

TEACHERS' MATHEMATICAL KNOWLEDGE FOR
TEACHING, INSTRUCTIONAL PRACTICES, AND STUDENT OUTCOMES

BY

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DISSERTATION

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Abstract

This dissertation examines the relationships among teachers' mathematical knowledge, their teaching practices, and student achievement. Quantitative and qualitative data collection techniques (content knowledge assessments, surveys, interviews, and classroom observations) were used to collect data from 21 teachers and 873 students. Twenty-one in-service teachers who enrolled in a master's program designed specifically for the needs of a partnership district were followed for 4 years to study how their mathematical knowledge as well as their teaching changed over time. Of the 21 teachers, 8 teachers were chosen for additional classroom observations and interviews. For the quantitative part of the study, two-level linear growth models were used to examine the effects of the mathematical knowledge of K-8 teachers on their instructional practices. After student-level data were added, three-level growth models were used to analyze the effects of teachers' knowledge and instructional practices on students' gain scores. Teachers' beliefs about teaching and learning mathematics were also included in some analyses. The results indicated that, compared with the initial baseline data, teachers' mathematical knowledge increased dramatically, and the teachers made statistically significant changes in their lesson design, mathematical agenda of the lessons, task choices, and classroom climate. The gains in teachers' mathematical knowledge predicted changes in the quality of their lesson design, mathematical agenda, and classroom climate. Teachers' beliefs were related to the quality of their lesson design, mathematical agenda, and the quality of the tasks chosen. However, only student engagement was significantly related to students' gain scores. Neither teachers' mathematical knowledge nor other aspects of instruction (inquiry-oriented teaching, the quality of task choices, and the classroom climate) were associated with students' gain scores. The qualitative analyses revealed particular strands of the complex relationship between teachers'

mathematical knowledge and their instructional practices. Teachers' beliefs played a mediating role in the relationship between teachers' mathematical knowledge and instructional practices. Teachers favoring standards-based views of mathematics tended to teach in more inquiry-oriented ways and ask more questions of students; however, among teachers with limited mathematical knowledge, these practices seemed superficial. Additionally, the teachers' task choices appeared to be confounded by teachers' current level of mathematical knowledge and their textbook use.

To my parents, husband, and siblings for their unending support and love

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Table of Contents

Chapter 1 Introduction.....	1
Chapter 2 Literature Review	7
Chapter 3 Methods of Data Collection and Analysis.....	35
Chapter 4 A Quantitative Analysis of the Relationship Among Teachers' Knowledge of Mathematics, Instructional Practices, Beliefs, and Student Achievement.....	86
Chapter 5 Qualitative Findings	119
Chapter 6 Discussion	222
References	238
Appendix A Adapted Classroom Observation Protocol.....	247
Appendix B Tentative Focus Teacher Interview Protocol	253
Appendix C Teacher Beliefs Survey.....	255

Chapter 1

Introduction

In the last decades of the 20th century, educational inequities, economic problems and the poor showing of U.S. students on the Second International Mathematics Study (McKnight et al., 1987) led the National Council Teachers of Mathematics (NCTM) to publish a series of standards documents (1989, 1991, 1995, 2000) laying out a new vision of mathematics instruction so that students would be equipped with the necessary knowledge required in the workplace and in the new technological age.

The content and, in particular, the way mathematics is taught as established by the NCTM (1989, 2000) were definitely a change from the traditional curriculum. Unlike the traditional method of teaching mathematics, which tended to rely on the assumption that students acquire knowledge and skills by observing a teacher's explanations and practices (Greeno, 2003; Schoen, Fey, Hirsch, & Coxford, 1999), the NCTM *Standards* viewed learning mathematics as an active process. It recognized that students construct their knowledge through experience by engaging in meaningful and purposeful activities. In *Standards*-based instruction, "knowing" mathematics is defined as "doing" mathematics (Greeno, 2003).

Concerning the methods of instruction, the *Standards* emphasized problem solving as a means of learning mathematics, rather than using problems to practice procedures that have already been learned (Hiebert, 2003; Klein, 2007; Schoenfeld, 2004). Students work on fewer but more complex problems that are often based on real-life situations and applications. Students are not discouraged from using alternative algorithms instead of the standard algorithms (Baroody, 2003). The *Standards*-based teaching emphasized what was called the development of "mathematical power," which involves learning procedures through understanding, reasoning,

problem solving, connecting mathematical ideas, and communicating mathematics to others (National Research Council [NRC], 2001).

The commonly used terms “standards-based” and “inquiry-oriented” teaching refer to the practices that are advocated in these documents. However, it should be noted that the use of a standards-based approach to instruction does not preclude the use of more traditional activities as well, such as explicit instruction. Multiple strategies are necessary to teach several aspects of mathematics. A primary emphasis of the reform-oriented teaching includes promoting teaching practices that are assumed to facilitate student learning. The *Standards* document argues that the traditional focus on facts and skills should be expanded to include conceptual understanding and engagement in a variety of mathematical processes (Hiebert, 2003; Sfard, 2003). Teachers are encouraged to create an environment in which students share their observations, propose conjectures, and justify their arguments. Students can develop complex cognitive skills and processes by actively participating in instruction. In reformed-based mathematics teaching, teachers are encouraged to devote more time to class discussions and group work. In this style of instruction, the teacher is neither the sole source of authority nor the primary source of knowledge. Teachers’ questioning strategies also play a vital role in the quality of instruction students receive. Teachers can foster students’ reasoning ability by asking questions that promote student thinking.

Creating and carrying out mathematics lessons as envisioned in these documents demands more of teachers than before. The teacher is responsible for creating opportunities for students to become mathematically proficient and, at the same time, fostering a classroom environment that supports the students (Ball, 1993). As stated in the NCTM’s *Principles and Standards for School Mathematics* (2000), “[T]eachers must know and understand deeply the

mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks” (p. 17).

However, the role of teachers’ mathematical knowledge in their teaching is not clear (Mewborn, 2003; National Mathematics Advisory Panel [NMAP], 2008). The field lacks an understanding of which instructional practices are related to teachers’ mathematical knowledge. Furthermore, the findings from earlier research on the relationships among teachers’ mathematical knowledge, their teaching, and student learning are mixed (e.g., Hamilton et al., 2003; Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; NRC, 2004; Schoen & Hirsch, 2003; Thompson, 1992; Webb, 2003). Prior research either has investigated a connection between teachers’ knowledge and student achievement in general without paying attention to teachers’ instruction or has focused on the relationship between teachers’ knowledge and their practices while ignoring the effects on student outcomes (e.g., Borko, Eisenhart, Brown, Underhill, Jones, et al., 1992; Hill, Blunk, Charalambous, Lewis, Phelps, et al., 2008; Leinhardt & Smith, 1985; Monk, 1994; Putnam, Heaton, Prawat, & Remillard, 1992; Rockoff, Jacob, Kane, & Staiger, 2008).

We currently lack a detailed understanding of how teachers’ knowledge affects student learning and how teachers’ instruction mediates the effects of their knowledge on student performance (Mason, 2008; Silverman & Thompson, 2008; Graeber & Tirosh, 2008). These limitations are partly due to the practice of capturing teachers’ knowledge at a particular point in time, whereas a longitudinal analysis of teachers’ knowledge would allow for a better understanding of the relationships among teachers’ mathematical knowledge, their teaching practices, and student learning.

Study Overview

In this study, using the data collected from 21 in-service teachers who were enrolled in a new master's degree program, I investigate the complex relationships among teachers' mathematical knowledge, their mathematics instruction, and student outcomes. In doing so, I seek to answer the following two questions:

1. How does teachers' mathematical knowledge affect their instruction? What factors, such as beliefs and the curriculum, mediate the expression of their knowledge of mathematics in instruction?
2. To what extent are changes in teachers' mathematical knowledge, instructional practices, or both associated with students' gain in achievement?

Conceptual Framework

Several factors play roles in the interplay among teacher knowledge, their instruction, and student learning. Modified from Porter and Brophy's (1988) "*a model of good teaching*," the framework in Figure 1 below illustrates the complex, dynamic, and interactive relationships among these factors. This conceptual framework is used to ground this study.

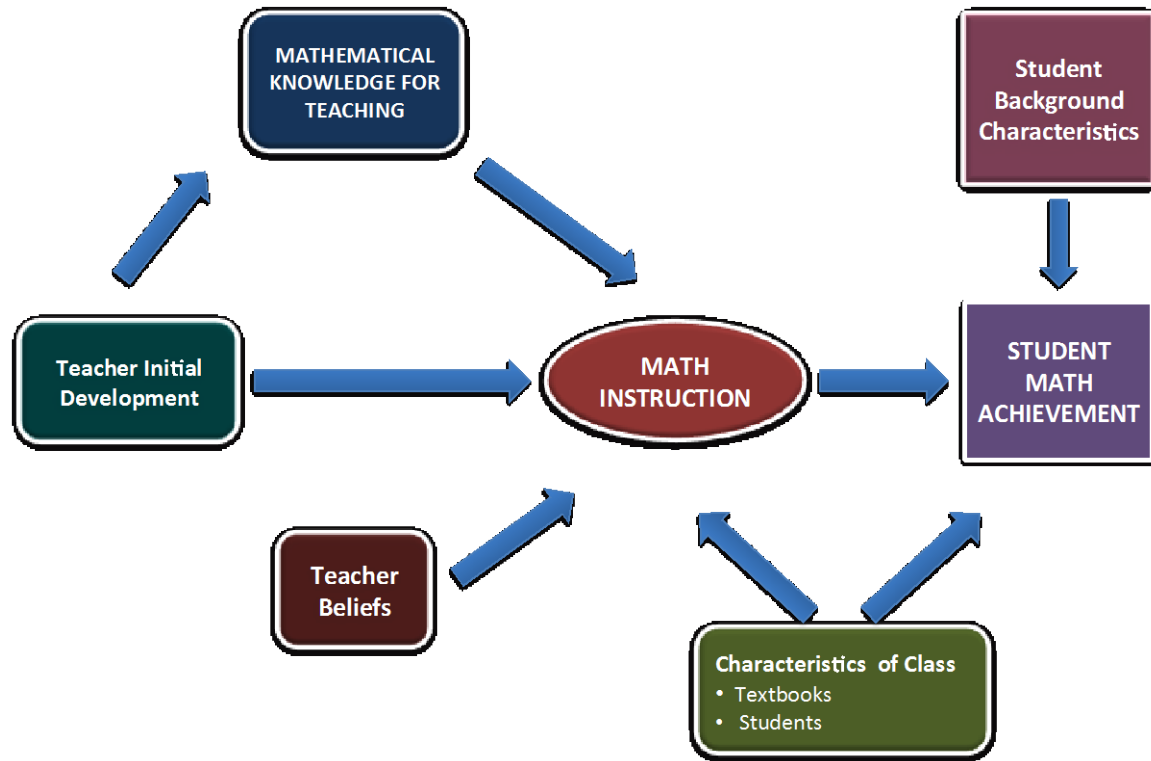


Figure 1. A framework for the factors representing the relationship between teachers' knowledge and their instruction as well as student performance (adapted from Porter & Brophy, 1988, p. 76).

This model highlights the fact that several factors such as teachers' initial preparation and mathematical knowledge, as well as their beliefs, have an impact on teachers' instructional practices. Another important factor is that the model includes contextual factors, indicating that teachers are likely to experience different influences, depending on resources, background of participants, and other factors (Fennema & Franke, 1992; Grossman, 1990; Porter & Brophy, 1988; Thompson, 1992).

Teachers' beliefs are also a part of the conceptual framework for this study. Certain beliefs that teachers hold seem to mediate the effects of teachers' knowledge on their teaching practices (e.g., Grossman, Wilson, & Shulman, 1989; Putnam et al., 1992; Stodolsky & Grossman, 1995; Thompson, 1984, 1992). In particular, in this study, special attention is given to

teachers' beliefs about teaching and learning mathematics (Borko et al., 1992; Hill et al., 2008; Putnam et al., 1992; Thompson, 1984, 1992).

However, teachers' beliefs are not the only factor besides teachers' knowledge that influences their instructional practices. Teachers' development, such as through certification, years of teaching experience, professional development activities, and completing mathematics content and methods courses, might have an effect on their teaching. Additionally, some aspects of instruction might be related to the curriculum in use or the characteristics of the students.

Finally, I used the MKT ("mathematical knowledge for teaching") theory developed by Ball and colleagues (2008) to define the mathematics that elementary and middle school teachers need to know. Using this framework, I explore the relationship among teachers' knowledge, their instruction, and student learning, taking into consideration other factors in the framework, such as teachers' beliefs.

In this chapter, I have briefly outlined the problem of interest, research questions, and conceptual framework that guided my study. In Chapter 2, I review previous work on the teachers' mathematical knowledge for teaching, beginning with an overview conceptualizing what mathematical knowledge seems necessary for effective teaching and ultimately student learning, followed by reviews of studies on teacher knowledge. In Chapter 3, I provide detailed information on the methods used in the study. The following two chapters, Chapters 4 and 5, present quantitative and qualitative findings of the study, respectively. Finally, in Chapter 6, I discuss the findings, limitations, and implications of the study.

Chapter 2

Literature Review

In this chapter, I review the literature on teachers' knowledge, with special attention given to its effect on teachers' instruction and student achievement. I begin the chapter with an overview of current conceptualizations of what mathematical knowledge is necessary for effective teaching and student learning. I then discuss the literature on teachers' mathematics knowledge focusing on instruction and student learning.

Current Conceptualization of Teacher Knowledge in Mathematics

Early studies dating from the 1960s reveal an assumption implicitly held regarding teacher knowledge: effectiveness in teaching resides simply in the mere subject matter knowledge a teacher has accrued. However, the results of the studies using proxy measures, such as the number of university-level mathematics and teaching method courses taken, were inconclusive, in part, due to the methodological complexity of measuring such variables as well as the variables' poor approximation of teachers' knowledge (e.g. Begle, 1979; Monk, 1994).

Shulman's presidential address delivered to the American Educational Research Association membership (1986) launched increased attention to subject matter knowledge unique to teaching. Shulman reframed the study of teacher knowledge in ways that attend to the role of content in teaching. He defined three categories related to teacher content knowledge (1986) and later on, in a related *Harvard Education Review* article (1987), he specified seven categories of a knowledge base for teaching: knowledge of content; knowledge of curriculum; pedagogical content knowledge; knowledge of pedagogy; knowledge of learners and learning; knowledge of contexts of schooling; and knowledge of educational philosophies, goals, and objectives.

Shulman's *content knowledge* component includes both the amount of the subject knowledge as well as the organizing structure of the subject (Shulman, 1986, 1987; Grossman, Wilson, & Shulman, 1989). It is "beyond knowledge of the facts or concepts of a domain" (p. 9). Teachers must know and be able to explain under what conditions a particular proposition can hold true. According to Shulman and his colleagues, teachers should have knowledge of the substantive structures of a discipline, "the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts," and of the syntactic structure, which is "the set of ways in which truth or falsehood, validity or invalidity, are established" (p. 9, 1986).

Later, in *Knowledge Base for the Beginning Teacher*, Shulman and his colleagues discuss two types of teachers' beliefs as another dimension of subject matter knowledge for teaching that influence novice teachers' teaching and learning. "No discussion of teacher knowledge would be complete without an accompanying discussion of teacher belief, for it is difficult sometimes to differentiate between the two" (Grossman, Wilson, & Shulman, 1989, p. 31). Teachers' beliefs about the subject matter they teach and their beliefs about teaching and learning seem to influence what and how they teach.

The second category, *curriculum knowledge*, consists of knowledge of different programs and corresponding materials available for teaching the given content. It goes beyond an awareness of the different programs and materials to also include knowledge of the effectiveness and implications of programs and materials for given contexts. It entails knowledge of content and corresponding materials in other subject areas of students and consists of knowledge of how topics are developed across a given program (Shulman, 1986).

According to Shulman (1986), the third category, *pedagogical content knowledge*, which has become of central interest to researchers and teacher educators alike, is “the category most likely to distinguish the understanding of the content specialist from that of the pedagogue”(1987, p. 8). It comprises

the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. . . . Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. (p. 9)

The initial call by Shulman (1986) launched scholars’ efforts to specify what body of knowledge is required for teaching. In particular, the term “*pedagogical content knowledge*” has been widely accepted and used by researchers since then. However, researchers differed in their definitions of the term and referred to different aspects of subject matter knowledge for teaching, which seems to have led to increased ambiguity and limited its usefulness (Ball, Thames, & Phelps, 2008; Mason, 2008; Graeber & Tirosh, 2008).

Grossman, one of Shulman’s research team members, sought to identify the domains of subject matter knowledge necessary for effective teaching. Her theoretical framework relied on case studies of six first-year English teachers in secondary school. Three beginning English teachers who lacked professional preparation for teaching and three teachers who graduated from a fifth-year teacher education program were chosen in order to explore the source and nature of pedagogical content knowledge in English. By contrasting these two groups of teachers, Grossman tried to generalize about teacher knowledge for teaching (Grossman, 1990). Based on her study, Grossman reorganized the seven categories defined by Shulman into four main categories: subject-matter knowledge, general pedagogical knowledge, pedagogical content knowledge, and knowledge of context (see Figure 2). The *subject matter content knowledge* was

composed of the same components Shulman and his colleagues defined: knowledge of content, syntactic structure of a discipline, and substantive structures (Shulman, 1987). Grossman placed Shulman’s third component of *curriculum knowledge* into the *pedagogical content knowledge* category (labeled as *curricular knowledge*). Aligned with the later work of Shulman and his colleagues, beliefs became a part of the knowledge base for teaching. Grossman’s component related to belief was listed as a part of pedagogical content knowledge and comprised “knowledge and beliefs about the purposes for teaching a subject at different grade levels” (Grossman, 1990, p. 8).

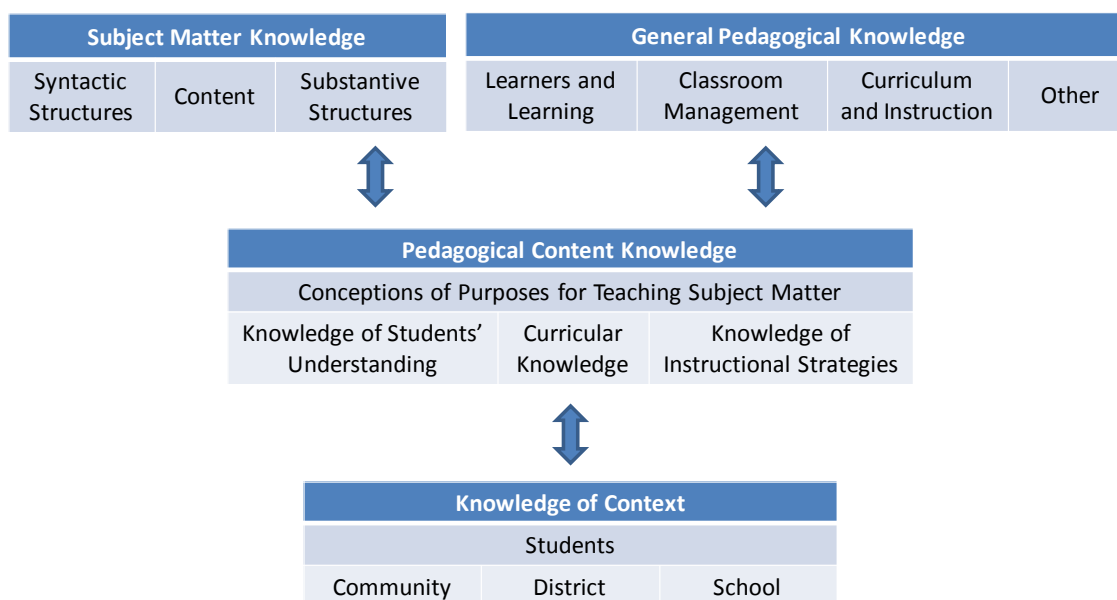


Figure 2. Grossman’s Model of Teacher Knowledge (p. 5, 1990).

Unlike Grossman, Leinhard and Smith (1985) used teachers’ experience as a contrasting point to identify dimensions of mathematics knowledge for teaching. The authors studied four expert and four novice fourth-grade teachers in mathematics. This intensive study included three months of observational field notes from mathematics, ten hours of videotaped lessons, and interviews on several topics, including the videotaped lessons. Comparing the fraction knowledge of these two groups of teachers favored the expert teachers. The authors chose three

expert teachers, whose knowledge of fractions and lesson coverage were similar, to study closely. Further analysis of these teachers' behaviors indicated that the details of their presentations to students were different. In light of these results, the authors identified two aspects of knowledge for teaching: subject matter knowledge and lesson structure knowledge. According to Leinhardt and Smith, subject matter knowledge consists of "concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number systems being drawn upon, the understanding of classes of student errors, and curriculum presentation" (p. 247). The latter includes planning and running a lesson smoothly and providing clear explanations of the materials covered. In their study, the authors broke down mathematical ideas into small component parts, which might lead to overlooking the overall understanding of mathematics (Franke & Fennema, 1992).

Several other scholars have attempted to identify components of teacher mathematics knowledge (e.g. Marks, 1990). In their review chapter in *Handbook of Research on Mathematics Teaching and Learning* (1992), Fennema and Franke proposed their own model of teachers' mathematics knowledge (see Figure 3). Their suggested model includes four categories related to knowledge: knowledge of mathematics, context specific knowledge, pedagogical knowledge, and knowledge of learners' cognition in mathematics. Teachers' beliefs were also part of their model, interacting with teacher knowledge. Additionally, the four components of teachers' knowledge each influenced one another. Another important characteristic of the model was that each component of teachers' knowledge was situated in a classroom context.

The first component, *knowledge of mathematics*, comprises:

knowledge of the concepts, procedures, and problem-solving processes within the domain they teach, as well as in related content domains. It includes knowledge of the concepts underlying the procedures, the interrelatedness of these concepts, and how these concepts and procedures are used in various types of problem solving. (p. 162)

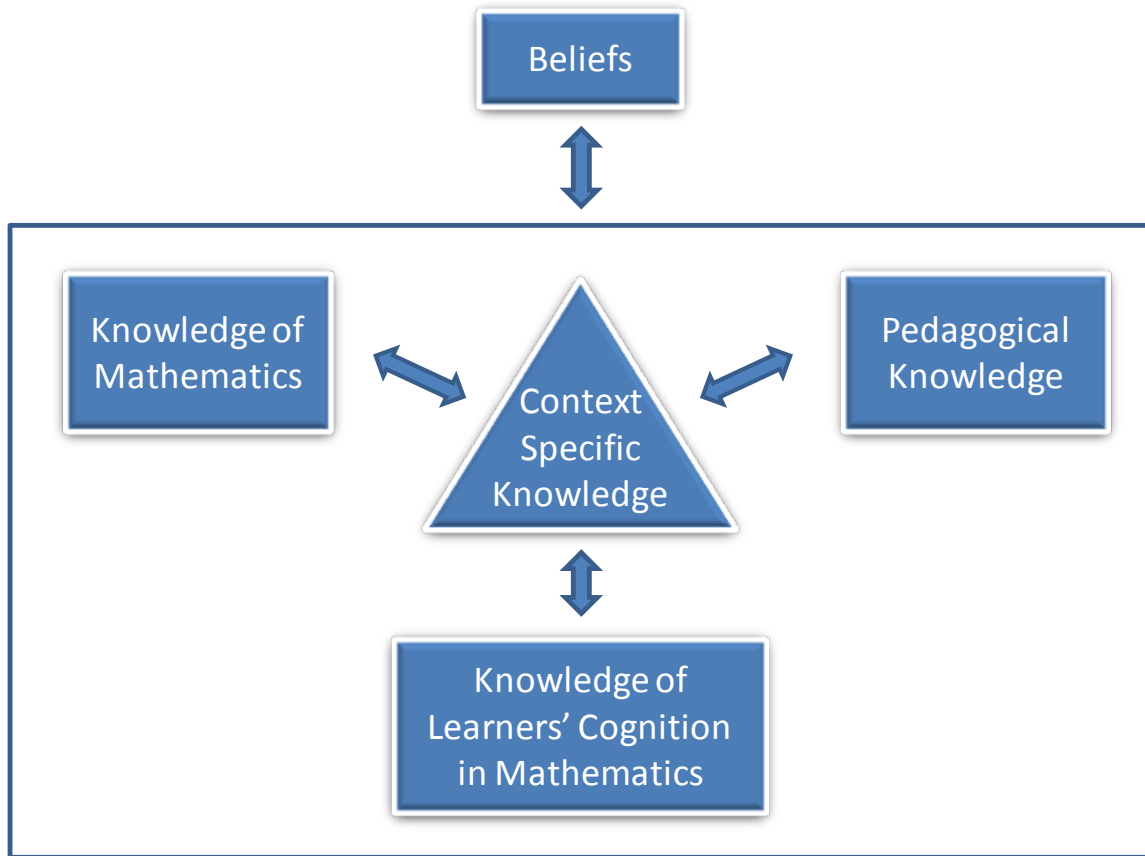


Figure 3. Fennema's and Franke's (1992) Model of Teacher Knowledge (p. 162).

Pedagogical knowledge includes knowledge of teaching procedures such as effective strategies for planning, classroom routines, behavior management techniques, classroom organizational procedures, and motivational techniques. The third category, *knowledge of learners' cognitions in mathematics*, includes knowledge of students' thinking and learning processes, particularly involving mathematics content. The last component of teachers' knowledge, *context specific knowledge*, is a unique set of knowledge that drives teachers' classroom behavior. "Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create" this knowledge (p. 162). Although the authors explained other elements of their model, they did not specify the role of teachers' beliefs.

Despite Fennema and Franke's effort to combine existing research studies and propose a model to help future studies, it seems that researchers are still influenced by Shulman's call to define what subject matter knowledge for teaching means and use his work as a starting point for their studies. In part due to the individualized and disjointed efforts towards defining subject matter knowledge for teaching, researchers are still trying to identify what is needed for effective teaching. Several studies managed to identify some aspects of mathematical knowledge necessary for teaching. Ma's (1999) study could be considered an example of such success. Ma compared 23 U.S. and 72 Chinese elementary teachers' mathematical knowledge in several elementary mathematics topics: subtraction with regrouping, multi-digit multiplication, division by fractions, and perimeter and area of a closed figure. Analysis of teachers' responses to interview items revealed that Chinese teachers had more coherent and connected knowledge of mathematics. In this work, Ma went on to describe another element of mathematics knowledge for teaching: "*profound understanding of fundamental mathematics (PUFM)*," a knowledge that goes beyond conceptual understanding. It refers to the capacity to see a connection between a given topic and other mathematical concepts and to be able to organize a set of related ideas.

With a similar approach, An, Kulm, and Wu (2004) compared 28 U.S. and 33 Chinese middle school teachers' pedagogical knowledge. Paralleling Ma's notion of PUFM, the authors constructed "*profound pedagogical content knowledge*," which is deep and broad knowledge of teaching and the curriculum. Like the model suggested by Fennema and Franke (1992), the authors highlighted the importance of taking into account context and teachers' beliefs. In addition, their framework indicated interactive relationships among the components of teacher knowledge.

Many of the researchers, as exemplified above, who have attempted to characterize mathematical knowledge for teaching have conducted qualitative studies of small numbers of teachers engaged in teaching practice. The qualitative focus of the studies tended to illuminate certain constructs of teachers' mathematical knowledge (Borko et al., 1992; Even, 1993; Fennema & Franke, 1992; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998; Swafford, Graham, & Carol, 1997). However, Ball and her colleagues (2008) developed a practice-based theory of teachers' mathematical knowledge for teaching and have been experimentally testing the components of their framework.

Ball and her colleagues analyzed the existing literature on mathematical knowledge for teaching at that time and identified the elements that seemed essential parts of mathematics knowledge for teaching. They studied teaching mathematics rather than teachers, in order to analyze the mathematical demands of teaching. They developed a set of hypotheses concerning the nature of mathematical knowledge for teaching elementary school mathematics (e.g., Ball, Thames, & Phelps, 2008). Although their practice-based theory is still under construction and some of the components of the model have not yet been empirically tested, several reports have already been published regarding discernible categories in the framework (e.g., Ball et al., 2008; Hill, Rowan, & Ball, 2005; see Figure 4). Ball and her colleagues created the term “mathematical knowledge for teaching” (MKT) to refer to a special kind of knowledge required only for teaching mathematics (Ball et al., 2008; Hill et al., 2005).

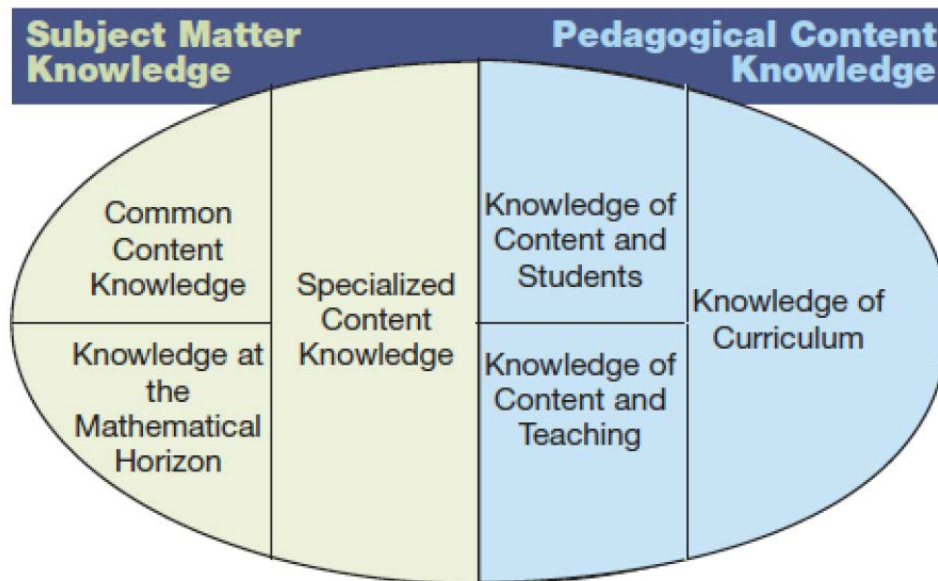


Figure 4. Model of Teachers' Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403).

As shown in Figure 4 above, the model has two main domains: subject matter knowledge and pedagogical content knowledge. According to Ball and her colleagues (2008), subject matter knowledge consists of three sub-domains: common content knowledge, specialized content knowledge, and knowledge at the mathematical horizon. The third component of teachers' subject matter knowledge, knowledge at the mathematical horizon, is described as a provisional category recognizing connections among topics throughout the curriculum. The pedagogical content knowledge is also composed of three sub-domains: knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Similarly, *knowledge of content and students (KCS)* and *knowledge of content and teaching (KCT)* were identified as two different constructs of pedagogical content knowledge.

The first component of teachers' content knowledge, *common content knowledge*, refers to the mathematical knowledge and skills that not only teachers but also others might have. This knowledge is not unique to teaching. Solving mathematics problems or knowing how to carry out a procedure as well as knowing the definition of a concept are examples of common content

knowledge. The second domain, *specialized content knowledge*, is mathematical knowledge specific to teaching. This knowledge differs both from knowledge of students or pedagogy and from Shulman's pedagogical content knowledge. When identifying patterns in student errors or assessing whether a nonstandard approach would work, teachers need to have a kind of mathematical knowledge that others do not. The last component of teachers' content knowledge, *horizon knowledge*, is "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball, Thames, & Phelps, 2008, p. 403). The place of *horizon knowledge* has not been tested yet, so the place of this category has not yet been fixed.

Another discernible domain, *knowledge of content and students (KCS)*, is a combination of knowledge of students and knowledge of mathematics. It requires familiarity with, and anticipation of, students' mathematical thinking and understanding for a given content. The last domain, *knowledge of content and teaching (KCT)*, brings knowing about teaching and knowing about mathematics together. It includes knowledge of how to choose

which examples to start with and which examples to use to take students deeper into the content. Teachers evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. (Ball et al., 2008, p. 401)

The two sub-domains defined within pedagogical content knowledge in this model match with Shulman's elements of pedagogical content knowledge (1986): "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" and "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Ball and her colleagues, like Grossman's (1990) framework, placed Shulman's third category, *curricular knowledge*, within their pedagogical content knowledge category.

Although the results of their current studies suggest multidimensional subject matter knowledge for teaching exists, it could be difficult at times to distinguish specialized content knowledge from knowledge of students and content. For example, while selecting a numerical example to examine whether students understand decimal numbers, a shift occurs across four domains; ordering a list of decimals requires common knowledge, while generating a list to be ordered would demand specialized content knowledge. Additionally, recognizing which decimals would cause students' difficulty to understand entails knowledge of students and content. Finally, decisions of what to do regarding students' difficulties involve knowledge of content and teaching (Ball et al., 2008).

As stated, Ball and her research team's approach to mathematical knowledge, the categorization of mathematical knowledge specific to teaching, is ongoing work and open to refinement and revision. Ball and her research team caution us against unknown territory, pointing out the extent to which their formulation of MKT is culturally specific. These scholars have not emphasized other contexts (e.g., schools, classroom environment) and teachers' beliefs have not been part of their specification process of MKT to date, but, as suggested by other scholars (Fennema & Franke, 1992; Grossman 1990), beliefs could also influence decisions and moves teachers make in teaching.

Hill and Ball, in work with a variety of colleagues (e.g., Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005), demonstrated that teaching mathematics demands mathematical understanding beyond the mathematical knowledge needed by other practitioners of mathematics. In addition, findings of other studies have contributed to the notion that majoring in mathematics or having strong subject matter content knowledge is insufficient for the mathematical knowledge necessary for teaching. However, the fact is that we still lack a

detailed understanding of what mathematical knowledge is necessary for teaching and to what extent other factors influence teachers' MKT (Graeber & Tirosh, 2008; Mason, 2008; Silverman & Thompson, 2008). Researchers in the field are still developing a framework for content knowledge for teaching by focusing on different elements of the body of mathematics knowledge for teaching. Researchers' different conceptions and beliefs about what mathematical knowledge is necessary to be an effective teacher (e.g., Silverman & Thompson, 2008) might be one explanation for the variety of models of mathematical knowledge for teaching.

However, there are some common conceptualizations in these frameworks. Teachers' knowledge and teachers' practices can and do interact with one another and change over time. Another important common ground is that the models are developed in context, indicating that teachers are likely to experience different influences, depending on locality, resources, participant background, and other factors (Fennema & Franke, 1992; Grossman, 1990; Porter & Brophy, 1988; Thompson, 1992).

The studies that were designed to identify elements of teachers' knowledge for effective teaching have not included student performance as a factor in their models in general. It could be partly due to the small scale nature of these studies, which might prevent them from studying the effects of teachers' knowledge as well as their teaching on student achievement. It could also be due to the implicit assumption that increasing teachers' knowledge and improving teaching practices should be enough to improve student achievement.

On the other hand, teachers' beliefs are a part of some researchers' models. Certain beliefs that teachers hold seem to mediate the effects of teachers' knowledge on teachers' practices. The most commonly studied beliefs are those about: the nature of mathematics (Grossman et al., 1989; Putnam et al., 1992; Stodolsky & Grossman, 1995; Thompson, 1984;

Thompson, 1992); teaching and learning mathematics (Borko et al., 1992; Hill et al., 2008; Putnam et al., 1992; Thompson, 1984; Thompson, 1992); content (Forgasz & Leder, 2008; Grossman et al., 1989; Putnam et al., 1992); and general pedagogy and learning (Borko et al., 1992; Forgasz & Leder, 2008). However, teachers' beliefs are not the only factor besides teachers' knowledge influencing teachers' instructional practices. Teachers' own experiences as students could influence teachers' instructional practices (Grossman, 1990; Porter & Brophy, 1988; Thompson, 1992). Also, teachers' experiences could affect their beliefs about mathematics teaching and learning (Grossman, 1990; Thompson, 1992). Additionally, teachers' perceptions about their students' abilities have an influence on their teaching (Forgasz & Leder, 2008; Porter & Brophy, 1988).

Some aspects of instruction could result from external pressures rather than teachers' own beliefs about what is appropriate. External factors on teacher thinking and action should be included in any framework that portrays the relationships between teachers' knowledge and their observed instruction, as well as between knowledge and student achievement, to better understand possible inconsistencies between those dimensions. For example, policies such as No Child Left Behind could influence teachers' practices without necessarily affecting their beliefs or reflecting teachers' knowledge for teaching.

Empirical Studies of Teachers' Mathematics Knowledge

Throughout the past four decades, numerous studies have been undertaken in an attempt to identify the relationships among teachers' mathematics knowledge, instructional practices, and student learning (Mewborn, 2003). Researchers have differed in their approaches to explore these relationships. Studies, especially earlier ones, sought to demonstrate the impact of teachers' knowledge on student standardized test scores by using proxy variables. Education level, number

of undergraduate mathematics and/or mathematics education courses, years of teaching experience, certification status, and majoring or minoring in mathematics were commonly used variables in this genre. Another approach taken in the literature employs more direct measures of teachers' knowledge, such as teachers' performance on certification exams or other tests of mathematics knowledge. Especially after Shulman's (1986) presidential call, researchers conducted more observational studies to identify elements of mathematical knowledge required for teaching in actual classroom settings.

Studies using proxy measures. The studies using proxy measures have been mainly quantitative and sought to demonstrate a relationship between teachers' knowledge and student achievement. Teacher education level, years of teaching experience, and number of university-level mathematics and teaching method courses taken were used as measures of teachers' mathematical knowledge. These studies failed to find any statistically significant correlation between measures of teacher knowledge and student achievement (e.g., Begle, 1979; Monk, 1994). Begle (1979) found no evidence to suggest a significant positive relationship between students' mathematics achievement and teachers' mathematical knowledge when estimated by teachers' mathematics credits beginning with calculus, credits in mathematics methods, and majoring or minoring in mathematics. However, a less often reported result from his study indicated that the number of credits in mathematics method courses was more strongly correlated with student performance than the total number of credits in mathematics courses a teacher had. Similarly, Monk (1994) found that the number of mathematics education courses had positive effects on the performance of the students in secondary schools and contributed more to student achievement gains than the number of mathematics courses. Additionally, it appeared that the relationship between the number of mathematics courses a teacher had taken and student

achievement was not linear. The effect of the number of mathematics courses on student performance diminished beyond five or more courses.

Another example of a study in this tradition is Rowan and his colleagues' (2002) analysis of teachers' effects on student achievement. Teachers' degrees in mathematics served as a proxy for teachers' knowledge. Two data sets were used to investigate the relationship: in the first longitudinal data set students were followed from first grade to third grade, while in the second data set students from grade three were followed for three years. The findings of the study implied that for both cohorts, students whose teachers held advanced degrees in mathematics did worse than students whose teachers did not have a mathematics degree. Wayne and Youngs (2003) reviewed 21 studies in which students' SES was controlled and found that additional course work and degrees in math significantly correspond to high school students' performance, but these results were not applicable to elementary school students. Similarly, the National Mathematics Advisory Panel (2008) indicated teachers having majors in mathematics had a positive effect on high school students' performance. According to the report, the same effects were not present in studies of elementary teachers.

Studies involving this approach have several limitations. Certain problems exist with the methods researchers applied, particularly in meta-analysis. Researchers' approaches varied in terms of which existing studies to include in their meta-analyses and which statistical methods to apply; as a result, they reached different conclusions (Greenwald, Hedges, & Laine, 1996; Hanushek, 1996; Rowan et al., 2002). Besides, the quality of the measures to assess teachers' mathematical knowledge is problematic. No information is available concerning the quality of the courses a teacher had taken (Monk, 1994). Beyond that, the number of courses and having a

major or minor in mathematics are poor proxies for the kind of knowledge that matters for teaching (Hill et al., 2005).

Studies using direct measures of teacher knowledge. Using gross measures such as number of courses taken was replaced or accompanied by more direct assessments of teachers' knowledge in this genre. Although some of the measures are indistinguishable from tests that were given to students, others pose tasks special to the practice of teaching. The studies in this tradition typically reveal a positive relationship between teachers' knowledge and student achievement (e.g., Hill et al., 2005; Hill, Sleep, Lewis, & Ball, 2007). However, what is assessed in these measures has differed across studies, which leads to important limitations on the interpretations of this work.

The first study involving this approach was the Coleman Report (1966), and it measured teachers' knowledge via a multiple-choice questionnaire. The report indicated that teachers' scores positively predicted student achievement in mathematics. However, none of the items on the measure were specifically related to mathematics.

Rowan, Chiang, and Miller (1997) used the National Education Longitudinal Study of 1988 (NELS: 88) data and tested the effects of teachers on student achievement in mathematics by using a general employees' performance model. This model suggested that teachers' abilities, motivation, and work situations could explain teachers' effects on students' performance in mathematics. Teacher ability was defined in terms of teachers' knowledge of subject matter and teaching strategies. Two separate measures were used as an indication of teachers' knowledge in mathematics. One was the teachers' response to a one-item math questionnaire, and the other was whether a teacher majored in math or not. The results indicated that students whose teachers answered the item correctly and students whose teachers had a math major had higher

achievement levels than those whose teachers' answer was wrong and whose teachers did not have a math major. However, in both cases the size of the effect was quite small.

Hill et al. (2005) investigated the effects of teachers' mathematical knowledge for teaching on student performance in mathematics. Researchers collected data from 115 elementary schools that were participants of one of the three leading reform programs and from 26 comparison schools similar to the reform oriented-schools in terms of district setting and district SES. Although the sample was different from a nationally representative sample of schools, 334 first-grade and 365 third-grade teachers participated in this study. Eight students from each teacher's classroom were chosen and followed for two periods of three years, from kindergarten to second grade and from third-grade to fifth grade. Parents were interviewed via phone to access students' academic history and other home background-related factors. The standardized assessment used for students did not match well with the areas in which teachers' knowledge was assessed and covered additional content domains besides the elementary school mathematics curriculum. These researchers assessed teachers' mathematical knowledge in two domains defined by Ball and her colleagues (2008): *common content knowledge* and *specialized content knowledge* in three content areas: number concepts, operations and patterns, functions and algebra. Although proxy variables, such as the average number of content and methods courses they had taken and teachers' certifications, were not significant contributors to student performance, the teachers' scores in this measure were the strongest teacher-level predictor for student achievement. Based on the findings of the study, one standard deviation increase in teachers' content knowledge measure predicted a one-half to two-thirds of a month of additional student growth in mathematics. However, the authors noted that teachers' related variables,

including their math score, only explained a small variation in students' achievement (8% for first grade and 2% for third grade).

Recently, Rockoff, Jacob, Kane, and Staiger (2008) investigated the relationship between students' math score gains and novice teachers' knowledge, measured by both traditional predictors and the assessment to capture MKT defined by Ball et al. (2008). Of 602 new elementary and middle school math teachers who teach mathematics from fourth to eighth grade, 333 completed the survey measuring teachers' MKT. A high percentage of the teachers who completed the survey are members of Teaching Fellows and/or TFA corps, which indicates non-randomness in respondent teachers and may limit the generalization of findings to all new teachers. Student math achievement was captured by using state-level standardized test scores, which also enabled the researchers to access student demographics and prior achievement. Aligned with some of the previous studies, majoring in mathematics and holding certification were not significant predictors for student performance. However, a teacher's score on the test for MKT was the most significant predictor for student performance with an effect-size of .03 standard deviations. It is important to remember that even teachers' MKT (with the largest coefficient) only explained less than 8% of the teacher-level variation. Another noteworthy point is that these researchers also analyzed the joint effects of teachers' variables on student performance. The researchers used factor analysis, resulting in two groups, one of which was labeled as cognitive skills and the other labeled as non-cognitive skills. The cognitive skill group included the variables such as attending a more selective college, SAT score, the IQ test, and math knowledge for teaching, while the non-cognitive skill group included measures such as teachers' extraversion, conscientiousness, and efficacy. Both factors seemed to have a modest association with student performance, while a single variable in each group might not be a

significant factor for teacher effectiveness and may not significantly predict student achievement. This finding suggests that some factors that are not directly related to teachers' subject matter could still play a part in teachers' effectiveness.

Observational Studies

Another line of inquiry concerning teachers' effectiveness focuses on investigating teachers' mathematical knowledge while they are teaching. According to this view, mathematical knowledge for teaching goes beyond that captured in measures of mathematics courses taken or teachers' scores on mathematics tests. Most of the work done in this genre has been qualitative in orientation and has used a variety of different approaches to address their question: case studies (e.g., Grossman, 1990), expert-novice comparisons (e.g., Leinhardt & Smith, 1985), international comparisons (e.g., Ma, 1999), and studies of novice teachers (e.g., Borko et al., 1992).

Several studies in this genre attempt to focus on links between a teacher's lack of mathematical understanding and quality of their instruction. These studies suggest that mathematics knowledge for teaching is different from mere subject matter knowledge in mathematics. For instance, Borko and her colleagues (1992) used a case study to illuminate the difficulty of one student teacher with a strong mathematical background. The student teacher struggled to conceptually explain the standard algorithm for division of fractions in a sixth grade mathematics classroom. Although the novice teacher seemed to be mathematically well-equipped by taking several college-level mathematics courses and to hold beliefs regarding teaching mathematics similar to the current views of effective mathematics teaching, her lack of strong knowledge of both elementary school mathematics and how to teach mathematics limited her teaching to explanations of the procedures. As noted by Putnam et al. (1992) "the limits of [the teachers'] knowledge of mathematics became apparent and their efforts fell short of

providing students with powerful mathematical experiences” (p. 221). In their study of four fifth-grade teachers, teachers holding limited mathematical knowledge tended to choose mathematically incorrect representations to interest their students, in part due to their conceptions that mathematics, by nature, is hard and boring.

Similar results were also obtained from the studies with secondary school mathematics teachers (e.g., Kahan, Cooper, & Bethea, 2003). For instance, Even (1993) used a two-phase study to investigate prospective secondary mathematics teachers’ conceptions of function and its relationship to their pedagogical knowledge. In the first phase of the study, 152 prospective students from several universities participated in the study and completed an open-ended questionnaire to capture their understanding of what function means and how to teach function. In the second phase, to make an in-depth analysis of the relationship between teachers’ understanding and their preferred way of teaching, ten more novice teachers were asked to participate in the study. In addition to the questionnaire used in the former phase of the study, these prospective teachers were interviewed. The findings indicated that novice teachers lacked an understanding of the current conceptualization of function, and this limited knowledge seemed to have an impact on how they intended to teach the concept of function.

One could argue that teachers’ lack of understanding of mathematics could be resolved by requiring them to increase their subject matter knowledge by taking more college mathematics courses. Ma (1999) challenges this assumption in her comparison study. She argued, in her comparison study between U.S. and Chinese elementary school teachers, that Chinese teachers built knowledge necessary for effective teaching while working as a teacher. Chinese teachers had two or three more years of teacher education in addition to their equivalent of a 9th grade mathematics education. Teaching only one subject and/or one other subject could

enable teachers to specialize in their mathematical knowledge in a way that they could see the connection between concepts and their relative importance.

