^0 and a ring map $E^0 \longrightarrow S$ let $S \otimes \mathbb{G}_E$ be the p-divisible group such that $(S \otimes \mathbb{G}_E)[p^k] = S \otimes (\mathbb{G}_E[p^k])$.

There are a few facts ([15]) regarding the p^k-series for the formal group law $\mathbb{F}_{E_n}(x, y)$ that we will need later that are best collected here. For $0 \leq h < n$

$$[p](x) = \quad [p]_h(x^{p^h}) = \quad u_h x^{p^h} + \dots \bmod (p, u_1, \dots u_{h-1})$$

$$[p^k](x) = \quad [p^k]_h(x^{p^{kh}}) = \quad (u_h)^k (x^{p^{kh}}) + \dots \bmod (p, u_1, \dots u_{h-1})$$

There is a localization map $E_n \longrightarrow L_{K(t)}E_n$ that induces $E_n^0 \longrightarrow L$ and $\mathbb{F}_{L_{K(t)}E_n}(x,y)$ is obtained from $\mathbb{F}_{E_n}(x,y)$ by applying this map to the coefficients. Proposition 3.1.2 implies that in $E_n^0[\![x]\!]$

$$[p^k](x) = f_k(x)w_k(x)$$

where $f_k(x)$ is a monic degree p^{kn} polynomial and $w_k(x)$ is a unit. In $L[\![x]\!]$

$$[p^k](x) = g_k(x)v_k(x)$$

where $g_k(x)$ is a monic degree p^{kt} polynomial and $v_k(x)$ is a unit. Sometimes, when there is no reason to be pedantic about notation, we will write $[p^k](x) = f(x)w(x)$ and ignore the k subscripts.

Now we focus our attention on the p-divisible group $L \otimes \mathbb{G}_{E_n}$.

Proposition 3.1.3. $L \otimes \mathbb{G}_{E_n}$ is a p-divisible group of height n with formal part of height t.

Proof. The idea of the proof is the following: we have the pullback square

$$\begin{array}{ccc} L \otimes \mathbb{G}_{E_n} & \longrightarrow & \mathbb{G}_{E_n} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(E_n) \end{array}$$

and we show at the level of p^k-torsion that $L \otimes \mathbb{G}_{E_n}[p^k]$ is a disjoint union by exhibiting $\mathcal{O}_{L\otimes\mathbb{G}_{E_n}[p^k]}$ as a product. We will see that the factor that contains the identity is isomorphic to the p^k-torsion of a formal group over L and thus connected. We prove this for the case $k = 1$. The other cases follow analogously.

The height of $L \otimes \mathbb{G}_{E_n}$ is an immediate consequence of Proposition 3.1.2. To discover the height of the formal part of $L \otimes \mathbb{G}_{E_n}$ we must work out the height of the connected component of the identity of $L \otimes \mathbb{G}_{E_n}[p]$.

$$L \otimes E_n^0(B\mathbb{Z}/p) \cong L \otimes E_n^0[\![x]\!]/([p](x)) \cong L \otimes E_n^0[x]/(f(x)) \cong L[x]/f(x)$$

where $f(x)$ is a monic degree p^n polynomial. The second isomorphism follows from the Weierstrass preparation theorem.

In $E_n^0[\![x]\!]$, $[p](x) = f(x)w(x)$ and in $L[\![x]\!]$, $[p](x) = g(x)v(x)$ with $g(x)$ a monic degree p^t polynomial and both power series $w(x)$ and $v(x)$ units. Both factorizations hold true in $L[\![x]\!]$ and thus $f(x) = g(x)h(x)$ as polynomials where $h(x) = v(x)/w(x)$.

$L[x]/f(x)$, $L[x]/g(x)$, and $L[x]/h(x)$ are all free as the polynomials are monic and thus the natural map induced by quotienting $L[x]/f(x) \longrightarrow L[x]/g(x) \times L[x]/h(x)$ has the correct rank on both sides. We must show that it is surjective.

Initially we work mod m_L. Mod m_L, $g(x) = x^{p^t}$ and $h(x)$ has constant term a unit, u_t, and smallest nonconstant term degree x^{p^t} thus the ideals $(g(x))$ and $(h(x))$ are coprime. The isomorphism mod m_L can be lifted by Corollary 2.2.3 to L by choosing generators for the free modules mod m_L and choosing lifts to the modules over L. For instance we could choose the basis consisting of powers of x for the domain and tensors of powers of x for the codomain.

Now $L[x]/g(x)$ is isomorphic to the p-torsion of the formal group associated to $L_{K(t)}E_n$ and thus contains the identity and is connected. Its height is as specified in the proposition. □

We conclude that the connected component of the identity of $L \otimes \mathbb{G}_{E_n}[p^k]$ is isomorphic to $\mathbb{G}_{L_{K(t)}E_n}[p^k]$. Let $\mathbb{G} = L \otimes \mathbb{G}_E$ and $\mathbb{G}_0 = \mathbb{G}_{L_{K(t)}E_n}$.

Recall that we are working to prove that the p-divisible group $\mathbb{G}$ lives in a short exact sequence

$$0 \longrightarrow \mathbb{G}_0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{et} \longrightarrow 0$$

where the first group is formal and the last is ind-etale. This will come from an exact sequence at each level

$$0 \longrightarrow \mathbb{G}_0[p^k] \longrightarrow \mathbb{G}[p^k] \longrightarrow \mathbb{G}_{et}[p^k] \longrightarrow 0.$$

Next we show that $\mathbb{G}_{et}[p^k]$ is in fact etale (as its nomenclature suggests). We begin by giving a description of the global sections of $\mathbb{G}_{et}[p^k]$.

$\mathbb{G}_{et}[p^k]$ is the quotient of $\mathbb{G}[p^k]$ by $\mathbb{G}_0[p^k]$. It can be described as the coequalizer of

$$\mathbb{G}_0[p^k] \times \mathbb{G}[p^k] \underset{\pi}{\overset{\mu}{\rightrightarrows}} \mathbb{G}[p^k]$$

where the two maps are the multiplication, μ, and the projection, π.

The following general discussion on norms and quotients of group schemes follows that of Strickland in [16] or Demazure-Gabriel in [4] and is included for completeness. Given a finite free map of affine schemes $f : X \longrightarrow Y$ and a $u \in \mathcal{O}_X$, multiplication by u is an $\mathcal{O}_Y$-linear endomorphism of $\mathcal{O}_X$. Thus its determinant is an element in $\mathcal{O}_Y$. Let $N_f : \mathcal{O}_X \to \mathcal{O}_Y$ be the multiplicative norm map

$$N_f(x) = \det(- \times x)$$

the map that sends $u \in \mathcal{O}_X$ to the determinant of multiplication by u. N_f is not additive.

Two important properties of the norm are the following:

Lemma 3.1.4. [16] Let

$$\begin{array}{ccc} V & \xrightarrow{s} & X \\ \downarrow{\scriptstyle g} & \ulcorner & \downarrow{\scriptstyle f} \\ W & \xrightarrow{t} & Y \end{array}$$

be a pullback square of affine schemes where f and thus g are finite and free then $N_g \circ s^* = t^* \circ N_f$.

Lemma 3.1.5. [16] Suppose that $s : Y \longrightarrow X$ is a section of f and that $s^*u = 0$. Then $N_f u = 0$.

Above we described $\mathbb{G}_{et}[p^k]$ as a coequalizer of group schemes, the global sections of the diagram gives a description of $\mathcal{O}_{\mathbb{G}_{et}[p^k]}$ as an equalizer

$$\mathcal{O}_{\mathbb{G}_{et}[p^k]} \longrightarrow \mathcal{O}_{\mathbb{G}[p^k]} \rightrightarrows \mathcal{O}_{\mathbb{G}[p^k]} \otimes \mathcal{O}_{\mathbb{G}_0[p^k]}.$$

Using this description and the lemmas about norms we can show that $y = N_\pi \mu^*(x)$, naturally an element of $\mathcal{O}_{\mathbb{G}[p^k]}$, in fact lives in $\mathcal{O}_{\mathbb{G}_{et}[p^k]}$ and generates it as an algebra.

