OBSTRUCTIONS TO THE EXISTENCE OF DISPLACEABLE LAGRANGIAN SUBMANIFOLDS

BY

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DISSERTATION

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Abstract

We utilize Floer theory and an index relation relating the Maslov index, Morse index and Conley-Zehnder index for a periodic orbit of the flow of a specific Hamiltonian function to state and prove some nonexistence results for certain displaceable Lagrangian submanifolds. We start with results under the assumption that the symplectic manifold (M, ω) is closed and symplectically aspherical and then generalize to the case when (M, ω) is weakly exact. The specific Lagrangian submanifolds in consideration are split hyperbolic submanifolds, spheres, products of spheres, Cayley projective plane and quaternionic projective spaces.

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Contents

1 Introduction

Gromov, in [Gr], proved that there are no Lagrangian spheres in $M = \mathbb{C}^n$ (for $n \geq 2$). [1] Whether similar statements hold for other symplectic manifolds (M, ω) and other Lagrangian submanifolds different than spheres are interesting questions. When we have the assumption that (M, ω) is closed and symplectically aspherical and without having any further assumptions on L and M, it is known that the analogous statement of Gromov doesn't hold: There are examples of Lagrangian spheres in symplectically aspherical manifolds. So we consider the more restricted class of displaceable Lagrangian submanifolds. For this class we obtain several new obstruction results described as Theorems 1.2, 1.5, 1.6, 1.7 and 1.8 in this section.

For the proof of all the theorems, among the main tools we utilize are Morse index theory, Hamiltonian dynamics, filtered Floer cohomology, and an index relation relating the Morse index, Conley-Zehnder index and Maslov index of a nondegenerate 1-periodic orbit together with a spanning disc. Given a displaceable Lagrangian submanifold, we specifically construct a Hamiltonian function and for this Hamiltonian we prove, using Floer theory, the existence of a periodic orbit in the cotangent bundle which projects to an orbit in a critical submanifold and which has a spanning disc such that the orbit together with its spanning disk has the desired Conley-Zehnder index. The displaceability assumption on the Lagrangian submanifold provides us with the property that the Hamiltonian flow is non length minimizing and hence this specific orbit is nonconstant. We also present computations of Morse index for critical submanifolds of the energy functional. Then we apply the index relation for the indices of this specific orbit and observe the cases which result in a contradiction. The first restrictions on the Maslov class for the case of tori in $\mathbb{R}^{2n}$ has been established in the work of C. Viterbo in [Vi] and our approach as described is motivated by the approach in this

[1] And more generally, $M = \mathbb{C}^n$ contains no closed exact Lagrangian submanifolds, see p. 330 of [Gr].

work.

Here are the statements of the results: [2]

The following is part of the main result from [KS] and its proof is presented in section 5.12.1.

Theorem 1.1. [KS] *Let (M^{2n}, ω) be a closed, rational and proportional symplectic manifold. Then there is no easily displaceable, split hyperbolic, Lagrangian submanifold in M^{2n} if $N_L > n+2\,(-1)$ where N_L is the minimal Maslov number of L and where (-1) contributes if L is orientable.*

The proofs of the rest of the theorems stated in this section are presented in section 5.12.

Theorem 1.2. *There are no displaceable Lagrangian spheres in a closed, symplectically aspherical, symplectic manifold (M^{2n}, ω) if $n > 3$.*

Example 1.3. We will see later from the proof of Theorem 1.2 that we can state the following for the case $n = 1$: There are no displaceable Lagrangian S^1 in a closed, symplectically aspherical, symplectic manifold (M^2, ω) if $\pi_1(L) \longrightarrow \pi_1(M)$ is injective. If M is simply connected,i.e. $M = S^2$, then it can have a displaceable Lagrangian S^1 in it. For example, let L be any non-equatorial cross section in (S^2, ω).

Example 1.4. There are examples of sympectically aspherical symplectic manifold (M, ω) for any dimension which contain displaceable Lagrangian spheres as submanifolds. For example, [Se] provides the procedure to obtain m Lagrangian n-spheres in the symplectic manifold (M, ω) given as the affine hypersurface in $\mathbb{C}^{n+1}$ defined by the equation $z_1^2 + z_2^2 + ... + z_n^2 = z_{n+1}^{m+1} + \frac{1}{2}$ and equipped with the standard symplectic form ω for any m and n, and the configuration of these spheres implies the existence of a displaceable Lagrangian sphere in (M, ω) for any n if $m \geq 3$. But this does not constitute a counterexample for our Theorem 1.2 since these symplectic manifolds are

[2]See next section for all the relevant definitions.

not closed. Also, by theorem 1.2, we can conclude that they can not be symplectically embedded into closed symplectically aspherical manifolds of the same dimension for $n > 3$.

Refining the Floer theoretic tools used in the proof of Theorem 1.2 we can relax the assumption that the ambient symplectic manifold is symplectically aspherical, and obtain:

Theorem 1.5. *Let (M^{2n}, ω) be a closed symplectic manifold with $\omega|_{\pi_2(M)} = 0$ and with minimal Chern number $N \geq n$. Then there are no displaceable Lagrangian spheres in (M, ω) if $n > 3$.*

Among the other Lagrangian submanifolds we have tried to use the method of proof described above are Lagrangian products of spheres:

Theorem 1.6. *Let $(M^{2(n+1)}, \omega)$ be a closed symplectic manifold with $\omega|_{\pi_2(M)} = 0$ and with minimal Chern number $N \geq n+1$. If $L = S^1 \times S^n$ is a displacable Lagrangian submanifold of $(M^{2(n+1)}, \omega)$ such that $\pi_1(L) \longrightarrow \pi_1(M)$ induced by inclusion is injective, then $n \in \{2, 3, 4\}$.*

Note that $\pi_1(L)$ is nontrivial in these examples.

Theorem 1.7. *Let $n > 2$ and $m > 2$ be integers and let $(M^{2(n+m)}, \omega)$ be a closed symplectic manifold with $\omega|_{\pi_2(M)} = 0$ and with minimal Chern number $N \geq n + m$. Then there are no displaceable Lagrangian $S^n \times S^m$ in $(M^{2(n+m)}, \omega)$.*

Finally, we consider compact rank 1 [3] symmetric spaces. The complete list of globally symmetric spaces of rank 1 is the sphere, real projective, complex projective and quaternionic projective spaces, and the Cayley projective plane ($\{S^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{QP}^n$, $Ca\mathbb{P}^2\}$), [Zi]. We see that using the same method of proof for these manifolds leads to the following statement for the Lagrangian embeddings of the quaternionic projective space and the Cayley projective plane, whereas it doesn't provide any such statement about the existence of Lagrangian real and complex projective spaces.

[3] Rank is defined as the maximum dimension of a subspace of the tangent space at any point on which the curvature is identically 0.

Theorem 1.8. *1. There is no displaceable Lagrangian quaternionic projective space in closed symplectic manifold* (M^{4n}, ω) *with* $\omega|_{\pi_2(M)} = 0$ *and with minimal Chern number* $N \geq 2n$ *for any* n.

2. There is no displaceable Cayley projective plane in closed symplectic manifold (M^{32}, ω) *with* $\omega|_{\pi_2(M)} = 0$ *and with minimal Chern number* $N \geq 16$.

1.1 Some known results

In this section we present some known results about displacability of Lagrangian submanifolds:

1.1.1 The case of M= $\mathbb{C}^n$

We do not know whether in general there are displacable Lagrangian submanifolds of the form $S^n \times S^m$ for any $n + m \leq 5$ in closed symplectically aspherical (M, ω). We know that for the case of $M = \mathbb{C}^3$ (for $n + m = 3$), there are Lagrangian embeddings of product of spheres ($S^1 \times S^2$), [Fu] . These are displacable since for L compact and M containing a factor of $\mathbb{C}$ we can always find a Hamiltonian diffeomorphism displacing L.
The following statement from [Po] is a result about the construction of Lagrangian submanifolds of $\mathbb{C}^n$ with given Maslov index k where $2 \leq k \leq n$.

Proposition 1.9. $[Po]$ *For every two integers* $2 \leq k \leq n$ *there exists a compact manifold* $L_{n,k}$ *which admits a monotone Lagrange embedding into* $\mathbb{C}^n$ *with Maslov index k. The manifold* $L_{n,k}$ *has the following structure:*

- $L_{n,n} = S^{n-1} \times S^1/\tau_{n-1} \times \tau_1$ *where* $\tau_j : S^j \longrightarrow S^j$ *is the standard antipodal involution.*

- $L_{n,k} = L_{k,k} \times S^{n-k}$ *where* $k < n$.

When n is even, $L_{n,n}$ *is diffeomorphic to* $S^{n-1} \times S^1$.

1.1.2 The cases of some other assumptions on M

The first statement of the next proposition is a result from [FS]. It is also quoted in [FS] that the second statement holds and that Lalonde and Polterovich proved the third statement in [LP].

Proposition 1.10. *Let (M, ω) be a symplectic manifold and $L \subset M - \partial M$ be a closed Lagrangian submanifold. Assume that either of the following cases hold:*

1. *(M, ω) is weakly exact and convex; the inclusion of L into M induces an injection $\pi_1(L) \longrightarrow \pi_1(M)$; and L admits a Riemannian metric none of whose closed geodesics is contractible.*

2. *(M, ω) is geometrically bounded; the symplectic area class restricted to $\pi_2(M, L)$ is zero.*

3. *the injection $L \subset M$ induces an injection $\pi_1(L) \longrightarrow \pi_1(M)$ and L admits a Riemannian metric of non-positive curvature.*

Then L is not displacable.

The following more general proposition which gives us conditions for nondisplaceability of a Lagrangian submanifold is from [MDS2] (see page 297):

Proposition 1.11. *(Gromov,[MDS2]) Let (M, ω) be a symplectic manifold without boundary and assume that (M, ω) is convex at infinity. Let $L \subset M$ be a compact Lagrangian submanifold such that $[\omega]$ vanishes on $\pi_2(M, L)$. Let $\phi : M \longrightarrow M$ be a Hamiltonian symplectomorphism. Then $\phi(L) \cap L \neq \emptyset$.*

2 Basic Concepts

[4] In this section we provide the basic definitions in symplectic geometry and Hamiltonian dynamics that we shall use in the following pages. One can refer to [MDS1] and [Ca] for the material in the subsections 2.1-2.4 and 2.8. We follow the presentation in [KS] in subsections 2.5, 2.6 and 2.7.

2.1 Symplectic Manifolds

A differential 2-form ω defined on a manifold M is called *symplectic* if it is closed (i.e. ω satisfies $d\omega = 0$) and non-degenerate (i.e. $\omega(u, v) = 0$ for all $v \in T_pM$ and for some $u \in T_pM$ implies that $u = 0$). A $symplectic\ manifold$ is a manifold together with a symplectic form. There are many examples of symplectic manifolds. The standard example is $(\mathbb{R}^{2n}, w)$ where w is the standard symplectic form defined, in local coordinates, as $w = \sum_i dx_i \wedge dy_i$. By Darboux's theorem, every symplectic structure is locally diffeomorphic to $(R^{2n},\ w)$. Hence there are no local symplectic invariants.

Another example of a symplectic manifold is (S^2, w) where w is the symplectic form defined by $w_p(u, v) =< p, u \times v >$ where $u, v \in T_p(S^2)$. Note that S^{2n} for $n > 1$ cannot be given any symplectic structure. This is due to the fact that the symplectic form ω should represent a nontrivial cohomology class whereas $H^2(S^n)$ is trivial for n >1.

2.2 Lagrangian Submanifolds

A submanifold L of a symplectic manifold (M^{2n}, ω) is called *Lagrangian* if it has dimension n and the symplectic form vanishes on L. That is if $\iota^*\omega = 0$ is satisfied where ι denotes inclusion map. For example, if the symplectic manifold is $(\mathbb{R}^2, \omega)$, any 1-dimensional submanifold is a Lagrangian submanifold. Other examples include the zero section of the cotangent bundle of

[4] This section and Section 3 includes previously published material from [KS] and permission to reprint is provided.

any manifold equipped with its canonical symplectic form, and the graph of a symplectomorphism in $(M \times M, \omega \oplus -\omega)$ where (M, ω) is any symplectic manifold. For further examples we can look at [AL].

2.3 Natural homomorphisms on $\pi_2(M)$

We shall define two homomorphisms. Let c_1 denote the first chern class of the tangent bundle of the manifold M. Define $I_{c_1} : \pi_2(M) \longrightarrow \mathbb{Z}$ by evaluation of c_1 on smooth representatives of classes in $\pi_2(M)$. That is, $I_{c_1}(A) = c_1(A) = \int_A c_1$ for A a smooth representative in $\pi_2(M)$. Similarly define $I_\omega : \pi_2(M) \longrightarrow \mathbb{R}$ by evaluation of ω on spheres, that is integration of the symplectic area class over a smooth representative of a class in $\pi_2(M)$.

A symplectic manifold (M, ω) is called *weakly exact* if the symplectic area class vanish on elements of $\pi_2(M)$, that is if $\omega|_{\pi_2(M)} = 0$.

A symplectic manifold (M, ω) is called *symplectically aspherical* if both the first Chern class and the symplectic area class vanish on elements of $\pi_2(M)$, that is if $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$. For example (CP^n, ω) and in particular $(S^2 = CP^1, \omega)$ where ω is the Fubini-Study symplectic form is not symplectically aspherical.

Any symplectic manifold with $\pi_2(M) = 0$ is trivially symplectically aspherical. For example $\mathbb{T}^{2n}$ for any n satifies this condition. Hence (T^{2n}, ω) for any n is symplectically aspherical.

One can ask whether there are examples of weakly exact symplectic manifolds which are not symplectically aspherical. An affirmative answer to this is given by Gompf in [Go] (see Introduction and page 4).

The *index of rationality* is defined as

$$inf_{A \in \pi_2(M)} \{\omega(A) | \omega(A) > 0\}. \tag{1}$$

A symplectic manifold (M, ω) is called *rational* if the *index of rationality* is positive.

A symplectic manifold (M, ω) is called *proportional* if there is a constant v such that $c_1(A) = v\omega(A)$ for all $A \in \pi_2(M)$.

We denote by N the *minimal Chern number* which is defined as the nonnegative integer such that the image of c_1 on $\pi_2(M)$ is $N\mathbb{Z}$.

2.4 Hamiltonian Flows

Let (M, ω) be a symplectic manifold and let H be any smooth, compactly supported function on $S^1 \times M$. We call H a *Hamiltonian*. Given any H, there is a vector field X_H defined by the equation

$$\iota_{X_{Ht}}\omega = -dH_t$$

where the left hand side of the equation denotes the contraction of the symplectic form along the time dependent vector field X_{H_t}. We let ϕ_H^t to denote the flow of this vector field and $\mathcal{P}(H)$ to denote the set of contractible (in M) 1-periodic orbits of ϕ_H^t.

Example 2.1. Consider $(M, \omega) = (\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i)$. Let $H = \frac{1}{2}(x_i^2 + y_i^2)$. Then X_H should satisfy

$$\iota_{X_H} \textstyle\sum_i dx_i \wedge dy_i = -(x_i dx_i + y_i dy_i).$$

Hence $X_H = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$.

