

© 2011 Daniel Morton

GKM MANIFOLDS WITH LOW BETTI NUMBERS

BY

DANIEL MORTON

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

Doctoral Committee:

Professor Eugene Lerman, Chair
Professor Ely Kerman
Professor Susan Tolman
Professor Hal Schenck

ABSTRACT

A GKM manifold is a symplectic manifold with a torus action such that the fixed points are isolated and the isotropy weights at the fixed points are linearly independent. Each GKM manifold has a GKM graph which contains much of the topological information of the manifold, in particular the equivariant cohomology and Chern classes. We are interested in the case where the torus action is Hamiltonian. In this thesis we will consider the case where the GKM graphs are complete. When the dimension of the torus action is sufficiently large, we can completely classify the complete, in the graph theoretic sense, GKM graphs, and thus completely describe the cohomology rings and Chern classes of the associated "minimal" GKM manifolds. For each possible cohomology ring and total Chern class we can find a well-known GKM manifold having that ring and class. If we put some restrictions on the allowable subgraph, and thus restrict the allowable submanifolds, then we can completely classify the possible cohomology rings and Chern classes of minimal GKM manifolds.

We will also consider one of the cases where the GKM graph is not complete. In the case of six dimensional symplectic manifolds whose GKM structure comes from a Hamiltonian 2-torus action we can also completely classify all the possible GKM graphs, and thus all the possible cohomology rings and Chern classes. Once again, for each possible cohomology ring and total Chern class, we can construct a manifold having that ring and class.

TABLE OF CONTENTS

LIST OF FIGURES	iv
CHAPTER 1 INTRODUCTION	1
CHAPTER 2 BACKGROUND	12
CHAPTER 3 MINIMAL GKM MANIFOLDS AND GRAPHS	21
3.1 Introduction	21
3.2 Dimension at Most 8	22
3.3 The Higher Dimensional Case	34
3.4 Other Results	38
3.5 Beyond Petrie	45
CHAPTER 4 CLASSIFICATION OF COMPLEXITY 1 GKM MANIFOLDS AND GRAPHS.	47
4.1 Classification of Graphs	47
4.2 Construction of GKM manifolds.	57
4.3 Cohomology	63
REFERENCES	70

LIST OF FIGURES

3.1	The two types of (3,2) graphs.	24
3.2	The three types of (4,2) graphs.	26
3.3	The two possible non-simplicial highly symmetric GKM graphs with six fixed points.	42
4.1	Construction the Twisted GKM manifolds.	58
4.2	Construction of the Flag-like GKM manifolds.	61

CHAPTER 1

INTRODUCTION

Symplectic Geometry originated with William Rowan Hamilton's study of classical mechanics. At its simplest, a symplectic form relates the trajectory of a particle (in phase space) to its energy function in much the same way that a Riemannian metric relates a function and its gradient flow. Unlike Riemannian metrics, however, symplectic forms do not exist on all manifolds.¹ Necessary (but certainly not sufficient) conditions for a manifold to have a symplectic form are that it be even dimensional, orientable, have an almost complex structure and, if compact, have nonzero even-dimensional cohomology groups. (i.e. $H^{2k}(M) \neq 0$ for $0 \leq k \leq \frac{1}{2} \dim M$.)

A **symplectic form** ω is a closed nondegenerate 2 form on a manifold. A **symplectic manifold** is a pair (M, ω) with $\omega \in \Omega^2(M)$ symplectic.

If $H : M \rightarrow \mathbb{R}$ is some (smooth) function then the **Hamiltonian vector field** of H is the unique vector field $X_H \in \Gamma(TM)$ such that $\iota_{X_H}\omega = -dH$. In mechanics, integrating this vector field gives us the trajectory of a particle whose energy function is H . Unsurprisingly, the value of H along a trajectory is constant.

An **integrable system** is a generalization of a Hamiltonian vector field. In fact, it is a family of Hamiltonians that “commute.” More specifically, if f_1, \dots, f_k are linearly independent functions (i.e. $df_1 \wedge \dots \wedge df_n \neq 0$) whose Poisson brackets are zero (i.e. $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j}) = 0$ for all i and j) then they and their vector fields X_{f_1}, \dots, X_{f_k} form an integrable system. The function

$$\mu = (f_1, \dots, f_k) : M \rightarrow \mathbb{R}^k$$

is a generalization of the Hamiltonian vector field of a single vector field and is called a **moment map**. Of particular interest is the case where $k = \frac{1}{2} \dim M$ (we must always have $k \leq \frac{1}{2} \dim M$.) In this case, which we call a **completely integrable system**, the compact

¹For any manifold M , the cotangent bundle T^*M is naturally symplectic, so symplectic forms are in some sense universal.

connected components of preimages of regular points are Lagrangian torii. The Arnold-Liouville theorem [Ar78] completely describes the manifold M in a neighborhood of one of these preimages. One consequence of this is that there is an action of an n -dimensional torus on this neighborhood that preserves the symplectic form.

By this rather roundabout route we are inaugurated to the study of torus actions on symplectic manifolds. In particular, we are interested in **Hamiltonian** torus actions. If (M, ω) is a compact $2n$ -dimensional symplectic manifold with an effective $\mathbb{T}^k \approx (S^1)^k$ action preserving the symplectic form, then we say the action is **Hamiltonian** if there is a function² $\mu : M \rightarrow (\mathbb{R}^k)^*$. such that $\iota_{X^\#}\omega = -d\langle\mu, X\rangle$ for all $X \in \mathbb{R}^k$ with $X^\#$, the vector field on M generated by $\{\exp(tX)\}_{t \in \mathbb{R}}$. The image of the moment map is invariant under the group action. The torus actions in the Arnold-Liouville theorem are Hamiltonian.

An early result in the study of symplectic manifolds with Hamiltonian torus actions was proved independently by Atiyah [At82] and Guillemin and Sternberg [GS82]. They show that the image of the moment map of a compact Hamiltonian symplectic manifold is a polytope; moreover, that it is the convex hull of the images of the connected components of the fixed point set.

It is generally impossible to reconstruct the manifold from the moment image. For example the moment image tells us next to nothing about the dimension of the manifold. If we know the dimension of the manifold in advance though, there is a case where we can go backwards. This is a consequence of Delzant's Theorem. In the case where the torus action is completely integrable (known as **symplectic toric manifolds**) the moment image of the manifold is an n dimensional simple polytope with edges of the form $a + tu$ where $u \in \mathbb{Z}^n$ and where, at each vertex, the n directional vectors u_1, \dots, u_n of the n edges adjacent to the vertex can be chosen to be a lattice basis for \mathbb{Z}^n . Delzant [D88] showed that each such polytope is the moment image of a compact symplectic manifold with Hamiltonian \mathbb{T}^n action, unique up to equivariant symplectomorphism.

Although it requires a bit more data than just the moment map, Karshon [K99] has proved a comparable classification theorem for four-dimensional manifolds with Hamiltonian circle actions. See also [KT01, KT03, T10].

Delzant's Theorem also shows that all the symplectic toric manifolds admit a Kähler structure, and are, in fact, the smooth toric varieties. These have been extensively studied in the algebraic category, and are closely tied to combinatorics. This shows that there is a close relationship between the study of completely integrable Hamiltonian manifolds and the

²For technical reasons the map is to the dual space.

algebraic and combinatorial study of polytopes. See also [Au03, CLS11, F93, M93, M96].

Classification becomes harder in the case of $2n$ -dimensional symplectic manifolds with Hamiltonian \mathbb{T}^k -actions where $k < n$. Lie group actions on a manifold can be seen as symmetries of the manifold, and the lower the dimension of the group, the less symmetrical the manifold needs to be. We can make some progress if we restrict to manifolds that are “like” toric manifolds.

If we further require that the fixed point set $M^{\mathbb{T}}$ be finite and that, for every subgroup $K \subset \mathbb{T}^k$ of codimension 1, $\dim M^K \leq 2$ (conditions that are satisfied automatically in the toric case) then we may use the techniques of Goresky, Kottwitz, and MacPherson [GKM98] to construct graphs that allow us to study these **Hamiltonian GKM manifolds** in some of the same ways that we use polytopes to study symplectic toric manifolds.

Definition 1 (Hamiltonian GKM manifold) *A Hamiltonian GKM manifold is a symplectic manifold with a Hamiltonian torus action such that the fixed point set is finite and for every subgroup $K \subset \mathbb{T}^k$ of codimension 1, $\dim M^K \leq 2$.*

A long standing problem in the study of G -manifolds, and spaces with group actions in general, is the computation of the equivariant cohomology ring $H_G^*(M) := H^*(EG \times_G M)$. In general, this is a hard problem, but in special cases progress can be made. Goresky, Kottwitz, and MacPherson [GKM98] developed a method of computing the equivariant cohomology of G -spaces provided certain conditions are met. The common conditions that generally make some form of GKM type analysis possible are equivariant formality and a condition on the orbit space, which, in the case of \mathbb{T}^k actions, reduces to isolated fixed points and isotropy submanifolds of codimension 1 subgroups having dimension at most 2 (namely, copies of S^2 .) A G -manifold is equivariantly formal if the bundle $EG \times_G M \rightarrow BG$ satisfies the Leray-Hirsch Theorem, that is, if $H_G^*(M) \rightarrow H^*(M)$ is a surjection. That symplectic manifolds with Hamiltonian torus actions are equivariantly formal was proved independently by Kirwan [K84] and Ginzburg [Gi87].

Since the publication of [GKM98] several notions of “GKM manifold” have appeared. Sometimes the manifold is assumed to be complex, with a complex torus action [BGT]. In other cases, only an almost complex structure is required [GZ01]. One paper [GHZ06] looks at a class of homogeneous spaces and shows they are amenable to GKM techniques. Others work in the more general category of spaces that are equivariantly formal [B]. In [GKM98] itself, Goresky, Kottwitz, and MacPherson list no fewer than fourteen categories of spaces (not necessarily manifolds) on which their techniques can be applied. We will confine

ourselves to the study of Hamiltonian GKM manifolds. (See Definition 1) Henceforth, we will drop the term “Hamiltonian” and refer to these manifolds as GKM manifolds.

In order to compute the equivariant cohomology of a GKM manifold we construct a labeled graph from the manifold data. The vertices of the graph are the fixed points of the action. Since the group action is Hamiltonian, the positive dimensional components of M^K (where $K \subset \mathbb{T}^k$ has codimension 1) are symplectic and the moment map restricted to M^K is still a moment map, now for the circle action on M^K . Thus the positive dimensional components of M^K must be Hamiltonian S^1 -manifolds, and thus must be copies of S^2 . Thus they each contain two points in $M^{\mathbb{T}}$. We connect two vertices by an edge if they lie in the same connected component of M^K for some $K \subset \mathbb{T}^k$. We also need a notion of a weight on each (oriented) edge. We are considering undirected graphs (each edge corresponds to an isotropy sphere) but it is necessary to consider a notion of direction on the edges. The formal definition follows:

Definition 2 (The GKM Graph of a GKM manifold) *Let M be a GKM manifold. If $V = M^{\mathbb{T}}$ is the set of vertices, then the set of edges E is the subset of the set of unordered pairs of elements of V such that $\{p, q\} \in E$ if p and q lie in the same connected component of M^K for some codimension 1 $K \subset \mathbb{T}$. We also define a set $I \subset V \times V$ of edges with all possible choices of orientation by saying $(p, q) \in I$ and $(q, p) \in I$ if and only if $\{p, q\} \in E$. Then we will say that (p, q) is the edge $\{p, q\}$ oriented to start at p and finish at q . At a fixed point $p \in M^{\mathbb{T}}$ the group action induces an action on $T_p M$. We define $\alpha : I \rightarrow (\mathbb{R}^k)^*$ by letting $\alpha(p, q)$ be the isotropy weight of $T_p S_{pq} \subset T_p M$ where S_{pq} is the isotropy sphere corresponding to the edge $\{p, q\}$. This is well-defined since all symplectic manifolds have almost complex structures and the space of almost complex structures on any given manifold is contractible. It is clear that $\alpha(p, q) = -\alpha(q, p)$. The graph with the function α is a **GKM graph of a GKM manifold**.*

A $2n$ -dimensional GKM manifold acted upon by \mathbb{T}^k with χ fixed points will be called an (n, k, χ) GKM manifold. An n -valent GKM graph with χ vertices and weights in $(\mathbb{Z}^k)^*$ will be called an (n, k, χ) GKM manifold. It is clear that an (n, k, χ) GKM manifold has an (n, k, χ) GKM graph.

If we look at GKM manifold acted upon by a subtorus we often get a GKM manifold with weights lying in a lower dimensional sublattice of the original lattice.

Definition 3 *Let M be a symplectic manifold with a Hamiltonian \mathbb{T}^k action that makes M a GKM manifold. Given a lower dimensional sub-torus $\mathbb{T}^l \subset \mathbb{T}^k$, if the \mathbb{T}^l action is effective*

then M with the \mathbb{T}^l action is also a GKM manifold. The GKM graph of the new GKM manifold has the same set of vertices and edges as the GKM graph of the GKM manifold with the \mathbb{T}^k action. The inclusion $\iota : \mathbb{T}^l \rightarrow \mathbb{T}^k$ induces a projection $\iota^* : (\mathbb{Z}^k)^* \rightarrow (\mathbb{Z}^l)^*$. The weights of the new GKM graph are the projections under this map of the weights of the old GKM graph. Thus we call the new (n, l, χ) GKM graph a **projection** of the (n, k, χ) GKM graph.

As with GKM manifolds, there are multiple ways to define GKM graphs. Some authors prefer to avoid the subtlety of dealing with two weights on each edge, and prefer to consider GKM graphs as directed graphs [Sa09, ST10]. Others [GKM98, GHZ06] define GKM graphs in more general contexts than Hamiltonian symplectic manifolds. There is also a way to define abstract GKM graphs without any notion of GKM manifold [GZ02, GZ99, GZ01]. We will introduce another definition of abstract GKM graph specifically tailored for studying the GKM graphs of GKM manifolds.

A natural place to start in the classification of GKM manifolds is to consider those with the minimum possible Betti numbers. These are manifolds where $\dim H^i(M) = 1$ for $0 \leq i \leq 2n$ even (and $\dim H^i(M) = 0$ for i odd.) Equivalently, the manifolds will have the same rational cohomology as $\mathbb{C}\mathbb{P}^n$. This corresponds to having one more fixed point than half the dimension. We refer to these manifolds as **minimal** GKM manifolds. The GKM graphs of minimal GKM manifolds are complete graphs, a subclass of GKM graphs that is reasonably tractable.

The study of manifolds with torus actions and the same cohomology as $\mathbb{C}\mathbb{P}^n$ naturally invites comparison with the work of Ted Petrie. Petrie [P72] conjectured that any manifold homotopy equivalent to $\mathbb{C}\mathbb{P}^n$ that has a nontrivial circle action has the same Pontryagin classes as $\mathbb{C}\mathbb{P}^n$. This conjecture has been proven when the manifold has dimension at most 8 [DE76, J85, Y77], and when the number of fixed point components is at most 4 [MA81, TW79, W75].

The conjecture is also known to be true in some cases where the S^1 action can be extended to a larger torus action. Petrie, [P73], proved the conjecture when \mathbb{T}^n acts smoothly and effectively on a homotopy $\mathbb{C}\mathbb{P}^n$. Wang, [W80], extended this to effective \mathbb{T}^k actions whose fixed point set has exactly $k + 1$ components. Masuda then showed that the conjecture holds if an effective \mathbb{T}^k ($k \geq 2$) action preserves an almost complex structure or if the action is smooth and all fixed point components have positive dimension [MA83]. More recently, Dessai has proven Petrie's Conjecture for effective \mathbb{T}^k actions with $k > \frac{n+1}{4}$. [D04].

Wang has also considered effective topological \mathbb{T}^n actions on spaces that are rationally

cohomologous to $\mathbb{C}\mathbb{P}^n$ (i.e. $H^*(M, \mathbb{Q}) = \mathbb{Q}[\alpha]/(\alpha^{n+1})$) and shows that Petrie's Conjecture holds in this case [W77].

In [T10] Tolman considers S^1 actions on spaces rationally cohomologous to $\mathbb{C}\mathbb{P}^n$, but restricts to the symplectic category. In dimension 6 these hypotheses are sufficient to get all the possible integral cohomology rings and Chern classes. We prove a similar result for minimal GKM manifolds acted upon by torii of sufficiently large dimension.

We can now state our main theorems.

Theorem 1 *Let M be a $2n$ dimensional minimal GKM manifold acted on by a k -dimensional torus. If $k > \frac{n+1}{2}$ or $k = \frac{n}{2}$ then the GKM graph of the manifold is the GKM graph of $\mathbb{C}\mathbb{P}^n$ acted upon by a subtorus of \mathbb{T}^n . The manifold has the same integral cohomology ring and Chern classes as $\mathbb{C}\mathbb{P}^n$. If $k = \frac{n+1}{2}$ then the GKM graph of the manifold is either the GKM graph of $\mathbb{C}\mathbb{P}^n$ acted upon by a subtorus of \mathbb{T}^n or the GKM graph of $G_+(2, n+2)$, the Grassmannian of oriented 2-planes in \mathbb{R}^{n+2} , acted upon by \mathbb{T}^k . The manifold has the same integral cohomology ring and Chern classes as either $\mathbb{C}\mathbb{P}^n$ or $G_+(2, n+2)$.*

Proof. This theorem is an immediate consequence of Propositions 4 and 5 in Section 3.3 of Chapter 3.

This theorem does not hold in the case where $k = \frac{n-1}{2}$. The known counter example is G_2/P , where G_2 is the complexification of the exceptional simple Lie group (also denoted G_2) and P is the parabolic subgroup generated by the Borel subgroup and the exponential of the short simple root, which is 10-dimensional and acted upon by \mathbb{T}^2 . We will see that if we put more conditions on the manifolds we will get stronger results.

The conditions necessary for GKM-type analysis are conditions on the type of submanifold that can be fixed by codimension 1 subgroups of \mathbb{T}^k . The proof of Theorem 1 consists of considering the possible subgraphs corresponding to submanifolds fixed by some larger subtorus. It is natural then to consider what can happen if we control what these manifolds can be. In particular, consider the case where we restrict what can happen to submanifolds fixed by codimension 2 subgroups.

Definition 4 *A flag-face of a polytope is an $(n+1)$ -tuple of face consisting of the polytope, a hyperface of the polytope, a hyperface of the hyperface, etc. A **lattice regular polytope** is polytope whose vertices lie on the points of a lattice and, given any two flag-faces, there exists a lattice-affine (preserving the lattice) transformation mapping one flag-face onto the other.*

It should be noted that the moment images of $\mathbb{C}\mathbb{P}^n$ acted upon by \mathbb{T}^n , $G_+(2, 2n - 1)$ acted upon by \mathbb{T}^n , and G_2/P acted upon by \mathbb{T}^2 are all lattice regular polytopes with all the fixed points mapped to vertices (i.e. not the interior) of the polytope. We show that all minimal GKM manifolds whose moment image is a lattice regular polytope and whose fixed points map to the vertices must have the cohomology ring and Chern classes of one of the aforementioned manifolds. Theorem 2 and Theorem 3 suggest that we have all the possible cohomology rings and Chern classes of minimal GKM manifolds by showing that any new GKM graph of a minimal GKM manifold must have radically different GKM subgraphs and must have a certain amount of asymmetry. All known GKM graphs of minimal GKM manifolds are projections of a lattice regular graph.

Theorem 2 *If the isotropy submanifolds of $k - 2$ dimensional subgroups of dimension ≥ 4 have GKM graphs that are GKM graphs of complex projective space or oriented Grassmannians acted upon by some copy of \mathbb{T}^2 then the GKM graph of the manifold is the GKM graph of $\mathbb{C}\mathbb{P}^n$ acted upon by \mathbb{T}^k for some $k \leq n$ or the GKM graph of $G_+(2, 2n + 1)$ acted upon by \mathbb{T}^k for some $k \leq n$. The manifold must then have the same integral cohomology ring and Chern classes as $\mathbb{C}\mathbb{P}^n$ or $G_+(2, 2n + 1)$.*

Proof. This theorem is an immediate consequence of Proposition 6 in Section 3.4 of Chapter 3.

Theorem 3 *If the image of the moment map of a minimal GKM manifold is a lattice-regular polytope and all the fixed points map to vertices of the polytope then the GKM graph of the manifold is the GKM graph of $\mathbb{C}\mathbb{P}^n$ acted upon by \mathbb{T}^k for some $k \leq n$, the GKM graph of $G_+(2, 2n + 1)$ acted upon by \mathbb{T}^k for some $k \leq n$, or the graph of the manifold G_2/P . The manifold must then have the integral cohomology ring and Chern classes of $\mathbb{C}\mathbb{P}^n$, $G_+(2, 2n + 1)$, or G_2/P , where P is the parabolic subgroup generated by the Borel subgroup and the exponential of the short simple root.*

Proof. This theorem is an immediate consequence of Proposition 7 in Section 3.4 of Chapter 3.

The same techniques used to prove Theorem 1 can also be used to prove a similar result in the case where the number of fixed points is two more than half the dimension of the manifold.

Theorem 4 *Let M be a $2n$ ($n \geq 2$) dimensional GKM manifold acted upon by a k -dimensional torus with $n + 2$ fixed points. Then $k \leq \frac{n}{2} + 1$ and if $k = \frac{n}{2} + 1$ then the*

GKM graph of the manifold is the same as the GKM graph of $G_+(2, n + 2)$. Thus the manifold has the same integral cohomology and Chern classes as $G_+(2, n + 2)$.

Proof. This theorem is an immediate consequence of Proposition 8 in Section 3.5 of Chapter 3.

This result no longer holds in the case where $k = \frac{n}{2}$. When $n = 6$ and $k = 3$ we know of three GKM manifolds with eight fixed points. The first is $G_+(2, 6)$ acted upon by a three-dimensional subtorus of the usual \mathbb{T}^4 action. The other two are the quotient of $SO(7)$ by the parabolic subgroup generated by the Borel subgroup and the exponential of the Lie algebra generated by all but the shortest negative simple weight and the quotient of $Sp(3)$ by the parabolic subgroup generated by the Borel subgroup and the exponential of the Lie algebra generated by all but the longest negative simple weight. The $SO(7)$ quotient has graph the one skeleton of a cube with simplicial faces, and the $Sp(3)$ quotient has graph the one skeleton of a cube with complete octaplexes as faces.

It should be noted that all the results have been stated in terms of the cohomology of the manifolds and say nothing about the uniqueness of the manifolds in question. We cannot say that $\mathbb{C}P^n$ and $G_+(2, 2n + 1)$ are the only GKM manifolds, up to equivariant symplectomorphism, with the cohomology rings described in Theorem 1.

Whether or not a GKM graph is the GKM graph of a unique (up to equivariant symplectomorphism) GKM manifold is in general not known. Indeed, the question of what invariants are needed to uniquely determine Hamiltonian symplectic manifolds is, in general, hard. The **complexity** of the GKM manifold, defined to be $\frac{1}{2} \dim(M) - \dim(\mathbb{T})$ determines how hard the problem is. The complexity 0 case is the only one that has been solved completely (see [D88],) as has the case of 4-manifolds with circle actions (see [K99].) This case is the easiest since the reduced spaces are always points. In the complexity 1 case progress is expected since the generic reduced spaces are symplectic surfaces, which are completely classified by their volume and genus. Karshon and Tolman have been able to prove uniqueness results in the case where the manifold is “centered” [KT01] and “tall” [KT03]. Complexity 2 Hamiltonian symplectic manifolds are harder to classify since the generic reduced spaces are symplectic 4-manifolds, the complete classification of which is intractable. Some special cases may be classifiable, however (see [Go11, McD09].) Complete classification for complexity 3 and higher is generally considered intractable.

In simple cases we can make some progress on the question of existence. In the case of six-dimensional GKM manifolds acted upon by \mathbb{T}^2 with six fixed points, we can find all the possible abstract GKM graphs, which allows us to compute all the (theoretically) possible

cohomology rings and Chern classes, and show that each of these graphs is the GKM graph of at least one GKM manifold. We state our result, followed by a discussion of the “right” notion of abstract GKM graphs for proving this and the previous theorems.

Theorem 5 *Let M be a six dimensional GKM manifold acted upon by a two dimensional torus with six fixed points.. Then one of the following statements is true:*

1. *M has the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ acted upon by a two dimensional subtorus of the Hamiltonian \mathbb{T}^3 action. Moreover*

$$H^*(M) = \mathbb{Z}[\alpha, \beta]/(\alpha^3, \beta^2 + n\alpha\beta) \text{ and}$$

$$c(M) = 1 + (3 + n)\alpha + 2\beta + 3(1 + n)\alpha^2 + 6\alpha\beta + 6\alpha^2\beta.$$

2. *M has the GKM graph of a $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ acted upon by a two dimensional subtorus of the Hamiltonian \mathbb{T}^3 action. This family has two parameters, n and m , with $0 \leq n \leq m$. Moreover*

$$H^*(M) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^3 - (m + n)\alpha\beta^2) \text{ and}$$

$$c(M) = 1 + (2 - m - n)\alpha + 3\beta + 2(3 - n - m)\alpha\beta + 3\beta^2 + 6\alpha\beta^2.$$

3. *If we let A, B, \dots, F be the vertices then there exist natural numbers $a, b > 0$ and $c > -a - b$ and an $SL(2, \mathbb{Z})$ transformation of the weights of the GKM graph of M such that $\alpha(A, B) = \alpha(F, C) = (1, 0)$, $\alpha(A, D) = \alpha(F, E) = (0, 1)$, $\alpha(B, E) = \alpha(C, D) = (-1, 1)$, $\alpha(A, B) = (-a, b)$, $\alpha(B, C) = (a + c, b)$, $\alpha(E, D) = (a, b + c)$. Moreover*

$$H^*(M) = \mathbb{Z}[\alpha, \beta]/(\alpha^3, \beta^2 - (2a + c)\alpha\beta + a(a + b + c)\alpha^2) \text{ and}$$

$$c(M) = 1 + (3 - 2a - c)\alpha + 2\beta + 3(1 - 2a - c)\alpha^2 + 6\alpha\beta + 6\alpha^2\beta.$$

4. *If we let A, B, \dots, F be the vertices then there exist natural numbers $a, b > 0$ and $c > a + b$ and an $SL(2, \mathbb{Z})$ transformation the weights of the GKM graph of M such that $\alpha(A, C) = \alpha(B, E) = (1, 0)$, $\alpha(A, F) = \alpha(B, D) = (0, 1)$, $\alpha(C, D) = \alpha(E, F) = (-1, 1)$, $\alpha(A, B) = (-a, b)$, $\alpha(C, E) = (a - c, b)$, $\alpha(D, F) = (a, b - c)$. Moreover*

$$H^*(M) = \mathbb{Z}[\alpha, \beta]/(\alpha^3, \beta^2 - (2a - c)\alpha\beta + a(a + b - c)\alpha^2) \text{ and}$$

$$c(M) = 1 + (3 - 2a + c)\alpha + 2\beta + 3(1 - 2a + c)\alpha^2 + 6\alpha\beta + 6\alpha^2\beta.$$

5. M has the GKM graph of the blow-up of $G_+(2, 5)$ with its standard \mathbb{T}^2 action along one of its large isotropy spheres. Each of the four fixed points of $G_+(2, 5)$ represent one of two planes in \mathbb{R}^5 with a choice of orientation. The large isotropy spheres connect the pairs of points that represent the same plane with opposite orientation. Moreover

$$H^*(M) = \mathbb{Z}[\alpha, \beta, \gamma, \delta]/(\alpha^2, \alpha\beta - 2\gamma, \beta^2 + \alpha\beta - 2\delta, \gamma\delta, \alpha\gamma, \beta(\gamma + \delta), \alpha\delta - \beta\gamma) \text{ and}$$

$$c(M) = 1 + 3\alpha + 2\beta + 2\beta^2 + 7\alpha\beta + 3\alpha\beta^2.$$

Proof. This theorem follows from Propositions 9, 10, 11, 12 in Section 4.1 of Chapter 4.