Let $\pi_{12} : \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] \times \mathbb{G}_0[p^k] \longrightarrow \mathbb{G}[p^k] \times \mathbb{G}_0[p^k]$ be the projection on the first two factors. By considering the functor of points it is clear that the following two diagrams are pullback squares:

$$\begin{array}{ccc} \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] \times \mathbb{G}_0[p^k] & \xrightarrow{1\times\mu} & \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] \\ \downarrow{\scriptstyle \pi_{12}} & \ulcorner & \downarrow{\scriptstyle \pi} \\ \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] & \xrightarrow{\pi} & \mathbb{G}[p^k] \end{array}$$

and

$$\begin{array}{ccc} \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] \times \mathbb{G}_0[p^k] & \xrightarrow{\mu\times 1} & \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] \\ \downarrow{\scriptstyle \pi_{12}} & \ulcorner & \downarrow{\scriptstyle \pi} \\ \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] & \xrightarrow{\mu} & \mathbb{G}[p^k]. \end{array}$$

As π is finite and free we have that $\pi^* N_\pi = N_{\pi_{12}}(1\times\mu)^*$ and $\mu^* N_\pi = N_{\pi_{12}}(\mu\times 1)^*$. Thus as $\mu(1\times\mu) =$

$\mu(\mu \times 1)$ we have

$$\begin{aligned}\mu^* y &= \mu^* N_\pi \mu^* x \\ &= N_{\pi_{12}}(\mu \times 1)^* \mu^* x \\ &= N_{\pi_{12}}(1 \times \mu)^* \mu^* x \\ &= \pi^* N_\pi \mu^* x \\ &= \pi^* y.\end{aligned}$$

It follows that y is an element of the equalizer.

Let $i : \mathbb{G}_0[p^k] \longrightarrow \mathbb{G}[p^k]$ be the inclusion.

Lemma 3.1.6. [16] $i^* y = 0$.

Proof. Let $j : \mathbb{G}_0[p^k] \longrightarrow \mathbb{G}[p^k] \times \mathbb{G}_0[p^k]$ be the map that sends $a \mapsto (0, a)$. Consider the following diagram:

$$\begin{array}{ccccc} \mathbb{G}_0[p^k] & \xrightarrow{j} & \mathbb{G}[p^k] \times \mathbb{G}_0[p^k] & \xrightarrow{\mu} & \mathbb{G}[p^k] \\ {\scriptstyle s}\downarrow & \ulcorner & \downarrow{\scriptstyle \pi} & & \\ \mathrm{Spec}(L) & \xrightarrow{0} & \mathbb{G}[p^k] & & \end{array}$$

We have that $\pi j = 0$ and $\mu j = i$. Thus $i^* y = j^* \mu^* y = j^* \pi^* y = 0^* y$. Also from the first lemma on norms we have that $0^* y = N_s j^* \mu^* x = N_s i^* x$. Now as $0 : \mathrm{Spec}(L) \longrightarrow \mathbb{G}_0[p^k]$ is a section of s and $0^*(i^* x) = 0$ the second lemma on norms implies that $N_s i^* x = 0$. □

Proposition 3.1.7. There is an isomorphism $\mathcal{O}_{\mathbb{G}_{et}[p^k]} \cong L[y]/(j_k(y))$ where $j_k(y)$ is a monic polynomial of degree $p^{k(n-t)}$.

Proof. For readability we will drop the k subscript from the polynomials g, f and j. Recall that we have given more explicit descriptions of $\mathcal{O}_{\mathbb{G}[p^k]}$ and $\mathcal{O}_{\mathbb{G}_0[p^k]}$:

$$\begin{aligned}\mathcal{O}_{\mathbb{G}[p^k]} &\cong L[x]/(f(x)) \\ \mathcal{O}_{\mathbb{G}_0[p^k]} &\cong L[x]/(g(x)).\end{aligned}$$

The previous proposition implies that $g(x)|y$ in $L[x]/(f(x))$.

It turns out to be easy to understand y mod m_L. This is because the norm commutes with quotienting. When working mod m_L, $g(x) = x^{p^{kt}}$. So $\mathcal{O}_{\mathbb{G}[p^k] \times \mathbb{G}_0[p^k]} \cong (L/m_L)[x, z]/(f(x), z^{p^{kt}})$ and $\mu^* x = x$ mod z because $\mu^* x$ is the image of the formal group law in $(L/m_L)[x, z]/(f(x), z^{p^{kt}})$. So the matrix for multiplication by $\mu^* x$ in the basis $1, z, \ldots, z^{p^{kt}-1}$ is upper triangular with diagonal entries x. Thus $y = N_\pi \mu^* x = x^{p^{kt}} \bmod m_L$.

$\mathcal{O}_{\mathbb{G}_{et}[p^k]}$ is a subalgebra of $\mathcal{O}_{\mathbb{G}[p^k]}$ that is free as an L-module. As $y \in \mathcal{O}_{\mathbb{G}_{et}[p^k]}$ so is $y^l = N_\pi \mu^* x^l$. Now as each of $\{1, y, \ldots, y^{p^{(n-t)k}-1}\}$ are linearly independent $\bmod\ \mathfrak{m}_L$, they are linearly independent in $L[x]/(f(x))$. Also Nakayama's lemma implies that they are part of a basis for $L[x]/(f(x))$, because the set is part of a basis mod $\mathfrak{m}_L$. Together these facts along with the fact that there are enough of them to span $\mathcal{O}_{\mathbb{G}_{et}[p^k]}$ implies that they in fact do span. Thus $\mathcal{O}_{\mathbb{G}_{et}[p^k]} \cong L[y]/(j(y))$ where $j(y)$ is the monic polynomial relation between the exponents of y. □

Strickland also shows that $0^*(y) = 0$, where $0 : \mathrm{Spec}(L) \longrightarrow \mathbb{G}_{et}[p^k]$ is the identity of the group, this implies that $x|y$. Thus for a ring R, $\mathbb{G}_{et}[p^k](R)$ is a group with identity the $0 \in R$. This in turn implies that $y|j(y)$ as 0 must be a root of $j(y)$.

We have shown that $\mathcal{O}_{\mathbb{G}_{et}[p^k]}$ is a free module of rank $p^{(n-t)k}$. In our final analysis of $\mathbb{G}_{et}[p^k]$ we would like to conclude that $j'(y)$ is a unit. This will imply $\mathbb{G}_{et}[p^k]$ is etale [11].

We begin with a trivial lemma.

Lemma 3.1.8. Given a ring of the form $R[x]/(p(x))$ where $p(x)$ is some monic polynomial and an ideal $I \subset R$ the following diagram commutes.

$$\begin{array}{ccc} R[x]/(p(x)) & \xrightarrow{\frac{\partial}{\partial x}} & R[x]/(p(x)) \\ \downarrow & & \downarrow \\ (R/I)[x]/(p(x)) & \xrightarrow{\frac{\partial}{\partial x}} & (R/I)[x]/(p(x)) \end{array}$$

Proof. □

We prove that $\mathbb{G}_{et}[p^k]$ is etale for the case $k = 1$ in order to ease the notational burden. The other cases follow almost identically. Let's recall and establish some notation.

Recall that $\mathcal{O}_{\mathbb{G}[p]} \cong L \otimes E_n^0[\![x]\!]/([p](x)) \cong L[x]/(f(x))$ because $[p](x) = f(x) \cdot w(x)$ where $w(x)$ is a unit. Also

$$[p](x) = [p]_t(x^{p^t}) = u_t x^{p^t} + \ldots \bmod m_L$$

Thus $[p]_t(x^{p^t}) = f^*(x^{p^t}) w^*(x^{p^t}) \bmod m_L$ where w^* is a unit. Studying $\mathbb{G}_{et}[p]$ above we showed that $j(y) = f^*(y) \bmod m_L$. Thus

$$\mathcal{O}_{\mathbb{G}_{et}[p]} \cong L[y]/(j(y)) \cong L \otimes E_n^0[\![y]\!]/[p]_t(y) \bmod m_L.$$

Recall that

$$[p](x) = [p]_{t+1}(x^{p^{t+1}}) \bmod m_L + (u_t).$$

Thus $[p]_t(y) = [p]_{t+1}(y^p) \bmod u_t$.

Lemma 3.1.9. Modulo m_L, $[p]_t'(y) = 1 \otimes [p]_t'(y) \in (L/m_L) \otimes E_n^0[\![y]\!]/[p]_t(y)$ is a unit

Proof. We show that $u_t | [p]_t'(y)$ in $E_n^0[\![y]\!]/([p]_t(y))$ or in other words that $[p]_t'(y) = 0 \bmod u_t$. From above we have that

$$[p]_t(y) = [p]_{t+1}(y^p) \bmod u_t$$

and the derivative of this is zero as we are working in characteristic p. Now the previous lemma (applied to module-finite power series rings) implies that $[p]_t'(y) = 0 \bmod u_t$. This implies that

$$1 \otimes [p]_t'(y) = u_t \otimes (1 + \ldots)$$

which is a unit. □

Proposition 3.1.10. $\mathbb{G}_{et}[p]$ is an etale group scheme over L.

