COLORINGS AND LIST COLORINGS OF GRAPHS AND HYPERGRAPHS

BY

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DISSERTATION

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Abstract

In this thesis we study some extremal problems related to colorings and list colorings of graphs and hypergraphs. One of the main problems that we study is: What is the minimum number of edges in an r-uniform hypergraph that is not t-colorable ? This number is denoted by $m(r,t)$. We study it for general r-uniform hypergraphs and the corresponding parameter for simple hypergraphs. We also study a version of this problem for conflict-free coloring of hypergraphs. Finally, we also look into list coloring of complete graphs with some restrictions on the lists.

Let t be a positive integer and $n = \lfloor \log_2 t \rfloor$. Generalizing earlier known bounds, we prove that there is a positive $\epsilon(t)$ such that for sufficiently large r, every r-uniform hypergraph with maximum edge degree at most

$$\epsilon(t)\, t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$$

is t-colorable. The above expression is also a lower bound for $m(r,t)$.

A hypergraph is *b-simple* if no two distinct edges share more than b vertices. Let $m(r,t,g)$ denote the minimum number of edges in an r-uniform non-t-colorable hypergraph of girth at least g. Erdős and Lovász [10] proved that

$$m(r,t,3) \geq \frac{t^{2(r-2)}}{16r(r-1)^2}$$

$$\text{and} \quad m(r,t,g) \leq 4 \cdot 20^{g-1} r^{3g-5} t^{(g-1)(r+1)}.$$

A result of Z. Szabó [30] improves the lower bound by a factor of $r^{2-\epsilon}$ for sufficiently large r. We improve the lower bound by another factor of r and extend the result to b-simple hypergraphs. We also get a new lower bound for hypergraphs with a given girth. Our results imply that for fixed b, t and ϵ and sufficiently large r, every r-uniform b-simple hypergraph $\mathcal{H}$ with maximum edge-degree at most $t^r r^{1-\epsilon}$ is t-colorable. Some results hold for list coloring, as well.

We also study the same problem for conflict-free coloring. A coloring of the vertices of a hypergraph $\mathcal{H}$ is called *conflict-free* if each edge e of $\mathcal{H}$ contains a vertex whose color does not get repeated in e. The smallest number of colors required for such a coloring is called the *conflict-free*

chromatic number of $\mathcal{H}$ and is denoted by $\chi_{CF}(\mathcal{H})$. Pach and Tardos studied this parameter for graphs and hypergraphs. Among other things, they proved that for a $(2r-1)$-uniform hypergraph $\mathcal{H}$ with m edges, $\chi_{CF}(\mathcal{H})$ has the order $m^{1/r}\log m$. They also asked whether the same result holds for r-uniform hypergraphs. In this thesis we show that this is not necessarily true. Furthermore, we provide lower and upper bounds on the minimum number of edges in an r-uniform simple hypergraph that is not conflict-free k-colorable.

Another topic we study is "choosability with separation" for complete graphs. For a graph G and a positive integer c, let $\chi_l(G,c)$ be the minimum value of k such that one can properly color the vertices of G from any lists $L(v)$ such that $|L(v)| = k$ for all $v \in V(G)$ and $|L(u) \cap L(v)| \le c$ for all $uv \in E(G)$. Kratochvíl, Tuza and Voigt [24] asked to determine $\lim_{n\to\infty} \chi_l(K_n,c)/\sqrt{cn}$, if it exists. We prove that the limit exists and equals 1. We also find the exact value of $\chi_l(K_n,c)$ for infinitely many values of n.

Section 2 deals with coloring of general hypergraphs. It is a joint work with A. Kostochka and V. Rödl and appears in [22]. Section 3 deals with coloring of simple hypergraphsa. It is a joint work with A. Kostochka and appears in [20]. In Section 4, we study conflict-free coloring of hypergraphs and it is a joint work with A. Kostochka and T. Łuczak. It appears in [21]. Section 5 deals with separated list coloring of complete graphs. It is a joint work with Z. Füredi and A. Kostochka and appears in [14].

To my parents

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Chapter 1

Introduction

1.1 Basic definitions for graphs and hypergraphs

In this section we review the basic terminology used throughout this thesis. For the most part we follow the text of West [31].

For every $x \in R$, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer not greater than x and the smallest integer not less then x, respectively. We sometimes use $[n]$ to denote the set $\{1, 2, ..., n\}$.

A *graph* G with n vertices and m edges consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge $e \in E(G)$ consists of two (possibly equal) vertices called its *endpoints*. We write uv for an edge $\{u, v\}$. We say that u and v are adjacent and that u and v are *incident* to e if $e = uv \in E(G)$. When two vertices are adjacent, they are *neighbors*.

A *loop* is an edge whose endpoints are identical. *Multiedges* are edges with the same pair of endpoints. A graph is *simple* if it has no loops or multiedges.

The set of neighbors of a vertex v in a graph G is the *neighborhood* of v, denoted by $N_G(v)$ or $N(v)$. The number of edges incident to v is the *degree* of v, denoted by $d_G(v)$ or $d(v)$. In other words, $|N_G(v)| = d_G(v)$. The minimum and maximum degree among the vertex degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Graph G is *regular* if the degrees of all the vertices in G are the same. G is *d-regular* if the degree of every vertex in G is d. The degree of an edge e is the number of edges adjacent to e. *Edge degree* of G is the maximum of the degrees of the edges among $E(G)$, denoted $D(G)$.

A *path* of *length* l in a graph G is an alternating sequence $v_0, e_0, v_1, e_1, v_2, ..., e_{l-1}, v_{l-1}$ of distinct vertices and edges in G such that v_i, v_{i+1} are incident to e_i, for $0 \leq i \leq l-1$. The *distance* between two vertices is the number of edges in a shortest path having them as endpoints. A *cycle* of *length* l in a graph G is an alternating cyclic sequence $v_0, e_0, v_1, e_1, ..., e_{l-1}, v_0$ of distinct edges and vertices in G such that v_i is incident to e_{i-1}, e_i for $1 \leq i \leq l-1$ and v_0 incident to e_0, e_{l-1}. The *girth* of a graph is the length of the smallest cycle in the graph.

A *subgraph* H of a graph G is a graph whose vertex set is a subset of $V(G)$ and edge set is a subset of $E(G)$. A subgraph of G is an *induced subgraph* if it is obtained by undefining a set of vertices. A *complete graph* on n vertices is a simple graph in which all vertices are pairwise adjacent.

The isomorphism class of which is denoted by K_n. An *independent set* of vertices in a graph G is a set $S \subseteq V(G)$ whose elements are pairwise non-adjacent. The size of a largest independent set of G is denoted by $\alpha(G)$. A *bipartite graph* is a graph whose vertices can be partitioned into two sets such that each edge has one endpoint in each of these two sets. A graph G is *connected* if it has a u, v-path whenever $u, v \in V(G)$. The *components* of a graph are its maximal connected subgraphs. A *forest* is a graph without cycles. A *tree* is a connected forest. A *matching of size k* is a forest with k components such that each component has two vertices.

A *t-coloring* of a graph is a labeling of its vertices from a set S of size t. A *proper t-coloring* of a graph is a t-coloring such that adjacent vertices are labeled by different elements. The elements of S are called *colors*. The smallest t for which a graph has a proper t-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$.

Given a graph G, a *list assignment L for G* is an assignment of a set $L(v)$ of colors for every $v \in V(G)$. We say that G is L-colorable, if there exists a proper coloring f of the vertices of G from L, i.e. if $f(v) \in L(v)$ for all $v \in V(G)$ and $f(u) \neq f(v)$ for all $uv \in E(G)$. The *list chromatic number* of G, denoted by $\chi_l(G)$, is the least k such that G is L-colorable whenever $|L(v)| = k$ for all $v \in V(G)$. It is also sometimes called the *choice number* or the *choosability* of G.

A *hypergraph* $\mathcal{H}$ is a pair (V, E), where V is a set of *vertices*, and E is a set of non-empty subsets of V called *edges*. In other words, a hypergraph is a set-system. The notion of a hypergraph is a generalization of that of a graph since an edge can be incident to any number of vertices. A hypergraph is *r-uniform* if all edges have size r. A 2-uniform hypergraph is a graph.

A *cycle* of *length l* in a hypergraph $\mathcal{H}$ is an alternating cyclic sequence $v_0, e_0, v_1, e_1, ..., e_{l-1}, v_0$ of distinct edges and vertices in $\mathcal{H}$ such that $v_i \in e_{i-1} \cap e_i$ for $1 \leq i \leq l-1$ and $v_0 \in e_0 \cap e_{l-1}$. The *girth* of a hypergraph is the length of its shortest cycle. A hypergraph is *simple* if its girth is at least 3, in other words, if every two distinct edges share at most one vertex.

The *degree* of a vertex is number of edges containing it. The *degree* of an edge is the number of edges intersecting it. The *maximum degree* of $\mathcal{H}$ is the maximum of the degree of the vertices among $V(\mathcal{H})$, denoted $\Delta(\mathcal{H})$. *Edge degree* of $\mathcal{H}$ is the maximum of the degrees of the edges among $E(\mathcal{H})$, denoted $D(\mathcal{H})$. A *complete r-uniform hypergraph* is a hypergraph where all possible $\binom{|V|}{r}$ edges are present. An independent set is a set of vertices such that no edge is entirely contained in that set.

A *proper t-coloring* of a hypergraph is a labeling of its vertices from a set S of size t such that every edge has vertices of at least two distinct colors. In other words, no edge is *monochromatic* if the coloring is proper. The smallest t for which a hypergraph can be properly t-colored is the *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$. The list chromatic number for $\mathcal{H}$ is defined similary to the list chromatic number of graphs. A *rainbow coloring* is a coloring of the vertices such that for every edge the colors of all the vertices in that edge are distinct. In literature, rainbow coloring is sometimes referred to as *strong coloring*. A rainbow coloring is clearly a proper coloring.

1.2 Main results

One of the well-studied problems in graph theory is that of graph coloring. A graph is *t-colorable* if it has a proper t-coloring; it is *t-chromatic* if $\chi(G) = t$.

When edges of a graph represent conflicts among its vertices, the chromatic number represents the minimum number of conflict-free classes (called *color classes*) needed to partition the vertex set. Consider the problem of finding the minimum number of time periods needed to schedule examinations. In this problem each course can be considered as a set of students and these sets having common members require different 'time slots', so we seek the chromatic number of the intersection graph of these sets. Among other applications, one of the most famous problems in graph coloring is determining the chromatic number of a planar graph. Given any planar map, what is the smallest number of colors needed to color the countries so that countries with common boundaries gets different colors ? This problem was open for about 120 years and was solved in 1976. It says that 4 colors suffice to properly color any planar graph.

In a proper coloring, each color class is an independent set. Therefore a graph is t-colorable if and only if it is t-partite. The 2-colorable graphs are precisely the graphs with no odd cycles (bipartite graphs). Using breadth-first search it is easy to test in polynomial time if a graph is 2-colorable. For larger t there is no such known characterization of t-chromatic graphs. In fact it is known that even testing whether a graph is 3-colorable is an NP-complete problem.

Even though it is computationally hard to say whether a graph is t-colorable, estimates for the chromatic number can be given as upper and lower bounds in terms of other graph parameters such as maximum degree, degeneracy, clique number, independence number, etc.

Many famous problems are associated with graph coloring. We have already seen one above, the Four-Color Problem. Stated in a slightly different way, it is equivalent to say (by Wagner's Theorem) that if a graph G is 5-chromatic, then G has a K_5-minor. This is a special case of the more general conjecture of Hadwiger, which states that if a graph G is t-chromatic then it has a K_t-minor. Hadwiger's conjecture is considered as one of the deepest unsolved problems in modern graph theory. Many extremal problems have also been considered in connection with graph coloring. Ramsey's theorem on edge-coloring of graphs states that given a copy of a complete graph K_t, there exists a copy of a large enough complete graph K_n such that every edge coloring of K_n with two colors contains a monochromatic copy of K_t. One is then interested in finding the smallest n for which this holds.

Another extremal problem in connection with graph coloring arises from Turán's Theorem. To understand the structure of t-chromatic graphs one might want to know beside other things the smallest and the largest size of t-chromatic graphs on n vertices. For the largest size of t-chromatic graphs on n vertices, it is easy to see it must be a complete multi partite graph with almost equal parts (also called Turán Graph). Turán showed that in an n-vertex graph, if the number of edges exceeds the size of Turán graph, then we are not only forced to use $t+1$ colors, but also have a copy of K_{t+1} as a subgraph. For the minimum size, it is not too hard to see that any connected graph with at most $\binom{t}{2}+n-t-1$ edges is t-colorable. Without the restriction on n, the smallest t-chromatic graph is K_t and has $\binom{t}{2}$ edges. The problem of finding the smallest size of a t-chromatic hypergraph is more interesting.

A proper t-coloring of a hypergraph is defined similarly. It is a coloring of the vertices of the hypergraph from a set of t colors such that every edge has vertices of at least two distinct colors. In other words, no edge is monochromatic. One of the very first results in this area was by Erdős about 50 years ago. Since then hypergraph coloring has been studied by various mathematicians. Some of the major tools in combinatorics such as the Lovász Local Lemma and the semirandom method have been developed to solve problems in this area. Most of this thesis revisits some classical problems in this area. We consider the following problems:

1. One of the main questions that we consider in this thesis is: Given the number of colors used (say t), what is the smallest number of edges in a hypergraph $\mathcal{H}$ such that $\mathcal{H}$ is not t-colorable ? Erdős was the first to study this problem. He defined $m(r,t)$ to be the minimum number of edges in an r-uniform hypergraph that is not t-colorable. For graphs ($r=2$), as mentioned earlier $m(2,t)=\binom{t}{2}$ and is achieved by the complete graph K_t. For hypergraphs it is not always the case that the size of the complete r-uniform hypergraph is the smallest size of an r-uniform non t-colorable hypergraph. For example, consider $r=3, t=2$. The smallest 3-uniform complete hypergraph which is not 2-colorable has 10 edges, but the Fano plane has 7 edges and is not 2-colorable. Erdős [8, 9] proved that

$$2^{r-1} \leq m(r,2) \leq r^2 2^r,$$

which was one of the first examples of the use of the Probabilistic Method in combinatorics. His result extends for t colors, i.e $t^{r-1} \leq m(r,t) \leq r^2 t^r$. Erdős and Lovász had in fact proved more general results regarding the minimum edge degree of an r-uniform hypergraph for which it is not k-colorable; we denote this value by $D(r,t)$. Erdős and Lovász showed that

$$\frac{1}{4} t^r < D(r,t) \leq 20 r^3 t^{r-1}.$$

This was done in their seminal paper [10] where they introduced the Local Lemma. The lower bound was improved over the years by Beck, by Spencer and by Radhakrishnan and Srinivasan [28], who showed that (for sufficiently large r,) $D(r,2) \geq 0.17 \cdot 2^r \sqrt{r/\ln r}$. In this thesis, using Kostochka's earlier results from [19], we extend Radhakrishnan and Srinivasan's results for general (but fixed) t. Let t be a positive integer and $n = \lfloor \log_2 t \rfloor$. We prove that there is an $\epsilon = \epsilon(t) > 0$ such that for sufficiently large r,

$$D(r,t) \geq \epsilon(t)\, t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}.$$

Erdős and Lovász mentioned in their paper [10] that "perhaps $r2^r$ is the correct order of magnitude of $m(2,r)$". Our result supports the intuition of Erdős and Lovász.

2. We also study the same problem for simple hypergraphs. One can define in general $m(r,t,g)$ to denote the minimum number of edges in an r-uniform non-t-colorable hypergraph of girth at least g. For simple hypergraphs ($g = 3$), Erdős and Lovász proved that

$$\frac{t^{2(r-2)}}{16r(r-1)^2} \leq m(r,t,3) \leq 1600 \cdot r^4 t^{2(r+1)}.$$

Szabó [30] improved the lower bound by a factor of $r^{2-\epsilon}$ for sufficiently large r. In this thesis we improve his lower bound by another factor of r by establishing a slightly more general result. We show that

$$m(r,t,2s+1) \geq \frac{t^{r(1+s)}}{r^{\epsilon}},$$

if r is large in comparison with t, s and $1/\epsilon$. Simple hypergraphs can be generalized in yet another way. We say a hypergraph is *b-simple*, if every two edges intersect in at most b vertices. Let $f(r,t,b)$ denote the minimum number of edges in an r-uniform non-t-colorable b-simple hypergraph. We show that for fixed t, b, and ϵ and sufficiently large r,

$$\frac{t^{r(1+1/b)}}{r^{\epsilon}} \leq f(r,t,b) \leq 40t^2 \left(t^r r^2\right)^{1+1/b}.$$

The upper bound was obtained by using techniques of Erdős and Lovász. These results for simple hypergraphs hold for list coloring as well.

3. We again consider the same extremal problem but now for *conflict-free coloring.* It is a generalized version of proper coloring in which each edge has a vertex whose color occurs exactly once in that edge. This kind of coloring was introduced by Even et al. [12] in a geometric setting with applications in a frequency allocation problem. It is an intermediate coloring between proper coloring and rainbow coloring. It turns out that conflict-free chromatic number of a hypergraph is related to another parameter called the *tree-depth* of a graph G, denoted by $td(G)$. The concept of tree-depth was introduced by Nešetřil and Ossona de Mendez [25]. In simple terms, the tree-depth of a graph G is the minimum height of a rooted forest F such that G occurs as a subgraph of closure of F, where the closure is defined in a certain way. Nešetřil and Ossona de Mendez showed that given a graph G, if $\mathcal{H}$ is the hypergraph with vertex set $V(G)$ whose edges are the vertex sets of connected subgraphs of G, then $td(G) = \chi_{CF}(\mathcal{H})$. Pach and Tardos [26] analyzed the conflict-free chromatic number for graphs and hypergraphs and studied its relationship with the number of edges. They proved that for a $(2r-1)$-uniform hypergraph $\mathcal{H}$ with m edges, $\chi_{CF}(\mathcal{H})$

is at most $m^{1/r}\log m$ (Solving for the number of edges m in terms of the number of colors t and r gives us a lower bound on the minimum number of edges in a $(2r-1)$-uniform hypergraph $\mathcal{H}$ that is not t-colorable.) They also raised the question whether the same result holds for r-uniform hypergraphs (i.e., can this lower bound be improved?). In this thesis, we show that this is not true. We establish this by showing that there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that

$$\chi_{CF}(\mathcal{H}) > C_r m^{2/(r+2)}/\log m.$$

Furthermore, we provide lower and upper bounds on the minimum number of edges of an r-uniform simple hypergraph that is not conflict-free t-colorable. For the lower bound, we show that

if $r \le t/8$ and $m \le \frac{4}{t^2 r}(\frac{t}{8(r-1)})^r$, then $\chi_{CF}(\mathcal{H}) \le t$ for every r-uniform simple hypergraph $\mathcal{H}$ with m edges. Moreover, for the upper bound, we show that

if $r \le t$, then there exists an r-uniform simple hypergraph $\mathcal{H}$ with $(1+o(1))(4t\ln t)^2(\frac{4e^2t}{r})^r$ edges such that $\chi_{CF}(\mathcal{H}) > t$.

4. Finally we consider list colorings of complete graphs. It is easy to see that $\chi_l(K_n) = \chi(K_n) = n$, with lists of size n needed when the lists are identical; in other words, when the lists intersect a lot. It is natural to ask what happens when the lists do not intersect too much. We say that a list assignment L for a graph G is a (k,c)-*list* if $|L(v)| = k$ for all $v \in V(G)$ and $|L(u) \cap L(v)| \le c$ for all $uv \in E(G)$, that is for every edge, the lists of its endpoints have at most c colors in common. Kratochvíl et al. [24] introduced $\chi_l(G,c)$ to be the least k such that G is L-colorable from each (k,c)-list L. Among other results, they showed that

$$\sqrt{\frac{cn}{2}} \le \chi_l(K_n,c) \le \sqrt{2ecn}.$$

Problem 1 in their paper asks whether $\lim_{n\to\infty} \chi_l(K_n,c)/\sqrt{cn}$ exists. Solving their problem, we prove that the limit exists and is equal to 1. We also find the exact value of $\chi_l(K_n,c)$ for infinitely many values of n by showing that if q is a prime power, $c < q-1$ and c divides $q-1$, then

$$\chi_l(K_n,c) = q+1, \quad \forall n \in [\frac{q^2-1}{c}+2, \frac{1}{c}(q^2+\frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})].$$

Chapter 2

Coloring hypergraphs with few edges

2.1 Introduction

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. Recall that $\mathcal{H}$ is called *r-uniform* if all the edges of $\mathcal{H}$ have size r. Also recall that a mapping $c : V(\mathcal{H}) \to \{1, 2, 3, ...\}$ of $V(\mathcal{H})$ is a *proper coloring of* $\mathcal{H}$ if no edge of $\mathcal{H}$ is monochromatic. The minimum number of colors required for such a coloring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$.

It is easy to see that if a hypergraph has very few edges, then one can properly color it with few colors. On the other hand if a hypergraph has many edges then it becomes harder to color with few colors. A natural question that arises is : Given the number of colors used (say t), what is the smallest number of edges in a hypergraph $\mathcal{H}$ such that $\mathcal{H}$ is no longer t-colorable ? Erdős was interested in studying this relationship between $\chi(\mathcal{H})$ and the number of edges of $\mathcal{H}$. Let $m(r, t)$ denote the minimum number of edges in an r-uniform hypergraph that is not t-colorable. The most studied case is the case $t = 2$ (In literature this property of a hypergraph being 2-colorable has also been referred to as Property B, where given a family of sets $\mathcal{F}$, one partitions the ground set into two sets X and Y in such a way that every set in $\mathcal{F}$ meets both X and Y). Erdős [8, 9] proved that $2^{r-1} \leq m(r, 2) \leq r^2 2^r$, which was one of the first examples of the use of the Probabilistic method in Combinatorics. Then Beck [6] improved the lower bound to $2^r r^{1/3-\epsilon}$ and Spencer [29] presented a simpler proof of Beck's bound based on random recoloring. Radhakrishnan and Srinivasan [28] further improved it by proving the following.

Theorem 1. **[28]** *For every $c < 1/\sqrt{2}$, there exists an $r_0 = r_0(c)$ such that for every $r > r_0$,*

$$m(r, 2) \geq c 2^r \sqrt{r/\ln r}.$$

Erdős [9] and Erdős and Lovász [10] said that "perhaps, the order of magnitude of $m(r, 2)$ is $r2^r$". Repeating the argument of Erdős [8, 9], one can see that for every $t \geq 2$, there exists $C = C(t)$ such that $t^{r-1} \leq m(r, t) \leq C r^2 t^r$.

Recall that $\Delta(H)$ is the maximum degree of vertices in $\mathcal{H}$ and $D(\mathcal{H})$ is the maximum of the edge degrees over all the edges of $\mathcal{H}$.

In Erdős and Lovász's seminal paper [10] (where the Lovász Local Lemma appeared), they proved the following bound.

Theorem 2 ([10]). *If $t, r \geq 2$, then every r-uniform hypergraph with $D(\mathcal{H}) \leq \frac{1}{4}t^r$ is t-colorable. In particular, if $\Delta(\mathcal{H}) \leq \frac{1}{4}t^r r^{-1}$, then $\mathcal{H}$ is t-colorable.*

The proof works also for list coloring. A remarkable feature of this result is that it works for all $t, r \geq 2$, and in many cases the bound is rather close to the best possible. In particular, Erdős and Lovász [10] showed that the bound cannot be significantly improved even if we consider only hypergraphs with high girth. A corollary from one of their results is the following.

Theorem 3 ([10]). *For each $t, r, g \geq 2$, there exists an r-uniform hypergraph of girth g with maximum edge degree at most $20r^3t^{r-1}$ that is not t-colorable.*

This last bound was recently slightly improved for $t < r$ by Kostochka and Rödl [23].

Theorem 4 ([23]). *For each $t, r, g \geq 2$, there exists an r-uniform hypergraph of girth g with maximum edge degree at most $r\lceil r\, t^{r-1} \ln t\rceil$ that is not t-colorable.*

Let us denote by $D(r,t)$ the minimum D such that there exists an r-uniform non-t-colorable hypergraph G with maximum edge degree D, then the above results can be summarized as follows.

$$\frac{1}{4}t^r < D(r,t) \leq \min\{20r^3t^{r-1}, r\lceil r\, t^{r-1} \ln t\rceil\}.$$

Elaborating the proof of a lower bound on $m(r,2)$, and using the Lovász Local Lemma, Radhakrishnan and Srinivasan [28] improved the lower bound on $D(r,t)$ for $t = 2$ and large r.

Theorem 5 ([28]). *If r is sufficiently large, then every r-uniform hypergraph with $D(r,t) \leq 0.17 \cdot 2^r\sqrt{r/\ln r}$ is 2-colorable.*

The main result of this chapter is the following extension of Theorem 5 for fixed t and large r.

Theorem 6. *For every integer $t \geq 2$, let $\epsilon = \epsilon(t) = \exp\{-4t^2\}$ and $n = n(t) = \lfloor \log_2 t\rfloor$. Then for every sufficiently large r, every r-uniform hypergraph with maximum edge degree at most $D = \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$ is t-colorable. In other words,*

$$D(r,t) > \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}.$$

The proof of Theorem 6 heavily uses the proof of the following result by Kostochka [19].

Theorem 7 ([19]). *For every positive integer t, let $\epsilon = \epsilon(t) = \exp\{-4t^2\}$ and $n = n(t) = \lfloor \log_2 t\rfloor$. Then for every $r > \exp\{2\epsilon^{-2}\}$,*

$$m(r,t) \geq \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}.$$

The proof also uses some ideas of Radhakrishnan and Srinivasan [28] and the Lovász Local Lemma.

In Section 2.2, a semi-random procedure Evolution is described and some of its simple properties are derived. In Section 2.3 we study the structure of so called *cause trees* arising in the analysis of Evolution. In the next two sections we define some auxiliary "bad" events and estimate their probabilities. Using the independence structure of these auxiliary events and the Lovász Local Lemma, in the final section we show that for hypergraphs satisfying the conditions of Theorem 6, with positive probability Evolution gives a proper t-coloring. This means that such a coloring exists.

