

© 2011 by Hehui Wu. All rights reserved.

EXTREMAL PROBLEMS ON CYCLES, PACKING, AND DECOMPOSITION OF
GRAPHS

BY

HEHUI WU

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

Doctoral Committee:

Professor Alexandr V. Kostochka, Chair
Professor Douglas B. West, Director of Research
Associate Professor József Balogh
Associate Professor Chandra S. Chekuri

Abstract

In this thesis, we study extremal problems concerning cycles and paths in graphs, graph packing, and graph decomposition. We use “graph” in the general sense, allowing loops and multi-edges.

The Chvátal–Erdős Theorem states that every graph whose connectivity is at least its independence number has a spanning cycle. In 1976, Fouquet and Jolivet conjectured an extension: If G is an n -vertex k -connected graph with independence number a , and $a \geq k$, then G has a cycle with length at least $\frac{k(n+a-k)}{a}$. In Chapter 2 we prove this conjecture.

Nash-Williams and Tutte independently characterized when a graph has k edge-disjoint spanning trees; a consequence is that $2k$ -edge-connected graphs have k edge-disjoint spanning trees. Kriesell conjectured a more general statement: defining a set $S \subseteq V(G)$ to be j -edge-connected in G if S lies in a single component of any graph obtained by deleting fewer than j edges from G , he conjectured that if S is $2k$ -edge-connected in G , then G has k edge-disjoint trees containing S . In Chapter 3, we show that it suffices for S to be $6.5k$ -edge-connected in G .

A *shortcutting operation* on a graph G replaces a path in G by an edge joining its endpoints. An S -connector of G is a subgraph of G from which after some shortcutting operations we get a connected graph with vertex set S . In Chapter 3, we also show that if S is $10k$ -edge-connected in G , then G has k edge-disjoint S -connectors.

Say that a graph with maximum degree at most d is d -bounded. In chapter 4, we prove a sharp sparseness condition for decomposability into k forests plus one d -bounded graph

when $d > k$. Consequences are that every graph with fractional arboricity at most $k + \frac{d}{k+d+1}$ has such a decomposition. When $d = k + 1$, and also in the case where $k = 1$ and $d \leq 6$, the d -bounded graph in the decomposition can also be required to be a forest. For $d \leq k + 1$, we prove that every graph with fractional arboricity at most $k + \frac{d}{2k+2}$ decomposes into k forests plus one d -bounded forest.

To my beloved wife Lifeng, my father and my mother.

Acknowledgments

I wish to thank my mentor and my friend Professor Douglas Brent West. His care, his patience, and his help has made my study in Illinois a joyful experience. I have benefitted greatly from him in my research in my teaching in my English, etc. Most importantly, from him, I learned to be a happy Mathematician.

I also wish to thank Professor Hong-Jian Lai, my M.S. advisor in West Virginia University. I was very lucky to be able to work with a great advisor, Professor Hong-Jian Lai. Without him, I would not be able to come to United States and start my study in Graph Theory. I wish to thank him for his unlimited support all the way.

I also thank Professors József Balogh, Alexandr Kostochka, and Chandra Chekuri for serving on my thesis committee. I wish to express special thanks to Professor Kostochka for his care and support during my study here. I also want to thank Professor Füredi for the wonderful courses he offered.

Many thanks also go to all my friends I met at the University of Illinois and West Virginia University. They make my life here full of fun. I especially want to thank Gexin Yu and Kun Mo for their friendship.

I would like to thank my family for their steady support and love. Especially, I thank my beloved wife Lifeng for her unfailing love. She has always been with me and provides great happiness and encouragement in times of difficulties.

Table of Contents

Chapter 1	Introduction	1
1.1	Extremal problem on cycles	2
1.2	Steiner Tree packing and local connectivity	4
1.3	Graph decomposition	7
1.4	Basic terminology and notation	11
Chapter 2	Extremal Problem for Longest Cycles	15
2.1	The Path Lemma	16
2.2	Finding a Good Cycle	19
Chapter 3	Extremal Problems for Packing of Graphs	24
3.1	(k, g) -family and the Strong Partition Condition	26
3.2	S -partitions and submodularity of f_g	32
3.3	Existence of (k, g) -families	38
3.4	Steiner tree packing	43
3.5	S -connector packing	54
Chapter 4	Extremal Problems for Decomposition of Graphs	59
4.1	(k, d) -decomposition for $d > k$	61
4.2	$(k, d)^*$ -decomposition for $d \leq k + 1$	66
4.3	Approach to $(k, d)^*$ -decomposition	73
4.4	Discharging argument and submodularity	80
4.5	Neighbors of 3-vertices when $k = 1$	82
4.6	The Strong NDT Conjecture for $(k, d) = (1, 2)$	85
References		88
Vita		91

Chapter 1

Introduction

The Hamiltonian cycle problem, determining whether a graph contains a spanning cycle, is a central problem in graph theory. A more general problem is finding the length of the longest cycle in a graph. In Section 1.1, we give some results on the Hamiltonian cycle problem and state our main results about cycles and paths in a graph with given connectivity and independence number.

Given a family \mathcal{F} of graphs, an \mathcal{F} -*packing* of a graph G is a set of edge-disjoint subgraphs of G such that each of them is in \mathcal{F} . Starting from the theorem of Nash-Williams and Tutte on tree packing, Section 1.2 will generalize to Steiner tree packing and S -connector packing. Also, our result on S -connector packing is related to Mader's Splitting Lemma on preserving local edge-connectivity.

Dual to packing, an \mathcal{F} -*decomposition* of a graph G consists of edge-disjoint subgraphs such that each of them is in \mathcal{F} and their union is G . Many problems in graph theory can be viewed as graph decomposition problems. Starting from Nash-William's Arboricity Theorem on forest decomposition, Section 1.3 will strengthen it to decomposition of graphs into forests plus one graph with bounded degree.

In Section 1.4, we give definitions of basic terminology we use in this thesis.

1.1 Extremal problem on cycles

A *Hamiltonian cycle* in a graph is a cycle covering all vertices of the graph. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. Named after Sir William Rowan Hamilton, Hamiltonian cycle problems date from the 1850s. As of now, there are more than ten survey papers, dozens of open problems, and a flood of papers on this topic.

Testing whether a graph is Hamiltonian is a fundamental problem in computer science. This problem is NP-complete. In 1952, Dirac [13] proved for $n \geq 3$ that any n -vertex graph with minimum degree at least $n/2$ is Hamiltonian. In 1960, Ore [36] strengthened Dirac's result to the following: if $d(u) + d(v) \geq n$ for any two nonadjacent vertices u and v in G , then G is Hamiltonian. Another well-known theorem, published by Chvátal and Erdős [12] in 1972, gives a sufficient condition in terms of the connectivity $\kappa(G)$ and independence number $\alpha(G)$ of the graph G .

Theorem 1.1.1. (Chvátal–Erdős [12]) *If G is a graph such that $\kappa(G) \geq \alpha(G)$, then G has a cycle through all its vertices.*

When a sufficient condition for a graph being Hamiltonian fails slightly, we may still expect that the graph has a long cycle. The long-cycle version of Dirac's Theorem states that every 2-connected graph has a cycle with length at least $\min\{n, 2\delta(G)\}$, where $\delta(G)$ denotes the minimum vertex degree of G . The long-cycle version of Ore's Theorem was published by Bondy [3] in 1971; it states that if $d(u) + d(v) \geq m$ whenever u and v are distinct nonadjacent in G , then G has a cycle with length at least $\min\{n, m\}$.

It is natural to seek a long-cycle version of the Chvátal–Erdős Theorem. That is, can we give a lower bound on the circumference of a graph in terms of its independence number and connectivity? In 1976, Fouquet and Jolivet conjectured an answer.

Conjecture 1.1.2. (Fouquet–Jolivet [15]) *If G is a k -connected n -vertex graph with independence number a , and $a \geq k$, then G has a cycle with length at least $\frac{k(n+a-k)}{a}$.*

The case $k = a$ simplifies to the Chvátal–Erdős Theorem. The conjecture is in fact sharp: infinitely often the circumference of G equals $k(n + a - k)/a$ as shown in Example 2.0.1.

The following result of Kouider [34] has been used in partial results toward Conjecture 1.1.2.

Theorem 1.1.3. (Kouider [34]) *If H is a subgraph of a k -connected graph G , then either $V(H)$ can be covered by a cycle in G , or there is a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - k$.*

A single application of Theorem 1.1.3 with $H = G$ implies the Chvátal and Erdős Theorem (Theorem 1.1.1) when $\kappa(G) \geq \alpha(G)$; a spanning cycle is guaranteed. When $\kappa(G) < \alpha(G)$, repeatedly applying Theorem 1.1.3 with H being the subgraph left by deleting the vertices of earlier cycles shows that the vertices of a graph G can be covered by at most $\lceil \frac{\alpha(G)}{\kappa(G)} \rceil$ cycles.

Inspired by Kouider’s result and her proof, we prove an analogous theorem about paths joining two specified vertices.

Theorem 1.1.4. *Let G be a k -connected graph. If $H \subseteq G$, and u and v are distinct vertices in G , then G contains a u, v -path P such that $V(H) \subseteq V(P)$ or $\alpha(H - V(P)) \leq \alpha(H) - (k - 1)$.*

To prove the Fouquet–Jolivet Conjecture, we will only need the case $k = 2$ of Theorem 1.1.4.

After we announced our proof for the Fouquet–Jolivet Conjecture, Fujita et al. proved an analogous result about paths joining two specified vertices.

Theorem 1.1.5 (Fujita–Halperin–Magnant [19]). *If G is a k -connected n -vertex graph with independence number a , and u and v are distinct vertices in G , then G has a u, v -path with length at least $\min\{\frac{(k-1)(n-k)}{a} + k, n\}$.*

Recently, Chen et al. [11] strengthened our lower bound for circumference to $k\lfloor \frac{n+2a-2k}{a} \rfloor$. Inspired by their result, we improve the bound on circumference, which is sharp when the vertex number, independence number and connectivity are given.

Theorem 1.1.6. *Let G be a k -connected n -vertex graph with independence number a . If m and d are the integers such that $n = k + ma + d$ and $0 \leq d \leq a - 1$, then either G is Hamiltonian or G has a cycle with length at least $k + mk + \min\{d, k\}$.*

The result is sharp: as show in Example 2.0.1, for any integers n, a, k with $n \geq a + k$ and n , we can construct a k -connected n -vertex graph with independence number a having the given circumference.

The results of Chapter 2 are joint work with Suil O and Douglas B. West and appear in [35].

1.2 Steiner Tree packing and local connectivity

Given a family \mathcal{F} of graphs, an \mathcal{F} -packing of a graph G is a set of edge-disjoint subgraphs of G such that each of them is in \mathcal{F} . For example, we may let \mathcal{F} be the family of trees having the same number of vertices as G . In 1961, Nash-Williams [32] and Tutte [38] independently obtained a necessary and sufficient condition for a graph to have k edge-disjoint spanning trees.

Theorem 1.2.1 (Tree Packing Theorem; Nash-Williams [32], Tutte [38]). *A graph G contains k edge-disjoint spanning trees if and only if $\sum_{A_i \in P} \delta(A_i) \geq 2k(|P| - 1)$ for every partition P of $V(G)$.*

An easy consequence is that every $2k$ -edge-connected graph has k edge-disjoint spanning trees. Given a specified subset S of the vertices, a tree T contained in G such that $S \subseteq V(T)$

is an S -Steiner-tree or simply S -tree in G . Kriesell [25] conjectured a generalization of the Tree Packing Theorem that seeks edge-disjoint S -trees.

Given a graph G , a vertex set S is *connected in G* if S lies in a single component of G . A set S is k -edge-connected in G if S remains connected in every graph obtained by deleting fewer than k edges from G . The *local edge-connectivity* $\kappa'_G(x, y)$ of a pair $\{x, y\}$ of vertices in G is the maximum number k such that $\{x, y\}$ is k -edge-connected in G .

Conjecture 1.2.2 (Kriesell's Conjecture [25]). *If S is $2k$ -edge-connected in G , then G contains k edge-disjoint S -trees.*

Finding the most such trees for given S is the *Steiner-Tree Packing Problem*. Lap Chi Lau [27] gave a partial result toward Kriesell's Conjecture, showing that S being $24k$ -edge-connected in G suffices for the existence of k edge-disjoint S -trees. In Chapter 3, we improve Lau's result.

Theorem 1.2.3. *If S is $6.5k$ -edge-connected in G , then G contains k edge-disjoint S -trees.*

To prove this result, we use a stronger concept called S -connector. In a graph G , let S be a set of distinguished vertices called *terminals*. An S -path is a path in G with both ends in S and no internal vertices in S . *Short-cutting* a u, v -path means replacing its edges with one edge uv . An S -connector in G is the union of a family of edge-disjoint S -paths such that short-cutting them yields a connected graph with vertex set S . We prove the following result:

Theorem 1.2.4. *If S is $10k$ -edge-connected in G , then G contains k edge-disjoint S -connectors.*

We also pose the following conjecture:

Conjecture 1.2.5. *If S is $3k$ -edge-connected in G , then G contains k edge-disjoint S -connectors.*

Furthermore, in Chapter 3, we define a concept called (k, g) -family, which is the union of edge-disjoint subgraphs in which k of them are S -connectors and the others are paths, with $g(v)$ such paths starting from each vertex v and ending in S . Our main result in Chapter 3 gives a necessary and sufficient condition for existence of a (k, g) -family. This result is a generalization of the Tree Packing Theorem, and the necessary and sufficient condition has a similar form to the one in the Tree Packing Theorem.

In our S -tree Packing Problem and S -connector Packing Problem, the hypothesis is a local edge-connectivity condition. How to preserve the local edge-connectivity of the graph after shortcutting is the key issue. In this topic, Mader's Splitting Lemma plays an important role.

Let uv and vw be two edges of G . The uv, vw -shortcut of G is the graph obtained from G by replacing uv and vw with an edge joining u and w , and we call it a *shortcut of G at v* . Shortcutting a path can be accomplished by shortcutting at all its internal vertices one by one. We call a graph a *shortcut of G* if it can be obtained from G by a succession of shortcutting of paths.

Theorem 1.2.6 (Mader's Splitting Lemma [28]). *Let x be a non-cut-vertex of G . If x has degree at least 2 (except when $d_G(x) = 3$ and x has three distinct neighbors), then there is a shortcut G' of G at x such that $\kappa'_G(u, v) = \kappa'_{G'}(u, v)$ whenever $u, v \in V(G) - \{x\}$.*

Mader's Splitting Lemma guarantees that we can preserve the local edge-connectivity between other vertices by shortcutting at any non-cut-vertex with degree at least 4. For any even vertex x , after iteratively applying Mader's Splitting Lemma on it until there is no edge incident to it, we get a new graph H on vertex set $V(G) - x$ such that the local edge-connectivity for any pair of vertices in H is the same as it is in G . This can be generalized to the following:

Theorem 1.2.7. *For any vertex set S of G , if every vertex in \overline{S} has even degree, then there*

is a shortcut H of G such that for any pair of vertex (x, y) in S ,

$$\kappa'_{H[S]}(x, y) = \kappa'_G(x, y).$$

The above theorem is not true when there are some odd vertices in \bar{S} . It is natural to ask what ratio of the local edge-connectivity we can preserve. Theorem 1.2.4 on S -connector Packing implies that if S is $10k$ -edge-connected graph, then there is a shortcut H of G such that S is k -edge-connected in $H[S]$.

More generally, we have the following conjecture:

Conjecture 1.2.8. *There exists some positive constant c such that, for any vertex set S of any graph G , there exists a shortcut H of G such that*

$$\kappa'_{H[S]}(x, y) \geq \lfloor c\kappa'_G(x, y) \rfloor, \forall x, y \in S,$$

The results in Chapter 3 are joint work with Douglas B. West and appear in [40].

1.3 Graph decomposition

A *decomposition* of a graph G consists of edge-disjoint subgraphs with union G . The *arboricity* of G , written $\Upsilon(G)$, is the minimum number of forests needed to decompose it. The famous Nash-Williams Arboricity Theorem states that a necessary and sufficient condition for $\Upsilon(G) \leq k$ is that no subgraph H has more than $k(|V(H)| - 1)$ edges. This is a sparseness condition. A slightly different sparseness condition places a bound on the average vertex degree in all subgraphs. The *maximum average degree* of a graph G , denoted $\text{Mad}(G)$, is $\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$; it is the maximum over subgraphs H of the average vertex degree in H . (Our model of “graph” allows multi-edges but no loops.)

Many papers have obtained various types of decompositions from bounds on $\text{Mad}(G)$.

Our results extend some of these and the Nash-Williams Theorem, which states that $\Upsilon(G) = \left\lceil \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} \right\rceil$. We consider the *fractional arboricity* $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$, introduced by Payan [37]; for this we use the notation $\text{Arb}(G)$, by analogy with $\text{Mad}(G)$.

Three forests are needed to decompose a graph with fractional arboricity $2 + \epsilon$, but since this is just slightly above 2 one may hope that some restrictions can be placed on the third forest. Say that a graph is *d-bounded* if it has maximum degree at most d . Montassier et al. [31] posed the *Nine Dragon Tree (NDT) Conjecture* (honoring a famous tree in Kaohsiung, Taiwan that is far from acyclic):

Conjecture 1.3.1 (NDT Conjecture). *If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into k forests plus one d -bounded forest.*

They proved the cases $(k, d) = (1, 1)$ and $(k, d) = (1, 2)$.

They also posed a weaker version of the Nine Dragon Tree Conjecture, which does not require the d -bounded subgraph to be a forest:

Conjecture 1.3.2 (Weak NDT Conjecture). *If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into k forests plus one d -bounded subgraph.*

They showed that no larger value of $\text{Arb}(G)$ is sufficient even for the weak NDT Conjecture.

Our main purpose in Chapter 4 is proving some partial results of the NDT Conjecture and the weak NDT Conjecture. Before showing our results, we will give some more background about decomposition of sparse graphs.

Another line of research considers decomposing a planar graph into a forest plus one d -bounded graph, following the seminal paper [22], which motivated the topic by its application to “game coloring number”. For a planar graph with girth g to decompose into a forest plus one matching, $g \geq 8$ suffices [31, 39] (earlier, sufficiency was proved for $g \geq 11$ in [22], for $g \geq 10$ in [2], and for $g \geq 9$ in [7]). Also, the graph left by deleting the edges of a forest can

be guaranteed to be 2-bounded when $g \geq 7$ [22] (improved to $g \geq 6$ in [24]) and 4-bounded when $g \geq 5$ [22]. Borodin, Ivanova, and Stechkin [4] disproved the conjecture from [22] that every planar graph G decomposes into a forest plus one $(\lceil \Delta(G)/2 \rceil + 1)$ -bounded graph. In [5], there are sufficient conditions for a planar graph with triangles to decompose into a forest plus one matching, and [6] shows that a planar graph without 4-cycles (3-cycles are allowed) decomposes into a forest plus one 5-bounded graph.

Many conclusions on planar graphs with large girth hold more generally when only the corresponding bound on $\text{Mad}(G)$ is assumed. If G is a planar graph with girth g , then G has at most $\frac{g}{g-2}(n-2)$ edges, by Euler's Formula. This holds for all subgraphs, so girth g implies $\text{Mad}(G) < \frac{2g}{g-2}$. Montassier et al. [30] posed the question of finding the weakest bound on $\text{Mad}(G)$ to guarantee decomposition into one forest plus one d -bounded graph. They proved that $\text{Mad}(G) < 4 - \frac{8d+12}{d^2+6d+6}$ is sufficient and that $\text{Mad}(G) = 4 - \frac{4}{d+2}$ is not (seen by subdividing every edge of a $(2d+2)$ -regular graph). The case $k = 1$ of our Theorem 1.3.3 completely solves this problem, implying that $\text{Mad}(G) < 4 - \frac{4}{d+2}$ suffices.

Our result also implies the previous girth results for decomposition of planar graphs into one forest plus one d -bounded graph. Girth 8, 6, and 5 imply that $\text{Mad}(G)$ is less than $8/3$, 3, and $10/3$, respectively, which are precisely the bounds that by our result guarantee decomposition into one forest plus one graph with maximum degree at most 1, 2, or 4, respectively.

Other work brought these problems closer together, requiring the leftover d -bounded graph to be a forest or considering the leftover after deleting more than one forest. For convenience, let a (k, d) -decomposition of a graph G be a decomposition of G into k forests plus one d -bounded graph, and let a $(k, d)^*$ -decomposition be a (k, d) -decomposition in which the "leftover" d -bounded graph also is a forest. Graphs having such decompositions are (k, d) -decomposable or $(k, d)^*$ -decomposable, respectively.

Examples of planar graphs with girth 7 having no $(1, 1)$ -decomposition and examples

with girth 5 having no $(1, 2)$ -decomposition appear in [31, 24]. Gonçalves [20] proved the conjecture of Balogh et al. [1] that every planar graph is $(2, 4)^*$ -decomposable. He also proved that planar graphs with girth at least 6 are $(1, 4)^*$ -decomposable and with girth at least 7 are $(1, 2)^*$ -decomposable.

The NDT Conjecture State that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ guarantees a $(k, d)^*$ -decomposition. The fractional arboricity of a planar graph can be arbitrarily close to 3, which is not small enough for the NDT Conjecture to guarantee $(2, d)^*$ -decomposability for any constant d . However, requiring girth at least 6 or 7 yields fractional arboricity less than $6/4$ or $7/5$, respectively, in which case the NDT Conjecture would guarantee $(1, 4)^*$ - or $(1, 2)^*$ -decompositions, respectively. Hence the NDT Conjecture implies the results of Gonçalves for $(1, d)^*$ -decomposition of planar graphs with large girth, but not his result on $(2, 4)^*$ -decomposition.

Let $\|A\|$ be the number of edges with both endpoints in A . That is $\|A\| = |E(G[A])|$. Instead of using the bound on $\text{Arb}(G)$ or $\text{Mad}(G)$ to describe the sparseness of the graph, we introduce the intermediate condition we call (k, d) -sparse: $(k + 1)(k + d) |A| - (k + d + 1) \|A\| - k^2 \geq 0$ for all $A \subseteq V(G)$. We obtain the following theorem, which holds whenever $d > k$:

Theorem 1.3.3. *For $d > k$, every (k, d) -sparse graph is (k, d) -decomposable. Furthermore, the condition is sharp.*

Since $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ implies that G is (k, d) -sparse, our Theorem 1.3.3 proves the weak NDT Conjecture for the case $d > k$. Further motivation for introducing the (k, d) -sparseness condition comes from the sharpness example in Section 3.0.7.

Meanwhile, Theorem 1.3.3 says nothing about the case $d \leq k$. In Chapter 4, we prove a result implying that a stronger condition on $\text{Arb}(G)$ than in the NDT Conjecture suffices to guarantee the stronger property of $(k, d)^*$ -decomposability when $d \leq k + 1$.

Theorem 1.3.4. *For $d \leq k + 1$, if $\text{Arb}(G) \leq k + \frac{d}{2k+2}$, then G is $(k, d)^*$ -decomposable.*

When $d = k + 1$, this bound equals $k + \frac{d}{k+d+1}$, so this theorem implies the case $d = k + 1$ of the NDT Conjecture.

Also, we prove the NDT Conjecture for $(k, d) = (1, d)$ with $d \leq 6$ by using discharging argument.

Meanwhile, Montassier et al. [31] also pose a stronger version of the NDT Conjecture.

Conjecture 1.3.5 (Strong NDT Conjecture). *If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G has a $(k, d)^*$ -decomposition in which every component of the d -bounded forest has at most d edges.*

We prove this for $(k, d) = (1, 2)$ as in the following statement:

Theorem 1.3.6. *If $\text{Mad}(G) < 3$, then G decomposes into one forest plus one graph in which each component has at most 2 edges.*

The results in Chapter 4 are joint work with Kim, Kostochka, West, and Zhu and appear in [23].

1.4 Basic terminology and notation

In this section we review basic terminology and standard elementary results used throughout this thesis. For other notions on graph theory not listed here, please refer to the introductory textbook on graph theory by Douglas B. West [41].

A graph G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and an *incidence relation* between $V(G)$ and $E(G)$. Elements of $V(G)$ and $E(G)$ are *vertices* and *edges*, respectively. Each edge is *incident* to two vertices (not necessary distinct). The vertices incident to an edge are the *endpoints* of the edge. We write xy for an edge with endpoints x and y , and we say x and y are *adjacent* to each other or are *neighbors* of each other. Given a

vertex set X , if an edge e has at least one endpoint in X , then we also say that e and X are incident. The *neighborhood* of a vertex v is the set of all neighbors of v , denoted by $N(v)$. The *closed neighborhood*, denoted by $N[x]$, is $N(x) \cup \{x\}$.

Parallel edges or a *multi-edge* are two or more edges incident to the same two vertices. A *loop* is an edge whose endpoints are identical. In this thesis, we use “graph” to mean the general model, which allows loops and multi-edges, except in Chapter 4, where we allow multi-edges but not loops.

The *degree* of v is the number of edges incident to it, denoted by $d(v)$, or $d_G(v)$ when we need to specify the graph G . When a vertex has even degree, we call it an *even vertex*; otherwise, we call it an *odd vertex*.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning subgraph* of G is a subgraph H such that $V(H) = V(G)$. Given $S \subseteq V(G)$, let $\bar{S} = V(G) - S$. The *induced subgraph* of G induced by S is the subgraph obtained by deleting the vertices of \bar{S} ; this may be written as $G[S]$ or $G - \bar{S}$. When $\bar{S} = \{v\}$, we write $G - v$ instead of $G - \{v\}$. We also write $G - e$ for the (non-induced) subgraph obtained by deleting an edge e . A *proper subgraph* of G is a subgraph of G not equal to G . Two graphs G and H are *isomorphic* if there is a bijection f from $V(G)$ to $V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H .

