THE GRADE CONJECTURE AND ASYMPTOTIC INTERSECTION MULTIPLICITY

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

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Abstract

In this thesis, we study Peskine and Szpiro's Grade Conjecture and its connection with asymptotic intersection multiplicity χ_∞. Given an A-module M of finite projective dimension and a system of parameters $x_1, \ldots, x_r$ for M, we show, under certain assumptions on M, that $\chi_\infty(M, A/\underline{x}) > 0$. We also give a necessary and sufficient condition on M for the existence of a system of parameters $\underline{x}$ with $\chi_\infty(M, A/\underline{x}) > 0$.

We then prove that if the Grade Conjecture holds for a given module M, then there is a system of parameters $\underline{x}$ such that $\chi_\infty(M, A/\underline{x}) > 0$. We also prove the Grade Conjecture for complete equidimensional local rings in any characteristic.

Acknowledgments

I would like to thank my thesis advisor, Sankar Dutta. He is an exceptional teacher, and his continued guidance and support have been vital in making me the mathematician I am today. I would also like to thank Phil Griffith, Hal Schenck, and Bill Haboush, who served on my thesis committee and have helped me throughout my graduate career, both personally and mathematically. Furthermore, I would like to thank Steven Bradlow, who served as graduate director during my tenure and supported me throughout.

My parents have been significant figures in my mathematical career since long before I can remember. From counting steps to counting dimension and grade, they have ceaselessly encouraged me, taught me, and inspired me to continue onwards. My father, in particular, has been an invaluable sounding board for all of my mathematical ideas, and has spent countless hours reading drafts of this thesis. Thank you.

Finally, I would like to thank Caroline for her continued love, support, and encouragement.

Table of Contents

1 Introduction

Throughout this thesis, A will be a local noetherian ring, and all modules considered will be finitely generated.

Our main focus is Peskine and Szpiro's Grade Conjecture. We will prove the Grade Conjecture for complete equidimensional local rings in any characteristic, and we will show how to reduce the conjecture from equicharacteristic to characteristic p.

We will then explore the connection with the asymptotic intersection multiplicity $\chi_\infty(M, A/\underline{x})$, where M is an A-module of finite projective dimension and $\underline{x} = x_1, \ldots, x_r$ is a system of parameters for M. We will give a necessary and sufficient condition on M for the existence of such a system of parameters with $\chi_\infty > 0$. We will then prove that if the Grade Conjecture holds, then there exists a system of parameters such that $\chi_\infty > 0$, and we will discuss some corollaries.

1.1 Detailed Summary

In 1965, Serre [Ser65] proved the following theorem about intersection multiplicities:

Theorem 1.1.1 (Serre)**.** *Let A be a regular local ring either containing a field or unramified over a discrete valuation ring, and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$. Letting*

$$\chi(M, N) = \sum_{i=0}^{\dim A} (-1)^i \ell(\mathrm{Tor}_i(M, N))$$

be the intersection multiplicity, we then have

1. $\dim M + \dim N \leq \dim A$
2. $\chi(M, N) \geq 0$
3. $\chi(M, N) = 0$ *if and only if* $\dim M + \dim N < \dim A$*.*

He also proved (1) for an arbitrary regular local ring, and conjectured that (2) and (3) hold as well. These statements can be further generalized to an arbitrary local ring when one of the modules has finite projective dimension:

The Dimension Inequality. $\dim M + \dim N \leq \dim A$*;*

Non-negativity. $\chi(M,N) \geq 0$*;*

Vanishing. $\chi(M,N) = 0$ *if* $\dim M + \dim N < \dim A$*; and*

Positivity. $\chi(M,N) > 0$ *if* $\dim M + \dim N = \dim A$*.*

When A is regular local, nonnegativity was proved by Gabber [Gab95], and vanishing was proved by Roberts [Rob85] and Gillet-Soulé [GS85] independently; positivity is still open when A is a ramified regular local ring.

When A is not regular, the Dimension Inequality is open, but nonnegativity, vanishing, and positivity are false: Dutta, Hochster, and McLaughlin [DHM85] gave an example of a hypersurface A and modules M and N, with $\operatorname{pd} M < \infty$, where $\chi(M,N) < 0$ even though $\dim M + \dim N < \dim A$.