This is a joint work with A. Kostochka and V. Rödl and appears in [22].

2.2 Coloring procedure Evolution and its properties

Let t, n and ϵ be as in the statement of the theorem. Let $r \geq \exp\{2\epsilon^{-2(n+1)}\}$. Throughout the chapter we will use the following notation: $c = -\ln\epsilon = 4t^2$, $z = \lfloor cr/\ln r \rfloor$. Fix some $0 < p < 2^{-t^r r}$. Then there is the unique positive integer s such that $sp \leq \frac{\ln r}{(n+1)r} < (s+1)p$. Let $G = (V, E)$ be an r-uniform hypergraph with maximum edge degree at most $D = \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$.

The coloring procedure Evolution described below consists of $n+1$ stages, and every stage apart from Stage 0 consists of s steps. For $1 \leq l \leq n$ and $1 \leq i \leq s$, Step $(l-1)s + i$ is the ith step in Stage l.

We also fix a linear order L on $V(G)$. Now, the procedure works as follows.

Stage 0. Color every vertex $v \in V(G)$ randomly and independently, with a color $\phi(v) \in \{0, 1, 2, \ldots, t-1\}$ chosen uniformly in this set. Also for every $v \in V(G)$, define the random variable $I(v)$ with the values in $\{1, 2, \ldots, sn\} \cup \{\infty\}$ as follows:

$$\mathbf{Pr}\{I(v) = x\} = \begin{cases} p, & \text{if } x \in \{1, 2, \ldots, sn\}; \\ 1 - psn, & \text{if } x = \infty. \end{cases} \tag{2.1}$$

Each random variable $I(v)$ is defined to be mutually independent of all other $I(w)$.

Stage l, $l = 1, \ldots, n$**.**

STEP $i + s(l-1)$, $1 \leq i \leq s$. Following order L, for one by one vertex $v \in V(G)$, check whether
(C1) $I(v) = (l-1)s + i$ and
(C2) v belongs to an edge that was monochromatic, say, of color α, before Stage l, and still is monochromatic at the current moment.
If both conditions (C1) and (C2) hold, then recolor v with color $\alpha + 2^{l-1}$ (modulo t). Otherwise, do nothing with v.

Remark 1. By Condition (C1), each vertex can be recolored at most once.

Remark 2. As it follows from the description of the procedure, every step consists of $|V(G)|$ smaller steps (one per vertex).

Lemma 8. *For every $w, q \geq 1$, every set $W \subseteq V$ with $|W| = w$, and every set $Q \subseteq \{1, 2, \ldots, sn\}$ with $|Q| = q$, the probability that for each vertex $v \in W$, $I(v) \in Q$ is at most $(qp)^w$.*

PROOF. For every vertex $v \in V(G)$ and every $1 \leq l \leq n$ and $1 \leq i \leq s$, $\mathbf{Pr}\{I(v) = s(l-1)+i\} = p$. Therefore, the probability that $I(v) \in Q$ is at most qp. The mutual independence of all $I(v)$ yields the lemma. ■

For an edge $e \in E$ and $1 \leq l \leq n$, let $M(e, l) = \{v \in e : I(v) \leq sl\}$.

Lemma 9. *For every $e \in E$ and $1 \leq l \leq n$,*

$$\mathbf{Pr}\{|M(e,l)| \geq z\} \leq \epsilon^{0.5r}.$$

PROOF. It is enough to prove the lemma for $l = n$. By Lemma 8, this probability is at most

$$\binom{r}{z}(nsp)^z \leq \left(\frac{er}{z}\right)^z \left(\frac{n \ln r}{(n+1)r}\right)^z \leq \left(\frac{ne \ln r}{z(n+1)}\right)^z.$$

Since r is large and $z = \lfloor cr/\ln r \rfloor > n$,

$$\frac{ne \ln r}{z(n+1)} \leq \frac{e \ln r}{z+1} \leq \frac{e \ln^2 r}{cr} \leq r^{-0.6}.$$

Thus

$$\left(\frac{ne \ln r}{z(n+1)}\right)^z \leq \left(r^{-0.6}\right)^{(cr/\ln r)-1} < e^{-0.5cr} = \epsilon^{0.5r}. \qquad ■$$

Lemma 10. *If a vertex is of color α at the end of Stage l, $l \geq 1$, then at the end of Stage 0 it can be colored only with colors α, $\alpha - 2^0, \alpha - 2^1, \ldots, \alpha - 2^{l-1}$ (modulo t).*

PROOF. By Remark 1, every vertex can be recolored at most once and by definition, a vertex of color β can be recolored during Stage j only with color $\beta + 2^{j-1}$ (modulo t). ■

Definition [Blaming edges]. If an edge e_0 becomes monochromatic of color α during Stage l, then it must contain at the end of Stage 0 a vertex of color $\alpha - 2^{l-1}$. Suppose that at the end of Stage 0 it contained vertices of colors $\alpha - 2^{l_1 - 1}, \ldots, \alpha - 2^{l_h - 1}$, where $l_h = l$ and $l_1 < l_2 < \ldots < l_h$. Then for every $1 \leq j \leq h$, there exists an edge e_j and a vertex $v_j \in e_0 \cap e_j$ such that
(a) e_j was monochromatic of color $\alpha - 2^{l_j - 1}$ at the end of Stage $l_j - 1$;
(b) v_j was recolored with α during Stage l_j and it was the last vertex of this color in e_0 recolored with α.
In this case we say that e_0 and v_j *l_j-blame* e_j.

Remark 3. Since in every step of procedure Evolution the vertices of G are considered consecutively, every edge e can blame only an edge sharing exactly one vertex with e.

Remark 4. It might be that an edge e_0 can blame more than one edge containing the same vertex v_j. On the other hand, by definition, e_0 cannot blame an edge containing another vertex $v \in e_0$ with $\phi(v) = \phi(v_j)$.

Definition [Cause trees]. If an edge e_0 is monochromatic of color α at the end of Stage l, then a *cause tree* $T = T(e_0, \alpha, l)$ is a subset of edges of G defined by induction on l as follows. The set T always contains e_0. If e_0 was monochromatic of color α already after Stage 0, then $T = \{e_0\}$ for every l. Suppose that at the end of Stage 0 edge e_0 contained vertices of colors $\alpha - 2^{l_1-1}, \ldots, \alpha - 2^{l_h-1}$, where $l_h = l$ and $l_1 < l_2 < \ldots < l_h$. Suppose further that for $j = 1, \ldots, h$, edge e_0 l_j-blames edge e_j. Then

$$T = T(e_0, \alpha, l) = \{e_0\} \cup \bigcup_{j=1}^{h} T(e_j, \alpha - 2^{l_j-1}, l_j - 1).$$

Remark 5. By Remark 4 and the definition of cause trees, it could be that in the same outcome of Evolution for the same triple (e_0, α, l), we can construct several distinct cause trees $T = T(e_0, \alpha, l)$.

Definition [Levels of edges]. If $T = T(e_0, \alpha, l)$ is defined as above, then we also say that $e_1, e_2, \ldots, e_h$ are the *edges of level* 1 of T, the edges blamed by the edges of level 1 are the *edges of level* 2 of T, and so on. Thus, if an edge e of a cause tree has vertices of exactly t distinct colors at the end of Stage 0, then e blames either $t - 1$ or t other edges.

2.3 Structure of cause trees

Since each vertex can be recolored at most once, each edge at different stages of Evolution can become monochromatic with at most two colors. Furthermore, if an edge e was monochromatic of a color α_1 after Stage l_1 and becomes monochromatic of a color $\alpha_2 \neq \alpha_1$ after Stage l_2, then e has to be monochromatic of color α_1 already after Stage 0 and all vertices of e change their color to $\alpha_2 = \alpha_1 + 2^{l_2-1}$ at Stage l_2. In this case, each cause tree for e considered after Stage l_2 has exactly one edge of level 1.

In view of this, if an edge e becomes monochromatic exactly once during Evolution, then the corresponding color α is called *the main color of* e and denoted by $\mu(e)$, and if e becomes monochromatic twice, then *the main color of* e, $\mu(e)$, is the first of these two colors.

If e is monochromatic of some color α after some Stage l, then we say that e is an *l-unlucky edge*.

Lemma 11. *If e_0 is an l-unlucky edge with a cause tree T, then the main colors of all the edges of T are distinct.*

PROOF. If e and e' are edges of T, then there exist two sequences $e_0, e_1, \ldots, e_q = e$ and $e'_0 = e_0, e'_1, \ldots, e'_{q'} = e'$ such that e_j l_j-blames e_{j+1} for $j = 0, 1, \ldots, q-1$ and e'_j i'_j-blames e'_{j+1} for $j = 0, 1, \ldots, q'-1$. Furthermore, $l_0 > l_1 > \cdots > l_{q-1}$, $i'_0 > i'_1 > \cdots > i'_{q-1}$, and the sequences $l_0, l_1, \ldots, l_q$ and $i'_0, i'_1, \ldots, i'_q$ are not identical. Thus, the numbers $2^{l_0-1} + 2^{l_1-1} + \ldots + 2^{l_{q-1}-1}$ and $2^{i'_0-1} + 2^{i'_1-1} + \ldots + 2^{i'_{q'-1}-1}$ are distinct and differ by less than t. On the other hand, by definition, the main color of e is $\alpha - 2^{l_0-1} - 2^{l_1-1} - \ldots - 2^{l_{q-1}-1}$ and the main color of e' is $\alpha - 2^{i'_0-1} - 2^{i'_1-1} - \ldots - 2^{i'_{q'-1}-1}$. This proves the lemma. ∎

Lemma 12. *Suppose that e_0 is an l-unlucky edge with a cause tree T. If e and e' are edges of T and neither of them blames the other, then e and e' are disjoint.*

PROOF. Assume that e and e' have a common vertex v and both belong to T. Then there exist two sequences $e_0, e_1, \ldots, e_q = e$ and $e'_0 = e_0, e'_1, \ldots, e'_{q'} = e'$ such that e_j l_j-blames e_{j+1} for $j = 0, 1, \ldots, q-1$ and e'_j i'_j-blames e'_{j+1} for $j = 0, 1, \ldots, q'-1$. Furthermore, $l_0 > l_1 > \cdots > l_{q-1}$, $i'_0 > i'_1 > \cdots > i'_{q-1}$.

Claim 1. $l_{q-1} \neq i'_{q'-1}$.

Proof of Claim. If $l_{q-1} = i'_{q'-1}$, then e and e' both were monochromatic at the end of Stage $l_{q-1} - 1$. But by Lemma 11, their main colors differ. This proves the claim.

Thus below we can assume that $l_{q-1} < i'_{q'-1}$. It follows that e ceased to be monochromatic before e' did. In particular, v was recolored from $\mu(e)$ to $\mu(e')$. This yields that

$$\mu(e') - \mu(e) \text{ (modulo } t\text{) is a power of 2.} \tag{2.2}$$

Claim 2. $\mu(e') - \mu(e) = 2^{l_{q-1}-1}$ *modulo t.*

Proof of Claim. Recall that

$$\mu(e') - \mu(e) = (\alpha - 2^{i'_0-1} - 2^{i'_1-1} - \ldots - 2^{i'_{q'-1}-1}) - (\alpha - 2^{l_0-1} - 2^{l_1-1} - \ldots - 2^{l_{q-1}-1}).$$

In this expression, α cancels out and every other summand apart from $2^{l_{q-1}-1}$ is divisible by $2^{l_{q-1}}$. Together with (2.2), this yields the claim.

Claim 2 implies that v was recolored during Stage l_{q-1} and thus $\mu(e') = \mu(e_{q-1})$. This contradicts Lemma 11. ∎

Lemma 13. *Let $\lambda(l)$ denote the maximal possible number of edges in a cause tree T for an unlucky edge e_0 under the condition that $\mu(e_1) - \mu(e_2) \in \{1, 2, \ldots, 2^{l-1}\}$ (modulo t) for every pair of edges (e_1, e_2) such that e_1 blames e_2. Then for every $l \geq 0$, $\lambda(l) \leq 2^l$. In particular, each cause tree has at most $2^n \leq t$ edges.*

PROOF. If e_1 l_1-blames e_2 and e_2 l_2-blames e_3, then $l_2 < l_1$. Thus, under conditions of the lemma, for the root e_0 and an arbitrary edge e of the tree, we have

$$\mu(e_0) - \mu(e) \in \{1, 2, \ldots, 2^{l-1} + 2^{l-2} + \cdots + 1\} = \{1, 2, \ldots, 2^l - 1\}.$$

Now, Lemma 11 implies that T has at most $1 + (2^l - 1)$ edges. ■

Below we will analyze which subsets of edges of G can form cause trees $T(e, \alpha, l)$ for some values of e, α and l. Lemma 12 implies that every cause tree $T = T(e, \alpha, l)$ is an r-uniform hypergraph tree in the ordinary sense rooted at e. Moreover, every vertex of such a tree belongs to at most two edges of this tree. In connection with this, let us fix some notation. Everywhere below, when we say *"r-tree"*, we mean an r-uniform hypergraph tree in which every vertex belongs to at most two edges of this tree. Often, we will consider *rooted* r-trees. *The root of an r-tree* will be an edge of this r-tree, and not a vertex. By *a sub-r-tree of* G we mean an r-tree that is a subhypergraph of G. Given an r-tree T with a root e_0, the *children* of e_0 are the edges adjacent to e_0, and for $e \in E(T)$ at distance d from e_0 (in T), the *children* of e are the edges adjacent to e that are at distance $d+1$ from e_0. Naturally, the *descendants* of an $e \in E(T)$ are its children, children of children and so on. If e_1 is a descendant of e_2, then e_2 is an *ancestor* of e_1. For an r-tree T with a root e_0 and another edge e_1 of T, by $T(e_1)$ we denote the subtree of T formed by e_1 and all its descendants. We will use the following fact on sub-r-trees of r-uniform hypergraphs.

Lemma 14. *Let H be an r-uniform hypergraph with maximum edge degree at most D. Let $e_0 \in E(H)$. Then e_0 belongs to at most $(4D)^{y-1}$ sub-r-trees of H with y edges.*

PROOF. Let T be a sub-r-tree of H containing e_0 with $|E(T)| = y$. Consider T as a rooted r-tree with root e_0. Order the edges of T $e_0, e_1, \ldots, e_{y-1}$ starting from e_0 using Breadth-First search. We say that T has *type* $(h_0, \ldots, h_{y-2})$ if for $i = 0, \ldots, y-2$, edge e_i has exactly h_i children. Since $h_0 + \ldots + h_{y-2} = y - 1$, the number of distinct types does not exceed the number of representation of $y-1$ as the sum of $y-1$ of ordered nonnegative summands, which equals $\binom{(y-1)+(y-1)-1}{y-2} < 4^{y-1}$. When we know the type of T, then for every edge e_i, $i \geq 1$, we know the immediate ancestor (father edge). So, we can embed a tree T of a given type, edge by edge into G. Furthermore, at each step i, $i \geq 1$, we have at most D choices for our edge among the edges of G adjacent to its father edge. Thus, e_0 belongs to at most D^{y-1} r-trees of given type with y edges. Since the number of distinct types is at most 4^{y-1}, this proves the lemma. ■

2.4 Auxiliary events

The goal of this section is to introduce auxiliary events that imply the "bad" events in Evolution and are easier to control. In the next section we estimate probabilities of these auxiliary events.

Let $e \in E(G)$, $\alpha \in [t]$, $l \in \{1, \ldots, n\}$ and T be a sub-r-tree of G rooted at e. Then let $W(e, \alpha, T, l)$ be the event that edge e is monochromatic of color α after Stage l of Evolution, and a cause tree for this is T. Also, let $e_1, \ldots, e_q$ be the edges of T of the first level, i.e., the edges of T sharing a vertex with e. For $j \in [q]$, let $e \cap e_j = \{v_j\}$. Let $Q \doteq \{v_1, \ldots, v_q\}$.

If $W(e, \alpha, T, l)$ occurs, then the following properties hold.

(W1) *For every $v \in e$, $\phi(v) \in \{\alpha\} \cup \phi(Q) \subseteq \{\alpha, \alpha - 2^0, \alpha - 2^1, \ldots, \alpha - 2^{l-1}\}$ (modulo t).*

Proof: By the definition of cause trees and Lemma 10.

(W2) *For $j \in [q]$, $\phi(v_j) \neq \alpha$, and for distinct j and j', $\phi(v_{j'}) \neq \phi(v_j)$. In particular, if $A_j = A_j(e, \phi) = \{v \in e : \phi(v) = \phi(v_j)\}$, then all sets A_j are disjoint subsets of e.*

Proof: By the definition of cause trees, for each $j \in [q]$, v_j is the last vertex of color $\phi(v_j)$ that changed its color to α. This implies both statements.

(W3) *$I(v) \leq ls$ for each $v \in \bigcup_{j=1}^{q} A_j$.* Moreover, for each $j \in [q]$, if v_j becomes of color α at Stage l_j, then

(W4) *$\alpha - \phi(v_j) = 2^{l_j - 1}$;*

(W5) *the event $W(e_j, \phi(v_j), T(e_j), l_j - 1)$ occurs;*

(W6) *for every $u \in e_j$ with $I(u) > (l_j - 1)s$, we have also $I(u) \geq I(v_j)$;* and

(W7) *for each $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq I(v_j)$.*

Proof: Since each $v \in \bigcup_{j=1}^{q} A_j$ has changed its color by Stage l, by condition (C1) in the definition of Evolution, (W3) follows. Statement (W4) also follows from the definition of Evolution. If e_j were not monochromatic of color $\phi(v_j)$ after Stage $l_j - 1$, then v_j would not obtain color α blaming e_j. This yields (W5). If some $u \in e_j$ would have $(l_j - 1)s < I(u) < I(v_j)$, then by the definition of Evolution, it would mean that u did not change its color before Stage l_j, and so it should change its color at the moment $I(u)$, i.e. earlier than v_j did, in which case v_j would not blame e_j. This contradiction proves (W6). Now (W7) follows from the facts that all vertices in A_j must change their colors in Stage l_j (in order to change it from $\phi(v_j)$ to α) and that v_j is the last vertex in A_j that changes its color.

(W8) *If e was already monochromatic after Stage $l - 1$, then for each $v \in e$, $I(v) \notin [s(l-1)+1, sl]$.*

Proof: If e were monochromatic after Stage $l - 1$, and $I(v) \in [s(l-1)+1, sl]$ for some $v \in e$, then v would change its color, and so $W(e, \alpha, T, l)$ would not happen.

Unfortunately, events $W(e, T, \alpha, l)$ and $W(e', T', \alpha', l')$ can be dependent even if $V(T')$ is disjoint from $V(T)$. So, for each $e_0 \in E(G)$, each sub-r-tree T of G with root e_0 and $|E(T)| \leq t$, and each color α, we will introduce the auxiliary event $\widetilde{W}(e_0, \alpha, T, l)$ that contains the event $W(e_0, \alpha, T, l)$, and in addition essentially possesses properties (W1)–(W8) above, but does not depend on the values of $\phi(u)$ and $I(u)$ for all $u \notin V(T)$. We define these events by induction on the number of edges in T.

If $E(T) = \{e_0\}$, then the event $\widetilde{W}(e_0, \alpha, T, l)$ means that all of the following holds
(i) $\phi(e_0)$ is monochromatic of color α,
(ii) $|M(e_0, n)| < z$, and
(iii) $I(v) > ls$ for every $v \in e_0$.

Suppose that the event $\widetilde{W}(e_0, \alpha, T, l)$ is defined for all parameters e_0, α, T, l such that $|E(T)| < y$. Let $e_0 \in E(G)$, $\alpha \in [t]$, and T be any sub-r-tree T of G with root e_0 and y edges. Let $e_1, \ldots, e_q$ be the edges of T sharing a vertex with e. For $j \in [q]$, let $e \cap e_j = \{v_j\}$. Let $Q \doteq \{v_1, \ldots, v_q\}$. We say that $\widetilde{W}(e_0, \alpha, T, l)$ occurs, if either $|M(e, n)| \geq z$ for at least one $e \in E(T)$ or all of the following holds:

($\widetilde{W}1$) *For every* $v \in e$, $\phi(v) \in \{\alpha\} \cup \phi(Q) \subseteq \{\alpha, \alpha - 2^0, \alpha - 2^1, \ldots, \alpha - 2^{l-1}\}$ *(modulo* t*).*

($\widetilde{W}2$) *For* $j \in [q]$, $\phi(v_j) \neq \alpha$, *and for distinct* j *and* j', $\phi(v_{j'}) \neq \phi(v_j)$. *In particular, if* $A_j = A_j(e, \phi) = \{v \in e : \phi(v) = \phi(v_j)\}$, *then all sets* A_j *are disjoint.*

($\widetilde{W}3$) $I(v) \leq ls$ *for each* $v \in \bigcup_{j=1}^{q} A_j$. *Moreover, for each* $j \in [q]$, *if* $(l_j - 1)s + 1 \leq I(v_j) \leq sl_j$, *then*
($\widetilde{W}4$) $\alpha - \phi(v_j) = 2^{l_j - 1}$;
($\widetilde{W}5$) *event* $\widetilde{W}(e_j, \phi(v_j), T(e_j), l_j - 1)$ *occurs;*
($\widetilde{W}6$) *for every* $u \in e_j$ *with* $I(u) > (l_j - 1)s$, *we have also* $I(u) \geq I(v_j)$, and
($\widetilde{W}7$) *for each* $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq I(v_j)$.

($\widetilde{W}8$) *If event* $\widetilde{W}(e_0, \alpha, T, l - 1)$ *occurs, then for each* $v \in e_0$, $I(v) \notin [s(l - 1) + 1, sl]$.

The following two lemmas justify the introduction of the events $\widetilde{W}(e, \alpha, T, l)$.

Lemma 15. *Let* $e_0 \in E(G)$, $\alpha \in [t]$, $l \in \{0, \ldots, n\}$ *and* T *be a sub-*r*-tree of* G *with root* e_0. *If the event* $W(e_0, \alpha, T, l)$ *occurs, then the event* $\widetilde{W}(e_0, \alpha, T, l)$ *also occurs.*

PROOF. Suppose that for some values of the parameters e_0, T, l, and α, $W(e_0, \alpha, T, l)$ occurs but $\widetilde{W}(e_0, \alpha, T, l)$ does not occur. We may choose this pair so that for all subtrees of T this does not happen, and that for given e_0, T, α and for $l' < l$, this does not happen.

Let us check which of the properties in the definition of $\widetilde{W}(e_0, \alpha, T, l)$ may fail. Since ($\widetilde{W}1$) and ($\widetilde{W}2$) coincide with (W1) and (W2), respectively, they hold. For the same reason, properties ($\widetilde{W}4$), ($\widetilde{W}6$), and ($\widetilde{W}7$) hold. Property ($\widetilde{W}5$) follows from (W5) and the minimality of our counterexample. The property $I(v) \leq ls$ for each $v \in \bigcup_{j \geq 1} A_j$ (first line of ($\widetilde{W}3$)) follows from the fact that otherwise, by the definition of Evolution, some vertex in $\bigcup_{j=1}^{q} A_j$ would not change its color to α.

Assume finally that ($\widetilde{W}8$) does not hold, in other words, that $\widetilde{W}(e_0, \alpha, T, l-1)$ occurs, and for some $v \in e_0$, $I(v) \in [s(l-1)+1, sl]$. By (W8), this implies that $W(e_0, \alpha, T, l-1)$ does not occur, i.e., after Stage $l-1$, e_0 is not monochromatic of color α. It follows that in order e_0 to become monochromatic of color α after Stage l, we need $I(u) \in [s(l-1)+1, sl]$ for some $u \in \bigcup_{j=1}^{q} A_j$. On the other hand, by ($\widetilde{W}3$) for the event $\widetilde{W}(e_0, \alpha, T, l-1)$, $I(u) \leq (l-1)s$ for each $u \in \bigcup_{j=1}^{q} A_j$. This contradiction finishes the proof of the lemma. ∎

Lemma 16. *Let $e_0 \in E(G)$, $\alpha_0 \in [t]$, $l_0 \in \{0, \ldots, n\}$, and T_0 be a sub-r-tree of G with root e_0. Then $\widetilde{W}(e_0, \alpha_0, T_0, l_0)$ is independent of all events $\widetilde{W}(e, \alpha, T, l)$ such that $V(T) \cap V(T_0) = \emptyset$.*

PROOF. By definition, the events $\widetilde{W}(e_0, \alpha_0, T_0, l_0, \psi(e_0))$ are completely defined when we know the values of $\phi(v)$ and $I(v)$ for all $v \in V(T_0)$. This yields the lemma. ∎

2.5 Probabilities of auxiliary events

Lemma 17. *Let $D := \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$ and G be an r-uniform hypergraph with maximum edge degree at most D. Let $e \in E(G)$, $\alpha \in [t]$, and $0 \leq l \leq n$. Let T be a rooted sub-r-tree of G with root e. If T has y edges, then*

$$\mathbf{Pr}(\widetilde{W}(e, \alpha, T, l)) \leq \epsilon D^{-y} \left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}}.$$

PROOF. We use induction on l. Consider first $l = 0$. If $\widetilde{W}(e, \alpha, T, 0)$ occurs, then by ($\widetilde{W}1$), $\phi(v) = \alpha$ for each $v \in e$. Thus, in this case

$$\mathbf{Pr}(\widetilde{W}(e, \alpha, T, 0)) = t^{-r} = \frac{\epsilon}{D} \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}.$$

This proves the case $l = 0$.

Now, suppose that the lemma holds for every $l' < l$. Consider the event $\widetilde{W}(e, \alpha, T, l)$ for some $e \in E(G)$, an r-tree T with y edges rooted at e, and $\alpha \in [t]$. Let $X(T)$ denote the event that $|M(e', n)| \geq z$ for at least one $e' \in E(T)$, and $\overline{X(T)}$ be its complement. Suppose that the event $\widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)}$ occurs.