The proofs of these theorems rely on the study of the GKM graphs of these manifolds. In fact, the proofs require that we define an abstract GKM graph independent of GKM manifolds. We do not use the definition given by Guillemin and Zara in [GZ02, GZ99, GZ01]. Although very general, and useful for studying many properties of GKM graphs in the widest possible setting, it has the drawback that it allows graphs that cannot be the GKM graphs of any GKM manifold, so is not well suited to our purposes. We want abstract GKM graphs to satisfy as many of the properties of GKM graphs of GKM manifolds as possible. Our definition does not guarantee that each abstract GKM graph is the GKM graph of a GKM manifold, but we know of no case where this fails to hold.

We define an **abstract GKM graph** as follows:

Definition 5 (Abstract GKM Graph) *An (n, k, χ) abstract GKM graph is a regular n -valent graph with vertices V , edges E , directed edges I , and a function $\alpha : I \rightarrow (\mathbb{Z}^k)^*$ such that:*

1. *Each vertex has the same number of edges.*
2. *The weights at each vertex span $(\mathbb{Z}^k)^*$ and are pairwise linearly independent.*
3. *If p and q are adjacent vertices then $\alpha(p, q) = -\alpha(q, p)$.*
4. *If p and q are adjacent vertices and $\{p_i\}_{i=1}^{n-1}$ and $\{q_i\}_{i=1}^{n-1}$ are the other vertices adjacent to p and q respectively, then we can order these vertices so that $\alpha(p, p_i) = \alpha(q, q_i) \pmod{\alpha(p, q)}$.*
5. *There exists a function $\mu : V \rightarrow (\mathbb{R}^k)^*$ such that:*
 - (a) *For all adjacent vertices p and q , $\mu(q) - \mu(p) = c\alpha(p, q)$ for some $c > 0$.*

(b) For all $p \in V$ there is a neighborhood $U \subset (\mathbb{R}^k)^*$ of $\mu(p)$ such that the cone of the weights at p (the set $\mu(p) + \sum_{(p,q) \in I} c_q \alpha(p,q)$ with $c_q \geq 0$) intersected with U is the same as the convex hull of $\mu(V)$ intersected with U .

6. Given any sublattice of $(\mathbb{Z}^k)^*$, the edges whose weights span that sublattice form a GKM graph with respect to that sublattice.

Remark. it should be obvious that the notion of a projection (see Definition 3) of a GKM graph still holds. Simply take any projection $(\mathbb{Z}^k)^* \rightarrow (\mathbb{Z}^l)^*$. If the images of the weights of the original abstract GKM manifold still satisfy condition 2 (the others are satisfied automatically,) then the graph with the new weights is still an abstract GKM graph. We will say that this is the **projection** of the abstract GKM graph.

CHAPTER 2

BACKGROUND

Theorem 5 is proved independently of Theorems 1, 2, 3, 4. In this chapter we discuss the preliminary results needed for both sets of theorems. We then prove one key result about GKM graphs and discuss some examples that we will use later.

A well-known theorem due to Darboux shows that there are no local invariants in symplectic geometry.

Theorem 6 (Darboux Theorem) *Let (M, ω) be a $2n$ -dimensional symplectic manifold, and $p \in M$. Then there is a neighborhood U of p such that there exists a symplectomorphism $f : U \rightarrow \mathbb{R}^{2n}$ and $f^*\omega_0 = \omega$ where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ is the standard symplectic form on \mathbb{R}^{2n} .*

Proof. See, for example, [Au03, dS01]. There are multiple proofs of this. One involves using the exponential map of some Riemannian metric as a local diffeomorphism $T_pM \rightarrow M$. Darboux's Theorem for T_pM then follows from an application of Moser's Trick. If the manifold has an action of a compact Lie group on it, then the metric in the proof of Darboux's Theorem can be taken to be equivariant, and we get the following result:

Theorem 7 (Equivariant Darboux Theorem) *Let (M, ω) be a $2n$ -dimensional symplectic manifold and G a compact Lie group acting on M preserving the symplectic form. Let $p \in M$ be a fixed point of G . Consider the induced symplectic G -action on T_pM . Then the exponential map $\exp : T_pM \rightarrow M$ (defined with respect to some G -invariant metric) induces an equivariant symplectomorphism between an equivariant neighborhood of $0 \in T_pM$ and an equivariant neighborhood of $p \in M$.*

Proof. See [Au03].

There are many similar results showing the lack of local invariants in symplectic geometry. These usually involve studying imbeddings of isotropic, Lagrangian, or coisotropic submanifolds. A submanifold is isotropic if the pullback of the symplectic form is 0, coisotropic if

the tangent space at each point contains its symplectic orthogonal, and Lagrangian if it is both isotropic and coisotropic. We will need the following result on coisotropic submanifolds later:

Theorem 8 (Coisotropic Embedding Theorem) *Let M be a manifold with symplectic forms ω_1 and ω_2 , and $N \subset M$ a submanifold coisotropic for both ω_1 and ω_2 . If $\omega_1|_N = \omega_2|_N$ then there exist open neighborhoods U_1 and U_2 of N in M and a symplectomorphism $\phi : (U_1, \omega_1) \rightarrow (U_2, \omega_2)$ with $\phi|_N = \text{id}$. If M is a symplectic G -manifold, with G a compact Lie group, the symplectomorphism can be chosen to be equivariant.*

Proof See [Go82].

One of the most important results on Hamiltonian torus actions on compact symplectic manifolds is the Atiyah Guillemin-Sternberg Theorem showing that the image of the moment map is a convex polytope. Our definition of abstract GKM (see Definition 5) graph is specifically designed to include a counterpart to this theorem.

Theorem 9 (Atiyah Guillemin-Sternberg Convexity Theorem) *Let (M, ω) be a compact connected symplectic manifold with a Hamiltonian k -torus action with moment map $\mu : M \rightarrow (\mathbb{R}^k)^*$. Then $\mu(M)$ is a convex polytope, namely the convex hull of $\mu(M^{\mathbb{T}})$. For each $x \in (\mathbb{R}^k)^*$, the set $\mu^{-1}(x)$ is empty or connected.*

Proof. See [At82] and [GS82].

We also need this fact about moment maps.

Theorem 10 *The moment map of a compact Hamiltonian manifold acted upon by a compact Lie group is an open map onto its image.*

Proof. See [Sj98].

Our main result in this section shows that every GKM graph that comes from a GKM manifold satisfies the conditions of an abstract GKM graph.

Theorem 11 *Every GKM graph of a GKM manifold is an abstract GKM graph.*

Proof. Since the manifold is symplectic and the group action is Hamiltonian, there is a \mathbb{T}^k -invariant almost complex structure on the manifold. At a fixed point p , the action of \mathbb{T}^k induces a linear action on $T_p M$. Since the action is compact, we can view \mathbb{T}^k as a subset of $U(n)$, and since the action is abelian we may, after a unitary change of basis, assume

that the action is diagonal. We may then split T_pM into a direct sum of two-dimensional subspaces on which elements of \mathbb{T}^k act by rotation. More specifically $T_pM = \bigoplus_{i=1}^n \oplus V_i$ and, if $\{t_1, \dots, t_k\}$ are generators of \mathbb{T}^k and $z_i \in V_i - \{0\}$, then $t_j \cdot z_i = t_j^{a_{ij}} z_i$ for some $a_{ij} \in \mathbb{Z}$. Then the vector $\alpha_i = (a_{i1}, \dots, a_{ik})$ is the isotropy weight of the \mathbb{T}^k action on V_i .

By the equivariant Darboux theorem (see Theorem 7,) there is an equivariant symplectomorphism from a neighborhood of the origin in T_pM to a neighborhood of $p \in M$. The intersection of this neighborhood with each V_i is mapped to a submanifold fixed by a codimension 1 subgroup of \mathbb{T}^k , namely an isotropy sphere. Thus each fixed point of M is the fixed point of n isotropy spheres, so each vertex will have n edges. The the graph of the manifold satisfies Condition 1 of the definition of GKM graphs.

The weights at p are the isotropy weights of T_pM . These must be pairwise linearly independent, since otherwise the same subgroup would fix two isotropy spheres at p . Thus Condition 2 of Definition 5 is satisfied.

The isotropy weights at the fixed points of a Hamiltonian circle action on a two-sphere are the negatives of each other. Thus the isotropy weight at p is the negative of the isotropy weight at q , so $\alpha(p, q) = -\alpha(q, p)$. Condition 3 of Definition 5 follows.

Let $K \subset \mathbb{T}$ be the codimension 1 subgroup fixing the isotropy sphere connecting p and q , and \mathfrak{k} be its Lie algebra. Then $\alpha(p, q)(\xi) = 0$ for all $\xi \in \mathfrak{k}$. But then K has an isotropy representation on the tangent space of each point of the sphere. This representation must be the same at each point, so in particular must be the same at the two fixed points. Thus $\rho_K \alpha(p, p_i) = \rho_K \alpha(q, q_i)$ where $\rho_K : \mathfrak{t}^* \rightarrow \mathfrak{k}^*$ is the usual projection. Condition 4 of Definition 5 follows.

The function μ is the restriction of the moment map ϕ to the fixed points of the manifold. The image of the moment map restricted to an isotropy sphere is a line segment in the direction of the corresponding isotropy weight, so $\mu(q) - \mu(p) = c\alpha(p, q)$ for some $c > 0$. Condition 5a follows.

By the Atiyah Guillemin-Sternberg Convexity Theorem (see Theorem 9) we know that the image of the moment map is a convex polytope, namely the convex hull of the image of the fixed points. By Theorem 11 the moment map is an open mapping onto its image. There is also a neighborhood W of p such that $\phi(W)$ is a subset of the cone of the weights at p . This subset is the intersection of the cone with an open set U of $\phi(M)$. Since $\phi(M)$ is the convex hull of $\mu(V)$ condition 5b follows.

Let Λ be a sublattice of $(\mathbb{Z}^k)^*$. Then the inclusion $\Lambda \rightarrow (\mathbb{Z}^k)^*$ induces a projection $\mathbb{T}^k \rightarrow \mathbb{T}^{\dim \Lambda}$, and the kernel of this projection fixes all the isotropy spheres corresponding

to the edges whose weights lie in the sublattice. There must then be a larger submanifold containing these spheres that is fixed by the same subgroup. This manifold will also be GKM and its GKM graph will consist of the vertices with adjacent edges lying in the sublattice. Condition 6 follows. \square

Remark. The function μ allows us to view the GKM graph of a GKM manifold as lying in the moment image of the manifold. The vertices are just the images of the fixed points, and the isotropy spheres map to line segments connecting adjacent vertices. In particular, if a subgraph has weights lying in some subspace of $(\mathbb{Z}^k)^*$ then the vertices of the subgraph are mapped to some affine subspace of $(\mathbb{R}^k)^*$. In this situation we adopt the convention that the vertices of the subgraph lie on a subspace.

Remark. If we did not have condition 5 we would be forced to allow many abstract GKM graphs that do not come from GKM manifolds. The simplest example would be the graph with vertices A and B and three edges connecting these two vertices. The weights of the oriented edges going from A to B can be taken to be $(1, 0)$, $(0, 1)$ and $(-1, -1)$. This is not the GKM graph of a Hamiltonian manifold, but is the the GKM graph, under the definition of Guillemin and Zara, of S^6 acted upon by \mathbb{T}^2 . See example 1.9.1 in [GZ01].

Let $S(\mathbb{Z}^k)$ be the space of symmetric forms on \mathbb{Z}^k with the obvious grading. We define the **equivariant cohomology ring** $HE^*(\Gamma, \alpha)$ of an abstract GKM graph to be the set of all maps $f : V \rightarrow S(\mathbb{Z}^k)$ such that $f(p) = f(q) \pmod{\alpha(p, q)}$ when p and q are adjacent and $(\mathbb{Z}^k)^*$ identified with $S(\mathbb{Z}^k)_1$, the space of linear forms.

Theorem 12 *Let M be a GKM manifold. Then the map*

$$i^* : H_{\mathbb{T}}^*(M, \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(M^{\mathbb{T}}, \mathbb{Z}) = \bigoplus_{p \in M^{\mathbb{T}}} H_{\mathbb{T}}^*(pt, \mathbb{Z})$$

is an injection. The image of the function i^ is determined by the GKM graph of the manifold; two GKM manifolds with the same GKM graph will have the same equivariant cohomology rings and equivariant Chern classes. If the subgroup $\Gamma_n = \{t \in \mathbb{T} | t^n = 1\}$ does not act trivially for any $n > 1$, then the equivariant cohomology ring of the GKM manifold is isomorphic to the equivariant cohomology ring of its GKM graph.*

Proof. See [T11]. See also [GKM98] for the proof with \mathbb{Z} replaced by \mathbb{Q} (which does not require the condition on Γ_n .)

Since GKM manifolds are equivariantly formal, when the condition of Theorem 12 are satisfied the cohomology ring of a GKM manifold is $H^*(M) = HE^*(\Gamma, \alpha) / S_+(\mathbb{Z}^k) HE^*(\Gamma, \alpha)$.

We then define $HE^*(\Gamma, \alpha)/S_+(\mathbb{Z}^k)HE^*(\Gamma, \alpha)$ to be the **cohomology ring** of an abstract GKM graph. Both the ordinary and equivariant cohomology rings have the natural grading.

For the graphs we consider in Theorems 1-5 the technical condition on Γ_n are rarely an issue. The only cases we must account for are those where the GKM graph is the projection of a GKM graph with weights lying in a higher dimensional lattice. In this case, since the GKM graph determines the cohomology ring of the GKM manifold, if we know the cohomology ring and Chern classes of one manifold with a given GKM graph we know the cohomology ring and Chern classes of all the GKM manifolds with the same GKM graph. In the cases where the GKM graph is the projection of another GKM graph the latter GKM graph is always the the GKM graph of a manifold satisfying the condition on Γ_n . Since the projection of the GKM graph of a GKM manifold is merely the GKM graph of the same manifold acted upon by a lower dimensional torus, this gives us the cohomology ring and Chern classes of the original GKM manifold. The GKM graphs in question are the projections of the GKM graphs of $\mathbb{C}\mathbb{P}^n$, $G_+(2, 5)$, $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$, and $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$.

The following lemma then holds:

Lemma 1 *The cohomology ring and Chern classes of the abstract GKM graphs described in items 3 and 4 of Theorem 5 and the GKM graph of the blowup of $G_+(2, 5)$ are the cohomology rings and Chern classes of the GKM manifolds that have the respective GKM graphs.*

For each $p \in V$ (the set of vertices,) let $c^i(p)$ be the i th elementary symmetric function on $\alpha(p, q_1), \dots, \alpha(p, q_n)$ (viewed as elements of $S(\mathbb{Z}^k)_1$.) The i th **equivariant Chern** class of the graph is defined to be the function $f : V \rightarrow S(\mathbb{Z}^k)_i$ defined by $f(p) = c^i(p)$. The **Chern classes** are defined to be the images of the equivariant Chern classes under the quotient map $HE^*(\Gamma, \alpha) \rightarrow HE^*(\Gamma, \alpha)/S_+(\mathbb{Z}^k)HE^*(\Gamma, \alpha)$.

Proposition 1 *When the abstract GKM graph comes from a GKM manifold, then the ordinary and equivariant Chern classes of the graph are the ordinary and equivariant Chern classes of the manifold.*

Proof. The images of the equivariant Chern classes under the map i^* in Theorem 12 is the element of $\bigoplus_{p \in M^\mathbb{T}} H_{\mathbb{T}}^*(pt, \mathbb{Z})$ consisting of the equivariant Chern classes of the restriction of the tangent bundle to each of the fixed points. If we choose a set of generators $\{t_1, \dots, t_k\}$ for \mathbb{T}^k then this determines elements $x_1, \dots, x_k \in H_{\mathbb{T}^k}^2(pt)$ such that $H_{\mathbb{T}^k}^*(pt) = \mathbb{Z}[x_1, \dots, x_k]$. If $\mathbb{C} \rightarrow \{pt\}$ is an equivariant S^1 bundle such that $s \cdot z = s^n z$ (s acts with speed n) then the equivariant Euler class is the Euler class of the bundle $ES^1 \times_{S^1} \mathbb{C} \rightarrow BS^1$ which is just

$\mathcal{O}(n) \rightarrow \mathbb{C}\mathbb{P}^\infty$. Thus if $x \in H^2(\mathbb{C}\mathbb{P}^\infty)$ is the generator of the cohomology of $\mathbb{C}\mathbb{P}^\infty$ then the equivariant Euler class of $\mathbb{C} \rightarrow \{pt\}$ is ax . Let $\mathbb{C} \rightarrow \{pt\}$ be a \mathbb{T}^k vector bundle such that $t_i \cdot z = t_i^{a_i} z$ for each generator t_i . Denote the same bundle with only the S^1 action generated by t_i as $\mathbb{C}_i \rightarrow \{pt\}$. Then the identify map $id : \mathbb{C}_i \rightarrow \mathbb{C}$ is equivariant, and the pullback of the equivariant Euler class of the second bundle is the equivariant Euler class of the first (i.e. $id^* e_{\mathbb{T}}(\mathbb{C}) = e_{S^1}(\mathbb{C}_i) = a_i x_i$). Thus the equivariant Euler class of the \mathbb{T}^k -equivariant line bundle is $\sum_{i=1}^k a_i x_i$. At each fixed point $p \in M$ the induced action on $T_p M$ is diagonalizable, and thus splits $T_p M$ into a direct sum of complex line bundles. Thus the total equivariant Chern class of $T_p M \rightarrow \{p\}$ is the product of the total equivariant Chern classes of the two-dimensional sub-bundles. But The Euler classes of these sub-bundles will be sum of the isotropy weights of the torus action on $T_p M$. Thus the i th equivariant Chern class will be the function $c^i : V \rightarrow S((\mathbb{Z}^k)^*)_i$, where $c^i(p)$ is the i th symmetric function on the weights of p . \square

The rest of this section is examples. The first two are examples of GKM graphs that we will use frequently in the proofs.

Example 1. The $(2, 2, 3)$ GKM graph with vertices A, B , and C with weights $\alpha(A, B) + \alpha(B, C) = \alpha(A, C)$. This graph comes from the standard \mathbb{T}^2 action on $\mathbb{C}\mathbb{P}^2$.

Example 2. The family of GKM graphs corresponding to the Hirzebruch surfaces. The graph corresponding to $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ has vertices A, B, C , and D with weights $\alpha(A, B) = \alpha(C, D)$ and $\alpha(B, D) = \alpha(A, C) + n\alpha(A, B)$.

The $(2, 2, 3)$ and $(2, 2, 4)$ GKM graphs are GKM graphs of toric manifolds. It is well known that these are the only four dimensional toric manifolds with at most 4 fixed points, and that all the other four dimensional toric manifolds can be constructed from $\mathbb{C}\mathbb{P}^2$ or $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ by a series of blow-ups (see [Au03].)

There are also some more complicated graphs that will be important for us later. These are the graphs corresponding to the six-dimensional toric manifolds with six fixed points. These two families of manifolds correspond to two of the families of $(3, 2, 6)$ GKM manifolds, namely the bundles $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ and $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$. We can construct these manifolds using existence part of the proof of Delzant's Theorem. ([D88, dS01].)

The GKM graph of a toric manifold can be viewed (via the function μ) as the one-skeleton of the Delzant Polytope of a toric manifold. Different choices of μ will give us different polytopes, but the set of normal vectors to the facets of the polytopes will remain the same. Thus the choice of polytope only determines the choice of symplectic form on the manifold, not the manifold itself. Since the choice of symplectic form is irrelevant (we only need that

one exits) we will choose one polytope per graph.

Suppose the $(3, 3, 6)$ graph has weights $\alpha(A, B) = \alpha(D, E) = (1, 0, 0)$, $\alpha(A, C) = \alpha(D, F) = (0, 1, 0)$, $\alpha(B, C) = \alpha(E, F) = (-1, 1, 0)$, $\alpha(A, D) = (0, 0, 1)$, $\alpha(B, E) = (-n, 0, 1)$, and $\alpha(C, F) = (0, -n, 1)$. We will show that this is the GKM graph of the toric manifold $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$. This GKM graph is the one-skeleton of the following polytope:

$$\Delta = \{x \in (\mathbb{R}^3)^* : \langle x, (-1, 0, 0) \rangle \leq 0, \langle x, (0, -1, 0) \rangle \leq 0, \\ \langle x, (0, 0, -1) \rangle \leq 0, \langle x, (0, 0, 1) \rangle \leq 1, \langle x, (1, 1, n) \rangle \leq n + 1\}.$$

We define a map $\pi : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ by

$$\pi = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & n \end{bmatrix}.$$

The kernel of this map is $i : \ker(\pi) \rightarrow \mathbb{R}^5$, and the dual map is

$$i^* = \begin{bmatrix} 1 & 1 & n & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Define $\phi : \mathbb{C}^5 \rightarrow (\mathbb{R}^5)^*$ to be

$$\phi(z) = -(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2, |z_5|^2) + (0, 0, 0, 1, n + 1).$$

This is the moment map for the standard \mathbb{T}^5 action on \mathbb{C}^5 that sends the origin to $(0, 0, 0, 1, n + 1)$. Then consider $Z = (i^* \circ \phi)^{-1}(0)$. This space is $\{z : |z_1|^2 + |z_2|^2 + n|z_3|^2 + |z_5|^2 = n + 1, |z_3|^2 + |z_4|^2 = 1\}$. The map π induces a surjection $\mathbb{R}^5/\mathbb{Z}^5 \rightarrow \mathbb{R}^3/\mathbb{Z}^3$ and these two spaces can be identified with \mathbb{T}^5 and \mathbb{T}^3 respectively. Denote the kernel of this map by N . Then N acts freely on Z and Z/N is the toric manifold with the Delzant polytope Δ .

We can show that this manifold is a $\mathbb{C}\mathbb{P}^1$ -bundle over $\mathbb{C}\mathbb{P}^2$. The group N is the subgroup of \mathbb{T}^5 generated by $\{(e^{i\theta_1}, e^{i\theta_1}, e^{ni\theta_1}, 1, e^{i\theta_1}), (1, 1, e^{i\theta_2}, e^{i\theta_2}, 1)\}$. Any point of Z/N has a representation by $(z_1, z_2, z_3, z_4, z_5)$ and there is a well-defined equivariant map $Z/N \rightarrow \mathbb{C}\mathbb{P}^2$ defined by:

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow [z_1, z_2, z_5] \in \mathbb{C}\mathbb{P}^2.$$

Given a point $p \in \mathbb{C}\mathbb{P}^2$ and a number $a \in [1, n + 1]$ we can pick a representation $[z_1, z_2, z_5]$

such that $|z_1|^2 + |z_2|^2 + |z_5|^2 = a$. Then $a = n + 1 - n|z_3|^2$. As we vary a this allows for every possible choice of (z_3, z_4) satisfying $|z_3|^2 + |z_4|^2 = 1$. Then the set of points (z_3, z_4) that can be in a representation of a point in the preimage of p is the whole 3-sphere $|z_3|^2 + |z_4|^2 = 1$. But when we quotient out by N we see that this reduces to $\mathbb{C}\mathbb{P}^1$.

In fact, this manifold is $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$. When we quotient out by the subgroup generated by $(1, 1, e^{i\theta_2}, e^{i\theta_2}, 1)$ we see that what remains is $S^5 \times S^1$ and when we quotient out by $N/\{(1, 1, e^{i\theta_2}, e^{i\theta_2}, 1)\}$ we see that we have $S^5 \times_{S^1} \mathbb{C}\mathbb{P}^1$ under the S^1 action $u \cdot (x, y) = (ux, u^n y)$ where $x \in S^5$ and $y \in \mathbb{C}\mathbb{P}^1$. Thus the manifold is $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$.

Now suppose the $(3, 3, 6)$ graph has weights $\alpha(A, B) = \alpha(C, D) = \alpha(E, F) = (0, 0, 1)$, $\alpha(A, C) = (1, 0, 0)$, $\alpha(A, E) = (0, 1, 0)$, $\alpha(C, E) = (-1, 1, 0)$, $\alpha(B, D) = (1, 0, m)$, $\alpha(B, F) = (0, 1, n)$, $\alpha(D, F) = (-1, 1, n - m)$. Without loss of generality, we may assume that both n and m are positive and $n > m$. We will show that this is the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$. Then this GKM graph is the one-skeleton of the following polytope:

$$\Delta = \{x \in (\mathbb{R}^3)^* : \langle x, (-1, 0, 0) \rangle \leq 0, \langle x, (0, -1, 0) \rangle \leq 0, \langle x, (0, 0, -1) \rangle \leq 0, \\ \langle x, (1, 1, 0) \rangle \leq 1, \langle x, (-m, -n, 1) \rangle \leq 1\}$$

Define $\pi : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ by

$$\pi = \begin{bmatrix} -1 & 0 & 0 & 1 & -m \\ 0 & -1 & 0 & 1 & -n \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The dual of the kernel of this map is

$$i^* = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ -m & -n & 1 & 0 & 1 \end{bmatrix}.$$

Define $\phi : \mathbb{C}^5 \rightarrow (\mathbb{R}^5)^*$ to be

$$\phi(z) = -(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2, |z_5|^2) + (0, 0, 0, 1, 1).$$

Then consider $Z = (i^* \circ \phi)^{-1}(0)$. This space is $\{z : |z_1|^2 + |z_2|^2 + |z_4|^2 = 1, |z_3|^2 + |z_5|^2 = 1 + m|z_1|^2 + n|z_2|^2\}$. Then $N = \ker(\mathbb{R}^5/\mathbb{Z}^5 \rightarrow \mathbb{R}^3/\mathbb{Z}^3)$ acts freely on Z and Z/N is the toric manifold with the Delzant polytope Δ . The group N is the subgroup of \mathbb{T}^5 generated by

$$\{(e^{i\theta_1}, e^{i\theta_1}, 1, e^{i\theta_1}, 1), (e^{-im\theta_2}, e^{-in\theta_2}, e^{i\theta_2}, 1, e^{i\theta_2})\}.$$

Any point of Z/N has a representation by $(z_1, z_2, z_3, z_4, z_5)$ and there is a well-defined equivariant map $Z/N \rightarrow \mathbb{C}\mathbb{P}^1$ defined by :

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow [z_3, z_5].$$

Given a point $p \in \mathbb{C}\mathbb{P}^1$ and a number $a \in [1, m + n + 1]$ we can pick a representation $[z_3, z_5]$ such that $|z_3|^2 + |z_5|^2 = a$. Then $a = 1 + m|z_1|^2 + n|z_2|^2$. As a varies this allows for every possible choice of (z_1, z_2, z_4) satisfying $|z_1|^2 + |z_2|^2 + |z_4|^2 = 1$. Then the set of points that can be in a representation of a point in the preimage of p is the whole 5-sphere $|z_1|^2 + |z_2|^2 + |z_4|^2 = 1$. But when we quotient out by N we see that this reduces to $\mathbb{C}\mathbb{P}^2$.