Ball and Wilson's (1990) findings also undermined the claim that more subject matter knowledge is the solution for better teaching. In their study, they compared the mathematical knowledge of 22 undergraduate students majoring in mathematics and education and 21 postbaccalaureate mathematics majors in alternate route programs at both the entry and exit points of their programs. The mathematics content of the study dealt with the relationship between perimeter and area, proof by example, division by zero, and division of fractions. Neither upon entry to the teacher education programs nor exit from their programs did these teacher trainees differ much in their understanding of underlying meanings for mathematical ideas. Even at the end of the program, beginning teachers in both programs lacked preparation; they still had limited understanding of choosing meaningful presentations to teach concepts.

The empirical studies cited above suggest there is knowledge used in classrooms beyond formal subject matter knowledge, a contention also supported by Shulman's (1986) notion of "pedagogical content knowledge." For example, Thompson and Thompson (1994, 1996) studied one middle school teacher as he was teaching the concept of rate to one of the mathematically-strong students in his sixth-grade classroom. The teacher in the study seemed to hold strong mathematical content knowledge, based on the test used by the researchers, and to hold general pedagogical knowledge, based on interview with the teacher. However, the language he chose to explain the multiplicative relationship between speed and distance, with the goal of teaching speed as rate, and his limited attention to the student's thinking process, reasoning, and interpretation of his explanation might have undermined his efforts to help the student correct her misunderstanding and understand rates conceptually.

Unlike the studies cited above, which tend to focus on teachers' limited mathematical knowledge and its relationship to their instruction, Fernández (1997) aimed to identify the situations where strong mathematical knowledge could play more significant roles in instructional practices. She studied nine secondary mathematics teachers who had strong mathematical backgrounds and held compatible beliefs regarding teaching mathematics with the ones the current *Standards* suggested. She focused on teachers' responses to students' unexpected answers and noted that these teachers were able to provide counter-examples to the students to show the errors in their thinking, see where students' thinking and solutions lead to, and include students' alternative methods in their instruction.

Like Fernandez, several other researchers have attempted to identify what mathematics knowledge matters in the work of teaching. Carpenter, Fennema, Peterson, Chiang, and Loef (1989) underlined the importance of knowledge of students' thinking. In their study, they randomly assigned half of the forty teachers to participate in a month-long training designed to help the teachers understand students' thinking while they solve addition and subtraction problems. The other 20 teachers participated in two-hour workshops focused on non-routine problem solving. The researchers observed all of the teachers in the following year. The researchers asked the teachers to predict how their 12 randomly chosen students would solve selected problems. The findings from classroom observations and students' performances on mathematics tests suggested that teachers in the experimental group increased their knowledge of individual students' problem-solving processes and changed their instructional practices. They seemed to teach problem solving significantly more and number facts significantly less than did the teachers in the control group. Additionally, experimental teachers encouraged students to use a wide range of problem-solving strategies, listened to their students' processes, and discussed

alternative methods more often than the teachers in the control group. Students of the teachers in the experimental group outperformed their counterparts in the areas of problem solving.

Although students' achievement in experimental classrooms differed modestly from the ones in control classrooms, the interviews with students indicated that students in the experimental classrooms reported a greater understanding of mathematics and more confidence in solving problems.

As indicated by Carpenter and his colleagues, teachers' knowledge of student thinking seems to be an important aspect of mathematics knowledge for teaching. Swafford et al. (1997) reported the effects of a four-week professional development course designed to increase teachers' geometry content knowledge and teachers' knowledge of students' cognition on teachers' instructional practices. Forty-nine middle grade (4-8) teachers attended this course and made significant gains in their content knowledge. The authors chose eight of them for follow-up observations and interviews to see the impact of their knowledge on their instruction. The teachers appeared to try new instructional practices, such as hands-on activities and manipulatives, and to take more risks to enhance student learning. They also seemed to devote more time to geometry instruction and to report more confidence in their abilities.

Although the teachers in the mentioned studies changed some of their practices, the relationship between teacher knowledge and teaching is not straightforward; beliefs and several other factors could mediate the effects of teachers' content knowledge on instruction. Sowder, Phillipp, Armstrong, and Schappelle (1998) reported on a two-year professional development project aimed to increase teachers' knowledge in areas of rational number, quantity, and proportional reasoning. Five middle school teachers participated in this study, and four of them attended the whole program. Sowder and her colleagues investigated teachers' knowledge in the

mentioned areas and its relationship to their instructional decisions and student achievement. The findings indicated that teachers' practices changed as their content knowledge increased and deepened. However, the researchers noted that change in teachers' instructional practices took time; even a year after the implementation of the program the changes the teachers made were limited. Another issue raised by the researchers that had influence on teachers' instructional decisions as well as their beliefs, was teachers' concerns regarding student performance on standardized tests, which constrained teachers' willingness to take risks to change their instructional practices.

Like Sowder and her colleagues, Hill, Blunk et al. (2008) investigated the complex relationship between teachers' mathematical knowledge and their instruction. Ten teachers participated in the study and several types of data were collected from them: a survey measuring teachers' mathematical knowledge for teaching, classroom observations, interviews, and debriefings regarding the observed lesson. The researchers used a rubric to evaluate the quality of the participant teachers' instruction based on six elements: mathematics errors, responding to students inappropriately, connecting classroom practice to mathematics, richness of mathematics, responding to students appropriately, and mathematical language. Correlation analysis indicated that teachers' scores on mathematics assessments were significantly positively associated with the element of instruction *responding to students appropriately* and were significantly negatively associated with *mathematics errors* teachers made. In the next phase of the study, Hill and her colleagues chose five of the teachers for further investigation of the interrelations between teachers' knowledge and their instruction. The teacher who scored high on both the mathematics test and lesson rubric and the teacher who scored low on both the mathematics test and lesson rubric were chosen to illustrate the role that mathematical knowledge plays in supporting or

hindering the quality of instruction. To uncover alternative explanations, two teachers (with low math score and high instruction score and with high math score and low instruction score) were selected. The final teacher was chosen for the case study due to her both divergent and convergent cases. Her mathematical knowledge was strongly similar to the one who scored high on both the mathematics test and lesson rubric, while her score for her instruction was in the middle of the range.

In-depth analysis of these teachers suggested that teachers with strong mathematical knowledge made fewer errors and provided rich examples of mathematics. The examples and activities they chose and their responses to the students also reflected their high level of mathematical knowledge. Teachers with lower-level content knowledge could exhibit some of these characteristics in their instruction, but it was not consistent across their lessons. This variability was in part due to the extent of the support they got from either their textbook or the professional development they received. The authors discussed three factors that seemed to mediate the effects of teachers' mathematical knowledge on their instruction. Teachers' beliefs regarding how mathematics should be learned and how mathematics should be made enjoyable for students appeared to be factors affecting teachers' instruction. Moreover, how teachers use the curriculum materials and their views of the curriculum materials seemed to affect their instruction; it could sometimes serve to degrade the quality of instruction or sometimes improve the instruction. Finally, the professional development teachers received seemed to have mixed effects on teachers' instruction. Teachers, especially ones with lower mathematical knowledge, appeared to have problems implementing supplementary materials and tasks learned in professional development in the intended way; as a consequence, these activities and materials could serve to lower the quality of the mathematics instruction. This finding highlights the

importance of Shulman's third category of teacher knowledge, "curriculum knowledge."

Although the authors provide the field insights about the complex relationship between teachers' knowledge and their instruction, we still do not know how these variations between teachers' knowledge and instruction differ in their effects on student performance.

What is Missing in the Literature?

Research on teachers' mathematical knowledge generally appears to either investigate a connection between teachers' knowledge and student achievement without paying attention to teachers' instruction or to focus on the relationship between teachers' knowledge and their practice but ignore their effects on students' outcomes. We lack a detailed understanding of how teachers' knowledge affects student learning and how their teaching mediates the effects of teachers' knowledge on student performance.

Another constraint in the literature dealing with teachers' knowledge is the quality of the assessments used to measure teachers' content knowledge (particularly earlier studies). As Hill and her colleagues argued what was assessed as teachers' knowledge in these measures remains questionable (Hill et al., 2007). New assessments designed to measure important aspects of teachers' mathematics knowledge, such as *Learning Mathematics for Teaching* (LMT) (Learning Mathematics for Teaching, 2004) and the *Diagnostic Teacher Assessment in Mathematics and Science* (DTAMS; Center for Research in Mathematics and Science Teacher Development, 2011), should be used in order to investigate teachers' knowledge more efficiently. It seems that teachers' MKT has not been adequately measured in several studies, which might dispute the findings of the existing research, not only regarding the magnitude of the effect of teachers' knowledge on student learning but also regarding the kinds of teacher knowledge that matter most in producing student learning (Hill et al., 2005).

Another limitation is researchers' tendency to address a relatively narrow range of mathematical content in the studies. It seems that numerous researchers have measured teachers' knowledge on the topics of place value, division, fractions, area and perimeter (Mewborn, 2001). Teachers' mathematical knowledge in areas other than these has not been studied as much. Given recent emphasis on more contemporary mathematical topics such as probability, data analysis, algebraic reasoning, and number theory, it seems important to investigate teachers' knowledge in broader domains.

Another constraint worth noting here is that several studies generally illustrate how knowledgeable teachers or less knowledgeable teachers teach (e.g., Putnam et al., 1992). However, the research conducted so far, which tends to focus on either teachers with strong mathematical knowledge or teachers with limited mathematical knowledge, has failed to include a wide range of teachers differing in their mathematical knowledge. The researchers attribute many characteristics of teachers' instruction to teachers' knowledge, making a wide range of teachers necessary. The field needs more studies that focus on teachers who work in similar contexts but vary in their mathematical knowledge, in order to identify what and how mathematical knowledge for teaching (MKT) is related to teachers' practices and student learning.

Additionally, although teachers' beliefs about the nature of mathematics, teaching mathematics, and particularly their views on what constitutes evidence of students' understanding seem to play a significant role in shaping teachers' instructional practices (e.g., Stodolsky & Grossman, 1995; Thompson, 1994), the studies have not analyzed the effects of beliefs systematically while looking at the relationship between teachers' knowledge and their practice. Teachers' beliefs generally appear to be mentioned in these studies when there is a

conflicting outcome between teachers' knowledge and expected instructional practices.

However, failure to recognize the role of teachers' beliefs in instructional practices might lead to misunderstanding about ways to improve the quality of mathematics instruction in schools.

Lastly, given the difficulty of disentangling the causal order in studies linking teacher knowledge to instruction and student outcomes, one of the most important questions that remain unanswered is how teachers' knowledge changes over time and how these changes affect their instructional practices as well as student learning. Only a few studies have provided a longitudinal analysis of teachers' knowledge, while the change in teachers' knowledge (if it exists) generally has not been measured by well-established assessments (e.g., Sowder et al., 1998). Rather than capturing teachers' knowledge at a particular point in time, investigating teachers' knowledge over time might shed new light on how teachers' mathematical knowledge relates to their practices and student learning.

In this study, using Ball and Hill's MKT construct, I explore how changes in teachers' MKT are related to changes in their instructional practices and student achievement. I discuss this process in detail in the next chapter.

Chapter 3

Methods of Data Collection and Analysis

Chapter 2 discussed the literature on teachers' knowledge of mathematics, focusing on its effects on instruction and student achievement. This chapter outlines the methods used to explore the associations among teachers' mathematical knowledge for teaching (MKT), their instructional practices, and the achievement of their students. I begin this chapter with an overview of the mixed methods approach, followed by the research context and methods of data collection and analysis.

Mixed Methods Approach

As mentioned above, in this study, I investigated whether a relationship existed among teachers' mathematical knowledge and their instructional practices as well as student achievement, and elaborated on how that association between teachers' mathematical knowledge and instruction actually occurred.

I used a mixed methods approach because using qualitative and quantitative methods together yielded a better understanding of the phenomena than using a single method. As stated by Greene and Caracelli (2003), "social reality is both casual and contextual, and social knowledge is both propositional and constructed. To respect all facets of realism, multiple methods are not only welcomed but required" (p. 99).

My primary intent was to generate a more elaborate and comprehensive understanding of the interplay between teachers' knowledge of mathematics and instructional practices. In particular, using quantitative data, I investigated the extent to which changes in teachers' MKT were related to changes in their instructional practices and their students' achievement gains. Using qualitative data, I investigated how the relationships occurred among teachers' knowledge,

instructional practices, and student learning. My primary intent in including qualitative methods in the study was not for the purpose of triangulation. My aim of using a mixed methods approach was for “complementarity, in which different research methods address different aspects of the phenomenon, and convergence is not necessarily expected. Findings from the separate components are then fitted together like a jigsaw puzzle” (Smith, 2006, p. 465). Hence, I aimed to achieve this goal by “collecting quantitative data and then collecting qualitative data to help explain or elaborate on the quantitative results” (Creswell, 2002, p. 566). As I explain in detail in the following section, I first started by analyzing the quantitative data, and based on a preliminary analysis of the data, I chose a subsample of teachers for the qualitative part of the data analysis.

More specifically, this study entailed addressing the following two research questions:

1. How does teachers’ MKT affect their instruction? What factors, such as beliefs and the curriculum, mediate the expression of MKT in instruction?
2. To what extent are changes in teachers’ MKT, instructional practices, or both related to students’ gains in achievement?

Research Context

Description of the program. The main data for this study resulted from the evaluation of a 2.5-year master’s program created with support by a federally funded, state-administered Math and Science Partnership (MSP) grant. Faculty from a state university designed a master’s degree program in collaboration with a partnering high-needs school district located in an ethnically diverse, midsized city in the Midwest. In the partnership district, roughly one half of the students were students of color and two-thirds of the students qualified for free or reduced lunch, which is 24% higher than the percentage of students who qualify for free or reduced lunch at the state level. Furthermore, as of 2008, 39% of the teachers in the district held a master’s degree, which

is 14% lower than at the state level (see Figure 5). The percentage expenditure spent on instruction was 38% at the district level, whereas the ratio is 48% at the state level. Finally, 77% of the students in the district met or exceeded the standards, whereas 85% of the students in the state reach that level.

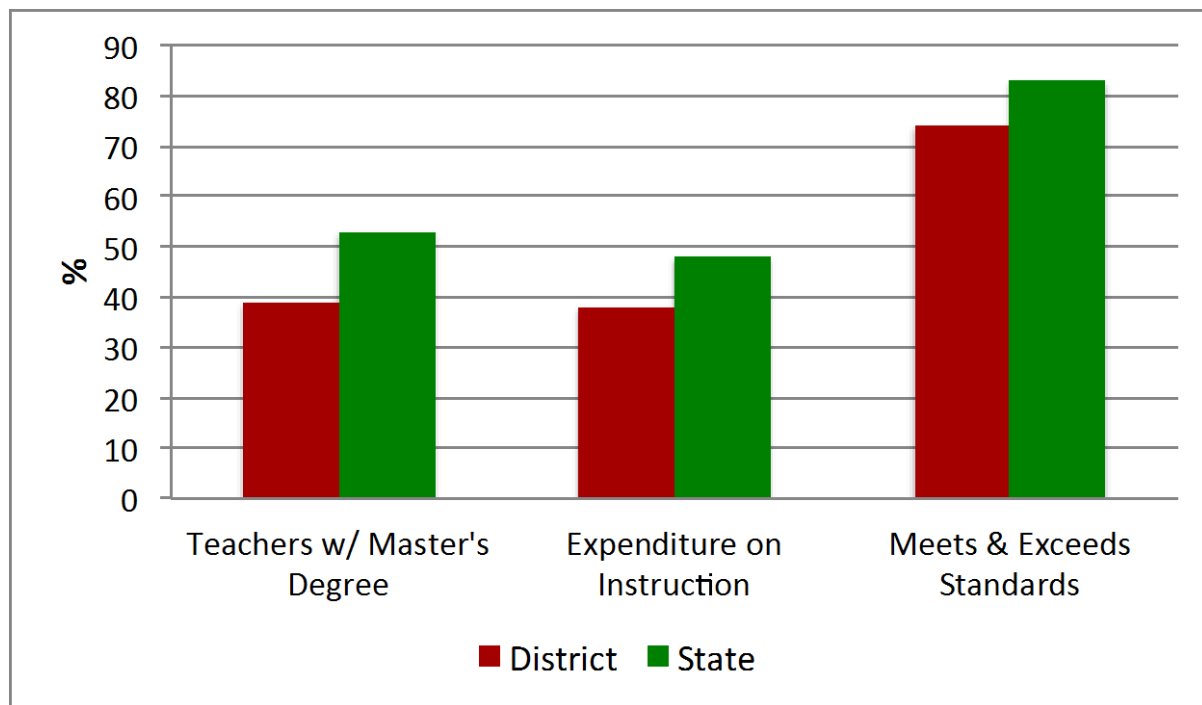


Figure 5. Comparison of the partnership district and state.

The program took place from August 2008 until December 2010. The program focused on “sense-making in mathematics and science” and was designed to meet the partner district’s needs and to deepen teachers’ content and pedagogical content knowledge in mathematics and science. This 32-credit-hour Ed.M. program consisted of nine courses, including mathematics, science, and education courses. Courses were offered at a local community college within the school district during the school year and on the university campus during the summer.

Two courses focused specifically on mathematics. The first course, which was a hybrid of mathematics content and methods, occurred during the first semester of the program (fall

2008). The second, which was a mathematics content course, occurred in the second year of the program (spring 2010). Both courses were four credits and met Thursday evenings. Teachers who both completed the program with at least a 3.25 GPA and participated in all the evaluation components (including repeated assessments of their mathematical knowledge) were given a \$1,500 stipend. This stipend encouraged teachers' serious engagement with the program components, including coursework and evaluation.

Description of the two mathematics courses.

Mathematics Content and Methods Hybrid Course. The hybrid course was a combination of typical mathematics methods and “mathematics for elementary teachers” course work. The instructor for this course organized it around the National Council of Teachers of Mathematics’ (NCTM, 2000) content strands (numbers, geometry, measurement, algebra and functions, and statistics and probability). The instructor drew primarily from the popular mathematics methods text *Elementary and Middle School Mathematics*, by Van De Walle (2006), organizing most weekly lessons around the assigned chapter(s) in the text. In addition to reading the text, teachers completed two major problem-solving assignments, conducted a small action research project in their classrooms, taught a lesson to the class, and participated in reading circles in which they discussed (electronically) what they were learning from the text and other assigned articles. The problem-solving assignments included a variety of challenging mathematics problems adapted from various sources, including the Van De Walle text, the mathematics methods text, *Fostering Children’s Mathematical Power, An investigative Approach to K-8 Mathematics Instruction*, by Baroody and Coslick (1998), *The Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), *Developing Mathematical Ideas* (Schifter, Bastable, Russel, Yaffee, Lester, & Cohen, 1999), and *Thinking*

Mathematically (Mason, Burton, & Stacey, 1985). Although collaborative group work was encouraged on the problem-solving assignments, two tests served as a measure of how well teachers learned what was intended from those assignments, as well as from the course readings and discussions. Overall, the course was designed to deepen teachers' knowledge of mathematics content specifically relevant to K-8 teaching, and to enhance teachers' ability to teach that content with a focus on student sense-making.

Mathematics Content Course. The second mathematics course focused specifically on mathematics content with the theme, "mathematics in the world around you." The course was taught by an award-winning mathematics department instructor. The instructor had prior experience teaching middle school mathematics, as well as "mathematics for elementary teachers" courses. This content course exposed teachers to a wide variety of interesting mathematics applications while enhancing their knowledge of algebra, probability and statistics, number theory, and other topics. The college textbook used was entitled *For All Practical Purposes: Mathematical Literacy in Today's World* by Consortium for Mathematics and Its Applications, 2003. Specific topics included networks, linear programming, random samples, probability, inference, voting systems, game theory, symmetry and tilings, geometric growth, codes, and data management. Additionally, the teachers were given homework via the Assessment and Learning in Knowledge Spaces system, to encourage review of algebraic and other fundamental mathematical ideas underlying the various real-world applications under study. The course followed a fairly traditional mathematics course format, with students completing mathematics problem sets and taking tests. Overall, the course was designed to enhance teachers' knowledge of mathematics and applications of mathematics so that they would

be better informed about the ways in which mathematics is used in the world and would have a richer variety of examples to draw on when teaching their own students.

Participant teachers. Participants consisted of 21 K-8 in-service teachers who were enrolled in the MSP-sponsored master's program. Although the cohort started with 26 teachers in total, five program participants were excluded from the study because three were middle school science teachers and the other two were not from the partnership district.¹ Of the 21 remaining teachers, three were middle school mathematics teachers, one was a special education teacher, and the rest were elementary school teachers.

The teachers were employed in 12 public elementary and middle schools across the district and surrounding areas. Only one teacher was male and the majority of the teachers were White (and non-Latino/a), with the exception of three African American teachers. All participant teachers held teacher certification before enrolling in the program, and they taught mathematics in grades ranging from first to seventh. The majority of these teachers (19) had been elementary education majors. One teacher had been a physical education major. In addition, one teacher had been a middle school education major. The teachers' years of experience in the classroom ranged from 1 to 12 years (mean = 5.4; median = 4).

Although these 21 teachers were teaching or were expecting to teach mathematics each year starting from the beginning of the program, some teachers switched between grades and subject areas for a number of reasons. For example, one teacher became a literacy coach in the first year of the master's program but taught mathematics in the second year of the program. Another teacher was teaching mathematics before the program started but did not teach

¹ I also observed several other teachers from surrounding districts because of my special interest in those teachers' instruction, but these two teachers were not in that group.

mathematics for the last 2 years. In addition, several teachers switched between grade levels because of school needs.

Target teachers. The results from the initial quantitative data sources, such as teachers' scores on mathematical knowledge tests, were used to identify target teachers for more in-depth data collection. Eight of these 21 teachers were chosen for the qualitative part of the analysis. In the Case Selection section, I explain in detail how and why I chose the target teachers.

Students. As shown in Table 1, a total of 873 students were included in the data analysis. It is important to note that I had access to only the partnership district Illinois Standards Achievement Test (ISAT) data, which reduced the number of teachers and students available for the data analysis. Each year, two thirds of the students were eligible for free or reduced lunch and just over half of the students were African American (see Table 1).

Table 1

Demographic Information on the Participant Students

Year	<i>N</i>	African American (%)	White (%)	Low-income eligible (%)
Year 0	231	53.3	46.8	70.6
Year 1	225	53.3	46.7	68.4
Year 2	225	58.2	41.8	72.9
Year 3	192	55.2	44.8	76
Overall	873	55	45	71.8

Data Collection

I collected a variety of data throughout the program. Hence, I used several data collection instruments. I used paper-and-pencil tests to measure teachers' MKT, a survey of teachers' beliefs about teaching and learning mathematics, and a classroom observation protocol to quantify the quality and frequency of teachers' practices. Data collection also included classroom observations (which included both a protocol and field notes), interviews with the target teachers,

and school-provided student achievement data. Because my major focus throughout the data collection period remained on teachers not their students, I considered the data on teachers' MKT and instructional practices my primary data sources, whereas I considered the other data, such as student achievement data and interviews, supplementary data used to improve understanding. Table 2 shows the data available for each teacher. In this section, I first briefly describe the instruments used and the data collected, followed by a detailed discussion of the data collection process.

Table 2

Available Data Sources From the Teachers

Source	Teachers	
	Yes	No
Assessment for MKT	21	0
Classroom observations	21	0
Beliefs survey	19	2
Student test scores	11 ²	0
Interviews	8 ³	13

Note. MKT = mathematical knowledge for teaching.

MKT Measure. In this study, LMT instruments were used to assess teachers' MKT. The LMT assessments, which were developed for the Study of Instructional Improvement at the University of Michigan, were specifically designed to capture elementary school teachers' MKT (e.g., Ball, Thames, & Phelps, 2008). The researchers developed multiple-choice items intended to assess teachers' knowledge of the mathematics most relevant to teaching in elementary

² Students' standardized tests on LMT scores were available for 14 teachers, but the gain score could be calculated for only 11 teachers who taught students at the fourth-grade level and above.

³ I conducted interviews with a subsample of teachers in whose classrooms I had also conducted extra observations.

classrooms, with items situating mathematics-related questions within teaching-specific scenarios (e.g., interpreting or evaluating student responses).

The sample item in Figure 6 provides a glimpse of the LMT measure even though this particular item was excluded from the LMT item bank because it was psychometrically problematic. The item requires specialized mathematical knowledge, and teachers must know more than just the standard multiplication algorithm to answer correctly. For instance, teachers need an understanding of place value and the distributive property of multiplication over addition to determine whether the three given methods are generalizable.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
$\begin{array}{r} 35 \\ \times 25 \\ \hline 125 \\ +75 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ +700 \\ \hline 875 \end{array}$	$\begin{array}{r} 35 \\ \times 25 \\ \hline 25 \\ 150 \\ 100 \\ +600 \\ \hline 875 \end{array}$

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 6. Sample item from the Learning Mathematics for Teaching (LMT) measure.

The LMT instruments used in the study were designed to capture both the common and specialized mathematical knowledge of the teachers. All the LMT items used were contextualized in scenarios pertaining to the teaching and learning of K-8 mathematics content.

Two parallel LMT forms (MSP_04A and MSP_04B) were used in this study to assess teachers' knowledge of three content areas: numbers and operations (K-6); patterns, functions, and algebra (K-6); and geometry (K-8). The two forms included 30 and 31 stems and 62 and 66 items for Forms A and B, respectively. (The sample question involved 1 stem and 3 items; see Table 3) Validity and reliability of the measures were established by the test developers. The retesting reliability coefficients were .75 and .76 for the forms, respectively.

Table 3

Number of Items on the Learning Mathematics for Teaching (LMT) Forms by Content Area

Item category	Numbers and operations	Geometry	Patterns, functions, and algebra	Total
Form A	26	19	17	62
Form B	25	23	18	66

Note. The 62 and 66 items were based on a total of 30 and 31 item stems for Form A and Form B, respectively.

During the development of the LMT forms, pilot teachers' raw LMT scores were converted to Item Response Theory (IRT) scales. The IRT scale scores can be interpreted as a measure called *logits*, ranging from approximately -3.00 to 3.00 , with a mean of 0. A higher score indicates a higher level of teacher knowledge. The forms were equated by the test developers so that we could compare teachers' scores across the forms. Individuals with the same scores on the test were assumed to have the same level of mathematical knowledge, regardless of when they were tested or which form was used.⁴

⁴ The LMT developers originally used a convenience sample to pilot the forms, and the mean for that sample was made to correspond with 0 when establishing the scale. Hence, a score below 0

Classroom observation protocol. Several available observation protocols were considered for use in the project, such as the Local Systemic Change (LSC) Classroom Observation Protocol (Horizon Research, Inc., 2011), the Reformed Teaching Observation Protocol (Piburn & Sawada, 2000), and the Oregon Mathematics Leadership Institute (OMLI) Classroom Observation Protocol (Weaver et al., 2005). However, no single protocol captured all aspects of instruction relevant to the project goals, such as quality of student discourse and tasks chosen. Hence, a modified version that was a combination of the LSC and OMLI protocols was created. The final form of the protocol provided an overall assessment of lesson planning and teaching as in the LSC protocol while also providing a deeper assessment of classroom discourse patterns and “sense-making,” similar to the OMLI protocol. All project staff (including myself) continued to train by using the finalized classroom observation protocol, and by the end of the training, there was substantial agreement among staff on ratings. Training included watching videos, scoring the video lessons using the observation protocol, and discussing where disagreements arose. Through these training sessions, we reached a common understanding of what each item on the protocol meant. Through multiple training exercises, we established an intraclass correlation coefficient of .80, which strengthened our confidence in having individual observers use this adapted protocol in the field.

The final protocol includes several categories. The first part of the protocol focuses on describing the classroom and lesson in general terms. Along with some basic descriptive information (e.g., subject, course title, and grade of the class), we documented the purpose of the

on the LMT scale is not necessarily “below average” among the general population of U.S. teachers. Indeed, Hill (2010) recently found that the performance of a nationally representative sample of elementary teachers was slightly lower than that of the original pilot sample on identical items.

lesson as described by the teacher: how the class time was spent, including the number of minutes spent on instructional activities as opposed to “housekeeping,” interruptions, and the like; and the percentage of instructional time spent as a whole class, in pairs or small group work, and in individual work. (The adapted protocol can be found in Appendix A.)

The focus of the protocol was on describing the quality of the observed lessons in each of four component areas: the lesson design and its implementation, the mathematics discourse and sense-making, the task implementation, and the classroom culture. In each case, we first rated the extent to which the lesson exhibited characteristics of the component area. For example, in the lesson design and implementation, we rated the extent to which mathematics was portrayed as a dynamic body of knowledge continually enriched by conjecture, investigative analysis, proof or justification, or their combination. While rating this item, we asked ourselves, “Do the children get an idea that conjecturing, exploring, and proving is what mathematics is all about? Or is it about following rules given by a teacher or book?” The items rated were on a 5-point frequency scale, ranging from 1 (*never*) to 5 (*consistently*).

The final section in the classroom observation protocol includes an overall rating of four key aspects of the observed lesson: depth of student knowledge and understanding; locus of mathematical or scientific authority; social support; and student engagement in mathematics or science. A 5-point scale was again used, with higher ratings being more positive.

It is also noteworthy to mention here that in addition to completing the classroom observation protocol for each observation, the project staff took detailed field notes during the observations, including describing what the teacher and students were doing throughout the lesson and recording the times various activities began and ended. The protocol enabled us to quantitatively compare the lessons, whereas field notes provided insights regarding aspects of the

qualities that the protocol captured. The field notes provided summaries of the lesson and the quality of the lesson, descriptions of what happened in the lesson, and enough rich detail that readers would have a sense of having been there. Moreover, starting from the second year of the program, the observed lessons were audiotaped.

Beliefs survey. The beliefs questionnaire was more specific about teachers' beliefs regarding teaching and learning mathematics. Teachers were asked to indicate the extent of their agreement on a 5-point Likert scale, ranging from 1 (*strongly disagree*) to 5 (*strongly agree*). I used the questionnaire modified by Beswick (2005). The survey was designed to capture the extent to which teachers held a traditional view or standards-based view of mathematics. I use *traditional view* of mathematics to refer to content-focused teaching with an emphasis on performance and skill mastery, whereas I use *standards-based view* of mathematics to describe more learner-focused teaching with an emphasis on mathematical sense-making and understanding of concepts and procedures. The two factors (traditional and reformed-oriented views of mathematics) were created based on factor analysis, and the corresponding reliability scores for these clusters were 0.78 and 0.77, respectively (Beswick, 2005). The survey had a total of 26 items, 11 of which were scored in reverse because they were designed to capture contrasting views of problem solving. The scores ranged from 1 to 5, with a higher score indicating greater consistency with a standards-based view of teaching and learning mathematics (see Appendix B).

ISAT. The ISAT is a standardized test administered by the state to students in Grade 3 and above annually between late February and early March. The test is administered in three 45-min sessions. Students' knowledge is assessed in five content areas: (a) number sense (representations and ordering; computation, operations, estimation, and properties; ratios,

proportions, and percentages); (b) measurement (units, tools, estimation, and applications); (c) algebra (representations, patterns, and expressions; connections using tables, graphs, and symbols; writing, interpreting, and solving equations); (d) geometry (properties of single figures and coordinate geometry; relationships between and among multiple figures; justifications of conjectures and conclusions); and (e) data analysis, statistics, and probability (Illinois State Board of Education, 2012).

The reliabilities gleaned from the published ISAT technical manuals are reported in Table 4. As shown in this table, reliability estimates for the ISAT tests are high.

Table 4

Reliability Estimates for the Illinois Standards Achievement Tests

Grade\Year	2007	2008	2009	2010	2010
3	.93	.94	.94	.94	.94
4	.92	.92	.93	.93	.93
5	.93	.93	.93	.94	.93
6	.93	.93	.94	.94	.94
7	.94	.92	.93	.94	.93

Students' scores are computed based on 65 multiple-choice items, 2 short-response items, and 1 extended-response item. Multiple-choice items contribute 85% to the total score, whereas short-response items and the extended-response item contribute 5 and 10%, respectively, to the total score. Scores are on an IRT scale, ranging from 120 to 400, with a standard deviation of 30. Students' ISAT scores for a given year were used as their posttest scores, whereas their ISAT scores for the year before were used as their pretest scores. More specifically, for students in the teachers' classrooms in 2008–2009, students' spring 2008 ISAT scale scores were used as the pretest (taken shortly before they were assigned to our teachers' classrooms), whereas their spring 2009 ISAT mathematics scale scores were used as their posttests. I had no data prior to Grade 4, because third graders had no pretest. Additionally, none of the teachers with available

student-level data taught eighth-grade mathematics. For these reasons, the student achievement data primarily included students from Grades 4 to 7.

Additional student-level data. In addition to students' ISAT data, the district provided me with information on free or reduced lunch status, gender, race, and grade level of all students in the district. Hence, I had access to this information for each student in the district who was tested from 2007 to 2011. I used student race, free lunch status, and grade level in the data analysis. Additionally, by using the ISAT scores of all students who were in Grades 4 to 7, I created a variable capturing the average ISAT gain for the district each year so that I could reduce the effects of some unobservable factors on my data analysis.

Interviews. As mentioned earlier, I chose a subgroup of teachers as target teachers for the qualitative analysis. I conducted semistructured interviews with the eight target teachers individually after I had completed my classroom visits. In the interviews, I intended to learn more about the teachers' instructional practices, the effects of the mathematics courses on their practices, and their views of teaching and learning mathematics. I also asked whether they had made any changes in their instructional practices and why they had made those changes, to capture how they perceived the changes in their content knowledge and whether those changes had any effect on their teaching. Most of the questions asked in the interviews can be found in Appendix C. I also developed two or three specific questions for each teacher. For instance, I knew one of the teachers used animated lessons on a Smart Board to carry out her lesson plans, so I asked her why she used those lessons. All the interviews were audiotaped and transcribed verbatim.

Extra classroom observations. I collected classroom observation data from the same target teachers over a span of roughly 1 week. I explain in detail in the following section how I

chose these teachers. During these observations, I audiotaped the mathematics lessons and took detailed field notes on the teachers' instruction. My primary focus was the teacher, rather than their students' responses. I also took general notes regarding students' behavior, including off-task behavior, behavior during group work and seatwork, and students' general manner in the observed lessons.

Although my intention was to observe teachers' mathematics lessons for an entire week, on some days, several teachers cancelled their scheduled observations for a number reasons, such as health problems, school programs, and conferences. I continued to observe teachers until I felt I had observed a sufficient number of lessons from which to draw conclusions. Additionally, I stayed longer in the classrooms of the teachers who had not been observed in previous years. For these reasons, the total number of lessons I observed in each classroom varied somewhat, ranging from four to nine.

In observing the lessons, I focused on teachers' instructional practices, particularly as captured in the observation protocol. I kept a log of each observed lesson. I recorded the problems and tasks teachers used in the lessons. I also paid close attention to how the teachers responded to their students' ideas and what they said in return. In addition, I used audio recordings to check my notes. I typed and edited these written field notes on the same day as the observed lesson to reduce data loss.

Timetable for data collection. Table 5 below represents a timeline for the data collection. I briefly discuss the data collection process in more detail in the following section.

Table 5

Schedule for Data Collection

Source	Year 0	Year 1			Year 2			Year 3
		Fall 2008	Spring 2009	Summer 2009	Fall 2009	Spring 2010	Summer 2010	Fall 2010
Course		<i>Mathematics hybrid course</i>	Science education course	Science and educational psychology courses	Educational psychology	<i>Mathematics content course</i>	Science education course	Action research course
MKT	August	December			January		June	December
Classroom observations	May to early June		April to May			March to May		October to November
ISAT	February to March		February to March			February to March		February to March
Teacher interview								
Teacher beliefs survey							August	

Note. MKT = mathematical knowledge for teaching; ISAT = Illinois Standards Achievement Test.

Data collection schedule for assessing teachers' MKT. The LMT instrument was first administered at the start of the program in August 2008. Given that two forms (Forms A and B) were available, teachers were randomly assigned to the forms; one half the teachers took Form A, and the rest took Form B. Teachers alternated forms in later assessments. In total, teachers took the LMT five times.

As shown in Table 5, teachers took the LMT test a second time in December 2008, just after they finished the content and methods hybrid course, to check on the immediate gains made during that first course. Then, in January 2010, one year after the completion of the first course but before the teachers began their second mathematics-focused course, teachers' mathematical knowledge was again assessed using the LMT. Teachers completed the mathematics content course in early May 2010. Approximately 7 weeks later, in June 2010, teachers' mathematical knowledge was assessed using the LMT measures. The final administration of the LMT then took place at the end of the program, in December 2010. Because I had four time points in the quantitative data analysis, I used teachers' scores on the LMT tests administered in August 2008, January 2010, June 2010, and December 2010 as an indicator of their mathematical knowledge in the years between 2008 and 2011.⁵

Timetable for classroom observations. Classroom observation data were first collected before the master's program started. Several research assistants conducted classroom observations, seeing at least two mathematics lessons from all participating teachers who taught mathematics during the first weeks of May in 2008. For each observation, a combined and edited

⁵ I did not use teachers' scores on the test administered in December 2008 as an indicator of their MKT for the year 2009 because that test was administered soon after they completed the mathematics hybrid course, and teachers' scores on that test might be an overestimation of the knowledge teachers retained. The difference in teachers' scores on the tests administered in December 2008 and January 2010 was not statistically significant, even though the teachers' scores were lower in January 2010.

version of the Horizon LSC protocol and the OMLI classroom observation protocol was completed. During the first 2 years that the program was implemented, teachers' mathematics lessons were observed two or three times a year in March to early May by project staff (including myself). In the final year, because the program ended in fall 2010, classroom observations were conducted primarily in October and November. In addition to those regular observations conducted for the master's program, I conducted at least four classroom observations of the target teachers in fall 2010. I used the same observation protocol that was used by the project for the regular classroom observations.

Timetable for teacher beliefs survey and interviews. The beliefs survey was administered once in August 2010. Teachers' beliefs were not measured earlier in the program because teachers were asked to complete the very detailed Surveys of Enacted Curriculum (The Council of Chief State School Officers, 2011) regarding their practices annually. In that teacher survey, several items were also designed to capture teachers' beliefs. However, an examination of the beliefs items on that survey indicated that those items did not adequately capture what I intended to measure, which was teachers' beliefs regarding how mathematics should be taught. Hence, I used the beliefs survey modified by Beswick (2005) to capture teachers' beliefs regarding teaching and learning mathematics.

I conducted interviews with the eight teachers individually after I had completed my classroom visits. Two of the teachers told me that they were not comfortable being interviewed and would not remember much about what they would like to say during the interview. They asked instead that I send them the questions via e-mail, and they sent me their typed responses. With the rest of the teachers, I conducted oral interviews, which lasted about 1 hour. To do this, I visited the teachers in their schools during lunchtime or after school.

Data collection from the students of participating teachers. As mentioned earlier, students' ISAT scores were used as a measure of their gain in mathematical knowledge. The ISAT tests were administered over the course of 3 weeks in late February and early March. The student data became available each year in mid-August. In addition to data from the students of participating teachers in the partnership district, I had access to all district student ISAT data.

Data Analysis

As mentioned earlier, my purpose in using mixed methods was to measure overlapping and distinct features of the phenomenon by elaborating or clarifying results using multiple methods (Greene, Caracelli, & Graham, 1989). Hence, I used both qualitative and quantitative data analysis methods to investigate my research questions. The results from initial quantitative data sources, such as teachers' scores on the MKT measure, were used to identify the target teachers for extra classroom observations and interviews (see Figure 7). Before introducing the methods I used to explore my research questions, I begin by explaining the steps taken to analyze the data. I first describe the measures developed from the collected data based on initial analysis of the data, followed by selection of the cases. I explain in detail how and why I chose those cases. Finally, I end the section with descriptions of the methods I used to answer my research questions.

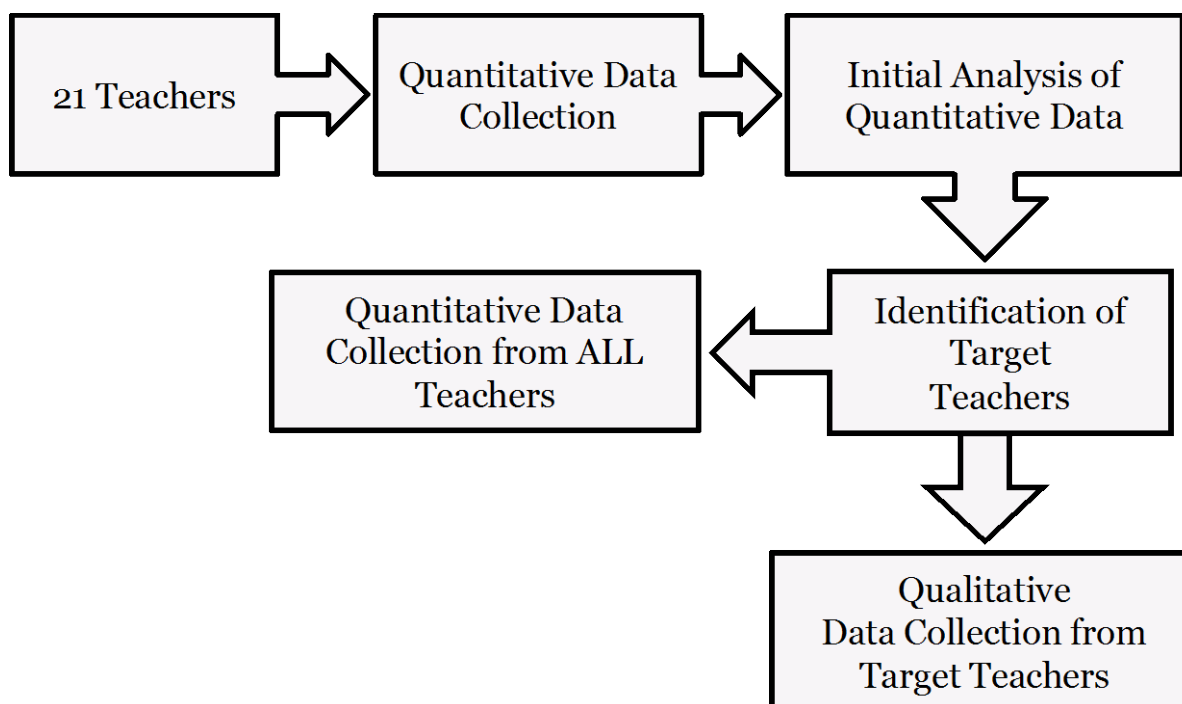


Figure 7. Illustration of the data collection process from participating teachers.

Measures developed from the collected data.

Scales created from the observation protocol. Given that two different observation protocols were combined, I used factor analysis to identify scales that captured different aspects of instruction. I conducted factor analysis by using more than 200 mathematics classroom observations.⁶ Factor analysis is a statistical procedure whereby items are grouped together according to the similarity of respondents' answers.

I included all but two items in the main section of the adapted classroom observation protocol. Those two items on the protocol were excluded from the analysis because of their small

⁶ I treated the lesson observation as the unit of analysis, which could have biased the results because it violated the independence of observations. However, this was necessary in order to run factor analyses on the relatively large number of items on the protocol. Because of both the lack of independence among lessons and the fact that teachers were not selected randomly for this study, the results should be viewed as suggestive of relationships that might hold in the general population of teachers and that merit further analysis.

standard deviations, a result of our agreeing on “default” ratings of the items⁷ ($\leq .5$). The remaining 37 items were analyzed using principal components analysis with varimax rotation. All factors with eigenvalues greater than 1.0 were selected to remain. All items except one (free-standing) were grouped into one of the five factors. Factor analysis of the current available data revealed five factors (eigenvalues of 16, 3.1, 2.1, 1.6, and 1.1); 64.5% of the total variation was explained by these factors. Factor loadings for these scales were also clear, with reasonably high factor loadings (ranging from .53 to .80, .51 to .86, 60 to .81, .41 to .73, and .59 to .81 on these five factors. The items on each scale, with corresponding reliability scores, are shown in Table 6.

⁷ The items excluded from the analysis were “the design of the lesson reflected careful planning and organization” and “the teacher appeared confident in his/her ability to teach mathematics.”

