INTERFERENCE MANAGEMENT IN WIRELESS NETWORKS

BY

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DISSERTATION

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ABSTRACT

The world is going wireless, and the availability of high-speed ubiquitous wireless connectivity is being taken for granted. Along with high bandwidth consuming applications such as high-definition video, mobile devices such as smartphones and tablets are becoming omnipresent. The legacy wireless systems are not designed to meet such an exponential growth in the demand for wireless connectivity. To meet both short- and long-term demands, we need to develop methods to maximize the spectral efficiency of existing wireless systems, and also understand the fundamental limits of various architectures to guide the design of future wireless networks.

Breaking the interference barrier is an important step in achieving higher throughput in both cellular and ad-hoc wireless networks. Towards this end, there has been a renewed interest in information-theoretic studies of Gaussian interference channels in recent years. The technique used by almost all legacy systems to handle interference in wireless networks is to separate the users signals as much as possible using the available time, frequency and spatial dimensions, and then to treat the residual interference as noise. We refer to this technique as simply *treating interference as noise*. In the first part of the dissertation, we consider the following two problems: (1) Suppose we restrict the strategy to treating interference as noise, then what is the best achievable sum-rate? (2) How sub-optimal is this strategy compared to the best possible strategy? Both of these problems have been widely studied for over three decades, and yet they remain open. We solve both of these problems under certain conditions on the channel parameters which are satisfied when the interference levels are low compared to the signal levels. In such a *low interference regime*, we show that the best sum-rate achievable with treating interference as noise is a solution to a convex maxmin optimization problem, and therefore the optimal transmit strategies and the corresponding best sum-rate can be efficiently computed using standard convex optimization algorithms. We also show that the corresponding best sum-rate is indeed equal to the sum capacity, thus proving that treating interference as noise is the best strategy in the low interference regime.

In the second part of the dissertation, we obtain insights into the problem of interference channel with coordinated multi-point (CoMP) transmission and

reception, where the transmitters cooperate to jointly transmit the messages, and the receivers cooperate to jointly receive the messages. The advanced cellular systems such as LTE-Advanced are likely to use CoMP as the physical layer interference management technique to enhance the capacity. Since determining the exact capacity of wireless systems is a difficult problem, often the coarser metric of degrees of freedom (DoF) is used to obtain first-order insights at high SNRs. We provide some insights into the benefits of CoMP by studying the DoF of interference channel with CoMP transmission and reception as a function of the transmit and receive cooperation orders.

To the memory of Sundar, my brother

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CHAPTER 1

INTRODUCTION

The demand for wireless broadband services is set to explode in the next few years. Along with high bandwidth consuming applications such as high-definition video, mobile devices such as smartphones and tablets are becoming omnipresent. In fact, according to the Cisco Visual Networking Index (VNI) forecast [1], the global mobile traffic grew 2.6-fold in 2010, nearly tripling for the third time in a row starting from 2008, and is expected to increase by 26-fold by 2015. These numbers pose a great challenge to the communication engineer who is responsible for designing the wireless systems. Since the radio frequency spectrum is a scarce and expensive resource, the emphasis is on building efficient wireless networks that can extract as much throughput as the physical wireless channel can offer. Interference is identified as one of the major bottlenecks limiting the throughput in a wireless system. Unlike the wired medium, the broadcast and multiple-access nature of the wireless medium ensures that every transmitter is heard by every neighboring receiver. The signal transmitted by a user is interference to all the neighboring users who operate on the same frequency spectrum. Since our objective is to support large numbers of users while providing the highest possible data rates using limited spectrum, it is imperative that we understand the best ways to handle interference in wireless networks.

1.1 Interference Management Techniques

1.1.1 Treating Interference as Noise

Historically speaking, communication engineers first studied the problem of communication through a wired medium where the capacity is limited by the thermal noise. Shannon, in his celebrated paper [2], established the fundamental limits of communication in the presence of noise. In the subsequent years, researchers have understood how to design good practical codes that achieve data rates close to the the capacity promised by Shannon for the point-to-point additive white Gaussian noise (AWGN) channel. With appropriate modifications these point-to-point codes can also be used to communicate in the presence of interference if the interference signal is treated as just noise. Thus emerged a natural and most popular interference management technique – *treat interfer-*

ence as noise while taking care to ensure that the level of interference is small compared to the signal level, by orthogonalizing the adjacent users. Typically, orthogonalization is achieved by scheduling the adjacent users in different resource units, where a resource unit denotes a square block in the two-dimensional grid with time and frequency as its axes. The advent of multiple-antenna, known as multiple-input multiple-output (MIMO), technology added an additional dimension to the notion of resource unit – the adjacent users can now be separated in *space* by appropriately choosing the transmit and receive beams along which the antenna arrays transmit and receive signals.

1.1.2 Partial Interference Cancellation

Unlike thermal noise generated by nature, interference has a definite structure since it is generated by other users. In general, the receivers can exploit this structure to decrease the uncertainty and thus achieve higher data rates. A canonical example that demonstrates the scenario where treating interference as noise is not a good strategy is pointed out by Carleial in [3]. Consider a scenario where two users share a wireless channel such that the transmitter of each user is close to the receiver of the other user resulting in a very strong interference setting. If the interference level is sufficiently strong compared to the signal level, then interference can be completely decoded while treating the desired signal as noise. Once the interference is perfectly decoded, the receivers can subtract the interference and achieve the same performance as if there was no interference. Even if the interference is not very strong, the idea of decoding a part of the interference and subtracting it from the received signal to partially cancel interference can still be applied. The Han-Kobayashi scheme [4] for the two-user interference channel is one such example, which is known to outperform treating interference as noise in the settings where interference level is comparable to the signal level.

1.1.3 Structured Codes

Shannon introduced the technique of *random coding* to prove that any rate less than capacity is achievable for the AWGN channel. It is not clear if the random coding technique is sufficient to determine the fundamental limits of communication in the presence of interference. Several researchers have considered using structured codes, in particular lattice codes, for the interference channels [5, 6, 7, 8, 9].

1.2 Sum-Rate with Treating Interference as Noise

Among all the interference management techniques discussed so far, treating interference as noise is the most popular and widely used technique for two reasons. Firstly, treating interference as noise is a low complexity strategy, and we have a good understanding of how to design practical point-to-point codes to achieve rates close to those promised by information-theoretic analyses. Secondly, it is not clear if the advanced techniques can outperform treating interference as noise in large systems such as cellular networks. For these reasons, determining the best achievable sum-rate with treating interference as noise is an important problem.

1.2.1 Nonconvex Optimization Problem

We consider the problem of determining the best achievable sum-rate with the transmitters using Gaussian inputs, and the receivers treating interference as noise. For any fixed transmit strategy, i.e., fixed transmit covariance matrices, the optimization problem becomes decoupled and each receiver can focus on maximizing its own rate since its action does not affect the performance at other receivers. We can exactly determine the receiver strategies for any fixed transmit strategy. Therefore, the problem boils down to determining the best transmit strategy that maximizes the achievable sum-rate. This problem has been widely studied, and yet it remains a challenging open problem to date, mainly due to interference. Interference not only limits performance of wireless networks, but it also makes the mathematical analysis difficult in a fundamental fashion. The achievable sum-rate is a concave function if there is no interference, leading to nice closed-form algorithms such as the water-filling algorithm for point-to-point channels. With interference, however, this nice property may not hold true, which makes the nonconvex optimization problem of determining the best sum-rate extremely difficult to solve. Several iterative algorithms, including the water-filling algorithm [10, 11], the gradient projection algorithm [12], and the interference pricing algorithm [13], have been proposed to find good lower bounds to the best achievable sum-rate. Not much can be said about the global optimality of the solutions provided by these algorithms due to the nonconvex nature of the optimization problem.

1.2.2 Interference Alignment and Degrees of Freedom

Recently, there has been renewed excitement about determining the best sum-rate, resulting in many new algorithms [14, 15], including the Min-Leakage algorithm and the Max-SINR algorithm. Since determining the best sum-rate exactly is a difficult optimization problem, these algorithms aim at maximizing a coarser metric called degrees of freedom (DoF). The DoF, also known as the multiplexing gain, is an asymptotic quantity that captures the behavior of the sum-rate at high SNRs. Roughly speaking, an algorithm is said to achieve d_Σ number of DoF if the achievable sum-rate is of the order $\Omega(d_\Sigma \log \mathrm{SNR})$ at high SNRs. The high SNR analysis makes sense for the interference channels because in the other extreme of low SNRs, the effect of interference can be ignored since the interference level is negligible compared to the noise level, and we already have a good understanding of communication in the presence of noise.

The primary driving force behind all these algorithms is the new idea called *interference alignment* introduced in [16, 17]. Interference alignment refers to the concept of aligning interference from multiple interferers in order to ensure that the total interference occupies fewer dimensions, leaving more dimensions for the signal. Since the DoF metric counts only the number of dimensions, and does not depend on how well the signal space is separated from the interference space, the algorithms inspired by interference alignment are observed to achieve the best DoF. The algorithms such as Max-SINR that approximate the standard water-filling algorithm at low SNRs and the interference alignment algorithm at high SNRs, are observed to perform well even at moderate SNRs.

1.2.3 Convex Relaxation

In this dissertation, we consider the problem of determining the best achievable sum-rate exactly. Since the main difficulty is the nonconvex nature of the optimization problem, we upper-bound the achievable sum-rate by a concave function, and solve the resulting convex optimization problem to obtain an upper bound to the best achievable sum-rate. We also obtain a lower bound by evaluating the sum-rate function at the transmit strategy given by the global optimal solution to the relaxed convex optimization problem. Since the sum-rate function is concave if the interference levels are exactly equal to zero, there should exist a concave function that is very close to the sum-rate function when the interference levels are nonzero but very close to zero. Thus, intuitively speaking, the convex relaxation approach should yield good upper and lower bounds if the interference levels are small compared to the signal levels. Amazingly, the upper bound exactly matches the lower bound if the interference levels are low enough, thus leading to an exact characterization of the best achievable sum-rate in a *low interference regime*. Since the upper bound and lower bounds are obtained by solving a convex optimization problem, we can even derive the corresponding necessary and sufficient conditions for the bounds to coincide.

1.2.4 Sum Capacity in Low Interference Regime

Among the known interference management techniques, treating interference as noise is attractive for many reasons, and hence we studied the problem of determining the best achievable sum-rate with treating interference as noise.

We now ask how sub-optimal the technique of treating interference as noise is compared to the best technique. Intuitively speaking, treating interference as noise should be close to optimal when the interference levels are low. The reason is that for any advanced technique to perform better than the treating interference as noise, the receivers should be able to exploit the structure in interference, and it should be difficult for the receivers to explore the structure in interference when the interference levels are low compared to the signal levels. Interestingly, we show that in the low interference regime, not only is treating interference as noise close to optimal, but it is indeed the optimal strategy. We prove this by showing that the convex relaxation upper bound is fundamental in the sense that the upper bound obtained by the convex relaxation technique is actually an upper bound to the sum capacity. Therefore, we have that, in the low interference regime when the bounds meet, the sum capacity is achievable by treating interference as noise; i.e., there exists no other scheme that can achieve a sum-rate better than that achieved by treating interference as noise.

1.3 CoMP Transmission and Reception

Even if we can determine and implement the best possible achievable schemes for the interference channel, the demand for wireless connectivity is likely to exceed what the physical channel can offer. For this and other reasons, there has been much interest in understanding the benifits of cooperation in interference networks. Typically cooperation requires additional infrastructure, but it could be cost-effective depending on the overall objective. In this dissertation, we study the particular cooperation technique called coordinated multi-point (CoMP) transmission and reception, also known as joint processing, network-MIMO, virtual-MIMO, multi-cell-MIMO. CoMP is best explained in the context of cellular networks, where the base stations are connected to each other through a high-speed backhaul network. Suppose the backhaul network is very strong; then the base stations can exchange and jointly transmit the messages in the case of downlink, and can exchange the quantized received signals and jointly receive the messages in the case of uplink. In fact CoMP transmission (for downlink) and CoMP reception (for uplink) are being considered as the physical layer interference management techniques to be included in the fourth generation cellular systems such as LTE-Advanced.

Observe that interference is completely eliminated in the extreme case where each message is transmitted jointly by all the transmitters and received jointly by all the receivers. Such perfect cooperation at the transmitters and receivers may not be feasible, and hence we consider the interference channel with partial cooperation at both transmitters and receivers. We capture the cost of cooperation through transmit cooperation order and receive cooperation order, which refer to the number of transmitters that jointly transmit a message, and the number of receivers that jointly receive a message, respectively. As the transmit and receive cooperation orders vary from 1 to the number of users, we cover all the cases from no cooperation to perfect cooperation. Since determining the exact sum capacity is a hard problem, we resort to studying the degrees of freedom (DoF) to obtain insights into the benefits of cooperation.

1.4 Dissertation Outline

The rest of the dissertation is organized as follows. In Chapter 2, we introduce the model for the Gaussian interference channel that we study. In Chapter 3, we study the problem of determining the best achievable sum-rate, obtained by using Gaussian inputs and treating interference as noise, in the two-user MIMO Gaussian interference channel. We propose a convex maxmin optimization problem the solution to which provides lower and upper bounds to the best achievable sum-rate. We show that if the bounds coincide, then the best achievable sum-rate is indeed equal to the sum capacity. We determine necessary and sufficient conditions for the bounds to coincide, leading to an exact characterization of the sum capacity. We then consider the special cases of symmetric MISO and SIMO interference channels and simplify the conditions. In Chapter 4, we extend the sum capacity characterization in the low interference regime to Gaussian interference channels with more than two users. In Chapter 5, we consider the Gaussian interference channel with coordinated multi-point (CoMP) transmission and reception, and derive lower and upper bounds to the DoF as a function of the number of users and the transmit and receive cooperation orders. In Chapter 6, we provide some concluding remarks.

1.5 Notation

We use the following notation. For deterministic objects, we use lowercase letters for scalars, lowercase letters in bold font for vectors, and uppercase letters in bold font for matrices. For example, we use h to denote a deterministic scalar and $\mathbf{h}$ to denote a deterministic vector, and $\mathbf{H}$ to denote a deterministic matrix. For random objects, we use uppercase letters for scalars, and underlined uppercase letters for vectors. Random objects with superscripts denote sequences of the random objects in time. For example, we use X to denote a random scalar, $\underline{X}$ to denote a random vector, and X^n and $\underline{X}^n$ to denote the sequences of length n of the random scalars and vectors, respectively. We use $\mathbf{\Sigma}_{\underline{X}}$ and $\mathrm{Cov}\left(\underline{X}\right)$ to denote the covariance matrix of a random vector $\underline{X}$. We use $\mathbf{\Sigma}_{\underline{Y}|\underline{X}}$ and $\mathrm{Cov}\left(\underline{Y}|\underline{X}\right)$ to denote the covariance matrix of the minimum mean square estimation error in estimating the random vector $\underline{Y}$ from the random vector $\underline{X}$, with similar notation for random scalars. We use $\mathcal{CN}(0, \mathbf{\Sigma})$ to denote the circularly symmetric complex Gaussian vector distribution with zero mean and covariance matrix $\mathbf{\Sigma}$, with similar notation for random scalars. We use $\mathsf{H}(.)$ to denote the entropy of a discrete random variable, $\mathsf{h}(.)$ to denote the differential entropy of a continuous random variable or vector and $\mathsf{I}(.;.)$ to denote the mutual information.

We use $\mathcal{K}$ to denote the set $\mathcal{K} = \{1, 2, \cdots, K\}$, where the number K will be obvious from the context. For any $m \leq K$, we use $k \uparrow m$ and $k \downarrow m$ to denote the sets

$$k \uparrow m = \{k, k+1, k+2, \cdots, k+m-1\}$$
$$k \downarrow m = \{k, k-1, k-2, \cdots, k-m+1\}.$$

The indices are taken modulo K such that $k \uparrow m, k \downarrow m \subseteq \mathcal{K}$. Observe that for any two indices i, j and $m \leq K$, $i \in j \uparrow m$ is true if and only if $j \in i \downarrow m$.

CHAPTER 2

CHANNEL MODEL

The K-user Gaussian interference channel, as described in Figure 2.1, consists of K transmitter-receiver pairs communicating over a wireless medium. In contrast to the wired medium, the broadcast and multiple-access nature of the wireless medium implies that every transmitter is heard by every receiver. The signal $\underline{Y}_i \in \mathbb{C}^{N_r}$ received by receiver i, is given by

$$\underline{Y}_i = \sum_{j=1}^{K} \mathbf{H}_{ij} \underline{X}_j + \underline{Z}_i, \ \forall i \in \mathcal{K} \tag{2.1}$$

where $\underline{X}_j \in \mathbb{C}^{N_t \times 1}$ denotes the signal transmitted by transmitter j, $\underline{Z}_i \in \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_r})$ denotes the additive white Gaussian noise at receiver i, and $\mathbf{H}_{ij} \in \mathbb{C}^{N_r \times N_t}$ denotes the channel transfer matrix from transmitter j to receiver i. Each transmitter is assumed to have N_t transmit antennas, and each receiver is assumed to have N_r receive antennas. The general case with N_t and N_r taking arbitrary values is referred to as the multiple-input multiple-output (MIMO) interference channel. The special cases with $N_t = 1$ or $N_r = 1$ or $N_t = N_r = 1$ are referred to as the single-input multiple-output (SIMO), multiple-input single-output (MISO) and single-input single-output (SISO) interference channels, respectively. The transmitters are assumed to operate under average power constraints; i.e., for each $j \in \mathcal{K}$, the power spent by transmitter j must not exceed P_j on an average.

2.1 Achievable Scheme

Consider the problem of communicating K messages over the interference channel (2.1). For each $k \in \mathcal{K}$, the message W_k is available at the transmitter k, and is desired by the receiver k. A communication scheme consists of an encoder-decoder pair per each message. The encoder at transmitter k performs the operation of mapping the message W_k onto the physical signal $\underline{X}_k$ to be transmitted. The decoder at receiver k performs the operation of reconstructing the message W_k from the received signal $\underline{Y}_k$. We say that the communication scheme is reliable if the messages can be reconstructed at the receivers with high probability. The genius of Shannon showed us that reliable communication is

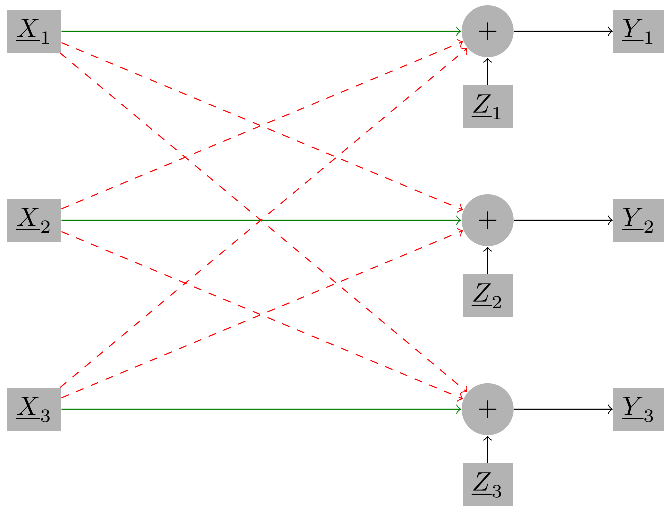

Figure 2.1: The 3-user Gaussian interference channel. The green solid lines indicate the links carrying signal, and the red dashed lines indicate the links carrying interference.

feasible in noisy channels by coding over multiple symbols. We consider the same block coding framework where the communication scheme operates over n symbols at a time. For a fixed rate tuple $(R_1, R_2, \cdots, R_K) \in \mathbb{R}_+^K$ and a block length $n \geq 1$, the message W_k takes values from the set $\mathcal{W}_k = \left\{1, 2, \cdots, \lceil 2^{nR_k} \rceil\right\}$. The block code consists of the encoders

$$\underline{X}_k^n : \mathcal{W}_k \to \mathbb{C}^{N_t \times n}, \ \forall k \in \mathcal{K}$$

and the decoders

$$\hat{W}_k : \mathbb{C}^{N_r \times n} \to \mathcal{W}_k, \ \forall k \in \mathcal{K}.$$

Assuming that the message W_k is a uniform random variable taking values in the set $\mathcal{W}_k$, the probability of decoding error is defined as

$$e_n = \max_{k \in \mathcal{K}} \mathsf{Pr}\left(\hat{W}_k\left(\underline{Y}_k^n\right) \neq W_k\right).$$

We say that the rate tuple $(R_1, R_2, \cdots, R_K)$ is achievable if and only if there exists a sequence of block codes satisfying the average power constraints

$$\mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} ||\underline{X}_k(t)||^2\right] \leq P_k, \ \forall k \in \mathcal{K}$$

such that the probability of error $e_n \to 0$ as $n \to \infty$.

2.2 Capacity and Degrees of Freedom

The capacity region $\mathcal{C}$ is defined as the closure of the set of achievable rate tuples. Except in some special cases, the capacity region of the Gaussian interference channel remains an elusive holy grail in the information theory literature. For this reason, we define the degrees of freedom (DoF) region which is relatively easy to characterize. The DoF region provides insights about the behavior of the capacity region at high SNRs. Roughly speaking, the DoF region is equal to the capacity region scaled by $\log \mathrm{SNR}$ at high SNRs. For the purpose of DoF analysis, we assume a symmetric power constraint, i.e., $P_k = P$ for all $k \in \mathcal{K}$. We say that the DoF tuple $(d_1, d_2, \cdots, d_K)$ is achievable if for every $P > 0$ there exists an achievable rate-tuple $(R_1(P), R_2(P), \cdots, R_K(P))$ such that

$$d_k = \limsup_{P \to \infty} \frac{R_k(P)}{\log P}, \ \forall k \in \mathcal{K}.$$

The DoF region $\mathcal{D}$ is defined as the closure of the set of achievable DoF tuples. The sum capacity and the sum DoF are defined as

$$\begin{aligned} \mathcal{C}_{\text{sum}} &= \max_{(R_1, R_2, \cdots, R_k) \in \mathcal{C}} R_1 + R_2 + \cdots + R_K \\ \mathcal{D}_{\text{sum}} &= \max_{(d_1, d_2, \cdots, d_k) \in \mathcal{D}} d_1 + d_2 + \cdots + d_K. \end{aligned}$$

2.3 Channel Knowledge

Throughout the thesis, we assume global channel knowledge, i.e., all the channel coefficients are assumed to be fixed and known at all the transmitters and at all the receivers. In practice, the channel knowledge is obtained by transmitting known signals, called pilots, at regular intervals and estimating the channel through some kind of filtering. This process is called channel estimation which enables local knowledge of the channel coefficients at the receivers. The estimated (local) channel coefficients are then distributed to other transmitters and receivers. Although the processes of channel estimation and distribution incur significant overhead, it is difficult to accommodate this overhead into the capacity analysis. The common practice is to perform capacity analysis assuming global channel knowledge, which is difficult as it is, and accounts for the overhead when designing practical achievable schemes. Since the information-theoretic capacity analysis only provides high level insights regarding the actual design of the achievable schemes, this kind of layered approach is generally acceptable.

2.4 CoMP Transmission and Reception

In Chapter 5, the transmitters and receivers are allowed to cooperate so that the messages are transmitted and received jointly by multiple transmitters and receivers, respectively. For each $k \in \mathcal{K}$, the message W_k is associated with a transmit set $\mathcal{T}_k \subseteq \mathcal{K}$ and a receive set $\mathcal{R}_k \subseteq \mathcal{K}$. The transmit set $\mathcal{T}_k$ denotes the set of transmitters that have access to the message W_k. The receive set $\mathcal{R}_k$ denotes the set of receivers whose received signals are available at the decoder k. The classical interference channel is recovered by setting $\mathcal{T}_k = \mathcal{R}_k = \{k\}, k \in \mathcal{K}$.

With CoMP transmission and reception, the definitions of the encoders and the decoders must be changed appropriately. Each transmitter is associated with an encoder, and each message is associated with a decoder. For each $j \in \mathcal{K}$, the encoder at transmitter j takes the available messages $\{W_k : j \in \mathcal{T}_k\}$ as input and outputs the physical signal $\underline{X}_j^n$ to be transmitted. For each $k \in \mathcal{K}$, the decoder of message W_k takes the available received signals $\{\underline{Y}_i^n : i \in \mathcal{R}_k\}$ as input and outputs the reconstructed message $\hat{W}_k$. All the other definitions in Sections 2.1 and 2.2 remain unchanged.

CHAPTER 3

TWO-USER INTERFERENCE CHANNEL

In this chapter, we consider the Gaussian interference channel (2.1) in the two-user case. The two-user MIMO Gaussian interference channel is given by

$$\begin{aligned} \underline{Y}_1 &= \mathbf{H}_{11}\underline{X}_1 + \mathbf{H}_{12}\underline{X}_2 + \underline{Z}_1 \\ \underline{Y}_2 &= \mathbf{H}_{21}\underline{X}_1 + \mathbf{H}_{22}\underline{X}_2 + \underline{Z}_2 \end{aligned} \tag{3.1}$$

where $\underline{Z}_i \in \mathcal{CN}(0, \mathbf{I})$, and the average power constraints at transmitters 1 and 2 are denoted by P_1 and P_2, respectively. Let N_{1t}, N_{2t} denote the number of transmit antennas at the transmitters 1 and 2, respectively, and N_{1r}, N_{2r} denote the number of receive antennas at the receivers 1 and 2, respectively. The dimensions of the channel matrices, the signal vectors, and the noise vectors are defined appropriately. We are interested in determining the best sum-rate achievable by using Gaussian inputs and treating interference as noise, and also the sum capacity of the two-user MIMO Gaussian interference channel.

We start by studying the problem of determining the best achievable sum-rate. The multiple-antennas at the transmitters and at the receivers present an opportunity to design the transmit and receive beams to suppress the interference and improve the achievable sum-rate. While it is easy to express the achievable sum-rate as a function of the beams, the design of the optimal beams that maximize the achievable sum-rate is known to be a difficult problem. The main difficulty stems from the fact that the sum-rate optimization problem cannot be posed as a convex optimization problem, which makes the optimization problem difficult to solve analytically or even numerically. In this chapter, we study the technique of convex approximation and optimization to solve this nonconvex optimization problem. Specifically, we upper-bound the achievable sum-rate with a concave function, and solve the corresponding convex optimization problem to obtain lower and upper bounds to the original sum-rate optimization problem. We show that if the channel parameters satisfy certain conditions, then the bounds coincide, leading to an exact characterization of the best achievable sum-rate by using Gaussian inputs and treating interference as noise.

The problem of determining the best achievable sum-rate by treating interference as noise is important from a practical perspective because communication engineers have a good understanding of designing the codes to achieve the rates promised by the information-theoretic analysis. However, it is also important to understand how far the achievable sum-rate by treating interference as noise is from the sum capacity. We show that if the lower and upper bounds on the achievable sum-rate coincide, then the best achievable sum-rate is indeed equal to the sum capacity. Using the theory of Karush-Kuhn-Tucker (KKT) conditions, we obtain necessary and sufficient conditions for the bounds to coincide leading to an exact characterization of the sum capacity. We observe that the conditions are satisfied in a *low interference regime* where the interfering signal levels are low compared to the desired signal levels. We end the chapter by providing some nontrivial examples of the two-user Gaussian interference channel in the low interference regime. In particular, we consider the special cases of symmetric MISO and SIMO interference channels, and derive a simple closed-form condition on the channel parameters for the channels to be in the low interference regime.

3.1 Related Work

The study of the two-user interference channel was initiated by Shannon in 1961 [18]. Carleial made an interesting and counter-intuitive observation that interference does not reduce the capacity of the two-user SISO Gaussian interference channel in the *very strong interference* regime [3]. If the interference level is very high compared to the signal level, the receivers can first decode the interfering message to subtract its contribution from the received signal, thus achieving the same rate as if there was no interference. Subsequently, the capacity region was determined in the *strong interference* regime [4, 19], where interference reduces the capacity but, like in the very strong interference regime, the optimal strategy requires the receivers to again decode the interfering message. Characterizing the capacity region of the two-user SISO Gaussian interference channel in the general setting still remains an open problem. The best known achievable region is based on the Han-Kobayashi (HK) scheme [4, 20]. The HK region was shown to be the capacity region for a class of discrete-memoryless deterministic interference channels in 1982 [21]. Little was known about the optimality of the HK region in the Gaussian case until recently. The concept of a genie giving side-information to the receivers was used in [22, 23] to derive outer bounds on the capacity region of two-user SISO Gaussian interference channel. The outer bound region in [23] is within one bit of the HK region, thus leading to an approximate characterization of the capacity region of the two-user SISO Gaussian interference channel. The genie-based outer bound technique was further extended in [24, 25, 26] to prove that treating interference as noise achieves the sum capacity in a low interference regime (referred to as noisy-interference regime in [24]).

The two-user MIMO Gaussian interference channel was studied in [10, 11, 12, 13, 14, 27, 28, 29] from the point of view of determining the best achievable rate region obtained by using Gaussian inputs and treating interference as noise. Several iterative algorithms including the water-filling algorithm [10, 11], the gradient projection algorithm [12], the interference pricing algorithm [13] and the Max-SINR algorithm [14], were proposed to find good lower bounds to the best achievable sum-rate. For the MISO Gaussian interference channel, it was proved in [27, 28, 29] that rank one covariance matrices are optimal. However, the problem of determining the best achievable sum-rate remains open and is known to be a difficult problem even in the SISO case [30, 31] due to the nonconvex nature of the sum-rate.

The two-user MIMO Gaussian interference channel was studied in [32, 33, 34] from the point of view of determining the capacity region. In [33], the authors showed that sum DoF of the MIMO Gaussian interference channel is equal to

$$\min\left(N_{1t}+N_{2t}, N_{1r}+N_{2r}, \max(N_{1t}, N_{2r}), \max(N_{2t}, N_{1r})\right)$$

and that the optimal sum DoF is achieved by treating interference as noise. In [34], the authors extended the approximate capacity characterization of the two-user SISO Gaussian interference channel in [23] to the MIMO case. The contents of this chapter are a result of our attempt to extend the low interference regime characterization of the two-user SISO Gaussian interference channel in [24, 25, 26] to the MIMO case. Some of the results in this chapter are published in [35], [36], and also in [37], [38] which are due to an independent and parallel work by Shang, Chen, Kramer and Poor.

3.2 Standard Form

In this section, we show that the following assumptions can be made about the two-user MIMO Gaussian interference channel (3.1) with out any loss of generality:

1. The direct channel matrices $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ have unit (Frobenius) norm.

2. The cross channel matrices $\mathbf{H}_{12}$ and $\mathbf{H}_{21}$ are diagonal with real and non-negative entires.

3. The numbers of transmit and receive antennas $(N_{1t}, N_{2t}, N_{1r}, N_{2r})$ satisfy

$$N_{1t} \leq \operatorname{rank}\begin{bmatrix} \mathbf{H}_{11} \\ \mathbf{H}_{21} \end{bmatrix},\ N_{2t} \leq \operatorname{rank}\begin{bmatrix} \mathbf{H}_{12} \\ \mathbf{H}_{22} \end{bmatrix}$$

   and

$$N_{1r} \leq \operatorname{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}],\ N_{2r} \leq \operatorname{rank}[\mathbf{H}_{21}\ \mathbf{H}_{22}].$$

The second assumption implies that the cross channel matrices can be expressed as

$$\mathbf{H}_{12} = \begin{bmatrix} \tilde{\mathbf{H}}_{12} \\ \mathbf{0} \end{bmatrix}, \ \mathbf{H}_{21} = \begin{bmatrix} \tilde{\mathbf{H}}_{21} \\ \mathbf{0} \end{bmatrix}$$

where $\tilde{\mathbf{H}}_{12}$ and $\tilde{\mathbf{H}}_{21}$ are diagonal matrices with full row rank. This is the only assumption we use in the development of the outer bound techniques presented in this chapter. The other two assumptions are used in Section 3.10 to simplify the presentation.

The first assumption can easily be justified by scaling the transmit power constraints P_1 and P_2 appropriately. We now justify the other two assumptions. First, consider the singular value decomposition of $\mathbf{H}_{12}$ and $\mathbf{H}_{21}$:

$$\begin{aligned} \mathbf{H}_{12} &= \mathbf{U}_1 \mathbf{\Lambda}_{12} \mathbf{V}_2^\dagger \\ \mathbf{H}_{21} &= \mathbf{U}_2 \mathbf{\Lambda}_{21} \mathbf{V}_1^\dagger \end{aligned}$$

where $\mathbf{\Lambda}_{12}, \mathbf{\Lambda}_{21}$ are diagonal matrices with real and nonnegative entries, and $\mathbf{V}_1, \mathbf{V}_2, \mathbf{U}_1, \mathbf{U}_2$ are unitary matrices. We obtain an equivalent Gaussian interference channel, satisfying the second assumption, by projecting the received signals along $\mathbf{U}_1, \mathbf{U}_2$, and the transmitted signals along $\mathbf{V}_1, \mathbf{V}_2$, i.e., by making the following substitutions:

$$\begin{aligned} \underline{X}_j &\leftarrow \mathbf{V}_j^\dagger \underline{X}_j \\ \underline{Y}_i &\leftarrow \mathbf{U}_i^\dagger \underline{Y}_i \\ \underline{Z}_i &\leftarrow \mathbf{U}_i^\dagger \underline{Z}_i \\ \mathbf{H}_{ij} &\leftarrow \mathbf{U}_i^\dagger \mathbf{H}_{ij} \mathbf{V}_j. \end{aligned}$$

Observe that the average transmit power constraint and the distribution of the receive noise terms remain unchanged because $\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1, \mathbf{V}_2$ are unitary matrices.

The third assumption can be justified by appropriately choosing the unitary matrices. For example, suppose $N_{1r} > \mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$. Consider the SVD of $\mathbf{H}_{12} = \mathbf{U}_1 \Lambda_{12} \mathbf{V}_2^\dagger$. Observe that the span of the first $\mathrm{rank}\,\mathbf{H}_{12}$ columns of $\mathbf{U}_1$ is equal to the column space of $\mathbf{H}_{12}$, and we have flexibility in choosing the remaining $N_{1r} - \mathrm{rank}\,\mathbf{H}_{12}$ columns. We may choose those columns such that the span of the first $\mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$ columns of $\mathbf{U}_1$ is equal to the column space of $[\mathbf{H}_{11}\ \mathbf{H}_{12}]$ so that the last $N_{1r} - \mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$ columns of $\mathbf{U}_1$ are orthogonal to the columns of $[\mathbf{H}_{11}\ \mathbf{H}_{12}]$. Therefore, the last $N_{1r} - \mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$ rows of the channel matrices $\mathbf{H}_{11}$ and $\mathbf{H}_{12}$ in the new channel are equal to zero, i.e., receiver 1 sees nothing but Gaussian noise from the last $N_{1r} - \mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$ antennas. Hence, we can ignore them and assume that $N_{1r} = \mathrm{rank}[\mathbf{H}_{11}\ \mathbf{H}_{12}]$. We can repeat the same argument at receiver 2, and also at transmitters 1 and 2 to justify the other inequalities in the third assumption.

3.3 Achievable Sum-Rate

For the two-user MIMO Gaussian interference channel (3.1), the sum-rate achievable by using (circularly symmetric) Gaussian inputs and treating interference as noise is given by

$$\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G}\right)+\mathsf{I}\left(\underline{X}_{2G};\underline{Y}_{2G}\right) \tag{3.2}$$

where the subscript G indicates that Gaussian inputs are used. Let $\mathbf{Q}_1 = \mathbf{\Sigma}_{\underline{X}_{1G}}$ and $\mathbf{Q}_2 = \mathbf{\Sigma}_{\underline{X}_{2G}}$ denote the covariance matrices of the Gaussian random vectors $\underline{X}_{1G}$ and $\underline{X}_{2G}$, respectively. To meet the average power constraints, the covariance matrices must belong to the feasible region

$$\mathcal{Q} = \{(\mathbf{Q}_1, \mathbf{Q}_2) : \mathbf{Q}_i \succeq 0,\ \mathsf{Tr}\left(\mathbf{Q}_i\right) \leq P,\ i = 1, 2\}.$$

We have the opportunity to design the covariance matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ to maximize the achievable sum-rate, leading us to the optimization problem

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} f(\mathbf{Q}_1, \mathbf{Q}_2). \tag{3.3}$$

where $f(\mathbf{Q}_1, \mathbf{Q}_2)$ denotes the sum-rate (3.2) as function of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, i.e.,

$$\begin{aligned} f(\mathbf{Q}_1, \mathbf{Q}_2) &= \mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G}\right)+\mathsf{I}\left(\underline{X}_{2G};\underline{Y}_{2G}\right) \\ &= \mathsf{h}\left(\underline{Y}_{1G}\right)-\mathsf{h}\left(\underline{Y}_{1G}|\underline{X}_{1G}\right)+\mathsf{h}\left(\underline{Y}_{2G}\right)-\mathsf{h}\left(\underline{Y}_{2G}|\underline{X}_{2G}\right) \\ &= \log\frac{\det \mathbf{\Sigma}_{\underline{Y}_{1G}}}{\det \mathbf{\Sigma}_{\underline{Y}_{1G}|\underline{X}_{1G}}}+\log\frac{\det \mathbf{\Sigma}_{\underline{Y}_{2G}}}{\det \mathbf{\Sigma}_{\underline{Y}_{2G}|\underline{X}_{2G}}}. \end{aligned} \tag{3.4}$$

The explicit dependence on $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be seen by substituting

$$\begin{aligned} \mathbf{\Sigma}_{\underline{Y}_{1G}} &= \mathbf{I}+\mathbf{H}_{11}\mathbf{Q}_1\mathbf{H}_{11}^{\dagger}+\mathbf{H}_{12}\mathbf{Q}_2\mathbf{H}_{12}^{\dagger} \\ \mathbf{\Sigma}_{\underline{Y}_{2G}} &= \mathbf{I}+\mathbf{H}_{22}\mathbf{Q}_1\mathbf{H}_{21}^{\dagger}+\mathbf{H}_{22}\mathbf{Q}_2\mathbf{H}_{22}^{\dagger} \\ \mathbf{\Sigma}_{\underline{Y}_{1G}|\underline{X}_{1G}} &= \mathbf{I}+\mathbf{H}_{12}\mathbf{Q}_2\mathbf{H}_{12}^{\dagger} \\ \mathbf{\Sigma}_{\underline{Y}_{2G}|\underline{X}_{2G}} &= \mathbf{I}+\mathbf{H}_{21}\mathbf{Q}_1\mathbf{H}_{21}^{\dagger}. \end{aligned}$$

The nonconvex nature of the objective function $f(\mathbf{Q}_1, \mathbf{Q}_2)$ makes the optimization problem (3.3) extremely difficult to solve. The following claim proves that $f(\mathbf{Q}_1, \mathbf{Q}_2)$ is not concave in general.

Claim 1. *The function $f(\mathbf{Q}_1, \mathbf{Q}_2)$ need not be concave in $(\mathbf{Q}_1, \mathbf{Q}_2)$.*

Proof. Consider the special case of symmetric SISO interference channel with $\mathbf{H}_{11} = \mathbf{H}_{22} = 1$, and $\mathbf{H}_{21} = \mathbf{H}_{12} = h$. Thus, we have

$$f(q_1, q_2) = \log\left(1+\frac{q_1}{1+h^2q_2}\right)+\log\left(1+\frac{q_2}{1+h^2q_1}\right).$$

Observe that

$$\frac{f(2q,0)+f(0,2q)}{2}-f(q,q)=\log(1+2q)-2\log\left(1+\frac{q}{1+h^2q}\right)\geq 0$$

whenever q and h satisfy

$$\begin{aligned}(1+2q)&\geq\left(1+\frac{q}{1+h^2q}\right)^2\\ \Leftrightarrow 2q&\geq\frac{2q}{1+h^2q}+\left(\frac{q}{1+h^2q}\right)^2\\ \Leftrightarrow 2h^2(1+h^2q)&\geq 1.\end{aligned}$$

This concludes that the function $f(\mathbf{Q}_1,\mathbf{Q}_2)$ is not concave in general. □

3.4 Local Optimal Solution

Since a global optimal solution has to be locally optimal, we can obtain insights into the structural properties of the optimal transmit covariance matrices by analyzing the necessary KKT conditions. Let $\lambda_1,\lambda_2\geq 0$ and $\mathbf{M}_1,\mathbf{M}_2\succeq 0$ denote the dual variables associated with the constraints $\mathsf{Tr}(\mathbf{Q}_1)\leq P_1, \mathsf{Tr}(\mathbf{Q}_2)\leq P_2$ and $\mathbf{Q}_1,\mathbf{Q}_2\succeq 0$, respectively. The Lagrangian associated with (3.3) is given by

$$\begin{aligned}&L(\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{M}_1,\mathbf{M}_2,\lambda_1,\lambda_2,)\\ &\quad=f(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)+\sum_{i=1}^{2}\mathsf{Tr}(\mathbf{M}_i\mathbf{Q}_i)-\lambda_i(\mathsf{Tr}(\mathbf{Q}_i)-P_i).\end{aligned}$$

The KKT conditions are given by

$$\begin{aligned}\nabla_{\mathbf{Q}_1}f(\mathbf{Q}_1,\mathbf{Q}_2)&=\lambda_1\mathbf{I}-\mathbf{M}_1\\ \nabla_{\mathbf{Q}_2}f(\mathbf{Q}_1,\mathbf{Q}_2)&=\lambda_2\mathbf{I}-\mathbf{M}_2\\ \lambda_1(\mathsf{Tr}(\mathbf{Q}_1)-P_1)&=0\\ \lambda_2(\mathsf{Tr}(\mathbf{Q}_2)-P_2)&=0\\ \mathsf{Tr}(\mathbf{M}_1\mathbf{Q}_1)&=0\\ \mathsf{Tr}(\mathbf{M}_2\mathbf{Q}_2)&=0.\end{aligned}\tag{3.5}$$

The following fact from matrix differential calculus is useful in deriving the expressions for gradients [12, 39]. Given matrices $\mathbf{\Sigma}=\mathbf{\Sigma}^\dagger$ and $\mathbf{H}$, we have

$$\nabla_{\mathbf{Q}}\log\det\left(\mathbf{\Sigma}+\mathbf{HQH}^\dagger\right)=\mathbf{H}^\dagger\left(\mathbf{\Sigma}+\mathbf{HQH}^\dagger\right)^{-1}\mathbf{H}.$$

Using the expression (3.4) for sum-rate, we obtain that

$$\begin{aligned}\nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1,\mathbf{Q}_2) &= \mathbf{H}_{11}^{\dagger}\mathbf{\Sigma}_{\underline{Y}_{1G}}^{-1}\mathbf{H}_{11} + \mathbf{H}_{21}^{\dagger}\left(\mathbf{\Sigma}_{\underline{Y}_{2G}}^{-1} - \mathbf{\Sigma}_{\underline{Y}_{2G}|\underline{X}_{2G}}^{-1}\right)\mathbf{H}_{21}\\ \nabla_{\mathbf{Q}_2} f(\mathbf{Q}_1,\mathbf{Q}_2) &= \mathbf{H}_{22}^{\dagger}\mathbf{\Sigma}_{\underline{Y}_{2G}}^{-1}\mathbf{H}_{22} + \mathbf{H}_{12}^{\dagger}\left(\mathbf{\Sigma}_{\underline{Y}_{1G}}^{-1} - \mathbf{\Sigma}_{\underline{Y}_{1G}|\underline{X}_{1G}}^{-1}\right)\mathbf{H}_{12}.\end{aligned} \tag{3.6}$$

3.5 Convex Approximation and Optimization

In this section, we outline the outer bound technique. We introduce a convex optimization problem, the solution to which provides lower and upper bounds on the best achievable sum-rate (3.3). Suppose we upper bound $f(\mathbf{Q}_1,\mathbf{Q}_2)$ with $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)$, i.e., $f(\mathbf{Q}_1,\mathbf{Q}_2) \le \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)$, such that $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)$ is concave in $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Then, we can solve the convex optimization problem

$$\bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*) = \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)$$

to obtain lower and upper bounds to the sum-rate optimization problem (3.3):

$$f(\mathbf{Q}_1^*,\mathbf{Q}_2^*) \le \text{Best achievable sum-rate} \le \bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*).$$

The tightness of the lower and upper bounds depends on the choice of the upper bound function $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)$. The upper bound function we use in this chapter is based on a genie giving side-information to the receivers. By treating the side-information as a part of the received signal, we obtain a *genie-aided* MIMO Gaussian interference channel. Let Ψ denote the genie parameters, which will be defined in Section 3.6. The achievable sum-rate $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ in the genie-aided channel is an obvious upper bound to the achievable sum-rate $f(\mathbf{Q}_1,\mathbf{Q}_2)$ in the original channel, i.e.,

$$f(\mathbf{Q}_1,\mathbf{Q}_2) \le \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).$$

We say that the genie Ψ is useful if the upper bound function $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ is concave. Let Ψ_u denote the usefulness set; i.e., $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ is concave in $(\mathbf{Q}_1,\mathbf{Q}_2)$ for all $\Psi \in \Psi_u$. We obtain the best upper bound to $f(\mathbf{Q}_1,\mathbf{Q}_2)$ by optimizing over $\Psi \in \Psi_u$; i.e,

$$f(\mathbf{Q}_1,\mathbf{Q}_2) \le \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2) = \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).$$

Therefore, the upper and lower bounds to the best sum-rate can be obtained by solving the maxmin optimization problem

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).$$

We will also show that the upper bound function $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ is convex in Ψ for every $(\mathbf{Q}_1, \mathbf{Q}_2)$. Therefore, we have a convex maxmin optimization problem which can be solved efficiently using standard convex optimization algorithms.