2.4.1 Displaceable Lagrangian Submanifolds

A submanifold L of (M^{2n}, ω) is called *displaceable* if there is a Hamiltonian H such that the intersection of the submanifold with its image under the Hamiltonian flow is empty, i.e.

$$L \cap \phi_H^1(L) = \emptyset \tag{2}$$

For example, any compact subset K of R^{2n} is displaceable by some Hamiltonian. Let $H = y_i$ where y_i is a standard local coordinate which is nonzero on boundary of K. Then $X_H = \frac{\partial}{\partial x_i}$. Let d denote the diameter of K. For any $k \in K$, $d + \epsilon + x_i^k \notin K$ for some $\epsilon > 0$. Hence

$$K \cap \phi_H^{d+\epsilon}(K) = \emptyset. \tag{3}$$

We can reparametrize H to get an H' such that $\phi_H^{d+\epsilon}(K) = \phi_H'^1(K)$.

A Lagrangian submanifold L of a rational symplectic manifold is called *easily displaceable* if there is a Hamiltonian H displacing it such that the Hofer norm of H is less than half of the index of rationality.

2.5 Geodesic flows

Assume L is a closed manifold. Let g be a Riemannian metric on L, and consider the energy functional of g which is defined on the space of smooth loops $C^\infty(S^1, L)$, by

$$\mathcal{E}_g(q(t)) = \int_0^1 \frac{1}{2} \|\dot{q}(t)\|^2 \, dt. \tag{4}$$

The critical points of $\mathcal{E}_g$, $Crit(\mathcal{E}_g)$, are the closed geodesics of g with period equal to one. The closed geodesics of g with any positive period $T > 0$

correspond to the 1-periodic orbits of the metric $\frac{1}{T}g$, and are thus the critical points of the functional $\mathcal{E}_{\frac{1}{T}g}$.

The Hessian of $\mathcal{E}_g$ at a critical point $q(t)$ will be denoted by $Hess(\mathcal{E}_g)_q$. As is well known, the space on which $Hess(\mathcal{E}_g)_q$ is negative definite is finite-dimensional. It's dimension is, by definition, the *Morse index* of q and will be denoted here by $I_{Morse}(q)$. The kernel of $Hess(\mathcal{E}_g)_q$ is also finite-dimensional, and is always nontrivial unless L is a point.

A submanifold $D \subset C^\infty(S^1, L)$ which consists of critical points of $\mathcal{E}_g$ is said to be $Morse - Bott$ *nondegenerate* if the dimension of the kernel of $Hess(\mathcal{E}_g)_q$ is equal to the dimension of D for every $q \in D$. An example of such a manifold is the set of constant geodesics of any metric on L. This is a Morse-Bott nondegenerate submanifold which is diffeomorphic to L. The energy functional $\mathcal{E}_g$ is said to be Morse-Bott if all the 1-periodic geodesics are contained in Morse-Bott nondegenerate critical submanifolds of $\mathcal{E}_g$. Note that if $\mathcal{E}_g$ is Morse-Bott, then so is $\mathcal{E}_{\frac{1}{T}g}$ for any $T > 0$.

2.6 Perturbed Geodesic Flows

It will be useful for us to perturb a Morse-Bott energy functional $\mathcal{E}_g$ so that the critical points of the resulting functional are nondegenerate. We restrict ourselves to perturbations of the following classical form

$$\mathcal{E}_{g,V}(q) = \int_0^1 \left(\frac{1}{2}\|\dot{q}(t)\|^2 - V(t, q(t))\right) dt \tag{5}$$

where the function $V\colon S^1 \times L \to \mathbb{R}$ is assumed to be smooth. The critical points of $\mathcal{E}_{g,V}$ are solutions of the equation

$$\nabla_t \dot{q} + \nabla_g V(t, q) = 0 \tag{6}$$

where ∇_t denotes covariant differentiation in the $\dot{q}$-direction with respect to the Levi-Civita connection of g, and $\nabla_g V$ is the gradient vector field of V with respect to g. We refer to solutions of this equation as *perturbed geodesics*.

There is a dense set ${}_{reg}(g) \subset C^\infty(S^1 \times L)$ such that for $V \in_{reg} (g)$ the critical points of $\mathcal{E}_{g,V}$ are nondegenerate, see [We2].

When a Morse-Bott functional ${}_g$ is perturbed, the critical submanifolds break apart into critical points. For a small perturbation V it is possible to relate each critical point of $\mathcal{E}_{g,V}$ to a specific critical submanifold of $\mathcal{E}_g$, and to relate their indices. Here is the precise statement.

Lemma 2.2. *Let $\mathcal{E}_g$ be Morse-Bott and let ${}^a(\mathcal{E}_g) = \amalg_{j=1}^{\ell} D_j$ be the (finite, disjoint) union of all critical submanifolds of $\mathcal{E}_g$ with energy less than a. Let $\epsilon > 0$ be small enough so that the ϵ-neighborhoods of the D_j in $C^\infty(S^1, L)$ are disjoint. If $V \in_{reg} (g)$ is sufficiently small, then each nondegenerate critical point $q(t)$ of $\mathcal{E}_{g,V}$, with action less than a, lies in the ϵ-neighborhood of exactly one component D_j of ${}^a(\mathcal{E}_g)$. Moreover,*

$$I_{Morse}(q) \in [I_{Morse}(D_j), I_{Morse}(D_j) + \dim(D_j)].$$

2.7 Hamiltonian Geodesic Flows

Consider the cotangent bundle of L, T^*L, equipped with the symplectic structure $d\theta$ where θ is the canonical Liouville 1-form. We will denote points in T^*L by (q,p) where q is in L and p belongs to T_q^*L. In these local coordinates, $\theta = pdq$ and so $d\theta = dp \wedge dq$.

The metric g on L induces a bundle isomorphism between TL and the cotangent bundle T^*L and hence a cometric on T^*L.

Let $K_g \colon T^*L \to \mathbb{R}$ be the function $K_g(q,p) = 2\|p\|^2$. The Legendre transform yields a bijection between the critical points of the perturbed energy functional $\mathcal{E}_{g,V}$ and the critical points of the action functional

$$\mathcal{A}_{K_g+V} \colon C^\infty(S^1, T^*L) \to \mathbb{R}$$

defined by

$$\mathcal{A}_{K_g+V}(x) = \int_0^1 (K_g + V)(t, x(t))\, dt - \int_{S^1} x^*\theta.$$

The critical points of $\mathcal{A}_{K_g+V}$ are the 1-periodic orbits of the Hamiltonian K_g+V on T^*L. If $x(t) = (q(t), p(t))$ belongs to $\mathcal{P}(K_g+V)$, then its projection to L is a closed 1-periodic solution of (6) with initial velocity $\dot{q}(0)$ determined by $g(\dot{q}(0), \cdot) = p(0)$. Moreover, $x(t)$ is nondegenerate if and only if $q(t)$ is nondegenerate.

2.8 Weinstein's Tubular Neighborhood Theorem

There is a standard theorem in symplectic geometry which allows one to identify a sufficiently small neighborhood of a Lagrangian L in a symplectic manifold with a neighborhood of the zero section in the cotangent bundle of L.

Theorem 2.3. *(Weinstein Tubular Neighborhood Theorem [Ca]) There are neighborhoods U of a Lagrangian L in (M, ω) and U_0 of the zero section of (T^*L, ω_{std}) and a diffeomorphism ψ from U_0 to U such that*

$$\psi^*(\omega) = \omega_{std} \text{ and } \iota_0 \circ \psi(U_0) = \iota(U) \tag{7}$$

where ι_0 and ι denote the embedding of L as the zero section into T^*L and embedding of L into M respectively.

Remark 2.4. Consider a neighborhood of the zero section in T^*L of the following type

$$U_r = \{(q, p) \in T^*L \mid \|p\| < r\}.$$

By Theorem 2.3, for sufficiently small $r > 0$, there is a neighborhood of L in (M, ω) which is symplectomorphic to U_r. We only consider values of r for which this holds, and will henceforth identify U_r with a neighborhood of L in (M, ω).

For a subinterval $I \subset [0, r)$, we will use the notation

$$U_I = \{(q, p) \in U_r \mid \|p\| \in I\}.$$

Remark 2.5. We can extend a Hamiltonian H defined on U_r to all of M. Choose a bump function $\hat{\sigma}\colon [0, +\infty) \to \mathbb{R}$ such that $\hat{\sigma}(s) = 1$ for s near 0 and $\hat{\sigma}(s) = 0$ for $s > r$. Then $\hat{\sigma}(\|p\|) \cdot H$ for $\|p\| \in I$ is the corresponding function with support in U_I and is defined on M by Theorem 2.3.

2.9 Basic indices

We define the Maslov index and the Maslov class following the presentation from [KS].

2.9.1 The Maslov index

For loops of Lagrangian subspaces:

The Maslov index can be defined for loops of Lagrangian spaces in Lagrangian Grassmannian Λ_{2n} of the space of all Lagrangian subspaces of $\mathbb{R}^{2n}$ equipped with standard symplectic structure as follows:

Let $\eta\colon S^1 \to \Lambda_{2n}$ be a loop of Lagrangian subspaces and let $V \in \Lambda_{2n}$ be a fixed reference space. One calls $t_0 \in S^1$ a **crossing** of η (with respect to V) if $\eta(t_0)$ and V intersect nontrivially. At a crossing t_0, one can define a crossing form $Q(t_0)$ on $\eta(t_0) \cap V$ as follows. Let $W \in \Lambda_{2n}$ be transverse to $\eta(t_0)$. For each v in $\eta(t_0) \cap V$ we define, for t near t_0, the path $w(t)$ in W by

$$v + w(t) \in \eta(t).$$

We then set

$$Q(t_0)(v) = \frac{d}{dt}\Big|_{t=t_0} \omega(v, w(t)).$$

The crossing t_0 is said to be **regular** if $Q(t_0)$ is nondegenerate. If all the crossings of η are regular then they are isolated and the Maslov index is defined by

$$\mu_{Maslov}(\eta; V) = \sum_{t \in S^1} sign(Q(t)), \tag{8}$$

where *sign* denotes the signature and the sum is over all crossings. This integer is independent of the choice of V (as well as the choices of W at each crossing).

(Recall that the signature of a quadratic form is the number of its positive eigenvalues minus the number of its negative eigenvalues.)

For paths of symplectic matrices:

We can also define the Maslov index for paths of symplectic matrices. It is defined as an intersection number :

We denote by $Sp^*(n)$ the set of all symplectic matrices Φ such that $det(\Phi - Id) \neq 0$ and let $C(n)$ be those with $det(\Phi - Id) = 0$. $C(n)$ is called the $Maslov$ *cycle*. We let $\mathcal{SP}(n)$ denote the set of paths $\gamma(t)$ of symplectic matrices that start at identity and end in $Sp^*(n)$. Then $\mu_{Maslov}(\gamma(t))$ for $\gamma(t) \in \mathcal{SP}(n)$ is defined to be the intersection number of $\gamma(t)$ with the Maslov cycle $C(n)$ (see [Sa] or [We2]).

2.9.2 The Maslov Class of a Lagrangian Submanifold

We will define the Maslov class of a Lagrangian submanifold L as a map from $\pi_2(M, L)$ to $\mathbb{Z}$. We specify the value it takes on an element $[w] \in \pi_2(M, L)$ as the following:

Let $q(t)$ be any loop in L, let D denote the closed unit disc in $\mathbb{R}^2$ and let $w : (D, \partial D) \longrightarrow (M, L)$ be a continuous representative of $[w] \in \pi_2(M, L)$ such that $w(e^{2\pi i t}) = q(t)$. We shall associate a loop of Lagrangian subspaces $\eta(t)$ to the spanning disc w as follows:

Given any metric on L, there is an induced cometric on T^*L and a Levi-Civita connection on T^*L which gives us the splitting of the tangent space of the cotangent space of L as

$$TT^*L = Hor(TT^*L) \oplus Vert(TT^*L). \tag{9}$$

where $Hor(TT^*L)$ and $Vert(TT^*L)$ denote the horizontal and vertical bundle components of TT^*L respectively. The vertical bundle $Vert(TT^*L)$ is a Lagrangian subbundle of $T(T^*L)$.

Let $\Phi_w(t)$ be a symplectic trivialization of $q^*(TT^*L)$.

Set $\eta(t) = \Phi_w^{-1}(Vert(q(t)))$ which is a loop of Lagrangian subspaces. Then we define the Maslov class of L by specifying the value that it takes on $[w]$ as

$$\mu^L_{Maslov}([w]) = \mu_{Maslov}(\eta(t)) \tag{10}$$

Note that the index is independent of the representative of [w]. A different representative of the same class would give a different symplectic trivialization, so we would have different loops of Lagrangian subspaces. But the loops would be homotopic. Hence their Maslov indices would be the same, and so the index is well-defined on $\pi_2(M, L)$.

Also note that any class in $\pi_2(M, L)$ can be realized as a spanning disk by making use of connected sum: To get a spanning disk which is an element of a specific class in $\pi_2(M, L)$, one can take the connected sum of a given spanning disk for x(t) and an element of $\pi_2(M)$ which will again be a spanning disk for x(t).

We denote by N_L the *minimal Maslov number* of L and it defined as the smallest nonnegative integer such that the image of μ^L_{Maslov} on $\pi_2(M, L)$ is $N_L\mathbb{Z}$.

2.9.3 The Conley-Zehnder Index

We shall give an axiomatic definition.

Let $\Sigma(n)$ denote the set of paths of symplectic matrices $\Phi(t)$ where $t \in [0,1]$ with $\Phi(0) = Id$ and $\Phi(1) \in Sp^*(2n)$. For each n, there is a unique function μ_{cz}, called the $Conley - Zehnder\ index$, which assigns an integer to any $\Phi(t) \in \Sigma(n)$ and these functions have the following properties:

1. homotopy: Two paths of symplectic matrices starting at identity and ending in $Sp^*(2n)$ have the same Conley-Zehnder index if and only if they are homotopic.

2. direct sum: For any $\Phi(t) \in \Sigma(n)$ and $\Psi(t) \in \Sigma(m)$, the path formed in $\Sigma(n+m)$ by $\Phi \oplus \Psi(t)$ satisfies $\mu_{cz}(\Phi \oplus \Psi) = \mu_{cz}(\Phi) + \mu_{cz}(\Psi)$

3. loop: If Φ is a path of symplectic matrices as above and Ψ is a loop of symplectic matrices with $\Psi(0) = \Psi(1) = Id$, then $\mu_{cz}(\Psi\Phi) = \mu_{cz}(\Phi) + 2\mu_{Maslov}(\Psi)$

4. inverse: $\mu_{cz}(\Phi^{-1}) = -\mu_{cz}(\Phi)$.

5. normalization: The index for the path $t \overset{\Phi_0}{\to} e^{\pi J t}$ where $t \in [0,1]$ is 1.

   We note that the normalization we use differs by a minus sign from the one in [Ke1] and [Ke2] and differs from the one in [MDS1] where Maslov index 1 is assigned to the path $t \longmapsto e^{2\pi i t}$ where $t \in [0,1]$.It is the same as the one used in [KS].