Let $e_1, \ldots, e_q$ be all the edges of T that share a vertex with e. For $j = 1, \ldots, q$, let $\{v_j\} = e \cap e_j$ and let y_j be the number of edges in $T(e_j)$. Let $Q \doteq \{v_1, \ldots, v_q\}$. By ($\widetilde{W}3$), for each $j \in [q]$, there exists an $l_j \in [l]$ such that $s(l_j - 1) < I(v_j) \leq sl_j$. Moreover, by ($\widetilde{W}4$) and ($\widetilde{W}2$), all l_j are distinct.

Let $\Theta_0 = \Theta_0(q, l)$ be the set of vectors $(x_1, \ldots, x_q)$ such that (a) $x_j \in [l]$ for each $j \in [q]$, and (b) all $x_1, \ldots, x_q$ are distinct. By the previous paragraph,

$$\widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)} = \widetilde{W}(e, \alpha, T, l) \cap \overline{X(T)} \cap \{(j_1, \ldots, j_q) \in \Theta_0\}. \tag{2.3}$$

Let $\Theta_1(q,l) = \Theta_0(q,l-1)$, i.e. the set of $(x_1,\dots,x_q) \in \Theta_0$ such that $x_j \le l-1$ for all $j \in [q]$. Let $\Theta_2 = \Theta_2(q,l) = \Theta_0(q,l) - \Theta_1(q,l)$. For $i = 1,2$, let

$$F_i(e,\alpha,T,l) = \widetilde{W}(e,\alpha,T,l) \cap \overline{X(T)} \cap \{(j_1,\dots,j_q) \in \Theta_i\}.$$

By (2.3),

$$\widetilde{W}(e,\alpha,T,l) \subseteq X(T) \cup F_1(e,\alpha,T,l) \cup F_2(e,\alpha,T,l). \tag{2.4}$$

Our goal is to prove that for $i = 1,2$,

$$\mathbf{Pr}(F_i(e,\alpha,T,l)) \le 0.4\epsilon D^{-y} \left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}}. \tag{2.5}$$

Since by Lemma 9, $\mathbf{Pr}(X(T)) \le t\epsilon^{0.5r} < 0.1\epsilon D^{-y}$, (2.4) and (2.5) will imply the lemma.

Observe that the condition "$x_j \le l-1$ for all $j \in [q]$" in the definition of $\Theta_1(\alpha,l)$ implies that if $\widetilde{W}(e,\alpha,T,l)$ occurs, then all conditions $(\widetilde{W}1)$–$(\widetilde{W}8)$ are satisfied for the event $\widetilde{W}(e,\alpha,T,l-1)$. By the induction assumption,

$$\mathbf{Pr}(\widetilde{W}(e,\alpha,T,l-1)) \le \epsilon D^{-y} \left(\frac{r}{\ln r}\right)^{(n-l+1)/(n+1)}. \tag{2.6}$$

Let $Z(e,l)$ be the event that for each $v \in e - M(e,l-1)$, $I(v) \notin \{s(l-1)+1,\dots,sl\}$. If $\widetilde{W}(e,\alpha,T,l) \cap \overline{X(T)}$ holds, then by $(\widetilde{W}8)$, $Z(e,l)$ occurs. Since all random variables $I(v)$ are mutually independent,

$$\mathbf{Pr}(\{Z(e,l) \mid \widetilde{W}(e,\alpha,T,l-1)\}) \le \left(\frac{1-lps}{1-(l-1)ps}\right)^{r-|M(e,l-1)|} \le (1-ps)^{r-|M(e,l-1)|}. \tag{2.7}$$

Therefore,

$$\mathbf{Pr}(\{Z(e,l) \mid \widetilde{W}(e,\alpha,T,l-1)\}) \le \sum_{M \subset e} \mathbf{Pr}\{M = M(e,l-1)\}(1-ps)^{r-|M|}.$$

By Lemma 9, $\mathbf{Pr}(|M(e,l-1)| \ge z) \le \epsilon^{0.5r}$. Hence

$$\sum_{M \subseteq e} \mathbf{Pr}\{M = M(e,l-1)\}(1-ps)^{r-|M|} \le \epsilon^{0.5r} + \sum_{M \subseteq e\,:\,|M|<z} \mathbf{Pr}\{M = M(e,l-1)\}(1-ps)^{r-|M|} \le$$

$$\le \epsilon^{0.5r} + (1-ps)^{r-z} \le \epsilon^{0.5r} + \exp\{-psr(1-\frac{c}{\ln r})\}.$$

Since $ps \ge \frac{\ln r}{(n+1)r} - p$, by the definition of p and s,

$$\exp\{-psr(1-\frac{c}{\ln r})\} \le \exp\{-(\frac{\ln r}{n+1} - pr)(1-\frac{c}{\ln r})\} \le \exp\{-\frac{\ln r}{n+1} + \frac{c}{n+1} + pr\}.$$

Recall that $c = -\ln \epsilon$. Since $p < 2^{-t^r r}$, $pr < \frac{c}{n+1}$ and hence

$$\exp\{-\frac{\ln r}{n+1} + \frac{c}{n+1} + pr\} \leq r^{\frac{-1}{n+1}} e^{2c/(n+1)} \leq \frac{1}{\epsilon} r^{\frac{-1}{n+1}}.$$

Recall that $\ln r \geq 2\epsilon^{-2(n+1)}$ and $\epsilon = \exp\{-4t^2\}$. So,

$$\frac{1}{\epsilon} r^{\frac{-1}{n+1}} \leq \frac{1}{\epsilon}\left(\frac{r}{\ln r}\right)^{\frac{-1}{n+1}} \epsilon^2 \leq \left(\frac{r}{\ln r}\right)^{\frac{-1}{n+1}} \exp\{-4t^2\} < 0.1 \left(\frac{r}{\ln r}\right)^{\frac{-1}{n+1}}.$$

By this and (2.6),

$$\mathbf{Pr}(F_1(e, \alpha, T, l)) \leq \mathbf{Pr}(\widetilde{W}(e, \alpha, T, l-1) \cap \overline{X(T)})\mathbf{Pr}(\{Z(e, l) \mid \widetilde{W}(e, \alpha, T, l-1) \cap \overline{X(T)}\}) \leq$$

$$\leq \epsilon^{0.5r} + 0.1\epsilon D^{-y} \left(\frac{r}{\ln r}\right)^{\frac{n-l+1}{n+1} - \frac{1}{n+1}}.$$

Since $\epsilon^{0.5r-1} < 0.01D^{-t} \leq 0.01D^{-y}$, this implies (2.5) for $i = 1$.

Now we will prove (2.5) for $i = 2$. Suppose that $F_2(e, \alpha, T, l)$ occurs. Then there exists $j^* \in [q]$ such that $l_{j^*} = l$. Also, for every $j \in [q]$, there exists $h_j \in [s]$ such that $I(v_j) = s(l_j - 1) + h_j$. By ($\widetilde{W}5$), for every $j \in [q]$, the event $\widetilde{W}(e_j, \alpha - 2^{l_j - 1}, T(e_j), l_j - 1)$ occurs. For $j \in [q]$, let $A_j = \{v \in e : \phi(v) = \phi(v_j)\}$ and $a_j = |A_j| - 1$. Let $\widetilde{W}6(j, h)$ be the event that for every $u \in e_j$ with $I(u) > (l_j - 1)s$, we have also $I(u) \geq (l_j - 1)s + h$, and $\widetilde{W}7(j, h)$ be the event for each $u \in A_j - v_j$, $(l_j - 1)s + 1 \leq I(u) \leq (l_j - 1)s + h$. By ($\widetilde{W}6$) and ($\widetilde{W}7$), for each $j \in [q]$, both $\widetilde{W}6(j, h_j)$ and $\widetilde{W}7(j, h_j)$ occur.

For a vector $(l_1, \ldots, l_q)$, let $\Psi(l_1, \ldots, l_q)$ be the set of colorings ψ of e such that all of the following holds:
(P1) $\psi(v_j) = \alpha - 2^{l_j - 1}$ for all $j \in [q]$.
(P2) $\psi(v) \in \{\psi(v_1), \ldots, \psi(v_q), \alpha\}$ for all $v \in e$.

Thus, in order $F_2(e, \alpha, T, l)$ to occur, all of the following should happen:

(F0) $\overline{X(T)}$ occurs.

(F1) For some $l_1, \ldots, l_q \in [l]$ and $h_1, \ldots, h_q \in [s]$, $I(v_j) = s(l_j - 1) + h_j$ for all $j \in [q]$.

(F2) For these $l_1, \ldots, l_q \in [l]$ and $h_1, \ldots, h_q \in [s]$, each of $\widetilde{W}(e_j, \alpha - 2^{l_j - 1}, T(e_j), l_j - 1)$ occurs and each of $\widetilde{W}6(j, h_j)$ occurs.

(F3) $\phi(e) \in \Psi(l_1, \ldots, l_q)$.

(F4) For each $j \in [q]$, $\widetilde{W}7(j,h_j)$ occurs.

So, we estimate

$$\mathbf{Pr}(F_2(e,\alpha,T,l)) \leq$$

$$\leq \sum_{(l_1,\ldots,l_q)\in\Theta_2} \sum_{h_1=1}^{s} \sum_{h_2=1}^{s} \ldots \sum_{h_q=1}^{s} \mathbf{Pr}\left(\bigcap_{j=1}^{q} \{I(v_j) = s(l_j-1)+h_j\}\right) \times \tag{2.8}$$

$$\times\, \mathbf{Pr}\left(\bigcap_{j=1}^{q}\left(\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap \widetilde{W}6(j,h_j)\right)\right) \times \tag{2.9}$$

$$\times \sum_{\psi\in\Psi(l_1,\ldots,l_q)} \mathbf{Pr}\left(\{\phi(e)=\psi \mid \bigcap_{j=1}^{q}\left(\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap \widetilde{W}6(j,h_j)\right)\}\right) \times \tag{2.10}$$

$$\times\, \mathbf{Pr}\Big(\{\widetilde{W}7(j,h_j) \mid \{\phi(e)=\psi\} \cap \bigcap_{j=1}^{q} \widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap \widetilde{W}6(j,h_j)\}\Big). \tag{2.11}$$

We first deal with (2.8). Since all $I(v)$ are independent, by (2.1),

$$\mathbf{Pr}(\bigcap_{j=1}^{q}\{I(v_j)=s(l_j-1)+h_j\}) = p^q. \tag{2.12}$$

Since the vertex sets of $T(e_j)$ for distinct j are disjoint and by Lemma 16, for every j the event $\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap\widetilde{W}6(j,h_j)$ depends only on the values of $I(v)$ and $\phi(v)$ for $v \in V(T(e_j))$,

$$\Gamma(j,h_j) \doteq \mathbf{Pr}\left(\bigcap_{j=1}^{q}\left(\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap \widetilde{W}6(j,h_j)\right)\right) =$$

$$= \prod_{j=1}^{q} \mathbf{Pr}\left(\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\cap \widetilde{W}6(j,h_j)\right).$$

If $T(e_j)$ has y_j edges, then by the induction assumption,

$$\mathbf{Pr}(\widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)) \leq t\epsilon D^{-y_j}\left(\frac{r}{\ln r}\right)^{(n-l_j+1)/(n+1)}.$$

Let us estimate $\gamma(j,h_j) \doteq \mathbf{Pr}\left(\{\widetilde{W}6(j,h_j) \mid \widetilde{W}(e_j,\alpha-2^{l_j-1},T(e_j),l_j-1)\}\right)$. If $v \in e_j - M(e_j,l_j -$

$1) - v_j$, then

$$\mathbf{Pr}(I(v) \geq (l_j-1)s+h_j) = \frac{1-p(l_j-1)s-p(h_j-1)}{1-p(l_j-1)s} \leq 1-p(h_j-1).$$

By the independence of $I(v)$ for distinct v, similarly to (2.7) and the argument following (2.7), we have

$$\gamma(j,h_j) \leq \sum_{M\subseteq e_j} \mathbf{Pr}\{M = M(e_j, l_j-1)\}(1-p(h_j-1))^{r-|M|-1} \leq \epsilon^{0.5r} +$$

$$+ \sum_{M\subseteq e:|M|<z} \mathbf{Pr}\{M = M(h_j-1)\}(1-p(h_j-1))^{r-|M|-1} \leq \epsilon^{0.5r} + (1-p(h_j-1))^{r-z}.$$

Since $\epsilon^{0.5r} \leq 0.1(1-ps)^r \leq 0.1(1-p(h_j-1))^{r-z}$, we conclude that

$$\Gamma(j,h_j) \leq \prod_{j=1}^{q} t\epsilon D^{-y_j}\left(\frac{r}{\ln r}\right)^{(n-l_j+1)/(n+1)} 1.1(1-p(h_j-1))^{r-z}. \tag{2.13}$$

Now we evaluate (2.10) and (2.11). Observe that each event $\widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j-1)$ already fixes the color of v_j in ϕ, but all other vertices of e are "free". So, for each $\psi \in \Psi(l_1,\ldots,l_q)$,

$$\mathbf{Pr}(\{\phi(e) = \psi \mid \bigcap_{j=1}^{q} \widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j-1) \cap \widetilde{W}6(j,h_j)\}) \leq t^{q-r}. \tag{2.14}$$

Next, observe that the event $\bigcap_{j=1}^{q} \widetilde{W}7(j,h_j)$ does not depend on

$$\bigcap_{j=1}^{q} \widetilde{W}(e_j, \alpha - 2^{l_j-1}, T(e_j), l_j-1) \cap \widetilde{W}6(j,h_j),$$

since it relates only to the values of $I(u)$ for $u \in e - Q$. For each $j \in [q]$ and each $u \in A_j - v_j$, we have $\mathbf{Pr}\big((l_j-1)s+1 \leq I(u) \leq (l_j-1)s+h_j\big) = ph_j$. Since each $\psi \in \Psi(l_1,\ldots,l_q)$ is completely defined when we choose disjoint sets $A_1 - v_1, \ldots, A_q - v_q$ in $e - Q$, the expression in the lines (2.10) and (2.11) does not exceed

$$t^{q-r} \sum_{a_1=0}^{r-q} \sum_{a_2=0}^{r-q-a_1} \cdots \sum_{a_q=0}^{r-q-a_1-\ldots-a_{q-1}} \binom{r-q}{a_1}\binom{r-q-a_1}{a_2}\cdots\binom{r-q-a_1-\ldots-a_{q-1}}{a_q} \prod_{j=1}^{q}(ph_j)^{a_j}. \tag{2.15}$$

Thus pugging (2.12), (2.13) and (2.15) into (2.8)–(2.11), we have that $\mathbf{Pr}(F_2(e,\alpha,T,l))$ does not exceed

$$\sum_{(l_1,\ldots,l_q)\in\Theta_2} \sum_{h_1=1}^{s} \sum_{h_2=1}^{s} \cdots \sum_{h_q=1}^{s} \sum_{a_1=0}^{r-q} \sum_{a_2=0}^{r-q-a_1} \cdots \sum_{a_q=0}^{r-q-a_1-\ldots-a_{q-1}} t^{q-r} \times \tag{2.16}$$

$$\times \prod_{j=1}^{q}\left(pt\epsilon D^{-y_j}\left(\frac{r}{\ln r}\right)^{(n-l_j+1)/(n+1)} 1.1(1-p(h_j-1))^{r-z}\binom{r}{a_1}\binom{r}{a_2}\cdots\binom{r}{a_q}(ph_j)^{a_j}\right). \quad (2.17)$$

We now will simplify and estimate the expressions in (2.16) and (2.17). First observe that $(1-p(h_j-1))^{r-z} \le (1-p)^{(r-z)(h_j-1)}$. Thus since $0 < p < 2^{-t^r r}$ and $h_j \le s \le \frac{\ln r}{p(n+1)r}$, we have

$$1.1(1-p(h_j-1))^{r-z} \le 1.1(1-p)^{(r-z)(h_j-1)} \le 1.2(1-p)^{(r-z)h_j}.$$

For $j = q, q-1, \ldots, 1$ (in this order), we can estimate

$$p\sum_{h_j=1}^{s}\sum_{a_j=0}^{r-q-a_1-\ldots-a_{j-1}}\binom{r}{a_j}1.2(1-p)^{(r-z)h_j}(h_jp)^{a_j} \le 1.2p\sum_{h_j=1}^{s}(1-p)^{rh_j-zs}\sum_{a_j=0}^{r}\binom{r}{a_j}(h_jp)^{a_j} \le$$

$$\le 1.2p(1-p)^{-zs}\sum_{h_j=1}^{s}(1-p)^{rh_j}(1+h_jp)^r \le 1.2pe^{\frac{pzs}{1-p}}\sum_{h_j=1}^{s}(1-p)^{rh_j}(1+h_jp)^r \le$$

$$\le 1.2pe^{\frac{z\ln r}{(1-p)(n+1)r}}\sum_{h_j=1}^{s}(1-p)^{rh_j}(1+p)^{rh_j} \le 1.2(ps)e^{\frac{c}{(1-p)(n+1)}}.$$

Since $ps \le \frac{\ln r}{r(n+1)}$, $n+1 \ge 2$, and $c = 4t^2 = -\ln\epsilon$, we have

$$1.2(ps)e^{\frac{c}{(1-p)(n+1)}} \le 1.2\frac{\ln r}{r(n+1)}\epsilon^{-1/2(1-p)} < e^{-3t^2/2}\frac{\ln r}{r(n+1)\epsilon}.$$

Thus,

$$\mathbf{Pr}(F_2(e,\alpha,T,l)) \le \sum_{(l_1,\ldots,l_q)\in\Theta_2} t^{-r}D^{-y_1-\ldots-y_q}\prod_{j=1}^{q}\left(\epsilon t^2\left(\frac{r}{\ln r}\right)^{\frac{n-l_j+1}{n+1}} e^{-3t^2/2}\frac{\ln r}{r(n+1)\epsilon}\right). \quad (2.18)$$

Note that

$$t^{-r}D^{-y_1-\ldots-y_q} = t^{-r}D^{-y+1} \le D^{-y}\epsilon\left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}. \quad (2.19)$$

Recall that by the definition of Θ_2, there is j^* such that $l_{j^*} = l$. For every other j, we estimate

$$\epsilon t^2\left(\frac{r}{\ln r}\right)^{\frac{n-l_j+1}{n+1}} e^{-3t^2/2}\frac{\ln r}{r(n+1)\epsilon} \le \frac{t^2}{n+1}e^{-3t^2/2} < \frac{1}{n+1}, \quad (2.20)$$

but for $j = j^*$ we gain more. By (2.18), (2.19), and (2.20), we have

$$\mathbf{Pr}(F_2(e,\alpha,T,l)) \le \sum_{(l_1,\ldots,l_q)\in\Theta_2}\left(D^{-y}\epsilon\left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}\right)\epsilon\left(\frac{r}{\ln r}\right)^{\frac{n-l_{j^*}+1}{n+1}} t^2e^{-3t^2/2}\frac{\ln r}{r\epsilon(n+1)}(n+1)^{-q+1}.$$

Note that since $l_{j^*} = l$, the summands in the last expression do not depend on the choice of $(l_1, \ldots, l_q) \in \Theta_2$. Since $|\Theta_2| \le (l+1)^q \le (n+1)^q$, we have

$$\mathbf{Pr}(F_2(e,\alpha,T,l)) \le \left(D^{-y}\epsilon\left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}\right)\epsilon\left(\frac{r}{\ln r}\right)^{\frac{n-l+1}{n+1}} t^2 e^{-3t^2/2}\frac{\ln r}{r\epsilon} =$$

$$= t^2 e^{-3t^2/2}\epsilon D^{-y}\left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}} \le 0.4\epsilon D^{-y}\left(\frac{r}{\ln r}\right)^{\frac{n-l}{n+1}}.$$

This proves (2.5) for $i = 2$ and thus the lemma. ∎

Applying Lemma 17 for $l = n$, we get the following immediate consequence.

Corollary 18. *Let $e \in E(G)$ and $\alpha \in [t]$. Let $D := \epsilon t^r \left(\frac{r}{\ln r}\right)^{\frac{n}{n+1}}$. Let T be a rooted sub-r-tree of G with root e. If T has y edges, then*

$$\mathbf{Pr}(\widetilde{W}(e,\alpha,T,n)) \le \epsilon D^{-y}.$$

2.6 Proof of Theorem 6

Recall the following version of the Lovász Local Lemma.

Theorem 1 ([2])**.** *Let $A_1, A_2, ..., A_N$ be any events. Let $S_1, S_2, ..., S_N$ be subsets of $[n]$ such that for each i, A_i is independent of the events $\{A_j : j \in ([N] - S_i)\}$. If there exist numbers $x_1, x_2, ..., x_N \in [0,1)$ such that for all $i \in [N]$, $\mathbf{Pr}[A_i] \le x_i \prod_{j\in S_i}(1 - x_j)$, Then,*

$$\mathbf{Pr}[\bigwedge_{i\in[N]} \overline{A_i}] \ge \prod_{i\in[N]}(1 - x_i) > 0.$$

Radhakrishnan and Srinivasan used it in the following form.

Lemma 19 ([28])**.** *Let $A_1, A_2, ..., A_N$ be any events. Let $S_1, S_2, ..., S_N$ be subsets of $[N]$ such that for each i, A_i is independent of the events $\{A_j : j \in ([N] - S_i)\}$. If for all $i \in [N]$, $\mathbf{Pr}(A_i) < \frac{1}{2}$ and $\sum_{j\in S_i}\mathbf{Pr}(A_j) \le \frac{1}{4}$, then $\mathbf{Pr}[\bigwedge_{i\in[N]} \overline{A_i}] > 0$.*

PROOF: We show that if the conditions of this lemma hold, then the conditions of Theorem 1 hold for $x_i = 2\mathbf{Pr}(A_i)$, $i \in [N]$. Indeed, with so defined x_i, inequality

$$\mathbf{Pr}[A_i] \le x_i \prod_{j\in S_i}(1 - x_j)$$

follows if $\prod_{j \in S_i} (1 - x_j) \geq \frac{1}{2}$ holds. Furthermore,

$$\prod_{j \in S_i} (1 - x_j) \geq 1 - \sum_{j \in S_i} x_j = 1 - 2 \sum_{j \in S_i} \mathbf{Pr}(A_j) \geq \frac{1}{2} \text{ (since } \sum_{j \in S_i} \mathbf{Pr}(A_j) \leq \tfrac{1}{4}).$$

Hence by Theorem 1, we have the result. ■

Lemma 20. *Let $0 < \epsilon \leq 4^{-t} t^{-4}$. If $\mathbf{Pr}(\widetilde{W}(e, \alpha, T, n)) \leq \epsilon D^{-y}$ for every $\alpha \in [t]$, every sub-r-tree T of G with $y \leq t$ edges and for every $e \in E(T)$, then with positive probability, none of these events occurs.*

PROOF: Consider the probability space of the outcomes of Evolution. Let the events $A_1, \ldots, A_N$ be the events $\widetilde{W}(e, \alpha, T, n)$ for all $e \in E(G)$, all $\alpha \in [t]$ and sub-r-trees T of G containing e with at most t edges. It is enough to verify that the conditions of Lemma 19 hold for our events $A_1, \ldots, A_N$. Each of the conditions $\mathbf{Pr}(\widetilde{W}(e, \alpha, T, n)) < 1/2$ immediately follows from Corollary 18. By Lemma 16, for the event $A_i = \widetilde{W}(e, \alpha, T, n)$, we we can take S_i equal to the set of all events $\widetilde{W}(e', \alpha', T', n)$ such that $V(T') \cap V(T) \neq \emptyset$.

Now, fix an event $A_i = \widetilde{W}(e, \alpha, T, n)$, where T has y edges, and estimate $\sum_{j \in S_i} \mathbf{Pr}(A_j)$. Let $\widetilde{W}(e', \alpha', T', n) \in S_i$ and suppose that the size of T' is y'. Then some edge e'' of T' intersects $V(T)$ (in particular, e'' can be an edge of T, too). The number of ways to choose an edge that intersects $V(T)$ is at most $D + 1$ if $y = 1$, and is at most yD, if $y > 1$. By Lemma 14, G contains at most $(4D)^{y'-1}$ r-trees of size y' containing edge e''. In each of such trees, there are y' ways to choose a root, e', and t ways to choose the color α'. Since $\mathbf{Pr}(\widetilde{W}(e', \alpha', T', n)) \leq \epsilon D^{-y'}$, it follows that

$$\sum_{j \in S_i} \mathbf{Pr}(A_j) \leq \sum_{y'=1}^{t} tD \cdot (4D)^{y'-1} y' t \epsilon D^{-y'} = \sum_{y'=1}^{t} t^2 y' 4^{y'-1} \epsilon \leq t^4 4^{t-1} \epsilon. \tag{2.21}$$

Since $0 < \epsilon \leq 4^{-t} t^{-4}$, the last expression in (2.21) is at most $1/4$. Thus we are done by Lemma 19. ■

Now we are ready to complete the proof of Theorem 6. Indeed, let G be a hypergraph satisfying the conditions of the theorem. Consider procedure Evolution. By Corollary 18, for each y-edge r-tree T, each edge $e \in E(T)$ and each $\alpha \in [t]$, $\mathbf{Pr}(\widetilde{W}(e, \alpha, T, n)) \leq \epsilon D^{-y}$. For $t \geq 2$, we have $\epsilon = \exp\{-4t^2\} < 4^{-t} t^{-4}$. So, by Lemma 20, with positive probability none of the events $\widetilde{W}(e, \alpha, T, n)$ occurs. It follows that in some outcome of Evolution none of the events $\widetilde{W}(e, \alpha, T, n)$ occurs. By Lemma 15, in this outcome none of the events $W(e, \alpha, T, n)$ occurs. But then the resulting t-coloring will be proper.