3.6 Genie-Aided Channel

Suppose the genie provides the receivers 1 and 2 with side-information $\underline{S}_1$ and $\underline{S}_2$, respectively. The signals $\underline{S}_1$ and $\underline{S}_2$ are defined as

$$\begin{aligned} \underline{S}_1 &= \tilde{\mathbf{H}}_{21}\underline{X}_1 + \underline{W}_1 \\ \underline{S}_2 &= \tilde{\mathbf{H}}_{12}\underline{X}_2 + \underline{W}_2 \end{aligned} \tag{3.7}$$

where $\tilde{\mathbf{H}}_{12}, \tilde{\mathbf{H}}_{21}$, defined in Section 3.2, represent the matrices containing the nonzero rows of $\mathbf{H}_{12}, \mathbf{H}_{21}$, respectively, and $\underline{W}_1, \underline{W}_2$ are random vectors denoting Gaussian noise. The genie chooses how the noise terms $\underline{W}_1$ and $\underline{W}_2$ are correlated to $\underline{Z}_1$ and $\underline{Z}_2$, respectively. We use Ψ as a shorthand notation to denote the genie parameters $\Psi = \{\mathbf{\Sigma}_{\underline{W}_1}, \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}, \mathbf{\Sigma}_{\underline{W}_2}, \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\}$ satisfying the positive semidefinite constraints

$$\mathrm{Cov}\left(\begin{bmatrix} \underline{Z}_i \\ \underline{W}_i \end{bmatrix}\right) = \begin{bmatrix} \mathbf{I} & \mathbf{\Sigma}_{\underline{Z}_i\underline{W}_i} \\ \mathbf{\Sigma}_{\underline{W}_i\underline{Z}_i} & \mathbf{\Sigma}_{\underline{W}_i} \end{bmatrix} \succeq 0,\ i = 1, 2.$$

We use the achievable sum-rate of the genie-aided interference channel as the upper bound function

$$\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) = \mathsf{I}\left(\underline{X}_{1G}; \underline{Y}_{1G}, \underline{S}_{1G}\right) + \mathsf{I}\left(\underline{X}_{2G}; \underline{Y}_{2G}, \underline{S}_{2G}\right). \tag{3.8}$$

Since the mutual information is nonnegative, we obtain that

$$\begin{aligned} \mathsf{I}\left(\underline{X}_{1G}; \underline{Y}_{1G}\right) &\le \mathsf{I}\left(\underline{X}_{1G}; \underline{Y}_{1G}\right) + \mathsf{I}\left(\underline{X}_{1G}; \underline{S}_{1G} | \underline{Y}_{1G}\right) = \mathsf{I}\left(\underline{X}_{1G}; \underline{Y}_{1G}, \underline{S}_{1G}\right) \\ \mathsf{I}\left(\underline{X}_{2G}; \underline{Y}_{2G}\right) &\le \mathsf{I}\left(\underline{X}_{2G}; \underline{Y}_{2G}\right) + \mathsf{I}\left(\underline{X}_{2G}; \underline{S}_{2G} | \underline{Y}_{2G}\right) = \mathsf{I}\left(\underline{X}_{2G}; \underline{Y}_{2G}, \underline{S}_{2G}\right) \end{aligned}$$

and hence

$$f(\mathbf{Q}_1, \mathbf{Q}_2) \le \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) \text{ for any } \Psi.$$

To utilize the idea in Section 3.5, we now define the usefulness set Ψ_u, and show the following properties:

1. The set Ψ_u is convex.
2. The function $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ is concave in $(\mathbf{Q}_1, \mathbf{Q}_2)$ for any $\Psi \in \Psi_u$.
3. The function $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ is convex in Ψ for any $\mathbf{Q}_1$ and $\mathbf{Q}_2$.

3.6.1 Useful Genie: Concavity Property

Let Ψ_u be the set of genie parameters $\Psi = \left(\mathbf{\Sigma}_{\underline{W}_1}, \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}, \mathbf{\Sigma}_{\underline{W}_2}, \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\right)$ satisfying the usefulness conditions

$$\begin{aligned} \begin{bmatrix} \mathbf{I} & \mathbf{\Sigma}_{\underline{Z}_1\underline{W}_1} \\ \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1} & \mathbf{\Sigma}_{\underline{W}_1} \end{bmatrix} &\succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \mathbf{I} & \mathbf{\Sigma}_{\underline{Z}_2\underline{W}_2} \\ \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2} & \mathbf{\Sigma}_{\underline{W}_2} \end{bmatrix} &\succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned} \tag{3.9}$$

It immediately follows that Ψ_u is a convex set because the convex combination of any two positive semidefinite matrices is also positive semidefinite.

Lemma 1. *The function* $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ *is concave and nondecreasing in* $(\mathbf{Q}_1, \mathbf{Q}_2)$ *for any* $\Psi \in \Psi_u$.

Proof. First, we expand the terms in $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$:

$$\begin{aligned} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) &= \mathsf{I}\left(\underline{X}_{1G}; \underline{Y}_{1G}, \underline{S}_{1G}\right) + \mathsf{I}\left(\underline{X}_{2G}; \underline{Y}_{2G}, \underline{S}_{2G}\right) \\ &= \mathsf{h}\left(\underline{Y}_{1G}, \underline{S}_{1G}\right) - \mathsf{h}\left(\underline{Y}_{1G}, \underline{S}_{1G} | \underline{X}_{1G}\right) \\ &\quad + \mathsf{h}\left(\underline{Y}_{2G}, \underline{S}_{2G}\right) - \mathsf{h}\left(\underline{Y}_{2G}, \underline{S}_{2G} | \underline{X}_{2G}\right) \\ &= \mathsf{h}\left(\underline{S}_{1G}\right) + \mathsf{h}\left(\underline{Y}_{1G} | \underline{S}_{1G}\right) - \mathsf{h}\left(\underline{S}_{1G} | \underline{X}_{1G}\right) - \mathsf{h}\left(\underline{Y}_{1G} | \underline{S}_{1G}, \underline{X}_{1G}\right) \\ &\quad + \mathsf{h}\left(\underline{S}_{2G}\right) + \mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}\right) - \mathsf{h}\left(\underline{S}_{2G} | \underline{X}_{2G}\right) - \mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}, \underline{X}_{2G}\right). \end{aligned}$$

The terms $\mathsf{h}\left(\underline{S}_{1G} | \underline{X}_{1G}\right)$ and $\mathsf{h}\left(\underline{S}_{2G} | \underline{X}_{2G}\right)$ do not depend on $\mathbf{Q}_1$ and $\mathbf{Q}_2$. From Lemma 12 in Appendix A, it immediately follows that the terms $\mathsf{h}\left(\underline{Y}_{1G} | \underline{S}_{1G}\right)$ and $\mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}\right)$ are concave and nondecreasing in $(\mathbf{Q}_1, \mathbf{Q}_2)$. The remaining terms contribute

$$\mathsf{h}\left(\underline{S}_{1G}\right) - \mathsf{h}\left(\underline{Y}_{1G} | \underline{S}_{1G}, \underline{X}_{1G}\right) + \mathsf{h}\left(\underline{S}_{2G}\right) - \mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}, \underline{X}_{2G}\right).$$

We show that $\mathsf{h}\left(\underline{S}_{1G}\right) - \mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}, \underline{X}_{2G}\right)$ is a concave and nondecreasing function in $(\mathbf{Q}_1, \mathbf{Q}_2)$.

$$\mathsf{h}\left(\underline{S}_{1G}\right) - \mathsf{h}\left(\underline{Y}_{2G} | \underline{S}_{2G}, \underline{X}_{2G}\right) = \mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G} + \underline{W}_1\right) - \mathsf{h}\left(\mathbf{H}_{21}\underline{X}_{1G} + \underline{Z}_2 | \underline{W}_2\right)$$

Recall from Section 3.2 that $\tilde{\mathbf{H}}_{21}$ is related to $\mathbf{H}_{21}$ as

$$\mathbf{H}_{21} = \begin{bmatrix} \tilde{\mathbf{H}}_{21} \\ \mathbf{0} \end{bmatrix}.$$

Hence, by appropriately dividing the vector $\underline{Z}_2$ into $\underline{Z}_{21}$ and $\underline{Z}_{22}$, we see that

$$\mathbf{H}_{21}\underline{X}_{1G} + \underline{Z}_2 = \begin{bmatrix} \tilde{\mathbf{H}}_{21}\underline{X}_{1G} + \underline{Z}_{21} \\ \underline{Z}_{22} \end{bmatrix}.$$

Therefore, we obtain that

$$
\begin{aligned}
&\mathsf{h}\left(\underline{S}_{1G}\right)-\mathsf{h}\left(\underline{Y}_{2G}|\underline{S}_{2G},\underline{X}_{2G}\right)\\
&\quad=\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{W}_1\right)-\mathsf{h}\left(\mathbf{H}_{21}\underline{X}_{1G}+\underline{Z}_2|\underline{W}_2\right)\\
&\quad=\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{W}_1\right)-\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{Z}_{21},\underline{Z}_{22}|\underline{W}_2\right)\\
&\quad=\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{W}_1\right)-\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{Z}_{21}|\underline{Z}_{22},\underline{W}_2\right)-\mathsf{h}\left(\underline{Z}_{22}\right).
\end{aligned}
$$

Obviously, the term $\mathsf{h}\left(\underline{Z}_{22}\right)$ is independent of $\mathbf{Q}_1$. From Lemma 14 in Appendix A, it follows that the remaining expression

$$
\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{W}_1\right)-\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_{1G}+\underline{Z}_{21}|\underline{Z}_{22},\underline{W}_2\right)
$$

is concave in $\mathbf{Q}_1$ if

$$
\mathrm{Cov}\left(\underline{W}_1\right)\preceq\mathrm{Cov}\left(\underline{Z}_{21}|\underline{Z}_{22},\underline{W}_2\right).
$$

From Lemma 9 in Appendix A, it follows that the above condition is equivalent to

$$
\begin{bmatrix}\mathbf{\Sigma}_{\underline{W}_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\preceq\mathrm{Cov}\left(\begin{bmatrix}\underline{Z}_{21}\\ \underline{Z}_{22}\\ \underline{W}_2\end{bmatrix}\right)=\mathrm{Cov}\left(\begin{bmatrix}\underline{Z}_{2}\\ \underline{W}_2\end{bmatrix}\right).
$$

From the definition (3.9) of the usefulness set, we observe that the above condition is satisfied for every $\Psi\in\Psi_u$. Similarly, we can show that $\mathsf{h}\left(\underline{S}_{2G}\right)-\mathsf{h}\left(\underline{Y}_{1G}|\underline{S}_{1G},\underline{X}_{1G}\right)$ is also concave and nondecreasing in $(\mathbf{Q}_1,\mathbf{Q}_2)$. □

3.6.2 Convexity Property

Lemma 2. *For any fixed* $(\mathbf{Q}_1,\mathbf{Q}_2)$*, the function* $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ *is convex in* $\Psi=\left(\mathbf{\Sigma}_{\underline{W}_1},\mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1},\mathbf{\Sigma}_{\underline{W}_2},\mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\right)$.

Proof. First, observe that

$$
\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)=\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)+\mathsf{I}\left(\underline{X}_{2G};\underline{Y}_{2G},\underline{S}_{2G}\right).
$$

We prove that $\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)$ is convex in Ψ. The convexity of $\mathsf{I}\left(\underline{X}_{2G};\underline{Y}_{2G},\underline{S}_{2G}\right)$ follows in a similar manner. Observe that

$$
\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)=\mathsf{h}\left(\underline{X}_{1G}\right)-\mathsf{h}\left(\underline{X}_{1G}|\underline{Y}_{1G},\underline{S}_{1G}\right).
$$

The first term $\mathsf{h}\left(\underline{X}_{1G}\right)$ is independent of Ψ. From Lemma 13 in Appendix A, it follows that the second term $\mathsf{h}\left(\underline{X}_{1G}|\underline{Y}_{1G},\underline{S}_{1G}\right)$ is concave in

$$
\mathrm{Cov}\left(\begin{bmatrix}\underline{X}_{1G}\\ \underline{Y}_{1G}\\ \underline{S}_{1G}\end{bmatrix}\right)=\begin{bmatrix}\times & \times & \times\\ \times & \times & \mathbf{\Sigma}_{\underline{Z}_1\underline{W}_1}\\ \times & \times+\mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1} & \times+\mathbf{\Sigma}_{\underline{W}_1}\end{bmatrix}
$$

where $\times$ denotes the terms that are independent of the genie parameters. From this, we conclude that $\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)$ is convex in $\left(\mathbf{\Sigma}_{\underline{W}_1},\mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1},\mathbf{\Sigma}_{\underline{W}_2},\mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\right)$.

In the above proof, the expansion

$$\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)=\mathsf{h}\left(\underline{X}_{1G}\right)-\mathsf{h}\left(\underline{X}_{1G}|\underline{Y}_{1G},\underline{S}_{1G}\right)$$

is not valid when $\mathbf{Q}_1$ is rank deficient because $\mathsf{h}\left(\underline{X}_{1G}\right)=\log\det\left(\pi e\mathbf{Q}_1\right)=-\infty$. The problem can be circumvented using the following trick. Let $r<N_t$ be the rank of $\mathbf{Q}_1$. Using the eigenvalue decomposition of $\mathbf{Q}_1$, we can compute $\mathbf{C}\in\mathbb{C}^{N_t\times r}$ such that $\mathbf{C}\mathbf{C}^\dagger=\mathbf{Q}_1$. We can now define a new random vector $\underline{\tilde{X}}_{1G}\sim\mathcal{CN}(0,\mathbf{I}_r)$ and have $\underline{X}_{1G}=\mathbf{C}\underline{\tilde{X}}_{1G}$ so that the distribution of $\underline{X}_{1G}$ remains unchanged. Clearly, the rank of $\mathbf{C}$ is equal to r and hence $\underline{\tilde{X}}_{1G}$ can be exactly reconstructed for any given $\underline{X}_{1G}$. Since the covariance matrix of $\tilde{\mathbf{x}}_{1G}$ has full rank, we can expand $\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)$ as

$$\mathsf{I}\left(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)=\mathsf{I}\left(\underline{\tilde{X}}_{1G};\underline{Y}_{1G},\underline{S}_{1G}\right)=\mathsf{h}\left(\underline{\tilde{X}}_{1G}\right)-\mathsf{h}\left(\underline{\tilde{X}}_{1G}|\underline{Y}_{1G},\underline{S}_{1G}\right).$$

The rest of the proof remains unchanged. □

3.6.3 Sum-Rate Upper Bound

Following the argument in Section 3.5, and using Lemmas 1 and 2, we obtain the following theorem.

Theorem 1. *The best sum-rate achievable by treating interference as noise is bounded above and below by*

$$f(\mathbf{Q}_1^*,\mathbf{Q}_2^*)\leq\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}}f(\mathbf{Q}_1,\mathbf{Q}_2)\leq\bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)$$

where $(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)$ is a solution to the following convex maxmin optimization problem

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}}\min_{\Psi\in\Psi_u}\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).$$

The utility of the above theorem is that we can use the standard convex optimization algorithms to efficiently solve for $(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)$, and thus obtain computable lower and upper bounds to the best achievable sum-rate.

3.7 Sum Capacity Upper Bound

In the previous section, we saw a technique to obtain computable lower and upper bounds on the best achievable sum-rate with treating interference as noise. Suppose the bounds meet and we have exactly determined the best achievable sum-rate and the corresponding optimal covariance matrices. Even then, we only have an achievable sum-rate and we cannot eliminate the possibility that there may exist other simple achievable schemes that could potentially outperform the best achievable sum-rate with treating interference as noise. Interestingly, this question can be resolved because the upper bound in Theorem 1 can be shown to be an upper bound to the sum capacity as well.

Theorem 2. *The sum capacity* ($\mathcal{C}_{\text{sum}}$) *of the two-user MIMO Gaussian interference channel satisfies*

$$f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) \leq \mathcal{C}_{\text{sum}} \leq \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$$

where $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ *is the solution to the convex maxmin optimization problem*

$$\max_{(\mathbf{Q}_1, \mathbf{Q}_2) \in \mathcal{Q}} \min_{\Psi \in \Psi_u} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi).$$

Proof. The lower bound is obvious since $f(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ is defined as the achievable sum-rate when the transmitters use Gaussian inputs with covariance matrices $\mathbf{Q}_1^*$ and $\mathbf{Q}_2^*$ and the receivers treat interference as noise. Note that the covariance matrices $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ satisfy the transmit power constraints. We now prove upper bound. Using the standard converse arguments involving Fano's inequality, we obtain that any achievable rate tuple (R_1, R_2) must satisfy

$$\begin{aligned} R_1 &\leq \frac{1}{n} \mathsf{I}\,(\underline{X}_1^n; \underline{Y}_1^n) + \epsilon_n \\ R_2 &\leq \frac{1}{n} \mathsf{I}\,(\underline{X}_2^n; \underline{Y}_2^n) + \epsilon_n \end{aligned}$$

for some $\epsilon_n \to 0$ as $n \to \infty$. For any genie $\Psi \in \Psi_u$, we can upper-bound the mutual information terms with the corresponding terms in the genie-aided channel to obtain

$$\begin{aligned} R_1 &\leq \frac{1}{n} \mathsf{I}\,(\underline{X}_1^n; \underline{Y}_1^n, \underline{S}_1^n) + \epsilon_n \\ R_2 &\leq \frac{1}{n} \mathsf{I}\,(\underline{X}_2^n; \underline{Y}_2^n, \underline{S}_2^n) + \epsilon_n. \end{aligned}$$

Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ denote the average covariance matrices at the transmitters 1 and 2, respectively:

$$\begin{aligned} \mathbf{Q}_1 &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} \underline{X}_{1i}\underline{X}_{1i}^{\dagger}\right] \\ \mathbf{Q}_2 &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} \underline{X}_{2i}\underline{X}_{2i}^{\dagger}\right]. \end{aligned} \tag{3.10}$$

Consider the problem of maximizing $\mathsf{I}\,(\underline{X}_1^n; \underline{Y}_1^n, \underline{S}_1^n) + \mathsf{I}\,(\underline{X}_2^n; \underline{Y}_2^n, \underline{S}_1^n)$ over all product input distributions $p(\underline{X}_1^n)p(\underline{X}_2^n)$ satisfying the covariance constraints (3.10). We show that if the genie is useful, i.e., $\Psi \in \Psi_u$, then i.i.d. Gaussian inputs are optimal; i.e.,

$$\mathsf{I}\left(\underline{X}_1^n;\underline{Y}_1^n,\underline{S}_1^n\right)+\mathsf{I}\left(\underline{X}_2^n;\underline{Y}_2^n,\underline{S}_1^n\right)\le n\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).$$

The proof is similar to the proof of concavity of $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ in Lemma 1. Let $\underline{X}_{1G}$ and $\underline{X}_{2G}$ denote independent Gaussian random vectors with covariance matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$, respectively, and $\underline{Y}_{1G},\underline{Y}_{2G},\underline{S}_{1G},\underline{S}_{2G}$ denote the corresponding Gaussian random vectors. We have

$$\begin{aligned}
&\mathsf{I}\left(\underline{X}_1^n;\underline{Y}_1^n\right)+\mathsf{I}\left(\underline{X}_2^n;\underline{Y}_2^n\right)\\
&\quad\le \mathsf{I}\left(\underline{X}_1^n;\underline{Y}_1^n,\underline{S}_1^n\right)+\mathsf{I}\left(\underline{X}_2^n;\underline{Y}_2^n,\underline{S}_2^n\right)\\
&\quad= \mathsf{h}\left(\underline{Y}_1^n,\underline{S}_1^n\right)-\mathsf{h}\left(\underline{Y}_1^n,\underline{S}_1^n|\underline{X}_1^n\right)+\mathsf{h}\left(\underline{Y}_2^n,\underline{S}_2^n\right)-\mathsf{h}\left(\underline{Y}_2^n,\underline{S}_2^n|\underline{X}_2^n\right)\\
&\quad= \mathsf{h}\left(\underline{S}_1^n\right)+\mathsf{h}\left(\underline{Y}_1^n|\underline{S}_1^n\right)-\mathsf{h}\left(\underline{S}_1^n|\underline{X}_1^n\right)-\mathsf{h}\left(\underline{Y}_1^n|\underline{S}_1^n,\underline{X}_1^n\right)\\
&\qquad\quad+\mathsf{h}\left(\underline{S}_2^n\right)+\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n\right)-\mathsf{h}\left(\underline{S}_2^n|\underline{X}_2^n\right)-\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n,\underline{X}_2^n\right).
\end{aligned}$$

From Lemma 16 in Appendix A, it follows that the terms $\mathsf{h}\left(\underline{Y}_1^n|\underline{S}_1^n\right)$ and $\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n\right)$ are maximized by i.i.d. Gaussian inputs. The remaining terms contribute

$$\mathsf{h}\left(\underline{S}_1^n\right)-\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n,\underline{X}_2^n\right)+\mathsf{h}\left(\underline{S}_2^n\right)-\mathsf{h}\left(\underline{Y}_1^n|\underline{S}_1^n,\underline{X}_1^n\right).$$

We now prove that $\mathsf{h}\left(\underline{S}_1^n\right)-\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n,\underline{X}_2^n\right)$ is maximized by i.i.d. Gaussian inputs. As explained in the proof of Lemma 1, we obtain that

$$\begin{aligned}
&\mathsf{h}\left(\underline{S}_1^n\right)-\mathsf{h}\left(\underline{Y}_2^n|\underline{S}_2^n,\underline{X}_2^n\right)\\
&\quad=\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_1^n+\underline{W}_1^n\right)-\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_1^n+\underline{Z}_{21}^n|\underline{Z}_{22}^n,\underline{W}_2^n\right)-\mathsf{h}\left(\underline{Z}_{22}^n\right).
\end{aligned}$$

Obviously, the term $\mathsf{h}\left(\underline{Z}_{22}^n\right)$ is independent of $p(\underline{X}_1^n)p(\underline{X}_2^n)$. From Lemma 17 in Appendix A, it follows that the remaining expression

$$\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_1^n+\underline{W}_1^n\right)-\mathsf{h}\left(\tilde{\mathbf{H}}_{21}\underline{X}_1^n+\underline{Z}_{21}^n|\underline{Z}_{22}^n,\underline{W}_2^n\right)$$

is maximized by i.i.d. Gaussian inputs if

$$\mathbf{\Sigma}_{\underline{W}_1}\preceq \mathrm{Cov}\left(\underline{Z}_{21}|\underline{Z}_{22},\underline{W}_2\right)$$

From Lemma 9 in Appendix A, it follows that the above condition is equivalent to

$$\begin{bmatrix}\mathbf{\Sigma}_{\underline{W}_1} & \\ & \mathbf{0}\end{bmatrix}\preceq \mathrm{Cov}\left(\begin{bmatrix}\underline{Z}_{21}\\ \underline{Z}_{22}\\ \underline{W}_2\end{bmatrix}\right)=\mathrm{Cov}\left(\begin{bmatrix}\underline{Z}_2\\ \underline{W}_2\end{bmatrix}\right)$$

which is satisfied for every $\Psi\in\Psi_u$. Similarly, we can show that $\mathsf{h}\left(\underline{S}_2^n\right)-\mathsf{h}\left(\underline{Y}_1^n|\underline{S}_1^n,\underline{X}_1^n\right)$ is also maximized by i.i.d. Gaussian inputs. Thus, we proved that for any $\Psi \in \Psi_u$,

$$\begin{aligned}\mathsf{I}(\underline{X}_1^n;\underline{Y}_1^n) + \mathsf{I}(\underline{X}_2^n;\underline{Y}_2^n) &\le \mathsf{I}(\underline{X}_1^n;\underline{Y}_1^n,\underline{S}_1^n) + \mathsf{I}(\underline{X}_2^n;\underline{Y}_2^n,\underline{S}_1^n)\\ &\le n\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi).\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\frac{1}{n}\mathsf{I}(\underline{X}_1^n;\underline{Y}_1^n) + \frac{1}{n}\mathsf{I}(\underline{X}_2^n;\underline{Y}_2^n) &\le \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)\\ &\le \min_{\Psi\in\Psi_u} \max_{(\mathbf{Q}_1,\mathbf{Q}_2\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)\\ &= n\bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*).\end{aligned}$$

Thus we proved that any achievable rate tuple (R_1, R_2) must satisfy

$$R_1 + R_2 \le \bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*) + \epsilon_n.$$

The proof is complete by letting $n \to \infty$. □

3.8 Smart Genie: Zero Gap

In the previous section, we derived computable lower and upper bounds to the sum capacity. A natural follow-up step is to check if the bounds ever meet. We start by obtaining a necessary and sufficient condition on $(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)$ for the gap to be zero. We say that a genie Ψ is $(\mathbf{Q}_1,\mathbf{Q}_2)$-smart if

$$\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi) = f(\mathbf{Q}_1,\mathbf{Q}_2).$$

The genie just gives side-information to the receivers, but it is smart enough not to any leak any additional information about the respective transmitted signals that the original received signals could not provide. The total amount of additional information that a genie leaks is equal to

$$\begin{aligned}&\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi) - f(\mathbf{Q}_1,\mathbf{Q}_2)\\ &\quad= \mathsf{I}(\underline{X}_{1G};\underline{Y}_{1G},\underline{S}_{1G}) + \mathsf{I}(\underline{X}_{2G};\underline{Y}_{2G},\underline{S}_{2G}) - \mathsf{I}(\underline{X}_{1G};\underline{Y}_{1G}) - \mathsf{I}(\underline{X}_{2G};\underline{Y}_{2G})\\ &\quad= \mathsf{I}(\underline{X}_{1G};\underline{S}_{1G}|\underline{Y}_{1G}) + \mathsf{I}(\underline{X}_{2G};\underline{S}_{2G}|\underline{Y}_{2G}).\end{aligned}$$

Since the conditional mutual information is always nonnegative, the genie is smart if and only if $\mathsf{I}(\underline{X}_{1G};\underline{S}_{1G}|\underline{Y}_{1G}) = \mathsf{I}(\underline{X}_{2G};\underline{S}_{2G}|\underline{Y}_{2G}) = 0$. The conditional mutual information $\mathsf{I}(\underline{X}_{1G};\underline{S}_{1G}|\underline{Y}_{1G})$ is equal to zero if and only if $\underline{X}_{1G} - \underline{Y}_{1G} - \underline{S}_{1G}$ forms a Markov chain. Since all the random variables are jointly Gaussian, $\underline{X}_{1G} - \underline{Y}_{1G} - \underline{S}_{1G}$ forms a Markov chain if and only if the MMSE estimate of $\underline{S}_{1G}$ given $(\underline{X}_{1G},\underline{Y}_{1G})$ is the same as the MMSE estimate of $\underline{S}_{1G}$ given $\underline{Y}_{1G}$, i.e.,

$$\mathbb{E}[\underline{S}_{1G}|\underline{Y}_{1G},\underline{X}_{1G}] = \mathbb{E}[\underline{S}_{1G}|\underline{Y}_{1G}]. \tag{3.11}$$

Let $\mathbf{T}_1\underline{Y}_{1G}$ be the MMSE estimate of $\underline{S}_{1G}$ given $\underline{Y}_{1G}$. Using the orthogonality principle, summarized in Section A.1, we know that the MMSE estimation error $\underline{E}_1 = \underline{S}_{1G} - \mathbf{T}_1\underline{Y}_{1G}$ is independent of the observation $\underline{Y}_{1G}$. Since (3.11) implies that $\mathbf{T}_1\underline{Y}_{1G}$ is also the MMSE estimate of $\underline{S}_{1G}$ given $\underline{Y}_{1G}$ and $\underline{X}_{1G}$, we obtan that $\underline{E}_1$ is independent of $\underline{X}_{1G}$ also. Therefore, we have that (3.11) is true if and only if there exists a matrix $\mathbf{T}_1$ such that

$$\begin{aligned} \underline{S}_{1G} &= \mathbf{T}_1\underline{Y}_{1G} + \underline{E}_1 \\ \Leftrightarrow \tilde{\mathbf{H}}_{21}\underline{X}_{1G} + \underline{W}_1 &= \mathbf{T}_1\left(\mathbf{H}_{11}\underline{X}_{1G} + \mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1\right) + \underline{E}_1 \\ \Leftrightarrow \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1\mathbf{H}_{11}\right)\underline{X}_{1G} + \underline{W}_1 &= \mathbf{T}_1\left(\mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1\right) + \underline{E}_1 \end{aligned}$$

with $\underline{E}_1$ being independent of $\underline{Y}_{1G}$ and $\underline{X}_{1G}$. Since $\underline{X}_{1G}$ is independent of all the other random vectors involved, the random vector $\left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1\mathbf{H}_{11}\right)\underline{X}_{1G}$ must be equal to zero almost surely, which is equivalent to saying

$$\left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1\mathbf{H}_{11}\right)\mathbf{Q}_1 = 0.$$

The remaining expression is equivalent to saying that $\mathbf{T}_1\left(\mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1\right)$ is the MMSE estimate of $\underline{W}_1$ given $\mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1$ since $\underline{E}_1$ is independent of $\underline{Y}_{1G} - \mathbf{H}_{11}\underline{X}_{1G} = \mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1$. Therefore, we obtain the following expression for $\mathbf{T}_1$:

$$\mathbf{T}_1 = \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}\left(\mathbf{I} + \mathbf{H}_{12}\mathbf{Q}_2\mathbf{H}_{21}^{\dagger}\right)^{-1}.$$

Thus, we can conclude that $\underline{X}_{1G} - \underline{Y}_{1G} - \underline{S}_{1G}$ forms a Markov chain if and only if the following condition is satisfied:

$$\left(\tilde{\mathbf{H}}_{21} - \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}\left(\mathbf{H}_{12}\mathbf{Q}_2\mathbf{H}_{12}^{\dagger} + \mathbf{I}\right)^{-1}\mathbf{H}_{11}\right)\mathbf{Q}_1 = \mathbf{0}.$$

We can derive a similar necessary and sufficient condition for $\mathsf{I}\left(\underline{X}_{2G};\underline{S}_{2G}|\underline{Y}_{2G}\right)$ to be equal to zero, and hence we obtain the following lemma.

Lemma 3. *The genie Ψ is $(\mathbf{Q}_1, \mathbf{Q}_2)$-smart, i.e., $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) = f(\mathbf{Q}_1, \mathbf{Q}_2)$, if and only if the following conditions are satisfied:*

$$\begin{aligned} \left(\tilde{\mathbf{H}}_{21} - \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}\left(\mathbf{H}_{12}\mathbf{Q}_2\mathbf{H}_{12}^{\dagger} + \mathbf{I}\right)^{-1}\mathbf{H}_{11}\right)\mathbf{Q}_1 &= \mathbf{0} \\ \left(\tilde{\mathbf{H}}_{12} - \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\left(\mathbf{H}_{21}\mathbf{Q}_1\mathbf{H}_{21}^{\dagger} + \mathbf{I}\right)^{-1}\mathbf{H}_{22}\right)\mathbf{Q}_2 &= \mathbf{0}. \end{aligned} \tag{3.12}$$

3.9 Low Interference Regime

We say that a two-user MIMO Gaussian interference channel is in the low interference regime if the sum capacity is achieved by using Gaussian inputs and treating interference as noise. Suppose the upper and lower bounds defined in Theorem 2 meet; then the channel (3.1) is in the low interference regime. In this section, we derive necessary and sufficient conditions for the bounds in Theorem 2 to meet. Recall that the bounds meet if and only if

$$f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) = \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) \tag{3.13}$$

where $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ is an optimal solution to the following convex maxmin optimization problem:

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi). \tag{3.14}$$

We start with the following claim which exploits the concave-convex property of $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ to simplify the above two conditions.

Claim 2. *The following two statements are equivalent:*

1. *A maxmin solution* $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ *to* (3.14) *satisfies* (3.13).
2. *There exist* $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ *and* $\Psi^* \in \Psi_u$ *satisfying*

$$f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) = \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*).$$

Before proving the claim, we state its relevance. The claim says that it is sufficient to consider only one instance of the genie (Ψ^*) to obtain the best upper bound instead of minimizing the upper bound function over all useful genies. It is easy to check that the second statement implies the first statement without invoking any special structural properties of $\bar{f}(\mathbf{Q}_1\,\mathbf{Q}_2, \Psi)$. Proving that the second statement is necessary requires the concave-convex property of $\bar{f}(\mathbf{Q}_1\,\mathbf{Q}_2, \Psi)$.

Proof. First, we prove that the second statement implies the first statement. Let $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ satisfy the conditions in the second statement. Then, we see that the conditions in the first statement are also satisfied:

$$\begin{aligned} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) &\leq \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) \\ &\overset{(a)}{\leq} \min_{\Psi\in\Psi_u} \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) \\ &\leq \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*) \\ &\overset{(b)}{=} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) \\ &\overset{(c)}{=} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) \end{aligned} \tag{3.15}$$

where step (a) follows from the standard minmax inequality, and steps (b) and (c) follow from the assumptions in second statement. Now, we prove that the first statement implies the second statement. Let $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ be a maxmin solution to (3.14) satisfying (3.13). Since the objective function $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$

is concave in $(\mathbf{Q}_1, \mathbf{Q}_2)$ and convex in Ψ, and the sets Ψ_u and $\mathcal{Q}$ are convex, we can apply Fan's minmax theorem [40] to conclude that step (a) in (3.15) holds true with equality. By appropriately modifying Ψ^*, we can assume that $(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*)$ is also a minmax and hence a saddle point solution; i.e.,

$$\min_{\Psi \in \Psi_u} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi) = \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \max_{(\mathbf{Q}_1, \mathbf{Q}_2) \in \mathcal{Q}} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*).$$

However, the condition (3.13) and the fact that $\bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi)$ is an upper bound to $f(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ immediately implies that

$$\min_{\Psi \in \Psi_u} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi) = \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*).$$

Hence the first equality in the saddle point equation can be replaced with (3.13) as stated in the claim. $\square$

We have already derived the necessary and sufficient conditions for (3.13) to be true in Lemma 3. We now derive the KKT conditions which are both necessary and sufficient conditions for $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ to be a global optimal solution to

$$\max_{(\mathbf{Q}_1, \mathbf{Q}_2) \in \mathcal{Q}} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*). \tag{3.16}$$

Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be the dual variables associated with the constraints $\mathsf{Tr}\left(\mathbf{Q}_1\right) \leq P_1$ and $\mathsf{Tr}\left(\mathbf{Q}_2\right) \leq P_2$. Let $\mathbf{M}_1 \succeq 0$ and $\mathbf{M}_2 \succeq 0$ be the dual variables associated with the constraints $\mathbf{Q}_1 \succeq 0$ and $\mathbf{Q}_2 \succeq 0$. The Lagrangian associated with the optimization problem (3.16) is given by

$$\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*) + \sum_{i=1}^{2} \mathsf{Tr}\left(\mathbf{M}_i \mathbf{Q}_i\right) - \lambda_i \left(\mathsf{Tr}\left(\mathbf{Q}_i\right) - P_i\right).$$

The corresponding KKT conditions are as given in (3.19). Thus, we obtain the following Theorem.

Theorem 3. *Suppose there exist transmit covariance matrices $\mathbf{Q}_1^* \succeq \mathbf{0}, \mathbf{Q}_2^* \succeq 0$, genie parameters $\Psi^* = \left(\mathbf{\Sigma}_{\underline{W}_1}, \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}, \mathbf{\Sigma}_{\underline{W}_2}, \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\right)$, and dual variables $\lambda_1 \geq 0, \lambda \geq 0, \mathbf{M}_1 \succeq 0, \mathbf{M}_2 \succeq \mathbf{0}$ satisfying the following conditions:*

1. *Transmit power constraints:* $\mathsf{Tr}\left(\mathbf{Q}_1\right) \leq P_1$ *and* $\mathsf{Tr}\left(\mathbf{Q}_2\right) \leq P_2$

2. *Useful genie conditions:*

$$\begin{aligned}
\begin{bmatrix} \mathbf{I} & \mathbf{\Sigma}_{\underline{Z}_1\underline{W}_1} \\ \mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1} & \mathbf{\Sigma}_{\underline{W}_1} \end{bmatrix} &\succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\begin{bmatrix} \mathbf{I} & \mathbf{\Sigma}_{\underline{Z}_2\underline{W}_2} \\ \mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2} & \mathbf{\Sigma}_{\underline{W}_2} \end{bmatrix} &\succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{3.17}$$

3. Smart genie conditions:

$$\begin{aligned}\left(\tilde{\mathbf{H}}_{21}-\mathbf{\Sigma}_{\underline{W}_1\underline{Z}_1}\left(\mathbf{H}_{12}\mathbf{Q}_2^*\mathbf{H}_{12}^\dagger+\mathbf{I}\right)^{-1}\mathbf{H}_{11}\right)\mathbf{Q}_1^*=\mathbf{0}\\ \left(\tilde{\mathbf{H}}_{12}-\mathbf{\Sigma}_{\underline{W}_2\underline{Z}_2}\left(\mathbf{H}_{21}\mathbf{Q}_1^*\mathbf{H}_{21}^\dagger+\mathbf{I}\right)^{-1}\mathbf{H}_{22}\right)\mathbf{Q}_2^*=\mathbf{0}\end{aligned} \tag{3.18}$$

4. KKT conditions:

$$\begin{aligned}\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)&=\lambda_1\mathbf{I}-\mathbf{M}_1\\ \nabla_{\mathbf{Q}_2}\bar{f}(\mathbf{Q}_1^*,\mathbf{Q}_2^*,\Psi^*)&=\lambda_2\mathbf{I}-\mathbf{M}_2\\ \lambda_1(\mathsf{Tr}\,(\mathbf{Q}_1^*)-P_1)&=0\\ \lambda_2(\mathsf{Tr}\,(\mathbf{Q}_2^*)-P_2)&=0\\ \mathsf{Tr}\,(\mathbf{M}_1\mathbf{Q}_1^*)&=0\\ \mathsf{Tr}\,(\mathbf{M}_2\mathbf{Q}_2^*)&=0.\end{aligned} \tag{3.19}$$

Then, the sum capacity of the two-user MIMO Gaussian interference channel (3.1) *is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}}=f(\mathbf{Q}_1^*,\mathbf{Q}_2^*).$$

Conversely, if there exist no such parameters satisfying the stated constraints, then the lower and upper bounds in Theorem 2 do not coincide.

The above theorem provides sufficient conditions for the two-user Gaussian interference channel (3.1). The problem now is to determine if there exists an algorithm to verify the feasibility of these conditions. Observe that the conditions in Theorem 3 are nothing but the necessary and sufficient conditions for the bounds in Theorem 2 to coincide. Therefore, we can use the standard convex optimization algorithms to solve the maxmin optimization problem in Theorem 2 efficiently, and thus verify the feasibility of conditions in Theorem 3. In the sections to follow, we explore the possibility of verifying the feasibility of the conditions of Theorem 3 analytically. We provide two corollaries with simpler conditions that are sufficient but may not be neccesary. In some special cases, such as symmetric MISO and SIMO interference channels, we actually simplify the conditions of Theorem 3 into a closed-form equation that depends only on the channel matrices and the power constraints. To achieve all these objectives, we first need to simplify the KKT conditions (3.19).