For an example of a computation of the Conley-Zehnder index of a path in $Sp(2,\mathbb{R})$ one can see [We1].

The Conley-Zehnder index can also be defined using spectral flow, see [Wh].

2.9.4 The Conley-Zehnder index of a periodic orbit with respect to a spanning disk

One can also define a Conley-Zehnder index for the contractible nondegenerate periodic orbits of a general Hamiltonian flow. Let H be a Hamiltonian on (M, ω) and let $x\colon S^1 \to M$ be a contractible and nondegenerate 1-periodic orbit of X_H. A spanning disc for x, $w\colon D^2 \to M$, determines a symplectic trivialization

$$\Phi_w\colon S^1 \times \mathbb{R}^{2n} \to x^*(TM).$$

The Conley-Zehnder index of x with respect to w is then defined by

$$\mu_{cz}(x, w) = \mu_{cz}\left(\Phi_w(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Phi_w(0)\right).$$

2.9.5 The effect of the first Chern class on the Maslov and Conley-Zehnder indices

We would like to describe how the Maslov index on $\pi_2(M, L)$ changes under the action of $\pi_2(M)$. This depends on the first Chern class of A, $c_1(A)$. Let $\#$ denote the connected sum. We have the following lemma (see page 4 of [Oh]):

Lemma 2.6. *Let w and $\tilde{w}$ be two spanning disks for a loop $q(t)$ in L. Let $A \in \pi_2(M)$ be the element formed by gluing the spanning disks along $q(t)$ as $w\#\overline{\tilde{w}}$ where $\overline{\tilde{w}}$ represents $\tilde{w}$ with opposite orientation. Then we have*

$$\mu_{Maslov}([w]) - \mu_{Maslov}([\tilde{w}]) = 2c_1(A). \tag{11}$$

We see that a similar formula holds for the Conley-Zehnder indices and we have the following lemma.

Lemma 2.7. *Let H be a Hamiltonian on (M,ω) and let $x\colon S^1 \to M$ be a contractible and nondegenerate 1-periodic orbit of X_H. Let w and $\tilde{w}$ be two spanning disks for a periodic orbit $x(t)$. Then we have*

$$\mu_{cz}(x,w) - \mu_{cz}(x,\tilde{w}) = 2c_1(A) \tag{12}$$

where $A \in \pi_2(M)$ is the element formed by gluing the spanning disks along $x(t)$ as $w\#\overline{\tilde{w}}$ where $\overline{\tilde{w}}$ represents $\tilde{w}$ with opposite orientation.

Proof. For gluing $A \in \pi_2(M)$ to the map w, we have the formula

$$\mu_{cz}(x, A\#w) = \mu_{cz}(x,w) + 2c_1(A)$$

(see [KS]). Now if we are given any two spanning disks w and $\tilde{w}$ with the same boundary and if A is the element of $\pi_2(M)$ formed by $w\#\overline{\tilde{w}}$, then

$$\mu_{cz}(x,w) = \mu_{cz}(x, w\#(\overline{\tilde{w}}\#\tilde{w})) = \mu_{cz}((x, w\#\overline{\tilde{w}})\#\tilde{w}) = \mu_{cz}(x, A\#\tilde{w})$$

$$= \mu_{cz}(x,\tilde{w}) + 2c_1(A) \tag{13}$$

where the last equality is by the above mentioned formula . Hence from the equality of the first term and the last term we get the result.

□

From this lemma we can see that when we have $c_1|_{\pi_2 M} = 0$, the Conley-Zehnder index becomes independent of the choice of spanning disc, so we shall denote it by $\mu_{cz}(x)$.

3 The relation between the Morse, Conley-Zehnder and Maslov indices

There is an index relation relating the Morse, Conley-Zehnder and Maslov indices for periodic orbits which satisfy certain conditions. (For example, one can see [Du], [We2] or [KS].) For our purpose of proving the theorems in the first section, we need to make use of specific periodic orbits for which we know the Conley-Zehnder index. Hence we shall state the index relation for a specific orbit of a specific Hamiltonian function the existence of which is guaranteed by the theorems we will state in the next subsection.

3.1 The propositions on the existence of an orbit with specific index

For the case of symplectically aspherical manifolds, the existence of a non-constant 1-periodic orbit of a Hamiltonian with Conley-Zehnder index n+1 follows from [Ke2] in the next proposition.

Proposition 3.1. $[Ke2]$ *Let L be a displacable Lagrangian submanifold of a closed and symplectically aspherical symplectic manifold* (M, ω). *There is an* $\epsilon > 0$ *and a Floer Hamiltonian* $H_L = H_L^\epsilon$ *such that there is a nonconstant contractible 1-periodic orbit* $x(t) = (q(t), p(t)) \in \mathcal{P}(H_L^\epsilon)$ *in* $U_{(\epsilon,2\epsilon)}$ *such that* $\mu_{cz}(x) = n + 1$.

Proof. This follows from Proposition 5.1, Lemma 6.1, Proposition 6.3 and the explanation on page 26 in [Ke2].

□

For the case of manifolds with first Chern class not necessarily zero, we shall state the analogue of the above proposition and see in section 3.3 that this statement holds:

Proposition 3.2. *Let L be a displacable Lagrangian submanifold of a closed weakly exact symplectic manifold (M,ω) with minimal Chern number $N \geq n$. There is an $\epsilon > 0$ and a Floer Hamiltonian $H_L = H_L^\epsilon$ such that there is a nonconstant contractible 1-periodic orbit $x(t) = (q(t), p(t)) \in \mathcal{P}(H_L^\epsilon)$ in $U_{(\epsilon,2\epsilon)}$ and a spanning disk w for $x(t)$ such that $\mu_{cz}(x,w) = n+1$.*

3.2 A Hamiltonian Function H_L with desired properties

In this section we construct a Hamiltonian H_L whose Hamiltonian flow is supported in a tubular neighborhood of L. The nonconstant contractible periodic orbits of this flow project to perturbed geodesics on L. As well, the Hamiltonian path $\phi^t_{H_L}$ fails to minimize the negative Hofer length in its homotopy class.

The propositions which are stated in section 3.1 are stated for this Hamiltonian H_L the construction of which is provided in this section and the properties of which are stated in Proposition 3.3.

3.2.1 Reparametrization of the Hamiltonian generating the cogeodesic flow

Let $\nu = \nu_{(\epsilon,C)} \colon [0,+\infty) \to \mathbb{R}$ be a smooth function with the following properties:

- $\nu(0) = 0$, $\nu'(0) = 0$, $\nu''(0) = 0$;
- $\nu', \nu'' > 0$ on $(0, 2\epsilon)$;
- $\nu = -\epsilon + Cs$ on $[2\epsilon, r - 2\epsilon]$;
- $\nu' > 0$ and $\nu'' < 0$ on $(r - 2\epsilon, r - \epsilon)$;
- $\nu = A < rC$ on $[r - \epsilon, +\infty)$.

Define a Hamiltonian K_ν on M by

$$K_\nu(q,p) = \begin{cases} \nu(\|p\|) & \text{if } (q,p) \text{ is in } U_r, \\ A & \text{otherwise.} \end{cases}$$

Location of the orbits:

The Hamiltonian flow of K_ν is trivial in both U_0 and the complement of $U_{r-\epsilon}$. Hence, each nonconstant 1-periodic orbit $x(t) = (q(t), p(t))$ of K_ν is contained in $U_{(0,r-\epsilon)}$. We have

$$X_{K_\nu}(q,p) = \left(\frac{\nu'(\|p\|)}{\|p\|}\right) X_{K_g}(q,p) \tag{14}$$

where K_g is the kinetic energy Hamiltonian generating the cogeodesic flow of g on the cotangent bundle.

We may also choose the positive constant C so that it is not the length of any closed geodesic of g and this implies that all nonconstant orbits of K_ν occur on the level sets contained in $U_{(0,2\epsilon)}$ or $U_{(r-2\epsilon,r-\epsilon)}$, where ν is convex or concave, respectively. In fact, these nonconstant orbits lie in

$$U_{(+\delta,2\epsilon-\delta)} \cup U_{(r-2\epsilon+\delta,r-\epsilon-\delta)}$$

for some $\epsilon > \delta > 0$. This follows from the fact that dK_ν equals zero along the boundary of $U_{(0,2\epsilon)} \cup U_{(r-2\epsilon,r-\epsilon)}$.

3.2.2 Statement of H_L and its properties

Proposition 3.3. *Let L be a closed, displaceable, Lagrangian submanifold of a closed symplectic manifold (M,ω) and let g be a metric on L whose energy functional is Morse-Bott. Fix a sufficiently small r such that we can identify U_r with a neighborhood of L in (M,ω). Let $\nu : [0,+) \to \mathbb{R}$ be a smooth* *function as defined in the previous subsection.*

Then, for sufficiently small $\epsilon > 0$, there is a Hamiltonian H_L with the following properties:

(H1) *The constant 1-periodic orbits of H_L correspond to the critical points of a Morse function F on M. Near these points the Hamiltonian flows of H_L and $c_0 F$ are identical for some arbitrarily small constant $c_0 > 0$;*

(H2) *The nonconstant 1-periodic orbits of H_L are nondegenerate and contained in $U_{(0,r-\epsilon)}$. If C is not the length of a closed geodesic of g on L, then they are contained in $U_{(0,2\epsilon)} \cup U_{(r-2\epsilon,r-\epsilon)}$. Each such orbit projects to a nondegenerate closed perturbed geodesic $q(t)$.*

Moreover, if T is the period of q, then q can be associated to exactly one critical submanifold D of $\mathcal{E}_{\frac{1}{T}g}$ and

$$I_{Morse}(q) \in [I_{Morse}(D), I_{Morse}(D) + \dim(D)]; \tag{15}$$

In either case, for every nonconstant $x(t) = (q(t), p(t)) \in \mathcal{P}(H_L)$, we have the uniform bound

$$\dot{q(t)} < 2C; \tag{16}$$

(H3) *There is a point $Q \in L \subset M$ which is the unique local minimum of $H_L(t, \cdot)$ for all $t \in [0, 1]$. Moreover,*

$$\|H_L\|^- = -\int_0^1 H_L(t, Q)\, dt > 2e(U_r); \tag{17}$$

(H4) *The flow of $\phi^t_{H_L}$ does not minimize the negative Hofer length in its homotopy class.*

Proof. **A Morse function isolating L**

Let $F_0\colon M \to \mathbb{R}$ be a Morse-Bott function with the following properties:

- The submanifold L is a critical submanifold with index equal to 0.
- On U_r, we have $F_0 = f_0(\|p\|)$ for some increasing function $f_0 \colon [0, r] \to \mathbb{R}$ which is strictly convex on $[0, 2\epsilon)$, linear on $[2\epsilon, r - 2\epsilon]$, and strictly concave on $(r - 2\epsilon, r - \epsilon]$.
- All critical submanifolds other than L are isolated nondegenerate critical points with strictly positive Morse indices.

Such a function is easily constructed by starting with the square of the distance function from L with respect to a metric which coincides, in the normal directions, with the cometric of g inside U_r. This distance function can then be deformed within U_r to obtain the first and second properties above. Perturbing the resulting function away from U_r, one can ensure that it is Morse.

Let $F_L \colon L \to \mathbb{R}$ be a Morse function with a unique local minimum at a point Q in L. Choose a bump function $\hat{\sigma} \colon [0, +\infty) \to \mathbb{R}$ such that $\hat{\sigma}(s) = 1$ for s near zero and $\hat{\sigma}(s) = 0$ for $s \geq r/5$. Let $\sigma = \hat{\sigma}(\|p\|)$ be the corresponding function on M with support in $U_{2\epsilon}$ and set

$$F = F_0 +_L \cdot \sigma \cdot F_L.$$

For a sufficiently small choice of $\epsilon_L > 0$, F is a Morse function whose critical points away from $U_{2\epsilon}$ agree with those of F_0 and whose critical points in $U_{2\epsilon}$ are precisely the critical points of F_L on $L \subset M$ (see, for example, [BH] page 87).

For $c_0 > 0$, consider the function

$$H_0 = K_\nu + c_0 F.$$

By construction, we have

$$H_0 = \begin{cases} c_0 F & \text{on } U_{[0,2\epsilon)}, \\ (\nu + c_0 f_0)(\|p\|) & \text{on } U_{[2\epsilon, r-\epsilon]}, \\ A + c_0 F & \text{elsewhere.} \end{cases}$$

From this expression it is clear that each H_0 is a Morse function with $Crit(H_0) = Crit(F)$. As well, Q is the unique local minimum of H_0.

Moreover, when c_0 is sufficiently small, the nonconstant 1-periodic orbits of H_0, like those of K_ν, are contained in

$$U_{(+\delta, 2\epsilon-\delta)} \cup U_{(r-2\epsilon+\delta, r-\epsilon-\delta)}$$

for some $\epsilon > \delta > 0$.

Proof of (H4):

According to Proposition 2.1. of [Ke2] we will have property (H4) if we have that U has finite displacement energy, that is $e(U) < \infty$ and if $\|H_L\|^- > 2e(U)$. Since (M, ω) is weakly exact, L is displaceable implies that it is easily displaceable and hence $e(U)$ is finite. (It is strictly less than half the index of rationality which is infinite.) We have that $\|K_\nu\|^- > 2e(U)$ and that $\|H_L\|^- > \|K_\nu\|^-$ so we have **(H4)**.

Nondegeneracy of orbits for (H2):

The function H_0 has properties **(H1)**, **(H3)** and **(H4)**. To obtain a function with property **(H2)** we must perturb H_0 so that the 1-periodic orbits of the resulting Hamiltonian are nondegenerate.

Let N^ρ be a critical submanifold of $\mathcal{A}_{H_0}$ which is contained in the level set $\|p\| = \rho$. Denote the projection of N^ρ to L by D^ρ. Then D^ρ is a Morse-Bott nondegenerate set of periodic geodesics with period $((\nu' + c_0 f_0')(\rho))^{-1}$. Alternatively, D^ρ can be viewed as a collection of 1-periodic geodesics of the metric $(\nu' + c_0 f_0')(\rho) g$. We will adopt this latter point of view.