Chapter 3

Coloring simple hypergraphs with few edges

3.1 Introduction

We now turn our attention from coloring of general hypergraphs (discussed in the previous chapter) to coloring of simple hypergraphs. Recall that a hypergraph is called *simple* if every two edges intersect in at most one vertex. A hypergraph is simple if the girth is at least three. As before, one can ask a similar question: Given the number of colors used (say t), what is the smallest number of edges in a simple hypergraph $\mathcal{H}$ such that $\mathcal{H}$ is no longer t-colorable ? More generally, one might also fix the girth of a hypergraph. Let $m(r,t,g)$ denote the smallest number of edges in an r-uniform hypergraph with girth at least g and chromatic number at least $t+1$. In their seminal paper [10], Erdős and Lovász gave the upper bound

$$m(r,t,g) \leq 4 \cdot 20^{g-1} r^{3g-5} t^{(g-1)(r+1)} \tag{3.1}$$

for all g and the lower bound

$$m(r,t,3) \geq \frac{t^{2(r-2)}}{16r(r-1)^2} \tag{3.2}$$

for simple hypergraphs. The ratio of the upper bound to the lower bound for simple hypergraphs is only r^7. The bound (3.2) was derived from the famous result stated in Theorem 2.

To derive the bound, they used an interesting trick of *trimming*. We discuss trimming in Subsection 3.3.1.

Szabó [30] refined the second part of the bound of Theorem 2 for simple hypergraphs as follows.

Theorem 21. *If $t \geq 2$ and $\epsilon > 0$ are fixed and r is sufficiently large, then every r-uniform simple hypergraph $\mathcal{H}$ with maximum degree at most $t^r r^{-\epsilon}$ is t-colorable.*

Actually, Szabó proved the theorem only for $t = 2$, since that was what he needed for his applications, but the technique works for any fixed t. Again, applying trimming and this theorem, one easily gets that for fixed t and ϵ and large r,

$$m(r,t,3) \geq \frac{t^{2r}}{r^{1+\epsilon}}. \tag{3.3}$$

Here we consider simple and so called b-simple hypergraphs. A hypergraph $\mathcal{H}$ is *b-simple* if $|e \cap e'| \leq b$ for every distinct $e, e' \in E(\mathcal{H})$. Sometimes, b-simple hypergraphs are called *partial Steiner systems.* A 1-simple hypergraph is a simple hypergraph.

The main result of this chapter (we state it in the next section) says that for fixed $t \geq 2$ and $\epsilon > 0$ and sufficiently large r, if a simple r-uniform hypergraph $\mathcal{H}$ cannot be colored with t colors, then either it has a vertex of degree greater than $r\,t^r$, or there are "many" vertices of degree greater than $t^r r^{-\epsilon}$. This will improve the bound (3.3) by a factor of r. Our result also yields an improvement of the edge-degree version of Theorem 2 for simple hypergraphs as follows.

Theorem 22. *If $b \geq 1$, $t \geq 2$ and $\epsilon > 0$ are fixed and r is sufficiently large, then every r-uniform b-simple hypergraph $\mathcal{H}$ with maximum edge-degree at most $t^r r^{1-\epsilon}$ is t-colorable.*

The theorem holds also for list colorings. In order to keep proofs easier to read, we give the proof for ordinary colorings and comment at the end of the chapter how to adapt the proofs to list coloring.

Let $f(r,t,b)$ denote the fewest possible number of edges in an r-uniform b-simple hypergraph that is not t-colorable. From our main result we deduce that for fixed t, b and $\epsilon > 0$ and sufficiently large r,

$$f(r,t,b) \geq \frac{t^{r(1+1/b)}}{r^{\epsilon}}. \tag{3.4}$$

It turns out that in terms of r the bound cannot be improved by more than a polynomial factor. Using the Erdős–Lovász technique [10] for proving (3.2), we show that for large r,

$$f(r,t,b) \leq 40t^2 \left(t^r r^2\right)^{1+1/b}. \tag{3.5}$$

We also use our main result and trimming to derive the following lower bounds on $m(r,t,g)$ for arbitrary fixed g (in [10], the bound was only for $g = 3$):

$$m(r,t,2s+1) \geq \frac{t^{r(1+s)}}{r^{\epsilon}}, \tag{3.6}$$

if r is large in comparison with t, s and $1/\epsilon$.

The structure of the rest of the chapter is as follows. In the next section we prove the main result. In Section 3.3, lower bounds on the size of non-t-colorable hypergraphs are given. In Section 3.4, bound (3.5) is derived. We conclude the chapter with some comments. In particular, we comment on list colorings of hypergraphs.

This is a joint work with A. Kostochka and appears in [20].

3.2 Coloring simple hypergraphs with bounded edge degrees

Szabó's theorem says that for large r, every r-uniform simple hypergraph with the degree of each vertex at most $t^r r^{-\epsilon}$ is t-colorable. Our result extends the conclusion to r-uniform simple (and b-simple) hypergraphs in which the degree of each edge is at most $t^r r^{1-\epsilon}$.

A vertex v of $\mathcal{H}$ is *low*, if $\deg(v) \le t^r r^{-\epsilon}$ and *high* otherwise. For an edge e, let $L(e)$ (respectively, $H(e)$) be the set of low (respectively, high) vertices in e. An edge e is *light*, if $|H(e)| \le 0.5r$ and *heavy* otherwise.

For a given $\epsilon > 0$, an r-uniform hypergraph $\mathcal{H}$ is (t, ϵ)-*sparse* if

$$\Delta(\mathcal{H}) \le t^r\, r, \quad \text{and} \tag{3.7}$$

$$\text{every vertex of } \mathcal{H} \text{ is in at most } t^r/r^{\epsilon} \text{ heavy edges.} \tag{3.8}$$

Our main result is the following.

Theorem 23. *If $b \ge 1$, $t \ge 2$ and $\epsilon > 0$ are fixed and r is sufficiently large, then every r-uniform b-simple (t, ϵ)-sparse hypergraph $\mathcal{H}$ is t-colorable.*

In order to derive Theorem 22 from our main result, we observe that for sufficiently large r, every not $(t, 0.5\epsilon)$-sparse hypergraph $\mathcal{H}$ has an edge of degree greater than $t^r r^{1-\epsilon}$. This is trivial if (3.7) does not hold. Suppose now that (3.8) does not hold, in particular that some edge e in $\mathcal{H}$ is heavy. Then the sum of degrees of vertices in e is greater than $0.5rt^r r^{1-0.5\epsilon}$. Since every edge $e' \ne e$ contributes at most b to this sum, e itself contributes r, and $r^{0.5\epsilon} > 4b$, the degree of e in $\mathcal{H}$ is greater than $t^r r^{1-\epsilon}$. This proves Theorem 22 (modulo Theorem 23).

3.2.1 Szabó's approach and the structure of the proof

We follow the ideas of Szabó [30]. He used the following lemma of Beck [6], who in turn used the Lovász Local Lemma.

Lemma 24 (Beck). *Let X be a finite set and B_1,B_2,...,B_s be not necessarily distinct subsets of X with $|B_i| \ge r$. For every i, let $f_i : B_i \to \{1, 2, ..., t\}$ be a given t-coloring of B_i. If*

$$\sum_{i:p \in B_i} (1 - \frac{1}{r})^{-|B_i|} t^{-|B_i|} \le \frac{1}{r} \tag{3.9}$$

for every $p \in X$, then there exists a t-coloring $f : X \to \{1, 2, ..., t\}$ such that $f|_{B_i} \ne f_i$.

Szabó's idea of the proof is the following. Let $\mathcal{H}$ be an r-uniform simple hypergraph satisfying the conditions of his theorem. Szabó starts from a t-coloring of vertices of $\mathcal{H}$ where each vertex is colored with a color uniformly at random chosen from the set $\{1, \ldots, t\}$ independently from all other vertices. He considers a special set of so called *configurations* that are pairs (B_i, f_i), where $B_i \subseteq V(\mathcal{H})$ and f_i is a given t-coloring of B_i. The meaning of configurations, is that they are bad situations that may cause some edges to become monochromatic after special recolorings in the future. He proved that

(a) if f is any (not necessarily proper) t-coloring of $V(\mathcal{H})$ and none of his configurations occurs, then some vertices of $\mathcal{H}$ can be recolored so that the resulting t-coloring of $\mathcal{H}$ is proper;

(b) Inequality (3.9) holds for every $p \in V(\mathcal{H})$.

Together with Lemma 24, this yields that $\mathcal{H}$ has a proper t-coloring. Observe that each configuration $B \subseteq V(\mathcal{H})$ contributes to the sum in (3.9) the amount $(1-\frac{1}{r})^{-|B|}t^{-|B|}$, and we will call this expression *the contribution of* B. To prove that (3.9) holds, for every "bad" configuration $B \subseteq V(\mathcal{H})$, Szabó estimated its contribution.

We will use the same scheme with somewhat changed rules of recoloring and somewhat different configurations.

Another idea of Szabó is that in each edge e of $\mathcal{H}$ he chooses a subset $R(e)$ such that later, if e is monochromatic, then he tries to recolor only vertices in $R(e)$ and does not touch other vertices. This choice allows to decrease the number of "bad" configurations whose contributions we need to estimate. The structure of our proof is the following. In the next subsection, we construct a subset $R(e)$ of each edge e. Later, if e becomes monochromatic, we will try to recolor only vertices in $R(e)$. In Subsection 3.2.3 we give the main proof assuming that we have some bounds on the contributions of "bad" configurations. In Subsections 3.2.4 and 3.2.5 we prove these bounds on contributions.

3.2.2 Choosing $R(e)$

Lemma 25. *Let $k \le r/3$. Then in every light edge e, we can choose a k-element set $R(e) \subseteq L(e)$ so that for each low vertex v,*

$$| \{e : v \in R(e)\} | \le \frac{t^r}{r^\epsilon} \frac{4k}{r}. \tag{3.10}$$

Proof. Consider the bipartite graph $G[X, Y]$, where X is the set of light edges in $\mathcal{H}$, Y is the set of low vertices in $\mathcal{H}$, and $xy \in E(G)$ if and only if edge x contains vertex y in $\mathcal{H}$. By the definition of light edges, each vertex in X has degree in G at least $r/2$. By the definition of low vertices, $d_G(v) \le t^r/r^\epsilon$ for every $v \in Y$. Let G_1 be the graph obtained from G by splitting every vertex $v \in Y$ into $\lceil 2d_G(v)/r \rceil$ vertices, each with degree at most $\lceil r/2 \rceil$.

Let G_2 be obtained from G_1 by deleting some edges so that the degree of every vertex $x \in X$ becomes $\lceil r/2 \rceil$. By Konig's theorem, there exists a proper edge-coloring ϕ of G_2 with $\lceil r/2 \rceil$ colors. Let G_3 be the subgraph of G_2 formed by the edges with colors $\{1, 2, ...k\}$ in ϕ. Finally, let G_4 be obtained from G_3 by gluing back all the split vertices in Y. By construction, G_4 is a spanning subgraph of G, and the degree of every vertex $x \in X$ in G_4 is exactly k. The degree in G_4 of every vertex $v \in Y$ is at most

$$k\lceil 2d_G(v)/r \rceil \leq k \left\lceil \frac{t^r}{r^\epsilon} \frac{2}{r} \right\rceil .$$

The last expression for large r does not exceed the RHS of (4.4).

For every edge e in $\mathcal{H}$, let $R(e)$ be the set of vertices adjacent to vertex e in G_4. By the properties of G_4, the lemma holds for these $R(e)$. ☐

Lemma 26. *Let $k \leq r/3$. Then in every heavy edge e, we can choose a k-element set $R(e) \subseteq H(e)$ so that for each heavy vertex v,*

$$| \{e : v \in R(e)\} | \leq \frac{t^r}{r^\epsilon} \frac{4k}{r}. \tag{3.11}$$

Proof. By (3.8), every vertex is in at most $t^r r^{-\epsilon}$ heavy edges. We essentially repeat the proof of Lemma 25, only replacing light edges with heavy and low vertices with high ones. ☐

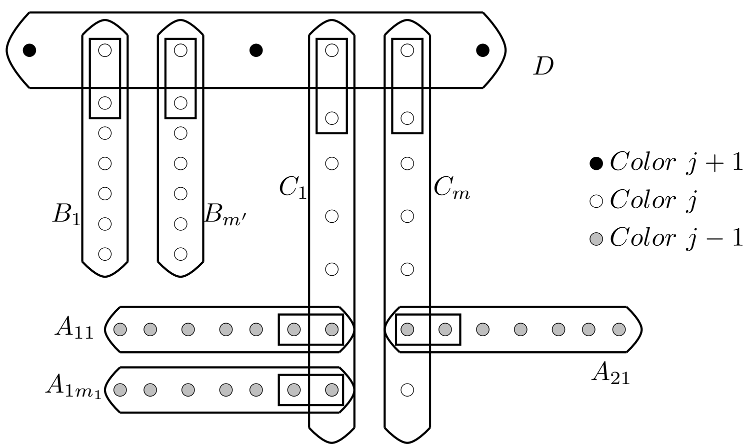

Figure 3.1: An example of a Configuration of Type 1.

3.2.3 Configurations and the main proof

We start from a random t-coloring f of vertices of $\mathcal{H}$ where each vertex v is colored with a color $f(v)$ uniformly at random chosen from the set $\{1, \ldots, t\}$ independently from all other vertices.

Configurations of Type 1: A *configuration of Type 1*, $\mathcal{C}(j, m, m', m_1, \ldots, m_m)$, *with parameters* $j, m, m', m_1, \ldots, m_m$ consists of $1 + m + m' + (m_1 + \ldots + m_m)$ (not necessarily distinct) edges $D, B_1, \ldots, B_{m'}, C_1, \ldots, C_m, A_{1,1}, \ldots, A_{1,m_1}, A_{2,1}, \ldots, A_{m,m_m}$ arranged and colored so that:

(α_1) There are m' distinct vertices $b_1, \ldots, b_{m'}$ in $H(D)$ such that $b_i \in R(B_i)$ for $i = 1, \ldots, m'$.

(α_2) There are m distinct vertices $c_1, \ldots, c_m$ in $L(D)$ such that $c_i \in R(C_i)$ for $i = 1, \ldots, m$.

(α_3) All $B_1, \ldots, B_{m'}, C_1, \ldots, C_m$ are distinct.

(α_4) All vertices in $D - \{b_1, \ldots, b_{m'}, c_1, \ldots, c_m\}$ are colored with color $j + 1$ (modulo t).

(α_5) All vertices in $B_1, \ldots, B_{m'}$ are colored with j.
(α_6) For $i = 1, \ldots, m$, $H(C_i)$ contains m_i distinct vertices $a_{i,1}, \ldots, a_{i,m_i}$ such that $a_{i,i'} \in R(A_{i,i'})$ for all $i' = 1, \ldots, m_i$.
(α_7) Vertex a_{i_1,i'_1} may coincide with a_{i_2,i'_2}, when $i_1 \neq i_2$, in which case A_{i_1,i'_1} *should* coincide with A_{i_2,i'_2}). If $a_{i_1,i'_1} \neq a_{i_2,i'_2}$, then $A_{i_1,i'_1} \neq A_{i_2,i'_2}$.
(α_8) For every $i = 1, \ldots, m$ all vertices in $C_i - \{a_{i,1}, \ldots, a_{i,m_i}\}$ are colored with j.
(α_9) All vertices in all $A_{1,1}, \ldots, A_{1,m_1}, A_{2,1}, \ldots, A_{m,m_m}$ are colored with $j-1$ (modulo t).

Comments. Since $b_1, \ldots, b_{m'} \in H(D)$, each of $B_1, \ldots, B_{m'}$ is a heavy edge. For the same reason, each of $A_{1,1}, \ldots, A_{1,m_1}, A_{2,1}, \ldots, A_{m,m_m}$ is heavy. Similarly, each of $C_1, \ldots, C_m$ is a light edge.

In a configuration of Type 1, edge D is called *the leading edge*, edges $B_1, \ldots, B_{m'}$ are *type B edges*, edges $C_1, \ldots, C_m$ are *type C edges*. Vertices $b_1, \ldots, b_{m'}$ and $c_1, \ldots, c_m$ are *special in* $\mathcal{C}$. The *size* of a configuration is the cardinality of the union of its edges.

Let $k = \lceil \frac{20}{\epsilon} \rceil$. In the next subsection, we will prove that for every vertex p in $\mathcal{H}$, the total contribution of configurations of Type 1 containing p such that at least one of $m, m', m_1, \ldots, m_m$ exceeds k is $o(1/r)$.

Configurations of Type 2a: There is a heavy edge B such that for each vertex $b \in R(B)$ there is a configuration $\mathcal{C}_b$ of Type 1 with $m = 0$ and $m' \leq k$ such that b is special and B is an edge of type B in $\mathcal{C}_b$.

Configurations of Type 2b: There is a light edge C such that for each vertex $c \in R(C)$ there is a configuration $\mathcal{C}_c$ of Type 1 with each of $m', m, m_1, \ldots, m_m$ at most k such that c is special and C is an edge of type C in $\mathcal{C}_c$.

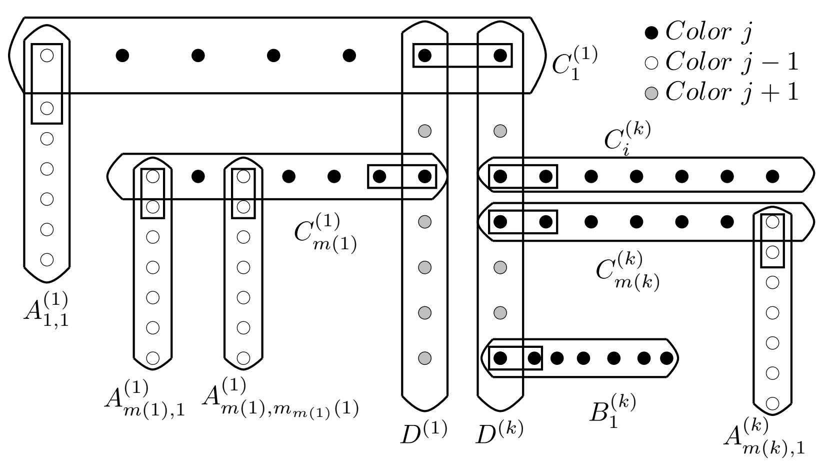

Figure 3.2: An example of a Configuration of Type 2b.

In Subsection 3.2.5 we prove that for every vertex p in $\mathcal{H}$, the total contribution of configurations of Types 2a and 2b containing p is $o(1/r)$. These facts together with Lemma 24 yield that there exists a t-coloring f' avoiding configurations of Type 1 with at least one of $m'm, m_1, \dots, m_m$ exceeding k and also avoiding all configurations of Type 2a and 2b. This coloring f' might have monochromatic edges, but we shall see that we can recolor some of the vertices and get a proper t-coloring.

First recoloring: Since configurations of Type 2a do not appear in f', for every heavy monochromatic edge B (say, of color $j(B)$), there exists a vertex $b(B) \in R(B)$ such that there is no configurations of Type 1 with a leading heavy edge D such that $b(B)$ is a special vertex in D and B is a Type B edge in this configuration. For every monochromatic heavy edge B, recolor $b(B)$ with color $j(B) + 1$ (modulo t). By the choice of $b(B)$, we recolored only some high vertices.

We claim that the new coloring f'' does not have monochromatic heavy edges. Indeed, suppose that some heavy edge D is monochromatic of color j in f''. This means that it was not monochromatic of color j in f', since in that case, a vertex of $R(D)$ would be recolored to $j+1$. So, there are vertices $b_1, \dots, b_{m'}$ in $H(D)$ that were recolored from color $j-1$, and for each b_i, there is a heavy edge B_i with $b_i \in R(B_i)$ that was monochromatic in f' and $b_i = b(B_i)$. So, we have a configuration of Type 1 in f' that contradicts the definition of the vertex $b(B_1)$.

Second recoloring: Let C be a monochromatic edge of color $j(C)$ in the new coloring f''. By above, it is a light edge, and in f' C either was monochromatic of the same color, or some vertices $b_1, \dots, b_{m'} \in H(C)$ were of color $j(C)-1$, and each b_i was in $R(B_i)$ for some heavy monochromatic edge B_i and was recolored because of this edge. Suppose that for every $c \in R(C)$, there is a configuration of Type 1 in coloring f'' with $m_1 = m_2 = \dots = m_m = 0 = m'$ and the leading edge containing c as a special vertex, where C is a Type C edge. Then each such configuration in f'' corresponds to some more general configuration of Type 1 in coloring f'. It follows that we encounter a configuration of Type 2b in f', a contradiction to the choice of f'. Thus, every monochromatic edge C in the new coloring f'' contains a vertex $c(C) \in R(C)$ such that there is no configuration of Type 1 in coloring f'' with the leading edge containing c as a special vertex such that $m_1 = m_2 = \dots = m_m = 0 = m'$ and C is a Type C edge in this configuration.

For every monochromatic edge C in f'', recolor $c(C)$ with color $j(C) + 1$. Observe that at this second recoloring, we recolored only low vertices. Assume that some edge D is monochromatic in the new coloring f (of color $j(D)$). If it was also monochromatic in f'', then D is light, and some vertex of $R(D)$ would be recolored; so this is not the case. Thus, there are vertices $c_1, \dots, c_m$ in $L(D)$ that were recolored from color $j(D)-1$, and for each c_i, there is a light edge C_i with $c_i \in R(C_i)$ that was monochromatic in f'' of color $j(D) - 1$ and $c_i = c(C_i)$. Furthermore, since C_i was monochromatic in f'', either it also was monochromatic in f' or there are vertices $a_{i,1}, \dots, a_{i,m_i} \in H(C)$ of color $j(D) - 2$ that were recolored in the first stage. In this case, in f' each $a_{i,i'}$ was in $R(A_{i,i'})$ for some heavy monochromatic edge $A_{i,i'}$ and was recolored in first stage because of this edge. Some vertices

$b_1, \ldots, b_{m'}$ in $H(D)$ also could be recolored in the first stage. Thus, we have a configuration of Type 1 in f', a contradiction to the choice of $c(C_1)$. Since we recolored high vertices in the first stage and low at the second, no vertex is recolored more than once.

Thus, the theorem will be proved when we show that for every vertex p in $\mathcal{H}$, the total contribution of configurations of Type 1 containing p such that at least one of $m, m', m_1, \ldots, m'_m$ exceeds k is $o(1/r)$ and that the total contribution of configurations of Types 2a and 2b containing p is $o(1/r)$.

3.2.4 Handling configurations of Type 1

We will first consider some partial cases.

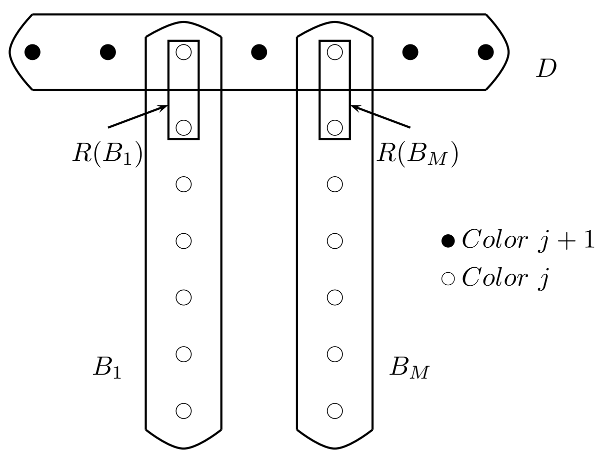

Figure 3.3: An example of a Configuration of Type 1a.

Configuration of Type 1a: This is a configuration of Type 1 in which $m_1 = m_2 = \ldots = m_m = 0$ and $k \leq m + m' \leq \frac{r}{10bk}$.

For convenience of notation in handling configurations of Type 1a, define $B_{m'+i} = C_i$ for $i = 1, \ldots, m$ and let $M = m + m'$. For $q = 1, \ldots, M$, call edge B_q *determined* if it intersects with $\cup_{i \leq q-1} B_i$ in at least $b + 1$ vertices. Let $p \in V(\mathcal{H})$ and z be a non-negative integer. Then the total contribution $\phi_{1a}(p, M, z, D)$ of all configurations of Type 1a, containing p such that $p \in D$ and exactly z edges in $\{B_1, \ldots, B_M\}$ are determined is estimated as follows:

(β_1) The number of candidates for D containing p is at most $\deg(p) \leq t^r r$.
(β_2) The number of ways to choose $b_1, \ldots, b_M$ in D is at most $\binom{r}{M}$.
(β_3) The number of choices of colors for vertices in D such that vertices $b_1, \ldots, b_M$ are colored with j and other are colored with $j + 1$ is t.

(β_4) The number of ways to choose which z edges B_i will be determined is $\binom{M}{z}$.
(β_5) By Lemmas 25 and 26, the number of ways to choose a non-determined B_i when we know the corresponding special vertex is at most $\frac{t^r}{r^\epsilon}\frac{4k}{r}$.
(β_6) Since every determined edge contains some $(b+1)$-tuple of vertices in the union of "previous" edges and these $(b+1)$-tuples should be distinct for different edges, the number of ways to choose a determined B_i when we know the corresponding special vertex is at most $\binom{Mr}{b+1} < \binom{r^2}{b+1} < r^{2b+2}$.
(β_7) Since

$$|\bigcup_{l=1}^{i} B_l - \bigcup_{l=1}^{i-1} B_l| \geq \begin{cases} r-b, & \text{if } B_i \text{ is non-determined,} \\ r-Mb, & \text{if } B_i \text{ is determined,} \end{cases} \tag{3.12}$$

and $Mb \leq r/10k$, the size of each such configuration is at least $r + (M-z)(r-b) + z\,\frac{9r}{10}$.