3.9.1 Simplified KKT Conditions

When the smart genie conditions (3.18) are satisfied, the gradient expressions in (3.19) can be greatly simplified. We first explain the intuition before proceeding to present the simplified expressions. Recall that $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)$ is an upper bound to $f(\mathbf{Q}_1, \mathbf{Q}_2)$. Let $g(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ denote the gap

$$g(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) = \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi) - f(\mathbf{Q}_1, \mathbf{Q}_2) \geq 0.$$

Suppose the genie Ψ^* is $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$-smart, i.e., $g(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = 0$. Then, we see that $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ is an optimal solution to

$$\min_{\mathbf{Q}_i : \mathbf{Q}_i \succeq 0} g(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*).$$

Therefore, $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ must satisfy the corresponding neccesary KKT conditions. Let $\mathbf{N}_1 \succeq \mathbf{0}$ and $\mathbf{N}_2 \succeq \mathbf{0}$ denote the dual variables corresponding to the constraints $\mathbf{Q}_1 \succeq \mathbf{0}$ and $\mathbf{Q}_2 \succeq \mathbf{0}$, respectively. The Lagrangian associated with the above minimization problem is given by

$$g(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*) - \sum_{i=1}^{2} \mathsf{Tr}\left(\mathbf{N}_i \mathbf{Q}_i\right).$$

Therefore, there must exist some dual variables $\mathbf{N}_1 \succeq \mathbf{0}$ and $\mathbf{N}_2 \succeq \mathbf{0}$ satisfying the KKT conditions

$$\begin{aligned}
\nabla_{\mathbf{Q}_1} g(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) &= \mathbf{N}_1 \\
\nabla_{\mathbf{Q}_2} g(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) &= \mathbf{N}_2 \\
\mathsf{Tr}\left(\mathbf{N}_1 \mathbf{Q}_1^*\right) &= 0 \\
\mathsf{Tr}\left(\mathbf{N}_2 \mathbf{Q}_2^*\right) &= 0.
\end{aligned}$$

Thus, we see that the gradients in (3.19) can be simplified as

$$\begin{aligned}
\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) &= \nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) + \mathbf{N}_1 \\
\nabla_{\mathbf{Q}_2} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) &= \nabla_{\mathbf{Q}_2} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) + \mathbf{N}_2
\end{aligned}$$

for some $\mathbf{N}_1 \succeq \mathbf{0}$ and $\mathbf{N}_2 \succeq \mathbf{0}$ satisfying $\mathsf{Tr}\left(\mathbf{N}_1 \mathbf{Q}_1^*\right) = \mathsf{Tr}\left(\mathbf{N}_2 \mathbf{Q}_2^*\right) = 0$. In the following Lemma, we obtain the exact expressions for $\mathbf{N}_1$ and $\mathbf{N}_2$ in terms of the channel matrices, genie parameters, and the transmit covariance matrices.

Lemma 4. *Suppose the smart genie conditions* (3.18) *are satisfied; i.e.,*

$$\begin{aligned}
\left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1 \mathbf{H}_{11}\right) \mathbf{Q}_1^* &= \mathbf{0} \\
\left(\tilde{\mathbf{H}}_{12} - \mathbf{T}_2 \mathbf{H}_{22}\right) \mathbf{Q}_2^* &= \mathbf{0}
\end{aligned}$$

where the matrices $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are defined as*

$$\begin{aligned}
\mathbf{T}_1 &= \mathbf{\Sigma}_{\underline{W}_1 \underline{Z}_1} \left(\mathbf{H}_{12} \mathbf{Q}_2^* \mathbf{H}_{12}^\dagger + \mathbf{I}\right)^{-1} \\
\mathbf{T}_2 &= \mathbf{\Sigma}_{\underline{W}_2 \underline{Z}_2} \left(\mathbf{H}_{21} \mathbf{Q}_1^* \mathbf{H}_{21}^\dagger + \mathbf{I}\right)^{-1}.
\end{aligned}$$

Then, we have

$$\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) + \mathbf{N}_1$$
$$\nabla_{\mathbf{Q}_2} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \nabla_{\mathbf{Q}_2} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) + \mathbf{N}_2$$

where the matrices $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *are given by*

$$\mathbf{N}_1 = \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1 \mathbf{H}_{11}\right)^\dagger \mathbf{\Sigma}^{-1}_{\underline{W}_1 | \underline{Y}_{1G}, \underline{X}_{1G}} \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1 \mathbf{H}_{11}\right)$$
$$\mathbf{N}_2 = \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_2 \mathbf{H}_{22}\right)^\dagger \mathbf{\Sigma}^{-1}_{\underline{W}_2 | \underline{Y}_{2G}, \underline{X}_{2G}} \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_2 \mathbf{H}_{22}\right).$$

Furthermore, $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *satisfy* $\mathsf{Tr}\left(\mathbf{N}_1 \mathbf{Q}_1^*\right) = \mathsf{Tr}\left(\mathbf{N}_2 \mathbf{Q}_2^*\right) = 0$.

Proof. Note that the genie-aided channel is also a MIMO interference channel with received signals $\underline{\bar{Y}}_1$ and $\underline{\bar{Y}}_2$ given by

$$\underline{\bar{Y}}_1 = \begin{bmatrix} \underline{Y}_1 \\ \underline{S}_1 \end{bmatrix} \text{ and } \underline{\bar{Y}}_2 = \begin{bmatrix} \underline{Y}_2 \\ \underline{S}_2 \end{bmatrix}$$

and the corresponding channel matrices given by

$$\bar{\mathbf{H}}_{11} = \begin{bmatrix} \mathbf{H}_{11} \\ \tilde{\mathbf{H}}_{21} \end{bmatrix}, \ \bar{\mathbf{H}}_{21} = \begin{bmatrix} \mathbf{H}_{21} \\ \mathbf{0} \end{bmatrix}$$
$$\bar{\mathbf{H}}_{22} = \begin{bmatrix} \mathbf{H}_{22} \\ \tilde{\mathbf{H}}_{12} \end{bmatrix}, \ \bar{\mathbf{H}}_{12} = \begin{bmatrix} \mathbf{H}_{12} \\ \mathbf{0} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi^*) &= \mathsf{I}\left(\underline{X}_{1G}; \underline{\bar{Y}}_{1G}\right) + \mathsf{I}\left(\underline{X}_{2G}; \underline{\bar{Y}}_{2G}\right) \\ &= \log \frac{\det \mathbf{\Sigma}_{\underline{\bar{Y}}_{1G}}}{\det \mathbf{\Sigma}_{\underline{\bar{Y}}_{1G} | \underline{X}_{1G}}} + \log \frac{\det \mathbf{\Sigma}_{\underline{\bar{Y}}_{2G}}}{\det \mathbf{\Sigma}_{\underline{\bar{Y}}_{2G} | \underline{X}_{2G}}}. \end{aligned}$$

Therefore, we obtain that

$$\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \bar{\mathbf{H}}_{11}^\dagger \mathbf{\Sigma}^{-1}_{\underline{\bar{Y}}_{1G}} \bar{\mathbf{H}}_{11} + \bar{\mathbf{H}}_{21}^\dagger \left(\mathbf{\Sigma}^{-1}_{\underline{\bar{Y}}_{2G}} - \mathbf{\Sigma}^{-1}_{\underline{\bar{Y}}_{2G} | \underline{X}_{2G}}\right) \bar{\mathbf{H}}_{21}.$$

Also, recall that

$$\nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) = \mathbf{H}_{11}^\dagger \mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}} \mathbf{H}_{11} + \mathbf{H}_{21}^\dagger \left(\mathbf{\Sigma}^{-1}_{\underline{Y}_{2G}} - \mathbf{\Sigma}^{-1}_{\underline{Y}_{2G} | \underline{X}_{2G}}\right) \mathbf{H}_{21}.$$

Since we are evaluating the gradients at $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$, the covariance matrices in the above expressions should also be evaluated at $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$. We implicitly assume this throughout the rest of the proof.

First, we explore the implications of the genie Ψ^* being $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$-smart. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be the matrices such that $\mathbf{T}_1 \underline{Y}_{1G}$ is the MMSE estimate of $\underline{S}_{1G}$

given $\underline{Y}_{1G}$, and $\mathbf{T}_2\underline{Y}_{2G}$ is the MMSE estimate of $\underline{S}_{2G}$ given $\underline{Y}_{2G}$. We have

$$\begin{aligned}
\mathbf{T}_1 &= \mathbf{\Sigma}_{\underline{S}_{1G}\underline{Y}_{1G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}} \\
\mathbf{T}_2 &= \mathbf{\Sigma}_{\underline{S}_{2G}\underline{Y}_{2G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{2G}} \\
\mathbf{\Sigma}_{\underline{S}_{1G}|\underline{Y}_{1G}} &= \mathbf{\Sigma}_{\underline{S}_{1G}} - \mathbf{\Sigma}_{\underline{S}_{1G}\underline{Y}_{1G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}}\mathbf{\Sigma}_{\underline{Y}_{1G}\underline{S}_{1G}} \\
\mathbf{\Sigma}_{\underline{S}_{2G}|\underline{Y}_{2G}} &= \mathbf{\Sigma}_{\underline{S}_{2G}} - \mathbf{\Sigma}_{\underline{S}_{2G}\underline{Y}_{2G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{2G}}\mathbf{\Sigma}_{\underline{Y}_{2G}\underline{S}_{2G}}.
\end{aligned}$$

Similarly, define

$$\begin{aligned}
\hat{\mathbf{T}}_1 &= \mathbf{\Sigma}_{\underline{S}_{1G}\underline{Y}_{1G}|\underline{X}_{1G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}|\underline{X}_{1G}} \\
\hat{\mathbf{T}}_2 &= \mathbf{\Sigma}_{\underline{S}_{2G}\underline{Y}_{2G}|\underline{X}_{2G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{2G}|\underline{X}_{2G}} \\
\mathbf{\Sigma}_{\underline{S}_{1G}|\underline{Y}_{1G},\underline{X}_{1G}} &= \mathbf{\Sigma}_{\underline{S}_{1G}|\underline{X}_{1G}} - \mathbf{\Sigma}_{\underline{S}_{1G}\underline{Y}_{1G}|\underline{X}_{1G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}|\underline{X}_{1G}}\mathbf{\Sigma}_{\underline{Y}_{1G}\underline{S}_{1G}|\underline{X}_{1G}} \\
\mathbf{\Sigma}_{\underline{S}_{2G}|\underline{Y}_{2G},\underline{X}_{2G}} &= \mathbf{\Sigma}_{\underline{S}_{2G}|\underline{X}_{2G}} - \mathbf{\Sigma}_{\underline{S}_{2G}\underline{Y}_{2G}|\underline{X}_{2G}}\mathbf{\Sigma}^{-1}_{\underline{Y}_{2G}|\underline{X}_{2G}}\mathbf{\Sigma}_{\underline{Y}_{2G}\underline{S}_{2G}|\underline{X}_{2G}}.
\end{aligned}$$

From the discussion in Section 3.8, we see that if the genie is $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$-smart, then we have

$$\begin{aligned}
\mathbf{T}_1 &= \hat{\mathbf{T}}_1 \\
\mathbf{T}_2 &= \hat{\mathbf{T}}_2 \\
\mathbf{\Sigma}_{\underline{S}_{1G}|\underline{Y}_{1G}} &= \mathbf{\Sigma}_{\underline{S}_{1G}|\underline{Y}_{1G},\underline{X}_{1G}} \\
\mathbf{\Sigma}_{\underline{S}_{2G}|\underline{Y}_{2G}} &= \mathbf{\Sigma}_{\underline{S}_{2G}|\underline{Y}_{2G},\underline{X}_{2G}}.
\end{aligned} \tag{3.20}$$

We now use the above expressions to simplify the gradient expressions. First, observe that

$$\mathbf{\Sigma}_{\bar{\underline{Y}}_{1G}} = \mathrm{Cov}\left(\begin{bmatrix} \underline{Y}_{1G} \\ \underline{S}_{1G} \end{bmatrix}\right) = \begin{bmatrix} \mathbf{\Sigma}_{\underline{Y}_{1G}} & \mathbf{\Sigma}_{\underline{Y}_{1G},\underline{S}_{1G}} \\ \mathbf{\Sigma}_{\underline{S}_{1G},\underline{Y}_{1G}} & \mathbf{\Sigma}_{\underline{S}_{1G}} \end{bmatrix}.$$

Using the blockwise matrix inversion formula, we have

$$\mathbf{\Sigma}^{-1}_{\bar{\underline{Y}}_{1G}} = \begin{bmatrix} \mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{T}_1^\dagger \\ \mathbf{I} \end{bmatrix}\mathbf{\Sigma}^{-1}_{\underline{S}_{1G}|\underline{Y}_{1G}}\begin{bmatrix} -\mathbf{T}_1 & \mathbf{I} \end{bmatrix}. \tag{3.21}$$

Similarly, we obtain

$$\mathbf{\Sigma}^{-1}_{\bar{\underline{Y}}_{1G}|\underline{X}_{1G}} = \begin{bmatrix} \mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}|\underline{X}_{1G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\hat{\mathbf{T}}_1^\dagger \\ \mathbf{I} \end{bmatrix}\mathbf{\Sigma}^{-1}_{\underline{S}_{1G}|\underline{Y}_{1G},\underline{X}_{1G}}\begin{bmatrix} -\hat{\mathbf{T}}_1 & \mathbf{I} \end{bmatrix}.$$

Using (3.20), we see that the difference is equal to

$$\mathbf{\Sigma}^{-1}_{\bar{\underline{Y}}_{1G}} - \mathbf{\Sigma}^{-1}_{\bar{\underline{Y}}_{1G}|\underline{X}_{1G}} = \begin{bmatrix} \mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}} - \mathbf{\Sigma}^{-1}_{\underline{Y}_{1G}|\underline{X}_{1G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Similarly, we also have

$$\boldsymbol{\Sigma}_{\bar{\underline{Y}}_{2G}}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{\underline{Y}_{2G}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\mathbf{T}_2^\dagger \\ \mathbf{I} \end{bmatrix} \boldsymbol{\Sigma}_{\underline{S}_{2G}|\underline{Y}_{2G}}^{-1} \begin{bmatrix} -\mathbf{T}_2 & \mathbf{I} \end{bmatrix}.$$

and

$$\boldsymbol{\Sigma}_{\bar{\underline{Y}}_{2G}}^{-1} - \boldsymbol{\Sigma}_{\bar{\underline{Y}}_{2G}|\underline{X}_{2G}}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{\underline{Y}_{2G}}^{-1} - \boldsymbol{\Sigma}_{\underline{Y}_{2G}|\underline{X}_{1G}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3.22}$$

Now, consider the difference

$$\begin{aligned} \mathbf{N}_1 &= \nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) - \nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) \\ &= \bar{\mathbf{H}}_{11}^\dagger \boldsymbol{\Sigma}_{\bar{\underline{Y}}_{1G}}^{-1} \bar{\mathbf{H}}_{11} - \mathbf{H}_{11}^\dagger \boldsymbol{\Sigma}_{\underline{Y}_{1G}}^{-1} \mathbf{H}_{11} \\ &\quad + \underbrace{\bar{\mathbf{H}}_{21}^\dagger \left(\boldsymbol{\Sigma}_{\bar{\underline{Y}}_{2G}}^{-1} - \boldsymbol{\Sigma}_{\bar{\underline{Y}}_{2G}|\underline{X}_{2G}}^{-1} \right) \bar{\mathbf{H}}_{21} - \mathbf{H}_{21}^\dagger \left(\boldsymbol{\Sigma}_{\underline{Y}_{2G}}^{-1} - \boldsymbol{\Sigma}_{\underline{Y}_{2G}|\underline{X}_{2G}}^{-1} \right) \mathbf{H}_{21}}_{= \mathbf{0}}. \end{aligned}$$

Observe that the second term is equal to zero. This follows from (3.22) and the definition of $\bar{\mathbf{H}}_{21}$. Therefore, we obtain that

$$\begin{aligned} \mathbf{N}_1 &= \bar{\mathbf{H}}_{11}^\dagger \boldsymbol{\Sigma}_{\bar{\underline{Y}}_{1G}}^{-1} \bar{\mathbf{H}}_{11} - \mathbf{H}_{11}^\dagger \boldsymbol{\Sigma}_{\underline{Y}_{1G}}^{-1} \mathbf{H}_{11} \\ &\overset{(b)}{=} \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1 \mathbf{H}_{11} \right)^\dagger \boldsymbol{\Sigma}_{\underline{S}_1|\underline{Y}_{1G}}^{-1} \left(\tilde{\mathbf{H}}_{21} - \mathbf{T}_1 \mathbf{H}_{11} \right) \end{aligned}$$

where the last step follows from (3.21) and the definition of $\bar{\mathbf{H}}_{11}$. A similar expression can be obtained for $\mathbf{N}_2$. We complete the proof by noting that

$$\boldsymbol{\Sigma}_{\underline{S}_1|\underline{Y}_{1G}} = \boldsymbol{\Sigma}_{\underline{S}_1|\underline{Y}_{1G}, \underline{X}_{1G}} = \boldsymbol{\Sigma}_{\underline{W}_1|\mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1}$$

and

$$\mathbf{T}_1 = \hat{\mathbf{T}}_1 = \boldsymbol{\Sigma}_{\underline{S}_1 \underline{Y}_{1G}|\underline{X}_{1G}} \boldsymbol{\Sigma}_{\underline{Y}_{1G}|\underline{X}_{1G}}^{-1} = \boldsymbol{\Sigma}_{\underline{W}_1 \underline{Z}_1} \left(\mathbf{H}_{12} \mathbf{Q}_2^* \mathbf{H}_{12}^\dagger + \mathbf{I} \right)^{-1}.$$

□

Remark 1. *Suppose the conditions of Theorem 3 are satisfied; then the sum capacity is given by* $f(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$*. As an obvious corollary, we also get that* $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ *are the optimal covariances matrices maximizing the achievable sum-rate* (3.3)*. This means that* $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ *must satisfy the corresponding necessary KKT conditions* (3.5)*. Therefore, it must be that the conditions in Theorem 3 imply the conditions* (3.5)*. Lemma 4 makes it easier to see this connection. Observe that the KKT conditions* (3.19)*, along with the smart genie conditions* (3.18)*, imply that*

$$\begin{aligned} \nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) &= \lambda_1 \mathbf{I} - \mathbf{M}_1 - \mathbf{N}_1 \\ \nabla_{\mathbf{Q}_2} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*) &= \lambda_2 \mathbf{I} - \mathbf{M}_2 - \mathbf{N}_2 \end{aligned}$$

where $\mathbf{N}_1 \succeq 0$, $\mathbf{N}_2 \succeq 0$ *and* $\mathsf{Tr}\left(\mathbf{N}_1\mathbf{Q}_1^*\right) = \mathsf{Tr}\left(\mathbf{N}_2\mathbf{Q}_2^*\right) = 0$. *Therefore, by replacing* $\mathbf{M}_1$ *by* $\mathbf{M}_1+\mathbf{N}_1$ *and* $\mathbf{M}_2$ *by* $\mathbf{M}_2+\mathbf{N}_2$*, we see that the KKT conditions* (3.5) *are satisfied.*

3.9.2 Full Rank Optimal Covariance Matrices

We use the insights from the previous section to simplify the conditions in Theorem 3 when the optimal covariance matrices $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ have full rank. Suppose the conditions of Theorem 3 are satisfied and $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ have full rank. Then, it must be that the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ defined in Lemma 4 must be equal to zero. This is because $\mathbf{N}_1$ and $\mathbf{N}_2$ are positive semidefinite matrices satisfying $\mathsf{Tr}\left(\mathbf{N}_1\mathbf{Q}_1^*\right) = \mathsf{Tr}\left(\mathbf{N}_2\mathbf{Q}_2^*\right) = 0$. Therefore, we have that

$$\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \nabla_{\mathbf{Q}_1} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$$
$$\nabla_{\mathbf{Q}_2}\bar{f}(\mathbf{Q}_1^*, \mathbf{Q}_2^*, \Psi^*) = \nabla_{\mathbf{Q}_2} f(\mathbf{Q}_1^*, \mathbf{Q}_2^*).$$

Hence, the KKT conditions (3.19) are identical to (3.5) which are satisfied if $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$ is a local optimal solution to the optimization problem (3.3). Thus, we obtain the following corollary to Theorem 3.

Corollary 1. *Suppose there exist a local optimal solution* $\mathbf{Q}_1^* \succ \mathbf{0}, \mathbf{Q}_2^* \succ 0$ *to*

$$\max_{(\mathbf{Q}_1, \mathbf{Q}_2) \in \mathcal{Q}} f(\mathbf{Q}_1, \mathbf{Q}_2)$$

and a genie Ψ^* *that is both useful and* $(\mathbf{Q}_1^*, \mathbf{Q}_2^*)$*-smart; i.e., the conditions* (3.17) *and* (3.18) *are satisfied. Then, the sum capacity of the two-user MIMO Gaussian interference channel* (3.1) *is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = f(\mathbf{Q}_1^*, \mathbf{Q}_2^*).$$

3.9.3 Concave Sum-Rate Function

As summarized in Section 3.5, the basic idea leading to the techniques developed in this chapter is that the achievable sum-rate function $f(\mathbf{Q}_1, \mathbf{Q}_2)$ is not necessarily concave in $(\mathbf{Q}_1, \mathbf{Q}_2)$ and so we used the genie-aided channel to develop a concave upper bound $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi)$ to handle the optimization problem. The best concave upper bound to $f(\mathbf{Q}_1, \mathbf{Q}_2)$ is given by

$$\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2) = \min_{\Psi \in \Psi_u} \bar{f}(\mathbf{Q}_1, \mathbf{Q}_2, \Psi).$$

Suppose $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2) = f(\mathbf{Q}_1, \mathbf{Q}_2)$ for every feasible $(\mathbf{Q}_1, \mathbf{Q}_2)$. Then, we see that $f(\mathbf{Q}_1, \mathbf{Q}_2)$ is a concave function within the region of interest, and hence we can just use the standard convex optimization algorithms to determine the global optimal solution to

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} f(\mathbf{Q}_1,\mathbf{Q}_2).$$

Clearly, this also implies the sum capacity is equal to $f(\mathbf{Q}_1^*,\mathbf{Q}_2^*)$. Observe that

$$\begin{aligned}\mathcal{C}_{\text{sum}} &\le \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \min_{\Psi\in\Psi_u} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2,\Psi)\\ &= \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} \bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)\\ &\overset{(a)}{=} \max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} f(\mathbf{Q}_1,\mathbf{Q}_2)\\ &= f(\mathbf{Q}_1^*,\mathbf{Q}_2^*)\end{aligned}$$

where the step (a) follows because we assumed that $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)=f(\mathbf{Q}_1,\mathbf{Q}_2)$ for every feasible $(\mathbf{Q}_1,\mathbf{Q}_2)$. Since the condition $\bar{f}(\mathbf{Q}_1,\mathbf{Q}_2)=f(\mathbf{Q}_1,\mathbf{Q}_2)$ is equivalent to the existence of a genie Ψ that is both useful and $(\mathbf{Q}_1,\mathbf{Q}_2)$-smart, we obtain the following corollary to Theorem 2.

Corollary 2. *Suppose for every feasible transmit covariance matrices* $(\mathbf{Q}_1,\mathbf{Q}_2)$*, there exists a genie* Ψ *that is both useful and* $(\mathbf{Q}_1,\mathbf{Q}_2)$*-smart; i.e., the conditions* (3.17) *and* (3.12) *are satisfied. Then, the achievable sum-rate function* $f(\mathbf{Q}_1,\mathbf{Q}_2)$ *is concave in* $(\mathbf{Q}_1,\mathbf{Q}_2)$ *in the feasible region* $\mathcal{Q}$*, and the sum capacity is achievable by using Gaussian inputs and treating interference as noise, and is given by* $f(\mathbf{Q}_1^*,\mathbf{Q}_2^*)$*, the optimal solution to the convex optimization problem*

$$\max_{(\mathbf{Q}_1,\mathbf{Q}_2)\in\mathcal{Q}} f(\mathbf{Q}_1,\mathbf{Q}_2).$$

3.9.4 SISO Interference Channel

In this section, we consider the SISO interference channel in the standard form

$$\begin{aligned}Y_1 &= X_1 + h_{12}X_2 + Z_1\\ Y_2 &= h_{21}X_1 + X_2 + Z_2\end{aligned} \tag{3.23}$$

with the transmit power constraints P_1 and P_2. We now simplify the low interference regime conditions of Theorem 3 to obtain a simple closed-form condition. In the SISO case, the KKT conditions of Theorem 3 are automatically satisfied. Recall that the role of the KKT conditions is to make sure (q_1^*,q_2^*) is the global optimal solution to

$$\max_{(q_1,q_2)\in\mathcal{Q}} \bar{f}(q_1,q_2,\Psi^*).$$

However, we have already proved in Lemma 1 that $\bar{f}(q_1,q_2,\Psi^*)$ is a nondecreasing function in (q_1,q_2). Therefore, $(q_1^*,q_2^*)=(P_1,P_2)$ must be a global optimal solution to the above optimization problem, and hence there must exist dual variables satisfying the KKT conditions (3.19). Therefore, it only remains to verify the existence of a genie $\Psi^*=(\Sigma_{W_1},\Sigma_{W_1Z_1},\Sigma_{W_2},\Sigma_{W_2Z_2})$ that is useful and (P_1, P_2)-smart, i.e., satisfying the conditions (3.17) and (3.18).

Theorem 4. *The sum capacity of the two-user SISO Gaussian interference channel* (3.23) *is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = \log\left(1 + \frac{P_1}{1 + |h_{12}|^2 P_2}\right) + \log\left(1 + \frac{P_2}{1 + |h_{21}|^2 P_1}\right)$$

if the channel parameters satisfy the condition

$$|h_{21}|\left(1 + |h_{12}|^2 P_2\right) + |h_{12}|\left(1 + |h_{21}|^2 P_1\right) \leq 1. \tag{3.24}$$

Proof. The smart genie conditions (3.18) are given by

$$\begin{aligned}\left(h_{21} - \Sigma_{W_1 Z_1}\left(h_{12} P_2 h_{12}^\dagger + 1\right)^{-1}\right) P_1 = 0\\ \left(h_{12} - \Sigma_{W_2 Z_2}\left(h_{21} P_1 h_{21}^\dagger + 1\right)^{-1}\right) P_2 = 0\end{aligned}$$

which are equivalent to

$$\begin{aligned}\Sigma_{W_1 Z_1} &= h_{21}\left(1 + |h_{12}|^2 P_2\right)\\ \Sigma_{W_2 Z_2} &= h_{12}\left(1 + |h_{21}|^2 P_1\right).\end{aligned}$$

The useful genie conditions (3.17) are given by

$$\begin{aligned}\begin{bmatrix} 1 & \Sigma_{Z_1 W_1} \\ \Sigma_{W_1 Z_1} & \Sigma_{W_1}\end{bmatrix} &\succeq \begin{bmatrix} \Sigma_{W_2} & 0 \\ 0 & 0\end{bmatrix}\\ \begin{bmatrix} 1 & \Sigma_{Z_2 W_2} \\ \Sigma_{W_2 Z_2} & \Sigma_{W_2}\end{bmatrix} &\succeq \begin{bmatrix} \Sigma_{W_1} & 0 \\ 0 & 0\end{bmatrix}\end{aligned}$$

which are equivalent to

$$\begin{aligned}0 \leq \Sigma_{W_1}, \Sigma_{W_2} &\leq 1\\ \Sigma_{W_1}(1 - \Sigma_{W_2}) &\leq |\Sigma_{W_1 Z_1}|^2\\ \Sigma_{W_2}(1 - \Sigma_{W_1}) &\leq |\Sigma_{W_2 Z_2}|^2.\end{aligned}$$

Substituting $\Sigma_{W_1} = \cos^2\phi_1$ and $\Sigma_{W_2} = \sin^2\phi_2$, where $\phi_1, \phi_2 \in [0, \pi/2]$, the above equations can be simplified as

$$\begin{aligned}\cos\phi_1 \cos\phi_2 &\leq |\Sigma_{W_1 Z_1}|\\ \sin\phi_1 \sin\phi_2 &\leq |\Sigma_{W_2 Z_2}|.\end{aligned}$$

It is easy to check that a solution $\phi_1, \phi_2 \in [0, \pi/2]$ exists if and only if

$$|\Sigma_{W_1 Z_1}| + |\Sigma_{W_2 Z_2}| \leq 1.$$

□

3.10 Symmetric MISO and SIMO Interference Channels

In this section, we consider the symmetric MISO and SIMO interference channels, and simplify the conditions in Theorem 3 to derive a simple closed-form equation for the low interference regime.

Symmetric MISO interference channel:

$$\begin{aligned} Y_1 &= \mathbf{d}^\dagger \underline{X}_1 + h\mathbf{c}^\dagger \underline{X}_2 + Z_1 \\ Y_2 &= \mathbf{d}^\dagger \underline{X}_2 + h\mathbf{c}^\dagger \underline{X}_1 + Z_2. \end{aligned}$$

Symmetric SIMO interference channel:

$$\begin{aligned} \underline{Y}_1 &= \mathbf{d}X_1 + h\mathbf{c}X_2 + \underline{Z}_1 \\ \underline{Y}_2 &= \mathbf{d}X_2 + h\mathbf{c}X_1 + \underline{Z}_2. \end{aligned}$$

In both the above cases, we assume that the transmitters satisfy an average transmit power constraint of P. We assume that $h \geq 0$ is a real number, and the vectors $\mathbf{d}$ and $\mathbf{c}$ have unit norm, and are defined as

$$\mathbf{d} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.25}$$

for some $\theta \in [0, \pi/2]$. See Section 3.2 for a justification for these assumptions. In particular, observe that the third assumption in Section 3.2 states that we can restrict the study of the MISO (resp. SIMO) Gaussian interference channels to the case with only two transmit (resp. receive) antennas.

Observe that both the MISO and SIMO channels with $\theta = 0$ are equivalent to the classical two-user SISO Gaussian interference channel. In Section 3.9.4, we showed that the channel is in low interference regime, and hence treating interference as noise achieves the sum capacity equal to

$$2\log\left(1 + \frac{P}{1+h^2P}\right)$$

if the parameters h and P satisfy $h(1+h^2P) \leq 0.5$. On the other extreme, with $\theta = \pi/2$, we obtain the scenario where the users do not interfere with each other, and hence the sum capacity is given by $2\log(1+P)$ for any h. We vary θ from 0 to $\pi/2$, and analyze the behavior of sum capacity and the low interference regime as a function of θ.

3.10.1 Achievable Sum-Rate

First, consider the SIMO interference channel. The achievable sum-rate (3.4) obtained by using Gaussian inputs and treating interference as noise is given by

$$f(q_1, q_2) = \log \frac{|\mathbf{I} + q_1 \mathbf{d}\mathbf{d}^\dagger + h^2 q_2 \mathbf{c}\mathbf{c}^\dagger|}{|\mathbf{I} + h^2 q_2 \mathbf{c}\mathbf{c}^\dagger|} + \log \frac{|\mathbf{I} + q_2 \mathbf{d}\mathbf{d}^\dagger + h^2 q_1 \mathbf{c}\mathbf{c}^\dagger|}{|\mathbf{I} + h^2 q_1 \mathbf{c}\mathbf{c}^\dagger|}$$

where q_1 and q_2 denote the transmit powers. Recall that q_1 and q_2 must satisfy the average power constraints of $q_1, q_2 \leq P$. The achievable sum-rate by using the maximum power, i.e., by setting $q_1 = q_2 = P$, is given by

$$\begin{aligned} f(P, P) &= 2 \log \frac{|\mathbf{I} + P\mathbf{d}\mathbf{d}^\dagger + h^2 P \mathbf{c}\mathbf{c}^\dagger|}{|\mathbf{I} + h^2 P \mathbf{c}\mathbf{c}^\dagger|} \\ &= 2 \log \left|\mathbf{I} + \mathbf{J}^{-1} P \mathbf{d}\mathbf{d}^\dagger\right| \\ &= 2 \log \left(1 + P\mathbf{d}^\dagger \mathbf{J}^{-1} \mathbf{d}\right) & \text{(3.26a)} \\ &= 2 \log \left(1 + \frac{P \cos^2 \theta}{1 + h^2 P} + P \sin^2 \theta\right) & \text{(3.26b)} \end{aligned}$$

where the matrix $\mathbf{J}$ denotes

$$\mathbf{J} = \mathbf{I} + h^2 P \mathbf{c}\mathbf{c}^\dagger = \begin{bmatrix} 1 + h^2 P & 0 \\ 0 & 1 \end{bmatrix}.$$

Step (a) follows from the identity $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$, and step (b) follows from the definitions (3.25) of $\mathbf{d}$ and $\mathbf{c}$. The above sum-rate can be shown to be achievable with the receivers projecting the received vector along a beamforming direction, denoted by a unit norm vector $\mathbf{b}$, i.e.,

$$\begin{aligned} \tilde{Y}_1 &= \mathbf{b}^\dagger \underline{Y}_1 = \mathbf{b}^\dagger \mathbf{d} X_1 + h \mathbf{b}^\dagger \mathbf{c} X_2 + \mathbf{b}^\dagger \underline{Z}_1 \\ \tilde{Y}_2 &= \mathbf{b}^\dagger \underline{Y}_2 = \mathbf{b}^\dagger \mathbf{d} X_2 + h \mathbf{b}^\dagger \mathbf{c} X_1 + \mathbf{b}^\dagger \underline{Z}_2. \end{aligned}$$

We choose the beamforming direction $\mathbf{b}$ as

$$\mathbf{b} = \frac{\mathbf{J}^{-1}\mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||} \tag{3.27}$$

in order to achieve the best SINR:

$$\text{SINR} = \frac{P|\mathbf{b}^\dagger \mathbf{d}|^2}{1 + h^2 P |\mathbf{b}^\dagger \mathbf{c}|^2} = \frac{P|\mathbf{b}^\dagger \mathbf{d}|^2}{\mathbf{b}^\dagger \mathbf{J} \mathbf{b}} = P\mathbf{d}^\dagger \mathbf{J}^{-1} \mathbf{d}.$$

This interpretation of receive beamforming helps in understanding the best achievable sum-rate of the dual MISO interference channel. Observe that the achievable sum-rate of the MISO interference channel is given by

$$f(\mathbf{Q}_1, \mathbf{Q}_2) = \log \left(1 + \frac{P\mathbf{d}^\dagger \mathbf{Q}_1 \mathbf{d}}{1 + h^2 P \mathbf{c}^\dagger \mathbf{Q}_2 \mathbf{c}}\right) + \log \left(1 + \frac{P\mathbf{d}^\dagger \mathbf{Q}_2 \mathbf{d}}{1 + h^2 P \mathbf{c}^\dagger \mathbf{Q}_1 \mathbf{c}}\right).$$

Using the insight from the SIMO interference channel, we let the transmitters to transmit along the beamforming direction $\mathbf{b}$; i.e., we set

$$\mathbf{Q}_1^* = \mathbf{Q}_2^* = \mathbf{Q}^* = P\mathbf{b}\mathbf{b}^\dagger.$$

The corresponding achievable sum-rate is given by

$$f(\mathbf{Q}^*, \mathbf{Q}^*) = 2\log\left(1 + \frac{P\mathbf{d}^\dagger\mathbf{Q}^*\mathbf{d}}{1 + h^2 P\mathbf{c}^\dagger\mathbf{Q}^*\mathbf{c}}\right) = 2\log\left(1 + \frac{P\cos^2\theta}{1+h^2P} + P\sin^2\theta\right).$$

3.10.2 Low Interference Regime

Theorem 5. *The sum capacity of the symmetric MISO and SIMO Gaussian interference channels described in Section 3.10 is achieved by using Gaussian inputs and treating interference as noise at the receivers, and is given by*

$$\mathcal{C}_{\text{sum}} = 2\log\left(1 + \frac{P\cos^2\theta}{1+h^2P} + P\sin^2\theta\right)$$

if the channel parameters satisfy the threshold condition $h \le h_0(\theta, P)$, where $h_0(\theta, P)$ is defined as the unique positive solution to the implicit equation

$$h^2 - \sin^2\theta = \left(\frac{\cos\theta}{1+h^2P} - h\right)_+^2. \tag{3.28}$$

(The notation x_+^2 is used to denote $(\max(0,x))^2$.)

The above theorem is obtained by specializing conditions in Theorem 3 for the special case of symmetric MISO and SIMO channels. Before we go into the proof details, we first prove some properties of the threshold function $h_0(\theta, P)$. The threshold $h_0(\theta, P)$ is plotted as a function of θ for different values of P in Figure 3.1. It can be observed that the threshold curve is always above the $\sin\theta$ curve and approaches the $\sin\theta$ curve as P becomes larger.

We summarize the observations from Figure 3.1 in the following claim.

Claim 3. *The threshold $h_0(\theta, P)$ satisfies*

1. *$h_0(\theta, P) > \sin\theta$ for all $P < P_0(\theta)$*
2. *$h_0(\theta, P) = \sin\theta$ for all $P \ge P_0(\theta)$*

where $P_0(\theta)$ is defined as

$$P_0(\theta) = \begin{cases} \dfrac{\cos\theta - \sin\theta}{\sin^3\theta} & \text{when } 0 \le \theta < \pi/4 \\ 0 & \text{when } \pi/4 \le \theta < \pi/2 \end{cases}.$$

Proof. Observe that the L.H.S. of (3.28) is strictly increasing in h, whereas the R.H.S. is decreasing in h. This verifies that (3.28) has a unique positive solution.

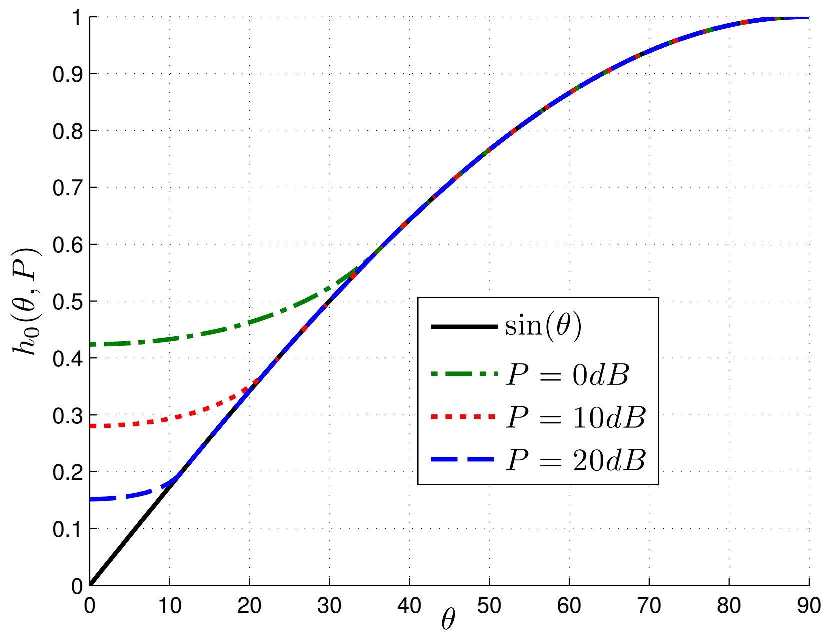

Figure 3.1: Threshold on h characterizing the low interference regime of the symmetric MISO and SIMO interference channels

Note that the L.H.S. is strictly negative when $h < \sin\theta$, whereas the R.H.S. is always nonnegative. This immediately implies that $h_0(\theta, P) \geq \sin\theta$. It can be easily checked that the R.H.S. is equal to zero at $h = \sin\theta$ when $P \geq P_0(\theta)$. Hence we obtain the second statement. Similarly, it can be easily checked that the R.H.S. is greater than zero at $h = \sin\theta$ when $P < P_0(\theta)$. Hence, we obtain the first statement. □

3.10.3 SIMO Interference Channel

Recall from Section 3.2 that $\tilde{\mathbf{H}}_{12}$ and $\tilde{\mathbf{H}}_{21}$ denote the nonzero rows of the respective matrices $\mathbf{H}_{12}$ and $\mathbf{H}_{21}$. Therefore, for the special case of SIMO interference channel, we have $\tilde{\mathbf{H}}_{12} = \tilde{\mathbf{H}}_{21} = h$, and the genie signals (3.7) are given by

$$\begin{aligned} S_1 &= hX_1 + W_1 \\ S_2 &= hX_2 + W_2. \end{aligned}$$

We now simplify the conditions in Theorem 3 and show that they are equivalent to the threshold condition in Theorem 5. As explained in Section 3.9.4 for the SISO interference channel, we do not have to explicitly check for the KKT conditions (3.19) because they are automatically satisfied at $(q_1^*, q_2^*) = (P, P)$. This is true because $\bar{f}(q_1, q_2, \Psi^*)$ is nondecreasing in (q_1, q_2). Therefore it only remains to verify the existence of a genie $\Psi^* = \left(\Sigma_{W_1}, \mathbf{\Sigma}_{W_1\underline{Z}_1}, \Sigma_{W_2}, \mathbf{\Sigma}_{W_2\underline{Z}_2}\right)$ that is useful and (P, P)-smart, i.e., satisfying the conditions (3.17) and (3.18). Since we are working with the symmetric interference channel, we restrict ourselves to a symmetric genie; i.e., we assume $\mathbf{\Sigma}_{W_1\underline{Z}_1} = \mathbf{\Sigma}_{W_2\underline{Z}_2} = \mathbf{\Sigma}_{W\underline{Z}} = [a_1 \ a_2]$ and $\Sigma_{W_1} = \Sigma_{W_2} = \Sigma_W$.

The usefulness condition (3.17) when specialized to the SIMO channel is given by

$$\begin{bmatrix} 1 & 0 & a_1^\dagger \\ 0 & 1 & a_2^\dagger \\ a_1 & a_2 & \Sigma_W \end{bmatrix} \succeq \begin{bmatrix} \Sigma_W & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1-\Sigma_W & 0 & a_1^\dagger \\ 0 & 1 & a_2^\dagger \\ a_1 & a_2 & \Sigma_W \end{bmatrix} \succeq 0.$$

Using the fact that a hermitian matrix is positive semidefinite if and only if all the principal minors are nonnegative, the above condition is equivalent to the conditions

$$\begin{aligned} 0 \leq \Sigma_W &\leq 1 \\ \Sigma_W - |a_2|^2 &\geq 0 \\ (1-\Sigma_W)(\Sigma_W - |a_2|^2) - |a_1|^2 &\geq 0. \end{aligned} \tag{3.29}$$

The smartness condition (3.18) when specialized to the SIMO channel is given by

$$\begin{aligned} \left(h - \mathbf{\Sigma}_{w\mathbf{z}} \left(h^2 P \mathbf{c}\mathbf{c}^\dagger + \mathbf{I}\right)^{-1} \mathbf{d}\right) P &= \mathbf{0} \\ \Leftrightarrow h - \mathbf{\Sigma}_{w\mathbf{z}} \begin{bmatrix} 1+h^2P & 0 \\ 0 & 1 \end{bmatrix}^{-1} \mathbf{d} &= \mathbf{0} \\ \Leftrightarrow h - \frac{a_1 \cos\theta}{1+h^2P} - a_2 \sin\theta &= 0. \end{aligned} \tag{3.30}$$

Therefore, we obtain that the SIMO interference channel is in the low interference regime if there exist parameters a_1, a_2, Σ_W satisfying the conditions (3.29) and (3.30). The following claim completes the proof of SIMO part of Theorem 5.