Since positivity is false in the general case, we consider instead the case where $N = A/\underline{x}$, where $x_1, \ldots, x_r$ is a system of parameters for M. *Positivity is even unknown in this case as well.*

Note that due to Serre [Ser65] and Lichtenbaum [Lic66], if $x_1, \ldots, x_r$ form an A-sequence, then $\chi(M, A/\underline{x}) > 0$; *however, outside of this special case, positivity is completely unknown.*

In 1982, Dutta [Dut83] introduced the notion of asymptotic multiplicity χ_∞, which we will define in Section 2.6, to investigate vanishing and positivity over a local ring of characteristic p. In particular, he proved the following:

1. $\chi_\infty(M,N) = 0$ if $\dim M + \dim N < \dim A$;
2. $\chi_\infty(M,N) > 0$ if $\dim M + \dim N = \dim A$ and M is Cohen-Macaulay;
3. There exist modules M and N with $\chi_\infty(M,N) < 0$.

Again, since positivity of χ_∞ is false in this general case, we consider the case where $N = A/\underline{x}$, where $x_1, \ldots, x_r$ is a system of parameters for M. *Positivity of χ_∞ is unknown in this case, even when $x_1, \ldots, x_r$ form an A-sequence, and our motivating question is: when is $\chi_\infty(M, A/\underline{x}) > 0$?*

In this thesis, we will consider $\chi_\infty(M, A/\underline{x})$ when $x_1, \ldots, x_r$ is a system of parameters for M. We prove the following:

Theorem 4.1.5. *Suppose* $\operatorname{pd} M < \infty$*, where* $d = \dim A$ *and* $r = \dim M$*. Then there is a system of parameters* $x_1, \ldots, x_r$ *for* M*, that is part of a system of parameters for* A*, such that*

$$\chi_\infty(M, A/\underline{x}) > 0 \textit{ if and only if } \dim \operatorname{Ext}^{d-r}(M,A) = r.$$

We will also show a special case of asymptotic positivity:

Theorem 4.1.7. *Let* $d = \dim A$ *and* $r = \dim M$*, and suppose that* $\operatorname{pd} M = d - r$*. Then any system of parameters* $x_1, \ldots, x_r$ *for* M *is part of a system of parameters for* A*, and*

$$\chi_\infty(M, A/\underline{x}) > 0.$$

Next, we will study the following conjecture of Peskine and Szpiro [PS73, Conjecture (f) of Chapter II]:

The Grade Conjecture. *Suppose that* $\operatorname{pd} M < \infty$. *Then*

$$\operatorname{grade} M + \dim M = \dim A.$$

The Grade Conjecture is known in some specific cases: when M is perfect or A is Cohen-Macaulay (these are due to Peskine and Szpiro [PS73]); and in the graded case, when $M = \bigoplus M_i$ is a graded module over a graded ring $A = \bigoplus A_i$ with A_0 artinian (this is also due to Peskine and Szpiro [PS74]). Foxby [Fox79] showed that the Grade Conjecture holds if A is complete and equidimensional in the equicharacteristic case. We will prove this result in any characteristic:

Theorem 3.2.4. *Suppose that A is complete and equidimensional and* $\operatorname{pd} M < \infty$. *Then*

$$\operatorname{grade} M + \dim M = \dim A.$$

After proving the main results about the positivity of χ_∞, we will also prove the following connection between the Grade Conjecture and χ_∞:

Theorem 4.2.1. *Let A be a local ring in characteristic p, and suppose that* $\operatorname{pd} M < \infty$. *Assume the Grade Conjecture holds for M. Then there is a system of parameters $x_1, \ldots, x_r$ for M, that is part of a system of parameters for A, such that*

$$\chi_\infty(M, A/\underline{x}) > 0.$$

We will also show how to translate characteristic p results relating to the Grade Conjecture to equicharacteristic zero. In particular, we will show:

Theorem 3.3.1. *Suppose that the Grade Conjecture holds over every local ring of characteristic p. Then it also holds for every local ring of equicharacteristic zero.*

Finally, as a consequence of the equivalence proved in Theorem 4.1.5, as well as Theorems 4.1.7 and 4.2.1 on the positivity of χ_∞, we can show several cases in which $\dim \operatorname{Ext}^{d-r}(M, A) = r$. We can use the same reduction-to-characteristic-p techniques as above to prove these results in any equicharacteristic local ring:

Theorem 4.2.3. *Let A be a local ring of equal characteristic, M a finitely generated A-module of finite projective dimension and let $d = \dim A$ and $r = \dim M$. If $\operatorname{pd} M = d - r$, then $\dim \operatorname{Ext}^{d-r}(M, A) = r$.*

Corollary 4.2.4. *Let A be a local ring of equal characteristic, M a finitely generated A-module of finite projective dimension, and let $d = \dim A$. If $\dim M = 1$, then $\dim \operatorname{Ext}^{d-1}(M, A) = 1$.*

2 Background

We will begin by presenting background material that will be needed in this thesis. More information and details can be found in Matsumura [Mat89] and Serre [Ser65].