Table 6

Factor Loadings of the Five Scales Obtained

Item	Scales with Cronbach's alpha reliability estimates					Communality h^2
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	
IIB-12 The teacher productively probed/"pushed on" the mathematics in students' responses.	.80	.18	.09	.16	.19	.75
IIA-9 The teacher's questioning strategies for eliciting student thinking promoted discourse around important concepts in mathematics.	.74	.21	.14	.20	.17	.69
IIA-5 The lesson design provided opportunities for student discourse around important concepts in mathematics.	.68	.23	.33	.22	-.13	.69
IIA-3 The lesson had a problem/investigation-centered structure.	.66	.07	.40	.08	-.01	.72
IID-7 The classroom climate encouraged students to engage in mathematical discourse.	.65	.28	.12	.37	.12	.75

(table continues)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	Communality h^2
IID-5 Wrong answers were treated as worthwhile learning opportunities.	.64	.17	.28	.13	.25	.62
IIA-6 Mathematics was portrayed as a dynamic body of knowledge continually enriched by conjecture, investigation analysis, and/or proof/justification.	.64	.06	.36	.25	.06	.70
IIA-8 The instructional strategies were consistent with investigative mathematics.	.63	.18	.30	.06	.14	.74
IIA-2 The design of the lesson incorporated tasks, roles, and interactions consistent with investigative mathematics.	.63	.05	.40	.12	.06	.68
IID-4 Interactions reflected a collaborative working relationship between the teacher and the students.	.62	.24	.12	.06	.30	.65

(table continues)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	Communality h^2
IIA-11 The teacher was flexible and able to take advantage of "teachable moments."	.61	.04	.16	.21	.27	.56
IIB-11 The teacher and students engaged in meaning making at the end of the activity/instruction.	.58	.04	-.01	.52	.08	.61
IIB-8 Students determined the correctness/sensibility of an idea and/or procedure based on the reasoning presented.	.55	.15	.22	.43	.10	.68
IIB-4 Students justified mathematical ideas and/or procedures.	.53	.29	.20	.39	.14	.71
IIA-13 The vast majority of the students were engaged in the lesson and remained on task.	-.04	.86	.11	.21	0	.81
IIB-5 Students listened intently and actively to the ideas and/or procedures of others for the purpose of understanding someone's methods or reasoning.	.21	.78	.22	.15	-.02	.75

(table continues)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	Communality h^2
IIA-12 The teacher's classroom management style/strategies enhanced the quality of the lesson.	-.08	.74	-.02	.08	.36	.72
IIB-2 Students shared their observations or predictions.	.28	.73	.06	.16	.21	.68
IID-6 Students were willing to openly discuss their thinking and reasoning.	.38	.65	.21	.23	.21	.72
IIB-3 Students explained mathematical ideas and/or procedures.	.43	.53	.29	.20	.12	.63
IID-3 Interactions reflected a productive working relationship among students.	.35	.51	.31	-.04	-.02	.52
IIC-5 Tasks encouraged students to employ multiple representations and tools to support their learning, ideas, and/or procedures.	.13	.19	.81	.18	.08	.75

(table continue)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	Communality h^2
IIB-10 Students drew upon a variety of methods (verbal, visual, numerical, algebraic, graphical, etc.) to represent and communicate their mathematical ideas and/or procedures.	.21	.34	.75	-.03	.01	.76
IIC-4 Tasks encouraged students to search for multiple solution strategies and to recognize task constraints that may limit solution possibilities.	.32	.12	.64	.20	-.03	.61
IIC-1 Tasks focused on an understanding of important and relevant mathematical concepts, processes, and relationships.	.45	.12	.60	.32	.19	.73
IIC-2 Tasks stimulated complex, nonalgorithmic thinking.	.36	-.05	.60	.23	.13	.77
IIB-9 Students made generalizations, or made generalized conjectures regarding mathematical ideas and procedures.	.12	.19	.11	.73	-.07	.61

(table continues)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					Communality h^2
	Inquiry-oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	
IIC-3 Tasks successfully created mathematically productive disequilibrium among students.	.31	.18	.34	.65	.09	.71
IIC-6 Tasks encouraged students to think beyond the immediate problem and make connections to other related mathematical concepts.	.24	-.06	.43	.55	.08	.68
IIA-4 The instructional objectives of the lesson were clear and the teacher was able to clearly articulate what mathematical ideas and/or procedures the students were expected to learn.	.05	.21	.46	.54	.39	.71
IIB-1 Students asked questions to clarify their understanding of mathematical ideas or procedures.	.24	.30	.05	.47	.07	.46
IIA-14 Appropriate connections were made to other areas of mathematics, to other disciplines, and/or to real-world contexts.	.43	.17	.06	.46	.10	.45

(table continues)

Table 6 (continued)

Item	Scales with Cronbach's alpha reliability estimates					Communality h ²
	Inquiry- oriented lesson (.95)	Student engagement (.89)	Worthwhile mathematical tasks (.87)	Mathematical sense-making Agenda (.84)	Classroom climate (.71)	
IIB-7 Students defended their mathematical ideas and/or procedures.	.40	.41	.12	.41	.18	.69
IIA-10 The pace of the lesson was appropriate for the developmental level/needs of the students and the purpose of the lesson.	.22	.18	.02	.19	.81	.79
IID-2 The teacher displayed respect for students' ideas, questions, and contributions.	.34	.12	.14	-.02	.64	.71
IID-1 Active participation of all students was encouraged and valued.	.20	.51	.06	-.08	.59	.66
Eigenvalues	15.96	3.07	2.1	1.62	1.13	
Percentage of variance explained	43.14	8.23	5.67	4.39	3.05	

As shown in Table 6, these five factors explained at least 45% of the variance in observation scores for each item, as indicated by the communality values. The value of the Kaiser-Meyer-Olkin criterion, which provides an index for the appropriateness of applying factor analysis to a given data set, was .896. This value was greater than 0.5, the cutoff point recommended for a satisfactory analysis to proceed (e.g., Kaiser, 1970; Westland & Clark, 1999). Additionally, Bartlett's test of sphericity was significant ($X^2 = 2,519.536$, $df = 666$, $p < .0001$), indicating the appropriateness of the factor model. Finally, Cronbach's alpha, an indicator of the internal consistency of the factors, ranged from .71 to .95, implying relatively high consistency within each scale.

Beliefs subscale. Initial analysis of teachers' responses to the beliefs survey (Beswick, 2005) indicated great agreement on several items in the survey. To differentiate teachers based on the differences in their beliefs, I created a subscale from the instrument when at least 20% of the teachers chose different responses from the two most common options. The items used to create the subscale are listed below. Similar to the total score, the sub-scale score is also computed by reversely coding the items that capture a traditional view of mathematics, and then averaging the teachers' scores on these items. The correlation between teachers' total score on the beliefs survey and on the subscale was .69.

- Discovery methods of teaching have limited value because students often get answers without knowing where they came from.
- Mathematics is a rigid discipline, which functions strictly according to inescapable laws.
- Each student should feel free to use any method for solving a problem that suits him or her best.
- Teachers should make assignments on just that which has been thoroughly discussed in class.
- There are often many different ways to solve a mathematics problem.
- The language of mathematics is so exact that there is no room for variety of expression.

Case selection. Decisions related to case selection are critical to understanding the phenomenon under investigation (Yin, 1994). Following the suggestion of Yin (1994) for selecting diverse cases, I chose eight teachers with a wide range of mathematical knowledge as well as a wide range in the MKT gains they made. These diverse cases can help us better understand the relationship between teachers' MKT and instructional practices.

Of the 21 participant teachers, I chose these 8 teachers for several reasons. First, as mentioned above, the teachers represented the full range of mathematical knowledge of the teachers in this study, as well as the full range of gains made during the program (see Tables 7 and 8). Although there are no certain rules about how many cases are necessary for multiple-case studies, Yin (1994) suggests 6 to 10 cases. To increase variation in teachers' MKT, I chose two of the teachers who had not been observed in previous years (which limited my knowledge of their instructional practices to the period before they enrolled in the program). I decided to include these two teachers because they had the highest scores on the mathematics tests and none of the other teachers who had

been observed before had mathematical knowledge as strong as these two teachers did at the beginning and end of the program.

Table 7

Mathematics Scores⁸ and the Mathematical Knowledge for Teaching (MKT) Gain of the Eight Participant Teachers Over Time

Teacher ⁹	Before the program	After the program	MKT gain
Stephanie	1.24	1.99	.75
Jacqueline	0.73	1.29	.56
Valerie	−0.24	0.85	1.09
Rebecca	−0.55	0.46	1.01
Sonya	−0.67	0.21	.88
Beth	−1.22	−0.39	.83
Ann	−0.68	−0.43	.25
Meg	−1.29	−0.61	.68

Note. Teachers are ordered based on their current level of MKT.

⁸ Possible scores ranged from −3 to 3, and a teacher with average mathematical knowledge would be expected to score 0.

⁹ All names are pseudonyms, and the number of syllables show the level of MKT (i.e. more syllables indicate a higher level of MKT).

Table 8

Change in Teachers' Mathematical Knowledge Over Time

Before the program	After the program				
	Very low ¹⁰	Low	Average	High	Very high
Very high					Stephanie
High					Jacqueline
Average				Valerie	
Low		Ann	Sonya	Rebecca	
Very low		Meg and Beth			

In addition to increasing variation in teachers' MKT, I tried to reduce the effects of confounding factors on the relationship between MKT and instruction. I chose teachers with diverse MKT but with similar teaching settings. Hence, I chose teachers with different MKT levels who taught similar grades because this would help reduce any grade-level effects (see Table 9). For this reason I included at least one strong and one low MKT teacher for both of the upper and lower elementary grades. For the same reason, I tried to choose teachers in the same school whenever possible. I also took into consideration school demographics, such as socioeconomic status and race, to reduce the effects of these factors on the relationship investigated. However, as earlier studies indicated (e.g., Hill & Lubienski, 2007), teachers with strong MKT were usually hired in more affluent schools. Given that the most of the schools in the partnership district serve

¹⁰ I created categories starting with the "average" category. I created one-half standard deviation intervals around 0 for the "average" category. Similarly, the widths of the "high" and "low" mathematical knowledge categories also were one half a standard deviation. As a result, the "very low" category ranged from the lowest possible score to $-.77$ logits, the "low" category ranged from $-.76$ to $-.26$, and the "average" category ranged from $-.25$ to $.25$. The "high" category ranged from $.26$ to $.76$, and the "very high" category ranged from $.77$ to the highest possible scores.

mainly students from low-income families, finding a teacher with strong MKT from the partnership district was not easy. Hence, I included Stephanie and Jacqueline with very strong MKT from a neighboring district. I also chose Valerie from the partnership district, given that she was the only teacher with strong MKT and had similar school demographics to that of Stephanie and Jacqueline. Finally, these eight teachers were honest and sincere during the interviews I conducted for the program, and they seemed to accept being observed several times.

Table 9

Teachers' Personal and School Information

Teacher	Personal information			School information ¹¹				
	BS major	Grade level	Years of teaching	Free/reduced lunch status	White, %	African American, %	School size	Curriculum used
Stephanie	Elementary education	7	6	42	87	5	228	CMP
Jacqueline	Junior high/middle school education	6	8	42	87	5	228	CMP
Valerie	Elementary education	4–6	14	47	62	25	370	
Rebecca	Elementary education	2	11	76	56	28	354	EnVision
Sonya	Elementary education	6	10	85	39	48	264	CMP
Beth	Elementary education	5	4	85	39	48	264	EnVision
Ann	Elementary education	3	8	73	26	66	326	EnVision
Meg	Education	2	10	79	48	40	285	EnVision

Note. CMP = Connected Math Project.

¹¹ School information was obtained from the State of Illinois Report Card and is based on the 2010 Report Card.

Method Used to Analyze the First Research Question Quantitatively

How does teachers' MKT affect their instruction? What factors, such as beliefs and the curriculum, mediate the expression of MKT in instruction? To investigate how MKT affects instruction, I examined the effects of a change in teachers' MKT on their instruction by using multilevel (hierarchical) multivariate growth modeling. The growth modeling approach explicitly models changes in individuals across time. Moreover, unlike repeated measures, it handles missing data efficiently (Hedeker, 2004; Hedeker & Gibbons, 2006). Teachers with missing data could be included in the data analysis and the results could be interpreted as if no data were missing, provided the data were missing at random. In this study, there was no systematic reason why some teachers were not observed each year. Of the 21 teachers, data were missing because of health problems, relocation, and school policies in the way teachers were assigned to subjects. Another advantage of using multilevel modeling is that it takes into account correlated errors and allows the partitioning of variance into within- and between-group components.

A two-level hierarchical (multilevel) multivariate growth model indicates that in a Level 1 model, each individual teacher's trajectory of instruction is estimated based on a set of parameters. In this study, time (years) and teachers' mathematical knowledge were used to predict the instructional practices of each teacher. These individual growth parameters then became outcomes in the Level 2 model, where they could be regressed on person-level characteristics. The Level 2 variables captured individual-level characteristics, such as being an experienced or novice teacher and the highest-grade level taught. More specifically, using the five scales derived from the classroom observation protocol as outcome variables, I investigated the association between teachers' practices and MKT, controlling for the effect of time. By using

the teachers' years of teaching experience and grade level, I could also adjust these differences between teachers.

The basic model was as follows:

$$\text{Level 1: } \text{Instruction}_{ij} = \beta_{0j} + \beta_{1j} \cdot \text{time}_{ij} + \beta_{2j} \cdot \text{math}_{ij} + r_{ij}; \text{ and}$$

$$\text{Level 2: } \beta_{0j} = \gamma_{00} + \gamma_{01} \cdot \text{grade}_j + \gamma_{02} \cdot \text{new teacher}_j + u_j,$$

$$\beta_{1j} = \gamma_{10},$$

$$\beta_{2j} = \gamma_{20}.$$

The Level 1 model indicates that the score of an individual teacher i on the instructional practice scale at time j is influenced by her initial level of instructional practice (β_{0j}), the effect of time (i.e., β_{1j}), and the effect of her mathematical knowledge at time j . The Level 2 model indicates that the initial score of an individual teacher i on that instructional practice scale is determined by the overall initial scores of teachers on that scale (γ_{00}), the teachers' grade level, experienced teacher status, and the individual teacher's difference from the other teachers (u_j), which is constant across time.

All models were fit using the mixed methods procedure (PROC MIXED) in SAS/STAT software (SAS Institute Inc., 2008), and all significance tests were conducted at a α level of 0.05. I preferred the PROC MIXED of SAS for data analysis because the program is flexible and suitable for multilevel model analysis (Singer, 1998). Additionally, given that the data were collected from the same teachers over time, residuals within teachers were correlated. Therefore, for each teacher, residuals at one time were correlated with residuals at another time, which is a violation of the independence of errors. Hence, the autocorrelated error needed to be taken into consideration in addition to the random error (e.g., measurement errors and missing variables;

Hedeker & Gibbons, 2006). The autoregressive structure was used to test whether serial correlation needed to be taken into account.

To answer the second part of the first research question, “What factors, such as beliefs and curriculum, mediate the expression of MKT in instruction?” I examined the relationship between teachers’ beliefs and instruction. Given that teachers’ beliefs were captured only in the second year of the program and could have changed throughout the program, I did not include beliefs in the growth model analysis; instead, I fit a linear regression model using only data from the final year. The reason behind this decision was that when teachers’ beliefs were treated as a time-varying covariate, it produced many missing data points (because teachers’ beliefs were captured once, and for the rest of the years, the teachers’ beliefs were missing). This led to a decrease in the power of analysis. Hence, I conducted a separate analysis for teachers’ beliefs. I regressed teachers’ scores for each instructional practices scale on teachers’ belief scores. Because not all 21 teachers were teaching mathematics in the final year of data collection, I included only teachers’ beliefs and instructional practices in the data analysis. I conducted this analysis to identify which instructional practices were related to teachers’ beliefs.

Coding Variables

In hierarchical linear modeling (HLM) growth models, the interpretation of the intercept differs depending on how the variables in the model are defined. The year variable was coded as 0 for the first wave of data to make the intercept more meaningful. Given that 0 was in the possible score range for the mathematics test, teachers’ mathematics scores were not centered. The grade level was dummy coded because two teachers in the study were Montessori teachers and were teaching students in Grades 1 to 3 or Grades 4 to 6. As a result, teachers were grouped

into two categories: teachers who were teaching in the lower grades (Grades 1 to 3), and teachers who were teaching in Grades 4 to 7. Because only two teachers were teaching Grade 7, these two teachers were included in the latter category. As a result, each year 29% of the teachers were teaching in the lower grades, whereas the rest were teaching in Grades 4 and above. Teachers were also grouped in two categories based on years of mathematics teaching experience. Teachers who had fewer than 3 years of teaching experience at the beginning of the program were grouped in a category called “novice teacher,” whereas the rest were grouped together.¹²

Teachers’ scores on instructional practice scales as well as their scores on the beliefs test were rescaled so that the intercept would become meaningful. Because both the scores on the instructional practice scales and the scores on the beliefs test ranged from 1 to 5, I made an adjustment by subtracting 1 from the teachers’ scores. The new range for both became 0 to 4.

Additionally, as mentioned earlier, teachers’ MKT was assessed five times over the duration of the program. For data analysis, I used teachers’ scores on the LMT test administered at the following time points: August 2008, January 2010, June 2010, and December 2010. I did not use teachers’ scores on the LMT test administered in December 2008 because it was administered soon after the methods course, which might not reflect the knowledge the teachers retained.

The linearity of the relationship between outcome variables (five instructional practice scales) and year as well as MKT was determined by using scatterplots. The general trend between the outcome variable and year (or MKT scores) for each teacher was used to determine

¹² One might question why I did not treat the dummy-coded variable “years of teaching experience” as a time-varying covariate (Level 1 variable). When I treated the dummy-coded years of teaching experience as a time-varying covariate, the results were not different from those obtained when I treated the dummy-coded years of teaching experience as a Level 2 variable.

whether a quadratic term was needed. This exploratory analysis suggested that no quadratic term was needed.

Exploring the Second Research Question Quantitatively

To what extent are changes in teachers' MKT, instructional practices, or both related to students' gain in achievement? To investigate how teachers' MKT, teaching practices, or both were related to their students' achievement gains, I followed an approach similar to that taken in the analysis of the first research question. For the analyses predicting student outcomes from teachers' knowledge scores and instructional practice scale scores, I fit a series of 3-level HLM models to account for the fact that students were nested within teachers and time. This indicated that different groups of students for each teacher were included in the study (see Figure 8).

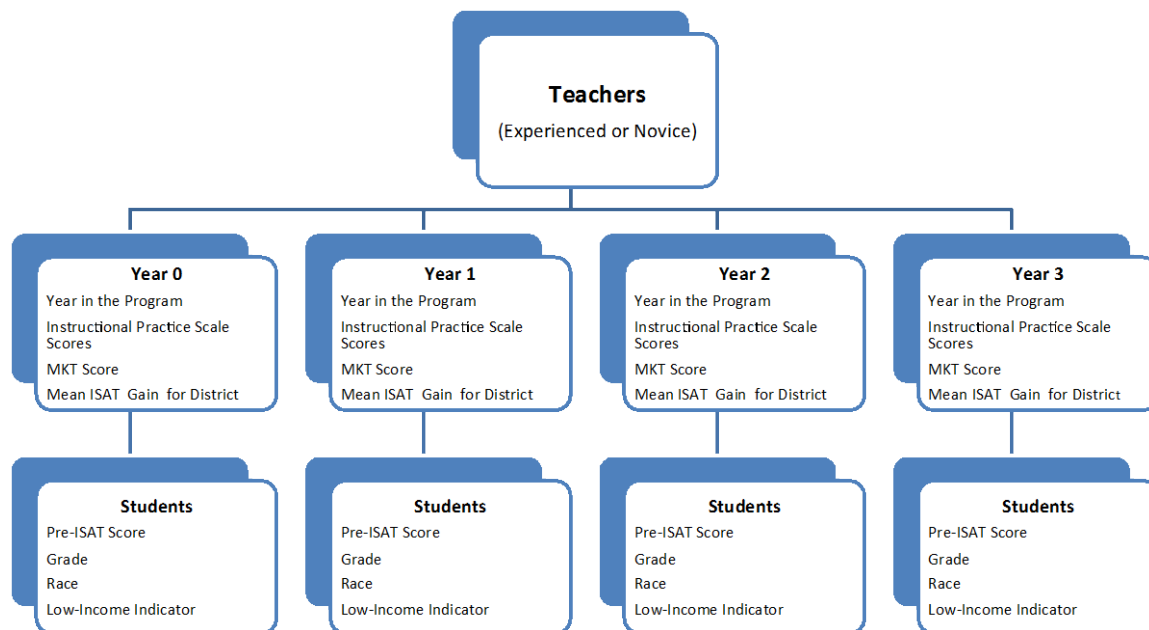


Figure 8. Illustration of the three-level hierarchical linear modeling (HLM) analysis.

I used gain scores (the difference between students' pre- and post-ISAT scores) to examine the relationship between teachers MKT and instructional practices and their students'

gain. Thus, I asked what the effects of teacher knowledge and teaching practice were on the change in students' ISAT scores. I also added four student-level predictors (race, free or reduced lunch status, grade, and pre-ISAT score). As illustrated in the models below, the differences in students' gains were modeled as a function of students' race, reduced or free lunch eligibility, grade, and pre-ISAT scores at Level 1; a function of teachers' MKT or instructional practices or both, the mean district ISAT gain, and time variables at Level 2; and a function of being a novice or experienced teacher at Level 3.¹³

The basic model was as follows:

$$\text{Level 1: StudentISATGain}_{itj} = \beta_{0tj} + \beta_{1tj} * \text{race}_{itj} + \beta_{2tj} * \text{lowincome}_{itj} + \beta_{3tj} * \text{preISAT}_{itj} + \beta_{4tj} * \text{grade}_{itj} + r_{itj};$$

$$\text{Level 2: } \beta_{0tj} = \gamma_{00j} + \gamma_{001} * \text{time}_{tj} + \gamma_{002} * \text{instructionalpractice}_{tj} + \gamma_{003} * \text{teachermathscore}_{tj} + \gamma_{004} * \text{meandistrictISATgain}_{tj} + u_{0tj},$$

$$\beta_{1tj} = \gamma_{10j},$$

$$\beta_{2tj} = \gamma_{20j},$$

$$\beta_{3tj} = \gamma_{30j}; \text{ and}$$

$$\text{Level 3: } \gamma_{00j} = \xi_{000} + \xi_{001} \text{ noviceteacher}_j + \omega_{00j},$$

$$\gamma_{10j} = \xi_{100},$$

$$\gamma_{20j} = \xi_{200},$$

$$\gamma_{30j} = \xi_{300}.$$

Because of the high correlation among the five instructional practice scales, I decided not to enter teachers' instructional practice scale scores together in the models (see Table 10).

¹³ With the same reasoning, I did not treat the dummy-coded variable "years of teaching experience" as a time-varying covariate in the analyses. I also looked at the interaction term between year in the program and being a novice or experienced teacher. The interaction term was not significant in any of the analyses.

Table 10

Correlation Between Instructional Practices Scales Across Years

Year	Scale	Inquiry	Engage	Task	Sense-Making
Year 0 <i>N</i> = 17	Engage	.52* .03			
	Task	.70** .002	.57* .017		
	Sense-Making	.72** .001	.58* .015	.65** .005	
	Climate	.45~ .069	.60* .01	.48* .049	.43~ .086
Year 1 <i>N</i> = 14	Engage	.52* .031			
	Task	.70** .002	.57* .017		
	Sense-Making	.72** .001	.58* .015	.65** (.005)	
	Climate	.45~ .069	.60* .01	.48* (.049)	.43~ (.086)
Year 2 <i>N</i> = 12	Engage	.65* .022			
	Task	.46 (.14)	.57~ (.05)		
	Sense-Making	.81** (.002)	.74** (.006)	.59* (.042)	
	Climate	.68* (.016)	.56~ (.06)	.28 (.37)	.52~ (.086)

(table continues)

Table 10 (continued)

Year	Scale	Inquiry	Engage	Task	Sense-Making
Year 3 N = 14	Engage	.58* .031			
	Task	.76** .002	.56* .04		
	Sense-Making	.86 ^a <.001	.73** .003	.63* .02	
	Climate	.89 ^a <.0001	.72** .004	.66* .01	.85*** <.001

Note. ~ p = < .10, * p<.05. ** p<.01. *** p<.001. ^a p<.0001

I also investigated the extent to which teachers' beliefs might be related to student achievement. I conducted separate analyses for teachers' beliefs and student achievement gains. Because teachers' belief scores were captured only once and students were nested within teachers, I used two-level HLM modeling by using data collected in the third year of the program. Only student-level variables and teachers' beliefs were entered into the model.

The basic model was as follows:

$$\text{Level 1: StudentISATGain}_{ij} = \beta_{0j} + \beta_{1j} \cdot \text{race}_{ij} + \beta_{2j} \cdot \text{lowincome}_{ij} + \beta_{3j} \cdot \text{preISAT}_{ij} + \beta_{4j} \cdot \text{grade}_{ij} + r_{ij}; \text{ and}$$

$$\text{Level 2}^{14}: \beta_{0j} = \gamma_{0j} + \gamma_{01} \cdot \text{teacherbeliefs}_j + u_{0j}.$$

Coding. In addition to the recoded variables for the two-level HLM analyses, several newly added variables in the study were recoded to make the intercept meaningful. In particular, students' pre-ISAT scores were grand centered. Students' race as coded by the district was regrouped under two categories: African American and White. The trends in the ISAT scores of

¹⁴ Due to the small sample size, teachers' MKT and instructional practices were not included in the data analysis.

students in the multiracial, Asian, and Hispanic categories followed trends more similar to those of White students than to those of African American students. Hence, I grouped all students except the African American students together. Given that the point of interest in this part of the study is students' ISAT gain, I treated grade level as a continuous variable and entered it in models at Level 1. Additionally, I recoded grade level by subtracting 4 so that 0 was in the interval range. I also included the overall ISAT gain at the district level each year to control for the effects of unobserved district-related factors to the extent possible. Mean ISAT gain for the district was grand centered.

Exploring the research questions qualitatively. The purpose of this mixed methods study was complementarity, meaning that qualitative data were collected to explore the findings from the quantitative data. As mentioned, I conducted extra classroom observations and interviews to better understand the effect of MKT on instruction. Whereas quantitative data were longitudinal, in-depth classroom observations and interviews were only conducted during the last year of the program, which limited the use of classroom observations to analyze how changes in MKT corresponded with changes in instruction. To overcome this problem, during the interviews, I asked teachers how their practices changed over the duration of the program. Qualitative data on classroom observations were mainly used to study how teachers' current level of MKT related to their current instructional practices.

My initial plan was to analyze the qualitative data by focusing on the instructional practice scales as themes and to examine how prevalent and useful those themes were as explanations of the effect of MKT on teachers' practices. However, when I was observing the teachers, I made memos in my field notes about additional noticeable patterns and ideas to pursue (Bogdan & Biklen, 2003).

After I completed data collection, while reading over my field notes and interview transcripts, I found I had jotted down similar memos. These ideas were mainly related to (a) the quality of teachers' analyses of students' responses, (b) the purpose of the lessons, (c) textbook use, and (d) the use of lesson time. I developed initial coding schemes for the five original instructional practice scales as well as for these additional subthemes. While analyzing interview data, I also noticed similarities and differences in teachers' reports on (e) the effects of the MKT gain on their teaching practices. These patterns seemed to be related to the level of teachers' mathematical knowledge, and were therefore included in the qualitative data analyses. I then developed rules that defined these themes, and I assigned coding categories to the data (Bogdan & Biklen, 2003).

By "testing" my codes against my field notes, I realized that the pattern for the Classroom Climate scale as well as some items in the other four instructional practice scales was weak, indicating that either the teachers did not mention the aspects captured in that scale during the interview or that the analysis of classroom observation data did not suggest a clear and consistent pattern. It seems that all the target teachers tried to create a welcoming environment to some extent, regardless of their MKT. Hence, I excluded the code for the Classroom Climate scale from further data analysis. Similarly, I excluded some of the items from each scale.

For the remaining four observation scales, I found that some of the specific observation protocol items were more salient than others to understanding how teachers' knowledge influenced instruction, and those are included in Table 11. For instance, the inquiry-oriented scale consisted of 14 items, but not all 14 items meaningfully varied by teacher. As one example, the item "wrong answers were treated as worthwhile learning opportunities" varied across the observed lessons of the same teacher due to several reasons such as lesson time.

Table 11

Themes Used to Analyze Qualitative Data

Themes	Data sources	Questions to guide my analysis
Inquiry-oriented lesson	Interviews and classroom observations	<ul style="list-style-type: none"> • To what extent did the lesson have a problem- or investigation-centered structure? • To what extent was mathematics portrayed as a dynamic body of knowledge continually enriched by conjecture, investigation, analysis, and proof or justification? • Were instructional practices consistent with investigative mathematics? • Did the design of the lesson incorporate tasks, roles, and interactions consistent with investigative mathematics? • Was the teacher flexible and able to take advantage of teachable moments? • Did students determine the correctness or sensibility of an idea or procedure based on the reasoning presented?
Student engagement	Interviews and classroom observations	<ul style="list-style-type: none"> • Were the vast majority of the students engaged in the lesson and did they remain on task? • To what extent did students share their observations? • To what extent did students explain their ideas or procedures? • Were students willing to openly discuss their thinking and reasoning? • Did interactions reflect a productive working relationship among students?
Worthwhile mathematical tasks	Interviews and classroom observations	<ul style="list-style-type: none"> • To what extent did tasks focus on understanding important and relevant mathematical concepts, processes, and relationships? • Did tasks stimulate complex, nonalgorithmic thinking? • Did the teacher choose mathematically appropriate tasks to teach concept? <ul style="list-style-type: none"> ○ Were the tasks appropriate for the students' level of understanding?

(table continues)

Table 11 (continued)

Themes	Data sources	Questions to guide my analysis
Textbook use	Interviews and classroom observations	<ul style="list-style-type: none"> • How closely did the teacher follow the textbook?
Mathematical sense-making agenda	Interviews and classroom observations	<ul style="list-style-type: none"> • What was the mathematical quality of the lesson? • Was the teacher able to clearly articulate what mathematical ideas and/or procedures that students were expected to learn? • Did the teacher create an environment that helped students made sense of the concepts that they were expected to learn? • To what extent did tasks create mathematically productive disequilibrium among students? • To what extent did tasks encourage the students to think beyond the immediate problem and make connections to other related mathematical concepts? • To what extent did students make generalizations regarding mathematical ideas? • What was the teacher's focus when she was analyzing her student's work? <ul style="list-style-type: none"> ○ Did the teacher productively probe the mathematics in students' responses?¹⁵
Purpose of the lesson	Classroom observations	<ul style="list-style-type: none"> • What was the focus of the lesson? <ul style="list-style-type: none"> ○ Did teachers focus on teaching only procedure, not meaning? ○ Did teachers focus on teaching procedure and meaning (i.e., why they were using the procedure, explaining the procedure, or both)

(table continues)

¹⁵ This item is part of inquiry-oriented scale; however, based on my classroom observations, I realized that there is a difference in teachers' focus while pushing on the mathematics in students, which seems more related to mathematical sense-making agenda. As a result, for qualitative analysis, this item was considered as part of mathematical sense-making agenda.

Table 11 (continued)

Themes	Data sources	Questions to guide my analysis
Use of lesson time	Classroom observations	<ul style="list-style-type: none"> • How did the teacher use the lesson time? <ul style="list-style-type: none"> ○ How much of the lesson time was really devoted to the teaching and learning of mathematics? ○ How much time was devoted to activities such cutting, coloring and pasting, or management?
Teachers' MKT perceptions	Interviews	<ul style="list-style-type: none"> • How did teachers' gain in MKT affect their teaching? <ul style="list-style-type: none"> ○ Did an MKT gain increase teachers' self-confidence? ○ Did teachers see how mathematical ideas were connected?

Note. MKT = mathematical knowledge for teaching.

Additionally, during the testing phase, I realized that my additional codes from the classroom observation data were closely related to some of the instructional practice scales. As part of the iterative process of qualitative research (Bogdan & Biklen, 2003), I subsequently revised the coding scheme, creating the two subcodes (purpose of the lessons and use of lesson time) as subcategories for the Mathematical Agenda of Sense-Making scale. Similarly, the sub-theme, *textbook use*, was related to the Worthwhile Mathematical Tasks scale.

The finished coding scheme included four of the instructional practice scales (inquiry-oriented lesson; mathematical agenda of sense-making, with two subcategories, namely, purpose of lessons and use of lesson time; worthwhile mathematical tasks, with a subcategory of textbook use; and student engagement), and one additional theme that captured teachers' perceptions of the effect of their MKT gain on teaching practices (hereafter called "teachers' MKT perceptions." Most of the additional themes emerged during my classroom observations and during the transcription of my field notes and interviews. Only "teachers' MKT perceptions" emerged from the interview data. Table 11 summarizes the main themes (four classroom observation scales) as well as the additional subthemes that emerged during my classroom observations and during the transcription of my field notes and interviews. Table 11 also includes the questions that ultimately guided my analysis of the target teachers' instructional practices. When I was analyzing teachers' interviews and classroom observations, I closely followed the relevant items on each of the four instructional practice scales. However, as mentioned earlier, some items were excluded or rephrased to capture the qualitative data more accurately.

While analyzing data from each teacher, I answered each of the questions listed in Table 11. I searched through my field notes and interview data to find information related to these questions. Because I observed these teachers' mathematics lessons several times and because

there could be some variation in individual teachers' instruction in each observed lesson, I only focused on the common aspects of each individual teacher in all the observed lessons. To increase the reliability of my approach, I provided a deep description of the observed lessons (Eisenhart, 1988) and used teachers' reports of their instructional practices from the interview. More specifically, if the teacher's typical evaluation of her students' work included only one-word comments, she might also mention during the interview how she provided feedback on her students' work. I then included her example from the interview to account for all the data in some way.

I followed a somewhat different approach for the three additional themes: use of lesson time, purpose of the lesson, and effect of MKT gain on instruction. For each individual teacher, I computed the percentage of lesson time devoted to mathematics-related activities. For each teacher, I computed the total amount of time used for mathematics and mathematics-related activities and divided that amount by the total amount of lesson time. The reason behind this quantification of qualitative data was to more concretely illustrate how teachers spent their mathematics lesson time and to allow for easier comparisons among teachers. For a similar reason, for the "purpose of the lesson" theme, I coded each lesson based on the primary purpose of the lesson, as well as whether it was designed to teach a procedure or teach the meaning behind the procedure. For each target teacher, all the observed lessons had a similar focus, meaning that each teacher consistently focused on procedures or on teaching meaning behind the procedures. Each lesson used in the portraits illustrated the purpose of the lesson. For the theme "teachers' MKT perceptions," I report only the most prevalent and common changes teachers listed because of the gain in their MKT.

As summarized here, my main focus on qualitative analysis was the relationship between teachers' mathematical knowledge and instructional practices. However, quantitative analyses (3-level MLM) also included the relationship among teachers' mathematical knowledge, instructional practices, beliefs, and student achievement gains. Unfortunately, I did not collect qualitative data from teachers' students, which limited my qualitative analysis on how teachers' mathematical knowledge and instructional practices were associated with student achievement. I generally took notes regarding overall students' behaviors whether most of the students were on task or participating the lessons or to what extent the major behavior problems caused interruption in observed lessons. I also visited earlier years' field notes to find information regarding students' behaviors and attention to the classroom discussion. I also used teachers' interview data to understand students' mathematical comprehension at the classroom level. Since I had limited data related to students, I only focused on the "significant quantitative results" while analyzing qualitative data for this part of the study. This approach is somewhat different from what I did for the qualitative analyses on MKT, beliefs, and instruction in that I analyzed quantitative and qualitative data separately, while reporting qualitative analysis, I typically focused on the patterns revealed in quantitative findings. However, for analysis on how teachers' MKT and instructional practices were associated with their student learning, I used qualitative data first and based on the quantitative findings, I revisited both classroom observation and interview data to understand the existing quantitative relationships.

In this chapter, I have detailed the methodological approaches of this study. In the following two sections, I present the findings of the study, starting with results of the quantitative analysis, followed by portraits of the eight teachers.

Chapter 4

A Quantitative Analysis of the Relationship Among Teachers' Knowledge of Mathematics, Instructional Practices, Beliefs, and Student Achievement

In this chapter, I present the study results pertaining to the two research questions, which focus on the relationships among teachers' mathematical knowledge, beliefs, instructional practices, and student achievement. Hence, the structure this chapter takes is as follows. I begin by presenting the results of the first research question which was aimed to address the extent to which the change in teachers' knowledge of mathematics was related to change in their instructional practices. Then I present results of the second research question, which was whether the changes in teachers' math knowledge and instructional practices would predict their students' ISAT gains. I also report the extent to which teacher beliefs corresponded with teachers' instructional practices as well as students' ISAT gains.

Part I—The Relationship between Teachers' MKT and Their Instructional Practices

In this section, I begin with descriptive statistics of the variables used in this part of the study. As mentioned in the Methods Chapter, I have five different instructional practices scales (inquiry-oriented lesson, student engagement, mathematical agenda of sense-making, worthwhile mathematical tasks, and classroom climate). To predict teachers' practice from teachers' knowledge of mathematics, I ran a separate analysis for each scale. Hence, I report the findings for each instructional practice scale separately.

Descriptive statistics. Table 12 presents the means and standard deviations of teachers' scores on the mathematics test and instructional practices scales for each year as well as for the beliefs survey administered in the final year. The teachers began the program with a mathematics test score of $-.35$ logits, indicating ample room for growth in their mathematical knowledge. For

the duration of the program, based on the results of the F-test, teachers' mathematical knowledge changed significantly ($F(3, 55) = 33.55, p < .0001$) as well as several aspects of their instructional practices did. Specifically, there were significant changes in teachers' inquiry-based teaching ($F(3,33) = 3.22, p = .035$), teachers' mathematical-sense making agenda ($F(3, 33) = 3.42, p = .028$), the use of worthwhile mathematical tasks ($F(3,33) = 4.70, p = .008$), and classroom climate ($F(3,33) = 4.72, p = .008$). Only teachers' scores on the student engagement scale did not change significantly ($F(3,33) = .80, p = .50$).

Table 12

Descriptive Statistics for the Variables

Variable	Year 0	Year 1	Year 2	Year 3
Mathematics score	-.35 (.69)	.31 (.78)	.19 (.68)	.26 (.75)
Inquiry-oriented lesson design	1.36 (.50)	1.77 (.55)	1.43 (.68)	1.68 (.95)
Student engagement	2.12 (.44)	2.32 (.51)	2.23 (.70)	2.17 (.69)
Worthwhile mathematical task	1.46 (.62)	1.79 (.51)	1.36 (.80)	2.18 (.83)
Mathematical sense-making agenda	1.39 (.39)	1.85 (.33)	1.42 (.65)	1.63 (.78)
Classroom climate	2.43 (.43)	2.87 (.52)	2.90 (.38)	2.62 (.50)
Beliefs				2.48 (.54)

Note. Numbers in parentheses are standard deviations.

Of the variables used in this study, only teachers' beliefs were captured at one point in time during the implementation of the program. In the summer of the second year of the program, the teachers were surveyed to capture the extent to which their views were aligned with the

problem-solving view of mathematics, in contrast to the traditional way of teaching. Given that the scores for this survey can range from 0 to 4 and that a higher score indicates more alignment with the problem-solving view of mathematics, the score of 2.48 on the Beliefs survey indicated that, on average, teachers' held a standards-based view of mathematics to a moderate extent.

In summary, teachers' mathematical knowledge and instructional practices changed during the program. Given that some teachers were not observed each year, the summary statistics do not provide sufficient information about how individual teachers' practices and mathematical knowledge changed over the years, but the descriptive statistics provide an overall idea of the general trends in the data each year. Finally, the teachers seemed to hold a moderate view of teaching through problem solving. In the following sections, I present how the change in teachers' instructional practices captured in the five scales mentioned was associated with their gains in mathematical knowledge and beliefs about teaching mathematics.

The relationship among teachers' inquiry-oriented lessons, MKT, and beliefs. In this section, I present the results of the relationship among teachers' scores on the inquiry-oriented lesson scale, mathematics test, and beliefs survey. As mentioned earlier, since teachers' beliefs were measured only at the end of second year, I run separate linear regression analyses for teachers' beliefs and each instructional practice scale.

Table 13 presents the results of the models in which teachers' scores on the inquiry-oriented lesson scale were predicted from teachers' MKT scores, year, grade level (dummy coded), and a dummy coded indicator of being an experienced teacher. Based on the results of the null model, 66% of the total variation in inquiry-oriented lesson scale, which captures the extent to which teachers created a problem-based oriented and student-investigation centered lessons, was attributable to differences between teachers while 34% of the variation was

attributable to differences within teachers. This finding suggests that, on average, there was important variation in lesson design across teachers. The residual correlation was not significant for any of the models for inquiry-oriented lesson design, which indicates that the correlation between errors is not significant. Results of Model 1 suggest that teachers' MKT scores were a significant predictor of their scores on the inquiry-oriented lesson design. Adding the dummy coded variables "grade level" and "being an experienced teacher" did not improve the model (Model 2), as neither was a significant predictor of inquiry-oriented lesson design. This could be partly due to the small number of teachers in these categories. Model 2 had the bigger AIC and BIC values, indicating that Model 1 would be preferred. The deviance test for Model 2 suggested that inclusion of grade level and experience or new teacher did not contribute significantly to the final model ($p = .78$).

Table 13

Results of the Linear Growth Models for Inquiry-Oriented Lesson

Variable	Null	Model 1	Model 2
Fixed effect			
Intercept	1.56*** (.14) (.0005)	1.59**** (.15) <.0001	1.44 (.29) $p = .148$
Year		-.04 (.05) .42	-.04 (.05) .44
MKT Score		.36** (.13) $p = .008$.36** (.13) .009
Experienced Teacher			.14 (.32) .67
Grade (1-3)			.14 (.29) .64
Variance component			
Level 1: Within-person	.16*** (.04) $p < .0001$.13*** (.03) <.0001	.13*** (.03) <.0001
Level 2: Between person	.31** (.12) $p = .005$.31** (.11) $p = .004$.30** .11 $= .003$
Fits statistic			
Deviance	93.2	85.6	85.1
AIC	99.2	95.6	99.1
BIC	102.4	100.8	106.4
		Level-1 Added	Level-2 Added

Note. AIC = Akaike's information criterion; BIC = Bayesian information criterion.

~ $p = < .10$, * $p < .05$, ** $p < .01$, *** $p < .001$, **** $p < .0001$.¹⁶

¹⁶ Due to small sample size, .10 was used as a cut-off value in this study.

Model 1, including only teachers' mathematics score and years, explained 18.8% of variation in individual teacher's scores on inquiry-oriented lesson scale. Teachers' math knowledge explained 14.7% of the variance in individual teachers' scores on the inquiry-based lesson scale. According to Model 1, after controlling for teachers' mathematical knowledge and year, the intercept for the inquiry-based lesson scale was still significantly random; indicating that teachers' scores on this scale were different from each other at the beginning of the program regardless of their math knowledge. A 1-logit increase in individual teachers' mathematics score was associated with a .36-point increase in their score on inquiry-oriented lesson design ($p = .008$).

The separate Year 3 linear regression analysis using only teachers' beliefs score as a predictor of the inquiry-based lesson scale score indicated that a one unit-increase in teachers' beliefs score was associated with a .85 increase in teachers' scores on inquiry-oriented lesson design, which was significant at $p = .046$ ($N = 14$). Additionally, 23.4% of the variation between teachers' scores on inquiry-oriented lesson design was explained by teachers' beliefs score.

Figure 9 displays the relationship between gains in teachers' mathematics knowledge as measured at 4 time points and their estimated linear regression scores on inquiry-oriented lesson design. As illustrated in Figure 9, each teacher is represented by one linear regression line that summarizes the relationship between their MKT gain score and their score for inquiry-oriented lesson design over time. Given that most of the lines slope upward, it is not surprising that there is an overall positive association between teachers' knowledge and their use of inquiry¹⁷.

¹⁷ One might wonder why I did not use random slope for teachers' MKT given that there seems some variation in slopes. For any of the instructional practices scales, either deviance test indicated no significant improvement in the model, or models did not converge when random slopes were included.

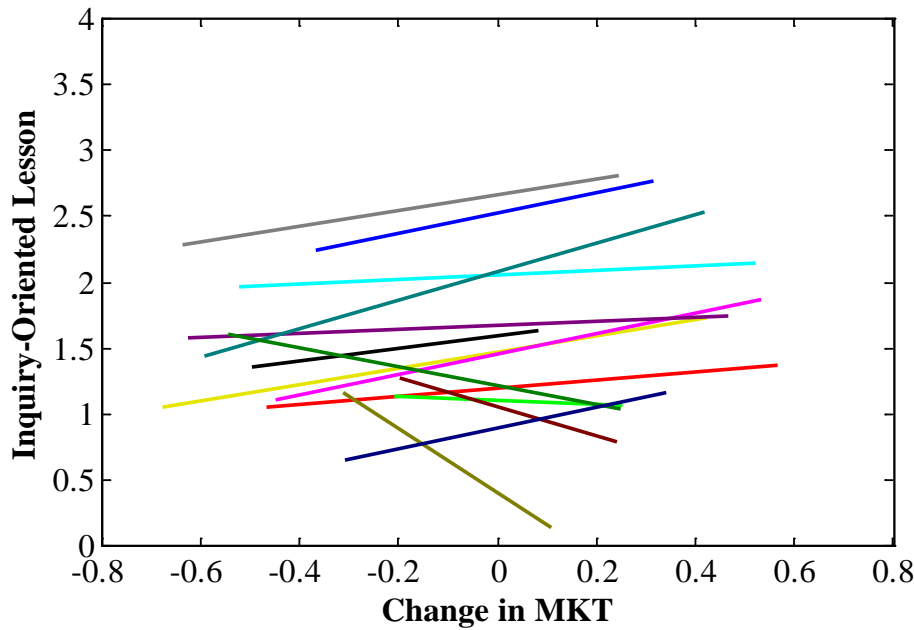


Figure 9. Individual linear regression models for inquiry-oriented lesson design by change in teachers' mathematical knowledge.

Note. Each color represents a different teacher.¹⁸

The relationship among teachers' student engagement, MKT, and beliefs. Table 14 presents the results of the models, in which teachers' scores for the Student Engagement scale served as the outcome. Recall that the student engagement scale captures the extent to which students shared and explained their thinking and productively worked with their peers. The null model indicates that, 68.3% of the total variation in student engagement was attributable to differences between teachers while 31.2% of the total variation was attributable to differences in individual teachers. This finding suggests that, on average, there was important variation in student engagement across teachers. Similar to the models for inquiry-oriented lesson design, autocorrelation was not significant for any of the models tested. Results of Model 1 and Model 2 suggest that none of the fixed effects were significant. Moreover, the deviance tests indicated

¹⁸ I used the same color for each teacher in all figures, so that the reader has a chance to see how individual teacher's MKT gain was associated with changes in their scores on different scales.

that Model 1 and 2 were not statistically different from the null model, suggesting that none of the predictors in the study helped explain the variation in the Student Engagement scale within and between teachers. In sum, students' engagement did not change significantly over the years, and none of the predictors, including teachers' mathematical knowledge, grade level, and experience level, were significantly related to student engagement. Furthermore, teachers' beliefs scores were also unrelated to their scores on the student engagement scale ($p = .31$).

Table 14

Results of the Linear Growth Models for Student Engagement

Variable	Null	Model 1	Model 2
Fixed effect			
Intercept	2.15*** (.12) <.001	2.17**** (.14) <.0001	1.70** (.25) .01
Year		-.03 (.05) .59	-.02 (.05) .71
MKT Score		.12 (.12) .33	.09 (.12) .46
Experienced Teacher			.45 (.27) .11
Grade (1-3)			.39 (.24) .12
Variance component			
Level 1: Within-person	.122*** (.03) <.0001	.117 (.029) <.0001	.117 (.029) <.0001

(table continues)

Table 14 (continued)

Variable	Null	Model 1	Model 2
Level 2: Between person	.263** (.109) .008	.277** (.114) .008	.199* (.087) .011
Fits statistic			
Deviance	80	79	73.4
AIC	86	89	87.4
BIC	89.1	94.3	94.7
		Level-1 Added	Level-2 Added

Note. AIC = Akaike's information criterion; BIC = Bayesian information criterion.

~ $p = < .10$, * $p < .05$, ** $p < .01$, *** $p < .001$, **** $p < .0001$.

The Figure 10 also illustrates the weak relationship between student engagement and teachers' mathematical knowledge gains. The slopes for individual teachers seem gentle slope.