There are finitely many critical submanifolds of $\mathcal{A}_{H_0}$. We label the submanifolds of 1-periodic closed geodesics which appear as their projections by

$$\{D_j^{\rho_j} \mid j = 1, \dots \ell\}.$$

Theorem 1.1 of [We2] (Theorem 3.3 of [KS]) implies that the set of potentials

$$\bigcap_{j=1,\dots,\ell} \mathcal{V}_{reg}((\nu' + c_0 f_0')(\rho_j)g)$$

is dense in $C^\infty(S^1 \times L)$. We can choose a V in this set which is arbitrarily small with respect to the C^∞-metric. By Lemma 2.2 , the projection, $q(t)$, of each 1-periodic orbit $x(t)$ of $\nu(K_g + V)$ is then nondegenerate and lies arbitrarily close to (within a fixed distance of) exactly one of the $D_j^{\rho_j}$ and satisfies

$$I_{Morse}(q) \in [I_{Morse}(D_j^{\rho_j}), I_{Morse}(D_j^{\rho_j}) + \dim(D_j^{\rho_j})]. \tag{18}$$

Given such a V we define $V_0 \colon S^1 \times M \to \mathbb{R}$ so that

$$V_0 = \begin{cases} V(t,q) & \text{, in } U_{(2\epsilon+\delta, r-2\epsilon-\delta)} \\ 0 & \text{, on the complement of } U_{(2\epsilon, r-2\epsilon)}. \end{cases}$$

Clearly, the function V_0, like V itself, can be chosen to be arbitrarily small. We then define H_L by

$$H_L(t,q,p) = \begin{cases} (\nu + c_0 f_0)\left(\sqrt{2\|p\|^2 + V_0(t,q,p)}\right) & \text{for } (q,p) \text{ in } U_{(2\epsilon, r-2\epsilon)}, \\ H_0(q,p) & \text{otherwise.} \end{cases}$$

Since H_L is a small perturbation of K_ν, this new Hamiltonian still has properties **(H1)** - **(H4)**.

□

3.3 Proof of Proposition 3.2

Let $H_L = H_L^\epsilon$ be a Hamiltonian as described in Proposition 3.3. We need to show that it has a periodic orbit with a spanning disk such that the Conley-Zehnder index is $n+1$ and the orbit lies in $U_{(0,2\epsilon)}$. Note that this neighborhood corresponds to the part of domain of ν on which it is convex. Let $x(t)$ be the orbit obtained from Proposition 5.9 in section 5. By Proposition on H_L, we have that $x(t)$ lies either in $U_{(0,2\epsilon)}$ or $U_{(r-2\epsilon,r-\epsilon)}$. Noting that the statements and proofs of Lemmas 6.2. and 6.3 of [Ke2] are independent of the assumption on the first chern class, we can conclude that Proposition 3.2 holds.

3.4 The statement of the index relation

We adapt the statement of [KS] about this index relation to the Hamiltonian H_L of the previous proposition of [Ke2]. The Hamiltonian described in [Ke2] and this work are slightly different than the Hamiltonian described in [KS] but we see that we can state the same index relation. The index relation actually holds for any Hamiltonian that is a reparametrization of the cogeodesic flow provided we take into account the shifts in indices due to convexity/concavity of the Hamiltonian, see [KS], [Th].

Proposition 3.4. *Let $x(t) = (q(t), p(t))$ be a nonconstant 1-periodic orbit of the H_L of the previous proposition which is in $U_{(0,2\epsilon)}$ and let w be a spanning disc for x. Then we have the relation*

$$\mu^L_{Maslov}([w]) = \mu_{cz}(x, w) - I_{Morse}(q)\ (+1) \tag{19}$$

where (+1) contributes only if q^*TL is nontrivial.

Proof. See [KS].

3.5 The effect of the first Chern class on the index relation

From the lemmas of section 2.9.5, we see that the Conley-Zehnder index and the Maslov index are effected by the same amount when the Chern class is different due to the choice of a different spanning disc. Since the Morse index of a geodesic is independent of choice of a spanning disk, it is independent of the chern class,and we see overall that the index relation is independent of the choice of a spanning disk and hence independent of our assumption that $c_1 = 0$ on $\pi_2(M)$.

4 Some computed indices

4.1 Two Lemmas on Maslov Class

Lemma 4.1. *Let* (M, ω) *be a symplectic manifold such that* $c_1(M)\big|_{\pi_2(M)} = 0$ *and* L *be a Lagrangian submanifold such that* $\pi_1(L) \longrightarrow \pi_1(M)$ *induced by inclusion is injective. Then* $\mu^L_{Maslov} = 0$.

Proof. By the assumption on Chern class, $\mu^L_{Maslov} : \pi_2(M, L) \longrightarrow \mathbb{Z}$ descends to a map on $ker(\pi_1(L) \longrightarrow \pi_1(M))$ induced by inclusion as in [Ke2] and the result follows by assumption. □

When we omit the assumption on Chern class of M but add an assumption on L, we can again obtain zero Maslov index.

Lemma 4.2. *Let* (M, ω) *be a symplectic manifold and* L *be a Lagrangian submanifold such that* $\pi_2(L) = 0$ *and* $\pi_1(L) \longrightarrow \pi_1(M)$ *induced by inclusion is injective. Then* $\mu^L_{Maslov}([w]) = 2c_1([w])$.

Proof. If we write the long exact sequence of homotopy for the pair (M,L) with the given assumptions on L, we see that $\pi_2(M, L)$ is isomorphic to $\pi_2(M)$. Hence we can consider μ^L_{Maslov} as a map on $\pi_2(M)$. Also, note that in the same way, we can consider c_1 as a map on $\pi_2(M, L)$. By the assumptions on L, choose $w^{‘}$ so that $w^{‘}$ lies in L and $\mu_{Maslov}([w^{‘}]) = 0$. Then the index relation for the difference in Maslov indices caused by different spanning disks (namely Lemma 2.6) gives us the statement of the lemma. □

Example 4.3. Note that the spheres and products of spheres satisfy these assumptions for dimensions greater than 2.

4.2 On Morse index of Critical Submanifolds

4.2.1 Morse index for the case of S^n

The main theorem of this section desribes the critical submanifolds of the energy functional on the loops on the sphere. We present the proof of computation of the Morse index of the critical submanifolds as in [Kl] :

Lemma 4.4. *([Kl], p. 58) The eigenvectors of the operator $A_c = id - (1 - \nabla^2)^{-1} \circ R + 1$ belonging to the eigenvalue $\lambda \in \mathbb{R}$ are the* ***periodic*** *solutions of the differential equation*

$$(\lambda - 1)(\nabla^2 - 1)\xi - (R+1)\xi = 0. \tag{20}$$

[5]

Let g denote the round metric on S^n. For example, in particular for n=2, the round metric in spherical coordinates is of the form $g = \begin{bmatrix} 1 & 0 \\ 0 & sin^2\theta \end{bmatrix}$.

Theorem 4.5. *([Kl], p. 71)*

The critical set $Cr\Lambda S^n$ of the energy functional $\mathcal{E}_g(q) = \int_0^1 \frac{1}{2}\|\dot{q}(t)\|^2 dt$ on the loop space $\Lambda(S^n)$ decomposes into the nondegenerate critical submanifolds

[5] A_c is a self adjoint operator $A_c : T_c\Lambda \longrightarrow T_c\Lambda$ obtained by the identity

$$< A_c\xi, \xi' >_1 = < \xi, A_c^T\xi' >_1 = D^2E(c)(\xi, \xi') = < \xi, \xi' >_1 - < (R + id)\xi, \xi' >_0 \tag{21}$$

where c is a critical point of E in ΛM and we have the definitions

$$< \xi, \xi >_0 = \int_S < \xi, \xi >_t dt \tag{22}$$

and

$$< \xi, \xi >_1 = < \xi, \xi >_0 + < \nabla\xi, \nabla\xi >_0 . \tag{23}$$

Also note that A_c needs to be periodic in t since $c(t)$ and hence the Hessian is periodic in t.

$\Lambda^0 S^n$ which is isomorphic to S^n and B_q consisting of the q-fold covered great circles $q = 1, 2, \ldots$. B_q is isomorphic to the Stiefel manifold $V(2, n+1)$ of orthonormal 2-frames in $\mathbb{R}^{n+1}$ and the index of B_q is $(2q-1)(n-1)$.

Proof. The special form of the curvature tensor of S^n yields, for $c \in B_q$, i.e. $|\dot{c}| = 2\pi q$,

$$R(\xi(t), \dot{c}(t), \dot{c}(t)) = - < \dot{c}(t), \xi(t) > \dot{c}(t) + < \dot{c}(t), \dot{c}(t) > \xi(t) = \tag{24}$$

[6]

$$- < \dot{c}(t), \xi(t) > \dot{c}(t) + 4\pi^2 q^2 \xi(t). \tag{25}$$

Thus $\lambda = 1$ is not an eigenvalue. With this, the formula of Lemma 4.4 is

$$(\nabla^2 - 1)\xi - \tfrac{(R+1)}{\lambda - 1}\xi = 0 \tag{26}$$

That is,

$$\nabla^2 \xi + \tfrac{(R+\lambda)}{1-\lambda}\xi = 0 \tag{27}$$

[6] Note that the curvature tensor produces a vector perpendicular to $\dot{c}$ when $< \dot{c}, \dot{c} >= |\dot{c}|^2 = 1$. One can observe this by first assuming that ξ and $\dot{c}$ are perpendicular. If not, then note that we get rid of the part/component of ξ which is in the direction of $\dot{c}$.

So,

$$\nabla^2\xi + \tfrac{(4\pi^2q^2+\lambda)}{1-\lambda}\xi + \tfrac{<\dot{c},\xi>}{\lambda-1}\dot{c} = 0$$

(28)

We decompose ξ into the subset of tangential vectors,

$\xi(t) = \alpha(t)\dot{c}(t)$, and vertical vectors, $\xi(t)\perp\dot{c}$.

Then last formula decomposes into

$$(tan)\ \ddot{\alpha}(t) + \frac{\lambda\alpha(t)}{(1-\lambda)} = 0,$$

(29)

$$(ver)\ \nabla^2\xi + \frac{(4\pi^2q^2+\lambda)}{(1-\lambda)}\xi = 0$$

(30)

For $\lambda < 0$, (tan) has no periodic solutions. The nontrivial solutions of (ver) occur for

$$\lambda = \frac{4\pi^2(p^2-q^2)}{(1+4\pi^2p^2)}, \quad p \in \{0, 1, ..., q-1\}$$

(31)

For $p = 0$, the solutions are

$$\xi(t) = \xi_0 \tag{32}$$

for $\xi_0 \perp \dot{c}$.

For $0 < p < q$, the solutions are

$$\xi(t) = \xi_0 cos 2\pi pt + \xi_1 sin 2\pi pt \tag{33}$$

for $\xi_0 \perp \dot{c}$ and $\xi_1 \perp \dot{c}$.

The dimension of $T_c^- \Lambda S^n$ for $c \in B_q$ therefore becomes

$$(n-1) + (q-1)2(n-1) = (2q-1)(n-1) \tag{34}$$

In the case $\lambda = 0$, (tan) has the solution $\alpha = \alpha_0$ and (ver) has the solution

$$\xi(t) = \xi_0 cos 2\pi qt + \xi_1 sin 2\pi qt \tag{35}$$

for $\xi_0 \perp \dot{c}$ and $\xi_1 \perp \dot{c}$.

Hence nullity of $B_q = 2n - 1 = \dim B_q$, i.e. B_q is nondegenerate.

□

4.2.2 An alternative computation of the index for spheres

We know that if we have a differential equation of the form

$$ay'' + by' + cy = 0 \tag{36}$$

and we seek for solutions of the form $y = e^{rt}$, then r has to be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{37}$$

(See pages 154-155 of [BD].)

If we have $b^2 - 4ac < 0$, then the roots of the characteristic equation are complex conjugate, say $r_1 = \mu_1 + i\mu_2$ and $r_2 = \mu_1 - i\mu_2$ where $\mu_1, \mu_2 \in \mathbb{R}$ and the general solution of the equation is

$$y = c_1 e^{\mu_1 t} cos(\mu_2 t) + c_2 e^{\mu_1 t} sin(\mu_2 t) \tag{38}$$

where c_1 and c_2 are arbitrary constants.

In our case of (30), we have $a = 1$, $b = 0$ and $c = \frac{(4\pi^2 q^2 + \lambda)}{(1-\lambda)}$. Hence the characteristic equation is

$$r^2 + \frac{(4\pi^2 q^2 + \lambda)}{(1-\lambda)} = 0. \tag{39}$$

- If we have $-4\pi^2 q^2 < \lambda \leq 0$, then $c > 0$. So $b^2 - 4ac = -4c < 0$ and so the roots are complex conjugate with $r_1 = i\sqrt{\frac{(4\pi^2 q^2+\lambda)}{(1-\lambda)}}$ $r_2 = -i\sqrt{\frac{(4\pi^2 q^2+\lambda)}{(1-\lambda)}}$, so $\mu_1 = 0$ and $\mu_2 = \sqrt{\frac{(4\pi^2 q^2+\lambda)}{(\lambda-1)}}$ and the general solution is

$$(40) \qquad y = c_1 cos(\sqrt{\frac{(4\pi^2q^2+\lambda)}{(1-\lambda)}}t) + c_2 sin(\sqrt{\frac{(4\pi^2q^2+\lambda)}{(1-\lambda)}}t)$$

- If we have $\lambda < -4\pi^2q^2$, then $c < 0$. So the characteristic equation has real roots $r_1 = \sqrt{\frac{(4\pi^2q^2+\lambda)}{(\lambda-1)}}$ and $r_2 = -\sqrt{\frac{(4\pi^2q^2+\lambda)}{(\lambda-1)}}$. So the general solution is

$$(41) \qquad y = c_1 e^{\sqrt{\frac{(4\pi^2q^2+\lambda)}{(\lambda-1)}}t} + c_2 e^{\sqrt{\frac{(4\pi^2q^2+\lambda)}{(\lambda-1)}}t}$$

  But noting that the solutions of this form are not periodic, we won't count them among the solutions.

These are all the solutions of the (ver) part of the equation.
Next we solve the (tan) part:

- If we have $\lambda < 0$, then the solutions of (tan) are not periodic.
- If we have $\lambda = 0$, the only solution is $\alpha(t) = \alpha_0$ for α_0 any constant.

Now we can count the solutions to find dim $T_c^0\Lambda M$ and dim $T_c^-\Lambda M$:

- The general solution of (28) corresponding to $\lambda = 0$ is of the form

$$(42) \qquad \xi = \alpha_0\dot{c}(t) + cos(2\pi qt)\xi_0 + sin(2\pi qt)\xi_1(t)$$

  where $\xi_0, \xi_1 \perp \dot{c}(t)$.

Then the number of solutions corresponding to $\lambda = 0$ is $1+(n-1)+(n-1)$ where 1, (n-1) and (n-1) are due to the possible choices of α_0, ξ_0 and ξ_1 respectively. Hence the nullity of $\mathrm{B}_q = 2n - 1$.

- The general solution of (28) corresponding to $\lambda < 0$ is of the form

$$\xi = cos(\sqrt{\frac{(4\pi^2 q^2 + \lambda)}{(1-\lambda)}}t)\xi_0 + sin(\sqrt{\frac{(4\pi^2 q^2 + \lambda)}{(1-\lambda)}}t)\xi_1 \tag{43}$$

where $\xi_0(t), \xi_1(t) \perp \dot{c}(t)$.