In fact, this manifold is $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$. When we quotient out by the subgroup generated by $(e^{i\theta_1}, e^{i\theta_1}, 1, e^{i\theta_1}, 1)$ we see that what remains is $S^3 \times \mathbb{C}\mathbb{P}^2$ and when we quotient out by $N/\{(e^{i\theta_1}, e^{i\theta_1}, 1, e^{i\theta_1}, 1)\}$ we see that we have $S^3 \times_{S^1} \mathbb{C}\mathbb{P}^2$ under the S^1 action $u \cdot (x, [x_1, x_2, x_4]) = (ux, [u^m x_1, u^n x_2, x_4])$. Thus the manifold is $\mathbb{P}(\mathcal{O}(m) \oplus \mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$.

CHAPTER 3

MINIMAL GKM MANIFOLDS AND GRAPHS

3.1 Introduction

There are two well-known families of examples of minimal GKM manifolds. The first is $\mathbb{C}\mathbb{P}^n$ itself, and the second is $G_+(2, 2n - 1)$, the Grassmannian of oriented 2-planes in \mathbb{R}^{2n-1} . For a maximal torus action their respective GKM graphs are the one skeleton of the simplex and the complete graph on the vertices of the octaplex. The weights of the simplicial graph are the differences of the images of the fixed points under the canonical moment map (which sends one fixed point to 0 and the others to $\{e_i\}$.) The weights of the octaplex are also the differences of the fixed points under the canonical moment map (which sends the fixed points to $\{\pm e_i\}$) unless the points are antipodal, in which case the weights are half the differences. The edges connecting antipodal points will be called interior edges, while the remaining edges are exterior edges. If a subtorus acts then the graph will be a projection of one of these GKM graphs onto a lower dimensional space. We will call any projection of a simplex a **simplicial graph** and any projection of a complete octaplex a **complete octaplex graph**.

Both of these families can be realized as quotients of Lie groups. It is well known that $\mathbb{C}\mathbb{P}^n$ can be realized as $U(n + 1)/U(1) \times U(n)$ and $G_+(2, 2n + 1)$ can be realized as $SO(2n + 1)/SO(2) \times SO(2n - 1)$. The only other known minimal GKM manifold is also a quotient of a Lie group, namely G_2/P where P is the parabolic subgroup generated by the Borel subgroup and the exponential of the short simple root.

Simplicial graphs are characterized by the fact that any complete subgraph on three vertices is a GKM subgraph. If A, B, C are three vertices then the complete graph on these vertices is simplicial if and only if $\alpha(A, B) + \alpha(B, C) = \alpha(A, C)$. The complete graph on three vertices (A, B , and C) of a complete octaplex graph is either simplicial or (up to relabeling) satisfies $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$.

Theorem 1 is a consequence of Propositions 4 and 5. Theorem 2 is a consequence of

Proposition 6. Theorem 3 is a consequence of proposition 7. Theorem 4 is a consequence of Proposition 8.

3.2 Dimension at Most 8

Our main result in this section is that when $n \leq 4$ all GKM manifolds with a minimal number fixed points have the same equivariant cohomology as complex projective space or an oriented Grassmannian. This follows from the fact that, under the same hypotheses, these are the only two complete GKM graphs.

We begin with a few lemmas.

Lemma 2 *Let ABC be a complete subgraph of the GKM graph Γ . In order for ABC to be a GKM subgraph it is sufficient to show that $\alpha(A, C) = \alpha(B, C) \pmod{\alpha(A, B)}$ and $\alpha(A, B) = \alpha(C, B) \pmod{\alpha(A, C)}$.*

Proof. Since $\alpha(A, C) - \alpha(B, C) = 0 \pmod{\alpha(A, B)}$ we have $\alpha(A, C) + \alpha(C, B) = n\alpha(A, B)$, and thus $\alpha(C, B) = n\alpha(A, B) \pmod{\alpha(A, C)}$. But then $(n - 1)\alpha(A, B) = 0 \pmod{\alpha(A, C)}$. Since $\alpha(A, B)$ and $\alpha(A, C)$ are linearly independent, $n = 1$, and thus ABC is a GKM subgraph. \square

Lemma 3 *To show that a complete GKM graph is simplicial it is sufficient to show that every triangular subgraph with a fixed vertex is a GKM subgraph.*

Proof. Pick a vertex A of the complete GKM graph and suppose every triangular subgraph containing A in its set of vertices is a GKM subgraph. Let BCD be another triangular subgraph. Since the graph is complete ABC , ACD , and ABD are subgraphs and thus GKM subgraphs. Then $\alpha(B, C) = \alpha(B, A) + \alpha(A, C)$, $\alpha(C, D) = \alpha(C, A) + \alpha(A, D)$, and $\alpha(B, D) = \alpha(B, A) + \alpha(A, D)$. Adding these together we see that

$$\alpha(B, C) + \alpha(C, D) = \alpha(B, A) + \alpha(A, D) = \alpha(B, D).$$

Thus the triangle BCD is a GKM subgraph. Since the choice of triangle was arbitrary, we see that all triangular subgraphs are GKM subgraphs, and thus the graph is simplicial. \square

Lemma 4 *Suppose a complete GKM graph lying in $(\mathbb{R}^k)^*$ with $n + 1$ vertices has n vertices lying in an affine subspace of $(\mathbb{R}^k)^*$. Then the whole graph is simplicial.*

Proof. Let A be the vertex not lying in the subspace, and B and C two vertices lying in the subspace. Then the 2-dimensional subspace spanned by A , B , and C does not contain any other vertices, so the complete graph on A , B , and C must be simplicial. But since B and C are arbitrary, we see that the any triangular subgraph containing A is a GKM subgraph. Thus, by Lemma 3 the whole graph is simplicial. \square

We can now classify all minimal GKM graphs with degree at most 4. When the degree is 1, the graph is a line segment, and when the degree is 2 the graph is the simplicial triangle of Example 1. Degrees 3 and 4 are classified in the next two propositions.

Proposition 2 *The only (3,2) GKM graphs are the simplicial graphs, and the complete octaplex graph.*

Proof. We have two cases to consider. First, suppose the image of the graph under μ has three exterior vertices and one interior vertex. We will label the vertices A , B , C , D , with A the interior vertex. The three interior edges then have weights $\alpha(A, B)$, $\alpha(A, C)$, and $\alpha(A, D)$. Each of these edges is contained in the border of two half-planes, and since these are interior edges there is only one pair of edges in each half plane. These edges must then have a common vertex. If we start with the edge AB then we see that $\alpha(A, C) = \alpha(B, C) \pmod{\alpha(A, B)}$. Similarly, $\alpha(A, B) = \alpha(C, B) \pmod{\alpha(A, C)}$. Thus ABC is a GKM subgraph. By symmetry, the same holds for ABD and ACD . By lemma 3 the the full graph is simplicial.

The second case is when there are four exterior vertices, labeled in clockwise order as in the picture. This case is split into two subcases.

The first subcase is when all four triangular subgraphs are simplicial subgraphs. But then it is clear that the whole graph is simplicial.

The second subcase is where one of the triangular subgraphs, say ABC , is not a simplicial subgraph. Thus there is no GKM subgraph containing the edges $\alpha(A, B)$ and $\alpha(A, C)$ so $\{\alpha(A, B), \alpha(A, C)\}$ span \mathbb{Z}^2 . Since $\alpha(A, B) = \alpha(C, B) \pmod{\alpha(A, C)}$ we see that $\alpha(A, B) + \alpha(B, C) = n\alpha(A, C)$ where $n > 1$. Thus

$$\det \begin{bmatrix} \alpha(B, A) \\ \alpha(B, C) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, A) \\ \alpha(A, B) + \alpha(B, C) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, A) \\ n\alpha(A, C) \end{bmatrix} = n$$

so $\{\alpha(B, A), \alpha(B, C)\}$ does not span \mathbb{Z}^2 . Thus there is some GKM subgraph that lies on the sublattice they do span; this must be the quadrilateral $ABCD$. Thus either $\alpha(A, B) = \alpha(D, C)$ or $\alpha(A, D) = \alpha(B, C)$. Without loss of generality we may suppose the former. Then

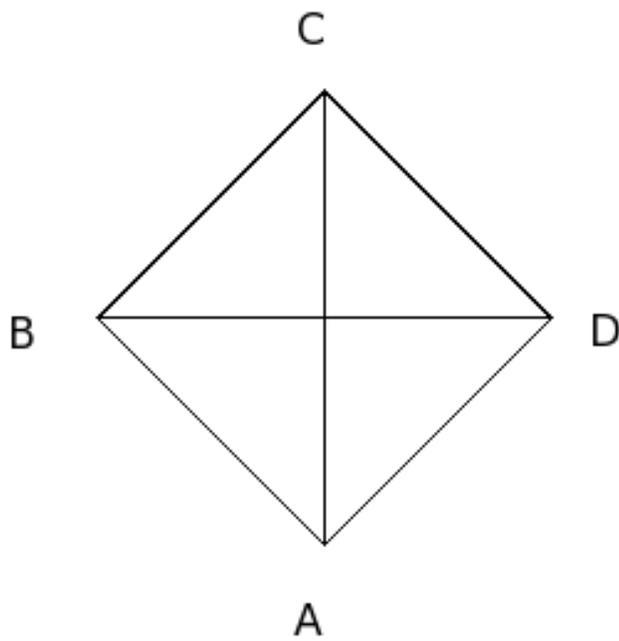
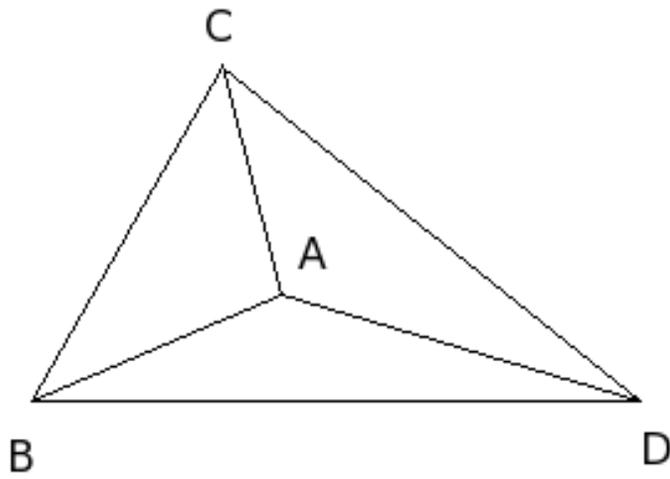


Figure 3.1: The two types of (3,2) graphs.

$\alpha(A, D) = \alpha(B, C) + n\alpha(A, B)$ for some n . But since $ABCD$ is a GKM subgraph we see that $\alpha(B, D) = \alpha(C, A) \pmod{\alpha(B, C)}$, and $\alpha(A, C) = \alpha(D, B) \pmod{\alpha(A, D)}$. Thus $\alpha(A, D)$ is parallel to $\alpha(B, C)$, so $n = 0$ and $\alpha(A, D) = \alpha(B, C)$.

Since $\{\alpha(A, B), \alpha(A, C)\}$ span and $\alpha(A, C) - \alpha(B, D)$ is parallel to $\alpha(A, B)$ we see that

$$\det \begin{bmatrix} \alpha(A, C) \\ \alpha(A, B) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, D) \\ \alpha(A, B) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, A) \\ \alpha(B, D) \end{bmatrix} = 1$$

and thus $\{\alpha(B, A), \alpha(B, D)\}$ span the lattice. Similarly, $\{\alpha(B, C), \alpha(B, D)\}$ span the lattice. Since $\{\alpha(A, B), \alpha(A, C)\}$ span the lattice, we see that $\alpha(B, D) = r\alpha(A, B) + s\alpha(A, C)$ for some $r, s \in \mathbb{Z}$. But

$$\det \begin{bmatrix} \alpha(B, A) \\ \alpha(B, D) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, A) \\ r\alpha(A, B) + s\alpha(A, C) \end{bmatrix} = s \det \begin{bmatrix} \alpha(B, A) \\ \alpha(A, C) \end{bmatrix} = s$$

so $s = 1$ and

$$\begin{aligned} 1 = \det \begin{bmatrix} \alpha(B, D) \\ \alpha(B, C) \end{bmatrix} &= \det \begin{bmatrix} r\alpha(A, B) + \alpha(A, C) \\ \alpha(B, A) + n\alpha(A, C) \end{bmatrix} \\ &= rn \det \begin{bmatrix} \alpha(A, B) \\ \alpha(A, C) \end{bmatrix} + \det \begin{bmatrix} \alpha(A, C) \\ \alpha(B, A) \end{bmatrix} = -rn - 1 \end{aligned}$$

so $rn = -2$. Since $n > 1$ we see that $r = -1$ and $n = 2$. Then $\alpha(A, B) = \alpha(A, C) + \alpha(D, B)$ and $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$. By symmetry $\alpha(B, C) + \alpha(C, D) = 2\alpha(B, D)$, $\alpha(A, D) + \alpha(D, C) = 2\alpha(A, C)$ and $\alpha(B, A) + \alpha(A, D) = 2\alpha(B, D)$. From this we see that the graph must be a complete octaplex. \square

Proposition 3 *The only $(4, 2)$ GKM graphs are the simplicial graphs.*

Proof. There are three cases to consider.

Case I

The first case is where there are three exterior vertices. The two interior vertices, labeled D and E are connected by an edge. There are two half-planes whose boundaries contain this edge. One half-plane will contain one exterior vertex (B) and the other half-plane will contain two exterior vertices (A and C). Consider the edge connecting D and B . It too lies

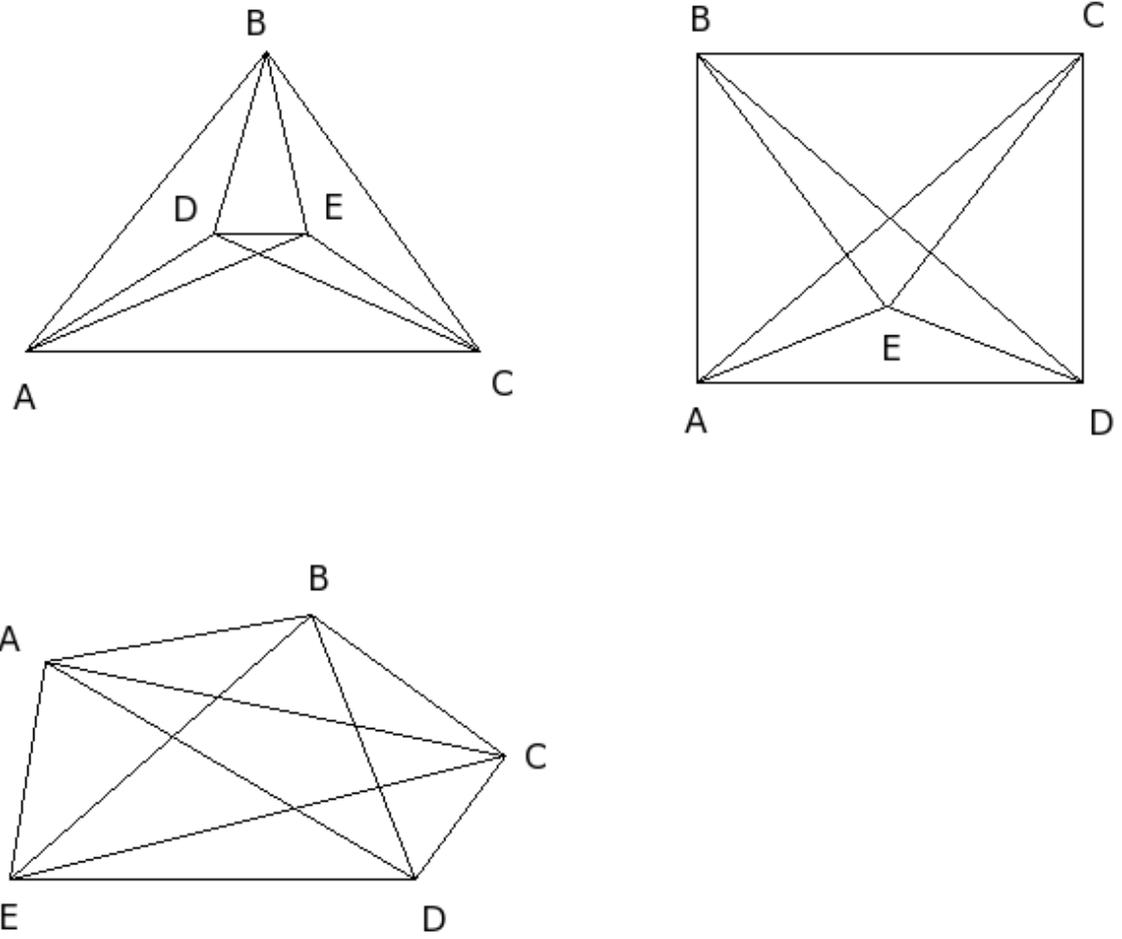


Figure 3.2: The three types of (4,2) graphs.

on the boundary of two half planes, one of which will contain A and one of which will contain C and E . Similarly, the edge connecting B and E divides the plane into half planes, one of which contains C and one of which contains A and D . One of the half planes whose boundary contains AD contains B but no other vertices. Thus $\alpha(A, B) = \alpha(E, D) \pmod{\alpha(A, E)}$ and $\alpha(D, A) = \alpha(B, A) \pmod{\alpha(B, D)}$. Then by Lemma 2, ABD is a simplicial subgraph. By symmetry BCE is a simplicial subgraph.

Suppose BDE is not a simplicial subgraph. Then since $\alpha(D, B) = \alpha(E, B) \pmod{\alpha(D, E)}$ we see that $\alpha(D, B) + \alpha(B, E) = n\alpha(D, E)$ where $n > 1$. Then

$$\det \begin{bmatrix} \alpha(B, D) \\ \alpha(B, E) \end{bmatrix} = \det \begin{bmatrix} \alpha(E, B) + \alpha(B, D) \\ \alpha(B, E) \end{bmatrix} = \det \begin{bmatrix} n\alpha(E, D) \\ \alpha(B, E) \end{bmatrix} = \det \begin{bmatrix} \alpha(E, B) \\ n\alpha(E, D) \end{bmatrix} > n,$$

so $\{\alpha(B, D), \alpha(B, E)\}$ cannot span the lattice \mathbb{Z}^2 . But then, by Property 6 of abstract GKM graphs there must be some GKM subgraph on the lattice they do span. This GKM subgraph can only be the complete graph on A, B, D , and E , or the complete graph on D, E, C , and B . But both of two options must be $(3, 2)$ graphs with three exterior vertices, and thus (by the proof of lemma 2) be simplicial, so BDE will be a simplicial subgraph. Thus BDE is a simplicial subgraph.

If ABE is not a simplicial subgraph then $\alpha(B, A) = \alpha(E, A) \pmod{\alpha(B, E)}$ implies that $\{\alpha(A, B), \alpha(AE)\}$ do not span the lattice \mathbb{Z}^2 . Then the complete graph on A, B, D , and E , or the complete graph on A, B, C , and E must be a GKM subgraph. But these must both be $(3, 2)$ graphs with three exterior vertices and thus simplicial, so ABE is a simplicial subgraph. Thus ABE is a simplicial subgraph. By symmetry, so is BCD .

We then see that $\alpha(B, A) = \alpha(C, A) \pmod{\alpha(B, C)}$ and $\alpha(A, C) = \alpha(B, C) \pmod{\alpha(A, B)}$, so ABC is a GKM subgraph by lemma 2. Thus every triangular subgraph containing the vertex B is a GKM subgraph, so by lemma 3 the whole graph is simplicial.

Case II

The second case is where there are four exterior vertices. These four vertices form a quadrilateral with two diagonals. We label the vertices A, B, C , and D so that the diagonals are AC and BD . Without loss of generality we may suppose that E is contained in the

half plane bordering AC and containing D and the half plane bordering BD containing A . One of the half planes bordered by AE only contains the vertex D , so $\alpha(A, D) = \alpha(E, D) \bmod \alpha(A, E)$. Similarly, $\alpha(E, A) = \alpha(D, A) \bmod \alpha(E, D)$. Thus ADE is a GKM subgraph. We will have shown that the whole graph is simplicial if we can show that every triangle with vertex E is a simplicial subgraph.

First we show that there can be no complete octaplex subgraphs.

Suppose the complete graph on A, B, C , and E is a complete octaplex subgraph. Then $\alpha(B, D) = \alpha(E, D) \bmod \alpha(B, E)$.

Suppose BDE is not a GKM subgraph. Then $\{\alpha(D, B), \alpha(D, E)\}$ does not span the lattice. Then there is some GKM subgraph whose weights lie on the sublattice spanned by $\{\alpha(D, B), \alpha(D, E)\}$. The only candidates are the complete graph on A, B, D , and E and the complete graph on B, C, D , and E . The first is a $(3, 2)$ graph with three exterior vertices, and thus is simplicial, so BCE is a simplicial subgraph. Thus the complete graph on B, C, D , and E must be a GKM subgraph. If it is simplicial then BDE is simplicial so this graph must be a complete octaplex subgraph. But then $\alpha(A, E) = \alpha(B, C) = \alpha(E, D)$, which violates Property 2 of GKM graphs. Thus BDE is a simplicial subgraph.

Suppose ABD is not simplicial. Since $\alpha(A, D) = \alpha(B, D) \bmod \alpha(A, B)$ we see that $\{\alpha(D, A), \alpha(D, B)\}$ cannot span the lattice. Then there must be a larger subgraph whose weights must lie on the sublattice spanned by $\{\alpha(D, A), \alpha(D, B)\}$. Since the GKM subgraph cannot be simplicial (otherwise ABD would be simplicial) it must be the complete subgraph on A, B, C , and D and must be a complete octaplex subgraph. But then $\alpha(A, D) = \alpha(B, C) = \alpha(A, E)$, which violates Property 2 of GKM graphs. Thus ABD is a simplicial subgraph.

But then

$$\begin{aligned} \alpha(B, E) &= \alpha(B, D) + \alpha(D, E) \\ &= \alpha(B, A) + \alpha(A, D) + \alpha(D, A) + \alpha(A, E) \\ &= \alpha(B, A) + \alpha(A, E) = 2\alpha(B, E). \end{aligned}$$

Thus the complete graph on A, B, C , and E cannot be a complete octaplex subgraph. By symmetry, the complete graph on B, C, D , and E cannot be a complete octaplex subgraph.

Suppose the complete graph on A, B, C , and D is a complete octaplex subgraph.

Suppose BDE is not a simplicial subgraph. Since $\alpha(B, E) = \alpha(D, E) \bmod \alpha(B, D)$ we see that $\{\alpha(E, B), \alpha(E, D)\}$ cannot span the lattice. Thus there must be some GKM

subgraph containing B , D , and E . Since this GKM subgraph cannot be simplicial it must be $BCDE$. But then $\alpha(B, A) = \alpha(C, D) = \alpha(B, E)$ or $\alpha(E, D) = \alpha(B, C) = \alpha(A, D)$, both of which violate Property 2 of GKM graphs. Thus BDE is simplicial.

Suppose ABE is not simplicial. Since $\alpha(A, E) = \alpha(B, E) \pmod{\alpha(A, B)}$ we see that $\{\alpha(E, A), \alpha(E, B)\}$ cannot span the lattice. But then either the complete graph on A , B , D , and E or the complete graph on A , B , C , and E must be a GKM subgraph. The first graph must be simplicial since it is a $(3, 2)$ graph with one interior vertex and the second graph must be simplicial since it cannot be a complete octaplex graph. Thus ABE must be simplicial.

Then

$$\begin{aligned}\alpha(B, D) &= \alpha(B, E) + \alpha(E, D) \\ &= \alpha(B, A) + \alpha(A, E) + \alpha(E, A) + \alpha(A, D) \\ &= \alpha(B, A) + \alpha(A, D) = 2\alpha(B, D).\end{aligned}$$

Thus the complete graph on A , B , C , and D cannot be a complete octaplex subgraph. Thus there are no complete octaplex subgraphs.

Suppose ABE is not a simplicial subgraph.

If $\{\alpha(E, A), \alpha(E, B)\}$ do not span the lattice then there must be some GKM subgraph containing A , B , and E with weights lying on the edges spanned by the lattice. The only possible non simplicial subgraph that satisfies these condition is the complete graph on A , B , C , and E , and it must be a complete octaplex subgraph, which is impossible. Thus $\{\alpha(E, A), \alpha(E, B)\}$ span the lattice.

Since $\alpha(E, A) = \alpha(B, A) \pmod{\alpha(B, E)}$ we see that

$$\det \begin{bmatrix} \alpha(B, A) \\ \alpha(B, E) \end{bmatrix} = \det \begin{bmatrix} \alpha(E, A) \\ \alpha(B, E) \end{bmatrix} = \det \begin{bmatrix} \alpha(E, B) \\ \alpha(E, A) \end{bmatrix} = 1$$

and thus $\{\alpha(B, A), \alpha(B, E)\}$ must also span \mathbb{Z}^2 . Since ABE is not a GKM subgraph, $\{\alpha(A, B), \alpha(A, E)\}$ cannot span \mathbb{Z}^2 , so the quadrilateral $ABCE$ must be a GKM subgraph.

Thus either $\alpha(B, C) = \alpha(A, E)$ or $\alpha(A, B) = \alpha(E, C)$.

Suppose $\alpha(B, C) = \alpha(A, E)$. Then $\{\alpha(B, E), \alpha(B, C)\}$ span the lattice. If the vectors $\{\alpha(E, B), \alpha(E, C)\}$ do not span then since BCE cannot be a GKM subgraph the complete graph on B , C , D , and E must be a complete octaplex graph, which is impossible. If $\{\alpha(E, B), \alpha(E, C)\}$ do span then $\alpha(B, C) = \alpha(E, C) \pmod{\alpha(B, E)}$. Since

$\alpha(E, C) = \alpha(A, B) + n\alpha(B, C)$ and $\alpha(A, B) = \alpha(B, C) \pmod{\alpha(B, E)}$ this implies $n = 0$ and $\alpha(A, B) = \alpha(E, C)$

Thus we can reduce to the case where $\alpha(A, B) = \alpha(E, C)$.

But then since $\alpha(C, A) = \alpha(E, B) \pmod{\alpha(C, E)}$ and $\alpha(E, A) = \alpha(C, B) \pmod{\alpha(C, E)}$ we see that $\alpha(A, C) = \alpha(B, E) \pmod{\alpha(A, B)}$ and $\alpha(A, E) = \alpha(B, C) \pmod{\alpha(A, B)}$. Thus $\alpha(A, D) = \alpha(B, D) \pmod{\alpha(A, B)}$.