Claim 4. *There exist parameters a_1, a_2, Σ_W satisfying the conditions* (3.29) *and* (3.30) *if and and only if the following condition is true*

$$h^2 - \sin^2\theta \leq \left(\frac{\cos\theta}{1+h^2P} - h\right)_+^2. \tag{3.31}$$

Proof. First, observe that the channel parameters h, θ, P are real, and hence we can restrict the genie parameters a_1, a_2 also to be real. We simplify the usefulness conditions (3.29) by eliminating the parameter Σ_W. We show that there exists Σ_W satisfying (3.29) if and only if $2|a_1| + a_2^2 \leq 1$. Observe that the last condition in (3.29) can be expressed as

$$\begin{aligned} &(1-\Sigma_W)(-\Sigma_W + a_2^2) + a_1^2 \leq 0 \\ &\quad \Leftrightarrow (1-\Sigma_W)a_2^2 + a_1^2 \leq (1-\Sigma_W)\Sigma_W. \end{aligned}$$

This automatically implies the second condition in (3.29). We can eliminate Σ_W by checking when the quadratic inequality

$$\Sigma_W^2 - \Sigma_W(1 + a_2^2) + 1 + a_1^2 \leq 0$$

has a solution between 0 and 1. We see that the above expression achieves a minimum of

$$1 + a_1^2 - (1 + a_2^2)^2/4 = a_1^2 - (1 - a_2^2)^2/4$$

at $\Sigma_W = \frac{1+a_2^2}{2}$. Since $|a_2| \leq 1$ must be satisfied, we see that $\frac{1+a_2^2}{2}$ is in between 0 and 1. Therefore, the above quadratic inequality has a solution between 0 and 1 if and only if a_1 and a_2 satisfy $a_1^2 - (1 - a_2^2)^2/4$ is nonnegative; i.e, $2|a_1| + a_2^2 \leq 1$.

Therefore, we are now left with the problem of determining conditions on h, θ, P such that there exist a_1 and a_2 exist satisfying $2|a_1| + a_2^2 \leq 1$ and

$$h = \frac{a_1 \cos\theta}{1 + h^2 P} + a_2 \sin\theta. \tag{3.32}$$

It can be easily checked that $2|a_1| + a_2^2 \leq 1$ implies $|a_1| + |a_2| \leq 1$. Therefore, a necessary condition is that h, θ, P must satisfy

$$h \leq \max\left(\frac{\cos\theta}{1 + h^2 P}, \sin\theta\right).$$

We now divide the proof into two cases.

Case 1: Channel parameters satisfy $h \leq \sin\theta$. By setting $a_1 = 0$ and $a_2 = h/\sin\theta$, we see that a_1 and a_2 exist for any $h \leq \sin\theta$.

Case 2: Channel parameters satisfy

$$\sin\theta < h \leq \frac{\cos\theta}{1 + h^2 P}. \tag{3.33}$$

Observe that if a_1 is negative, then $h \leq a_2 \sin\theta \leq \sin\theta$. Therefore, we can restrict a_1 to be nonnegative. We now obtain a neccesary condition for a_1 and a_2 to exist. We can rewrite the condition (3.32) as

$$h(1 - a_1) = a_1\left(\frac{\cos\theta}{1 + h^2 P} - h\right) + a_2 \sin\theta.$$

Applying Cauchy-Schwarz inequality, we obtain that h, θ, P must satisfy

$$h^2(1 - a_1)^2 \leq \left(\left(\frac{\cos\theta}{1 + h^2 P} - h\right)^2 + \sin^2\theta\right)(a_1^2 + a_2^2)$$

and hence

$$h^2 - \sin^2\theta \leq \left(\frac{\cos\theta}{1 + h^2 P} - h\right)^2$$

because the inequality $2a_1 + a_2^2 \leq 1$ is equivalent to the inequality $a_1^2 + a_2^2 \leq (1 - a_1)^2$. Conversely, we note that a_1 and a_2 exist for any h, θ, P satisfying

(3.33) and the above condition. The corresponding expressions for a_1 and a_2 are obtained by solving the system of linear equations comprised of (3.32) and the following equality which makes the Cauchy-Schwarz inequality tight

$$a_1 \sin\theta = a_2 \left(\frac{\cos\theta}{1+h^2P} - h \right).$$

Therefore, combining both the cases, we see that a_1, a_2 exist if and only if the channel parameters satisfy condition (3.31). $\square$

3.10.4 MISO Interference Channel

Observe that Theorem 5 is obtained by specializing Theorem 3 for the special case of symmetric MISO interference channel. The genie signals (3.7) are given by

$$\begin{aligned} S_1 &= h\mathbf{c}^\dagger \underline{X}_1 + W_1 \\ S_2 &= h\mathbf{c}^\dagger \underline{X}_2 + W_2. \end{aligned}$$

Theorem 5 follows if we show that there exist genie parameters and dual variables satisfying the conditions of Theorem 3 at

$$\mathbf{Q}_1^* = \mathbf{Q}_2^* = \mathbf{Q}^* = P\mathbf{b}\mathbf{b}^\dagger.$$

Since the channel is symmetric across the users, we restrict the genie parameters $\Psi^* = (\Sigma_{W_1}, \Sigma_{W_1Z_1}, \Sigma_{W_2}, \Sigma_{W_2Z_2})$ and the dual variables $\lambda_1, \lambda_2, \mathbf{M}_1, \mathbf{M}_2$ to be symmetric, i.e.,

$$\begin{aligned} \Sigma_{W_1Z_1} &= \Sigma_{W_2Z_2} = \Sigma_{WZ} \\ \Sigma_{W_1} &= \Sigma_{W_2} = \Sigma_W \\ \lambda_1 &= \lambda_2 = \lambda \\ \mathbf{M}_1 &= \mathbf{M}_2 = \mathbf{M}. \end{aligned}$$

Therefore, we need to prove the existence of the parameters $\Sigma_W, \Sigma_{WZ}, \lambda \geq 0, \mathbf{M} \succeq 0$ satisfying the following conditions:

1. Useful genie condition:

$$\begin{bmatrix} 1 & \Sigma_{ZW} \\ \Sigma_{WZ} & \Sigma_W \end{bmatrix} \succeq \begin{bmatrix} \Sigma_W & 0 \\ 0 & 0 \end{bmatrix}$$

2. Smart genie condition:

$$\left(h\mathbf{c}^\dagger - \frac{\Sigma_{WZ}}{1+h^2P|\mathbf{b}^\dagger\mathbf{c}|^2}\mathbf{d}^\dagger \right) \mathbf{b} = 0$$

3. KKT conditions:

$$\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}^*, \mathbf{Q}^*, \Psi^*) = \lambda \mathbf{I} - \mathbf{M}$$
$$\mathsf{Tr}\,(\mathbf{M}\mathbf{Q}^*) = 0$$

Note that the useful genie condition is equivalent to

$$\Sigma_W(1 - \Sigma_W) \geq |\Sigma_{WZ}|^2$$

and that the smart genie condition is equivalent to

$$\begin{aligned} h\mathbf{c}^\dagger \mathbf{b} &= \frac{\Sigma_{WZ}}{\mathbf{b}^\dagger \mathbf{J} \mathbf{b}} \mathbf{d}^\dagger \mathbf{b} \\ \Leftrightarrow \Sigma_{WZ} &= h\, \frac{\mathbf{c}^\dagger \mathbf{b}}{\mathbf{d}^\dagger \mathbf{b}}\, \mathbf{b}^\dagger \mathbf{J} \mathbf{b}. \end{aligned}$$

Recall that the matrix $\mathbf{J}$ is defined as

$$\mathbf{J} = \mathbf{I} + h^2 P \mathbf{c}\mathbf{c}^\dagger = \begin{bmatrix} 1 + h^2 P & 0 \\ 0 & 1 \end{bmatrix}$$

and the beamforming vector $\mathbf{b}$ is defined as the unit norm vector in the direction of

$$\mathbf{b} = \frac{\mathbf{J}^{-1}\mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||} = \frac{1}{||\mathbf{J}^{-1}\mathbf{d}||} \begin{bmatrix} \frac{\cos^2\theta}{1+h^2P} \\ \sin\theta \end{bmatrix}.$$

Thus, we see that the smart genie condition is equivalent to

$$\begin{aligned} \Sigma_{WZ} &= h\, \frac{\mathbf{c}^\dagger \mathbf{b}}{\mathbf{d}^\dagger \mathbf{b}}\, \mathbf{b}^\dagger \mathbf{J} \mathbf{b} \\ &= h\, \frac{\mathbf{c}^\dagger \mathbf{J}^{-1} \mathbf{d}}{\mathbf{d}^\dagger \mathbf{J}^{-1} \mathbf{d}}\, \frac{\mathbf{d}^\dagger \mathbf{J}^{-1} \mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||^2} \\ &= h\, \frac{\mathbf{c}^\dagger \mathbf{J}^{-1} \mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||^2} \end{aligned}$$

i.e.,

$$\Sigma_{WZ} = \frac{h\, \dfrac{\cos\theta}{1 + h^2 P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2} + \sin^2\theta}. \tag{3.34}$$

We now simplify the KKT conditions. Observe that $\mathsf{Tr}\,(\mathbf{M}\mathbf{Q}^*) = 0$ is equivalent to saying that $\mathbf{M}\mathbf{b} = 0$. Therefore, we see that $\mathbf{b}$ is an eigenvector of the gradient matrix $\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}^*, \mathbf{Q}^*, \Psi^*)$ with $\lambda \geq 0$ as the eigenvalue value. The condition $\mathbf{M} \succeq 0$ implies that the other eigenvalue is smaller than λ. Thus, we see that the KKT conditions are equivalent to saying that

$\mathbf{b}$ is the dominant eigenvector of $\nabla_{\mathbf{Q}_1} \bar{f}(\mathbf{Q}^*, \mathbf{Q}^*, \Psi^*)$ with eigenvalue $\lambda \geq 0$.

We can use the Lemma 4 to simplify the derivation of the gradient matrix. Observe that

$$\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}^*,\mathbf{Q}^*,\Psi^*) = \nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*,\mathbf{Q}^*) + \mathbf{N}$$

where the matrix $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{(h\mathbf{c}-t\mathbf{d})^\dagger\,(h\mathbf{c}-t\mathbf{d})}{\Sigma_{W_1|Y_{1G},\underline{X}_{1G}}}$$

and the parameter t is given by

$$t = \frac{\Sigma_{WZ}}{h^2P|\mathbf{b}^\dagger\mathbf{c}|^2+1} = \frac{\Sigma_{WZ}}{\mathbf{b}^\dagger\mathbf{J}\mathbf{b}} = h\,\frac{\mathbf{c}^\dagger\mathbf{b}}{\mathbf{d}^\dagger\mathbf{b}}. \tag{3.35}$$

Observe that the matrix $\mathbf{N}$ satisfies $\mathbf{N}\mathbf{b}=0$. Therefore, we need to show that $\mathbf{b}$ is an eigenvector of $\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*,\mathbf{Q}^*)$. Note that

$$\begin{aligned}
\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*,\mathbf{Q}^*) &= \frac{\mathbf{d}\mathbf{d}^\dagger}{\Sigma_{Y_{1G}}} + \frac{h^2\mathbf{c}\mathbf{c}^\dagger}{\Sigma_{Y_{2G}}} - \frac{h^2\mathbf{c}\mathbf{c}^\dagger}{\Sigma_{Y_{2G}|\underline{X}_{2G}}} \\
&= \frac{1}{\Sigma_{Y_{1G}}}\left(\mathbf{d}\mathbf{d}^\dagger - h^2\left(\frac{\Sigma_{Y_{2G}}}{\Sigma_{Y_{2G}|\underline{X}_{2G}}}-1\right)\mathbf{c}\mathbf{c}^\dagger\right) \\
&= \frac{1}{\Sigma_{Y_{1G}}}\left(\mathbf{d}\mathbf{d}^\dagger - h^2\,\mathrm{SINR}\,\mathbf{c}\mathbf{c}^\dagger\right).
\end{aligned} \tag{3.36}$$

Recall that the expression for SINR is given by

$$\mathrm{SINR} = \frac{P\cos^2\theta}{1+h^2P} + P\sin^2\theta.$$

Using the expansion

$$\begin{aligned}
\mathbf{d}\mathbf{d}^\dagger - h^2\,\mathrm{SINR}\,\mathbf{c}\mathbf{c}^\dagger &= \begin{bmatrix} \cos^2\theta - h^2\mathrm{SINR} & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \\
&= \begin{bmatrix} \frac{\cos^2\theta}{1+h^2P} - h^2P\sin^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}
\end{aligned}$$

we can easily check that $\mathbf{b}$ is an eigenvector of the matrix $\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*,\mathbf{Q}^*)$ with the corresponding eigenvalue given by

$$\lambda = \frac{1}{\Sigma_{Y_{1G}}}\left(\frac{\cos^2\theta}{1+h^2P} + \sin^2\theta\right). \tag{3.37}$$

Since $\mathbf{N}\mathbf{b}=0$, we also obtain that the $\mathbf{b}$ is an eigenvector of the matrix $\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}^*,\mathbf{Q}^*,\Psi^*)$ with the same eigenvalue λ. Since the sum of eigenvalues is equal to the trace of the matrix, the other eigenvalue is equal to

$$\mathsf{Tr}\left(\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}^*,\mathbf{Q}^*,\Psi^*)\right) - \lambda.$$

Therefore, we have that the $\mathbf{b}$ is the dominant eigenvector if and only if

$$\begin{aligned}2\lambda &\geq \mathsf{Tr}\left(\nabla_{\mathbf{Q}_1}\bar{f}(\mathbf{Q}^*,\mathbf{Q}^*,\Psi^*)\right)\\&=\mathsf{Tr}\left(\nabla_{\mathbf{Q}_1}f(\mathbf{Q}^*,\mathbf{Q}^*)\right)+\mathsf{Tr}\left(\mathbf{N}\right).\end{aligned}$$

Note that

$$\mathsf{Tr}\left(\mathbf{N}\right)=\frac{||h\mathbf{c}-t\mathbf{d}||^2}{\Sigma_{W_1|Y_{1G},\underline{X}_{1G}}}=\frac{||h\mathbf{c}-t\mathbf{d}||^2}{\Sigma_W-\dfrac{\Sigma_{WZ}^2}{1+h^2|\mathbf{b}^\dagger\mathbf{c}|^2}}.$$

Therefore, $\mathbf{b}$ is the dominant eigenvector if and only if

$$\Sigma_W\geq\frac{\Sigma_{WZ}^2}{1+h^2|\mathbf{b}^\dagger\mathbf{c}|^2}+\frac{||h\mathbf{c}-t\mathbf{d}||^2}{2\lambda-\mathsf{Tr}\left(\nabla_{\mathbf{Q}_1}f(\mathbf{Q}^*,\mathbf{Q}^*)\right)}. \tag{3.38}$$

Observe that all the variables other than Σ_W are known and can be expressed as a function of h,θ and P. By substituting the corresponding expressions, we can simplify the R.H.S. of (3.38).

Claim 5. *The condition* (3.38) *is equivalent to*

$$\Sigma_W\geq\frac{\dfrac{h^2(1+P\sin^2\theta)}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2}+\sin^2\theta}. \tag{3.39}$$

Therefore, it remains to verify the existence of the parameter Σ_W satisfying the usefulness condition

$$\begin{aligned}\Sigma_W(1-\Sigma_W)&\geq\Sigma_{WZ}^2\\\Sigma_W^2-\Sigma_W+\Sigma_{WZ}^2&\leq 0\end{aligned}$$

and (3.39), where Σ_{WZ} is given by the smartness condition (3.34). Observe that the above quadratic inequality has a solution if and only if $\Sigma_{WZ}\leq 0.5$, and that the largest possible value for Σ_W satisfying the quadratic inequality is given by $0.5(1+\sqrt{1-4\Sigma_{WZ}^2})$. Since the condition (3.39) requires Σ_W to be larger than a threshold, without any loss of generality, we can set $\Sigma_W=0.5(1+\sqrt{1-4\Sigma_{WZ}^2})$. Therefore, it remains to verify if h,θ and P satisfy the following two conditions:

$$\begin{aligned}\Sigma_{WZ}&=\frac{h\dfrac{\cos\theta}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2}+\sin^2\theta}\leq 0.5\\\Sigma_W&=\frac{-1+\sqrt{1-4\Sigma_{WZ}^2}}{2}\geq\frac{\dfrac{h^2(1+P\sin^2\theta)}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2}+\sin^2\theta}.\end{aligned}$$

We now show that the above two conditions are satisfied when $h \le h_0(\theta, P)$ by dividing the proof into two cases.

Case 1: Channel parameters satisfy $h \le \sin\theta$. Recall that Σ_{WZ} is given by

$$\Sigma_{WZ} = \frac{h\,\dfrac{\cos\theta}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2} + \sin^2\theta} \le \frac{\sin\theta\,\dfrac{\cos\theta}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2} + \sin^2\theta} = \frac{ab}{a^2+b^2}$$

where we used a and b to denote $\cos\theta/(1+h^2P)$ and $\sin\theta$, respectively. Using the fact that $2ab \le a^2+b^2$, we obtain that $\Sigma_{WZ} \le 0.5$. Using the fact that $(a^2+b^2)^2 - 4a^2b^2 = (a^2-b^2)^2$, we see that the corresponding Σ_W satisfies

$$\begin{aligned}\Sigma_W &= \frac{1+\sqrt{1-4\Sigma_{WZ}^2}}{2} \\ &\ge \frac{\max(a^2,b^2)}{a^2+b^2} \\ &\ge \frac{b^2}{a^2+b^2}.\end{aligned}$$

Now, observe that

$$b^2 = \sin^2\theta \ge \sin^2\theta - \frac{\sin^2\theta - h^2}{1+h^2P} = \frac{h^2(1+P\sin^2\theta)}{1+h^2P}.$$

Thus, we see that the KKT condition (3.39) is satisfied.

Case 2: The channel parameters satisfy

$$\sin\theta < h \le \frac{\cos\theta}{1+h^2P}. \tag{3.40}$$

Observe that the condition

$$\Sigma_W = \frac{2h\,\dfrac{\cos\theta}{1+h^2P}}{\dfrac{\cos^2\theta}{(1+h^2P)^2} + \sin^2\theta} \le 1$$

is equivalent to

$$h^2 - \sin^2\theta \le \left(\frac{\cos\theta}{1+h^2P} - h\right)^2$$

and hence is satisfied because $h \le h_0(\theta, P)$. Observe that the condition (3.39) is satisfied because

$$\Sigma_W = 0.5(1+\sqrt{1-4\Sigma_{WZ}^2}) \ge 0.5 \ge \Sigma_{WZ}$$

and

$$h\cos\theta \ge h^2(1+h^2P) \ge h^2(1+P\sin^2\theta).$$

3.11 Summary and Future Directions

We studied the nonconvex optimization problem of determining the best achievable sum-rate, using Gaussian inputs and treating interference as noise, in the two-user MIMO Gaussian interference channel. We used the idea of genie-aided channel to relax the nonconvex optimization problem and proposed a related convex maxmin optimization problem. The corresponding saddle point solution provides lower and upper bounds to the best achievable sum-rate. We then showed that the resulting upper bound is indeed an upper bound to the sum capacity as well. We also derived necessary and sufficient conditions for the bounds to coincide, leading to an exact characterization of the the best achievable sum-rate with treating interference as noise, and the sum capacity. We then simplified the conditions in the special cases of symmetric MISO and SIMO Gaussian interference channels, and showed that the conditions are equivalent to a threshold condition on the cross-channel gain. Interestingly, the threshold is identical for the symmetric MISO and the dual SIMO interference channels.

3.11.1 Beyond Low Interference Regime

We have derived lower and upper bounds to the sum capacity and the best achievable sum-rate with treating interference as noise, and showed that the bounds coincide in the low interference regime. The upper bound in Theorem 2 is good when the interference levels are low, and can be very loose when the interference levels are high. It is of interest to obtain good lower and upper bounds beyond the low interference regime. Recall that the basic idea in this chapter is to upper-bound the achievable sum-rate $f(\mathbf{Q}_1, \mathbf{Q}_2)$ by a concave function. We now argue that the genie-aided upper bound function $\bar{f}(\mathbf{Q}_1, \mathbf{Q}_2)$ presented in this chapter may not be the best concave upper bound to the sum-rate function $f(\mathbf{Q}_1, \mathbf{Q}_2)$, and present a trick used in [23, 34] to obtain a better upper bound.

The basic idea is to apply Theorem 2 to the interference channels obtained by removing some of the interfering links. Since removing the contribution of an interfering message at a receiver can only improve the sum capacity and the achievable sum-rate, the resulting upper bounds also serve as upper bounds to the original interference channel. There are a total of two interfering links, and hence we have four scenarios corresponding to whether each interfering link is present or not. Therefore, we obtain the following outer bound to the sum capacity:

$$\mathcal{C}_{\mathrm{sum}} \leq \max_{(\mathbf{Q}_1, \mathbf{Q}_2) \in \mathcal{Q}} \min_{0 \leq i \leq 3} \min_{\Psi_i \in \Psi_{iu}} \bar{f}_i(\mathbf{Q}_1, \mathbf{Q}_2, \Psi_i)$$

where

- $\bar{f}_0(\mathbf{Q}_1, \mathbf{Q}_2, \Psi_0)$ and Ψ_{0u} denote the sum-rate of the genie-aided channel and the usefulness set, respectively.

- $\bar{f}_1(\mathbf{Q}_1, \mathbf{Q}_2, \Psi_1)$ and Ψ_{1u} denote the sum-rate of the genie-aided channel and the usefulness set, respectively, obtained by setting $\mathbf{H}_{21} = 0$.
- $\bar{f}_2(\mathbf{Q}_1, \mathbf{Q}_2, \Psi_2)$ and Ψ_{2u} denote the sum-rate of the genie-aided channel and the usefulness set, respectively, obtained by setting $\mathbf{H}_{12} = 0$.
- $\bar{f}_3(\mathbf{Q}_1, \mathbf{Q}_2, \Psi_3)$ and Ψ_{3u} denote the sum-rate of the genie-aided channel and the usefulness set, respectively, obtained by setting both $\mathbf{H}_{12} = \mathbf{H}_{21} = 0$. In this case, since both the interfering links are removed, the minimization over the genie parameters is not required since the corresponding sum-rate function itself is concave.

For each $0 \leq i \leq 3$, let $f_i(\mathbf{Q}_1, \mathbf{Q}_2)$ denote the sum-rate of the interference channels obtained by removing some of the interfering links as explained above, and let $\bar{f}_i(\mathbf{Q}_1, \mathbf{Q}_2)$ denote the corresponding concave upper bound obtained by minimizing over all useful genies. Observe that

$$f_0(\mathbf{Q}_1, \mathbf{Q}_2) \leq f_i(\mathbf{Q}_1, \mathbf{Q}_2), \, \forall i = 1, 2, 3.$$

However, the corresponding upper bounds may not satisfy the same relations. This implies that the $\min_{0 \leq i \leq 3} \bar{f}_i(\mathbf{Q}_1, \mathbf{Q}_2)$ could be smaller than $\bar{f}_0(\mathbf{Q}_1, \mathbf{Q}_2)$, which proves that the genie-aided upper bound presented in this chapter is in general not the best concave upper bound to the sum-rate function. An interesting line of research is to explore other ways of obtaining concave upper bounds to the sum-rate function $f(\mathbf{Q}_1, \mathbf{Q}_2)$.

3.11.2 Symbol Extensions

The concept of symbol extensions is exploited in [41] as a means to improve the achievable rates. But the same idea can be used to obtain better upper bounds as well. For example, by considering two symbols as one super-symbol, we obtain the *multi-letter interference channel*:

$$\begin{aligned}
\underline{\bar{Y}}_1 &= \bar{\mathbf{H}}_{11}\underline{\bar{X}}_1 + \bar{\mathbf{H}}_{12}\underline{\bar{X}}_2 + \underline{\bar{Z}}_1 \\
\underline{\bar{Y}}_2 &= \bar{\mathbf{H}}_{21}\underline{\bar{X}}_1 + \bar{\mathbf{H}}_{22}\underline{\bar{X}}_2 + \underline{\bar{Z}}_2
\end{aligned}$$

where the channel matrices $\bar{\mathbf{H}}_{11}, \bar{\mathbf{H}}_{12}, \bar{\mathbf{H}}_{21}, \bar{\mathbf{H}}_{22}$ are defined as

$$\bar{\mathbf{H}}_{ij} = \begin{bmatrix} \mathbf{H}_{ij} & \\ & \mathbf{H}_{ij} \end{bmatrix}, \, i, j = 1, 2,$$

and the power constraints are given by $2P_1$ and $2P_2$. Observe that the sum capacity of the single-letter interference channel is equal to half of the multi-letter interference channel. However, the upper bound obtained by applying Theorem 2 to the multi-letter interference channel can only be better than that by considering the single-letter interference channel. The reason is as follows.

The genie for the single-letter interference channel has i.i.d. Gaussian noise. By considering the multi-letter interference channel, we essentially are allowing for the Gaussian noise in genie signals to be correlated in time. Since we are considering a larger class of genie parameters, the outer bound can only improve.

3.12 Proof of Claim 5

In this section, we prove that the R.H.S. of (3.38) is equal to

$$\frac{\Sigma_{WZ}^2}{1+h^2|\mathbf{b}^\dagger\mathbf{c}|^2}+\frac{||h\mathbf{c}-t\mathbf{d}||^2}{2\lambda-\mathsf{Tr}\left(\nabla_{\mathbf{Q}_1}f(\mathbf{Q}^*,\mathbf{Q}^*)\right)}=\frac{h^2(1+h^2P)(1+P\sin^2\theta)}{\cos^2\theta+(1+h^2P)^2\sin^2\theta}.$$

Recall that $\mathbf{c}=[1\ 0]^\top$, $\mathbf{d}=[\cos\theta\ \ \sin\theta]^\top$,

$$\mathbf{J}=\mathbf{I}+h^2P\mathbf{c}\mathbf{c}^\dagger=\begin{bmatrix}1+h^2P & 0\\ 0 & 1\end{bmatrix}$$

and that $\mathbf{b}$ is the unit norm vector in the direction of $\mathbf{J}^{-1}\mathbf{d}$; i.e.,

$$\mathbf{b}=\frac{\mathbf{J}^{-1}\mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||}.$$

From (3.35), we obtain that

$$t=h\,\frac{\mathbf{c}^\dagger\mathbf{b}}{\mathbf{d}^\dagger\mathbf{b}}=\frac{\dfrac{h\cos\theta}{1+h^2P}}{\dfrac{\cos^2\theta}{1+h^2P}+\sin^2\theta}=\frac{h\cos\theta}{1+h^2P\sin^2\theta}.$$

Therefore,

$$\begin{aligned}||h\mathbf{c}-t\mathbf{d}||^2&=t^2+h^2-2ht\cos\theta\\&=(t-h\cos\theta)^2+h^2\sin^2\theta\\&=\frac{h^2\cos^2\theta(h^2P\sin^2\theta)^2}{(1+h^2P\sin^2\theta)^2}+h^2\sin^2\theta\\&=h^2\sin^2\theta\left(\frac{h^4P\sin^2\theta\cos^2\theta}{(1+h^2P\sin^2\theta)^2}+1\right)\\&=h^2\sin^2\theta\,\frac{1+2h^2P\sin^2\theta+h^4P^2\sin^2\theta}{(1+h^2P\sin^2\theta)^2}\\&=h^2\sin^2\theta\,\frac{\cos^2\theta+(1+h^2P)^2\sin^2\theta}{(1+h^2P\sin^2\theta)^2}.\end{aligned}$$

From (3.36), we obtain that

$$\begin{aligned}\mathsf{Tr}\left(\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*, \mathbf{Q}^*)\right) &= \frac{1}{\Sigma_{Y_{1G}}}\left(1 - h^2\,\mathrm{SINR}\right)\\ &= \frac{1}{\Sigma_{Y_{1G}}}\left(1 - \frac{h^2 P\cos^2\theta}{1+h^2P} - h^2 P\sin^2\theta\right).\end{aligned}$$

Substituting the expression (3.37) for λ, we obtain

$$\begin{aligned}2\lambda - \mathsf{Tr}\left(\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*, \mathbf{Q}^*)\right) &= \frac{1}{\Sigma_{Y_{1G}}}\left(\frac{\cos^2\theta}{1+h^2P} + \sin^2\theta(1+h^2P)\right)\\ &= \frac{\cos^2\theta + (1+h^2P)^2\sin^2\theta}{(1+h^2P)\Sigma_{Y_{1G}}}.\end{aligned}$$

Therefore,

$$\frac{||h\mathbf{c} - t\mathbf{d}||^2}{2\lambda - \mathsf{Tr}\left(\nabla_{\mathbf{Q}_1} f(\mathbf{Q}^*, \mathbf{Q}^*)\right)} = \frac{h^2\sin^2\theta(1+h^2P)\,\Sigma_{Y_{1G}}}{(1+h^2P\sin^2\theta)^2}.$$

From (3.35), we obtain that

$$\frac{\Sigma_{WZ}^2}{1+h^2P|\mathbf{c}^\dagger\mathbf{b}|^2} = \frac{\Sigma_{WZ}^2}{\mathbf{b}^\dagger\mathbf{J}\mathbf{b}} = t^2\,\mathbf{b}^\dagger\mathbf{J}\mathbf{b} = \frac{h^2\cos^2\theta\,\mathbf{b}^\dagger\mathbf{J}\mathbf{b}}{(1+h^2P\sin^2\theta)^2}.$$

Using the fact that

$$\begin{aligned}\Sigma_{Y_{1G}} &= (\mathbf{b}^\dagger\mathbf{J}\mathbf{b})(1+\,\mathrm{SINR})\\ &= (\mathbf{b}^\dagger\mathbf{J}\mathbf{b})\left(1 + \frac{P\cos^2\theta}{1+h^2P} + P\sin^2\theta\right)\\ &= (\mathbf{b}^\dagger\mathbf{J}\mathbf{b})\frac{1+P+h^2P+h^2P^2\sin^2\theta}{1+h^2P}\end{aligned}$$

we obtain that the R.H.S. of (3.38) is equal to

$$\begin{aligned}&\mathbf{b}^\dagger\mathbf{J}\mathbf{b}\left(\frac{h^2\cos^2\theta}{(1+h^2P\sin^2\theta)^2} + \frac{h^2\sin^2\theta(1+P+h^2P+h^2P^2\sin^2\theta)}{(1+h^2P\sin^2\theta)^2}\right)\\ &= \mathbf{b}^\dagger\mathbf{J}\mathbf{b}\left(\frac{h^2(1+P\sin^2\theta + h^2P\sin^2\theta + h^2P^2\sin^4\theta)}{(1+h^2P\sin^2\theta)^2}\right)\\ &= \mathbf{b}^\dagger\mathbf{J}\mathbf{b}\left(\frac{h^2(1+P\sin^2\theta)(1+h^2P\sin^2\theta)}{(1+h^2P\sin^2\theta)^2}\right)\\ &= \mathbf{b}^\dagger\mathbf{J}\mathbf{b}\left(\frac{h^2(1+P\sin^2\theta)}{1+h^2P\sin^2\theta}\right).\end{aligned}$$

We complete the proof by noting that

$$\mathbf{b}^\dagger\mathbf{J}\mathbf{b} = \frac{\mathbf{d}^\dagger\mathbf{J}^{-1}\mathbf{d}}{||\mathbf{J}^{-1}\mathbf{d}||^2} = \frac{\dfrac{\cos^2\theta}{1+h^2P} + \sin^2\theta}{\dfrac{\cos^2\theta}{(1+h^2P)^2} + \sin^2\theta} = \frac{(1+h^2P)(1+h^2P\sin^2\theta)}{\cos^2\theta + (1+h^2P)^2\sin^2\theta}.$$

Hence the R.H.S. of (3.38) is equal to

$$\frac{h^2(1+h^2P)(1+P\sin^2\theta)}{\cos^2\theta+(1+h^2P)^2\sin^2\theta}.$$

This completes the proof of the claim.

CHAPTER 4

K-USER INTERFERENCE CHANNELS

In the previous chapter, we established the sum capacity of the two-user MIMO Gaussian interference channel in a low interference regime. The intuition is that if the interference is low enough, the receiver will not be able to exploit the structure in the interference, and hence treating interference as noise achieves the sum capacity. In this chapter, we extend the low interference regime results from the two-user case to the K-user case. We focus on the SISO case, where each transmitter and receiver is equipped with single antenna. The K-user SISO Gaussian interference channel is given by

$$Y_i = \sum_{j=1}^{K} h_{ij} X_j + Z_i, i \in \mathcal{K} \tag{4.1}$$

with P_j denoting the average transmit power constraint on transmitters j. Without any loss of generality, by appropriately scaling the power constraints, we assume that the direct channel gains are equal to unity; i.e.,

$$h_{ii} = 1, \forall i \in \mathcal{K}.$$

Since we assumed single transmit antenna, it is easy to determine the lower bound on the sum capacity obtained by using Gaussian inputs (with maximum power) and treating interference as noise:

$$\mathcal{C}_{\text{sum}} \geq \sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}\right) = \sum_{i=1}^{K} \log\left(1 + \frac{P_i}{1 + \sum_{j \neq k} |h_{ij}|^2 P_j}\right)$$

where the subscript G indicates that the inputs are Gaussian distributed (with maximum power). We say that the K-user Gaussian interference channel (4.1) is in low interference regime if the above lower bound is equal to the sum capacity. The objective of this chapter is to derive conditions such that the K-user interference channel belongs to the low interference regime.

We first consider two special cases of the K-user interference channel, introduced in [42, 5]: the *many-to-one interference channel*, where only one user experiences interference, and the *one-to-many interference channel*, where the interference is generated by only one user. For these two special cases, we derive conditions under which the channels belong to the low interference regime.

The genie-aided channel concept, used in the previous chapter, is not required in the upper bound proofs of these two special cases. For the general K-user interference channel, however, the upper bounds are based on the genie-aided channel concept. The basic idea behind the proof is the same as in the previous chapter, and can be summarized as follows:

$$\begin{aligned} \mathcal{C}_{\text{sum}} &\leq \text{sum capacity of the genie-aided channel} \\ &\overset{(a)}{=} \sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}, S_{iG}\right) \\ &\overset{(b)}{=} \sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}\right) \end{aligned} \tag{4.2}$$

where S_i denotes the side-information given to the receiver i. The subscript G indicates that Gaussian inputs (with maximum power) are used. We use the same terminology as in the previous chapter. We say that a genie is useful if step (a) is satisfied; i.e., treating interference as noise with Gaussian inputs achieves the sum capacity of the genie-aided channel. We say that a genie is smart if step (b) is satisfied; i.e., the genie does not improve the achievable sum-rate when Gaussian inputs are used. Therefore, the objective is to determine conditions under which there exists a genie that is both useful and smart.

The structure of the side-information signals provided by the genie is crucial in determining if the genie is useful and smart. When specialized to the SISO case, the two-user genie used in the previous chapter is given by

$$\begin{aligned} S_1 &= h_{21} X_1 + W_1 \\ S_2 &= h_{12} X_2 + W_2. \end{aligned}$$

An interpretation of the two-user genie is that it provides each receiver with a noisy and interference-free observation of the desired signal. In [43], Shang et al. generalized the two-user genie to the K-user case along this observation, and derived sufficient conditions for low interference regime. We refer to their genie as *scalar genie* because it provides each receiver with a scalar signal.

The two-user genie can also be interpreted in a different way. Observe that the side-information S_1 given to the receiver 1 has the same structure as the interference observed by the receiver 2, and the side-information S_2 given to the receiver 2 has the same structure as the interference observed by the receiver 1. By generalizing the two-user genie along this observation, we obtain a *vector genie* that provides each receiver with multiple signals. We show that the vector genie also results in a nontrivial low interference regime. We compare the low interference regimes obtained by the scalar genie and the vector genie in the special case of symmetric interference channels, and observe that neither construction is uniformly better than the other.

4.1 Many-to-One Interference Channel

Consider the many-to-one Gaussian interference channel, where only one user experiences the interference:

$$Y_1 = X_1 + \sum_{j=2}^{K} h_{1j} X_j + Z_1$$
$$Y_i = X_i + Z_i,\ \ i = 2, 3, \cdots, K.$$

Theorem 6. *The sum capacity of the many-to-one interference channel is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = \log\left(1 + \frac{P_1}{\sum_{i=2}^{K} |h_{1i}|^2 P_i}\right) + \sum_{i=2}^{K} \log\left(1 + P_i\right)$$

if the channel parameters satisfy the low interference regime condition:

$$\sum_{i=2}^{K} |h_{1i}|^2 \leq 1.$$

Proof. The achievability is based on the transmitters using Gaussian inputs and the receivers treating interference as noise. We now prove the converse. Using Fano's inequality, we have

$$n(\mathcal{C}_{\text{sum}} - K\epsilon_n) \leq \sum_{i=1}^{K} I(X_i^n; Y_i^n).$$

Therefore, it is sufficient to prove that the R.H.S. of the above equation is maximized by i.i.d. Gaussian inputs (with maximum power). Observe that

$$\begin{aligned}
\sum_{i=1}^{K} I(X_i^n; Y_i^n) &= I(X_1^n; Y_1^n) + \sum_{i=2}^{K} I(X_i^n; Y_i^n) \\
&= \mathsf{h}\left(Y_1^n\right) - \mathsf{h}\left(Y_1^n | X_1^n\right) + \sum_{i=2}^{K} \mathsf{h}\left(Y_i^n\right) - \sum_{i=2}^{K} \mathsf{h}\left(Z_i^n\right).
\end{aligned}$$

The terms $\mathsf{h}\left(Z_i^n\right)$ are independent of the input distributions. From Lemma 16, it follows that the term $\mathsf{h}\left(Y_1^n\right)$ is maximized by i.i.d. Gaussian inputs with maximum power. The remaining terms contribute

$$\sum_{i=2}^{K} \mathsf{h}\left(Y_i^n\right) - \mathsf{h}\left(Y_1^n | X_1^n\right) = \sum_{i=2}^{K} \mathsf{h}\left(X_i^n + Z_i^n\right) - \mathsf{h}\left(\sum_{i=2}^{K} h_{1i} X_i^n + Z_1^n\right).$$

From Lemma 19, it follows that the above expression is maximized by i.i.d. Gaussian inputs with maximum power when the condition $\sum_{i=2}^{K} |h_{1i}|^2 \leq 1$ is satisfied. $\square$

4.2 One-to-Many Interference Channel

Consider the one-to-many Gaussian interference channel, where only one user causes the interference:

$$\begin{aligned} Y_1 &= X_1 + Z_1 \\ Y_i &= X_i + h_{i1}X_1 + Z_i, \;\; i = 2, 3, \cdots, K. \end{aligned} \tag{4.3}$$

Theorem 7. *The sum capacity of the one-to-many interference channel* (4.3) *is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = \log\left(1 + P_1\right) + \sum_{i=2}^{K} \log\left(1 + \frac{P_i}{1 + |h_{i1}|^2 P_1}\right)$$

if the channel parameters satisfy the low interference regime condition:

$$\sum_{i=2}^{K} \frac{h_{i1}^2 P_1 + h_{i1}^2}{h_{i1}^2 P_1 + 1} \leq 1. \tag{4.4}$$

Proof. The achievability is based on the transmitters using Gaussian inputs and the receivers treating interference as noise. We now prove the converse. Using Fano's inequality, we have

$$n(\mathcal{C}_{\text{sum}} - K\epsilon_n) \leq \sum_{i=1}^{K} I(X_i^n; Y_i^n).$$

Therefore, it is sufficient to prove that the R.H.S. of the above equation is maximized by i.i.d. Gaussian inputs (with maximum power). Observe that

$$\begin{aligned} \sum_{i=1}^{K} I(X_i^n; Y_i^n) &= I(X_1^n; Y_1^n) + \sum_{i=2}^{M} I(X_i^n; Y_i^n) \\ &= \mathsf{h}\left(Y_1^n\right) - \mathsf{h}\left(Z_1^n\right) + \sum_{i=2}^{K} \mathsf{h}\left(Y_i^n\right) - \sum_{i=2}^{K} \mathsf{h}\left(Y_i^n | X_i^n\right). \end{aligned}$$

The term $\mathsf{h}\left(Z_1^n\right)$ is independent of the input distributions. From Lemma 16, it follows that the terms $\mathsf{h}\left(Y_i^n\right)$ are maximized by i.i.d. Gaussian inputs with maximum power. The remaining terms contribute

$$\begin{aligned}\mathsf{h}\left(Y_1^n\right) - \sum_{i=2}^{K} \mathsf{h}\left(Y_i^n | X_i^n\right) &= \mathsf{h}\left(X_1^n + Z_1^n\right) - \sum_{i=2}^{K} \mathsf{h}\left(h_{i1} X_1^n + Z_i^n\right) \\ &= \sum_{i=2}^{K} \left(\lambda_i \mathsf{h}(X_1^n + Z_1^n) - \mathsf{h}(h_{i1} X_1^n + Z_i^n)\right)\end{aligned}$$

where λ_i's are nonnegative real numbers satisfying $\sum_{i=2}^{K} \lambda_i = 1$. If the condition (4.4) is satisfied, then we can choose λ_i's satisfying

$$\lambda_i \geq \frac{h_{i1}^2 P_1 + h_{i1}^2}{h_{i1}^2 P_1 + 1}.$$

Observe that the condition (4.4) immediately implies that $|h_{i1}| \leq 1$ for each $i \in \mathcal{K}$. Therefore, from Lemma 19, it follows that the expression

$$\lambda_i \mathsf{h}(X_1^n + Z_1^n) - \mathsf{h}(h_{i1} X_1^n + Z_i^n)$$

is maximized by Gaussian inputs with maximum power. This completes the proof. □

4.3 Scalar Genie

As mentioned in the introduction, the two-user genie provides each receiver with a noisy and interference-free observation of the desired signal. Generalizing this observation to the K-user case, we obtain the genie

$$S_k = X_k + W_k,\ \forall k \in \mathcal{K}$$

where the genie controls how the Gaussian noise random variable W_i is correlated to Z_i. We call this genie a scalar genie because it provides each receiver a scalar signal. Shang et al. [43] derived the conditions under which there exists a scalar genie that is both useful and smart.

Theorem 8 (Shang, Kramer and Chen [43]). *The sum capacity of the K-user SISO Gaussian interference channel* (4.1) *is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = \sum_{i=1}^{K} \log\left(1 + \frac{P_i}{1 + \text{INR}_i}\right)$$

if the channel parameters satisfy the conditions

$$\begin{aligned}
&\sum_{j\neq i}|h_{ij}|^2\frac{(1+\mathrm{INR}_j)^2}{\rho_j^2}+\rho_i^2\leq 1,\ \forall i\in\mathcal{K}\\
\left(\sum_{i\neq j}\frac{|h_{ij}|^2}{\mathrm{INR}_i+1-\rho_i^2}\right)&\left(P_j+\frac{(1+\mathrm{INR}_j)^2}{\rho_j^2}\right)\leq 1,\ \forall j\in\mathcal{K}
\end{aligned} \tag{4.5}$$

for some $\{\rho_i\in[0\ 1]\}_{i=1}^K$. *Here we used* INR_i *to denote the total interference-to-noise ratio at receiver* i*:*

$$\mathrm{INR}_i=\sum_{j\neq i}|h_{ij}|^2P_j.$$

We can verify that Theorem 8, when specialized to the many-to-one and the one-to-many interference channels, simplifies to Theorem 6 and Theorem 7, respectively. For the K-user symmetric interference channel obtained by setting $h_{ij}=h,\ \forall i\neq j$ and $P_j=P,\forall j$, Theorem 8 simplifies to the following corollary.