Note that we are interested in 1-periodic solutions. That is, we should have $\xi(0) = \xi(1)$. Hence

$$\sqrt{\frac{(4\pi^2 q^2 + \lambda)}{(1-\lambda)}} = 2\pi p \tag{44}$$

where $p \in \mathbb{Z}$ should be satisfied.
Solving for λ, we get :

$$\lambda = \frac{4\pi^2(p^2 - q^2)}{(1 + 4\pi^2 p^2)}, \quad p \in \mathbb{Z} \tag{45}$$

The condition that $\lambda < 0$ forces $p < q$ so we should have

$$\lambda = \frac{4\pi^2(p^2 - q^2)}{(1 + 4\pi^2 p^2)}, \quad p \in \{0, 1, ..., q-1\}. \tag{46}$$

Hence we can rewrite the general solution of (28) corresponding to $\lambda < 0$ in the form

$$\xi = cos(2\pi p t)\xi_0 + sin(2\pi p t)\xi_1 \tag{47}$$

where $p \in \{0, 1, ..., q-1\}$ and $\xi_0(t), \xi_1(t) \perp \dot{c}(t)$.
Then the number of solutions corresponding to $\lambda < 0$ is $(n-1)+(q-1)((n-1)+(n-1))$ where the first (n-1) stands for the possible choices of directions in the case of $p = 0$, the second and third (n-1)'s stand for the possible choices of directions for ξ_0 and ξ_1 respectively and (q-1) is for the number of different values of p other than 0. Hence

$$dimT_c^- \Lambda M = (2q-1)(n-1). \tag{48}$$

□

4.2.3 Computation of the Morse indices for some other submanifolds

Next we present the statement from [Hi] of Morse index of the critical submanifolds of simply connected compact rank 1 symmetric spaces (M= S^n, $\mathbb{CP}^n$, $\mathbb{H}P^n$, CaP^2):

Theorem 4.6. *([Hi] , p. 103) The standard metric on M is normalized so that the maximal sectional curvature is 1. Then all geodesics are closed and of length 2π; the Poincare map is the identity and the critical set of the energy function on ΛM consists of the critical manifolds A_m of geodesics of length $2\pi m$, $m \geq 0$. We have $A_0 \cong M$ and $A_m \cong STM$ the unit tangent bundle of M for $m \geq 1$. By counting zeros of the Jacobi fields, one can see that A_m ($m > 0$) is a nondegenerate critical submanifold of index*

$$\lambda(A_m) = (2m-1)(a-1) + (m-1)(r-1)a \tag{49}$$

where $a = n$, 2, 4 or 8 respectively for $M = S^n$, $\mathbb{CP}^n$, $\mathbb{H}P^n$, CaP^2 and $r = \frac{dimM}{a}$ (= 1, n , n , n = 2 respectively).

Remark 4.7. The same indices have been obtained by Ziller in [Zi].

4.3 Morse index of a geodesic for the case of S^n

One can compute the Morse index of the perturbed closed geodesics in the critical submanifolds of the energy functional on the loop space of the S^n using Theorem 6.4 of section 6.2. Here is the statement of this result:

Lemma 4.8. *For a perturbed closed geodesic $q(t)$ in B_r of the previous proposition,*

$$I_{Morse}(q) = (2r-1)(n-1) + d \tag{50}$$

where $d \in \{0, n-1, n, 2n-1\}$

Proof. For details of the proof, one can refer to Appendix A. Here we present a sketch of proof: A calculation shows that we can construct a Morse-Bott function g on the Stiefel manifold B_r with $Crit(g)$ consisting of 4 points having Morse indices in $\{0, n-1, n, 2n-1\}$. We find this by starting with the height function on the Stifel manifold having two critical submanifolds isomorphic to S^{n-1}. Then, as in the technique described in [HuBa], we perturb this function to get a Morse function F on B_r. Then we use the same technique once again to perturb the energy functional by adding epsilon times g to it on a neighborhood of B_r. By construction, it turns out that the critical points $q(t)$ of this perturbed energy functional are exactly the same as the critical points of F constructed on the Stiefel manifold B_r. By the above lemma we already have that the Morse-Bott index of a critical submanifold B_r is $(2r-1)(n-1)$. Then, by construction, Morse index of the geodesic $q(t)$ in B_r is the sum of these two indices. Hence we have the result.

□

5 Existence of a periodic orbit with Conley-Zehnder index $n + 1$

In the case of symplectically aspherical manifolds,the existence of a periodic orbit with Conley-Zehnder index $n + 1$ is proved in [Ke2] and we will follow the same construction. What differs in the weakly exact case is that the Conley-Zehnder index is no longer defined as an integer but is defined modulo $2N$ where N is the minimal Chern number, that is the positive integer such that $N\mathbb{Z} = c_1(\pi_2(M))$. Hence the Floer complex is no longer graded by the Conley-Zehnder index. We will modify the construction by considering pairings of orbits and disks in the complex.

5.1 Action Functional

The action functional is defined on the loop space of the manifold as the following:

$$\mathcal{A}(x(t)) = \int_0^1 H(t, x(t))\,dt - \int_D u^*\omega, \tag{51}$$

where x(t) is a loop and $t \in S^1 = \mathbb{R}/\mathbb{Z} = \mathbb{R}/\mathbb{Z}$ and $u : D \longrightarrow M$ is the map from the unit disk in $\mathbb{R}^2$ to the manifold such that its boundary maps to x(t) , i.e. $u(e^{2\pi it}) = x(t)$. The action functional is well-defined when $\omega|_{\pi_2(M)} = 0$ since the action becomes independent of the choice of the spanning disk for $x(t)$.

The critical points of the action functional correspond to the contractible 1-periodic orbits, the elements of $\mathcal{P}(H)$.

5.2 Moduli space of Cylinders

Let $J(t)$ be a time-dependent ω-compatible almost complex structure on M. Consider the space of smooth maps $w : \mathbb{R}^1 \times S^1 \longrightarrow M$ such that w satisfies the Floer equation

$$\partial_s w + J(t)(\partial_t w - X_H(t, w)) = 0 \tag{52}$$

and the asymptotic conditions

$$lim_{s \to -\infty} w(s,t) = x(t) \tag{53}$$

$$lim_{s \to \infty} w(s,t) = y(t) \tag{54}$$

for two given periodic orbits x(t) and y(t) in $\mathcal{P}(H)$. We denote the space of all such w by $\mathcal{M}(x, y, H, J)$ or by $\mathcal{M}(x, y)$ when we have fixed H and J.

For symplectically aspherical manifolds and for a generic J, $\mathcal{M}(x, y, H, J)$ is a finite dimensional smooth manifold of dimension $\mu_{cz}(y) - \mu_{cz}(x)$, see [MDS2] p.454. For the case where the first Chern class no longer needs to be 0, since the Conley-Zehnder index is defined only modulo $2N$, the dimension of the moduli space will vary depending on the class of the choice of the spanning disk. More specifically, from [HS], we have that the dimension of the moduli space near $u \in \mathcal{M}(x, y, H, J)$ is

$$dim_u \mathcal{M}(x, y, H, J) = ind_H(x) - ind_H(y) + 2 \int u^* c_1 \tag{55}$$

We shall denote by $\mathcal{M}^1(x, y, H, J)$ the one-dimensional component of $\mathcal{M}(x, y, H, J)$.

5.3 Floer Cohomology in the symplectically aspherical case

Let H be a Floer Hamiltonian, that is a Hamiltonian all of whose contractible 1-periodic orbits are nondegenerate. We define the Floer Complex $CF^*(H)$ to consist of the vector space of formal sums of periodic orbits of the Hamiltonian with Conley-Zehnder index k at each k and with $\mathbb{Z}_2$ coefficients. That is

$$\mathrm{CF}^k(H) = \{\xi \ \mid \xi = \textstyle\sum_{x \in \mathcal{P}(H) s.t. \mu_{cz}(x) = k} \xi_x x \text{ and } \xi_x \in \mathbb{Z}_2\} \text{ for all k} \tag{56}$$

where by $\mu_{cz}(x)$ we mean the Conley-Zehnder index of x with respect to any spanning disk.

We define the coboundary map on elements of $\mathcal{P}(H)$ which generate the Floer Complex as

$$\delta(x) = \sum_{y \in \mathcal{P}(H) \ s.t. \ \mu_{cz}(y) = \mu_{cz}(x) + 1} \#_2\{\mathcal{M}(x, y)/\mathbb{R}\} \ y \tag{57}$$

where $\#_2\{\mathcal{M}(x, y)/\mathbb{R}\}$ denotes the number of elements of the 0-dimensional moduli space $\mathcal{M}(x,y)/\mathbb{R}$ modulo 2.

The coboundary map δ defined on elements of $\mathcal{P}(H)$ extends to the Floer Complex as

$$\delta(\xi) = \delta(\textstyle\sum_{x\in\mathcal{P}(H)}\xi_x x) = \sum_{x\ s.t.\ \xi_x\neq 0}\delta(x) \tag{58}$$

Since $\delta \circ \delta = 0$ ([Fl]) , $CF^*(H)$ is a cochain complex and we define the Floer cohomology groups as

$$HF^k = \frac{ker\delta^k}{im\delta^{k-1}} \tag{59}$$

5.4 Floer Cohomology with $\mathbb{Z}_N$ -grading

In the case $c_1 \neq 0$, we can define Floer cohomology in a way that the cohomology groups are ungraded, see [FS]. Or we can also define them to have $\mathbb{Z}_N$ - grading, see [BH] and [HS].

In [FS], $CF^k = CF$ is defined to consist of formal sums of elements of $\mathcal{P}(H)$ for all k. The coboundary operator is defined as a *sum over all periodic orbits* and the coefficients of periodic orbits are the number of the 0-dimensional connected components of the moduli space modulo 2. This way we get ungraded Floer cohomology groups.

Alternatively, we can define CF^k to consist of formal sums of periodic orbits which have Conley-Zehnder index k modulo $2N$ and the coboundary operator can be defined on a generator of CF^k as a *sum over generators of* CF^{k+1} where the coefficients of the periodic orbits are $\#_2\mathcal{M}^1(x,y)$. That is, we define

$$\mathrm{CF}^k(H) = \{\xi \mid \xi = \textstyle\sum_{x\in\mathcal{P}(H),\mu_{cz}(x)=k(mod 2N)} \xi_x x \text{ and } \xi_x \in \mathbb{Z}_2\} \tag{60}$$

and the coboundary map can be defined on a generator $x \in CF^k$ as

$$\delta^k(x) = \textstyle\sum_{y\in\mathcal{P}(H)\ s.t.\ \mu_{cz}(y)=\mu_{cz}(x)+1\ (mod 2N)} \#_2\{\mathcal{M}^1(x,y)/\mathbb{R}\}\ y \tag{61}$$

With this definition, the CF and HF have a grading.

5.5 Floer Cohomology for pairings of orbits and disks

With the Z_N-graded Floer cohomology, if we are given the Conley-Zehnder index of the image of an orbit under the boundary operator, we can say the Conley-Zehnder index of the orbit up to modulo 2N. But for our purposes, we would like to have a construction so that we can tell the exact index of a periodic orbit if we know the index of its image. For the symplectically aspherical case, this is already achieved and for the case that $c_1|_{\pi_2(M)}$ is not necessarily zero, we need to consider the contributions of the spanning discs to the index. Hence we define a complex as the following:

Let

$\mathrm{D}_k = \{[\mathrm{x(t), w}] | x(t)$ is an orbit with $\mu_{cz}(x) = k \; mod \; 2N$

and w is a spanning disc for x(t) $\}$
(62)

where $[x(t), w]$ represents the homotopy class of w relative to x(t).

Define

$\overline{CF}^k(H) = \{\xi \mid \xi = \sum_{\mathcal{D}_k} \xi_{[x,w]}[x,w]$ and $\xi_{[x,w]} \in \mathbb{Z}_2\}$
(63)

and we define the coboundary map δ on generators of $\overline{CF}^*(H)$ as

$\delta^k([x,w]) = \sum_{\mathcal{S}} \#_2\{\mathcal{M}^1([x,w],[y,\widetilde{w}])/\mathbb{R}\} \; [y,\widetilde{w}]$
(64)

where $\mathcal{S}$ is defined as the set of $[y,\widetilde{w}]$ satisfying the following:

- $\mathrm{y} \in \mathcal{P}(H)$
- $\widetilde{w} \in \pi_2(M,L)$ *such that* $\partial(\widetilde{w}) \; = \; y(t)$
- $\mu_{cz}[y,\widetilde{w}] - \mu_{cz}[x,w] = 1$
- $\widetilde{w} = w\#u$ *represents an extension of* w.

Remark 5.1. Note that using the dimension formula for the moduli space $\mathcal{M}^1$ would give us

$$\mu_{cz}([x,w]) - \mu_{cz}([y,\widetilde{w}]) + 2\int u^* c_1 = 1 \tag{65}$$

and the third condition would require $\int u^* c_1 = 0$. Hence $c_1(\widetilde{w}) = c_1(w)$ should hold for the extensions of w that we are considering. So we can alternatively define $\mathcal{S}$ by omiting the third condition and adding $c_1(\widetilde{w}) = c_1(w)$ to the fourth condition.

5.6 Filtration

Since action of a periodic orbit is independent of c_1, filtered Floer cohomology is defined in the same way for $\mathbb{Z}$ -grading and $\mathbb{Z}_N$-grading.

Define $\mathcal{P}_{(a,b)}(H)$ as

$$\mathcal{P}_{(a,b)}(H) = \{x(t) \in \mathcal{P}(H) | a < \mathcal{A}(x(t)) < b\} \tag{66}$$

We define the filtered complex $CF_{(a,b)}{}^*(H)$ as

$$\mathrm{CF}^k_{(a,b)}(H) = CF_{(a,b)}(H) = \{\xi \mid \xi = \textstyle\sum_{x\in\mathcal{P}_{(a,b)}(H)} \xi_x x \text{ and } \xi_x \in \mathbb{Z}_2\}\ forallk.$$

We define the coboundary operator as the restriction of the unfiltered coboundary operator to the filtered chain complex and denote as $\delta|_{CF_{(a,b)}}$. $\delta|_{CF_{(a,b)}}$ again satisfies $\delta|_{CF_{(a,b)}} \circ \delta|_{CF_{(a,b)}} = 0$. We define the filtered Floer cohomology groups in the usual way as

$$HF_{(a,b)}{}^{k} = \frac{ker\delta|_{CF_{(a,b)}}{}^{k}}{im\delta|_{CF_{(a,b)}}{}^{k-1}} \tag{67}$$

$\overline{CF}^*$ and $\overline{HF}^*$ are defined analogously.

5.7 Isomorphism between cohomologies

The following theorem states the isomorphism between Floer cohomology and singular homology. It includes the case of manifolds with minimal Chern number at least n in addition to the statements of Theorem 3.1 and 3.2 of [Ke2] .

Theorem 5.2. $[HS]$ *Let (M^{2n}, ω) be a compact symplectic manifold. Assume either that (M, ω) is monotone or $c_1|_{(\pi_2(M))} = 0$ or the minimal Chern number $N \geq n$. Then for every pair (H,J) there exists a natural isomorphism*

$$\phi^k : HF^\alpha : HF^k(H) \longrightarrow \oplus_{j=k+n(mod2N)} H^j(M) \tag{68}$$

.