If ABD is not a simplicial subgraph then $\{\alpha(D, A), \alpha(D, B)\}$ cannot span the lattice. Thus there must be some larger GKM subgraph containing A, B , and D . The only possibility that is not simplicial is the complete graph on A, B, C , and D which must be a complete octaplex graph. But this is impossible. Thus ABD is a simplicial subgraph.

But then since $\alpha(B, A) = \alpha(D, A) \pmod{\alpha(B, D)}$ holds, we have $\alpha(B, E) = \alpha(D, E) \pmod{\alpha(B, D)}$.

If BDE is a simplicial subgraph then $\alpha(B, D) = \alpha(B, E) + \alpha(E, D) = \alpha(B, E) + \alpha(E, A) + \alpha(A, D)$ so $\alpha(B, A) = \alpha(B, E) + \alpha(E, A)$ and ABE is simplicial, which contradicts our assumption. Thus BDE is not a simplicial subgraph and $\{\alpha(D, B), \alpha(D, E)\}$ must span the lattice.

But then $\{\alpha(B, E), \alpha(B, D)\}$ cannot span the lattice, so $BCDE$ must be a GKM subgraph. Then since $\alpha(D, C) = \alpha(E, B) \pmod{\alpha(D, E)}$, we see that $\alpha(D, B) = \alpha(E, C) \pmod{\alpha(D, E)}$, and thus $\{\alpha(E, C), \alpha(E, D)\}$ spans the lattice, just as $\{\alpha(E, A), \alpha(E, B)\}$ does. Then by symmetry $\alpha(B, E) = \alpha(C, D)$ and ACD is a simplicial subgraph. But then $\alpha(A, E) = \alpha(C, E) \pmod{\alpha(A, C)}$ and $\alpha(A, B) = \alpha(C, B) \pmod{\alpha(A, C)}$. But since $\alpha(A, E) = \alpha(B, C) + n\alpha(A, B)$, we see that $n\alpha(A, B) = 0 \pmod{\alpha(A, C)}$ so $n = 0$. Thus $\alpha(A, E) = \alpha(B, C)$ and by symmetry $\alpha(E, D) = \alpha(B, C)$. But then $\alpha(A, E) = \alpha(E, D)$, which is impossible.

Thus ABE , and by symmetry CDE are simplicial subgraphs.

Now suppose that BCE is not a GKM subgraph. Then $\{\alpha(E, B), \alpha(E, C)\}$ must span the lattice. If $\{\alpha(B, C), \alpha(B, E)\}$ does not span the lattice then the quadrilateral $BCDE$ is a GKM subgraph. If $\alpha(B, E) = \alpha(C, D)$ then $\{\alpha(C, E), \alpha(C, D)\}$ span the lattice, so $\{\alpha(D, C), \alpha(D, E)\}$ span the and thus $\{\alpha(B, C), \alpha(B, E)\}$ span, so $\alpha(B, C) = \alpha(E, D)$. If $\{\alpha(C, B), \alpha(C, E)\}$ does not span the lattice then similarly, $\alpha(B, C) = \alpha(A, E)$ so $\alpha(A, E) = \alpha(E, D)$. Thus one of $\{\alpha(B, C), \alpha(B, E)\}$ or $\{\alpha(C, B), \alpha(C, E)\}$ must span the lattice; we may choose the first pair. If the second pair does not span then $ABCE$ is a subgraph. But $\alpha(A, E) = \alpha(B, C)$ so $\{\alpha(E, A), \alpha(E, B)\}$ span, and thus $\{\alpha(A, B), \alpha(A, E)\}$ span, so $\{\alpha(C, B), \alpha(C, E)\}$ span. Thus all three pairs of weights on the triangle BCE span the

lattice, so BCE is a GKM subgraph. Thus BCE must be a GKM subgraph.

It now remains to show that ACE is a GKM subgraph. Then by symmetry BDE will also be a GKM subgraph. Since $\alpha(E, A) = \alpha(C, A) \bmod \alpha(E, C)$ and $\alpha(A, C) = \alpha(E, C) \bmod \alpha(A, E)$, ACE (and thus BDE) are GKM subgraphs, so the whole graph is simplicial. Thus every triangular subgraph containing the vertex E is a GKM subgraph. By Lemma 3, the whole graph is simplicial.

Case III

The final case to consider is when all five vertices are exterior vertices. We label these vertices clockwise. Thus (for example) one of the half planes whose boundary contains AC contains the vertex B and the other contains D and E .

We first show that there can be no complete octaplex subgraphs. Suppose the complete graph on A, B, C , and D is a complete octaplex graph. (The choice of vertices is arbitrary.) Then $\alpha(A, E) = \alpha(B, E) \bmod \alpha(A, B)$ and $\alpha(E, A) = \alpha(B, A) \bmod \alpha(B, E)$, so ABE , and by symmetry CDE , is a GKM subgraph. Also $\alpha(B, E) = \alpha(D, E) \bmod \alpha(B, D)$. If $\{\alpha(E, B), \alpha(E, D)\}$ span then BDE is a subgraph and $\alpha(B, D) = \alpha(B, E) + \alpha(E, D) = \alpha(B, A) + \alpha(A, E) + \alpha(E, D) = \alpha(B, A) + n\alpha(A, D)$ for some n since $\alpha(A, E) = \alpha(D, E) \bmod \alpha(A, D)$. But $\alpha(B, D) = \frac{1}{2}(\alpha(B, A) + \alpha(A, D))$, so $\frac{1}{2}\alpha(B, A) = (n - \frac{1}{2})\alpha(A, D)$, which is impossible. Thus $\{\alpha(E, B), \alpha(E, D)\}$ do not span the lattice so either the complete graph on A, B, D, E is a GKM subgraph or $BCDE$ is a GKM subgraph. In the first case $\alpha(E, D) = \alpha(A, B) = \alpha(D, C)$, in the second either $\alpha(E, D) = \alpha(B, C) = \alpha(A, D)$ or $\alpha(B, E) = \alpha(C, D) = \alpha(B, A)$, all of which contradict Property 2 of GKM graphs. Thus there are no complete octaplex subgraphs.

We then consider the case where ABC, BCD, CDE, ABE , and ADE are all GKM subgraphs. Suppose ACD is not a GKM subgraph. Then if $\{\alpha(A, C), \alpha(A, D)\}$ do not span the lattice the complete graph on A, B, C , and D must be a complete octaplex subgraph. We have just shown that this is impossible, so $\{\alpha(A, C), \alpha(A, D)\}$ span. Then since $\alpha(D, B) = \alpha(C, B) \bmod \alpha(C, D)$ and $\alpha(C, E) = \alpha(D, E) \bmod \alpha(C, D)$ we must have $\alpha(A, C) = \alpha(A, D) \bmod \alpha(D, C)$, so $\alpha(C, D) = \alpha(C, A) + \alpha(A, D)$. Thus ACD , and by symmetry ABD and ACE are GKM subgraphs. Thus every triangle containing vertex A is a GKM subgraph, and thus a graph is simplicial.

Now consider the case where one outer triangle, say ABC , is not a GKM subgraph. Then since $\alpha(A, B) = \alpha(C, B) \bmod \alpha(A, C)$, we see that the pair $\{\alpha(B, A), \alpha(B, C)\}$ cannot span

\mathbb{Z}^2 . Thus one of $ABCD$, $ABCE$ or $ABCDE$ must be a GKM subgraph.

Suppose $ABCD$ is a subgraph. Then we must have either $\alpha(B, A) = \alpha(C, D)$ or $\alpha(B, C) = \alpha(A, D)$.

First suppose $\alpha(B, A) = \alpha(C, D)$. Then, since $\{\alpha(A, B), \alpha(B, C)\}$ cannot span the lattice, $\{\alpha(C, A), \alpha(C, D)\}$ must span. If $\{\alpha(A, C), \alpha(A, D)\}$ span then since

$$\det \begin{bmatrix} \alpha(A, D) \\ \alpha(A, C) \end{bmatrix} = \det \begin{bmatrix} \alpha(C, A) \\ \alpha(C, D) \end{bmatrix} = \det \begin{bmatrix} \alpha(D, C) \\ \alpha(A, C) \end{bmatrix} = 1$$

we see that

$$\det \begin{bmatrix} \alpha(A, D) - \alpha(C, D) \\ \alpha(A, C) \end{bmatrix} = 0$$

and thus $\alpha(A, D) = \alpha(C, D) \pmod{\alpha(A, C)}$.

But then $\alpha(A, D) = \alpha(B, C)$. Thus we may reduce to the case where $\alpha(A, D) = \alpha(B, C)$.

Then $\{\alpha(A, C), \alpha(A, D)\}$ span the lattice. Since $\alpha(A, C) = \alpha(D, B) \pmod{\alpha(A, D)}$, we see that $\{\alpha(D, B), \alpha(D, A)\}$ must span the lattice. Since $\alpha(B, C) = \alpha(A, D)$ we see that $\{\alpha(B, C), \alpha(B, D)\}$ span the lattice, and since $\alpha(B, C) = \alpha(D, C) \pmod{\alpha(B, D)}$ we see that $\{\alpha(D, B), \alpha(D, C)\}$ span the lattice.

Suppose that $\{\alpha(B, A), \alpha(B, D)\}$ does not span the lattice. Then we have $\alpha(C, D) = \alpha(B, A) + n\alpha(C, B)$ where $n \neq 0$ and thus

$$\begin{aligned} \det \begin{bmatrix} \alpha(C, A) \\ \alpha(C, D) \end{bmatrix} &= \det \begin{bmatrix} \alpha(C, A) \\ \alpha(B, A) + n\alpha(C, B) \end{bmatrix} \\ &= \det \begin{bmatrix} \alpha(A, C) \\ \alpha(A, B) \end{bmatrix} + \det \begin{bmatrix} \alpha(A, C) \\ n\alpha(C, B) \end{bmatrix} = 1 + n, \end{aligned}$$

so $\{\alpha(C, A), \alpha(C, D)\}$ does not span the lattice. Then the quadrilaterals $ABDE$ and $ACDE$ are GKM subgraphs. Both of these subgraphs contain $\{\alpha(E, A), \alpha(E, D)\}$ so must lie in the same sublattice. Thus there is a GKM subgraph with degree at least 3 whose weights lie on this sublattice. But no such subgraph can exist, so $\{\alpha(B, A), \alpha(B, D)\}$ and $\{\alpha(C, A), \alpha(C, D)\}$ must both span the lattice. Thus $n = 0$.

Then we have $\alpha(A, B) = \alpha(D, C)$ and $\alpha(B, C) = \alpha(A, D)$. Now since $\{\alpha(A, B), \alpha(A, C)\}$ and $\{\alpha(B, A), \alpha(B, D)\}$ both span we see that $\alpha(A, C) = \alpha(B, D) \pmod{\alpha(A, B)}$, and thus

$\alpha(C, A) = \alpha(D, B) \pmod{\alpha(D, C)}$. We also have

$$\begin{aligned}\alpha(A, C) &= \alpha(D, B) \pmod{\alpha(A, D)} \\ \alpha(C, A) &= \alpha(B, D) \pmod{\alpha(B, C)} \\ \alpha(A, B) &= \alpha(C, B) \pmod{\alpha(A, C)} \\ \alpha(A, D) &= \alpha(C, D) \pmod{\alpha(A, C)} \\ \alpha(B, A) &= \alpha(D, A) \pmod{\alpha(B, D)} \\ \alpha(B, C) &= \alpha(D, C) \pmod{\alpha(B, D)}.\end{aligned}$$

Thus the complete graph on A, B, C , and D must be a GKM subgraph, and since it is not simplicial it must be a complete octaplex subgraph. But this is impossible. Thus $ABCD$, and by symmetry $ACDE$ cannot be GKM subgraphs.

Next suppose $ABCDE$ is a GKM subgraph. Then $\{\alpha(A, B), \alpha(A, E)\}$ do not span the lattice and $\alpha(A, E) = \alpha(B, C) \pmod{\alpha(A, B)}$.

If ABE is a GKM subgraph then $\alpha(A, E) = \alpha(B, E) \pmod{\alpha(A, B)}$ so $\{\alpha(B, A), \alpha(B, E)\}$ span the same sublattice as $\{\alpha(B, A), \alpha(B, C)\}$. Then there is some subgraph containing $ABCDE$ all of whose weights lie in this sublattice. Since this subgraph must be regular it can only be the complete graph, which is impossible. Similarly ADE , BCD , and CDE are not simplicial subgraphs.

If $ABDE$ is a GKM subgraph then $\alpha(A, E) = \alpha(B, D) \pmod{\alpha(A, B)}$. Thus the set $\{\alpha(B, A), \alpha(B, D)\}$ span the same sublattice as $\{\alpha(B, A), \alpha(B, C)\}$. Then there is some regular subgraph containing $ABCDE$ all of whose weights lie in this sublattice. Since the only such graph is the complete graph, we see that $ABDE$ cannot be a GKM subgraph. By symmetry, $ACDE$, $ABCD$, and $BCDE$ cannot be GKM subgraphs.

Since $\{\alpha(A, C), \alpha(A, B)\}$ and $\{\alpha(B, A), \alpha(B, E)\}$ both span the lattice we must have $\alpha(A, C) = \alpha(B, E) \pmod{\alpha(A, B)}$. Since $\alpha(A, E) = \alpha(B, C) \pmod{\alpha(A, B)}$ we see that $\alpha(A, D) = \alpha(B, D) \pmod{\alpha(A, B)}$. Thus if ACD is a GKM subgraph then we see that $\alpha(A, C) = \alpha(D, C) \pmod{\alpha(A, D)}$, and thus $\alpha(A, B) = \alpha(D, B) \pmod{\alpha(A, D)}$. But then, by Lemma 2, we see that ABD is a GKM subgraph. Similarly, ACE , BCE , and BDE are GKM subgraphs.

Now $\alpha(A, B) + \alpha(B, C) = m\alpha(A, C)$ and $\alpha(A, E) + \alpha(E, D) = l\alpha(A, D)$. But

$$\begin{aligned}\alpha(A, C) &= \alpha(A, E) + \alpha(E, C) = \alpha(A, E) + \alpha(E, B) + \alpha(B, C) \\ &= \alpha(A, E) + \alpha(E, D) + \alpha(D, B) + \alpha(B, C) \\ &= \alpha(A, E) + \alpha(E, D) + \alpha(D, A) + \alpha(A, B) + \alpha(B, C) \\ &= (l-1)\alpha(A, D) + m\alpha(A, B).\end{aligned}$$

Thus $(m-1)\alpha(A, C) = (l-1)\alpha(A, D)$ so $m = l = 1$, which contradicts the fact that ABC and ADE are not GKM subgraphs.

Thus ACE is not a GKM subgraphs. In particular $\{\alpha(A, E), \alpha(A, C)\}$ must span the lattice.

Since $\{\alpha(B, A), \alpha(B, C)\}$ and $\{\alpha(A, B), \alpha(A, E)\}$ span the same sublattice we see that $\alpha(A, E) = n\alpha(A, B) + m\alpha(B, C)$ for some n and m . But $\{\alpha(A, E), \alpha(A, C)\}$ span \mathbb{Z}^2 so

$$\begin{aligned}1 = \det \begin{bmatrix} \alpha(A, E) \\ \alpha(A, C) \end{bmatrix} &= \det \begin{bmatrix} n\alpha(A, B) + m\alpha(B, C) \\ \alpha(A, C) \end{bmatrix} \\ &= n \det \begin{bmatrix} \alpha(A, B) \\ \alpha(A, C) \end{bmatrix} + m \det \begin{bmatrix} \alpha(B, C) \\ \alpha(A, C) \end{bmatrix} = -n + m\end{aligned}$$

so $\alpha(A, E) = n\alpha(A, B) + (1+n)\alpha(B, C)$. Also

$$\det \begin{bmatrix} \alpha(A, E) \\ \alpha(A, B) \end{bmatrix} = \det \begin{bmatrix} n\alpha(A, B) + (1+n)\alpha(B, C) \\ \alpha(A, B) \end{bmatrix} = (1+n) \det \begin{bmatrix} \alpha(B, C) \\ \alpha(A, B) \end{bmatrix}$$

so $n = 0$ and $\alpha(A, E) = \alpha(B, C)$. But then by symmetry $\alpha(E, D) = \alpha(A, B)$ and $\alpha(D, C) = \alpha(E, A) = \alpha(C, B)$, which is impossible.

Thus ABC , and by symmetry any outer triangle, must be a simplicial subgraph, and the whole graph must then be simplicial.

Thus every $(4, 2)$ GKM subgraph is simplicial. \square

3.3 The Higher Dimensional Case

In this section we prove that for $k > \frac{1}{2}n + \frac{1}{2}$ and $k = \frac{n}{2}$ the only possible GKM graphs are simplicial, and for $k = \frac{1}{2}n + \frac{1}{2}$ the only possible GKM graphs are simplicial graphs

and complete octaplexes. Since these graphs determine the integral cohomology and Chern classes, Theorem 1 follows.

Proposition 4 *Every complete GKM graph with $n + 1$ vertices whose weights span $(\mathbb{R}^k)^*$, where $k > \frac{1}{2}n + \frac{1}{2}$, is simplicial. Every complete GKM graph with $n + 1$ vertices whose weights span $(\mathbb{R}^k)^*$ and where $k = \frac{1}{2}n + \frac{1}{2}$ is either simplicial or a complete octaplex graph.*

Proof. We will prove the two statements separately. We first consider the case where $k > \frac{1}{2}n + \frac{1}{2}$.

We proceed by induction. The base case is $k = 2$. We proved that the $(2, 2)$ graph has to be simplicial in Example 1. Now suppose this statement is true for all (n, k) graphs with $k > \frac{1}{2}n + \frac{1}{2}$, and consider a $(n + 2, k + 1)$ graph.

We first assume that all but one vertex lies on a k -dimensional subspace V . Then lemma 4 shows that the graph must be simplicial.

Now suppose that no k -dimensional subspace has more than $n + 1$ vertices. We can then apply the induction hypothesis. Any subgraph lying on a k -dimensional subspace will be complete with at most $n + 1$ vertices and thus will be simplicial since $n < 2k - 1$. Any three vertices must lie in some such subgraph and thus, with the edges connecting them, form a GKM subgraph. Thus the total graph is simplicial.

The second case is where $k = \frac{1}{2}n + \frac{1}{2}$. The base case is now $(3, 2)$, which, by lemma 2, can be either simplicial or a complete octaplex graph. We suppose by induction that the theorem holds for $(n, \frac{n}{2} + \frac{1}{2}, n + 1)$, and consider a $(n + 2, \frac{n}{2} + \frac{3}{2}, n + 3)$ graph. If all but one vertex lies on a k -dimensional subspace V then, by lemma 4, the graph is simplicial. Similarly, if every k -dimensional subgraph is simplicial then the total graph must be simplicial.

This leaves us with the case where at least one k -dimensional GKM subgraph Γ is a complete octaplex subgraph. Let E and F be the two vertices not in this subgraph, and let $ABCD$ be a four vertex complete octaplex subgraph of the k -dimensional complete octaplex subgraph, where the vertices are labeled A, B, C , and D clockwise around the graph¹. We will say that an edge XY of $ABCD$ is **orthodox** if the complete graph on X, Y, E , and F is a complete octaplex graph. An edge that is not orthodox will be called **heterodox**. If an edge XY is heterodox then the complete graph on X, Y, E , and F is simplicial.

If two adjacent edges are orthodox then their corresponding quadrilaterals must have three vertices in common: E, F , and the vertex common to both edges. But their corresponding

¹In this and the next proposition we will always label vertices in sequence.

quadrilaterals must lie in different planes, so can only have one common edge, and thus only two vertices in common. Thus we cannot have adjacent orthodox edges.

Suppose one of the exterior edges, say AB , is orthodox. Then AC is heterodox. The three dimensional space containing $ABCD$ and E and F can be split into two half-spaces by the plane containing $ABCD$. Either E and F both lie in the same half-space, or they lie in different half-spaces

Suppose E and F both in the same half-space. Then either $\alpha(A, B) = 2\alpha(A, E) + \alpha(E, B)$ or $\alpha(A, B) = \alpha(A, F) + 2\alpha(F, B)$. Without loss of generality we may assume the former. Then

$$\begin{aligned} 2\alpha(A, E) + 2\alpha(E, C) &= 2\alpha(A, C) \\ &= \alpha(A, B) + \alpha(B, C) = \alpha(A, B) + \alpha(B, E) + \alpha(E, C) \\ &= \alpha(A, E) + \alpha(E, C). \end{aligned}$$

Thus $\alpha(E, C) = 0$, which is impossible.

Now suppose that E and F lie on opposite sides of $ABCD$. Then we have $2\alpha(A, B) = \alpha(A, E) + \alpha(E, B)$. We still have $2\alpha(A, E) + \alpha(E, C) = \alpha(A, B) + \alpha(B, E)$. But then

$$2\alpha(A, E) + \alpha(E, C) = \alpha(A, B) + 2\alpha(B, A) + \alpha(A, E)$$

and thus $\alpha(A, C) = \alpha(A, E) + \alpha(E, C) = \alpha(B, A)$ which contradicts Property 2 of Theorem GKM graphs.

Thus only the interior edges can be orthodox. Suppose one interior edge is heterodox, say AC . Then, since all the exterior vertices are heterodox, $\alpha(A, C) = \alpha(A, E) + \alpha(E, C) = \alpha(A, B) + \alpha(B, E) + \alpha(E, B) + \alpha(B, C) = \alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$, so $\alpha(A, C) = 0$, which is impossible.

Thus both interior edges must be orthodox. Thus the complete graph on A, B, C, D, E , and F is a complete octaplex subgraph. Since the choice of $ABCD$ was arbitrary we see that the whole graph is a complete octaplex graph. \square

Proposition 5 *Every complete GKM graph on $2k + 1$ vertices in $(\mathbb{R}^k)^*$ is simplicial.*

Proof. The proof is by induction. Proposition 3 is the base case of the induction. We assume the theorem is true for $(2k, k)$, and consider a $(2k + 2, k + 1)$ graph.

Let Γ be a k -dimensional subgraph. If this subgraph contains all but one vertex then Lemma 4 shows that the graph is simplicial. If it contains all but two vertices then it is a

$(2k, k)$ graph, and thus simplicial by induction. If it contains all but three vertices, it is a $(2k - 1, k)$ graph, and thus may be simplicial or a complete octaplex graph by Proposition 4. If there are more vertices lying off of this subgraph, then by Proposition 4 it must be simplicial.

Thus the total graph must be simplicial unless there is a k -dimensional complete octaplex subgraph. There will then be three vertices outside this subgraph. The remainder of the proof will consist of possible ways to configure these three points, and calculations showing that these positions do not give us a GKM graph.

Let $ABCD$ be a four vertex complete octaplex subgraph of the complete octaplex subgraph. Let E, F , and G be the three vertices which do not lie on the larger complete octaplex subgraph. An edge XY of $ABCD$ will be called **orthodox** if the complete graph on X, Y and two of three vertices off the subgraph is a complete octaplex graph. An edge that is not orthodox will be called **heterodox**.

Suppose we have a triangle of heterodox edges, say AB, BC , and AC . Then

$$\begin{aligned} \alpha(A, C) &= \alpha(A, E) + \alpha(E, C) \\ &= \alpha(A, B) + \alpha(B, E) + \alpha(E, B) + \alpha(B, C) \\ &= \alpha(A, B) + \alpha(B, C) = 2\alpha(A, C) \end{aligned}$$

which implies $\alpha(A, C) = 0$. Since this is impossible, we see that there cannot be a triangle of heterodox edges.

Suppose we have a triangle, say AB, BC , and AC , with one orthodox edge. Suppose AB is the orthodox edge. Let E , and F be the other two vertices of the complete octaplex subgraph. Then ABG, ACG , and BCG are simplicial triangles, and $\alpha(A, C) = \alpha(A, G) + \alpha(G, C) = \alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$, which is impossible. Similarly, if BC or AC is the orthodox edge and E and F are the other vertices of the complete octaplex subgraph, then ABG, ACG , and BCG are simplicial triangles and $\alpha(A, C) = 2\alpha(A, C)$.

Thus each triangle must have at least two orthodox edges. We note that this implies we cannot have adjacent heterodox edges.

Thus, if one of the interior edges of $ABCD$ is heterodox, then all the exterior edges must be orthodox. Suppose AB is an exterior edge of a complete octaplex graph whose other vertices are E and F . If CD is an edge of a complete octaplex graph containing G then C, D, G and one of E or F must be on the same plane. But since EF is parallel to AB , and thus to CD , we see that C, D, E, F and G all lie on the same plane, so must be the

vertices of simplicial $(4, 2)$ graph. Thus CD must be an exterior edge of a complete octaplex graph containing E and F . Then BCE and ADF must be parallel triangles, so cannot both be coplanar with G , and thus one of BC or AD must be heterodox. Thus each of the four exterior edges must be an interior edge of a complete octaplex subgraph containing two of E, F , or G . Without loss of generality, we can suppose E is on one side of $ABCD$ and F and G are on the other. But then each of the four complete octaplex subgraphs must have EF or EG as an interior edge. But then EF and EG can each intersect one edge of $ABCD$. Thus only two exterior edges can be orthodox.

Suppose AB is heterodox. Then AD and BC are both orthodox.

Suppose AD is an exterior edge of a complete octaplex graph containing the vertices E and F . Then, since EF is parallel to AD , it is also parallel to BC . As we saw in the preceding paragraph, if BC is an edge of a complete octaplex graph containing G then B, C, E, F , and G are coplanar and thus make a simplicial $(4, 2)$ graph. Thus $BCEF$ is the other complete octaplex graph, and BC is an exterior edge. Then $2\alpha(A, F) = \alpha(A, E) + \alpha(E, F) = \alpha(A, B) + \alpha(B, E) + \alpha(E, F) = \alpha(A, B) + 2\alpha(B, F)$, and thus $2\alpha(A, F) + 2\alpha(F, B) = \alpha(A, B)$. But $2\alpha(A, F) + 2\alpha(F, B) = 2\alpha(A, B)$ so $\alpha(A, B) = 2\alpha(A, B)$ which is impossible.

Now suppose AD is the interior edge of a complete octaplex graph. Then by the previous paragraph we see that BC must also be the interior edge of a complete octaplex graph. These graphs must have a common vertex E . But then neither F nor G can lie on the plane containing AEC , and thus AEC is simplicial. Thus $\alpha(A, C) = \alpha(A, E) + \alpha(E, C) = \alpha(A, B) + \alpha(B, E) + \alpha(E, C) = \alpha(A, B) + 2\alpha(B, C) = 2\alpha(A, C) + \alpha(B, C)$. Thus $\alpha(A, C) + \alpha(B, C) = 0$, which contradicts Property 2 of Theorem GKM graphs, which says $\alpha(C, A)$ and $\alpha(C, B)$ must be linearly independent.