All rings are assumed to be commutative Noetherian rings with identity. $(A, \mathfrak{m}, k)$ will denote a local Noetherian ring, where $\mathfrak{m}$ is the unique maximal ideal and $k = A/\mathfrak{m}$. If $x_1, \ldots, x_n$ is a sequence of elements of a ring A, we will often use $\underline{x}$ to denote either the sequence or the ideal generated by the sequence. Usually there is no confusion; if there is ambiguity, we will specify which we mean.

If $(A, \mathfrak{m}, k)$ is a local ring, we write $\widehat{A}$ for the completion of A with respect to $\mathfrak{m}$, and likewise $\widehat{M}$ for the completion of an A-module M. If $\mathfrak{p}$ is a prime ideal of A, we write $A_{\mathfrak{p}}$ for the localization of A at $\mathfrak{p}$.

2.1 Dimension and Multiplicity

Definition 2.1.1. Let A be a ring. The **spectrum** of A, denoted $\operatorname{Spec}(A)$ is the set of all prime ideals of A.

Definition 2.1.2. Let A be a ring and let M be an A-module. The **annihilator** of M is

$$\operatorname{ann}(M) = \{a \in A \mid aM = 0\}.$$

The **support** of M is

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \operatorname{ann} M\}.$$

The **associated primes** of M is

$$\operatorname{Ass}(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \mathfrak{p} = \operatorname{ann}(x) \text{ for some } x \in M\}.$$

If I is an ideal, then we write

$$V(I) = \operatorname{Supp}(A/I).$$

Definition 2.1.3. If A is a ring, the **Krull dimension** of A, denoted by

$\dim(A)$, is defined as

$$\dim(A) = \sup \left\{ d \,\middle|\, \begin{array}{l} \text{there exists } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d \\ \text{where each } \mathfrak{p}_i \text{ is a prime ideal of } A \end{array} \right\}.$$

The Krull dimension of a ring is simply referred to as the **dimension** of a ring. If M is an A-module, then

$$\dim M = \dim(A/\operatorname{ann}(M)).$$

The **codimension** of a module M is

$$\operatorname{codim} M = \dim A - \dim M.$$

If $\mathfrak{p} \in \operatorname{Spec} A$, then the **height** of $\mathfrak{p}$ is defined to be $\operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{p}}$. If I is a proper ideal of A, then we define

$$\operatorname{ht} I = \inf \{ \operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \in V(I) \}.$$

We say that A is **equidimensional** if for all minimal primes $\mathfrak{p}$ of A, $\dim A/\mathfrak{p} = \dim A$.

Definition 2.1.4. Let A be a local ring and M be a finitely generated A-module. Let

$$s(M) = \inf \left\{ d \,\middle|\, \begin{array}{l} \text{there exists } x_1, \ldots, x_d \in \mathfrak{m} \text{ such} \\ \text{that } \ell(M/(x_1, \ldots, x_d)M) < \infty \end{array} \right\}.$$

If $s(M) = d$, then any sequence $x_1, \ldots, x_d \in \mathfrak{m}$ such that

$$\ell(M/(x_1, \ldots, x_d)M) < \infty$$

is called a **system of parameters** of M.

Definition 2.1.5. Let A be a ring and let M be a finitely generated A-module. Let I be an ideal of A such that $\ell(M/IM) < \infty$. The **Hilbert-Samuel polynomial** of M with respect to I is defined to be the unique polynomial $P_I(M, X) \in \mathbb{Q}[X]$ such that

$$P_I(M, n) = l(M/I^n M) \text{ for } n \gg 0.$$

We write

$$P_I(M, X) = \frac{a_d}{d!} X^d + \text{lower degree terms},$$

and we set

$$d(I; M) = d$$

and

$$e(I; M) = a_d.$$

We call $e(I;M)$ the **Hilbert-Samuel multiplicity** of M with respect to I. If $(A,\mathfrak{m})$ is local, we write $d(M)$ for $d(\mathfrak{m};M)$ and $e(M)$ for $e(\mathfrak{m};M)$.

The above three definitions are tied together by the Dimension Theorem.