Hence by Poincare Duality, in the case of $N \geq n$, for each k there are isomorphisms

$$\Phi_H : H_{n-k}(M) \longrightarrow HF^k(M, H) \tag{69}$$

5.7.1 Extending the isomorphism

As a direct consequence of Theorem 5.2 we can state an isomorphism for $\overline{CF}^*$:

Lemma 5.3. *If (M, ω) is a compact symplectic manifold such that the minimal Chern number $N \geq n$. Then for every (H, J) and for every k, there are* *isomorphisms*

$$\overline{\Phi}_H : H_{n-k}(M) \oplus \mathcal{S} \stackrel{\Phi_H \oplus Id}{\rightarrow} HF^k(M,H) \oplus \mathcal{S} \longrightarrow \overline{HF}^k(M,H) \tag{70}$$

where $\mathcal{S}$ is the space of spanning disks and Φ_H is defined as in the previous section.

5.8 Continuation maps

Let H and G be Hamiltonians that satisfy the condition

$$\phi^1_H = \phi^1_G \tag{71}$$

and let $x(t) \in \mathcal{P}(G)$ and $y(t) \in \mathcal{P}(H)$.

For Z_N-graded cohomology, the continuation map

$$\Psi^G_H : CF^*(G) \longrightarrow CF^*(H) \tag{72}$$

is defined on a generator x of $CF^*(G)$ as

$$\Psi^G_H(x) = \sum_{\mu_{cz}(y)=\mu_{cz}(x) mod\, 2N} \#_2 \mathcal{M}^0(x,y) y \tag{73}$$

where to define $\mathcal{M}(x,y)$ we use any smooth homotopy between (H, J_H) and (G, J_G).

Similarly, one can define the continuation map

$$\overline{\Psi^G_H} : \overline{CF}^*(G) \longrightarrow \overline{CF}^*(H) \tag{74}$$

on a generator $[x,w]$ of $\overline{CF}^*(G)$ as

$$\overline{\Psi_H^G}([x,w]) = \sum_{\mu_{cz}(y)=\mu_{cz}(x) mod\, 2N} \#_2\mathcal{M}^0([x,w],[y,\tilde{w}])[y,\tilde{w}] \tag{75}$$

where to define $\mathcal{M}([x,w],[y,\tilde{w}])$ we use any smooth homotopy between (H, J_H) and (G, J_G).

5.9 Three Classes in Floer Cohomology

5.9.1 The fundamental Class in Floer Cohomology

Let h be a Morse function on M which has exactly one local maximum, p. Consider the class of p, [p], in the singular homology. Then $\Phi_h([p])$ is an element of $HF^{-n}(M,h)$ where Φ_h is the isomorphism between singular homology and and Floer cohomology defined two sections earlier. We call $\Phi_h([p])$ the fundamental class in Floer cohomology. We shall denote it by [M].

Similarly, we can consider $\Phi_H \oplus Id([p],w) = ([M],w)$ where w is any spanning disk for p. So a fundamental class in $\overline{HF}^{-n}$ is $[[M],w]$ and we can denote it by $[M_w]$.

5.9.2 The class $[M]_a$ in Filtered Floer Cohomology

In [Ke2], two cohomology classes are defined: $[M]_a$ and $[\widehat{M}]_a$ in filtered Floer cohomology HF_a^* that maps to the fundamental class $[M]$ in unfiltered Floer cohomology under the inclusion map

$$\iota_a : HF_a^*(H) \longrightarrow HF^*(H). \tag{76}$$

We see that these definitions directly extend to the Floer cohomology for pairings of orbits and spanning disks and we can denote these classes in $\overline{HF}$ by $[M_w]_a$ and $[\widehat{M_w}]_a$ respectively:

Let

$$\overline{\iota_a} : \overline{HF}^*_a(H) \longrightarrow \overline{HF}^*(H). \tag{77}$$

denote the inclusion map.

Then the following proposition defines one these two classes. The statements and methods of proofs are as in [Ke2].

Proposition 5.4. *[Ke2]For every $a > \rho^+([\phi^t_H]) + \int_0^1 H_t dt$ which lies outside $\mathcal{S}(H)$, there is a well-defined and nontrivial class $[M_w]_a$ in $\overline{HF}^{-n}_a(H)$ such that*

$$\iota_a([M_w]_a) = [M_w] = [[M], w]$$

and satisfies the following property:

If H is a Floer Hamiltonian and J is regular, then $[M_w]_a$ is represented by a cycle $[\alpha, w_\alpha]$ in $\overline{CF}^{-n}_a(H)$ with $\mathcal{A}_H(\alpha) \leq \rho^+([\phi^t_H]) + \int_0^1 H_t dt$.

Proof. Let b denote the minimum element in the action spectrum of H that is greater than $\rho^+([\phi^t_H]) + \int_0^1 H_t dt$. Let G be a Hamiltonian such that $G \in C([\Phi^t_H])$, $\int_0^1 H_t dt = \int_0^1 G_t dt$ and $|||G|||^+$ is less than a and b. Let f be a smooth Morse function on M which has exactly one local maximum p and such that

$$||f|| < \frac{1}{2}min\{a - |||G|||^{+}, b - |||G|||^{+}\} \tag{78}$$

Let $F_s = (1-b(s))f + b(s)G$ be a linear homotopy from f to G with $b(s): \mathbb{R} \to [0,1]$ a nondecreasing function which equals 0 on $(-\infty, -1]$ and equals 1 on $[1, \infty)$ and $0 \leq \dot{b} \leq 1$.

Let $\bar{\phi}^f_G : CF^*(f) \longrightarrow CF^*(G)$ and $\bar{\bar{\phi}}^f_G : \overline{CF}^*(f) \longrightarrow \overline{CF}^*(G)$ denote the chain maps induced by this linear homotopy and let $\bar{\Phi}^f_G$ and $\bar{\bar{\Phi}}^f_G$ be the respective maps between Floer cohomologies induced by it.

Let

$$\bar{\phi}^f_G(p) = \alpha'. \tag{79}$$

and so

$$\bar{\bar{\phi}}^f_G([p, w]) = [\alpha', w]. \tag{80}$$

Let $\mathcal{M}_s([x, w], [y, \tilde{w}], F_s, J_s)$, defined when $\tilde{w}$ is an extension of w as $\tilde{w} = w\#u$, be the space of maps $u : \mathbb{R} \times S^1 \to M$ satisfying

$$\partial_s u + J_s(t, u)(\partial_t u - X_{F_s}(t, u)) = 0 \tag{81}$$

where (F_s, J_s) is a smooth homotopy of data from (f, J) to (G, J^G_s) , and satisfies the asymptotic conditions

$$lim_{s \to -\infty} u(s, t) = x(t) \in \mathcal{P}(f) \tag{82}$$

and

(83) $$lim_{s\to+\infty}u(s,t) = y(t) \in \mathcal{P}(G).$$

For $u \in \mathcal{M}_s([x,w],[y,\tilde{w}],F_s,J_s)$, we have the formula

(84) $$0 \leq \mathcal{A}_f(x) - \mathcal{A}_G(y) + \int_{\mathbb{R}\times S^1} \partial_s F_s(s,t,u(s,t))dtds$$

and we have that $\partial_s F_s \leq 0$ for F_s a monotone homotopy.

Then we have

(85) $$\mathcal{A}_G(\alpha') < \mathcal{A}_f(p) + \int_{\mathbb{R}\times S^1} max_{p\in M}\partial_s F_s(s,t,u(s,t))dtds$$

and hence

(86) $$\mathcal{A}_G(\alpha') < \mathcal{A}_f(p) + \int_{\mathbb{R}\times S^1} max_{p\in M}\dot{b}(s)(G-f)dtds$$

and

(87)
$$\mathcal{A}_G(\alpha') < \mathcal{A}_f(p) + |||G-f|||^+ \leq f(p) + |||G-f|||^+ < |||G|||^+ + ||f|| < min\{a,b\}.$$

Hence by the definition of a and b, we have

(88) $$\mathcal{A}_G(\alpha') < \rho^+([\phi_H^t]) + \int_0^1 H_t dt.$$

Note that $\alpha' \in CF_a^{-n}(G)$ since it has action less than a. So, by the definition of fundamental class, we have that

(89) $$\iota_a([\alpha']) = [\bar{\phi}_G^f([p])] = [M] \in HF(G).$$

and

$$\bar{\iota}_a([\alpha', w]) = [\bar{\bar{\phi}}^f_G([p, w])] = [M_w] \in \overline{HF}(G). \tag{90}$$

We also have the following identifications between Floer complexes for H and G:

$$(CF^*(G), \delta_{\tilde{J}}) = (CF^*(H), \delta_J) \tag{91}$$

and

$$(\overline{CF}^*(G), \delta_{\tilde{J}} = \overline{CF}^*(H), \delta_J) \tag{92}$$

where

$$\tilde{J} = d(\phi^t_H \circ (\phi^t_G)^{-1}) \circ J \circ (\phi^t_G \circ (\phi^t_H)^{-1}). \tag{93}$$

So, with this identification, there is a class $\alpha \in HF^{-n}_a(H)$ such that

$$\iota_a([\alpha]) = [\bar{\phi}^f_H([p])] = [M] \in HF(H) \tag{94}$$

and so, for any spanning disc w for α, there is a class $[\alpha, w] \in \overline{HF}^{-n}_a(H)$ such that

$$\bar{\iota}_a([\alpha, w]) = [\bar{\bar{\phi}}^f_H([p, w])] = [M_w] \in \overline{HF}(H) \tag{95}$$

We set

$$[M]_a = [\alpha] \in HF^{-n}_a(H) \tag{96}$$

and we set

$$[M_w]_a = [\alpha, w] \in \overline{HF}^{-n}_a(H). \tag{97}$$

and observe that the class $[M_w]_a$ satisfies the properties of the proposition. What remains to show is that this class is well-defined and it suffices to show that α is well-defined. So we need to show that

1. the class $[\alpha]$ is independent of the choice of f.
2. the class $[\alpha]$ is independent of the choice of G.

For 1. , we will be done if we show that the choice of $[\alpha']$ is independent of the choice of f. Let g be another Morse function with the same properties as f. Let

$$\overline{\phi}_G^g(p) = \alpha''. \tag{98}$$

All the arguments about the action to achieve (88) now hold for $\alpha'' \in CF_a^{-n}(G)$ and we have

$$\iota_a([\alpha'']) = [\overline{\phi}_G^g([p])] = [M] \in HF(G). \tag{99}$$

Together with (89), we have

$$\iota_a([\alpha']) = \iota_a([\alpha'']) = [M] \tag{100}$$

Hence $[\alpha'] = [\alpha'']$.

For 2. , we need to show that if F is another choice of a Hamiltonian satisfying the same properties as that of G and we set

$$\overline{\phi}_F^f([p]) = \gamma' \tag{101}$$

then we have

$$[\gamma'] = [\overline{\phi}_G^f([p])] = [\alpha']. \tag{102}$$

Let $x(t)$ be an orbit in CF_a^{-n} and set $y(t) = \phi_F^t {\phi_G^t}^{-1}(x(t))$. Let G_s and F_s be linear homotopies from f to G and from f to F respectively and let G_s and J_s^G be regular families of almost complex structures. So (G_s, J_s^G) and (F_s, J_s^F) are smooth homotopies of data. To show the above equality, we need to show that

$$\mathcal{M}(p, x, G_s, J_s^G) = \mathcal{M}(p, y, F_s, J_s^F) \tag{103}$$

Define

$$C^\infty([\psi_t]) = \{H \in C^\infty(S^1 \times M) | [\phi_H^t] = [\psi]\} \tag{104}$$

Since $\phi_G^1 = \phi_F^1$, both F and G belong to $C^\infty([\phi_H^t])$. So we construct a homotopy from G to F, extending to a homotopy from f to G to F as the following: Define $\tilde{F}$ as

$$\tilde{F} = \begin{cases} G_s & \text{if } s \le 1; \\ H_{s-2} & \text{if } 1 \le s \le 3; \\ F & \text{if } s \ge 3. \end{cases}$$

where H_s is a family of functions in $C^\infty([\phi_H^t])$ such that $H_s = G$ for $s < -1$ and $H_s = F$ for $s > 1$. Using $\tilde{F}_s$, for each s, we get a Hamiltonian loop based at identity as

$$\varsigma_{s,t} = \phi_{\tilde{F}_s}{}^t \circ (\phi_{G_s^t})^{-1} = \begin{cases} id & \text{if } s \le 1; \\ \phi_{Hs-2} \circ (\phi_G^t)^{-1} & \text{if } 1 \le s \le 3; \\ \varsigma_t & \text{if } s \ge 3. \end{cases}$$

Then we obtain two families of Hamiltonians A_s and B_s as

$$\partial_s \varsigma_{s,t} = X_{A_s}(\varsigma_{s,t})$$

and

$$\partial_t \varsigma_{s,t} = X_{B_s}(\varsigma_{s,t}).$$

Note that

$$A_s = 0 \; for \; s \geq 3. \tag{105}$$

By definition, $\mathcal{M}_s(p, x; G_s, J_s^G)$ is the space of maps $u : \mathbb{R} \times S^1 \longrightarrow M$ satisfying

$$\partial_s u + J_s(t, u)(\partial_t u - X_{F_s}(t, u)) = 0 \tag{106}$$

where (F_s, J_s) is a smooth homotopy of data and the asymptotic conditions

$$lim_{s \to -\infty} u(s, t) = p \in \mathcal{P}(f) \tag{107}$$

and

$$lim_{s \to +\infty} u(s, t) = x(t) \in \mathcal{P}(G). \tag{108}$$

We can see that $\varsigma_{s,t}$ maps $u(s, t) \in \mathcal{M}_s(p, x; G_s, J_s^G)$ to $\varsigma_{s,t}(u) = v(s, t) : \mathbb{R} \times S^1 \to M$ satisfying the asymptotic conditions

$$lim_{s \to -\infty} v(s, t) = lim_{s \to -\infty} \varsigma_{s,t}(u(s, t)) = id(p) = p \in \mathcal{P}(f) \tag{109}$$

and

(110)
$$lim_{s\to+\infty}v(s,t) = lim_{s\to+\infty}\varsigma_{s,t}(u(s,t)) = \varsigma_t(y(t)) = \phi_F^t \circ (\phi_G^t)^{-1}(y(t)) \in \mathcal{P}(F)$$

and satisfies the equation

$$\partial_s v + \tilde{J}_s(v)(\partial_t v - X_{\tilde{F}_s}(v)) = X_{A_s}(v) \tag{111}$$

where $(\tilde{F}_s, A_s, \tilde{J}_s)$ is a triple of smooth homotopies of data from (G_s, A_s, J_s^G) to $(F \circ G^{-1}, A_s, \tilde{J}_s)$.