A *path* with n vertices is a graph whose vertices can be named v_1, \dots, v_n so that the edges are $\{v_i v_{i+1} : 1 \leq i \leq n - 1\}$. In terms of the vertices, we use $\langle v_1, \dots, v_n \rangle$ to denote the path with these edges and we say the *length of the path* is $n - 1$. Without vertex names, P_n denotes the isomorphism class of paths with n vertices; we think of P_n as a single “unlabeled” graph. A path with endpoints x and y is an *x, y -path*.

A *cycle* consists of a path plus an edge consisting of its endpoints. That is, the vertices can be named v_1, \dots, v_n so that the edges are $\{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n v_1\}$. In terms of the vertices, we use $[v_1, \dots, v_n]$ to denote the cycle with these edges, and we say the *length*

of the cycle is n . Without vertex names, C_n denotes the isomorphism class of cycles with n vertices; again, we think of C_n as a single “unlabeled” graph.

A graph G is *connected* if for each pair $x, y \in V(G)$, there is an x, y -path in G . A graph G is *k -connected* if it has more than k vertices and every subgraph obtained by deleting fewer than k vertices is connected; the *connectivity* of G , written $\kappa(G)$, is the maximum k such that G is k -connected. Similarly, a graph G is *k -edge-connected* if every subgraph obtained by deleting fewer than k edges is connected; the *edge-connectivity* of G , written $\kappa'(G)$, is the maximum k such that G is k -connected.

An *independent set* is a set of pairwise nonadjacent vertices, and the *independence number* of G , written $\alpha(G)$, is the maximum size of such a set. The *circumference* is the maximum length of a cycle in G . A *component* of G is a maximal connected subgraph.

Given a function f and a set A in its domain, let $f(A) = \sum_{a \in A} f(a)$.

A *hereditary system* M is a pair (E, \mathcal{I}) , where E is a finite set (called the ground set) and \mathcal{I} is a collection of subsets of E (called the *independent sets*) with the following two properties:

1. \emptyset is an independent set.
2. Every subset of an independent set is an independent set.

A hereditary system is a *matroid* if it also satisfy the following property:

3. If A and B are two elements of \mathcal{I} and $|A| \geq |B|$, then there exists an element x in A that is not in B such that adding x to B still gives an independent set.

The *rank function* r_M of a hereditary system M is the function on 2^E defined by $r(X) = \max\{|Y|: Y \subseteq X, Y \in \mathcal{I}\}$.

A *partial order* is a binary relation “ \leq ” over a set P that is reflexive, antisymmetric, and transitive. That is, for all a, b , and c in P , we have:

1. $a \leq a$ (reflexivity);
2. If $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry);

3. If $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A set equipped with a partial order is a *partially ordered set*

If $x \leq y$ in a poset \mathcal{P} , then x is a *lower bound* for y and y is an *upper bound* for x . If some common upper bound z for x and y satisfies $z \leq w$ for every common upper bound w , then z is the *least upper bound* or *join* of x and y , written $x \vee y$. Similarly, the *meet* $x \wedge y$, if it exists, is the *greatest lower bound* of x and y . A *lattice* is a poset in which meets and joins exist for all pairs of elements; a finite lattice has a unique maximal element and a unique minimal element. The *rank* of an element in a poset is one less than the size of a largest chain on which it is the top element.

Chapter 2

Extremal Problem for Longest Cycles

In this chapter, we present our result on longest cycles in graphs, we prove the following theorem:

Theorem 1.1.6. *If G is a k -connected n -vertex graph with independence number a , and m, d are the integers such that $n = k + ma + d$ and $0 \leq d \leq a - 1$, then either G is Hamiltonian or G has a cycle with length at least $k + mk + \min\{d, k\}$.*

$$\text{Since } \frac{k(n+a-k)}{a} = \frac{k(a+ma+d)}{a} = k + mk + \frac{kd}{a} \leq k + mk + \frac{kd}{\max\{d,k\}} = k + mk + \min\{d, k\},$$

Theorem 1.1.6 implies the Fouquet-Jolivet Conjecture:

Theorem 1.1.2. *If G is a k -connected n -vertex graph with independence number a , and $a \geq k$, then G has a cycle with length at least $\frac{k(n+a-k)}{a}$.*

Theorem 1.1.6 is sharp, as shown by the following example:

Example 2.0.1. *Given any n, k, a such that $a > k$ and $n \geq k + a$, let m and d be the integers such that $n = k + ma + d$ and $0 \leq d \leq a - 1$. Construct a graph G as follows: form G from $a + 1$ complete graphs, of which one has k vertices, d have $m + 1$ vertices, and $a - d$ have m vertices, by making every vertex in the copy of K_k adjacent to all the other vertices. Now G has n vertices, $\alpha(G) = a$, $\kappa(G) = k$, and the maximum cycle length is $k + mk + \min\{d, k\}$.*

In 1982, Fournier [16] proved Conjecture 1.1.2 for $a \in \{k + 1, k + 2\}$. Two years later, he also proved it for $k = 2$ [17], using the fact that if C_1 and C_2 are distinct cycles in a 2-connected graph G , then there are distinct cycles C'_1 and C'_2 in G such that $V(C_1) \cup V(C_2) \subseteq$

$V(C'_1) \cup V(C'_2)$ and $|V(C'_1) \cap V(C'_2)| \geq 2$. In 2009, Manoussakis [29] proved the case $k = 3$ using a similar fact. This leads to a general conjecture.

Conjecture 2.0.2. (Chen–Chen–Liu) *If C_1 and C_2 are distinct cycles in a k -connected graph G , then there are distinct cycles C'_1 and C'_2 in G such that $V(C_1) \cup V(C_2) \subseteq V(C'_1) \cup V(C'_2)$ and $|V(C'_1) \cap V(C'_2)| \geq k$.*

Recently, Chen, Hu, and Wu [9] proved Conjecture 1.1.2 for $k = 4$. In another paper [10], they proved that Conjecture 2.0.2 implies Conjecture 1.1.2, and they also proved Conjecture 1.1.2 for $a < 2k - 1$.

In Section 2.1, we will prove the Path Lemma (Theorem 1.1.4) which is analogous to Kouider’s result(Theorem 1.1.3).

In Section 2.2, we prove Conjecture 1.1.2 in full by proving Theorem 1.1.6. To get our main result, instead of proving the stronger Conjecture 2.0.2, we prove two theorems on cycles, Theorems 2.2.1 and 2.2.2.

2.1 The Path Lemma

Recall that Kouider gave the following result about cycles:

Theorem 1.1.3. *If H is a subgraph of a k -connected graph G , then either $V(H)$ can be covered by a cycle in G or there is a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - k$.*

In this section, we prove our theorem that is analogous to Kouider’s result.

Theorem 1.1.4. *If H is a subgraph of a k -connected graph G , and u and v are distinct vertices in G , then G contains a u, v -path P such that $V(H) \subseteq V(P)$ or $\alpha(H - V(P)) \leq \alpha(H) - (k - 1)$.*

Our proof of Theorem 1.1.4 is obtained by slightly modifying Kouider’s proof of Theorem 1.1.3. First, we define notation for subpaths of a path. Let u and v be distinct

vertices in a graph G . A u, v -path is a path with first vertex u and last vertex v . Given a path P and vertices $a, b \in V(P)$, let $P[a, b]$ be the a, b -path contained in P . Similarly, let $P(a, b) = P[a, b] - \{a, b\}$, let $P[a, b) = P[a, b] - b$, and let $P(a, b] = P[a, b] - a$.

Proof. Suppose that no u, v -path P contains $V(H)$. For each u, v -path P , let F_P be a smallest component of $G - V(P)$ that intersects H . Choose a u, v -path P such that:

- (i) $\alpha(H - V(P))$ is smallest;
- (ii) subject to (i), F_P has the fewest vertices.

Let p_1, \dots, p_m be the vertices of P (in order) having neighbors in $V(F_P)$. Since G is k -connected, $m \geq k$. For $1 \leq i < m$, let Q_i be a p_i, p_{i+1} -path whose internal vertices lie in F_P , and let $U_i = V(P(p_i, p_{i+1}))$; note that $U_i \subset V(P)$.

Claim 1: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for $1 \leq i < m$. Let P' be the u, v -path obtained from P by deleting U_i and adding Q_i . If $\alpha(H - V(P - U_i)) = \alpha(H - V(P))$, then $V(F_P) \cap V(H) \subseteq V(P')$ would yield $\alpha(H - V(P')) < \alpha(H - V(P))$, because F_P is a component of $G - V(P)$ that intersects H . The resulting inequality violates (i). We may therefore assume that P' does not cover $V(F_P) \cap V(H)$. Since $V(P - U_i) \subseteq V(P')$, we have $\alpha(H - V(P')) \leq \alpha(H - V(P - U_i))$. By hypothesis, the latter value equals $\alpha(H - V(P))$. Since there remains a vertex of $F_P \cap H$ outside P' , we have $|V(F_{P'})| < |V(F_P)|$, which contradicts (ii). This proves the claim.

By Claim 1, restoring U_i to the induced subgraph $H - V(P)$ increases the independence number. Restoring the vertices of U_i in order, starting from p_i , let q_i be the first vertex at which the independence number increases. That is, with $U'_i = V(P(p_i, q_i))$, we have $\alpha(H - V(P - U'_i)) = \alpha(H - V(P)) + 1$, but $\alpha(H - V(P - U'_i) - q_i) = \alpha(H - V(P))$.

Claim 2: For $1 \leq i < j < m$, no path with internal vertices outside P joins U'_i and U'_j . Otherwise, let $r_i \in U'_i$ and $r_j \in U'_j$ be the endpoints of such a path \hat{P} , with r_i and r_j chosen closest to p_i and p_j along P , respectively. Since F_P is a component of $G - V(P)$, and no

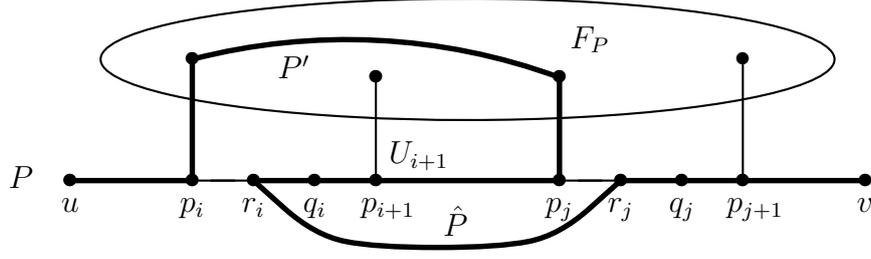


Figure 2.1: Finding a better path (Claim 2)

vertices of U'_i or U'_j have neighbors in F_P , the path \hat{P} does not visit F_P . Let P' be a path obtained from P by deleting $V(P(p_i, r_i))$ and $V(P(p_j, r_j))$ and adding \hat{P} and a p_i, p_j -path through F_P .

Since $r_i \in U'_i$ and $r_j \in U'_j$, restoring the vertices in $P(p_i, r_i)$ or $P(p_j, r_j)$ to $H - V(P)$ does not produce a larger independent set than exists in $H - V(P)$. Furthermore, the choice of r_i and r_j implies that these sets lie in different components of the subgraph obtained from G by deleting all of $V(P)$ except these sets. Hence both sets can be restored without increasing the independence number.

We conclude that $\alpha(H - V(P')) \leq \alpha(H - V(P))$. If $V(F_P) \cap V(H) \subseteq V(P')$, then strict inequality holds, violating (i). Hence $V(F_P) \cap V(H) \not\subseteq V(P')$ and equality holds; now choosing P' instead of P violates (ii). Hence P' must not exist, which completes the proof of the claim.

By the choice of q_i , we have $\alpha(H - V(P - U'_i)) \geq \alpha(H - V(P)) + 1$. Let $G' = G - V(P - \bigcup_{i=1}^{m-1} U'_i)$. By Claim 2, the sets U'_1, \dots, U'_{m-1} lie in different components of G' . Hence $\alpha(H - V(P - \bigcup_{i=1}^{m-1} U'_i)) \geq \alpha(H - V(P)) + m - 1$. Since $\alpha(H) \geq \alpha(H - V(P - \bigcup_{i=1}^{m-1} U'_i))$ and $m \geq k$, we have $\alpha(H - V(P)) \leq \alpha(H) - k + 1$ for the chosen path P . \square

Theorem 1.1.4 implies a conjecture stated in Chen, Hu, and Wu [9].

Corollary 2.1.1. *Given a graph G , if G admits no vertex partition (V_1, V_2) such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, then G is 2-connected or $G \in \{K_1, K_2\}$. Also, for distinct vertices*

$u, v \in V(G)$, there is a u, v -path P such that $\alpha(G - V(P)) < \alpha(G)$.

Proof. If G is disconnected, then such a partition exists. Suppose that G is connected and has a cut-vertex x . Let A be a component of $G - x$, and let $B = G - x - V(A)$. Let $A' = G - V(B)$ and $B' = G - V(A)$. If $\alpha(A) = \alpha(A')$, then

$$\alpha(G) \leq \alpha(A') + \alpha(B) = \alpha(A) + \alpha(B) \leq \alpha(G).$$

Equality holds throughout, and $(V(A'), V(B))$ is the required partition.

The remaining alternative is $\alpha(A) = \alpha(A') - 1$. Now there is an independent set S of size $\alpha(A)$ that contains no neighbor of x . We compute

$$\alpha(G) \leq \alpha(A) + \alpha(B') = |S| + \alpha(B') \leq \alpha(G),$$

and $(V(A), V(B'))$ is the required partition.

We conclude that G is 2-connected when G has at least three vertices and no such partition exists. Now Theorem 1.1.4 applies with $k = 2$ and $H = G$. \square

The sufficient condition given is not a necessary condition, as shown by the union of two complete graphs sharing one vertex. Examples where the conclusion fails include graphs consisting of two disjoint complete graphs plus one edge joining them.

2.2 Finding a Good Cycle

Given disjoint subgraphs F and H of a graph G , let an F, H -path in G be a path with endpoints in $V(F)$ and $V(H)$ and no internal vertex in $V(F) \cup V(H)$; this generalizes “ u, v -path”. Given a specified orientation of a cycle C and vertices $a, b \in V(C)$, let $C[a, b]$ be the a, b -path on C in the given orientation. Similarly, let $C(a, b) = C[a, b] - \{a, b\}$. A *block* in a

graph is a maximal subgraph having no cut-vertex; a graph is the union of its blocks.

Theorem 2.2.1. *Let k be an integer greater than 1. If C is a cycle with length at least k in a k -connected graph G , then for any nonempty subgraph H of $G - V(C)$, there exists a cycle C' in G such that $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.*

Proof. Consider a minimal counterexample H for some graph G and cycle C . Let $L = |V(C)|$. If H is disconnected or has a cut-vertex, then $\alpha(H) = \alpha(H[V_1]) + \alpha(H[V_2])$ for some partition (V_1, V_2) of $V(H)$, by Corollary 2.1.1. By the minimality of H , there is a cycle C' in $H[V_1]$ such that $|V(C) - V(C')| \leq (L/k) - 1$ and $\alpha(H[V_1 - V(C')]) \leq \alpha(H[V_1]) - 1$. Now $\alpha(H - V(C')) \leq \alpha(H[V_1 - V(C')]) + \alpha(H[V_2]) \leq \alpha(H[V_1]) - 1 + \alpha(H[V_2]) = \alpha(H) - 1$.

We may therefore assume that H is 2-connected or $H \in \{K_1, K_2\}$. Let B be the block of $G - V(C)$ that contains H . For B, C -paths P_1 and P_2 , define the C -distance between P_1 and P_2 to be the distance in C between the endpoints of P_1 and P_2 in C .

For $b \in V(B)$, a standard consequence of Menger's Theorem yields k paths from b to C that pairwise share only b ; call this a b, C -fan. By the pigeonhole principle, the C -distance between some two paths in a b, C -fan is at most L/k . If b is the only vertex of B (and hence $H = B$), then using those two paths to replace the part of C between their endpoints yields the desired cycle C' . Hence we may assume $|V(B)| > 1$.

Let P_1 and P_2 be two disjoint B, C -paths, with P_i having endpoints $u_i \in B$ and $v_i \in C$. Since B is connected and has no cut-vertex, Theorem 1.1.4 guarantees a u_1, u_2 -path P in B such that $\alpha(H - V(P)) \leq \alpha(H) - 1$. If $|C(v_1, v_2)| \leq L/k - 1$, then $(C - C(v_1, v_2)) \cup P_1 \cup P \cup P_2$ is the desired cycle C' (see Figure 2.2). Hence we may assume (*) *the C -distance between any two disjoint B, C -paths is more than L/k* . Note also that B, C -paths with distinct endpoints in B are internally disjoint, since B is a block in $G - V(C)$.

Let c_1, \dots, c_m be the endpoints in C of B, C -paths, indexed so that c_1, \dots, c_m appear in that order along a fixed orientation of C . Let $P_i = C[c_i, c_{i+1}]$ (indices modulo m); call P_i the

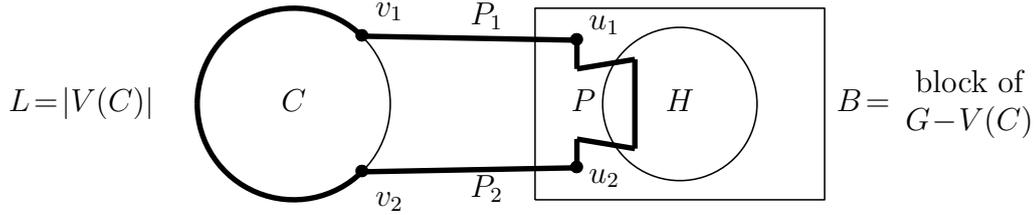


Figure 2.2: A detour to reduce $\alpha(H)$

i th segment of C . Let t be the number of indices i (modulo m) such that c_i and c_{i+1} are the endpoints of B, C -paths from distinct vertices of B . By (*), each such segment has length more than L/k , and hence $t < k$.

For $b \in V(B)$, a b, C -fan has k endpoints in C . Some $k - t$ of the paths along C joining consecutive endpoints of the fan must not contain endpoints of B, C -paths from other vertices of B . Hence these paths are distinct for distinct vertices of B . Consider a segment within each such path.

Since these segments avoid the t excluded segments, their total length is less than $L - t(L/k)$, which equals $L(k - t)/k$. For each vertex of B , choose a shortest among these $k - t$ segments. The total length of the union of the chosen segments is less than L/k .

Form C' from C by deleting the chosen segments and adding, for each $b \in B$, the two paths in the b, C -fan whose endpoints are the ends of the segment chosen for b (see Figure 2.3). The subgraph C' is a cycle, because B, C -paths from distinct vertices of B are internally disjoint. Since the total length of the chosen segments is less than L/k and $V(H) \subseteq V(B) \subseteq V(C')$, the cycle C' has the desired properties. \square

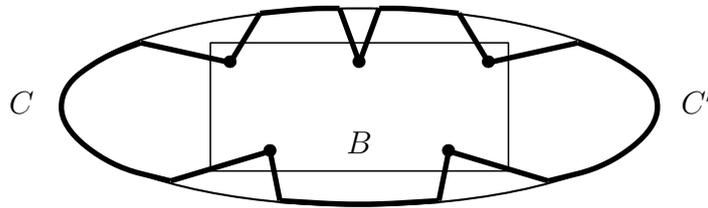


Figure 2.3: Skipping the chosen segments

Lemma 2.2.2. *If G is a k -connected graph with independence number a , and $0 \leq l \leq a - k$, then there exist cycles C_0, \dots, C_l satisfying the following conditions:*

- (1) $\alpha(G - \bigcup_{i=0}^l V(C_i)) \leq a - k - l$,
- (2) $\left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

Proof. We prove the claim by induction on l . For $l = 0$, Theorem 1.1.3 with $H = G$ provides a cycle C_0 such that $\alpha(G - V(C_0)) \leq a - k$. For the induction step, consider l with $0 < l \leq a - k$, and suppose that cycles C_0, \dots, C_{l-1} exist satisfying the claim for $l - 1$. We observe first that $|V(C_0)| \geq k$; when $l = 1$ this holds because the case $l = 0$ of (1) states that $\alpha(G - V(C_0)) \leq a - k$, and when $l > 1$ it holds because the left side of (2) is nonnegative.

Let $H = G - \bigcup_{i=0}^{l-1} V(C_i)$; by hypothesis, $\alpha(H) \leq a - k - (l - 1)$. We may assume $\alpha(H) \geq 1$; otherwise, just let $C_l = C_0$. Since $|V(C_0)| \geq k$, we can apply Theorem 2.2.1 using C_0 as C to obtain a cycle C' in G such that $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1 \leq a - k - l$. Now adding C' to the list as C_l satisfies (1), but we must also satisfy (2).

Case 1: $|V(C')| \leq |V(C_0)|$. Note that

$$\left| V(C') - \bigcup_{j=0}^{l-1} V(C_j) \right| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1.$$

In this case it suffices to add C' as C_l .

Case 2: $|V(C')| > |V(C_0)|$. Define a new list C'_0, \dots, C'_l of cycles by letting $C'_0 = C'$ and letting $C'_i = C_{i-1}$ for $1 \leq i \leq l$. Now $\alpha(G - \bigcup_{i=0}^l V(C'_i)) = \alpha(H - V(C')) \leq a - k - l$, satisfying (1). Also, for $i = 1$ we have $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) = V(C'_1) - V(C'_0) = V(C_0) - V(C')$, and for $2 \leq i \leq l$ we have $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$. In both cases,

$$\left| V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \right| \leq \frac{|V(C_0)|}{k} - 1 \leq \frac{|V(C'_0)|}{k} - 1.$$

Hence C'_0, \dots, C'_l satisfies the required conditions. \square

We can now prove the Theorem 1.1.6, which implies the conjecture of Fouquet and Jolivet (Conjecture 1.1.2).

Theorem 1.1.6. *If G is a k -connected n -vertex graph with independence number a , and m, d are the integers such that $n = k + ma + d$ and $0 \leq d \leq a - 1$, then either G is Hamiltonian or G has a cycle with length at least $k + mk + \min\{d, k\}$.*

Proof. Consider $l = a - k$ in Lemma 2.2.2. By (1), the resulting cycles C_0, \dots, C_l cover $V(G)$. Using this and then summing the inequalities in (2), we obtain

$$n = |V(C_0)| + \sum_{i=1}^l \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \leq |V(C_0)| + (a - k) \lfloor \frac{|V(C_0)|}{k} - 1 \rfloor.$$

Suppose $|V(C_0)| < k + mk + \min\{d, k\}$, then $\lfloor \frac{|V(C_0)|}{k} - 1 \rfloor \leq m$. Hence $n \leq |V(C_0)| + (a - k)m < k + mk + d + (a - k)m = k + ma + d = n$. Contradictions!

So we have $|V(C_0)| \geq k + mk + \min\{d, k\}$.

\square

Chapter 3

Extremal Problems for Packing of Graphs

In This chapter, given a set S of terminals in a graph G , we consider packing of edge-disjoint paths, S -trees, and S -connectors into G when G is highly edge-connected.

Recall that Kriesell gave the following conjecture:

Conjecture 1.2.2 (Kriesell's Conjecture [25]). *If S is $2k$ -edge-connected in G , then G contains k edge-disjoint S -trees.*

In this chapter, always $|S| \geq 2$.

Known partial results toward Kriesell's Conjecture include the following.

Theorem 3.0.3 (Kriesell [25]). *If S is $2k$ -edge-connected in G , and every vertex outside S has even degree, then G contains k edge-disjoint S -trees.*

Theorem 3.0.4 (Frank–Király–Kriesell [18]). *If S is $3k$ -edge-connected in G , and $G - S$ has no edges, then G contains k edge-disjoint S -trees.*

Theorem 3.0.5 (Lau [27]). *If S is $24k$ -edge-connected in G , then G has k edge-disjoint S -trees.*

We obtain the following improvements.

Theorem 1.2.3. *Given a vertex set S of a graph G , if S is $6.5k$ -edge-connected in G , then G contains k edge-disjoint S -trees.*

Theorem 1.2.4. *Given a vertex set S of a graph G , if S is $10k$ -edge-connected in G , then G contains k edge-disjoint S -connectors.*

An S -tree need not be an S -connector. For example, when $|S| \geq 3$, a star whose leaf set is S is an S -tree but not an S -connector. Thus stricter conditions may be needed to guarantee S -connectors. We pose an analogue for S -connectors of Kriesell's Conjecture.

Conjecture 3.0.6. *If S is $3k$ -edge-connected in G , then G contains k edge-disjoint S -connectors.*

We will show that Conjecture 3.0.6 holds when $G - S$ has no edges; this strengthens Theorem 3.0.4. For each of these conjectures, infinitely many examples prove sharpness. Sharpness examples for Kriesell's Conjecture are well known. Let G be the graph obtained from $K_{2k,2k}$ by deleting a perfect matching. With $S = V(G)$, the set S is $(2k - 1)$ -edge-connected in G , since $\kappa'(G) = 2k - 1$. However, G does not have k edge-disjoint S -trees, since k spanning trees would need $k(4k - 1)$ edges, while $|E(G)| = (2k)^2 - 2k$. Sharpness for Conjecture 3.0.6 takes a bit more work.

Example 3.0.7. To show that Conjecture 3.0.6 is sharp, we construct an infinite family of graphs G with specified sets S such that S is $(3k - 1)$ -edge-connected in G but G does not contain k edge-disjoint S -connectors. For $b \in \mathbb{N}$, let S be a set of size $3b$. For $1 \leq i < k$, let G_i be a 3-connected 3-regular bipartite graph with partite sets S and T_i . Form the graph G_k by subdividing every edge in a 2-connected 3-regular graph with vertex set T_k of size $2b$, using S as the set of $3b$ vertices of degree 2 added to subdivide the edges.

The graphs G_1, \dots, G_k all contain the vertex set S ; let $G = \bigcup_{i=1}^k G_i$. Note that G is bipartite with partite sets S and T , where $T = \bigcup_{i=1}^k T_i$. Every vertex of T has degree 3 in G ; vertices of S have degree $3k - 1$. Any two vertices of S are joined by three internally disjoint paths in G_1, \dots, G_{k-1} and two in G_k , so S is $(3k - 1)$ -edge-connected in G .