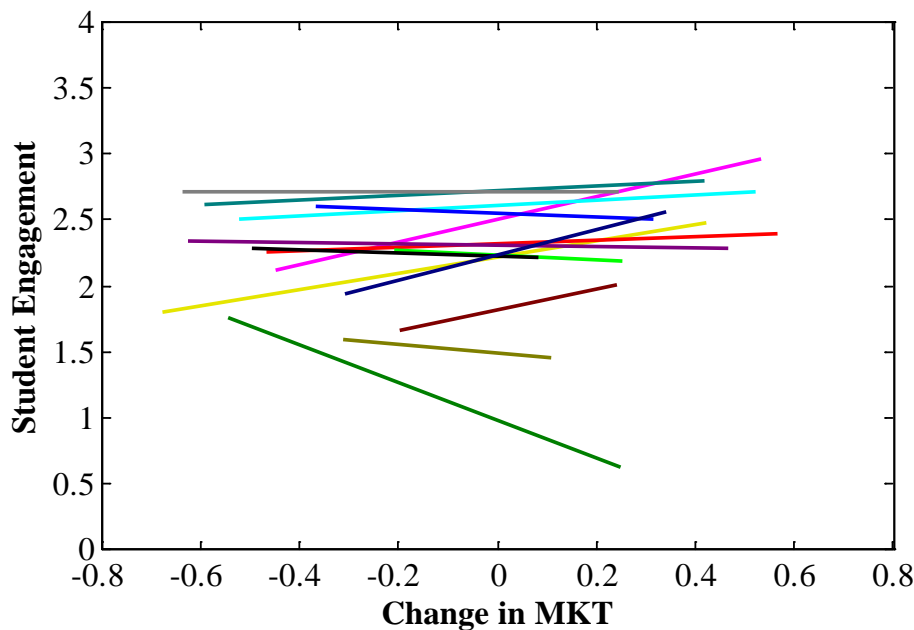


Figure 10. Individual linear regression models for student engagement by change in teachers' mathematical knowledge.

The relationship among teachers' worthwhile tasks choices, MKT, and beliefs. As shown in Table 15, the results of the models, in which teachers' scores for the Worthwhile Mathematical Task scale serve as the outcome, suggest no relationship between teachers' MKT

scores and their scores on the worthwhile mathematical tasks scale, which captures the extent of tasks' stimulating non-algorithmic thinking. 28.9% of the total variation on the Worthwhile Mathematical Task scale lay between the teachers (in contrast to 34% for lesson structure and 31.2% for student engagement)¹⁹. Results of Model 1 and 2 suggest that teachers' MKT score, year, grade level, and dummy coded variable of being a novice or experienced teacher were not significant predictors of teachers' task choice. Furthermore the deviance test indicates that the random intercept was significant ($p = .068$), indicating that after controlling teachers' MKT, grade level taught, and experience level, teachers' scores on worthwhile mathematical tasks were found to be different from one another at the beginning of the program.

Table 15

Results of the Linear Growth Models for Worthwhile Mathematical Task Use

Variable	Null	Model 1	Model 2
	Fixed effect		
Intercept	1.68 ^a (.13) <.0001	1.51*** (.16) .006	1.59~ (.29) .061
Year		.13 (.08) .13	.12 (.08) .15
MKT Score		.16 (.15) .31	.19 (.16) .25
Experienced Teacher			-.16 (.30) .59
Grade (1-3)			.14 (.26) .58
(table continues)			

¹⁹ When the findings from the scales examined previously were aligned, the autocorrelation was not significant in any of the models tested for the Worthwhile Mathematical Task scale.

Table 15 (continued)

Variable	Null	Model 1	Model 2
Variance component			
Level 1: Within-person	.402 (.099) ^a <.0001	.358 (.085)*** <.001	.358 (.085)*** <.001
Level 2: Between person	.163~ (.124) .095	.155~ (.104) .068	.147~ (.10) .072
Fits statistic			
Deviance	124.6	118.7	118.2
AIC	130.6	128.7	132.2
BIC	133.7	133.9	139.5
		Level-1 Added	Level-2 Added

Note. AIC = Akaike's information criterion; BIC = Bayesian information criterion.

~ $p = <.10$, * $p < .05$, ** $p < .01$, *** $p < .001$, ^a $p < .0001$.

The regression of teachers' scores on the Worthwhile Mathematical Task on teachers' belief scores indicated that a one-point increase in teachers' beliefs scores corresponded with a .64-point increase in their task scale ($p = .092$). Teachers' beliefs explained 21.9% of the variation in the worthwhile task scores between teachers.

Figure 11 below illustrates the lack of relationship between the change in teachers' knowledge of mathematics and their quality of task choice. It seems that for some teachers, there is a negative relationship between their math score and their worthwhile mathematical task scale score, while this relationship is positive for some other teachers²⁰.

²⁰ As mentioned earlier, mostly due to small sample size, I could not include a random slope for the effect of teachers' MKT.

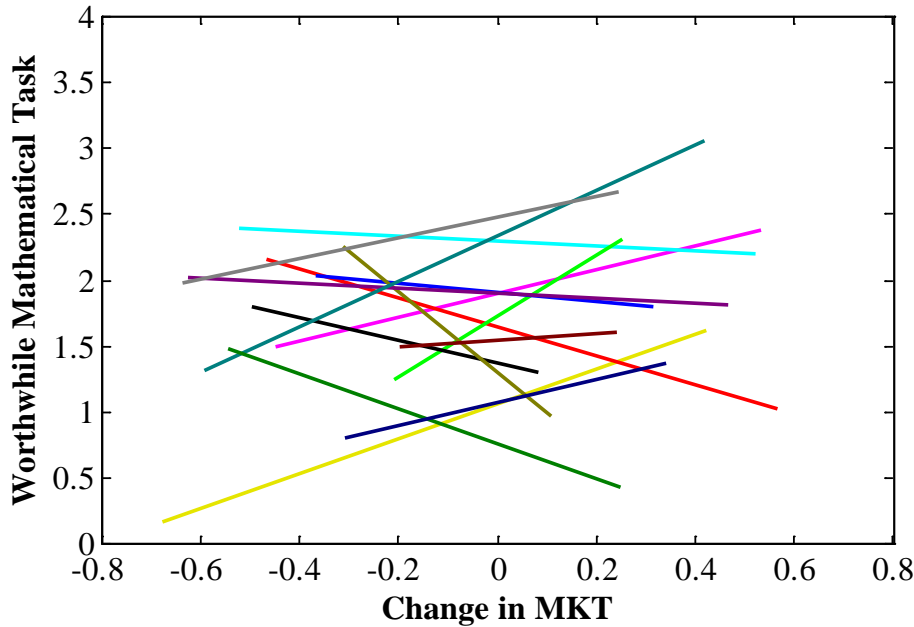


Figure 11. Individual linear regression models for worthwhile mathematical task by change in teachers' mathematical knowledge.

The relationship among teachers' mathematical sense-making agenda, MKT, and beliefs. Results of the models tested for the Mathematical Sense-Making Agenda scale are presented in Table 16.²¹ Based on results of the null model, 30.7% of the total variation in teachers' scores on the Mathematical Agenda scale lay between teachers. Recall that mathematical agenda measures the "mathematical quality" of the observed lessons, which is the extent of which teachers were able to articulate what mathematical ideas students were expected to learn and students were able to see connection between and make generalizations regarding the ideas. The results of Model 1 suggest that the effect of teachers' MKT score on the Mathematical Agenda scale was marginally significant ($p = .069$). Furthermore, results of Model 2 suggest that only the teachers' MKT score was a significant predictor of their score on the sense-making agenda scale. As shown in Table 16, Models 1 and 2 were not statistically

²¹ The autocorrelation was not significant for any of the models, so it was removed from the analysis and reports.

different from the null model. However, the model that included only teachers' MKT score was statistically different from the null model ($p = .08$).

Table 16

Results of the Linear Growth Models for Mathematical Sense-Making Agenda

Variable	Null	Model 1	Model 2
Fixed effect			
Intercept	1.57 ^a (.10) <.0001	1.61 ^a (.13) <.0001	1.51* (.24) .049
Year		-.04 (.06) .54	-.03 (.06) .58
MKT Score		.23~ (.12) .069	.22~ (.13) .094
Experienced Teacher			.12 (.25) .64
Grade (1-3)			.03 (.21) .88
Variance component			
Level 1: Within-person	.228 ^a (.005) <.0001	.202 ^a (.048) <.0001	.202 ^a (.048) <.0001
Level 2: Between person	.101~ (.07) .077	.121* (.07) .046	.119* (.071) .045

(table continues)

Table 16 (continued)

Variable	Null	Model 1	Model 2
Fits statistic			
Deviance	93.3	89.9	89.6
AIC	99.3	99.9	103.6
BIC	102.4	105.1	111
		Level-1 Added	Level-2 Added

Note. AIC = Akaike's information criterion; BIC = Bayesian information criterion.

$\sim p = < .10$, $*p < .05$, $***p < .001$, $^a p < .0001$.

Model 1 explained only 11% of the variation within individual teachers' scores on the sense-making agenda scale. When teachers' grade level, their status as a novice or experienced teacher, their mathematical knowledge, and the effects of time (year) were controlled for, the intercept differed significantly for teachers, indicating that teachers' scores on the Mathematical Sense-Making Agenda were different at the beginning of the program. With this model, a 1-logit increase in teachers' mathematical knowledge was related to an increase of .23 on the sense-making agenda scale.

Separate analysis of the relationship between teachers' beliefs and the Mathematical Agenda of Sense-Making scale indicated that a 1-point increase in teachers' belief scores was marginally associated with a .58-point increase the Mathematical Agenda of Sense-Making scale ($p = .10$). Teachers' belief scores explained 21% of the variation in teachers' scores on the Mathematical Agenda of Sense-Making scale.

The relationship between individual teachers' gain in mathematical knowledge and the change in their Mathematical Sense-Making Agenda scores is represented in Figure 12. For most

of the teachers, the gain in mathematical knowledge appeared to be positively related to the change in their scores on the sense-making agenda scale.²²

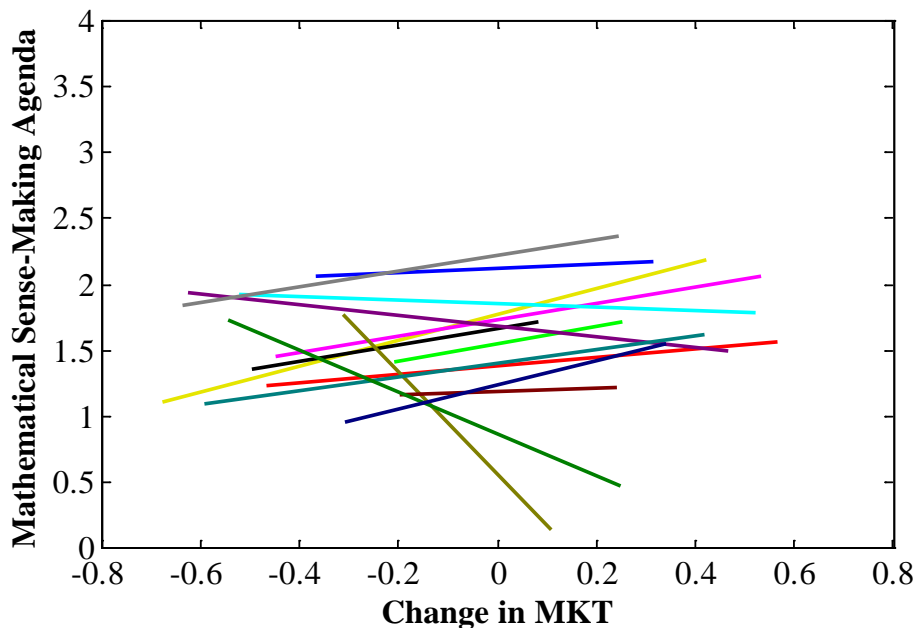


Figure 12. Individual linear regression models for the Mathematical Agenda of Sense-Making scale by the change in teachers' mathematical knowledge for teaching (MKT).

The relationship among teachers' classroom climate, MKT, and beliefs. Results of the models tested for the Classroom Climate scale, capturing the extent to which teachers created

²² As seen in Figure 12 and the figures for the other four instructional practice scales, the same two teachers were outliers. A negative relationship was observed between the gain in these teachers' MKT scores and their instructional practices. From the perspective of faculty and RAs involved with the master's degree program, these two teachers stood out, appearing less engaged and less conscientious than most other teachers in the program. The teaching position of one of these two teachers was terminated in the third year of the program because of her students' limited progress on the ISAT test. Her class tended to be dominated by serious student behavioral problems. She told me that she had decided not to interfere with her students' behaviors after she had been hit while trying to stop two students from fighting. I cannot speculate about the reasons underlying the other teacher's lack of engagement; she often stood out because of our difficulty in scheduling observations with her and her relatively limited engagement in the master's program courses. Due to both teachers' lack of enthusiasm for the data collection process, I was unable to collect further data to examine possible reasons for their lack of engagement as teachers.

a welcoming environment and respected students' ideas, are presented in Table 17.²³ Results of the null model indicated that 21% of the total variation in teachers' scores on the Classroom Climate scale lay within teachers. Including the year and teachers' MKT scores as predictors in the null model resulted in teachers' MKT scores being significantly related to an increase in their scores on the Classroom Climate scale ($p = .015$). Moreover, grade level and the dummy-coded variable for the indication of teachers' experience level did not contribute significantly to the model ($p = .16$).

Within the individual teachers' scores, Model 1 explained 21% of the variation in their scores on the Classroom Climate scale. Models 1 and 2 did not noticeably help explain the difference between teachers; however, it is important to mention that the variation between teachers on the Classroom Climate scale was subtle. With this model, a 1-logit increase in individual teachers' knowledge was related to an increase of .26 on the teachers' Classroom Climate scale.

Separate analysis of teachers' scores on the Classroom Climate scale for teachers' beliefs regarding the teaching and learning of mathematics indicated no significant relationship between teachers' belief score and their Classroom Climate score ($p = .124$).

²³ The autocorrelation was not significant for any of the models, so it was removed from the analysis and reports.

Table 17

Results of the Linear Growth Models for Classroom Climate

Variable	Null	Model 1	Model 2
Fixed effect			
Intercept	2.67*** (.08) < .001	2.64 ^a (.11) < .0001	2.45 ^a (.18) < .0001
Year		.01 (.05) .87	.01 (.05) .78
MKT Score		.26* (.10) .015	.24* (.10) .023
Experienced Teacher			.12 (.18) .51
Grade (1-3)			.28 (.16) .092
Variance component			
Level 1: Within-person	.188*** (.043) < .0001	.149*** (.036) < .0001	.149*** (.036) < .0001
Level 2: Between person	.05 (.041) .11	.072~ (.05) .07	.05~ (.04) .10
Fits statistic			
Deviance	77.4	69.9	66.2
AIC	83.4	79.9	80.2
BIC	86.5	85.1	87.5
		Level-1 Added	Level-2 Added

Note. AIC = Akaike's information criterion; BIC = Bayesian information criterion.

~ $p = < .10$, * $p < .05$, *** $p < .001$, ^a $p < .0001$.

The relationship between each individual teacher's gain in mathematical knowledge and the change in teachers' scores on the Classroom Climate scale is represented in Figure 13. For most of the teachers, the gain in mathematical knowledge seemed to be positively related to the change in their scores on the Classroom Climate scale.

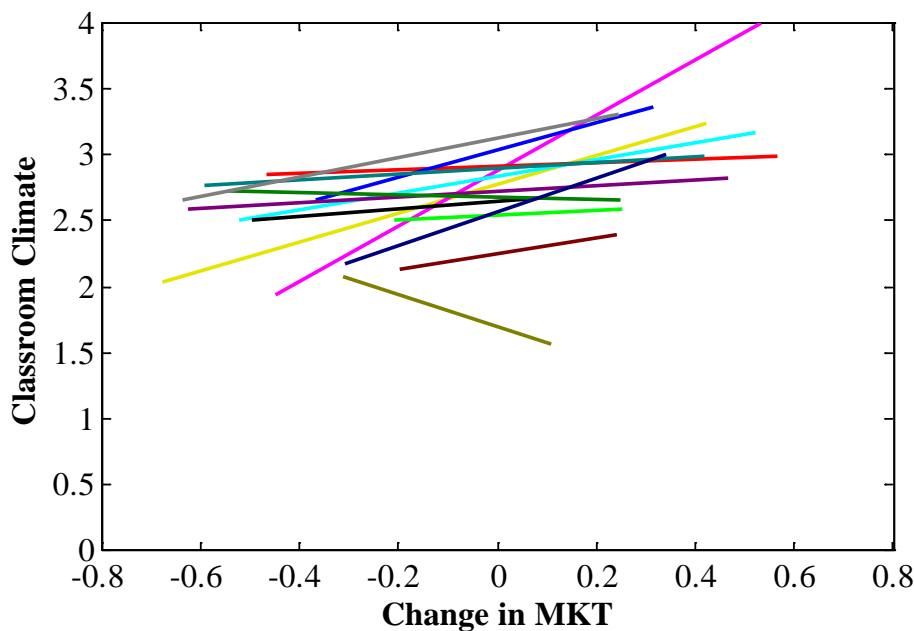


Figure 13. Individual linear regression models for classroom climate by change in teachers' mathematical knowledge.

Summary of the relationship among teachers' instructional practices, MKT, and beliefs. The results indicated that, compared with the initial baseline data, teachers made significant changes on the Mathematical Knowledge, Inquiry-Oriented Lesson Plans, Worthwhile Mathematical Tasks, Mathematical Agenda of Sense-Making, and Classroom Climate scales. Only teachers' scores on the Student Engagement scale did not change significantly over the duration of the program. The gain in teachers' mathematical knowledge seemed to be associated with the quality of the inquiry-oriented lesson design, mathematical sense-making agenda, and classroom climate (see Figure 14). Teachers with higher relative gains in their mathematical knowledge seemed to design their lessons plans to be more closely aligned

with inquiry-based teaching ($\beta = .39$), create a more positive classroom environment ($\beta = .40$), and have a clearer mathematical agenda that involved pushing their students to think hard about the mathematical ideas being taught ($\beta = .30$). Additionally, teachers' beliefs played a positive and significant role in their scores on the Inquiry-Oriented Lesson Plans ($\beta = .55$), Worthwhile Mathematical Tasks ($\beta = .47$), and Mathematical Agenda of Sense-Making scales ($\beta = .45$). Of the five instructional scales, only the Student Engagement scale could not be explained by either the teachers' mathematical knowledge or their beliefs.

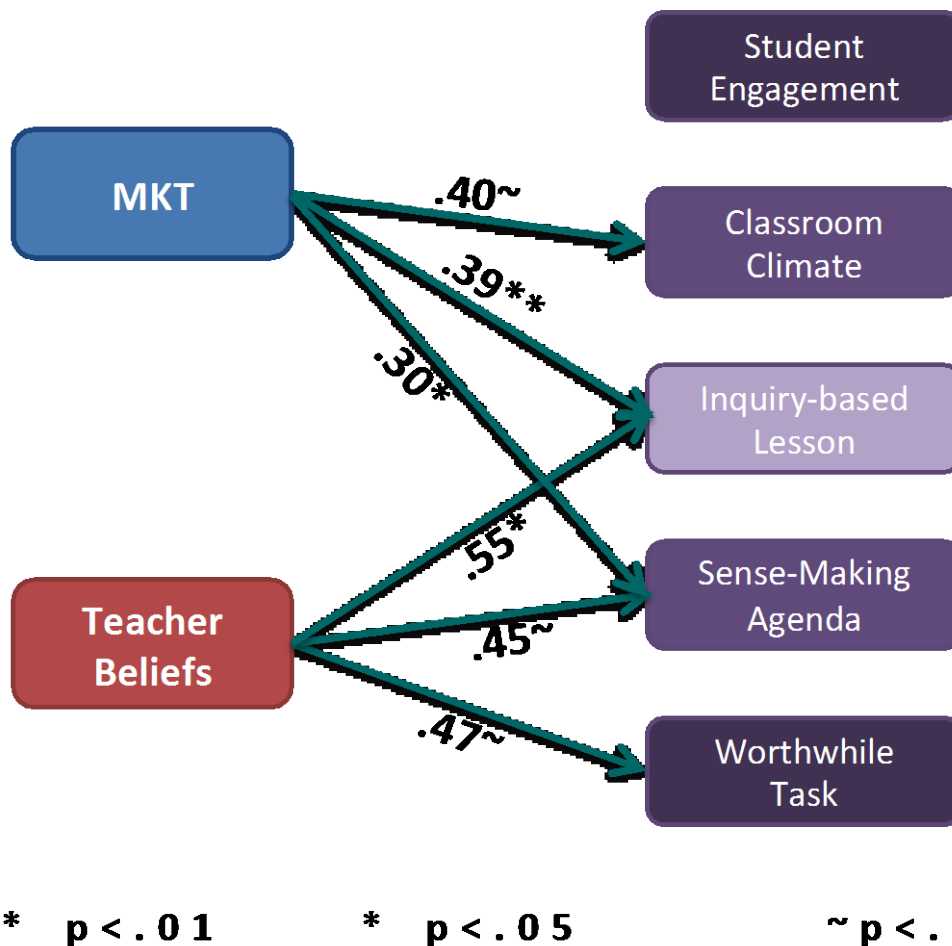


Figure 14. Standardized regression coefficients for the significant relationships among mathematical knowledge for teaching (MKT), beliefs, and instructional practices.

Part II—Linking Teachers’ MKT and Instructional Practices to Their Students’ Achievement Gains

The previous section provided findings regarding the relationship among teachers’ knowledge of mathematics, beliefs, and instructional practices. In this section, I present results of the analysis of the extent to which teachers’ knowledge of mathematics, instruction, and beliefs were related to student achievement gains (if at all). As discussed in the first section, I analyzed data using 2-level linear growth models with time-varying covariates. Including student-level achievement data in the analysis added one more level to the growth model.

This section is also organized under two main headings. First, I provide descriptive statistics regarding the student-level variables used in the study, such as students’ ethnic background and income level indicators. To better understand trends in student achievement, I also report the average ISAT gain for all students in the district over time. Finally, I present the models in which teachers’ instructional practices, mathematical knowledge, and beliefs were used as predictors of student achievement gains.

Descriptive statistics for the student-level variables. As seen in Table 18, a total of 873 students were included in the data analysis. Again, only students in the fourth grade and above were included in the data analysis because ISAT is administered to students in the third grade and above, and third graders as well as their teachers were not included in data analysis because these students did not have baseline ISAT scores. Additionally, I had access to only the partnership district ISAT data, which limited me from including two other teachers from a surrounding district in this part of the analysis. Another point that should be clarified is that each year, the teachers had a different body of students, so the results should be interpreted as average gains associated with the teachers.

Table 18

Demographic Information for the Participant Students

Time	<i>N</i>	African American (%)	White (%)	Low-income eligible (%)
Year 0	231	53.3	46.8	70.6
Year 1	225	53.3	46.7	68.4
Year 2	225	58.2	41.8	72.9
Year 3	192	55.2	44.8	76.0
Overall	873	55.0	45.0	71.8

Each year, more than two thirds of the students were eligible for reduced-price lunches, and almost half of the students were African American (see Table 18). As mentioned in the Methods chapter, I grouped White, Asian, Hispanic, and multiracial students into one category and African American students into another. As explained previously, this was done because the ISAT gains of Asian, Hispanic, and multiracial students were more similar to those of White students than to those of African American students.

The average ISAT gain of the participant teachers' students in the year before the program began was 13 points. As seen in Table 19, the average gain for the teachers in the first and third years of the program was less than in the year before the program started. However, all students in the district had gained fewer points on average in those years (see Figure 15). Additionally, as shown in Figure 15, the mean gain of the participant teachers' students increased more than did the mean gain of the students in the district after the first year of the program.

Table 19

Average Illinois Standards Achievement Test (ISAT) Gains of the Students of the Teachers in the Program

Time	<i>N</i>	Mean (<i>SD</i>)	Minimum	Maximum
Year 0	231	13 (16.03)	-29	61
Year 1	225	11.3 (16.09)	-31	74
Year 2	225	15.68 (13.83)	-21	41
Year 3	192	12.31 (13)	-30	58
Overall	873	13.1 (14.93)	-31	74

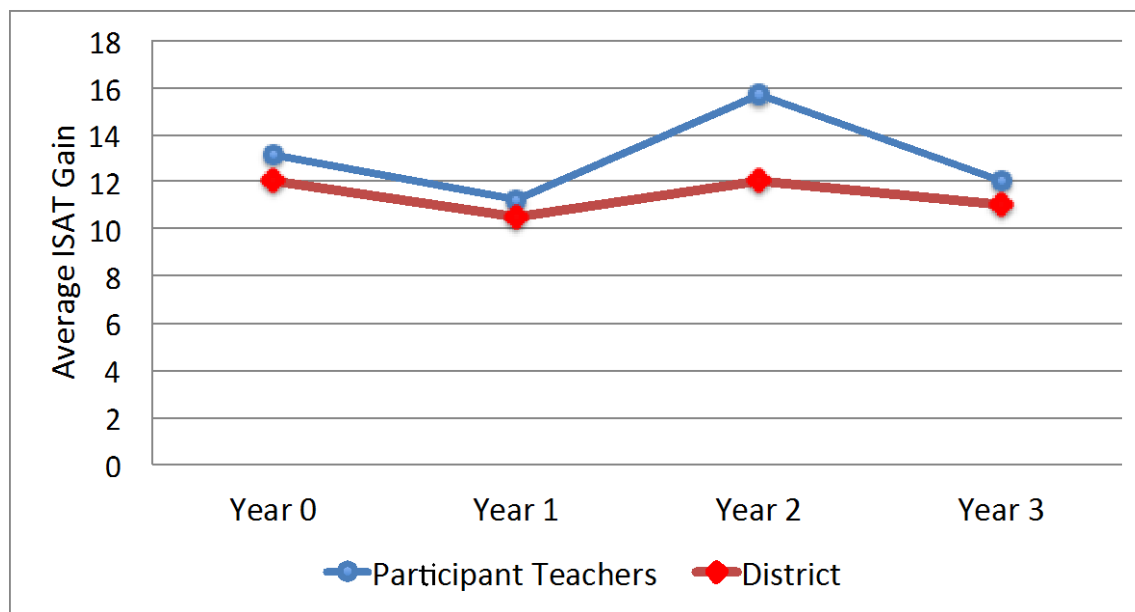


Figure 15. Average Illinois Standards Achievement Test (ISAT) gains of the students of teachers participating in the program compared with the gains of students in the partnership district.

The relationships among teachers' MKT, instructional practices, and student achievement gains. Before presenting the results of the analysis of students' ISAT gains in

relation to teachers' mathematical knowledge and instructional practices, it is important to clarify what variables were used in the study. As discussed in Chapter 3, 3-level multivariate growth models were used to analyze the data. Students were nested within time because the students were different each year. Time (i.e., year in the program) was nested within teacher because the teachers were observed and tested annually (see Figure 16). For Level 1, four variables were used: students' pre-ISAT scores, grade level, race (dummy coded), and a low-income indicator (dummy coded as well). At the time level (Level 2), eight variables were used: year in the program, teachers' scores on the five instructional practice scales, teachers' MKT scores, and mean ISAT gain for the district. As discussed previously, because of the high correlations among the instructional practice scales, each scale was entered into the models individually. Finally, at the teacher level (Level 3), only the dummy-coded variable for the indicator of being an experienced or novice teacher was included.²⁴

²⁴ Although teachers' years of experience changed over time, the interaction term between their dummy-coded experience and year was not significant. Furthermore, treating years of teaching experience as a time-varying covariate (i.e., Level-2 variable) did not noticeably change the results.

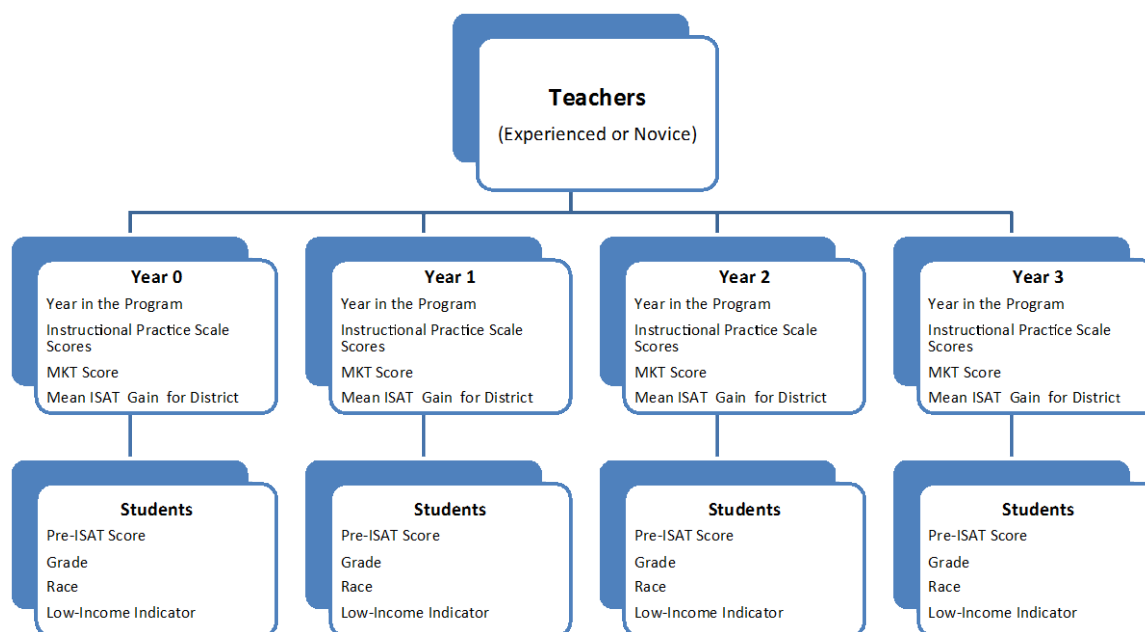


Figure 16. Illustration of the 3-level hierarchical linear modeling analysis. MKT = mathematical knowledge for teaching; ISAT = Illinois Standards Achievement Test.

Given that teachers' beliefs were measured only once, linear regression analyses were conducted separately for teachers' beliefs and students' ISAT gains. Only student-level variables and teachers' belief scores were included in those linear regression analyses because of the small sample size.

Table 20 presents the models tested to predict students' ISAT gains. Model 1, which includes only student-level variables, indicates that students' pre-ISAT scores and their race were significantly and negatively related to their gain scores. In particular, a 1-point increase in students' pre-ISAT scores was related to a decrease of .14 in their gain scores ($p < .001$). White students gained 2.3 points more than African American students ($p = .054$) on average. Students' grade level and their income level were not significant predictors of their ISAT gains.

Table 20

Effects of Teachers' Mathematical Knowledge for Teaching (MKT) and Instructional Practices on Students' Illinois Standards Achievement Test (ISAT) Gain

Variable	Null model	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
Fixed effects													
Intercept	14.04 ^a (1.64) < .0001	14.00 ^a (2.18) < .0001	-22.50~ (11.46) .076	-23.33~ (12.99) .10	-43.04** (13.14) .007	-25.08~ (13.36) .09	-26.46 (12.22) .053	-24.15 (13.72) .11	-23.33~ (13.02) .10	-41.72* (13.5) .01	-23.26 (13.63) .12	-26.05~ (12.36) .06	-24.08 (13.72) .11
Pre-ISAT		-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001	-.14*** (.02) < .0001
African American		2.35~ (1.07) .053	2.32~ (1.07) .054	-2.66* (1.09) .035	-2.56* (1.09) .041	-2.60* (1.09) .038	-2.57* (1.09) .041	-2.62* (1.09) .04	-2.66* (1.09) .035	-2.55* (1.09) .04	-2.60* (1.09) .038	-2.57* (1.09) .041	-2.62* (1.09) .038
Ineligible for lunch subsidy		.21 (1.18) .85	.26 (1.17) .83	.22 (1.19) .86	.23 (1.19) .85	.24 (1.19) .85	.27 (1.19) .83	.24 (1.19) .85	.22 (1.19) .86	.25 (1.19) .84	.26 (1.19) .83	.28 (1.19) .82	.23 (1.19) .85
Grade		.48 (1.07) .65	.32 (1.01) .75	1.02 (1.04) .33	.38 (.98) .70	.96 (1.05) .36	.71 (1.05) .50	.82 (1.04) .43	1.02 (1.05) .33	.36 (.99) .72	.91 (1.05) .39	.69 (1.05) .51	.84 (1.04) .42
Year			5.53* (2.17) .01	5.13** (1.93) .008	5.27* (2.04) .01	5.19* (2.02) .01	5.55** (1.96) .005	6.42** (2.22) .004	5.12* (2.2) .02	5.59* (2.18) .01	5.74* (2.22) .01	5.74** (2.17) .008	6.36** (2.29) .006

(table continues)

Table 20 (continued)

Variable	Null model	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
Year ²			-1.58* (.69) .02	-1.51* (.64) .02	-1.46* (.67) .03	-1.53* (.67) .02	-1.61* (.65) .01	-1.87* (.72) .01	-1.51* (.68) .03	-1.52* (.69) .03	-1.64* (.70) .02	-1.65* (.68) .02	-1.86* (.73) .01
MKT			-1.36 (1.67) .42						.02 (1.82) .99	-.67 (1.46) .65	-1.0 (1.74) .57	-.37 (1.74) .83	.22 (1.91) .91
IOL				-2.97 (1.91) .12					-2.98 (1.99) .14				
SE					3.03* (1.37) .027					3.00* (1.37) .03			
MASM						-1.88 (1.39) .18					-1.93 (1.39) .17		
WMT							-1.83 (1.18) .12					-1.78 (1.2) .14	
CC								-2.68 (2.07) .20					-2.81 (2.32) .23
District Mean ISAT			3.0** (.96) .002	3.41*** (.97) < .0005	4.28^a (1.01) < .0001	3.42** (1.03) .001	3.56*** (.96) .0002	3.66*** (.10) .0002	3.42*** (.99) .0006	4.16^a (1.05) < .0001	3.25** (1.07) .003	3.51*** (.98) .0004	3.69*** (1.01) .0003

(table continues)

Table 20 (continued)

Variable	Null model	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7	Model 8	Model 9	Model 10	Model 11	Model 12
Random effects													
Teacher level	20.80* (12.34) .046	17.35~ (11.48) .065	19.84* (10.83) .03	26.94*~ (14.67) .03	11.48 (8.38) .09	24.65* (13.3) .032	24.93* (13.13) .029	21.53* (11.42) .03	27.00* (15.15) .04	11.00~ (8.14) .09	23.77 (12.83) .29	24.41~ (13.1) .03	21.91* (12.08) .035
Time level	14.46* (7.2) .022	10.48* (5.91) .038	2.47 (3.65) .25	1.04 (3.01) .37	2.22 (3.94) .29	1.86 (3.18) .28	1.20 (2.94) .34	1.93 (3.42) .29	1.04 (3.01) .37	2.32 (4.01) .28	1.76 (3.16) .32	1.21 (2.97) .34	1.90 (3.41) .29
Student level	193.27 ^a (9.87) < .0001	184.14 ^a (9.43) < .0001	184.56 ^a (9.47) < .0001	183.10 ^a 9.564 < .0001	183.92 ^a (9.71) < .0001	183.00 ^a (9.63) < .0001	183.2 ^a (9.64) < .0001	183.26 ^a (9.66) < .0001	183.13 ^a (9.64) < .0001	183.90 ^a (9.71) < .0001	183.03 ^a (9.64) < .0001	183.20 ^a (9.64) < .0001	183.20 ^a (9.66) < .0001
Fit Statistic													
Deviance	6,519.6	6,476.1	6,466	6,104.3	6,102.4	6,104.5	6,104.2	6,104.6	6,104.3	6,102.2	6,104.2	6,104.1	6,104.6
AIC	6,527.6	6,492.1	6,490	6,128.3	6,126.4	6,128.5	6,128.1	6,128.6	6,130.3	6,128.2	6,130.2	6,130.1	6,130.6
BIC	6,529.5	6,495.9	6,495.8	6,134.1	6,132.2	6,134.4	6,134.0	6,134.4	6,136.6	6,134.5	6,136.5	6,136.4	6,136.9

Note. IOL = Inquiry-Oriented Lesson; SE = Student Engagement; MASM = Mathematical Agenda of Sense-Making ; WMT = Worthwhile Mathematical Task; CC = Classroom Climate; AIC = Akaike's information criterion; BIC = Bayesian information criterion.

~ $p < .10$. * $p < .05$. ** $p < .01$. *** $p < .001$. ^a $p < .0001$.

In the Models 2 to 7, I examined whether the gain in teachers' MKT or changes in their instructional practices would predict their students' ISAT gains. I entered teachers' scores on the MKT test and instructional practice scales separately to see the effects of each predictor individually. The change in teachers' mathematical knowledge did not seem to be significantly related to their students' gains on the ISAT (Model 2; $p = .42$). Of the five instructional practice scales, only Student Engagement was significantly associated with students' ISAT gains. Specifically, a one-point increase in teachers' score on the Student Engagement scale was related to a 3.03-point increase in students' ISAT gain ($p = .027$). For each model, students' initial ISAT scores were negatively related to the gains they made on the ISAT. African American students gained 2.5 points on average less than did White students. Additionally, the district mean ISAT gain was positively and significantly related to students' ISAT gain ($p < .001$). As shown in the models, the effect of year was quadratic, meaning that the effect of year was positive but that it decreased with time.²⁵

In Models 8 to 12, I entered teachers' MKT score and one of the five instructional practice scales together. As shown in Table 20, adding the MKT and instructional practice scales together did not noticeably change the results. Nevertheless, Student Engagement was the only significant and positive predictor of students' ISAT gain ($p = .03$). Additionally, the effects of the student-level variables, year, and the district mean ISAT score did not change noticeably. I did not report the models in which the variable of being an experienced or novice teacher was included because of its trivial contribution to the models ($p = .70$).

The models in which teachers' MKT scores and each of the instructional practice scales were entered together were also not statistically different from the models in which only the

²⁵ For the first 2 years, the effect of the gain was higher than in the previous year, but for the third year, the effect was lower than the effect for the second year.

instructional practice scale was included. Furthermore, the goodness-of-fit indicators, the AIC and BIC values, were smaller for the models with only the instructional practice scales than for the models with only the MKT scores or for the models in which the MKT scores and instructional practice scales were entered together, indicating that the models with only the instructional practice scales were more favorable.

Of these models, the models with predictors (i.e., Models 2 to 12) explained only about 5% of the variation in individual students' ISAT gains. Given that 85% of the variability in students' gain scores was within the students (i.e., in Level 1), whereas only 9% of the variability in students' gain scores resided within the teachers, the results were not surprising. Earlier studies also indicated that teacher-level variables did not contribute much to explaining the differences in students' achievement (e.g., Baumert et al., 2010; Hill, Rowan, & Ball, 2005). Teachers' MKT scores and the instructional practice scales did not contribute much to explaining individual students' ISAT gains across the years. However, teachers' MKT scores and their scores on the instructional practice scales explained at least 84% of the variation at the time level. The models with Level-2 predictors (year, mean district ISAT gain, teachers' MKT scores, instructional practices, or their combination) revealed diminished differences in the average ISAT gain for individual teachers across years, meaning that after controlling for these variables, students' average ISAT gains for individual teachers were not statistically different over the years. However, differences between teachers' average ISAT gains were still significant.

The relationship between teachers' beliefs and students' ISAT gains. As mentioned earlier, because teachers' beliefs were captured once throughout the program and their beliefs could have changed during the implementation of the program, I did not assume that the teachers' beliefs were the same over the years. However, treating teachers' beliefs as time-

varying covariates reduced the power of the analysis by creating many missing cases. As a result, I conducted separate analyses for teachers' beliefs by using the data in which teachers' beliefs were measured. Because of similar concerns (small sample size) while analyzing the relationship between teachers' beliefs and the outcome measures, I included only a limited number of variables. More specifically, in this part of the analysis, only student-level variables and teachers' beliefs were entered together to predict students' ISAT gains. Teachers' belief scores were not significantly related to their students' ISAT score gains ($p = .20$).

Summary of the multivariate analyses. The results indicated that several variables predicted students' ISAT gains. White students' ISAT gains were statistically higher than those of African American students, and students with low pre-ISAT scores gained more points on the posttest than did students with high pre-ISAT scores. However, neither the students' grade level nor their income level predicted students' gain on the ISAT tests.

Of the five scales, only the Student Engagement scale was positively and significantly related to a gain in students' ISAT scores. The other four instructional scales (Inquiry-Oriented Lesson Plans, Worthwhile Mathematical Task, Mathematical Agenda of Sense-Making, and Classroom Climate) were not significantly related to students' gain scores. In addition, teachers' MKT scores were not significantly associated with gains in their students' post-ISAT. Similarly, teachers' beliefs were not related to their students' ISAT gains. The results also indicated that students' average gain changed nonlinearly over time. Students' ISAT gains increased nonlinearly for the first 2 years of the program and decreased in the third year of the program. Even in the third year, students' ISAT gains were higher than that in the year before the program started. Finally, only a small percentage of the variation in individual students' ISAT gains was explained by the variables in the study.

Model 4 had the smallest AIC and BIC values, and the deviance tests indicated that other models with more predictors did not noticeably improve the model. In this model, student engagement was significantly related to students' ISAT gains. It is important to note that the Student Engagement scale captures students' overall level of engagement in the mathematics lessons. This finding implies that students' engagement (remaining on task and explaining their thinking and their ideas) was a predictor of individual students' ISAT gains, even after controlling for individual students' pre achievement scores.

Summary of Chapter 4

In Chapter 4, I examined the relationships among teachers' MKT, beliefs, instructional practices, and student achievement. First, I investigated the role of teachers' MKT and beliefs in their teaching practices. As illustrated in Figure 17, teachers' MKT was significantly related to their Inquiry-Oriented Lesson Plans, Mathematical Agenda of Sense-Making, and Classroom Climate scales. A 1-standard deviation (SD) increase in teachers' MKT scores was related to a .39-SD increase on teachers' Inquiry-Oriented Lesson Plans scale, a .30-SD increase on the Mathematical Agenda of Sense-Making scale, and a .40-SD increase on the Classroom Climate scale. Teachers' MKT scores were not significantly related to their scores on the Student Engagement and Worthwhile Mathematical Tasks scales.

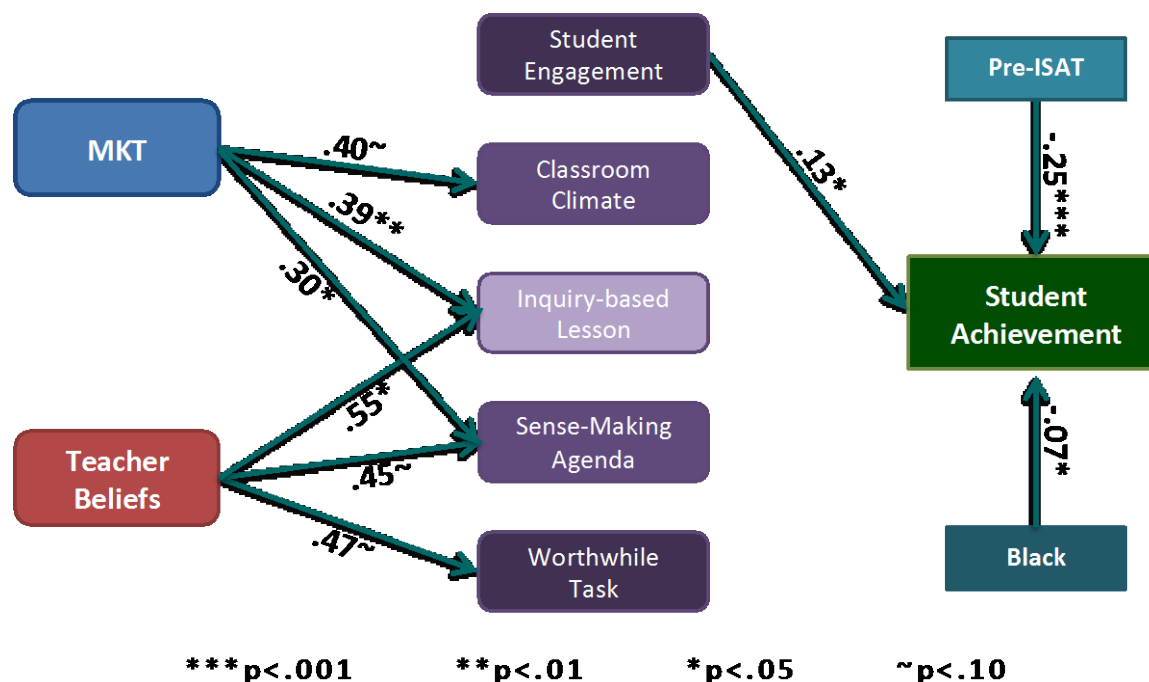


Figure 17. Standardized regression coefficients for the significant relationships among teachers' mathematical knowledge for teaching (MKT), beliefs, instructional practices, and student achievement.²⁶

Separate analyses of the relationship between teachers' beliefs and teaching practices indicated that teachers' beliefs were significantly related to teachers' scores on the Inquiry-Oriented Lesson Plans, Mathematical Agenda of Sense-Making, and Worthwhile Mathematical Tasks scales. More specifically, a 1-SD increase in teachers' scores on the beliefs survey was associated with a .55-SD increase on the Inquiry-Oriented Lesson Plans scale, a .45-SD increase on the Mathematical Agenda of Sense-Making scale, and a .47-SD increase on the Worthwhile Mathematical Tasks scale. Neither teachers' MKT scores nor their beliefs were related to their scores on the Student Engagement scale.

In the second part of this chapter, I investigated the extent to which teachers' MKT scores, teaching practices, and beliefs were related to students' ISAT gain after several factors, such as

²⁶ Year and mean district ISAT gain are not reported here because these variables were not the main focus of the research.

students' earlier achievement and income level, were taken into account. The findings indicated that neither teachers' MKT scores nor their beliefs were related to students' ISAT gains. Of the five instructional practice scales, only teachers' scores on the Student Engagement scale were significantly related to students' ISAT gains. A 1-SD increase in teachers' scores on the Student Engagement scale was related to a .13-SD increase in students' ISAT gains. Given that students' pre-ISAT scores were related to a .25-SD decrease in students' ISAT gain and that African American students had a .07-SD lower ISAT gain on average than did White students, the magnitude of the Student Engagement scale was substantial in comparison.