Theorem 2.1.6 ([Mat89, Theorem 13.4])**.** *Let A be a local ring and M a finitely generated A-module. Then*

$$\dim(M) = s(M) = d(M).$$

Note that if $\ell(M) < \infty$, then $e(I;M) = \ell(M)$ for any proper ideal I. The following theorem describes a deeper connection between multiplicity and length.

Theorem 2.1.7 ([Mat89, Theorem 14.7])**.** *Let A be a local ring and M a finitely generated A-module, and let I be an ideal of A such that $\ell(M/IM) < \infty$. Let $\{\mathfrak{p}_1,\dots,\mathfrak{p}_t\}$ be the minimal primes of $\operatorname{Ass}(M)$ such that $\dim A/\mathfrak{p}_i = \dim M$. Then*

$$e(I;M) = \sum_{i=1}^{t} e(\overline{I};A/\mathfrak{p}_i)\ell(M_{\mathfrak{p}_i}),$$

where $\overline{I}$ are the images of I modulo $\mathfrak{p}_i$.

Next we will define intersection multiplicity as formulated by Serre.

Definition 2.1.8. Let A be a ring and M an A-module. A **projective resolution** of M is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective A-module. The **projective dimension** of M, denoted $\operatorname{pd} M$, is defined to be the smallest h, if it exists, such that there is a projective resolution

$$0 \to P_h \to P_{h-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

If no such h exists, we define $\operatorname{pd} M = \infty$.

Similarly, an **injective resolution** of M is an exact sequence

$$0 \to E_0 \to E_1 \to E_2 \to \cdots ,$$

where each E_i is an injective A-module. The **injective dimension** of M, denoted $\operatorname{id} M$, is defined to be the smallest h, if it exists, such that there is an injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{h-1} \to E_h \to 0.$$

If no such h exists, we define $\operatorname{id} M = \infty$.

Lemma 2.1.9. *Let A be a ring and let M and N be finitely generated A-modules, and let $I = \operatorname{ann} M$ and $J = \operatorname{ann} N$. Then $I + J \subseteq \operatorname{ann} \operatorname{Tor}_i(M,N)$ for* *all* $i \geq 0$.

Proof. Let $\mathbf{P}_\bullet$ be a projective resolution for M. Then $\mathrm{Tor}_i(M, N)$ is the homology of the complex $\mathbf{P}_\bullet \otimes N$, and so it is annihilated by $\operatorname{ann} N = J$. Similarly, by using a projective resolution for N, we see that $\mathrm{Tor}_i(M, N)$ is annihilated by I. Therefore, it is annihilated by $I + J$, as desired. □

Definition 2.1.10. Let A be a ring and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$, such that either M or N has finite projective dimension. Writing $h = \min(\operatorname{pd} M, \operatorname{pd} N)$, we define, following Serre, the **intersection multiplicity**

$$\chi(M, N) = \sum_{i=0}^{h} (-1)^i \ell(\mathrm{Tor}_i(M, N)).$$

The condition $\ell(M \otimes N) < \infty$ implies that $\ell(\mathrm{Tor}_i(M, N)) < \infty$ for all $i \geq 0$ by Lemma 2.1.9, so the definition of intersection multiplicity makes sense.

Definition 2.1.11. Let A be a ring, let $\mathbf{C}_\bullet$ be a complex of A-modules with homology of finite length, and assume that $C_i = 0$ for $i > h$. We define the **Euler characteristic** of the complex $\mathbf{C}_\bullet$ by

$$\chi(\mathbf{C}_\bullet) = \sum_{i=0}^{h} (-1)^i \ell(H_i(\mathbf{C}_\bullet)).$$

We note that using the above notation,

$$\chi(M, N) = \chi(M \otimes \mathbf{Q}_\bullet) = \chi(\mathbf{P}_\bullet \otimes N),$$

where $\mathbf{P}_\bullet$ (resp., $\mathbf{Q}_\bullet$) is a finite projective resolution of M (resp., N).

The following lemma follows immediately from the above definition.

Lemma 2.1.12. *Let A be a ring, and let $\mathbf{C}_\bullet$ be a complex of A-modules of finite length, and assume that $C_i = 0$ for $i > h$. Then*

$$\chi(\mathbf{C}_\bullet) = \sum_{i=0}^{h} (-1)^i \ell(C_i).$$

2.2 Depth and Grade

Definition 2.2.1. Let A be a ring and M an A-module. A sequence $x_1, \ldots, x_n \in A$ is called a **regular sequence** on M (or M-**regular**, or an M-**sequence**) if

1. $(x_1, \ldots, x_n)M \neq M$

2. For each i, x_i is a **nonzerodivisor** on $M/(x_1, \ldots, x_{i-1})M$; equivalently,

the "multiplication by x_i" map

$$\frac{M}{(x_1,\dots,x_{i-1})M} \xrightarrow{x_i} \frac{M}{(x_1,\dots,x_{i-1})M}$$

is injective.