Finding k edge-disjoint S -connectors in G would require $k(|S| - 1)$ edge-disjoint paths passing through vertices of T . Each vertex of T has degree 3 and hence lies in at most one such path. Hence there are at most $|T|$ such paths. We compute $|T| = (k - 1)3b + 2b = (3k - 1)b$.

Comparing $(3k - 1)b$ and $k(3b - 1)$, we find that not enough paths exist when $b > k$.

In contrast, there is an S -tree in each G_i , so G does have k -edge-disjoint S -trees. \square

In Section 3.1, we define the notion of (k, g) -family: this is the union of edge-disjoint subgraphs of k are S -connectors, and the others are paths ending in S , with $g(v)$ of them starting from each vertex v . Theorem 3.1.2 gives a necessary and sufficient condition, called the Strong Partition Condition, for the existence of a (k, g) -family. In Section 3.2 and Section 3.3, we prove Theorem 3.1.2. We prove Theorem 1.2.3 in Section 3.4 and we prove Theorem 1.2.4 in Section 3.5.

3.1 (k, g) -family and the Strong Partition Condition

To obtain our results, we will prove a theorem that generalizes the Tree Packing Theorem of Nash-Williams and Tutte. Stating it requires some terminology and notation.

Definition 3.1.1. For $S \subseteq V(G)$, write \bar{S} for $V(G) - S$. Write $[A, B]$ for the set of edges in G having endpoints in A and B . When A or B has only one vertex v , we write v instead of $\{v\}$ in this notation. Following Lovász, let $\delta(S) = |[S, \bar{S}]|$.

A partition A_1, \dots, A_l of a set containing S in $V(G)$ is an S -partition if each A_i intersects S . For an S -partition P , we generally write $P = \{A_1, \dots, A_l\}$ and let $B_P = V(G) - \bigcup_{i=1}^l A_i$. Also let T_P be the set of vertices in S that are in blocks of P containing only one vertex of S . We write $|P|$ for the number of blocks in an S -partition P , since P is a set of blocks. Let $\mathcal{P}(S)$ be the set of all S -partitions of G .

Let \mathbb{N}_0 be the set of nonnegative integers. Given a graph G , an S -parity function is a function $g: V(G) \rightarrow \mathbb{N}_0$ such that $g(v) \equiv d_G(v) \pmod{2}$ for all $v \in \bar{S}$ (there is no restriction on $g(v)$ for $v \in S$). For any vertex set A and function h , let $h(A) = \sum_{v \in A} h(v)$.

In a graph G with terminal set S and S -parity function g , a g -family is a set of $g(V(G))$ positive-length paths that can be oriented (from beginning to end) to satisfy the following

two properties: (1) each path ends in S , and (2) for each $v \in V(G)$, there are $g(v)$ paths in the family starting at v . A (k, g) -family is a set of $k + g(V(G))$ edge-disjoint subgraphs such that k are S -connectors and the others form a g -family. \square

Our main result gives a necessary and sufficient condition for existence of a (k, g) -family.

Theorem 3.1.2. *Let S be a set of terminals in G . If g is an S -parity function on G , then G has a (k, g) -family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$, where f_g is defined by*

$$f_g(P) = \left(\sum_{A_i \in P} \delta(A_i) \right) - 2k(|P| - 1) - g(B_P) - 2g(T_P). \quad (3.1)$$

We call the condition that $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$ the *Strong Partition Condition (SPC)*. The notion of S -parity function enables us to generalize the problem of packing S -connectors in a way (existence of (k, g) -families) that permits a characterization of existence and facilitates the proof of our results about packing of S -trees and S -connectors. The statement of Theorem 3.1.2 is the reason why we *restrict to $|S| \geq 2$ throughout the paper*. If $|S| = 1$, then every S -partition has one block, so we can make k arbitrarily large without affecting the SPC. However, when $S = \{v\}$ there is only one subgraph that is an S -connector, namely the one subgraph consisting of the vertex v and no edges. We also use the condition $|S| \geq 2$ in Proposition 3.1.4.

Proposition 3.1.3. *The SPC is a necessary condition for existence of a (k, g) -family.*

Proof. Consider a (k, g) -family \mathcal{F} in G . For an S -partition P , let $t = \sum_{A_i \in P} \delta(A_i)$. Each S -connector in \mathcal{F} contributes at least $2k(|P| - 1)$ to t . For each vertex v in B_P , the paths starting from v reach S and hence contribute at least $g(v)$ to t . Finally, for $v \in T_P$, the oriented paths starting from v contribute at least $2g(v)$ to t , since they end in some other block of P . Thus $t \geq 2k(|P| - 1) + g(B_P) + 2g(T_P)$, so $f_g(P) \geq 0$. \square

The content of Theorem 3.1.2 is the converse: the Strong Partition Condition suffices for the existence of a (k, g) -family. We show next that the SPC implies a property that is obviously necessary for the existence of a (k, g) -family; hence we will be able to assume this property when we are proving Theorem 3.1.2. (The stronger inequality $d(v) \geq k + g(v)$ that we obtain in the case $v \in S$ is also necessary for a (k, g) -family.)

Proposition 3.1.4. *If the SPC holds for an S -parity function g on a graph G , then $g(v) \leq d(v)$ for all $v \in V(G)$, where $d(v)$ denotes the degree of v in G .*

Proof. For $v \notin S$, let P be the single-block S -partition $\{V(G) - \{v\}\}$. With $|S| \geq 2$, we have $d(v) - 0 - g(v) - 0 = f_g(P) \geq 0$, so $g(v) \leq d(v)$. For $v \in S$, let $P = \{\{v\}, V(G) - \{v\}\}$ (using $|S| \geq 2$). Now $2d(v) - 2k - 0 - 2g(v) \geq f_g(P) \geq 0$, so $d(v) \geq k + g(v)$. \square

A natural S -parity function yields a notable application of Theorem 3.1.2. Given a vertex set $A \subseteq V(G)$, let $n_o(A)$ be the number of vertices of A having odd degree in G .

Theorem 3.1.5. *Let S be a set of terminals in a graph G . If each $P \in \mathcal{P}(S)$ satisfies*

$$\sum_{A_i \in P} \delta(A_i) \geq 2k(|P| - 1) + n_o(B_P),$$

then G contains k edge-disjoint S -connectors.

Proof. Define an S -parity function by $g(v) = 1$ when v is a vertex of \bar{S} having odd degree in G and otherwise $g(v) = 0$. For $P \in \mathcal{P}(S)$, always $B_P \subseteq \bar{S}$, and hence $g(B_P) = n_o(B_P)$. Also, $g(T_P) = 0$. Hence the difference between the two sides of the specified inequality is $f_g(P)$, and the assumption that it holds is precisely the assumption that the SPC holds for this S -parity function. By Theorem 3.1.2, G has a (k, g) -family, and hence there are k edge-disjoint S -connectors. \square

The condition in Theorem 3.1.5 is sufficient but not necessary, as seen by adding to such a graph G a large component in which every vertex has odd degree. The case of Theo-

rem 3.1.5 when no vertex of \bar{S} has odd degree implies Theorem 3.0.3 in the same way that the Tree Packing Theorem implies that $2k$ -edge-connected graphs have k edge-disjoint spanning trees. Indeed, we obtain S -connectors instead of S -trees with the same hypothesis, thereby strengthening Theorem 3.0.3. Theorem 3.1.5 also enables us to strengthen Theorem 3.0.4.

Theorem 3.1.6. *If S is $3k$ -edge-connected in G , and $G - S$ has no edges, then G contains k edge-disjoint S -connectors.*

Proof. Deleting a vertex of degree 1 outside S does not affect the hypothesis, so we may assume that every vertex in \bar{S} has degree at least 2. By Theorem 3.1.5, it suffices to prove that $\sum_{A_i \in P} \delta(A_i) - n_o(B_P) \geq 2k(|P| - 1)$ for every S -partition P . Since $G - S$ has no edges, $\delta(B_P) \leq \sum \delta(A_i)$. Hence $n_o(B_P) \leq \frac{1}{3}\delta(B_P) \leq \frac{1}{3}\sum \delta(A_i)$, and we have $\sum \delta(A_i) - n_o(B_P) \geq 2/3 \sum \delta(A_i) \geq 2k|P| > 2k(|P| - 1)$. \square

Two other special cases are classical results.

Theorem 1.2.1 (Nash-Williams [32], Tutte [38]). *A graph G contains k edge-disjoint spanning trees if and only if $\sum_{A_i \in P} \delta(A_i) \geq 2k(|P| - 1)$ for every partition P of $V(G)$.*

Proof. Set $S = V(G)$, and make g identically 0. The S -partitions are the partitions of $V(G)$, and the terms in the SPC involving g are always 0. Hence the stated hypothesis is just the SPC for this S and g , and the resulting S -connectors are the spanning trees. \square

Theorem 3.1.7 (Hakimi [21]). *Given a graph G and a function $g: V(G) \rightarrow \mathbb{N}_0$, there is an orientation D of G such that each vertex v has outdegree at least $g(v)$ in D if and only if for all $T \subseteq V(G)$ there are at least $g(T)$ edges incident to T .*

Proof. Set $S = V(G)$ and $k = 0$. Every S -partition P satisfies $B_P = \emptyset$. Hence the only requirement imposed on $\sum_{i=1}^l \delta(A_i)$ in the SPC is from the singleton blocks; the sum must be at least $2g(T_P)$. In fact, the sum counts edges leaving singleton blocks twice, and it counts nothing else when the remainder of $V(G)$ is in one block.

Hence Hakimi's condition implies the SPC, and by Theorem 3.1.2 a $(0, g)$ -family exists. Since $S = V(G)$, the paths can be single edges. Obtain the desired orientation by orienting the $g(v)$ edges chosen for each v outward from v (orient non-chosen edges arbitrarily). \square

The special case of Theorem 3.1.2 when $S = V(G)$ generalizes the Tree Packing Theorem and can be proved using only the Matroid Union Theorem. No special results about S -partitions are needed when S -partitions are just partitions of $V(G)$. We present this proof first because it is needed for the proof of Theorem 3.1.2, needs no further lemmas, and provides motivation for the definition of f_g .

Given matroids M_1, \dots, M_ℓ defined on the same set E of elements, their *union* M is the hereditary system whose independent sets are $\{\bigcup_{i=1}^t I_i : I_i \text{ is an independent set in } M_i\}$. The Matroid Union Theorem (Edmonds [14]) states that M is a matroid on E and that the maximum size of an independent set in M is $\min_{X \subseteq E(G)} |\overline{X}| + \sum_{i=1}^h r_i(X)$, where $\overline{X} = E - X$ and $r_i(X)$ denotes the maximum size of a subset of X that is independent in M_i .

In the conclusion of the next theorem, reducing H_1, \dots, H_n to stars and directing them outward from the centers yields a g -family. When $S = V(G)$, every spanning tree is an S -connector, so H_1, \dots, H_{k+n} is a (k, g) -family.

Theorem 3.1.8. *Let $S = V(G) = \{v_1, \dots, v_n\}$. If the Strong Partition Condition holds for a function $g: V(G) \rightarrow \mathbb{N}_0$, then G contains edge-disjoint subgraphs H_1, \dots, H_{n+k} such that $d_{H_i}(v_i) = g(v_i)$ for $1 \leq i \leq n$ and H_{n+1}, \dots, H_{n+k} are spanning trees.*

Proof. For $v_i \in V(G)$, let $E(v_i)$ denote the set of edges incident to v_i in G . We introduce matroids M_1, \dots, M_{k+n} on $E(G)$. Let M_{n+1}, \dots, M_{n+k} be copies of the cycle matroid of G . For $1 \leq i \leq n$, let M_i be the matroid on $E(G)$ whose independent sets are $\{X \subseteq E(v_i) : |X| \leq g(v_i)\}$ (edges not incident to v_i are loops in M_i).

Let $M = \bigcup_{i=1}^{k+n} M_i$; a subset of $E(G)$ is independent in M if and only if it is the disjoint union of sets X_1, \dots, X_{n+k} such that X_i is independent in M_i for each i . The desired sets

exist if and only if M has an independent set of size $k(n-1) + g(V(G))$, in which case the independent sets X_1, \dots, X_{n+k} decomposing it are the edge sets of the desired subgraphs.

By the Matroid Union Theorem, the maximum size of an independent set in M is $\min_{X \subseteq E(G)} t(X)$, where $t(X) = |\overline{X}| + \sum_{i=1}^{k+n} r_i(X)$. Hence it suffices to show for each $X \subseteq E(G)$ that $t(X) \geq k(n-1) + g(V(G))$.

If $0 < r_i(X) < g(v_i)$, then deleting $X \cap E(v_i)$ from X shifts the amount $r_i(X)$ from the term for M_i to the term for \overline{X} without increasing other terms. Hence we may restrict our attention to sets X such that $r_i(X) \in \{0, g(v_i)\}$ for $1 \leq i \leq n$. Given such X , let P be the partition of $V(G)$ whose blocks are the vertex sets of the components of the spanning subgraph of G with edge set X . We express $t(X)$ in terms of P and then apply the SPC.

The set \overline{X} consists of all edges joining blocks of P and possibly some edges within blocks of P . Hence $|\overline{X}| \geq \frac{1}{2} \sum_{A_i \in P} \delta(A_i)$. Note that $B_P = \emptyset$, since $S = V(G)$.

A vertex v_i is a singleton block of P if and only if it has no incident edge in X . Thus $T_P = \{v_i : r_i(X) = 0\}$. With $r_i(X) \in \{0, g(V(G))\}$, we have $\sum_{i=1}^n r_i(X) = g(V(G)) - g(T_P)$. For $i > n$, the rank function of the cycle matroid yields $r_i(X) = n - |P|$.

By these computations, $2t(X) \geq \sum_{A_i \in P} \delta(A_i) - 2k(|P| - n) - 2g(T_P) + 2g(V(G))$. Thus $2t(X) \geq f_g(P) + 2k(n-1) + 2g(V(G))$. By the SPC, $f_g(P) \geq 0$, so the desired independent set and desired subgraphs exist. \square

The proof of Theorem 3.1.2 (Section 3.3) has many ingredients, including a submodularity inequality for f_g (Section 3.2), a variant of Mader's Splitting Lemma, and Theorem 3.1.8. Proving the S -tree result (Theorem 1.2.3) in Section 3.4 uses the characterization of (k, g) -families (Theorem 3.1.2) and Mader's Splitting Lemma. Section 3.5 presents the analogous argument to prove the S -connector result (Theorem 1.2.4).

3.2 S -partitions and submodularity of f_g

As mentioned, we will need a submodularity lemma for f_g to complete our inductive proof of Theorem 3.1.2. Recall that a lattice is a poset in which meets and joins exist for all pairs of elements. A function ϕ defined on a lattice is *submodular* if $\phi(x \wedge y) + \phi(x \vee y) \leq \phi(x) + \phi(y)$ for all elements x and y . For any S -parity function g , we will prove that f_g is submodular for special pairs in this poset (the poset is a lattice).

The partition lattice Π_G on $V(G)$ is the poset of all partitions of $V(G)$, ordered by refinement. That is, when Q and Q' are partitions of $V(G)$, we put $Q \leq Q'$ in Π_G if for every block $A_i \in Q$, there is a block $A'_j \in Q'$ such that $A_i \subseteq A'_j$. The unique minimal element is the partition into singleton blocks, and in general the rank of a partition Q in $\Pi(G)$ is $|V(G)| - |Q|$, where $|Q|$ denotes the number of blocks of a partition Q .

To define the order relation on $\mathcal{P}(S)$, we map an S -partition P to a partition Q_P of $V(G)$ by defining $Q_P = \{A_1, \dots, A_l, \{b_1\}, \dots, \{b_{|B_P|}\}\}$, where $P = \{A_1, \dots, A_l\}$ and $B_P = \{b_1, \dots, b_{|B_P|}\}$. This mapping is injective; it simply splits B_P into singleton sets and adds them as blocks to P . Define the order relation on $\mathcal{P}(S)$ by putting $P \leq P'$ if and only if $Q_P \leq Q_{P'}$ in Π_G . This makes $\mathcal{P}(S)$ isomorphic to a subposet $\mathcal{Q}(S)$ of Π_G .

We will study meet and join in $\mathcal{P}(S)$ by relating it to meet and join in $\mathcal{Q}(S)$ as a subposet of Π_G . Let \wedge_{Π} and \vee_{Π} denote the meet and join operations in Π_G . We use two well-known properties of the partition lattice (after subtracting each term from $|V(G)|$, statement (2) becomes the statement that the rank function of Π_G is submodular).

Proposition 3.2.1. *For partitions Q and Q' of $V(G)$,*

- (1) $Q \wedge_{\Pi} Q' = \{A_i \cap A_j : A_i \in Q, A_j \in Q'\}$;
- (2) $|Q \wedge_{\Pi} Q'| + |Q \vee_{\Pi} Q'| \geq |Q| + |Q'|$.

Let the symbols \wedge and \vee without subscripts denote the meet and join in $\mathcal{P}(S)$.

Proposition 3.2.2. For $P, P' \in \mathcal{P}(S)$, the meet and join of P and P' are well defined, with

$$(1) P \wedge P' = \{A_i \cap A'_j : A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \neq \emptyset\};$$

$$(2) Q_{P \vee P'} = Q_P \vee_{\Pi} Q_{P'};$$

$$(3) B_{P \vee P'} = B_P \cap B_{P'}.$$

Proof. (1) Let $\hat{P} = \{A_i \cap A'_j : A_i \in P, A'_j \in P', A_i \cap A'_j \cap S \neq \emptyset\}$. By definition, $\hat{P} \in \mathcal{P}(S)$ and $\hat{P} \leq P, P'$. For any block A'' in any common lower bound P'' , exist $A_i \in P$ and $A'_j \in P'$ such that $A'' \subseteq A_i \cap A'_j$. Since $A'' \cap S \neq \emptyset$, we have $A_i \cap A'_j \in \hat{P}$. Hence $P'' \leq \hat{P}$.

(2) Let $Q'' = Q_P \vee_{\Pi} Q_{P'}$. If $Q'' \notin \mathcal{Q}(S)$, then there exists $A \in Q''$ such that $A \cap S = \emptyset$ and $|A| \geq 2$. For $a \in A$, the block C containing a in Q_P is contained in A . Since $A \cap S = \emptyset$ and P is an S -partition, C must be $\{a\}$. Similarly, $\{a\} \in Q_{P'}$. Now $\{a\}$ is a block in $Q_P \vee_{\Pi} Q_{P'}$, contradicting $|A| \geq 2$.

Hence $Q'' \in \mathcal{Q}(S)$, making Q'' the least upper bound in $\mathcal{Q}(S)$ for Q_P and $Q_{P'}$. Since $\mathcal{P}(S)$ and $\mathcal{Q}(S)$ are isomorphic, also $P \vee P'$ exists, with $Q_{P \vee P'} = Q_P \vee_{\Pi} Q_{P'}$.

(3) follows immediately from (2). □

Common lower bounds in $\mathcal{P}(S)$ do not always translate so nicely to $\mathcal{Q}(S)$. Fortunately, they do for the pairs of S -partitions we will need. Two S -partitions $\{A_1, \dots, A_l\}$ and $\{A'_1, \dots, A'_l\}$ form a *good pair* if $A_i \cap A'_j \neq \emptyset$ implies $A_i \cap A'_j \cap S \neq \emptyset$.

Proposition 3.2.3. If S -partitions P and P' form a good pair, then:

$$(1) Q_{P \wedge P'} = Q_P \wedge_{\Pi} Q_{P'};$$

$$(2) B_{P \wedge P'} = B_P \cup B_{P'};$$

$$(3) |P \wedge P'| + |P \vee P'| \geq |P| + |P'|.$$

Proof. (1) Since P and P' form a good pair, the expression for their meet simplifies to $P \wedge P' = \{A_i \cap A'_j : A_i \in P, A'_j \in P', A_i \cap A'_j \neq \emptyset\}$, which maps to $Q_P \wedge_{\Pi} Q_{P'}$.

(2) $B_{P \wedge P'}$ and $B_P \cup B_{P'}$ both equal the set of elements outside all $A_i \cap A'_j$.

(3) Note that $|P| = |Q_P| - |B_P|$ and $|P'| = |Q_{P'}| - |B_{P'}|$. Using (2) and Proposition 3.2.2(3),

$$|B_P| + |B_{P'}| = |B_P \cap B_{P'}| + |B_P \cup B_{P'}| = |B_{P \wedge P'}| + |B_{P \vee P'}|.$$

Now the claim follows from $|Q_P \wedge_{\Pi} Q_{P'}| + |Q_P \vee_{\Pi} Q_{P'}| \geq |Q_P| + |Q_{P'}|$ (Proposition 3.2.1(2)). \square

Definition 3.2.4. For two sets $A, B \subseteq V(G)$, write $[A, B]$ for the set of edges with endpoints in both A and B . When A or B consists of one vertex v , we write v instead of $\{v\}$ in this notation. Let $G[A]$ denote the subgraph induced by A . Given an S -partition P with blocks A_1, \dots, A_l , assign each edge $e \in E(G)$ a weight $h_P(e)$ by

$$h_P(e) = \begin{cases} 2, & \text{if } e \in [A_i, A_j] \text{ for some } i \text{ and } j; \\ 1, & \text{if } e \in [A_i, B_P] \text{ for some } i; \\ 0, & \text{otherwise.} \end{cases}$$

Grouping the sum by edges yields $\sum_{A_i \in P} \delta(A_i) = \sum_{e \in E(G)} h_P(e)$ for any S -partition P . \square

Proposition 3.2.5. *If $S \subseteq V(G)$ and P and P' form a good pair in $\mathcal{P}(S)$, then*

$$h_{P \wedge P'}(e) + h_{P \vee P'}(e) \leq h_P(e) + h_{P'}(e)$$

for all e in $E(G)$. Also, if the endpoints of e lie in different blocks in both P and P' , but in the same block in $P \vee P'$, then the two sides of the inequality differ by 2.

Proof. For $uv \in E(G)$, let $W = \{u, v\}$. Note that $h_P(uv) = 2 - |W \cap B_P| - 2t_P(uv)$, where $t_P(uv) = 1$ if $W \in A_i$ for some $A_i \in P$, and otherwise $t_P(uv) = 0$. Since $B_{P \wedge P'} = B_P \cup B_{P'}$ and $B_{P \vee P'} = B_P \cap B_{P'}$, we have $|W \cap B_P| + |W \cap B_{P'}| = |W \cap B_{P \vee P'}| + |W \cap B_{P \wedge P'}|$. Therefore $h_{P \wedge P'}(uv) + h_{P \vee P'}(uv) \leq h_P(uv) + h_{P'}(uv)$ if and only if $t_{P \wedge P'}(uv) + t_{P \vee P'}(uv) \geq$

$t_P(uv) + t_{P'}(uv)$. This holds when P and P' form a good pair, since $\max\{t_P(uv), t_{P'}(uv) = 1\}$ implies $t_{P \vee P'}(uv) = 1$.

If u and v lie in different blocks in P and P' but in the same block in $P \vee P'$, then $t_{P \wedge P'}(uv) + t_{P \vee P'}(uv) = t_P(uv) + t_{P'}(uv) + 1$, so the difference between the two sides of the claimed inequality is then 2. \square

Lemma 3.2.6. *Let g be a S -parity function. If P and P' form a good pair in $\mathcal{P}(S)$, then*

$$f_g(P \wedge P') + f_g(P \vee P') \leq f_g(P) + f_g(P'). \quad (3.2)$$

Proof. Let Q be an S -partition. From the definition of f_g and the observation in Definition 3.2.4 that $\sum_{A_i \in P} \delta(A_i) = \sum_{e \in E(G)} h_P(e)$, we have

$$f_g(Q) = \sum_{e \in E(G)} h_Q(e) - 2k(|Q| - 1) - g(B_Q) - 2g(T_Q). \quad (3.3)$$

We consider the contributions of these terms to (3.2). Proposition 3.2.5 yields

$$\sum_{e \in E(G)} [h_{P \wedge P'}(e) + h_{P \vee P'}(e)] \leq \sum_{e \in E(G)} [h_P(e) + h_{P'}(e)].$$

By Proposition 3.2.3(3),

$$2k(|P \wedge P'| - 1) + 2k(|P \vee P'| - 1) \geq 2k(|P| - 1) + 2k(|P'| - 1).$$

Since $B_{P \wedge P'} = B_P \cup B_{P'}$ and $B_{P \vee P'} = B_P \cap B_{P'}$,

$$g(B_{P \wedge P'}) + g(B_{P \vee P'}) = g(B_P) + g(B_{P'}).$$

For the last term, recall the definition: $T_P = \{v \in S : |C_P(v) \cap S| = 1\}$, where $C_P(v)$ is

the block containing v in P . If $v \in T_P \cup T_{P'}$, then $v \in T_{P \wedge P'}$; if $v \in T_P \cap T_{P'}$, then since P and P' form a good pair, $v \in T_{P \vee P'}$. Summing the contributions made by each vertex yields

$$g(T_{P \wedge P'}) + g(T_{P \vee P'}) \geq g(T_P) + g(T_{P'}).$$

Summing the formulas for all four terms completes the proof of (3.2). \square

Sometimes we will need a stronger inequality than (3.2), ensuring a difference of 4. For $x \in V(G)$, let $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. We write $G - uv$ to mean the graph obtained from G by deleting one copy of the edge uv when uv has multiplicity at least 1.

Lemma 3.2.7. *Let P and P' be S -partitions that form a good pair. Let uv be an edge such that u and v lie in different blocks in both P and P' but in the same block in $P \vee P'$. If $N_{G-uv}(v)$ intersects both $C_P(u)$ and $C_{P'}(u)$, then $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P') \geq 4$.*

Proof. We showed in proving Lemma 3.2.6 that the terms in (3.3) involving g make a non-negative contribution to $f_g(P) + f_g(P') - f_g(P \wedge P') - f_g(P \vee P')$. Hence it suffices to gain 4 from the other terms.