In sum, quantitative analyses indicated that the Student Engagement scale, which was the only scale not predicted by teachers' MKT or beliefs, was the only significant predictor of students' ISAT gains. The quantitative analysis did not allow for an explanation of the reasons behind these results. In qualitative analyses, I explored the relationships among teachers' MKT scores, beliefs, and teaching practices by using teacher interviews and classroom observations. Special attention was given to the patterns observed in the quantitative analysis to shed light on these findings.

Chapter 5

Qualitative Findings

In Chapter 4, I presented results of the quantitative data analysis. The quantitative analysis provided the magnitude and direction of the relationships between teachers' mathematical knowledge for teaching (MKT) and teaching practices, but it did not help clarify why and how this relationship occurred. Given that earlier studies highlighted the importance of beliefs on instructional practices (e.g., Grossman Wilson, & Shulman, 1989; Putnam et al., 1992; Stodolsky & Grossman, 1995; Thompson, 1984, 1992), this study also helped identify which of the instructional practices were related to teachers' mathematical knowledge and which of them were related to teachers' beliefs.

Longitudinal analyses of teachers' mathematical knowledge and instructional practices uniquely contributed to existing literature by revealing in what ways changes in teachers' mathematical knowledge correspond to changes in teachers' practices. The results indicated that teachers' MKT is closely related to how teachers designed the lesson (inquiry-oriented lesson), how welcoming the environment was that they created for their students (classroom climate), how they implemented the lesson they designed, and the extent to which they made clear and pushed students to think about mathematical ideas the students were expected to learn from the lesson (mathematical sense-making agenda). The cross-sectional analysis of teachers' beliefs about teaching and learning mathematics also suggested that three aspects of instructional practices—lesson design, mathematical quality of lessons, and teachers' task choices—were related to their beliefs. The qualitative analysis of the eight portraits will help clarify the relationships among teachers' MKT, instruction, and beliefs. In this chapter, I aim to elaborate on the quantitative findings using qualitative data analyses (Creswell, 2002). As explained

previously, in-depth classroom observations were conducted only in the third year of the program. Hence, classroom observation data are used to illuminate how teachers' current level of MKT is related to their teaching. Teachers' interview data were used to see how changes in MKT could correspond to changes in their instruction.

One might wonder why knowing the relationship between MKT and instruction matters, since in this study neither teachers' MKT nor many aspects of their instructional practices envisioned in the *Standards* (NCTM, 1991, 2000) were related to their students' achievement gains. The lack of a relationship between teachers' mathematical knowledge and their students' achievement is not new (e.g., NMAP, 2008; Wayne & Young, 2003). Even in the studies suggesting a significant relationship between teachers' MKT and their students' standardized test scores, the magnitude of the relationship was small (e.g., Hill, Rowan, & Ball, 2005). Although using standardized achievement tests as a proxy of student learning might not be the ideal way to capture student learning (e.g., NRC, 2001), the results still have value. First, teachers were accountable for the results of these tests, and the quantitative part of the study indicated that students' Illinois Standards Achievement Test (ISAT) gains were not related to how much teachers increased their MKT or the extent to which they taught in the way envisioned in the *Standards* (NCTM, 1991, 2000). This study suggests that what matters the most is the extent to which the majority of the students remained on task and shared their thinking and ideas with each other. This result was also aligned with earlier studies indicating that the only significant and persistent predictor of student learning was students' remaining on task and their engagement in mathematics lessons (NRC, 2001). Even though the quantitative results regarding teachers' MKT and student achievement gains did not tell a different story, analyses of the qualitative data collected from a subsample of the teachers from whom the quantitative data were

collected provides insights into the lack of a relationship between MKT and student learning as well as between the teachers' teaching practices and student learning. This is especially important given that these earlier studies (e.g., Wayne & Young, 2003) were mostly quantitative and failed to provide reasons for the lack of a relationship. Telling a similar story as found in former, large-scale quantitative studies but providing insights into the reasons for the results could contribute to the field of education by elucidating how teachers' MKT plays a role in instruction. While analyzing the qualitative data, I tried to investigate why their MKT or instruction did not matter in terms of students' ISAT gains.

It is worth noting here that I did not collect relevant data from the students of the participating teachers. Hence, I can only elaborate on the lack of relationships among MKT, instruction, and student learning to some extent. Furthermore, I did not have access to ISAT data from all my target teachers; two of them were not from the partnership district, and two of them were teaching second graders, which limited the number of teachers with available data sets. However, the second-grade teachers were also required to take some district-mandated tests, and I also had some indicators of the ISAT gains for the two teachers from the neighboring district. Furthermore, because student engagement is a predictor of student achievement gains, I focused on the relationship between teachers' MKT and student engagement.

The Portraits of Eight Teachers

In this chapter, I provide portraits of the eight target teachers to allow the reader a closer look at the possible effects of teachers' MKT on their instructional practices. While investigating the effects of MKT on teaching, I analyzed the data based on the themes mentioned in the Methods chapter: inquiry-oriented lesson, mathematical sense-making agenda (with two subthemes of the purpose of the lesson and the use of lesson time), worthwhile mathematical

tasks (with the subtheme of textbook use), student engagement, and teachers' MKT perceptions; however, while reporting findings, special attention was given to the patterns observed in the quantitative analysis. Although individual differences were apparent in the instructional practices among these teachers, commonalities also existed among teachers with similar levels of MKT. The portraits are organized in a way that provides several examples of the instructional practices captured by the classroom observation scales and the additional themes listed in the Methods section. These portraits, which are drawn from interview and classroom observation data, will help the reader understand how teachers' MKT is related to their lesson design, quality of lesson implementation, quality of their task choices, and student engagement. It also provides insights into how teachers perceived that an MKT gain affected their teaching practices.

As mentioned earlier, my purpose in mixing methods was to elaborate on the quantitative results. The quantitative results suggested that the change in teachers' knowledge of mathematics was positively related to the change in their inquiry-oriented lesson design, mathematical sense-making agenda, and classroom climate. Teachers' beliefs regarding teaching and learning mathematics were also significantly related to their inquiry-oriented lesson design, mathematical sense-making agenda, and task choice. On the other hand, the quantitative data also suggested that the change in teachers' mathematical knowledge was not related to their task choice or student engagement. Although my quantitative analysis also included the effects of teachers' instructional practices and mathematical knowledge on student achievement, I did not have detailed qualitative data to conduct an in-depth analysis of this aspect of my quantitative findings. However, I have focused on the elements of the Student Engagement scale, which was the only significant and positive predictor of students' gains on the ISAT test. Readers need to remember several points. First, in-depth classroom observations of the target teachers were

collected only in the third year of the program; hence, teachers' interviews were used to examine teachers' perceptions on how the change in teachers' MKT knowledge was related to the changes in their practices. Classroom observations were mainly used to investigate the relationship between teachers' current level of mathematical knowledge and their instructional practices. Second, the qualitative data analysis was not designed to address the classroom climate scale due to limited qualitative data related to this scale.

Eight Teachers: Stephanie, Jacqueline, Valerie, Rebecca, Sonya, Ann, Beth, and Meg

I explained in the Methods chapter why I chose to focus on these eight teachers in particular. As discussed previously, there was a wide range in these eight teachers' knowledge of mathematics at the beginning and end of the program (see Figure 18 and Table 21). The wide variation in these teachers' initial and current level of knowledge of mathematics would allow me to see how the change in their mathematical knowledge was related to the changes in their practices (the longitudinal effect of mathematical knowledge on instruction) as well as how their current level of mathematical knowledge was associated with their current instructional practices (cross-sectional effect of mathematical knowledge on instruction). As mentioned earlier, the effects of MKT gains on instruction were analyzed by using teachers' interviews, and the effects of teachers' current level of MKT on instruction were analyzed by using classroom observations, field notes, and teacher interviews.

Before the Program	After the Program					
		Very Low	Low	Average	High	Very High
	Very Low		Meg Beth			
	Low		Ann	Sonya	Rebecca	
	Average				Valerie	
	High					Jacqueline
	Very High					Stephanie

Figure 18. Change in teachers' mathematical knowledge over time.

Table 21

The Eight Teachers' Scores on the Measures and Their Students' Average Illinois Standards Achievement Test (ISAT) Gains Over Time

MKT Year 3	Teachers	MKT Gain	Beliefs Score Year 3	Grade Year 3	Inquiry- Oriented Lesson Year		Student Engagement Year		Mathematical Sense- Making Agenda Year		Worthwhile Mathematical Tasks Year		Classroom Climate Year		Mean Pre- ISAT Scores (Average ISAT Gains) Year	
					0	3	0	3	0	3	0	3	0	3	0	3
Very High	Stephanie	.8	4.2	7	3.2		3.3		3.2		3		3.7			
	Jacqueline	.6	4.2	6	4.3		4.4		4.3		4.7		4.2			
High	Valerie	1.1	4.0	4-6	2.0	2.9	2.8	3.6	2	2.9	1.1	2.4	3	4.0	232 (10)	229 (13)
	Rebecca	1.1	4.0	2	2.1	3.3	3.4	3.8	1.8	2.9	2.4	4.0	3.6	4.0		
Average	Sonya	.9	4.0	6	3.1	4.4	3.6	4.0	2.8	4.1	3.5	4.2	3.7	4.6	229 (21)	229 (7) ²⁷
Low	Ann	.3	3.0	3	2.1	1.9	2.1	2.7	1.8	2.0	2.6	2.6	3	3.4	206 (14)	216 ²⁸ (23)
	Beth	.8	3.0	5	2.4	2.6	3.1	3.4	2.4	2.4	2.8	2.5	3.5	3.7	220 (16)	222 (17)
	Meg	.7	3.7	2	3.4	4.2	3.5	3.6	2.9	2.9	3.5	3.6	3.7	4.3		

²⁷ Although the other teachers' average ISAT gain followed a consistent pattern, Sonya's students' average ISAT gains dropped drastically in the final year of the program (Average ISAT gains were 17 and 18 for year 1 and 2, respectively).

²⁸ Ann taught 3rd grades in year 3; hence the gain was not available for the year 2; instead her year 2 data were reported.

I also provided personal and school information for these eight teachers in the Methods chapter. As seen in Table 21, the eight teachers followed similar curricula; early elementary school teachers followed the EnVisionMath Series, whereas the upper elementary and middle school teachers followed the Connected Math Series. Only Valerie did not follow a certain curriculum. In addition to teachers' personal and school information, Table 21 provides information on teachers' scores on Instructional Practice scales and difference in their MKT scores before and after the program, their scores on the beliefs test, and their students' mean ISAT gain, as well as their mean pre-ISAT scores. The reason I provided teachers' scores on these measures is to give readers a better understanding of where teachers were in terms of their teaching before they enrolled in the program. It is also crucial to highlight that teachers' scores on the Instructional Practice scales are based on several items, and the reader should not expect a perfect alignment between teachers' scores and the qualitative findings, given that I focused on only certain aspects of each scale in the qualitative analysis.

As seen in Table 21, several rows are missing. In particular, Stephanie and Jacqueline were not observed at the beginning of program; hence, I have only self-reports regarding their earlier practices. Additionally, I did not have access to Stephanie's or Jacqueline's students' ISAT scores. However, according to the State Report Card, 87 and 91% of the sixth- and seventh-grade students in their school met or exceeded the *Standards* in 2010. Meg's and Samantha's students were not tested because they were second graders and ISAT is administered only to students in third grade and higher. Finally, as mentioned earlier, Ann taught third graders in 2011, which prevented me from reporting her students' average ISAT gain because her students' pre-ISAT scores were unavailable. Instead, I have reported her students' average gain in the second year of the program.

Organization of Portraits

I first introduce the teacher by providing information about the school, her academic background, and her classroom environment. I also provide information regarding teachers' knowledge of mathematics. Second, I summarize how teachers reported the changes in their practices, which could help readers better understand the teachers' current instructional practices. I then illustrate their instructional practices in the last year of the program mainly by using data from the lessons I observed. I also draw on interview data in which teachers described their practices and why they used certain practices, as well as their beliefs about teaching and learning mathematics. My intention in drawing on the teacher interview data was to explain teachers' perspectives on the lessons I describe. After that, I analyze the characteristics of these eight teachers' instructional practices by using the Instructional Practice scales as well as additional themes that emerged from the data. Finally, the chapter ends with a summary of key findings.

I summarized the lessons I observed in a way that the reader could get a feeling for the teachers' overall lessons as well as specific aspects of their teaching. When I portrayed these eight teachers' teaching, I only focused on consistent aspects of their teaching across the lessons I observed. As explained in detail in the Methods chapter, I first searched through my field notes to find patterns in their teaching practices across teachers. I followed a general-to-specific approach in that I first analyzed the overall lesson structure and then looked more closely at aspects of the instruction captured by the Instructional Practice scales and each theme. I focused only on the scales and themes that were typical parts of teachers' practices. As mentioned earlier, not all items in each scale stood out noticeably across observed lessons and teachers; hence I focused on the items with strong "evidence" across teachers. However, although I did not use all items from the scales, I still focused on the main aspects captured in each scale. Additionally,

based on the analysis of classroom observation data, several sub-themes were revealed, which highlighted the differences in teachers' practices related to their MKT. I listed the questions that guided my analysis in the Methods Chapter. As a reminder, each theme and what is captured in each theme is briefly summarized here.

The Inquiry-oriented lesson scale captured the extent to which teachers designed a lesson consistent with standard based teaching, including lesson structure (the extent of having a problem-centered structure) and instructional strategies consistent with investigative mathematics. Mathematical sense-making agenda is more about “mathematical quality” of the implemented lesson. It captures the extent to which teachers were able to articulate what mathematical ideas students were expected to learn and students were able to see connections between and make generalizations regarding the ideas. Of the two sub-themes related to mathematical agenda scale, *purpose of the lesson* captured the primary focus of the lesson (teaching procedure vs. teaching meaning and procedure), and *use of lesson time* was the percentage of lesson time allocated for math and math related activities. Worthwhile mathematical tasks captured the extent to which tasks stimulated non-algorithmic thinking and focused on understanding of important and relevant mathematical concepts. A related sub-theme, *textbook use*, was more about how closely teachers followed their curriculum. Student engagement captured students' participation in sharing and explaining their thinking and working productively with their peers. The theme, *teachers' MKT perception* captured, according to teacher interviews, the impact of MKT gains on them and their instruction, such as causing an increase in their self-confidence.

I have maintained the teachers' lesson format to demonstrate how they carried out typical mathematics lessons in their classes. I then chose excerpts that best illustrated how these teachers

typically carried out certain elements of instruction, such as how they handled whole-class discussions and analyzed students' responses. I also inserted teachers' beliefs about teaching and learning mathematics into the relevant parts of their instruction. This was done to account for all the data in some way. Including teachers' perspectives and beliefs also allowed me to depict how teachers' beliefs were related to their teaching practices.

The eight portraits are presented according to each teacher's level of mathematical knowledge, beginning with the teachers with the highest level of mathematical knowledge. The chapter ends with an overall analysis of the portraits.

Stephanie

Background. Stephanie had the highest mathematics scores on the tests over the three years of the study. At the beginning of the study, she had been teaching mathematics for 6 years. Her undergraduate major was elementary education. Like the other teachers, she was a certified teacher. She was teaching seventh-grade mathematics in a middle school that enrolled approximately 225 students each year. She and Jacqueline were working in a different district than the rest of the teachers. Their students were economically more advantaged compared with those in the other teachers' classes. Forty-two percent of the students in Stephanie's school were eligible for free or reduced-price lunches in the 2009–2010 academic year. Eighty-seven percent of the students were White, and 5% were African American.

Stephanie was teaching both general mathematics and pre-algebra. Her classes were small, having 15 students at most. Stephanie did not have a SMART Board in her classroom (although Jacqueline had one in her classroom). She explained to me that she did not have one because she did not request one. Her classroom was generally not rich in resources. Students' desks were aligned in four rows, limiting student interaction during the lesson.

Stephanie began the program with high mathematical knowledge. Her mathematics score on the first test administered before the program was 1.24²⁹, indicating she already had a solid background in mathematics. She was also aware of this. When I asked whether she thought her mathematical knowledge had increased during the program, she responded, “*Well, what I teach, I feel like I had mastery of before. So I don’t feel like I gained content knowledge for that.*”

However, she was able to increase her knowledge of mathematics further. She answered almost all questions correctly on the test administered at the end of the program, giving her a score of 2.

Stephanie’s self-report on changes in her teaching practices. Stephanie reported what instructional changes she made throughout the program. During her undergraduate education, she was trained to teach through a problem-solving approach. However, she stated in her interview that she changed her practices toward more inquiry-based teaching:

I went to school with these basic understandings and learning about the National Council of Teacher Mathematics and the principles and standards taught there. And I always knew that I loved problem solving and that was the best way to challenge students’ minds. But I did not feel as capable of teaching this way. I didn’t feel like I had the tools. And I didn’t feel informed enough about how. And I felt very tied to my curriculum and my textbooks. And now I just feel much freer to expand on those and to use those as a base, but to do so much more than I felt I could before. . . . I think it’s pushed me to ask more of the students, for them to develop their own ways of thinking.

She then elaborated further about how she teaches mathematics now:

Today in class, we were working on changing fractions to decimals. So I got out the 100 grid and I had them show one fifth and then write the decimal equivalent and kind of show it both as a fraction and a decimal and then do the same with the decimals. They had to do the decimal, and then they had to divide it into pieces so that they were all the same. . . . So instead of just showing them, take the top number and divide it by the bottom, actually showing them fraction equivalents that are decimals and fractions that made sense in a 100th type of a term. And that, I explained to them later as we went along, this only worked because all the fractions that I chose were—the denominator was 5 or 10 or 25 or 20 or one fourth. But I didn’t choose any that were three twelfths or four twelfths. I chose something that was divisible by 100....and the kids did it in different ways, the way that they divided it. Some people, when they did one fifth, they took the

²⁹ MKT scores range from -3 to 3, and 0 indicates an average level of MKT knowledge.

100s and they would show one fifth using—they took the 100, they would divide it into five pieces and color in one of the five, and another five pieces and color in one five. And then other people divided the whole thing into five pieces. So either way, it's one fifth, but it's one fifth 20 times or it's—the whole thing's one fifth. So the whole thing overall still showed a fifth of the thing, but it was a different way of looking at the whole thing.

She also explained why she chose these methods:

I like to try to get kids to be challenged as much as possible and to try to understand what's going on mathematically, rather than just learning algorithms straightforward, and to try to get them to think outside the box and to be able to explain and defend their thinking and what they're doing.

Furthermore, she said that she had begun to include activities from outside her textbook.

I definitely was pretty much just using the textbook and went through it chapter by chapter and taught as much as I could. I do still use the textbook daily, or at least often. . . . [Vande Walle] also had a lot of good problems and rich problems to bring into the classroom as well. . . . And now I do try to do a lot more Vande Walle types of things than I did before. . . . I do still use the textbook daily, or at least often, but I definitely pull from more resources and the Illuminations and try to have more activities and games and problems. And sometimes the problems are from the textbook. . . . I'll just have problems and problems and problems.

As shown in the preceding excerpts from the interview, Stephanie mentioned teaching more towards inquiry-based teaching and choosing examples outside of her textbook, and pressing students to make-sense of the mathematics they learned. The following section briefly illustrates Stephanie's teaching of mathematics during the last year of the program.

Observation of Stephanie's teaching. Stephanie planned to teach addition and subtraction with integers during my observations. She had only 14 students in her pre-algebra class. She used chips and began the lesson introducing what the red and yellow chips represented. She first asked her students to present positive, negative numbers, and then 0 using chips. She continued to ask her students to solve basic addition (later subtraction) problems using chips.

In her interview, she explained how she organized the lessons I observed:

I was presenting, “These are positive, these are negatives, and when you combine them, then what do you have?” And recognizing that the yellow and the red cancel each other out—and then having students try to come up with algorithms from that. Instead of telling them, “Same signs, you add; different signs, you subtract,” they had to look at several that were the same and come up with a strategy that worked for all of them to simplify it because we can’t use chips for everything. But they had a model so they could see what it is, and then they had to come up with the rules. . . . As I go along, I try and get them to predict what the next problem I’m going to put on the board is so that they can see the pattern and that I didn’t just randomly choose numbers, that there was a sequence to my numbers so that we can see what’s happening and kind of look at those specifically.

Consistent with what she said in the interview, she did not tell her students the rule, rather waited for her students to find a rule for addition and subtraction with integers. Her students needed more time to conceptualize subtraction—especially how they would represent subtraction with different signs of integers. Hence, she spent two more lessons on subtraction. Even during her instruction, she adjusted her lesson to ensure that her students were ready to move to the next step. Particularly, she saw that students realized the pattern in addition for integers of the same sign, and then she moved to the addition problems with integers of different signs. She also mentioned flexibility in her teaching in her interview:

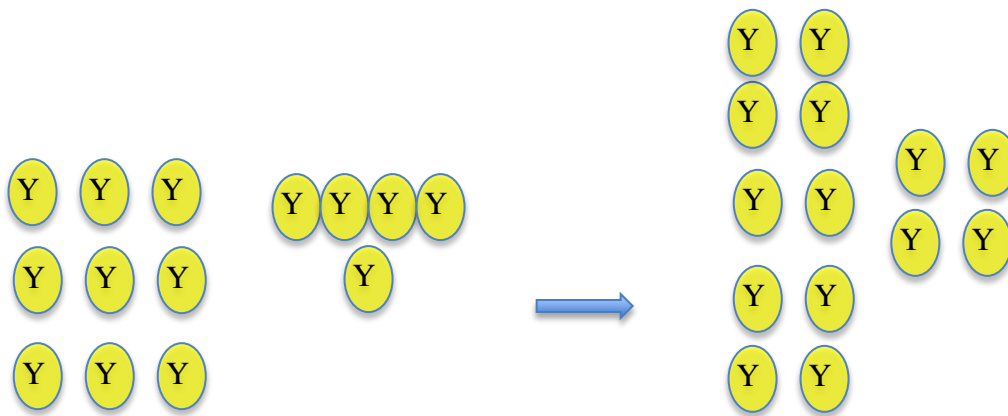
I have to be a little bit fluid in my teaching, and I don’t know what I’m going to do tomorrow. I have a general outline of where I want to go, but I have to sort of listen to what the students have done to be able to be flexible enough to change what I’m doing tomorrow based on what they understood today and what they did today and what they said. So what the students are telling me kind of does guide instruction. I can’t just be rigid in what I’m teaching.

Stephanie encouraged her students to share their thinking and to analyze their peers’ work critically. She explained this aspect of her teaching in the interview as well:

What I have is the doc cam, and so I have them bring it up and we show different examples and we can show what the students have done and try to make sense of what their thinking is and see if other students can understand what they did versus what—if someone’s done it in different ways. So we can look at things from a different perspective. It’s not all just, “This is the way to solve a problem.”

The following excerpt illustrates her typical teaching. From the beginning of the lesson, Stephanie asked her students to solve subtraction problems. In this excerpt, Stephanie asked her

students to solve 5 take away 9 using chips. She first allowed some time for the students to find a solution, and then she picked one student to show her presentation. The student put down two sets of yellow chips, one with 9 yellow chips and the other with 5 yellow chips, and then she combined them. Stephanie asked the class their opinions about her presentation:



Stephanie: Can anybody tell me what she did?

Student 1: She had a group of 5 and a group of 9. And she put them together.

Stephanie: So what did she show?

Student 2: Addition.

Stephanie: That's right. She showed us addition. She started with 9 and then she added 5 to that.

(The student at the board realized her mistake, and she put 4 red chips on the document camera.)

Stephanie: Where do red ones come from though? I see you throwing up 4 chips, but all your yellow chips are still there. Ok, what I think is she knows the answer, which is negative 4, but she does not know how to do it. How many of you know the answer to this problem?

Several students: Not me!

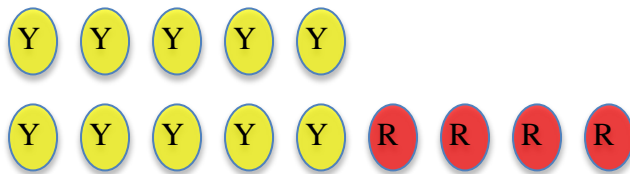
Stephanie: Ok, some of us know the answer, and she's right. The answer to this problem is going to be negative 4. But just like before, you knew the answer to this was positive 4. Can you show how the subtraction worked? What does it mean to subtract more than I have. I just don't have 9 yellows. I only have 5 yellows. How on earth can I take away 9

yellows if only 5 yellows exist. That's what you gotta think about. Put 5 yellows in front of you.

(The student put down 5 yellow chips, and Stephanie continued.)

Stephanie: You have to take away 9 yellow chips. How are you going to take away 9 when you only have 5?

The students were given some more time to figure out the correct representation, and Stephanie picked another student to show how she did it. The student put down one set of 5 yellow chips and another set of 5 yellow and 4 red chips. Stephanie started to analyze her work:



Stephanie: Ok, let's take a look at this. Can you explain to us what you're doing?

Student at the board: I had 5 and then I had 9, but I put 5. That's the number we had, and that's 4 more we need.

Stephanie: I see more than negative 4 here. If your answer is negative 4, shouldn't you end up with negative 4? What I see you doing is starting with 5 and then adding 5. . .

(The student at the board then took out 10 yellows. Stephanie commented on this action.)

Stephanie: You put up 5 and put up another 5 and put up negative 4 and take away 10.

Stephanie mathematically expressed what she did on the board. She wrote,
 $5 + 5 + (-4) - 10 = (-4)$

Stephanie asked the class whether they could also see mathematically what the student did. She also picked one volunteer to show his representation and told the class that she would give them a hint if they still could not figure it out. The student first put down 5 yellow chips and then took off 5 chips and said that he needed to take off 4 more, so he put down 4 red chips.

Stephanie: This is an awesome idea. You guys think what he did. Very, very smart. Not exactly right. Pretty close. That was really close. Let's see! What did he do? Can you walk me through what he did? What did he do?

Student: He had five yellows.

Stephanie: That's right.

Student: He took 'em away.

Stephanie: He took away five?

Student: Hmm, and he added four reds.

Stephanie: And he added four reds. This is Brad's problem.

Stephanie wrote the corresponding mathematical expression on the board,

$$5 - 5 + (-4)$$

Stephanie: 5 take away 5, and plus negative 4. When you're subtracting, what you guys are showing me is that you have this understanding of subtracting 1 is the same as adding a negative 1. Because this is kind of the same idea. Ok, I'm going to give a clue as to how to show positive 5 take away positive 9. In your workspace, can you put 1 yellow? What number is that?

Class: (altogether) 1.

Stephanie then asked them to put down 1 yellow and 1 red chip, and again she asked the value of the number. The class answered, "1." Stephanie continued to ask similar questions, using different representations of the same number. She then continued.

Stephanie: Is there more than one way to show the number 4?
(The students agreed with her.)

Stephanie: I want to think about this question again. You just displayed the number of 4 in a lot of ways. You displayed the other numbers in a lot of ways, correct?

Students: Yes.

Stephanie: I would like to start with the number of 5, and I want you to be able to take away 9. Ok? You remember what 9 take away 5 looks like? You start with 9 yellows and you physically remove 5. I want you to start with the number of 5 and take away 9.

Stephanie gave the students some more time to work on the problem, and then she selected a student to show his work to the class. The student put down a group of 9 yellow chips and another group of 4 red chips. He then took away 9 yellow chips.

Stephanie: You did it! Watch him really close. How many did he move? What is the really tricky part of this problem?

(The students were talking, but most seemed confused. Stephanie explained.)

Stephanie: He took away 5, but he also took away the extras. We have to show the number 5. (She put down 5 yellow chips.) Is that the number 5?



Stephanie: Is that the only way to show the number 5?

Several students: No.

(Stephanie then put down two pairs of chips and asked whether this was still the number 5. Many students agreed it was still the number 5.)

Stephanie: I can keep doing that; I can keep adding 0s. Do you see these are 0s? By adding them, I'm not changing the value at all, but the reason I'm putting those 0s in there in the first place is because I can actually do the subtraction problem. I have to physically have 9 there in order to take it away, and this makes sense to you guys? Yes? Now I can remove them, right? These positives don't really exist because they're part of the 0s, but you can still physically see them. They are right there. So when I take them away, I am left with the negative 4. Do you guys get this? Does that make sense to you?

Stephanie spent two more lessons on subtraction. She continued to solve similar problems, and for each problem, she asked her students to find a corresponding addition problem. The activities continued until her students found the rule for subtraction.

Jacqueline

Background. Jacqueline was a sixth-grade mathematics teacher. She was the only teacher in my focus group whose undergraduate major was junior high/middle school education. In the initial year of the program, she had been teaching mathematics for 4 years.

Jacqueline taught in the same school as Stephanie. She had been teaching all the sixth graders in the school. Her classroom was richer in terms of resources compared with Stephanie's

classroom. She had a SMART Board, which she used daily. Her students' desks were arranged in four columns. However, during lesson time, Jacqueline let them bring their desks together.

Jacqueline was one of the few teachers who began the program with a high level of mathematical knowledge. Nevertheless, she was able to increase her mathematical knowledge considerably (.58 SD). Because she was not teaching in the partner district, I did not have the opportunity to observe her mathematics lessons until the last year of the program, 2010.

Jacqueline's self-report on changes in her teaching practices. Jacqueline also described the changes she made in her teaching. Similar to Stephanie, Jacqueline was trained to teach through a problem-solving approach.

In my undergrad program, I had a lot of the same, like problem-solving. . . . And so I became even more so driven to teach that way. . . . So it kind of helped me to really tweak and refine the lessons better . . . a little bit of tell, but a lot more discovery. It really forced me to think of a real-life and a concrete concept for almost every topic I taught. . . . I use examples now instead of just like, "Here's a random topic that we're going to learn about." I might use examples like, "If I have this many Oreos and this many milks and need to share them among you guys or whatever, how could I do that and how many different ways?" . . . I learned by doing and by discovering this for myself. . . . It was more like, okay, being told students learn better this way versus now I know because I've taught this way. . . . It made me want to give them almost a gift of that level of understanding.

Jacqueline also mentioned what materials she used in her math lessons. The following excerpts from the interview illustrate the changes she made in her task choices:

In terms of materials, I don't necessarily prefer the textbook. . . . I like to come up with, like, real-life ideas that I can apply and introduce topics with. . . . I've always used the Fibonacci problem or pattern. But before, I might have given them the pattern and expected them to find out what was happening and find the next number. Now I give them the Fibonacci bunny problem. . . . I'll expect them to come up with the pattern and then look at the pattern to figure out what's happening. So it's just more from the ground up.

Jacqueline listed several modifications she did in her teaching related to the Mathematical Sense-Making Agenda. In particular, she mentioned that she started to ask better questions.

My questioning has changed through the program for sure, and I now know, like, what kinds of questions to ask instead of just—surface ones. . . . And I get them thinking a little bit more deeply with questioning.

She gave specific examples regarding her teaching, which illustrate how she had created an environment in which students could discuss and analyze their ideas and discover new ones:

With fractions, I've used the different examples of how to draw like one third. So, like, I would give them a triangle and a rectangle, and we would discuss the ways of finding a third of a triangle, like the different ways. And so if you draw it and they draw two horizontal lines, is the top little part a third just like the bottom? Are they both thirds? And so that really, I mean, we spent a whole class drawing thirds. And it's, like, great discussion...

Observation of Jacqueline's teaching. Jacqueline started the new topic, Algebra. She chose a Fibonacci Problem to introduce the concept. She taught the same material to multiple classes at different levels. Her classes had a small number of students, less than 20. Jacqueline's SMART Board was always on, and what they were going to do during the lesson was listed on the screen. Jacqueline introduced the class to the new chapter they were going to study.

Jacqueline: We're going to talk about variables like x and y and unknown quantities; we solve equations. . . . I get really excited because of the problem you'll work, the Fibonacci problem. It's famous. It's been around hundreds of years. It's an advanced problem, but I feel really excited because I do it, you do it, in sixth grade every year. It is great way to introduce algebra and the algebra concept.

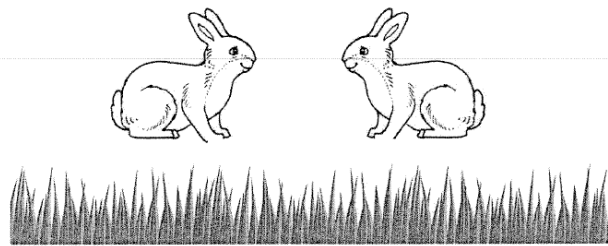
As mentioned in the interview, she changed how she used the Fibonacci Problem to introduce algebra, before Jacqueline had given them the pattern but asked them what the next number was. But after participating in the program, she expected her students to come up with the pattern. In her interview, she explained why she preferred asking them to find the pattern:

Because I feel like the students are more invested if they found the pattern and then to ask how many rabbits would there be in a year, it's like they're in it. It's not just, like, there's no investment. Like "Here's a pattern; find the 12th term." That's totally irrelevant. Who cares? But if it's about rabbits or this guy or even if you made it bees or whatever, they have something to kind of hang it on and it's more meaningful. Plus they feel more pride, I think, when they discover the pattern. They're like, "Oh, look," and they really get it. And there's like camaraderie and even a little bit of a challenge. And there's so many more things that come along with it.

Jacqueline told the class that they were going to work on the Fibonacci problem for two lessons. The question shown in Figure 19 was on the SMART Board. Jacqueline highlighted some of the assumptions in the questions that might not happen in real life, such as having one boy and one girl every time. Jacqueline first asked them to brainstorm how they were going to record their results. The students would work in groups of two or three. The first task was to figure out the answer to the second month. The students were working very seriously and asking questions to clarify their understanding. Jacqueline then asked each group to share their system of tracking the bunnies. Some students went up to the third month, and some made mistakes. Instead of Jacqueline telling them that they were wrong, she waited for their classmates to comment on it. After each group presented their system of recording their solutions, Jacqueline showed the class other students' methods in different classrooms, and she introduced another way of representations of the solution. She then showed how they could use chips to represent their results.

Fibonacci's Bunny Problem

Suppose a newly-born pair of rabbits, (one male and one female) are put in a field. Assume the pair of rabbits will mate so that by the end of the pair's second month the female rabbit will produce a new pair of baby rabbits. Suppose that our rabbits **never die** and that the female **always** produces one new pair (including one male, one female) **every month** from the second month on.



By the end of the first month, there is ONE pair of rabbits. How many TOTAL rabbits OR pairs of rabbits will there be at the end of the SECOND month?

Figure 19. The Fibonacci problem.

The new task was to find out how many bunnies there would be at the end of the fourth month. Students worked in their groups about 10 minutes and then shared their work with their

classmates. After group presentations, Jacqueline showed the class the solution using chips. For each task, each group presented their solutions. Each student had a chance to present their group's results, and each student had a right to challenge others' ideas. The first lesson ended after each group presented their solutions for the number of bunny pairs at the end of the fourth month. The following excerpt illustrates her typical way of prompting all of her students to share their ideas. As mentioned earlier, each group presented their solutions for each question. One group said that there were 10 bunny pairs in Month 4, but one student in this group said that he disagreed with them and said that the response should be 16 for the fourth month. Jacqueline responded.

Jacqueline: Do you want to show us what you have?

(The student presented his answer.)

Jacqueline: More than 10, what do you guys think?

Class: No.

Jacqueline: Why do you think yours is coming up with something a little different? Do you see anything in his work that we could guide him on? Comments, questions?

(Some students shared their reasoning about why he got different results.)

Jacqueline: So they're questioning your number of adults. That might be somewhere to look. I'm not saying you're right or wrong. We're just sharing our thoughts.

In the following lesson, students were supposed to find the number of bunny pairs at the end of the sixth month. After students presented their solutions for the sixth month, Jacqueline asked them to record the number of bunny pairs for each month. She made a chart of months and number of pairs. Then she asked them to find the number of bunny pairs at the end of first year. Jacqueline did not allow them to use chips for this task. She encouraged them to focus on the data chart.

Jacqueline: Now we're going to take it to the next level, and the real problem, the real Fibonacci problem, asks this: "How many bunny rabbits are there after a year?" We've done 6 months' work to get a year. Can I just double this and say there is going to be 26?

Class: No.

Jacqueline: Are you sure?

Several students: Yes!

Jacqueline: Is that happening in this pattern? We were just doubling?

(Some of the students agreed that the pattern was doubling, but others disagreed.)

Jacqueline: Some of us said, "Maybe"; some of us said, "No." We'll have to find out. You can't just assume right? . . . I want to encourage you, this is algebra coming out, ok. It's where we're taking a situation like this, and we look at what's happening. We try to see, ok, can I make a conclusion? Is there something that happened each time that maybe it could help me? So I want to point your attention to the numbers because you guys see what's happening each new month? . . . Looking at your picture, looking at your chips, just pay real close attention... Here's what I would like from you: You're going to have an "aha" moment at some point. I live for that. I live for you guys having that moment and clarity when you look, "Oooh I get it." There's nothing more exciting to me here. When you have that moment, I would like you to keep it to yourself and don't share it with anybody until tomorrow. Can you promise me?

Her students spent one lesson together attempting to find the pattern. Their homework was also to find the pattern. In the following lesson, "Did you find Fibonacci's pattern?" was on the screen. Jacqueline introduced what they were going to do in that lesson:

Jacqueline: You were supposed to go all the way to Month 12. That was your assignment yesterday, and I really encouraged you to look at the numbers. Because that was the idea behind algebra. Because you look at a real scenario, a real-life situation like bunnies or cells reproducing or whatever—lots of situations—and you watch what is happening until you realize, "Ooh," and then you can always say now anywhere along this pattern, you can figure it out. You don't have to have bunnies in front of you to figure out Month 24. That's what it's all about, coming up with a general idea of these patterns.

Some of the students had not yet been able to find the pattern, and Jacqueline gave them extra time to work with their group-mates to find it, but she did not let them use chips. Jacqueline then asked how many pairs were in each month and they filled out the rest of the chart up to the

12th month. Jacqueline asked them whether they could see a pattern. After students found the pattern, she explained why the pattern works.

Jacqueline shared in her interview why she did not explain a pattern until the students found it:

It's usually general, though, like I'll think about 'Okay, don't tell them this.' Like, 'Hold onto that; let them struggle.' That was a big thing as a teacher that I didn't used to like to do. It was uncomfortable, but I let them struggle because there's value in that.

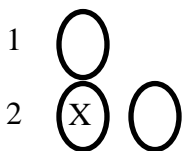
When Jacqueline introduced the problem, she told them they were going to work on it for 2 days, but it took three lessons for them to arrive at the answer. In her interview, she commented, "I always misjudge the time it's going to take, always. I always underestimate how much time it's going to take."

After her students found the pattern, Jacqueline made a change in the problem:

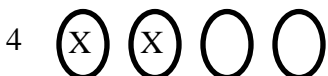
Jacqueline: What if bunnies could have babies in a month? Do you think that you will get the same pattern?

Several students: No.

Jacqueline asked the class to find how many pairs in the second month and drew the figure below:



After drawing the first 2 months, Jacqueline asked the class to predict how many bunnies would be in the third month:



She continued to ask the same question till her students began to guess easily. The class had found that pattern after the fifth month. Jacqueline highlighted the fact that they had only

changed from 2 months to 1 month and the pattern changed. She also connected the lesson with science and cell reproduction. After that, she turned on the SMART Board to show the class the PowerPoint presentation about the history of the problem and Fibonacci himself. She gave many examples from nature and daily life to exemplify Fibonacci numbers. Jacqueline listed several flower petal examples. She also gave examples from leaves and branching plants. She then asked them to find examples of Fibonacci numbers. In the next two lessons, they worked on different patterns, including Pascal's triangle. She used the same pattern but starting with a different number.

Valerie

Background. Valerie, a Montessori teacher, taught upper elementary students who were in grades 4 to 6. She was a certified and experienced teacher who had been teaching for 14 years. Like many other teachers in this study, she had also majored in elementary education.

Valerie had a very large classroom. Students had their own desks, but during the lesson time, they sat on the floor around the teacher. She usually taught a small number of students, up to 10 even though she had around 30 students in her classroom. While she was teaching a small group of students, the rest worked on other things quietly. Her classroom was rich in manipulatives, with many different types present. She also had a document camera and a SMART Board. Her school enrolled 370 students in 2010, and 47% of the students were eligible for free or reduced-price lunches. Compared with other teachers in the same district, the students in her school were relatively more affluent. Sixty-two percent of the students were White, and 25% of them were African American.

Valerie increased her mathematical knowledge by more than 1 standard deviation. When she began the program, her mathematical knowledge score was negative .40 logits and her

mathematical knowledge after the program was .85 logits, indicating that she had a rather high level of mathematical knowledge at the end of the program.

Valerie's self-report on changes in her teaching practices. Valerie explained how she taught before and after the program:

I [was] just teaching duh, duh, duh, duh. . . . [Now] I start showing the reason behind it. [It was like] I'll show you the shortcut to be able to do it. And then it's missing that deep why it is the way it is that is going to be important as they move up into, like, high school math for example . . . I . . . start[ed] showing the reason behind it. . . . Many of the things in Montessori are often computational. And so adding some problems like the locker problem, problems to where they really have to do deep thinking, . . . so I have to figure ways to incorporate more of those kinds of questions in. . . . And I was, like, okay, how can I sit there and get that concept a little bit deeper and add to what I'm already doing with my kids. So I go back to that textbook [Vande Walle] a lot and bring those things back. . . . And just trying some of the different strategies that we learned. You know, just trying little things here and there and just incorporating things in.

She also described her teaching approach at the end of the program:

So, like, when I teach the lesson I teach today, I'm going to show them with materials. You know, I have to come up with the same denominator. But I can't add; I'm, like, "There is a rule out there that says I can't add—you don't add or subtract fractions without the same denominator. So how am I going to get the same denominator?" And they may come up with an idea like 6 and 3—they may come up with 12. Well, I can do that. I have materials that will let me do that. It's, like, okay, yeah, maybe so. . . . And then I can say, "Well, is there anything else that we can come up with?" And then I'll sort of guide them into those kinds of things.

Observation of Valerie's teaching. Valerie usually prepared PowerPoint presentations, including mathematical terms and their definitions. She usually mentioned what questions might appear in the ISAT and what was new in the test related to the topic covered. She used the document camera in each lesson to project her and her students' work. She also used the SMART Board to show students the PowerPoint presentation she had prepared for each lesson. She usually wrote down all the key vocabulary words and their meanings and asked the class to record them in their notebooks. Valerie described in her own words how she taught mathematics, and her description was consistent with my observations:

In Montessori we start out with the concrete. So I'm going to teach them concretely; I'm going to have materials out that are going to show them why things are the way that they are. And then after I do the concrete, then I'm going to move into abstraction. Oftentimes, for example, I do a problem, show them how to work the problem both in practice... And then they work...I do as much small group teaching as I can.

As she mentioned in her interview, after doing one or two examples on the board, she asked the class to answer several questions on their own. She then chose one student to show how the student had found the solution. During that time, Valerie asked the class what they thought about the work of the student at the board. Sometimes students commented, and usually the student at the board responded. If the student at the board did not explain how she or he arrived at the answer, Valerie sometimes asked the student to show how she or he arrived at the solution. She used manipulatives to show them why the rule or procedure made sense.

In the following episode, Valerie planned to teach her students greatest common factor. She started with reviewing several related concepts such as prime and composite numbers. She also asked the meaning of multiples. She used manipulatives to teach the concept, consistent with what she described in the interview.

Valerie: We have been working with multiples. Who can tell me what is a multiple, what does it mean to be a multiple?

Student: For 2 times 2 is 4 and 4 would be multiple of 2.

Valerie: I agree, it's the answer of a multiplication. Factor times factor is a multiple.

Valerie used beads and pegboard to help students visualize the factors of numbers. She put 18 beads on the pegboard.

Valerie: One way to get 18 is I can multiply 1 times 18. So we can say (writing) $1 * 18 = 18$. 1 and 18 what? Factors of . . . ? (waiting)

Student: 18.

Valerie: Right. 1 times 18 equals 18, and we say that a factor times a factor equals a multiple.

Valerie wrote,

a factor * a factor = multiple

Valerie: Do we remember all that? Does 2 go evenly into 18?

Class: Yes.

Valerie: Let's see. We're going to make it into two columns.

After Valerie divided the 18 beads into two rows, she asked the class what number the 2 and 9 were factors of. The whole class gave the answer. Valerie then asked the class, "How about 3?" and she chose one student to divide 18 into 3 rows. They followed the same procedure up to the number 11. They continued to make rows up to number 11. During this review, she asked the student at the board to make another row. The students were sharing their observations during this activity. For instance, when they were making 3 rows of 6, one student pointed out the similarity to 6 rows of 3. Valerie commented, saying, "It is, but it's visually looking different right? We were vertically in geometry wise and now we're horizontally."

After this review, Valerie listed the factors of 18 horizontally. She began with 1 and 18 and put these two numbers farther away from each other and continued to place the smallest factor next to the number 1 and the larger number next to the number 18. The students did the same thing for the number 24. This time, she asked the class what number times what number was equal to 24. She was recording students' responses on the board. Valerie asked in what direction they listed the factors. The class said horizontally. Valerie then told them that they should do it vertically so that they could see the common factors better. The students listed the factors vertically on their own. Valerie asked them what factors these numbers had in common. She then gave the class the definition of the highest common factor (HCF) and asked them which factor was the HCF. Then as she said in the interview, she asked her students to solve

problems on their own. After solving a couple of similar types of problems, Valerie asked whether they knew how to find the HCF. Some students said that they did not think that they could do it. Then Valerie asked the group to find the greatest common factor of 9 and 15. Students worked on the problem on their own while Valerie helped the struggling students figure it out. Valerie asked the class:

Valerie: How can I get 9?

The students listed factors of 9 and Valerie wrote them on the board vertically.

Valerie: How can I get 15?

Valerie again listed the factors her students named vertically.

Valerie: What are my common factors?

She picked one of the struggling students to answer.

Student: 3

Valerie: How do I know that? Because they are in both columns, right? See you can do that. Rather than saying I can't you can say I can ok? Let's try one more. This time no one says I can't do.

When her students said, "I can't do it" or "I don't understand," she would not let them say these kinds of negative comments. She also mentioned this attitude during her interview:

I sort of teach with this cheerleading kind of thing. I want them to sit there like, "You can do it. Believe you can do it." You know? And I think that's another thing—like in math, when they find that they're not successful, maybe that's not their strongest strength. Show them that math could be fun. Math is nothing but patterns . . . like, all of this cool stuff..

The lesson continued with her students' solving similar problems. After each problem, one student shared how she/he arrived at the solution and students analyzed the student's work. In subsequent lessons, she taught factor trees.