Proof. We show that $j'(y)$ is a unit. Recall that

$$[p]_t(y) = j(y)w^*(y) \qquad \bmod m_L$$
$$[p]_t'(y) = j'(y)w^*(y) + j(y)(w^*)'(y) \qquad \bmod m_L$$

but $j(y)(w^*)'(y) = 0$ in $(L/m_L)[y]/(j(y))$ and now we see that $j'(y) = [p]_t'(y)/w^*(y)$ is a unit $\bmod m_L$. The previous lemma now tells us that working in $L[y]/(j(y))$ (no longer working modulo m_L) $j'(y)$ maps to a unit and as m_L is in the Jacobson radical of the ring $j'(y)$ must be a unit. □

3.2 Splitting the Exact Sequence

Our goal is to algebraically construct the initial extension of L over which the p-divisible group $L \otimes \mathbb{G}_{E_n}$ splits as the sum of the connected part and a constant etale part. This is similar to work of Katz-Mazur in Section 8.7 of [7]. Although we often suppress the notation, all groups in this section are considered to be constant group schemes.

Initially we want to find the ring that represents $\hom(\mathbb{Q}_p/\mathbb{Z}_p^{n-t}, \mathbb{G})$. This was done for $t = 0$ in [5] and the construction here is analogous but stated more algebro-geometrically. It turns out to be convenient for working with the coordinate and for reasons of variance to use the duals of groups as well as the groups themselves.

Let $\Lambda_k = (\mathbb{Z}/p^k)^{n-t}$. It is a corollary of Theorem 3.1.1 that

Corollary 3.2.1. Given Λ_k and a set $\beta_1, \ldots, \beta_{n-t}$ of generators of Λ_k^* there is an isomorphism $E_n^0(B\Lambda_k) \cong E_n^0[\![x_1, \ldots, x_{n-t}]\!]/([p^k](x_1), \ldots, [p^k](x_{n-t}))$.

In this case one uses the map to the product $\beta_1 \times \ldots \times \beta_{n-t} : \Lambda_k \longrightarrow S^1 \times \ldots \times S^1$ to obtain the result using the fixed coordinate.

Given a sequence of epimorphisms $\Lambda_1 \xleftarrow{\rho_2} \Lambda_2 \xleftarrow{\rho_3}$, let a coherent set of generators for the dual sequence be, for each i, a set of generators $\{\beta_1^i, \ldots, \beta_{n-t}^i\}$ for Λ_i^* such that $p \cdot \beta_h^{i+1} = \rho_{i+1}^*(\beta_h^i)$. It is clear that a coherent system of generators for the dual sequence exists for any sequence of epimorphisms of the form above.

Proposition 3.2.2. Given a coherent system of generators for the dual sequence of the above sequence of epimorphisms the map $E_n^0(B\rho_k) : E_n^0(B\Lambda_k) \longrightarrow E_n^0(B\Lambda_{k+1})$ is induced by $x_i \mapsto [p](x_i)$.

Proof. This follows immediately from the proof of the previous corollary and the definition of a coherent system of generators. □

Given $\beta_i : \Lambda_k \longrightarrow S^1$ a generator of the dual group and $\beta^k : \mathbb{Z}/p^k \longrightarrow S^1$ as defined earlier, there exists a unique $f_i : \Lambda_k \longrightarrow \mathbb{Z}/p^k$ making the triangle commute. Using $\{\beta_i\}_{i\in\{1,\ldots,n-t\}}$, this provides an isomorphism $E_n^0(B\mathbb{Z}/p^k)^{\otimes n-t} \xrightarrow{\cong} E_n^0(B\Lambda_k)$.

Next consider the functor from L-algebras to sets given by

$$\hom(\Lambda_k^*, \mathbb{G}[p^k]) : R \mapsto \hom_{gp-scheme}(R \otimes \Lambda_k^*, R \otimes \mathbb{G}[p^k])$$

Lemma 3.2.3. There is an isomorphism of functors between $\hom(L \otimes E_n(B\Lambda_k), -)$ and $\hom(\Lambda_k^*, \mathbb{G}[p^k])$ for every choice of generators for the group Λ_k^*.

Proof. Let $\{\beta_1, \ldots, \beta_{n-t}\}$ be generators of Λ_k^*. Recall that these generators determine $L \otimes E_n(B\Lambda_k) \cong L \otimes E_n(B\mathbb{Z}/p^k)^{\otimes n-t} = \mathcal{O}_{\mathbb{G}[p^k]}^{\otimes(n-t)}$.

Let $f : \Lambda_k^* \longrightarrow \mathbb{G}[p^k]$, then $f^* : \mathcal{O}_{\mathbb{G}[p^k]} \longrightarrow \prod_{\Lambda_k^*} L$. The generators $\{\beta_1, \ldots, \beta_{n-t}\}$ induce $n-t$ maps $g_i : \mathcal{O}_{\mathbb{G}[p^k]} \longrightarrow L$ which induces a map $L \otimes E_n(B\Lambda_k) \longrightarrow L$. □

Now we permanently fix a sequence of epimorphisms

$$\Lambda_1 \xleftarrow{\rho_2} \Lambda_2 \xleftarrow{\rho_3} \Lambda_3 \longleftarrow \ldots$$

and a coherent set of generators for the duals, $\{\beta_i^k\}_{i \in 1,\ldots,(n-t)} \in \Lambda_k^*$.

Let $C_t' = \underset{k}{\mathrm{colim}} L \otimes E_n(B\Lambda_k)$ where the colimit is over the maps $L \otimes E_n(B\rho_k)$.

Proposition 3.2.4. Over C_t' there is a canonical map of p-divisible groups $\mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}$.

Proof. We show this at one level of torsion at a time. Because C_t' is a colimit there is a canonical map $L \otimes E_n^0(B\Lambda_k) \longrightarrow C_t'$ inducing $\Lambda_k^* \longrightarrow \mathbb{G}[p^k]$. We must show that these maps are compatible with each other. This follows from our choice of generators. The following square commutes for all k

$$\begin{array}{ccc} \Lambda_{k-1}^* & \xrightarrow{\rho_k^*} & \Lambda_k^* \\ \downarrow & & \downarrow \\ \mathbb{G}[p^{k-1}] & \xrightarrow{i_k} & \mathbb{G}[p^k] \end{array}$$

We can show this easily with the coordinate. Fix two generators β_i^{k-1} and β_i^k. Then for β_i^k the map $\mathcal{O}_{\mathbb{G}[p^k]} \cong C_t'[\![x]\!]/[p^k](x) \longrightarrow C_t'$ maps $x \mapsto x_i \in L \otimes E_n(B\Lambda_k) \hookrightarrow C_t'$. Thus x maps to $[p]x_i$ for $p \cdot \beta_i^k$, but this is the element of Λ_k^* that β_i^{k-1} maps to under ρ_k^*. □

Using the same reasoning it is clear that C_t' represents the functor

$$\hom(\mathbb{Q}_p/\mathbb{Z}_p^{n-t}, \mathbb{G}) : R \mapsto \hom_{p\text{-divisible}}(R \otimes \mathbb{Q}_p/\mathbb{Z}_p^{n-t}, R \otimes \mathbb{G})$$

and the previous proposition describes the map associated to $\mathrm{Id}_{C_t'}$.

Because over C_t' there is a canonical map $\mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}$ there is also a canonical map $\mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}$ using the natural inclusion $\mathbb{G}_0 \longrightarrow \mathbb{G}$.

$\mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t}$ is a p-divisible group of height n with etale quotient the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p^{n-t}$. Over C_t' the map $\mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}$ induces a map $\mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}_{et}$; our next goal is to find the minimal ring extension of C_t' over which this map is an isomorphism. To understand this we must analyze $\mathbb{G}_{et}$ and prove an analogue of Proposition 6.2 in [5].

We move on to analyzing $\mathbb{G}_{et}$ over C_t', that is, we study the canonical map $\mathbb{Q}_p/\mathbb{Z}_p^{n-t} \longrightarrow \mathbb{G}_{et}$ and determine the minimal ring extension of C_t' over which it is an isomorphism. We begin with a fact about $\mathbb{G}_{et}$ and some facts about finite group schemes.

Proposition 3.2.5. Let K be an algebraic closure of the fraction field of L/m_L. Then $K \otimes \mathbb{G}_{et} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t}$.

Proof. We have shown that over L/m_L, $\mathcal{O}_{\mathbb{G}_{et}[p^k]} \cong L/m_L[y]/(j(y))$. As $\mathbb{G}_{et}[p^k]$ is etale $j(y)$ and $j'(y)$, the derivative of $j(y)$, are coprime. This implies that they have no common roots over an algebraically closed field K, which implies that $K \otimes \mathbb{G}_{et}[p^k]$ is constant. Thus as the pullback of p-divisible groups is a p-divisible group we see that $K \otimes \mathbb{G}_{et}$ is constant of height $n-t$ which implies it is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p^{n-t}$. For further information see Demazure [3]. $\square$

Prior to proving our analogue of Prop 6.2 in [5] we need a key lemma.