For each edge e , let $\hat{h}(e) = h_P(e) + h_{P'}(e) - h_{P \wedge P'}(e) - h_{P \vee P'}(e)$. Proposition 3.2.5 implies that always $\hat{h}(e) \geq 0$ and that the locations of u and v yield $\hat{h}(uv) \geq 2$. It suffices to find another edge e with $\hat{h}(e) \geq 2$ or gain 2 from the term involving the number of blocks.

By the hypothesis on $N(v)$, deleting (one copy of) the edge vu leaves v with a neighbor in each of $C_P(u)$ and $C_{P'}(u)$. Suppose that v still has a neighbor w in $C_P(u) - C_{P'}(v)$ or $C_{P'}(u) - C_P(v)$ (possibly $w = u$). In either case, w and v lie in different blocks in both P and P' , and w and u lie in the same block of $P \vee P'$. By hypothesis, this block of $P \vee P'$ also contains v , so Proposition 3.2.5 yields $\hat{h}(wv) \geq 2$, which suffices.

Therefore, we may assume that the given vertices $w, w' \in N_{G-uv}(v)$ are in $C_P(u) \cap C_{P'}(v)$ and $C_{P'}(u) \cap C_P(v)$, respectively. Since u and v lie in distinct blocks in both P and P' , we have $w \neq w'$ (and neither of them is u).

Obtain P'' from P by splitting $C_P(v)$ into $C_P(v) - C_{P'}(u)$ and $C_P(v) \cap C_{P'}(u)$. Since P and P' form a good pair, P'' is an S -partition. Since all intersections of blocks in P'' and P' are intersections of blocks in P and P' , also P'' and P' form a good pair, and $P'' \wedge P' = P \wedge P'$. Furthermore, $P'' \vee P' = P \vee P'$, since $C_{P'}(v)$, $C_P(u)$, and $C_{P'}(u)$ successively put the pairs $\{v, w\}$, $\{w, u\}$, and $\{u, w'\}$ into the same block of $P'' \vee P'$ (using $C_{P''}(u) = C_P(u)$).

Now, since $|P'' \wedge P'| + |P'' \vee P'| - |P''| - |P'| \geq 0$ (by Proposition 3.2.3(3)) and $|P''| = |P| + 1$, we obtain $|P \wedge P'| + |P \vee P'| - |P| - |P'| \geq 1$. Since it has the coefficient $2k$, this term now provides the additional contribution of 2 that completes the proof. \square

Proposition 3.2.8. *If P is an S -partition and g is an S -parity function, then $f_g(P)$ is even.*

Proof. For $A \subseteq V(G)$, recall that $n_o(A)$ is the number of vertices of A having odd degree in G . Using $B_P \subseteq \bar{S}$ and the definition of S -parity function,

$$\begin{aligned} f_g(P) &= \left(\sum_{A_i \in P} \delta(A_i) \right) - 2k(l-1) - g(B_P) - 2g(T_P) \\ &\equiv \left[\sum_{i=1}^l \left(\sum_{v \in A_i} d_G(v) \right) - 2|E(G[A_i])| \right] + n_o(B_P) \\ &\equiv \left[\sum_{i=1}^l n_o(A_i) \right] + n_o(B_P) \equiv n_o(V(G)) \equiv 0 \pmod{2}. \end{aligned} \quad \square$$

For $X \subseteq \bar{S}$ and $P = (A_1, \dots, A_l)$, let $P - X = (A_1 - X, \dots, A_l - X)$. Note that if P is an S -partition, then so is $P - X$. Recall that $[A, B] = \{xy \in E(G) : x \in A, y \in B\}$.

Proposition 3.2.9. *If P is an S -partition and $X \subseteq A_i \cap \bar{S}$, where A_i is a block of P , then*

$$f_g(P) - f_g(P - X) \geq |[X, \bar{A}_i]| - |[X, A_i - X]|.$$

Proof. Since $f_g(P) = \sum_{i=1}^l \delta(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P)$, we have

$$\begin{aligned} f_g(P) - f_g(P - X) &= \delta(A_i) - \delta(A_i - X) + g(X) \\ &\geq \delta(A_i) - \delta(A_i - X) = |[X, \bar{A}_i]| - |[X, A_i - X]| \end{aligned} \quad \square$$

3.3 Existence of (k, g) -families

The goal of this section is to prove Theorem 3.1.2, which states that a (k, g) -family exists if and only if the Strong Partition Condition holds for (G, S, k, g) . After proving further properties of good pairs of S -partitions, our inductive proof of the main theorem will use Theorem 3.1.8 as the basis and a variant of Mader's Splitting Lemma in the induction step.

Let uv and vw be two edges of G . The uv, vw -shortcut of G is the graph obtained from G by replacing uv and vw with uw . When u is already adjacent to w , an extra copy of uw is added; when $u = w$, a double-edge is replaced with a loop. Fix an edge uv with $u \in S$. For $w \in N_{G-uv}(v)$, let G_w denote the uv, vw -shortcut of G . By $G - uv$, we mean the graph obtained from G by deleting one copy of uv ; this means that $w = u$ is possible when uv has multiplicity greater than 1 in G .

In order to prove Theorem 3.1.2 inductively, we will show that if uv is an edge in G with $u \in S$ and $v \notin S$, and G satisfies the Strong Partition Condition (SPC) for an S -parity function g such that $d_G(v) > g(v)$, then there exists $w \in N_{G-uv}(v)$ such that G_w also satisfies the SPC. This is the main technical result of our paper. Mader's Splitting Lemma (Lemma 1.2.6) is analogous; it guarantees shortcuts that preserve local connectivity conditions.

Definition 3.3.1. Given $S \subseteq V(G)$, suppose that G satisfies the SPC for an S -parity function g . Fix an edge $uv \in E(G)$ with $u \in S$ and $v \notin S$ such that $d_G(v) > g(v)$. A vertex w is *dangerous* for an S -partition P (relative to uv) if $f_g(P) < 0$ for the graph G_w . Let $D(P) = \{w \in V(G) : f_g(P) < 0 \text{ for } G_w\}$. □

When $w \in D(P)$, we have $f_g(P) \leq -2$ for G_w and $f_g(P) \geq 0$ for G , since $f_g(P)$ is always even (Proposition 3.2.8). The contributions to $f_g(P)$ for G and G_w differ only in $\sum_{A_i \in P} \delta(A_i)$, which decreases when replacing uv and vw with uw only if $u, w \notin C_P(v)$ (recall that $C_P(x)$ is the member of $\{A_1, \dots, A_l, B_P\}$ containing x , where A_1, \dots, A_l are the blocks of P). Since $u \in S$ and $v \notin S$, the ways a decrease can occur are shown in Figure 3.1. The shortcut

decreases $f_g(P)$ by 2 if $v \in B_P$ and $w \in C_P(u)$, by 2 if $v \notin B_P$ and $w \notin C_P(v) \cup C_P(u)$, and by 4 if $v \notin B_P$ and $w \in C_P(u)$. Otherwise, $f_g(P)$ does not change.

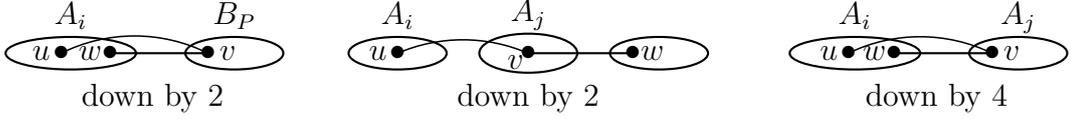


Figure 3.1: Dangerous locations for w

Vertex w will be dangerous with a decrease of 2 when $f_g(P) = 0$ or a decrease of 4 when $f_g(P) \in \{0, 2\}$. We group the cases as “Types” by the value of $f_g(P)$ and the location of v in P . These types determine the location of all w such that $f_g(P) < 0$ for G_w . For simplicity, write $N'(v)$ for $N_{G-uw}(v)$; thus $N'(v) = N_G(v) - \{u\}$ when uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$. The distinction between Type 2 and Type 3 is that decreasing $f_g(P)$ by 2 instead of 4 is enough when $f_g(P) = 0$, so vertices in all of $N'(v) - C_P(v)$ are dangerous instead of just those in $C_P(u)$. If P is none of these types, then $D(P) = \emptyset$.

Type	$f_g(P)$ for G	location of v	dangerous set $D(P)$
1	0	$v \in B_P$	$N'(v) \cap C_P(u)$
2	0	$v \notin B_P \cup C_P(u)$	$N'(v) - C_P(v)$
3	2	$v \notin B_P \cup C_P(u)$	$N'(v) \cap C_P(u)$

Our goal is to find $w \in N'(v)$ such that w is outside $D(P)$ for every P . Define an S -parity function by $g(v) = 1$ when v is a vertex of \bar{S} having odd degree in G and otherwise $g(v) = 0$. For $P \in \mathcal{P}(S)$, always $B_P \subseteq \bar{S}$, and hence $g(B_P) = n_o(B_P)$. Also, $g(T_P) = 0$. Hence the left side of the assumed equality is $f_g(P)$, and we have assumed that the SPC holds for this S -parity function. By Theorem 3.1.2, G has a (k, g) -family, and hence there are k edge-disjoint S -connectors. S -partition P ; in that case, G_w satisfies the SPC. We will need two lemmas about S -partitions.

With $D(P)$ defined relative to a fixed edge uv , let \mathcal{M} be the set of minimal S -partitions among those with maximal dangerous sets. That is, $P \in \mathcal{M}$ when there is no S -partition P' such that $D(P) \subset D(P')$ or such that $D(P) = D(P')$ and $P' < P$ in $\mathcal{P}(S)$. The next lemma will help us find an S -partition whose dangerous set contains $D(P)$ for all $P \in \mathcal{P}(S)$.

Lemma 3.3.2. *If $P, P' \in \mathcal{M}$, then P and P' form a good pair.*

Proof. We prove the contrapositive. When P and P' do not form a good pair, there exist $A_i \in P$ and $A'_j \in P'$ such that $\emptyset \neq A_i \cap A'_j \subseteq \overline{S}$. Let $X = A_i \cap A'_j$; we have remarked that $P - X \in \mathcal{P}(S)$. Changing P to $P - X$ splits elements of X from blocks in P (and in $Q(P)$) to become singletons in $Q(P - X)$, so $P - X \leq P$ (also, $P' - X \leq P'$). Hence it suffices to prove $D(P) \subseteq D(P - X)$ or $D(P') \subseteq D(P' - X)$, since then P and P' are not both in \mathcal{M} .

Claim ():* *If P is Type 1 or 3 and $f_g(P - X) \leq f_g(P)$, then $D(P) \subseteq D(P - X)$ unless $u \in A_i$ and $P - X$ is not Type 2 (and similarly for P'). Since $v \notin C_P(u)$, also $v \notin C_{P-X}(u)$. If $u \notin A_i$, then $C_{P-X}(u) = C_P(u)$, so $D(P) = N'(v) \cap C_P(u) = N'(v) \cap C_{P-X}(u) \subseteq D(P - X)$. Hence $u \in A_i$, so $v \notin A_i$ and $C_{P-X}(v) = C_P(v)$. If $P - X$ is Type 2, then $D(P) \subseteq N'(v) - C_P(v) = N'(v) - C_{P-X}(v) = D(P - X)$.*

If $|[X, A_i - X]| < \delta(X)/2$, then $|[X, \overline{A}_i]| > |[X, A_i - X]|$, so $f_g(P) > f_g(P - X)$, by Proposition 3.2.9. However, the SPC yields $f_g(P - X) \geq 0$, so $f_g(P - X) = 0$ and P is Type 3. By (*), we have $u \in A_i$ and $P - X$ is Type 1. Since P is Type 3, $v \notin B_P$, so $P - X$ being Type 1 requires $v \in X$, which contradicts $u \in A_i$.

This eliminates the case $|[X, A_i - X]| < \delta(X)/2$, and similarly for A'_j . Since $|[X, A_i - X]| + |[X, A'_j - X]| \leq \delta(X)$, the remaining case is $|[X, A_i - X]| = |[X, A'_j - X]| = \delta(X)/2$, and $[X, \overline{X}] = [X, (A_i \cup A'_j) - X]$. Also $f_g(P - X) \leq f_g(P)$ and $f_g(P' - X) \leq f_g(P')$ for G , by Proposition 3.2.9. Since $X \subseteq \overline{S}$, we know $u \notin A_i \cap A'_j$. By symmetry, we may take $u \notin A_i$, and hence P is Type 2 by (*). Thus $f_g(P - X) = f_g(P) = 0$.

If $v \in X$, then $v \notin C_P(u) \cup C_{P'}(u)$ yields $u \notin A_i \cup A'_j$. Since all edges leaving X go to

$A_i - X$ or $A'_i - X$, now $[X, \{u\}] = \emptyset$, which contradicts the existence of uv . Hence we may assume $v \notin X$. Since $f_g(P - X) = 0$ and P is Type 2, $v \notin X$ implies $P - X$ is Type 2, so $D(P) = N'(v) - C_P(v) \subseteq N'(v) - C_{P-X}(v) = D(P - X)$. \square

We now obtain a single S -partition whose dangerous set contains all dangerous sets.

Lemma 3.3.3. *There exists an S -partition whose dangerous set contains $\bigcup_{P \in \mathcal{P}(S)} D(P)$.*

Proof. If the dangerous sets for all S -partitions in \mathcal{M} are the same, then every member of \mathcal{M} has the desired property. Suppose $P, P' \in \mathcal{M}$ exist with $D(P) \neq D(P')$. By Lemma 3.3.2, P and P' form a good pair. Let $\check{P} = P \vee P'$ and $\hat{P} = P \wedge P'$. If \hat{P} is a Type 2 partition, then $D(P) \subseteq N'(v) - C_P(v) \subseteq N'(v) - C_{\hat{P}}(v) = D(\hat{P})$, which contradicts $P \in \mathcal{M}$.

Case 1: u and v lie in the same block of \check{P} . By Lemma 3.2.7 and the SPC, $f_g(P) + f_g(P') \geq f_g(\hat{P}) + f_g(\check{P}) + 4 \geq 4$. Since $D(P), D(P') \neq \emptyset$ requires $f(P), f(P') \leq 2$, we have $f_g(\hat{P}) = f_g(\check{P}) = 0$. Also $f_g(P) = f_g(P') = 2$, so P and P' are both Type 3, and $v \notin B_P \cup B_{P'} = B_{\hat{P}}$. We conclude that \hat{P} is Type 2.

Case 2: u and v do not lie in the same block of \check{P} . Suppose first that $f_g(\check{P}) \geq 4$, so both P and P' are Type 3 and $f_g(\hat{P}) = 0$. Also $v \notin B_P \cup B_{P'} = B_{\hat{P}}$, so \hat{P} is Type 2.

Next suppose that $f_g(\check{P}) = 2$. By submodularity, P or P' must be Type 3; let P be Type 3. Hence $v \notin B_P$. Since always $B_{\hat{P}} = B_P \cap B_{P'}$ (Proposition 3.2.2), we obtain $v \notin B_{\hat{P}}$.

Hence we may assume that $f_g(\check{P}) = 0$ or that $f_g(\check{P}) = 2$ and $v \notin B_{\hat{P}}$. Now $D(\check{P}) \supseteq N'(v) \cap C_{\check{P}}(u) \supseteq N'(v) \cap (C_P(u) \cup C_{P'}(u))$. If neither P nor P' is Type 2, then this last set is $D(P) \cup D(P')$. Since $D(P) \neq D(P')$ and $P, P' \in \mathcal{M}$, neither of $D(P)$ and $D(P')$ contains the other. Hence $D(\check{P})$ strictly contains both, which contradicts $P, P' \in \mathcal{M}$.

If both P and P' are Type 2, then submodularity yields $f_g(\hat{P}) = 0$. Also $v \notin B_P \cup B_{P'} = B_{P \wedge P'}$, so \hat{P} is Type 2. If P (and not P') is Type 2, then $D(P) = N'(v) - C_P(v)$ and $D(P') = N'(v) \cap C_{P'}(u)$. Since u and v are not in the same block of \check{P} , the sets $C_P(v)$ and $C_{P'}(u)$ are disjoint. Hence have $D(P') \subset D(P)$, contradicting $P' \in \mathcal{M}$. \square

Next we prove an analogue of Mader's Splitting Lemma (Lemma 1.2.6). Recall that $N'(v) = N_G(v) - \{u\}$ if uv has multiplicity 1, and otherwise $N'(v) = N_G(v)$. When A or B has only one vertex v , we write v instead of $\{v\}$ in the notation $[A, B]$.

Theorem 3.3.4. *If G satisfies the Strong Partition Condition and has an edge uv with $u \in S$, $v \notin S$, and $d_G(v) > g(v)$, then there is a vertex $w \in N'(v)$ such that G_w satisfies the SPC.*

Proof. By Lemma 3.3.3, there exists an S -partition P whose dangerous set contains the dangerous sets (relative to uv) for all S -partitions. If no desired vertex w exists, then $D(P) = N'(v)$. Thus $|[v, C_P(v)]| = 0$. Let P' be the S -partition obtained from P by moving v to $C_P(u)$; note that $|P'| = |P|$ and $T_{P'} = T_P$.

Using the expression for f_g in (3.1), we have $f_g(P) - f_g(P') = d_G(v) - g(v) > 0$ when P is Type 1, and $f_g(P) - f_g(P') = 2|[v, C_P(u)]| - 2|[v, C_P(v)]| > 0$ when P is Type 2 or Type 3. Since $f_g(P') \geq 0$, this yields $f_g(P) > 0$. Hence P is Type 3.

Since $N'(v) = D(P)$, now $N_G(v) \subseteq C_P(u)$. Since g is an S -parity function, $v \notin S$, and $d_G(v) > g(v)$, we also have $|[v, C_P(u)]| = d_G(v) \geq g(v) + 2 \geq 2$. Now $2 \geq f_g(P) - f_g(P') = 2|[v, C_P(u)]| \geq 4$, a contradiction. We conclude that the desired vertex w exists. \square

We can now prove our main result.

Theorem 3.1.2. *Let S be a set of terminals in G . If g is an S -parity function for G , then G has a (k, g) -family if and only if $f_g(P) \geq 0$ for all $P \in \mathcal{P}(S)$.*

Proof. Proposition 3.1.3 proves necessity. For sufficiency, we use induction on the total number of vertices and edges, with trivial basis. Theorem 3.1.8 is the case $S = V(G)$, so we may assume $\bar{S} \neq \emptyset$. We will reduce the claim to a special case where Theorem 3.1.8 applies.

Let $R = \bar{S} \cap N(S)$. We may assume $R \neq \emptyset$; otherwise, the induction hypothesis applies to $G - \bar{S}$. If $d_G(v) > g(v)$ for some $v \in R$, then choose $u \in N(v) \cap S$. Theorem 3.3.4 provides $w \in N'(v)$ (for this u and v) such that G_w satisfies the SPC. Since G_w is smaller than G , it

has a (k, g) -family. If any of the resulting S -connectors or paths contain the edge uw that is not in G , then replacing that edge with the original uv and vw yields a (k, g) -family in G .

Hence we may assume $d_G(v) = g(v)$ for $v \in R$, by Proposition 3.1.4. We next reduce to the case $N(v) \subseteq S$ for all $v \in R$. Let $P = \{S\}$; that is, $|P| = 1$ and $B_P = \bar{S}$. Since always $|S| \geq 2$, we have $T_P = \emptyset$, and hence $f_g(P) = |[S, \bar{S}]| - g(\bar{S})$. By the SPC, $|[S, \bar{S}]| \geq g(\bar{S}) \geq \sum_{v \in R} d_G(v)$. However, $|[S, \bar{S}]| \leq \sum_{v \in R} d_G(v)$. We conclude that R is an independent set whose neighbors all lie in S and that $g(v) = 0$ for $v \in \bar{S} - R$.

We argue that in this remaining case $G[S]$ satisfies the SPC. Let \hat{P} be an S -partition of $G[S]$; note that $B_{\hat{P}} = \emptyset$. We may also view \hat{P} as an S -partition of G , in which case we denote it by P , so $B_P = \bar{S}$. Comparing values of f_g for $G[S]$ and G , we have $f_g(\hat{P}) - f_g(P) = g(B_P) - |[S, \bar{S}]|$. Since $g(B_P) = g(R) = |[S, \bar{S}]|$, we have $f_g(\hat{P}) = f_g(P) \geq 0$.

Since $G[S]$ satisfies the SPC, Theorem 3.1.8 yields $k + g(S)$ edge-disjoint subgraphs of $G[S]$ such that k are S -connectors in $G[S]$ and the others combine into disjoint sets of $g(v)$ edges at v for each $v \in S$. Since $g(v) = 0$ for $v \in \bar{S} - R$ and $g(v) = d_G(v)$ for $v \in R$, adding the edges from R to S as directed paths completes a (k, g) -family for G . \square

3.4 Steiner tree packing

In this section we apply Theorem 3.1.2 to the problem of packing S -trees. Recall that $E(v)$ denotes the set of edges incident to a vertex v and that a vertex set S is j -edge-connected in a graph G when deleting any set of fewer than j edges leaves S in a single component. Our sufficient condition for k edge-disjoint S -trees uses the following theorem, which is the main technical result of this section and is proved using Theorem 3.1.2.

Theorem 3.4.1. *Let k and λk be positive integers $\lambda \geq 6.5$. Let S be a vertex set that is λk -edge-connected in a graph G . Fix a vertex $v \in S$ with $d_G(v) = \lambda k$. Let E_0, \dots, E_k be a partition of $E(v)$, and let $N_i = \{w : vw \in E_i\}$. If $|E_0| \geq k$, then G has edge-disjoint*

subgraphs H_0, \dots, H_k such that

- (1) $E_i \subseteq E(H_i)$ for $0 \leq i \leq k$;
- (2) $d_{H_0}(s) \geq k$ for all $s \in S$; and
- (3) for $1 \leq i \leq k$, the vertex set $(S - \{v\}) \cup N_i$ is connected in $H_i - v$.

We say Graphs H_0, \dots, H_k satisfying the requirements in Theorem 3.4.1 *properly extend* E_0, \dots, E_k or *form a proper extension* of E_0, \dots, E_k in G . By the meaning of “partition”, each E_i is nonempty. This notion of proper extension refines the “extension property” used by Lau in [27]. Lau had no special subgraph H_0 , and he required $d_{H_i}(s) \geq 2$ for each i and each $s \in S$. In the special case where S is independent, distributing the edges of our H_0 to the other subgraphs yields H_1, \dots, H_k satisfying his conditions. Lau used only the Nash-Williams Theorem, which we have extended to a condition for (k, g) -families.

Theorem 3.4.1 immediately yields Theorem 1.2.3.

Theorem 1.2.3 *If S is $6.5k$ -edge-connected in G , then G contains k edge-disjoint S -trees.*

Proof. Form \hat{G} by adding to G a vertex v and any $\lceil 6.5k \rceil$ edges joining v to S . Let $\hat{S} = S \cup \{v\}$; note that \hat{S} is $\lceil 6.5k \rceil$ -edge-connected in \hat{G} . Partition $E(v)$ into E_0, \dots, E_k with $|E_0| \geq k$. Applying Theorem 3.4.1 to \hat{G} and \hat{S} instead of G and S yields subgraphs H_0, \dots, H_k . By property (3) in Theorem 3.4.1, H_1, \dots, H_k contain the desired S -trees. \square

Definition 3.4.2. *Minimal counterexample G_0 .* If Theorem 3.4.1 is not true, then there is a graph G_0 with fewest edges such that S, v, λ, k and E_0, \dots, E_k satisfy the hypotheses of Theorem 3.4.1 (where λk is an integer) and yet no proper extension of E_0, \dots, E_k exists. Among such structures, choose one such that \bar{S} is smallest, where $\bar{S} = V(G_0) - S$. Henceforth let G_0 be such a minimal counterexample. In the lemmas of this section, we obtain properties that G_0 must satisfy, eventually obtaining a contradiction. Minimality implies that G_0 is connected. Also, a λk -edge-connected set of size at least 2 cannot have a loop at a vertex of degree λk , so we may assume there is no loop at the fixed vertex v . \square

Lemma 3.4.3. *In G_0 , the set \overline{S} of non-terminal vertices is independent.*

Proof. Let e be an edge with endpoints in \overline{S} . If S is λk -edge-connected in $G_0 - e$, then by the minimality of G_0 there exist H_0, \dots, H_k that properly extend E_0, \dots, E_k in $G_0 - e$. These subgraphs also properly extend E_0, \dots, E_k in G_0 .

Hence S is not λk -edge-connected in $G_0 - e$. Let F be a subset of $E(G_0)$ with exactly λk edges (including e) such that S is not connected in $G_0 - F$. Exactly two components of $G_0 - F$ contain vertices of S , since S is λk -edge-connected in G_0 . Let G_1 and G_2 be the graphs obtained by contracting one of these components to a single vertex, calling that vertex v_j in G_j . For $j \in \{1, 2\}$, let $S_j = (S \cap V(G_j) \cup \{v_j\})$; note that S_j is λk -edge-connected in G_j . By symmetry, we may assume that the special vertex v in S lies in $V(G_1)$.

Since the endpoints of e are in \overline{S} , the cut F does not isolate a vertex, so G_1 and G_2 are smaller than G_0 . Hence there exist H_0^1, \dots, H_k^1 that properly extend E_0, \dots, E_k in G_1 . Let $E_i^2 = E(H_i^1) \cap F$ for $0 \leq i \leq k$. In G^2 , we obtain H_0^2, \dots, H_k^2 that properly extend E_0^2, \dots, E_k^2 . For $0 \leq i \leq k$, let H_i be the subgraph of G with $E(H_i) = E(H_i^1) \cup E(H_i^2)$. Now H_0, \dots, H_k properly extend E_0, \dots, E_k in G_0 , a contradiction. \square

For $x, y \in V(G)$, let $\kappa'(x, y; G)$ denote the *local edge-connectivity* of x and y in G , which is the minimum number of edges whose deletion leaves x and y in different components. Mader's Splitting Lemma is a powerful inductive tool involving local edge-connectivity.