Corollary 3. *The sum capacity of the K-user symmetric Gaussian interference channel is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\mathrm{sum}}=K\log\left(1+\frac{P}{1+|\hat{h}|^2P}\right)$$

if the channel parameters satisfy the low interference regime condition:

$$|\hat{h}|(1+|\hat{h}|^2P)\leq 0.5$$

where $\hat{h}=\sqrt{K-1}h$.

4.4 Vector Genie

In this section, we explore an alternative way of generalizing the two-user genie to the K-user Gaussian interference channels. For simplicity, we restrict the presentation to the SISO case, but the genie construction can be extended to the MIMO case in a straightforward fashion. First, we provide the intuition behind the choice of our genie. Mathematically speaking, the reason for employing a genie is to combat interference. As explained in Appendix A, the positive differential entropy and conditional differential entropy terms are always maximized by i.i.d. Gaussian inputs (with maximum power), and are concave in the covariance matrices. On the other hand, the negative terms, which arise whenever there is interference, are minimized by i.i.d. Gaussian inputs, and are convex in the covariance matrices. Therefore, it is not clear in general if the sum of the positive and negative terms is maximized by i.i.d. Gaussian inputs or not. However, using the worst-case noise lemma (Lemma 17), the negative terms can be shown to be maximized by i.i.d. Gaussian inputs, and concave in the covariance matrices, if they are coupled with appropriate positive terms. However, the worst-case noise lemma requires the positive terms to have the same signal structure in both the positive and negative terms. For this reason, the two-user genie in Chapter 3 was chosen as

$$\underline{S}_1 = \mathbf{H}_{21}\underline{X}_1 + \underline{W}_1$$
$$\underline{S}_2 = \mathbf{H}_{12}\underline{X}_2 + \underline{W}_2$$

so that the genie signal S_1 provides the positive term to combat the interference seen at receiver 2, and the genie signal S_2 provides the positive term to combat the interference seen at receiver 1. Generalizing this idea, we can provide a signal similar to the interference seen at receiver $i-1$, as side-information to the receiver i:

$$S_i = \sum_{j \neq (i-1)} h_{i-1,j} X_j + W_{i-1} \qquad (\sim Y_{i-1}|X_{i-1}).$$

For the three-user case, the corresponding genie signals are given by

$$S_1 = h_{31}X_1 + h_{32}X_2 + W_1 \qquad (\sim Y_3|X_3)$$
$$S_2 = h_{12}X_2 + h_{13}X_3 + W_2 \qquad (\sim Y_1|X_1)$$
$$S_3 = h_{23}X_3 + h_{21}X_1 + W_3 \qquad (\sim Y_2|X_2).$$

Unlike in the two-user case, the above construction does not suffice in the K-user case because the genie signals are not interference-free. The genie signal S_{i+1} helps in combating the interference seen at receiver i, but in the process it also creates a new (negative) interference term at receiver $i+1$. We can fix this problem by repeating the above process $K-1$ times, which results in the following *vector genie*. For each receiver i, the genie signal $\underline{S}_i$ is a vector of length $K-1$:

$$S_{i,\ell} = \sum_{j \notin \{i-1, i-2, \cdots, i-\ell\}} h_{i-\ell,j} X_j + W_{i,\ell},\ 1\ell \leq K-1.$$

Observe that the $S_{i,\ell}$ has the same structure as $Y_{i-\ell}|X_{i-\ell}, X_{i-\ell+1}, \cdots, X_{i-1}$. For the three-user Gaussian interference channel, the vector genie is given by

$$\begin{aligned}
S_{11} &= h_{31}X_1 + h_{32}X_2 + W_{11} && (\sim Y_3|X_3) \\
S_{12} &= h_{21}X_1 + W_{12} && (\sim S_{31}|X_3 \sim Y_2|X_2X_3) \\
S_{21} &= h_{12}X_2 + h_{13}X_3 + W_{21} && (\sim Y_1|X_1) \\
S_{22} &= h_{32}X_2 + W_{22} && (\sim S_{11}|X_1 \sim Y_3|X_3X_1) \\
S_{31} &= h_{23}X_3 + h_{21}X_1 + W_{31} && (\sim Y_2|X_2) \\
S_{32} &= h_{13}X_3 + W_{32} && (\sim S_{21}|X_2 \sim Y_1|X_1X_2).
\end{aligned}$$

The genie controls how the Gaussian noise random vector $\underline{W}_i$ is correlated to Z_i. As in Chapter 3, we use Ψ to denote the genie parameters collectively:

$$\Psi = \left(\mathbf{\Sigma}_{\underline{W}_1}, \mathbf{\Sigma}_{\underline{W}_1 Z_1}, \mathbf{\Sigma}_{\underline{W}_2}, \mathbf{\Sigma}_{\underline{W}_2 Z_2}, \cdots, \mathbf{\Sigma}_{\underline{W}_K}, \mathbf{\Sigma}_{\underline{W}_K Z_K}\right).$$

4.4.1 Useful Genie

As in Chapter 3, we say that a genie is useful if the sum capacity of the genie-aided channel is achieved by using Gaussian inputs and treating interference as noise. The following lemma provides conditions on the genie parameters such that the genie is useful.

Lemma 5. *If the genie parameters Ψ satisfy the usefulness conditions*

$$\begin{bmatrix} 1 & \mathbf{\Sigma}_{Z_i \underline{W}_i} \\ \mathbf{\Sigma}_{\underline{W}_i Z_i} & \mathbf{\Sigma}_{\underline{W}_i} \end{bmatrix} \succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}_{i-1}} & \mathbf{0}_{K-1\times 1} \\ \mathbf{0}_{1\times K-1} & 0 \end{bmatrix}, \forall i \in \mathcal{K} \tag{4.6}$$

then the genie is useful; i.e., the sum capacity of the genie-aided interference channel is achieved by using Gaussian inputs with maximum power and treating interference as noise; i.e.,

$$\mathcal{C}_{\text{sum}} \le \mathcal{C}_{\text{sum}}^{\text{ga}-\text{ic}} = \sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}, \underline{S}_{iG}\right)$$

where $X_{iG} \sim \mathcal{CN}(0, P_i)$, and Y_{iG} and $\underline{S}_{iG}$ are the corresponding received signal and genie signal, respectively.

Proof. The proof is similar to the proof of Theorem 2 in Chapter 3. Using Fano's inequality, we have

$$n(\mathcal{C}_{\text{sum}}^{\text{ga}-\text{ic}} - K\epsilon_n) \le \sum_{i=1}^{K} \mathsf{I}\left(X_i^n; Y_i^n, \underline{S}_i^n\right).$$

Therefore, it is sufficient to show that the R.H.S. of the above equation is maximized by i.i.d. Gaussian inputs with maximum power. Let $\underline{\tilde{Y}}_i$ denote the vector consisting of all the signals received by receiver i in the genie-aided channel except for $S_{i,K-1}$, and $\underline{\tilde{Z}}$ denote the corresponding noise vector

$$\underline{\tilde{Y}}_i = \begin{bmatrix} Y_i \\ S_{i,1} \\ S_{i,2} \\ \vdots \\ S_{i,K-2} \end{bmatrix}, \underline{\tilde{Z}}_i = \begin{bmatrix} Z_i \\ W_{i,1} \\ W_{i,2} \\ \vdots \\ W_{i,K-2} \end{bmatrix}.$$

Observe that

$$\begin{aligned}
&\sum_{i=1}^{K} \mathsf{I}\left(X_i^n; Y_i^n, \underline{S}_i^n\right) \\
&\quad = \sum_{i=1}^{K} \mathsf{h}\left(Y_i^n, \underline{S}_i^n\right) - \mathsf{h}\left(Y_i^n, \underline{S}_i^n | X_i^n\right) \\
&\quad = \sum_{i=1}^{K} \mathsf{h}\left(Y_i^n, \underline{S}_i^n\right) - \mathsf{h}\left(\underline{\tilde{Y}}_i^n, S_{i,K-1}^n | X_i^n\right) \\
&\quad = \sum_{i=1}^{K} \mathsf{h}\left(\underline{S}_i^n\right) + \mathsf{h}\left(Y_i^n | \underline{S}_i^n\right) - \mathsf{h}\left(S_{i,K-1}^n | X_i^n\right) - \mathsf{h}\left(\underline{\tilde{Y}}_i^n | S_{i,K-1}^n, X_i^n\right).
\end{aligned}$$

By construction, the signal $S_{i,K-1} = h_{i+1,i}X_i + W_{i,K-1}$ is interference-free and hence the term $\mathsf{h}\left(S_{i,K-1}^n | X_i^n\right)$ does not depend on the input distribution. From Lemmas 16 and 13 in Appendix A, it follows that the term $\mathsf{h}\left(Y_i^n | \underline{S}_i^n\right)$ is maximized by i.i.d. Gaussian inputs with maximum power. The remaining terms contribute

$$\sum_{i=1}^{K} \mathsf{h}\left(\underline{S}_i^n\right) - \mathsf{h}\left(\underline{\tilde{Y}}_i^n | S_{i,K-1}^n, X_i^n\right).$$

We complete the proof by showing that the following expression is maximized by i.i.d. Gaussian inputs for each $i \in \mathcal{K}$:

$$\mathsf{h}\left(\underline{S}_{i+1}^n\right) - \mathsf{h}\left(\underline{\tilde{Y}}_i^n | S_{i,K-1}^n, X_i^n\right).$$

From construction of the genie signal, we have that

$$\begin{aligned}
&S_{i+1,1} \sim Y_i | X_{i-1} \\
&S_{i+1,\ell} \sim S_{i,\ell-1} | X_i \text{ for } \ell = 2, 3, \cdots, K.
\end{aligned}$$

Hence the two entropy terms above differ only in the noise terms. Let $\mathbf{G}$ be the matrix such that

$$\underline{S}_{i+1} = \mathbf{G}\underline{X} + \underline{W}_{i+1}$$

where $\underline{X}$ denotes the vector consisting of the transmit signals $X_1, X_2, \cdots, X_K$.

We obtain that

$$\begin{aligned} &\mathsf{h}\left(\underline{S}_{i+1}^n\right) - \mathsf{h}\left(\underline{\tilde{Y}}_i^n | S_{i,K-1}^n, X_i^n\right) \\ &\quad = \mathsf{h}\left(\mathbf{G}\underline{X}^n + \underline{W}_{i+1}^n\right) - \mathsf{h}\left(\mathbf{G}\underline{X}^n + \underline{\tilde{Z}}_i^n | W_{i,K-2}^n\right). \end{aligned}$$

Using Lemmas 17 and 14, it follows that the above expression is maximized by i.i.d. Gaussian inputs with maximum power if

$$\text{Cov}\left(\underline{W}_{i+1}\right) \preceq \text{Cov}\left(\underline{\tilde{Z}}_i | W_{i,K-2}\right).$$

From Lemma 9, the above condition is equivalent to

$$\left[\begin{array}{cc} \mathbf{\Sigma}_{\underline{W}_{i+1}} & \mathbf{0}_{K-1\times 1} \\ \mathbf{0}_{1\times K-1} & 0 \end{array}\right] \preceq \text{Cov}\left(\left[\begin{array}{c} \underline{\tilde{Z}}_i \\ W_{i,K-2} \end{array}\right]\right) = \text{Cov}\left(\left[\begin{array}{c} Z_i \\ \underline{W}_i \end{array}\right]\right)$$

where the last equality follows from the definition of $\underline{\tilde{Z}}_i$. □

4.4.2 Smart Genie

As in Chapter 3, we say that a genie is smart if the achievable sum-rate in the genie-aided channel is equal to the achievable sum-rate in the original interference channel. In this section, we derive conditions on the genie parameters such that the genie is smart. Before we present the smartness conditions, we summarize the discussion in Section 3.8 in the following lemma.

Lemma 6. *Suppose $\underline{X}_G \sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{X}_G})$ and $\underline{Y}_G$ and $\underline{S}_G$ are noisy-observations of $\underline{X}_G$:*

$$\begin{aligned} \underline{Y}_G &= \mathbf{H}_1\underline{X}_G + \underline{N}_1 \\ \underline{S}_G &= \mathbf{H}_2\underline{X}_G + \underline{N}_2 \end{aligned}$$

where $\underline{N}_1$ and $\underline{N}_2$ are jointly circularly symmetric, and jointly Gaussian complex random vectors. Then,

$$\mathsf{I}\left(\underline{X}_G; \underline{Y}_G, \underline{S}_G\right) = \mathsf{I}\left(\underline{X}_G; \underline{Y}_G\right)$$

if and only if the following condition is satisfied:

$$\left(\mathbf{H}_2 - \mathbf{\Sigma}_{\underline{N}_2\underline{N}_1}\mathbf{\Sigma}_{\underline{N}_1}^{-1}\mathbf{H}_1\right)\mathbf{\Sigma}_{\underline{X}_G} = 0.$$

Proof. The lemma follows immediately by replacing $\underline{X}_{1G}, \underline{Y}_{1G}, \underline{S}_{1G}, \mathbf{H}_{12}\underline{X}_{2G} + \underline{Z}_1, \underline{W}_1$ in the discussion of Section 3.8 by $\underline{X}_G, \underline{Y}_G, \underline{S}_G, \underline{N}_1, \underline{N}_2$ respectively. □

We now use the above lemma to derive the smartness conditions.

Lemma 7. *The genie is smart, i.e.,*

$$\sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}, \underline{S}_{iG}\right) = \sum_{i=1}^{K} \mathsf{I}\left(X_{iG}; Y_{iG}\right)$$

if the genie parameters Ψ *satisfy the smartness conditions*

$$\Sigma_{W_{i\ell} Z_i} = h_{i-\ell,i}\left(1 + \mathrm{INR}_i\right) - \sum_{j \notin \{i, i-1, \cdots, i-\ell\}} h_{i-\ell,j} h_{ij}^{\dagger} P_j \tag{4.7}$$

for all $1 \le i \le K$ *and* $1 \le \ell \le K-1$. *Recall that* INR_i *to denote the total interference-to-noise ratio at receiver* i*:*

$$\mathrm{INR}_i = \sum_{j \ne i} |h_{ij}|^2 P_j.$$

Proof. First, observe that the genie is smart if and only if

$$\mathsf{I}\left(X_{iG}; Y_{iG}, \underline{S}_{iG}\right) = \mathsf{I}\left(X_{iG}; Y_{iG}\right), \ \forall i \in \mathcal{K}.$$

Recall that

$$Y_{iG} = X_{iG} + \underbrace{\sum_{j \ne i} h_{ij} X_{jG} + Z_i}_{N_1}$$

and that

$$S_{i,\ell G} = h_{i-\ell,i} X_{iG} + \underbrace{\sum_{j \notin \{i, i-1, \cdots, i-\ell\}} h_{i-\ell,j} X_{jG} + W_{i\ell}}_{N_{2\ell}}.$$

Using Lemma 6, we see that $\mathsf{I}\left(X_{iG}; Y_{iG}, \underline{S}_{iG}\right) = \mathsf{I}\left(X_{iG}; Y_{iG}\right)$ if and only if

$$h_{i-\ell,i} = \frac{\Sigma_{W_{i\ell} Z_i} + \sum_{j \notin \{i, i-1, \cdots, i-\ell\}} h_{i-\ell,j} h_{ij}^{\dagger} P_j}{1 + \mathrm{INR}_i}$$

is satisfied for each $1 \le \ell \le K-1$. □

4.4.3 Low Interference Regime

Combining Lemmas 5 and 7, we obtain the following theorem.

Theorem 9. *The sum capacity of the* K*-user SISO Gaussian interference channel is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\mathrm{sum}} = \sum_{i=1}^{K} \log\left(1 + \frac{P_i}{1 + \sum_{j \ne i} |h_{ij}|^2 P_j}\right)$$

if there exist genie parameters

$$\Psi = \left(\mathbf{\Sigma}_{\underline{W}_1}, \mathbf{\Sigma}_{\underline{W}_1 Z_1}, \mathbf{\Sigma}_{\underline{W}_2}, \mathbf{\Sigma}_{\underline{W}_2 Z_2}, \cdots, \mathbf{\Sigma}_{\underline{W}_K}, \mathbf{\Sigma}_{\underline{W}_K Z_K}\right)$$

satisfying the usefulness conditions (4.6) *and the smartness conditions* (4.7).

4.4.4 Symmetric Interference Channel

In this section, we consider the K-user symmetric interference channel

$$Y_i = X_i + h \sum_{j \neq i} X_j + Z_i$$

with symmetric power constraint, i.e., $P_j = P$, $\forall j \in \mathcal{K}$, and simplify the conditions in Theorem 9. We restrict our attention to only symmetric genie parameters; i.e., we assume

$$\begin{aligned} \mathbf{\Sigma}_{\underline{W}_i} &= \mathbf{\Sigma}_{\underline{W}} \\ \mathbf{\Sigma}_{\underline{W}_i Z_i} &= \mathbf{\Sigma}_{\underline{W} Z}. \end{aligned}$$

We now proceed to simplify the conditions in Theorem 9. The smartness conditions (4.7) determine the parameter $\mathbf{\Sigma}_{\underline{W}Z} = \mathbf{a}$, where

$$a_\ell = h(1 + (K-1)|h|^2 P) - (K - \ell - 1)|h|^2 P,\ 1 \leq \ell \leq K-1. \tag{4.8}$$

Therefore, it remains to verify the existence of $\mathbf{\Sigma}_{\underline{W}} \succeq 0$ satisfying the usefulness conditions

$$\begin{bmatrix} 1 & \mathbf{a}^\dagger \\ \mathbf{a} & \mathbf{\Sigma}_{\underline{W}} \end{bmatrix} \succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{W}} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

Thus, we obtain the following theorem.

Theorem 10. *The sum capacity of the three-user symmetric Gaussian interference channel is achieved by using Gaussian inputs and treating interference as noise, and is given by*

$$\mathcal{C}_{\text{sum}} = K \log\left(1 + \frac{P}{1 + (K-1)|h|^2 P}\right)$$

if there exists a $(K-1) \times (K-1)$ *positive semidefinite matrix* $\mathbf{\Sigma}$ *satisfying the condition*

$$\begin{bmatrix} 1 & \mathbf{a}^\dagger \\ \mathbf{a} & \mathbf{\Sigma} \end{bmatrix} \succeq \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \tag{4.9}$$

where the vector $\mathbf{a} \in$ *is as defined in* (4.8).

We now assume that $K = 3$ and h is a real number, and determine the range of h such that the three-user symmetric Gaussian interference channel is in the low interference regime. Let $\mathcal{A}$ denote the set of (a_1, a_2) such that there exists $\mathbf{\Sigma} \succeq 0$ satisfying (4.9). Clearly, the set $\mathcal{A}$ must be convex. We would like to determine the implicit equations in a_1 and a_2 characterizing the boundary of the the set $\mathcal{A}$ so that the conditions in Theorem 10 can further be simplified. It is not clear if it is possible to do so. Since we assumed h is a real number, we have that a_1 and a_2 are also real numbers. In Figure 4.1, we plot the boundary of the feasible set $\mathcal{A}$.

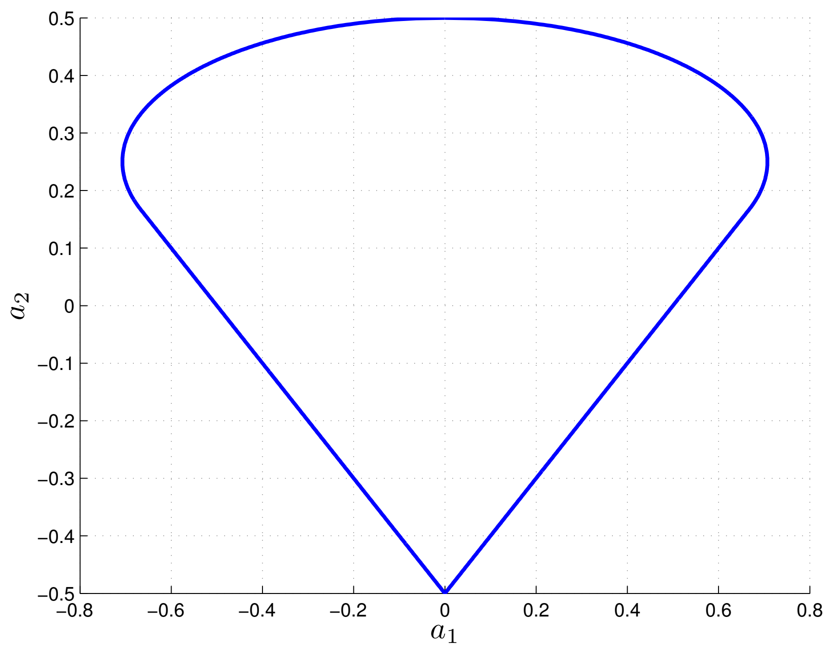

Figure 4.1: The boundary of the feasible set $\mathcal{A}$, defined as the set of parameters (a_1, a_2) for which there exists $\mathbf{\Sigma} \succeq 0$ satisfying condition (4.9) in Theorem 10.

Using this region, we numerically determine the range of real h such that the three-user symmetric Gaussian interference channel is in the low interference regime. We observe that for every fixed $P \geq 0$, the channel is in low interference regime if h satisfies the threshold criterion

$$-h_0^-(P) \leq h \leq h_0^+(P).$$

Interestingly, the positive and negative thresholds $h_0^+(P)$ and $h_0^-(P)$ are not equal. The positive threshold $h_0^+(P)$ is in general greater than the negative threshold $h_0^-(P)$. This is in contrast to the threshold criterion obtained by the scalar genie. Recall from Section 4.3 that the scalar genie provides a threshold criterion of the form

$$|h| \leq h_0(P)$$

where $h_0(P)$ is the (unique) positive solution to the equation

$$2\sqrt{2}h(1 + 2h^2 P) = 1.$$

The results are summarized in Figure 4.2, where we plot the three curves corresponding to $\mathrm{INR}^+ = 2\left(h_0^+\right)^2 P$, $\mathrm{INR}^- = 2\left(h_0^-\right)^2 P$ and $\mathrm{INR} = 2h_0^2 P$ as a function of $\mathrm{SNR} = P$. It can be observed that neither the vector genie nor the scalar genie is strictly better compared to the other.

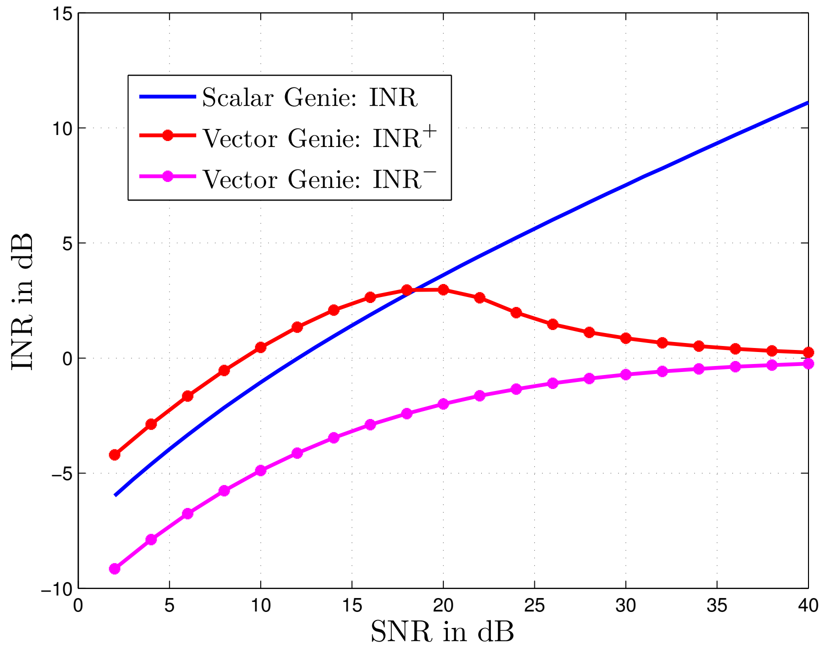

Figure 4.2: The INR threshold as a function of the SNR, below which treating interference as noise achieves the sum capacity of the three-user symmetric Gaussian interference channel.

4.5 Summary and Future Directions

We extended the sufficient conditions for the low interference regime, presented in the previous chapter for the two-user case, for the K-user case. We proposed the vector genie construction in order to obtain tight upper bounds on the sum capacity of the K-user SISO Gaussian interference channels. The advantage of the vector genie construction is that it is intuitive, and can be easily generalized to the MIMO case. Following the steps in the previous chapter, such a generalization can be used to obtain a convex maxmin optimization problem which facilitates the numerical computation of the optimal covariance matrices in the low interference regime. The disadvantage of the vector genie construction is that, in the symmetric case, the low interference regime condition depends on the phase of the cross-channel coefficient, which is counter-intuitive. Also, the low interference regime condition obtained by the vector genie is not uniformly better than that obtained by the scalar genie. A hybrid genie construction combining the good features of the scalar and vector genies may lead to further insights on the optimality of treating interference as noise in the K-user interference channels. An interesting question that remains to be answered is: *How does the optimal interference threshold scale as a function of the number of users in the symmetric K-user Gaussian interference channel?*

CHAPTER 5

COMP CHANNEL

In the previous chapters, we studied the problem of determining the sum capacity of Gaussian interference channels. Even if we can determine and implement the best possible achievable schemes, the demand for wireless connectivity is likely to exceed what the physical channel can offer. For this and other reasons, there has been much interest in understanding the fundamental limits of cooperative interference networks. Typically cooperation requires additional infrastructure, but it could be cost-effective depending on the overall objective. It should be noted that implementing advanced beamforming algorithms, obtained by studying the interference channels, would anyway require coordination between the base stations in order to exchange the required channel knowledge and compute the beams in a distributed or centralized fashion. Typically, this coordination is achieved through low latency and high speed fiber optic or microwave backhaul networks. In the context of cellular downlink, we could go one step further and assume that the backhaul is strong enough for the base stations to exchange and jointly transmit the individual messages to efficiently mitigate the interference at unintended receivers. This procedure is referred to as Coordinated Multi-Point (CoMP) transmission, which has been considered as the physical layer interference management technique in the fourth generation cellular standards such as LTE-Advanced. Similarly, in the context of cellular uplink, we can assume that the receivers cooperate over the backhaul to jointly receive and decode the messages. This procedure is referred to as CoMP reception. In this chapter, we combine both the techniques and consider the CoMP channel where the messages are jointly transmitted and jointly received by multiple transmitters and receivers, respectively.

We capture the cost of cooperation through numbers M_t and M_r which are referred to as the *transmit cooperation order* and the *receive cooperation order*, respectively. The transmit cooperation order M_t denotes the number of transmitters that jointly transmit each message, and the receive cooperation order M_r denotes the number of receivers that jointly decode each message. Observe that CoMP channel is a generalization of the interference channel, which can be recovered by setting $M_t = M_r = 1$. Our objective in this chapter is to understand the benefits from cooperation. As we have observed in the previous chapters, determining the sum capacity is a difficult problem, even in the case of the interference channels. Therefore we study the DoF, which provides insights about the high SNR behavior of sum capacity, as a function of the transmit and receive cooperation orders. Before we state the main results, we need to explain the channel model in more detail.

CoMP transmission (also known as network-MIMO, virtual-MIMO and multi-cell-MIMO) has been identified as one of the study items for fourth generation cellular systems such as LTE-Advanced. There has been considerable interest in devising practical cooperative schemes that improve on uncoordinated schemes, and in estimating the tradeoff between the performance benefits and the additional overhead due to cooperation [44, 45, 46]. Also, we note that CoMP transmission and reception is just one of the many possible ways for partial transmitter and receiver cooperation in the interference channel. In [47, 48], it is assumed that the nodes can both transmit and receive in full-duplex. In [49, 50], the presence of noise-free finite-capacity links between the transmitter nodes or the receiver nodes is assumed. In [51], the receivers are allowed to exchange the decoded messages over a backhaul link to enable interference cancelation.

5.1 Channel Model

We consider transmitting K independent messages over the SISO Gaussian interference channel with K transmitters and K receivers

$$Y_i = \sum_{j=1}^{K} h_{ij} X_j + Z_i, \ \forall i \in \mathcal{K}. \tag{5.1}$$

In fact, we consider L such parallel Gaussian interference channels, providing the encoders and decoders an opportunity to jointly encode and jointly decode the messages over the L parallel channels. We can combine the L parallel channels and express them together as one MIMO Gaussian interference channel

$$\underline{Y}_i = \sum_{j=1}^{K} \mathbf{H}_{ij} \underline{X}_j + \underline{Z}_i, \ \forall i \in \mathcal{K} \tag{5.2}$$

such that the channel transfer matrices are square and diagonal. The channel transfer matrix $\mathbf{H}_{ij}$ is given by

$$\mathbf{H}_{ij} = \begin{bmatrix} h_{ij}(1) & & & \\ & h_{ij}(2) & & \\ & & \ddots & \\ & & & h_{ij}(L) \end{bmatrix}$$

where $h_{ij}(\ell)$ denotes the channel coefficient from transmitter j to receiver i in the ℓth parallel channel. The reason for considering the parallel channels will be clear at a later stage.

In the CoMP setup, the messages are jointly transmitted and jointly received by multiple transmitters and multiple receivers, respectively. For each $k \in \mathcal{K}$, the message W_k is transmitted jointly by the transmitters from the *transmit set* $\mathcal{T}_k$ given by

$$\mathcal{T}_k = k \uparrow M_t = \{k, k+1, \cdots, k+M_t-1\} \tag{5.3}$$

and is received jointly by the receivers from the *receive set* $\mathcal{R}_k$ given by

$$\mathcal{R}_k = k \uparrow M_r = \{k, k+1, \cdots, k+M_r-1\}. \tag{5.4}$$

Thus the CoMP channel is specified by the parameters K, M_t, M_r and L, denoting the number of users, transmit cooperation order, receive cooperation order, and the number of parallel channels, respectively. Recall from Section 2.4 that the DoF is defined as the supremum number d_Σ such that a sum-rate of $d_\Sigma \log(P/\Delta)$ is achievable where Δ is a constant that is independent of the power constraint P. In general, this number can depend on the specific realizations of channel coefficients

$$h_{ij}(\ell) : i, j \in \mathcal{K}, 1 \leq \ell \leq L.$$

However, we ignore this dependency because the DoF, in all the known cases, is the same for all *generic channel coefficients*. We refer the reader to Appendix B for a precise definition of the generic property. Let $\mathrm{DoF}(K, M_t, M_r, L)$ denote the sum DoF of the CoMP channel normalized over the number of parallel channels, and let $\mathrm{DoF}(K, M_t, M_r)$ denote the asymptotic normalized sum DoF, i.e.,

$$\mathrm{DoF}(K, M_t, M_r) = \limsup_{L\to\infty} \mathrm{DoF}(K, M_t, M_r, L).$$

We say that the DoF is independent of the number of parallel channels L and is equal to some number d_Σ if and only if $\mathrm{DoF}(K, M_t, M_r, L) = d_\Sigma$ for all $L \geq 1$.

5.2 Related Work

Observe that the K-user Gaussian interference channel is a special case of the CoMP channel, and can be recovered by setting $M_t = M_r = 1$ so that no cooperation is allowed either at the transmitters or at the receivers. In [41], Cadambe and Jafar exploited the channel diversity obtained by considering the parallel channels and proposed a scheme that achieves $K/2$ DoF in an asymptotic fashion. It was already known that the DoF is upper-bounded by $K/2$ [52]. The Cadambe-Jafar achievable scheme is a linear beamforming scheme that operates on L-parallel Gaussian interference channels simultaneously to create d

interference-free channels per user such that $d \to L/2$ as $L \to \infty$, thus proving that

$$\mathrm{DoF}(K, 1, 1) = \lim_{L\to\infty} Kd/L = K/2.$$

Other special cases of the the CoMP channel have been studied in the past under different names such as cognitive interference channel [53, 54, 55, 56, 57], interference channel with local or partial side-information [58, 59], interference channel with clustered decoding [60], or a combination thereof [61]. However, the DoF of the CoMP channel has not been determined except in the following special cases:

1. $(M_t, M_r) = (K, 1)$: With perfect cooperation at the transmitters, we see that each parallel channel is equivalent to the K-user MISO broadcast channel with K transmit antennas. Therefore, we obtain that the DoF is independent of L and is equal to K.

2. $(M_t, M_r) = (1, K)$: With perfect cooperation at the receivers, we see that each parallel channel is equivalent to the K-user SIMO multiple access channel with K receive antennas. Therefore, we obtain that the DoF is independent of L and is equal to K.

3. $(M_t, M_r) = (K, K)$: With perfect cooperation at the transmitters and at the receivers, we see that each parallel channel is equivalent to the point-to-point MIMO channel with K transmit antennas and K receive antennas. Therefore, we obtain that the DoF is independent of L and is equal to K.

4. $(M_t, M_r) = (K-1, 1)$: It is shown in [53] that the DoF is independent of L and is equal to $K-1$. The achievable scheme is again based on linear transmit beamforming. Since each message is transmitted jointly using $K-1$ transmit antennas, a zero-forcing beam vector can be used to perfectly null out the interference at $K-1$ receivers. By only scheduling $K-1$ users, it is clear that a sum DoF of $K-1$ can be achieved per each parallel channel. It is easy to see that a similar argument holds true when $M_t = 1$ and $M_r = K-1$.

To summarize, we know the following results:

$$\mathrm{DoF}(K, M_t, M_r) = \begin{cases} K/2 & (M_t, M_r) = 1 \\ K-1 & (M_t, M_r) = (K-1, 1) \text{ or } (1, K-1) \\ K & \max(M_t, M_r) = K. \end{cases}$$

5.3 Outer Bounds

In this section, we derive an outer bound on the DoF as function of K, M_t and M_r. First, we present an outer bound on the DoF region of the CoMP channel with arbitrary transmit and receive sets, i.e., without explicitly using the structure of the transmit sets (5.3) and the receiver sets (5.4).

5.3.1 Outer Bound on DoF Region

Theorem 11. *Any point $(d_1, d_2, \cdots, d_K)$ in the normalized (by the number of parallel channels) DoF region of the CoMP channel with generic channel coefficients satisfies the inequalities:*

$$\sum_{k:\mathcal{T}_k \subseteq \mathcal{A} \text{ or } \mathcal{R}_k \subseteq \mathcal{B}} d_k \leq \max(|\mathcal{A}|, |\mathcal{B}|), \forall \mathcal{A}, \mathcal{B} \subseteq \mathcal{K}. \tag{5.5}$$

Proof. Without any loss of generality, we can assume $|\mathcal{A}| = |\mathcal{B}|$. Otherwise, the smaller set can be blown up to add more terms on the L.H.S. of (5.5) without affecting the R.H.S., resulting in an inequality that is stricter than what we need to prove. Now, the objective is to show that

$$\sum_{k:\mathcal{T}_k \subseteq \mathcal{A} \text{ or } \mathcal{R}_k \subseteq \mathcal{B}} d_k \leq |\mathcal{B}|. \tag{5.6}$$

Define the subsets

$$\begin{aligned} \mathcal{W}_t &= \{W_k : \mathcal{T}_k \subseteq \mathcal{A}\} \\ \mathcal{W}_r &= \{W_k : \mathcal{R}_k \subseteq \mathcal{B}, \mathcal{T}_k \nsubseteq \mathcal{A}\} \end{aligned}$$

and $\mathcal{W}_f$ as the set of free messages that do not appear in either of the sets $\mathcal{W}_r$ and $\mathcal{W}_t$. The proof idea is to start with the signals received by the receivers $\mathcal{B}$, and show that the messages $\mathcal{W}_t$ and $\mathcal{W}_r$ can be decoded using these $|\mathcal{B}|$ received signals with $\mathcal{W}_f$ as side-information. For any given subset $\mathcal{S} \subseteq \mathcal{K}$, we use the notation $\underline{X}_{\mathcal{S}}$ to denote the vector made up of the signals transmitted by the transmitters in the set $\mathcal{S}$, with a similar notation used for $\underline{Y}_{\mathcal{S}}$ and $\underline{Z}_{\mathcal{S}}$.

For each k, using Fano's inequality and the definition of the receive set $\mathcal{R}_k$, we have that any reliable communication scheme must satisfy

$$\mathsf{H}\left(W_k | \underline{Y}^n_{\mathcal{R}_k}\right) \leq n\epsilon_n \tag{5.7}$$

where $\epsilon_n \to 0$, as $n \to \infty$. Therefore, we immediately have

$$\mathsf{H}\left(\mathcal{W}_r | \underline{Y}^n_{\mathcal{B}}\right) \leq \sum_{k:\mathcal{R}_k \in \mathcal{B}} \mathsf{H}\left(W_k | \underline{Y}^n_{\mathcal{B}}\right) \leq |\mathcal{W}_r| n\epsilon_n \tag{5.8}$$

i.e., the messages $\mathcal{W}_r$ can be decoded by the receivers $\mathcal{B}$. Similarly, the messages $\mathcal{W}_t$ can be decoded using all the received signals:

$$\mathsf{H}\left(\mathcal{W}_t | \underline{Y}^n_{\mathcal{K}}, \mathcal{W}_r, \mathcal{W}_f\right) \leq \mathsf{H}\left(\mathcal{W}_t | \underline{Y}^n_{\mathcal{K}}\right) \leq |\mathcal{W}_t| n\epsilon_n. \tag{5.9}$$

But, we need to show that the messages $\mathcal{W}_t$ can also be decoded by the receivers

$\mathcal{B}$ with $\mathcal{W}_f$ as side-information. We do so by arguing that the signal contribution in $\underline{Y}_{\mathcal{K}}^n$ can be reconstructed using $\mathcal{W}_f, \mathcal{W}_r$ and $\underline{Y}_{\mathcal{B}}^n$:

$$\begin{aligned}
&\mathsf{H}\left(\mathcal{W}_t|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_r, \mathcal{W}_f\right)\\
&\quad\leq \mathsf{H}\left(\mathcal{W}_t|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_r, \mathcal{W}_f\right) - \mathsf{H}\left(\mathcal{W}_t|\underline{Y}_{\mathcal{K}}^n, \mathcal{W}_r, \mathcal{W}_f\right) + |\mathcal{W}_t|n\epsilon_n\\
&\quad= \mathsf{I}\left(\mathcal{W}_t; \underline{Y}_{\mathcal{B}^c}^n|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f, \mathcal{W}_r\right) + |\mathcal{W}_t|n\epsilon_n\\
&\quad= \mathsf{h}\left(\underline{Y}_{\mathcal{B}^c}^n|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f, \mathcal{W}_r\right) - \mathsf{h}\left(\underline{Y}_{\mathcal{B}^c}^n|\mathcal{W}_f, \mathcal{W}_r, \mathcal{W}_t\right) + |\mathcal{W}_t|n\epsilon_n\\
&\quad= \mathsf{h}\left(\underline{Y}_{\mathcal{B}^c}^n|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f, \mathcal{W}_r\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}^c}^n\right) + |\mathcal{W}_t|n\epsilon_n\\
&\quad\leq \mathsf{h}\left(\underline{Y}_{\mathcal{B}^c}^n|\underline{Y}_{\mathcal{B}}^n, \underline{X}_{\mathcal{A}^c}^n\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}^c}^n\right) + |\mathcal{W}_t|n\epsilon_n.
\end{aligned}$$

Observe that, over each symbol, we have

$$\begin{aligned}
\underline{Y}_{\mathcal{B}^c} &= \mathbf{H}(\mathcal{B}^c, \mathcal{A})\underline{X}_{\mathcal{A}} + \mathbf{H}(\mathcal{B}^c, \mathcal{A}^c)\underline{X}_{\mathcal{A}^c} + \underline{Z}_{\mathcal{B}^c}\\
\underline{Y}_{\mathcal{B}} &= \mathbf{H}(\mathcal{B}, \mathcal{A})\underline{X}_{\mathcal{A}} + \mathbf{H}(\mathcal{B}, \mathcal{A}^c)\underline{X}_{\mathcal{A}^c} + \underline{Z}_{\mathcal{B}}
\end{aligned}$$

where we used $\mathbf{H}$ to denote the $KL \times KL$ channel transfer matrix from all the K transmitters to the K receivers, i.e.,

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \cdots & \mathbf{H}_{1K}\\ \vdots & \ddots & \vdots\\ \mathbf{H}_{K1} & \cdots & \mathbf{H}_{KK} \end{bmatrix}$$

and $\mathbf{H}(\mathcal{B}, \mathcal{A})$ to denote the $|\mathcal{B}|L \times |\mathcal{A}|L$ channel transfer matrix from transmitters $\mathcal{A}$ to the receivers $\mathcal{B}$, and $\mathbf{H}(\mathcal{B}^c, \mathcal{A}^c)$, $\mathbf{H}(\mathcal{B}, \mathcal{A}^c)$ and $\mathbf{H}(\mathcal{B}^c, \mathcal{A})$ to denote appropriate submatrices. For generic channel coefficients, since we assumed that $|\mathcal{A}| = |\mathcal{B}|$, the matrix $\mathbf{H}(\mathcal{B}, \mathcal{A})$ is invertible, and hence we have

$$\begin{aligned}
\underline{\tilde{Z}} = \underline{Y}_{\mathcal{B}^c} &- \mathbf{H}(\mathcal{B}^c, \mathcal{A})\underline{X}_{\mathcal{A}^c}\\
&- \mathbf{H}(\mathcal{B}^c, \mathcal{A})\mathbf{H}(\mathcal{B}, \mathcal{A})^{-1}\left(\underline{Y}_{\mathcal{B}} - \mathbf{H}(\mathcal{B}, \mathcal{A})\underline{X}_{\mathcal{A}^c}\right)\\
= \underline{Z}_{\mathcal{B}^c} &- \mathbf{H}(\mathcal{B}^c, \mathcal{A})\mathbf{H}(\mathcal{B}, \mathcal{A})^{-1}\underline{Z}_{\mathcal{B}}.
\end{aligned}$$

Thus, we get

$$\mathsf{H}\left(\mathcal{W}_t|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_r, \mathcal{W}_f\right) \leq \mathsf{h}\left(\underline{\tilde{Z}}^n\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}^c}^n\right) + |\mathcal{W}_t|n\epsilon_n.$$

Therefore, we have

$$\begin{aligned}
\mathsf{H}\left(\mathcal{W}_r, \mathcal{W}_t\right) &\leq \mathsf{H}\left(\mathcal{W}_r, \mathcal{W}_t|\mathcal{W}_f\right)\\
&= \mathsf{I}\left(\mathcal{W}_r, \mathcal{W}_t; \underline{Y}_{\mathcal{B}}^n|\mathcal{W}_f\right) + \mathsf{H}\left(\mathcal{W}_r, \mathcal{W}_t|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f\right)\\
&= \mathsf{h}\left(\underline{Y}_{\mathcal{B}}^n|\mathcal{W}_f\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}}^n\right) + H(\mathcal{W}_r|\underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f) + H(\mathcal{W}_t|\mathcal{W}_r, \underline{Y}_{\mathcal{B}}^n, \mathcal{W}_f)\\
&\leq \mathsf{h}\left(\underline{Y}_{\mathcal{B}}^n\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}}^n\right) + \mathsf{h}\left(\underline{\tilde{Z}}^n\right) - \mathsf{h}\left(\underline{Z}_{\mathcal{B}^c}^n\right) + (|\mathcal{W}_t| + |\mathcal{W}_r|)n\epsilon_n.
\end{aligned}$$

Observe that all the terms, except for $\mathsf{h}\left(\underline{Y}_{\mathcal{B}}^n\right)$, are independent of the power constraint P. Furthermore, the sequence $\underline{Y}_{\mathcal{B}}^n$ denotes a vector of length $n|\mathcal{B}|L$. Therefore, there must exist a constant c that may depend on the channel coefficients, but is independent of the power constraint P and the block length n such that

$$\mathsf{H}(\mathcal{W}_r, \mathcal{W}_t) \leq n|\mathcal{B}|L\log P + nc + (|\mathcal{W}_t| + |\mathcal{W}_r|)n\epsilon_n.$$

Therefore, any achievable rate tuple $(R_1, R_2, \cdots, R_K)$ must satisfy

$$\sum_{k:\mathcal{T}_k\subseteq\mathcal{A} \text{ or } \mathcal{R}_k\subseteq\mathcal{B}} R_k \leq |\mathcal{B}|L\log P + c$$

which immediately implies that any achievable DoF vector (normalized by the number of parallel channels L) must satisfy (5.6). □

5.3.2 Outer Bound on Sum DoF

We use Theorem 11 to obtain an outer bound on $\text{DoF}(K, M_t, M_r, L)$. Observe that an obvious outer bound given by

$$\text{DoF}(K, M_t, M_r, L) \leq K$$

can be obtained by setting $\mathcal{A} = \mathcal{B} = \mathcal{K}$. The following theorem provides a nontrivial outer bound when $M_t + M_r \leq K$.