Thus all configurations of orthodox and heterodox edges are impossible, so all k -dimensional subgraphs of the total graph are simplicial, and thus the whole graph is simplicial. \square

3.4 Other Results

The GKM category consists of manifolds whose isotropy submanifolds of $k - 1$ dimensional subgroups are discrete points or spheres. In the case of minimal GKM manifolds if we impose certain conditions on the possible isotropy submanifolds of $k - 2$ dimensional subgroups then we can describe all possible cohomology rings and Chern classes. Imposing conditions on isotropy submanifolds of $k - 2$ dimensional subgroups is tantamount to imposing conditions on two-dimensional subgraphs. Theorem 2 follows immediately from the following

proposition.

Proposition 6 *If every two dimensional subgraph of a complete GKM graph is simplicial or a complete octaplex graph, then the graph is simplicial or a complete octaplex graph.*

Proof. The case where every two dimensional subgraph is simplicial is trivial.

We now consider the case where at least one two dimensional subgraph is a complete octaplex. There are two possible triangular subgraphs in this case. One possibility is simplicial triangular subgraphs. The other is a subgraph ABC where $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$. If the subgraph is simplicial then we say that all three edges are **short**. In the other case we say that AB and BC are short and AC is **long**. We first want to show that an edge that is long on one triangle cannot be short on another. We can then partition the graph into short and long edges.

Suppose some edge is long for one complete octaplex subgraph but short for another. We then have two subgraphs ABC and ACD where $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$ and $\alpha(A, C) + \alpha(C, D) = 2\alpha(A, D)$. We note that complete subgraph on A, B, C , and D cannot lie in a plane. The triangle ABD can be simplicial or have weights satisfying one of the following three equations: $\alpha(A, B) + \alpha(B, D) = 2\alpha(A, D)$, $\alpha(A, B) + 2\alpha(B, D) = \alpha(A, D)$, $2\alpha(A, B) + \alpha(B, D) = \alpha(A, D)$. Similarly, there are four possibilities for the triangle BCD . Thus there are sixteen possible configurations to check.

We will not write out the calculations for all sixteen configurations. We will give two examples; the other fourteen cases will all be similar to one of these.

First suppose ABD and BCD are both simplicial. Then

$$\begin{aligned}\alpha(A, D) &= \alpha(A, B) + \alpha(B, D) = \alpha(A, B) + \alpha(B, C) + \alpha(C, D) \\ &= 2\alpha(A, C) + \alpha(C, D) = \alpha(A, C) + 2\alpha(A, D)\end{aligned}$$

and thus $0 = \alpha(A, C) + \alpha(A, D)$, which contradicts Property 2 of Theorem GKM graphs, since $\alpha(A, C)$ and $\alpha(A, D)$ are linearly independent.

Next suppose that $\alpha(A, B) + \alpha(B, D) = 2\alpha(A, D)$ and $\alpha(B, C) + \alpha(C, D) = 2\alpha(B, D)$. Then

$$\begin{aligned}4\alpha(A, D) &= 2\alpha(A, B) + 2\alpha(B, D) = 2\alpha(A, B) + \alpha(B, C) + \alpha(C, D) \\ &= \alpha(A, B) + 2\alpha(A, C) + \alpha(C, D) = \alpha(A, B) + \alpha(A, C) + 2\alpha(A, D)\end{aligned}$$

and thus $2\alpha(A, D) = \alpha(A, B) + \alpha(A, C)$ so $\alpha(B, D) = \alpha(A, C)$. But since AC and BD are parallel, $A, B, C,$ and D must lie on a plane, which is impossible.

Now suppose that one edge is long for a complete octaplex subgraph but is also part of a simplicial subgraph. We then have two subgraphs ABC and ACD where $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$ and $\alpha(A, C) + \alpha(C, D) = \alpha(A, D)$. Once again the complete subgraph on $A, B, C,$ and D cannot lie on a plane. Once again there are sixteen possibilities for the subgraphs ABD and BCD . If $\alpha(A, D) + \alpha(D, B) = 2\alpha(A, B)$ or $\alpha(B, D) + \alpha(D, C) = 2\alpha(B, C)$ then we can reduce to the case where one edge can be both long and short in complete octaplex subgraphs.. If ABD and BCD are both simplicial then $2\alpha(A, C) = \alpha(A, B) + \alpha(B, C) = \alpha(A, D) + \alpha(D, C) = \alpha(A, C)$. If ABD is simplicial and $\alpha(B, C) + \alpha(C, D) = 2\alpha(B, D)$ then $2\alpha(B, D) = 2\alpha(B, A) + 2\alpha(A, D)$ so

$$\begin{aligned}\alpha(B, C) + \alpha(C, D) &= 2\alpha(B, A) + 2\alpha(A, D) \\ \alpha(C, D) + 2\alpha(D, A) &= \alpha(C, B) + 2\alpha(B, A) \\ \alpha(C, A) + \alpha(D, A) &= 2\alpha(C, A) + \alpha(B, A) \\ \alpha(D, A) &= \alpha(C, A) + \alpha(B, A) \\ \alpha(D, B) &= \alpha(C, A)\end{aligned}$$

and thus $A, B, C,$ and D lie on a plane. Both of these are impossible. Most of the remaining cases can be eliminated by similar calculations.

The only case that remains is that of $\alpha(A, B) + \alpha(B, D) = 2\alpha(A, D)$ and $\alpha(C, B) + \alpha(B, D) = 2\alpha(C, D)$. Each of ABC, ABD and BCD must be part of complete octaplex subgraphs, the fourth vertices of which will be denoted E, F, G respectively. Since $\alpha(E, A) + \alpha(A, B) = 2\alpha(E, B)$ and $\alpha(B, A) + \alpha(A, F) = 2\alpha(B, F)$ we see that if BEF is not a simplicial subgraph then one of BE or BF would be both long and short in complete octaplex subgraphs. Thus BEF is simplicial. Since $\alpha(E, A) + \alpha(A, F) = 2\alpha(E, B) + \alpha(B, A) + \alpha(A, B) + 2\alpha(B, F) = 2\alpha(E, B) + 2\alpha(B, F) = 2\alpha(E, F)$, we see that AEF is not simplicial. Thus there is a vertex H such that the complete graph on $A, E, F,$ and H is a complete octaplex subgraph parallel to the complete graph on B, C, D, G . Similarly, there will be complete octaplex subgraphs parallel to the complete graphs on A, B, D, F and A, B, C, E . Thus we have a parallelepiped whose faces are complete octaplex subgraphs.

Consider the complete graph on $A, B, C,$ and G . We know already that $\alpha(A, B) + \alpha(B, C) = 2\alpha(A, C)$ and $\alpha(B, C) + \alpha(C, G) = 2\alpha(B, G)$. If $\alpha(A, G) + \alpha(G, C) = 2\alpha(A, C)$ then ACG is part of a complete octaplex subgraph. Call the fourth vertex J . Then $\alpha(J, A) =$

$\alpha(C, G) = \alpha(A, F)$, which contradicts Property 2 of Theorem GKM graphs. If $\alpha(A, G) + 2\alpha(G, C) = \alpha(A, C)$ or $2\alpha(A, G) + \alpha(G, C) = \alpha(A, C)$ then AC is both long and short in a complete octaplex subgraph. Thus ACG must be simplicial. Similarly, ABG must be simplicial. But then

$$\begin{aligned} 2\alpha(A, C) &= 2\alpha(A, G) + 2\alpha(G, C) = 2\alpha(A, B) + 2\alpha(B, G) + 2\alpha(G, C) \\ &= 2\alpha(A, B) + \alpha(B, C) + \alpha(G, C) = \alpha(A, B) + 2\alpha(A, C) + \alpha(G, C) \end{aligned}$$

so $\alpha(A, B) = \alpha(C, G)$ and A, B, C , and G lie on a plane, which is impossible.

Thus we have a well-defined partition between long and short edges on the complete graph. This also shows that no two long edges can be adjacent.

We next need to show that every subgraph with four vertices, four short edges and two long edges is a complete octaplex subgraph. Suppose not. Consider the complete graph on A, B, C , and D where AB, BC, CD , and AD are short and AC and BD are long. Then ABC must be part of a complete octaplex subgraph so there is long edge adjacent to B . But since long edges cannot be adjacent this has to be BD . Thus the complete graph on A, B, C , and D is a complete octaplex subgraph.

It remains to show that every subgraph with four vertices, five long edges, and one short edge is part of a complete octaplex graph with 6 vertices. Suppose A, B, C , and D are the vertices and AD is the long edge. Then there are vertices E and F such that the complete graphs on A, B, D, E and A, C, D, F are complete octaplex subgraphs. But then the complete graph on B, C, E , and F has four short and two long edges (BE and CF) so is also a complete octaplex subgraph, and thus the complete graph on A, B, C, D, E , and F is a complete octaplex subgraph.

Now suppose we have a complete graph all of whose two dimensional subgraphs are simplicial or complete octaplex subgraphs, and at least one two dimensional subgraph is a complete octaplex. Let A be a vertex off of the subgraph. Any two-dimensional subspace containing this vertex and a long edge BC must contain a fourth vertex, and the complete graph on these four vertices must be a complete octaplex. From this we see that the whole graph must be a complete octaplex graph. \square

Another case we can classify completely is highly symmetric GKM graphs. We define a highly symmetric GKM graph to be a complete GKM graph whose vertices are the vertices of a lattice regular polytope in $(\mathbb{R}^k)^*$. Using the classification of lattice regular polytopes we can completely classify highly symmetric GKM graphs. Theorem 3 follows.

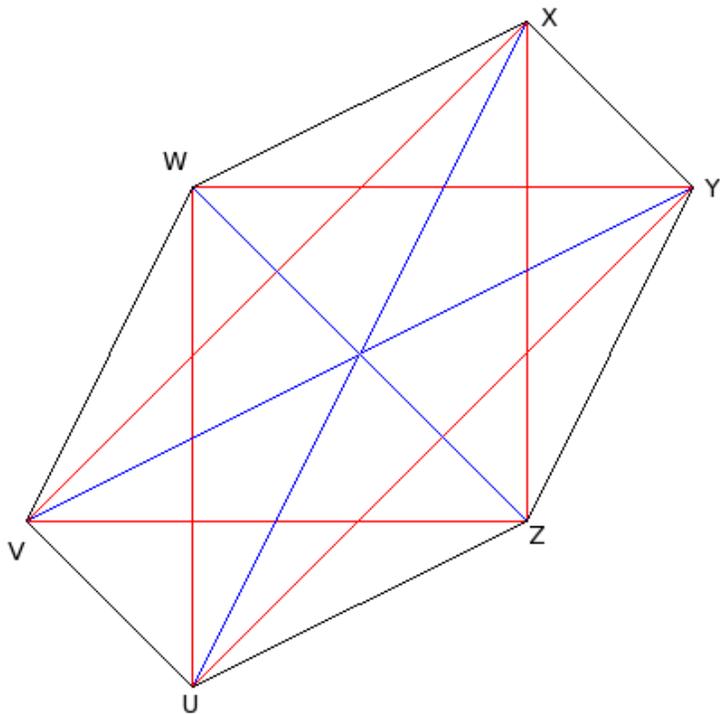
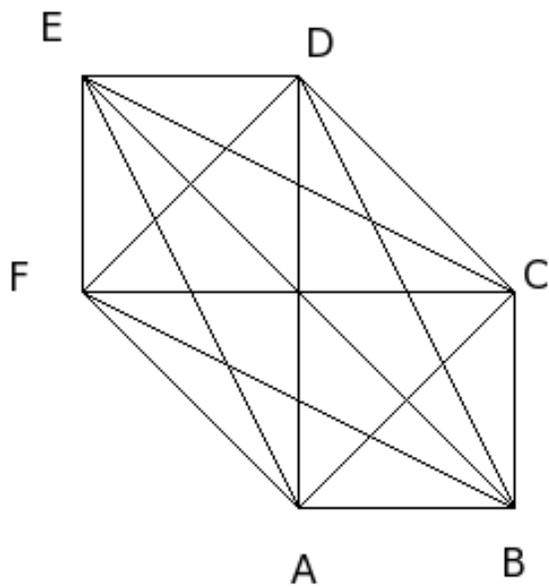


Figure 3.3: The two possible non-simplicial highly symmetric GKM graphs with six fixed points.

Proposition 7 *There are two families of highly symmetric GKM graphs and one exceptional highly symmetric GKM graph. The two families are the simplicial graphs and the complete octaplex graphs. The exceptional graph is the GKM graph corresponding to G_2/P .*

Proof. Lattice regular polytopes are completely classified in [K06] and [MR09].

In dimension three and dimension five and higher, the only lattice regular polytopes are simplicies, hypercubes, and octaplexes. In dimension four, the lattice regular polytopes are simplicies, hypercubes, octaplex, and 24-cells. (The last is a lattice regular four-dimensional polytope with no corresponding regular polytope in any other dimension, see [C69].) Each two dimensional subgraph must have at most four vertices. Thus by Proposition 2 all the two dimensional subgraphs are simplicial or complete octaplexes so by Proposition 6 the total graphs must be simplicial or complete octaplexes.

This leaves us with the case of dimension two. By [K06] and [MR09] there are, up to scale and $SL(2, \mathbb{Z})$ transformation, six lattice regular polygons: two lattice regular triangles, two lattice regular quadrilaterals, and two regular hexagons.

The triangles must be simplicial and the quadrilaterals must be simplicial or complete octaplexes since these are the only complete GKM graphs with at most four vertices.

Up to scale and $SL(2, \mathbb{Z})$ transformation there are two lattice regular hexagons. The first has vertices $A = (0, -1)$, $B = (1, -1)$, $C = (1, 0)$, $D = (0, 1)$, $E = (-1, 1)$, and $F = (-1, 0)$.

We know that $\alpha(A, B) = (n, 0)$ for some positive integer n , $\alpha(A, C) = (m, m)$ for some positive integer m , and $\alpha(A, D) = (0, r)$ for some positive integer r .

If n and m are not both 1 then

$$\det \begin{bmatrix} \alpha(A, B) \\ \alpha(A, C) \end{bmatrix} = \det \begin{bmatrix} n & 0 \\ m & m \end{bmatrix} = nm \neq 1.$$

Thus there must be some GKM subgraph containing $\alpha(A, B)$ and $\alpha(A, C)$. This subgraph has at most five vertices so by Propositions 2 and 3 must be simplicial or a complete octaplex. But the edge parallel to AB and adjacent to BC and the edge parallel to BC and adjacent to AB are not adjacent to a common vertex so this subgraph must be simplicial. Thus $\alpha(B, A) + \alpha(A, C) = \alpha(B, C)$. But $\alpha(B, A) + \alpha(A, C) = (m - n, m)$ and $\alpha(B, C) = (0, l)$ for some $l > 0$. Thus $l = m$ and $l = n$. A similar argument shows that $\alpha(E, D) = (n, 0)$, $\alpha(F, E) = (0, n)$, $\alpha(D, C) = \alpha(F, A) = (n, -n)$, $\alpha(F, D) = (n, n)$, $\alpha(E, C) = \alpha(F, B) = (2n, -n)$ and $\alpha(E, A) = \alpha(D, B) = (n, -2n)$.

Since $\alpha(A, D) = \alpha(C, F) \pmod{\alpha(A, C)}$ we see that $\alpha(C, F) = (-r, 0)$ and n divides r (similarly $\alpha(B, E) = (-r, r)$.) Thus we must have $n = 1$ in order to have the weights at each

vertex span the lattice. If $\alpha(A, B) = \alpha(D, B) \pmod{\alpha(A, D)}$ then $(0, 2) = 0 \pmod{(0, r)}$ and if $\alpha(A, B) = \alpha(D, C) \pmod{\alpha(A, D)}$ then $(0, 1) = 0 \pmod{\alpha(0, r)}$. Thus $r = 1$ or $r = 2$. If $r = 1$ the graph is a complete octaplex; if $r = 2$ the graph is simplicial.

The second lattice regular hexagon has, up to scale and $SL(2, \mathbb{Z})$ transformation, vertices at $U = (-1, -2)$, $V = (-2, -1)$, $W = (-1, 1)$, $X = (1, 2)$, $Y = (2, 1)$, and $Z = (1, -1)$.

We know that $\alpha(U, V) = (-n, n)$, $\alpha(U, W) = (0, m)$, and $\alpha(U, X) = (r, 2r)$ for some positive integers n , m and r .

If n and m are not both 1 then

$$\det \begin{bmatrix} \alpha(U, W) \\ \alpha(U, V) \end{bmatrix} = \det \begin{bmatrix} 0 & m \\ -n & n \end{bmatrix} = nm \neq 1.$$

Thus there must be some GKM subgraph containing $\alpha(U, V)$ and $\alpha(U, W)$. This subgraph has at most five vertices so by Propositions 2 and 3 must be simplicial or a complete octaplex. But the edge parallel to UV adjacent to VW and the edge parallel to VW and adjacent to UV are not adjacent to a common vertex so this subgraph must be simplicial. Thus $\alpha(V, U) + \alpha(U, W) = \alpha(V, W)$. But $\alpha(V, U) + \alpha(U, W) = (n, m - n)$ and $\alpha(V, W) = (l, 2l)$ for some $l > 0$. Thus $l = n$ and $m = 3n$. A similar argument shows that $\alpha(X, Y) = (n, -n)$, $\alpha(Z, Y) = (n, 2n)$, $\alpha(U, Z) = \alpha(W, X) = (2n, n)$, $\alpha(Z, X) = (0, 3n)$, $\alpha(U, Z) = \alpha(W, Y) = (3n, 0)$, and $\alpha(V, X) = \alpha(U, Y) = (3n, 3n)$.

Since $\alpha(U, X) = \alpha(W, Z) \pmod{\alpha(U, W)}$ we see that $\alpha(W, Z) = (r, -r)$ (and similarly $\alpha(V, Y) = (2r, r)$) and $3n$ divides $3r$. Thus n divides r , so $\alpha(U, X)$ lies in the proper sublattice spanned by $\alpha(U, V)$, $\alpha(U, W)$, $\alpha(U, Y)$, $\alpha(U, Z)$. Thus the weights at U do not span the lattice, so this is not a GKM graph.

Thus $m = n = 1$, and by symmetry $\alpha(X, Y) = (1, -1)$, $\alpha(Z, Y) = (1, 2)$, $\alpha(U, Z) = \alpha(W, X) = (2, 1)$, $\alpha(Z, X) = (0, 1)$, $\alpha(U, Z) = \alpha(W, Y) = (1, 0)$, and $\alpha(V, X) = \alpha(U, Y) = (1, 1)$. The same argument as above shows us that $\alpha(U, X) = (r, 2r)$ and $\alpha(V, Y) = (2r, r)$. Then since $\alpha(U, W) \neq \alpha(X, W) \pmod{\alpha(U, X)}$ (since $(2, 2) \neq 0 \pmod{(r, 2r)}$) we see that $\alpha(U, W) = \alpha(X, V) \pmod{\alpha(U, X)}$. But then $(1, 2) = 0 \pmod{(r, 2r)}$ so $r = 1$. This is the GKM graph of G_2/P . \square

Corollary 1 *The only minimal GKM graphs of coadjoint orbits of Lie groups are the simplicial graphs, the complete octaplex graphs, and the GKM graph corresponding to G_2/P .*

Proof. The GKM graphs of coadjoint orbits are lattice regular. \square

3.5 Beyond Petrie

We can also say something about the case where $\chi = n + 2$.

If we start with a complete octaplex graph and remove the interior edges then we get an incomplete octaplex graph. These graphs are the GKM graphs of oriented Grassmannians of even dimensional space. Our last proposition shows that these when $k = \frac{n+2}{2}$ and $\chi = n + 2$ these are the only possible GKM graphs. Theorem 4 follows.

Proposition 8 *Every degree n GKM graph with $n + 2$ ($n \geq 4$) vertices whose weights span $\mathbb{R}^{\frac{n+2}{2}}$ is a $G_+(2, n + 2)$ graph.*

Proof. Consider a $(4, 3, 6)$ graph. This must have a two-dimensional subgraph. Since the graph is not simplicial, there must be at least one such subgraph that is not simplicial. Thus it must have at least four vertices, and must contain at least two non-adjacent vertices.

If the subgraph contained five vertices, then the sixth vertex would be adjacent to four of the five vertices. Let A be the vertex off the subgraph and B the vertex on the subgraph not adjacent to A . Then B is adjacent to all four other vertices on the subgraph, while any other vertex of the subgraph is adjacent to only three. Thus the subgraph is not regular, so the subgraph cannot have five vertices.

Thus the subgraph must contain four vertices, each with two adjacent edges, so must be the graph of a Hirzebruch surface. We can label the vertices of the graph A, B, C, D , with A, D , and B, C the nonadjacent pairs, such that $\alpha(A, B) = \alpha(C, D)$ and $\alpha(B, D) = n\alpha(A, B) + \alpha(A, C)$ for some $n \geq 0$. If E is one of the vertices off the subspace, then each of A, B, C , and D must be adjacent to E . Thus E must be a vertex of a simplicial subgraph with each of the adjacent pairs on the subgraph. Thus

$$\begin{aligned} \alpha(B, D) &= \alpha(B, E) + \alpha(E, D) = \alpha(B, A) + \alpha(A, E) + \alpha(E, D) \\ &= \alpha(B, A) + \alpha(A, E) + \alpha(E, C) + \alpha(C, D) \\ &= \alpha(B, A) + \alpha(A, C) + \alpha(C, D) = \alpha(A, C) \end{aligned}$$

since $\alpha(B, A) = \alpha(D, C)$. Since there is no edge connecting A and D we see that the subgraph containing A, D , and E must contain a fourth vertex F , and that F must also be the fourth vertex of the subgraph containing B, C , and E . By the same argument as above $AEDF$ and $BECE$ must each have two pairs of parallel weights. Thus the graph is an incomplete octaplex.

Since $\alpha(E, A) + \alpha(A, F) = \alpha(E, B) + \alpha(B, F)$ and $\alpha(A, F) = \alpha(E, D)$ and $\alpha(B, F) = \alpha(E, C)$ we see that $\alpha(E, A) + \alpha(E, D) - \alpha(E, B) = \alpha(E, C)$. Thus, in order to have all four weights at the vertex E span the lattice, we must have that $\{\alpha(E, A), \alpha(E, B), \alpha(E, D)\}$ span the lattice. Thus we may take $\alpha(E, A) = (1, 0, 0)$, $\alpha(E, B) = (0, 1, 0)$, $\alpha(E, D) = (0, 0, 1)$, and thus $\alpha(E, C) = (1, -1, 1)$.

This graph cannot be the projection of a $(4, 4, 6)$ GKM graph. Suppose there were such a graph. Then it would have to have a three-dimensional subgraph with fewer vertices. But this would have to be a $(3, 3, 4)$ GKM graph, and thus simplicial. Since every set of four vertices would have to lie in a three-dimensional subgraph, we see that the whole graph must be simplicial. But this is impossible since the graph is not complete. Thus there are no $(4, 4, 6)$ GKM graphs.

The statement we will now induct on is that the only $(n, \frac{n+2}{2}, n+2)$ GKM graph is the $G_+(2, n+2)$ graph and there are no $(n, k, n+2)$ GKM graphs with $k > \frac{n+2}{2}$.

Suppose we have a $(n, \frac{n+2}{2}, n+2)$ graph with $n > 4$. There must be a $\frac{n}{2}$ -dimensional subgraph that is not minimal, and thus has two vertices which are not adjacent. Both of these vertices must be adjacent to all the other vertices of the subgraph, so each vertex of the subgraph must be adjacent all but one other vertex. Thus we must have an even number of vertices. If there are two vertices not lying on the subgraph then the subgraph is a $(n-2, \frac{n}{2}, n)$ graph. By induction it must be a $G_+(2, 2n-4)$ subgraph. If there are $2l$ vertices not lying on the subgraph ($l > 1$) then the subgraph is a $(n-2l, \frac{n}{2}, n-2l+2)$ graph, which by induction cannot exist. Thus there must be 2 vertices lying off the subgraph and the subgraph is an incomplete octaplex graph.

Let A and B be the two vertices lying off of the subgraph, and let C and D be two nonadjacent vertices on the subgraph. Then A , C , and D must be vertices of some two-dimensional subgraph, and, since C and D are not adjacent, this subgraph must contain at least one other vertex, which can only be B . Thus A and B must lie on opposite sides of the subgraph. Then each vertex on the subgraph must be adjacent to both A and B , and if E , and F are two adjacent vertices on the subgraph, then AEF , and BEF are triangular subgraphs, so $\alpha(E, A) + \alpha(A, F) = \alpha(E, F)$. Thus the whole graph is a incomplete octaplex graph.

If there were a $(n, k, n+2)$ graph with $k > \frac{n+2}{2}$, then this would have a $k-1$ dimensional subgraph. This subgraph would have to be a $(n-2l, k-1, n-2l+2)$ graph for some positive l , which is impossible by induction since $k-1 > \frac{n}{2} > \frac{n-2l}{2}$. Thus there are no $(n, k, n+2)$ GKM graphs with $k > \frac{n+2}{2}$. \square

CHAPTER 4

CLASSIFICATION OF COMPLEXITY 1 GKM MANIFOLDS AND GRAPHS.

We say that a GKM manifold is minimal if it has the smallest possible number of fixed points (one more than half the dimension of the manifold.) The corresponding GKM graphs are complete graphs. When the torus action is sufficiently large we can determine all the possible sets of isotropy weights of minimal GKM manifolds. It is natural to try to answer the same question for GKM manifolds with more fixed points. In this paper we look at six dimensional GKM manifolds with six fixed points (five fixed points is impossible) acted upon by a two-dimensional torus. Theorem 5 follows immediately from Propositions 9, 10, 11, and 12 the construction of the manifolds in section 4.2 and the cohomology computations in section 4.3.

4.1 Classification of Graphs

In this section we completely classify all possible three valent GKM graphs with six vertices and weights spanning \mathbb{Z}^2 . These are the $(3, 2, 6)$ GKM graphs.

Proposition 9 *All of the $(3, 2, 6)$ graphs with three exterior vertices are the GKM graphs of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ acted upon by two-dimensional subtorus of \mathbb{T}^3 acting by the standard Hamiltonian action.*

Proof. Any graph with three exterior vertices must have all three exterior vertices adjacent. Then, since the graph is 3-valent, each vertex must be adjacent to one interior vertex. If any interior vertex is adjacent to two exterior vertices then there must be an interior vertex not adjacent to any exterior vertex. But then we would have a vertex with three adjacent edges but only two adjacent vertices. Similarly, if one interior vertex were adjacent to all three exterior vertices, then the other two interior vertices could only be adjacent to each other. Thus each interior vertex must be adjacent to one exterior vertices, and then all the interior vertices must be adjacent to each other.

We label the exterior vertices A , B , and C and we label the interior vertices D , E , F so that A and D , B and E , and C and F are adjacent. Since B and E lie in one half-plane bordered by the line containing A and D and the convex hull of the weights at D must be \mathbb{R}^2 we see that C and F lie in the other half-plane. Thus $\alpha(A, B) = \alpha(D, E) \pmod{\alpha(A, D)}$. Similarly $\alpha(B, A) = \alpha(E, D) \pmod{\alpha(B, E)}$. Since $\alpha(A, D)$ cannot equal $\alpha(B, E)$ (otherwise C could not be adjacent to both A and B) we see that $\alpha(A, B) = \alpha(D, E)$. By the same reasoning $\alpha(A, C) = \alpha(D, F)$ and $\alpha(B, C) = \alpha(E, F)$.