Lemma 3.2.6. Let $\mathbb{G}$ be a finite free commutative group scheme over a ring R such that $\mathcal{O}_{\mathbb{G}} \cong R[x]/(f(x))$ where $f(x)$ is a monic polynomial such that $x|f(x)$. Then in $\mathcal{O}_{\mathbb{G}\times\mathbb{G}} \cong R[x]/(f(x)) \otimes R[y]/(f(y))$ the two ideals $(x-y)$ and $(x -_{\mathbb{G}} y)$ are equal. That is $x -_{\mathbb{G}} y = (x-y)\cdot u$ where u is a unit.

Proof. Consider the two maps, $\Delta : \mathbb{G} \longrightarrow \mathbb{G}\times\mathbb{G}$ and $i : \ker(-) \longrightarrow \mathbb{G}\times\mathbb{G}$ the inclusion of the kernel of $\mathbb{G}\times\mathbb{G} \xrightarrow{-} \mathbb{G}$. By considering the functor of points it is clear that both are the equalizer of

$$\mathbb{G}\times\mathbb{G} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbb{G}.$$

Thus we have the commutative triangle

$$\begin{array}{ccc} \ker(-_{\mathbb{G}}) & \xrightarrow{\cong} & \mathbb{G} \\ & \searrow \quad \swarrow & \\ & \mathbb{G}\times\mathbb{G} & \end{array}$$

After applying global sections it suffices to find the generators of the kernels of Δ^* and i^*. For a ring S, $\Delta(S) : \mathbb{G}(S) \longrightarrow \mathbb{G}(S)\times\mathbb{G}(S) : a \mapsto (a,a)$ for $a \in \mathbb{G}(s)$ thus $\Delta^* : R[x]/(f(x)) \otimes R[y]/(f(y)) \longrightarrow R[x]/(f(x))$ must send $x \mapsto x$ and $y \mapsto x$, so $(x-y)$ must be in $\ker(D^*)$ and as Δ^* is surjective and the quotient $R[x]/(f(x)) \otimes R[y]/(f(y))/(x-y) \cong R[x]/(f(x))$, $(x-y)$ must be the whole kernel.

To understand i^* we note that $\ker(-)$ is the pullback

$$\begin{array}{ccc} \ker(-) & \longrightarrow & \mathbb{G}\times\mathbb{G} \\ \downarrow & \lrcorner & \downarrow - \\ e & \longrightarrow & \mathbb{G} \end{array}$$

Global sections gives $\mathcal{O}_{\ker(-)} \cong R \otimes_{R[x]/(f(x))} (R[x]/(f(x)) \otimes R[y]/(f(y)))$ where x is sent to $0 \in R$ and $x -_{\mathbb{G}} y$ in $R[x]/(f(x)) \otimes R[y]/(f(y))$. Thus the kernel of i^* is the ideal $(x -_{\mathbb{G}} y)$. $\square$

The following is our analogue of Prop 6.2 in [5]. Given a homomorphism

$$\phi : \Lambda_k^* \longrightarrow R \otimes \mathbb{G}_{et}[p^k],$$

for $\alpha \in \Lambda_k^*$ let $\phi(\alpha)$ be the image of $y \in R[y]/j_k(y)$ in the R corresponding to the factor of α in $\prod_{\Lambda_k^*} R$.

Proposition 3.2.7. Let R be an L-algebra. The following conditions on a homomorphism

$$\phi : \Lambda_k^* \longrightarrow R \otimes \mathbb{G}_{et}[p^k]$$

are equivalent:

i. For all $\alpha \neq 0 \in \Lambda_k^*$, $\phi(\alpha)$ is a unit.

ii. The Hopf algebra homomorphism

$$R[y]/(j(y)) \cong R \otimes_L \mathcal{O}_{G_{et}[p^k]} \longrightarrow R^{\Lambda_k^*}$$

is an isomorphism.

Proof. The proof of this proposition follows the proofs of Proposition 6.2 and Lemma 6.3 in [5]. With respect to the bases consisting of the powers of x and the obvious basis of the product ring corresponding to the elements of Λ_k^*, the matrix of the Hopf algebra map is the Vandermonde matrix of the set $\phi(\Lambda_k^*)$.

Assuming i. we must show that the determinant, Δ of the Vandermonde matrix is a unit. As in [5], for elements x, y of a ring S, we will write $x \sim y$ if x and y are associates, that is, if $x = uy$ for u a unit. As the matrix is Vandermonde, $\Delta \sim \prod_{\alpha_i \neq \alpha_j \in \Lambda_k^*} (\phi(\alpha_i) - \phi(\alpha_j))$.

Using Prop 3.2.6 we have

$$\begin{aligned}\prod(\phi(\alpha_i) - \phi(\alpha_j)) &\sim \prod(\phi(\alpha_i) -_{\mathbb{G}_{et}} \phi(\alpha_j)) \\ &= \prod(\phi(\alpha_i - \alpha_j)) \\ &= \prod_{\alpha_i - \alpha_j = \alpha} \prod_{\alpha \neq 0} \phi(\alpha) \\ &= \prod_{\alpha \neq 0} \phi(\alpha)^{|\Lambda_k^*|}\end{aligned}$$

In a ring a product of elements is a unit if and only if each of the elements is a unit. Thus the formulas above imply the reverse implication, ii. implies i.. □

As an aside, in [5] it is also shown that p must be inverted for ϕ to be an isomorphism. This is not the case in our situation. The analagous statement is that u_h must be inverted, and it was already inverted in order to form $\mathbb{G}_{et}$.

Prop 3.2.7 seems to imply that, in order to make the canonical map $\mathbb{Q}_p/\mathbb{Z}_p^{n-t} \xrightarrow{\phi} \mathbb{G}_{et}$ an isomorphism, we must invert $\phi(\alpha)$ for all $\alpha \in \mathbb{Q}_p/\mathbb{Z}_p^{n-t}$. This is essentially what we do.

Proposition 3.2.8. The functor from L-algebras to sets given by

$$\mathrm{Iso}(\mathbb{G}_0[p^k] \oplus \Lambda_k^*, \mathbb{G}[p^k]) : R \mapsto \mathrm{Iso}(R \otimes \mathbb{G}_0[p^k] \oplus \Lambda_k^*, R \otimes \mathbb{G}[p^k])$$

is representable by a nonzero ring C_t^k with the property that the map $L/m_L \xrightarrow{i} C_t^k/(m_L \cdot C_t^k)$ is faithfully flat.

Proof. Let S_k be the multiplicative subset of $L \otimes E_n^0(B\Lambda_k)$ generated by $\phi(\Lambda_k^*)$ for the canonical map $\phi : \Lambda_k^* \longrightarrow (L \otimes E_n^0(B\Lambda_k)) \otimes \mathbb{G}_{et}[p^k]$. Let $C_t^k = S_k^{-1}(L \otimes E_n^0(B\Lambda_k))$. For an L-algebra R, a map from C_t^k to R is a map $\Lambda_k^* \xrightarrow{\phi} R \otimes \mathbb{G}_{et}[p^k]$ such that $\phi(\alpha)$ is a unit in R for all $\alpha \neq 0 \in \Lambda_k^*$, by Prop 3.2.7 above this means precisely that ϕ is an isomorphism. Then

$$\mathrm{Hom}(L \otimes E_n^0(B\Lambda_k), R) \cong \mathrm{Hom}(\Lambda_k^*, R \otimes \mathbb{G}[p^k])$$

and

$$\mathrm{Hom}(C_t^k, R) \cong \mathrm{Iso}_{\mathbb{G}_0[p^k]/}(R \otimes \mathbb{G}_0[p^k] \oplus \Lambda_k^*, R \otimes \mathbb{G}[p^k]),$$

the isomorphisms under $\mathbb{G}_0[p^k]$. The last isomorphism is due to the 5-lemma applied to (see [12] for embedding categories of group schemes in abelian categories)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R \otimes \mathbb{G}_0[p^k] & \longrightarrow & R \otimes \mathbb{G}_0[p^k] \oplus \Lambda_k^* & \longrightarrow & \Lambda_k^* & \longrightarrow & 0\,. \\
 & & \downarrow = & & \downarrow & & \downarrow \cong & & \\
0 & \longrightarrow & R \otimes \mathbb{G}_0[p^k] & \longrightarrow & R \otimes \mathbb{G}[p^k] & \longrightarrow & R \otimes \mathbb{G}_{et}[p^k] & \longrightarrow & 0
\end{array}$$

Thus over C_t^k there is a canonical isomorphism $\mathbb{G}_0[p^k] \oplus \Lambda_k^* \longrightarrow \mathbb{G}[p^k]$.

It is vital that we show that C_t^k is nonzero. We will do this by showing that $L/m_L \xrightarrow{i} C_t^k/m_L$ is faithfully flat and thus an injection. The map i is flat because $(L \otimes E_n^0(B\Lambda_k))/m_L$ is a finite module over L/m_L and localization is flat. To prove that it is faithfully flat we use the same argument found in [5]. Consider a prime $\mathcal{P} \subset L/m_L$. Let $L/m_L \xrightarrow{\theta} K$ be a map to an algebraically closed field with kernel exactly $\mathcal{P}$. This can be achieved by taking the algebraic closure of the fraction field of the integral domain $(L/m_L)/\mathcal{P}$.