Theorem 12. *The (normalized sum) DoF of the CoMP channel with generic channel coefficients satisfies*

$$DoF(K, M_t, M_r, L) \leq \left\lceil \frac{K + M_t + M_r - 2}{2} \right\rceil.$$

When $K + M_t + M_r$ is odd, the above outer bound can be improved to obtain

$$DoF(K, M_t, M_r, L) \leq \frac{K}{K-1}\frac{K + M_t + M_r - 3}{2}.$$

Proof. First, observe that the stated outer bounds are weak compared to the obvious outer bound $\text{DoF}(K, M_t, M_r, L) \leq K$ if $M_t + M_r \geq K + 1$. Therefore, we assume that $M_t + M_r \leq K$ in proving the theorem. The best outer bound on $\text{DoF}(K, M_t, M_r)$ that we can obtain using Theorem 11 is obtained by solving the linear program

$$\max_{(d_1,\cdots,d_K)} d_1 + d_2 \cdots + d_K \tag{5.10}$$

subject to the constraints (5.5), given by

$$\sum_{k\in\mathcal{K}:\mathcal{T}_k\subseteq\mathcal{A} \text{ or } \mathcal{R}_k\subseteq\mathcal{B}} d_k \leq r$$

for every $\mathcal{A}, \mathcal{B} \subseteq \mathcal{K}$ such that $|\mathcal{A}| = |\mathcal{B}| = r$. Since the the transmit sets (5.3) and receive sets (5.4) are symmetric across the transmitter and receiver indices, by appropriately averaging the above upper bound by fixing r, and rotating the sets $\mathcal{A}$ and $\mathcal{B}$, we obtain the following upper bound on the normalized sum DoF:

$$\mathrm{DoF}(K, M_t, M_r, L) \leq \frac{Kr}{|k \in \mathcal{K} : \mathcal{T}_k \subseteq \mathcal{A} \text{ or } \mathcal{R}_k \subseteq \mathcal{B}|}.$$

Therefore, the objective is to choose the sets $\mathcal{A}$ and $\mathcal{B}$ so that the ratio on the R.H.S. of the above inequality is minimized. Since $\mathcal{T}_k = k \uparrow M_t$, and $|\mathcal{A}| = r$, we have that

$$|k \in \mathcal{K} : \mathcal{T}_k \subseteq \mathcal{A}| = (r - M_t + 1)_+.$$

Similarly, we have that

$$|k \in \mathcal{K} : \mathcal{R}_k \subseteq \mathcal{B}| = (r - M_r + 1)_+.$$

Clearly, r must satisfy $r \leq K$. It can be easily argued that, without any loss of generality, we can also restrict r so that $r - M_t + 1 \geq 1$ and $r - M_r + 1 \geq 1$ and $2r - M_t - M_r + 2 \leq K$. For any such value of r, we can choose the sets $\mathcal{A}$ and $\mathcal{B}$ to be

$$\begin{aligned} \mathcal{A} &= \{1, 2, \cdots, r\} \\ \mathcal{B} &= \{r - M_t + 2, r - M_t + 3, \cdots, 2r - M_t + 1\}. \end{aligned}$$

so that the sets $\{k \in \mathcal{K} : \mathcal{T}_k \subseteq \mathcal{A}\}$ and $\{k \in \mathcal{K} : \mathcal{R}_k \subseteq \mathcal{B}\}$ do not intersect. This results in the outer bound

$$\mathrm{DoF}(K, M_t, M_r, L) \leq \frac{Kr}{2r - M_t - M_r + 2}.$$

To obtain the best possible outer bound, it is clear that we should choose r to be as high as possible while satisfying the conditions $2r - M_t - M_r + 2 \leq K$ and $r \leq K$. When $K + M_t + M_r$ is even, the best is to set

$$r = \frac{K + M_t + M_r - 2}{2}$$

resulting in the required outer bound $\mathrm{DoF}(K, M_t, M_r, L) \leq r$. When $K + M_t + M_r$ is odd, the best is to set

$$r = \frac{K + M_t + M_r - 3}{2}$$

resulting in the required outer bound $\mathrm{DoF}(K, M_t, M_r, L) \leq Kr/(K-1)$. $\square$

5.4 Full DoF with Partial Cooperation

Recall from Section 5.2 that the DoF of the CoMP channel is equal to K if perfect cooperation is allowed at either the transmitter side or the receiver side, i.e.,

$$\mathrm{DoF}(K, M_t, M_r) = K \text{ if } \max(M_t, M_r) = K.$$

In this section, we obtain a necessary and sufficient condition on M_t and M_r such that the DoF is equal to K. First, we can obtain some intuition on the condition from outer bound in Section 5.3. Observe that Theorem 12 says that the DoF is strictly less than K whenever $M_t + M_r \leq K$. We show that the DoF is equal to the maximum value K whenever $M_t + M_r \geq K + 1$.

Theorem 13. *The DoF of the CoMP channel with generic channel coefficients is independent of L, and is equal to K, if and only if M_t and M_r satisfy $M_t + M_r \geq K + 1$; i.e.,*

$$DoF(K, M_t, M_r) = K \Leftrightarrow M_t + M_r \geq K + 1. \tag{5.11}$$

The achievable scheme is based on the linear transmit and receive beamforming strategy over each parallel channel. We prove the theorem assuming $L = 1$, and the general case follows by treating each parallel channel separately. Let $\mathbf{V}$ and $\mathbf{U}$ be the $K \times K$ matrices representing the transmit and receive beams respectively. The k^{th} column of $\mathbf{V}$ (resp. $\mathbf{U}$) represents the beam along which the message W_k is transmitted (resp. received). To comply with the physical constraints imposed by the transmit sets (5.3) and the receive sets (5.4), the matrices $\mathbf{V}$ and $\mathbf{U}$ must satisfy

$$\begin{aligned} v_{ik} \neq 0 &\Rightarrow i \in \mathcal{T}_k = k \uparrow M_t \\ u_{ik} \neq 0 &\Rightarrow i \in \mathcal{R}_k = k \uparrow M_r. \end{aligned} \tag{5.12}$$

Let $\mathbf{H}$ denote the $K \times K$ channel transfer matrix. If M_t and M_r satisfy $M_t + M_r \geq K + 1$, then we prove the existence of $\mathbf{V}$ and $\mathbf{U}$ satisfying (5.12), and

$$\mathbf{U}^\top \mathbf{H} \mathbf{V} = \mathbf{I} \tag{5.13}$$

for a generic matrix $\mathbf{H}$. Observe that the above choice for beamfroming matrices $\mathbf{V}$ and $\mathbf{U}$ achieves K DoF since they create K interference-free AWGN channels, one per each message, with each channel having a nonzero SNR. Since $\mathbf{U}$ and $\mathbf{V}$ are square matrices, it is easy to see that (5.13) is equivalent to

$$\mathbf{H}^{-1} = \mathbf{V} \mathbf{U}^\top. \tag{5.14}$$

Thus, it remains to show that the $\mathbf{H}^{-1}$ admits the matrix decomposition in (5.14) for a generic $\mathbf{H}$. We now prove a more general result.

5.4.1 Structural Matrix Decomposition

Observe that the above matrix decomposition problem(5.14) is similar to the LU decomposition in the sense that we are interested in expressing a matrix $\mathbf{A} = \mathbf{H}^{-1}$ as a product of two matrices $\mathbf{V}$ and $\mathbf{U}^\top$ with structural constraints on $\mathbf{V}$ and $\mathbf{U}$. In the case of LU decomposition, we require that both $\mathbf{V}$ and $\mathbf{U}$ are lower triangular matrices, whereas in (5.14) we require $\mathbf{V}$ and $\mathbf{U}$ to satisfy the structural conditions (5.12). In this section, we consider the general problem of structural matrix decomposition (SMD) that generalizes both (5.12) and LU decomposition. We need the following definition to formulate the SMD problem.

Definition 1 (S-matrix)**.** *Given a matrix $\mathbf{V}$ and a $(0,1)$-matrix $\bar{\mathbf{V}}$ of the same size, we say that $\bar{\mathbf{V}}$ is a structural matrix (or S-matrix) of $\mathbf{V}$ if $\bar{v}_{ij} = 1$ for all i, j such that $v_{ij} \neq 0$.*

Example 1. *Suppose $\mathbf{V}$ and $\mathbf{U}$ be transmit and receive beamforming matrices satisfying the conditions (5.12) corresponding to the setting $K = 3$ and $(M_t, M_r) = (2, 2)$. Then, the S-matrices of $\mathbf{V}$ and $\mathbf{U}$ are given by*

$$\bar{\mathbf{U}} = \bar{\mathbf{V}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{5.15}$$

where the ones in the kth column of $\bar{\mathbf{V}}$ correspond to the transmit set $\mathcal{T}_k$, and the ones in the kth column of $\bar{\mathbf{U}}$ correspond to the receive set $\mathcal{R}_k$.

Definition 2 (SMD)**.** *Let $\mathbf{A}$ be a square matrix, and $\bar{\mathbf{V}}, \bar{\mathbf{U}}$ be $(0,1)$-matrices of same size. We say that the matrix $\mathbf{A}$ admits a structural matrix decomposition (SMD) with respect to $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ if $\mathbf{A}$ can be factorized as*

$$\mathbf{A} = \mathbf{V}\mathbf{U}^\top$$

with $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ being S-matrices of $\mathbf{V}$ and $\mathbf{U}$ respectively.

To prove that $\mathrm{DoF}(3, 2, 2) = 3$, we need to show that a generic 3×3 matrix $\mathbf{A}$ admits SMD with respect to $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ defined in (5.15). The LU decomposition can be seen as a special case of the SMD with $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ given by

$$\bar{\mathbf{U}} = \bar{\mathbf{V}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{5.16}$$

We know that a generic matrix $\mathbf{A}$ admits an LU decomposition, i.e., a generic matrix $\mathbf{A}$ admits an SMD if $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are given by (5.16). We shall show that the same holds true even if (5.16) is replaced with (5.15). The following theorem provides a sufficient condition on $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ such that a generic matrix admits SMD.

Theorem 14. *Suppose the $K \times K$ $(0,1)$-matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ satisfy the conditions*

1. *The diagonal entries of $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ are nonzero.*
2. *The matrix $\bar{\mathbf{V}} + \bar{\mathbf{U}}^\top$ is a full matrix, i.e., all of its entries are nonzero.*

Then, a generic $K \times K$ matrix $\mathbf{A}$ admits SMD $\mathbf{A} = \mathbf{V}\mathbf{U}^\top$ with respect to the S-matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$.

Proof. Suppose a matrix $\mathbf{A}$ admits SMD $\mathbf{A} = \mathbf{V}\mathbf{U}^\top$; then the decomposition is not unique since for any full rank diagonal matrix Λ, we have

$$\mathbf{A} = \mathbf{V}\mathbf{U}^\top = (\mathbf{V}\Lambda)\left(\mathbf{U}\Lambda^{-1}\right)^\top. \tag{5.17}$$

To avoid such degeneracy, we set $u_{kk} = 1$ for all $k \in \mathcal{K}$. We now interpret $\mathbf{A} = \mathbf{V}\mathbf{U}^\top$ as a system of polynomial equations

$$a_{ij} = f_{ij}(\mathbf{t}), \forall i, j \in \mathcal{K} \tag{5.18}$$

where $\mathbf{t}$ represents those elements of $\mathbf{V}$ and $\mathbf{U}$ that can take arbitrary values, i.e., $\mathbf{t}$ contains the variables

$$\{v_{ij} : \bar{v}_{ij} = 1\} \cup \{u_{ij} : i \neq j \text{ and } \bar{u}_{ij} = 1\}. \tag{5.19}$$

Let N_v denote the number of variables so that $\mathbf{t} \in \mathbb{C}^{N_v}$. Our objective is show that the system of equations (5.18) has a solution $\mathbf{t} \in \mathbb{C}^{N_v}$ for a generic matrix $\mathbf{A}$. From Lemma 21 in Section B, it follows that (5.18) admits a solution for generic $\mathbf{A}$ if and only if the Jacobian matrix $\mathbf{J}_f$ of the polynomial map

$$\mathbf{f} : \mathbb{C}^{N_v} \to \mathbb{C}^{K \times K} \tag{5.20}$$

has full row rank at some point $\mathbf{t}^*$.

We now prove that $\mathbf{J}_f$ has full row rank, equal to K^2, by explicitly computing the Jacobian matrix $\mathbf{J}_f$ at the point $\mathbf{t}^*$ corresponding to $\mathbf{U}^* = \mathbf{V}^* = \mathbf{I}$. Observe that the two conditions in the theorem statement ensure that for every $i, j \in \mathcal{K}$, either v_{ij} or u_{ji} is a variable. Thus, $N_v \geq K^2$, which is a necessary condition for the Jacobian matrix to be a fat matrix, and to have full row rank. Observe that $\mathbf{J}_f$ has full row rank if any $K^2 \times K^2$ submatrix has full rank. We consider the submatrix corresponding to the K^2 variables $\{t_{ij} : i, j \in \mathcal{K}\}$ defined such that t_{ij} is equal to either v_{ij} or u_{ji} for each $i, j \in \mathcal{K}$. Consider the partial derivative

$$\begin{aligned} \frac{\partial a_{pq}}{\partial t_{ij}} &= \frac{\partial f_{pq}(\mathbf{t})}{\partial t_{ij}} \\ &= \frac{\partial \sum_{\ell=1}^{K} v_{p\ell} u_{q\ell}}{\partial t_{ij}} \\ &= \sum_{\ell=1}^{K} \frac{\partial (v_{p\ell} u_{q\ell})}{\partial t_{ij}}. \end{aligned} \tag{5.21}$$

Suppose $t_{ij} = v_{ij}$; then we see that

$$\begin{aligned}\frac{\partial a_{pq}}{\partial t_{ij}} &= \sum_{\ell=1}^{K} \frac{\partial(v_{p\ell} u_{q\ell})}{\partial t_{ij}} \\ &= \delta_{pi} u_{qj}^* \\ &= \delta_{pi}\delta_{qj}\end{aligned} \tag{5.22}$$

where δ_{ij} is the Kronecker delta function, and in the last step we used the fact that the derivative is taken at the point t^* corresponding to $\mathbf{U}^* = \mathbf{V}^* = \mathbf{I}$. We obtain the same even if $t_{ij} = u_{ji}$. Therefore, we get

$$\frac{\partial a_{pq}}{\partial t_{ij}} = \begin{cases} 1 & \text{if } (p,q) = (i,j) \\ 0 & \text{otherwise.} \end{cases} \tag{5.23}$$

Thus, we see that the submatrix of $\mathbf{J}_f$ corresponding to the variables $\{t_{ij}\}$ is equal to the identity matrix. Hence from Lemma 21 in Section B, we conclude that a solution to (5.18) exists for a generic $\mathbf{A}$. $\square$

5.4.2 Proof of Theorem 13

To complete the proof of Theorem 13, we need to show that the conditions of Theorem 14 are satisfied when $M_t + M_r \geq K + 1$. Recall from (5.12) that the S-matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ of the beamforming matrices $\mathbf{V}$ and $\mathbf{U}$ are given by

$$\begin{aligned}\bar{v}_{ij} &= 1 \Leftrightarrow i \in j \uparrow M_t \\ \bar{u}_{ij} &= 1 \Leftrightarrow i \in j \uparrow M_r.\end{aligned}$$

Clearly, the diagonal entries of $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ are equal to one satisfying the first condition of Theorem 14. Since $M_t + M_r \geq K + 1$, for any (i,j) either

$$i \in j \uparrow M_t \Rightarrow \bar{v}_{ij} = 1$$

or

$$i \in j \downarrow M_r \Rightarrow j \in i \uparrow M_r \Rightarrow \bar{u}_{ji} = 1.$$

This verifies that the second condition of Theorem 14 is also satisfied. Therefore, we see that the matrix $\mathbf{H}^{-1}$ admits SMD (5.14) for a generic $\mathbf{H}$. This completes the proof of Theorem 13.

5.4.3 Relation to MIMO Interference Channel and Alignment

The condition $M_t + M_r \geq K + 1$ is similar to the condition obtained in [62] for the MIMO interference channel. The MIMO interference channel with $N_t = M_t$ antennas per transmitter and $N_r = M_r$ antennas per receiver is similar to the

CoMP channel, in the sense that each message is transmitted and received using M_t and M_r antennas, respectively. The difference is that the messages in the MIMO interference channel have dedicated antennas, whereas the messages in the CoMP channel share antennas to mimic the MIMO interference channel. In [62], Yetis et al. studied the feasibility of transforming the MIMO interference channel into K interference-free channels using transmit and receive beamforming strategies. They used Bernstein's theorem from algebraic geometry to prove that the beams exist if and only if $M_t + M_r \geq K + 1$.

The common theme that leads to these results in both the cases, i.e., MIMO interference channel and CoMP channel, is interference alignment. It is easy to see interference alignment in action in the special case $M_t = K - 1$ and $M_r = 2$ where each decoder has access to two received signals. Out of these two dimensions, one must be reserved for the desired signal, meaning that the remaining $K-1$ interfering signals must align and appear in the direction. This process of packing the interfering signals into a smaller number of dimensions is the essence of interference alignment.

The role of interference alignment can be better understood by considering the two extreme cases: $(M_t, M_r) = (K, 1)$ and $(M_t, M_r) = (1, K)$. Recall that the objective is to construct beamforming matrices satisfying the structural constraints and

$$\mathbf{U}^\top \mathbf{H} \mathbf{V} = \mathbf{I}.$$

When $M_t = K$, then $\mathbf{V}$ can be full matrix. Therefore, we can choose the beamforming matrices as $\mathbf{V} = \mathbf{H}^{-1}$ and $\mathbf{U} = \mathbf{I}$ corresponding to transmit zero-forcing. Similarly, if $M_r = K$, then we can choose the beamforming matrices as $\mathbf{V} = \mathbf{I}$ and $\mathbf{U} = \mathbf{H}^{-1}$ corresponding to receive zero-forcing. The concepts of transmit zero-forcing and receive zero-forcing are well understood in the communication theory literature. The reason why $M_t = K$ or $M_r = K$ works is the following. In both the cases, there are $K - 1$ additional antennas at each transmitter or at each receiver to avoid interference. Essentially either the transmitters or the receivers take the burden to avoid interference. The condition $M_t + M_r \geq K + 1$ says that this burden to avoid interference does not have to be taken solely either by the transmitters or the receivers, but can be shared by both. In other words, interference alignment can be thought of as a generalized zero-forcing strategy that allows the burden of interference avoidance to be shared by the transmitters and receivers by carefully designing the beams. The disadvantage of doing so is that, while the design of transmit or receive zero-forcing beams requires only local channel knowledge, the design of interference alignment beams requires global channel knowledge and even the computational aspects become more complicated. Since the existence proofs are nonconstructive, it is not clear if there is any closed-form algorithm or even iterative algorithm to numerically compute the interference alignment beams.

5.4.4 Closed-Form Algorithm

We showed that a linear beamforming strategy based on interference alignment achieves K DoF whenever M_t and M_r satisfy $M_t + M_r \geq K + 1$. The proof of Theorem 13 is not constructive. In this section, we consider the problem of numerical computation of interference alignment beams, i.e., computation of matrices $\mathbf{V}$ and $\mathbf{U}$ that satisfy the structural constraints imposed by transmit sets and receive sets, and diagonalize the channel matrix $\mathbf{H}$

$$\mathbf{U}^\top \mathbf{H}\mathbf{V} = \mathbf{I}. \tag{5.24}$$

In the previous section, we have seen that the problem is easy if either $M_t = K$ or $M_r = K$, where the beamforming matrices correspond to either transmit zero-forcing or receive zero-forcing. In this section, we show that there exists a closed form solution when $M_t = K - 1$ or $M_r = K - 1$. Without any loss of generality, we consider the case $M_t = K - 1$ and $M_r = 2$, and show that the closed-form solution described in Algorithm 1 satisfies the structural constraints and (5.24). The rest of this section focuses on justifying the steps in Algorithm 1.

The usual approach to solve for $\mathbf{U}$ and $\mathbf{V}$ is by first eliminating $\mathbf{U}$ by obtaining the necessary and sufficient conditions on $\mathbf{V}$ for an appropriate $\mathbf{U}$ to exist, and then solving for $\mathbf{V}$. Let $\mathbf{M}$ denote the matrix $\mathbf{HV}$. We now obtain the necessary and sufficient conditions on the matrix $\mathbf{M}$ so that its inverse $\mathbf{M}^{-1} = \mathbf{U}^\top$ satisfies the structural constraints imposed by the receive sets. For example, if $M_r = 2$, then the receive beamforming matrix should have the following structure:

$$\mathbf{U} = \begin{bmatrix} \times & & & & \times \\ \times & \times & & & \\ & \times & \times & & \\ & & \ddots & \ddots & \\ & & & \times & \times \end{bmatrix}. \tag{5.25}$$

The nullity theorem [63, 64] from linear algebra is useful in obtaining the neccesary and sufficient conditions on $\mathbf{M}$.

Lemma 8 (Nullity Theorem)**.** *Complementary submatrices of a matrix and its inverse have the same nullity.*

Two submatrices are *complementary* when the row numbers not used in one are the column numbers used in the other. For any subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{K}$, applying the Nullity Theorem to $\mathbf{M}$ and $\mathbf{U}^\top = \mathbf{M}^{-1}$, we have that

$$\begin{aligned} \operatorname{nullity} \mathbf{M}(\mathcal{A}, \mathcal{B}) &= \operatorname{nullity} \mathbf{U}^\top(\mathcal{B}^c, \mathcal{A}^c) \\ \Leftrightarrow |\mathcal{B}| - \operatorname{rank} \mathbf{M}(\mathcal{A}, \mathcal{B}) &= |\mathcal{A}^c| - \operatorname{rank} \mathbf{U}(\mathcal{A}^c, \mathcal{B}^c) \\ \Leftrightarrow \operatorname{rank} \mathbf{M}(\mathcal{A}, \mathcal{B}) &= \operatorname{rank} \mathbf{U}(\mathcal{A}^c, \mathcal{B}^c) + |\mathcal{A}| + |\mathcal{B}| - K. \end{aligned}$$

Observe that the structural constraints on the matrix $\mathbf{U}$ can be described as

$$\operatorname{rank} \mathbf{U}(\mathcal{R}_k^c, k) = 0, \forall k \in \mathcal{K}. \tag{5.26}$$

By choosing $\mathcal{A} = \mathcal{R}_k$ and $\mathcal{B} = \{k\}^c$, we observe that structural constraints on $\mathbf{U}$ are equivalent to the following constraints on $\mathbf{M}$:

$$\operatorname{rank} \mathbf{M}(\mathcal{R}_k, \{k\}^c) = M_r - 1, \forall k \in \mathcal{K}. \tag{5.27}$$

Note that the above conditions are nothing but the interference alignment conditions. The matrix $\mathbf{M} = \mathbf{HV}$ should be interpreted as the matrix containing the receive directions as the columns

$$\mathbf{M} = \begin{bmatrix} \mathbf{H}(\mathcal{K}, \mathcal{T}_1)\mathbf{v}_1 & \mathbf{H}(\mathcal{K}, \mathcal{T}_2)\mathbf{v}_2 & \cdots & \mathbf{H}(\mathcal{K}, \mathcal{T}_K)\mathbf{v}_K \end{bmatrix} \tag{5.28}$$

where $\mathbf{v}_k \in \mathbb{C}^{M_t \times 1}$ denotes the beamforming vector corresponding to the message W_k, i.e., $\mathbf{v}_k = \mathbf{V}(\mathcal{T}_k, k)$. Consider the decoder of message W_k which has access to the signals received by the receivers $\mathcal{R}_k$. The submatrix

$$\mathbf{M}(\mathcal{R}_k, \mathcal{K}) = \begin{bmatrix} \mathbf{H}(\mathcal{R}_k, \mathcal{T}_1)\mathbf{v}_1 & \mathbf{H}(\mathcal{R}_k, \mathcal{T}_2)\mathbf{v}_2 & \cdots & \mathbf{H}(\mathcal{R}_k, \mathcal{T}_K)\mathbf{v}_K \end{bmatrix}$$

represents the matrix with the column denoting the directions along which the signals appear at the decoder k. Thus, we see that the condition (5.27) is equivalent to saying that the interfering signals should occupy only $M_r - 1$ dimensions out of the available M_r dimensions at decoder k, leaving one dimension for the signal. With this intuition, we could have arrived at the alignment conditions (5.27) directly without invoking the nullity theorem. However, the constraints (5.27) do not directly lead to a closed-form solution.

We now demonstrate the usefulness of the nullity theorem by deriving another set of equivalent conditions on $\mathbf{M}$ that immediately lead to the closed-form solution described in Algorithm 1. The crucial observation is the following. In the description (5.26), we noticed that each column of $\mathbf{U}$ has $K - M_r$ zeros. Alternatively, we can use the fact that each row of $\mathbf{U}$ has $K - M_r$ zeros to arrive at an alternate description of the structural constraints on $\mathbf{U}$:

$$\operatorname{rank} \mathbf{U}(k-1, k \uparrow (K - M_r)) = 0, \forall k \in \mathcal{K}.$$

By choosing $\mathcal{A} = \{k-1\}^c$ and $\mathcal{B} = \{k \uparrow K - M_r\}^c = (k-1) \downarrow M_r$, we observe that the structural constraints on $\mathbf{U}$ are equivalent to following constraints on $\mathbf{M}$:

$$\operatorname{rank} \mathbf{M}(\{k-1\}^c, (k-1) \downarrow M_r) = M_r - 1, \forall k \in \mathcal{K}.$$

For the special case of $M_t = K - 1$ and $M_r = 2$, we have that $\mathcal{T}_k = \{k-1\}^c$ and $(k-1) \downarrow M_r = \{k-1, k-2\}$. Using the expression (5.28) for $\mathbf{M}$, we see that the above conditions can be written as

$$\operatorname{rank}\left[\begin{array}{cc} \mathbf{H}(\mathcal{T}_k, \mathcal{T}_{k-1})\mathbf{v}_{k-1} & \mathbf{H}(\mathcal{T}_k, \mathcal{T}_{k-1})\mathbf{v}_{k-2} \end{array}\right] = 1, \forall k \in \mathcal{K}.$$

For a generic $\mathbf{H}$, the submatrix $\mathbf{H}(\mathcal{T}_k, \mathcal{T}_{k-1})$ is invertible, and hence the above conditions can equivalently be expressed as

$$\mathbf{v}_{k-1} \propto \mathbf{B}_{k-2}\mathbf{v}_{k-2}$$

where $\mathbf{B}_{k-2} = \mathbf{H}(\mathcal{T}_k, \mathcal{T}_{k-1})^{-1}\mathbf{H}(\mathcal{T}_k, \mathcal{T}_{k-2})$. Therefore, the transmit beams must be designed to satisfy

$$\begin{aligned} \mathbf{v}_2 &\propto \mathbf{B}_1\mathbf{v}_1 \\ \mathbf{v}_3 &\propto \mathbf{B}_2\mathbf{v}_2 \\ &\vdots \\ \mathbf{v}_K &\propto \mathbf{B}_{K-1}\mathbf{v}_{K-1} \\ \mathbf{v}_1 &\propto \mathbf{B}_K\mathbf{v}_K. \end{aligned}$$

The above conditions are satisfied if and only if $\mathbf{v}_1$ is an eigenvector of the matrix $\mathbf{B}_K\mathbf{B}_{K-1}\cdots\mathbf{B}_1$, and $\mathbf{v}_{k+1} \in \mathbf{B}_k\mathbf{v}_k$ for $k = 2, 3, \cdots, K$. We can then compute the receive beamforming vectors by computing $\mathbf{M} = \mathbf{HV}$ and setting $\mathbf{U} = \mathbf{M}^{-\top}$. The choice of transmit beams and the nullity theorem ensures that the resulting receive beamforming matrix $\mathbf{U}$ has the required structure (5.25).

Algorithm 1 Closed Form Solution: $M_t = K - 1$ and $M_r = 2$

1: For each $k \in \mathcal{K}$, define the alignment matrix

$$\mathbf{B}_k = \mathbf{H}(\mathcal{T}_{k+2}, \mathcal{T}_{k+1})^{-1}\mathbf{H}(\mathcal{T}_{k+2}, \mathcal{T}_k)$$

2: Choose $\mathbf{v}_1$ as an eigenvector of the matrix

$$\mathbf{B}_K\mathbf{B}_{K-1}\cdots\mathbf{B}_1$$

3: For $k = 1, 2, \cdots, K-1$, compute

$$\mathbf{v}_{k+1} = \mathbf{B}_k\mathbf{v}_k$$

5: Compute the transmit beamforming matrix $\mathbf{V}$ such that

$$\mathbf{v}_k = \mathbf{V}(\mathcal{T}_k, k),\ \forall k \in \mathcal{K}.$$

6: Compute the receive beamfoming matrix $\mathbf{U} = (\mathbf{HV})^{-\top}$.

5.4.5 Numerical Results

In this section, we consider the three-antenna system, i.e., $K = 3$. From Theorem 13, we have that the maximum 3 DOF is achievable if and only if $M_t + M_r \geq 4$. We numerically verify the achievability part of the theorem by showing that 3 DoF is achievable when $M_t + M_r \geq 4$. Without any loss of generality, we only consider the two settings $(M_t, M_r) = (3, 1)$ and $(M_t, M_r) = (2, 2)$ because the other settings can be shown to follow from these two settings. In Figure 5.1, we plot the average achievable sum-rate, where the averaging is performed over the multiple realizations of the channel coefficients which are generated independently according to complex normal distribution. When $(M_t, M_r) = (3, 1)$, the system is equivalent to a broadcast channel, and so we use the zero forcing transmit beams described in Section 5.4.3. When $(M_t, M_r) = (2, 2)$, we have that $M_r = K - 1$, and so we use the alignment scheme described in Algorithm 1 to compute the transmit and receive beams. In step 2 of Algorithm 1, the computation of the transmit beam $\mathbf{v}_1$ involves computing an eigenvector of the 2×2 matrix. In Figure 5.1, we plot the two curves for the setting $(M_t, M_r) = (2, 2)$: one corresponds to arbitrary eigenvector and the other corresponds to best eigenvector over each channel realization.

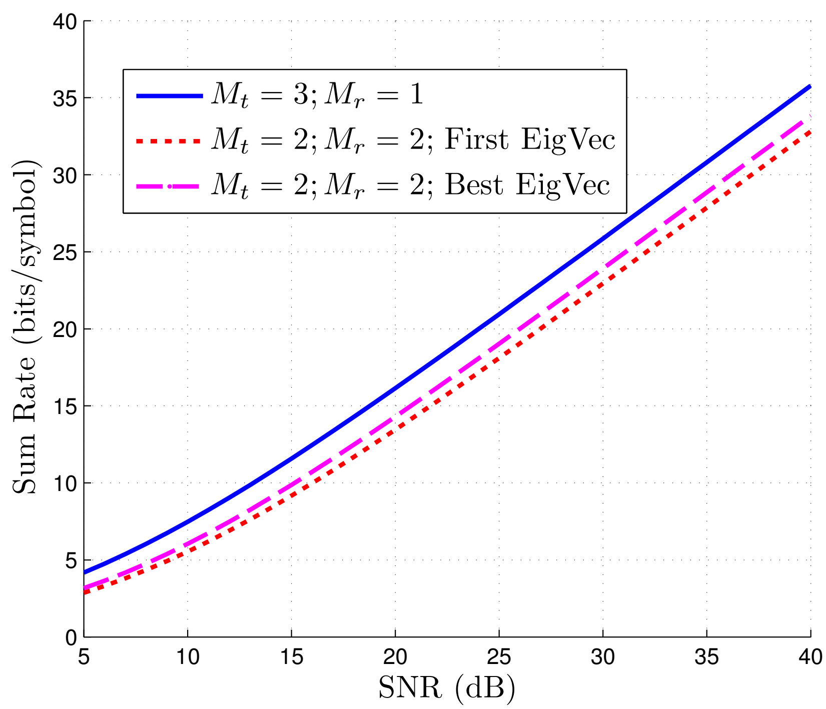

Figure 5.1: Achievable sum-rates in a three-antenna system with alignment schemes.

The plots numerically verifiy that the achievable scheme described in Algorithm 1 indeed achieves 3 DoF with $(M_t, M_r) = (2, 2)$. Indeed, a linear growth of 10 bits/symbol in sum-rate for every 10 dB improvement in SNR corresponds to

$$\frac{10}{\log_2 10} \approx 3 \text{ DoF}.$$

It is also interesting to see that $(M_t, M_r) = (3, 1)$ achieves better sum-rate when compared to $(M_t, M_r) = (2, 2)$. The performance gap is roughly 3 dB at high SNRs when arbitrary eigenvector is used, and is roughly 2 dB when best eigenvector is used.

5.5 DoF with CoMP Transmission

In the previous sections, we derived an outer bound on the DoF and showed that the DoF is equal to the maximum value K if and only if $M_t + M_r \geq K + 1$. In this section, we set $M_r = 1$, and consider the problem of characterizing $\mathrm{DoF}(K, M_t, 1)$, the DoF of interference channel with CoMP transmission, as a function of K and M_t. From the outer bound in Section 5.3.2, we obtain that $\mathrm{DoF}(K, M_t, 1)$ is upper bounded as

$$\mathrm{DoF}(K, M_t, 1) \leq \begin{cases} \dfrac{K + M_t - 1}{2}, & K + M_t \text{ is odd} \\ \dfrac{K}{K-1} \dfrac{K + M_t - 2}{2} \leq \left\lceil \dfrac{K + M_t - 1}{2} \right\rceil, & K + M_t \text{ is even.} \end{cases}$$

For the achievability part, we prove the following two theorems. For any K and M_t, we propose a scheme that aims at achieving a DoF of $(K + M_t - 1)/2$. A crucial part of the proof involves checking that a certain Jacobian matrix has full row rank. We could provide an analytical proof when $M_t = 2$. For $M_t > 2$, we could verify in MATLAB that the Jacobian matrix has full row rank for all the values of K we checked. Specifically, we checked till $K \leq 9$, but we conjecture that the result holds true for any K and M_t. For more discussion on the problematic issue, we refer the reader to Section 5.7.4.

Theorem 15. *The DoF of interference channel with CoMP transmission satisfies*

$$DoF(K, M_t, 1) \geq \frac{K + M_t - 1}{2}$$

for all K if $M_t = 2$ and for all $M_t \leq K < 10$.

Combining the above theorem with the outer bound, we have determined the DoF exactly when $K + M_t$ is odd, and approximately when $K + M_t$ is even (for all K if $M_t = 2$ and for all $M_t \leq K < 10$). For the special case of $M_t = K - 2$, we propose an achievable scheme that exactly meets the outer bound.

Theorem 16. *The DoF of interference channel with CoMP transmission with $M_t = K - 2$ satisfies*

$$DoF(K, K-2, 1) = \frac{K M_t}{M_t + 1} = \frac{K(K-2)}{K-1}.$$

Before proving the above theorems, we first explain the connection to the DoF of the MISO interference channel.

5.5.1 Relation to MISO Interference Channel

The MISO interference channel with $N_t = M_t$ antennas per transmitter and the cellular uplink channel with M_t number of users per cell are similar to the interference channel with CoMP transmission in the sense that, in all the three channels, each message is transmitted using M_t antennas and received using only one antenna. The difference is that the messages share the antennas in the CoMP channel, whereas the messages have dedicated antennas in the other two channels. Both the MISO interference channel and the cellular uplink channel have the same DoF, equal to $KM_t/(M_t+1)$ for all $M_t < K$. In comparison, we see that the interference channel with CoMP transmission has a smaller DoF except in the special cases where $M_t \in \{1, K-1, K-2, K\}$.

Claim 6. *For all $M_t \notin \{1, K-2, K-1, K\}$,*

$$DoF(K, M_t, 1) < \frac{KM_t}{M_t+1}.$$

Proof. Suppose $M_t + K$ is odd. Then we see that

$$\begin{aligned}
\mathrm{DoF}(K, M_t, 1) = \frac{K+M_t-1}{2} &< \frac{KM_t}{M_t+1} \\
\Leftrightarrow K(M_t+1) + (M_t-1)(M_t+1) &< 2KM_t \\
\Leftrightarrow (M_t-1)(M_t+1) &< K(M_t-1) \\
\Leftrightarrow M_t + 1 &< K
\end{aligned}$$

which is true since we assumed that $M_t < K-2$. Suppose $M_t + K$ is even; then

$$\begin{aligned}
\mathrm{DoF}(K, M_t, 1) \leq \frac{K}{K-1}\frac{K+M_t-2}{2} &< \frac{KM_t}{M_t+1} \\
\Leftrightarrow K(M_t+1) + (M_t-2)(M_t+1) &< 2KM_t - 2M_t \\
\Leftrightarrow (M_t-1)(M_t+2) &< K(M_t-1) \\
\Leftrightarrow M_t + 2 &< K
\end{aligned}$$

which is true since we assumed that $M_t < K-2$. □

5.6 CoMP Transmission: Proof of Theorem 16

In this section, we show that the DoF of the interference channel with CoMP transmission and a transmit cooperation order of $M_t = K-2$ and a receive cooperation order $M_r = 1$ is equal to

$$\mathrm{DoF}(K, K-2, 1) = \frac{KM_t}{M_t+1}.$$

The achievable scheme is based on transmit and receive beamforming. As summarized in Figure 5.2, the beam design process is broken into two steps. First, we transform each parallel CoMP channel into a derived channel. Then, we design an asymptotic interference alignment scheme over the derived channel achieving the requied DoF in an asymptotic fashion with the number of parallel channels $L \to \infty$.

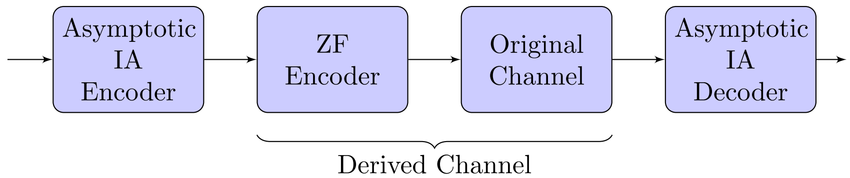

Figure 5.2: Summary of the achievable scheme.

5.6.1 Derived Channel

Recall from Section 5.5.1, that the cellular uplink channel with M_t transmitters per cell has $KM_t/(M_t+1)$ DoF. Therefore, we first transform the CoMP channel into a derived channel that mimics the cellular uplink channel. For each k, the transmit set $\mathcal{T}_k = k \uparrow M_t$ of user k consists of M_t transmitters. We use the M_t transmitters in $\mathcal{T}_k$ to create M_t virtual transmit nodes with inputs $X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(M_t)}$. The channel inputs of the CoMP channel are related to the channel inputs of the derived channel through a linear transformation. The contribution of the derived channel inputs $X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(M_t)}$ in the real transmit signals $X_k, X_{k+1}, \cdots, X_{k+M_t-1}$ is defined by a $M_t \times M_t$ beamforming matrix; i.e.,

$$\begin{bmatrix} X_k \\ X_{k+1} \\ \vdots \\ X_{k+M_t-1} \end{bmatrix} = (*) + \mathbf{V}_k \begin{bmatrix} X_k^{(1)} \\ X_k^{(2)} \\ \vdots \\ X_k^{(M_t)} \end{bmatrix}$$

where $*$ represents the contribution from the derived channel inputs of other users. Thus, we see that the beamforming matrices $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_K$, which will be specified later, define the transformation from the original channel to the derived channel. The message W_k of user k is divided into M_t parts

$$W_k = \left(W_k^{(1)}, W_k^{(2)}, \cdots, W_k^{(M_t)} \right)$$

such that the mth part controls the derived channel input $X_k^{(m)}$. Thus we can treat the virtual transmit nodes as non-cooperative transmitters communicating to the same receiver and so this system is similar to a cellular uplink system with M_t trasmitters per cell:

$$Y_i = \sum_{k=1}^{K} \sum_{m=1}^{M_t} g_{ik}^{(m)} X_k^{(m)} + Z_i, \; i \in \mathcal{K} \tag{5.29}$$

where $g_{ik}^{(m)}$ represents the derived channel coefficient from transmitter m in cell k to the receiver in cell i. It is easy to see that the derived channel coefficients are related to the original channel coefficients as

$$\begin{bmatrix} g_{ik}^{(1)} & g_{ik}^{(2)} & \cdots & g_{ik}^{(M_t)} \end{bmatrix} = \mathbf{H}(i, \mathcal{T}_k)\mathbf{V}_k$$

for all $i, k \in \mathcal{K}$, where $\mathbf{H}$ denotes $K \times K$ channel transfer matrix of the CoMP channel.