Since A and B lie in one half plane bordered by the line containing D and E , we see that, since the convex hull of the weights at both D and E must be \mathbb{R}^2 , C and F lie in the other. Thus $\alpha(D, F) = \alpha(E, F) \pmod{\alpha(D, E)}$. Similarly, $\alpha(D, E) = \alpha(F, E) \pmod{\alpha(D, F)}$. Thus $\alpha(D, E) + \alpha(E, F) = \alpha(D, F)$ so DEF is a GKM subgraph. Then ABC is also a GKM subgraph.

Since $\alpha(A, D) = \alpha(B, E) \pmod{\alpha(A, B)}$ and $\alpha(D, A) = \alpha(E, B) \pmod{\alpha(D, E)}$ we see that $ABED$ is the GKM graph of a Hirzebruch surface. Thus $\alpha(B, E) = \alpha(A, D) + n\alpha(A, B)$ for some $n \in \mathbb{Z}$. Similarly $ACFD$ and $BCFE$ are Hirzebruch surfaces. Thus $\alpha(C, F) = \alpha(A, D) + m\alpha(A, C)$ for some m . But since $\alpha(B, E) = \alpha(C, F) \pmod{\alpha(B, C)}$ we see that $n\alpha(B, A) = m\alpha(C, A) \pmod{\alpha(B, C)}$, and thus $n = m$. Thus $ABED$, $ACFD$, and by symmetry $BCFE$, are all copies of the GKM graph of the n th Hirzebruch surface.

The graph is then a projection of the graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ acted upon by the Hamiltonian \mathbb{T}^3 action. This graph has vertices A' , B' , C' , D' , E' , F' with $\alpha(A', B') = \alpha(D', E') = (1, 0, 0)$, $\alpha(A', C') = \alpha(D', F') = (0, 1, 0)$, $\alpha(B', C') = \alpha(E', F') = (-1, 1, 0)$, $\alpha(A', D') = (0, 0, 1)$, $\alpha(B', E') = (-n, 0, 1)$, $\alpha(C', F') = (0, -n, 1)$. The projection takes $\alpha(A', B') \rightarrow \alpha(A, B)$, $\alpha(A', C') \rightarrow \alpha(A, C)$, etc.

Thus every graph with three exterior edges is the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C})$ for some n acted upon by a two dimensional subtorus of \mathbb{T}^3 acting by the standard Hamiltonian action. \square

Proposition 10 *There are two classes of $(3, 2, 6)$ GKM graphs with four exterior vertices. The first class is GKM graphs of $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ acted upon by a two-dimensional subtorus of \mathbb{T}^3 acting by the standard Hamiltonian action. The second class is the twisted GKM graphs.*

Proof. Any graph with four exterior vertices must have two interior vertices. Each of the four exterior vertices must be adjacent to at least two other exterior vertices, and each of the interior vertices must be adjacent to at least two exterior vertices. If an interior vertex

were adjacent to three exterior vertices then the other interior vertex could only be adjacent to one exterior vertex. Thus the interior vertices are adjacent to each other each interior vertex is adjacent to two of the exterior vertices.

Label the exterior vertices $A, B, C,$ and D so that $AB, AC, BD,$ and CD are the exterior edges. Label the interior vertices E and F . Now E and F are either both adjacent to exterior vertices that are adjacent to each other or E and F are both adjacent to exterior vertices that are not adjacent to each other. Since the convex hull of the weights of the edges adjacent to E and the convex hull of the weights of the edges adjacent to F must span \mathbb{R}^2 we see that we cannot have two pairs of edges intersect. We may suppose that C and D are positioned so that EC and FD do not intersect. Then $\alpha(C, D) = \alpha(E, F) \pmod{\alpha(C, E)}$ and $\alpha(D, C) = \alpha(F, E) \pmod{\alpha(D, F)}$. Thus either $\alpha(C, D) = \alpha(E, F)$ or $\alpha(C, E)$ and $\alpha(D, F)$ are parallel. But since $\alpha(E, C) = \alpha(F, D) \pmod{\alpha(E, F)}$ we see that if $\alpha(C, E)$ and $\alpha(D, F)$ are parallel, then they must be equal. Thus $\alpha(C, E) = \alpha(D, F) \pmod{\alpha(C, D)}$ and $CDEF$ is GKM subgraph.

If the other two edges adjacent to E and F do not intersect then A is adjacent to E and B is adjacent to F . We also see that $ABFE$ must be a GKM subgraph for the same reason that $CDEF$ is a GKM subgraph. If the other two edges adjacent to E and F do intersect then A is adjacent to F and B is adjacent to E .

Case I

Suppose A is adjacent to E and B is adjacent to F . Then $ABFE$ and $CDFE$ are both GKM subgraphs. Then consider the half planes whose borders contain the vertices A and E . By Property 5 of GKM graphs, one of these half-planes contains the vertex C , and the other contains the vertices B and F . Thus $\alpha(A, C) = \alpha(E, C) \pmod{\alpha(A, E)}$. Similarly, $\alpha(C, A) = \alpha(E, A) \pmod{\alpha(C, E)}$. Thus ACE is a GKM subgraph. By the same reasoning, so is BDF . If $\alpha(A, C) = \alpha(B, D)$ then $\alpha(A, E) \neq \alpha(B, F)$, and thus $\alpha(E, C) \neq \alpha(F, D)$, and if either $\alpha(A, E) = \alpha(B, F)$ or $\alpha(E, C) = \alpha(F, D)$ then $\alpha(A, C) \neq \alpha(B, D)$. Since $\alpha(A, B) = \alpha(C, D) \pmod{\alpha(A, C)}$ and $\alpha(B, A) = \alpha(D, C) \pmod{\alpha(B, D)}$ we see that either $\alpha(A, C)$ and $\alpha(B, D)$ are parallel or $\alpha(A, B) = \alpha(C, D)$. But since $\alpha(A, B) = \alpha(E, F) \pmod{\alpha(A, E)}$ and $\alpha(B, A) = \alpha(F, E) \pmod{\alpha(B, F)}$, and $\alpha(C, D) = \alpha(E, F) \pmod{\alpha(C, E)}$ and $\alpha(D, C) = \alpha(F, E) \pmod{\alpha(D, F)}$, and at most one pair of $\alpha(A, C)$ and $\alpha(B, D)$, $\alpha(A, E)$ and $\alpha(B, F)$, and $\alpha(C, E)$ and $\alpha(D, F)$ can be parallel we must have $\alpha(A, B) = \alpha(C, D) = \alpha(E, F)$.

Now $ABFE$ and $CDEF$ must be the GKM graphs of Hirzebruch surfaces. Since $\alpha(A, B) = \alpha(C, D)$ and $\alpha(A, C) = \alpha(B, D) \pmod{\alpha(A, B)}$ we see that $ABDC$ is also a GKM graph of a Hirzebruch surface. If $\alpha(B, F) = \alpha(A, E) + n\alpha(A, B)$ and $\alpha(B, D) = \alpha(A, C) + m\alpha(A, B)$ then $\alpha(D, F) = \alpha(C, E) + (n - m)\alpha(E, F)$.

Thus we see that the GKM graph is the projection of a GKM graph for $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ acted upon by the standard Hamiltonian \mathbb{T}^3 action. In particular, it is the projection of the the graph with vertices $A', B', C', D', E',$ and F' and weights $\alpha(A', B') = \alpha(C', D') = \alpha(E', F') = (0, 0, 1)$, $\alpha(A', C') = (1, 0, 0)$, $\alpha(A', E') = (0, 1, 0)$, $\alpha(C', E') = (-1, 1, 0)$, $\alpha(B', D') = (1, 0, m)$, $\alpha(B', F') = (0, 1, n)$ and $\alpha(D', F') = (-1, 1, n - m)$.

Case II

Now suppose A is adjacent to F and B is adjacent to E .

If $CDFE$ is not a GKM subgraph then by property 6 of GKM graphs $\{\alpha(C, D), \alpha(C, E)\}$ must span the lattice, as must the three pairs $\{\alpha(D, C), \alpha(D, F)\}$, $\{\alpha(E, C), \alpha(E, F)\}$, and $\{\alpha(F, D), \alpha(F, E)\}$. But then

$$\alpha(C, E) = \alpha(D, F) \pmod{\alpha(C, D)}$$

$$\alpha(C, D) = \alpha(E, F) \pmod{\alpha(C, E)}$$

$$\alpha(D, C) = \alpha(F, E) \pmod{\alpha(D, F)}$$

and

$$\alpha(E, C) = \alpha(F, D) \pmod{\alpha(E, F)}$$

and thus $CDFE$ is a GKM subgraph. If $\alpha(C, E) = \alpha(D, F)$ then, since the weights of the edges adjacent to E and F must both span \mathbb{R}^2 we see that AF and BE cannot intersect. Thus $\alpha(C, D) = \alpha(E, F)$.

Since $A, C,$ and F cannot be part of a GKM subgraph by property 6 $\{\alpha(A, C), \alpha(A, F)\}$ must span the lattice. Thus we can make an $SL(2, \mathbb{Z})$ transformation of all the weights so that $\alpha(A, C) = (1, 0)$ and $\alpha(A, F) = (0, 1)$. Then since $\alpha(A, C) = \alpha(F, E) \pmod{\alpha(A, F)}$ we see that $\alpha(F, E) = \alpha(D, C) = (1, -x)$ for some $x > 0$.

If $ABEC$ is not a GKM subgraph then $\{\alpha(C, A), \alpha(C, E)\}$ must span the lattice so $\alpha(C, E) = (-y, 1)$ for some $y > 0$. Also $\{\alpha(E, B), \alpha(E, C)\}$ must span the lattice, as must $\{\alpha(B, A), \alpha(B, E)\}$. If $\{\alpha(A, C), \alpha(A, B)\}$ spanned the lattice then $ABEC$ would be a GKM

subgraph, so $\{\alpha(A, C), \alpha(A, B)\}$ cannot span. By property 6, we see that $\{\alpha(E, B), \alpha(E, F)\}$ must span. Then if $\alpha(B, E) = (s, t)$ ($s > 0$) we have

$$\det \begin{bmatrix} \alpha(E, F) \\ \alpha(E, B) \end{bmatrix} = \det \begin{bmatrix} -1 & x \\ -s & -t \end{bmatrix} = t + xs = 1$$

so $\alpha(B, E) = (s, 1 - xs)$ and

$$\det \begin{bmatrix} \alpha(E, B) \\ \alpha(E, C) \end{bmatrix} = \det \begin{bmatrix} -s & sx - 1 \\ y & -1 \end{bmatrix} = s + y - sxy = 1$$

so $x = \frac{s+y-1}{sy}$. But the only way x can be an integer is if one of s or y is 1, and then $x = 1$. If $y = 1$ then $\alpha(E, C) = -\alpha(E, B)$ which contradicts Property 2. Thus $s = 1$ so $\alpha(B, E) = (1, 0)$. But then $\{\alpha(A, C), \alpha(A, B)\}$ span because

$$\det \begin{bmatrix} \alpha(A, C) \\ \alpha(A, B) \end{bmatrix} = \det \begin{bmatrix} \alpha(B, A) \\ \alpha(B, E) \end{bmatrix} = 1$$

since $\alpha(A, C) = \alpha(B, E)$. Thus $ABEC$ is a GKM subgraph, and by symmetry so is $ABDF$.

At least one of $\alpha(C, E)$ or $\alpha(F, D)$ cannot be equal to $\alpha(A, B)$. Thus at least one pair of $\alpha(A, C)$ and $\alpha(B, E)$ or $\alpha(A, F)$ and $\alpha(B, D)$ must be equal. Without loss of generality we may assume $\alpha(B, E) = \alpha(A, C) = (1, 0)$. But then since $B, E,$ and F cannot be part of a GKM subgraph, $\alpha(E, F)$ and $\alpha(E, B)$ must span the lattice. Thus $\alpha(E, F) = (-1, 1)$, and thus $\alpha(C, D) = (-1, 1)$.

Since $ABDF$ is a GKM subgraph we have $\alpha(B, E) = \alpha(D, C) \pmod{\alpha(B, D)}$ so $(0, 1) = 0 \pmod{\alpha(B, D)}$. Thus $\alpha(B, D) = (0, 1)$.

Finally, we can take $\alpha(A, B) = (-a, b)$, $\alpha(C, E) = (a - c, b)$ and $\alpha(D, F) = \alpha(a, b - c)$. This is a twisted GKM graph. \square

Proposition 11 *There are two classes of $(3, 2, 6)$ GKM graphs with five exterior vertices. The first class is GKM graphs of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{CP}^1$ acted upon by a two-dimensional subtorus of \mathbb{T}^3 acting by the standard Hamiltonian action. The second class is GKM graphs of $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{CP}^2$ acted upon by a two-dimensional subtorus of \mathbb{T}^3 acting by the standard Hamiltonian action.*

Proof. If there are five exterior vertices then there must be one interior vertex, and it must be adjacent to three of the exterior vertices. Label the exterior vertices $A, B, C, D,$ and

E so that AB , BC , CD , DE , and EA are the exterior edges. Label the interior vertex F . Without loss of generality we may suppose that F is adjacent to A , C , and E , and that B and D are adjacent.

Consider the half-planes bordered by the line containing E and F . One of these half-planes must contain A and, since the convex hull of the weights at F must be \mathbb{R}^2 , the other half-plane must contain C . Thus $\alpha(E, A) = \alpha(F, A) \bmod \alpha(E, F)$, and similarly $\alpha(A, E) = \alpha(F, E) \bmod \alpha(A, F)$. Thus AEF must be a GKM subgraph.

One of the half-planes bordered by the line containing B and D must contain the vertex C and the other half-plane must contain the vertices A and E . Thus $\alpha(B, C) = \alpha(D, C) \bmod \alpha(B, D)$.

Suppose BCD is not a GKM subgraph. Then by property 6, we see that the pairs $\{\alpha(B, C), \alpha(B, D)\}$ and $\{\alpha(D, B), \alpha(D, C)\}$ must both span the lattice. We can then take an $SL(2, \mathbb{Z})$ transformation to get $\alpha(B, C) = (1, 1)$ and $\alpha(B, D) = (0, 1)$. Then $\alpha(D, C) = (1, -x)$ for some positive integer $x > 1$. Now $\alpha(F, C) = (a, b)$ for some integers $a > 0$ and b . Since BCD is not a GKM subgraph we see that $\alpha(B, D) = \alpha(C, F) \bmod \alpha(B, C)$ so $\alpha(F, C) = (b+1, b)$. But $\alpha(C, F) = \alpha(D, B) \bmod \alpha(C, D)$ so $(b+1, b-1) = 0 \bmod (1, -x)$. Then $x = \frac{1-b}{b+1}$ so $b = 0$ and $x = 1$ (if $b = 1$ and $x = 0$ then BCD is a GKM subgraph.) Thus $\alpha(F, C) = (1, 0)$ and $\alpha(D, C) = (1, -1)$.

If $ABCF$ were a GKM subgraph then we would have $\alpha(B, A) = \alpha(C, F) \bmod \alpha(B, C)$ and thus $\alpha(B, D) = \alpha(C, D) \bmod \alpha(B, C)$. Thus $ABCF$ cannot be a subgraph, so by property 6 we see that $\{\alpha(F, A), \alpha(F, C)\}$ must span the lattice. Thus $\alpha(A, F) = (c, 1)$ for some integer c . But then $\alpha(A, F) = \alpha(B, C) \bmod \alpha(A, B)$ or $\alpha(A, F) = \alpha(B, D) \bmod \alpha(A, B)$. In either case $\alpha(A, B) = (r, 0)$ for some r . But since $\{\alpha(A, B), \alpha(A, F)\}$ must also span the lattice (since $ABCF$ is not a GKM subgraph) we see that $\alpha(A, B) = (1, 0)$. But then $ABCF$ is a GKM subgraph.

Thus BCD must be a GKM subgraph.

Suppose AEF and BCD have the same weights (i.e $\alpha(A, E) = \alpha(B, D)$, $\alpha(A, F) = \alpha(B, C)$, and $\alpha(E, F) = \alpha(D, C)$.) Then we have $\alpha(A, F) = \alpha(B, C) \bmod \alpha(A, B)$ and $\alpha(F, A) = \alpha(C, B) \bmod \alpha(C, F)$, $\alpha(A, B) = \alpha(F, C) \bmod \alpha(A, F)$, and $\alpha(B, A) = \alpha(C, F) \bmod \alpha(B, C)$. Thus $ABCF$ is a GKM subgraph. Similarly, $ABDE$ and $CDEF$ are also GKM subgraphs. Thus we see that $\alpha(F, C) = \alpha(A, B) + n\alpha(A, F)$, $\alpha(E, D) = \alpha(A, B) + m\alpha(A, E)$, and $\alpha(F, C) = \alpha(E, D) + k\alpha(E, F)$. for some n , m , and k . But $\alpha(F, C) - \alpha(E, D) = n\alpha(A, F) - m\alpha(A, E) = k\alpha(E, F)$ so $k = m = n$. Thus $ABCF$, $ABDE$, and $CDEF$ are all copies of the GKM graph of the n th Hirzebruch surface.

The graph is then a projection of the graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ with the standard \mathbb{T}^3 action. This graph has vertices A', B', C', D', E', F' with $\alpha(A', F') = \alpha(B', C') = (1, 0, 0)$, $\alpha(A', E') = \alpha(B', D') = (0, 1, 0)$, $\alpha(F', E') = \alpha(C', D') = (-1, 1, 0)$, $\alpha(A', B') = (0, 0, 1)$, $\alpha(F', C') = (-n, 0, 1)$, $\alpha(E', D') = (0, -n, 1)$. The projection takes $\alpha(A', F') \rightarrow \alpha(A, F)$, $\alpha(A', E') \rightarrow \alpha(A, E)$, etc.

Now suppose AEF and BCD do not have the same weights. Then since $\alpha(A, B) = \alpha(F, C) \pmod{\alpha(A, F)}$ and $\alpha(B, A) = \alpha(C, F) \pmod{\alpha(B, C)}$, $\alpha(A, B) = \alpha(E, D) \pmod{\alpha(A, E)}$ and $\alpha(B, A) = \alpha(D, E) \pmod{\alpha(B, D)}$, and $\alpha(F, C) = \alpha(E, D) \pmod{\alpha(E, F)}$ and $\alpha(C, F) = \alpha(D, E) \pmod{\alpha(C, D)}$, and at most one pair of $\alpha(A, F)$ and $\alpha(B, C)$, $\alpha(A, E)$ and $\alpha(B, D)$, and $\alpha(E, F)$ and $\alpha(D, C)$ can be equal, we see that $\alpha(A, B) = \alpha(F, C) = \alpha(E, D)$.

Then since $\alpha(F, A) = \alpha(C, B) \pmod{\alpha(C, F)}$ we see that $\alpha(A, F) = \alpha(B, C) \pmod{\alpha(A, B)}$. Thus $ABCF$ is a GKM subgraph. Similarly, $ABDE$ and $CDEF$ are GKM subgraphs. Thus $\alpha(B, C) = \alpha(A, F) + n\alpha(A, B)$, $\alpha(B, D) = \alpha(A, E) + m\alpha(A, B)$, and thus $\alpha(C, D) = \alpha(F, E) + (m - n)\alpha(A, B)$.

Thus we see that the GKM graph is the projection of a GKM graph for a $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ acted upon by the standard Hamiltonian \mathbb{T}^3 action. In particular, it is the projection of the the graph with vertices $A', B', C', D', E',$ and F' and weights $\alpha(A', B') = \alpha(E', D') = \alpha(F', C') = (0, 0, 1)$, $\alpha(A', F') = (1, 0, 0)$, $\alpha(A', E') = (0, 1, 0)$, $\alpha(F', E') = (-1, 1, 0)$, $\alpha(B', C') = (1, 0, n)$, $\alpha(B', D') = (0, 1, m)$ and $\alpha(C', D') = (-1, 1, m - n)$. \square

Proposition 12 *There are two classes of $(3, 2, 6)$ GKM graphs and one exceptional $(3, 2, 6)$ GKM graph with six exterior vertices. The two classes are GKM graphs of $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ and the flag-like GKM graphs. The exceptional GKM graph is the the graph of the blow-up of $G_+(2, 5)$ along one of it's large isotropy spheres.*

Proof. Label the vertices $A, B, C, D, E,$ and F so that $AB, BC, CD, DE, EF,$ and FA are the exterior edges. We will consider A and D, B and $E,$ and C and F to be pairs of antipodal vertices.

Suppose no pair of antipodal vertices are adjacent. Then without loss of generality A must be adjacent to $C,$ and B must be adjacent to $D.$ But then there must be two edges connecting E and $F,$ which contradicts Property 2 of GKM graphs. Thus at least one pair of antipodal vertices must be adjacent. Without loss of generality suppose they are A and $D.$ We then have two cases to consider. The first is when B and $F,$ and C and E are adjacent pairs; the second is when B and $E,$ and C and F are adjacent pairs.

Case I

Suppose B and F , and C and E are adjacent pairs.

Suppose that ABF is not a GKM subgraph. We see that BA and BF cannot be part of some GKM subgraph so, by Property 6, $\{\alpha(B, A), \alpha(B, F)\}$, and similarly $\{\alpha(F, A), \alpha(F, B)\}$, must span the lattice. Then we can take an $SL(2, \mathbb{Z})$ transformation so that $\alpha(A, B) = (-1, 1)$ and $\alpha(B, F) = (1, 0)$, and then $\alpha(A, F) = (a, 1)$ for some integer a . Since ABF is not a GKM subgraph we see that $\alpha(B, F) = \alpha(A, D) \pmod{\alpha(A, B)}$ and $\alpha(A, D) = \alpha(F, B) \pmod{\alpha(A, F)}$. If $\alpha(A, D) = (b, c)$ then $(b, c) = (1, 0) \pmod{(1, -1)}$ so $c = 1 - b$. Then $(b, 1 - b) = (-1, 0) \pmod{(a, 1)}$ so $a = \frac{b+1}{1-b}$. But then $b = 0$ and $a = 1$, so $\alpha(A, F) = (1, 1)$ and $\alpha(A, D) = (0, 1)$. (If $b = -1$ then $a = 0$ and ABF would be a GKM subgraph.)

Now $\alpha(A, F) = \alpha(D, E) \pmod{\alpha(A, D)}$, so $\{\alpha(D, A), \alpha(D, E)\}$ span the lattice. Similarly, $\{\alpha(D, A), \alpha(D, C)\}$ span the lattice. Thus $\alpha(C, D) = (1, d)$ and $\alpha(D, E) = (1, -e)$ for some $d, e > 0$.

If CDE is a GKM subgraph then $\alpha(D, A) = \alpha(C, B) \pmod{\alpha(C, D)}$ and thus the pair $\{\alpha(C, B), \alpha(C, D)\}$ spans the lattice. Let $\alpha(B, C) = (s, t)$ where $t > 0$. Then $\alpha(B, C) = \alpha(A, F) \pmod{\alpha(A, B)}$ so $(s - 1, t - 1) = 0 \pmod{(1, -1)}$ and thus $s = 2 - t$. Then $\alpha(B, C) = (2 - t, t)$. Since $\alpha(D, A) = \alpha(C, B) \pmod{\alpha(C, D)}$ we see that $(2 - t, t - 1) = 0 \pmod{(1, d)}$ so $d = \frac{t-1}{2-t}$. Thus $t = 1$ and $d = 0$. But by symmetry $e = 0$ as well, so $\alpha(C, D) = \alpha(D, E)$. This contradicts Property 2 of GKM graphs. Thus CDE is not a GKM subgraph.

Thus $\{\alpha(C, E), \alpha(C, D)\}$ must span the lattice, as must $\{\alpha(E, C), \alpha(E, D)\}$. Then if $\alpha(C, E) = (p, q)$ ($p > 0$) we see that

$$\det \begin{bmatrix} \alpha(C, E) \\ \alpha(C, D) \end{bmatrix} = \det \begin{bmatrix} p & q \\ 1 & d \end{bmatrix} = pd - q = 1$$

so $q = pd - 1$ and $\alpha(C, E) = (p, pd - 1)$. Then

$$\det \begin{bmatrix} \alpha(E, D) \\ \alpha(E, C) \end{bmatrix} = \det \begin{bmatrix} -1 & e \\ -p & 1 - pd \end{bmatrix} = pd - 1 + pe = 1$$

so $p(d + e) = 2$. The only solution that does not make CDE a GKM subgraph is $d = e = p = 1$. Thus $\alpha(C, D) = (1, 1)$, $\alpha(D, E) = (1, -1)$ and $\alpha(C, E) = (1, 0)$.

Then $\alpha(B, F) = \alpha(C, E) \pmod{\alpha(B, C)}$ so $\alpha(B, A) = \alpha(C, D) \pmod{\alpha(B, C)}$. Thus $(0, 2) = 0 \pmod{\alpha(B, C)}$. Then $\alpha(B, C) = (0, 1)$ or $\alpha(B, C) = (0, 2)$. Since $\alpha(B, C) = \alpha(A, F) \pmod{\alpha(A, B)}$ we see that $\alpha(B, C) = (0, 2)$. By symmetry $\alpha(F, E) = (0, 2)$. We will show in

the next section that this is the GKM graph of the blow-up of $G_+(2, 5)$ along one of its large isotropy spheres.

Now suppose ABF is a GKM subgraph. From the last case we see that if either of the two triangles is not a GKM subgraph then the other is not a GKM subgraph either. Thus CDE must also be a GKM subgraph. Then we have $\alpha(A, D) = \alpha(B, C) \bmod \alpha(A, B)$ and $\alpha(D, A) = \alpha(C, B) \bmod \alpha(C, D)$, $\alpha(A, D) = \alpha(F, E) \bmod \alpha(A, F)$ and $\alpha(D, A) = \alpha(E, F) \bmod \alpha(D, E)$, and $\alpha(B, C) = \alpha(F, E) \bmod \alpha(B, F)$ and $\alpha(C, B) = \alpha(E, F) \bmod \alpha(C, E)$. Since at most one of the pairs $\alpha(A, B)$ and $\alpha(C, D)$, $\alpha(A, F)$ and $\alpha(D, E)$, and $\alpha(B, F)$ and $\alpha(C, E)$ can be equal, we see that $\alpha(A, D) = \alpha(B, C) = \alpha(F, E)$.

Then since $\alpha(A, B) = \alpha(D, C) \bmod \alpha(A, D)$ we have $\alpha(B, A) = \alpha(C, D) \bmod \alpha(B, C)$. Thus $ABCD$ is a GKM subgraph. Similarly, $ADEF$ and $BCEF$ are GKM subgraphs. Thus $\alpha(C, D) = \alpha(B, A) + n\alpha(A, D)$, $\alpha(D, E) = \alpha(A, F) + m\alpha(A, D)$ and $\alpha(C, E) = \alpha(B, F) + (n + m)\alpha(B, C)$.