We have shown in Prop 3.2.5 that $\mathbb{G}_{et}[p^k](K) \cong \Lambda_k^*$, fixing an isomorphism provides a map $C_t^k/m_L \xrightarrow{\Psi} K$

that extends θ. We have

$$\begin{array}{ccc} C_t^k/m_L & \xrightarrow{\Psi} & K \\ \uparrow & \nearrow_{\theta} & \\ L/m_L & & \end{array}$$

and $\ker(\Psi)$ is a prime ideal of C_t^k that restricts to (or is a lift of) $\mathcal{P}$. The map i is a flat map that is surjective on Spec. This implies that it is faithfully flat. $\square$

The localization in the above proposition can be applied to both sides of $L \otimes E_n(B\rho_k)$ and the map is well-defined. Thus over the colimit $C_t = \operatorname*{colim}_k C_t^k$, using the same reasoning as with C_t', there is a canonical isomorphism $C_t \otimes \mathbb{G} \cong C_t \otimes (\mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t})$.

It follows that there is a canonical map

$$i_k : E_n^0(B\Lambda_k) \longrightarrow L \otimes E_n^0(B\Lambda_k) \longrightarrow C_t.$$

Corollary 3.2.9. The ring C_t is the initial ring extension of L over which $\mathbb{G}$ splits as a sum $\mathbb{G}_0 \oplus \mathbb{Q}_p/\mathbb{Z}_p$.

Proof. This follows from Lemma 3.2.3. Corresponding to a map $R \otimes \Lambda_k^* \xrightarrow{f} R \otimes \mathbb{G}[p^k]$ there is a map $L \otimes E_n^0(B\Lambda_k) \longrightarrow R$ and we have that the following diagram commutes

$$\begin{array}{ccc} R \otimes (L \otimes E_n^0(B\Lambda_k)) \otimes \Lambda_k^* & \longrightarrow & R \otimes (L \otimes E_n^0(B\Lambda_k)) \otimes \mathbb{G}[p^k] \\ \downarrow \cong & & \downarrow \cong \\ R \otimes \Lambda_k^* & \xrightarrow{f} & R \otimes \mathbb{G}[p^k] \end{array}.$$

The top arrow is $R \otimes -$ the map corresponding to $\mathrm{Id}_{L \otimes E_n^0(B\Lambda_k)}$ in 3.2.3. The result follows. $\square$

Chapter 4

Transchromatic Generalized Character Maps

We move on to defining the character map and we show that it induces an isomorphism over C_t. The point of all of the preceding discussion and the construction of C_t is that we are going to use C_t to construct a map of equivariant cohomology theories for every finite group G

$$\Phi_G : E_n^*(EG \times_G X) \longrightarrow C_t^*(EG \times_G \mathrm{Fix}(X)).$$

The domain of Φ_G is Borel equivariant E_n and the codomain is Borel equivariant C_t applied to $\mathrm{Fix}(X)$. It is constructed in such a way that if $G \cong \mathbb{Z}/p^k$ the map of theories on a point is the global sections of the map on p^k-torsion $C_t \otimes (\mathbb{G}_0[p^k] \oplus (\mathbb{Z}/p^k)^{n-t}) \longrightarrow \mathbb{G}[p^k]$.

The map Φ_G can be split into two parts, a topological part and an algebraic part. We will begin by describing the topological part. It is topological because it is induced by a map of topological spaces. After some preliminary discussion on the Borel construction and transport categories we will describe the map of topological spaces.

4.1 The Topological Part

Let G be a finite group and X a left G-space. Associated to X as a topological space is a category, X, that has objects the points of X and only the identity morphisms (we remember the topology on the set of objects). Including the action of G we arrive at the transport category, TX, of X, that is the category that has objects the points of X and a morphism $g : x_1 \longrightarrow x_2$ when $gx_1 = x_2$. This process associates to a group action on a topological space a category object in topological spaces.

Let EG be the category with objects the elements of G and a unique isomorphism between any two objects representing left multiplication in G. The realization of the nerve of this groupoid is a model for the classical space EG, a contractible space with a free G-action.

There are both left and right G actions on the category EG. Let $g_1 \xrightarrow{k} g_2$ be a morphism in EG, that is $kg_1 = g_2$. Then for $g \in G$, the action is given by $g \cdot (g_1 \xrightarrow{k} g_2) = gg_1 \xrightarrow{gkg^{-1}} gg_2$ and $(g_1 \xrightarrow{k} g_2) \cdot g =$

$(g_1 g \xrightarrow{k} g_2 g)$. When viewing G as the category with objects the elements of G and only identity morphisms, the multiplication for G makes G a monoidal category and the two actions above are left and right actions of the monoidal category G on the category EG.

Proposition 4.1.1. As categories, $EG \times_G X \cong TX$ where the left G-action on the objects of X is the G-action on the points of X. The realization of either of these categories is a model for the classical Borel construction.

Proof. We view $EG \times_G X$ as a quotient of the product category (in fact a coequalizer). We have

$$(g_1, x) \xrightarrow{(k, id_x)} (g_2, x) = (e, g_1 x) \xrightarrow{(k, id_x)} (e, g_2 x) \mapsto (g_1 x \xrightarrow{k} g_2 x) \in \mathrm{Mor}(TX)$$

which is clearly an isomorphism. □

The category EG is monoidal as well with multiplication $m : EG \times EG \longrightarrow EG$ using the group multiplication for objects and sending unique morphisms to unique morphisms. Explicitly:

$$m : (g_1, h_1) \xrightarrow{(k,l)} (g_2, h_2) \mapsto g_1 h_1 \xrightarrow{g_2 l g_1^{-1}} g_2 h_2.$$

$EG \times_G X$ has a left action by G induced by the left action of G on EG. This action can be uniquely extended to a left action of EG as a monoidal category. This leads to

Proposition 4.1.2. $EG \times_{EG} (EG \times_G X) \simeq EG \times_G X$

Proof. We may view $EG \times_G X$ as TX. On objects $(g, x) = (e, gx) \mapsto gx$. On morphisms

$$((g_1, x_1) \xrightarrow{(k,h)} (g_2, x_2)) = ((e, g_1 x_1) \xrightarrow{(1, g_2 h g_1^{-1})} (e, g_2 x_2)) \mapsto (g_1 x_1 \xrightarrow{g_2 h g_1^{-1}} g_2 x_2).$$

The equivalence is clear as every morphism $(g_1, x_1) \xrightarrow{(k,h)} (g_2, x_2)$ can be put in a canonical form $(e, g_1 x_1) \xrightarrow{(1, g_2 h g_1^{-1})} (e, g_2 x_2)$. □

Let X be a finite G-space. Let $G_p = \mathrm{Hom}(\mathbb{Z}_p^{n-t}, G)$. Also for each G fix a $k \geq 0$ so that any map $\alpha : \mathbb{Z}_p^{n-t} \to G$ factors through $\Lambda_k = (\mathbb{Z}/p^k)^{n-t}$. Define $\mathrm{Fix}(X) = \coprod_{\alpha \in G_p} X^{\mathrm{im}\,\alpha}$. Note that G_p and $\mathrm{Fix}(X)$ both depend on t.

Lemma 4.1.3. $\mathrm{Fix}(X)$ is a G-space.

Proof. Let $x \in X^{\mathrm{im}(\alpha)}$ then for $g \in G$, $gx \in X^{g\,\mathrm{im}(\alpha) g^{-1}}$. □

Consider the inclusion

$$X^{\operatorname{im}\alpha} \hookrightarrow X.$$

Using α we may define

$$E\Lambda_k \times_{\Lambda_k} X^{\operatorname{im}\alpha} \to EG \times_G X.$$

As the action of Λ_k on $X^{\operatorname{im}\alpha}$ through G is trivial, $E\Lambda_k \times_{\Lambda_k} X^{\operatorname{im}\alpha} \cong B\Lambda_k \times X^{\operatorname{im}\alpha}$. This provides us with a map $\coprod_{\alpha \in G_p} B\Lambda_k \times X^{\operatorname{im}\alpha} \to EG \times_G X$.

Proposition 4.1.4. The map $\coprod B\Lambda_k \times X^{\operatorname{im}\alpha} \to EG \times_G X$ extends to a map $EG \times_G \coprod B\Lambda_k \times X^{\operatorname{im}\alpha} \to EG \times_G X$.