Theorem 1.2.6 (Mader's Splitting Lemma [28]). *Let x be a non-cut-vertex of G . If x has degree at least 2 (except when $d_G(x) = 3$ and x has three distinct neighbors), then there is a shortcut \hat{G} of G at x such that $\kappa'(u, v; G) = \kappa'(u, v; \hat{G})$ whenever $u, v \in V(G) - \{x\}$.*

To simplify our subsequent proofs, we need a slightly stronger version of Mader's Lemma that is less well known.

Theorem 3.4.4 (Mader's Splitting Lemma, variation). *If $x \in V(G)$ and x is not incident to a cut-edge of G , then there is a shortcut \hat{G} of G at x that preserves local edge-connectivity*

in $V(G) - \{x\}$ unless $d_G(x) = 3$ and x has three distinct neighbors.

Proof. By Lemma 1.2.6, we may assume that x is a cut-vertex of G . Since x is not incident to a cut-edge, x has at least two neighbors in each component of $G - x$. Let G_1, \dots, G_t be the components of $G - x$. Let y and y' be neighbors of x in G_1 , and let z and z' be neighbors of x in G_2 . Form G' from G by the shortcut replacing yx and xz with yz . We show that $\kappa'_{G'}(u, v) \geq \kappa'_G(u, v)$ for $u, v \in V(G)$.

Suppose first that $u, v \in V(G_i) \cup \{x\}$. Any family of edge-disjoint u, v -paths in G lies in the subgraph induced by $V(G_i) \cup \{x\}$ and remains in G' unless it uses one of the shortcut edges. Hence we may assume $i = 1$, by symmetry. In that case, the shortcut edge yx can be replaced by a path through the edge yz , a zz' -path in G_2 , and the edge z_2x to obtain a family of the same size in G' .

Hence we may assume that u and v lie in different components of $G - x$. Let $\ell = \min\{\kappa'_G(u, x), \kappa'_G(v, x)\}$. We showed in the previous paragraph that no set of $\ell - 1$ edges separates x from u or v in G' . Hence also no set of $\ell - 1$ edges separates u from v in G' . Since u and v lie in different components of $G - x$, all u, v -paths in G pass through x , and hence $\kappa'_G(u, v) = \ell$, which completes the proof. \square

Since Theorem 3.4.1 trivially holds for a graph that has only two vertices (both in S), the next structural property of G_0 allows us to assume henceforth that $|S| \geq 3$.

Lemma 3.4.5. *In G_0 , every vertex of \bar{S} has degree 3, with three distinct neighbors in S (and hence $|S| \geq 3$).*

Proof. Consider $x \in \bar{S}$. If x is incident to a cut-edge e , then S is contained within one component of $G - e$, since S is λk -edge-connected in G . In this case, we can apply minimality in the choice of G_0 , restricting the graph to that component.

We may therefore assume that x is not incident to a cut-edge. Except when $d_{G_0}(x) = 3$ and x has three distinct neighbors, Mader's Splitting Lemma now implies that S is λk -edge-

connected in some shortcut of G_0 at x . By minimality in the choice of G_0 , that shortcut of G_0 has a proper extension of E_0, \dots, E_k , which implies that G_0 does also.

We may therefore assume that $d_{G_0}(x) = 3$ and x has three distinct neighbors. By Lemma 3.4.3, those three distinct neighbors lie in S . \square

Definition 3.4.6. *The modified set S' of terminals.* Within G_0 , pick a vertex u_i from N_i for $1 \leq i \leq k$. These vertices need not be distinct and may lie in S . Let $U = \{u_1, \dots, u_k\}$, $S' = S - \{v\}$, $N'_i = N_i - u_i - S'$ and $X = \bigcup_{i=1}^k N'_i$ (see Figure 3.2). Let M be the maximal bipartite subgraph of G_0 with partite sets X and S' . Note that $|S'| \geq 2$. \square

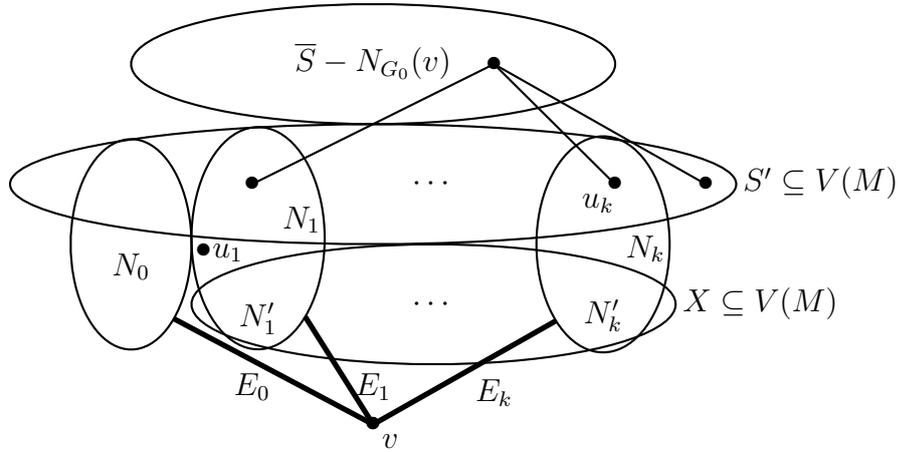


Figure 3.2: Vertices and vertex sets in G_0 ; let $G' = G_0 - v - X$

Lemma 3.4.7. *In G_0 , there exists a subgraph M' of M such that:*

- (1) $d_{M'}(x) = 1$ for all $x \in X$; and
- (2) $d_{M'}(s) \geq \lfloor d_M(s)/2 \rfloor$ for all $s \in S'$.

Proof. By Definition 3.4.6, $X \subseteq \overline{S} \cap N_{G_0}(v)$. Hence every vertex in X has two distinct neighbors in M , by Lemma 3.4.5. By adding one vertex adjacent to all vertices of odd degree in M and following an Eulerian circuit in each component of the resulting graph, we obtain an orientation D of M (ignoring the edges added to M) in which every vertex $s \in S'$ has outdegree $\lfloor d_M(s)/2 \rfloor$ or $\lceil d_M(s)/2 \rceil$ and every vertex of M has indegree 1. The subgraph of M whose edges are those oriented from S' to X in D is the desired subgraph M' . \square

Definition 3.4.8. *The derived graph G' and special parity function.* Given G_0 as in Definition 3.4.2, let $G' = G_0 - v - X$. Using S' as the set of terminals, where $S' = S - \{v\}$ as in Definition 3.4.6, we define a special S' -parity function g as follows

$$g(u) = \begin{cases} 0, & u \in (N_0 \cup U) - S'; \\ 1, & u \in \bar{S} - N_{G_0}(v); \\ \max\{k - d_{M'}(u) - |E(u) \cap E_0|, 0\}, & u \in S'. \end{cases}$$

□

We will prove that G' has a (k, g) -family for the terminal set S' and this S' -parity function g . Because the proof is lengthy, we first motivate it by using such a (k, g) -family to complete the proof of Theorem 3.4.1. Obtaining a proper extension of E_0, \dots, E_k contradicts the definition of G_0 , thus forbidding counterexamples and proving Theorem 3.4.1.

Lemma 3.4.9. *If the graph G' derived from G_0 has a (k, g) -family for the S' -parity function g in Definition 3.4.8, then there is a proper extension of E_0, \dots, E_k in G_0 .*

Proof. We will use a (k, g) -family in G' to extend E_0, \dots, E_k in G_0 , adding edges to E_i to form H_i , thereby satisfying (1) in Theorem 3.4.1. For $1 \leq i \leq k$, we will add to E_i the edges of one S' -connector and additional edges needed to ensure (3) in Theorem 3.4.1. To extend E_0 , we will use the oriented paths in the (k, g) -family, suitably adjusted.

In order to handle the vertices of $U - S'$ (recall that $U = \{u_1, \dots, u_k\}$), we first adjust the (k, g) -family in G' . We are given S' -connectors H'_1, \dots, H'_k and oriented paths $P_1, \dots, P_{g(V(G'))}$. We may assume that H'_1, \dots, H'_k are minimal S' -connectors. Thus each path joining vertices of S' in H'_j is an edge or has length 2 with internal vertex in \bar{S} .

Minimality also implies that short-cutting the paths forming H'_j turns H'_j into a tree T'_j with vertex set S' . *Mark* an edge in T'_j with label i if it arises by short-cutting the two-edge path through u_i for some $u_i \in U - S'$. Since such a vertex u_i has degree 2 in G' , and H'_1, \dots, H'_k are edge-disjoint, each label marks an edge in at most one tree. We will modify

T'_1, \dots, T'_k so that each T'_j contains at most one marked edge.

If some such tree T has two marked edges, then let e be one of them. At most k edges are marked, so some tree T' in the list has none. Adding e to T' completes a unique cycle via a path that crosses from one component of $T - e$ to the other using an edge e' of T' . Replacing T and T' with $T - e + e'$ and $T' - e' + e$ yields a new set of trees in which fewer have more than one marked edge. The edge switch corresponds in G' to switching paths in the edge-disjoint S' -connectors.

Repeat the switching argument until no tree has more than one marked edge. Re-index the resulting S' -connectors so that each $u_i \in U - S'$ occurs in none of H'_1, \dots, H'_k other than H'_i . For $1 \leq i \leq k$, let \hat{H}_i be the spanning subgraph of G_0 with edge set $E_i \cup E(H'_i) \cup B_i$, where B_i is the set of edges in $E(M) - E(M')$ incident to N'_i . Let \hat{H}_0 be the spanning subgraph of G_0 with edge set $E_0 \cup E(M') \cup \bigcup_{j=1}^{g(V(G))} E(P_j)$.

Since H'_i is an S' -connector in G' , all of $S - \{v\}$ is connected in $\hat{H}_i - v$. If $x \in N'_i$, then x has two incident edges in M ; one is in M' (by Lemma 3.4.7) and the other connects x to $S - \{v\}$ in $\hat{H}_i - v$. Now all of $(S - \{v\}) \cup N_i$ is connected in $\hat{H}_i - v$, except possibly u_i if $u_i \in U - S'$. In this case, u_i is not in M but is in G' . By the switching argument given above, if the two edges incident to u_i in G' are in $\bigcup_{j=1}^k H'_j$, then they are in H'_i , and we let $H_i = \hat{H}_i$. Otherwise, we add those two edges to \hat{H}_i to form H_i , unless they form some path P_r in the g -family (note that $g(u_i) = 0$), in which case we add the edge leaving u_i in P_r to \hat{H}_i to form H_i . In each case, u_i is now connected to S' , and we have satisfied (3) in Theorem 3.4.1.

In forming H_i , we may have removed one edge of one path P_r from \hat{H}_0 . Let H_0 be the subgraph of \hat{H}_0 that remains after all such edges have been removed. No edges of E_0 were removed, so $d_{H_0}(v) \geq k$, and we need only check that H_0 has enough edges at each $s \in S'$ to satisfy (2) in Theorem 3.4.1. There remain at least $g(s)$ edges from the paths in the g -family, since we removed only edges leaving vertices of $U - S'$. Adding $E(s) \cap E_0$ and the edges of M' yields $d_{H_0}(s) \geq g(s) + |E(s) \cap E_0| + d_{M'}(s) \geq k$. \square

By Lemma 3.4.9, the next lemma completes the proof of Theorem 3.4.1 and hence also Theorem 1.2.3. This is where we use $\lambda \geq 6.5$. Although introducing the vertex set U complicates the construction in Lemma 3.4.9, it enables us to improve our result from $\lambda \geq 10$ to $\lambda \geq 6.5$ by reducing the requirement on $d_{H_0}(s)$ in (2) of Theorem 3.4.1 from $2k$ to k .

Lemma 3.4.10. *Given G_0 , the derived graph G' has a (k, g) -family for the S' -parity function g in Definition 3.4.8.*

Proof. By Theorem 3.1.2, it suffices to prove that the SPC holds for G' and g . That is, $f_g(P) \geq 0$ for each S' -partition P of G' . Recall the definition:

$$f_g(P) = \sum_{A_i \in P} \delta_{G'}(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P). \quad (3.4)$$

Our discussion of P and the sets B_P and T_P is always with respect to G' . It suffices to prove $f_g(P) \geq 0$ for a S' -partition P with special properties among those that minimize f_g .

By Lemma 3.4.5, every vertex of $V(G_0) - S$ has degree 3 in G_0 , with three distinct neighbors in S . If $w \in A_i - S'$ for some block A_i in P , and w has no neighbor in A_i , then w has a neighbor in some block A_j other than A_i , and switching w from A_i to A_j produces an S' -partition P' of G' with $f_g(P') < f_g(P)$. Hence we may assume that every vertex of $V(G') - S'$ in a block of P has a neighbor in that block.

Next, the definition of g immediately yields $g(B_P) = n_o(B_P)$ (computed in G'). If $w \in B_P$, then $d_{G'}(w) \in \{2, 3\}$, and the neighbors of w are distinct vertices of S' . If $d_{G'}(w) = 2$, or if $d_{G'}(w) = 3$ and w has two neighbors in one block of P , then let P' be the S' -partition formed from P by moving w into a block containing at least half of $N_{G'}(w)$. Regardless of whether $d_{G'}(w)$ is 2 or 3, we obtain $f_g(P') \leq f_g(P)$. Iterating this operation yields P minimizing f_g such that every vertex in B_P has neighbors in three different blocks of P , and $g(B_P) = |B_P|$. Hence also v has no neighbor in B_P .

We can now exclude $|P| = 1$. If $|P| = 1$, then $|S'| \geq 2$ implies $T_P = \emptyset$. Since vertices of

B_P must have neighbors in three blocks, also $B_P = \emptyset$. Hence $\delta(A_1) = 0$ and $f_g(P) = 0$.

To prove $f_g(P) \geq 0$ when $|P| > 1$, we need lower bounds on $\delta_{G'}(A_i)$. We obtain these using the λk -edge-connectedness of S in G_0 . Vertices of X are not in G' , but in G_0 they have exactly two neighbors in S' . For $x \in X$ and $j \in \{1, 2\}$, put $x \in X_j$ when $N(x) \cap V(G')$ intersects exactly j blocks in P ; thus X_1 and X_2 partition X . Add each vertex of X_1 to the block of P containing its neighbors, forming $A'_1, \dots, A'_{|P|}$ from $A_1, \dots, A_{|P|}$; we have $\delta_{G'}(A_i) = \delta_{G_0}(A'_i) - |[A'_i, X_2 \cup \{v\}]|$. Since S is λk -edge-connected in G_0 , its subset S' is also λk -edge-connected in G_0 . Since $|P| > 1$, we thus have $\delta_{G_0}(A'_i) \geq \lambda k$ for $1 \leq i \leq |P|$. Since each vertex of X_2 is adjacent to v and two vertices of S' , and v has no neighbor in B_P , in G_0 we have $\sum_{i=1}^{|P|} |[A'_i, X_2 \cup \{v\}]| = d_{G_0}(v) + |X_2|$. These computations yield

$$\begin{aligned} \sum_{A_i \in P} \delta_{G'}(A_i) &= \sum_{i=1}^{|P|} \left(\delta_{G_0}(A'_i) - |[A'_i, X_2 \cup \{v\}]| \right) \\ &\geq \lambda k |P| - d_{G_0}(v) - |X_2| = \lambda k(|P| - 1) - |X_2|. \end{aligned} \quad (3.5)$$

Also $3|B_P| = \delta_{G'}(B_P) \leq \sum_{A_i \in P} \delta_{G'}(A_i)$, so $g(B_P) \leq \frac{1}{3} \sum_{A_i \in P} \delta_{G'}(A_i)$. Using (3.5),

$$\sum_{A_i \in P} \delta_{G'}(A_i) - g(B_P) - 2k(|P| - 1) \geq \frac{2}{3} [(\lambda - 3)k(|P| - 1) - |X_2|]. \quad (3.6)$$

Now, to prove $f_g(P) \geq 0$, using the definition in (3.4) and applying (3.6), it suffices to prove

$$(\lambda - 3)k(|P| - 1) - |X_2| - 3g(T_P) \geq 0. \quad (3.7)$$

Our last preliminary computation bounds $|X_2|$. Since $X \subseteq \overline{S}$, vertices of X have no incident multi-edges. Hence $X \cap N_0 = \emptyset$, and we explicitly discarded u_1, \dots, u_k to form the sets comprising X . Hence E_0 and the k edges from v to U do not reach X . Since

$d_{G_0}(v) = \lambda k$, we conclude

$$|X_2| \leq |X| \leq (\lambda - 2)k. \quad (3.8)$$

Let $T'_P = \{s \in T_P : g(s) > 0\}$; note that $g(T'_P) = g(T_P)$. We complete the proof by considering four cases in terms of $|P|$ and $|T'_P|$, showing in each that $f_g(P) \geq 0$.

Case 1: $|P| = 2$ and $|T'_P| = 0$. Since $|P| < 3$, we have $B_P = \emptyset$. Using (3.5) and (3.8) instead of (3.7) yields $f_g(P) \geq \lambda k(|P| - 1) - (\lambda - 2)k - 2k(|P| - 1) = (\lambda - 2)k(|P| - 2) = 0$.

Case 2: $|T'_P| \leq |P| - 2$. We may assume $|P| \geq 3$. Let L denote the left side of (3.7). Using $g(s) = k - d_{M'}(s) - |E_0 \cap E(s)|$ for $s \in T'_P$, we have

$$L \geq (\lambda - 3)k(|P| - 2) + (\lambda - 2)k - |X_2| - k - 3k|T'_P| + 3 \sum_{s \in T'_P} d_{M'}(s)$$

If $|P| \geq 4$ and $|T'_P| \leq |P| - 2$, then (3.8) and $\lambda \geq 6.5$ yield $L \geq (\lambda - 6)k(|P| - 2) - k \geq 0$.

Hence we may assume $|P| = 3$. We obtain $L \geq (\lambda - 4)k \geq 0$ if $|T'_P| = 0$, so we may also assume $|T'_P| = 1$. Now let s be the one vertex of T'_P . The computation simplifies to

$$L \geq -0.5k + (\lambda - 2)k - |X_2| + 3d_{M'}(s).$$

Now $|X_2| \leq (\lambda - 2.5)k$ or $d_{M'}(s) \geq k/6$ suffices. If both fail, then $|[s, v]| \leq d_{G_0}(v) - |X_2| < 2.5k$ (since $d_{G_0}(v) = \lambda k$) and $|[s, X_2]| \leq d_M(s) \leq 2d_{M'}(s) + 1 < k/3 + 1$.

Now index the blocks of P so that $s \in A_1$. Focusing on A_1 , we compute

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 4k - |B_P| - 2g(s) \geq 2|[A_1, A_2 \cup A_3]| + 3|B_P| - 4k - |B_P| - 2k \\ &= 2\delta_{G'}(A_1) - 6k = 2(\delta_{G_0}(A_1) - |[s, X_2 \cup \{v\}]|) - 6k \\ &> 13k - 2(k/3 + 1 + 2.5k) - 6k > 0. \end{aligned}$$

Case 3: $|T'_P| = |P| - 1 \geq 1$. Each $x \in X_2$ has neighbors in S in two blocks of P ; hence

x has a neighbor in T'_P . Thus $\sum_{s \in T'_P} d_M(s) \geq |X_2|$. Also, $g(T'_P) \leq k|T'_P| - \sum_{s \in T'_P} d_{M'}(s)$. Starting again from L , the left side of (3.7), and using $\lambda k \geq 6k + 1$, we have

$$\begin{aligned} L &\geq (\lambda - 3)k(|P| - 1) - |X_2| - 3k(|P| - 1) + 3 \sum_{s \in T'_P} d_{M'}(s) \\ &\geq (|P| - 1) + \sum_{s \in T'_P} d_{M'}(s) + \sum_{s \in T'_P} (d_M(s) - 1) - |X_2| \geq 0. \end{aligned}$$

Case 4: $|T'_P| = |P| \geq 2$. Here $T'_P = S'$, and each block of P contains just one vertex of S' , so $X_1 = \emptyset$ and $X = X_2$. Also, $d_{M'}(T'_P) = d_{M'}(S') = d_{M'}(X) = |X|$. Hence $g(T'_P) = k|P| - |X| - |[v, S'] \cap E_0|$.

We need to strengthen the lower bound on $\sum_{A_i \in P} \delta_{G'}(A_i)$ and upper bound on $|B_P|$ used in (3.5). Let $W = \{w \in \bar{S} : vw \in E_0\}$. Note that $|[v, W]| = |W|$, since $W \subseteq \bar{S}$. If $w \in W \cap A_i$, then w is adjacent to the vertex of S' in A_i (by our initial reduction of P) and to a vertex of S' in another block A_j (by Lemma 3.4.5). Hence $\delta_{G'}(A_i) = \delta_{G'}(A_i - W)$. Since $|[S', X]| = 2|X|$, and $X \subseteq N(v)$, and S' is λk -edge-connected in G_0 , we have

$$\begin{aligned} \sum_{A_i \in P} \delta_{G'}(A_i) &= \sum_{A_i \in P} (\delta_{G_0}(A_i - W) - |[A_i - W, X \cup \{v\}]|) \\ &\geq \lambda k|P| - d(v) - |X| + |W| = \lambda k(|P| - 1) - |X| + |W|. \end{aligned}$$

Each vertex of B_P supplies three of the edges leaving blocks of P , but not the edges leaving blocks of P to or from vertices of W ; hence $3|B_P| \leq (\sum_{A_i \in P} \delta_{G'}(A_i)) - 2|W|$. Now

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - |B_P| - 2k(|P| - 1) - 2g(T'_P) \\ &\geq \frac{2}{3}(\lambda k(|P| - 1) - |X|) + \frac{4}{3}|W| - 2k(|P| - 1) - 2k|P| + 2|X| + 2|[v, S'] \cap E_0| \\ &= \left(\frac{2}{3}\lambda - 4\right)k(|P| - 1) + \frac{4}{3}|W| + 2|[v, S'] \cap E_0| + \frac{4}{3}|X| - 2k \\ &\geq \frac{1}{3}k(|P| - 1) + \frac{4}{3}k - 2k. \end{aligned}$$

In the last step, we used $|W| + |[v, S'] \cap E_0| = |E_0| \geq k$, along with $\lambda \geq 6.5$ and $|X| \geq 0$. The final expression is nonnegative when $|P| \geq 3$.

This leaves the case $|T'_P| = |P| = 2$. As in Case 2, $B_P = \emptyset$, and we have

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 2k - 2g(T'_P) \\ &\geq \lambda k - |X| + |W| - 2k - 4k + 2|X| + 2|[v, S'] \cap E_0| > 0. \end{aligned}$$

□

3.5 S -connector packing

To prove Theorem 1.2.4, we prove a theorem for S -connectors analogous to Theorem 3.4.1. Note that Theorem 3.5.1 immediately yields Theorem 1.2.4 in the way that Theorem 3.4.1 yields Theorem 1.2.3, by applying it to a graph obtained from the given graph by adding one vertex. The difference from Theorem 3.4.1 is that, because we seek connectors instead of trees in (3) and (4), the threshold we need in (2) is $2k$ instead of k . This leads to the later computations needing $\lambda \geq 10$ instead of $\lambda \geq 6.5$.

Theorem 3.5.1. *Fix $k \in \mathbb{N}$ and $\lambda k \in \mathbb{N}$ such that $\lambda \geq 10$. Consider $S \subseteq V(G)$ and $v \in S$ such that S is λk -edge-connected in G and $d_G(v) = \lambda k$. If E_0, \dots, E_k is a partition of $E(v)$ such that $|E_0| \geq 2k$, then there exist edge-disjoint subgraphs H_0, \dots, H_k such that*

- (1) $E_i \subseteq E(H_i)$;
- (2) $d_{H_0}(s) \geq 2k$ for any $s \in S$;
- (3) For $i > 0$, H_i is an S -connector; and
- (4) For $i > 0$, deleting from the family of paths forming H_i the paths that use edges of E_i leaves an $(S - v)$ -connecting family.

The proof of Theorem 3.5.1 is similar to the proof of Theorem 3.4.1; we describe the differences without repeating the full argument.

As in Section 3.4, we consider a minimal counterexample G_0 . The arguments of Lemmas 3.4.3 and 3.4.5 show that the non-terminal vertices in G_0 form an independent set in which every vertex has degree 3, with three distinct neighbors in S . This time we do not choose special vertices u_1, \dots, u_k . With $S' = S - \{v\}$, $N'_i = N_i - S'$, and $X = \bigcup_{i=1}^l N'_i$, we let M be the maximal bipartite subgraph of G_0 with partite sets X and S' . The argument of Lemma 3.4.7 yields the subgraph M' such that $d_{M'}(x) = 1$ for $x \in X$ and $d_{M'}(s) \geq \lfloor d_M(s)/2 \rfloor$ for $s \in S$.

Again let $G' = G_0 - v - X$. This time we define a slightly different S' -parity function on G' : there is no set U , and for $u \in S'$ we replace k with $2k$ in the definition.

$$g(u) = \begin{cases} 0, & u \in N_0 - S'; \\ 1, & u \in \bar{S} - N_{G_0}(v); \\ \max\{2k - d_{M'}(u) - |E(u) \cap E_0|, 0\}, & u \in S'. \end{cases} \quad (3.9)$$

We reduce the problem to showing that G' has a (k, g) -family for S' and this g , by proving as in Lemma 3.4.9 that E_0, \dots, E_k extend in G_0 as specified in Theorem 3.5.1 when G' has a (k, g) -family with g as in (3.9). This time the reduction is easier, since we have no chosen vertices u_1, \dots, u_k to complicate the construction.

Lemma 3.5.2. *If the graph G' derived from G_0 has a (k, g) -family for the S' -parity function g defined by (3.9), then E_0, \dots, E_k extend in G_0 as specified in Theorem 3.5.1.*

Proof. Given a (k, g) -family in G' , let H'_1, \dots, H'_k be the S' -connectors and $P_1, \dots, P_{g(V(G'))}$ be the oriented paths. Constructing H_i by augmenting E_i yields (1) in Theorem 3.5.1.