5.6.2 Generic Channel Coefficients

The derived channel (5.29) is similar to the cellular uplink channel with K cells and M_t transmitters in each cell, which has $KM_t/(M_t+1)$ DoF with generic channel coefficients [65]. A naive argument is to conclude from here that the derived channel, and hence the CoMP channel with generic channel coefficients, also has the same DoF. However, from Claim 6 in Section 5.5.1, we know that the DoF of the CoMP channel is strictly smaller than $KM_t/(M_t+1)$, which means that the above naive argument has to be incorrect.

The reason for the failure of the above naive argument is related to the subtle concept of generic channel coefficients. Indeed, the derived channel has $KM_t/(M_t+1)$ DoF with generic channel coefficients, which means that there exists a nonzero polynomial $f_g(\mathbf{g})$ in the derived channel coefficients

$$\mathbf{g} = \{g_{ij}^{(m)}(m) : 1 \le i, j \le K, 1 \le m \le M_t, 1 \le \ell \le L\}$$

such that the achievable scheme works for all $\mathbf{g}$ such that $f_g(\mathbf{g}) \neq 0$. In the case of the cellular uplink channel, this statement makes sense since the coefficients $\mathbf{g}$ are generated by nature and hence can be assumed to be generic. However, in the case of the CoMP channel, nature generates the original channel coefficients $\{h_{ij}(m)\}$, denoted by $\mathbf{h}$. The coefficients $\mathbf{g}$ are derived from $\mathbf{h}$ using rational transformations. Suppose we expand the polynomial f_g in terms of the coefficients $\mathbf{h}$ to obtain the rational function $f_h(\mathbf{h}) = f_g(\mathbf{g}(\mathbf{h}))$. There are two possibilities: the function f_h is either identically equal to zero or it is nonzero. If $f_h = 0$, then the achievable scheme designed for the derived channel with generic $\mathbf{g}$ may fail for all realizations of $\mathbf{h}$, in which case the DoF result of the derived channel with generic channel coefficients *cannot* be directly applied to CoMP channel with generic channel coefficients. On the other hand, if f_h is a nonzero function, then we see that the achievable scheme works for generic $\mathbf{h}$, in which case the DoF result of the derived channel with generic channel coefficients *can* be directly applied to CoMP channel with generic channel coefficients.

In summary, we need to be careful in applying the DoF result of the cellular uplink channel to the CoMP channel, and the applicability of the result depends on how the derived channel coefficients are related to the original channel coefficients.

5.6.3 Zero-Forcing Step

We now specify our choice of the beamforming matrices $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_K$, that define the relation of the derived channel coefficients to the original channel coefficients. As we shall notice later during the design of the asymptotic interference alignment scheme, the beamforming matrices should be chosen to minimize the number of nontrivial derived channel coefficients, where we say that a derived channel coefficient is trivial if it is equal to either zero or one. Therefore, the objective is to set as many derived channel coefficients as possible to zeros or ones. Consider the derived channel coefficients

$$\begin{bmatrix} g_{k+1,k}^{(1)} & g_{k+1,k}^{(2)} & \cdots & g_{k+1,k}^{(M_t)} \\ g_{k+2,k}^{(1)} & g_{k+2,k}^{(2)} & \cdots & g_{k+2,k}^{(M_t)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{k+M_t,k}^{(1)} & g_{k+M_t,k}^{(2)} & \cdots & g_{k+M_t,k}^{(M_t)} \end{bmatrix} = \mathbf{H}(\mathcal{T}_{k+1}, \mathcal{T}_k)\mathbf{V}_k.$$

By choosing $\mathbf{V}_k = \mathbf{H}(\mathcal{T}_{k+1}, \mathcal{T}_k)^{-1}$, we can set all the above mentioned derived channel coefficients to either zero or one. In particular, we see that for each $i \in \mathcal{T}_{k+1}$

$$g_{ik}^{(m)} = \begin{cases} 1 & i = k+m \\ 0 & \text{Otherwise.} \end{cases}$$

Since we assumed that $M_t = K-2$, the set $\mathcal{T}_{k+1}$ contains all the receiver indices except for $k-1$ and k. Therefore, we see that each transmitter $X_k^{(m)}$ in the derived channel causes interference to only two receivers, i.e., receivers $k+m$ and $k-1$. Thus, the derived channel (5.29) can be simplified as

$$Y_i = \sum_{m=1}^{M_t} g_{ii}^{(m)} X_i^{(m)} + \sum_{m=1}^{M_t} g_{i,i+1}^{(m)} X_{i+1}^{(m)} + \sum_{m=1}^{M_t} X_{i-m}^{(m)} + Z_i \tag{5.30}$$

where the coefficients $g_{ii}^{(m)}$ and $g_{i,i+1}^{(m)}$ are given by

$$\begin{aligned} \begin{bmatrix} g_{i,i+1}^{(1)} & \cdots & g_{i,i+1}^{(M_t)} \end{bmatrix} &= \mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{V}_{i+1} = \mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{H}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1} \\ \begin{bmatrix} g_{ii}^{(1)} & \cdots & g_{ii}^{(M_t)} \end{bmatrix} &= \mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{V}_i = \mathbf{H}(i, \mathcal{T}_i)\mathbf{H}(\mathcal{T}_{i+1}, \mathcal{T}_i)^{-1}. \end{aligned} \tag{5.31}$$

Figure 5.3 provides a description of the derived channel for the special case of $K = 4$ and $M_t = 2$.

5.6.4 Asymptotic Interference Alignment

In this section, we consider L parallel derived channels and propose a scheme achieving a DoF that is arbitrary close to $KM_t/(M_t+1)$ in the limit $L \to \infty$.

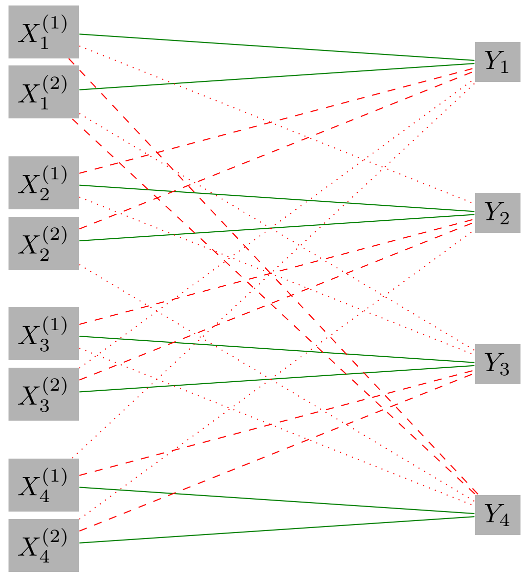

Figure 5.3: The derived channel in Section 5.6.3 when $K = 4$ and $M_t = 2$. The thick green lines indicate the links carrying signal. The dashed and dotted red lines indicate the links carrying interference. Dotted lines indicate that the corresponding coefficients are equal to 1.

We can combine the L parallel channels of (5.30) and express them together as

$$\underline{Y}_i = \sum_{m=1}^{M_t} \mathbf{G}_{ii}^{(m)} \underline{X}_i^{(m)} + \sum_{m=1}^{M_t} \mathbf{G}_{i,i+1}^{(m)} \underline{X}_{i+1}^{(m)} + \sum_{m=1}^{M_t} \underline{X}_{i-m}^{(m)} + \underline{Z}_i \tag{5.32}$$

where $\underline{X}_j^{(m)}, \underline{Y}_i$ and $\underline{Z}_i$ are $L \times 1$ column vectors and $\mathbf{G}_{ij}^{(m)}$ is $L \times L$ diagonal channel transfer matrix given by

$$\mathbf{G}_{ij}^{(m)} = \begin{bmatrix} g_{ij}^{(m)}(1) & & & \\ & g_{ij}^{(m)}(2) & & \\ & & \ddots & \\ & & & g_{ij}^{(m)}(L) \end{bmatrix}.$$

The achievable scheme that we propose is based on the asymptotic alignment scheme introduced by Cadambe and Jafar in [41].

Definition 3 (Cadambe-Jafar (CJ) subspace)**.** *The order-n CJ subspace generated by the diagonal matrices*

$$\mathbf{G}_1, \mathbf{G}_2, \cdots, \mathbf{G}_N$$

is defined as the linear subspace spanned by the vectors

$$\{\mathbf{G}_1^{a_1} \mathbf{G}_2^{a_2} \cdots \mathbf{G}_N^{a_N} \mathbf{1} : \mathbf{a} \in \mathbb{Z}_+^N \text{ and } \sum_i a_i \leq n\}. \tag{5.33}$$

The matrix containing these $\binom{N+n}{n}$ vectors as columns is said to be the order-n CJ matrix.

Let $\mathbf{V}$ denote the order-n CJ supspace (and the corresponding matrix) generated by the nontrivial channel matrices carrying interference:

$$\{\mathbf{G}_{i,i+1}^{(m)} : i \in \mathcal{K}, 1 \leq m \leq M_t\}.$$

We use $\mathbf{V}$ as the transmit beamforming matrix at every transmitter. The nice property about the CJ subspace is that the interference seen at any receiver is limited to the order-$(n+1)$ CJ subspace, denoted by $\mathbf{INT}$. At receiver k, the desired signal streams appear along the directions

$$\begin{bmatrix} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{G}_{kk}^{(2)}\mathbf{V} & \cdots & \mathbf{G}_{kk}^{(M_t)}\mathbf{V} \end{bmatrix}. \tag{5.34}$$

The proposed scheme works if the receivers are able to extract out the desired signal streams free of interference, which is true if the matrix

$$\mathbf{M}_k = \begin{bmatrix} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{G}_{kk}^{(2)}\mathbf{V} & \cdots & \mathbf{G}_{kk}^{(M_t)}\mathbf{V} & \mathbf{INT} \end{bmatrix} \tag{5.35}$$

has full column rank for every $k \in \mathcal{K}$. For the matrix $\mathbf{M}_k$ to have full column rank, the number of rows, equal to the number of parallel channels (L), must be greater than or equal to the number of columns. The number of columns in $\mathbf{V}$ and $\mathbf{INT}$, respectively, is given by

$$\begin{aligned} |\mathbf{V}| &= \binom{KM_t + n}{KM_t} \\ |\mathbf{INT}| &= \binom{KM_t + n + 1}{KM_t}. \end{aligned} \tag{5.36}$$

Hence the number of columns in $\mathbf{M}_k$ is equal to $M_t|\mathbf{V}| + |\mathbf{INT}|$. We set $L = M_t|\mathbf{V}| + |\mathbf{INT}|$ so that $\mathbf{M}_k$ is a square matrix for each $k \in \mathcal{K}$. Note that the matrix $\mathbf{M}_k$ depends on the derived channel coefficients

$$g_{ii}^{(m)}(\ell), g_{i,i+1}^{(m)}(\ell) : 1 \leq m \leq M_t, 1 \leq i \leq K, 1 \leq \ell \leq L.$$

We need to prove that the matrices $\mathbf{M}_1, \cdots, \mathbf{M}_k$ have full rank for generic (original) channel coefficients

$$h_{ij}(\ell) : 1 \leq i, j \leq K, 1 \leq \ell \leq L.$$

The proof uses techniques from algebraic geometry summarized in Section B. Using Corollary 4, we see that the matrices $\mathbf{M}_1, \mathbf{M}_2, \cdots, \mathbf{M}_k$ have full column rank if the rational transformation (5.31) from the original channel coefficients to the derived channel coefficients is such that the rational functions denoted by the variables

$$g_{kk}^{(m)}, g_{i,i+1}^{(m)} : 1 \leq m \leq M_t, 1 \leq i \leq K \tag{5.37}$$

are algebraically independent. Before we prove the algebraic independence, we show that the proposed scheme achieves the required DoF. Since the derived channel has a total of KM_t number of transmitters, and the proposed interference alignment scheme creates $|\mathbf{V}|$ number of interference-free AWGN channels per each transmitter, we obtain the following lower bound on the (normalized) DoF:

$$\mathrm{DoF}(K, K-2, 1, L) \geq \frac{KM_t|\mathbf{V}|}{L} = \frac{KM_t|\mathbf{V}|}{M_t|\mathbf{V}| + |\mathbf{INT}|} = \frac{KM_t}{M_t + 1 + \frac{KM_t}{n+1}}.$$

Therefore, we obtain that

$$\begin{aligned}
\mathrm{DoF}(K, K-2, 1) &= \limsup_{L\to\infty} \mathrm{DoF}(K, K-2, 1, L) \\
&\geq \lim_{n\to\infty} \frac{KM_t}{M_t + 1 + \frac{KM_t}{n+1}} \\
&= \frac{KM_t}{M_t + 1}.
\end{aligned}$$

5.6.5 Proof of Algebraic Independence

Since the achievable scheme is symmetric across the user indices, it is sufficient to prove the claim for $k = 1$. The $(K+1)M_t$ variables (5.37) are rational functions of the K^2 variables $\{h_{ij} : 1 \leq i, j \leq K\}$. Let $\mathbf{J}$ denote the corresponding $(K+1)M_t \times K^2$ Jacobian matrix. From Lemma 20, the variables (5.37) are algebraically independent if and only if the Jacobian matrix $\mathbf{J}$ has full row rank equal to $(K+1)M_t$. Let

$$\mathbf{J}[\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_K; \mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_K] \tag{5.38}$$

denote the $(K+1)M_t \times (K+1)M_t$ submatrix of $\mathbf{J}$ with rows corresponding to the variables $\mathbf{g}_0, \mathbf{g}_1, \cdots, \mathbf{g}_K$ and columns corresponding to the variables $\mathbf{h}_0, \mathbf{h}_1, \cdots, \mathbf{h}_K$, where

$$\begin{aligned}
\mathbf{g}_0 &= \left(g_{11}^{(1)}, g_{11}^{(2)}, \cdots, g_{11}^{(M_t)}\right) \\
\mathbf{g}_i &= \left(g_{i,i+1}^{(1)}, g_{i,i+1}^{(2)}, \cdots, g_{i,i+1}^{(M_t)}\right) \\
\mathbf{h}_0 &= (h_{11}, h_{22}, \cdots, h_{M_tM_t}) \\
\mathbf{h}_i &= (h_{i,i+1}, h_{i,i+2}, \cdots, h_{i,K}, h_{i,1}, \cdots, h_{i,i-2}).
\end{aligned}$$

We complete the claim by showing that square matrix (5.38) has full rank. This is easy to verify using the symbolic toolbox of MATLAB for any fixed K. An analytical proof involves computing the submatrix (5.38) at a specific point $\mathbf{H} = \mathbf{A}$, and showing that it has full rank. Although this is true at any randomly generated $\mathbf{A}$, certain choices can simplify the proof. We choose $\mathbf{A}$ to be the circulant matrix given by

$$a_{ij} = \begin{cases} 1 & \text{if } j = i \text{ or } j = i-1 \\ 0 & \text{otherwise} \end{cases}. \tag{5.39}$$

For the special case of $K = 4$ and $M_t = 2$, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.40}$$

The following claim, whose proof is relegated to Section 5.9, completes the proof of Theorem 16.

Claim 7. *The determinant of the submatrix* (5.38) *evaluated at the point* $\mathbf{H} = \mathbf{A}$ *is equal to* ± 1.

5.6.6 Discussion

We end the section by explaining why the proposed scheme does not extend for arbitrary $M_t < K - 2$. Observe that a straightforward extension of the achievable scheme involves the same choice of ZF transmit beams in Section 5.6.3. However, since $M_t < K - 2$, each transmitter in the derived channel now causes interference at $K - M_t$ receivers, i.e., the transmitter $X_k^{(m)}$ causes interference at the receivers $k+m, k+M_t+1, k+M_t+2, \cdots, k+K-1$. Since the asymptotic interference alignment scheme requires that we use all the nontrivial channel matrices in generating the CJ subspace, we can verify that the achievable scheme works if the rational functions defined by the variables

$$g_{kk}^{(m)}, g_{i,i+1}^{(m)}, g_{i,i+2}^{(m)}, \cdots, g_{i,i+K-M_t-1}^{(m)} : 1 \leq m \leq M_t, 1 \leq i \leq K$$

are algebraically independent for each $k \in \mathcal{K}$. The total number of rational functions is given by

$$(1 + (K - M_t - 1)K)M_t.$$

If the above number were to be greater than K^2, then we can end this discussion since $m > n$ rational functions in n variables cannot be algebraically independent. But that is not the case. For example, when $M_t = 2$ and $K = 5$, we have 22 rational functions in 25 variables. If these rational functions were to be algebraically independent, then the achievable scheme generalizes achieving a DoF of $KM_t/(M_t + 1)$, but we know from the discussion in Section 5.5.1 that the DoF is strictly less than $KM_t/(M_t + 1)$ for all $1 < M_t < K - 2$. Therefore, it must be that these rational functions are algebraically dependent.

5.7 CoMP Transmission: Proof of Theorem 15

In this section, we show that the DoF of the interference channel with CoMP transmission and with transmit cooperation order of M_t and a receive cooperation order of $M_r = 1$ is lower-bounded by

$$\mathrm{DoF}(K, M_t, 1) \geq \frac{K + M_t - 1}{2}. \tag{5.41}$$

We prove this by arguing that the DoF vector

$$d_i = \begin{cases} 1 & 1 \leq i \leq M_t - 1 \\ 0.5 & M_t \leq i \leq K \end{cases} \tag{5.42}$$

is achievable; i.e., the first $M_t - 1$ users benefit from cooperation and achieve 1 degree of freedom, whereas the remaining $K - M_t + 1$ users achieve $1/2$ degree of freedom just like in the interference channel without cooperation. Conceptually, the achievable scheme in this section is identical to the achievable scheme in Section 5.6 for the special case when $M_t = K - 2$; i.e., the achievable scheme is again based on converting the CoMP channel into a derived channel and then employing the asymptotic interference alignment scheme on the derived channel, as summarized in Figure 5.2.

5.7.1 Derived Channel

As in Section 5.6, we convert the CoMP channel into a derived channel that mimics the cellular uplink channel. Since our objective is to achieve a DoF vector that is asymmetric, the derived channel is also chosen to be asymmetric. The derived channel we consider in this section has two transmitters in each of the first $M_t - 1$ cells, and one transmitter in the remaining $K - M_t + 1$ cells.

$$Y_i = \sum_{j=1}^{K} g_{ij}^{(1)} X_j^{(1)} + \sum_{j=1}^{M_t - 1} g_{ij}^{(2)} X_j^{(2)} + Z_i. \tag{5.43}$$

As in Section 5.6, we assume that the channel inputs of the CoMP channel are related to the channel inputs of the derived channel through a linear transformation. The contribution of the derived channel input $X_j^{(m)}$ in the real transmit signals $X_j, X_{j+1}, \cdots, X_{j+M_t-1}$ is defined by a $M_t \times 1$ beamforming vector, i.e.,

$$\begin{bmatrix} X_j \\ X_{j+1} \\ \vdots \\ X_{j+M_t-1} \end{bmatrix} = (*) + \mathbf{v}_j^{(m)} X_j^{(m)}$$

where $*$ represents the contribution from other derived channel inputs. It is easy to see that the derived channel coefficients are related to the original channel coefficients as

$$g_{ij}^{(m)} = \mathbf{H}(i, \mathcal{T}_j)\mathbf{v}_j^{(m)}$$

for all $i, j \in \mathcal{K}$ and appropriate m. Since we are designing the achievable scheme to achieve 1 degree of freedom for the first $M_t - 1$ users, it must be that the first $M_t - 1$ receivers in the derived channel should not see any interference.

5.7.2 Zero-Forcing Step

We now explain our choice of the beamforming vectors that ensures that the first $M_t - 1$ receivers do not see any interference.

ZF beam design:

We first describe the general idea of constructing a zero-forcing beam. Consider the problem of designing a zero-forcing beam $\mathbf{v}$ to be transmitted by n transmit antennas indexed by the set $\mathcal{T} \subseteq \mathcal{K}$ such that it does not cause interference at $n-1$ receive antennas indexed by the set $\mathcal{I} \subseteq \mathcal{K}$, i.e.,

$$\mathbf{H}(\mathcal{I}, \mathcal{T})\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{H}(\mathcal{I}, \mathcal{T})$ is a $n-1 \times n$ matrix, the choice for $\mathbf{v}$ is unique up to a scaling factor. For any arbitrary row vector $\mathbf{a}$ of length n, we can use the Laplace expansion to expand the determinant

$$\det \begin{bmatrix} \mathbf{H}(\mathcal{I}, \mathcal{T}) \\ \mathbf{a} \end{bmatrix} = \sum_{j=1}^{n} a_j c_j$$

where c_j is the cofactor of a_j, that depends only on the channel coefficients in $\mathbf{H}(\mathcal{I}, \mathcal{T})$, and is independent of $\mathbf{a}$. By setting the beamforming vector $\mathbf{v}$ as $\mathbf{v} = [c_{n1} \ c_{n2} \ \cdots \ c_{nn}]$, we see that an arbitrary receiver i sees the signal transmitted along the beam $\mathbf{v}$ with a strength equal to

$$\mathbf{g} = \mathbf{H}(i, \mathcal{T})\mathbf{v} = \det \begin{bmatrix} \mathbf{H}(\mathcal{I}, \mathcal{T}) \\ \mathbf{H}(i, \mathcal{T}) \end{bmatrix} = \det \mathbf{H}(\mathcal{I} \cup i, \mathcal{T}).$$

Clearly, this satisfies the zero-forcing condition $\mathbf{H}(i,\mathcal{T})\mathbf{v} = 0$ for all $i \in \mathcal{I}$.

Design of transmit beam $\mathbf{v}_j^{(1)}$ for $j \geq M_t$:

The signal $X_j^{(1)}$ is transmitted by the M_t transmitters from the transmit set $\mathcal{T}_j = j \uparrow M_t$. The corresponding beam $\mathbf{v}_j^{(1)}$ is designed to avoid the interference at the first $M_t - 1$ receivers $\mathcal{I} = 1 \uparrow (M_t - 1)$. Therefore, we see that the contribution of $X_j^{(1)}$ at receiver i is given by

$$g_{ij}^{(1)} = \det \mathbf{H}(\mathcal{A}, \mathcal{B}) \tag{5.44}$$

where

$$\begin{aligned} \mathcal{A} &= \{1, 2, \cdots, M_t - 1, i\} \\ \mathcal{B} &= \{j, j+1, \cdots, j + M_t - 1\}. \end{aligned}$$

Design of transmit beams $\mathbf{v}_j^{(1)}$ and $\mathbf{v}_j^{(2)}$ for $j < M_t$:

The signals $X_j^{(1)}$ and $X_j^{(2)}$ are transmitted by the M_t transmitters from the transmit set $\mathcal{T}_j = 1 \uparrow M_t$. They must avoid interference at the $M_t - 2$ receivers

$$\mathcal{I} = \{1, 2, \cdots, j-1, j+1, \cdots, M_t - 1\}. \tag{5.45}$$

Since we only need to avoid interference at $M_t - 2$ receivers, it is sufficient to transmit each signal from $M_t - 1$ number of transmitters. We use the first $M_t - 1$ antennas of the transmit set $\mathcal{T}_j$ to transmit $X_j^{(1)}$, and the last $M_t - 1$ antennas of the transmit set $\mathcal{T}_j$ to transmit $X_j^{(2)}$. Thus, we obtain

$$\begin{aligned} g_{ij}^{(1)} &= \det \mathbf{H}(\mathcal{A}, \mathcal{B}_1) \\ g_{ij}^{(2)} &= \det \mathbf{H}(\mathcal{A}, \mathcal{B}_2) \end{aligned} \tag{5.46}$$

where

$$\begin{aligned} \mathcal{A} &= \{1, 2, \cdots, j-1, j+1, M_t - 1, i\} \\ \mathcal{B}_1 &= \{j, j+1, \cdots, j + M_t - 2\} \\ \mathcal{B}_2 &= \{j+1, j+1, \cdots, j + M_t - 1\}. \end{aligned}$$

Thus, the derived channel (5.43) can be simplified as

$$\begin{aligned} Y_i &= g_{ii}^{(1)} X_j^{(1)} + g_{ii}^{(2)} X_j^{(2)} + Z_i,\ 1 \leq i < M_t \\ Y_i &= \sum_{j=1}^{K} g_{ij}^{(1)} X_j^{(1)} + \sum_{j=1}^{M_t - 1} g_{ij}^{(2)} X_j^{(2)} + Z_i,\ M_t \leq i \leq K \end{aligned} \tag{5.47}$$

where the derived channel coefficients are as described in (5.44) and (5.46). Figure 5.4 provides a description of the derived channel for the special case of $K = 4$ and $M_t = 2$. We note that the derived channel in this section is a not a generalization, and does not specialize to the derived channel in Section 5.6 when $M_t = K - 2$. In fact, the achievable scheme in this section achieves fewer DoF compared to the optimal $\frac{KM_t}{M_t+1}$ DoF achieved in Section 5.6.

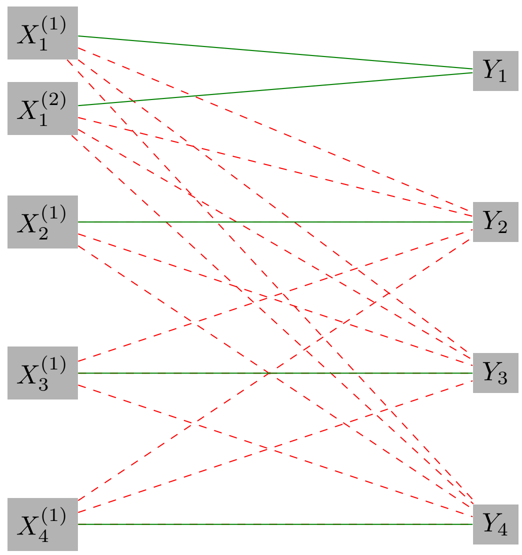

Figure 5.4: The derived channel in Section 5.7 when $K = 4$ and $M_t = 2$. The thick green lines indicate the links carrying signal. The dashed red lines indicate the links carrying interference.

5.7.3 Asymptotic Interference Alignment

In this section, we consider L parallel derived channels, and propose a scheme achieving a DoF arbitrary close to $(K + M_t - 1)/2$ in the limit $L \to \infty$. As in Section 5.6.4, we can combine L parallel derived channels (5.47) and express them together as

$$\underline{Y}_i = \mathbf{G}_{ii}^{(1)} \underline{X}_j^{(1)} + \mathbf{G}_{ii}^{(2)} \underline{X}_j^{(2)} + \underline{Z}_i, \; 1 \leq i < M_t$$
$$\underline{Y}_i = \sum_{j=1}^{K} \mathbf{G}_{ij}^{(1)} \underline{X}_j^{(1)} + \sum_{j=1}^{M_t-1} \mathbf{G}_{ij}^{(2)} \underline{X}_j^{(2)} + \underline{Z}_i, \; M_t \leq i \leq K$$

where $\underline{X}_j^{(m)}, \underline{Y}_i$ and $\underline{Z}_i$ are $L \times 1$ column vectors and $\mathbf{G}_{ij}^{(m)}$ is $L \times L$ diagonal channel transfer matrix given by

$$\mathbf{G}_{ij}^{(m)} = \begin{bmatrix} g_{ij}^{(m)}(1) & & & \\ & g_{ij}^{(m)}(2) & & \\ & & \ddots & \\ & & & g_{ij}^{(m)}(L) \end{bmatrix}.$$

As in Section 5.6.4, we use $\mathbf{V}$, defined as the order$-n$ CJ subspace generated by the channel matrices carrying interference

$$\{\bar{\mathbf{G}}_{ij}^{(1)}, \bar{\mathbf{G}}_{ij}^{(2)} : i \geq M_t, j < M_t\} \cup \{\bar{\mathbf{G}}_{ij}^{(1)} : i \neq j \geq M_t\} \tag{5.48}$$

as the transmit beamforming matrix at every transmitter of the derived channel. The first M_t-1 receivers do not see any interference. Therefore, for each $k < M_t$, the receiver k can decode all the desired streams free of interference if the matrix

$$\mathbf{M}_k = \left[\begin{array}{cc} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{G}_{kk}^{(2)}\mathbf{V} \end{array}\right] \tag{5.49}$$

has full column rank. Assuming that the number of rows in $\mathbf{M}_k$, equal to the number of parallel channels L, is greater than or equal to the number of columns, i.e., $L \geq 2|\mathbf{V}|$, the matrix $\mathbf{M}_k$ has full column rank for generic (original) channel coefficients $\{h_{ij}(\ell)\}$ if the following claim is true. See Corollary 4 in Section B for an explanation.

Claim 8. *For each $k < M_t$, the polynomials denoted by the variables*

$$\{g_{kk}^{(1)}, g_{kk}^{(2)}\} \cup \{g_{ij}^{(1)}, g_{ij}^{(2)} : i \geq M_t, j < M_t\} \cup \{g_{ij}^{(1)} : i \neq j \geq M_t\} \tag{5.50}$$

are algebraically independent.

For each $k \geq M_t$, the interference seen at receiver k is limited to the order$-(n+1)$ CJ subspace, denoted by $\mathbf{INT}$. Therefore, the receiver k can decode all the desired streams free of interference if the matrix

$$\mathbf{M}_k = \left[\begin{array}{cc} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{INT} \end{array}\right] \tag{5.51}$$

has full column rank. Assuming that the number of rows is greater than or equal to the number of columns, i.e., $L \geq |\mathbf{V}| + |\mathbf{INT}|$, the matrix $\mathbf{M}_k$ has full column rank for generic (original) channel coefficients $\{h_{ij}(t)\}$ if the following claim is true.

Claim 9. *For each $k \geq M_t$, the polynomials denoted by the variables*

$$\{g_{kk}^{(1)}\} \cup \{g_{ij}^{(1)}, g_{ij}^{(2)} : i \geq M_t, j < M_t\} \cup \{g_{ij}^{(1)} : i \geq M_t, j \geq M_t\} \tag{5.52}$$

are algebraically independent.

To satisfy the requirements on L, we choose L as

$$L = \max(2|\mathbf{V}|, |\mathbf{V}| + |\mathbf{INT}|) = |\mathbf{V}| + |\mathbf{INT}|. \tag{5.53}$$

Observe that

$$|\mathbf{V}| = \binom{N+n}{n} \text{ and } |\mathbf{INT}| = \binom{N+n+1}{n+1} \tag{5.54}$$

where N is the number of matrices (5.48) used to generate the CJ subspace, and is given by

$$\begin{aligned} N &= 2(K - M_t + 1)(M_t - 1) + (K - M_t + 1)(K - M_t) \\ &= (K - M_t + 1)(K + M_t - 2). \end{aligned} \tag{5.55}$$

Therefore, the achievable DoF is given by

$$\begin{aligned} \mathrm{DoF}(K, M_t, 1, L) &\geq \frac{2(M_t - 1)|\mathbf{V}| + (K - M_t + 1)|\mathbf{V}|}{L} \\ &= \frac{(K + M_t - 1)|\mathbf{V}|}{|\mathbf{V}| + |\mathbf{INT}|} \\ &= \frac{K + M_t - 1}{2 + \frac{N}{n+1}}. \end{aligned} \tag{5.56}$$

Therefore, we obtain that

$$\begin{aligned} \mathrm{DoF}(K, K-2, 1) &= \limsup_{L\to\infty} \mathrm{DoF}(K, K-2, 1, L) \\ &\geq \lim_{n\to\infty} \frac{K + M_t - 1}{2 + \frac{N}{n+1}} \\ &= \frac{K + M_t - 2}{2}. \end{aligned}$$

5.7.4 Proof of Algebraic Independence

As in Section 5.6.5, we use the Jacobian criterion to prove Claims 8 and 9. Recall that each derived channel coefficient is a polynomial in K^2 variables $\{h_{ij} : 1 \leq i, j, \leq K\}$. Let $\mathbf{g}$ denote the vector consisting of the polynomials specified by the derived channel coefficients in the respective claims. The exact description of the polynomials can be obtained from (5.44) and (5.44) in Section 5.7.2. The number of polynomials in Claims 8 and 9 is equal to $N + 2$ and $N + 1$, respectively, where N is given by (5.55). From Lemma 20 in Appendix B, we see that a collection of polynomials is algebraically independent if and only if the corresponding Jacobian matrix has full row rank. It can be easily verified that $N + 2 \leq K^2$, and hence $N + 1 \leq K^2$, for any K and M_t, which is a necessary condition for the corresponding Jacobian matrices to have full row rank. It is easy to verify that the Jacobian matrices corresponding to the polynomials in Claims 8 and 9 have full row rank using symbolic toolbox of MATLAB for any fixed K and M_t. In particular, we verified that the Jacobian matrices have full row rank for all values of $M_t < K \leq 9$.

5.8 Summary and Future Directions

We considered the problem of determining the DoF of K-user CoMP channel with a transmit cooperation order of M_t and a receive cooperation order of M_r. We showed that the DoF is equal to the maximum value of K if and only if $M_t + M_r \geq K + 1$. We related the problem to the problem of matrix decomposition with structural constraints. We proved a new theorem, that provides sufficient conditions for a generic matrix to admit SMD, generalizing the LU decomposition. We then set the receive cooperation order $M_r = 1$, and considered the problem of determining the DoF of interference channel with CoMP transmission. We determined $\mathrm{DoF}(K, M_t, 1)$ exactly in some cases, and approximately in other cases.

5.8.1 Transmit Set Selection

Recall that interference channel with CoMP transmission, i.e., with $M_r = 1$, is a good model for cellular downlink. One of the problems in the design of practical downlink CoMP schemes is the problem of transmit set selection. The transmit set of a user is the set of base stations that jointly transmit its message. In this chapter, we assumed a specific spiral structure for the transmit sets so that the mathematical analysis is simple. Instead, if we allow for arbitrary transmit sets satisfying the cooperation order constraints given by

$$|\mathcal{T}_k| \leq M_t,\ k \in \mathcal{K}$$

and study the problem of maximizing the DoF, the resulting insights can provide rough guidelines for the transmit set selection problem. Since we did not make any assumptions on the structure of transmit sets and receive sets in Theorem 11, we can obtain an upper bound on the sum DoF for any given choice of transmit sets. Solving the combinatorial optimization problem of maximizing the resulting upper bound on DoF subject to cooperation order constraint could also lead to insights on the problem of transmit set selection. However, solving this combinatorial optimization problem seems to be difficult.

An immediate insight from the outer bound in Theorem 11 is that clustering does not improve DoF. To illustrate the point, suppose $K = 2M_t r$ for some integer r, and suppose the transmit sets are chosen such that the K transmitters are divided into $2r$ clusters with each cluster containing M_t transmitters. The sets $\mathcal{A}$ and $\mathcal{B}$ in Theorem 11 can be appropriately chosen to conclude that the DoF is outer-bounded by $K/2$. Since $K/2$ DoF is achievable even without cooperation, we see that clustering is not a good transmit set selection algorithm from the DoF perspective.

5.8.2 Achievable DoF using Beamforming Strategies

In the case of CoMP transmission, the proposed asymptotic achievable schemes in Sections 5.6 and 5.7 are based on the CJ subspace, which is more of a proof technique than a practical achievable scheme. The nice feature of this approach is that asymptotic DoF in the limit $L \to \infty$ matches with the information-theoretic upper bound. For finite L such as $L = 1$, the achievable DoF with beamforming strategies does not, in general, match with the information-theoretic outer bound. Since the beamforming strategies are practical, it is of interest to determine the best achievable DoF using beamforming strategies for finite values of L. Recently, algebraic geometry techniques have been used in [66, 67] to determine the best achievable DoF in the K-user MIMO Gaussian interference channels without cooperation. Extending such analysis to the CoMP channel would provide good insights into the usefulness of CoMP within the class of beamforming strategies.

5.9 Proof of Claim 7

In this section, we complete the proof of Theorem 16 by show that the determinant of the submatrix (5.38) evaluated at the point $\mathbf{H} = \mathbf{A}$ is equal to ± 1. Recall that

$$\begin{aligned}
\mathbf{g}_0 &= \left(g_{11}^{(1)}, g_{11}^{(2)}, \cdots, g_{11}^{(M_t)}\right) = \mathbf{H}(1, \mathcal{T}_1)\mathbf{H}(\mathcal{T}_2, \mathcal{T}_1)^{-1} \\
\mathbf{g}_i &= \left(g_{i,i+1}^{(1)}, g_{i,i+1}^{(2)}, \cdots, g_{i,i+1}^{(M_t)}\right) = \mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{H}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1} \\
\mathbf{h}_0 &= (h_{11}, h_{22}, \cdots, h_{M_t M_t}) \\
\mathbf{h}_i &= (h_{i,i+1}, h_{i,i+2}, \cdots, h_{i,K}, h_{i,1}, \cdots, h_{i,i-2}) = \mathbf{H}(i, \mathcal{T}_{i+1})
\end{aligned}$$

where the transmit set $\mathcal{T}_i$ is given by

$$\mathcal{T}_i = i \uparrow (K-2) = \{i, i+1, \cdots, i+K-2\}.$$

Let $\mathbf{J}[\mathbf{g}_i; \mathbf{h}_j]$ denote the submatrix of the Jacobian matrix with rows corresponding to the variables $\mathbf{g}_i$ and columns corresponding to the variables $\mathbf{h}_j$. Then, the submatrix (5.38) can be expressed as

$$\begin{bmatrix} \mathbf{J}[\mathbf{g}_0; \mathbf{h}_0] & \cdots & \mathbf{J}[\mathbf{g}_0; \mathbf{h}_K] \\ \vdots & \ddots & \vdots \\ \mathbf{J}[\mathbf{g}_K; \mathbf{h}_0] & \cdots & \mathbf{J}[\mathbf{g}_K; \mathbf{h}_K] \end{bmatrix}. \tag{5.57}$$

Differentiating $\mathbf{g}_i = \mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{H}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1}$ at $\mathbf{H} = \mathbf{A}$, we get

$$\begin{aligned}
d\mathbf{g}_i &= d\mathbf{H}(i, \mathcal{T}_{i+1})\mathbf{A}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1} \\
&\quad - \mathbf{A}(i, \mathcal{T}_{i+1})\mathbf{A}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1} d\mathbf{H}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})\mathbf{A}(\mathcal{T}_{i+2}, \mathcal{T}_{i+1})^{-1}.
\end{aligned} \tag{5.58}$$

The matrix $\mathbf{A}$ is chosen to satisfy

$$\mathbf{A}(i,\mathcal{T}_{i+1}) = \mathbf{0}$$
$$\mathbf{A}(\mathcal{T}_{i+2},\mathcal{T}_{i+1}) = \mathbf{B},$$

where $\mathbf{B}$ is the $M_t \times M_t$ matrix with all the diagonal and the superdiagonal entries being equal to 1. Note that $\det \mathbf{B} = 1$. For the special case of $K = 4$ and $M_t = 2$, the matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{5.59}$$

Therefore, (5.58) can be simplified as

$$d\mathbf{g}_i = d\mathbf{H}(i,\mathcal{T}_{i+1})\mathbf{B}^{-1} = d\mathbf{h}_i\mathbf{B}^{-1}.$$

Equivalently, for each $i \geq 1$, we have

$$\mathbf{J}[\mathbf{g}_i;\mathbf{h}_i] = \mathbf{B}^{-\top}$$
$$\mathbf{J}[\mathbf{g}_i;\mathbf{h}_j] = \mathbf{0}, \forall j \neq i.$$

Hence, the determinant of the submatrix (5.57) is equal to

$$\det \mathbf{J}[\mathbf{g}_0;\mathbf{h}_0]/(\det \mathbf{B})^K = \det \mathbf{J}[\mathbf{g}_0;\mathbf{h}_0].$$

We now show that $\det \mathbf{J}[\mathbf{g}_0;\mathbf{h}_0] = \pm 1$. Recall from Section 5.6.3 that $\mathbf{g}_0$ is related to $\mathbf{H}$ as

$$\mathbf{g}_0 = \left(g_{11}^{(1)}, g_{11}^{(2)}, \cdots, g_{11}^{(M_t)}\right) = \mathbf{H}(1,\mathcal{T}_1)\mathbf{H}(\mathcal{T}_2,\mathcal{T}_1)^{-1}.$$

Differentiating $\mathbf{g}_0 = \mathbf{H}(1,\mathcal{T}_1)\mathbf{H}(\mathcal{T}_2,\mathcal{T}_1)^{-1}$ at $\mathbf{H} = \mathbf{A}$, we get

$$\begin{aligned} d\mathbf{g}_0 &= d\mathbf{H}(1,\mathcal{T}_1)\mathbf{A}(\mathcal{T}_2,\mathcal{T}_1)^{-1} \\ &\qquad - \mathbf{A}(1,\mathcal{T}_1)\mathbf{A}(\mathcal{T}_2,\mathcal{T}_1)^{-1}d\mathbf{H}(\mathcal{T}_2,\mathcal{T}_1)\mathbf{A}(\mathcal{T}_2,\mathcal{T}_1)^{-1} \\ &= d\mathbf{H}(1,\mathcal{T}_1)\mathbf{B}^{-1} - \mathbf{A}(1,\mathcal{T}_1)\mathbf{B}^{-1}d\mathbf{H}(\mathcal{T}_2,\mathcal{T}_1)\mathbf{B}^{-1}. \end{aligned}$$

Now, observe that

$$\begin{aligned} \mathbf{A}(1,\mathcal{T}_1)\mathbf{B}^{-1} &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}\mathbf{B}^{-1} \\ &= \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \end{bmatrix}. \end{aligned} \tag{5.60}$$

Therefore, we get

$$
\begin{aligned}
d\mathbf{g}_0\mathbf{B} = d\mathbf{H}&(1,\mathcal{T}_1) - \mathbf{A}(1,\mathcal{T}_1)\mathbf{B}^{-1}d\mathbf{H}(\mathcal{T}_2,\mathcal{T}_1) \\
&= \begin{bmatrix} dh_{11} & dh_{12} & \cdots & dh_{1,K-2} \end{bmatrix} \\
&\quad - \begin{bmatrix} dh_{21} & dh_{22} & \cdots & dh_{2,K-2} \end{bmatrix} \\
&\quad + \begin{bmatrix} dh_{31} & dh_{32} & \cdots & dh_{3,K-2} \end{bmatrix} \\
&\quad \vdots \\
&\quad (-1)^{K-1} \begin{bmatrix} dh_{K-2,1} & dh_{K-2,2} & \cdots & dh_{K-2,K-2} \end{bmatrix} \\
&\quad (-1)^{K} \begin{bmatrix} dh_{K-1,1} & dh_{K-1,2} & \cdots & dh_{K-1,K-2} \end{bmatrix}.
\end{aligned}
$$

To determine $\mathbf{J}[\mathbf{g}_0;\mathbf{h}_0]$, we are only interested in the partial derivatives with respect to the variables $h_{11}, h_{22}, \cdots, h_{K-2,K-2}$. The contribution of $d\mathbf{h}_0$ in $d\mathbf{g}_0$ is given by

$$
\begin{aligned}
&\begin{bmatrix} dh_{11} & -dh_{22} & dh_{33} & -dh_{44} & \cdots \end{bmatrix}\mathbf{B}^{-1} \\
&= d\mathbf{h}_0 \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}\mathbf{B}^{-1}
\end{aligned} \tag{5.61}
$$

which implies that

$$
\mathbf{J}[\mathbf{g}_0;\mathbf{h}_0] = \mathbf{B}^{-\top} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}. \tag{5.62}
$$

Hence, $\det \mathbf{J}[\mathbf{g}_0;\mathbf{h}_0] = \pm \det \mathbf{B} = \pm 1$.