Hence

$$\phi_{1a}(p,M,z,D) \leq t^r r\, t \binom{r}{M}\binom{M}{z}\left(\frac{4k\,t^r}{r^{1+\epsilon}}\right)^{M-z} r^{2z(b+1)} \left(\frac{r}{t(r-1)}\right)^{r+(M-z)(r-b)+0.9zr}.$$

Since $\binom{r}{M}\binom{M}{z} \leq r^M$ and $\left(\frac{r}{r-1}\right)^{r+(M-z)(r-b)+0.9zr} \leq 3^{1+M}$, the last expression is at most

$$3^{1+M}(4k)^{M-z}\, t^{r+1+r(M-z)-r-(M-z)(r-b)-0.9zr}\, r^{1+M-(M-z)(1+\epsilon)+2z(b+1)}.$$

Denoting the last expression by $\psi_{1a}(M,z)$, we have

$$\frac{\psi_{1a}(M,z+1)}{\psi_{1a}(M,z)} \leq \frac{1}{4k}\, t^{-r+(r-b)-0.9r}\, r^{(1+\epsilon)+2(b+1)} = \frac{1}{4k}\, t^{-b-0.9r}\, r^{2b+3+\epsilon},$$

which is less than $1/4$ for large r. Therefore,

$$\sum_{z=0}^{M} \phi_{1a}(p,M,z,D) < 2\psi_{1a}(M,0) =$$

$$= 3^{1+M} 2(4k)^M\, t^{r+1+rM-r-M(r-b)}\, r^{1+M-M(1+\epsilon)} = 6t\,r \left(\frac{12k\,t^b}{r^\epsilon}\right)^M. \tag{3.13}$$

Since for large r, $12k\,t^b < r^{\epsilon/2}$, the last expression is less than $6t\,r^{1-0.5M\epsilon}$. Since $M \geq k \geq \frac{20}{\epsilon}$, this is less than $6t\,r^{-9} = o(r^{-8})$. Thus, the total contribution $\phi_{1a}(p,D)$ of all configurations of Type 1a such that $p \in D$ is less than

$$\sum_{M=k}^{r} 2\psi_{1a}(M,0) < r \cdot o(r^{-8}) = o(r^{-7}).$$

Now we calculate the contribution of configurations of Type 1a containing p such that $p \notin D$. In this case fix an edge B_i containing p in at most $t^r r$ ways. Then we can choose vertex $b_i \in R(B_i)$ in at most k ways and the edge D containing B_i in at most $k\,t^r r$ ways. To choose the remaining $M-1$ special vertices in D there are only $\binom{r-1}{M-1}$ ways. Then using the same argument and almost the same calculations as above we get that the total contribution here is at most $k\,r$ times greater than $\sum_{M=1}^{r} 2\psi_{1a}(M,0)$. Hence the total contribution, $\phi_{1a}(p)$ of all configurations of Type 1a containing p is $o(\frac{1}{r^6})$.

Configuration of Type 1b: We need this structure to handle configurations of Type 1 in which $m_1 = m_2 = \ldots = m_m = 0$ and $m+m' \geq \frac{r}{10bk}$. But we consider a somewhat different situation: it is a configuration of Type 1 in which $m_1 = m_2 = \ldots = m_m = 0$ and $m+m' = \lfloor \frac{r}{10bk} \rfloor$, but non-special vertices in D also allowed to be colored with color j (and not only with $j+1$). We will estimate the contributions of such new configurations.

As in case of Type 1a, define $B_{m'+i} = C_i$ for $i = 1, \ldots, m$ and let $M = m + m'$.

For $p \in V(\mathcal{H})$ and an integer z, let $\phi_{1b}(p,z,D)$ denote the total contribution of all configurations of Type 1b, containing p such that $p \in D$ and exactly z edges among B_i are determined. We repeat the first half of the argument for Type 1a, replacing (β_3) by the following:
(β_3') The number of choices of colors for vertices in D such that vertices $b_1, \ldots, b_M$ are colored with j and other are colored with j or $j+1$ is $t\,2^{r-M}$.

Because of the extra factor of 2^{r-M}, instead of (3.13), we get

$$\sum_{z=0}^{M} \phi_{1b}(p,z,D) < 6t\,r \left(\frac{12k\,t^b}{r^{\epsilon}}\right)^M 2^{r-M}.$$

Again, for large r, $12k\,t^b < r^{\epsilon/2}$, and the last expression is at most $2^{r-M} 6t\,r^{1-0.5M\epsilon}$. Since $M = \lfloor \frac{r}{10bk} \rfloor$ this is $o(r^{-8})$.

Similarly to the argument for configurations of Type 1a, the contribution of configurations of Type 1b containing p such that $p \notin D$ cannot exceed the last expression more than r^2 times. Thus, the total contribution $\phi_{1a}(p) + \phi_{1b}(p)$ of all configurations of Types 1a and 1b containing p is $o(r^{-6})$.

From now on, we consider only t-colorings of $V(\mathcal{H})$ such that no configurations of types 1a or 1b occur.

Configuration of Type 1c: This is a configuration of Type 1 in which $k \leq m + m' \leq \frac{r}{5bk}$.

By (α_8) and (α_9) in the definition configurations of Type 1, for every $i = 1, \ldots, m$, the set $C_i \cup \bigcup_{i'=1}^{m_i} A_{i'}$ with our coloring form a configuration of Type 1a or 1b if $m_i \geq k$. Since such configurations are forbidden, we assume that $m_i < k$ for every i. Similarly, if $m' \geq \frac{r}{10bk}$, then the set $B_1 \cup \ldots B_{m'} \cup D$ with our coloring forms a configuration of Type 1b, and so we consider only the case $m' < \frac{r}{10bk}$.

In order to calculate carefully the contributions of configurations of Type 1c, let $\hat{m}_i$ denote the number of edges in $\{A_{i,1}, \ldots, A_{i,m_i}\}$ that are distinct from all $A_{l,l'}$ for all $l < i$.

Let $p \in V(\mathcal{H})$, $k \leq m + m' \leq \frac{r}{5bk}$, and $z, z', \hat{m}_1, \ldots, \hat{m}_m$ be non-negative integers. Let $\hat{M} = \hat{m}_1 + \ldots + \hat{m}_m$. Let $\phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ denote the total contribution of all configurations of Type 1c with parameters $m', m, \hat{m}_1, \ldots, \hat{m}_m$ containing p such that $p \in D$, exactly z edges among $B_1, \ldots, B_{m'}, C_1, \ldots, C_m$ are determined, and exactly z' other edges are determined. We can estimate it as follows:

(γ_1) The number of candidates for D containing p is at most $\deg(p) \leq t^r r$.
(γ_2) The number of ways to choose $b_1, \ldots, b_{m'}$ and $c_1, \ldots, c_m$ in D is at most $\binom{r}{m+m'}\binom{m+m'}{m}$.
(γ_3) The number of choices of colors for vertices in D such that vertices $b_1, \ldots, b_{m'}$ and $c_1, \ldots, c_m$ are colored with j and all others are colored with $j+1$ is t.
(γ_4) The number of ways to choose which z edges among $B_1, \ldots, B_{m'}, C_1, \ldots, C_m$ will be determined is $\binom{m'+m}{z}$.
(γ_5) By Lemmas 25 and 26, the number of ways to choose a non-determined B_i when we know $b_i \in R(B_i)$ or C_i when we know $c_i \in R(C_i)$ is at most $\frac{t^r}{r^\epsilon} \frac{4k}{r}$.
(γ_6) The number of ways to choose a determined edge B_i when we know $b_i \in R(B_i)$ or C_i when we know $c_i \in R(C_i)$ is at most $\binom{(m+m')r}{b+1} < \binom{r^2}{b+1} < r^{2b+2}$.
(γ_7) The number of ways to choose all vertices $a_{i,i'}$ in $C_1 \cup \ldots C_m$ that will be colored with $j-1$ is at most $\binom{mr}{\hat{M}}$.
(γ_8) The number of ways to choose which z' edges among $A_{i,i'}$ will be determined is $\binom{\hat{M}}{z'}$.
(γ_9) The number of ways to choose a non-determined $A_{i,i'}$ when we know $a_{i,i'} \in R(A_{i,i'})$ is at most $\frac{t^r}{r^\epsilon} \frac{4k}{r}$.
(γ_{10}) The number of ways to choose a determined edge $A_{i,i'}$ when we know $a_{i,i'} \in R(A_{i,i'})$ is at most $\binom{mrk}{b+1} < \binom{r^2}{b+1} < r^{2b+2}$.
(γ_{11}) To estimate the size of such a configuration, recall that $m'+m \leq \frac{r}{5bk}$ and that $\hat{m}_i \leq m_i \leq k-1$ for each i. Therefore, $m' + m + \hat{M} \leq k(m' + m) \leq \frac{r}{5b}$. Similarly to (3.12), when we add edges one by one to the configuration, every non-determined edge adds at least $r - b$ vertices and every determined edge adds at least $r - b(m' + m + \hat{M}) \geq r - r/5 = 4r/5$ vertices to the union. It follows that the size of each such configuration is at least $r + (m+m'+\hat{M}-z-z')(r-b) + (z+z')\frac{4r}{5}$.

Hence $\phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ is at most

$$t^r r \binom{r}{m+m'} \binom{m+m'}{m} t \binom{m'+m}{z} \left(\prod_{i=1}^{m} \binom{r}{\hat{m}_i} \right) \binom{\hat{M}}{z'} \times$$

$$\times \left(\frac{4kt^r}{r^{1+\epsilon}}\right)^{m+m'+\hat{M}-z-z'} r^{(z+z')(2b+2)} \left(\frac{r}{t(r-1)}\right)^{r+(m+m'+\hat{M})(r-b)-(z+z')(0.2r-b)}.$$

Since $\frac{r}{r-1} < 3^{1/r}$ and

$$\binom{r}{m+m'}\binom{m+m'}{m}\binom{m'+m}{z}\left(\prod_{i=1}^{m}\binom{r}{\hat{m}_i}\right)\binom{\hat{M}}{z'} \leq$$

$$r^{m+m'}(m+m')^z r^{\hat{M}}\hat{M}^{z'} \leq r^{m+m'+\hat{M}+z+z'},$$

$\phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ is at most

$$t^{r+1+r(m+m'+\hat{M}-z-z')-r-(m+m'+\hat{M})(r-b)+(z+z')(0.2r-b)}(4k)^{m+m'+\hat{M}-z-z'}\times$$

$$\times r^{1+m+m'+\hat{M}+z+z'-(1+\epsilon)(m+m'+\hat{M}-z-z')+(z+z')(2b+2)}3^{1+m+m'+\hat{M}} =$$

$$= t^{1+b(m+m'+\hat{M})-(z+z')(0.8r+b)}(4k)^{m+m'+\hat{M}-z-z'}r^{1-\epsilon(m+m'+\hat{M})+(z+z')(2b+4+\epsilon)}3^{1+m+m'+\hat{M}}.$$

Denoting the last expression by $\psi_{1c}(m+m'+\hat{M}, z+z')$, we have

$$\frac{\psi_{1c}(m+m'+\hat{M}, z+z'+1)}{\psi_{1c}(m+m'+\hat{M}, z+z')} \leq \frac{1}{4k}\, t^{-0.8r-b}\, r^{2b+4+\epsilon},$$

which is less than $1/4r$ for large r. Therefore,

$$\sum_{z=0}^{m'+m}\sum_{z'=0}^{\hat{M}} \phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D) < 2\psi_{1c}(m+m'+\hat{M}, 0) =$$

$$= 6t\, r\left(\frac{12k\, t^b}{r^\epsilon}\right)^{m+m'+\hat{M}}. \tag{3.14}$$

Observe that the last bound depends only on $\hat{M}$ and not on the values of particular $\hat{m}_1, \ldots, \hat{m}_m$. Let

$$\phi_{1c}(p, m', m, \hat{M}, D) = \sum_{(\hat{m}_1,\ldots,\hat{m}_m)\,:\,\hat{m}_1+\ldots+\hat{m}_m=\hat{M}}\ \sum_{z+z'=0}^{m+m'+\hat{M}} \phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D).$$

Since the number of m-tuples $(\hat{m}_1, \ldots, \hat{m}_m)$ with $\hat{m}_1 + \ldots + \hat{m}_m = \hat{M}$ is $\binom{m+\hat{M}-1}{m-1} < 2^{m+\hat{M}}$, for large r, by (3.14),

$$\phi_{1c}(p, m', m, \hat{M}, D) \leq 6t\, r\left(\frac{12k\, t^b}{r^\epsilon}\right)^{m+m'+\hat{M}} 2^{m+\hat{M}} \leq 6t\, r^{1-0.5\epsilon(m+m'+\hat{M})}.$$

Since $m + m' \geq k \geq \frac{20}{\epsilon}$, the last expression is $o(r^{-8})$. Since $m' < k$, $m < r/k$ and $\hat{M} < mk < r$, the total contribution of all configurations of Type 1c containing p such that $p \in D$, is at most

$$o(r^{-8})\, k\frac{r}{k}r = o(r^{-6}).$$

Similarly to the case of configurations of Type 1a, the total contribution of all configurations of Type 1c containing p such that $p \in B_i$, is at most r^2 times greater than our bound above. The bound for the total contribution of all configurations of Type 1c containing p such that p is in a light edge C_i or in $A_{i,i'}$ is only k times greater than the bound above, since $R(C_i)$ and $R(A_{i,i'})$ consist only of low vertices. Hence, the total contribution, $\phi_{1c}(p)$, of all configurations of Type 1c containing p is $o(r^{-4})$.

Configuration of Type 1d: We need it to handle configurations of Type 1 in which $m+m' \geq \frac{r}{5bk}$. Since the situation with $m' \geq \frac{r}{10bk}$ is covered by configurations of Type 1b, it is enough to consider the following situation: Configuration of Type 1d is a configuration of Type 1 in which $m = \lfloor \frac{r}{10bk} \rfloor$ and $m' = 0$ but non-special vertices in D also allowed to be colored with color j. We will estimate the contributions of such configurations.

Let $p \in V(\mathcal{H})$, and $z, z', \hat{m}_1, \ldots, \hat{m}_m$ be non-negative integers. Let $\hat{M} = \hat{m}_1 + \ldots + \hat{m}_m$. Let $\phi_{1d}(p, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ denote the total contribution of all configurations of Type 1d with parameters $\hat{m}_1, \ldots, \hat{m}_m$ containing p such that $p \in D$, exactly z edges among $C_1, \ldots, C_m$ are determined, and exactly z' other edges are determined. The ingredients for an upper bound on $\phi_{1d}(p, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ are almost the same as for $\phi_{1c}(p, m', m, \hat{m}_1, \ldots, \hat{m}_m, z, z', D)$ above with $m' = 0$; the only difference is that Item (γ_3) is replaced with

(γ_3') The number of choices of colors for vertices in D such that vertices $c_1, \ldots, c_m$ are colored with j and all others are colored either with j or with $j+1$ is $t\, 2^{r-m}$.

Thus, repeating the argument for configurations of Type 1c, instead of (3.14), we will obtain

$$\sum_{z=0}^{m} \sum_{z'=0}^{\hat{M}} \phi_{1d}(p, \hat{m}_1, \ldots, \hat{m}_m, z, z', D) < 2^{r-m} 6t\, r \left(\frac{12k\, t^b}{r^\epsilon} \right)^{m+\hat{M}}. \tag{3.15}$$

Since $m = \lfloor \frac{r}{10bk} \rfloor$, the extra factor of 2^{r-m} does not hurt our upper bounds, and we essentially repeat the argument from (3.14) above for configurations of Type 1c.

Forbidding configurations of Types $1c$ and $1d$ forbids all configurations of Type 1 with $m+m' \geq k$.

3.2.5 Handling configurations of Type 2

Configuration of Type 2a: Let $j \in \{1, 2, ..., t\}$. Suppose that there exist k configurations of Type 1a (for the same j) with edge sets (for $l = 1, \dots, k$) $\{D^{(l)}, B_1^{(l)}, \dots, B_{m'(l)}^{(l)}\}$ such that
(i) $B_1^{(1)} = B_1^{(2)} = \dots = B_1^{(k)}$,
(ii) all $b_1^{(1)}, b_1^{(2)}, \dots, b_1^{(k)}$ are distinct vertices, so that $\{b_1^{(1)}, b_1^{(2)}, \dots, b_1^{(k)}\} = R(B_1^{(1)})$'
(iii) all edges $D^{(l)}$ and $B_i^{(l)}$ are heavy.
Then the union of these k configurations is a configuration of Type 2a.

It is possible that $D^{(l)} = D^{(l')}$ for $l \neq l'$, but in this case, (since both $b_1^{(l)}$ and $b_1^{(l')}$ are colored with j) $b_1^{(l)}$ coincides with some $b_i^{(l')}$ such that $B_i^{(l')}$ is distinct from $B_1^{(1)}$. Thus, in any case, there are at least k distinct edges among $D^{(l)}$ and $B_i^{(l)}$. On the other hand, since large configurations of Type 1a are forbidden, m' and each of $m'(l)$ is at most k. So, the total number of involved edges is at most $(k+1)^2$. Since we have so few edges, in calculations we will not care about determined edges, our only concern will be repetitions of edges.

Given a configuration of Type 2a, let x denote the number of distinct $D^{(l)}$. Order the edges of our configuration so that first edge is $B_1^{(1)}$ followed by all of the $D^{(l)}$, and then all the other edges. With a given ordering, for all suitable l, let $\hat{m}(l)$ denote the number of corresponding edges that do not appear earlier in the order. Let $M = \sum_{l=1}^{k} \hat{m}(l)$.

Let $p \in V(\mathcal{H})$. Let $\Phi = \phi_{2b}(p, \hat{m}(1), \dots, \hat{m}(k), x, B_1^{(1)})$ denote the total contribution of all configurations of Type 2a with the corresponding given parameters containing p such that $p \in B_1^{(1)}$.

We can estimate Φ as follows:

(δ_1) The number of candidates for $B_1^{(1)}$ containing p is at most $t^r r$.
(δ_2) The number of partitions of $R(B_1^{(1)})$ into x non-empty sets is less than k^x.
(δ_3) The number of ways to choose for every of the x parts in the partition an edge containing this class is at most $(t^r r^{-\epsilon})^x$, since the number of heavy edges containing any given vertex is at most $t^r r^{-\epsilon}$. These edges will be our edges $D^{(1)}, \dots, D^{(k)}$.
(δ_4) The number of choices of color j is t.
(δ_5) The number of ways to choose for every $l \in \{1, \dots, k\}$, vertices $b_1(l), \dots, b_{\hat{m}(l)}(l)$ is at most $\prod_{l=1}^{k} \binom{r}{\hat{m}(l)} \leq \binom{kr}{M} \leq (kr)^M$.
(δ_6) By Lemma 26, the number of ways to choose a $B_i^{(l)}$ when we know $b_i(l) \in R(B_i^{(l)})$ is at most $\frac{t^r}{r^\epsilon} \frac{4k}{r}$.
(δ_7) To estimate the size of such a configuration, recall that in total we have at most $(k+1)^2$ edges. Therefore, each edge has at most $(k+1)^2 b$ vertices that are common with any other edge. It follows that the size of each such configuration is at least $(r - (k+1)^2 b)(1 + x + M)$.

Hence

$$\Phi \leq t^r r \left(\frac{kt^r}{r^\epsilon}\right)^x t\,(kr)^M \left(\frac{4kt^r}{r^{1+\epsilon}}\right)^M \left(\frac{r}{t(r-1)}\right)^{(r-(k+1)^2 b)(1+x+M)} \leq$$

$$\leq k^{x+M+M} 4^M \, r^{1-\epsilon x+M-(1+\epsilon)M} \, t^{r+rx+1+rM-(r-(k+1)^2b)(1+x+M)} 3^{1+x+M} =$$

$$= k^{x+2M} 4^M \, r^{1-\epsilon(x+M)} \, t^{1+(k+1)^2b(1+x+M)} 3^{1+x+M} \leq tr^{1+\epsilon} \left(\frac{12k^2 t^{(k+1)^2 b}}{r^\epsilon} \right)^{1+x+M}.$$

The number of different presentations of M in the form $M = \sum_{l=1}^k \hat{m}(l)$ is at most $\binom{M+k-1}{M} < 2^{M+k-1}$. Therefore, the total contribution, $\phi_{2b}(p, x, M, B_1^{(1)})$, of all configurations of Type 2a with given x and M containing p such that $p \in B_1^{(1)}$ for large r is at most

$$2^{M+k-1} tr^{1+\epsilon} \left(\frac{12k^2 t^{(k+1)^2 b}}{r^\epsilon} \right)^{1+x+M} \leq 2^{k-x} tr^{1+\epsilon-0.5\epsilon(1+x+M)}.$$

By construction, $M + x \geq k$. Hence, since $k \geq 20/\epsilon$,

$$\phi_{2a}(p, x, M, B_1^{(1)}) \leq 2^k tr^{-8} = o(r^{-7}).$$

Since $x \leq k$ and $M \leq k^2$, the total contribution of all configurations of Type 2a containing p such that $p \in B_1^{(1)}$ is also $o(r^{-7})$. The total contribution of all configurations of Type 2a containing p such that $p \in D^{(l)}$ for some l is estimated in practically the same steps and also is $o(r^{-7})$. The same holds for the total contribution of all configurations of Type 2a containing p such that $p \in B_i^{(l)}$ for some l and i. Thus the total contribution, $\phi_{2a}(p)$, of all configurations of Type 2a containing p is $o(r^{-6})$.

Configuration of Type 2b: Let $j \in \{1, 2, ..., t\}$. Suppose that there exist k configurations of Type 1c (for the same j) with edge sets (for $l = 1, \ldots, k$)

$$\{D^{(l)}, B_1^{(l)}, \ldots, B_{m'(l)}^{(l)}, C_1^{(l)}, \ldots, C_{m(l)}^{(l)}, A_{1,1}^{(l)}, \ldots, A_{1,m_1(l)}^{(l)}, \ldots, A_{m(l),m_{m(l)}(l)}^{(l)}\}$$

such that $C_1^{(1)} = C_1^{(2)} = \ldots = C_1^{(k)}$ and all $c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(k)}$ are distinct vertices, so that $\{c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(k)}\} = R(C_1^{(1)})$. Then the union of these k configurations is a configuration of Type 2b. As in configurations of Type 1c, some representative vertices can coincide, in which case the corresponding edges also should coincide.

It is possible that $D^{(l)} = D^{(l')}$ for $l \neq l'$, but in this case, (since both $c_1^{(l)}$ and $c_1^{(l')}$ are colored with j) $c_1^{(l)}$ coincides with some $c_i^{(l')}$ such that $C_i^{(l')}$ is distinct from $C_1^{(1)}$. Thus, in any case, there are at least k distinct edges among $D^{(l)}$ and $C_i^{(l)}$. On the other hand, since large configurations of Type 1c are forbidden, each of $m(l), m'(l), m_i(l)$ is at most k. So, the total number of involved edges is at most $k(k+1)^2$.

Given a configuration of Type 2b, let x denote the number of distinct $D^{(l)}$. Order the edges of our configuration so that first is listed the edge $C_1^{(1)}$, then all $D^{(l)}$, then all $B_i^{(l)}$ (in any order), then all $C_i^{(l)}$, and then all other edges. With a given ordering, for all suitable i and l, let $\hat{m}(l), \hat{m}'(l)$, and $\hat{m}_i(l)$ denote the number of corresponding edges that do not appear earlier in the order. Let $M = \sum_{l=1}^{k}(\hat{m}(l) + \hat{m}'(l))$ and

$$\hat{M} = \sum_{l=1}^{k}\sum_{i=1}^{m} \hat{m}_i(l). \tag{3.16}$$

Let $p \in V(\mathcal{H})$. Let

$$\Phi = \phi_{2b}(p, \hat{m}'(1), \ldots, \hat{m}'(k), \hat{m}(1), \ldots, \hat{m}(k), \hat{m}_1(1), \ldots, \hat{m}_1(k), \ldots, \hat{m}_m(k), x, C_1^{(1)})$$

denote the total contribution of all configurations of Type 2b with the corresponding parameters containing p such that $p \in C_1^{(1)}$.

We can estimate Φ as follows:

(κ_1) The number of candidates for $C_1^{(1)}$ containing p is at most $t^r r$.
(κ_2) The number of partitions of $R(C_1^{(1)})$ into x non-empty sets is less than k^x.
(κ_3) The number of ways to choose for every of the x parts in the partition an edge containing this class is at most $(t^r r^{-\epsilon})^x$, since every vertex in $R(C_1^{(1)})$ is a low vertex. These edges will be our edges $D^{(1)}, \ldots, D^{(k)}$.
(κ_4) The number of choices of color j is t.
(κ_5) The number of ways to choose for every $l \in \{1, \ldots, k\}$, vertices $b_1(l), \ldots, b_{\hat{m}'(l)}(l)$ and $c_2(l), \ldots, c_{\hat{m}(l)}(l)$ in $D^{(l)}$ is at most $\prod_{l=1}^{k} \binom{r}{\hat{m}(l)+\hat{m}'(l)}\binom{\hat{m}(l)+\hat{m}'(l)}{\hat{m}'(l)}$.
(κ_6) By Lemmas 25 and 26, the number of ways to choose a $B_i^{(l)}$ when we know $b_i(l) \in R(B_i^{(l)})$ or $C_i^{(l)}$ when we know $c_i(l) \in R(C_i^{(l)})$ is at most $\frac{t^r}{r^\epsilon}\frac{4k}{r}$.
(κ_7) The number of ways to choose all vertices $a_{i,i'}(l)$ in $\cup_{l=1}^{k} \cup_{i=1}^{\hat{m}(l)} C_i^{(l)}$ that will be colored with $j-1$ is at most $\binom{k^2 r}{\hat{M}}$.
(κ_8) The number of ways to choose an $A_{i,i'}^{(l)}$ when we know $a_{i,i'}(l) \in R(A_{i,i'}^{(l)})$ is at most $\frac{t^r}{r^\epsilon}\frac{4k}{r}$.
(κ_9) To estimate the size of such a configuration, recall that in total we have at most $(k+1)^3$ edges. Therefore, each edge has at most $(k+1)^3 b$ vertices that are common with any other edge. It follows that the size of each such configuration is at least $(r-(k+1)^3 b)(1+x+M+\hat{M})$.