Let H_0 be the spanning subgraph of G with edge set $E_0 \cup E(M') \cup \bigcup_{j=1}^{g(V(G))} E(P_j)$. For $1 \leq i \leq k$, let H_i be the spanning subgraph of G with edge set $E_i \cup E(H'_i) \cup B_i$, where B_i is the set of edges in $E(M) - E(M')$ incident to N'_i .

For (3), note for $1 \leq i \leq k$ that $E_i \cup B_i$ is a nonempty set of paths that join v to vertices

of S' . We do not require H_0 to be an S -connector.

For (4), when we delete the paths formed by $E_i \cup B_i$, we return to H'_i , which is an S' -connector in G' and hence is an $(S' - \{v\})$ -connector in $G - v$.

For (2), we check that H_0 gains enough edges at each vertex of S' . For $s \in S'$, in H'_0 there are at least $g(s)$ edges incident to s , provided explicitly by the paths in the (k, g) -family. Adding $E_0 \cap E(s)$ and the edges of M' yields $d_{H_0}(s) \geq g(s) + |E_0 \cap E(s)| + d_{M'}(s) \geq 2k$. Also $d_{H_0}(v) \geq 2k$, since $|E_0| \geq 2k$. \square

Finally, we prove the analogue of Lemma 3.4.10.

Lemma 3.5.3. *Given G_0 , the derived graph G' has a (k, g) -family for the S' -parity function g defined by (3.9).*

Proof. By Theorem 3.1.2, it suffices to prove that the SPC holds for G' and g . That is, $f_g(P) \geq 0$ for each S' -partition of G' , where

$$f_g(P) = \sum_{A_i \in P} \delta(A_i) - 2k(|P| - 1) - g(B_P) - 2g(T_P).$$

As in Lemma 3.4.10, we may assume that every vertex of \bar{S} has degree 3 in G_0 , that every vertex outside S' in a block of P has a neighbor in that block, that every vertex in B_P has neighbors in three different blocks of P , and that $g(B_P) = |B_P|$. Similarly, vertices of X have exactly two neighbors in S' . Again let X_2 be the subset of X whose vertices having neighbors in distinct blocks of P . Arguing exactly as in Lemma 3.4.10 yields (3.5), (3.6), (3.7), (3.8), except that now we use $|[v, N_0]| = E_0 \geq 2k$ instead of $|[v, N_0 \cup U]| \geq 2k$, since there is no U and instead we increased the requirement on $|E_0|$ to $2k$.

There remain only the computations in the Cases. Again let $T'_P = \{s \in T_P : g(s) > 0\}$. The computations for $|P| = 1$ and Case 1 ($|T'_P| = |P| - 2 = 0$) are unchanged.

Case 2: $|T'_P| \leq |P| - 2$ and $|P| \geq 3$. Again let L be the left side of (3.7). Define an

S -parity function by $g(v) = 1$ when v is a vertex of \overline{S} having odd degree in G and otherwise $g(v) = 0$. For $P \in \mathcal{P}(S)$, always $B_P \subseteq \overline{S}$, and hence $g(B_P) = n_o(B_P)$. Also, $g(T_P) = 0$. Hence the left side of the assumed equality is $f_g(P)$, and we have assumed that the SPC holds for this S -parity function. By Theorem 3.1.2, G has a (k, g) -family, and hence there are k edge-disjoint S -connectors. Using (3.8) and $\lambda \geq 10$ and $g(T'_P) \leq 2k|T'_P|$,

$$L/k \geq (\lambda - 3)(|P| - 2) - 1 - 6|T'_P| \geq -1 + (|P| - 2) + 6(|P| - 2 - |T'_P|) \geq 0.$$

Case 3: $|T'_P| = |P| - 1 \geq 1$. With $g(T'_P) \leq 2k|T'_P| - \sum_{s \in T'_P} d_{M'}(s)$ and $\lambda k \geq 10k \geq 9k + 1$, the computation becomes

$$\begin{aligned} L &\geq (\lambda - 3)k(|P| - 1) - |X_2| - 6k(|P| - 1) + 3 \sum_{s \in T'_P} d_{M'}(s) \\ &\geq (|P| - 1) + \sum_{s \in T'_P} (d_{M'}(s) - 1) + \sum_{s \in T'_P} d_M(s) - |X_2| \geq 0. \end{aligned}$$

Case 4: $|T'_P| = |P| \geq 2$. As in Case 4 of Lemma 3.4.10, the computation starts with $\sum_{A_i \in P} \delta_{G'}(A_i) \geq \lambda k(|P| - 1) - |X| + |W|$ and $3|B_P| \leq (\sum_{A_i \in P} \delta_{G'}(A_i)) - 2|W|$. It ends with

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - |B_P| - 2k(|P| - 1) - 2g(T'_P) \\ &\geq \frac{2}{3}(\lambda k(|P| - 1) - |X|) + \frac{4}{3}|W| - 2k(|P| - 1) - 4k|P| + 2|X| + 2|[v, S'] \cap E_0| \\ &= \left(\frac{2}{3}\lambda - 6\right)k(|P| - 1) + \frac{4}{3}|W| + 2|[v, S'] \cap E_0| + \frac{4}{3}|X| - 4k \\ &\geq \frac{2}{3}k(|P| - 1) + \frac{8}{3}k - 4k. \end{aligned}$$

In the last step, we used $|W| + |[v, S'] \cap E_0| \geq |E_0| \geq 2k$, along with $\lambda \geq 10$ and $|X| \geq 0$. The final expression is nonnegative when $|P| \geq 3$.

This leaves the case $|T'_P| = |P| = 2$. As in Case 2, $B_P = \emptyset$, and $\lambda \geq 10$ is enough to give

$$\begin{aligned} f_g(P) &= \sum_{A_i \in P} \delta_{G'}(A_i) - 2k - 2g(T'_P) \\ &\geq \lambda k - |X| + |W| - 2k - 8k + 2|X| + 2|[v, S'] \cap E_0| \geq 0. \end{aligned}$$

□

Chapter 4

Extremal Problems for Decomposition of Graphs

In this chapter, we consider decomposition of sparse graphs into $k + 1$ subgraphs, where the first k are forests and the last subgraph has bound degree. We will give results relevant to the Nine Dragon Tree Conjecture and its weaker version which does not require the d -bounded graph to be forest.

Conjecture 1.3.1 (NDT Conjecture). *If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into k forests plus one d -bounded forest.*

Conjecture 1.3.2 (Weak NDT Conjecture). *If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into k forests plus one d -bounded subgraph.*

Our model of “graph” in this chapter allows multi-edges but no loops.

Recall that the fractional arboricity $\text{Arb}(G)$ is defined by $\text{Arb}(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$, and the maximum average degree $\text{Mad}(G)$ is defined by $\text{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. To compute $\text{Arb}(G)$ or $\text{Mad}(G)$, it suffices to perform the maximization only over induced subgraphs. Letting $G[A]$ denote the subgraph of G induced by a vertex set A , we write $\|A\|$ for the number of edges in $G[A]$ (and $|A|$ for the number of vertices). We restate the conditions as integer inequalities and introduce an intermediate condition called (k, d) -sparseness. Since $k(k + d + 1) + d = (k + 1)(k + d)$, we have the following comparison,:

Condition	Equivalent constraint (when imposed for all $A \subseteq V(G)$)
$\text{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k+1)(k+d) A - (k+d+1)\ A\ - (k+1)(k+d) \geq 0$
$\text{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k+1)(k+d) A - (k+d+1)\ A\ - 1 \geq 0$
(k, d) -sparseness	$(k+1)(k+d) A - (k+d+1)\ A\ - k^2 \geq 0$

Since $(k+1)(k+d) > k^2 \geq 1$, the condition on $\text{Arb}(G)$ implies (k, d) -sparseness, which in turn implies the condition on $\text{Mad}(G)$. By showing that (k, d) -sparseness suffices, Theorem 1.3.3 thus implies that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ suffices for G to be (k, d) -decomposable, but $\text{Mad}(G) < 2k + \frac{2d}{k+d+1}$ might not. However, since $k^2 = 1$ when $k = 1$, the $(1, d)$ -sparseness condition is the same as the desired condition $\text{Mad}(G) < 4 - \frac{4}{d+2}$ for the problem in [30].

In Section 4.1, we give the proof of Theorem 1.3.3, which implies the case $d > k$ of the Weak NDT Conjecture.

Theorem 1.3.3. *For $d > k$, every (k, d) -sparse graph is (k, d) -decomposable. Furthermore, the condition is sharp.*

In Section 4.2, we give the proof of Theorem 1.3.4, which implies the case $d = k + 1$ of the NDT Conjecture.

Theorem 1.3.4. *For $d \leq k + 1$, if $\text{Arb}(G) \leq k + \frac{d}{2k+2}$, then G is $(k, d)^*$ -decomposable.*

In Sections 4.3–4.5, we prove the NDT Conjecture for $(k, d) = (1, d)$ with $d \leq 6$, in a form that requires only (k, d) -sparseness as long as small graphs violating $\text{Arb}(G) \leq k + 1$ are forbidden. Meanwhile, the stronger version of the NDT Conjecture asserts that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ guarantees a $(k, d)^*$ -decomposition in which every component of the d -bounded forest has at most d edges. We prove this for $(k, d) = (1, 2)$ in Section 4.6 (the result of [31] implies it for $(k, d) = (1, 1)$). The results of Sections 4.3–4.6 use reducible configurations and discharging.

4.1 (k, d) -decomposition for $d > k$

We begin with a general example showing that Theorem 1.3.3 is sharp. This example also motivates the constant in the condition for (k, d) -sharpness. In studying (k, d) -decomposability of a graph G , define $\beta(A) = (k + 1)(k + d)|A| - (k + 1 + d)\|A\| - k^2$ for $A \subseteq V(G)$. The (k, d) -sparseness condition is that $\beta(A) \geq 0$ for all nonempty A .

Example 4.1.1. We construct a bipartite graph G with partite sets X and Y of sizes s and t , respectively. Let $s = t(k + d) - k + 1$, so $|V(G)| = t(k + d + 1) - k + 1$. With $X = \{x_1, \dots, x_s\}$ and $Y = \{y_1, \dots, y_t\}$, make x_i adjacent to y_i, \dots, y_{i+k} , where indices are taken modulo t . Every vertex in X has degree $k + 1$, so $|E(G)| = (k + 1)(k + d)t - k^2 + 1$.

A d -bounded subgraph of G has at most dt edges. Deleting a d -bounded subgraph thus leaves at least $k(k + d)t + kt - k^2 + 1$ edges. However, k forests in G cover at most $k[t(k + d + 1) - k]$ edges. Hence G is not (k, d) -decomposable.

On the other hand, G just barely fails to be (k, d) -sparse. If $|A| = 1$, then $\beta(A) = kd + k + d$. Now choose A to minimize $\beta(A)$ among subsets of $V(G)$ with size at least 2. If some vertex $v \in A$ has at most k neighbors in A , then $\beta(A - v) \leq \beta(A) - d$, contradicting the choice of A . Therefore, all $k + 1$ neighbors of each vertex in $A \cap X$ are also in A . Let $s' = |A \cap X|$ and $t' = |A \cap Y|$. Now

$$\begin{aligned} \beta(A) &= (k + 1)(k + d)(s' + t') - (k + d + 1)(k + 1)s' - k^2 \\ &= (k + 1)(k + d)t' - s'(k + 1) - k^2 = (k + 1)[(k + d)t' - s' - k + 1] - 1. \end{aligned}$$

We conclude that $\beta(A) \geq 0$ if and only if $s' \leq (k + d)t' - k$. When $t' = t$, this yields $\beta(A) < 0$ if and only if $A = V(G)$.

If $t' < t$, then each vertex of $Y - A$ forbids all its neighbors from A . For fixed t' , we maximize s' and minimize $\beta(A)$ for such A by letting $Y \cap A = \{y_1, \dots, y_{t'}\}$ (this makes the

forbidden subsets of X overlap as much as possible). Writing $i = qt + r$ with $q \geq 0$ and $1 \leq r \leq t$, this allows $x_i \in A$ only when $1 \leq r \leq t' - k$. With $s = t(k + d) - k + 1$, we have $s' \leq (k + d)(t' - k) < (k + d)t' - k$.

We conclude that $\beta(A) \geq 0$ except when $A = V(G)$. The choice of the constant k^2 in the definition of β has enabled us to construct a graph that fails to be (k, d) -decomposable with the slightest possible failure of (k, d) -sparseness. \square

We prove Theorem 1.3.3 in a seemingly more general form to facilitate the inductive proof, but we will show at the end of this section that the more general form is equivalent to Theorem 1.3.3. Prior results in this area have been proved by the discharging method, which uses properties of a minimal counterexample G to contradict the hypothesized sparseness. Replacing the constant bound d on vertex degrees by an individual bound for each vertex permits a simple inductive proof without using discharging.

Definition 4.1.2. Fix positive integers d and k . A *capacity function* on a graph G is a function $f: V(G) \rightarrow \{0, \dots, d\}$. A (k, f) -*decomposition* of G decomposes it into k forests and a graph D such that each vertex v has degree at most $f(v)$ in D . For each vertex set A in G , let

$$\beta_f(A) = (k + 1) \sum_{v \in A} (k + f(v)) - (k + d + 1) \|A\| - k^2.$$

A capacity function f on G is *feasible* if $\beta_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$.

The idea is to reserve an edge uv for use in D by deleting it and reducing the capacity of its endpoints (when both have positive capacity). If the reduced function f' is feasible on $G - uv$, then the induction hypothesis will complete a (k, f) -decomposition. We will use this idea to reduce to the case where the vertices with positive capacity form an independent set.

To prove feasibility for f' , we must show $\beta_{f'}(A) \geq 0$ for $A \neq \emptyset$. The endpoints of the deleted edge may be both outside A (no problem), both in A (still easy), or just one in A . The latter case is problematic when $\beta_f(A) \leq k$, since $\beta_{f'}(A) = \beta_f(A) - (k + 1)$. In this

situation we will assemble a (k, f) -decomposition inductively by combining a decomposition of $G[A]$ with a decomposition of the subgraph obtained by contracting A to one vertex. We begin with the definitions and lemmas needed to do that.

Definition 4.1.3. For $B \subseteq V(G)$, let G_B denote the graph obtained by contracting B into a new vertex z . The degree of z in G_B is the number of edges joining B to $V(G) - B$ in G ; edges of G with both endpoints in B disappear.

Lemma 4.1.4. *If f is a feasible capacity function on G , and B is a proper subset of $V(G)$ such that $|B| \geq 2$ and $\beta_f(B) \leq k$, then f^* is a feasible capacity function on G_B , where $f^*(z) = 0$ and f^* agrees with f on $V(G) - B$.*

Proof. For $A \subseteq V(G_B)$, we have $\beta_{f^*}(A) = \beta_f(A) \geq 0$ if $z \notin A$. When $z \in A$, we compute $\beta_{f^*}(A)$ by comparison with $\beta_f(A')$, where $A' = (A - \{z\}) \cup B$. Every edge in $G[A']$ appears in $G_B[A]$ or $G[B]$; hence the edges contribute the same to both sides of the equation below. Comparing the terms for constants and the terms for vertices (using $f^*(z) = 0$) yields

$$\beta_f(A') = \beta_{f^*}(A) - (k + 1)k + \beta_f(B) + k^2.$$

If $\beta_f(B) \leq k$, then $\beta_{f^*}(A) \geq \beta_f(A') \geq 0$. □

Lemma 4.1.5. *Let f be a capacity function on a graph G , and let B be a proper subset of $V(G)$. If $G[B]$ is $(k, f|_B)$ -decomposable and G_B is (k, f^*) -decomposable, where f^* is defined from f as in Lemma 4.1.4, then G is (k, f) -decomposable.*

Proof. Let (F, D) be a $(k, f|_B)$ -decomposition of $G[B]$, where F is the union of k forests. Let (F', D') be a (k, f^*) -decomposition of G_B , where F' is the union of k forests. Each edge of G is in $G[B]$ or G_B , becoming incident to z in G_B if it joins B to $V(G) - B$ in G . View $(F \cup F', D \cup D')$ as a decomposition of G by viewing the edges incident to z in F' as the corresponding edges in G .

The resulting decomposition is a (k, f) -decomposition of G . Since $f^*(z) = 0$, vertex z has degree 0 in D' , and all edges joining B to $V(G) - B$ lie in F' . Hence the restrictions from f are satisfied by $D \cup D'$. For each forest F_i among the k forests in F , its union with the corresponding forest F'_i in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F'_i when viewed as a forest in G' . \square

Theorem 4.1.6. *If $d > k$ and G is a graph with a feasible capacity function f , then G is (k, f) -decomposable.*

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most k edges. For the induction step, suppose that G is larger.

If $\beta_f(B) \leq k$ for some proper subset B of $V(G)$ with $|B| \geq 2$, then the capacity function f^* on G_B that agrees with f except for $f^*(z) = 0$ is feasible, by Lemma 4.1.4. Since $G[B]$ is an induced subgraph of G , the restriction of f to B is feasible on $G[B]$. Since G_B and $G[B]$ are smaller than G , by the induction hypothesis G_B is $(k, f|_B)$ -decomposable and G_B is (k, f^*) -decomposable. By Lemma 4.1.5, G is $(k\mathcal{F}, \mathcal{D}_f)$ -decomposable.

Hence we may assume that $\beta_f(B) \geq k+1$ for all such B . Let $S = \{v \in V(G) : f(v) > 0\}$. If S has adjacent vertices u and v , then let f' be the capacity function on $G - uv$ that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. If f' is feasible, then since $G - uv$ is smaller than G , it has a (k, f') -decomposition, and we add uv to the degree-bounded subgraph to obtain a (k, f) -decomposition of G .

To show that f' is feasible, consider $A \subseteq V(G') = V(G)$. If $u, v \notin A$, then $\beta_{f'}(A) = \beta_f(A)$. If $u, v \in A$, then the reduction in f and loss of one edge yield $\beta_{f'}(A) = \beta_f(A) - 2(k+1) + (k+d+1) \geq \beta_f(A)$, where the last inequality uses $d > k$. If exactly one of $\{u, v\}$ is in A , then A is a proper subset of $V(G)$. If $|A| \geq 2$, then $\beta_{f'}(A) = \beta_f(A) - (k+1) \geq 0$. If $|A| = 1$, then $\beta_{f'}(A) \geq k$, since G' has no loops.

Hence we may assume that S is independent. In this case, we show that G decomposes into k forests, yielding a (k, f) -decomposition of G in which the last graph has no edges. If

$\Upsilon(G) > k$, then $V(G)$ has a minimal subset A such that $\|A\| \geq k(|A| - 1) + 1$ (note that $|A| \geq 2$). By this minimality, every vertex of A has at least $k + 1$ neighbors in A . Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k + 1)|A'|$. Taking $k + 1$ times the first lower bound on $\|A\|$ plus d times the second yields

$$(k + 1 + d) \|A\| \geq (k + 1)k(|A| - 1) + (k + 1) + d(k + 1)|A'|.$$

Now we compute

$$\begin{aligned} \beta_f(A) &= (k + 1)k|A| + (k + 1) \sum_{v \in A'} f(v) - (k + d + 1) \|A\| - k^2 \\ &\leq (k + 1)k|A| + (k + 1)d|A'| - (k + 1)k(|A| - 1) - (k + 1) - d(k + 1)|A'| - k^2 \\ &= (k + 1)k - (k + 1) - k^2 = -1. \end{aligned}$$

This contradicts the feasibility of f , and hence the desired decomposition of G exists. \square

The generality of the capacity function facilitates the inductive proof, and the desired statement about (k, d) -decomposition is a special case, but in fact the special case with capacity d for all v implies the general statement, making Theorem 4.1.6 and Theorem 1.3.3 equivalent. The equivalence uses the notion of “ghost” that will be helpful in Sections 4.3.

Definition 4.1.7. When considering (k, d) -decomposition, a *ghost* is a vertex of degree $k + 1$ having only one neighbor (via all $k + 1$ incident edges). A neighbor of v that is a ghost is a *ghost neighbor* of v .

Proposition 4.1.8. *Theorem 1.3.3 implies Theorem 4.1.6.*

Proof. Assume Theorem 1.3.3 and consider a feasible capacity function f on G . Form G' by giving $d - f(v)$ ghost neighbors to each vertex v .

We claim that G' is (k, d) -sharp. Adding a ghost neighbor of a vertex in a set adds 1 to the size of the set and $k + 1$ to the number of edges induced. Hence it changes the value of β by $(k + 1)(k + d) - (k + d + 1)(k + 1)$, which equals $-(k + 1)$. It therefore suffices to prove that $\beta(A') \geq 0$ for subsets A' of $V(G')$ that contain all the ghost neighbors of their vertices. Let $A = A' \cap V(G)$. Counting the increase in capacity from $f(v)$ to d and the cost of the ghost neighbors, we have

$$\beta(A') = \beta_f(A) + (k + 1) \sum_{v \in A} (d - f(v)) - \sum_{v \in A} (k + 1)(d - f(v)) = \beta_f(A) \geq 0,$$

where the last inequality holds because f is feasible. By Theorem 1.3.3, G' has a (k, d) -decomposition. Deleting the ghost vertices yields a (k, f) -decomposition of G . \square

In essence, we have shown that ghosts have the same effect as reduced capacity on the existence of decompositions.

4.2 $(k, d)^*$ -decomposition for $d \leq k + 1$

The capacity function f in Section 4.1 does the job of controlling vertex degrees to facilitate inductive construction of a (k, d) -decomposition. However, it cannot control the creation of cycles when we return a deleted edge to a decomposition satisfying reduced capacity. To do this, we impose another condition on the decomposition.

Definition 4.2.1. A *strong $(k, f)^*$ -decomposition* is a $(k, f)^*$ -decomposition in which each component of the degree-bounded forest contains at most one vertex v such that $f(v) < d$.

The strong decomposition condition will control the introduction of cycles. We will apply the induction hypothesis to $G - uv$ with reduced capacity function f' only when at least one endpoint of uv has capacity d . In $G - uv$, both endpoints have capacity less than d and will be the only such vertex in their components in D , so they will be in different components.

We can thus add uv to D ; since one endpoint returns to capacity d , the strong condition continues to hold. This inductive approach will allow us to assume that no edge joins a vertex with capacity d to a vertex with positive capacity. For such graphs, the hypotheses will yield a decomposition into k forests, as in the final step of Theorem 1.3.3.

We will also need to strengthen the sparseness condition; feasibility of f is not sufficient. For example, if G consists of two vertices and an edge of multiplicity $k + 2$, and $f(u) = f(v) = d$, then $\beta(A) \geq 0$ for all A , but G does not decompose into $k + 1$ forests. We will need another auxiliary function that excludes such examples. Also, in order to impose a stronger sparseness condition, we introduce a modified version of β_f .

Definition 4.2.2. Given a capacity function f on $V(G)$ using capacities at most d , let $S = \{v \in V(G) : f(v) = d\}$. For $A \subseteq V(G)$, let $f(A) = \sum_{v \in A} f(v)$ and $\hat{f}(A) = \min\{f(x) : x \in A\}$. Define α_f and β_f^* on subsets of G as follows:

$$\alpha_f(A) = k|A| - k - \|A\| + |A \cap S|,$$

$$\beta_f^*(A) = (2k + 2 - d)k|A| + (k + 1)[f(A) - 2\|A\|] - (k - 1)(2k + 2 - d).$$

Say that f is *strongly feasible* when $\beta_f^*(A) > 0$ and $\alpha_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$, with $\alpha_f(A) > 0$ whenever $A \subseteq S$.

With these definitions, we can state the main result of this section.

Theorem 4.2.3. *If $d \leq k + 1$ and f is a strongly feasible capacity function on a graph G , then G has a strong $(k, f)^*$ -decomposition.*

Once again the sparseness condition is motivated by and weaker than the desired fractional arboricity condition. The condition $\text{Arb}(G) \leq k + \frac{d}{2k+2}$ is equivalent to

$$(2k + 2 - d)k|A| + (k + 1)[d|A| - 2\|A\|] - k(2k + 2) - d \geq 0 \quad \text{for } A \subseteq V(G).$$

When $f(v) = d$ for all v , this is the same as $\beta_f^*(A) \geq 0$, except that we are subtracting $k(2k+2) + d$ instead of $(k-1)(2k+2-d)$.

Corollary 4.2.4. $\text{Arb}(G) \leq k + \frac{d}{2k+2}$ guarantees $(k, d)^*$ -decomposability. In particular, the NDT Conjecture holds when $d = k + 1$.

Proof. Since the constant subtracted in the inequality for $\text{Arb}(G)$ is larger, $\text{Arb}(G) \leq k + \frac{d}{2k+2}$ implies $\beta_f^*(A) > 0$ for all A when $f(v) = d$ for all v . With this capacity function, $|A \cap S| = |A|$ for all $A \subseteq V(G)$, and the condition $\alpha_f(A) \geq 1$ (since $A \subseteq S$) becomes $\|A\| \leq (k+1)(|A| - 1)$, true for all A when $\text{Arb}(G) < k + 1$. Hence Theorem 4.2.3 applies.

When $d = k + 1$, we have $d + k + 1 = 2k + 2$, and $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ is sufficient. \square

The condition on α_f is necessary for a strong $(k, f)^*$ -decomposition. Nonnegativity of $\alpha_f(A)$ states that A has at most $|A \cap S|$ edges plus the number that k forests can absorb. Each vertex of A in S permits one more edge in a degree-bounded forest D , by allowing an edge joining two components. If $A \subseteq S$, then we reach the allowable spanning tree in $G[A]$ before the last vertex, so the requirement must increase to $\alpha(A) \geq 1$ when $A \subseteq S$.

We prove a useful bound on β_f^* in terms of α_f .

Lemma 4.2.5. For a capacity function f on a graph G and a set $A \subseteq V(G)$ with $|A| \geq 2$,

$$\beta_f^*(A) \leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S| + 1].$$

In particular, if $\alpha_f(A) \leq 0$ and $\beta_f^*(A) > 0$ with $A \not\subseteq S$, then $f(x) \geq |A \cap S|$ for all $x \in A$.