The G-action on $\coprod B\Lambda_k \times X^{\operatorname{im}\alpha}$ comes from the action of G on $\operatorname{Fix} X$ together with the trivial action on $B\Lambda_k$. With this action the G-space $\coprod B\Lambda_k \times X^{\operatorname{im}\alpha}$ is G-homeomorphic to $B\Lambda_k \times \operatorname{Fix} X$.

Proof. We will use the categorical formulation developed above. Applying the functor $EG \times_G (-)$ gives the map

$$EG \times_G \coprod B\Lambda_k \times X^{\operatorname{im}\alpha} \to EG \times_G (EG \times_G X).$$

Now the inclusion $G \hookrightarrow EG$ induces

$$EG \times_G (EG \times_G X) \longrightarrow EG \times_{EG} (EG \times_G X) \simeq EG \times_G X.$$

The composite of the two maps is the required extension. Explicitly:

$$((g_1, e) \overset{(k,a)}{\longrightarrow} (g_2, e), x \in X^{\operatorname{im}\alpha}) \mapsto (g_1 \overset{g_2\alpha(a)g_1^{-1}}{\longrightarrow} g_2\alpha(a), x \in X).$$

□

We can do some explicit computations of this map that will be useful in the sequel. Let $X = *$ and G be a finite abelian group. Then we have that

$$EG \times_G \coprod B\Lambda_k \times X^{\operatorname{im}\alpha} \cong \coprod BG \times B\Lambda_k$$

and $EG \times_G X$ is just BG. For a given α we can compute explicitly the map defined in Prop 4.1.4.

Proposition 4.1.5. For a fixed $\alpha : \Lambda_k \longrightarrow G$, $X = *$, G abelian and $+ : \Lambda_k \times G \longrightarrow G$ the addition in G, the map $t : B\Lambda_k \times BG \longrightarrow BG$ is just $B+$. In other words B of the map that sends $(a, g) \mapsto \alpha(a) + g$.

Proof. The map $t : BG \times B\Lambda_k \simeq EG \times_G B\Lambda_k \longrightarrow EG \times_G BG \longrightarrow EG \times_{EG} BG \simeq BG$ sends on morphisms (all that is important here)

$$\begin{aligned}(e,e) \overset{(g,a)}{\longrightarrow} (e,e) &\mapsto (e,e) \overset{(g,a)}{\longrightarrow} (g,e) \\ &\mapsto ((e,e) \overset{(g,\alpha(a))}{\longrightarrow} (g,e)) \\ &= ((e,e) \overset{g+\alpha(a)}{\longrightarrow} (e,e)) \\ &\mapsto g+\alpha(a).\end{aligned}$$

□

Next we compute the map with $X = G/H$ for H an abelian subgroup of a finite group G. These computations will be used in our discussion of complex oriented descent.

When the notation $\mathrm{Fix}(X)$ may be unclear we will use $\mathrm{Fix}_G(X)$ to clarify that we are using X as a G-space. We begin by analyzing $\mathrm{Fix}(G/H)$ as a G-set.

Proposition 4.1.6. For $H \subseteq G$ abelian, $EG \times_G \mathrm{Fix}_G(G/H) \simeq EH \times_H \mathrm{Fix}_H(*)$.

Proof. Fix an $\alpha : \mathbb{Z}_p^{n-t} \longrightarrow G$. For $(G/H)^{\mathrm{im}\,\alpha}$ to be non empty $\mathrm{im}\,\alpha \subseteq g^{-1}Hg$ for some $g \in G$. Why? Let $a \in \mathrm{im}\,\alpha$ assume that gH is fixed by a, then $agH = gH$ so $g^{-1}ag \in H$. Thus for gH to be fixed by all $a \in \mathrm{im}\,\alpha$, $\mathrm{im}\,\alpha$ must be contained in $g^{-1}Hg$.

We will show the equivalence in the proposition by considering both spaces in terms of their transport categories. Thus $EG \times_G \mathrm{Fix}_G(G/H)$ is the groupoid with objects the elements of $\mathrm{Fix}_G(G/H)$ and morphisms coming from the action of G.

Every object in $\mathrm{Fix}_G(G/H)$ is isomorphic to one of the form eH. Indeed, let $gH \in (G/H)^{\mathrm{im}\,\alpha}$ then $g^{-1}gH = eH \in (G/H)^{g^{-1}\,\mathrm{im}\,\alpha g}$. The only objects of the form eH come from maps α that are contained in H, thus we have one connected component of the groupoid $\mathrm{Fix}_G(G/H)$ for every $\alpha : \mathbb{Z}_p^{n-t} \longrightarrow H$.

Now to determine the groupoid up to equivalence it suffices to work out the automorphism group of $eH \in (G/H)^{\mathrm{im}\,\alpha}$. Clearly the only possibilities for $g \in G$ that fix eH are the $g \in H$. All of these fix eH. For if $g \in H$, $geH \in (G/H)^{g\,\mathrm{im}\,\alpha g^{-1}}$, but since H is abelian this is just $(G/H)^{\mathrm{im}\,\alpha}$. So $\mathrm{Aut}(eH) \cong H$ for any $eH \in \mathrm{Fix}_G(G/H)$.

The equivalence is now clear. We can, for example, send $* \in *^{\mathrm{im}\,\alpha}$ to $eH \in (G/H)^{\mathrm{im}\,\alpha}$ for the same α as $\mathrm{im}\,\alpha \in H$. □

Proposition 4.1.7. For $H \subseteq G$ abelian the following diagram commutes:

$$\begin{array}{ccc} EH \times_H B\Lambda_k \times \mathrm{Fix}_H(*) & \longrightarrow & EH \times_H * \\ \simeq \downarrow & & \downarrow \simeq \\ EG \times_G B\Lambda_k \times \mathrm{Fix}_G(G/H) & \longrightarrow & EG \times_G G/H \end{array}$$

Proof. We will represent a morphism in $EH \times_H B\Lambda_k \times \mathrm{Fix}_H(*)$ as a triple $(h_1 \xrightarrow{h} h_2, z_1 \xrightarrow{z} z_2, *)$. Checking commutativity on morphisms suffices (checking on identity morphisms checks it on objects). Fix an α as above. We have the following diagram morphism-wise:

$$\begin{array}{ccc} ((h_1, e) \xrightarrow{(h,z)} (h_2, e), * \in *^{\mathrm{im}\,\alpha}) & \longrightarrow & (h_1 \xrightarrow{h_2\alpha(z)h_1^{-1}} h_2\alpha(z), *) \\ \downarrow & & \downarrow \\ ((h_1, e) \xrightarrow{(h,z)} (h_2, e), eH \in (G/H)^{\mathrm{im}\,\alpha}) & \longrightarrow & (h_1 \xrightarrow{h_2\alpha(z)h_1^{-1}} h_2\alpha(z), eH \in (G/H)) \end{array}$$

□

The map $B\Lambda_k \times EG \times_G \mathrm{Fix}(X) \simeq EG \times_G \coprod B\Lambda_k \times X^{\mathrm{im}\,\alpha} \to EG \times_G X$ is the map of spaces that is used to define the first part of the character map. Applying E_n we get

$$E_n^*(EG \times_G X) \longrightarrow E_n^*(B\Lambda_k \times EG \times_G \mathrm{Fix}(X)).$$

4.2 The Algebraic Part

The algebraic part of the character map begins with the codomain above. The description of this part of the character map is much simpler. However we must begin with a word on gradings.

Until now we have done everything in the ungraded case. This is somewhat more familiar and it is a bit easier to think about the algebraic geometry in the ungraded situation. This turns out to be acceptable because E_n and $L_{K(t)}E_n$ are even periodic theories. We need two facts to continue.

Proposition 4.2.1. The ring extension $E_n^0 \longrightarrow C_t$ is flat implies the graded ring extension $E_n^* \longrightarrow C_t^*$ is flat.

Proof. Here C_t^* means the graded ring with C_t in even dimensions and the obvious multiplication.

There is a pushout of graded rings

$$\begin{array}{ccc} E_n^0 & \longrightarrow & C_t \\ \downarrow & & \downarrow \\ E_n^* & \longrightarrow & C_t^* \end{array}$$

where E_n^0 and C_t are taken to be trivially graded. As flatness is preserved under pushouts the proposition follows. □

Proposition 4.2.2. $E_n^*(B\Lambda_k)$ is an even periodic ring.

Proof. $E_n^*(B\Lambda_k)$ is a free E_n^*-module [5]. Even more, the function spectrum $E_n^{B\Lambda_k}$ is a free E_n-module as a spectrum. □

This is necessary to know because we will lift the map $E_n^0(B\Lambda_k) \longrightarrow C_t$ to a map of graded rings $E_n^*(B\Lambda_k) \longrightarrow C_t^*$. And now that we have discussed this point we will suppress the $*$ in C_t^* and let context decide if by C_t we mean the periodic graded ring, the classical ring, or the cohomology theory obtained by flat extension from E_n.