Rebecca

Background. Rebecca was a second-grade mathematics teacher. She had been teaching elementary school mathematics for 11 years. She held teacher certification and had an elementary education major during her undergraduate degree. Rebecca had 20 students in her class. Desks were placed in groups of four to increase student interaction. Her classroom was rich in resources. She had a document camera and a SMART Board, which she used on a daily basis. Her school enrolled more than 350 students in 2010, 76% of whom were eligible for free or reduced-price lunches. Fifty-six percent of the students were White, and 28% were African American.

She increased her mathematical knowledge noticeably over the duration of the program. She began with limited mathematical knowledge and ended the program with strong mathematical knowledge. The following section outlines her report of the changes in her instructional practices, followed by a brief illustration of her teaching at the end of program.

Rebecca's self-report on changes in her teaching practices.

Rebecca like Valerie increased her LMT scores more than 1-standard deviation unit over the duration of the program. She explained the changes in her practices.

I lectured before. I did step by step. If a student didn't get it, then I would pull them to the side and continue to show them one method over and over. . . . When we did story problems, I taught them how to use cheating words, you know, how much in all would mean adding and how much less or how much more and—I would have them circle the words, and “When you see these words, you know that you're probably going to do this,” and just giving them those kind of tools so that they never even really thought about what the question was asking. . . . I didn't really put a lot of validity in their ability to problem solve in second grade. . . . Even though I knew that there was more than one way to come up with a math answer, I thought that . . . if I showed them this way and then I showed them another way . . . or if someone—if even one of the other students came up with it another way, that that would confuse them.

[Now] teaching, allowing the kids to share their ideas, even if it's not the traditional way that they come up with an answer, I've done that a lot. . . . You know, you would think that a child would feel really defeated if someone says, “I disagree with your answer.”

But you've been in here. They do that. They say, you know, "I disagree with so and so. I think this." I can remember, you know, the kids coming up here and saying I agree with that answer. I never ever, in any of my classes before, didn't let them agree or disagree. . . . But to actually—to allow them to think through the problems . . . because now I see how they're connected and I see how the—it's not just an algorithm; I know then more of the meaning behind it, so then I can do that.

She also illustrated how she taught before and now:

Before, I taught lesson by lesson. What I mean is, like, if it was 13-2, the next day I taught 13-3, and the next day I taught 13-4. If they learned it, great. If they didn't, I did the next lesson. . . . I would just teach that lesson, and I didn't have them connected. So if they had never done something before, then—but now I see how they're connected and I see how the—it's not just an algorithm; I know . . . more of the meaning behind it. . . . So now if I'm doing something like fractions or measurement and we do it, and I give them a problem and they don't come up with the correct answer that day, or if they're totally completely lost, we do it again the next day, and we talk about it and we do it again the next day. . . . Now when we do addition, the kids know how to use the number chart going down and over. They know how to do—like, some of the kids will add the 100s and then they'll add the 10s and then they'll add the ones, and they'll put it together—different methods that they have come up with, that sometimes we make posters of their methods and we'll place them around the room. They share it.

Rebecca's current teaching: Rebecca's two different mathematics lessons. Rebecca

created two different mathematics lesson times for her students. One was to teach the regular mathematics curriculum, and the other was to improve her students' problem-solving abilities.

She felt considerable pressure to cover the curriculum, but at the same time, she wanted to improve her students' ability to solve problems:

Our school is very mobile . . . almost half of my class does not finish from the beginning to the end of the class, the end of the year. But they usually stay within [the same district]. Therefore, we all have to teach the same lesson every single day so that if they move to another school . . . they make sure that they learn that material. And you're only given, . . . 1 week to teach it. . . . So what I've been doing, I try to have math class where they're doing problem solving and they're working in groups and they're trying to figure it out. . . . And then other times, then in the afternoon, then whatever I am required by the school district to teach.

She also explained why she felt the need to create another lesson time for mathematics:

I notice whenever we have this, what's called a ThinkLink test for our district, . . . I notice that . . . they could only answer questions that they had been taught to answer a certain way. Any problem outside of knowing how to do or knowing key words or knowing how

to do—following directions, they had no idea how to even set it up. And so I started doing this.

I observed her teaching her regular mathematics lessons as well as her problem-solving lessons. This section illustrates Rebecca teaching in her problem-solving lessons because she preferred my observing her in her problem-solving lessons rather than in her regular lessons. Beyond that, her teaching in her regular and problem-solving lessons was similar to some extent. In her regular mathematics lessons, Rebecca also gave her students time to figure things out on their own, but not as much time as they were given during the problem-solving lessons. She paid attention to students' responses and provided counterexamples or challenged them if necessary. The main difference was that during her regular mathematics lessons, she followed her textbook. She sometimes showed some parts of the animated lessons in the EnVisionMATH curriculum. She commented on her teaching in these lessons:

[In my regular lesson,] I do start off with some kind of problem solving that they have that's related to that. Like if we—like right now, we're on measuring, so I'll give them some like uh, like we did a paper clip and a cube today, and they were measuring different things in the classes. So I give them freedom of what to measure and how to compare it and how to show it. But it's more geared towards the same material, whereas when I do the problem solving, it can be one thing one day and another thing the next day, so they're not actually learning repeatedly about just multiplication. . . . They might be learning one day about, when I'm doing the other, they do measuring and then they do measuring and they do measuring and they're doing measuring so that they're kind of mastering it more than sporadically.

Observation of Rebecca's teaching. Rebecca began each lesson reminding the students of the rules for group work. She then posed questions on the screen. For each lesson I observed, she had two or three questions, but the students were usually able to answer only one question in one or two lesson times. The questions assigned to the students during my classroom visits are listed below. Rebecca explained how she chose the problems if they were outside of her textbook:

You should do things that are applicable to their lives as second graders and things that they like. And I do sometimes—honestly, . . . I just give them ..problems where it takes more than one step, I think, are really good problems . . . because then they're learning more than one kind of math method.

Problems Rebecca Used in Her Problem-Solving Lessons

1. There are 326 injured animals that need to be rescued. If there are four rescue teams, how many will each team rescue.
2. If each team rescues an animal every 20 minutes, how long will it take to rescue all of them?
3. In the month of November, there were 7 candles sold every day. How many candles were sold for November?
4. If each candle sold gave the school 25 cents, how much money did the school receive from the candle sale?
5. If you sold pencils for your class trip and you made 30 cents every 20 minutes for 8 hours, how much money would you make?
6. If you had to give 10 cents of every 30 cents to the school, how much would the school receive? How much would you get to keep?
7. If you had a box of 12 pencils for every student in your room, how many pencils would you have?

In the following excerpt, Rebecca projected the first questions on the screen. As usual, she put out several manipulatives. Specifically, for the first two questions, she provided place-value blocks, cubes, clocks, and paper strips for them. The students were working in groups of three or four she had assigned.

For Question 1, Rebecca chose the number 326, which is not divisible by 4. She also did not mention that an equal number of animals should be rescued by each team. Partly due to the ambiguity in the first question, at first some groups were not dividing the animals into equal groups. For instance, one group told her that three groups rescued 100 animals and one group rescued 26 injured animals. Students also had a hard time finding a way to represent 326. Several

students made mistakes while counting. For instance, many students did not know how to count correctly to a number higher than 100.

While visiting each group, Rebecca saw that the students were having a hard time solving the problem. During group work time, Rebecca asked similar questions in each group visit: “If there were 100 sick animals, only 100, and there were 2 teams, how many would each team get?” After the students answered, Rebecca changed the question by saying, “If there were 4 groups.” First, the student said 50. Rebecca then challenged him by saying, “50 plus 50 plus 50 plus was more than 100.” The student then said, “25.” Rebecca began to explain: “25, four quarters, equals a dollar, remember. If there was 100 animals and 4 teams, each team would get 25. Now, make it bigger. Now you know, if there was 100, each team gets 25. If you have another 100, then each team will get another 25, 25, 25, and 25. If you have another 100, because there is 300, each team will get . . .” The student interrupted the teacher and said, “75!” Rebecca continued, “And then you take 26.”

Rebecca did not stop the group work until they had arrived a solution or came close to arriving at one. The students did not find a solution during the first lesson. They continued working on the first problem in the second lesson as well. As she said in the interview, she waited for her students to come up with a solution:

You have to give them the freedom to try to come up with it and be willing to wait. . . . I think that's the really big thing. . . . I really think that if you have a teacher who is pouring knowledge into a child's head and you have a teacher that's teaching problem solving, the teacher who is teaching problem-solving's kids will be better in math in the long run . . . and I think when you're pouring knowledge in, you see the results much faster. . . . So when you feel like you're up against the clock, you have to be willing to trust the method, even though it's not paying off at the beginning. . . . I really don't think that my class had it until the end of this year.

Group work was followed by a classroom presentation. Rebecca picked one volunteer group to present. Each group had a chance to present their work, regardless of whether they

found a solution. Rebecca tried to understand what they did and why they did it, and she asked questions during their presentation. She also encouraged her students to ask questions. The following excerpts illustrate her attitude during the presentations. On the second day of a 45-min lesson, the students shared their solutions to Question 1.

Rebecca: You have to explain how it worked.

Student 1: We started by fourths. We counted by fours.

Rebecca: Show me how you counted by fours. Then what did you do? Why did you count by fours?

Student 1: So we'd know, . . . because there are four teams.

Rebecca: Because there's four teams, so you counted by four. Then counting by four, what did you do with it? What's that 25? Can you explain to me and the class what you're doing?

After the group explained what they did, Rebecca turned to the class and asked whether they had any questions or comments. The lesson continued with other groups' presentations.

Sonya

Background. Sonya was an elementary school teacher who was teaching sixth-grade students. She had been teaching for 10 years. She was also a certified teacher with an elementary education major.

She and Beth worked in the same school. Their school enrolled 264 students, 85% of whom were eligible for reduced-price or free lunches in 2010. As indicated by this number, the majority of students in the school were from economically disadvantaged families. Forty-eight percent of the students were African American, and 49% were White. Her classroom walls were full of posters. She had a SMART Board and a document camera, which she used actively on a

daily basis. She sometimes used flip cameras. Desks were arranged in groups of 5 so that the students could work collaboratively.

Sonya's mathematical knowledge increased significantly over the duration of the program. She began with limited mathematical knowledge (-.60 logits) but increased her score to .20 logits. The next section illustrates Sonya's perspective on the changes over the course of the program.

Sonya's self-report on changes in her teaching practices. Sonya reported that even when she began the program, she had been teaching math using similar teaching strategies emphasized by the NCTM. She reported how she taught before compared with now:

Teachers should be facilitators of the lesson, meaning that they act as guides during the lesson to help students understand concepts through questioning, as opposed to telling the students how to approach a problem or giving them the answers. . . . I had already been trained to use this approach; however, the program encouraged me to continue to teach math in this manner. In that respect, the program has swayed me to continue teaching with the goal of students truly understanding the math concepts that are being taught, as opposed to just memorizing rules. . . . It encouraged me to continue teaching with student understanding as the primary focus. . . . [It] helped me differentiate sense making in the math class from telling students how to do math. It reinforced the approach to teaching math that allows students to work together to solve problems to create an environment of understanding math with a focus on learning from one another in math class is an acceptable way to learn math.

Observation of Sonya's teaching. I asked each teacher how she taught mathematics, and Sonya was the only teacher who mentioned her textbook to describe her teaching practices. One reason might be that she strictly followed her school-mandated curriculum, and another reason might be that she was enthusiastic about her curriculum, Connected Math:

When teaching math, Connected Math encourages the method of Launch, Explore, and then Summarize. I launch math lessons by letting students know the lesson goal first. Then I model the lesson expectation or pose questions or situations that students can connect to in the problem. During the Explore part of the lesson, students work with a partner or in groups to solve the problems posed. During this time, I facilitate the lesson by guiding students through questioning. Finally, during the Summarize phase of the lesson, the lesson is wrapped up or ended by students and/or me sharing how they made sense of the lesson goal.

She then further explained why she used those strategies:

Because it allows students to have some time to work through problems using their own thinking. They get to talk with their peers about their own ideas and are able to learn from each other in a variety of ways, as opposed to following the teacher's way of doing math. Students come to see that they have math knowledge and are capable of solving math problems in their own way at their own pace, and it is okay to do so. Students also realize that there are so many different ways to approach the same problem. It also allows me the opportunity to work with smaller groups of students that really need the extra attention and time to build their confidence about a math lesson. . . . As a student of the "old school" way of learning math, I memorized a lot of rules and did not have an understanding of math. I knew how to DO math. I could figure out a problem, but I really did not KNOW what I was doing. It did not make sense. For example, I knew how to add fractions, but I really did not know what it meant to do so. Making connections between fractions, decimals, and percents was nonexistent to me. I never knew they were related. Now that I am teaching through understanding, my students not only know how to DO math, but they understand WHY the math they are doing makes sense.

Sonya's summary of how she taught mathematics lessons was consistent with my observation of her teaching. As she said in the interview, she closely followed her curriculum. All activities and questions were from her textbook. In the following excerpt, I illustrate her teaching. Sonya was teaching factors, primes, and composite numbers. In the previous lesson, they started to make paper rectangles for the numbers up to 30 (see Figure 20). Sonya was doing the Investigation part from their textbook. Using tiles, they were trying to make all possible rectangles. Sonya asked them to write the dimensions of the rectangles. Sonya spent an entire lesson helping her students discover the factors of numbers up to 30. First, the students worked in their groups to finish their work.

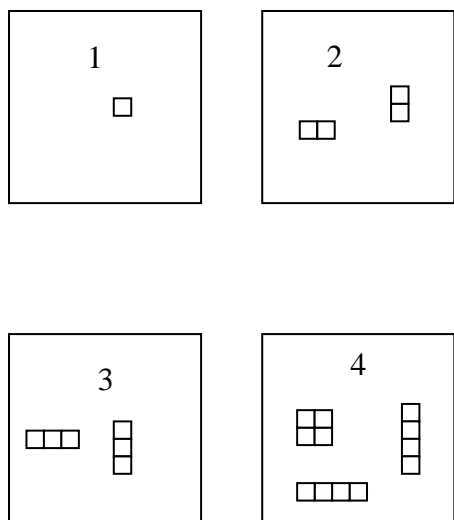
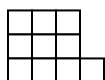


Figure 20. Paper rectangle models for the numbers 1 to 4.

Sonya began the discussion with an observation.

Sonya: One group had this dilemma.

She drew the figure below on the SMART Board:



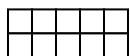
Sonya: What's wrong with this representation?

Student 1: Three is not a factor of 10.

Sonya rephrased what he said and then asked where he got the number 3. Again, Sonya rephrased what he said to the class.

Sonya: So he's saying there's 3 in each row going in this direction except this one. If we have 10 tiles, and I know that I can't make equal groups with these 3 sets of 3. All right, they ended up with a rearrangement, so they can have equal numbers in each row. So that led us into talking about the dimensions of each row. So we ended up counting sides to get our dimensions. So how many tiles are on this side of the rectangle?

She drew the 2×5 rectangle below and asked the dimensions of the rectangle:



Sonya: This is a representation of 2 by 5.

One student said he had an observation, and Sonya asked him to share it with the class. He said when he turned it around, the dimensions were reversed. Sonya rotated the rectangle and he said, “5 times 2.” Sonya asked for another example. Another student said 1 by 10, and Sonya rotated the rectangle, and the class said 10 by 1.

Sonya: What did you notice? We had 10 tiles in all. What do these numbers have in common with 10?

Student 3: Factors.

Sonya: (pushing her question) What do you mean by factors?

Student 3: Factors of 10.

Sonya: What numbers are the factors of 10? Can we list them from the lowest to greatest?

(Altogether, the students listed the factors of 10.)

Sonya: Why do we need to repeat this number over and over again? Because, they’re repeating here when we flip the dimensions. We took it vertically 2 by 5, and then we flipped it horizontally then it gave us 5 by 2. . . . Here’s what I want you to do today: We’re going to look at the rectangular shapes, and for each one, if making rectangular shapes will help you know all the factors of the numbers 1 to 30, I want you to write down the factors of the numbers from 1 to 30, and I want you to look at the rectangular shapes that we have made.

(After the students listed the factors of the numbers up to 30 by using the rectangular shapes they made, the whole-class discussion started with 1.)

Sonya: What are the factors of 1?

Class: One.

Sonya: According to these two shapes, what are the factors of 2? From least to greatest.

Class: One and 2.

Sonya: All right, now, here comes the part [where] we have to put in some thought. So we have made all possible rectangular models for the numbers 1 up to 30. We have looked at the models we made, and we have written all the factors in each of those numbers based on the rectangles we made. So now, in your groups, first you are going to work on $b - 1$ and $b - 2$. In order to work on $b - 1$ and $b - 2$, you have to look at the models we made.

(Sonya projected the book on the screen and read the question aloud.)

Sonya: “1–Which of these models has the most rectangles?” What kinds of numbers are these? They are asking you, are these prime numbers or composite numbers?

“2–Which numbers have the fewest number of factors?”

Sonya asked the students to work in their groups to answer these two questions. As a typical part of her teaching, during the group work time, she visited each group and asked questions. The following whole-class discussion that occurred for the second question illustrates her teaching.

Sonya: Which number has the fewest rectangles?

Student 1: One.

Sonya: I know 1 has the fewest, but give me a few more numbers.

Student 1: 1, 2, 3, 5, 7,

Another student: (interrupting) You can’t!

Sonya: Wait, tell me why you think we can’t? Tell me more.

Student 2: Because 1 has only one rectangle. Then if you try to go, like, 2 or something, you can’t pick 2 because it has 2 rectangles. So does 5, and so does 3, and so does 7. Every couple of odd numbers only have 2 rectangles. If you pick 2, you have to pick all the rest of them that have only 2 rectangles.

Sonya: He says if you pick 2, you have to pick all the rest of them that have only 2 rectangles. So, umm, that’s quite an interesting point that you have made. Let’s leave 2 out, and just let’s leave all of the other numbers that have only 2 rectangles. Then we’ll talk about 2 later.

Sonya asked them to analyze the numbers with only 2 rectangles. She asked them to make a list of all the numbers that have only 2 rectangles except for 2. She then checked whether this list included 3, 5, and so on, up to 29.

Sonya: What kind of numbers are these numbers?

Class: Prime numbers.

Sonya: (reading the definition of prime numbers) “Prime numbers are numbers with exactly 2 factors. One and the number of itself.” Let’s look at 2. Does 2 fit the definition?”

Several students: Yes.

Student 3: (challenging the point) But 2 is even.

Sonya: Chris made a good statement. He said 2 is an even number. It may be true: 2 is an even number. That doesn’t mean that it can’t be a prime number. It’s prime because it has only two factors, 1 and itself; therefore, it’s a prime. Does that make sense?

In that lesson, they also learned square numbers, and Sonya ended the lesson by asking the students what patterns they noticed. She asked, for example, what statement they could make about the numbers that have 2 as a factor, or what they noticed about square numbers and their factors. These questions were also from the textbook.

Beth

Background. Beth was a novice teacher when she began the program. She decided to become a teacher late in her career. She was working in the same school as Sonya. Since becoming a teacher, she had taught fifth-grade mathematics. She had also majored in elementary education and was a certified teacher.

Her classroom was different from Sonya’s in that she had fewer posters decorating the walls. She placed student desks together in groups of 4 or 5, enabling student interaction and collaboration. Although she also had a SMART Board and a document camera, she was uncomfortable using them and was relatively inefficient in using these tools, especially the SMART Board. She used it mainly for showing visual animations. She had 18 students in her mathematics classroom. Her students were different from Sonya’s students in that they were more respectful both to Beth and to their classmates. Classroom management was not a big problem in her mathematics class.

Beth began the program with very limited mathematical knowledge. Even though she increased her mathematics score by .83 logits, her score on the final mathematics test was $-.39$ logits, indicating that she still had relatively limited mathematical knowledge.

Beth's self-report on changes in her teaching practices. Beth compared how she had taught before the program and how she has started to teach now:

Before, . . . I taught each lesson as a whole group and then gave the students several problems to complete and turn in for a grade. . . . Now my teaching is driven by students' interest, problem solving, sharing, and just a few problems to determine that a concept has been mastered or that additional interventions are needed. . . . One of the most significant changes I have made is to teach less as a whole group. [Now] I do less whole group, more hands on, and allow them to work together to solve problems and find and share how they solved a problem. . . . For instance, this week our intent was to learn about graphs and data. The students paired, chose a survey question, conducted a survey, and created a line plot with interpretation to present to the class . . . (.83 gain).

Observation of Beth's teaching. As I had done with the other teachers, I asked Beth how she organized her mathematics lessons. Her description was also quite similar to what I had observed in her teaching:

During most lessons, I do a quick mini-lesson to introduce the math concept. Then the students usually pair or form small groups to complete specific problems that they will later present to the whole class. During other lessons, I might pose a problem before presenting the lesson to encourage the students to work together, to use prior knowledge, and share different ways to solve a problem.

The following episode illustrates how she organized her mathematics lessons. Students had been learning about multiplying whole numbers. In that lesson, Beth was planning to teach how to record repeated multiplication. She first distributed a worksheet and then introduced the lesson.

Beth: Suppose you want to multiply 4 five times. If you want to multiply 4 five times, pretend that your 4 equals y in your table; record your answer in this box. Remember we're talking about multiplication, not addition. What does the first column say?

Student 1: Expanded form.

Beth: Expanded form. How many times have I said? Five times, put it five times: 4 times 4 times 4 times 4. You want to put 4 five times, that's called expanded form. Is that an equation or is that an expression?

Student 2: Expression.

Beth: It's an expression. It doesn't have a what?

Students: Equal sign.

Beth: Yes, how can we write it in a different way?

One student knew how to write in exponential form and shared her thinking with the class, and then Beth wrote "Exponential Notations" and the equation below on the board:

Exponential Notations

$$4*4*4*4*4 = 4^5$$

Beth: Four is the base and 5 is the exponent.

Then Beth asked the class to find the answer. The students worked as a group to find the solution. Beth recorded each group's solution on the board. One group said, "20."

Beth: She said 20. What's 4 times 4? We want it 5 times.

(Other students also shared their answers, which were 64 and 1,024.)

Beth: We keep doing it till we find an answer that we all agree on.

(Beth wrote all the students' answers on the board.)

Beth: How many got 1,024? (The students raised their hands.) Guys, that's the correct answer. Let's take a simpler one. Let's do 2. This is my number 2^3 .

The students in their groups worked on similar types of problems. The following excerpt illustrates how she explained exponential form.

On the board, Beth wrote, " $7*7*7*7*7$."

Beth: This is expanded form. How can I change it to the exponential form? What is my base number?

Students: (altogether) 7.

Beth: Yes. What is my exponent?

Student 1: 5.

Beth: Are we getting a better idea? The big number is called the base and the little number is the exponent. It tells you the base number is how many times.

Although Beth did not mention animated lessons from EnVisionMath as a consistent part of her teaching, in almost every lesson I observed over a 2-year period, she showed the visual lessons from EnVisionMath. After teaching a brief lesson, she turned on the SMART Board, and EnVisionMath was on the screen. The following segment is the typical way she used animated lessons.

After a quick lesson and working on similar problems, Beth began the visual lesson. There was a story problem in the video. Beth paused the video and repeated the questions asked on the video. The answer to the question in the video was “ 5^3 ,” which was one of the questions they had just solved. One of her students told Beth, “We already did this problem!” Beth agreed with her but still continued the video. She repeated the questions from the visual lesson: “Can you use exponential notation to write 5 times 5 times 5, and can you use exponential notation to use 5 times 4 times 2?” She paused the video for students to answer, and then she hit the video again for the response from the book. She usually repeated the book response as well. For example, a mathematical expression like the one below appeared on the video:

$$2^5 \neq 10 \text{ (Why?)}$$

Beth: Listen to this question. She said (referring to the voice on the video) 2 to the fifth power is equal to 32, not 10.

(They then continued to watch the video. Several times, Beth paused the video for her students to answer the questions, and then she resumed the video to hear the book’s response.)

Beth: (What is the difference between standard form and expanded form?” (reading a question from the book)

Student: Standard form is the answer.

Beth: Yes, the standard form is the answer.

(They watched the video.)

Beth: She (referring to the voice on the video) said the standard form is the way numbers are written in everyday situations. 125 is just standard form, ok.

The video then explained, “Expanded form is the number of written products of factors.”

The following question was, “When might we use exponential notation rather than expanded notation?” Beth repeated the same question and picked one student to share his thoughts.

Student: To shorten it out.

Beth: He said to shorten it out. Any more ideas? (Beth then resumed the video.)

After having the students watch the visual lessons from EnVisionMath, Beth picked students to solve similar problems. “Your base is 3 and your exponent is 5. Write it in the expanded form” was a typical question type during this phase of the lesson. While the students on the board were solving the assigned problems, the rest of the class was also working on the problems in their groups. After the students completed their work, she checked their answers. During this period, the students found either the expanded or the exponential form. Beth’s explanation was typically: “You hear ‘exponent,’ that means that your answer should have an exponent in it.” After the students solved problems in their seats, she devoted some time to group work. The students were supposed to answer similar types of problems with their group mates. During this phase of the instruction, she also visited the groups. She solved similar questions with the students having difficulty completing them on their own. Her explanations during group work focused on the procedure, showing students what steps they needed to follow to find expanded or exponential forms.

Ann

Background. Ann was the one of the few teachers whose mathematical knowledge did not noticeably increase during the program. She had been teaching mathematics for 8 years, and she had begun to teach third-grade mathematics in the 2010–2011 academic year because of her school’s rotation policy. Previously, she had taught fifth-grade mathematics. Like the other teachers in my group, Ann also held a teaching certificate and had majored in elementary education.

Ann had 45 students across the two classes. In 2010, she was teaching two third-grade mathematics classes. Her classroom was rich in resources: She had a SMART Board and a document camera. Posters dotted each wall, and most served instructional purposes. The school in which she worked had economically disadvantaged students. In 2010, 73% of the students in her school were eligible for free or reduced-price lunches. Sixty-six percent of the students were African American, and 26% were White.

Ann’s initial mathematics score was $-.68$ logits, indicating that she had limited mathematical knowledge, and her final score on the test was $-.43$. Although her mathematical knowledge did not change drastically, some of her practices did change. In the initial year of the program, Ann’s classroom had been designed to minimize student interaction. The students sat in individual chairs, and Ann had not been tolerant of any noise during her instruction. However, after she enrolled in the program, student desks were placed in groups of four, and students were allowed to work with other students. The next section outlines her report on the changes in her teaching.

Ann’s self-report on changes in her teaching practices. Unlike other teachers, Ann did not noticeably increase her math knowledge over the duration of the program. When Ann

was asked how her instructional practices had changed over time, she described shifts in her teaching as follows:

I've taken myself out of it [the lesson] more. . . . I'm trying to be more of a facilitator to the learning, not the lecturer. . . . I've kind of listened to them more to kind of guide the lesson and my instruction. . . . If I notice, we're not getting this, well then, we're going to go this direction. . . . Before it was like, okay, we're going to do this and this is what we're doing today and this is what I'm doing this year. And they didn't really have a say.

[Now]I've realized I don't have to have all the answers and I shouldn't have all the answers. So, "Oh, that's, very interesting," "let's go with that. I'd like to know the answer to that, too." And kind of let them kind of—then we go that way. I can explore it with them . . . because before, I thought that I had to know everything and I had to do it all. . . . I've found that it's actually kind of nice to just walk around and listen to them and, do a little more questioning.

She further illustrated her teaching, which will help the reader visualize her instructional practices better:

Like right now we're doing division. . . . They've learned all these different strategies. Now they pick; whichever one they want to use is the one that they're going to use, the one that makes it easier for them. . . . [In] today's lesson . . . we started off with writing the division sentence. Somebody shared that. And then, okay, did anybody do it any different? I drew a picture. So they come up and they show the picture they drew. And then did—you know, I used multiplication. Somebody said they used adding, you know, which—whatever. But they—I mean, it made sense to them. They were just doing it a little backwards. And then they—you know, I used subtraction. Well, come up and show us. So they all—you know, if they all would have drawn a picture, that was fine, too. But that works for them. So they share and show us.

Observation of Ann's teaching. Consistent with my observation, Ann described how she organized her mathematics lessons:

I definitely think it should not be me at the board lecturing, saying, "Okay, this is the algorithm. This is how you solve the problem." . . . So I think it should definitely be fun. . . . You have to find a way to hook them into the learning. So sometimes it's a book, it might be a song. I always start the lesson, or most days, with an interactive learning activity. So we pose a question on the board. Sometimes I might have them work on it by themselves and then we talk about it. But a lot of times, they are talking about it in their group. . . . I just want to see—it's kind of, they have the floor. . . . And then I pull sticks and . . . they get to come up and show me on the board what were their ideas. Did anybody do it differently? And, . . . we get several different ways of solving the same problem that way. So I start off with that. Then I do use the computer technology to kind of guide a lesson. But I would say in the hour of math, it's probably maybe 10, 15

minutes tops that they're just listening to me, if even that. . . . So then after, you know, we've kind of given them the lesson, we do sometimes a little guided practice and practice problems together. And then I let them do their partner activity.

Ann summarized how she organized a mathematics lesson, and below I illustrate how she proceeded with it. Ann had been teaching subtraction, and in earlier lessons, her students had used place-value blocks. In this lesson, she wanted her students to know subtraction without using manipulatives or place-value drawings. The EnVisionMath lesson was on the screen. She started the lesson with a review. She wrote down the question below on the board:

$$\begin{array}{r} 60 \\ -32 \\ \hline \end{array}$$

Ann then asked her students what she needed to do to solve the problem.

Ann: 60 minus 32. What's the first thing I need to do?

Student 1: Draw.

Ann: How many 10s do I have here? How would I draw six 10s?

Student 1: Six strips.

Ann: Do we have any ones? (She waited several seconds.) No, no ones.

Ann: Can we take 2 ones away over here? There is no ones. What might we need to do? What is our big R word?

Student 2: Regrouping.

Ann then drew



Ann: So how much should we have? Let's count and see.

(They all counted, "10, 20, 30, 40, 50, 51, 52, . . . , 59, 60.")

Ann: So we still have 60, but we just draw it differently. Our subtraction problem tells us we need to take away how much?

(Nobody responded, and Ann asked the question again.)

Ann: What are we subtracting?

(Ann picked a student, but the student could not respond. She continued.)

Ann: So we're going to subtract 32. How many 10s in 32?

Student 3: Three.

(Ann then crossed out three 10s.)

Ann: How many ones do we need to take away?

(Some students said, "Two," and she crossed out two ones. She asked the class how many ones and 10s left.)

After reviewing subtraction with two-digit numbers, she told the class they were going to subtract two-digit numbers by using paper and pencil. Ann prepared a story problem. First, she read the problem, and then the students worked in groups to solve it. While she was reading the problem, she explained the words *refuge* and *crane* to ensure that the students understood the question. The question was, "A bird refuge had 35 cranes. It released 19 of the cranes back into the wild. How many cranes are left at the refuge?" Ann asked them to talk with their partners and she gave them 30 to 40 seconds.

Ann: What operation is needed to solve this problem?

Student 1: Subtraction.

Ann: Subtraction, ok. Does everybody agree that subtraction is what we need?

(Many students said, "Yes.")

Ann: Was there any key word that helped you realize it was subtraction?

Student 2: Crane.

Ann: “Crane” told you it was subtraction. A type of bird? Let me read the question again. (She read the question one more time.) What key word helped you figure out it was subtraction?

Student 4: Release. (hesitantly)

Ann: Possibly.

Student 5: How many birds are left?

Ann: How many are left? How many are left?

Ann picked one student to solve the problem. The student at the board drew the three strips and five dots below:



Ann: Explain what you did.

Student: I drew 3 tallies and 5 dots.

(The student then crossed out one of the tallies.)

Ann: What do you need to do next? (waited a second and then) Put 10 dots.

Ann: How many 10s are there in 19? How many ones in 19?

The student responded to her, and then Ann asked her how many ones and how many 10s were left. After the student at the board solved the problem, Ann asked the class, “Did anybody solve it differently?” One said he did it in his head. Another one said he did it differently, and Ann asked him to show how he did it to the class. He solved the problem as shown below:

$$\begin{array}{r} 21 \\ \cancel{35} 4 \\ -19 \\ \hline \end{array}$$

Ann: Why did this 5 have to become 4? Are you sure about that?

(He then erased 4 and made it 5 and Ann described the procedure to the class.)

Ann: I have 35, right? Tom said 9 is larger than 5. We cannot take 9 from 5; we can't do that. What he did is, he's going next door; he's borrowing 10. Ok? He has three 10s but he's borrowing one 10 and he made it two 10s. What we do with the place-value model is instead of making one 10, when we borrow it, we're regrouping it and we're drawing 10 more ones or 10 ones. It is still the same number, but we're distributing differently. Ok? He has two 10s and 10 ones just like here (pointing to the place-value drawing), so now if you have 15 ones, you take away nine—1, 2, 3, 4, 5, 6, 7, 8, 9—how many are left Tom?

Student: Six.

Ann: Yeah, 6 are left. Then you have two 10s and you're going to take away one 1. How many is left? So it's 16. Basically, it's a different way of doing the problem using paper and pencils.

Ann also followed the textbook closely. However, she also found different activities outside the EnVisionMath textbook. Ann preferred visual tasks so that she could use the SMART Board. She explained how she chose activities in the interview:

I really work hard at trying to make it fun, make it where they understand it, just bringing in all those components like I talked about, using technology anytime I can. Because that's the key here in this, with these kids. They have to have fun. And they have to be engaged and under control.

Consistent with how she described her teaching in the interview, after solving a story problem on subtraction together, Ann asked her class to open their textbooks. While the students were opening their textbooks, she turned on the animated lesson, which focused on the question, "How can you use subtraction?" As a whole class, they went over the questions in the visual lesson. Ann asked which key words in the questions told them they needed to do subtraction. Ann mentioned several key words, such as *remainder*, *difference*, *less than*, *fewer*, *how many more*, and *minus*. She paused the video several times and picked one student to tell what operation they needed to do, and the student identified the key word. Ann then picked another student to solve the problem in the visual lesson on the board. The Guided Practice from EnVisionMath then appeared on the SMART board. The questions in the Guided Practice were

practice problems. Ann pulled out a stick on which a student's name was written. Students went to the SMART Board to solve the problems similar to the one below.

Subtract

$$\begin{array}{r} 35 \\ - 19 \\ \hline \end{array}$$

30

Student participation was high during the SMART Board activities. However, her students could easily veer off task, and they had some behavioral problems as well. In each lesson, Ann spent time getting them back on track. Sometimes she would pause the lesson and turn off the lights and warn the class.

After solving several questions like the one above, the students worked in groups to write a corresponding story problem for each numerical expression such as $20 - 13$. She prepared sample questions and read them to the class: "I have \$20. I bought a hat that cost \$13. How much money do I have left?" The students worked on the problems in the following lesson and presented their story problem to the class.

Meg

Background. Meg was a second-grade teacher. As of 2011, she had been teaching mathematics for 10 years. Her undergraduate major was elementary education, and she held a teaching certificate.

The school where she had been working enrolled approximately 300 students. Eighty percent of the students in her school were eligible for free or reduced-price lunches in 2010.

³⁰ As the reader may notice, this question was the same question Ann used at the beginning of the lesson. Both Beth and Ann usually used a question from animated lessons when they started to teach a concept and then when they turned on animated lessons, they solved the same question again.

Forty-eight percent of the students in this school were White, and 40% were African American. She had approximately 25 students in her class. Her classroom was rich in terms of resources. The walls were full of posters, most of which served instructional purposes. She also had a SMART Board and a document camera in her classroom. Three or four students' desks were grouped together, facilitating group work and collaboration, which occurred consistently in her lessons.

She began the program with a very limited knowledge of mathematics (-1.3 logits). Even though she increased her mathematical knowledge significantly, her mathematical knowledge score was still -.68 logits at the end of the program, indicating that she has a limited knowledge of mathematics.

Meg's Self-Report on Changes in Her Teaching Practices. Meg described the changes in her practice in this way:

I'm more cognizant of what I'm saying, and I'm more cognizant of what they're saying to me. I'm trying to think of why they said what they said, whereas before, I kind of sort of [did]. But now I'm even more listening to them, letting them kind of teach each other, working more in groups and talking about what they're doing and how they're thinking, helping them, each other to work through problems and talk about problems and find solutions to the problems. I think I'm more aware of common mistakes that they make. It made me more aware of that and why they made them and helping them to think through the process more to help correct the mistakes. . . .

I needed them to talk through it more, more writing. Before I didn't do a lot of writing in math. Now they do—they put their thoughts on paper so I can track them. Before I started the program, . . . we did sharing, but now we do a lot of sharing. A lot. They discuss and they'll correct each other and talk to each other more. I think my classroom has been more—is more open now, where I allow a lot of conversation to go on. I'll allow them to say, "That's not right. We don't—we disagree with it," and a lot of discussion, whereas before, I wasn't prone to do that.

Observation of Meg's teaching. During my classroom visits, Meg and her students were working on adding money. In each lesson, Meg reminded the students why learning money was important. In addition, in one lesson they watched a video about how money was made.

Meg usually began each lesson with a review of the previous lesson. After a quick review, she introduced the task. During that time, to ensure that the students understood the task, Meg asked her students to paraphrase what they were expected to do for that activity. They solved one or two similar problems together. The students then worked in groups to solve problems. While the students were working together, Meg visited each group and asked some questions about what reasoning they had used to arrive at the solution. The students then shared their solutions and discussed each other's ideas. Finally, her lessons ended by summarizing what they learned in the lesson if any time remained.

Meg also followed the EnVisionMath Series very closely. Almost all her tasks were from the textbook. She also showed animated lessons from EnVisionMath. The following excerpt illustrates a typical whole-class discussion occurred. Students were supposed to find two different ways to make the amounts of money on the card given to them. Each group was assigned to make different amounts of money. After introducing the task, Meg distributed cards and coins (dimes, nickels, and coins) to each group. While the students were working in their groups, Meg visited each group and asked what they were doing and why. After group work was done, the groups went up to the document camera to present their work. The following excerpt of a group presentation gives a glimpse of Meg's approach.

The first group to present had been instructed to find two ways to make 71 cents. First, they put down 7 dimes and 1 penny. Meg asked them to show the class how 7 dimes and 1 penny was equal to 71 cents. They counted it up. As usual, Meg asked the class whether they agreed or disagreed with this response. Meg also encouraged other students to comment on the solution on the document camera. When Meg asked the group to present their second way, the group presenter only changed the order of the dimes and the penny.

Meg: What do you have there?

Student 1: One penny and 7 dimes.

Meg: One penny and 7 dimes. Is that another way to make 71 cents?

Student: Yes.

(Meg asked the class whether they agreed.)

Meg: How many dimes in the first place?

Student: Seven dimes and 1 penny.

Meg: How many dimes do you have?

Student: Seven dimes.

Meg: Is this a different way? Thumbs up if you think it's different.

(Meg picked a student who disagreed.)

Student 2: They switched it around.

Meg: What could you have done?

Student 2: Six dimes and 12 pennies.

Meg: Do you agree with him? Thumbs up or down? (to the class) You have a thumb down, what do you think?

The students who either agreed or disagreed shared their thoughts. One student used 6 dimes, 2 nickels, and 1 penny, whereas another student used 6 dimes and 11 pennies to make 71 cents. They all had to count and show that the amount was equal to 71 cents. Several other students tried to make 71 cents as well, but their responses were not complete, and even though Meg pushed them farther to help them think it through, they did not continue to explain their thought processes.

After that, students in their groups solved the problems in their textbook. In the following lesson, students were still working on the problems in the textbook. The following excerpt from

the whole-class discussion illustrates how Meg facilitated the lesson. Meg projected the worksheet “Counting Collections of Coins” on the screen. The students needed to determine which coin was the least and which coin was the greatest, and then find the total amount. Meg asked them to determine which coin had the least value, a penny or a nickel. The student said the penny had greater than nickel. After Meg asked her to justify her response, the student changed her answer to a nickel.

Meg: Why 5 cents, though? Why did you change your mind?

Student 1: I could use a number line.

Meg: What did you count on your number line? How did you use the number line to help you?

Student 1: Because, umm,

Meg: How did you use the number line to help you? Our question is...is a nickel greater than a penny? Is 5 cents greater than one cent? Who thinks one cent has the greatest value? Who thinks one cent is bigger than 5 cents? I need everyone voting. Thumps up or down.

(Meg picked one student to explain his response.)

Student: Five cents is bigger than 1 cent.

Meg: Why? How can you prove it to me 5 cents is bigger than one cent, just out of curiosity? Is this something you just know?

Student 2: I’ve counted by fingers, 1, 2, 3, 4, 5... (inaudible)

Meg: What goes next?

Student 2: Five

Meg: 5 cents, ok. Do we agree with that?

Class: Yes.

In the following lessons, the students continued to work on the problems in the textbook. Meg visited each group to help them. As usual, the students presented their work to the class.

Cross-Case Analysis

The qualitative analysis was conducted to shed light on the quantitative results, to explore how teachers' MKT knowledge affected their instructional practices. The portraits of the eight target teachers depict how the teachers perceived the changes (if any) in their teaching practices, and it gives their reports of their practices at the end of program. It also includes my observations of their teaching at the end of the program. In this section, I present a cross-case analysis of teachers' current practices and the changes in their practices under the themes mentioned in Chapter 3.

As a reminder, I focused only on four scales from the Classroom Observation scales: Inquiry-Oriented Lesson, Mathematical Sense-Making Agenda, Worthwhile Mathematical Tasks, and Student Engagement. The quantitative findings suggested that gains in teachers' MKT are related to changes in their inquiry-oriented lessons, their mathematical sense-making agendas, and the classroom climate.³¹ The quantitative results also indicated no relationship between teachers' MKT and their task choices as well as their students' engagement in mathematics lessons. With the help of qualitative data (interviews and classroom observations), I elaborate on the quantitative findings. As explained in the Methods chapter, analysis of the field notes and interviews suggested four additional subthemes: teachers' MKT perceptions, use of lesson time, purpose of the lesson, and textbook use. Because three of the additional themes were closely related to the scales from the observation protocol, I present the findings under the relevant scales.

³¹ As mentioned in Chapter 3, teachers' did not mention any changes in practices captured in the Classroom Climate scale during the interviews, which limited inclusion of the scale to analyze the effect of MKT gain on instruction. Furthermore, classroom observation data did not suggest a consistent pattern across teachers with different MKT levels. It seems that all teachers tried to respect students' ideas and paced the lesson based on their students' needs. Hence, I decided not to include Classroom Climate in the qualitative data analysis.

I begin this section with the theme “*teachers’ MKT perceptions*,” which emerged from teachers’ self-reports on how the gain in their mathematical knowledge and the mathematics courses in the program affected their practices. By including this theme, I hope to provide a more complete picture of their stories. I continue with patterns that appeared in the four instructional practice scales: the Inquiry-Oriented Lesson scale, the Mathematical Agenda of Sense-Making scale (with two additional themes, the use of lesson time and the purpose of the lesson), the Worthwhile Mathematical Task scale (with an additional subtheme of textbook use), and the Student Engagement scale.

Teachers’ MKT perceptions. To better understand how the gain in teachers’ MKT affected their instruction, I asked the teachers how they viewed this change. Table 22 summarizes the target teachers’ responses.

Table 22

Summary of Teachers' Reports on the Effect of MKT Gain

MKT level	Teacher	Lesson	MKT Gain	The effect of gain
Very high	Stephanie	Subtraction with integers (using chips)	.8	Did not think her MKT increased
Very high	Jacqueline	Fibonacci problem	.6	Saw connections between concepts more
High	Valerie	Factors and highest common factor	1.1	Had an increase in self-confidence
High	Rebecca	Problem solving	1.1	Had an increase in self-confidence; saw connections between concepts more
Average	Sonya	Factors	.9	Had an increase in self-confidence
Low	Beth	Multiplication with whole numbers	.3	Had an increase in self-confidence
Low	Ann	Subtraction with integers	.8	Had an increase in self-confidence
Low	Meg	Addition with money	.7	Did not think her MKT increased; sympathized more with struggling students

Only two teachers thought their mathematical knowledge had not increased. Stephanie did not think her knowledge of mathematics had increased because she thought she already had a firm understanding of what she taught. Given that her score was already high, even on the pretest, this comment regarding no change in her mathematical knowledge was understandable. Meg also did not think her mathematical knowledge had increased. Similarly, given that her last mathematics score was still quite low, it made sense why she thought this. Meg explained how she struggled during the mathematics courses:

Some of the time when I was . . . going over the material, I'm thinking, "You know what? My kids must feel like this when they see things and they're like, 'Huh?'" And I'm more sympathetic to them about . . . when they first see it and they go, "I have no idea what this is . . ." And I'll say, because I can relate, I'll say, "Maybe not right now, but you will. Just hang in there and you will. We'll help each other through the process." So it's made me—that has really, really changed.

The other six teachers thought their mathematical knowledge had changed. Five of them also reported an increase in their confidence level as a result:

Ann: I'm definitely, definitely more comfortable teaching math. . . . It's given me more confidence.

Beth: I feel more confident when teaching fractions or better equipped to give the students a variety of hands-on lessons and small-group interaction to learn fractions.

Valerie: [Before,] I didn't feel confident teaching that to the kids. . . . I didn't feel strong on, like, ratios, proportions, and all those kind of things. And then go into the class and, you know, doing some of those...then my confidence gained. So then I felt more confident to teach that.

Rebecca: It's been 20 years since I've been out of school. In that amount of time, I'd been teaching lower elementary, and if you don't use it, you lose it. So I did not remember how to do even basic algebra. . . . So [now] whenever I would come across something, I didn't panic as much. . . . If you don't feel like you know how to find the answers yourself, you're not going to teach that. You stay away from it. You know? So I think that having a wider math knowledge basis that it has affected my teaching because now, the kids can go off on trails and it doesn't frighten me and I can—to do that.

Sonya: My fraction and percent knowledge were math concepts that I had always had trouble with as a student through school. . . . [What I learned] helped build my confidence to do a better job of teaching it to my students.

Jacqueline and Rebecca also reported that they began to see the connections between mathematical ideas and concepts more clearly:

Jacqueline: It's kind of like I was under a microscope before and now I'm removed more so. I see kind of the bigger picture of it instead of just, like, living in my little box of algebra or geometry. It's like a bigger picture, and I can see more connections than before.

Rebecca: I knew . . . second-grade math even before . . . , but even though I did know that, having more information in math, having a wider basis of it, I think that it helps me because then, first of all, I see that—how things are connected.

In sum, most of the teachers mentioned the same effect of MKT gain on their teaching. Their self-confidence increased and they became more comfortable with teaching mathematics. It is interesting that the only two teachers who reported seeing connections between mathematical ideas were the teachers whose current level of MKT was high.