Hence

$$\Phi \le t^r r \left(\frac{kt^r}{r^\epsilon}\right)^x t \prod_{l=1}^{k} \binom{r}{\hat{m}(l)+\hat{m}'(l)}\binom{\hat{m}(l)+\hat{m}'(l)}{\hat{m}'(l)}\binom{k^2 r}{\hat{M}}\left(\frac{4kt^r}{r^{1+\epsilon}}\right)^{M+\hat{M}} \times$$

$$\times \left(\frac{r}{t(r-1)}\right)^{(r-(k+1)^3 b)(1+x+M+\hat{M})}.$$

Since $\left(\frac{r}{r-1}\right)^{(r-(k+1)^3b)(1+x+M+\hat{M})} \leq 3^{1+x+M+\hat{M}}$ and

$$\prod_{l=1}^{k} \binom{r}{\hat{m}(l)+\hat{m}'(l)} \binom{\hat{m}(l)+\hat{m}'(l)}{\hat{m}'(l)} \binom{k^2 r}{\hat{M}} \leq r^M \left(k^2 r\right)^{\hat{M}},$$

we have

$$\Phi \leq k^{x+M+3\hat{M}}\, 4^{M+\hat{M}} r^{1-x\epsilon-\epsilon(M+\hat{M})} t^{r+rx+1+r(M+\hat{M})-(r-(k+1)^3b)(1+x+M+\hat{M})} 3^{1+x+M+\hat{M}}$$

$$= k^{x+M+3\hat{M}}\, 4^{M+\hat{M}} r^{1-\epsilon(x+M+\hat{M})} (3t^{(k+1)^3b})^{1+x+M+\hat{M}} \leq r^{1+\epsilon} \left(\frac{12k^3 t^{(k+1)^3b}}{r^\epsilon}\right)^{1+x+M+\hat{M}}.$$

The last bound does not depend on values of $\hat{m}(l), \hat{m}'(l)$ and $m_i(l)$, but only on x, M, and M'. The number of different presentations of M in the form $M = \sum_{l=1}^{k}(\hat{m}(l)+\hat{m}'(l))$ is at most $\binom{M+2k-1}{M} < 2^{M+2k-1}$. Similarly, the number of different presentations of $\hat{M}$ in the form (3.16) is at most the number of different presentations of $\hat{M}$ as a sum of at most k^2 nonnegative summands, which is at most

$$\sum_{q=1}^{k^2} \binom{\hat{M}+q}{q} \leq \sum_{q=1}^{k^2} \binom{\hat{M}+k^2}{q} \leq 2^{k^2+\hat{M}}.$$

Therefore, the total contribution, $\phi_{2b}(p, x, M, \hat{M}, C_1^{(1)})$, of all configurations of Type 2b with given x, M, and $\hat{M}$ containing p such that $p \in C_1^{(1)}$ is at most

$$2^{M+2k-1}\, 2^{k^2+\hat{M}}\, r^{1+\epsilon} \left(\frac{12k^3 t^{(k+1)^3b}}{r^\epsilon}\right)^{1+x+M+\hat{M}} < 2^{(k+1)^2}\, r^{1+\epsilon} \left(\frac{12k^3 t^{(k+1)^3b}}{r^\epsilon}\right)^{1+x+M+\hat{M}}$$

For large r, this does not exceed $2^{(k+1)^2}\, r^{1+\epsilon-0.5\epsilon(1+x+M+\hat{M})}$. As observed above, $x+M \geq k \geq 20/\epsilon$. Thus for large r, $\phi_{2b}(p, x, M, \hat{M}, C_1^{(1)}) = o(r^{-8})$. Since $x \leq k$, $M \leq 2k^2$, and $\hat{M} \leq k^3$, the total contribution, of all configurations of Type 2b containing p such that $p \in C_1^{(1)}$ is also $o(r^{-8})$. Similarly to the argument for configurations of Type 1c, the total contribution, of all configurations of Type 2b containing p such that $p \in D(l)$, or $p \in B_i(l)$, or $p \in C_i(l)$, or $p \in A_{i,i'}(l)$ does not exceed the obtained bound more than r^2 times. Thus for large r, the total contribution, $\phi_{2b}(p)$, of all configurations of Type 2b containing p is $o(r^{-6})$.

3.3 Lower bounds on the number of edges

3.3.1 Trimming

In order to get lower bound on the number of edges in an r-uniform $(t+1)$-chromatic simple hypergraph, Erdős and Lovász in [10] applied a simple but quite useful technique of *trimming*. A *trimming* of a hypergraph $\mathcal{H}$ is the hypergraph $F(\mathcal{H})$ obtained from $\mathcal{H}$ by deleting from each edge a vertex of maximum possible degree. Trimming has two useful properties: (a) if $\mathcal{H}$ is not t-colorable, then $F(\mathcal{H})$ also is; and (b) if $\mathcal{H}$ is simple and $F(\mathcal{H})$ has a vertex of degree at least d, then $\mathcal{H}$ has at least $d+1$ vertices of degree at least d. We will somewhat elaborate upon the notion of trimming.

For positive integers x and D, an edge A of a hypergraph $\mathcal{H}$ is (x,D)-*heavy*, if at least x vertices in A have degree at least D in $\mathcal{H}$. An (x,D)-*trimming* of a hypergraph $\mathcal{H}$ is the hypergraph $F_{x,D}(\mathcal{H})$ obtained from $\mathcal{H}$ in two steps: first choose in each edge A a vertex $a(A)$ that is contained in the most (x,D)-heavy edges; then replace each edge A with $A-a(A)$. The ordinary trimming above can be considered as a $(1,1)$-trimming.

Let $F^{(m)}_{x,D}(\mathcal{H})$ denote the hypergraph obtained from $\mathcal{H}$ by applying (x,D)-trimming m times.

Lemma 27. *Let $b,x,y,d,s,$ and D be positive integers and $\mathcal{H}$ be a hypergraph.*
(a) If $\mathcal{H}$ is b-simple, $F^{(b)}_{x,D}(\mathcal{H})$ has a vertex that belongs to at least d (x,D)-heavy edges and $\mathcal{H}$ has y vertices that belong to at least d (x,D)-heavy edges each, then $\binom{y}{b}\geq d$.
(b) If $\mathcal{H}$ has girth at least $2s+1$, $b\leq s$, and $F^{(b)}_{x,D}(\mathcal{H})$ has a vertex that belongs to at least d (x,D)-heavy edges, then $\mathcal{H}$ has at least $(d-1)^b$ vertices at distance exactly b from v that belong to at least d (x,D)-heavy edges each.

Proof. For convenience, denote $F^{(0)}_{x,D}(\mathcal{H})=\mathcal{H}$. By definition, every edge $A\in E(\mathcal{H})$ contains distinct vertices $a^{(1)}(A),\dots,a^{(b)}(A)$ such that for $i=1,\dots,b$,

$$E(F^{(i)}_{x,D}(\mathcal{H}))=\{A-\{a^{(1)}(A),\dots,a^{(i)}(A)\}\ :\ A\in E(\mathcal{H})\}$$

and vertex $a^{(i)}(A)$ is contained in the most of (x,D)-heavy edges of the hypergraph $F^{(i-1)}_{x,D}(\mathcal{H})$ among the vertices in $A^{(i-1)}:=A-\{a^{(1)}(A),\dots,a^{(i-1)}(A)\}$.

Suppose that $\mathcal{H}$ is b-simple and v is a vertex in $F^{(b)}_{x,D}(\mathcal{H})$ that belongs to at least d (x,D)-heavy edges. Suppose that the edges $A^{(b)}_1,\dots,A^{(b)}_d$ of $F^{(b)}_{x,D}(\mathcal{H})$ contain v. By the definition of (x,D)-trimming, each of the vertices $a^{(b)}(A_1),\dots,a^{(b)}(A_d)$ is contained in at least d (x,D)-heavy edges in $F^{(b-1)}_{x,D}(\mathcal{H})$ (otherwise, v would be the corresponding $a^{(b)}(A_i)$). Similarly, each of the vertices $a^{(b-1)}(A_1),\dots,a^{(b-1)}(A_d)$ is contained in at least d (x,D)-heavy edges in $F^{(b-2)}_{x,D}(\mathcal{H})$, and so on.

Let Y be the set of vertices in $\mathcal{H}$ that are contained in at least d (x,D)-heavy edges. By the previous paragraph, each of the vertices $a^{(j)}(A_i)$ for $i=1,\dots,d$ and $j=1,\dots,b$ is in Y. Vertices $a^{(j)}(A_i)$ and $a^{(j')}(A_{i'})$ may coincide for distinct i and i', but the sets $\{a^{(1)}(A_i),\dots,a^{(b)}(A_i)\}$ should be distinct for distinct i, since $\mathcal{H}$ is b-simple. Thus, the number of b-element subsets of Y is at least d. This proves (a).

Suppose now that the girth of $\mathcal{H}$ is at least $2s+1$, $b\leq s$, and v is a vertex in $F^{(b)}_{x,D}(\mathcal{H})$ that belongs to at least d (x,D)-heavy edges. Suppose that the edges $A^{(b)}_1,\dots,A^{(b)}_d$ of $F^{(b)}_{x,D}(\mathcal{H})$ contain v. As above, each of the vertices $a^{(b)}(A_1),\dots,a^{(b)}(A_d)$ is contained in at least d (x,D)-heavy edges in $F^{(b-1)}_{x,D}(\mathcal{H})$. Moreover, since the girth of $\mathcal{H}$ is at least three, all $a^{(b)}(A_1),\dots,a^{(b)}(A_d)$ are distinct and each of them is a neighbor of v. Thus, each of $a^{(b)}(A_i)$ is contained in some d (x,D)-heavy edges $A_{i,1}^{(b-1)},\dots,A_{i,d}^{(b-1)}$. If $b=1$, then we are done.

Suppose $b\geq 2$. Then the girth of $\mathcal{H}$ is at least five and for each $1\leq i\leq d$, exactly one edge among $A_{i,1}^{(b-1)},\dots,A_{i,d}^{(b-1)}$ (namely, $A_i^{(b)}$) contains v, and all others are almost disjoint amongst them and are disjoint from all other $A_{j,1}^{(b-1)},\dots,A_{j,d}^{(b-1)}$ for $j\neq i$. It follows that for all $i_1,i_2=1,\dots,d$ such that $A_{i_1,i_2}^{(b-1)}\neq A_{i_1}^{(b)}$, all vertices $a^{(b-1)}(A_{i_1,i_2}^{(b-1)})$ are distinct and each such $a^{(b-1)}(A_{i_1,i_2}^{(b-1)})$ belongs to at least d (x,D)-heavy edges $A_{i_1,i_2,1}^{(b-2)},\dots,A_{i_1,i_2,d}^{(b-2)}$ in $F_{x,D}^{(b-2)}(\mathcal{H})$ and is at distance 2 from v in $\mathcal{H}$. In particular, there are at least $d(d-1)$ of them. Again, if $b=2$, then we are done. Otherwise the girth of $\mathcal{H}$ is at least seven and for all triples (i_1,i_2,i_3) such that $A_{i_1,i_2,i_3}^{(b-2)}\neq A_{i_1,i_2}^{(b-1)}$, the sets $A_{i_1,i_2,i_3}^{(b-2)}-a^{(b-1)}(A_{i_1,i_2}^{(b-1)})$ are disjoint from each other and from all edges (j_1,j_2,j_3) for $(j_1,j_2)\neq(i_1,i_2)$.

Continuing in this way, finally, we construct $d(d-1)^{b-1}$ distinct vertices $a^{(1)}(A_{i_1,i_2,\dots,i_b})$ such that each of them belongs to at least d (x,D)-heavy edges in $F_{x,D}^{(0)}(\mathcal{H})$ and is at distance exactly b from v. □

3.3.2 Size of $(t+1)$-chromatic b-simple hypergraphs

Theorem 28. *Let t and b be positive integers, $\epsilon>0$, and r be sufficiently large in comparison with t,b and ϵ. Let $\mathcal{H}$ be a $(t+1)$-chromatic r-uniform b-simple hypergraph. Then $\mathcal{H}$ has at least $t^{r(1+1/b)}r^{-\epsilon}$ edges.*

Proof. Let $x=\lceil (r-b)/2\rceil$ and $D=\lceil t^{r-b}/r^{\epsilon/3}\rceil$. Let $\mathcal{H}_1=F_{1,1}^{(b)}(\mathcal{H})$. By construction, $\mathcal{H}_1$ is $(r-b)$-uniform and b-simple. Since $\mathcal{H}$ is not t-colorable, $\mathcal{H}_1$ is also not t-colorable. So, by Theorem 23, either

(i) $\mathcal{H}_1$ has a vertex of degree at least $t^{r-b}\,(r-b)$, or

(ii) $\mathcal{H}_1$ has a vertex contained in at least D (x,D)-heavy edges.

If (i) holds, then by Lemma 27(a), $\mathcal{H}$ has at least $(t^{r-b}\,(r-b))^{1/b}$ vertices of degree at least $t^{r-b}\,(r-b)$. Hence the number of edges in $\mathcal{H}$ is at least

$$\frac{1}{r}\big(t^{r-b}\,(r-b)\big)^{1+1/b}\geq t^{r(1+1/b)}.$$

Suppose now that (ii) holds. Let Y be the set of vertices of degree at least D in $\mathcal{H}_1$. Each (x,D)-heavy edge containing v interests $Y-v$ in at least $x-1$ vertices. No b-tuple of vertices of $Y-v$ is contained in more than one edge containing v. Therefore

$$\binom{|Y|}{b}\geq D\binom{x-1}{b}.$$

For large r, this implies $|Y|^b\geq D(r/3)^b$, so that the number of edges in $\mathcal{H}_1$ is at least

$$\frac{1}{r}D^{1+1/b}\,\frac{r}{3}\geq\frac{1}{3}\big(\frac{t^{r-b}}{r^{\epsilon/3}}\big)^{1+1/b}\geq t^{r(1+1/b)}r^{-\epsilon}.$$

$\square$

3.3.3 Size of $(t+1)$-chromatic hypergraphs of girth $2s+1$ and $2s+2$

Theorem 29. *Let t and s be positive integers, $\epsilon > 0$, and r be sufficiently large in comparison with t, s and ϵ. Let $\mathcal{H}$ be a $(t+1)$-chromatic r-uniform hypergraph of girth at least $2s+1$. Then $\mathcal{H}$ has at least $t^{r(1+s)}r^{-\epsilon}$ edges. Moreover, if the girth of $\mathcal{H}$ is at least $2s+2$, then $\mathcal{H}$ has at least $t^{r(1+s)}r^{1-\epsilon}$ edges.*

Proof. Let $x = \lceil (r-2s+1)/2 \rceil$ and $D = \lceil t^{r-2s+1}/r^{\epsilon/3s} \rceil$. Let $\mathcal{H}_1 = F_{1,1}^{(s)}(\mathcal{H})$ and $\mathcal{H}_2 = F_{x,D}^{(s-1)}(\mathcal{H}_1)$. By construction, $\mathcal{H}_2$ is $(r-2s+1)$-uniform. Since $\mathcal{H}$ is not t-colorable, $\mathcal{H}_2$ is also not t-colorable. So, by Theorem 23, either
(i) $\mathcal{H}_2$ has a vertex of degree at least $t^{r-2s+1}(r-2s+1)$, or
(ii) $\mathcal{H}_2$ has a vertex contained in at least D (x, D)-heavy edges.

If (i) holds, then $\mathcal{H}_1$ also has a vertex of degree at least $t^{r-2s+1}(r-2s+1)$. By Lemma 27(b), $\mathcal{H}$ has at least $(t^{r-2s+1}(r-2s+1)-1)^s$ vertices of degree at least $t^{r-2s+1}(r-2s+1)$. Hence the number of edges in $\mathcal{H}$ is at least

$$\frac{1}{r}\left(t^{r-2s+1}(r-2s+1)-1\right)^{1+s} \geq t^{r(1+s)}r.$$

Suppose now that (ii) holds. By Lemma 27(b), $\mathcal{H}_1$ contains a set $F(s,v)$ of at least $(D-1)^{s-1}$ vertices at distance exactly $s-1$ from v such that each of them is contained in at least D (x, D)-heavy edges. Since the girth of $\mathcal{H}_1$ is at least $2s+1$, each $u \in F(s,v)$ is contained in exactly one edge $M(u)$ on the unique shortest path from u to v. Also, if for $u \in F(s,v)$ an edge $A(u) \neq M(u)$ meets or coincides with any edge containing any vertex w at distance at most $s-1$ from v, then $\mathcal{H}_1$ contains a cycle of length at most $2s$, a contradiction.

Thus, we have a set $F'(s,v)$ of at least $(D-1)^s$ (x, D)-heavy edges such that each edge in $F'(s,v)$ contains exactly one of our $(D-1)^{s-1}$ special vertices at distance $s-1$ from v and no other vertices at distance at most $s-1$ from v. This means that each of the edges in $F'(s,v)$ contains at least $x-1$ vertices of degree at least D that do not belong to any other edge in $F'(s,v)$. Since x is about $r/2$ and r is much larger than s and t, it follows that the number of edges in $\mathcal{H}_1$ is at least

$$\frac{1}{r}(D-1)^{s+1}(x-1) \geq \frac{1}{3}D^{s+1} \geq \frac{1}{3}\left(\frac{t^{r-2s+1}}{r^{\epsilon/3s}}\right)^{1+s} \geq t^{r(1+s)}r^{-\epsilon}.$$

This proves the statement for girth $2s+1$.

Suppose that the girth of $\mathcal{H}$ is at least $2s+2$. It (i) holds, then the statement is already proved above. Suppose that (ii) holds. Then we construct $F'(s,v)$ exactly as in the previous paragraph. Consider an edge $A \in F'(s,v)$ and any vertex $z \in A$ of degree at least D that does not belong to other edges in $F'(s,v)$. If any of the at least $D-1$ distinct from A edges containing z contains also a vertex from any other edge in $F'(s,v)$, then $\mathcal{H}$ has a cycle of length $2s+1$ or less. Thus all these edges are distinct and the total number of them is at least

$$|F'(s,v)|(x-1)(D-1) \geq \frac{r}{3}D^{s+1} \geq \frac{r}{3}(\frac{t^{r-2s+1}}{r^{\epsilon/3s}})^{1+s} \geq t^{r(1+s)}r^{1-\epsilon}.$$

$\square$

3.4 Upper bound on $f(r,t,b)$

The Erdős–Lovász [10] bound (3.1) can be easily extended to b-simple hypergraphs as follows.

Theorem 30. *If $b \geq 1$ and $t \geq 2$ are fixed and r is sufficiently large, then*

$$f(r,t,b) \leq 10t^2\left(2t^r r^2\right)^{(b+1)/b}.$$

Proof. We follow the lines of the proof of Theorem 1′ in [10] by Erdős and Lovász.

Let

$$n = \left\lceil 4t\left(2t^r r^{2(b+1)}\right)^{1/b}\right\rceil \quad \text{and} \quad m = 4n\,t^{r+1} \sim 8t^2\left(2t^r r^2\right)^{(b+1)/b}. \tag{3.17}$$

We let $\mathcal{H}_0$ be the edgeless hypergraph with $|V(\mathcal{H}_0)| = tn$ and for $i = 1,\ldots,m$ will obtain $\mathcal{H}_i$ from $\mathcal{H}_{i-1}$ by adding an edge e_i so that

(a) $\mathcal{H}_i$ remains b-simple and

(b) $x_i \leq (1-1/4t^r)x_{i-1}$, where x_i is the number of n-element subsets of $V(\mathcal{H}_0) = V(\mathcal{H}_i)$ not containing edges of $\mathcal{H}_i$.

As in [10], if we manage (a) and (b) until $i = m$, then

$$x_m \leq x_0\left(1-\frac{1}{4t^r}\right)^m = \binom{tn}{n}\left(1-\frac{1}{4t^r}\right)^{4nt^{r+1}} \leq \frac{(te)^n}{e^{tn}} = \left(\frac{te}{e^t}\right)^n < 1.$$

Suppose that (a) and (b) hold for $i = 0,1,\ldots,j$. Let S be an n-element subset of $V(\mathcal{H}_j)$ not containing any edge of $\mathcal{H}_j$. If an r-tuple $R \subset S$ cannot be added to $\mathcal{H}_j$ because (a) would fail, then R has $b+1$ elements in common with some e_i, $i \leq j$. The number of such $R \subset S$ is at most

$$j\binom{r}{b+1}\binom{n-b-1}{r-b-1} \leq \frac{mr^{b+1}}{(b+1)!}\binom{n}{r}\frac{r^{b+1}}{(n-b)^{b+1}} = \frac{mr^{2(b+1)}}{(b+1)!(n-b)^{b+1}}\binom{n}{r}.$$

By (3.17), for fixed b and t and for large r, we have

$$m\frac{r^{2(b+1)}}{(b+1)!(n-b)^{b+1}} \leq 4nt^{r+1}\frac{2r^{2(b+1)}}{(b+1)!n^{b+1}} \leq 4t^{r+1}\frac{2r^{2(b+1)}}{2!(4t)^b 2t^r r^{2(b+1)}} \leq \frac{1}{2}.$$

It follows that every n-element $S \subset V(\mathcal{H}_j)$ not containing any edge of $\mathcal{H}_i$ contains at least

$0.5\binom{n}{r}$ candidates for e_{j+1}. Therefore, some r-element subset e_{j+1} of $V(\mathcal{H}_i)$ is a candidate for at least

$$x_j \cdot 0.5\binom{n}{r}\binom{tn}{r}^{-1} \geq \frac{x_j\, n(n-1)\dots(n-r+1)}{2(tn)^r}$$

n-element subsets of $V(\mathcal{H}_j)$ not containing any edge of $\mathcal{H}_j$. Since $n \geq 8r^2$,

$$\frac{x_j\, n(n-1)\dots(n-r+1)}{2(tn)^r} \geq \frac{x_j}{2t^r}\left(\frac{n-r+1}{n}\right)^r \geq \frac{x_j}{4t^r}.$$

Thus, if we choose this e_{j+1}, then (a) and (b) hold for $i = j+1$.

□

3.5 Comments

1. While all b, t, s and ϵ are considered fixed, they also can be viewed as very slowly growing functions of r. For example, it is possible to consider $\epsilon = c\,\frac{\log\log\log r}{\log\log r}$ for a small positive constant c.

2. Condition (3.7) in the definition of (t,ϵ)-sparse r-uniform hypergraphs can be weakened by any polynomial factor of r. The problem in sharpening our results is in (3.8).

3. The proofs of Theorems 23 and 22 and inequalities (4) and (5) can be adapted to list coloring. In particular, the following statement holds (and implies the other results).

Theorem 31. *If $b \geq 1$, $t \geq 2$, and $\epsilon > 0$ are fixed and r is sufficiently large, then every r-uniform b-simple (t,ϵ)-sparse hypergraph $\mathcal{H}$ is list t-colorable.*

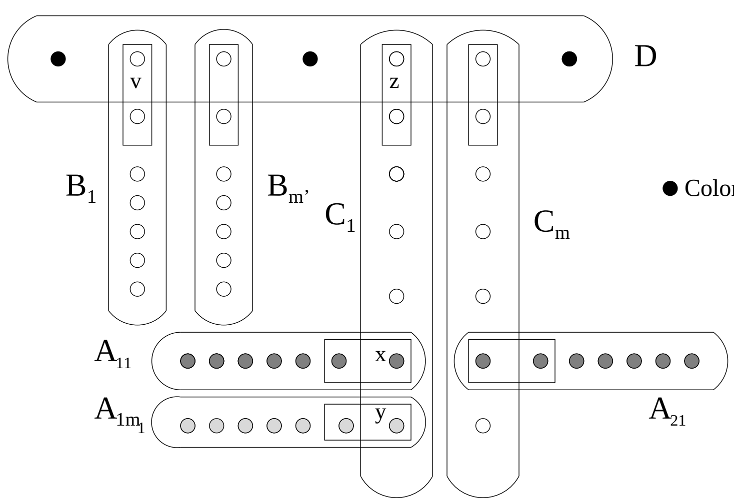

Figure 3.4: Configuration of Type 1 for list colorings.

We start from a random coloring f of vertices of $\mathcal{H}$ where each vertex v is colored with a color $f(v)$ uniformly at random chosen from its list $List(v)$ independently from all other vertices. To adapt the proof of Theorem 23, for each vertex $v \in V(\mathcal{H})$, fix any bijection ν_v of the list $List(v)$ onto itself with $\nu_v(\alpha) \neq \alpha$ for each $\alpha \in List(v)$. In all recolorings during the proof, each vertex v of color j will be tried to be recolored (if at all) into the color $\nu_v(j)$ (instead of color $j+1$, as it was in the proof in Section 3.2). So, the Configuration of Type 1 in Figure 1 will look more like in Figure 3.4. In this picture, if the "main color" of the edge D is j, then $f(v) = \nu_v^{-1}(j)$, $f(z) = \nu_z^{-1}(j)$, $f(x) = \nu_x^{-1}(f(z)) = \nu_x^{-1}(\nu_z^{-1}(j))$, $f(y) = \nu_y^{-1}(f(z)) = \nu_y^{-1}(\nu_z^{-1}(j))$, and so on. So, the colors of vertices v and z (likewise, of x and y) can be different, but the structure remains the same, and for each vertex, only one color is "dangerous" for the configuration. Similarly, we define the other configurations. After these definitions and before any recoloring is done, all the calculations will be the same as in Section 3.2, and the result follows.

Chapter 4

Conflict-free coloring of hypergraphs with few edges

4.1 Introduction

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. We recall that a coloring $c : V(\mathcal{H}) \to \{1, 2, 3, ...\}$ of $V(\mathcal{H})$ is a *proper coloring of* $\mathcal{H}$ if no edge of $\mathcal{H}$ is monochromatic. The minimum number of colors required for such a coloring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$. A *rainbow* coloring of $\mathcal{H}$ is a proper coloring of $\mathcal{H}$ such that for every edge e, the colors of all vertices of e are distinct. The minimum number of colors required for a rainbow coloring is called the *rainbow chromatic number* of $\mathcal{H}$, and is denoted by $\chi_R(\mathcal{H})$.

Even, Lotker, Ron and Smorodinsky [12] introduced (in a geometric setting) the following intermediate coloring. A proper coloring of $\mathcal{H}$ is *conflict-free* if for each edge e of $\mathcal{H}$, some color occurs on exactly one vertex of e. In other words, every edge has a color that does not get repeated in that edge. The minimum number of colors required for a conflict-free coloring is called the *conflict-free chromatic number* of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. This concept was introduced in connection with the following frequency assignment problem for cellular networks.