Proof. Substituting $\|A\| = k|A| - k - \alpha_f(A) + |A \cap S|$ into the formula for $\beta_f^*(A)$ yields

$$\beta_f^*(A) = -dk|A| + (k+1)[2\alpha_f(A) + f(A) - 2|A \cap S|] + (2k+2) + d(k-1).$$

Summing capacities over $x \in A$ yields $f(A) \leq (d-1)|A| + |A \cap S| + \hat{f}(A) - (d-1)$ (the inequality is strict when $A \subseteq S$). Substituting this into the formula above yields

$$\begin{aligned} \beta_f^*(A) &\leq -dk|A| + (k+1)(d-1)|A| + (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + 3k + 3 - 2d \\ &= (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + (d-k-1)|A| + 3k + 3 - 2d \\ &\leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + k + 1, \end{aligned}$$

where the last inequality uses $|A| \geq 2$. □

We need an analogue of Lemma 4.1.4, with G_B as defined there.

Lemma 4.2.6. *For $d \leq k+1$, let f be a strongly feasible capacity function on G , and let B be a proper subset of $V(G)$ with $|B| \geq 2$. Define f^* and \bar{f} on G_B by $f^*(z) = \hat{f}(B) - |B \cap S|$ and $\bar{f}(z) = 0$, letting both functions agree with f on $V(G) - B$. If $\alpha_f(B) = 0$, then f^* is strongly feasible. If $\beta_f^*(B) \leq k+1$, then \bar{f} is strongly feasible.*

Proof. First consider the case $\alpha_f(B) = 0$. As observed in Lemma 4.2.5, $\hat{f}(B) \geq |B \cap S|$ when $\alpha_f(B) = 0$. Hence $f^*(z) \geq 0$, so f^* is a capacity function. Since f is strongly feasible and $\alpha_f(B) = 0$, we have $\beta_f^*(B) > 0$ and $B \not\subseteq S$. Since $\hat{f}(B) = d$ only if $B \subseteq S$, we must have $f^*(z) < d$, so the set S is the same for f^* and f .

If $z \notin A \subseteq V(G_B)$, then $\beta_{f^*}^*(A) = \beta_f^*(A)$ and $\alpha_{f^*}(A) = \alpha_f(A)$. When $z \in A$, we compute $\alpha_{f^*}(A)$ and $\beta_{f^*}^*(A)$ from $\alpha_f(A')$ and $\beta_f^*(A')$, where $A' = (A - \{z\}) \cup B$. As in Lemma 4.1.4, $|A'| = |A| - 1 + |B|$ and $\|A'\| = \|A\| + \|B\|$, where $\|A\|$ counts edges in G_B . Hence

$$\begin{aligned} \alpha_f(A') &= \alpha_{f^*}(A) + \alpha_f(B); \\ \beta_f^*(A') &= \beta_{f^*}^*(A) + \beta_f^*(B) - (k+1)f^*(z) - (2k+2-d). \end{aligned}$$

Since $\alpha_f(B) = 0$, we obtain $\alpha_{f^*}(A) = \alpha_f(A') \geq 0$, as desired since $f^*(z) < d$. By

Lemma 4.2.5, $\alpha_f(B) = 0$ implies $\beta_f^*(B) \leq (k+1)[\hat{f}(B) - |B \cap S| + 1] = (k+1)f^*(z) + (k+1)$.

Now $\beta_{f^*}^*(A) \geq \beta_f^*(A') + k + 1 - d \geq \beta_f^*(A') > 0$.

For \bar{f} , again it suffices to check A with $z \in A \subseteq V(G_B)$ and let $A' = (A - \{z\}) \cup B$. Now

$$\beta_{\bar{f}}^*(A') = \beta_{\bar{f}}^*(A) + \beta_{\bar{f}}^*(B) - (2k + 2 - d) \leq \beta_{\bar{f}}^*(A),$$

where we have used $\beta_{\bar{f}}^*(B) \leq k + 1$, $\bar{f}(z) = 0$, and $k + 1 - d \geq 0$. We also need $\alpha_{\bar{f}}(A) \geq 0$.

With $\beta_{\bar{f}}^*(A) \geq \beta_{\bar{f}}^*(A') > 0$ and $\bar{f}(z) = 0$, this follows from Lemma 4.2.5. \square

Lemma 4.2.7. *Let f be a capacity function on G , and let B be a proper subset of $V(G)$ with $|B| \geq 2$. If $G[B]$ is strongly $(k, f|_B)^*$ -decomposable and G_B is strongly $(k, f^*)^*$ -decomposable, with f^* defined from f as in Lemma 4.2.6, then G is strongly $(k, f)^*$ -decomposable.*

Proof. Let (F, D) be a strong $(k, f|_B)^*$ -decomposition of $G[B]$, and let (F', D') be a strong $(k, f^*)^*$ -decomposition of G_B , where F and F' are unions of k forests. Each edge of G is in $G[B]$ or G_B , becoming incident to z in G_B if it joins B to $V(G) - B$ in G . Viewing F' and D' as subgraphs of G , we show that $(F \cup F', D \cup D')$ is a strong $(k, f)^*$ -decomposition of G .

As in Lemma 4.1.5, the union of any forest F_i in F with the corresponding forest F'_i in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F'_i when viewed as a forest in G' . This argument applies also to $D \cup D'$.

Recall that $S = \{v \in V(G) : f(v) = d\}$. If $\hat{f}(B) = d$, then $B \subseteq S$; we conclude that $f^*(z) < d$. Since (F', D') is a strong $(k, f^*)^*$ -decomposition, $f^*(z) < d$ implies that vertices other than z in its component in D' lie in S . Therefore, each component of $D \cup D'$ in G has at most one vertex outside S .

Since $D \subseteq G[B]$ and each component of D has at most one vertex outside S , each vertex v of B has at most $|B \cap S|$ neighbors in D . By the definition of $f^*(z)$, vertex v gains at most $\hat{f}(B) - |B \cap S|$ neighbors in D' ; together it has at most $f(v)$ neighbors in $D \cup D'$. \square

Proof of Theorem 4.2.3: *If $d \leq k + 1$ and f is a strongly feasible capacity function on a graph G , then G has a strong $(k, f)^*$ -decomposition.*

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most k edges. For the induction step, suppose that G is larger.

Recall that $S = \{v \in V(G) : f(v) = d\}$. Let $R = \{v \in V(G) : f(v) = 0\}$, and let $T = V(G) - S - R$. We prove the structural claim that if G has no strong $(k, f)^*$ -decomposition, then S is independent and no edge joins S and T .

Suppose that G has an edge uv such that $u \in S$ and $v \in S \cup T$. We choose such an edge with $v \in T$ if one exists; otherwise, $v \in S$. Let $G' = G - uv$, and let f' be the capacity function on G' that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. Note that $u \notin \{x : f'(x) = d\}$. If f' is strongly feasible, then since $G - uv$ is smaller than G , it has a strong $(k, f')^*$ -decomposition (F, D) . Since $f'(u) < d$ and $f(u) = d$, adding the edge uv to D yields a strong $(k, f)^*$ -decomposition of G .

To prove the structural claim, it thus suffices to show that f' is strongly feasible. We consider $\alpha_{f'}(A)$ and $\beta_{f'}^*(A)$. If $|A| = 1$, then $\alpha_{f'}(A) = |A \cap S|$ (positive if $A \subseteq S$). Also, $\beta_{f'}^*(A) = (2k + 2 - d) + (k + 1)f(A) \geq 2k + 2 - d > 0$, since $d \leq k + 1$.

Next consider $A = V(G)$. Since $u, v \in A$, we have $\beta_{f'}^*(A) = \beta_f^*(A) > 0$. Also, $\alpha_{f'}(A) < \alpha_f(A)$ requires $u, v \in S$. Not all vertices satisfy $f'(x) = d$, since $f'(u) < d$. Therefore, having $\alpha_f(A) \geq 1$ and $\alpha_{f'}(A) \geq 0$ suffices, so we may assume that $\alpha_f(A) = 0$. With $A = V(G)$ and $u, v \in S$, the choice of uv in defining f' implies that no edges join S and T . Since $\alpha_f(A) = 0$ implies $A \not\subseteq S$, we have $R \cup T \neq \emptyset$. If $R \neq \emptyset$, then $\hat{f}(A) = 0$, contradicting Lemma 4.2.5. Hence $R = \emptyset$. Since no edges join S and T , now G is disconnected, and we can combine strong decompositions of the components obtained from the induction hypothesis.

Finally, suppose $2 \leq |A| < |V(G)|$. If $\alpha_f(A) = 0$, then the capacity function f^* on G_A that agrees with f except for $f^*(z) = \hat{f}(A) - |A \cap S|$ is strongly feasible, by Lemma 4.2.6. Also, the restriction of f to A is strongly feasible on $G[A]$. Since G_A and $G[A]$ are smaller

than G , by the induction hypothesis $G[A]$ is strongly $(k, f|_A)^*$ -decomposable and G_A is strongly $(k, f^*)^*$ -decomposable. By Lemma 4.2.7, G is strongly $(k, f)^*$ -decomposable.

Hence we may assume that $\alpha_f(A) > 0$. For $\alpha_{f'}(A) < \alpha_f(A)$, we must have u or v in $A \cap S$, and the decline can only be by 1. Hence $\alpha_{f'}(A) \geq 0$, which is good enough since $f'(u), f'(v) < d$. If $\beta_{f'}^*(A) > 0$, then A causes no problem.

Otherwise, $\beta_f^*(A) \leq k + 1$, since reduction of β^* requires $|A \cap \{u, v\}| = 1$, and the reduction is then by $k + 1$. Now Lemma 4.2.6 implies that \bar{f} is strongly feasible on G_A , where $\bar{f}(z) = 0$ and otherwise \bar{f} agrees with f . By the induction hypothesis, G_A has a strong $(k, \bar{f})^*$ -decomposition (F, D) , and $G[A]$ has a strong $(k, f|_A)^*$ -decomposition (F', D') . As in Lemma 4.2.7, $(F \cup F', D \cup D')$ is a strong $(k, f)^*$ -decomposition of G ; since z is isolated in D , the components of D' do not extend.

Hence we may assume that S is independent and that no edge joins S and T . As in Theorem 4.1.6, we claim that G decomposes into k forests, completing the desired decomposition. Otherwise, we find a set A such that $\beta_f^*(A) \leq 0$, contradicting strong feasibility. Note that $\beta_f^*(A) = (2k + 2 - d)g(A) + h(A)$, where $g(A) = k(|A| - 1) - \|A\| + 1$ and $h(A) = (k + 1)f(A) - d\|A\|$. It suffices to find A such that $g(A) \leq 0$ and $h(A) \leq 0$.

If $\Upsilon(G) > k$, then $V(G)$ has a minimal subset A such that $\|A\| \geq k(|A| - 1) + 1$; that is, $g(A) \leq 0$. Minimality implies that every vertex of A has at least $k + 1$ neighbors in A .

If $A \cap T = \emptyset$, then $\|A\| \geq (k + 1)|A \cap S| = (k + 1)f(A)/d$, which simplifies to $h(A) \leq 0$. If $A \subseteq T$, then $|A \cap S| = 0$, so $\alpha_f(A) = g(A) - 1 < 0$, contradicting strong feasibility of f .

Hence we may assume that $A \cap T$ is a nonempty proper subset of A . The minimality of A implies that $\|A - T\| \leq k(|A - T| - 1)$, and hence more than $k|A \cap T|$ edges of $G[A]$ are incident to T . From the independence of S and the absence of edges joining S and T , we now have $\|A\| > (k + 1)|A \cap S| + k|A \cap T|$. Since $f(v) = d$ for $v \in S$ and $f(v) \leq d - 1$ for

$v \in T$, this yields $\|A\| \geq (k+1)\frac{f(A \cap S)}{d} + k\frac{f(A \cap T)}{d-1}$. Multiplying by d , we obtain

$$d\|A\| \geq (k+1)f(A \cap S) + kf(A \cap T)\frac{d}{d-1} \geq (k+1)f(A),$$

using $d/(d-1) \geq (k+1)/k$ and $f(R) = 0$. Thus $h(A) \leq 0$, which as we noted suffices to complete the proof. \square

4.3 Approach to $(k, d)^*$ -decomposition

For our remaining stronger conclusions in which the “leftover” subgraph D must also be a forest, the highly local approach of Section 4.1 that reserves one edge for D by reducing the degree capacity of its endpoints is not adequate. When $d > k+1$, it becomes harder to avoid creating a cycle when replacing a reserved edge.

We use the inductive approach of obtaining *reducible configurations* (structures that are forbidden from minimal counterexamples) and then the *discharging method*, showing that the average degree in any graph avoiding the reducible configurations is too high. This method can also be used to prove Theorem 1.3.3, but such a proof would be lengthier than that in the previous section. On the other hand, it may settle the case $k = d$ for (k, d) -decomposition.

For this discussion, we modify β by removing the term independent of A , and we drop the notation for the capacity function because each vertex will have capacity d .

Definition 4.3.1. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For a set A of vertices in a graph G , the *sparseness* $\beta_G(A)$ is defined by $\beta_G(A) = (k+1)(k+d)|A| - (k+d+1)\|A\|$.

The term “sparseness” here is natural, because if $\beta_G(A)$ is sufficiently large for all A , then G is sufficiently sparse to satisfy the relevant bound on $\text{Mad}(G)$ or $\text{Arb}(G)$. Sparseness also distinguishes between the conditions on $\text{Mad}(G)$ and $\text{Arb}(G)$. As mentioned previously, $\text{Arb}(G) \leq m_{k,d}/2$ may fail when $\text{Mad}(G) < m_{k,d}$ holds. The former requires a set A such

that $\beta_G(A) < (k+1)(k+d)$, while the latter requires only that $\beta_G(A) \geq 1$ for all A .

Example 4.3.2. Let H be the (multi)graph consisting of $q+1$ vertices in which one vertex has degree $(k+1)q$ and the others have degree $k+1$ and form an independent set. We have $\text{Arb}(H) = k+1$, but $\text{Mad}(H) = 2q(k+1)/(q+1)$. If $d < q < k+d$, then $\text{Mad}(H) < m_{k,d}$, but H has no $(k, d)^*$ -decomposition.

This graph H can be excluded by requiring (k, d) -sparseness (note that $d < q < k+d$ requires $k \geq 2$, which is where (k, d) -sparse and $\text{Mad}(G) < m_{k,d}$ differ). For H , we have $(k+1)(k+d)|V(H)| - (k+d+1)\|V(H)\| = (k+1)(k+d-q)$, which violates (k, d) -sparseness if and only if $q > d$. Furthermore, $q > d$ if and only if H has no (k, d) -decomposition.

Even $\beta_G(A) \geq k^2$ ((k, d) -sparseness) allows $\Upsilon(G) \leq k+1$ to fail, but only on a small subgraph. Violating $\Upsilon(G) \leq k+1$ requires an r -vertex subgraph with at least $(k+1)(r-1)+1$ edges. If such a graph is also (k, d) -sparse, then

$$(k+1)(k+d)r - (k+d+1)[(k+1)(r-1)+1] \geq k^2,$$

which simplifies to $r \leq \frac{k}{k+1}(d+1)$. □

In the cases where we can guarantee a $(k, d)^*$ -decomposition, we obtain a stronger statement than the case (k, d) of the NDT Conjecture by weakening the hypothesis to require only (k, d) -sparseness, while excluding multigraphs with at most $(d+1)k/(k+1)$ vertices that satisfy this bound but fail to decompose into $k+1$ forests.

Definition 4.3.3. Fix $k, d \in \mathbb{N}$. A graph G is *feasible* if $\beta_G(A) \geq k^2$ for all nonempty $A \subseteq V(G)$. A set $A \subseteq V(G)$ is *overfull* if $\|A\| > (k+1)(|A|-1)$.

Now that we are fixing (k, d) , “feasible” is a convenient abbreviation for “ (k, d) -sparse”. Theorem 1.3.3 showed that feasible graphs are (k, d) -decomposable (when $d > k$), and by Example 4.1.1 this condition on β_G is sharp. Graphs with overfull sets are not $(k, d)^*$ -decomposable. We have noted that the bound $\text{Arb}(G) \leq m_{k,d}/2$ both implies feasibility

and prohibits overfull sets. Furthermore, feasibility prohibits overfull sets with more than $(d+1)k/(k+1)$ vertices. Hence the conjecture below is equivalent to the NDT Conjecture.

Conjecture 4.3.4. *Fix $k, d \in \mathbb{N}$. If G is feasible and has no overfull set with at most $(d+1)k/(k+1)$ vertices, then G is $(k, d)^*$ -decomposable.*

We will prove Conjecture 4.3.4 when $k = 1$ and $d \leq 6$. The advantage we gain when $k = 1$ is that $k^2 = 1$, so the feasibility condition reduces to $\beta_G(A) > 0$ for all A . We can then bring a variety of techniques to bear, including properties of submodular functions.

The basic framework of the proof holds for general k , so we maintain the general language throughout this section before specializing to $k = 1$. We do this to suggest the generalization to larger k and because the proofs of these lemmas are as short for general k as for $k = 1$.

We typically use (F, D) to denote a $(k, d)^*$ -decomposition of G , where F is a disjoint union of k forests and D is a forest with maximum degree at most d . Note that the hypotheses of Conjecture 4.3.4 remain satisfied under discarding edges or vertices.

Definition 4.3.5. A j -vertex is a vertex of degree j . Among the non- $(\mathcal{F}, \mathcal{F}_d)$ -decomposable graphs satisfying the hypotheses of Conjecture 4.3.4, a *minimal counterexample* is one that has the fewest ghosts among those with the fewest non-ghosts.

Ghosts help control $(k, d)^*$ -decompositions, because such a decomposition must put one edge at a ghost into D . Without loss of generality, the other k edges at the ghost may be placed arbitrarily into the forests in F .

Lemma 4.3.6. *A minimal counterexample G is $(k+1)$ -edge-connected (and hence has minimum degree at least $k+1$).*

Proof. If G has an edge cut Q with size at most k , then $(k, d)^*$ -decompositions of the components of $G - Q$ combine to form an $(k, d)^*$ -decomposition of G by allowing each forest to acquire at most one edge of the cut. □

Corollary 4.3.7. *In a minimal counterexample G , a vertex with degree at most $2k+1$ cannot be a neighbor of a ghost.*

Proof. If such a vertex v is also a ghost, then G has two vertices and is $(k, d)^*$ -decomposable. Otherwise, the edges incident to v and not incident to the neighboring ghost form an edge cut of size at most k , contradicting Lemma 4.3.6. \square

Definition 4.3.8. A j -neighbor of a vertex is a neighbor that is a j -vertex. A ghost neighbor of a vertex is a neighbor that is a ghost. Adding a ghost neighbor at a vertex v means adding to the graph a vertex of degree $k+1$ whose only neighbor is v . For a vertex set A in a graph G , contracting A to a vertex v^* means deleting all edges within A and replacing A with a single vertex v^* incident to all edges that joined A to $V(G) - A$. Let G_A denote the graph obtained from G by contracting A to v^* and adding d ghost neighbors at v^* .

Lemma 4.3.9. *If G is feasible and $\beta_G(A) \leq k(k+1)$, then G_A is feasible.*

Proof. For $X \subseteq V(G_A)$, we show that $\beta_{G_A}(X) \geq k^2$. Let S be the set of d ghost neighbors added at v^* . If $v^* \notin X$, then the inequality is hardest when $S \cap X = \emptyset$, since each vertex of S adds $(k+1)(k+d)$ to the sparseness of $X - S$. With $S \cap X = \emptyset$, we have $\beta_{G'}(X) = \beta_G(X) \geq k^2$.

If $v^* \in X$, then the inequality is hardest when $S \subseteq X$, since each addition of a ghost to a set containing its neighbor reduces the sparseness by $k+1$. Before adding S , contracting A to v^* loses $|A| - 1$ vertices and $\|A\|$ edges. Let $X' = A \cup (X - S - v^*)$; note that $X' \subseteq V(G)$.

We compute

$$\begin{aligned} \beta_{G_A}(X) &= \beta_G(X') - (k+1)(k+d)(|A| - 1) + (k+d+1)\|A\| - d(k+1) \\ &= \beta_G(X') + k(k+1) - \beta_G(A) \geq \beta_G(X') \geq k^2. \end{aligned}$$

\square

Lemma 4.3.10. *For $A \subseteq V(G)$, if $G[A]$ and G_A are $(k, d)^*$ -decomposable, then G is $(k, d)^*$ -decomposable.*

Proof. Let (F, D) be a $(k, d)^*$ -decomposition of G_A . Since v^* has d ghost neighbors in G_A , its neighbors in D are only those ghosts; no edges of D join v^* to vertices of G . Let (F', D') be a $(k, d)^*$ -decomposition of $G[A]$.

Combining (F', D') and (F, D) (after deleting the ghost neighbors of v^*) forms a $(k, d)^*$ -decomposition of G . All edges joining v^* to $V(G) - A$ lie in F and are incident to various vertices of A . Since v^* lies on no cycle in F , adding the edges of F' does not complete a cycle. That is, each forest in a $k\mathcal{F}$ -decomposition of F can be combined with any one of the forests in a $k\mathcal{F}$ -decomposition of F' . □

Definition 4.3.11. Let $d_G(v)$ denote the degree of a vertex v in a graph G . A set $A \subseteq G$ is *nontrivial* if A contains at least two non-ghosts but not all non-ghosts in G .

We avoid confusion between the overall parameter d and the degree function by always using the relevant graph as a subscript when discussing individual vertex degrees.

Lemma 4.3.12. *Let A be a vertex set in a minimal counterexample G . If A is nontrivial, then $\beta_G(A) > k(k + 1)$. If A is trivial with exactly one non-ghost vertex v , and $\beta_G(A) \leq k(k + 1)$, then $d_G(v) \geq (k + 1)(d + 1)$.*

Proof. Suppose that $\beta_G(A) \leq k(k + 1)$. By Lemma 4.3.9, G_A is feasible. If A is nontrivial, then G_A has fewer non-ghosts than G . The minimality of G then implies that both G_A and $G[A]$ are $(k, d)^*$ -decomposable. By Lemma 4.3.10, also G would be $(k, d)^*$ -decomposable.

Hence we may assume that A is trivial with non-ghost vertex v , so A consists of v and some number h of ghost neighbors of v . Now $\beta_G(A) = (k + 1)(k + d - h)$, so $\beta_G(A) \leq k(k + 1)$ requires $h \geq d$. If $h > d$, then already $d_G(v) \geq (k + 1)(d + 1)$. If $h = d$ and $A = V(G)$, then G is explicitly $(k, d)^*$ -decomposable. In the remaining case, G has vertices outside A ,

and the only vertex of A with outside neighbors is v . Since G is $(k + 1)$ -edge-connected (by Lemma 4.3.6), we again have $d_G(v) \geq (k + 1)(d + 1)$. \square

Lemma 4.3.13. *If v is a vertex in a minimal counterexample G , and $d_G(v) < (k + 1)(k + d)$, then v has no non-ghost $(k + 1)$ -neighbor.*

Proof. Let u be a non-ghost $(k + 1)$ -neighbor of v , and let W be the set of other neighbors of u . Since $d_G(u) = k + 1$, no vertex in $W \cup \{v\}$ is a ghost. Form G' from G by deleting the edges incident to u and then adding $k + 1$ edges joining u to v ; this makes u a ghost neighbor of v in G' . Note that G' and G have the same numbers of edges and vertices, but G' has fewer non-ghost vertices than G , since u and its neighbors are non-ghosts in G and at least u becomes a ghost in G' .

If G' is feasible, then the choice of G implies that G' has an $(k, d)^*$ -decomposition (F, D) . Now modify (F, D) : delete the copies of uv in F (keeping the copy in D), and add the k other edges at u in G to the k forests in F . This yields a $(k, d)^*$ -decomposition of G .

It thus suffices to show that G' is feasible. We need only consider A such that $u, v \in A$ and $W \not\subseteq A$; otherwise, $\beta_{G'}(A) \geq \beta_G(A) \geq k^2$, since G is feasible. With $u \in A$, we have $\beta_{G'}(A) = \beta_G(A - u) - (k + 1)$, since adding a ghost neighbor costs $k + 1$. We worry only if $\beta_G(A - u) \leq k(k + 1)$. Since $W \not\subseteq A$, the set A does not contain all non-ghosts in G . If v is the only non-ghost in $A - u$, then $d_G(v) \geq (k + 1)(k + d)$, by Lemma 4.3.12. Since our hypothesis is $d_G(v) < (k + 1)(k + d)$, we conclude that $A - u$ is nontrivial, and now Lemma 4.3.12 yields $\beta_G(A - u) > k(k + 1)$. \square

Lemma 4.3.14. *If a minimal counterexample G has a vertex v with q ghost neighbors, where $q \geq 1$, then $d_G(v) > kq + k + d$.*

Proof. Form G' from G by deleting the ghost neighbors of v . Since G' is an induced subgraph of G , it is feasible. Forming G' does not increase the number of non-ghost vertices, but it decreases the numbers of vertices and edges, so G' has an $(k, d)^*$ -decomposition (F', D') .

By Lemma 4.3.6, $d_{G'}(v) \geq k + 1$. We may assume that $d_{D'}(v) \leq d_{G'}(v) - k$, since edges of D' at v can be moved arbitrarily to F' until F' has at least k edges at v . Now restore each ghost vertex by adding one incident edge to each forest in F' and the remaining incident edge to D' , yielding (F, D) .

Since $F \in k\mathcal{F}$ and $D \in \mathcal{F}$, it suffices to check $d_D(v)$. We have $d_D(v) = d_{D'}(v) + q \leq d_{G'}(v) - k + q = d_G(v) - kq - k$. Thus $d_D(v) \leq d$ unless $d_G(v) > kq + k + d$. \square

If v has q ghost neighbors, then $d_G(v) \geq (k+1)q$. Hence the lower bound in Lemma 4.3.14 strengthens the trivial lower bound when $q \leq k + d$.