Inquiry-oriented lessons

Changes in teachers' inquiry-oriented lesson. All the teachers reported some changes in their practices toward more inquiry-based teaching and changes seemed to correspond with the level of MKT gains. Table 23 briefly presents teachers' reports on how they taught before the program and what changes they had made in their practices relevant to inquiry-based teaching. Valerie and Rebecca were the two teachers with greater than 1-point increases in their LMT scores over the duration of the program. They both said that they had only lectured before they were enrolled in the program. According to their self-reports, Valerie was showing shortcuts, whereas Rebecca was showing one method step by step and teaching key words for problems. Valerie observed that she had started to use concrete materials to show the meaning behind algorithms. She had also added some problems that required thinking and included some of the strategies they had learned in the program. Rebecca became more open to students' use of different methods, and she let her students share their ideas and comment on each other's work.

She also mentioned focusing on the meaning behind the algorithm and on seeing the connections between mathematical concepts:

In reading, you're always saying, "Think about what you're reading. You're not just reading the words. If you just read the words, that it's not really reading." You can read really fluently, and if you're not thinking about it, then you're not really reading. Okay? But—but we never tell kids to think in math.

In agreement with their self-reports, these teachers' scores on the Inquiry-Oriented Lesson scale also increased, as shown in Table 23.

Table 23

Changes in Teachers' Practices Toward More Inquiry-Based Teaching

Teacher	MKT gain	Teachers' beliefs	Lesson	Change in inquiry-oriented lesson scale score	Teaching before program	Change in teaching based on teachers' self-reports
Stephanie	.8	4.2	Subtraction with integers (using chips)	—	Already teaching inquiry-based	More inquiry-based teaching More letting students develop their understanding
Jacqueline	.6	4.2	Fibonacci problem	—	Already teaching inquiry-based	More discovery
Valerie	1.1	4.0	Factors and highest common factor	Increased	Lecturing Showing shortcuts	Using concrete materials more to show meaning Adding problems required complex thinking
Rebecca	1.1	4.0	Problem solving	Increased	Lecturing Teaching key words	More students' sharing their ideas and analyzing their peer work Focusing on understanding
Sonya	.9	4.0	Factors	Increased	Already using inquiry-based teaching	More learning from one another Focusing on understanding
Ann	.3	3.0	Subtraction with integers	No change	Lecturing	More student sharing their responses

(table continues)

Table 23 (continued)

Teacher	MKT gain	Teachers' beliefs	Lesson	Change in inquiry-oriented lesson scale score	Teaching before program	Change in teaching based on teachers' self-reports
Beth	.8	3.0	Multiplication with whole numbers	No change	Lecturing	More small group More hands-on
Meg	.7	3.7	Addition with money	Increased	Already using inquiry-based teaching	More listening to students More learning from one another More discussion

Sonya, whose mathematics score increased by .88 points, reported that she had been teaching using an inquiry-oriented approach even before she enrolled in the program. Her description regarding the changes in her practices was more about the quality of the practices, such as “create an environment of understanding math with a focus on learning from one another in math class.” She also mentioned that her teaching focus was on “teaching with the goal of students truly understanding the math concepts that are being taught, as opposed to just memorizing rules.” Meg, who had a .68 increase in her mathematics score, also mentioned more profound changes in her practices, such as

even more listening to them, letting them kind of teach each other. . . . My classroom . . . is more open now, where I allow a lot of conversation to go on. I’ll allow them to say, “That’s not right. We don’t—we disagree with it,” and a lot of discussion, whereas before, I wasn’t prone to do that.

In agreement with their self-reports, their scores on the Inquiry-Oriented Lesson scale also increased.

Unlike these teachers, Beth’s score on the Inquiry-Oriented Lesson scale at the end of the program was not different from her score at the beginning. However, her mathematics score increased by .83 points. In the interview, she reported some changes in her practices. She reported that the most noticeable change in her practice was “to teach less as a whole group.” She continued, “I do less whole group, more hands on, and I allow time for students to problem solve or practice the concept in small groups.” Unlike the other teachers, Beth did not mention any in-depth changes other than superficial ones. The Inquiry-Oriented Lesson scale did not capture the changes she reported in the interview (Table 23).

Ann was the only target teacher whose mathematical knowledge did not change noticeably. Her scores on the Inquiry-Oriented Lesson scale at the beginning and end of the program also did not change markedly. She reported that she had begun to let her students guide

her lessons and that she listened to her students' ideas more. However, the changes she mentioned in the interview were not captured on the Inquiry-Oriented Lesson scale. A closer look at the examples Ann gave to illustrate the changes in her teaching might shed light on the reasons behind this. During the interview, Ann did not mention any changes in her teaching toward more conceptual understanding, and all the examples she provided were about general teaching strategies, without a specific emphasis on student thinking or learning. As illustrated in her portrait, her lesson was designed to teach strategies for subtraction. Although she asked her students to share their responses, she did not analyze the students' responses or ask questions to reveal their thinking.

Given that Jacqueline and Stephanie were not observed at the beginning of the program, no numbers were available to compare their scores on the instructional practice scales at the beginning and end of the program. According to their reports in the interview, Stephanie and Jacqueline were both trained to teach through problem solving during their undergraduate education. Stephanie reported an increase in her confidence to teach in a more inquiry-based manner: "And I didn't feel informed enough about how [to teach using a problem-solving approach]. And I felt very tied to my curriculum and my textbooks. And now I just feel much freer to expand on those and to use those as a base." Additionally, she reported that she would let their students develop their own understandings. Jacqueline reported that she learned to refine her lessons better so her students would discover more. She further mentioned that she wanted her students to learn by doing and discovering so that she could "give them almost a gift of that level of understanding."

Overall, all teachers reported some changes in their practices, but some of these changes were not captured by the Inquiry-Oriented Lesson scale. It seems that the structure of their

inquiry-oriented lessons underwent both quantitative and qualitative changes. Except for Beth, the gain in teachers' mathematical knowledge seemed to be related to changes in their inquiry-oriented lesson structure. Another striking difference is that only teachers with strong mathematical knowledge (including Sonya) at the end of the program mentioned more conceptual changes in their practices.

Teachers' current level of inquiry-based teaching. Based on teachers' self-reports, six of the eight teachers started to move toward more inquiry-based teaching.³² As reported in the previous section, most of the teachers changed their practices to be more aligned with inquiry-based teaching, which is in accordance with the gain in their MKT. However, cross-sectional analysis of classroom observation data conducted at the end of program suggests that teachers' existing level of MKT might not be related to their existing level of inquiry-oriented teaching. As I explain in detail in the "mediating factors" part, teachers favoring standards-based views of mathematics designed their lessons to be more aligned with inquiry-based teaching regardless of their existing level of MKT.

I provide several contrasting examples to illustrate the lack of relationship between teachers' current level of MKT and inquiry-based teaching. Although both teachers, Stephanie and Jacqueline, had a high level of MKT and used inquiry-based teaching to some extent, Jacqueline's teaching was more inquiry-oriented than Stephanie's. For instance, Stephanie's lessons did not have a problem-centered structure; rather, she asked her students to present problems with chips so that they could understand how subtraction worked. On the other hand,

³² Of all the target teachers, six teachers mentioned moving toward more inquiry-oriented teaching. Given that Ann's MKT did not change noticeably throughout the program, no change in her practices was expected.

Jacqueline's lesson had a problem-centered structure; she posed a question and expected her students to discover the pattern.

Sonya's teaching was more aligned with inquiry-based teaching than Valerie's, which was more direct. Valerie's lessons were designed more like a quick presentation of the concept the students were learning, and at that time, she would explain what they were supposed to be learning. Investigation and analyses of mathematical ideas were more prominent aspects of Sonya's mathematics lessons, as captured in the inquiry-based lesson scale.

As illustrated in the portraits of the teachers, Meg, with limited MKT, seemed to teach in an inquiry-oriented manner, as did Rebecca at the end of the program. They both posed a question and asked their students to investigate a solution. The students needed to justify their solutions and determine the sensibility of an idea or procedure based on the reasoning provided.

In contrast to Meg's teaching, the instruction of the other two teachers with low MKT (Ann and Beth) had very limited aspects of inquiry-based instruction. They first introduced a procedure or a rule, and then the students worked on similar problems. The students worked in groups to practice what they had learned in the lesson.

In sum, the analysis of cross-sectional data did not indicate a relationship between teachers' MKT and inquiry-based teaching. This might be the reason why many cross-sectional studies have failed to find a significant relationship between teachers' mathematical knowledge and their lesson structure. However, it seems that when teachers increased their mathematical knowledge, they seemed more inclined to use inquiry-oriented teaching. This cross-sectional and longitudinal analysis of teachers' practices suggests that the gain in their mathematical knowledge had a more similar effect on their inquiry-oriented lesson structure than did their current level of mathematical knowledge.

Mediating factor: Teachers' beliefs. Analysis of teachers' current practices indicated that teachers' current level of mathematical knowledge did not seem to be related to the extent of their inquiry-based teaching. Quantitative data analysis also indicated that teachers' beliefs about teaching and learning mathematics could play a role in their use of inquiry-based teaching. I also analyzed interview data to understand the extent to which differences in the teachers' inquiry-based teaching could be related to differences in their beliefs or their perspectives on teaching and learning mathematics.

In Table 23, as mentioned in the previous section, Stephanie and Jacqueline, the two teachers with very strong mathematical knowledge, implemented different levels of inquiry-based teaching. Their scores on the beliefs test suggest that these two teachers held similar beliefs about teaching and learning. However, during the interview, Stephanie commented that she could not teach using a completely inquiry-based method: "I just don't have the patience for that. And at some point, I'm going to do instruction, and especially when kids start getting off task, because they do." On the other hand, Jacqueline expressed how much she valued inquiry-based teaching: "I value inquiry and like questioning . . . and I just really saw the value in that." Furthermore, Jacqueline and Rebecca were the only teachers who mentioned the importance of letting students "struggle" to understand mathematical ideas. That could be a possible reason why their lesson had a problem-centered structure and their students worked in groups to solve problems.

As mentioned earlier, another teacher with high MKT, Valerie, started to move toward more inquiry-based teaching; however, her current level of MKT and her use of inquiry-based teaching did not match very well. As illustrated in her portrait, she talked about setting up a situation using concrete materials so that the students could "figure out" the rules. Her

explanation of how she created a discovery environment illustrates her view of inquiry-based teaching, which was somewhat different from the conventional view of reformed-based teaching. In her illustration of her students' finding a common denominator by using manipulatives, she thought that she had created a discovery environment because, as she said in the interview, "I've set everything up . . . because you have to have this (referring to concrete materials) to be able to go to this. . . . And so I wait for them to discover it. And once they discover it, then it's like—then everybody was like, it's that 'Oh' moment. And then it's just wonderful. Then I'm, like, "I love your 'Ah-ha' moments when you guys get something."

Furthermore, of the three teachers with low MKT, only Meg taught using inquiry-oriented methods. Of these three teachers, only Meg had high scores on the beliefs test (see Table 23). In addition, during the interview, only Meg mentioned that mathematics was more than "facts": "The kids are more than capable of doing it, of doing word problems, of thinking of their own problems, writing in their journals. Math is not just computation. Math is a whole umbrella of things." However, neither Beth nor Ann mentioned anything specific about inquiry-based teaching. Beth mentioned only using more group work, whereas Ann commented on how mathematics should be taught: "Well, I definitely think it should not be me at the board lecturing, saying, 'Okay, this is the algorithm. This is how you solve the problem.' . . . So I think it should definitely be fun."

In sum, it seems that teachers' beliefs and perspectives on teaching and learning mathematics could play a mediating role in the effects of teachers' mathematical knowledge on their instructional practices. Differences in the teachers' practices could be explained by differences in their beliefs, to some extent.

Summary of inquiry-oriented lessons. Separate analyses of the qualitative data on inquiry-based teaching shed light on the quantitative findings. As the quantitative data suggested, when teachers' MKT increased, they tended to move toward more inquiry-based teaching. Analysis of the target teachers' interviews supported the fact that teachers' made changes in their practices in alignment with the gain in their MKT. However, it also indicated that there was variation in the changes teachers made toward more inquiry-oriented teaching. Only teachers with strong MKT mentioned focusing on more conceptual changes.

Analysis of the interview and classroom observation data also indicated that at both the beginning and the end of the program, teachers' practices varied. Cross-sectional analysis of teachers' practices related to inquiry-oriented teaching suggested that teachers' current level of MKT did not seem to be closely related to their inquiry-based teaching practices. Teachers' beliefs and perspectives on their teaching practices seemed to be important factors influencing teachers' practices.

Mathematical agenda of sense-making. In this section, I first summarize teachers' self-reports on changes in their practices toward a more mathematical sense-making agenda. Using classroom observations and teachers' interviews, I then summarize teachers' practices captured in their sense-making agenda at the end of the program. As mentioned in the Methods chapter, I also report teachers' practices under two subthemes: purpose of the lesson and use of lesson time. I then briefly report teachers' beliefs and perspectives on teaching and learning mathematics as a possible mediating factor in the relationship between teachers' MKT and having a mathematical sense-making agenda. I end this part by summarizing the findings from the cross-case analysis.

Before presenting the results, it is important to note that the Mathematical Agenda of Sense-Making scale is used to capture the *mathematical* quality of lesson implementation. As laid out in the Methods chapter, the scale captures the extent to which teachers created an environment so that students could make sense of the concepts they were expected to learn. This included discussing and analyzing ideas mathematically and using tasks and real-life connections to make sense of the concept being taught, rather than simply sharing ideas and making real-world connections without making any connections to the mathematical concepts. As the reader may notice, having a mathematical sense-making agenda is similar to conducting inquiry-oriented lessons to some extent. The Inquiry-Oriented Lesson scale captures the extent to which teachers created an environment in which students could explain their ideas and asked students to investigate and analyze those ideas. The mathematical sense-making agenda differs from the Inquiry-Oriented Lesson scale in that it focuses on the mathematical strength of the observed lesson. More specifically, whereas the Inquiry-Oriented Lesson scale captures students' explanations and discussions of ideas; the mathematical sense-making agenda captures how teachers build their students' responses and justifications so that the students can see mathematics in their responses and justifications. The mathematical sense-making agenda also goes beyond to the extent to which mathematics is portrayed as a dynamic body of knowledge, and it captures the extent to which students make connections to other related mathematical ideas and generalizations regarding those ideas. A lesson could be taught using an inquiry-oriented approach but the mathematical aspects of that lesson could be weak. Or the mathematical sense-making agenda of a more traditionally designed lesson could be stronger than that of a lesson taught using an inquiry-oriented approach if the teacher using the former lesson type made more explicit what mathematical ideas the students were expected to learn.

Changes in teachers' mathematical agenda of sense-making. Findings from the quantitative data analysis indicated a positive relationship between the change in teachers' scores on the Mathematical Agenda of Sense-Making scale and their scores on the mathematics test. A closer look at these eight teachers' scores on the Mathematical Agenda scale before and after the program showed that the scores of the teachers with limited knowledge (Meg, Ann, and Beth) did not seem to increase noticeably compared with those of teachers with strong mathematical knowledge (Table 24).

Table 24

Change in Teachers' Practices Toward a More Sense-Making Agenda

Teacher	MKT gain	Teachers' beliefs	% qualified for lunch	Change in sense-making scale	Change in teaching based on teachers' reports	Use of lesson time (%)	Purpose of the lesson
Stephanie	.8	4.2	42	—	More focus on sense-making by analyzing students' responses More focus on students' developing their thinking and mathematics in their responses	96	Teaching procedure and meaning
Jacqueline	.6	4.2	42	—	More in-depth questioning to get students thinking Using more real-life connections	88	Teaching procedure and meaning
Valerie	1.1	4.0	47	Increased	More focusing on meaning behind a procedure	96	Teaching procedure and meaning
Rebecca	1.1	4.0	76	Increased	More analysis of student work More focusing on meaning behind a procedure	93	Teaching procedure and meaning
Sonya	.9	4.0	85	Increased	More questioning	95	Teaching procedure and meaning

(table continues)

Table 24 (continued)

Teacher	MKT gain	Teachers' beliefs	% qualified for lunch	Change in sense-making scale	Change in teaching based on teachers' reports	Use of lesson time (%)	Purpose of the lesson
Ann	.3	3.0	85	No change	Real-life connections	73	Teaching procedure
Beth	.8	3.0	73	No change	Real-life connections	72	Teaching procedure
Meg	.7	3.7	79	No change	Real-life connections	72	Teaching procedure

Teachers with high MKT mentioned changes in their practices captured in the Mathematical Agenda of Sense-Making scale. Stephanie had changed her practices “to ask more of the students, for them to develop their own ways of thinking.” She also mentioned learning from other students’ responses: “we show different examples and we can show what the students have done and try to make sense of what their thinking is. . . . So we can look at things from a different perspective.” Jacqueline listed several modifications she did in her teaching related to the Mathematical Agenda of Sense-Making scale. For instance, she mentioned using real-life situations more often to help students make sense of a concept. She also mentioned making changes in her questioning strategies: “I now know, like, what kinds of questions to ask instead of just—surface ones. . . . And I get them thinking a little bit more deeply with questioning.” She also reported waiting for her students to discover the concept.

Valerie mentioned setting up a situation with concrete materials so that students could make sense of the concept being taught. She mentioned that she had “start[ed] showing the reason.” Rebecca also provided more specific details regarding the changes captured on the Mathematical Agenda of Sense-Making scale. As mentioned in the Inquiry-Based Lesson section, Rebecca mentioned that students discussed and analyzed their peers’ work. In addition, as captured on the Mathematical Agenda of Sense-Making scale, she would “allow [her students] to think through the problems” because, as she stated,

I can see how the numbers are related. . . . and I see how the—it's not just an algorithm; I know then more of the meaning behind it, so then I can do that.

As mentioned earlier, Sonya commented that “[teachers] act as guides during the lesson to help students understand concepts through questioning, as opposed to telling the students how to approach a problem or giving them the answers.” She mentioned continuing to teach

mathematics in that way. She further commented, “The program helped me differentiate sense-making in the math class from telling students how to do math.”

As shown in the preceding excerpts, the teachers who increased their knowledge of mathematics significantly and ended the program with strong mathematical knowledge gave specific explanations regarding aspects captured on the Mathematical Agenda scale.³³ However, teachers who ended the program with limited mathematical knowledge failed to report more specific modifications in their practices captured on the Mathematical Agenda scale, regardless of their MKT gain. For instance, Beth, who increased her mathematical knowledge by .88 points but still had limited knowledge at the end of the program, mentioned using real-world examples to help students “better understand a concept.” She did not provide any details or examples regarding how real-world examples would help her students understand the concept.

Like Beth, Ann also reported that she had begun to use real-life scenarios to help students make sense of the concept and she listened to the students’ ideas, but she did not mention any specific information regarding building on her students’ responses or making the mathematics in their answers explicit, as captured on the Mathematical Agenda of Sense-Making scale. She further illustrated how she used real-life situations to help students understand mathematical concepts:

So we're not just doing 48 divided by 6 equals 8. It's, you know, we have 48 dogs. They need to be put in groups of 6, you know, and make it real to them so it's not just 48 divided by 6 equals 8. . . . I have realized if they bring up—if I don't know, “Hey, that's a great question. Let's find that out.” . . . I've realized I don't have to have all the answers and I shouldn't have all the answers. You know? So, “Oh, that's, very interesting,” you know, “let's go with that. I'd like to know the answer to that, too.” And kind of let them kind of—then we go that way. So yeah, I don't have to know everything for us to explore that. I can explore it with them.

³³ Although Sonya has an average level of MKT, her teaching was similar to high-MKT. Hence, unless I said otherwise, when I refer to high-MKT teachers, I also refer to Sonya.

In this example, there is no indication of her analyzing or pressing students' ideas. In agreement with her report, her score on that scale at the end of program did not change.

Like Beth and Ann, Meg also did not increase her score noticeably over the duration of the program. Although Meg's explanation of the importance of making connections to real-life situations was more specific, she also did not focus on the mathematical importance of using real-life connections:

it makes sense to them, as opposed to just giving them numbers and just giving them formulas. If when they can make a connection with how they live, where they live, or, for example, in math and science, the things around them, they're more apt to remember and to build upon what they know.

As shown in the preceding excerpts, the teachers with strong mathematical knowledge (Stephanie, Jacqueline, Rebecca, and Valerie) and Sonya gave more specific explanations regarding aspects captured on the Mathematical Sense-Making Agenda scale focusing on more conceptual changes to make sense of the concepts being taught. Teachers with limited MKT did not seem to promote "making sense of the lessons" by focusing on mathematical concepts; rather they assumed that using real-life scenarios would be enough for students to understand mathematical ideas. On the other hand, teachers with strong mathematical knowledge were more specific and provided more detailed explanations of how they used tasks and classroom discussions to promote conceptual understanding and illustrations of real-life examples, which revealed the interplay between teachers' focus and their MKT-level.

Teachers' current level of mathematical sense making agenda. In the previous section, I focused only on the changes in teachers' practices based on teachers' interview data. In this section, using both classroom observation and interview data, I investigate the relationship between teachers' current level of MKT and their practices, as captured on the Mathematical Agenda of Sense-Making scale.

As stated earlier, although the themes were derived from the classroom observation scales, I did not focus on each item in the scales. As already explained, the rationale behind this decision was that the qualitative data did not provide enough information concerning each item on the scales. Furthermore, the qualitative data can help us understand the quantitative findings, rather than simply measuring the same aspects captured in the quantitative analysis. More specifically, although there were no specific items on the scale that captured teachers' focus on students' work, the qualitative data analysis indicated that there seemed to be relationship between teachers' MKT and the mathematical quality of teachers' analysis of students' responses. In addition, the qualitative data analysis suggested two additional themes related to the sense-making agenda: the use of lesson time and the purpose of the lesson.

As indicated earlier, the Mathematical Agenda of Sense-Making scale is designed to capture the extent of the mathematical quality of the lessons. It seemed, based on the analysis of classroom observation data, that teachers' current level of MKT played an important role in their analysis of students' work and the mathematical quality of the discussion.

As illustrated in the portraits of the target teachers, the mathematical quality of the observed lessons was stronger for the teachers with strong MKT. Their focus on students' responses was mainly on the mathematical aspects of the responses. In integer subtraction lessons using chips, Stephanie evaluated her students' work mathematically, and she mathematically represented her students' responses. When Stephanie was evaluating her students' work, she focused on the mathematics behind their responses. In the episode depicting her teaching, she wrote the corresponding mathematical expressions for her students' responses:

Yes, and I do that too. As I go along, I try and get them to predict what the next problem I'm going to put on the board is so that they can see the pattern and that I didn't just randomly choose numbers, that there was a sequence to my numbers so that we can see what's happening and kind of look at those specifically.

Jacqueline spent three lessons helping her students find the pattern in the Fibonacci series. After the students found the pattern, she revised the same problem so that her students could see how the patterns were dependent on the rule. Furthermore, each student had a chance to present her or his work, and the focus during the discussion was on mathematical aspects of the students' responses. Although Valerie's task was mathematically problematic, she was able to push mathematically productive conversations among the students. The students made generalizations regarding those mathematical ideas or procedures. Several connections were made to other concepts in mathematics. She set up the lesson activity so that her students could learn what they were supposed to learn through this activity. Despite the questionable difficulty of Rebecca's task choices, her problems successfully created a mathematically productive disequilibrium among students. Rebecca and her students analyzed each group's work to understand their thinking, and their focus was on the mathematical aspects of students' responses.

Sofia's focus in discussions and in students' responses was also mathematics. With the help of her curriculum, she was able to create mathematically rich discussions in her classrooms regarding factors of the numbers. All these teachers asked students to explain their thinking, and they expected their students to provide some explanations, rather than simply stating the procedures they followed.

On the other hand, teachers with limited MKT had difficulty making the mathematics explicit in students' responses, and they failed to build on students' responses. Beth's lack of questioning strategies was consistent throughout all the lessons I observed. She did not ask in-depth questions, and as a result, she failed to probe students' thinking. As seen in the excerpt illustrating her teaching, Beth's students shared their answers, but Beth did not seem to analyze her students' work. When her students were finding the answer for 4^5 , Beth recorded each

group's responses, such as 20, 64, 1,024, and commented, "We[']ll keep doing it till we find an answer that we all agree on." However, after she recorded each group's answer on the board, she did not analyze why 20 or 64 did not make sense. She focused on how students arrived at an answer not in terms of their thinking, but more on the procedure they used. No in-depth analysis of students' ideas occurred in her mathematics lessons. Her questions mainly required one-word answers to give a solution or express a procedure.

Similar to Beth, Ann also sometimes asked the students to defend their ideas, but her questions did not go beyond asking for clarification of the steps they used. She listened to her students and allowed them to share their ideas with their classmates, but there was no mathematically significant discussion. When students were supposed to solve a story problem, she asked the class to identify "key words," and her comments on students' responses were not mathematically rich. For instance, one student at the board solved the subtraction problem $35 - 19$ using tallies. Ann asked the class, "Did anybody solve it differently?" Ann asked another student who did it using the standard algorithm to present how he did it and no connection was made between these two methods.

Unlike Beth and Ann, Meg attempted to investigate her students' ideas, and she encouraged her students to do the same. As illustrated in her portrait, even so, Meg failed to draw out the mathematics in her students' responses. However, as illustrated in her teaching episodes, although there was some discussion going on in her classrooms, the mathematical aspect of the discussion was not as clear or strong as in the classrooms of Sonya and Stephanie. For instance, although Meg tried to understand her students' ideas by asking "how" and "why" questions, she did not push her students' ideas mathematically. In particular, when Meg asked her students to decide which coin, a penny or a nickel, had greater value, one student said he did it "counting by

fingers,” and Meg asked the class whether they agreed with the student’s response. When another student said she “used a number line” but failed to explain how, Meg did not focus on how the number line could be used.

In sum, the teachers’ current level of mathematical understanding seemed to have an effect on their mathematical agenda of sense-making. As illustrated in their teaching, teachers with limited mathematical knowledge appeared to have a more difficult time revealing the mathematics in their students’ responses. On the other hand, teachers with more mathematical knowledge seemed to create mathematically richer classroom discourse, and more in-depth analysis of students’ responses took place in their mathematics lessons.

Use of lesson time. Another important pattern was the use of lesson time. I computed the percentages of lesson time devoted to mathematics and mathematics-related activities. On average, teachers with limited mathematical knowledge spent 72% of their lesson time on mathematics and mathematics-related activities, whereas teachers with a strong mathematical knowledge devoted 94% of the lesson time to mathematics and mathematics-related activities.

Beth spent only 73% of the observed lessons on mathematics-related activities, and Ann spent only 72% of the lesson time. In one of my classroom visits, I found only eight students in the classroom. Ann explained to me that the other students had failed the exam the day before, and they were practicing the concepts they had missed on the exam. She used only 20 minutes of the 65-minute lesson time, and during that time, students played a game in which they were supposed to learn mathematics, but Ann did not leave time for discussion. It was not clear what the students were expected to learn from this activity. Meg also spent only 72% of her lesson time on mathematics and mathematics-related activities. She showed a video on how money was made, and in another lesson, she announced that as a prize, half of the mathematics lesson time

would be used for mathematics and the other half would be used for fun because her students had behaved well that week.

On the other hand, teachers with high mathematical knowledge seemed to use more of the lesson time for mathematics-related activities. Only Jacqueline spent 88% of her lesson time on mathematics-related activities. Other teachers with strong mathematical knowledge as well as Sonya spent at least 93% of their lesson time on mathematics-related activities.

The purpose of lessons. Regarding the primary purpose of the lessons observed, another important pattern seemed to be the focus of the teaching. Ann, Meg, and Beth focused on teaching only an algorithm or procedure without the meaning behind it. On the other hand, the teachers with strong mathematical knowledge focused on the meaning behind the procedure. For instance, as illustrated in the teachers' portraits, Ann taught only subtraction, and her explanation did not go beyond explaining the steps. Beth spent an entire 60-minute lesson teaching exponential forms. Meg spent an entire week on addition with money, and the mathematical concepts that students were expected to learn were not clear.

On the contrary, Rebecca aimed to boost her students' reasoning and ability to solve problems. Stephanie intended to teach the concept behind the rules for addition and subtraction with integers by using chips. Valerie also used concrete materials so that her students could understand the concepts of factors and highest common factors. Sonya used paper rectangles to make sense of factors, primes, and composite numbers. Jacqueline used the Fibonacci problem to introduce the new chapter, algebra, so that students could make sense of the patterns. It seemed that teachers with limited mathematical knowledge focused on teaching procedure, an algorithm, or both without focusing on the underlying meaning, whereas teachers with strong mathematical knowledge as well as Sonya aimed to teach a concept by focusing on the meaning behind it.

Mediating factor: Teachers' beliefs. The quantitative data analysis indicated a positive relationship between teachers' beliefs about teaching and learning mathematics and their scores on the mathematical sense-making agenda. Similar to the quantitative findings, teachers with higher MKT mentioned the importance of teaching the meaning behind the procedures, whereas teachers with limited MKT did not highlight the importance of focusing on the meaning behind the procedures. Furthermore, during the interview, the teachers with high MKT and Sonya provided mathematically rich and purposeful examples to illustrate their teaching, whereas those with low MKT did not give any specific examples to show the mathematical aspects of their teaching.

Stephanie mentioned that she wanted her students to develop their understanding, rather than trying to emulate what she was doing. She wanted her students to “be mathematicians rather than learn about the content area of math. . . .” She then illustrated how she purposefully chose her examples to create an environment so that students could make sense of the concept. Jacqueline also highlighted the importance of teaching the meaning behind the rules. Similar to Stephanie, the example she provided was also mathematically rich:

Like I had always been told area of a circle is πR^2 . So like this forced me to want to show them how to take a circle and make it a parallelogram and see like “look, we know area is length times width, so how can we make that circle a parallelogram and find the formula?”

Valerie also valued teaching the meaning behind the procedure:

Math is more than rules. . . . And I really want them to get in there and understand really the concept behind it . . . I don't want to teach that . . . here's the rule. Follow the rule. I don't know why—it's that rule. . . . I want them to discover those things on their own, rather than me giving them that rule.

Similarly, she also provided an example of teaching that illustrated how she created an environment using concrete materials and appropriate fractions so that students could learn

adding fractions by finding common multiples. Rebecca's comment on her beliefs about focusing on the conceptual understanding was also clear:

In reading, you're always saying, "Think about what you're reading." You're not just reading the words. If you just read the words, that it's not really reading. You can read really fluently, and if you're not thinking about it, then you're not really reading. Okay? But—but we never tell kids to think in math.

Sonya explained why it is important for her to teach the meaning behind the procedures and algorithms. Unlike teachers with strong MKT, teachers with limited MKT did not refer to the importance of teaching the meaning behind the procedures. Beth did not provide any examples of teaching a particular concept to illustrate her teaching. Ann also did not mention anything related to conceptual understanding. Most of the examples she provided were about teaching different strategies.

Meg is the only teacher with limited MKT who mentioned that mathematics is more than knowing the facts. However, she did not refer to mathematical reasoning or thinking, nor did she provide specific examples like teachers with high MKT did to illustrate the importance of teaching for conceptual understanding. Rather, she explained that "kids are more than capable of doing it, of doing word problems, of thinking of their own problems, writing in their journals . . . thinking through the process of how to solve problems."

Summary of mathematical sense-making agenda. Analysis of the classroom observations and interview data supplemented findings from the quantitative analysis. The quantitative analysis suggested that changes in teachers' MKT were related to changes in the mathematical agenda of sense-making, and teachers' beliefs were also related to their scores on that scale. The qualitative analysis provided further support for the view that teachers' current level of mathematical knowledge played a role in their mathematical sense-making agenda.

Teachers with limited mathematical knowledge failed to facilitate discussions in which students

were prompted to make sense of the ideas. On the other hand, teachers with more mathematical knowledge seemed to use mathematically richer classroom discourse, and more in-depth analysis of students' responses took place in their mathematics lessons. In addition, the teachers analyzed students' ideas more efficiently in terms of their mathematical quality. Their lesson activities were designed so that the students could learn related activities. Like the teachers with strong mathematical knowledge, they asked their students to share their ideas, but they failed to analyze these ideas in depth.

Another striking difference among teachers with different levels of mathematical knowledge was the use of lesson time. Teachers with limited mathematical knowledge seemed to spend a considerable amount of time on activities unrelated to mathematics. This did not seem to be a pattern among teachers with stronger mathematical knowledge. Teachers with a high level of mathematical knowledge not only taught their students the rules, but they also provided explanations for why certain rules worked. On the other hand, teachers with limited mathematical knowledge focused only on teaching procedures, and they failed to provide explanations for why certain rules worked. Furthermore, none of the teachers with limited MKT provided perspectives on the importance of conceptual understanding, whereas all teachers with strong MKT highlighted the importance of understanding the mathematical ideas behind the rules and algorithms.

Worthwhile mathematical tasks. In this section, I elaborate on the quantitative findings by using classroom observation and interview data. First, I examine the relationship between changes in teachers' task choices and MKT gain, and then I analyze how their current level of MKT is related to their current level of task choices. Although quantitative data also indicated a relationship between teachers' beliefs and task choices, analysis of the interview data indicated

that teachers did not refer to their beliefs when they talked about their task choices. Hence, I cannot provide any qualitative insights into the relationship between teachers' beliefs and their task choices.

Change in teachers' worthwhile mathematical tasks. The quantitative findings indicated no association between changes in teachers' mathematical knowledge and the Worthwhile Mathematical Tasks scale. As seen in Table 25, only Valerie and Rebecca's scores noticeably increased and Sonya's score slightly increased on the scale. On the other hand, teachers with low MKT did not increase their scores on that scale. Analysis of the interview data from these eight teachers shed light on the lack of a relationship between the change in teachers' MKT and their task choices.

Table 25

Change in Teachers' Worthwhile Mathematical Task Choices

Teacher	MKT gain	Teachers' beliefs	Change in worthwhile mathematical tasks scale	Change in teaching	Textbook use
Stephanie	.8	4.2	—	Added problems from Vande Walle Started to follow her textbook loosely	Followed textbook loosely
Jacqueline	.6	4.2	—	More open-ended problems required higher level thinking	Did not follow a textbook
Valerie	1.1	4.0	Increased	Added problems from Vande Walle	Did not follow a textbook
Rebecca	1.1	4.0	Increased	Found activities outside her textbook	Followed her textbook loosely
Sonya	.9	4.0	Slightly increased	Added problems from Vande Walle	Followed her textbook closely
Ann	.3	3.0	No change	Added “fun” activities	Followed her textbook closely
Beth	.8	3.0	No change	No change	Followed her textbook closely
Meg	.7	3.7	No change	No change	Followed her textbook closely

As seen in Table 25, the change in teachers' scores on the Worthwhile Mathematical Tasks scale was somewhat positively related to teachers' MKT gain. Of the teachers whose MKT increased noticeably, only two teachers with high MKT (Valerie and Rebecca) noticeably increased their scores on the Worthwhile Mathematical Tasks scale. Sonya also modestly increased her score on that scale. Although Beth and Meg increased their MKT drastically, their task choices did not change noticeably. It appears, based on analysis of the qualitative data, that teachers with limited MKT did not make any noticeable changes in their practices, whereas teachers with strong MKT included more problems or modified the problems they had used.

Of the four teachers with high MKT, two teachers, Stephanie and Valerie, reported adding some problems similar to those used in the methods course in the program as well as some problems from Vande Walle, which was the textbook used in the same methods course. Stephanie also reported starting to use her textbook more loosely. Stephanie described the changes in her task choice, in which she used activities from Vande Walle:

And now I do try to do a lot more Vande Walle types of things than I did before. . . . I do still use the textbook daily, or at least often, but I definitely pull from more resources and the Illuminations and try to have more activities and games and problems.

Rebecca also reported beginning to add more problems in her teaching. She explained that she chose problems that "are applicable to [her students] lives as second graders . . . and . . . take more than one step [to solve]." Jacqueline did not mention adding new activities or problems; rather, as mentioned before, she changed how she used the tasks. Previously, she had provided the pattern for the Fibonacci problem, but then she asked the students to find the pattern themselves.

Sonya and the three teachers with limited mathematical knowledge appeared to follow their curricula very closely. However, Sonya reported making some modest changes in her task

choice. Although she followed her curriculum very closely, she also reported that she had started to modify the activities from her textbook and include some activities from Vande Walle:

I used [the professor]'s ideas a lot because some of the problems she presented in class are from the Connected Mathematics curriculum I teach. The way she presented some lessons to us, I used. Van deWalle's book was another tool.

Of the three teachers with limited mathematical knowledge, only Ann reported adding activities from outside her textbook. Ann illustrated the activities she chose:

You have to find a way to hook them into the learning. So sometimes, it's a book. It might be a song. I always start the lesson, or most days, with an interactive learning activity. So we pose a question on the board.

As exemplified by these teachers, the teachers with high mathematical knowledge began to include more activities outside their textbooks or modify the activities they had used, whereas Sonya and the teachers with limited mathematical knowledge used their textbooks as the main source of the problems and activities they chose. Additionally, none of the teachers with limited mathematical knowledge reported using the book by Vande Walle as a resource, whereas some of the teachers with high MKT (including Sonya) reporting adding activities from the book. On the other hand, Ann chose activities based on the criteria of being “fun and engaging.” As a result, it seems that teachers' current level of MKT had an impact on the relationship between MKT gain and task choices. Low-MKT teachers' quality of task choice did not change much regardless of their MKT gain, while high-MKT teachers changed the quality of task choice corresponding to changes in their MKT.

Teachers' current level of worthwhile mathematical task choices. In the previous section, I looked only at the relationship between changes in teachers' task choices and the gain in their MKT using their self-reports. In this section, I analyze how teachers' task choices were related to their current level of MKT. Cross-sectional analysis of classroom observations suggested that the textbook had an impact on teachers' task choices. In this section, I first report

how teachers' task choices were related to their textbook use, and then I focus on the cross-sectional relationship between teachers' task choices and their current level of MKT.

Textbook use. Cross-sectional analysis of the qualitative data suggested that teachers' task choices were confounded by the curriculum they used. As indicated in the previous section, the teachers with low MKT and Sonya followed their curriculum very closely. Sonya and the three teachers with limited mathematical knowledge appeared to follow their curricula very closely, and almost all the problems and activities were from their textbooks. For instance, Beth said, "I use the Visual Learning Animation of the enVisionMath program. . . . The animation uses lots of ways to grasp the students' attention and challenges them with questions all through the lesson." Meg also used animated lessons and her textbook on a daily basis. As noted earlier, although Ann started to include activities outside her curriculum, she also used her textbook daily.

On the other hand, only two teachers with high MKT, Jacqueline and Valerie, reported that they did not follow a textbook closely. Jacqueline also reported not using her textbook as the main resource. She said, "I don't necessarily prefer the textbook. . . . I like to come up with, like, real-life ideas that I can apply and introduce topics with." Valerie noted,

We're not doing that "teach a whole book." . . . I have textbooks in my classroom, but not a whole set of anything. They're there for a resource. . . . It's not something that I'm going to always pull from and that.

As mentioned earlier, Stephanie and Rebecca, who had been using their textbook very closely, reported adding more problems and activities from outside their textbook. For instance, Stephanie commented, "I definitely was pretty much just using the textbook and went through it chapter by chapter and taught as much as I could. I do still use the textbook daily, or at least often."

In sum, it seems that teachers with limited MKT (including Sonya) seemed to follow their curriculum closely, whereas teachers with high MKT seemed either to not follow their textbook as the main source or to include more activities from outside their curriculum.

Revisiting teachers' current level of worthwhile mathematical task choices. Teachers' current level of MKT did not seem to be associated with their task choices. As mentioned in the previous section, one reason was that the target teachers used either the Connected Math Project (CMP) or EnvisionMath curriculum, and most of the activities and problems (especially for the teachers with low MKT) came from the textbooks. Although the teachers with high MKT included activities from outside their curriculum or did not have a curriculum, as indicated by Stephanie and Rebecca, some of their activities were still from their textbook. Given that all teachers with low MKT and Rebecca were using the same curriculum, EnVisionMath, it is not surprising that there was a lack of relationship between teachers' MKT and their task choices.

Another possible reason for this lack of relationship seemed to be that the teachers had some difficulty finding mathematically appropriate or challenging tasks when they included problems from outside their curriculum. Based on teacher reports, those teachers with strong mathematical knowledge tended to include activities and problems from outside their textbook. As illustrated in the portraits of the target teachers, some problems teachers found might not have required higher order thinking. For instance, by using similar types of problems, Stephanie got her students to notice patterns in addition and subtraction with integers. However, the problems she used for students to find patterns did not require complex and nonalgorithmic thinking in many instances.

Rebecca found the problems by herself, and sometimes she failed to find ones appropriate for her students' level of understanding. More specifically, many of her students did not know

how to count correctly to a number higher than 100. Moreover, she chose a number that could not be divided by 4 for the first question posed. Similarly, not having a textbook might have limited the effectiveness of Valerie. As illustrated in her teaching, the problems she chose did not require deep thinking, so mathematically rich conversation did not happen so often. Although Sonya's MKT was lower than Stephanie's, Valerie's, and Rebecca's, with the help of her curriculum, the Connected Math curriculum, Sonya's tasks were designed to connect several mathematical ideas and concepts.

Summary of worthwhile mathematical tasks. Analysis of classroom observations and interview data provided insights into the lack of a relationship between the change in teachers' MKT and their task choices. The quantitative analysis suggested that teachers with limited MKT did not noticeably change the quality of their task choices. On the other hand, teachers with strong MKT (including Sonya) made some changes in their task choices. Either the teachers modified the activities they had used or they included activities from outside their curriculum. It is interesting that although all the teachers in the program were using the same book by Vande Walle, only the teachers with high MKT reported including activities from that book.

Cross-data analysis of classroom observation data suggested that the main reason behind the lack of relationship between MKT gains and task choice in the quantitative analyses was textbook use. Although teachers with high MKT reported using their curriculum less often or using no curriculum, teachers with strong MKT with a certain curriculum still used some activities from their textbooks. Given that upper elementary grade teachers used the same curriculum, Connected Math, and the lower elementary teachers used the same curriculum, EnVisionMath, it was predictable that teachers' task choices would be somewhat similar regardless of their MKT.

Another important point the qualitative data suggested was that even teachers with high MKT had difficulty finding the complex problems required higher order thinking on their own. When teachers created their own problems or found problems outside their curriculum, it seems that they tended to fail to find mathematically advanced problems so that their students could see connections among several mathematical ideas.

Student engagement. Results of the quantitative data analysis indicated a positive relationship between teachers' scores on the Student Engagement scale and students' gain on the ISAT, whereas no relationship was observed between teachers' scores on the Student Engagement scale and on the teachers' mathematics test. One point that should be mentioned before presenting the findings of the qualitative data is that some aspects captured on the Student Engagement scale were dependent on the students' past experiences, skills, and interests. For instance, the Student Engagement scale captured students' willingness to discuss their thinking and reasoning, the quality of interactions among the students, whether the students were paying attention to the lessons, and students' engagement with the tasks and the lessons. However, this did not mean that student engagement was completely out of the teachers' control. Student engagement is certainly influenced by both the teachers' ability to create an environment that promotes sharing and discussing ideas, as well as students' past experiences and their consequent willingness to be part of this environment.

In this section, similar to the previous section, I briefly summarize how changes in student engagement were related to changes in teachers' MKT, and then look at how the current level of student engagement was related to teachers' existing MKT. In this section, I did not provide information regarding the teachers' beliefs and their association with student

engagement because the teachers did not provide any specific information during the interview regarding student engagement.

Change in student engagement. As presented in Table 26, most of the teachers mentioned similar changes in their practices regardless of the gain in their MKT. Furthermore, in accordance with the quantitative results, a change in the average ISAT gain of the target teachers also indicated no relationship between the average ISAT gains for the teachers and their MKT gains, whereas there was a somewhat positive relationship between a change on the Student Engagement scale and the students' average ISAT gain.

On the basis of the analysis of interview data, two teachers (Valerie and Stephanie) did not mention any changes in their practices captured in the Student Engagement scale, whereas the remaining six teachers mentioned that they devoted more time to having students share and discuss ideas and work in groups. Jacqueline mentioned how much she had learned from other teachers during classroom discussions in her mathematics methods course in the program, and she had begun having her students share their thinking. She said that the professor was

allowing us to discuss as teachers our understanding. Like when I heard Stephanie talk in class, for example, like she thought about it way different than me. But, like, that was an eye-opening thing too, like my students could think of it this way.

Given that most of the teachers started to devote more time to students' sharing their ideas, classroom discussion, and group work, regardless of the change in their MKT, it was not surprising to find no significant relationship. Although there were no qualitative data to shed light on the relationship between students' engagement and their ISAT gain, as seen in Table 26, there seemed to be some similarities in the target teachers' changes on the Student Engagement scale and their students' average ISAT gain. In the following section, I investigate possible reasons for this by using classroom observation and interview data.

Table 26

Change in Teachers' Student Engagement Scale and Average Students' ISAT Gain

Teacher	MKT gain	% qualified for lunch	Change in student engagement scale	Change in teaching	ISAT gain
Stephanie	.8	42	—	No change	
Jacqueline	.6	42	—	More time for students to share their ideas More time More group work	
Valerie	1.1	47	Increased	No change	Slightly increased
Rebecca	1.1	76	Slightly increased	More time for students to share their ideas More time More group work	
Sonya	.9	85	Slightly increased	More time for students to share their ideas More time More group work	Decreased
Ann	.3	85	Increased	More time for students to share their ideas More time More group work	Increased

(table continues)

Table 26 (continued)

Teacher	MKT gain	% qualified for lunch	Change in student engagement scale	Change in teaching	ISAT gain
Beth	.8	73	No change	More time for students to share their ideas More time More group work	No change
Meg	.7	79	No change	More time for students to share their ideas More time More group work	

Current level of student engagement. My classroom observations at the end of the last year of the program as well as the teachers' self-reports regarding their student engagement indicated that the teachers had different levels of student engagement regardless of their level of mathematical knowledge or use of instructional practices.

During my classroom visits, Stephanie's students did not seem engaged in her mathematics lessons. Some students seemed to be unwilling to participate in classroom discussions. Stephanie was disappointed with her students' lack of interest in learning more mathematics:

I often don't have a lot of student interest when it comes to giving them homework and outside assignments. They're not highly motivated to continue to learn the math that I've got going on. And so it's all that I can do to keep them in it during the 40 minutes I've got them.