Consider a cellular network which has nodes as base stations (that act as servers). Clients are connected to base stations; connections between clients and base stations are implemented by radio links. Fixed frequencies are assigned to base stations to enable links to clients. Clients, on the other hand, continuously scan frequencies in search of a base station with good reception. The fundamental problem of frequency-assignment in cellular networks is to assign frequencies to base stations so that every client, located within the receiving range of at least one station, can be served by some base station, in the sense that the client is located within the range of the station and no other station within its reception range has the same frequency. The goal is to minimize the number of assigned frequencies since the frequency spectrum is limited and costly. In terms of hypergraphs, if we form a hypergraph with vertices as base stations and edges correspond to the sets of frequencies accessible to the clients, then the problem reduces to find the conflict-free chromatic number of that hypergraph.

It turns out that conflict-free chromatic number of a hypergraph is related to another parameter called the *tree-depth* of a graph G, denoted by $td(G)$. The concept of tree-depth was introduced by J. Nešetřil and Ossona de Mendez [25]. To define tree-depth, we need to introduce a few other concepts. A *rooted forest* is a disjoint union of rooted trees. The height of a vertex x in a rooted forest F is the number of vertices of a path from the root to x. The height of F is the maximum height of the vertices of F. A vertex x is an ancestor of y in F if x belongs to the path linking y and the root of the tree of F to which y belongs. The closure $clos(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set (x, y) : x is an ancestor of y in F. The tree-depth $td(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq clos(F)$. They proved the following:

Theorem 32 ([25])**.** *Given a graph G if $\mathcal{H}$ be the hypergraph with vertex set being $V(G)$ and the edge set being the vertices of connected subgraphs of G, then $td(G) = \chi_{CF}(\mathcal{H})$.*

This kind of coloring was referred as *centered coloring* in their paper. They used the concept of centered coloring and tree depth to determine whether a particular kind of generalized chromatic number is bounded for any proper minor closed class of graphs.

Clearly, $\chi(\mathcal{H}) \leq \chi_{CF}(\mathcal{H}) \leq \chi_R(\mathcal{H})$ for every $\mathcal{H}$ with equalities when $\mathcal{H}$ is an ordinary graph. Also for 3-uniform hypergraphs $\chi(\mathcal{H}) = \chi_{CF}(\mathcal{H})$. However, for general hypergraphs, the behavior of χ_{CF} may significantly differ from that of χ and of χ_R. For example, if we truncate an edge of a hypergraph, then χ cannot decrease, χ_R cannot increase, while χ_{CF} may increase, decrease, or stay the same. As yet another example we mention that if $\mathcal{H}$ is a 10^6-uniform hypergraph with 10 edges, then $\chi(\mathcal{H}) = 2$ and $\chi_{CF}(\mathcal{H})$ can be 2, 3, or 4.

Recall that a hypergraph is called *simple*, if any two distinct edges share at most one vertex. Also recall that the *edge degree* of an edge e in a hypergraph $\mathcal{H}$ is the number of other edges intersecting e. The maximum of the edge degrees over all the edges of $\mathcal{H}$ is denoted by $D(\mathcal{H})$.

Because of applications and interesting behavior, the parameter $\chi_{CF}(\mathcal{H})$ attracted considerable attention (see, e.g. [3, 4, 5, 7, 12, 16, 27, 26]). The chromatic number of hypergraphs is discussed in many papers. Some of them discuss the bounds on the chromatic number of uniform hypergraph in terms of their size or maximum (edge) degree as discussed in the previous sections. Pach and Tardos [26] analyzed the conflict-free colorings for graphs and hypergraphs. They proved the following.

Theorem 33 ([26])**.** $\chi_{CF}(\mathcal{H}) \leq 1/2 + \sqrt{2m + 1/4}$, *for every hypergraph $\mathcal{H}$ with m edges. This bound is tight.*

They also showed the following result.

Theorem 34 ([26])**.** *Let $\mathcal{H}$ be a hypergraph with m edges such that the size of every edge is at least $2r - 1$. Then $\chi_{CF}(\mathcal{H}) \leq C_r m^{1/r} \log m$, where C_r is a positive constant depending only on r.*

In fact, they proved that the same bound holds for hypergraphs $\mathcal{H}$ in which the size of every edge is at least $2r - 1$ and $D(\mathcal{H}) \leq m$. They also posed the following question:

Question: Is $\chi_{CF}(\mathcal{H}) \leq C_r m^{1/r} \log m$, when every edge has size at least r and intersects at most m others ?

We show that this is not necessarily true.

The goal is to give reasonable upper bounds on $\chi_{CF}(\mathcal{H})$ for r-uniform hypergraphs $\mathcal{H}$ with given number of edges or maximum edge degree. It will turn out that for a given m, the nature of the bounds for r-uniform hypergraphs with m edges strongly depends on whether r is small or large with respect to m. We also derive similar bound for r-uniform simple hypergraphs. It turns out that for positive integers r, t with $r \leq t/8$, both upper and lower bounds on the minimum number of edges in an r-uniform simple hypergraph that have no conflict-free colorings with t colors are roughly squares of the corresponding bounds for hypergraphs without the restriction of being simple.

In Section 4.2 we find how few edges may r-uniform hypergraphs have with χ_{CF} equal to 2 or 3. In particular, for arbitrarily large r for r-uniform hypergraphs $\mathcal{H}$ with just 7 edges we have $\chi_{CF}(\mathcal{H}) \leq 4$ and this bound is attained for some hypergraphs. In Section 4.3, we find upper bounds on $\chi_{CF}(\mathcal{H})$ in terms of the size/maximum edge degree of $\mathcal{H}$ and present some constructions showing that our bounds are reasonable. In Section 4.4, we do the same for simple r-uniform hypergraphs.

This is a joint work with A. Kostochka and T. Łuczak and appears in [21].

4.2 Conflict-free coloring of hypergraphs with very few edges

We define the *s-blow up* of a graph G to be the hypergraph formed by replacing every vertex v of G with an s element set B_v. The set B_v is called a *blob*. If uv is an edge in G, then $B_u \cup B_v$ is an edge in the blow-up.

Observation 35. *For a hypergraph $\mathcal{H}$, if either the degree of every vertex of H is at most 1, or if there is a vertex contained in every edge of H, then $\chi_{CF}(\mathcal{H}) = 2$.* □

Observation 36. *Let $r \geq 2$. If $\mathcal{H}$ is an r-uniform hypergraph which is not conflict-free 2-colorable, then it has at least 3 edges and the only such graph with 3 edges is the $(r/2)$-blow up of K_3.*

Proof. By Observation 35, every hypergraph with 2 edges is conflict-free 2-colorable. Moreover, a blow-up of K_3 is not. Now assume that $\mathcal{H}$ is an r-uniform hypergraph with 3 edges e_1, e_2, e_3 which is not conflict-free 2-colorable. If every vertex has degree at most 1 or if there is a vertex of degree 3, then by Observation 35 it is conflict-free 2-colorable. So assume that the maximum degree is 2. Without loss of generality assume that $v \in e_1 \cap e_2$. If there exists $u \in e_3 - e_1 - e_2$, then we color v and u with color 1 and all the remaining vertices with color 2. This would give a conflict-free 2-coloring of $\mathcal{H}$, a contradiction. Hence $e_3 \subseteq \{e_1 - e_2\} \cup \{e_2 - e_1\}$. Since $\mathcal{H}$ is r-uniform, we have that $e_3 \nsubseteq e_1$ and $e_3 \nsubseteq e_2$. Thus, $e_1 \cap e_3 \neq \emptyset$ and the above argument holds if v is replaced by a vertex $w \in e_3$. Consequently, $e_1 \subseteq \{e_2 - e_3\} \cup \{e_3 - e_2\}$ and similarly $e_2 \subseteq \{e_1 - e_3\} \cup \{e_3 - e_1\}$. Moreover, since $\mathcal{H}$ is r-uniform, it must be the $(r/2)$- blow up of K_3. In particular, r is even. □

Lemma 37. *Let $r \geq 3$. If $\mathcal{H}$ is an r-uniform hypergraph with at most 6 edges, then it is always conflict-free 3-colorable. Moreover, if $r \geq 4$ and r is divisible by 4, then there exists an r-uniform hypergraph with 7 edges which is not conflict-free 3-colorable.*

Proof. We first show that if $\mathcal{H}$ has at most 6 edges then $\chi_{CF}(\mathcal{H}) \leq 3$. Let $\Delta(\mathcal{H})$ be the maximum degree of $\mathcal{H}$.

Case 1. $\Delta(\mathcal{H}) \geq 4$. Let v be a vertex of degree at least 4. We color v with color 1. By Observation 36, there is a conflict-free coloring of $\mathcal{H} - v$ with colors 2 and 3. This gives a conflict-free 3-coloring of $\mathcal{H}$.

Case 2. $\Delta(\mathcal{H}) \leq 2$. Since $\chi_{CF}(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1$ for every hypergraph $\mathcal{G}$ we can conflict-free 3-color $\mathcal{H}$ (see [26]).

Case 3. $\Delta(\mathcal{H}) = 3$. Let v be a vertex of degree 3 contained in the edges e_1, e_2 and e_3. If $\mathcal{H} - \{e_1, e_2, e_3\} = \{e_4, e_5, e_6\}$ is conflict-free 2-colorable, then we color them conflict-free with colors 2 and 3, color v with color 1 and arbitrarily color the remaining vertices with colors 2 and 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. If not, then by Observation 36, $\{e_4, e_5, e_6\}$ forms the $(r/2)$-blow up of K_3. We may assume that $e_4 \cup e_5 \cup e_6 = B_4 \cup B_5 \cup B_6$, where B_4, B_5 and B_6 are the blobs $e_5 \cap e_6$, $e_4 \cap e_6$ and $e_4 \cap e_5$, respectively. Now, suppose that there is a vertex $u \in (e_4 \cup e_5 \cup e_6) - (e_1 \cup e_2 \cup e_3)$. Without loss of generality assume that $u \in B_6$. Let w be a vertex in B_5. We now color v and u with color 1, w with color 2 and the rest of the vertices with color 3. This gives a conflict-free 3-coloring of $\mathcal{H}$. Hence $\{e_4 \cup e_5 \cup e_6\} \subseteq \{e_1 \cup e_2 \cup e_3\}$. Thus every vertex in $\{e_4 \cup e_5 \cup e_6\}$ has degree 3.

The above argument holds for each vertex $u \in e_4 \cup e_5 \cup e_6$ by replacing v with u and e_1, e_2, e_3 with the three edges containing u. Hence by symmetry, the degree of every vertex of $\mathcal{H}$ is 3. We also know that deleting any vertex, leaves a copy of the $(r/2)$-blow up of K_3. Moreover, since $\mathcal{H}$ is r-uniform, $\mathcal{H}$ must be the $(r/2)$-blow up of K_4. A blow-up of K_4 can be conflict-free 3-colored as follows. In the first blob we color a vertex with color 1 and another with color 2 and the rest with 3. In the second blob we color one vertex with 2 and the rest with 3. In the third blob we color one vertex with 1 and the rest with 3 and in the fourth blob we color everything with color 3.

Now to show that there exists a hypergraph $\mathcal{H}$ with 7 edges which is not 3-conflict-free colorable, we consider the $(r/4)$-blow up of the Fano plane and take the complement of every edge. The resulting hypergraph $\mathcal{H}$ has seven blobs $B_1, B_2, ...B_7$ and the following edges: $e_1 = B_1 \cup B_2 \cup B_6 \cup B_7$, $e_2 = B_2 \cup B_3 \cup B_4 \cup B_7$, $e_3 = B_4 \cup B_5 \cup B_6 \cup B_7$, $e_4 = B_1 \cup B_2 \cup B_4 \cup B_5$, $e_5 = B_1 \cup B_3 \cup B_4 \cup B_6$, $e_6 = B_2 \cup B_3 \cup B_5 \cup B_6$, and $e_7 = B_1 \cup B_3 \cup B_5 \cup B_7$. Suppose that there is a conflict-free 3-coloring f of $\mathcal{H}$ with colors $1, 2$ and 3.

Claim 1: No color can appear in exactly one blob.

Proof: Assume that a color, say 1, appears in exactly one blob. Consider the three edges e_2, e_3, e_6 not containing B_1. They must be conflict-free 2-colorable with colors $2, 3$. But they form the $(r/2)$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction.

Claim 2: No color can appear in exactly two blobs.

Proof: Suppose that color 1 appears in exactly two blobs. Let B_1, B_2 be the blobs containing vertices of color 1. Consider the two edges e_1, e_4 containing both B_1 and B_2 and the edge e_3 containing neither B_1 nor B_2. These three edges form the $(r/2)$-blow up of K_3 with at least two vertices of color 1 present in a single blob. All other vertices gets color 2 or 3. With these restrictions there exists no conflict-free 3-coloring of the blow up of K_3.

Hence by the above claims, every color appears in at least three blobs.

Since f is a conflict-free 3-coloring of $\mathcal{H}$ which has seven edges, some color is unique for at least three edges. Assume that this color is 1.

Claim 3: A vertex with color 1 cannot be unique for more than one edge.

Proof: If not, then without loss of generality, assume that a vertex with color 1 belonging to B_1 is unique for edges e_4 and e_5. Hence the blobs B_2, B_3, B_4, B_5, B_6 do not have any vertices of color 1. So color 1 appears only in at most two blobs. This contradicts Claims 1 and 2.

Assume that a vertex of color 1 in B_1 is unique for e_1. So the blobs B_2, B_6, B_7 do not have vertices of color 1. Again without loss of generality assume that a vertex of color 1 in B_3 is unique for the edge e_2. So the blob B_4 does not have any vertex of color 1. Now there must be a vertex of color 1 in B_5 which is unique for e_3. We now consider the edges e_4, e_5, e_6. Each of these edges contains exactly two vertices of color 1. We delete these vertices and consider the new edges e'_4, e'_5, e'_6. The hypergraph formed by these edges must be conflict-free 2-colorable with colors $2, 3$. The edges e'_4, e'_5, e'_6 form the $((r/2) - 1)$-blow up of K_3 which is not conflict-free 2-colorable, a contradiction. □

4.3 Conflict-free coloring of hypergraphs with few edges

Having dealt with small cases, now we study the bounds for the conflict-free chromatic number for a general case. We start with a simple probabilistic fact we shall use later on.

Lemma 38. *Color a set T of q points, randomly, with s colors, so that each of s^q colorings is equally likely. Let $p_{q,s}$ be the probability that no color appears exactly once on T and let $\hat{p}_{q,s}$ be the probability that at most one color appears exactly once on T. Then*

$$p_{q,s} \leq \left(\frac{2q}{s}\right)^{\lceil q/2 \rceil} \tag{4.1}$$

and

$$\hat{p}_{q,s} \le \left(\frac{8q}{s}\right)^{\lceil (q-1)/2 \rceil}. \tag{4.2}$$

Proof. To prove (1), let us randomly color all elements of T, one by one. Note that if no color appears exactly once we shall use at most $\lfloor q/2 \rfloor$ of them, and the set T' of the elements that are colored with a color which we have already used has at least $\lceil q/2 \rceil$ elements. Thus, since the number of ways to choose T' is at most 2^q, we get

$$p_{q,s} < 2^q \left(\frac{q}{2s}\right)^{\lceil q/2 \rceil} \le \left(\frac{2q}{s}\right)^{\lceil q/2 \rceil}.$$

In order to show (2), we again randomly color all elements of T one by one. Note that we shall use at most q colors. Furthermore, in this case the set T' of the elements that are colored with a color which we have already used has at least $\lceil (q-1)/2 \rceil$ elements and the number of ways to choose T' is at most 2^q. Hence

$$\hat{p}_{q,s} < 2^q \left(\frac{q}{s}\right)^{\lceil (q-1)/2 \rceil} \le \left(\frac{8q}{s}\right)^{\lceil (q-1)/2 \rceil}.$$

□

Now we can bound the $\chi_{CF}(\mathcal{H})$ for a general r-uniform hypergraph with m edges.

Theorem 39. *Let $\mathcal{H}$ be a r-uniform hypergraph with m edges and maximum edge degree $D(\mathcal{H})$.*

(i) If $D(\mathcal{H}) \le 2^{r/2}$, and $D(\mathcal{H})$ (and thus r) is large enough, then there exists a vertex coloring of $\mathcal{H}$ with $120 \ln D(\mathcal{H})$ colors such that each edge has at least one color appearing exactly once. In particular,

$$\chi_{CF}(\mathcal{H}) \le 120 \ln D(\mathcal{H}) \le 120 \ln m.$$

(ii) If $m \ge 2^{r/2}$, then $\chi_{CF}(\mathcal{H}) \le 4r(16m)^{2/(r+2)}$.

Proof. In order to show (i) we set $p = 1.34 \ln D(\mathcal{H})/r$, choose a subset $\hat{T}$ of vertices of $\mathcal{H}$ independently with probability p, and then color each vertex of $\hat{T}$ independently with one of $s = 120 \ln D(\mathcal{H})$ colors. Let A_e be the event that no color appears exactly once in the edge e. Then, by Lemma 38,

$$\begin{aligned}
\Pr(A_e) &\le \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i}{s}\right)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i} \\
&\le \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i_0}{s}\right)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i},
\end{aligned}$$

where here and below $i_0 = \lfloor 2.5 \cdot 1.34 \ln D(\mathcal{H}) \rfloor$.

Since $p \le 1.34(r/2)(\ln 2)/r \le 0.47$, for $i \ge i_0 + 1$ we have

$$\frac{\binom{r}{i+1}p^{i+1}(1-p)^{r-i-1}}{\binom{r}{i}p^{i}(1-p)^{r-i}} \le \frac{r}{i}\frac{p}{1-p} \le \frac{1}{2.5(1-p)} < \frac{1}{1.325}.$$

so the second sum can be bounded from above by a geometric series and consequently

$$\sum_{i=i_0+1}^{r}\binom{r}{i}p^i(1-p)^{r-i} \le 4.08\binom{r}{i_0+1}p^{i_0+1}(1-p)^{r-i_0-1}.$$

Since $\binom{r}{j} \le (\frac{er}{j})^j$ and $(1-p)^{r-j} \le (1-p)^r \le (e^{-pr/j})^j$ we have

$$\begin{aligned}\Pr(A_e) &\le \Big(1+\Big(\sqrt{\frac{2i_0}{s}}-1\Big)p\Big)^r + 4.08\Big(\frac{erp}{i_0+1}\cdot e^{-pr/(i_0+1)}\Big)^{i_0+1}\\ &\le \exp(-0.76pr) + 4.08\exp\big(-0.79\cdot 1.34\ln D(\mathcal{H})\big)\\ &\le D(\mathcal{H})^{-1.01} + 4.08D(\mathcal{H})^{-1.05} \le 1/(4D(\mathcal{H}))\end{aligned}$$

for sufficiently large $D(\mathcal{H})$. Consequently, $D(\mathcal{H})\Pr(A_e) < 1/4$ and by Lovász Local Lemma, there exists a conflict-free coloring of $\mathcal{H}$ with $s = 120\ln D(\mathcal{H})$ colors, so (i) follows.

Now let $s = 2r(16m)^{2/(r+2)}$ and $k = 2s$. We shall show that $\mathcal{H}$ has a conflict-free coloring with at most k colors. Let v be a vertex of maximum degree in $\mathcal{H}$. Reserve a color c for v and delete v along with all the edges containing it. Repeat this procedure and reserve a different color every time we delete a vertex of maximum degree in the remaining hypergraph. This procedure is repeated $k/2$ times. Let $\mathcal{H}_1$ denote hypergraph obtained by $k/2$ repetitions of this procedure. We consider the following two cases.

Case 1. $D(\mathcal{H}_1) < m^{r/(r+2)}$. Color each vertex of $\mathcal{H}_1$ by a color chosen randomly among s colors. Let A_e be the event that no color appears exactly once in the edge e. By Lemma 38, $\Pr(A_e) < (2r/s)^{r/2}$. Thus for $r \ge 2$,

$$4\cdot D(\mathcal{H}_1)\cdot \Pr(A_e) < 4\cdot m^{r/(r+2)}\cdot(2r/s)^{r/2} = 4\cdot m^{r/(r+2)}\cdot(2r/2r(16m)^{2/(r+2)})^{r/2} \le 1.$$

Hence by Lovász Local Lemma, there exists a conflict-free coloring of $\mathcal{H}_1$ with $k/2$ colors. Together with the other $k/2$ colors, we have a conflict-free coloring of $\mathcal{H}$ with $k = 2s = 4r(16m)^{2/(r+2)}$ colors.

Case 2. $D(\mathcal{H}_1) \ge m^{r/(r+2)}$. Note that since each time we have deleted a vertex of maximum degree in the remaining hypergraph, we have removed at least $\Delta(\mathcal{H}_1) \ge \frac{D(\mathcal{H}_1)}{r} \ge \frac{m^{r/(r+2)}}{r}$ edges $k/2$ times. Thus, $m \ge km^{r/(r+2)}/(2r)$ which implies $k \le 2rm^{2/(r+2)}$, a contradiction. This completes the proof of *(ii)*. $\square$

It is not hard to see that the bound given by Theorem 39(i) is tight up to a constant factor. Indeed, the following holds.

Proposition 40. *For all $m \geq 1$ and for all even $r \geq 2$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > \frac{1}{2}\log_2 m$.*

Proof. If $1 \leq m \leq 4$, then $\frac{1}{2}\log_2 m \leq 1$, and the statement follows. Let $m \geq 5$ and let n be the largest integer such that $\binom{n}{2} \leq m$. Let $\mathcal{H}'$ be the $(r/2)$-blow up of K_n, where the blobs are $B_1, \dots, B_n$. Consider the hypergraph $\mathcal{H}$ obtained from $\mathcal{H}'$ by adding $m - \binom{n}{2}$ isolated edges. By construction, $\mathcal{H}$ has m edges.

Let $k = \lfloor \log_2 n \rfloor$. Suppose that $\mathcal{H}$ has a conflict-free coloring f with k colors. For $i = 1, \dots, n$, let S_i be the set of colors that appear in the blob B_i. Since there are $2^k - 1$ nonempty distinct subsets of the set $\{1, \dots, k\}$ and $n > 2^k - 1$, there are some $1 \leq i < j \leq n$ with $S_i = S_j$. Then each color occurs in the edge $B_i \cup B_j$ an even number of times, a contradiction. So, $\chi_{CF}(\mathcal{H}) \geq 1 + k$.

Since $m \leq \binom{n+1}{2} - 1 = \frac{n^2+n-2}{2} < n^2$, we have $\log_2 m < 2\log_2 n < 2(1+k) \leq 2\chi_{CF}(\mathcal{H})$. □

To construct a matching bound for Theorem 39(ii), when m is much larger than r, is a harder task. Pach and Tardos [26] showed that if $\mathcal{H}$ is a r-uniform hypergraph with m edges, then $\chi_{CF}(\mathcal{H}) \leq rm^{2/(r+1)} \log m$, and they ask whether $\chi_{CF}(\mathcal{H}) \leq rm^{1/r} \log m$. We answer their question in the negative. More precisely, we show that if r is much smaller than m, then there exists r-uniform hypergraph $\mathcal{H}$ such that $\chi_{CF}(\mathcal{H}) \geq C_r m^{2/(r+2)}/\log m$. Let us start with a simple observation.

Observation 41. *Given any coloring f of an n-element set with t colors, we can choose a family $\mathcal{A}_f$ of t disjoint sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2t \rfloor$ and is monochromatic.*

Proof. Consider the color classes $A_1, A_2, ...A_t$. For each color class A_i we partition it into subclasses B_{i_j} of size equal to $\lfloor n/2t \rfloor$ until we cannot anymore. The last subclass say $B_{i_{j'}}$ for each i will have size less than $\lfloor n/2t \rfloor$. Summing the sizes of these $B_{i_{j'}}s$ we get at most $n/2$ vertices. The remaining at least $n/2$ vertices give us a family of t sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2t \rfloor$ and is monochromatic. □

Theorem 42. *For each positive even fixed r, there exists a constant $c_r \leq 4(8e^2/r)^{r/2}$ such that for every integer $t \geq r/2$, there exists an r-uniform hypergraph $\mathcal{H}$ with less than $1 + c_r t^{(r+2)/2} \log t$ edges such that $\chi_{CF}(\mathcal{H}) > t$.*

Proof. Consider a vertex set V of size n, a multiple of $4t$. Let

$$m = \left\lceil 4(8e^2/r)^{r/2} t^{(r+2)/2} \log t \right\rceil . \tag{4.3}$$

We form a random r-uniform hypergraph $\mathcal{H}$ with m edges by choosing m subsets $F_1, F_2, ..., F_m$ of V of size r randomly with equal probability and repetitions allowed. We will prove that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is larger than t.

Let f be any fixed t-coloring of V. By Observation 41, there exists a family $\mathcal{A}_f$ of k sets $\{A_1, A_2, ..., A_t\}$ such that each of these sets has size $\lfloor n/2t \rfloor$ and is monochromatic. So the probability that edge F_i has a conflict is bounded from below by the probability that it has exactly 2 or 0 vertices from each of the sets in $\mathcal{A}_f$ and no vertices outside $\mathcal{A}_f$, which, in turn, is equal to $\binom{t}{r/2}\binom{\lfloor n/(2t) \rfloor}{2}^{r/2}\binom{n}{r}^{-1}$. Since

$$\left(\frac{n}{t}\right)^t \leq \binom{n}{t} \leq \left(\frac{en}{t}\right)^t,$$

we get

$$\Pr(\text{edge } F_i \text{ has a conflict}) \geq \left(\frac{t}{r/2}\right)^{r/2}\left(\frac{n^2}{16t^2}\right)^{r/2}\left(\frac{r^2}{e^2n^2}\right)^{r/2} = \left(\frac{r}{8e^2t}\right)^{r/2}.$$

Consequently,

$$\begin{aligned}\Pr(f \text{ is a conflict-free coloring of } \mathcal{H}) &\leq \left(1 - \left(\frac{r}{8e^2t}\right)^{r/2}\right)^m \\ &< \exp\left(-m\left(\frac{r}{8e^2t}\right)^{r/2}\right).\end{aligned}$$

There are t^n distinct colorings of $V(\mathcal{H})$, so

$$\begin{aligned}\Pr(\mathcal{H} \text{ is conflict-free colorable with } t \text{ colors}) &< t^n \exp\left(-m\left(\frac{r}{8e^2t}\right)^{r/2}\right) \\ &\leq \exp\left(-m\left(\frac{r}{8e^2t}\right)^{r/2} + n\log t\right).\end{aligned}$$

If $n = 4t$, then by (4.3) the probability that $\mathcal{H}$ is conflict-free colorable is strictly smaller than 1. Hence there exists an r-uniform hypergraph $\mathcal{G}$ with m edges such that $\chi_{CF}(\mathcal{G}) > t$. □

Remark: Solving (4.3) for t, we get $t \sim C_r m^{2/(r+2)}/\log m$, where C_r is a function of r. Thus, Theorem 42 shows that for a given m and $r \leq C_r m^{2/(r+2)}/\log m$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > C_r m^{2/(r+2)}/\log m$.