CHAPTER 6

CONCLUSIONS

In Chapters 3 and 4, we solved the following two problems exactly in a low interference regime:

1. Find the best sum-rate of Gaussian interference channels with transmitters using Gaussian inputs and receivers treating interference as noise.
2. Find the sum capacity of Gaussian interference channels.

We used the concept of a genie-aided channel to solve both of these problems in one shot. We chose the side-information signals carefully, so that the resulting sum-rate function of the genie-aided channel is concave in the transmit covariance matrices, and i.i.d. Gaussian inputs achieve the sum capacity of the genie-aided channel. Interestingly, both the concavity of the sum-rate function and the optimality of Gaussian inputs hold under the same conditions on the genie parameters, which we refer to as the usefulness conditions. It is worth exploring if this is coincidental or if the concavity of the achievable sum-rate function and the optimality of i.i.d. Gaussian inputs are fundamentally related to each other.

We solved both of the above problems in a *low interference regime* by minimizing the upper bound over all useful genies. We showed that the notion of smart genie is crucial in arriving at succinct expressions for low interference regime. It appears that the proposed genie-aided channel approach, or in general the convex relaxation and approximation approach, has untapped potential in both proving low interference regime theorems, and also in obtaining efficient algorithms to obtain good lower and upper bounds to the sum capacity, and best achievable sum-rate with treating interference as noise. We refer the reader to Sections 3.11 and 4.5, where we outlined a few interesting ideas that are not explored in this dissertation but could further our understanding of the sum capacity of Gaussian interference channels even in the classical two-user symmetric SISO case.

In Chapter 5, we studied the problem of determining the DoF of the K-user interference channel with CoMP transmission and reception as a function of the transmit and receive cooperation orders. We determined the exact DoF in some special cases, and derived lower and upper bounds in the other cases. We showed that the DoF is exactly equal to K if and only if the cooperation orders satisfy $M_t + M_r \geq K + 1$, and the achievability proof is based on the decomposition of a matrix as a product to two matrices with structural constraints. So far we have not found any other applications for the structural matrix decomposition, but we are hopeful that Theorem 14 can be useful in some signal processing applications. We then considered the problem of determining the DoF of interference channel with CoMP transmission, i.e., with $M_r = 1$. While the achievable schemes of Theorems 15 and 16 are based on the Cadambe-Jafar scheme and are easy to construct, proving that the achievable schemes work for generic channel coefficients is highly nontrivial. The notion of algebraic independence of rational functions, and the Jacobian criterion to test algebraic independence, turned out to be extremely crucial. We believe that these connections to algebraic geometry are useful in understanding the achievable DoF with linear beamforming strategies of any wireless channel. We refer the reader to Section 5.8 for more details on the usefulness of the algebraic geometry techniques.

APPENDIX A

INFORMATION THEORY

In this appendix, we state some results in information theory that are useful in Chapters 3 and 4. We refer the reader to the standard textbook [68] for basic definitions of information-theoritic quantities such as entropy, differential entropy, conditional entropy and mutual information, and to Section 1.5 for an explanation about the notation that we follow.

A.1 Minimum Mean Squared Error Estimation

Suppose $\underline{X}_G$ and $\underline{Y}_G$ are jointly circularly symmetric, and jointly Gaussian complex random vectors with zero mean. See [69] and [70] for the definitions and properties of circularly symmetric complex Gaussian random vectors. Since the random vectors are jointly circularly symmetric and Gaussian, the minimum mean squared error (MMSE) estimate of $\underline{X}_G$ given $\underline{Y}_G$

$$\underline{\hat{X}}_G = \mathbb{E}\left[\underline{X}_G|\underline{Y}_G\right]$$

is linear in $\underline{Y}_G$. The *orthogonality principle* says that $\underline{\hat{X}}_G = \mathbf{T}\underline{Y}_G$ is the MMSE estimate if and only if the MMSE estimation error $(\underline{X}_G - \underline{\hat{X}}_G)$ is orthogonal (and independent in this case since the random variables are jointly Gaussian) to the observation $\underline{Y}_G$; i.e.,

$$\mathbb{E}\left[(\underline{X}_G - \mathbf{T}\underline{Y}_G)\underline{Y}_G^{\dagger}\right] = 0 \Leftrightarrow \mathbf{\Sigma}_{\underline{X}_G\underline{Y}_G} = \mathbf{T}\mathbf{\Sigma}_{\underline{Y}_G}.$$

Suppose $\mathbf{\Sigma}_{\underline{Y}_G}$ is nonsingular; then $\mathbf{T}$ is uniquely determined and hence is given by

$$\mathbf{T} = \mathbf{\Sigma}_{\underline{X}_G\underline{Y}_G}\mathbf{\Sigma}_{\underline{Y}_G}^{-1}$$

and the conditional covariance matrix of $\underline{X}_G$ given $\underline{Y}_G$, defined as the covariance matrix of MMSE estimation error, is given by

$$\mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G} = \mathbf{\Sigma}_{\underline{X}_G} - \mathbf{\Sigma}_{\underline{X}_G\underline{Y}_G}\mathbf{\Sigma}_{\underline{Y}_G}^{-1}\mathbf{\Sigma}_{\underline{Y}_G\underline{X}_G}.$$

Note that $\mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G}$ is the Schur complement of $\mathbf{\Sigma}_{\underline{X}_G}$ in the matrix

$$\operatorname{Cov}\left(\begin{bmatrix} \underline{X}_G \\ \underline{Y}_G \end{bmatrix}\right) = \begin{bmatrix} \mathbf{\Sigma}_{\underline{X}_G} & \mathbf{\Sigma}_{\underline{X}_G,\underline{Y}_G} \\ \mathbf{\Sigma}_{\underline{Y}_G,\underline{X}_G} & \mathbf{\Sigma}_{\underline{Y}_G} \end{bmatrix}.$$

The following lemma is useful.

Lemma 9. *Suppose $\underline{X}_G$ and $\underline{Y}_G$ are jointly circularly symmetric, and jointly Gaussian complex random vectors, and $\mathbf{\Sigma}$ is a positive semidefinite matrix. Then,*

$$\mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G} \succeq \mathbf{\Sigma} \Leftrightarrow \operatorname{Cov}\left(\begin{bmatrix} \underline{X}_G \\ \underline{Y}_G \end{bmatrix}\right) \succeq \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be column vectors of the same length as $\underline{X}_G$ and $\underline{Y}_G$, respectively, and $\mathbf{T}\underline{Y}_G$ be the MMSE estimate of $\underline{X}_G$ given $\underline{Y}_G$. Observe that the first condition is equivalent to

$$\mathbf{a}^\dagger \mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G} \mathbf{a} = \mathbb{E}\left[\left|\mathbf{a}^\dagger \left(\underline{X}_G - \mathbf{T}\mathbf{y}_G\right)\right|^2\right] \geq \mathbf{a}^\dagger \mathbf{\Sigma} \mathbf{a},\ \forall \mathbf{a} \tag{A.1}$$

and the second condition is equivalent to

$$\mathbb{E}\left[\left|\mathbf{a}^\dagger \underline{X}_G + \mathbf{b}^\dagger \underline{Y}_G\right|^2\right] \geq \mathbf{a}^\dagger \mathbf{\Sigma} \mathbf{a},\ \forall \mathbf{a}, \mathbf{b}. \tag{A.2}$$

We complete the proof by showing that the equations (A.1) and (A.1) imply each other. First, observe that

$$\begin{aligned}
\mathbb{E}\left[\left|\mathbf{a}^\dagger \underline{X}_G + \mathbf{b}^\dagger \underline{Y}_G\right|^2\right] &= \mathbb{E}\left[\left|\mathbf{a}^\dagger \left(\underline{X}_G - \mathbf{T}\underline{Y}_G + \mathbf{T}\underline{Y}_G\right) + \mathbf{b}^\dagger \underline{Y}_G\right|^2\right] \\
&= \mathbb{E}\left[\left|\mathbf{a}^\dagger \left(\underline{X}_G - \mathbf{T}\underline{Y}_G\right) + \left(\mathbf{a}^\dagger \mathbf{T} + \mathbf{b}^\dagger\right) \underline{Y}_G\right|^2\right].
\end{aligned}$$

Since the MMSE estimation error $\underline{X}_G - \mathbf{T}\underline{Y}_G$ is independent of the observation $\underline{Y}_G$, we have that

$$\mathbb{E}\left[\left|\mathbf{a}^\dagger \underline{X}_G + \mathbf{b}^\dagger \underline{Y}_G\right|^2\right] = \mathbb{E}\left[\left|\mathbf{a}^\dagger \left(\underline{X}_G - \mathbf{T}\underline{Y}_G\right)\right|^2\right] + \mathbb{E}\left[\left|\left(\mathbf{a}^\dagger \mathbf{T} + \mathbf{b}^\dagger\right) \underline{Y}_G\right|^2\right].$$

Substituting the above expression in (A.2) it is clear that (A.1) implies (A.2). It is also clear that (A.2) implies (A.1) by setting $\mathbf{b} = -\mathbf{T}^\dagger \mathbf{a}$. □

A.2 Basic Extremal Inequalities

In this section, we review two basic extremal inequalities regarding the optimality of circularly symmetric complex Gaussian distributions. The following lemma says that among all continuous distributions with a fixed covariance matrix, the circularly symmetric complex Gaussian distribution maximizes the differential entropy.

Lemma 10 (Theorem 2 in [69])**.** *Let $\underline{X}$ be a complex and continuous random vector. Then,*

$$\mathsf{h}\left(\underline{X}\right) \leq \mathsf{h}\left(\underline{X}_G\right) = \log\det\left(\pi e \mathbf{\Sigma}_{\underline{X}_G}\right)$$

where $\underline{X}_G$ is a circularly symmetric complex Gaussian random vector with zero mean and covariance matrix equal to the covariance matrix of $\underline{X}$.

We now extend the lemma to conditional differential entropy.

Lemma 11. *Let $\underline{X}$ and $\underline{Y}$ be complex and continuous random vectors. Then,*

$$\mathsf{h}\left(\underline{X}|\underline{Y}\right) \leq \mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right) = \log\det\left(\pi e \mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G}\right)$$

where $\underline{X}_G$ and $\underline{Y}_G$ are jointly circularly symmetric and jointly Gaussian complex random vectors with zero mean and joint covariance matrix equal to the joint covariance matrix of $\underline{X}$ and $\underline{Y}$.

Proof. Let $\mathbf{T}\underline{Y}_G$ be the MMSE estimate of $\underline{X}_G$ given $\underline{Y}_G$. Since the estimation error $\underline{X}_G - \mathbf{T}\underline{Y}_G$ is independent of the observation $\underline{Y}_G$, we have

$$\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right) = \mathsf{h}\left(\underline{X}_G - \mathbf{T}\underline{Y}_G|\underline{Y}_G\right) = \mathsf{h}\left(\underline{X}_G - \mathbf{T}\underline{Y}_G\right) = \log\det\left(\pi e \mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G}\right).$$

Also, observe that

$$\mathsf{h}\left(\underline{X}|\underline{Y}\right) = \mathsf{h}\left(\underline{X} - \mathbf{T}\underline{Y}|\underline{Y}\right) \leq \mathsf{h}\left(\underline{X} - \mathbf{T}\underline{Y}\right) \leq \mathsf{h}\left(\underline{X}_G - \mathbf{T}\underline{Y}_G\right)$$

where the first inequality follows because conditioning can only reduce entropy, and the second inequality follows from Lemma 10 because the covariance matrix of $\underline{Y} - \mathbf{T}\underline{X}$ is the same as the covariance matrix of $\underline{Y}_G - \mathbf{T}\underline{X}_G$. $\square$

A.3 Concave Functions

In this section, we show that the differential entropy and the conditional differential entropy of circularly symmetric complex Gaussian random vectors are concave functions in the corresponding covariance matrices.

Lemma 12. *Suppose $\underline{X}_G$ is a circularly symmetric complex Gaussian random vector. The differential entropy $\mathsf{h}\left(\underline{X}_G\right) = \log\left(|\pi e \mathbf{\Sigma}_{\underline{X}_G}|\right)$ is concave and nondecreasing in $\mathbf{\Sigma}_{\underline{X}_G}$.*

We now extend the lemma to conditional differential entropy.

Lemma 13. *Suppose $\underline{X}_G$ and $\underline{Y}_G$ are jointly circularly symmetric, and jointly Gaussian complex random vectors. Then, the conditional differential entropy $\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right)$ is concave and nondecreasing in*

$$\mathrm{Cov}\left(\begin{bmatrix} \underline{X}_G \\ \underline{Y}_G \end{bmatrix}\right).$$

Proof. Let $(\underline{X}_{1G}, \underline{Y}_{1G})$, $(\underline{X}_{2G}, \underline{Y}_{2G})$ and $(\underline{X}_{2G}, \underline{Y}_{2G})$ be jointly circularly symmetric, and jointly Gaussian complex random vectors such that

$$\text{Cov}\left(\begin{bmatrix} \underline{X}_G \\ \underline{Y}_G \end{bmatrix}\right) = \lambda \text{Cov}\left(\begin{bmatrix} \underline{X}_{1G} \\ \underline{Y}_{1G} \end{bmatrix}\right) + (1-\lambda)\text{Cov}\left(\begin{bmatrix} \underline{X}_{2G} \\ \underline{Y}_{2G} \end{bmatrix}\right).$$

From Lemma 9, we know that the conditional covariance matrices $\mathbf{\Sigma}_{\underline{X}_{1G}|\underline{Y}_{1G}}$ and $\mathbf{\Sigma}_{\underline{X}_{2G}|\underline{Y}_{2G}}$ satisfy

$$\text{Cov}\left(\begin{bmatrix} \underline{X}_{iG} \\ \underline{Y}_{iG} \end{bmatrix}\right) \succeq \begin{bmatrix} \mathbf{\Sigma}_{\underline{X}_{iG}|\underline{Y}_{iG}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \text{ for } i = 1, 2.$$

Therefore, we obtain that

$$\text{Cov}\left(\begin{bmatrix} \underline{X}_G \\ \underline{Y}_G \end{bmatrix}\right) \succeq \begin{bmatrix} \lambda \mathbf{\Sigma}_{\underline{X}_{1G}|\underline{Y}_{1G}} + (1-\lambda)\mathbf{\Sigma}_{\underline{X}_{2G}|\underline{Y}_{2G}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Applying Lemma 9 again, we obtain that

$$\mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G} \succeq \lambda \mathbf{\Sigma}_{\underline{X}_{1G}|\underline{Y}_{1G}} + (1-\lambda)\mathbf{\Sigma}_{\underline{X}_{2G}|\underline{Y}_{2G}}.$$

Now, applying Lemma 12, we obtain that

$$\begin{aligned}
\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right) &= \log\left(\left|\pi e \mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G}\right|\right) \\
&\geq \log\left(\left|\pi e \left(\lambda \mathbf{\Sigma}_{\underline{X}_{1G}|\underline{Y}_{1G}} + (1-\lambda)\mathbf{\Sigma}_{\underline{X}_{2G}|\underline{Y}_{2G}}\right)\right|\right) \\
&\geq \lambda \log\left(\left|\pi e \mathbf{\Sigma}_{\underline{X}_{1G}|\underline{Y}_{1G}}\right|\right) + (1-\lambda)\log\left(\left|\pi e \mathbf{\Sigma}_{\underline{X}_{2G}|\underline{Y}_{2G}}\right|\right) \\
&= \lambda \mathsf{h}\left(\underline{X}_{1G}|\underline{Y}_{1G}\right) + (1-\lambda)\mathsf{h}\left(\underline{X}_{2G}|\underline{Y}_{2G}\right).
\end{aligned}$$

This completes the proof of concavity of $\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right)$ in the joint covariance matrix. The proof that $\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right)$ is nondecreasing in the joint covariance matrix follows directly from Lemma 9. $\square$

Lemma 14. *Suppose $\underline{X}_G, \underline{Z}, \underline{W}$ are circularly symmetric, and indepenent complex Gaussian random vectors such that $\mathbf{\Sigma}_{\underline{Z}} \preceq \mathbf{\Sigma}_{\underline{W}}$. Then,*

$$\mathsf{h}\left(\underline{X}_G + \underline{Z}\right) - \mathsf{h}\left(\underline{X}_G + \underline{W}\right)$$

is concave in $\mathbf{\Sigma}_{\underline{X}_G}$.

Proof. Let $\underline{V} \sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{W}} - \mathbf{\Sigma}_{\underline{Z}})$ be independent of $\underline{X}$ and $\underline{Z}$. Then, $\underline{Z} + \underline{V}$ has the same distribution as $\underline{W}$. Therefore, it is sufficient to prove that

$$\mathsf{h}\left(\underline{X}_G + \underline{Z}\right) - \mathsf{h}\left(\underline{X}_G + \underline{Z} + \underline{V}\right) = -\mathsf{I}\left(\underline{V}; \underline{X}_G + \underline{Z} + \underline{V}\right)$$

is concave in $\mathbf{\Sigma}_{\underline{X}_G}$. Observe that

$$-\mathsf{I}\left(\underline{V}; \underline{X}_G + \underline{Z} + \underline{V}\right) = -\mathsf{h}\left(\underline{V}\right) + \mathsf{h}\left(\underline{V}|\underline{X}_G + \underline{V} + \underline{W}\right).$$

The first term is independent of $\mathbf{\Sigma}_{\underline{X}_G}$, and from Lemma 13, it follows that the second term is concave in $\mathbf{\Sigma}_{\underline{X}_G}$. □

A.4 More Extremal Inequalities

In this section, we show that among all the sequences of random vectors with a fixed average covariance matrix, the circularly symmetric i.i.d. Gaussian random vectors maximize certain objective functions involving multi-letter differential entropy and conditional differential entropy terms.

Lemma 15. *Suppose $\underline{X}^n$ is a sequence of complex and continuous random vectors. Then,*

$$\mathsf{h}\left(\underline{X}^n\right) \leq n\mathsf{h}\left(\underline{X}_G\right) = n \log\det\left(\pi e \mathbf{\Sigma}_{\underline{X}_G}\right)$$

where $\underline{X}_G$ is a circularly symmetric Gaussian random vector with the covariance matrix equal to the average covariance matrix of the random sequence $\underline{X}^n$; i.e.,

$$\mathbf{\Sigma}_{\underline{X}_G} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{\Sigma}_{\underline{X}_i}.$$

Proof. Observe that

$$\begin{aligned}
\mathsf{h}\left(\underline{X}^n\right) &= \sum_{i=1}^{n} \mathsf{h}\left(\underline{X}_i|\underline{X}^{i-1}\right) \\
&\overset{(a)}{\leq} \sum_{i=1}^{n} \mathsf{h}\left(\underline{X}_i\right) \\
&\overset{(b)}{\leq} \sum_{i=1}^{n} \mathsf{h}\left(\underline{X}_{iG}\right) \\
&\overset{(c)}{\leq} n\mathsf{h}\left(\underline{X}_G\right)
\end{aligned}$$

where step (a) follows because conditioning can only reduce entropy, step (b) follows from Lemma 10, and step (b) follows from Lemma 12 that says that $\mathsf{h}\left(\underline{X}_G\right)$ is concave in $\mathbf{\Sigma}_{\underline{X}_G}$. □

We now extended the above lemma to conditional differential entropy.

Lemma 16. *Suppose $\underline{X}^n$ and $\underline{Y}^n$ are sequences of complex and continuous random vectors. Then,*

$$\mathsf{h}\left(\underline{X}^n|\underline{Y}^n\right) \leq n\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right) = n \log\det\left(\pi e \mathbf{\Sigma}_{\underline{X}_G|\underline{Y}_G}\right)$$

where $(\underline{X}_G, \underline{Y}_G)$ are jointly circularly symmetric and Gaussian complex random *vectors with the same joint covariance matrix equal to*

$$\mathrm{Cov}\left(\left[\begin{array}{c}\underline{X}_G\\ \underline{Y}_G\end{array}\right]\right)=\frac{1}{n}\sum_{i=1}^{n}\mathrm{Cov}\left(\left[\begin{array}{c}\underline{X}_i\\ \underline{Y}_i\end{array}\right]\right).$$

Proof. Let $(\underline{X}_{iG}, \underline{Y}_{iG})$ be jointly circularly symmetric, and jointly Gaussian random vectors with the jointly covariance matrix equal to the joint covariance matrix of $(\underline{X}_i, \underline{Y}_i)$. Then, we have

$$\begin{aligned}\mathsf{h}\left(\underline{X}^n|\underline{Y}^n\right) &= \sum_{i=1}^{n}\mathsf{h}\left(\underline{X}_i|\underline{Y}^{i-1},\underline{Y}^n\right)\\ &\overset{(a)}{\leq}\sum_{i=1}^{n}\mathsf{h}\left(\underline{X}_i|\underline{Y}_i\right)\\ &\overset{(b)}{\leq}\sum_{i=1}^{n}\mathsf{h}\left(\underline{X}_{iG}|\underline{Y}_{iG}\right)\\ &\overset{(c)}{\leq} n\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right)\end{aligned}$$

where step (a) follows because conditioning can only reduce entropy, step (b) follows from Lemma 11, and step (b) follows from Lemma 13 that says that $\mathsf{h}\left(\underline{X}_G|\underline{Y}_G\right)$ is concave in the joint covariance matrix of $\underline{X}_G$ and $\underline{Y}_G$. $\square$

Lemma 17. *Suppose $\underline{X}^n$ is a sequence of complex and continuous random vectors, and $\underline{Z}^n \sim$ i.i.d. $\sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{Z}})$, and $\underline{W}^n \sim$ i.i.d. $\sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{W}})$ such that $\mathbf{\Sigma}_z \preceq \mathbf{\Sigma}_w$. Then,*

$$\mathsf{h}\left(\underline{X}^n+\underline{Z}^n\right)-\mathsf{h}\left(\underline{X}^n+\underline{W}^n\right)\leq n\mathsf{h}\left(\underline{X}_G+\underline{Z}\right)-n\mathsf{h}\left(\underline{X}_G+\underline{W}\right)$$

where $\underline{Z} \sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{Z}})$, $\underline{W} \sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{W}})$, and $\underline{X}_G$ is a circularly symmetric Gaussian random vector with the covariance matrix equal to the average covariance matrix of the random sequence $\underline{X}^n$; i.e.,

$$\mathbf{\Sigma}_{\underline{X}_G}=\frac{1}{n}\sum_{i=1}^{n}\mathbf{\Sigma}_{\underline{X}_i}.$$

Proof. Let $\underline{V}^n \sim$ i.i.d. $\sim \mathcal{CN}(0, \mathbf{\Sigma}_{\underline{W}} - \mathbf{\Sigma}_{\underline{Z}})$ be independent of $\underline{Z}^n$. Then, the sequence $\underline{Z}^n + \underline{V}^n$ has the same distribution as $\underline{W}^n$. Therefore, it is sufficient to prove that

$$\begin{aligned}&\mathsf{h}\left(\underline{X}^n+\underline{Z}^n\right)-\mathsf{h}\left(\underline{X}^n+\underline{Z}^n+\underline{V}^n\right)\leq n\mathsf{h}\left(\underline{X}_G+\underline{Z}\right)-n\mathsf{h}\left(\underline{X}_G+\underline{Z}+\underline{V}\right)\\ &\Leftrightarrow -\mathsf{I}\left(\underline{V}^n;\underline{X}^n+\underline{Z}^n+\underline{V}^n\right)\leq -n\mathsf{I}\left(\underline{V};\underline{X}_G+\underline{Z}+\underline{V}\right).\end{aligned}$$

Observe that

$$
\begin{aligned}
-\mathsf{I}\left(\underline{V}^n;\underline{X}^n+\underline{Z}^n+\underline{V}^n\right) &= -\mathsf{h}\left(\underline{V}^n\right)+\mathsf{h}\left(\underline{V}^n|\underline{X}^n+\underline{Z}^n+\underline{V}^n\right)\\
&\stackrel{(a)}{\leq} -n\mathsf{h}\left(\underline{V}\right)+n\mathsf{h}\left(\underline{V}|\underline{X}_G+\underline{Z}+\underline{V}\right)\\
&= -n\mathsf{I}\left(\underline{V};\underline{X}_G+\underline{Z}+\underline{V}\right).
\end{aligned}
$$

where step (a) follows from Lemma 16. □

The above lemma is referred to as the worst-case noise lemma. Indeed, observe that $\mathsf{I}\left(\underline{V}^n;\underline{X}^n+\underline{Z}^n+\underline{V}^n\right)$ can be interpreted as the multi-letter mutual information of an additive noise channel with $\underline{V}$ as input and $\underline{X}+\underline{Z}$ as noise. The above result argues that i.i.d. Gaussian noise is the worst-case noise minimizing $\mathsf{I}\left(\underline{V}^n;\underline{X}^n+\underline{Z}^n+\underline{V}^n\right)$. In the scalar case, as explained in the mutual information game problem (see Exercise 9.21 in [68]), an alternative proof can be given using the entropy power inequality (EPI).

Lemma 18 (EPI)**.** *Suppose X^n and Z^n are independent sequences of complex random variables. Then,*

$$
e^{\frac{1}{n}\mathsf{h}(X^n+Z^n)} \geq e^{\frac{1}{n}\mathsf{h}(X^n)}+e^{\frac{1}{n}\mathsf{h}(Z^n)}.
$$

We use EPI to prove a generalized version of worst-case noise lemma in the scalar case.

Lemma 19. *Suppose $\{X_i^n : 1\leq i\leq M\}$ are independent sequences of complex random vectors satisfying an average power constraint P_i; i.e.,*

$$
\frac{1}{n}\sum_{j=1}^{n}\Sigma_{X_{ij}} \leq P_i,\ 1\leq i\leq M
$$

and $Z^n \sim$ i.i.d. $\sim \mathcal{CN}(0,\Sigma_Z)$. Let $\mu_1,\mu_2,\cdots,\mu_M$ be real numbers satisfying the conditions

$$
\mu_i \geq \frac{P_i}{\sum_{j=1}^{M}P_j+\Sigma_Z},\ 1\leq i\leq M.
$$

Then, we have

$$
\sum_{i=1}^{M}\mu_i\mathsf{h}\left(X_i^n\right)-\mathsf{h}\left(\sum_{i=1}^{M}X_i^n+Z^n\right) \leq n\sum_{i=1}^{M}\mu_i\mathsf{h}\left(X_{iG}\right)-n\mathsf{h}\left(\sum_{i=1}^{M}X_{iG}+Z\right)
$$

where $X_{iG}\sim\mathcal{CN}(0,P_i)$.

Proof. We will prove the lemma for

$$
\mu_i = \frac{P_i}{\sum_{j=1}^{M}P_j+\Sigma_Z}.
$$

The result with

$$\mu_i > \frac{P_i}{\sum_{j=1}^{M} P_j + \Sigma_Z}$$

follows because the additional positive entropy quantities are easily seen to be maximized by i.i.d. Gaussian random vectors. Let t_i denote the average differential entropy

$$t_i = \frac{\mathsf{h}\left(X_i^n\right)}{n}.$$

Applying the EPI (Lemma 18) repeatedly, we obtain that

$$\mathsf{h}\left(\sum_{i=1}^{M} X_i^n + Z^n\right) \geq n \log\left(\sum_{i=1}^{M} e^{t_i} + \pi e \Sigma_Z\right).$$

Therefore, we have

$$\sum_{i=1}^{M} \mu_i \mathsf{h}\left(X_i^n\right) - \mathsf{h}\left(\sum_{i=1}^{M} X_i^n + Z^n\right) \leq n \sum_{i=1}^{M} \mu_i t_i - n \log\left(\sum_{i=1}^{M} e^{t_i} + \pi e \Sigma_Z\right).$$

Let $f(\mathbf{t}) = \sum_{i=1}^{M} \mu_i t_i - \frac{1}{2} \log\left(\sum_{i=1}^{M} e^{t_i} + \Sigma_Z\right)$. The second term is called the *log-sum-exp* function [71], which is convex in $\mathbf{t}$. Therefore, it follows that f is concave in $\mathbf{t}$. Now, using

$$\frac{\partial f}{\partial t_i} = \mu_i - \frac{e^{t_i}}{\sum_{j=1}^{M} e^{t_j} + \pi e \Sigma_Z}$$

it can be easily checked that $\{t_j = \log\left(\pi e P_j\right)\}_{j=1}^{M}$ satisfy $\frac{\partial f}{\partial t_i} = 0$ for all i. Since $f(\mathbf{t})$ is concave in $\mathbf{t}$, we obtain that $f(\mathbf{t})$ achieves its maximum at $\{t_j = \log\left(\pi e P_j\right)\}_{j=1}^{M}$, and hence

$$\begin{aligned} f(\mathbf{t}) &\leq \sum_{i=1}^{M} \mu_i \log\left(\pi e P_i\right) - \log\left(\pi e \sum_{i=1}^{M} P_i + \pi e \Sigma_Z\right) \\ &= \sum_{i=1}^{M} \mu_i \mathsf{h}\left(X_{iG}\right) - n \mathsf{h}\left(\sum_{i=1}^{M} X_{iG} + Z\right). \end{aligned}$$

□

APPENDIX B

ALGEBRAIC GEOMETRY

In this appendix, we present some results in algebraic geometry that are essential in proving the main results in Chapter 5. We start by recalling some basic terminology in algebraic geometry. We refer the reader to the book [72] for an excellent introduction.

B.1 Varities and Ideals

Let $\mathbb{C}[t_1, t_2, \cdots, t_n]$ and $\mathbb{C}(t_1, t_2, \cdots, t_n)$ denote the set of multivariate polynomials and rational functions, respectively, in the variables $t_1, t_2, \cdots, t_n$. For any polynomials $f_1, f_2, \cdots, f_m \in \mathbb{C}[t_1, t_2, \cdots, t_n]$, the *affine variety* generated by $f_1, f_2, \cdots, f_m$ is defined as set of points at which the polynomials vanish:

$$V(\mathbf{f}) = \{\mathbf{t} \in \mathbb{C}^n : \mathbf{f}(\mathbf{t}) = \mathbf{0}\}. \tag{B.1}$$

Any subset $I \subseteq \mathbb{C}[t_1, t_2, \cdots, t_n]$ is called an *ideal* if it satisfies the three properties

- $0 \in I$.
- If $f_1, f_2 \in I$, then $f_1 + f_2 \in I$.
- If $f_1 \in I$ and $f_2 \in \mathbb{C}[t_1, t_2, \cdots, t_n]$, then $f_1 f_2 \in I$.

For any set $\mathcal{A} \subseteq \mathbb{C}^n$, the ideal generated by $\mathcal{A}$ is defined as

$$I(\mathcal{A}) = \{f \in \mathbb{C}[t_1, t_2, \cdots, t_n] : f(\mathbf{t}) = 0 \ \forall \mathbf{t} \in \mathcal{A}\}. \tag{B.2}$$

For any ideal I, the affine variety generated by I is defined as

$$V(I) = \{\mathbf{t} \in \mathbb{C}^n : f(\mathbf{t}) = 0 \ \forall f \in I\}. \tag{B.3}$$

The *Zariski topology* on the affine space $\mathbb{C}^n$ is obtained by taking the affine varieties as closed sets. For any set $\mathcal{A} \in \mathbb{C}^n$, the Zariski closure $\bar{\mathcal{A}}$ is defined as

$$\bar{\mathcal{A}} = V(I(\mathcal{A})). \tag{B.4}$$

A set $\mathcal{A} \subseteq \mathbb{C}^n$ is said to be *constructible* if it is a finite union of locally closed sets of the form $U \cap Z$ with U closed and Z open. If $\mathcal{A} \subseteq \mathbb{C}^n$ is constructible and $\bar{\mathcal{A}} = \mathbb{C}^n$, then $\mathcal{A}$ must be dense in $\mathbb{C}^n$, i.e., $\mathcal{A}^c \subseteq W$ for some non-trivial variety $W \subsetneq \mathbb{C}^n$.

B.2 Algebraic Independence and Jacobian Criterion

The rational functions $f_1, f_2 \cdots, f_m \in \mathbb{C}(t_1, t_2, \cdots, t_n)$ are called algebraically dependent (over $\mathbb{C}$) if there exists a nonzero polynomial $F \in \mathbb{C}[s_1, s_2 \cdots, s_m]$ such that $F(f_1, f_2, \cdots, f_m) = 0$. If there exists no such annihilating polynomial F, then $f_1, f_2, \cdots, f_m$ are algebraically independent.

Lemma 20 (Theorem 3 on page 135 of [73])**.** *The rational functions* $f_1, f_2 \cdots, f_m \in \mathbb{C}(t_1, t_2, \cdots, t_n)$ *are algebraically independent if and only if the Jacobian matrix*

$$\mathbf{J}_f = \left(\frac{\partial f_i}{\partial t_j}\right)_{1 \le i \le m, 1 \le j \le n} \tag{B.5}$$

has full row rank equal to m*.*

The Jacobian matrix is a function of the variables $t_1, t_2, \cdots, t_n$, and hence the Jacobian matrix can have different ranks at different points $\mathbf{t} \in \mathbb{C}^n$. The above lemma refers to the *structural rank* of the Jacobian matrix which is equal to m if and only if there exists at least one realization $\mathbf{t} \in \mathbb{C}^n$ where the Jacobian matrix has full row rank.

B.3 Dominant Maps and Generic Properties

A polynomial map $\mathbf{f} : \mathbb{C}^n \to \mathbb{C}^m$ is said to be *dominant* if the Zariski closure of the image $\mathbf{f}(\mathbb{C}^n)$ is equal to $\mathbb{C}^m$. The image of a polynomial map is constructible. Therefore, the image of a dominant polynomial map is dense, i.e., the complement of $\mathbf{f}(\mathbb{C}^n)$ is contained in a non-trivial variety $W \subsetneq \mathbb{C}^m$. The implication of this is that the system of polynomial equations

$$\begin{aligned} s_1 &= f_1(t_1, t_2, \cdots, t_n) \\ s_2 &= f_2(t_1, t_2, \cdots, t_n) \\ &\vdots \\ s_m &= f_m(t_1, t_2, \cdots, t_n) \end{aligned} \tag{B.6}$$

has a solution $\mathbf{t} \in \mathbb{C}^n$ for generic $\mathbf{s}$, where the notion of a generic property is defined below.

Definition 4. *A property is said to true for generic* $\mathbf{s} \in \mathbb{C}^m$ *if the property holds true for all* $\mathbf{s} \in \mathbb{C}^m$ *except on a non-trivial affine variety* $W \subsetneq \mathbb{C}^m$*. Such* *a property is said be a generic property.*

For example, a generic square matrix $\mathbf{A}$ has full rank because $\mathbf{A}$ is rank deficient only when it lies on the affine variety generated by the polynomial $f(\mathbf{A}) = \det \mathbf{A}$. If the variables are generated randomly according to a continuous joint distribution, then any generic property holds true with probability 1.

Observe that the Zariski closure of the image $\mathbf{f}(\mathbb{C}^n)$ is equal to $\mathbb{C}^m$ if and only if the ideal I generated by the image set is equal to $\{0\}$. Since I is equal to the set of annihilating polynomials

$$\begin{aligned} I &= \{F \in \mathbb{C}[s_1, s_2, \cdots, s_m] : F(\mathbf{s}) = 0 \ \forall \mathbf{s} \in \mathbf{f}(\mathbb{C}^n)\} \\ &= \{F \in \mathbb{C}[s_1, s_2, \cdots, s_m] : F(f_1, f_2, \cdots, f_m) = 0\}, \end{aligned} \tag{B.7}$$

the map $\mathbf{f}$ is dominant if and only if the polynomials $f_1, f_2, \cdots, f_m$ are algebraically independent. Thus we obtain the following lemma.

Lemma 21. *The system of polynomial equations* (B.6) *admits a solution for a generic* $\mathbf{s} \in \mathbb{C}^m$ *if and only if the polynomials* $f_1, f_2, \cdots, f_m$ *are algebraically independent, i.e., if and only if the Jacobian matrix* (B.5) *has full row rank.*

B.4 A Lemma on Full-Rankness of Certain Random Matrix

Let $\mathbf{t} \in \mathbb{C}^n$ be a set of original variables, and let $\mathbf{s} \in \mathbb{C}^m$ be a set of derived variables obtained through polynomial transformation $\mathbf{s} = \mathbf{f}(\mathbf{t})$ for some rational map $\mathbf{f}$. Suppose we generate q instances of $\mathbf{t}$

$$\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q) \tag{B.8}$$

and the corresponding q instances of $\mathbf{s}$

$$\mathbf{s}(1), \mathbf{s}(2), \cdots, \mathbf{s}(q)$$

and generate the $q \times p$ matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{s}(1)^{\mathbf{a}_1} & \mathbf{s}(1)^{\mathbf{a}_2} & \cdots & \mathbf{s}(1)^{\mathbf{a}_p} \\ \mathbf{s}(2)^{\mathbf{a}_1} & \mathbf{s}(2)^{\mathbf{a}_2} & \cdots & \mathbf{s}(2)^{\mathbf{a}_p} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}(q)^{\mathbf{a}_1} & \mathbf{s}(q)^{\mathbf{a}_2} & \cdots & \mathbf{s}(q)^{\mathbf{a}_p} \end{bmatrix} \tag{B.9}$$

for some exponent vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p \in \mathbb{Z}_+^m$ and $q \geq p$. We are interested in determining the set of variables (B.8) such that the matrix $\mathbf{M}$ has full column rank. If there exists an annihilating polynomial $F \in \mathbb{C}[s_1, s_2, \cdots, s_m]$ of the form

$$F(\mathbf{s}) = \sum_{i=1}^{p} c_i \mathbf{s}^{\mathbf{a}_i} \tag{B.10}$$

such that $F(f_1, f_2, \cdots, f_m) = 0$, then the matrix $\mathbf{M}$ satisfies $\mathbf{Mc} = \mathbf{0}$, and hence the matrix $\mathbf{M}$ does not have full column rank for any realizations of the variables (B.8). Interestingly, even the converse holds true.

Lemma 22. *The matrix* $\mathbf{M}$ *has full column rank for generic realizations of the variables* (B.8) *if and only if there does not exist an annihilating polynomial* F *of the form* (B.10) *satisfying* $F(f_1, f_2, \cdots, f_m) = 0$.

Proof. We have already proved that $\mathbf{M}$ does not have full column rank if there exists an annihilating polynomial F of the form (B.10). We now prove the converse; i.e., we assume that there does not exist an annihilating polynomial of the form (B.10), and prove that the matrix $\mathbf{M}$ has full column rank for generic realizations of the variables (B.8). Without any loss of generality, we assume that $p = q$. Otherwise, we can work with the $q \times q$ submatrix obtained after deleting the last $q - p$ rows.

Consider expanding the determinant $\det \mathbf{M}$ in terms of the variables (B.8). Since the variables $\mathbf{s}(1), \mathbf{s}(2), \cdots, \mathbf{s}(q)$ are rational functions of $\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q)$ respectively, the determinant is also a rational function; i.e.,

$$\det \mathbf{M} = \frac{d_1(\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q))}{d_2(\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q))}. \tag{B.11}$$

The determinant can either be identically equal to zero, or a nonzero function. If the determinant is a nonzero function, then $\mathbf{M}$ has full column rank for generic realizations of the variables (B.8) because $\mathbf{M}$ is rank deficient only when $d_1(\mathbf{t}(1), \mathbf{t}2), \cdots, \mathbf{t}(q)) = 0$ or when $(\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q))$ belongs to the affine variety $V(d_1) \subsetneq \mathbb{C}^{nq}$ generated by the polynomial d_1.

Therefore, it remains to prove that $\det M$ is not identically equal to zero under the assumption no annihilating polynomial F of the form (B.10) exists. We prove this claim by induction on q. The claim is trivial to check for $q = 1$. We now prove the induction step. We may assume that the determinant of the $(q-1) \times (q-1)$ submatrix $\tilde{\mathbf{M}}$, obtained after deleting the last row and column, is a nonzero function in $(\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q-1))$. Therefore, there must exist specific realizations

$$(\mathbf{t}(1), \mathbf{t}(2), \cdots, \mathbf{t}(q-1)) = (\mathbf{a}(1), \mathbf{a}(2), \cdots, \mathbf{a}(q-1)) \tag{B.12}$$

such that $\tilde{\mathbf{M}}$ has full rank. Consider the matrix $\mathbf{M}^*(\mathbf{t})$ obtained from $\tilde{\mathbf{M}}$ by setting $\mathbf{t}(q) = \mathbf{t}$ for each $\mathbf{t} \in \mathbb{C}^n$. If $\det \mathbf{M}$ is identically equal to zero, then the matrix $\mathbf{M}^*(\mathbf{t})$ must be rank deficient for all $\mathbf{t}$; i.e., there must exist $\mathbf{c}(\mathbf{t}) \neq \mathbf{0}$ such that $\mathbf{M}^*(\mathbf{t})\mathbf{c}(\mathbf{t}) = \mathbf{0}$ for each $\mathbf{t} \in \mathbb{C}^n$. Since the first $q-1$ rows are linearly independent and do not depend on $\mathbf{t}$, the vector $\mathbf{c}(\mathbf{t}) = \mathbf{c}^*$ is unique (up to a scaling factor) and is determined by (B.12). Therefore, we have that $\mathbf{M}^*(\mathbf{t})\mathbf{c}^* = \mathbf{0}$ for each $\mathbf{t} \in \mathbb{C}^n$. By expanding the last row of $\mathbf{M}^*(\mathbf{t})\mathbf{c}^* = \mathbf{0}$, we obtain

$$\sum_{i=1}^{q} c_i^* \mathbf{f}(\mathbf{t})^{\mathbf{a}_i} = 0. \tag{B.13}$$

This is a contradiction since we assumed that no annihilating polynomial of the form (B.10) exists. Therefore, $\det \mathbf{M}$ is not identically equal to zero and hence $\mathbf{M}$ has full rank for generic realizations of the variables (B.8). □

If the rational functions $f_1, f_2, \cdots, f_m$ are algebraically independent, then there cannot exist an annihilating polynomial F (of any form) satisfying

$$F(f_1, f_2, \cdots, f_m) = 0.$$

Thus, we immediately have the following corollary.

Corollary 4. *The matrix* $\mathbf{M}$ *has full column rank for generic realizations of the variables* (B.8) *if the rational functions* $f_1, f_2, \cdots, f_m$ *are algebraically independent, i.e., if the Jacobian matrix* (B.5) *has full row rank.*

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