Thus we see that the GKM graph is the projection of a GKM graph for a $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ with the standard Hamiltonian \mathbb{T}^3 action. In particular, it is the projection of the graph with vertices $A', B', C', D', E',$ and F' and weights $\alpha(A', D') = \alpha(B', C') = \alpha(F', E') = (0, 0, 1)$, $\alpha(A', B') = (1, 0, 0)$, $\alpha(A', F') = (0, 1, 0)$, $\alpha(B', F') = (-1, 1, 0)$, $\alpha(C', D') = (-1, 0, n)$, $\alpha(D', E') = (0, 1, m)$ and $\alpha(C', E') = (-1, 1, m + n)$.

Case II

Now suppose that B is adjacent to E and C is adjacent to F .

Suppose one of the quadrilaterals is not a GKM subgraph. Without loss of generality, we may suppose it to be $CDEF$. Consider the half-planes bordering the line containing C and F . One of these planes must contain E and D . Thus there can be no GKM subgraph containing CF and CD , so $\{\alpha(C, D), \alpha(C, F)\}$ must span the lattice, as must $\{\alpha(F, C), \alpha(F, E)\}$. Thus by an $SL(2, \mathbb{Z})$ transformation we can take $\alpha(F, C) = (1, 0)$, $\alpha(F, E) = (0, 1)$, and $\alpha(C, D) = (-a, 1)$ for some integer a .

Since $CDEF$ is not a GKM subgraph we must have that one of $\{\alpha(D, C), \alpha(D, E)\}$ or $\{\alpha(E, D), \alpha(E, F)\}$ does not span the lattice. Without loss of generality we may suppose $\{\alpha(E, D), \alpha(E, F)\}$ does not span.

If $ADEF$ is not a GKM subgraph then the exterior edges of the graph form a GKM subgraph. Thus $\alpha(D, C) = \alpha(E, F) \bmod \alpha(D, E)$, so $(a, 0) = 0 \bmod \alpha(D, E)$. But then $\alpha(E, D) = (x, 0)$ for some x . Since $ADEF$ is not a GKM subgraph we see that

$\{\alpha(D, A), \alpha(D, E)\}$ must span the lattice. Thus $\alpha(E, D) = (1, 0)$, and $CDEF$ is a GKM subgraph. Thus $ADEF$ is a GKM subgraph. By symmetry $BCDE$ is also a GKM subgraph.

Then $\alpha(E, B) = \alpha(F, C) \pmod{\alpha(F, E)}$ so $\alpha(B, E) = (-1, b)$ for some $b > 0$.

Since $BCDE$ is a GKM subgraph we must have $\alpha(E, D) = \alpha(B, C)$ or $\alpha(B, E) = \alpha(C, D)$.

If $\alpha(E, D) = \alpha(B, C)$ then $\alpha(E, D) = (e, f)$ where $f > 0$ (since B lies below the line containing F and C .) Thus $\alpha(E, D) \neq \alpha(F, A)$ so $\alpha(A, D) = (0, 1)$. But then $\alpha(C, F) = \alpha(D, A) \pmod{\alpha(C, D)}$ so $(1, -1) = 0 \pmod{(-a, 1)}$ and $a = 1$. Since $\alpha(B, E) - \alpha(C, D) = (0, b - 1) = n\alpha(E, D)$ for some n we see that $\alpha(E, D)$ and $\alpha(E, F)$ are linearly dependent, which contradicts property 1 of GKM graphs, or $b = 1$. Thus we may reduce to the case where $\alpha(B, E) = \alpha(C, D)$, in which case $\alpha(B, E) = \alpha(C, D) = (1, -1)$. By symmetry $\alpha(A, D) = (0, 1)$.

Since $\alpha(C, B) = \alpha(F, A) \pmod{\alpha(C, F)}$ we see that $\alpha(A, F) = (r, s)$ and $\alpha(B, C) = (t, s)$ for some r, s, t with $s > 0$. If $\alpha(A, F) = \alpha(B, E) \pmod{\alpha(A, B)}$ and $\alpha(A, D) = \alpha(B, C) \pmod{\alpha(A, B)}$ then $(r + 1, s - 1) = 0 \pmod{\alpha(A, B)}$ and $(t, s - 1) = 0 \pmod{\alpha(A, B)}$ thus $r = t - 1$. But then $\alpha(A, F) \neq \alpha(B, C)$. Since $\alpha(A, B) = \alpha(F, C) \pmod{\alpha(A, F)}$ and $\alpha(B, A) = \alpha(C, F) \pmod{\alpha(B, C)}$ we see that $\alpha(A, B) = \alpha(F, C) = (1, 0)$.

Thus we can reduce to the case where $\alpha(A, F) = \alpha(B, C) \pmod{\alpha(A, B)}$ and $\alpha(A, D) = \alpha(B, E) \pmod{\alpha(A, B)}$. The latter equation gives us $(1, 0) = 0 \pmod{\alpha(A, B)}$ so $\alpha(A, B) = (1, 0)$, and thus $ABCF$ is a GKM subgraph.

We can then take $\alpha(B, C) = (a + c, b)$, $\alpha(E, D) = (a, b + c)$ and $\alpha(A, F) = (-a, b)$, where $a, b > 0$ and $c > -a - b$. This is a flag-like GKM graph.

Now suppose that all the quadrilaterals are GKM subgraphs.

If $\alpha(C, D) = \alpha(F, E)$ then $\alpha(F, E) \neq \alpha(A, D)$. Thus $\alpha(A, F) = \alpha(D, E)$, and similarly $\alpha(A, B) = \alpha(F, E)$. But then $\alpha(A, B) = \alpha(D, C)$ so $\alpha(C, D) = \alpha(D, C)$, which implies $\alpha(C, D) = 0$.

Thus $\alpha(C, F) = \alpha(E, D) = \alpha(A, B)$, $\alpha(A, D) = \alpha(B, C) = \alpha(F, E)$, and $\alpha(B, E) = \alpha(A, F) = \alpha(C, D)$. Since $\alpha(A, B) = \alpha(D, C) \pmod{\alpha(A, D)}$ we see that $\alpha(A, B) = \alpha(D, C) + n\alpha(A, D)$. But $\alpha(A, D) = \alpha(F, C) = \alpha(A, B) \pmod{\alpha(D, C)}$ so $n = 1$. Thus we see that in order for $\{\alpha(A, B), \alpha(A, D), \alpha(A, F)\}$ to span the lattice, we must have $\{\alpha(A, B), \alpha(A, D)\}$ span the lattice. Thus we can take $\alpha(A, B) = (1, 0)$, $\alpha(A, D) = (0, 1)$ and $\alpha(A, F) = (-1, 1)$. This is the GKM graph of the complete flag on \mathbb{C}^3 , which is (obviously) a flag-like GKM graph. \square

4.2 Construction of GKM manifolds.

Each of the GKM graphs in section 3 is the GKM graph of at least one GKM manifold. We have already constructed the manifolds in the cases where the GKM graphs are projections of three-valent GKM graphs with six fixed points and weights spanning \mathbb{Z}^3 (see section 2.) They are the six-dimensional toric manifolds acted upon by a two-dimensional subgroup of the three-dimensional torus that acts on them in the standard way.

Now we construct the manifolds corresponding to the twisted graphs. Assume $\alpha(A, C) = \alpha(B, E) = (1, 0)$, $\alpha(A, F) = \alpha(B, D) = (0, 1)$, $\alpha(C, D) = \alpha(E, F) = (-1, 1)$, $\alpha(A, B) = (-a, b)$, $\alpha(C, E) = (a - c, b)$ and $\alpha(D, F) = (a, b - c)$ and choose the following moment map of the graph:

$$\begin{aligned}\mu(A) &= (a, 0), \quad \mu(B) = (0, b), \quad \mu(C) = (c + 1, 0) \\ \mu(D) &= (0, c + 1), \quad \mu(E) = (a + 1, b), \quad \mu(F) = (a, b + 1).\end{aligned}$$

We will construct a manifold for this graph by gluing together parts of two other GKM manifolds.

The graph of the first manifold is the GKM graph with vertices A', B', C', D', E', F' and weights $\alpha(A', C') = \alpha(B', E') = (1, 0)$, $\alpha(A', F') = \alpha(B', D') = (0, 1)$, $\alpha(C', F') = \alpha(E', D') = (-1, 1)$ and $\alpha(A', B') = \alpha(C', E') = \alpha(F', D') = (-a, c - b)$. The graph of the second is the GKM graph with vertices $A^*, B^*, C^*, D^*, E^*, F^*$ and weights $\alpha(A^*, C^*) = \alpha(B^*, E^*) = (1, 0)$, $\alpha(A^*, F^*) = \alpha(B^*, D^*) = (0, 1)$, $\alpha(C^*, F^*) = \alpha(E^*, D^*) = (-1, 1)$ and $\alpha(A^*, B^*) = (-a, b)$, $\alpha(C^*, E^*) = (a - c, b)$, and $\alpha(D^*, F^*) = (a, c - b - 2a)$.

We define a moment map for each graph. Let μ' be the moment map of the first graph, and μ^* the moment map of the second graph. Then we can take μ' and μ^* so that

$$\begin{aligned}\mu'(A') &= (a, 0), \quad \mu'(B') = (0, b), \quad \mu'(C') = (c + 1, 0) \\ \mu'(D') &= (0, a + b + 1), \quad \mu'(E') = (a + 1, b), \quad \mu'(F') = (a, c + 1 - a).\end{aligned}$$

and

$$\begin{aligned}\mu^*(A^*) &= (a, 2b - c), \quad \mu^*(B^*) = (0, b), \quad \mu^*(C^*) = (a - b + c + 1, 2b - c) \\ \mu^*(D^*) &= (0, c + 1), \quad \mu^*(E^*) = (c + 1 - b, b), \quad \mu^*(F^*) = (a, b + 1).\end{aligned}$$

The first graph is a GKM graph of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^2$ and the second graph is the GKM graph

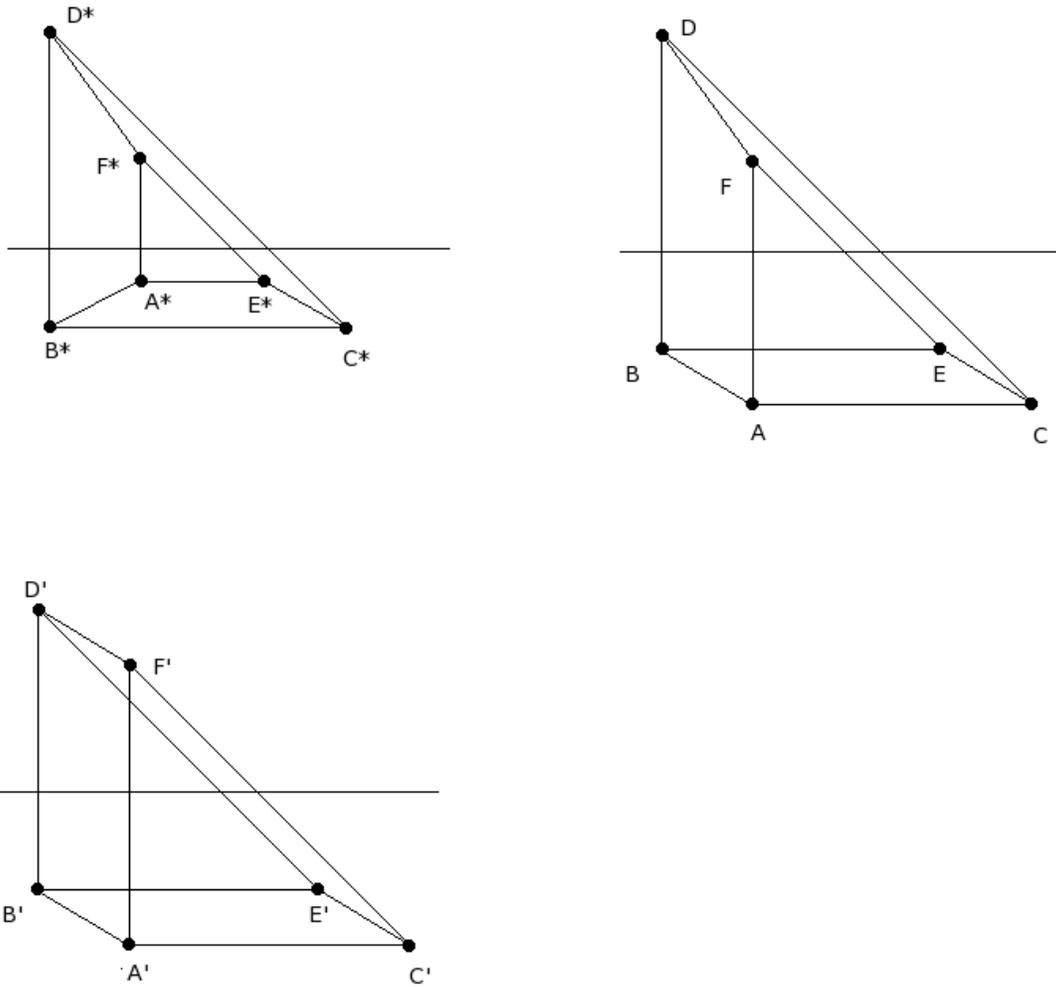


Figure 4.1: Construction the Twisted GKM manifolds.

of a $\mathbb{C}\mathbb{P}^1$ bundle over $\mathbb{C}\mathbb{P}^2$ both acted upon by some copy of \mathbb{T}^2 in \mathbb{T}^3 , and the moment maps of the manifolds, which we will also denote μ' and μ^* respectively, are the convex hulls of the moment maps of the respective graphs. We note that both moment maps have the same intersection with the line $y = b + \frac{1}{2}$. Consider $M' = \mu'^{-1}(\{y = b + \frac{1}{2}\})$ and $M^* = (\mu^*)^{-1}(\{y = b + \frac{1}{2}\})$. The preimages of the points $(0, b + \frac{1}{2})$, $(a, b + \frac{1}{2})$, $(a + \frac{1}{2}, b + \frac{1}{2})$ and $(c - b + \frac{1}{2}, b + \frac{1}{2})$ each contains a cross section (i.e. a circle) of an isotropy sphere. Every other point of M' and M^* is acted upon freely by $S^1 \times S^1$, and every point is acted upon freely by the subgroup $H = \{e\} \times S^1$. Thus M'/H and M^*/H are compact symplectic manifolds with semifree Hamiltonian S^1 actions. The common image of the moment maps of these actions is $[0, c - b + \frac{1}{2}] \times \{b + \frac{1}{2}\}$ and the fixed points map to $(0, b + \frac{1}{2})$, $(a, b + \frac{1}{2})$, $(a + \frac{1}{2}, b + \frac{1}{2})$ and $(c - b + \frac{1}{2}, b + \frac{1}{2})$. Thus by Karshon's classification of 4-dimensional symplectic manifolds with S^1 actions ([K99], see also [AH91] and [Au88]) we see that M'/H and M^*/H are equivariantly symplectomorphic.

Both $M' \rightarrow M'/H$ and $M^* \rightarrow M^*/H$ are S^1 -bundles over S^1 -equivariantly symplectomorphic manifolds. If we can show that their equivariant Chern classes are the same then we can say that there is a \mathbb{T}^2 -equivariant diffeomorphism between M' and M^* . Let F be the fixed point set of M'/H . Then by a theorem of Kirwan ([K84], see also [TW99]) the natural map $H_{S^1}(M'/H, \mathbb{Z}) \rightarrow H_{S^1}(F, \mathbb{Z})$ is injective and the equivariant Chern class of a bundle maps to the equivariant Chern class of its restriction to the fixed points. From the GKM graphs we see that the group $K = S^1 \times \{e\}$ fixes the fiber of the two leftmost fixed points of both M'/H and M^*/H and K acts on the fibers of the two rightmost fixed points of both M'/H and M^*/H in the same manner as H . Thus the equivariant Chern classes of the two circle bundles are equal. There is then an equivariant diffeomorphism between M' and M^* that preserves the pullbacks of the symplectic forms on M'/H and M^*/H . The coisotropic embedding theorem then allows us to find equivariantly symplectomorphic neighborhoods W' and W^* of M' and M^* respectively. Let $U = W' \cup \mu'^{-1}(\mathbb{R} \times (-\infty, b + \frac{1}{2}])$ and $V = W^* \cup (\mu^*)^{-1}(\mathbb{R} \times [b + \frac{1}{2}, \infty))$. Then $M = U \amalg V / \sim$, where \sim is the identification between points in W' and W^* , is the GKM manifold with the desired GKM graph.

The construction of the manifolds corresponding to the flag-like graphs is similar. Assume $\alpha(A, B) = \alpha(F, C) = (1, 0)$, $\alpha(A, D) = \alpha(F, E) = (0, 1)$, $\alpha(B, E) = \alpha(C, D) = (-1, 1)$, and $\alpha(F, A) = (a, -b)$, $\alpha(B, C) = (a + c, b)$ and $\alpha(E, D) = (a, b + c)$ where $a, b > 0$ and

$c > -a - b$. Then choose the moment map μ defined by

$$\begin{aligned}\mu(A) &= (a, 0), \quad \mu(B) = (a + b + 1, 0), \quad \mu(C) = (2a + b + c + 1, b) \\ \mu(D) &= (a, a + 2b + c + 1), \quad \mu(E) = (0, a + b + 1), \quad \mu(F) = (0, b).\end{aligned}$$

We will again construct a GKM manifold for this GKM graph from two other GKM manifolds.

The first graph is the GKM graph with vertices A', B', C', D', E', F' and weights $\alpha(A', B') = \alpha(F', C') = (1, 0)$, $\alpha(A', E') = \alpha(F', D') = (0, 1)$, $\alpha(B', E') = \alpha(C', D') = (-1, 1)$ and $\alpha(A', F') = \alpha(B', C') = \alpha(E', D') = (a, b + c)$. The second is the GKM graph with vertices $A^*, B^*, C^*, D^*, E^*, F^*$ and weights $\alpha(A^*, B^*) = \alpha(F^*, C^*) = (1, 0)$, $\alpha(A^*, E^*) = \alpha(F^*, D^*) = (0, 1)$, $\alpha(B^*, E^*) = \alpha(C^*, D^*) = (-1, 1)$ and $\alpha(A^*, F^*) = (-a, b)$, $\alpha(B^*, C^*) = (a + c, b)$, and $\alpha(D^*, E^*) = (a, -c - b - 2a)$.

We define a moment map for each graph. Let μ' be the moment map of the first graph, and μ^* the moment map of the second graph. Then we can take μ' and μ^* so that

$$\begin{aligned}\mu'(A') &= (0, -c), \quad \mu'(B') = (a + b + c + 1, -c), \quad \mu'(C') = (2a + b + c + 1, b) \\ \mu'(D') &= (a, a + 2b + c + 1), \quad \mu'(E') = (0, a + b + 1), \quad \mu'(F') = (a, b)\end{aligned}$$

and

$$\begin{aligned}\mu^*(A^*) &= (a, 0), \quad \mu^*(B^*) = (a + b + 1, 0), \quad \mu^*(C^*) = (2a + b + c + 1, b) \\ \mu^*(D^*) &= (0, 2a + 2b + c + 1), \quad \mu^*(E^*) = (a, b + 1), \quad \mu^*(F^*) = (0, b)\end{aligned}$$

The first graph is a GKM graph of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^2$ and the second graph is the GKM graph of a $\mathbb{C}\mathbb{P}^1$ bundle over $\mathbb{C}\mathbb{P}^2$ both acted upon by some copy of \mathbb{T}^2 in \mathbb{T}^3 , and the moment maps of the manifolds, which we will also denote μ' and μ^* respectively, are the convex hulls of the moment maps of the respective graphs. We note that both moment maps have the same intersection with the line $y = b + \frac{1}{2}$. Then we can use the same argument above to find equivariantly symplectomorphic open sets W' and W^* containing $\mu'^{-1}(\{y = b + \frac{1}{2}\})$ and $(\mu^*)^{-1}(\{y = b + \frac{1}{2}\})$. Let $U = W' \cup \mu'^{-1}(\mathbb{R} \times [b + \frac{1}{2}, \infty))$ and $V = W^* \cup (\mu^*)^{-1}(\mathbb{R} \times (-\infty, b + \frac{1}{2}])$. Then $M = U \amalg V / \sim$, where \sim is the identification between points in W' and W^* , is the GKM manifold with the desired GKM graph.

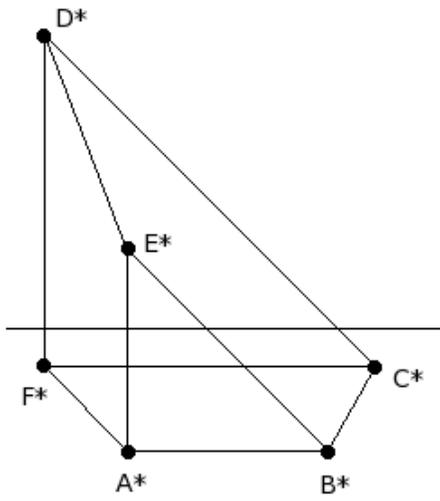
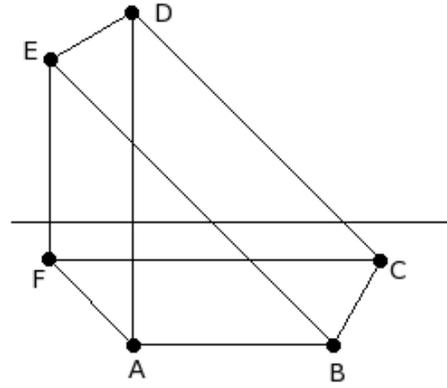
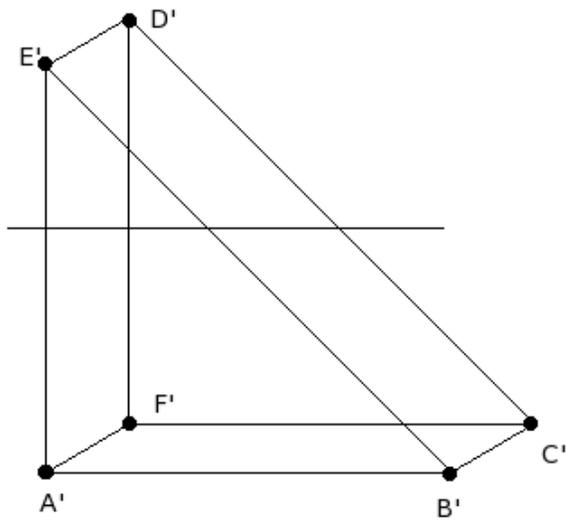


Figure 4.2: Construction of the Flag-like GKM manifolds.

The final $(3, 2, 6)$ GKM graph has weights $\alpha(A, B) = \alpha(E, D) = (1, 1)$, $\alpha(A, F) = \alpha(C, D) = (-1, 1)$, $\alpha(F, B) = \alpha(E, C) = (1, 0)$, $\alpha(B, C) = \alpha(F, E) = (0, 2)$ and $\alpha(A, D) = (0, 1)$. We can show that the blow-up of $G_+(2, 5)$ over one of its large isotropy spheres has this GKM graph. The torus action on $G_+(2, 5)$ is induced by the torus action on $\mathbb{C}^2 \times \mathbb{R}$ defined by

$$(\lambda_1, \lambda_2)(z_1, z_2, x) = (\lambda_1 z_1, \lambda_2 z_2, x).$$

The fixed points are then the subspaces $\{z_2 = 0, t = 0\}$ and $\{z_1 = 0, t = 0\}$ with either orientation. There are two possible volumes for the isotropy spheres connecting these fixed points. The isotropy spheres connecting the fixed points that represent the same plane with opposite orientation are larger than the other four isotropy spheres. Both of these spheres are naturally copies of $G_+(2, 3)$; one is the set of planes with $z_1 = 0$, the other is the set of planes with $z_2 = 0$.

The standard GKM graph for $G_+(2, 5)$ has four vertices A', B', C', D' and weights $\alpha(A', B') = \alpha(D', C') = (1, 1)$, $\alpha(A', D') = \alpha(B', C') = (-1, 1)$, $\alpha(A', C') = (0, 1)$, and $\alpha(D', B') = (1, 0)$. The segments AC and BD represent the large isotropy spheres. In a neighborhood of the fixed point D' we see that the manifold is equivariantly symplectomorphic to a neighborhood of $0 \in \mathbb{C}^3$ with \mathbb{T}^2 -action defined by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) = (e^{i(\theta_1+\theta_2)} z_1, e^{i\theta_1} z_2, e^{i(\theta_1-\theta_2)} z_3)$. This symplectomorphism maps the large isotropy sphere to $\{z_1 = z_3 = 0\}$. We then see that (in a neighborhood of the fixed point) the blow-up along this sphere is equivariantly symplectomorphic to a subset of the subvariety of $\mathbb{C}^3 \times \mathbb{C}\mathbb{P}^1$ (with coordinates $(z_1, z_2, z_3) \times [l_1, l_3]$) defined by $z_1 l_3 = z_3 l_1$. Then the group action is

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) \times [l_1, l_3] = (e^{i(\theta_1+\theta_2)} z_1, e^{i\theta_1} z_2, e^{i(\theta_1-\theta_2)} z_3) \times [e^{2i\theta_2} l_1, l_3].$$

There are now two fixed points which map, under the equivariant symplectomorphism between the blown-up spaces, to $(0, 0, 0) \times [0, 1]$ and $(0, 0, 0) \times [1, 0]$, with weights $(1, 1)$, $(1, 0)$, $(0, -2)$ and $(1, -1)$, $(1, 0)$, $(0, 2)$ respectively. Thus these points correspond to E , and F on the $(3, 2, 6)$ GKM graph. Similarly, we see that the blow-up replaces B' with two fixed points with weights $(-1, 1)$, $(-1, 0)$, $(0, -2)$ and $(-1, -1)$, $(-1, 0)$, $(0, 2)$. Thus we see that the blow-up of $G_+(2, 5)$ along a large isotropy sphere has the desired GKM graph.

Remark. If we blow up along one of the small isotropy spheres we get a manifold with a flag-like GKM graph. All four cases are the same, so without loss of generality we may take the sphere with fixed points C' and D' . After the blow-up we have a manifold whose GKM graph has six vertices $(A', B', C'', C^*, D'', D^*)$ and weights $\alpha(A', B') = \alpha(D'', C'') =$

$\alpha(D^*, C^*) = (1, 1)$, $\alpha(A', C^*) = \alpha(D'', D^*) = (0, 1)$, $\alpha(D^*, B') = \alpha(C^*, C'') = (1, 0)$, and $\alpha(A', D'') = \alpha(B', C'') = (-1, 1)$.

Remark. If we blow up $G_+(2, 5)$ at a fixed point we get a manifold that no longer satisfies the GKM properties. The blow-up at the point D' is the subspace of $\mathbb{C}^3 \times \mathbb{C}\mathbb{P}^2$ (with coordinates $(z_1, z_2, z_3) \times [l_1, l_2, l_3]$) defined by $z_1 l_2 = z_2 l_1$, $z_1 l_3 = z_3 l_1$, and $z_2 l_3 = z_3 l_2$. The group action is then $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2, z_3) \times [l_1, l_2, l_3] = (e^{i(\theta_1+\theta_2)} z_1, e^{i\theta_1} z_2, e^{i(\theta_1-\theta_2)} z_3) \times [e^{i\theta_2} l_1, l_2, e^{-i\theta_2} l_3]$. At the fixed point $(0, 0, 0) \times [0, 1, 0]$ we then have weights $(1, 0)$, $(0, 1)$, and $(0, -1)$. But then the subspace fixed by the subgroup generated by $e^{i\theta_1}$ has a component with dimension 4, so the manifold cannot be GKM. This shows that the GKM condition is not closed under blowing up.