Next, we extend the triple of smooth homotopies $(\tilde{F}_s, A_s, \tilde{J}_s)$ further to a triple of homotopies $(\tilde{F}_s^\lambda, A_s^\lambda, \tilde{J}_s^\lambda)$ which ends at $(F_s, 0, J_s^F)$:

$$(\tilde{F}_s^\lambda, A_s^\lambda, \tilde{J}_s^\lambda) = \begin{cases} (\tilde{F}_s, A_s, \tilde{J}_s) & \text{if } \lambda \leq -1; \\ H_{s-2} & \text{if } -1 \leq \lambda \leq 1; \\ (F_s, 0, J_s^F) & \text{if } 1 \leq \lambda. \end{cases}$$

where A_s^λ is a family of functions also having property (105) of A_s Then we observe that $\mathcal{M}_{s,\lambda}(p, y, \tilde{F}_s^\lambda, A_s^\lambda, \tilde{J}_s^\lambda)$ is a cobordism of moduli spaces between $\mathcal{M}_s(p, y, \tilde{F}_s, A_s, \tilde{J}_s)$ and $\mathcal{M}_s(p, y, F_s, J_s^F)$. Since $\mathcal{M}_s(p, y, \tilde{F}_s, A_s, \tilde{J}_s)$ is the image of $\mathcal{M}_s(p, x, G_s, J_s^G)$ under $\varsigma_{s,t}$, we achive (103).

□

5.9.3 The class $[\widehat{M}]_a$ in Filtered Floer Cohomology

Definition 5.5. A Hamiltonian H is called *pinned* if there is $Q \in M$ such that the following two conditions are satisfied:

- $H(t,Q) \geq H(t,p)$ for all $(t,p) \in [0,1] \times M$ with equality only along $[0,1] \times Q$
- For all $t \in [0,1]$, the hessian of H_t at Q is nondegenerate and the linearized flow $d\phi_H^t : T_Q M \to T_Q M$ has no nonconstant periodic orbits with period less than or equal to 1.

Lemma 5.6. $[Ke2]$ *For any Hamiltonian H which is pinned at Q, there is a Morse function f such that the following conditions are satisfied:*

- *The maximum value of f is zero and is only achieved at Q*
- *P(f) = Crit(f)*
- *The function $f_H = f(p) + H(t,Q)$ is greater than or equal to H with equality only along $[0,1] \times Q$.*

Lemma 5.7. $[Ke2]$ *Let H be a Floer Hamiltonian which is pinned at Q and let f and f_H be functions as in the above lemma. Then for the Floer continuation map defined by a monotone linear homotopy from f_H to H we have*

$$\overline{\phi}_H^{f_H}(Q) = Q + \beta \tag{112}$$

for some β in $CF^{-n}(H)$ with

$$\mathcal{A}_H(\beta) < \mathcal{A}_H(Q) = |||H|||^+. \tag{113}$$

Define a specified subset of Hamiltonians, $\mathcal{H}(Q)$, pinned Hamiltonians (pinned at Q) with positive Hofer length bounded below, as the following:

$$\text{H(Q)} = \{\text{H} \in C^\infty(S^1 \times M) | H \textit{ is pinned at } Q \textit{ and } |||H|||^+ > \rho^+([\phi_H^t])\}. \tag{114}$$

Proposition 5.8. $[Ke2]$ *Let $H_Q \in \mathcal{H}(Q)$ be a Floer Hamiltonian. For every $a > |||H_Q|||^+$ which lies outside $\mathcal{S}(H_Q)$, there is a well-defined and nontrivial class $[\widehat{M_w}]_a$ in $\overline{HF}_a^{-n}(H_Q)$ such that*

$\bar{\iota}_a([\widehat{M_w}]_a) = [M_w]$

and satisfies the following property:

If H_Q is a Floer Hamiltonian and J is regular, then $[\widehat{M_w}]_a$ is represented by a cycle $[\gamma, w] = [Q + \beta, w]$ in $\overline{CF}_a^{-n}(H_Q)$ with $\mathcal{A}_H(\beta) < |||H_Q|||^+$.

Proof. Let b denote the minimum of the elements that are greater than a in the action spectrum of H. Let f and f_H be functions as described by Lemma 5.6 such that $||f|| < \frac{1}{2}(b-a)$ and set

$$\bar{\Phi}_H^{f_H}([Q]) = [\widehat{M}]_a \tag{115}$$

and so

$$\bar{\bar{\Phi}}_H^{f_H}([Q, w]) = [\widehat{M_w}]_a. \tag{116}$$

We see that $[\widehat{M_w}]_a$ satisfies the first property of the proposition:

$$\bar{\iota}_a([\widehat{M_w}]_a) = \bar{\iota}_a(\bar{\bar{\Phi}}_H^{f_H}([Q, w])) = [\overline{M_w}]_a. = [M_w] \tag{117}$$

The second property follows from the Lemma 5.7.

What remains to show is that this class is well defined, so we need to show that $[\widehat{M_w}]_a$ is independent of the choice of f. Let g be another Morse function with the same properties as f. Then we have

$$\bar{\Phi}_H^{g_H}([Q]) = \bar{\Phi}_H^{f_H} \circ \bar{\Phi}_{f_H}^{g_H}([Q]) = \bar{\Phi}_H^{f_H}([Q]) = [\widehat{M}]_a \tag{118}$$

And so

$$\overline{\bar{\Phi}}_H^{g_H}([Q, w]) = [\widehat{M_w}]_a. \tag{119}$$

Hence $[\widehat{M_w}]_a$ is independent of the choice of the Morse function f.

□

5.10 The Action Selector

We define an action selector $\sigma(H)$ for a Floer Hamiltonian H pinned at Q as the following:

$$\sigma(H) = inf\{a > |||H|||^+ | [\alpha] = [\gamma] \in HF_a^{-n}(H)\} \tag{120}$$

Also for pairings of orbits and disks we can define an action selector as

$$\overline{\sigma}(H) = inf\{a > |||H|||^+ | [\overline{\alpha}] = [\overline{\gamma}] \in \overline{HF}_a^{-n}(H)\} \tag{121}$$

where $\overline{\alpha} = [\alpha, w_\alpha]$ and $\overline{\gamma} = [\gamma, w_\gamma]$ in $\overline{CF}_a^{-n}$

5.11 The main proposition

We can use the same method of proof in [Ke2]. We make the relevant changes in the proof by using Floer cohomology for pairings of disks and orbits to have the desired statement :

Proposition 5.9. *Let (M, ω) be a weakly exact symplectic manifold with minimal Chern number $N \geq n$. If H is a Floer Hamiltonian which is pinned at Q, then there is a periodic orbit $x(t)$ and a spanning disc w such that $\mu_{cz}(x(t), w) = -n-1$ and action $\mathcal{A}(x(t)) = \overline{\sigma}(H)$.*

Proof. For any $a > \overline{\sigma}(H)$, by definition of the action selector, $[\gamma - \alpha] = [Q+\beta-\alpha] = 0$ in $HF_a^{-n}(H)$. That means $Q+\beta-\alpha$ is in image of δ^{-n-1}. That is, there exists an element $[\eta(t), w] \in \overline{CF}_a^{-n-1}$ such that $\delta^{-n-1}([\eta(t), w]) = [Q + \beta - \alpha, w]$. We have that $\mu_{cz}([\eta(t), w]) = -n - 1$ since $[\eta(t), w] \in \overline{CF}^{-n-1}(H)$. We also have that $\mathcal{A}(\eta(t)) \in [\overline{\sigma}(H), a)$. Since there are finitely many periodic orbits, we can pick an a small enough so that there is only one periodic orbit with action in the interval $[\overline{\sigma}(H), a)$. Hence we get the result. $\square$

5.12 Proofs of statements in Section 1

5.12.1 Proof of Theorem 1.1

Existence of a periodic orbit with Conley-Zehnder index n

Let $\mathcal{J}$ be the set of smooth almost compatible complex structures on (M, ω). Define

$$\hbar(J) = inf_{\mathcal{A}}\{\omega(A) | A \in \mathcal{A}\} \tag{122}$$

where $\mathcal{A}$ is the set of nonconstant J-holomorphic spheres and define

$$\hat{\hbar(J)} = sup_{J\in\mathcal{J}}\{\hbar(J)\} \tag{123}$$

For a proof of the following theorem, see [Ke1].

Theorem 5.10. $[Ke1]$ *Let (M^{2n}, ω) be a closed symplectic manifold. Let H be a Floer Hamiltonian on M such that Hofer length of H, $||H||$, is less than $\hat{\hbar(J)}$. If Φ_H^t doesn't minimize the negative Hofer seminorm in its homotopy* *class, then there is a 1-periodic orbit x of H which admits a spanning disk w such that*

$$\mu_{cz}(x,w) = n \tag{124}$$

and

$$-||H||^- < \mathcal{A}(x,w) \leq ||H||^+. \tag{125}$$

Main result of [KS]

Lemma 5.11. *Let $\mathcal{E}_g$ be a Morse-Bott energy functional and let $Crit(\mathcal{E}_g) = \coprod_j D_j$. Then*

$$I_{Morse}(q) \in [I_{Morse}(D_j), I_{Morse}(D_j) + dim(D_j)]. \tag{126}$$

A manifold L is called *split hyperbolic* if it is diffeomorphic to a product manifold

$$L = P_1 \times ... \times P_k \tag{127}$$

such that each factor P_j admits a metric with negative sectional curvature. Note that for a split hyperbolic manifold, the bounds on Morse index satisfies, by Lemma above,

$$I_{Morse}(q) \in [0, 1 + dimP_2 + ... + dimP_k] \tag{128}$$

The following proposition is part of the main result of [KS][7]:

[7] The result also holds for convex symplectic manifolds with an admissable Hamiltonian, see [KS].

Proposition 5.12. *Let (M^{2n}, ω) be a rational and proportional symplectic manifold which is closed. If L is an easily displaceable Lagrangian submanifold of (M^{2n},ω) which is split hyperbolic, then*

$$N_L \leq n + 2(-1) \tag{129}$$

where (-1) contributes if L is orientable.

Proof. It is a property of our specifically constructed Hamiltonian that $\phi^t_{H_L}$ does not minimize the negative Hofer seminorm in its homotopy class. H_L also a Floer Hamiltonian and satisfies the bound on the Hofer norm by construction. Hence by Theorem 5.10, there is a pair (x, w) with Conley-Zehnder index n and with action

$$-||H||^- < \mathcal{A}(x, w) \leq ||H||^+. \tag{130}$$

Again by a property of H_L, x is nonconstant. Then the index relation

$$\mu^L_{Maslov}([w]) = \mu_{cz}(x, w) - I_{Morse}(q)(+1)(+1) \tag{131}$$

where the first (+1) contributes if q*TM is nonorientable and the second (+1) contributes if x is contained in $U_(3r/5 + \delta, 4r/5 - \delta)$. The bounds on Morse index

$$I_{Morse}(q) \in [0, 1 + dimP_2 + ... + dimP_k] \tag{132}$$

implies that the Maslov index of [w] has to satisfy

$$0 < dimP_1 - 1 \leq \mu^L_{Maslov}([w]) \leq n(+1)(+1) \tag{133}$$

This proves that $N_L \leq n + 2(-1)$.

$\square$

One can see that Theorem 1.1 immediately follows from Proposition 5.12.

5.12.2 Proof of Theorems 1.2 and 1.5

We use the index relation of Proposition 3.4. In our case, where $L = S^n$, we have that q^*TS^n is trivial. So the $+1$ doesn't contribute to the equality (19). Then, in equality (19), we use the statements of Proposition 3.1, Lemma 4.1 and Theorem 4.5 and get the following:

$$0 = (n+1) - (2r-1)(n-1) - d \tag{134}$$

where $d \in \{0, n-1, n, 2n-1\}$.

Let $k = 2r - 1$. So k is an odd positive integer (since r is a positive integer). Now, to determine all possible values of n, we solve the following 4 equalities for n:

1. $k(n-1) = n+1$.
   That is $(k-1)n = k+1$. If $k = 1$, we get a contradiction. So $k \neq 1$ and so $n = \frac{k+1}{k-1}$. k must be an odd positive integer such that n should also be a nonnegative integer. The only possibility is that $k = 3$. Hence n can only be 2.
2. $k(n-1) = 2$.
   That is $kn = k+2$. Since $k \neq 0$, we have $n = \frac{k+2}{k} = 1 + \frac{2}{k}$. The only possibility is $k = 1$, hence n can only be 3.
3. $k(n-1) = 1$.
   That is $n = \frac{k+1}{k}$. We have that k can only be 1 and n can only be 2.
4. $k(n-1) = 2 - n$.
   That is $n = \frac{k+2}{k+1}$. For no odd positive integer k, we can get an integer value n.

Altogether, we must have $n \in \{2, 3\}$ and the statement of Theorem 1.2 follows.

For Theorem 1.5, we use Proposition 3.2, Lemma 4.2 and Theorem 4.5 in (19) and the same computation gives the statement.

5.12.3 Proof of Theorem 1.6

We use the same arguments as in the proof of the previous theorem for $S^n \times S^m$ in $M^{2(n+m)}$. We also use that the Morse index of a geodesic in the product manifold is sum of the Morse indices of its components. [8] We get the following identity:

$$0 = (n + m + 1) - (2r_1 - 1)(n - 1) - d_1 - (2r_2 - 1)(m - 1) - d_2 \tag{135}$$

where $d_1 \in \{0, n - 1, n, 2n - 1\}$ and $d_2 \in \{0, m - 1, m, 2m - 1\}$.

We have $n = 1$. Also, let $d = d_1 + d_2$ and $k_2 = 2r_2 - 1$. Then the above equality becomes:

$$0 = (m + 2) - k_2(m - 1) - d \tag{136}$$

where $d \in \{0, 1, m - 1, m, m + 1, 2m - 1, 2m\}$.

That is

$$(k_2 - 1)m = k_2 + 2 - d \tag{137}$$

where $d \in \{0, 1, m - 1, m, m + 1, 2m - 1, 2m\}$

[8] For this and the next theorem we are using these statements: $Cr\Lambda(S^n \times S^m)$ is isomorphic to $Cr\Lambda(S^n) \times Cr\Lambda(S^m)$. That is, the geodesics of the product space are products of geodesics in S^n and S^m. And secondly, the Morse index of a geodesic $q(t) = (q_1(t), q_2(t))$ in $S^n \times S^m$ is the sum of the Morse indices of $q_1(t)$ in S^n and $q_2(t)$ in S^m.

Next, we analyse all the possible cases:

- If $k_2 = 1$:
  The equality becomes d= 3. Then $m \in \{4, 3, 2\}$.
- If $k_2 > 1$:
  The equality becomes $m = \frac{k_2+2-d}{k_2-1}$.
  Remembering that k_2 is an odd positive integer, we look at all the possibilities.
  - d=0 : no integer m is a solution.
  - d=1 : only solution is m=2.
  - d=m-1 : only solution is m=2.
  - d=m : no solution
  - d=m+1 : no solution
  - d=2m-1 : no solution
  - d=2m : no solution

  Altogether, we have $m \in \{2, 3, 4\}$ and the statement follows.

5.12.4 Proof of Theorem 1.7

As we did in the proof of Theorem 1.6, we analyse all the possible cases for d in the equation (135). First, we write (135) in the following form where we let $k_1 = 2r_1 - 1$ and $k_2 = 2r_2 - 1$:

$$(k_1 - 1)n + (k_2 - 1)m = k_1 + k_2 + 1 - d \tag{138}$$

where d $\in \{0, n-1, n, 2n-1, m-1, n+m-2, n+m-1, 2n+m-2, m, n+m, 2n+m-1, 2m-1, n+2m-2, n+2m-1, 2n+2m-2\}$.