On the other hand, Valerie's, Jacqueline's, and Rebecca's students were engaged in their mathematics lessons. Almost all Valerie's students were participating in classroom discussions. Her students not only shared their thinking, but also listened carefully to other students' ideas. Similar to Valerie's students, Jacqueline and Rebecca's students were actively part of the mathematics lessons. They were paying attention to their peers' work. They carefully listened to other students and challenged them if they thought their ideas were wrong. In particular, in Rebecca's mathematics lessons, I observed several times that even when the students ran out of time, they continued to work on the question. Rebecca also mentioned in the interview, "I know that a lot of times they'll be working in math and I'll say, 'Oh, it's recess time.' And they'll say, 'Oh,' and they don't want to leave to go to recess." Jacqueline also talked about how their students were excited about their mathematics lessons: "I have a lot of the times kids will tell me like, 'I didn't like math before and now I do.'"

Similar to Valerie's, Jacqueline's, and Stephanie's students, Beth's students seemed to be very interested in classroom activities. Almost all her students were participating in classroom discussions and commenting on their peers' work. I observed several times that they asked for extra homework.

Although Beth and Sonya were in the same school, Sonya had a difficult time with her students. Her students had difficulty focusing on the lesson, and they easily veered off task. Some students were not nice to each other or did not display respect for others, and some did not follow Sonya's directions. In each lesson I observed, she had to pause the lesson because of one or more behavioral problems. The problems with students were similar for Ann and Meg. Ann's students would easily veer off task, and they had some behavioral problems as well. In each lesson, Ann spent time getting them back on task. Sometimes she had to go farther by pausing the lesson and warning her students. She told me in the interview, "I really work hard at trying to make it fun, make it where they understand it. . . . Because that's the key here in this, with these kids. They have to have fun. And they have to be engaged and under control." Meg also had similar problems with her students. During group work, some students were fighting. She also had a difficult time keeping her students on task.

One reason the results showed no relationship between student engagement and teachers' mathematical knowledge but a positive relationship between student engagement and the students' gain on ISAT could be that each year, the teachers had different students, so the students' interest and engagement varied. What Valerie said in the interview describes this variation in student interest from year to year: "Every fourth grader that's going to . . . be great like the one I have this year. I don't know. . . . Probably not. Next year will probably—who

knows what I'll get." As she predicted, her students' average ISAT gain was 13 points that year compared with the 10-point gain the year before the program was initiated.

Summary of student engagement. Qualitative analysis of the classroom observation and interview data provided some insights into the lack of a relationship between teachers' MKT and the Student Engagement scale as well as the students' average ISAT gains. It seems that regardless of the change in teachers MKT, the majority of the target teachers allocated more time for their students to share and discuss their ideas and work in groups.

However, as illustrated by both data sets, student engagement was also dependent on students' willingness to participate in classroom activities, which might be the reason for the positive relationship between student engagement and the ISAT gain. Unfortunately, because I did not collect data from the students, the qualitative data did not provide further explanation for the lack of relationships among teachers' MKT, student engagement, and ISAT gain.

Summary of Chapter 5

Separate analyses of these eight teachers' reports of the changes in their instructional practices as well as analyses of their teaching at the end of the program pointed to a potential explanation for the findings from the quantitative analysis. Similar to the quantitative analysis, the qualitative analysis indicated a positive relationship between the teachers' gain in mathematical knowledge and a change in their practices toward an *inquiry-based lesson*. Of the seven teachers who increased their MKT knowledge, six of them also reported changes in their practices toward more inquiry-based teaching. Furthermore, longitudinal and cross-sectional analyses of teachers' practices indicated that the effect of the *gain* in mathematical knowledge seemed to have a similar effect on teachers' practices toward more inquiry-based teaching, whereas when looking at teachers' practices at certain time points, there seemed to be a weak

relationship, or no relationship, between teachers' knowledge of mathematics and their use of inquiry-based teaching.

The ways in which the teachers engaged in mathematical discourse varied with the teachers' knowledge of mathematics. The teachers with limited mathematical knowledge did not seem to draw out the mathematics in students' responses. The teachers listened to their students and allowed them to share their ideas with their classmates, but no mathematically significant discussion appeared to occur. Teachers with limited mathematical knowledge failed to promote discussions that pushed students to make sense of the ideas. On the other hand, teachers with more mathematical knowledge seemed to create mathematically richer classroom discourse and more in-depth analysis of students' responses. A closer look at aspects of the instruction captured by the *Mathematical Agenda of Sense-Making* scale indicated that teachers' mathematical knowledge seemed to be associated with the purpose of their lessons and their use of lesson time. The teachers with strong mathematical knowledge focused on making sense of the concepts behind the mathematics being taught. They showed the meaning behind a procedure and explained steps in the procedure. On the other hand, teachers with limited mathematical knowledge taught the procedure without teaching the underlying meaning. Furthermore, teachers with limited mathematical knowledge seemed to spend a considerable amount of time on activities unrelated to mathematics.

Qualitative analyses of the Worthwhile Mathematical Task and Student Engagement scales also provided insights into why no quantitative relationship was observed between teachers' mathematical knowledge gain and the aspects of their teaching practices captured on these scales. A closer look at aspects of the *Worthwhile Mathematical Tasks* scale indicated that teachers with strong mathematical knowledge included mathematically more advanced tasks in

their teaching. However, teachers with limited mathematical knowledge continued to follow their textbooks closely. For example, Sonya shaped her lessons around her curriculum, and her activities came from her textbook while none of the teachers with strong mathematical knowledge used their textbooks to guide their lessons. Cross-sectional analysis of teachers' MKT and the Worthwhile Mathematical Tasks scale indicated that *teachers' textbook use* had confounding effects on the quality of their task choices. Although teachers with high MKT did not follow the curriculum as closely as did their colleagues with low MKT, they still used some problems and activities from their textbook. Furthermore, teachers, even the teachers with high MKT, had difficulty finding mathematically advanced and high-level thinking problems.

Analyses of the qualitative data pertaining to *Student Engagement* suggest two potential reasons for the lack of a quantitative relationship between the gain in teachers' mathematical knowledge and scores on the Student Engagement scale. First, probably due to the emphases in the master's program, most teachers began to ask their students to share their ideas regardless of how much gain the teachers made in their content knowledge. Additionally, although teachers were responsible for creating an environment in which all students effectively engage in learning, students' interests and motivation played an important role in the success of that environment, and, students varied from year to year in some ways beyond the teachers' control.

Finally, as suggested by the quantitative data, *teachers' beliefs* played an important role in their use of inquiry-oriented lessons, having a mathematical agenda of sense-making, and task choices. Although the qualitative data could not elaborate on the findings regarding their task choices, analysis of the data for inquiry-oriented lesson plans and having a mathematical agenda of sense-making indicated that the differences in teachers' inquiry-oriented teaching could be explained by differences in their beliefs. Only teachers who favored reformed-based teaching

tried to implement inquiry-oriented lessons. Similarly, only teachers who valued the meaning behind the mathematical procedures and algorithms focused on teaching the meaning behind the procedures as well.

Chapter 6

Discussion

Prior research has yielded inconclusive results regarding the relationships between elementary and middle school teachers' mathematical knowledge and their instructional practices, and between their mathematical knowledge and student learning (e.g., Greenwald, Hedges, & Laine, 1996; Hanushek, 1996; National Mathematics Advisory Panel, 2008; Rockoff et al., 2008; Rowan et al., 2002). The majority of studies have been cross-sectional and teachers' mathematical knowledge has not been measured by well-established assessments. By using a mixed-methods approach, this study sought to address more thoroughly the extent to which teachers' mathematical knowledge for teaching (MKT) affects instructional practices. In particular, mixed methods were used to develop a more comprehensive assessment of teachers' knowledge and instructional practices, leading to a more differentiated understanding of the role of teachers' MKT on their instruction at the elementary/middle school level.

This longitudinal study monitored the growth in mathematical knowledge and instructional practices of 21 in-service teachers as they participated in a master's degree program. At the beginning of the program, the teachers had very different levels of mathematical knowledge as well as different instructional practices. However, teachers' scores on both the mathematics test and four of five instructional practice scales changed substantially during the program. In particular, teachers' scores on the MKT measure and on the Inquiry-Oriented Lesson, the Mathematical Agenda of Sense-Making, Worthwhile Mathematical Task, and Classroom Climate scales changed significantly. Only teachers' scores on the Student Engagement scale did not change noticeably over the duration of the program.

Results of the quantitative analysis indicated that changes in teachers' MKT scores were able to predict changes in their scores on the Inquiry-Oriented Lesson, Mathematical Agenda of Sense-Making, and Classroom Climate scales. Results also indicated that changes in teachers' MKT scores did not correspond to changes in their task choice and their level of student engagement. Teachers' beliefs also appeared to be positively related to their scores on the Inquiry-Oriented Lesson, Worthwhile Mathematical Task, and Mathematical Agenda of Sense-Making scales. Further analysis also suggested that students' gain scores were positively associated with only the Student Engagement scale. The other four instructional practice scales and teachers' MKT scores were not related to students' gain scores. Teachers' beliefs about teaching and learning did not seem to be related to students' gain scores.

Results of the qualitative analysis shed light on the complex relationship among teachers' knowledge, beliefs, and instruction. Although teachers' beliefs appeared to be an important factor affecting teacher practices, beliefs alone were not enough for teachers to make substantial changes in their practices; teachers also need strong mathematical knowledge to make more pronounced changes in their practices. Looking across both data sets indicate that teachers favoring standards-based view of mathematics tended to take more initiatives to make changes in their practices; however, without strong mathematical knowledge, these changes were superficial. For instance, the portraits of the eight teachers' instructional practices at the end of the program indicated important differences in their lesson designs, suggesting that teachers' beliefs tended to complicate a direct relationship between teachers' knowledge and their use of an inquiry-oriented lesson design at a particular point in time. However, although most teachers reported some changes toward more inquiry-based teaching, some of the changes were superficial (e.g.,

more group work and more student presentation with no discussion) and only high-MKT teachers mentioned focusing on more conceptual changes in their teaching.

The results of this study also highlight the importance of teachers' existing level of mathematical knowledge on the "mathematical quality" of lessons. When teachers' mathematical knowledge increased, the teachers appeared to create an environment in which their students could make more sense of the concept being taught. This result is in agreement with the findings of Baumert et al. (2010) regarding the positive relationship between teachers' pedagogical content knowledge and the quality of teacher-student interaction around the mathematics tasks. However, an examination of individual teachers' scores suggested that the teachers who ended the program with limited mathematical knowledge did not increase their scores on that scale. Earlier work (e.g., Webb, 1991) indicated that the quality of the discourse affects student learning. Similarly, qualitative analysis of the teachers' instructional practices in this study suggested that teachers with limited knowledge could ask "how" and "why" questions to reveal their students' mathematical thinking, but when it came to building on the students' responses, the teachers' current level of MKT might have hindered their effectiveness. On the other hand, teachers with strong mathematical knowledge appeared to encourage students to think productively and could draw out the mathematics in students' responses (Charalambous, 2010). This study contributes to the existing literature in that it used both quantitative and qualitative data and compared some teachers who were using the same curricula and whose students had similar demographics to reduce the effects of contextual factors. Furthermore, this study, assessing teachers' beliefs, suggested that teachers' beliefs about teaching and learning mathematics also affected the teachers' mathematical sense-making agenda; however, their level of mathematical knowledge hindered or increased the effect of their beliefs about sense-making.

Looking across both data sets suggest that teachers with strong mathematical knowledge make more mathematically pronounced changes in their practices while teachers with limited mathematical knowledge were not able to reach that level of change in their practice. For instance, as indicated in the interview data, only teachers with strong mathematical knowledge reported including problems and activities that required higher order thinking. Teachers with limited mathematical knowledge, regardless of their beliefs, seemed to continue to follow their textbooks on a daily basis, and most of the problems and tasks were taken from their textbooks. On the other hand, teachers with stronger mathematical knowledge seemed to include activities from outside their curriculum. Although my research interest was not in the teachers' use of the curriculum, it seemed that those with high MKT scores followed their curriculum more loosely. Another important point that the qualitative data indicated was that although teachers with high MKT scores seemed to focus on mathematical aspects of the activities, it appeared that even teachers with strong mathematical knowledge, such as Rebecca, had difficulty finding appropriate tasks that promoted conceptual understanding. As illustrated in Sonya's case, the standards-based curriculum could help teachers create a mathematically more engaging environment, which might lead to a better conceptual understanding (McCaffrey et al., 2001).

The quantitative analysis also indicated that teachers created a more positive classroom climate when their mathematical knowledge increased. As reported by several teachers in their case studies, this might have been due to the increase in their self-confidence. Or, as Meg and Valerie mentioned, "struggling" during their mathematics courses in the program might have allowed them to put themselves in their students' shoes and empathize with them.

Finally, the Student Engagement scale, which captured the degree to which students shared and explained their ideas and worked collaboratively with their peers, did not seem to

correspond to an increase in teachers' MKT scores. This result seemed somewhat surprising; however, classroom observations of the teachers might help explain this finding. First, regardless of the change in the teachers' MKT, many teachers reported devoting more time to having their students share their ideas and work in groups. Second, as mentioned previously, regardless of the teachers' existing level of mathematical knowledge, some teachers had problems with their students' participation in classroom activities. The students in their classrooms played an important role in how the lesson was carried out, and their engagement was related to many factors both inside and outside of their current mathematics classroom. Group work was not productive when students fought within their groups, and sharing answers was not easy when other students did not respect the student at the board. However, as highlighted several times, this does not mean that student engagement did not depend on the teachers. Although this study could not explain why student engagement corresponded neither to teachers' beliefs nor to teachers' mathematical knowledge, it is important to draw attention to the fact that only two teachers with strong mathematical knowledge, Rebecca and Jacqueline, reported that their students were enthusiastic about learning more mathematics. Both teachers challenged their students by asking questions that required them to think. They both waited for their students to discover solutions. These aspects of their teaching might have been related to their students' eagerness to participate in mathematics lessons.

Using students' scores on the standardized tests, I also analyzed relationships among the teachers' MKT, instructional practices, and student achievement gains. I acknowledge that "standardized achievement tests, in particular, are exceedingly blunt instruments for measuring what students might learn in a given year from a given curriculum" (NRC, 2001, p. 479), and standardized test scores do not always reflect the extent to which a student has a good

understanding of mathematical concepts (Erlwanger, 1973; Schoenfeld, 1988). Student engagement was the only positive and significant predictor of student achievement gains. The positive relationship between student engagement and achievement is also supported by existing research. Providing opportunities for students to engage in and spending time on activities is “widely considered the single most important predictor of student achievement” (National Research Council [NRC], 2001, p. 334).

Teachers’ scores on the other scales and on the MKT test did not seem to correspond to their students’ ISAT gains.³⁴ Although the findings are discouraging, the results of this study are important, given that value-added models of teaching and pay-for-performance have become widely used and popular in research and hiring decisions. An earlier study indicated that teacher performance changes from year to year, especially for elementary school teachers (e.g., McCaffrey, Sass, Lockwood, & Mihaly, 2009). This study further suggests that neither teachers’ mathematical knowledge nor instructional practices—as measured in this study—can explain the variation in their students’ performances in a given year. Similarly, Hill, Kapitula, and Umland’s (2011) value-added study with middle school teachers indicated a nonsignificant correlation between teachers’ MKT and the mathematical quality of their instruction when teachers’ value-added scores in the models were adjusted for students’ background. Partly because the researchers used a composite score to present the mathematical quality of teacher instruction, the authors could not identify which aspects of instruction corresponded to student gains. Using qualitative data from two outlier teachers (with a low quality of mathematical instruction and high value-added scores), the authors speculated that one reason could be the level of student participation in mathematics lessons. The findings of this study support their conjecture

³⁴ I also analyzed student data by using subscales of students’ scores for the content areas in which teachers’ MKT was measured. This analysis yielded the same results.

suggesting that that student engagement is a significant predictor of student achievement gains, and teachers' instructional practices and their MKT do not correspond to gains in student achievement.

However, one lingering question remains unanswered: Why did teachers' mathematical knowledge specific to teaching and the instructional practices envisioned in the *Standards* (i.e., NCTM, 2000) not correspond to student achievement gains? One reason for the lack of relationship between teachers' instructional practices and their students' gains in achievement could be that teaching in an inquiry-oriented manner, choosing cognitively demanding tasks, and creating an environment in which students could make sense of mathematics is time consuming and might not be the most efficient means of test preparation. As Rebecca mentioned in the interview, seeing the positive effects of teaching when using an inquiry-based approach took more time, but the students retained the knowledge better. Earlier studies have indicated that because of this dilemma, teachers often report relying on traditional instructional methods owing to time constraints (Hiebert & Carpenter, 1992; Pesek & Kirshner, 2000). Hence, teachers are faced with many tensions and dilemmas in their teaching practice (Adler, 1998). In fact, Jacqueline mentioned during the interview that she had begun to teach using a completely inquiry-oriented approach only that year. She explained her reasons and worries about this decision:

Because I want my kids to learn and retain it. Like I'm all about that and I'm not about the ISAT and I kind of just—This is the first year I actually enjoyed my teaching more than ever. And I didn't care about ISAT, which is sad to say because I do care. But I care more that they're learning. And I kind of feel like if they're truly learning, they'll do fine on ISAT.

Another important factor that might have affected these findings was the student population in teachers' classrooms. Noting that teacher evaluation methods tend to ignore

student-related factors because of the assumption that student gain scores implicitly eliminate the effect of student background factors (e.g., Ballou, Sanders, & Wright, 2004), the results of this study challenges this assumption. As suggested by the quantitative analysis results, overall student engagement at the classroom level is an important factor predicting students' test score gains. Furthermore, classroom observations of the same teachers over 4 years indicated that even for the same teachers, students' level of interest varied from year to year. As indicated in earlier studies, teacher effects on students' test scores were related to the student population in the classroom (Hill et al., 2011). Furthermore, McCaffery, Lockwood, Koretz, Louis, and Hamilton (2004) point out the importance of including both student- and school-level demographics, indicating that school-level lunch eligibility could predict students' gains even after controlling for individual-level lunch eligibility.

The case of one particular teacher, Stephanie helps illustrate the effects of student-related factors on achievement. Stephanie had taught in one of the low-achieving schools in the partnership district the year before she enrolled in the program. Seventy-four percent of the students in that school were eligible for subsidized lunch. Even before she enrolled in the program, Stephanie's MKT score indicated that she had strong mathematical knowledge. However, Stephanie's students' average gain was not promising the year before the program started despite her high MKT, only 70% of her students met or exceed the state standards.³⁵ In contrast, in her current school in which only 39% of the students were eligible for subsidized

³⁵ As mentioned earlier, because I had access to data for that particular district, I was able to see her students' ISAT scores in 2007; however, I did not have her students' ISAT data for the last year of the program.

lunch, her students had one of the very highest ISAT scores in the following years³⁶. As she explained during the interview, the main reason behind this difference was the change in student characteristics. Her comparison of the students in her current school with those in her previous school seemed to illustrate the effects of confounding factors on student ISAT gains. She commented on her students' ISAT scores in her current school: "My ISAT scores are high. . . . Since I started the program, my scores have been higher than anyone else's in the school." She then mentioned that she had left her prior school before officially receiving her students' ISAT scores:

I have no doubt they were very low. I did not feel successful there, and I did not have the curriculum that they have now [referring to her old school]. . . . It's like broken from the top. It's broken from the bottom. It's just so fractured, and it's sad because there are so many good teachers there. . . . And it doesn't matter how much money you give them. They're not going to be successful in this environment because we have broken kids and we have some administration problems, and it's not an effective place to teach. . . . And I thought all I would have to do is go in there and love them and be sweet and kind and show that I care and have a passion for math and I would make a difference, and that's not what happened. So I got out as fast as I could. It was not the experience for me.

Stephanie's explanation of her lack of success in a low-achieving, less affluent school also highlights the importance of school-related factors that affect teacher performance. I would also like to draw attention to the effect of standardized tests on teachers' practices. Some teachers' decisions about the instructional practices they used appeared to be influenced by their students' scores on the standardized test. Beth reported using group work more frequently, and she explained the reason for this: "My students have learned the true meaning of group interaction, and test scores show that all of my students are functioning at or above the fifth-grade level in math." Valerie explained why she chose particular instructional methods:

³⁶ As mentioned earlier, I did not have access to Stephanie's students' ISAT scores in her current school; however, based on the state report cards, more than 90% of the seventh graders met or exceeded the state standards in 2010.

“Because I’ve found that it works. Our school scores, like, top in standardized testing, at the top in the district, because what we do works.” In addition, it is important to highlight that all teachers, regardless of the grade level they taught, were under pressure because of the standardized tests. For instance, both Rebecca and Meg mentioned feeling pressure because of the district-wide tests (ThinkLink) that their students were required to take. Meg mentioned “[I have] concerns, definitely, because we are held accountable for those results.” Rebecca said,

I do have ThinkLink, and I’m very much held accountable to it. Like my scores are plastered on the wall downstairs for all the teachers to see. . . . I guess I would probably continue to teach this, but honestly, I mean, if my kids’ scores were so low, then I probably, you know—that would play a factor in it.

Given that assessment of teachers’ performance is dependent on the skills measured on students’ tests (e.g., Lockwood, McCaffrey, Hamilton, Stecher, Le, & Martinez, 2007), this suggests that if the standardized tests were not designed to capture student conceptual understanding, the teachers might be discouraged from teaching in the way envisioned in the *Standards* (NCTM, 1991, 2000).

Limitations

This study has several limitations worth noting. First, the sample of teachers in the study was nonrandom because the teachers chose to participate in a master’s program focused on mathematics and science. Hence, the findings shown here and the *p*-values reported are intended to be suggestive of relationships that may be present in a more general population, as opposed to definitive measures of relationships for all teachers.

A second potential limitation is the small sample size of teachers, which may limit the sensitivity to finding some existing relationships. In addition, the sample size prevented me from adding more teacher-level variables. However, in a multilevel longitudinal analysis, sample size

should be considered separately for each level, indicating that increasing the number of teachers may not increase the power of analysis as much as increasing the number of time points (years). Because I was interested in the change in teachers' practices, having more time points increased the power of the study more than having a greater number of teachers. Still, the small number of teachers involved means that some relationships that might be significant in the general population might have appeared insignificant in this analysis.

A third limitation is the quality of the measures. All measures used in this study, that is, the ISAT and the LMT Test, the classroom observation protocol, and the beliefs survey, included some measurement error, which reduced the possibility of finding and measuring relationships accurately. In addition to errors in the measures themselves, other errors may have been present, such as ones related to the testing conditions, however tests of teacher knowledge were carefully administered to minimize potential problems. That is, I tried to administer the tests in the same type of room (college classroom) during the same time of day, with the same directions and time allotment provided. Additionally, the fact that teachers were paid for their time to complete the assessments seemed to encourage them to take the repeated tests seriously and without complaint. Another potential source of measurement error relates to the limited number of observations conducted in teachers' classrooms. That is, two or three observations of teachers might not have sufficiently captured their teaching. To overcome this problem, I augmented the observation data with teachers' self-reports of how they taught. I also specifically asked to observe their typical mathematics lessons; the teachers were asked to refrain from preparing special lessons for observations because they might not have reflected their typical teaching.

Finally, the absence of audio recordings during the first 2 years of classroom observations was another limitation, which prevented me from using the qualitative data collected in those

years. This study could have been more powerful had I and other researchers in the program taken more detailed field notes during those earlier classroom visits instead of relying primarily on the observation protocol to capture the important details of the lessons.

Implications

The findings of this study have several implications spanning a variety of areas, including research, teacher education, professional development, and education policy.

Implications for research. This study informs researchers and teacher educators about which aspects of instructional practices are most closely related to teacher knowledge and which aspects are related to teacher beliefs, as opposed to (or in addition to) their mathematical knowledge. The fact that the teachers who had greater gains in MKT scores tended to move toward more inquiry-oriented lessons—despite the fact that all teachers were encouraged to move in that direction throughout the program—suggests that such knowledge may be linked in important ways to inquiry-based teaching. The mathematical quality of lessons was related to teachers’ level of mathematical knowledge. However, the results of this study were obtained from a limited, non-random sample of teachers, most of whom taught elementary school. Further studies are needed to examine whether the relationships identified here hold up in a larger sample, as well as whether similar relationships exist for middle and high school mathematics teachers.

The findings of this study also inform researchers regarding which instructional practices are related to teachers’ beliefs. This study suggests that teachers’ beliefs correspond to the extent to which the teachers design inquiry-based lessons, what tasks and activities they choose, and how they create a mathematically powerful environment. Since teachers’ beliefs were captured

once in this study, studies capturing changes in teachers' beliefs as well as knowledge are needed to investigate how beliefs and knowledge interact and affect teaching practices over time.

Implications for teacher education and professional development. The study also informs teacher educators and professional development designers about what activities could be used to improve teachers' MKT scores. The teachers in this master's degree program increased their MKT significantly and retained it. As illustrated in earlier sections, teachers reported that their confidence increased and that they created a more positive classroom environment in their mathematics lessons. In addition, they reported beginning to teach concepts they had not been comfortable with before. It seems that challenging courses that include both mathematical concepts and pedagogy could be beneficial for in-service teachers. Since the teachers increased specialized mathematical knowledge for teaching more in the program's content/pedagogy hybrid course as opposed to a mathematics content course (Copur-Gencturk & Lubienski, under review), teacher educators and professional development designers might consider creating more hybrid courses.

This study also brings attention to interactions between teachers' beliefs and mathematical knowledge as they shape teacher practices. In this, teachers who did not hold beliefs aligned with standards-based teaching made fewer prominent changes to their practices. Teacher education programs should focus on teacher beliefs as well as teachers' mathematical knowledge. This study also suggests that teachers might perceive standards-based teaching differently than what teacher educators envision. Teachers might have the misconception that asking "how" and "why" questions is enough for helping students make sense of concepts. Teacher educators should confront this notion and find ways to support teachers in making more effective use of students' responses to those questions.

Qualitative analysis of the data also suggested that teachers had difficulty choosing mathematically appropriate tasks for their students. The findings indicate that even teachers with strong mathematical knowledge were not able to choose mathematically rich problems. In teacher education programs, special attention should be given to what the teacher should take into consideration when choosing problems and activities to carry out in a lesson. Furthermore, teachers should be well informed regarding what “math should be fun” means and how math could be made fun by finding mathematically engaging activities.

The findings also imply that detecting the effect of teachers’ knowledge and instructional practices on student learning captured by standardized tests is not an easy task. Using state-mandated tests as student outcome measures limited the number of teachers who could be included in the study because students of the lower elementary school teachers were usually not tested. Additionally, the content focus of professional development programs might not completely align with that assessed by the state-mandated tests. Hence, professional development designers striving to detect student gains should consider using alternative (well-established) tests that are more specific to content covered in the professional development programs.

Implications for education policy. In agreement with earlier work (e.g., NRC, 2001), this study suggests that students being on task, working collaboratively, and sharing and explaining their ideas are related to student gains. On the other hand, teaching students in an inquiry-oriented manner and pressing students to probe their thinking do not seem to be related to their scores on the standardized tests. As mentioned earlier, the state-mandated tests are taken in the middle of the spring semester, and teaching in an inquiry-oriented manner might not allow teachers to cover all the concepts measured on the standardized tests. Prior research has indicated

that teachers tend to rely on traditional ways of teaching because of this dilemma (e.g., Hiebert & Carpenter, 1992; Pesek & Kirshner, 2000). However, the present study also suggests that the teachers themselves use state scores as a criterion for the effectiveness of certain teaching methods. For example, could Jacqueline have continued to focus on student learning if her students had failed the test? Or, as Rebecca confessed, if students' scores did not increase, could teachers continue to use inquiry-based teaching? If the standardized tests are not designed to capture students' conceptual understanding or if they "punish" teachers who strive to create a mathematically rich environment so that their students can build a deep understanding of mathematical concepts, students' understanding of mathematics will never reach the desired level.

Another important implication of this study is related to the evaluation of teachers based on the value added to their students' test scores. Echoing the concerns related to not controlling for student-related factors adequately in teacher evaluations (e.g., Amrein-Beardsley, 2008; Hill et al., 2011), this study further suggests that the student populations in teachers' classrooms confound measures of teacher effectiveness. By highlighting the important of, and differences in the mathematical engagement of cohorts each year, this study suggests that the variability in teachers' performance (e.g., Goldhaber & Hansen, 2010; McCaffrey et al., 2009) is related not only to "noise" in student test scores and to student demographics as traditionally measured, but also to differences in the "personalities" of student cohorts assigned to the teachers. Although one would hope that a skillful teacher can create a climate of high engagement within any cohort, the data from this study reveal that such a climate is dependent upon both the students and the teacher. Without controlling for students' characteristics, interests, motivation, and prior experiences at the individual and classroom levels, traditional evaluations of teacher

effectiveness are insufficient to consistently and fairly identify teachers with the most effective teaching skills. Using student trajectories rather than covariate-adjusted models could control students' related factors to some extent. However, given that existing student trajectory models do not control individual student's demographics (e.g., Education Value Assessment System (EVAAS) Sanders, 1998), more advanced student trajectory models are needed for more accurate measures of teacher performances.

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Appendix A

Adapted Classroom Observation Protocol

Name of observer_____ Code of teacher observed:_____ Date of observation_____

Pre Observation Interview Questions:

1. What has this class been covering recently? (what unit are you working on)?
2. What would you like the students to learn during this class?
3. In a paragraph or two, describe the lesson you observed. Include where this lesson fits in the overall unit of study. Include the general lesson structure and enough detail to provide a context for your ratings of this lesson.

I. Classroom Demographics and Context

A. What is the total number of students in the class? _____ (give exact count)

B. What is the number of students in the class at the time of the observation who are White?____ Af. American?____ Latino/a?____ Other?____

C. Indicate the *primary content area* of this lesson or activity. (*In general, choose just one.*)

- ☐ 1. Numeration and number theory
- ☐ 2. Computation (please specify: _____)
- ☐ 3. Estimation
- ☐ 4. Measurement (please specify: _____)
- ☐ 5. Patterns and relationships
- ☐ 6. Pre-algebra
- ☐ 7. Algebra
- ☐ 8. Geometry and spatial sense
- ☐ 9. Functions (including trigonometric functions) and pre-calculus concept
- ☐ 10. Data collection and analysis
- ☐ 11. Probability
- ☐ 12. Statistics (e.g., hypothesis tests, curve-fitting, and regression)
- ☐ 13. Topics from discrete mathematics (e.g., combinatorics, graph theory, recursion)
- ☐ 14. Mathematical structures (e.g., vector spaces, groups, rings, fields)
- ☐ 15. Calculus

- ☐ 16. None of the above (please explain)

D. Indicate the *primary intended purpose(s)* of this lesson or activity based on the pre- and/or post-observation interviews with the teacher. (In general, choose just one.)

- ☐ 1. Identifying prior student knowledge
- ☐ 2. Introducing new concepts
- ☐ 3. Developing conceptual understanding
- ☐ 4. Reviewing mathematics concepts
- ☐ 5. Developing problem-solving skills
- ☐ 6. Learning mathematics processes, algorithms, or procedures
- ☐ 7. Learning vocabulary/specific facts
- ☐ 8. Practicing computation for mastery
- ☐ 9. Developing appreciation for core ideas in mathematics
- ☐ 10. Developing students' awareness of contributions of mathematicians of diverse backgrounds
- ☐ 11. Assessing student understanding

E. Indicate the *major way(s)* in which student activities were structured.

- ☐ As a whole group
- ☐ As small groups
- ☐ As pairs
- ☐ As individuals

F. Indicate the *major way(s)* in which students engaged in class activities.

- ☐ Entire class was engaged in the same activities at the same time.
- ☐ Groups of students were engaged in different activities at the same time (e.g., centers).

G. Please provide specific times for each lesson component:

___# minutes whole group instruction/discussion (generally teacher-led instruction)

___# minutes small group work on experiments/tasks that are part of lesson/instruction

___# minutes individual work on experiments/tasks that are part of lesson/instruction

___# minutes for homework in small groups (most students collaborating substantially)

___# minutes for homework as individuals (most work done individually without collaboration)

H. Rate the adequacy of the physical environment.

a. Classroom resources:

1	2	3	4	5
Sparsely equipped				Rich in resources

b. Classroom Space:

1	2	3	4	5
Crowded				Adequate space

c. Room arrangement:

1 2 3 4 5
Inhibited interactions among students Facilitated interactions among students

II.

A. Lesson Design and Implementation (1 = never, 2 = very little, 3 = some, 4 = mostly, 5 = consistently)

1. The design of the lesson reflected careful planning and organization.	1	2	3	4	5
2. The design of the lesson incorporated tasks, roles, and interactions consistent with investigative mathematics.	1	2	3	4	5
3. The lesson had a problem/investigation-centered structure (e.g., teacher launched a problem/investigation, students explored, and teacher led a synthesizing discussion.)	1	2	3	4	5
4. The instructional objectives of the lesson were clear and the teacher was able to clearly articulate what mathematical ideas and/or procedures the students were expected to learn.	1	2	3	4	5
5. The lesson design provided opportunities for student discourse around important concepts in mathematics.	1	2	3	4	5
6. Mathematics was portrayed as a dynamic body of knowledge continually enriched by conjecture, investigation analysis, and/or proof/justification.	1	2	3	4	5
7. The teacher appeared confident in his/her ability to teach mathematics.	1	2	3	4	5
8. The instructional strategies were consistent with investigative mathematics.	1	2	3	4	5
9. The teacher's questioning strategies for eliciting student thinking promoted discourse around important concepts in mathematics.	1	2	3	4	5
10. The pace of the lesson was appropriate for the developmental level/needs of the students and the purpose of the lesson.	1	2	3	4	5
11. The teacher was flexible and able to take advantage of "teachable moments," (including building from students' ideas – both mathematical and non-mathematical).	1	2	3	4	5
12. The teacher's classroom management style/strategies enhanced the quality of the lesson.	1	2	3	4	5
13. The vast majority of the students were engaged in the lesson and remained on task.	1	2	3	4	5
14. Appropriate connections were made to other areas of mathematics, to other disciplines, and/or to real-world contexts.	1	2	3	4	5

B. Mathematical Discourse and Sensemaking

1. Student asked questions to clarify their understanding of mathematical ideas or procedures. <i>Logistical questions – "may I sharpen my pencil?" don't count.</i>	1	2	3	4	5
2. Students shared their observations or predictions.	1	2	3	4	5
3. Students explained mathematical ideas and/or procedures.	1	2	3	4	5
4. Students justified mathematical ideas and/or procedures.	1	2	3	4	5

5. Students listened intently and actively to the ideas and/or procedures of others for the purpose of understanding someone's methods or reasoning.	1	2	3	4	5
6. Students challenged each other's and their own ideas that did not seem valid.	1	2	3	4	5
7. Students defended their mathematical ideas and/or procedures.	1	2	3	4	5
8. Students determine the correctness/sensibility of an idea and/or procedure based on the reasoning presented.	1	2	3	4	5
9. Students made generalizations, or made generalized conjectures regarding mathematical ideas and procedures.	1	2	3	4	5
10. Students drew upon a variety of methods (verbal, visual, numerical, algebraic, graphical, etc.) to represent and communicate their mathematical ideas and/or procedures.	1	2	3	4	5
11. The teacher and students engaged in meaning making at the end of the activity/instruction. (There was a synthesis or discussion about what was intended to be learned from doing the activity.)	1	2	3	4	5
12. The teacher productively probed/"pushed on" the mathematics in students' responses (including both correct and incorrect responses).	1	2	3	4	5

C. Task Implementation

1. Tasks focused on understanding of important and relevant mathematical concepts, processes, and relationships.	1	2	3	4	5
2. Tasks stimulated complex, nonalgorithmic thinking.	1	2	3	4	5
3. Tasks successfully created mathematically productive disequilibrium among students.	1	2	3	4	5
4. Tasks encouraged students to search for multiple solution strategies and to recognize task constraints that may limit solution possibilities.	1	2	3	4	5
5. Tasks encouraged students to employ multiple representation and tools to support their learning, ideas and/or procedures.	1	2	3	4	5
6. Tasks encouraged students to think beyond the immediate problem and make connections to other related mathematical concepts.	1	2	3	4	5

D. Classroom Culture

1. Active participation of all students was encouraged and valued.	1	2	3	4	5
2. The teacher displayed respect for students' ideas, questions, and contributions.	1	2	3	4	5
3. Interactions reflected a productive working relationship among students.	1	2	3	4	5
4. Interactions reflected a collaborative working relationship between the teacher and the students.	1	2	3	4	5
5. Wrong answers were treated as worthwhile learning opportunities	1	2	3	4	5
6. Students were willing to openly discuss their thinking and reasoning.	1	2	3	4	5
7. The classroom climate encouraged students to engage in mathematical discourse.	1	2	3	4	5

E. Overall Rating—For each section below, mark the choice that best describes your overall summary of the lesson based on the observation.

1. Depth of Student Knowledge and Understanding—This scale measures the depth of the students' mathematical knowledge as evidenced by the opportunities students had to produce new knowledge by discovering relationships, justifying their hypotheses, and drawing conclusions.

A. Knowledge was very superficial. Mathematical concepts were treated trivially or presented as non-problematic. Students were involved in the coverage of information that they are to remember, but no attention was paid to the underlying mathematical concepts. For example, students applied an algorithm for factoring binomials or used the FOIL method of multiplication—in either case with no attention to the underlying concepts.

B. Knowledge was superficial or fragmented. Underlying or related mathematical concepts and ideas were mentioned or covered, but only a superficial acquaintance with or trivialized understanding of these ideas was evident. For example, a teacher might have explained why binomials are factored or why the FOIL method works, but the focus remained on students mastering these procedures.

C. Knowledge was uneven; a deep understanding of some mathematics concepts was countered by a superficial understanding of other concepts. At least one idea was presented in depth and its significance was grasped by some students, but in general the focus was not sustained.

D. Knowledge was relatively deep because the students provide information, arguments, or reasoning that demonstrate the complexity of one or more ideas. The teacher structured the lesson so that many (20% to 50%) students did at least one of the following: sustain a focus on a topic for a significant period of time; demonstrate their understanding of the problematic nature of a mathematical concept; arrive at a reasoned, supported conclusion with respect to a complex mathematical concept; or explain how they solved a relatively complex problem. Many (20% to 50%) students clearly demonstrated understanding of the complexity of at least one mathematical concept.

E. Knowledge was very deep. The teacher successfully structured the lesson so that almost all (90% to 100%) students did at least one of the following: sustain a focus on a topic for a significant period of time; demonstrate their understanding of the problematic nature of a mathematical concept; arrive at a reasoned, supported conclusion with respect to a complex mathematical concept; or explain how they solved a complex problem. Most (51% to 90%) students clearly demonstrated understanding of the complexity of more than one mathematical concept.

2. Locus of Mathematical Authority—This scale determines the extent to which the lesson supported a shared sense of authority for validating students' mathematical reasoning.

A. Students relied on the teacher or textbook as the legitimate source of mathematical authority. Students accepted an answer as correct only if the teacher said it was correct or if it was found in the textbook. If stuck on a problem, students almost always asked the teacher for help.

B. Students relied on the teacher and some of their more capable peers (who were clearly recognized as being better at math) as the legitimate sources of mathematical authority. The teacher often relied on the more capable students to provide the right answers when pacing the lesson or to correct erroneous answers. As a result, other students often relied on these students for correct solutions, verification of right answers, or help when stuck.

C. Many (20% to 50%) students shared mathematical authority among themselves. They tended to rely on the soundness of their own arguments for verification of answers, but, they still looked to the teacher as the authority for making final decisions. The teacher intervened with answers to speed things up when students seemed to be getting bogged down in the details of an argument.

D. Most (51% to 90%) students shared in the mathematical authority of the class. Though the teacher intervened when the students got bogged down, he or she did so with questions that focused the students' attention or helped the students see a contradiction that they were missing. The teacher often answered a question with a question, though from time to time he or she provided the students with an answer.

E. Almost all (90% to 100%) of the students shared in the mathematical authority of the class. Students relied on the soundness of their own arguments and reasoning. The teacher almost always answered a question with a question. Many (20% to 50%) students left the class still arguing about one or more mathematical concepts.

3. Social Support—This scale measures the extent to which the teacher supported the students by conveying high expectations for all students.

A. Social support was negative. Negative teacher or student comments or behaviors were observed. The classroom atmosphere was negative.

B. Social support was mixed. Both negative and positive teacher or student comments or behaviors were observed.

C. Social support was neutral or mildly positive. The teacher expressed verbal approval of the students' efforts. Such support tended, however, to be directed to students who were already taking initiative in the class and tended not to be directed to students who were reluctant participants or less articulate or skilled in mathematical concepts.

D. Social support from the teacher was clearly positive and there was some evidence of social support among students. The teacher conveyed high expectations for all, promoted mutual respect, and encouraged the students try hard and risk initial failure.

E. Social support was strong. The class was characterized by high expectations, challenging work, strong effort, mutual respect, and assistance for all students. The teacher and the students demonstrated these attitudes by soliciting contributions from all students, who were expected to put forth their best efforts. Broad participation was an indication that low-achieving students received social support for learning.

4. Student Engagement in Mathematics—This scale measures the extent to which students engaged in the lesson (e.g., attentiveness, doing the assigned work, showing enthusiasm for work by taking initiative to raise questions, contributing to group tasks, and helping peers).

A. Students were disruptive and disengaged. Students were frequently off task as evidenced by gross inattention or serious disruptions by many (20% to 50%).

B. Students were passive and disengaged. Students appeared lethargic and were only occasionally on task. Many (20% to 50%) students were either clearly off task or nominally on task but not trying very hard.

C. Students were sporadically or episodically engaged. Most (51% to 90%) students were engaged in class activities some or most of the time, but this engagement was uneven, mildly enthusiastic, or dependent on frequent prodding from the teacher.

D. Student engagement was widespread. Most (51% to 90%) students were on task pursuing the substance of the lesson most of the time. Most (51% to 90%) students seemed to take the work seriously and try hard. Or virtually all (90-100%) students are on task, but they do not seem genuinely interested in the subject at hand. They might be engaged with activities but not the mathematical ideas.

E. Students were seriously engaged. Almost all (90% to 100%) students were deeply engaged in pursuing the substance of the lesson almost all (90% to 100%) of the time. Kids are actually *interested* in the *mathematics* – they are taking initiative, grappling with questions/problems/ideas.

Appendix B

Tentative Focus Teacher Interview Protocol

1-) Can you briefly describe what you planned to teach in the lessons I observed? (Can you tell me more about what you mean by that?)

2-) How do you think the lessons went? (can push on “what do you mean by that? Or “What do you mean by “good”? Can segue to #3 below)

3-) I’m interested in what and how students learn in your classroom.

- What do you think students learned from these lessons? How can you tell what students learned?
- What do you think students had difficulty learning – or were there things they didn’t learn? How can you tell?
- What did you notice about student engagement during the lessons?

4-) How do you think students learn mathematics? Has this program influenced your thinking about how students’ learn mathematics? How does your student learning influence your teaching

5-) What do you think went particularly well in these lessons?

- What would you change next time you teach it? Why?
- Would you spend more or less time on this unit (or topic) next time?

6-) I’m interested in understanding the instructional strategies that teachers use.

- What instructional strategies did you use? (If they ask what it means -- Particular methods or structures you used)
- Why did you use those strategies?

7-) What do you think about how mathematics should be taught? Has this program affected the way you think about teaching mathematics? If yes, how has your view on teaching mathematics changed?

8-) What are your beliefs about the nature of mathematics? (Not just do you like it or not, but more like is it easy or hard, is it a bunch of rules you have to memorize or something that makes sense, etc.

9- Has this masters' program affected your views of mathematics? What, specifically, about the program has helped you change your mind about student learning, or how to teach math, or what math is?

10- In what ways has this program changed your teaching of math?

11-How has the program affected your knowledge of math? How does this change have an impact on your teaching (e.g., are there any particular mathematical topics or lessons that you are more confident teaching now than before the program?) Why/how did that change for you?

13-What math topics did you learn the most about in this program? Do you ever have moments when teaching math where you think specifically about a math idea you learned in the program? Describe some of those moments...

Appendix C

Teacher Beliefs Survey

Teacher ID: _____

Please circle the response that best indicates how you feel about the each statement in the following questionnaire.

1- Strongly disagree 2-Disagree 3-Undecided 4- Agree 5- Strongly agree

1. A vital task for the teacher is motivating children to solve their own mathematical problems.	1	2	3	4	5
2. Ignoring the mathematical ideas that children generate themselves can seriously limit their learning.	1	2	3	4	5
3. It is important for children to be given opportunities to reflect on and evaluate their own mathematical understanding	1	2	3	4	5
4. It is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other.	1	2	3	4	5
5. Effective mathematics teachers enjoy learning and ‘doing’ mathematics themselves.	1	2	3	4	5
6. Knowing how to solve a mathematics problem is as important as getting the correct solution.	1	2	3	4	5
7. Teachers of mathematics should be fascinated with how children think and intrigued by alternative ideas.	1	2	3	4	5
8. Providing children with interesting problems to investigate in small groups is an effective way to teach mathematics.	1	2	3	4	5
9. Mathematics is a beautiful, creative and useful human endeavor that is both a way of knowing and a way of thinking.	1	2	3	4	5
10. Allowing a child to struggle with a mathematical problem, even a little tension, can be necessary for learning to occur.	1	2	3	4	5
11. Children always benefit by discussing their solutions to mathematical problems with each other.	1	2	3	4	5

12. Persistent questioning has a significant effect on children's mathematical learning.	1	2	3	4	5
13. Justifying the mathematical statements that a person makes is an extremely important part of mathematics.	1	2	3	4	5
14. As a result of my experience in mathematics classes, I have developed an attitude of inquiry.	1	2	3	4	5
15. Teachers can create, for all children, a non-threatening environment for learning mathematics.	1	2	3	4	5
16. It is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems.	1	2	3	4	5
17. There is an established amount of mathematical content that should be covered at each grade level.	1	2	3	4	5
18. It is important that mathematics content be presented to children in the correct sequence.	1	2	3	4	5
19. Mathematical material is best presented in an expository style: demonstrating, explaining and describing concepts and skills.	1	2	3	4	5
20. Mathematics is computation.	1	2	3	4	5
21. Telling the children the answer is an efficient way of facilitating their mathematics learning.	1	2	3	4	5
22. I would feel uncomfortable if a child suggested a solution to a mathematical problem that I hadn't thought of previously.	1	2	3	4	5
23. It is not necessary for teachers to understand the source of children's errors; follow-up instruction will correct their difficulties.	1	2	3	4	5
24. Listening carefully to the teacher explain a mathematics lesson is the most effective way to learn mathematics.	1	2	3	4	5
25. It is important to cover all the topics in the mathematics curriculum in the textbook sequence.	1	2	3	4	5
26. If a child's explanation of a mathematical solution doesn't make sense to the teacher it is best to ignore it.	1	2	3	4	5