We return to the character map. A Kunneth theorem available in this situation gives

$$E_n^*(B\Lambda_k \times EG \times_G \mathrm{Fix}(X)) \cong E_n^*(B\Lambda_k) \otimes E_n^*(EG \times_G \mathrm{Fix}(X))$$

Now we have maps from Section 3.2

$$i_k : E_n^*(B\Lambda_k) \longrightarrow L \otimes E_n^*(B\Lambda_k) \longrightarrow C_t$$

also there is a map of cohomology theories $E_n \longrightarrow C_t$ coming from base extension and using the flatness of C_t over E_n^0. Together these induce

$$E_n^*(B\Lambda_k) \otimes E_n^*(EG \times_G \mathrm{Fix}(X)) \longrightarrow C_t^*(EG \times_G \mathrm{Fix}(X)).$$

Precomposing with the topological part we get the character map:

$$\Phi_G : E_n^*(EG \times_G X) \longrightarrow C_t^*(EG \times_G \mathrm{Fix}(X)).$$

It is a result of Kuhn's in [8] that the codomain is in fact an equivariant cohomology theory. Several things must be proved to verify the original claims.

Recall that Λ_k is defined so that all maps $\mathbb{Z}_p^{n-t} \longrightarrow G$ factor through Λ_k. First we show that this map does not depend on k.

Proposition 4.2.3. The character map does not depend on the choice of k in Λ_k.

Proof. Let $j > k$ and let $s = \rho_{k+1} \circ \ldots \circ \rho_j$ where ρ_i is the fixed epimorphism from Section 3.2. Precomposition with s provides an isomorphism $\hom(\Lambda_k, G) \cong \hom(\Lambda_j, G)$. We can use s to create a homeomorphism

$$EG \times_G \coprod_{\alpha \in \hom(\Lambda_k, G)} X^{\mathrm{im}\,\alpha} \cong EG \times_G \coprod_{\alpha \in \hom(\Lambda_j, G)} X^{\mathrm{im}\,\alpha}$$

that we quite reasonably (although just slightly incorrectly) call the identity map Id. Begin by noting that the following two diagrams commute.

$$\begin{array}{ccc}
 & & B\Lambda_k \times EG \times_G \mathrm{Fix}(X) \\
 & \swarrow & \uparrow {\scriptstyle Bs \times \mathrm{Id}} \\
EG \times_G X & & \\
 & \nwarrow & \\
 & & B\Lambda_j \times EG \times_G \mathrm{Fix}(X)
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_n^*(B\Lambda_k) & & \\
 & \searrow^{i_k} & \\
{\scriptstyle E_n^*(Bs)} \downarrow & & C_t \\
 & \nearrow_{i_j} & \\
E_n^*(B\Lambda_j) & &
\end{array}$$

where the diagonal arrows in the left hand diagram come from the topological part of the character map and the diagonal arrows in the right hand diagram come from the definition of C_t. The right hand diagram commutes by definition.

Putting these diagrams together gives the commutative diagram

$$\begin{array}{ccccc}
 & & E_n^*(B\Lambda_k) \otimes E_n^*(EG \times_G \mathrm{Fix}(X)) & & \\
 & \nearrow & & \searrow & \\
E_n^*(EG \times_G X) & & \downarrow & & C_t^*(EG \times_G \mathrm{Fix}(X)) \\
 & \searrow & & \nearrow & \\
 & & E_n^*(B\Lambda_j) \otimes E_n^*(EG \times_G \mathrm{Fix}(X)) & &
\end{array}$$

that shows the map is independent of k. □

Proposition 4.2.4. For $G \cong \mathbb{Z}/p^k$ and $X = *$ the codomain of the character map is the global sections of $C_t \otimes \mathbb{G}[p^k] \cong C_t \otimes (\mathbb{G}_0[p^k] \oplus \Lambda_k^*)$.

Proof. Let $G \cong \mathbb{Z}/p^k$ and $X = *$, as G is abelian it acts on $\mathrm{Fix}(X)$ component-wise. As $X = *$,

$$\begin{aligned} EG \times_G \mathrm{Fix}(X) &= EG \times_G \coprod *^{\mathrm{im}\,\alpha} \\ &\cong \coprod_{\mathrm{Hom}(\mathbb{Z}_p^{n-t},G)} BG. \end{aligned}$$

Applying cohomology and using $\beta^k \in (\mathbb{Z}/p^k)^* = G^*$ to identify $\mathrm{Hom}(\mathbb{Z}_p^{n-t}, G)$ and Λ_k^* gives

$$\begin{aligned} C_t^0(\coprod_{\mathrm{Hom}(\mathbb{Z}_p^{n-t},G)} BG) &\cong \prod_{\mathrm{Hom}(\mathbb{Z}_p^{n-t},G)} C_t^0(BG) \\ &\cong \prod_{\Lambda_k^*} C_t^0(BG). \end{aligned}$$

Spec of which is precisely $\mathbb{G}_0[p^k] \oplus \Lambda_k^*$. □

The next step is to compute the character map on cyclic p-groups. We begin by giving an explicit description, with the coordinate, of the global sections of the canonical map $C_t \otimes (\mathbb{G}_0[p^k] \oplus \Lambda_k^*) \longrightarrow \mathbb{G}_{E_n}[p^k]$. We describe the map from each summand of the domain separately.

The global sections of the map $C_t \otimes \mathbb{G}_0[p^k] \longrightarrow \mathbb{G}_{E_n}[p^k]$ are clearly given by

$$E_n^0[\![x]\!]/([p^k](x)) \xrightarrow{x \mapsto x} C_t[\![x]\!]/[p^k](x).$$

The global sections of the canonical map $\phi[p^k] : \Lambda_k^* \longrightarrow \mathbb{G}_{E_n}[p^k]$ were essentially described in Section 3.1. For $\beta = c_1 \cdot \beta_1 + \ldots + c_{n-t} \cdot \beta_{n-t} \in \Lambda_k^*$ the map

$$E_n^0[\![x]\!]/([p^k](x)) \longrightarrow C_t$$

factors through $L \otimes E_n^0(B\Lambda_k) \xrightarrow{i_k} C_t$ mapping $x \mapsto [c_1](x_1) +_{\mathbb{G}_{E_n}} \ldots +_{\mathbb{G}_{E_n}} [c_{n-t}](x_{n-t}) = \phi[p^k](\beta)$.

Putting these maps together for all $\beta \in \Lambda_k^*$ gives

$$E_n^0[\![x]\!]/([p^k](x)) \longrightarrow C_t[\![x]\!]/([p^k](x)) \otimes C_t^{\Lambda_k^*} \cong \prod_{\Lambda_k^*} C_t[\![x]\!]/([p^k](x))$$

mapping

$$x \mapsto x +_{\mathbb{G}} (\phi[p^k](l))_{l \in \Lambda_k^*} \mapsto (x + \phi[p^k](l))_{l \in \Lambda_k^*}.$$

Proposition 4.2.5. For $G \cong \mathbb{Z}/p^k$ and $X = *$ the character map is the global sections of $\mathbb{G}_0[p^k] \oplus \Lambda_k^* \longrightarrow$

$\mathbb{G}_{E_n}[p^k]$ described above.

Proof. Choose an $\alpha : \Lambda_k \longrightarrow G$, postcomposing with our fixed generator of $(\mathbb{Z}/p^k)^* = G^*$ we get an element $c_1 \cdot \beta_1 + \ldots + c_{n-t} \cdot \beta_{n-t} \in \Lambda_k^*$. By Prop 4.1.5 the topological part of the character map is induced by B of the addition map $\Lambda_k \times G \xrightarrow{+} G$. Using the coordinate and applying E_n^0 we see that

$$E_n^0[\![x]\!]/([p^k](x)) \longrightarrow E_n^0[\![x_1, \ldots, x_{n-t}]\!]/([p^k](x_1), \ldots, [p^k](x_{n-t})) \otimes E_n^0[\![x]\!]/([p^k](x)).$$

is the map sending

$$x \mapsto [c_1](x_1) +_{\mathbb{G}_{E_n}} \ldots +_{\mathbb{G}_{E_n}} [c_{n-t}](x_{n-t}) +_{\mathbb{G}_{E_n}} x$$

which maps via the algebraic map

$$E_n^0[\![x_1, \ldots, x_{n-t}]\!]/([p^k](x_1), \ldots, [p^k](x_{n-t})) \otimes E_n^0[\![x]\!]/([p^k](x)) \longrightarrow C_t[\![x]\!]/([p^k](x))$$

to $(x +_{\mathbb{G}_{E_n}} \phi[p^k](\alpha))$, where $\phi[p^k]$ is the same as above. Putting these together for all α gives a map

$$E_n^0[\![x]\!]/([p^k](x)) \longrightarrow \prod_{\Lambda_k^*} C_t[\![x]\!]/([p^k](x))$$

which is precisely the one shown to be the global sections prior to the proposition.