Lemma 4.3.15. *If G is a minimal counterexample, then two vertices in G are joined by $k + 1$ edges only when one of them is a ghost.*

Proof. Since G has no overfull set, edge-multiplicity is at most $k + 1$. If two ghosts are adjacent, then G has two vertices and is $(k, d)^*$ -decomposable.

Suppose that non-ghosts u and v are joined by $k + 1$ edges. Obtain G' from G by contracting these edges into a single vertex v^* and adding a ghost neighbor w to v^* .

We claim that G' is feasible and has no overfull set. If $A \subseteq V(G') - \{v^*\}$, then $\beta_{G'}(A) \geq \beta_G(A - \{w\}) \geq k^2$. If $v^* \in A \subseteq V(G')$, then $\beta_{G'}(A) \geq \beta_{G'}(A \cup \{w\}) = \beta_G(A') \geq k^2$, where $A' = (A - \{v^*, w\}) \cup \{u, v\}$. Hence G' is feasible.

Since G has no overfull set, an overfull set in G' must contain v^* , and a smallest such set A does not contain w . Let $A' = (A - \{v^*\}) \cup \{u, v\}$. Now A' has one more vertex than A and induces $k + 1$ more edges in G than A induces in G' . Hence A' is overfull if and only if A is overfull. We conclude that G' has no overfull set.

Since G' has the same numbers of vertices and edges as G , but G' has fewer non-ghosts than G , minimality of G now implies that G' has a $(k, d)^*$ -decomposition (F, D) . At w there is one edge in each forest in F and one edge in D . Replacing these with the edges joining u and v (one in each forest) yields a $(k, d)^*$ -decomposition of G , since the new degree of u or

v in D is at most $d_D(v^*)$, and an edge joining u and v completes a cycle in its forest only if contracting that edge yields a cycle in the corresponding forest in (F, D) . \square

4.4 Discharging argument and submodularity

The lemmas of Section 4.3 provide a framework for a discharging argument. We would like to show that if G has the structural properties of a minimal counterexample, then $\text{Mad}(G) \geq m_{k,d}$; this would prove the conjecture. We have not yet proved sufficient structural properties to complete the argument. By outlining a discharging argument, we will suggest what else is needed. Section 4.5 will complete the proof for $k = 1$ and $d \leq 6$.

Let G be a minimal counterexample. Since G is feasible, $\text{Mad}(G) < m_{k,d} = 2k + \frac{2d}{k+d+1}$. Give each vertex an initial charge equal to its degree in G (by Lemma 4.3.6, each vertex has degree at least $k + 1$). We aim to redistribute charge to obtain a final charge $\mu(v)$ for each vertex v such that $\mu(v) \geq m_{k,d}$. This motivates our first discharging rule.

Rule 1: A vertex of degree $k + 1$ takes charge $m_{k,d}/(k + 1) - 1$ along each incident edge from the other endpoint of that edge. This amount equals $\frac{k+d-1}{k+d+1}$.

In particular, a ghost takes total charge $m_{k,d} - (k + 1)$ from its neighbor. By force, Rule 1 increases the charge of each $(k + 1)$ -vertex to $m_{k,d}$, since Lemma 4.3.13 implies that $(k + 1)$ -vertices are not adjacent unless G has just two vertices.

If all neighbors of v have degree $k + 1$, then $\mu(v) = d_G(v) \frac{2}{k+d+1}$, since each edge takes $\frac{k+d-1}{k+d+1}$. In this case, $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k + 1)(k + d)$.

The problem is how to handle vertices with degree between $k + 1$ and $(k + 1)(k + d)$. Vertices with degree at most $2k$ need additional charge (as do vertices with degree $2k + 1$ when $d > k + 1$), though they do not need as much as $(k + 1)$ -vertices need. Vertices with degree less than $(k + 1)(k + d)$ cannot afford to give away too much. The principle we need

to quantify is that lower-degree vertices must have higher-degree neighbors.

A vertex v with degree less than $(k+1)(k+d)$ cannot be adjacent only to $(k+1)$ -vertices. By Lemma 4.3.13, v has no non-ghost $(k+1)$ -neighbor. If v has only ghost neighbors, then G consists of one vertex plus ghost neighbors, but such a graph has the desired decomposition or is infeasible (see Example 4.3.2). Hence v has some neighbors with higher degrees and will not need to give away as much. More information is needed about the degrees of neighboring vertices to complete a proof.

When $(k, d) = (1, 1)$, only 2-vertices need charge. By Lemma 4.3.13, their neighbors have high enough degree that Rule 1 suffices to complete the discharging argument. Since a forest with maximum degree 1 is a matching, this proves the result of [31] that the Strong NDT Conjecture holds when $(k, d) = (1, 1)$.

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition for completing the discharging argument.

Theorem 4.4.1. *For $d > k = 1$, let G be a minimal counterexample in the sense of Section 4.3. If each 3-vertex in G has a neighbor with degree at least $d + 2$, then $\text{Mad}(G) \geq m_{1,d} = 2 + \frac{d}{d+2}$.*

Proof. In addition to the special case for $k = 1$ of Rule 1 stated above, in which each 2-vertex receives $\frac{d}{d+2}$ along each edge, we add a rule to satisfy 3-vertices.

Rule 2: If $d_G(v) = 3$, and v has neighbor u with $d_G(u) \geq d + 2$, then v receives $\frac{d-2}{d+2}$ from u .

We show that the final charge of each vertex is at least $m_{1,d}$. Rules 1 and 2 ensure that $\mu(v) \geq m_{1,d}$ when $d_G(v) \in \{2, 3\}$ (since $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d + 2$ also remains valid.

If $4 \leq d_G(v) \leq 2d + 1$, then v has no non-ghost 2-neighbor, by Lemma 4.3.13. If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q + d + 2$, by Lemma 4.3.14. Hence $\mu(v) = d_G(v) > m_{1,d}$ if $4 \leq d_G(v) \leq d + 1$, since Rule 2 takes no charge from v .

If $d + 2 \leq d_G(v) \leq 2d + 1$, then v may give charge to q ghost neighbors (to each along two edges) and to $d_G(v) - 2q$ neighbors of degree 3. Using Lemma 4.3.14,

$$\mu(v) \geq d_G(v) - \frac{d}{d+2}2q - [d_G(v) - 2q]\frac{d-2}{d+2} = \frac{4(d_G(v) - q)}{d+2} \geq \frac{4(d+2)}{d+2} = 4 > m_{1,d}.$$

The final charge at each vertex is at least $m_{1,d}$, so no minimal counterexample is feasible. \square

This reduces Conjecture 4.3.4 for the case $k = 1$ to proving that in a minimal counterexample G , each 3-vertex has a neighbor with degree at least $d + 2$. Our proofs of this fact depend on d . In each case, we will use submodularity properties of the function β_G .

Definition 4.4.2. A function β on the subsets of a set is *submodular* if $\beta(X \cap Y) + \beta(X \cup Y) \leq \beta(X) + \beta(Y)$ for all subsets X and Y . When G' is an induced subgraph of G , define the *potential function* $\rho_{G'}$ by $\rho_{G'}(X) = \min\{\beta_G(W) : X \subseteq W \subseteq V(G')\}$.

Lemma 4.4.3. *For any graph G and any induced subgraph G' of G , the sparseness function β_G on the subsets of $V(G)$ is submodular.*

Proof. To compare $\beta_G(X \cap Y) + \beta_G(X \cup Y)$ with $\beta_G(X) + \beta_G(Y)$, note first that $|X \cup Y| + |X \cap Y| = |X| + |Y|$. Hence it suffices to show that $\|X \cup Y\| + \|X \cap Y\| \geq \|X\| + \|Y\|$. All edges contribute equally to both sides except edges joining $X - Y$ and $Y - X$, which contribute 1 to the left side but 0 to the right. \square

4.5 Neighbors of 3-vertices when $k = 1$

Throughout this section, $k = 1$. For $k = 1$, feasibility reduces to the statement that $\beta_G(A) = (2d + 2)|A| - (d + 2)\|A\| \geq 1$ for $A \subseteq V(G)$. When G is a minimal counterexample, Lemma 4.3.12 implies that $\beta_G(A) \geq 3$ when A is nontrivial (contains at least two non-ghosts but not all non-ghosts). Furthermore, if d is even, then always $\beta_G(A)$ is even, so in that

case we may assume $\beta_G(A) \geq 4$ when A is nontrivial. By Theorem 4.4.1, to prove the NDT Conjecture when $k = 1$ it suffices to prove that every 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$.

Lemma 4.5.1. *Fix d with $2 \leq d \leq 6$, and let G be a minimal counterexample. If v is a 3-vertex in G and has no neighbor with degree at least $d + 2$, then v has two neighbors u and u' such that $\rho_{G'}(\{u, u'\}) \geq d + 3$, where $G' = G - v$.*

Proof. Together, Corollary 4.3.7 and Lemma 4.3.15 imply that every 3-vertex has three distinct neighbors. Let U be the neighborhood of v , with $U = \{u_1, u_2, u_3\}$. Let $Z_i = U - \{u_i\}$. Suppose that $\rho_{G'}(U_i) \leq d + 2$ for all i .

For each i , let X_i be a subset of $V(G')$ such that $\rho_{G'}(Z_i) = \beta_G(X_i)$. For any permutation i, j, k of $\{1, 2, 3\}$,

$$2d + 4 \geq \beta_G(X_i) + \beta_G(X_j) \geq \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j).$$

For $X' \subseteq V(G')$, let $X = X' \cup \{v\}$. If $U \subseteq X' \subseteq V(G')$, then $\beta_G(X') = \beta_G(X) + d + 4$. If $X' \neq V(G')$, then $X \neq V(G)$, and X is nontrivial if it has at least two non-ghosts, which by Lemma 4.3.12 would yield $\beta_G(X') \geq d + 7 + \epsilon$, where $\epsilon = 1$ if d is even and $\epsilon = 0$ if d is odd. However, if $X' = V(G')$, then we only have $\beta_G(X') \geq d + 5 + \epsilon$.

Since each edge vu_i has multiplicity 1, no vertex in U is a ghost, and neither is v . Since $u_k \in X_i \cap X_j$ and $d_G(u_k) < d + 2$, Lemma 4.3.12 implies $\beta_G(X_i \cap X_j) \geq 3 + \epsilon$. Since $U \subseteq X_i \cup X_j$, we also conclude $\beta_G(X_i) + \beta_G(X_j) \geq d + 8 + 2\epsilon$ for all d , and the lower bound increases by 2 if $X_i \cup X_j \neq V(G')$.

Thus $\rho_{G'}(X_i) + \rho_{G'}(X_j) \geq d + 8 + 2\epsilon$. If $d \leq 4$, then $d + 8 + 2\epsilon > 2d + 4$, and the desired conclusion follows. Hence we may assume $d \in \{5, 6\}$; furthermore, $X_i \cup X_j = V(G')$ for all i, j , since otherwise the lower bound on $\beta_G(X_i) + \beta_G(X_j)$ again exceeds $2d + 4$.

In more detail, the computation of Lemma 4.4.3 is

$$\beta_G(X_i) + \beta_G(X_j) = \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j) + (k + d + 1)m,$$

where m is the number of edges joining $X_i - X_j$ and $X_j - X_i$. If $m \geq 1$, then we obtain $\beta_G(X_i) + \beta_G(X_j) \geq 2d + 10 > 2d + 4$, which yields the desired conclusion. Hence $m = 0$ in each case. That is, each $X_i \cap X_j$ is a separating set in G' . (If G' is disconnected, then some edge incident to v is a cut-edge, which contradicts Lemma 4.3.6.) Furthermore,

$$\beta_G(X_i \cap X_j) = \beta_G(X_i) + \beta_G(X_j) - \beta_G(X_i \cup X_j) \leq 2d + 4 - (d + 5 + \epsilon) = d - 1 - \epsilon.$$

Now let $Z = X_1 \cap X_2 \cap X_3$. Since $X_i \cup X_j = V(G')$, any vertex of $V(G') - Z$ misses exactly one of the three sets, so $\{Z, \overline{X}_1, \overline{X}_2, \overline{X}_3\}$ is a partition of $V(G')$. Since $\beta_G(X_i) \leq d + 2$ and $\beta_G(V(G')) \geq d + 5$, each \overline{X}_i is nonempty, so $Z \neq V(G')$. If Z contains only one non-ghost, then feasibility requires it to have at most d ghost neighbors, and $\beta_G(Z) \geq 2$. Otherwise, since $v \notin Z$, we conclude that Z is nontrivial, and hence $\beta_G(Z) \geq 3$.

Now, since $\overline{X}_i \subseteq X_j \cap X_k$, submodularity yields

$$2d + 1 - \epsilon \geq \beta_G(X_i) + \beta_G(X_j \cap X_k) \geq \beta_G(V(G')) + \beta_G(Z) \geq d + 7.$$

We conclude that $d \geq 6 + \epsilon$, which completes the proof for $d \leq 6$. □

Lemma 4.5.2. *If $3 \leq d \leq 6$ and G is a minimal counterexample, then every 3-vertex has a neighbor with degree at least $d + 2$.*

Proof. Let u_1, u_2, u_3 be the neighbors of a 3-vertex v , and let $U = \{u_1, u_2, u_3\}$. Suppose that $d_G(u) \leq d + 1$ for $u \in U$. Since each edge vu_i has multiplicity 1, no vertex in U is a ghost vertex, and any edge induced by U has multiplicity 1 (Lemma 4.3.15).

Let $G' = G - v$. By Lemma 4.5.1, we may assume by symmetry that $\rho_{G'}(\{u_1, u_2\}) \geq d+3$. Form H from G' by adding an extra edge joining u_1 and u_2 . For $A \subseteq V(H) = V(G')$, we have $\beta_H(A) = \beta_{G'}(A)$ unless $u_1, u_2 \in A$, but in the remaining case $\rho_{G'}(\{u_1, u_2\}) \geq d+3$ yields $\beta_H(A) \geq 1$.

Hence H is feasible, and it has fewer non-ghosts than G . In order to have an $(\mathcal{F}, \mathcal{F}_d)$ -decomposition of H , we need only exclude overfull sets of size at most $(d+1)/2$, which is at most 3. There are no triple-edges in H , since G has no double-edges within U . An overfull triple must include u_1 and u_2 , since G has no overfull triple. The third vertex w must be adjacent to u_1 or u_2 by two edges in G . Since those vertices are also adjacent to v , we have contradicted $d_G(u_1) = d_G(u_2) = 3$.

Let (F, D) be an $(\mathcal{F}, \mathcal{F}_d)$ -decomposition of H . Obtain a decomposition of G by (1) replacing the added edge u_1u_2 with vu_1 and vu_2 in whichever of F and D contains it, and (2) placing vu_3 in the other subgraph. The degree in D of u_1 and u_2 is the same as a subgraph of H or G , and cycles through v would correspond to cycles in the decomposition of H . The only worry is $d_D(u_3)$, since we have increased this by 1 if the added edge in H belonged to F . If $d_D(u_3)$ has increased to $d+1$, then we have the desired conclusion unless $d_G(u_3) = d+1$, but now we can move any one edge incident to u_3 from D to F to complete a $(\mathcal{F}, \mathcal{F}_d)$ -decomposition of G . \square

4.6 The Strong NDT Conjecture for $(k, d) = (1, 2)$

In this section we prove our strongest conclusion for our most restrictive hypothesis. Many of the steps are quite similar to our previous arguments, so we put them all together in a single proof.

Theorem 4.6.1. *The Strong NDT Conjecture holds when $(k, d) = (1, 2)$. That is, if G is feasible, then G has an $(\mathcal{F}, \mathcal{F}_2)$ -decomposition (F, D) in which every component of D has at*

most two edges (a strong decomposition).

Proof. Since $m_{1,2} = 3$, feasibility is equivalent to $\text{Mad}(G) < 3$. Let G be a counterexample with the fewest non-ghosts. By the argument of Lemma 4.3.6, G is 2-edge-connected.

If G has adjacent 2-vertices u and v , then at least one is not a ghost. Letting $G' = G - \{u, v\}$, the minimality of G yields a strong decomposition (F, D) of G' . Adding the edge uv to D and the other edges incident to u and v to F yields a strong decomposition of G .

If G has a vertex with three ghost neighbors, then G is infeasible, so every vertex has at most two ghost neighbors. If G has only one non-ghost, then G explicitly has a strong decomposition. Hence we may assume that G has at least two non-ghosts.

Since d is even, always β_G is even, so feasibility can be stated as $\beta_G(A) \geq 2$ for $A \subseteq V(G)$ (here $\beta_G(A) = 6|A| - 4\|A\|$). A set A is *tight* if $\beta_G(A) = 2$. A set consisting of a vertex with two ghost neighbors is a trivial tight set.

By Lemma 4.3.9, if A is a tight set, then G_A is feasible. The same argument as in Lemma 4.3.10 shows that if G is a minimal counterexample, $A \subseteq V(G)$, and G_A has a strong decomposition, then G has a strong decomposition. Hence we may assume, as in the earlier proofs, that $\beta_G(A) \geq 4$ for every nontrivial set A .

Suppose that G has a non-ghost 2-vertex v . Each neighbor of v has degree at least 3. If a neighbor u of v has at most one ghost neighbor, then form G' from $G - v$ by giving u one additional ghost neighbor w . Now G and G' have the same numbers of vertices and edges, but G' has fewer non-ghost vertices.

We claim also that G' is feasible. If $u \notin A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \notin A$, and then $\beta_{G'}(A) = \beta_G(A) \geq 2$. If $u \in A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \in A$, and then $\beta_{G'}(A) \geq \beta_G(A - \{w\} \cup \{v\}) - 2 \geq 2$, since $A - \{w\} \cup \{v\}$ is nontrivial.

We conclude that G' has a strong decomposition (F, D) , by the minimality of G . Each of F and D must have one edge incident to w . We obtain a strong decomposition of G by deleting w , adding vu to D , and adding the other edge at v to F .

We may therefore assume that every neighbor of a non-ghost 2-vertex has at least two ghost neighbors. Since G is 2-edge-connected, a q -vertex cannot have $(q - 1)/2$ ghost neighbors. In particular, a vertex with at least two ghost neighbors must have degree at least 6, so every neighbor of a non-ghost 2-vertex has degree at least 6.

Once again we have derived many properties of a minimal counterexample. We complete the proof by using discharging to show that if G has these properties, then $\text{Mad}(G) \geq 3$. This contradicts feasibility, which is equivalent to $\text{Mad}(G) < 3$; hence there is no minimal counterexample.

The initial charge of each vertex is its degree; we manipulate charge so that the final charge $\mu(v)$ of each vertex v is at least 3. The only discharging rule is that a 2-vertex takes charge $1/2$ along each incident edge from the other endpoint of that edge. Hence the final charge of a 2-vertex is 3.

Since each neighbor of a non-ghost 2-vertex has degree at least 6, vertices of degree 3, 4, or 5 give charge only to ghosts. If $d_G(v) = 3$, then v has no ghost neighbors, and $\mu(v) = 3$. If $d_G(v) \in \{4, 5\}$, then v has at most one ghost neighbor, and $\mu(v) \geq d_G(v) - 1 \geq 3$. If $d_G(v) \geq 6$, then v gives at most $1/2$ along each edge, so $\mu(v) \geq d_G(v) - d_G(v)/2 \geq 3$. \square

References

- [1] J. Balogh, M. Kochol, A. Pluhár, and X. Yu, Covering planar graphs with forests. *J. Combin. Theory Ser. B* 94 (2005), 147–158.
- [2] A. Bassa, J. Burns, J. Campbell, A. Deshpande, J. Farley, M. Halsey, S. Michalakis, P.-O. Persson, P. Pylyavskyy, L. Rademacher, A. Riehl, M. Rios, J. Samuel, B. Tenner, A. Vijayasarathy, L. Zhao, and D. J. Kleitman, Partitioning a planar graph of girth ten into a forest and a matching, manuscript (2004).
- [3] J. A. Bondy. Pancyclic graphs. I. *J. Combinatorial Theory Ser. B* 11 1971 80C84.
- [4] O.V.Borodin, A.O. Ivanova, B.S. Stechkin, Decomposing a planar graph into a forest and a subgraph of restricted maximum degree, *Sib. Elektron. Mat. Izv.*, 4 (2007), 296–299.
- [5] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, and N.N. Sheikh, Planar graphs decomposable into a forest and a matching, *Discrete Math.* 309 (2009), 277–279.
- [6] O.V.Borodin, A.O. Ivanova, A.V. Kostochka and N.N.Sheikh, Decompositions of quadrangle-free planar graphs, *Discussiones Math. Graph Theory* 29 (2009), 87–99.
- [7] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, and G. Yu. Decomposing a planar graph with girth 9 into a forest and a matching. *European J. Combinatorics*, 29(5) (2008), 1235–1241.
- [8] G. Chen, R. J. Faudree, and R. J. Gould, Intersections of longest cycles in k -connected graphs. *J. Combin. Theory (B)* 72 (1998), 143–149.
- [9] G. Chen, Z. Hu, and Y. Wu, Circumferences of k -connected graphs involving independence numbers. Manuscript, 2010.
- [10] G. Chen, Z. Hu, and Y. Wu, Circumferences of k -connected graphs involving independence numbers, II. Manuscript, 2010.
- [11] G. Chen, Z. Hu, and Y. Wu, A lower bound of circumferences involving connectivities and independence numbers of subgraphs, Manuscript, 2010.

- [12] V. Chvatál and P. Erdős, A note on hamiltonian circuits. *Discr. Math.* **2** (1972) 111–113.
- [13] G. A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc.* (3) **2**, (1952). 69C81.
- [14] J. Edmonds, Matroid partition. *Mathematics of the Decision Sciences, Part I (Seminar, Stanford, Calif., 1967)* (1968), 335–345.
- [15] J.L. Fouquet and J.L. Jolivet, Problème 438, in *Problèmes combinatoires et théorie des graphes, Orsay*, (1976).
- [16] I. Fournier, Thesis, University Parix-XI, Orsay (1982).
- [17] I. Fournier, Longest cycles in 2-connected graphs of stability α , in *Cycles in Graphs* (B. Alspach and C.D. Godsil, eds.), (Elsevier/North-Holland, 1984).
- [18] A. Frank, T. Király, and M. Kriesell, On decomposing a hypergraph into k connected subhypergraphs, *Discrete Appl. Math.* **131** (2003), 373–383.
- [19] S.Fujita, A. Halperin, C. Magnant. Long path lemma concerning connectivity and independence number, to appear.
- [20] D. Gonçalves. Covering planar graphs with forests, one having bounded maximum degree. *J. Combin. Theory Ser. B* **99** (2009), 314–322.
- [21] S. L. Hakimi, On the degrees of the vertices of a directed graph. *J. Franklin Inst.* **279** (1965), 290–308.
- [22] W. He, X. Hou, K.-W. Lih, J. Shao, W. Wang, and X. Zhu. Edge-decompositions of planar graphs and their game coloring numbers. *Journal of Graph Theory* **41** (2002), 307–317.
- [23] S. Kim, A.V. Kostochka, D.B. West, H. Wu, and X. Zhu. Decomposition of sparse graphs into forests and a graph with bounded degree (submitted)
- [24] D. J. Kleitman, Partitioning the edges of a girth 6 planar graph into those of a forest and those of a set of disjoint paths and cycles, Manuscript, 2006.
- [25] M. Kriesell, Edge-disjoint trees containing some given vertices in a graph. *J. Combin. Theory Ser. B* **88** (2003), 53–65.
- [26] M. Kriesell, Edge disjoint Steiner trees in graphs without large bridges. *J. Graph Theory* **62** (2009), 188–198.
- [27] L. C. Lau, An approximate max-Steiner-tree-packing min-Steiner-cut theorem. *Combinatorica* **27** (2007), 71–90.

- [28] W. Mayer, A reduction method for edge-connectivity in graphs. *Ann. Discrete Math.* 3 (1978), 145–164.
- [29] Y. Manoussakis, Longest cycles in 3-connected graphs with given independence number. *Graphs & Combin.* (2009), 377–384.
- [30] M. Montassier, A. Pêcher, A. Raspaud, D. B. West, and X. Zhu. Decomposition of sparse graphs, with application to game coloring number. *Discrete Mathematics* 310 (2010), 1520–1523.
- [31] M. Montassier, P. Ossona de Mendez, A. Raspaud, and X. Zhu. Decomposing a graph into forests. Manuscript, 2010.
- [32] C. St. J. A. Nash-Williams, Edge disjoint spanning trees of finite graphs, *J. London Math. Soc.* 36 (1961), 445–450.
- [33] C. St. J. A. Nash-Williams. Decompositions of finite graphs into forests. *J. London Math. Soc.* 39 (1964), 12.
- [34] M. Kouider, Cycles in graphs with prescribed stability number and connectivity, *J. Combin. Th (B)* 60 (1994), 315–318.
- [35] S. O, D.B. West, H. Wu. Longest cycles in k-connected graphs with given independence number. to appear in *J. Combin. Theory Ser. B*.
- [36] O. Ore, Note on Hamiltonian circuits, *American Mathematical Monthly* 67: 55.
- [37] C. Payan, Graphes equilibre et arboricité rationnelle. *European J. Combinatorics*, 7 (1986), 263–270.
- [38] W. T. Tutte, On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* 36 (1961), 221–230.
- [39] Y. Wang and Q. Zhang, Decomposing a planar graph with girth 8 into a forest and a matching, (to appear).
- [40] D.B. West, H. Wu. Packing of Steiner trees and S-connectors in graphs, to appear.
- [41] D.B. West, Introduction to graph theory, 2nd ed., Prentice Hall, Upper Saddle River, 2001.

Vita

Hehui Wu was born in the summer of 1980 in Guangdong, China. After he completed his Bachelor degree in Mathematics from the University of Science and Technology of China in July 2004, he came to pursue his graduate degree at West Virginia University(WVU). At WVU, he studied Graph Theory under the supervision of Professor Hong-Jian Lai. In December 2006, after getting a Master degree from WVU, he moved to Champaign, IL, and continue his Ph.D study at the University of Illinois at Urbana-Champaign, under the supervision of Professor Douglas B. West. In summer 2011, Hehui completed his Ph.D degree from the University of Illinois.