We only need to consider the following cases and we will be done with the other cases by symmetry:

- d=0 :
  The left hand side of the equation is an even number, whereas the right hand side is odd. So there are no solutions in this case.

- d=n-1 :
  The equation becomes $k_1 n + (k_2 - 1)m = k_1 + k_2 + 2$. This forces n to be an even number. Next, since $k_1 \neq 0$, we get
  $n = \frac{k_1+k_2+2-(k_2-1)m}{k_1} = 1 + \frac{k_2+2-(k_2-1)m}{k_1} < 1 + \frac{k_2+2-(k_2-1)2}{k_1} = 1 + \frac{4-k_2}{k_1} \leq 1 + \frac{3}{k_1} \leq 4.$
  The first, second and third inequalities are because $m > 2$, $k_2 \geq 1$ and $k_1 \geq 1$ respectively. So we have $n < 4$. Since we have the assumption that $n > 2$ and by the above argument n has to be an even number, there are no solutions in this case.

- d=n : We use exactly the same steps as in the case d=n-1. This time we conclude that n has to be odd and $n < 3$. Hence there are no solutions.

- d=2n-1 : we get $n < 2$. no solution.

- d=n+m-2 : we get $n < 3$. no solution.

- d=n+m-1 : $n < 2$. no soln.

- d=2n+m-2 : $n < 2$. no soln.

- d=n+m : $n < 1$.

- d=2n+m-1 : $n < 1$.

- d=2n+2m-2 : $n < 0$.
  So the statement of the theorem holds.

5.12.5 Proof of Theorem 1.8

We again use the index relation of Proposition 3.4. First we note that we don't know whether q^*TL is trivial or not for both the case of the Cayley (octanionic) projective plane CaP^2 and for the case of the Quaternionic projective space $\mathbb{Q}P^n$. So we shall check both cases of when +1 contributes and when it doesn't contribute to the equality (19).

The case of CaP^2

We have n=16 and the Morse index of the critical submanifold is 22m-15 according to Theorem 4.6. (One can also obtain this index from the table on page 11 of [Zi].) Using this in equality (19) and the statements of Proposition 3.2 and Lemma 4.2 we get the following two equalities:

$$0 = 16 + 1 - (22m - 15) - d(+1) \tag{139}$$

where d is the Morse index of a geodesic in a critical submanifold of CaP^2.

By the discussion in Appendix A.2 of the generalization of the the Morse index computation of the sphere to any Lagrangian submanifold, we can write

d= k + s

where k is an element of the set of Morse indices of a Morse function on CaP^2 and s is an element of the set of indices of a Morse function on the unit tangent bundle to CaP^2 at a critical point of CaP^2.

For k, we consider the homology of CaP^2 and see that $H_i(\mathcal{O}P^2) = \mathbb{Z}$

for $i \in \{0, 8, 16\}$ and 0 otherwise ,[Zi]. Hence we can find a Morse function which satisfies $k \in \{0, 8, 16\}$.

For s, the unit tangent bundle to CaP^2 will be isomorphic to S^{15}, so we have $s \in \{0, 15\}$.

Hence altogether, we need to check whether one of the following equalities can hold:

1. $17 = 22m + d$
   where d $\in \{0, 8, 16\}$.
2. $17 = 22m - 15 + d$
   where d $\in \{0, 8, 16\}$.
3. $17 + 1 = 22m + d$
   where d $\in \{0, 8, 16\}$.
4. $17 + 1 = 22m - 15 + d$
   where d $\in \{0, 8, 16\}$.

We see that there is no integer m satisfying any of these equalities and hence we conclude that there exists no Lagrangian Cayley projective planes in closed, weakly exact, symplectic (M^{32}, ω) with minimal Chern number $N \geq 16$.

The case of $\mathbb{Q}P^n$

In the case of the Quaternionic projective space, we have from theorem 4.6 that the Morse index of a critical submanifold is $2(m-1)(2n+1)+3$.

Then the index relation becomes the following two equalities:

$$0 = 4n + 1 - (2(m-1)(2n+1)+3) - d\,(+1) \tag{140}$$

where d is the Morse index of a geodesic in a critical submanifold of $\mathbb{Q}P^n$.

The homology of $\mathbb{Q}P^n$is $H_{4i}(\mathbb{Q}P^n) = \mathbb{Z}$ for $i \in \{0, ..., n\}$ and 0 otherwise, [Zi].

As we have done in the previous section, we shall write d as

d= k + s

where k is an element of the set of Morse indices of a Morse function on $\mathbb{Q}P^n$ and s is an element of the set of indices of a Morse function on $UT\mathbb{Q}P^n$ at a critical point of $\mathbb{Q}P^n$.

We can find a Morse function which satisfies $k \in \{0, ..., 4j, ..., 4n\}$.

Since $UT\mathbb{Q}P^n$ at a critical point is isomorphic to S^{4n-1}, we have $s \in \{0, 4n-1\}$.

Hence altogether, we need to check whether one of the following equalities can hold:

1. 4n+1 = (2(m-1)(2n+1) + 3) + k
   where k $\in \{0, ..., 4j, ..., 4n\}$.
2. 4n+1 = (2(m-1)(2n+1) + 3) + k + 4n-1
   where k $\in \{0, ..., 4j, ..., 4n\}$.
3. 4n+2 = (2(m-1)(2n+1) + 3) + k
   where k $\in \{0, ..., 4j, ..., 4n\}$.
4. 4n+2 = (2(m-1)(2n+1) + 3) + k + 4n-1
   where k $\in \{0, ..., 4j, ..., 4n\}$.

In the second and fourth equalities, we see that $4n's$ cancel and the right hand side of the equations become too big for the equalities to hold for any positive integer m. In the third equality, we see that the right hand side is even whereas the right hand side is even hence the equality cannot hold for any m. We write the first equality in the following form:

$$4n-2=(m-1)(4n+2)+k \tag{141}$$

Then we have

$$(m-1)=\frac{4n-2-4j}{4n+2} \tag{142}$$

which we can write as

$$m=1+\frac{2n-1-2j}{2n+1}=2-\frac{2j+2}{2n+1}. \tag{143}$$

We see that $2j+2$ is even whereas $2n+1$ is odd, so m can never be an integer and hence the index relation cannot hold for any m. So we can conclude that there are no Lagrangian Quaternion projective spaces in closed, weakly exact, symplectic manifold (M^{4n},ω) for minimal Chern number $N \geq 2n$.

A Appendix

A.1 Morse index computations for Product of Spheres and $V_2(R^n)$

A.1.1 A Morse function on $S^n \times S^m$

Let $h_1 : S^n \to\mathbb{R}$ and $h_2 : S^m \to\mathbb{R}$ be height functions on the spheres, i.e.

$$h_1(x_1, ..., x_n, x_{n+1}) = x_{n+1} \text{ and } h_2(y_1, ..., y_m, y_{m+1}) = y_{m+1}.$$

Let $M(x_1, ..., x_n, x_{n+1}, y_1, ..., y_m, y_{m+1})$ be the Morse function defined on the product $S^n \times S^m \to \mathbb{R}$ by

$$M = h_1h_2 + 2h_1 + 2h_2.$$

Then

$$\nabla M = [\partial M/\partial x_1, ..., \partial M/\partial x_{n+1}, \partial M/\partial y_1, ..., \partial M/\partial y_{m+1}] = 0$$

when

$$(h_2 + 2)\partial h_1/\partial x_{n+1} = 0 \text{ and } (h_1 + 2)\partial h_1/\partial y_{m+1} = 0.$$

Note that (h_2+2) and (h_1+2) are never zero. Hence the first and second equations are satisfied when x_{n+1} and y_{m+1} are the north or south poles of S^n (denote by n_1 and n_2) and S^m (denote by m_1 and m_2), respectively. Hence M has 4 critical points: (n_1, m_1),(n_1, m_2),(n_2, m_1),(n_2, m_2) with $n+m$, n, m, 0 as their respective Morse indices.

A.1.2 A Morse function on $V_2(R^{n+1})$

We can identify $V_2(R^{n+1})$ with the unit tangent bundle to S^n, that is

$UT(S^n) = \{\, \text{(x,v)} : \text{x} \in S^n, v \in T_x(S^n) \text{ and } \|v\|_x = 1 \,\}$

Consider the Morse-Bott function $h(x, v) = x_{n+1}$ on $UT(S^n)$. We have that

$$\nabla h = [0, ..., 0, \partial h/\partial x_{n+1}, 0......., 0, ..., 0] = 0$$

when

$$\partial h/\partial x_{n+1} = 0.$$

Hence there are two critical submanifolds:

$$\begin{gathered} \text{C}_1 = \{(n_1, v) | v \in T_{n_1}(S^n) \text{ and } \|v\|_{n_1} = 1\,\} \cong S^{n-1} \\ \text{C}_2 = \{(n_2, v) | v \in T_{n_2}(S^n) \text{ and } \|v\|_{n_2} = 1\,\} \cong S^{n-1} \end{gathered}$$

We will construct a Morse function on $UT(S^n)$ that is arbitrarily close to the Morse-Bott function h as suggested in [BH] :

Let T_i be a tubular neighborhood around C_i for $i = 1, 2$. That is,

$T_i \cong U_{n_i} \times S^{n-1}$ where U_{n_i} is a neighborhood of the pole n_i of S^n.

Let $(x_1, ..., x_{n+1}, y_1, ..., y_n)$ be the coordinates on T_i.

Let f_i on C_i be the height function y_n on S^{n-1}. Extend f_i to a function on T_i by making f_i constant in the $(x_1, ..., x_{n+1})$ directions. Let ρ_i be a bump function which is 0 outside T_i and is equal to 1 in a neighborhood of C_i. Let N_i denote the neighborhood where $\rho_i = 1$.

For $\epsilon > 0$, define a function g on $UT(S^n)$ as

$$g = h + \epsilon(\rho_1 f_1 + \rho_2 f_2)\ .$$

Then

$$\nabla g = 0$$

if and only if

$$\nabla h + \epsilon(f_1 \nabla \rho_1 + \rho_1 \nabla f_1 + f_2 \nabla \rho_2 + \rho_2 \nabla f_2) = 0.$$

We will find all the critical points of g by considering ∇g on different regions of $UT(S^n)$:

1) Outside T_1 and T_2 :
$\nabla h \neq 0$
$\rho_1 = 0$
$\rho_2 = 0$
$\nabla \rho_1 = 0$
$\nabla \rho_2 = 0$
which altogether imply that $\nabla g \neq 0$. Hence there are no critical points of g in this region.

2) On $T_1 - N_1$:
$\nabla h \neq 0$
$\rho_1 \neq 0$
$\rho_2 = 0$
$\nabla \rho_1 \neq 0$
$\nabla \rho_2 = 0$
Hence critical points occur when

$$-\nabla h = \epsilon(f_1 \nabla \rho_1 + \rho_1 \nabla f_1).$$

Now note that ∇h , ∇f_1 and $\nabla \rho_1$ are of the form

$$\nabla h = [0, ..., 0, \partial h/\partial x_{n+1}, 0...., 0]$$

$$\nabla f_1 = [0, ..., 0, \partial f_1/\partial y_1, ..., \partial f_1/\partial y_n]$$

$$\nabla \rho_1 = [\partial \rho_1/\partial x_1, ..., \partial \rho_1/\partial x_{n+1}, 0...., 0].$$

Hence

$$\nabla g = 0$$

if and only if

$$\nabla f_1 = 0 \text{ and } -\partial h/\partial x_{n+1} = \epsilon(f_1 \partial\rho_1/\partial x_{n+1}).$$

Now note that on $T_1 - N_1$, $-\partial h/\partial x_{n+1}$ is bounded away from 0, say by d. We can pick ϵ small enough so that $d/\epsilon < f_1\partial\rho_1/\partial x_{n+1}$ never holds. Hence there are no critical points of g on $T_1 - N_1$.

3) On N_1 :
$\nabla h = 0$ only when $x = n_1$
$\rho_1 = 1$
$\rho_2 = 0$
$\nabla\rho_2 = 0$
$\nabla\rho_1 = 0$
Hence critical points occur when $\nabla h + \epsilon\nabla f_1 = 0$.
So we have $-\nabla h = \epsilon\nabla f_1$. That is,

$$-[0, ..., 0, \partial h/\partial x_{n+1}, 0...., 0] = [0, ..., 0, \epsilon\partial f_1/\partial y_1, ..., \epsilon\partial f_1/\partial y_n].$$

This implies that both

$$\partial h/\partial x_{n+1} = 0 \text{ and } \epsilon\nabla f_1 = 0$$

has to hold at a critical point.

The first condition holds if and only if x is a pole of S^n . The second condition, $\nabla f_1 = 0$, holds if and only if v is a pole of S^{n-1}. Hence the critical points of g on N_1 are (n_1, p_1) and (n_1, p_2) where p_1 and p_2 are the north and south poles of S^{n-1} respectively.

So the only critical points on N_1 are (n_1, p_1) and (n_1, p_2).

4) On $T_2 - N_2$:
By a similar computation as done for T_1 N_1, there are no critical points of g on this region.

5) On N_2 :

By a similar computation as done for N_1, the critical points on N_2 are (n_2, p_1) and (n_2, p_2).

Hence Crit(g) consists of 4 points (n_1, p_1), (n_1, p_2), (n_2, p_1) and (n_2, p_2) with $2n - 1$, n , $n - 1$, 0 as their respective Morse indices.

A.2 Generalization of the Morse index computation of critical submanifolds to the case of any Lagrangian submanifold

We shall compute $I_{Morse}(q(t))$ for the case of any closed Lagrangian submanifold L.

First, note that similar to Lemma 4.8, the index will be of the form

$$\mathrm{I}_{Morse}(q) = I(B_r) + d$$

where $I_{Morse}(q(t))$ is the Morse index of the geodesic q in a critical submanifold B_r of the energy functional on ΛL, $I(B_r)$ is the Morse index of the critical submanifold B_r and d is the index of a Morse function on B_r. We note that, by the analogue of the discussion in Appendix A.1, d can be written as

d= k + s

where k is an element of the set of Morse indices of a Morse function on L and s is an element of the set of indices of a Morse function on the unit tangent bundle to L, UT_pL , at a critical point p of L.
[9]

For k, we consider the homology of L. If $H_i(L) \neq 0$ for $i \in \mathcal{A}$ and 0 otherwise , we can find a Morse function which satisfies $k \in \{A\}$.

For s, the unit tangent bundle to L^n will be isomorphic to S^{n-1}, so we have $s \in \{0, n-1\}$.

Hence altogether, we have will have one of the following cases:

1. $I_{Morse}(q) = I(B_r) + k$.
   where $k \in \mathcal{A}$.
2. $I_{Morse}(q) = I(B_r) + (k + n - 1)$
   where $k \in \mathcal{A}$.

[9] We can see from Appendix A.1 that in the case of S^n, we compute d by

d= k + s

where $k \in \{0, n\}$ and $s \in \{0, n-1\}$. Note that k corresponds to the index of the critical points of the height function on S^n (i.e. nonzero homologies of the sphere) and s corresponds to the index of the critical point of a height function on unit tangent bundle to the sphere at a critical point of the sphere.

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