□

4.3 The Isomorphism

We continue to prove that the map of cohomology theories defined above

$$\Phi_G : E_n^*(EG \times_G X) \longrightarrow C_t^*(EG \times_G \mathrm{Fix}(X)).$$

is in fact an isomorphism when the domain is tensored up to C_t. We follow the steps outlined in [5] with some added complications.

Given a finite G-CW complex X, let $G \hookrightarrow U(n)$ be a faithful complex representation of G. Let T be a maximal torus in $U(n)$. Then $F = U(n)/T$ is a finite G-space with abelian stabilizers. This means that it has fixed points for every abelian subgroup of G but no fixed points for non-abelian subgroups of G. We first show that the cohomology of X is determined by the cohomology of the spaces $X \times F^{\times h}$ so we can reduce to the case of spaces with abelian stabilizers. This is called complex oriented descent. Using Mayer-Vietoris for the cohomology theories we can then reduce to spaces of the form $G/H \times D^n \simeq G/H$ where H is abelian. Then induction implies that we only need to check the isomorphism on finite abelian groups. This will follow from our previous work.

We begin by proving the descent property for finite G-CW complexes. Thus we assume that the map is an isomorphism for spaces with abelian stabilizers and show that this implies it is an isomorphism for all finite G-spaces.

Proposition 4.3.1. F is a space with abelian stabilizers.

Proof. Let $A \subseteq G$ be an abelian subgroup. Then under the faithful representation above $A \subset uTu^{-1}$ for some $u \in G$. Thus for $a \in A$, $a = utu^{-1}$ for some $t \in T$ and now it is clear that A fixes the coset uT. □

Proposition 4.3.2. As F is a space with abelian stabilizers the realization of the simplicial space where the arrows are just the projections

$$EF = \left| F \leftleftarrows F \times F \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; F \times F \times F \dots \right|$$

is a space such that for $H \subseteq G$

$$EF^H \simeq \begin{cases} \emptyset & \text{if } H \text{ not abelian} \\ * & \text{if } H \text{ is abelian} \end{cases}$$

Proof. Because realization commutes with finite limits we just need to check that for F a non-empty space, EF is contractible. Then it is a basic fact that there is a contracting homotopy. □

Now $EG \times_G X \simeq EG \times_G (X \times EF)$ and exchanging homotopy colimits gives

$$EG \times_G X \simeq \left| EG \times_G (X \times F) \leftleftarrows EG \times_G (X \times F \times F) \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; \dots \right|$$

It is important to know that Fix preserves realizations.

Proposition 4.3.3. Fix preserves realizations. That is, given a simplicial G-space $X_\bullet$, $\mathrm{Fix}(|X_\bullet|) \simeq |\mathrm{Fix}(X_\bullet)|$.

Proof. Recall that for a G-space X, $\mathrm{Fix}(X) = \coprod_{\alpha \in \mathrm{Hom}(\mathbb{Z}_p^{n-t}, G)} X^{\mathrm{im}\,\alpha}$.

Also recall that geometric realization as a functor from simplicial G-spaces to G-spaces is a colimit (in fact a coend), geometric realization commutes with finite limits, and that the following diagram commutes:

$$\begin{array}{ccc} \text{G-Spaces}^{\Delta^{op}} & \xrightarrow{|\ |} & \text{G-Spaces} \\ \downarrow & & \downarrow \\ \text{Spaces}^{\Delta^{op}} & \xrightarrow{|\ |} & \text{Spaces} \end{array}$$

where the vertical arrows are the forgetful functor. Thus it suffices to check that Fix commutes with the realization of simplicial spaces as we already know that it lands in G-spaces.

As colimits commute with colimits we only need to check the fixed points. But for $H \subseteq G$ and a G-space X, $X^H \cong \lim_H X$ and as H is finite so is the limit. □

We will use the Bousfield-Kan Spectral Sequence. For a cosimplicial spectrum $S^\bullet$ it is a spectral sequence

$$E_2^{s,t} = \pi^s \pi_t S^\bullet \Rightarrow \pi_{t-s} \operatorname{Tot} S^\bullet$$

As $\Sigma^\infty_+ : \text{Top} \longrightarrow \text{Spectra}$ is a left adjoint it commutes with colimits and so preserves realizations. We work in a spectral model category of spectra. Let E be a cohomology theory, then $\operatorname{Hom}(|\Sigma^\infty_+ X_\bullet|, E) \cong \operatorname{Tot}\operatorname{Hom}(\Sigma^\infty_+ X_\bullet, E)$. The Bousfield-Kan spectral sequence begins with the homotopy of the cosimplicial spectrum $\operatorname{Hom}(\Sigma^\infty_+ X_\bullet, E)$ and abuts to the homotopy of $\operatorname{Tot}\operatorname{Hom}(\Sigma^\infty_+ X_\bullet, E)$.

This applies to our situation. We want to resolve

$$\begin{aligned} C_t^*(EG \times_G \operatorname{Fix}(X)) &\cong \pi_{-*} \operatorname{Hom}(\Sigma^\infty_+ EG \times_G \operatorname{Fix}(X), C_t) \\ &\cong \pi_{-*} \operatorname{Hom}(\Sigma^\infty_+ EG \times_G \operatorname{Fix}(|X \times F^\bullet|), C_t) \\ &\cong \pi_{-*} \operatorname{Hom}(|\Sigma^\infty_+ EG \times_G \operatorname{Fix}(X \times F^\bullet)|, C_t) \\ &\cong \pi_{-*} \operatorname{Tot}\operatorname{Hom}(\Sigma^\infty_+ EG \times_G \operatorname{Fix}(X \times F^\bullet), C_t). \end{aligned}$$

It follows from Prop 2.4 and 2.6 in [5] that $E_n^*(EG \times_G (X \times F^{\times h}))$ is a free $E_n^*(EG \times_G X)$-module for all h. Now as

$$E_n^*(EG \times_G (X \times F \times F)) \cong E_n^*(EG \times_G (X \times F) \times_{(EG \times_G X)} EG \times_G (X \times F))$$

the cosimplicial graded E_n^*-module

$$E_n^*(EG \times_G X \times F) \rightrightarrows E_n^*(EG \times_G X \times F \times F) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

is in fact the Amitsur complex of the faithfully flat (even free) map $E_n^*(EG \times_G X) \longrightarrow E_n^*(EG \times_G (X \times F))$ induced by the projection. This implies that its homology is concentrated in the zeroeth degree and isomorphic to $E_n^*(EG \times_G X)$. In other words the associated chain complex is exact everywhere but at the first arrow.

This is the E_1 term for the Bousfield-Kan spectral sequence and we have shown that it collapses. Tensoring with C_t retains this exactness as C_t is flat over E_n^*. Using our assumption regarding spaces with abelian stabilizers we now have a map of E_1-terms that is an isomorphism

$$\begin{array}{ccccc} C_t \otimes E_n^*(EG \times_G X \times F) & \rightrightarrows & C_t \otimes E_n^*(EG \times_G X \times F \times F) & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & \cdots \\ \downarrow \cong & & \downarrow \cong & & \\ C_t(EG \times_G \mathrm{Fix}(X \times F)) & \rightrightarrows & C_t(EG \times_G \mathrm{Fix}(X \times F \times F)) & \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} & \cdots . \end{array}$$

As the homology of these complexes is the $E_2 = E_\infty$ page of the spectral sequence and the spectral sequence does converge (Ch. 9, Section 5, [2]) to an associated graded (in this case with one term), this implies that $C_t \otimes E_n^*(EG \times_G X)$ and $C_t^*(EG \times_G \mathrm{Fix}(X))$ are isomorphic. This gives us complex oriented descent.

We are reduced to proving the isomorphism for spaces with abelian stabilizers. Using an equivariant cell decomposition Mayer-Vietoris reduces this to spaces of the form $G/H \times D^n$ where H is abelian and D^n is the n-disk. Now homotopy invariance reduces this to spaces of the form G/H with H abelian.

Proposition 4.3.4. The induction property holds for G/H where $H \subseteq G$ is abelian. That is the following diagram commutes:

$$\begin{array}{ccc} C_t \otimes E_n^*(EG \times_G G/H) & \xrightarrow{C_t \otimes \Phi_G} & C_t(EG \times_G \mathrm{Fix}_G(G/H)) \\ \downarrow \cong & & \downarrow cong \\ C_t \otimes E_n^*(EH \times_H *) & \xrightarrow{C_t \otimes \Phi_H} & C_t(EH \times_H \mathrm{Fix}_H(*)) \end{array}$$

Proof. This follows from Prop 4.1.7 and the independence of the character map on k. □

We are left having to show it is an isomorphism for finite abelian groups, but we can use the Kunneth theorem to reduce to cyclic p-groups and the isomorphism there has already been proved in Prop 4.2.5.

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