4.3 Cohomology

In this section we use the GKM graphs to compute the cohomology and Chern classes of all possible $(3, 2, 6)$ GKM manifolds. Since the cohomology and Chern classes are determined by the GKM graphs, and we have completely classified the $(3, 2, 6)$ GKM graphs, we can give a complete classification of all the possible cohomology rings and Chern classes for $(3, 2, 6)$ GKM manifolds. In this section, we will view the weights as elements in $S(\mathbb{Z}^2)_1$ (or $S(\mathbb{Z}^3)_1$) and we will denote elements of equivariant cohomology by Greek letters (i.e. ϕ) and elements of regular cohomology by Greek letters in brackets (i.e. $[\phi]$.)

Manifolds with the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$

Consider a $(3, 2, 6)$ GKM graph associated to $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$ acted upon by some two-dimensional subtorus of \mathbb{T}^3 . Since we are only interested in the regular cohomology ring, we may consider the $(3, 3, 6)$ graph of which the $(3, 2, 6)$ graph must be a projection. We identify $S(\mathbb{Z}^3)$ with $\mathbb{Z}[x, y, z]$ by the map determined by sending the dual basis vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ to x , y , and z respectively. This graph has vertices A, B, C, D, E, F with $\alpha(A, B) = \alpha(D, E) = x$, $\alpha(A, C) = \alpha(D, F) = y$, $\alpha(B, C) = \alpha(E, F) = y - x$, $\alpha(A, D) = z$, $\alpha(B, E) = z - nx$, $\alpha(C, F) = z - ny$. Each element of the equivariant cohomology is a function $\phi : V \rightarrow S(\mathbb{Z}^3)$ such that the difference of the values of ϕ at two adjacent vertices is divisible by the weight of the edge (i.e. $\alpha(A, B)$ divides $\phi(B) - \phi(A)$.)

We can thus write each element of the equivariant cohomology as a vector

$$(\phi(A), \phi(B), \phi(C), \phi(D), \phi(E), \phi(F)).$$

Addition and multiplication of equivariant cohomology elements. Since GKM manifolds are equivariantly formal, the kernel of the map $H_{\mathbb{T}^3}^*(M) \rightarrow H^*(M)$ will consist of all elements of the equivariant cohomology where each element of the vector has a common factor. Each element of regular cohomology will then have multiple representatives in equivariant cohomology.

The basis element of $H^0(M)$ is obviously $[1] = (1, 1, 1, 1, 1, 1)$. Since $\dim H^2(M) = 2$ we must find two basis elements for $H^2(M)$. These will be represented by vectors of linear polynomials. We need only consider representations with $\phi(A) = 0$. If ϕ is a vector with $\phi(A) \neq 0$, then we may replace it with $\phi - \phi(A)[1]$; these will represent the same element in regular cohomology. We then see that we can choose $\phi = (0, -x, -y, 0, -x, -y)$ and $\chi = (0, 0, 0, -z, nx - z, ny - z)$ to represent the basis elements of $H^2(M)$. Then

$$\begin{aligned} [\phi]^2 &= [(0, x^2, y^2, 0, x^2, y^2)] = [\phi^2] \\ &= [\phi^2 + x\phi] = [(0, 0, y(y-x), 0, 0, y(y-x))] \end{aligned}$$

$$\begin{aligned} [\chi]^2 &= [(0, 0, 0, z^2, (z-nx)^2, (z-ny)^2)] = [\chi^2] \\ &= [\chi^2 + z\chi] = [(0, 0, 0, 0, nx(nx-z), ny(ny-z))] \end{aligned}$$

and

$$[\phi\chi] = [(0, 0, 0, 0, x(z-nx), y(z-ny))].$$

Thus $[\phi]^2$ and $[\phi\chi]$ are a basis for $H^4(M)$, and $[\chi]^2 + n[\phi\chi] = 0$. Since $\dim H^6(M) = 1$ and

$$\begin{aligned} [\phi][\phi\chi] &= [(0, 0, 0, 0, x^2(nx-z), y^2(ny-z))] \\ &= [\phi^2\chi + x\phi\chi] = [(0, 0, 0, 0, 0, y(y-x)(ny-x))] \end{aligned}$$

we see that $[\phi^2\chi]$ is a basis for $\dim H^6(M)$. Also $[\phi^3] = [y\phi^2] = 0$, and $[\chi^3] = -n[\phi\chi^2] = n^2[\phi^2\chi]$. Thus $H^*(M) = \mathbb{Z}[\phi, \chi]/(\phi^3, \chi^2 + n\phi\chi)$.

The equivariant Chern classes are the vectors whose entries are the symmetric polynomials of the weights at each vertex. The regular Chern classes are just the images of the equivariant

Chern classes under the map $H_{\mathbb{T}^2}^*(M) \rightarrow H^*(M)$, and thus are represented by the equivariant Chern classes. For the GKM manifolds with the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^2$, the first Chern class is

$$\begin{aligned}
c_1(M) &= [(x + y + z, -x + (y - x) + (z - nx), -y + (x - y) + (z - ny), \\
&\quad x + y - z, -x + (y - x) + (-z + nx), -y + (x - y) + (-z + ny))] \\
&= [(x + y + z, -(2 + n)x + y + z, x - (2 + n)y + z, \\
&\quad x + y - z, (n - 2)x + y - z, x + (n - 2)y - z)] \\
&= [(0, -(3 + n)x, -(3 + n)y, -2z, (n - 3)x - 2z, (n - 3)y - 2z)] + (x + y + z)[1] \\
&= (3 + n)[\phi] + 2[\chi]
\end{aligned}$$

since $(x + y + z)[1] = 0$. The second Chern class is

$$\begin{aligned}
c_2(M) &= [(xy + xz + yz, x(x - y) + x(nx - z) + (x - y)(nx - z), \\
&\quad y(y - x) + y(ny - z) + (y - x)(ny - z), xy - xz - yz, \\
&\quad x(x - y) + x(z - nx) + (x - y)(z - nx), y(y - x) + y(z - ny) + (y - x)(z - ny))] \\
&= (xy + xz + yz)[1] + ((1 + 2n)x - (2 + n)y - 3z)[\phi] - 2(x + y)[\chi] \\
&\quad + [(0, 0, 3(1 + n)y(y - x), 0, 6x(z - nx), -3y((1 + n)x + (-1 + n)y - 2z))] \\
&= 3(1 + n)[\phi^2] + 6[\phi\chi]
\end{aligned}$$

since $(xy + xz + yz)[1] + ((1 + 2n)x - (2 + n)y - 3z)[\phi] - 2(x + y)[\chi] = 0$. The generator of $H^6(M)$ can be represented as a vector with five 0 entries and the sixth entry the product of the weights at the corresponding fixed point. The third Chern class is represented by a vector each of whose entries contains the product of the negatives of the weights at the corresponding fixed point. Thus $c_3(M) = 6[\phi^2\chi]$, six times the generator of the top cohomology group.

Remark. The argument for the computation of the top Chern class (aka the Euler class) holds in general. The equivariant Euler class will always have the product of the weights at each fixed point in the corresponding entry of the vector. This means it will be equal to the number of fixed points times the generator of the top cohomology group.

Manifolds with the GKM graph of $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$

Consider a GKM graph associated to $\mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}(m) \oplus \mathbb{C}) \rightarrow \mathbb{C}\mathbb{P}^1$ acted upon by some two-dimensional subtorus of \mathbb{T}^3 . Again, we may consider the $(3, 3, 6)$ graph of which the $(3, 2, 6)$ graph must be a projection, and identify $S(\mathbb{Z}^3)$ with $\mathbb{Z}[x, y, z]$ in the same way as before. This graph has vertices A, B, C, D, E , and F and weights $\alpha(A, B) = \alpha(C, D) = \alpha(E, F) = z$, $\alpha(A, C) = x$, $\alpha(A, E) = y$, $\alpha(C, E) = -x + y$, $\alpha(B, D) = x + mz$, $\alpha(B, F) = y + nz$ and $\alpha(D, F) = -x + y + (n - m)z$.

The basis element of $H^0(M)$ is obviously $[1] = (1, 1, 1, 1, 1, 1)$. The basis of $H^2(M)$ is $[\phi] = [(0, -z, 0, -z, 0, -z)]$ and $[\chi] = [(0, 0, -x, -x - mz, -y, -y - nz)]$. The same techniques used above show that the basis of $H^4(M)$ is $[\phi\chi] = [(0, 0, 0, z(x + mz), 0, z(y + nz))]$ and $[\chi^2] = [(0, 0, 0, mz(x + mz), y(y - x), (y + nz)(-x + y + nz))]$ with $[\phi^2] = 0$. We can also show that $[\phi\chi^2] = [(0, 0, 0, 0, 0, z(y + mz)(x - y + (n - m)z))]$ generates $H^6(M)$ and $[\chi^3] = (m + n)[\phi\chi^2]$. Thus $H^*(M) = \mathbb{Z}[\phi, \chi]/(\phi^2, \chi^3 - (m + n)\phi\chi^2)$.

The Chern classes are then

$$\begin{aligned} c_1(M) &= [(x + y + z, -z + (x + mz) + (y + nz), -x + (y - x) + z, \\ &\quad -z + (-x - mz) + (y - x + (n - m)z), -y + (x - y) + z, -z + (-y - nz) + (x - y + (m - n)z))] \\ &= (x + y + z)[1] + (2 - m - n)[\phi] + 3[\chi] \end{aligned}$$

$$\begin{aligned} c_2(M) &= [(xy + xz + yz, z(-x - mz) + z(-y - nz) + (-x - mz)(-y - nz), \\ &\quad x(x - y) - xz - (x - y)z, z(x + mz) + z(x - y + (m - n)z) + (x + mz)(x - y + (m - n)z), \\ &\quad y(y - x) - yz - (y - x)z, z(y + nz) + z(-x + y + (n - m)z) + (y + nz)(-x + y + (n - m)z)] \\ &= 2(3 - m - n)[\phi\chi] + 3[\chi^2] \end{aligned}$$

and $c_3(M) = 6[\phi\chi^2]$.

Manifolds with twisted GKM graphs.

These manifolds have GKM graph with $\alpha(A, C) = \alpha(B, E) = x$, $\alpha(A, F) = \alpha(B, D) = y$, $\alpha(C, D) = \alpha(E, F) = -x + y$, $\alpha(A, B) = -ax + by$, $\alpha(C, E) = (a - c)x + by$, $\alpha(D, F) =$

$ax + (b - c)y$. Then $H^0(M)$ has basis $[1] = (1, 1, 1, 1, 1, 1)$, $H^2(M)$ has basis

$$[\phi] = [(0, 0, -x, -y, -x, -y)],$$

$$[\chi] = [(0, ax - by, 0, a(x - y), (c - a)x - by, (c - a - b)y)].$$

Calculations similar to those above show that $[\phi^2] = [(0, 0, 0, y(y - x), 0, y(y - x))]$, and $[\phi\chi] = [(0, 0, 0, ay(y - x), x((a - c)x + by), (a + b - c)y^2)]$ are a basis for $H^4(M)$ and $[\chi^2] = (2a - c)[\phi\chi] + a(a + b - c)[\phi^2]$. Combined with $[\phi^3] = 0$ this shows us that $H^*(M) = \mathbb{Z}[\phi, \chi]/(\phi^3, \chi^2 - (2a - c)\phi\chi - a(a + b - c)\phi^2)$ and that $[\phi^2\chi] = [(0, 0, 0, 0, 0, y(x - y)(ax + (b - c)y))]$ generates $H^6(M)$.

The Chern classes are then

$$\begin{aligned} c_1(M) &= [(x + y - (ax - by), x + y + (ax - by), -x + (y - x) + (a - c)x + by, \\ &\quad -y + (x - y) + ax + (b - c)y, -x + (y - x) + (c - a)x - by, \\ &\quad -y + (x - y) - ax + (c - b)y)] \\ &= (3 - 2a - c)[\phi] + 2[\chi] \end{aligned}$$

$$\begin{aligned} c_2(M) &= [(xy + x(ax - by) - y(ax - by), xy + x(ax - by) + y(ax - by), \\ &\quad x(x - y) + x((c - a)x - by) + (x - y)((c - a)x - by), \\ &\quad y(y - x) + y(-ax + (c - b)y) + (y - x)(-ax + (c - b)y), \\ &\quad x(x - y) + x((a - c)x + by) + (x - y)((a - c)x + by), \\ &\quad y(y - x) + y(ax + (b - c)y) + (y - x)(ax + (b - c)y)] \\ &= 3(1 - 2a - c)[\phi^2] + 6[\phi\chi] \end{aligned}$$

and $c_3(M) = 6[\phi^2\chi]$.

Manifolds with flag-like GKM graphs.

These manifolds have GKM graphs with $\alpha(A, B) = \alpha(F, C) = x$, $\alpha(A, D) = \alpha(F, E) = y$, $\alpha(B, E) = \alpha(C, D) = -x + y$, $\alpha(A, F) = -ax + by$, $\alpha(B, C) = (a + c)x + by$, $\alpha(E, D) =$

$ax + (b + c)y$. Then $H^0(M)$ has basis $[1] = (1, 1, 1, 1, 1, 1)$, $H^2(M)$ has basis

$$[\phi] = [(0, -x, -x, -y, -y, 0)],$$

$$[\chi] = [(0, 0, -(a + c)x - by, -(a + b + c)y, a(x - y), ax - by)].$$

Calculations similar to those above show that $[\phi^2] = [(0, 0, 0, y(y - x), y(y - x), 0)]$, and $[\phi\chi] = [(0, 0, 0, (a + b + c)y(y - x), a(y - x)^2, x(ax - by))]$ are a basis for $H^4(M)$ and $[\chi^2] = (2a + c)[\phi\chi] - a(a + b + c)[\phi^2]$. This combined with $[\phi^3] = 0$ shows us that $H^*(M) = \mathbb{Z}[\phi, \chi]/(\phi^3, \chi^2 - (2a + c)\phi\chi - a(a + b + c)\phi^2)$ and that $[\phi^2\chi] = [(0, 0, 0, 0, 0, xy(ax - by))]$ generates $H^6(M)$.

The Chern classes are then

$$\begin{aligned} c_1(M) &= [(x + y - (ax - by), -x + (y - x) + (a + c)x + by, \\ &\quad -x + (y - x) - (a + c)x - by, -y + (x - y) - (ax + (b + c)y), \\ &\quad -y + (x - y) + (ax + (b + c)y), x + y + (ax - by))] \\ &= (3 + c - 2a)[\phi] + 2[\chi] \end{aligned}$$

$$\begin{aligned} &[(xy - x(ax - by) - y(ax - by), x(x - y) + x((-a - c)x - by) + (x - y)((-a - c)x - by), \\ &\quad x(x - y) + x((a + c)x + by) + (x - y)((a + c)x + by), \\ &\quad y(y - x) + y(ax + (b + c)y) + (y - x)(ax + (b + c)y), \\ &\quad y(y - x) + y(-ax - (b + c)y) + (y - x)(-ax - (b + c)y), \\ &\quad xy - x(-ax + by) - y(-ax + by))] = 3(1 - 2a + c)[\phi^2] + 6[\phi\chi] \end{aligned}$$

and $c_3(M) = 6[\phi^2\chi]$.

Manifolds with the GKM graph of a blow-up of $G_+(2, 5)$ along a large isotropy sphere.

These manifolds have GKM graphs with weights $\alpha(A, B) = \alpha(E, D) = x + y$, $\alpha(A, F) = \alpha(C, D) = -x + y$, $\alpha(B, C) = \alpha(F, E) = 2y$, $\alpha(F, B) = \alpha(E, C) = x$, and $\alpha(A, D) = y$. Then, as always, $H^0(M)$ is generated by $[1] = (1, 1, 1, 1, 1, 1)$. The generators of $H^2(M)$ are then $[\phi] = [(0, 0, -2y, -2y, -2y, 0)]$ and $[\chi] = [(0, -x - y, y - x, 0, x + y, x - y)]$. (Since

$2y$ must divide $\phi(C) - \phi(B)$ we must have a factor of 2 in each nonzero term of ϕ .) Then $[\phi\chi] = [(0, 0, 0, 2y(y-x), -4xy, 0)]$ and $[\chi^2] = [(0, 0, 0, 2y(y-x), 2x(x+3y), 2x(x-y))]$. Both of these are an element of $H^4(M)$ multiplied by 2, so they cannot be a basis of $H^4(M)$. We can, however, take a basis $[\psi] = [(0, 0, 0, y(y-x), -2xy, 0)]$ and $[v] = [(0, 0, 0, 0, x(x+y), x(x-y))]$ with $[\phi\chi] = 2[\psi]$ and $[\beta^2] = -2[\psi] + 2[v]$. We then see that $[\psi\chi] = [(0, 0, 0, y(x-y)(y+x), 0, 0)]$ is a basis for $H^6(M)$. Because GKM manifolds with this graph do not have cohomology generated in $H^2(M)$, the relations between the generators are more complicated. The cohomology ring is

$$H^*(M) = \mathbb{Z}[\phi, \chi, \psi, v]/(\phi^2, \phi\chi - 2\psi, \chi^2 + \phi\chi - 2v, \psi v, \phi\psi, \chi(\psi + v), \phi v - \chi\psi).$$

The Chern classes are then

$$\begin{aligned} c_1(M) &= [((x+y) + y - (x-y), -(x+y) - x + 2y, -x + (y-x) - 2y, \\ &\quad - (x+y) - y + (x-y), -2y + (x+y) + x, 2y + (x-y) + x)] \\ &= 3[\phi] + 2[\chi] \end{aligned}$$

$$\begin{aligned} c_2(M) &= [((-x-y)(-y) + (-x-y)(x-y) + (-y)(x-y), \\ &\quad (x+y)x + (x+y)(-2y) - 2xy, 2xy + x(x-y) + 2y(x-y), \\ &\quad (x+y)y + (x+y)(y-x) + y(y-x), -2yx + 2y(-x-y) - x(-x-y), \\ &\quad (y-x)(-x) - 2(y-x)y + 2xy)] \\ &= 2[\chi^2] + 7[\phi\chi] \end{aligned}$$

and $c_3(M) = 3[\phi\chi^2]$.

REFERENCES

- [AH91] Ahara, K., Hattori, A, 4 dimensional symplectic S^1 -manifolds admitting moment map. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math* **38** (1991), 251-298
- [Ar78] Arnold, V. Mathematical methods in classical mechanics. *Graduate Texts in Mathematics* Springer, New York (1978)
- [At82] Atiyah, M. Convexity and commuting Hamiltonians *Bull. London Math. Soc.* **14** (1982), 1-15.
- [Au88] Audin, M. Hamiltoniens périodiques sur les varétés symplectiques compactes de dimension 4. In: Géométrie symplectique et mécanique, Proceedings 1988, C. Albertm ed. *Springer Lecture Notes in Math.* **1416** (1990)
- [Au03] Audin, M. Torus Actions on Symplectic Manifolds. *Progress in Mathematics* Birkhäuser, Boston (2003)
- [B] Baird, T. GKM sheaves and nonorientable surfaces *Preprint*
- [BGT] Bolker, E., Guillemin, V., Holm, T. How is a graph like a manifold? *Preprint*
- [CLS11] Cox, D., Little, J., Schenk, H. Toric Varieties *Graduate Studies in Mathematics* Amer Mathematical Society, Providence (2011)
- [C69] Coxeter, H. S. M. *Introduction to Geometry, 2nd. ed.* New York: Wiley, p. 404, 1969.
- [dS01] da Silva, A. Lectures On Symplectic Geometry. *Lecture Notes in Mathematics*, Springer, New York (2001)
- [DE76] Dejter, I. J. Smooth S^1 manifolds in the homotopy type of $\mathbb{C}P^3$. *Michigan Math. J.* **23** (1976), 83-95.
- [D88] Delzant, T. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. math. France* **116** (1988), 315-339.
- [D00] Dessai, A. Homotopy complex projective spaces with $Pin(2)$ -action *Topology and its Applications.* **122** (2002), 487-499.
- [D98] Dessai, A. $Spin^c$ -manifolds with $Pin(2)$ -action. *Math. Ann.* **315** (1998), 511-528.

- [D04] Dessai, A. Torus actions on homotopy complex projective spaces. *Math. Z.* **247** (2004), 505-511.
- [DM90] Dovermann, K. H., Masuda, M. Exotic cyclic actions on homotopy complex projective spaces. *J. Fac. Sci. Univ. Tokyo.* **37** (1990), 335-376.
- [F93] Fulton, W., Introduction to Toric Varieties *Princeton University Press* Princeton, (1993)
- [Gi87] Ginzburg, V. Equivariant cohomology and Kähler geometry. *Functional Anal. Appl.* **21** (1987) no. 4, 271-283.
- [Go11] Gonzalez, E. Classifying semi-free Hamiltonian S^1 -manifolds. *Int. Math. Res. Not. IMRN* no. 2 (2011) 387-418.
- [GKM98] Goresky, M., Kottwitz, R., MacPherson, R. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.* **13**, no. 1(1998) 25-83.
- [Go82] Gotay, M. On coisotropic imbeddings of presymplectic manifolds. *Proc. Amer. Math. Soc.* **84** (1982), 111-114.
- [GS82] Guillemin, V., Sternberg, S. Convexity properties of the moment mapping. *Invent. Math.* **67** (1982), 491-513.
- [GHZ06] Guillemin, V., Holm, T., Zara, C. A GKM description of the equivariant cohomology ring of a homogeneous space. *J. Alg. Combinatorics.* **23** (2006), no. 1, 21-41.
- [GZ02] Guillemin, V., Zara, C. Combinatorial Formulas for Products of Thom Classes. *Proceedings of the Geometry, Mechanics, and Dynamics Workshop*, Toronto, August 7-12 2002, Springer-Verlag, NY
- [GZ99] Guillemin, V., Zara, C. Equivariant de Rham Theory and Graphs. *Asian Journal of Mathematics.* **3**, no. 1, (1999) 49-76.
- [GZ01] Guillemin, V., Zara, C. One-Skeleta, Betti Numbers and Equivariant Cohomology. *Duke Math J.* **107**, no. 2, (2001), 283-349.
- [H78] Hattori, A. $Spin^c$ -Structures and S^1 -actions. *Invent. Math.* **48** (1978), 7-31.
- [J85] James, D. M. Smooth S^1 actions on homotopy $\mathbb{C}P^4$'s. *Michigan Math. J.* **32** (1985), 259-266.
- [K06] Karpenkov, O. Classification of lattice-regular lattice convex polytopes. *Funct. Anal. Other Math.* **1** (2006), 17-35.
- [K99] Karshon, Y. Periodic Hamiltonian flows on four dimensional manifolds, *Memoirs Amer. Math. Soc.* **672**, (1999).

- [KT01] Karshon, Y., Tolman, S. Centered complexity one Hamiltonian torus actions. *Trans. Amer. Math. Soc.* **353** (2001), 4831-4861.
- [KT03] Karshon, Y., Tolman, S. Complete Invariants for Hamiltonian Torus Actions with Two Dimensional Quotients *J. Symplectic Geom.* **2** No. 1 (2003), 25-82.
- [K84] Kirwan, F. C. The cohomology of quotients in symplectic and algebraic geometry. *Princeton University Press* (1984)
- [MA83] Masuda, M. Integral weight system of torus actions on cohomology complex projective spaces. *Japan. J. Math.* **9**, (1983), 55-86.
- [MA81] Masuda, M. On smooth S^1 -action on cohomology complex projective spaces. The case where the fixed point set consists of four connected components. *J. Fac. Sci. Univ. Tokyo* **28** (1981), 127-167
- [McD09] McDuff, D. Some 6-dimensional Hamiltonian S^1 -manifolds. *Journal of Topology* **2**, no. 3 (2009) 589-623.
- [M93] McMullen, P. On simple polytopes. *Invent. Math.* **113** (1993), 419-444.
- [M96] McMullen, P. Weights on polytopes *Discrete and Computational Geometry.* **15** (1996), 363-388.
- [MR09] Montagard, P., Ressayre, N. Regular lattice polytopes and root systems. *Bull. London Math. Soc.* **41** (2009), 227-241.
- [M78] Musin, O. R. Unitary actions of S^1 on complex projective spaces. *Russian Math. Surveys.* **33:6** (1978) 249-250.
- [P74] Petrie, T. A setting for smooth S^1 -actions with applications to real algebraic actions on $\mathbb{C}\mathbb{P}^{4n}$. *Topology.* **13** (1974), 363-374.
- [P72] Petrie, T. Smooth S^1 -actions on cohomology complex projective spaces and related topics. *Bull. Amer. Math. Soc.* **78** (1972), 105-153.
- [P73] Petrie, T. Torus actions on homotopy complex projective space. *Invent. Math.* **20** (1973), 139-146.
- [Sa09] Sabatini, S. The topology of GKM spaces and GKM fibrations *Thesis*
- [ST10] Sabatini, S., Tolman, S. New techniques for obtaining schubert-type Formulas for Hamiltonian manifolds. *Preprint*
- [Sj98] Sjamaar, R. Convexity properties of the moment mapping re-examined. *Adv. Math.* **138** (1998), 46-91.

- [T98] Tolman, S. Examples of Non-Kähler Hamiltonian Torus Actions. *Invent. Math.* **131** no. 2 (1998), 299-310.
- [T11] Tolman, S. On Integral Cohomology and the Chang-Skjelbred Lemma. Forthcoming.
- [T10] Tolman, S. On a symplectic generalization of Petrie's conjecture. *Trans. Amer. Math. Soc.* **362** (2010), 3963-3996.
- [TW99] Tolman, S., Weitsman, J. On the cohomology rings of Hamiltonian T -Spaces. *Northern California Symplectic Geometry Seminar*, 251-258, Amer. Math. Soc. Transl. Ser. 2. **196**, Amer. Math. Soc., Providence RI, 1999.
- [TW79] Tsukada, E., Washiyama, R. S^1 -actions on cohomology complex projective space with three components of the fixed point set. *Hiroshima Math J.* **9** (1979), 41-46.
- [W75] Wang, K. Differentiable circle group actions on homotopy complex projective spaces. *Math. Ann.* **214** (1975), 73-80.
- [W77] Wang, K. Torus actions on cohomological complex projective spaces. *Proceedings of the American Mathematical Society.* **65**, no. 2 (1997), 355-356.
- [W80] Wang, K. Torus actions on cohomology complex projective spaces. *Bull. Inst. Math. Acad. Sinica.* **8**, (1980), no. 2-3, part 2, 439-445.
- [Y76] Yoshida, T. On smooth semi-free S^1 -action on cohomology complex projective space, *Publ. Res. Inst. Math. Sci.* **11** (1976), 483-496.
- [Y77] Yoshida, T. On S^1 actions on cohomology complex projective spaces. *Sûgaku.* **29** (1977), no. 2, 154-164.