4.4 Conflict-free coloring of simple hypergraphs

Although one can show that there exist simple r-uniform hypergraphs $\mathcal{H}$ with $m = C^r$ such that $\chi(\mathcal{H}) = \Theta(r)$, the second part of Theorem 39(ii) can be improved in the case of simple hypergraphs. Let us start with the following simple consequence of Lemma 38.

Lemma 43. *Let $r \leq t/8$ and let $\mathcal{H}$ be an r-uniform hypergraph. If $D(\mathcal{H}) < \frac{1}{4}(\frac{t}{8r})^{\lceil (r-1)/2 \rceil}$, then there exists a vertex coloring of $\mathcal{H}$ with t colors such that each edge has at least two colors appearing exactly once.*

Proof. Consider a random t-coloring of $\mathcal{H}$ and let A_e be the event that the edge e has at most one color appearing exactly once. By Lemma 38, the probability of A_e, $\Pr(A_e) \leq (\frac{8r}{t})^{\lceil (r-1)/2 \rceil}$. Now note that for a given edge e, the event A_e is independent of all but at most $D(\mathcal{H})$ other events $A_{e'}$. Thus, for $D(\mathcal{H}) < \frac{1}{4}(\frac{t}{8r})^{\lceil (r-1)/2 \rceil}$, we have $4 \cdot \Pr(A_e) \cdot D(\mathcal{H}) < 1$, and so by Lovász Local Lemma there exists a coloring where none of the events A_e occur. Consequently, there exists a coloring of $\mathcal{H}$ with t colors such that every edge has at least two colors appearing exactly once. □

Remark. By Lemma 40, for a given m, even if r is arbitrarily large (but even), there is an r-uniform hypergraph $\mathcal{H}$ with m edges and $\chi_{CF}(\mathcal{H}) > 0.5 \log_2 m$. There is no similar statement for simple hypergraphs. Indeed, if the maximum edge degree of a simple r-uniform hypergraph $\mathcal{H}$ is less than r, then we can choose in each edge e a vertex v_e that belongs only to e. Then we color each v_e with 1, and every other vertex with 2. So, such a hypergraph has a conflict-free coloring with just 2 colors.

Theorem 44. *Let $r \leq t/8$ and let $\mathcal{H}$ be an r-uniform simple hypergraph with m edges. If $m \leq \frac{1}{16r(r-1)^2}(\frac{t}{8(r-1)})^{r-2}$, then $\chi_{CF}(\mathcal{H}) \leq t$.*

Proof. Assume that $\chi_{CF}(\mathcal{H}) > t$. Let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by truncating each edge e by a vertex v_e of maximum degree. Observe that $\mathcal{H}_1$ is an $(r-1)$-uniform simple hypergraph and if f is a k-coloring of $\mathcal{H}_1$, then there exists an edge of $\mathcal{H}_1$ which has at most one color appearing exactly once, otherwise $\mathcal{H}$ would be conflict-free t-colorable. Now by Lemma 43, $D(\mathcal{H}_1) \geq \frac{1}{4}(\frac{t}{8(r-1)})^{\lceil (r-2)/2 \rceil}$. Furthermore, $\mathcal{H}_1$ has a vertex of degree at least $D(\mathcal{H}_1)/(r-1)$. If $\mathcal{H}_1$ has a vertex v of degree at least d, then every edge e in $\mathcal{H}_1$ containing v must have a vertex v_e whose degree in $\mathcal{H}$ is at least d. Moreover, since $\mathcal{H}$ is simple, all these d vertices are distinct. Hence $\mathcal{H}$ has at least $D(\mathcal{H}_1)/(r-1)$ vertices of degree at least $D(\mathcal{H}_1)/(r-1)$. So by the degree-sum formula,

$$m \geq D(\mathcal{H}_1)^2/r(r-1)^2 > \frac{1}{16r(r-1)^2}\left(\frac{t}{8(r-1)}\right)^{r-2}.$$

□

Note that if we solve the equation $m = \frac{1}{16r(r-1)^2}(\frac{t}{8(r-1)})^{r-2}$ with respect to t we get $t \sim C'_r m^{1/(r-2)}$ so, for large r, the upper bound for the conflict-free chromatic number for simple hypergraphs provided by Theorem 44 is roughly a square of the bound given by Theorem 39 for the general case. The following result shows that, at least for large r, this estimate is not very far from being optimal.

Lemma 45. *Let $r \leq t$. Then, there exists an r-uniform simple hypergraph $\mathcal{H}$ with $(1+o(1))(4t \ln t)^2(\frac{4e^2 t}{r})^r$ edges such that $\chi_{CF}(\mathcal{H}) > t$.*

Proof. We first construct an auxiliary $4t$-uniform simple hypergraph $\mathcal{H}_1$ as follows. Let q be a prime which will be chosen later. The vertex set of $\mathcal{H}_1$ is $S = S_1 \cup ... \cup S_{4t}$ where all S_i are disjoint copies of $GF(q) = \{0, 1, ..., q-1\}$. The edges of $\mathcal{H}_1$ are $4t$-tuples $(x_1, ..., x_{4t}) \in S_1 \times ... \times S_{4t}$ that are solutions of the system of linear equations

$$\sum_{i=1}^{4t} i^j x_i = 0, \qquad j = 0, 1, ..., 4t-3, \tag{4.4}$$

over $GF(q)$.

For any fixed pair of variables in (4.4), we have a $(4t-2) \times (4t-2)$ system of linear equations with Vandermond's determinant which has a unique solution over $GF(q)$. This means that $\mathcal{H}_1$ is $4t$-uniform simple hypergraph with $4tq$ vertices in which each vertex is contained in q edges, so $|E(\mathcal{H}_1)| = q^2$.

Now from each edge e of $\mathcal{H}_1$ we choose an r-subset A_e randomly and independently. Let $\mathcal{H}$ be the r-uniform simple hypergraph obtained from $\mathcal{H}_1$ by taking the subsets A_e as its edges. Our goal is to show that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is large.

To this end, fix a coloring f. Let B_e denote the event that the edge e has a conflict in the coloring f, and $p = \Pr(B_e)$. Arguing as in the proof of Theorem 42, one can show that

$$p \geq \left(\frac{r}{8e^2 t}\right)^{r/2}.$$

Since the edges of $\mathcal{H}$ were chosen independently, the probability that f is a conflict-free coloring of $\mathcal{H}$ is $(1-p)^{q^2}$. Moreover, the total number of colorings is t^{4tq}, so the probability that there exists a conflict-free coloring of $\mathcal{H}$ with t colors is at most $t^{4tq} \cdot (1-p)^{q^2}$. This probability is less than 1, provided

$$t^{4tq} \cdot e^{-pq^2} < 1\,,$$

which holds whenever

$$q > \frac{4t \ln t}{p}.$$

Now if we take the smallest prime q such that $q > q_0 = 4t \ln t (\frac{8e^2 t}{r})^{r/2}$, then we have an r-uniform simple hypergraph with q^2 edges and $\chi_{CF}(\mathcal{H}) > t$. It is known (see, for instance, [17]) that one can take $q = (1 + o(1))q_0$. Hence

$$|E(\mathcal{H})| = (1 + o(1))(4t \ln t)^2 \left(\frac{8e^2 t}{r}\right)^r.$$

□

Remark: Finally, let us remark that if we take $t = r$, then we get a simple r-uniform hypergraph $\mathcal{H}$ with $m = 2^{O(r)}$ edges such that $\chi(\mathcal{H}) > r = \Omega(\ln m)$, so Theorem 39(i) cannot be significantly improved in the case of simple hypergraphs, at least when m grows exponentially with r.

Chapter 5

Choosability with separation in complete graphs

5.1 Introduction

List coloring of graphs is a generalized version of the ordinary vertex coloring problem. As in ordinary vertex coloring, we pick a single color for each vertex, but the sets of colors available at different vertices may be different. This model was introduced independently by Vizing [32] and Erdős-Rubin-Taylor [11].

Given a graph $G(V, E)$, a *list L for G* is an assignment to every $v \in V(G)$ of a set $L(v)$ of colors that may be used for the coloring of v. We say that G is L-colorable, if there exists a proper coloring f of the vertices of G from L, i.e. if $f(v) \in L(v)$ for all $v \in V(G)$ and $f(u) \neq f(v)$ for all $uv \in E$. An extensively studied parameter is the *list chromatic number* of G, $\chi_l(G)$, which is the least k such that G is L-colorable, whenever $|L(v)| = k$ for all $v \in V(G)$. It is also sometimes called by *choice number*, or *choosability* of G.

It is easy to see that $\chi_l(G) \geq \chi(G)$. Moreover, the list chromatic number for an n-vertex graph can be as large as n, namely for complete graphs K_n when the lists are identical. It is natural to ask what happens when the lists do not intersect too much.

We say that a list L for a graph G is a *(k, c)-list* if $|L(v)| = k$ for all $v \in V(G)$ and $|L(u) \cap L(v)| \leq c$ for all $uv \in E(G)$, that is for every edge, the lists of its end points have at most c colors in common. Kratochvíl, Tuza and Voigt [24] introduced $\chi_l(G, c)$, the least k such that G is L-colorable from each (k, c)-list L. Among other results, they showed the following.

Theorem 46. [24] $\sqrt{\frac{cn}{2}} \leq \chi_l(K_n, c) \leq \sqrt{2ecn}$.

They also asked the following problem.

Problem: Does $\lim_{n\to\infty} \chi_l(K_n, c)/\sqrt{cn}$ exist ?

We prove that the limit exists and is equal to 1. We also find the exact value of $\chi_l(K_n, c)$ for infinitely many values of n.

This is a joint work with Z. Füredi and A. Kostochka and appears in [14].

5.2 Upper Bound

We start by citing two known facts.

For the complete graph K_n and a list L, the *vertex-color adjacency graph*, $F = F(V(K_n), U_L)$, is the bipartite graph whose partite sets are $V(K_n)$ and $U_L = \bigcup_{v \in V(K_n)} L(v)$ with $u \in V(K_n)$ being adjacent to $\alpha \in U_L$ if and only if $\alpha \in L(u)$.

Observation 47 (Vizing)**.** **[32]** *For every list assignment L for K_n, K_n has an L-coloring if and only if the vertex-color adjacency graph $F(V(K_n), U_L)$ has a matching saturating $V(K_n)$.*

Lemma 48 (Johnson's bound)**.** **[18]** *Let $E_1, ...E_m$ be sets such that $|E_i| \geq k$ and $|E_i \cap E_j| \leq c$. Then $|\bigcup_{i=1}^{m} E_i| \geq \frac{mk^2}{mc+k-c}$.*

The following lemma is a slight improvement of Lemma 48 when $m = q + 2$ and $k = q$.

Lemma 49. *Let $c \geq 1$. If $\mathcal{L}(V, E)$ is a q-uniform hypergraph such that $|E| = q+2$ and $|e \cap e'| \leq c-1$, then $|V| \geq \frac{1}{c}(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})$.*

Proof. Let d_v be the degree of vertex v, then we have

$$\sum_v \binom{d_v}{2} = \sum_{e,e' \in E} |e \cap e'| \leq (c-1)\binom{q+2}{2}. \tag{5.1}$$

$$\sum_v d_v = \sum_e |e| = (q+2)q. \tag{5.2}$$

Let $t \geq 2$ be an integer. We multiply (5.1) by $\frac{-1}{\binom{t}{2}}$, (5.2) by $\frac{2}{t}$ and sum them up. We also note the fact that $\frac{2}{t}d - \frac{1}{\binom{t}{2}}\binom{d}{2} \leq 1$ when $t \in \{2, 3, ...\}$ and $d \in \{1, 2, ...\}$. We now choose $t = c+1$ and we have

$$\sum_v 1 \geq \sum_v \frac{2}{c+1}d_v - \frac{1}{\binom{c+1}{2}}\binom{d_v}{2} \geq \frac{2}{c+1}q(q+2) - \frac{1}{\binom{c+1}{2}}(c-1)\binom{q+2}{2} = \frac{q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}}{c}.$$

Hence $|V| \geq \frac{1}{c}(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})$. □

Lemma 50. *Let L be a list assignment such that $|L(v)| \geq q+1$ and $|L(v_1) \cap L(v_2)| \leq c$. Then $F(V(K_n), U_L)$ has a matching saturating $V(K_n)$ if $n \leq \frac{1}{c}(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})$.*

Proof. We need to show that Hall's condition holds in F, that is, $|S| \leq |N(S)|$, for all $S \subseteq V$. For this we consider the subgraph F_S induced by vertices of S and $N(S)$.

Case 1 : $\deg_{F_S}(\alpha) \leq q+1$, for all $\alpha \in N(S)$.
Counting edges in F_S we have

$$|S|(q+1) = \textstyle\sum_{v \in S} |L(v)| = \sum_{\alpha \in N(S)} \deg_{F_S}(\alpha) \leq |N(S)|(q+1),$$

which implies $|S| \leq |N(S)|$.

Case 2: $\deg_{F_S}(\alpha) \geq q+2$, for some $\alpha \in N(S)$.
Suppose $\alpha \in L(v_1) \cap L(v_2) \cap ... \cap L(v_{q+2})$, where $v_1, ..., v_{q+2} \in S$. Consider the sets $L'_i = L(v_i) \backslash \{\alpha\}$. Then $|L'_i| \geq q$ and $|L'_i \cap L'_j| \leq c-1$, for all $1 \leq i, j \leq q+2, i \neq j$. We now consider a hypergraph $\mathcal{L}$ with L'_i as its edges. By Lemma 49,

$$|\cup_i L(v_i)| \geq 1 + \tfrac{1}{c}(q^2 + \tfrac{c+3}{c+1}q - \tfrac{2(c-1)}{c+1}).$$

Now if Hall's condition fails to hold, then $|S| > |N(S)|$, and thus

$$n \geq |S| > |N(S)| \geq |\cup_i L(v_i)| \geq 1 + \frac{1}{c}\left(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1}\right),$$

which is a contradiction.

$\square$

Remark: One could also use Lemma 48 in Case 2 of the Lemma 50 to obtain a slightly weaker upper bound of $\frac{q^2(q+2)}{c(q+1)-1}$.

Observation 47 and Lemma 50 now yield the following Proposition.

Proposition 51. $\chi_l(K_n, c) \leq q+1$ *for* $n \leq \frac{1}{c}(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})$.

5.3 Lower Bound

In this section we will obtain a lower bound on $\chi_l(K_n, c)$ and then use it to yield the main theorem stated below

Theorem 52. *Let* $c \geq 1$, *then*
(i) $\lim_{n\to\infty} \chi_l(K_n, c)/\sqrt{cn} = 1$.
(ii)If q *is a prime power,* $c < q-1$ *and* c *divides* $q-1$, *then* $\chi_l(K_n, c) = q+1$, *for all* $n \in [\frac{q^2-1}{c} + 2, \frac{1}{c}(q^2 + \frac{c+3}{c+1}q - \frac{2(c-1)}{c+1})]$.

To obtain a lower bound, we need to show that there is a particular (k, c) list assignment for K_n for which it is not list-colorable. This particular list assignment will come from an auxiliary hypergraph. The construction of this hypergraph is based on [13] and is shown below.

Let q be a prime power and c an integer such that $c < q-1$ and c divides $q-1$. Let $\mathbf{F}$ be the q-element finite field $\mathbf{GF(q)}$ and let h be an element of order c in the multiplicative group $\mathbf{F}\backslash\{0\}$. Set $H = \{1, h, h^2, ..., h^{c-1}\}$. Then H is a c-element subgroup of $\mathbf{F}\backslash\{0\}$. Let $(a,b), (a',b') \in \mathbf{F} \times \mathbf{F}\backslash\{(0,0)\}$. We say that $(a,b) \sim (a',b')$, if there exists $h^\alpha \in H$ such that

$a' = h^\alpha a$ and $b' = h^\alpha b$. Note that $\sim$ is an equivalence relation and each equivalence class is a collection of c elements in $\mathbf{F} \times \mathbf{F} \backslash \{(0,0)\}$. Hence there are $\frac{q^2-1}{c}$ equivalence classes. The equivalence class containing (a,b) will be denoted by $\langle a,b \rangle$.

Consider the set $L\langle a,b\rangle = \{\langle x,y\rangle : ax + by \in H\}$. Since H is a group, $ax + by \in H$ implies $(h'a)x + (h'b)y \in H$, for all $h' \in H$. Hence $L\langle a,b\rangle$ is well-defined.

Claim 1: Let $(a,b),(a',b') \in \mathbf{F} \times \mathbf{F} \backslash \{(0,0)\}$. Then $|L\langle a,b\rangle| = q$. Moreover, if $(a,b) \nsim (a',b')$, then either $|L\langle a,b\rangle \cap L\langle a',b'\rangle| = c$ or $|L\langle a,b\rangle \cap L\langle a',b'\rangle| = 0$.

Proof: Let $(a,b) \in \mathbf{F} \times \mathbf{F} \backslash \{(0,0)\}$. By symmetry we assume $b \neq 0$. Then for any given x and h^α, there is a unique solution of $ax + by = h^\alpha$. Hence there are exactly qc solutions. These solutions come in equivalence classes and hence $|L\langle a,b\rangle| = q$.
Now consider $(a,b) \nsim (a',b')$. Then for given α and β, if the system of equations

$$\begin{aligned} ax + by &= h^\alpha \\ a'x + b'y &= h^\beta \end{aligned}$$

has a solution, then it has a unique solution, since $\det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0$. Hence for c^2 values of α and β there are c^2 possible solutions. Since the solutions come in equivalence classes, either $|L\langle a,b\rangle \cap L\langle a',b'\rangle| = c$ or $|L\langle a,b\rangle \cap L\langle a',b'\rangle| = 0$.

Applying Claim 1 we obtain the following theorem.

Theorem 53 (Füredi)**.** **[13]** *Let $\mathcal{L}(V,E)$ be a hypergraph with the vertex set $V = \{\langle a,b\rangle : a,b \in \boldsymbol{F}, (a,b) \neq (0,0)\}$ and the edge set $E = \{L\langle a,b\rangle : a,b \in \boldsymbol{F}, (a,b) \neq (0,0)\}$. Then $\mathcal{L}$*
(i) is a q-uniform hypergraph,
(ii) has $\frac{q^2-1}{c}$ vertices,
(iii) has $\frac{q^2-1}{c}$ edges such that every two edges intersect in at most c vertices.

We use the construction given in Theorem 53 to obtain another hypergraph which we shall use to get our desired lower bound.

Claim 2: If $V_m = \{\langle x,y\rangle : y = mx\}$, then $|V_m| = \frac{q-1}{c}$ and $|V_m \cap L\langle a,b\rangle| \leq 1$, for each $\langle a,b\rangle \in V$.

Proof: Given m, $y = mx$ has q solutions in $\mathbf{F} \times \mathbf{F}$. Since the solutions come in equivalence classes $y = mx$ has $\frac{q-1}{c}$ solutions in V, where V is the vertex set of the hypergraph $\mathcal{L}$ from Theorem 53.

To prove the next claim, we might see the hypergraph $\mathcal{L}$ arising out of the affine plane geometry $AG(2,q)$ by combining c parallel lines of the form $ax + by = h^\alpha$ for c values of α to get the set $L\langle a,b\rangle$. Now in $AG(2,q)$ every two lines intersect in at most 1 point in $\mathbf{F} \times \mathbf{F}$. Hence for each α the line $ax + by = h^\alpha$ meets the line $y = mx$ in at most 1 point in $V(\mathcal{L})$ and thus $|V_m \cap L\langle a,b\rangle| \leq 1$. $\square$

Consider the hypergraph $\mathcal{H}(V', E')$ with

$$V' = V \cup \{x\}, \text{ where } x \notin V$$

$$\text{and } E' = E \cup \{\{x\} \cup V_1 \cup V_2 \cup ... \cup V_c, \{x\} \cup V_{c+1} \cup V_{c+2} \cup ... \cup V_{2c}, ...\},$$

where $V, E, V_i s$ sets considered in Claim 2. By Claim 2 and Theorem 53, $|\{x\} \cup V_1 \cup V_2 \cup ... \cup V_c| = 1 + c\frac{q-1}{c} = q$ and every two edges intersect in at most c vertices.

Thus $\mathcal{H}(V', E')$ is a q-uniform hypergraph such that $|V'| = \frac{q^2-1}{c} + 1$ and $|E'| > |V'|$ such that $|e \cap e'| \leq c$ for every $e, e' \in E'$.

Proposition 54. *Let q be a prime power. If $c < q - 1$ and c divides $q - 1$, then $\chi_l(K_n, c) \geq q + 1$ for $n \geq \frac{q^2-1}{c} + 2$, that is $\chi_l(K_n, c) \geq \sqrt{c(n-2)+1} + 1$.*

Proof. Let $n = \frac{q^2-1}{c} + 2$. Consider the hypergraph $\mathcal{H}$ constructed above. Let $f : V(K_n) \to E'$ be a bijective mapping. For every $v \in V(K_n)$ we let its list be $L(v) = f(v)$. Then L is a (q, c)-list assignment in which the total number of colors $|V'| < n$. Hence there is no proper coloring of K_n with this list assignment. Hence $\chi_l(K_n, c) \geq q + 1$ for $n = \frac{q^2-1}{c} + 2$ and thus $\chi_l(K_n, c) \geq q + 1$ for $n \geq \frac{q^2-1}{c} + 2$. □

We shall use the following lemma to give a general lower bound for any n.

Lemma 55. **[17]** *Let $c \geq 1$, n be sufficiently large. Then the interval $[\sqrt{c(n-2)+1} + 1 - n^{1/3}, \sqrt{c(n-2)+1} + 1]$ contains a prime q such that c divides $q - 1$.*

Proposition 56. *Let $c \geq 1$. Then for every sufficiently large positive integer n, $\chi_l(K_n, c) \geq \lfloor \sqrt{c(n-2)+1} + 1 \rfloor - n^{1/3}$.*

Proof. Given a sufficiently large n, consider the interval $[\sqrt{c(n-2)+1}+1-n^{1/3}, \sqrt{c(n-2)+1}+1]$. By Lemma 55, this interval contains a prime q such that c divides $q - 1$. Let $n' = \frac{(q^2-1)}{c} + 2$. By Proposition 54, $\chi_l(K_{n'}, c) \geq \sqrt{c(n-2)+1} + 1 = q + 1$. Hence

$$\chi_l(K_n, c) \geq \chi_l(K_{n'}, c) \geq q + 1 \geq \lfloor \sqrt{c(n-2)+1} + 1 \rfloor - n^{1/3}.$$

□

Propositions 51, 56 and 54 now imply Theorem 52.

For a fixed $c \geq 1$, one might be interested in knowing what is the maximum value of $\chi_l(G, c)$ over all n-vertex graphs G. Note that if H is an induced subgraph of G, then $\chi_l(H, c) \leq \chi_l(G, c)$, but this may not hold true for non-induced subgraphs. Below are two examples of hypergraphs that illustrate this fact for hypergraphs (We do not know of any examples for graphs yet).

Example 1: Let the graph $H = K_{3,27}$, where $V(H) = A \cup B$, with $|A| = 3, |B| = 27$. Consider the hypergraph $G = H \cup B$. Note that $\chi(H, 1) \geq 4$ (consider disjoint lists of size 3 on each vertex in A and all the 27 transversals of size 3 on the vertices in B; this gives a improper coloring). Now to show that $\chi(G, 1) \leq 3$. For that we see that if the lists of vertices in A, then one can give a proper coloring by choosing two colors on A such that there is always a third color present in the lists of vertices in B. Finally, if the lists of vertices in the A are disjoint, then a bad coloring can arise only if all 27 transversals occur on the vertices in B. But then it is not a 1-separated list assignment (since B forms an edge in G).

Example 2: The above example had edges of size 2 and one big edge. One might want the hypergraph to be more uniform. For that we consider the graph H as above, and let $B = B_1 \cup B_2$, where $|B_1| = 25, |B_2| = 2$. Consider the hypergraph $G = K_{3,27} \cup$ 25 edges of size 3 formed by taking one vertex from B_1 and the two vertices of B_2. Now to show that $\chi(G, 1) \leq 3$.

Case 1: If the lists of vertices in the A are disjoint and all 27 transversals occur on the vertices in B: Lists in B_1 can have at most one intersection with lists in B_2. This make it impossible to assign all the 27 lists to vertices in B.

Case 2: If the lists of vertices in the A are disjoint and not all 27 transversals occur on the vertices in B: We then color the vertices of A with one of the missing lists. The vertices in B_2 since they are in an edge have at least 5 distinct colors in the union of their lists. Taking away the colors used on A leaves at least two distinct colors in their union and can thus be properly colored.

Case 3: If the lists of vertices in the A are not disjoint: then one can give a coloring by choosing two colors on A such that there is always a third color present in the lists of vertices in B. Moreover, with argument similar to case 2, vertices of B_2 can be colored with distinct colors, giving G a proper coloring.

One might still suspect that in the case of graphs, the complete graph requires the most number of colors. We have the following conjecture.

Conjecture 57. *If $c, n \geq 1$ and G is an n-vertex graph, then $\chi_l(G, c) \leq \chi_l(K_n, c)$.*

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