The existence of a regular sequence can be detected by Ext modules as follows.

Theorem 2.2.2 ([Mat89, Theorem 16.6])**.** *Let A be a ring. Let M be a finitely generated A-module, and let I be an ideal of A such that $IM \neq M$. Then the following are equivalent:*

1. $\operatorname{Ext}^i(N,M)=0$ *for all $i<n$ and all finitely generated A-modules N such that* $\operatorname{Supp}(N)\subseteq V(I)$.
2. $\operatorname{Ext}^i(A/I,M)=0$ *for all $i<n$.*
3. $\operatorname{Ext}^i(N,M)=0$ *for all $i<n$ and some finitely generated A-module N such that* $\operatorname{Supp}(N)=V(I)$.
4. *There is an M-sequence of length n contained in I.*

This allows us to make the following definition.

Definition 2.2.3. Let A be a ring. Let M be a finitely generated A-module, and let I be an ideal of A such that $IM \neq M$. We define $\operatorname{depth}(I,M)$ to be the length of the longest M-sequence contained in I. If $(A,\mathfrak{m})$ is local, then we write $\operatorname{depth} M = \operatorname{depth}(\mathfrak{m},M)$.

Definition 2.2.4. Let A be a ring and let I be a proper ideal of A. We define $\operatorname{grade} I = \operatorname{depth}(I,A)$. If M is a finitely-generated A-module, we define $\operatorname{grade} M = \operatorname{grade} \operatorname{ann} M$. (For convenience, we sometimes define $\operatorname{grade} A = \infty$.)

Remark 2.2.5. Since one can view an ideal I as an A-module, the notation $\operatorname{grade} I$ could be ambiguous; in practice, whenever we write $\operatorname{grade} I$, we will always mean the grade as an ideal, not as a module.

Note that by Theorem 2.2.2, $\operatorname{grade} M$ is the smallest integer i such that $\operatorname{Ext}^i(M,A)\neq 0$, and so $\operatorname{grade} M \le \operatorname{pd} M$.

Definition 2.2.6. Let A be a ring and M a finitely generated A-module of finite projective dimension. We say that M is **perfect** if $\operatorname{grade} M = \operatorname{pd} M$.

Depth is related to projective dimension as follows.

Theorem 2.2.7 (Auslander-Buchsbaum Formula, [BH93, Theorem 1.3.3])**.** *Let A be a local ring and M a nonzero finitely-generated A-module of finite projective dimension. Then*

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A.$$

One can show that, in general, $\operatorname{depth} M \le \dim M$ for any finitely generated A-module M. When equality holds, we say the module is Cohen-Macaulay.

Definition 2.2.8. Let A be a local ring and M a finitely generated A-module. We say M is **Cohen-Macaulay** if $\dim M = \operatorname{depth} M$. If A is Cohen-Macaulay as a module over itself, then we say that A is **Cohen-Macaulay**.

Theorem 2.2.9 ([Mat89, Theorem 17.4])**.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring.*

1. *For a proper ideal I, $\operatorname{ht} I = \operatorname{depth}(I, A) = \operatorname{grade} I$.*
2. *A is **catenary**; that is, given any primes $\mathfrak{p} \subseteq \mathfrak{q}$, every chain of primes*

$$\mathfrak{p} = \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq p_n = \mathfrak{q}$$

   has the same length.

3. *For any sequence $x_1, \ldots, x_r \in \mathfrak{m}$, the following conditions are equivalent:*
   a. *$x_1, \ldots, x_r$ is an A-sequence;*
   b. *$\operatorname{ht}(x_1, \ldots, x_i) = i$ for every $1 \leq i \leq r$;*
   c. *$\operatorname{ht}(x_1, \ldots, x_r) = r$;*
   d. *$x_1, \ldots, x_r$ is part of a system of parameters for A.*

Definition 2.2.10. Let $(A, \mathfrak{m})$ be a local ring. If A has a system of parameters $x_1, \ldots, x_n$ that generates $\mathfrak{m}$, then we say that A is a **regular local ring**. If A is a (not necessarily local) ring, then we say that A is a **regular ring** if $\dim A < \infty$ and $A_{\mathfrak{m}}$ is a regular local ring for each maximal ideal $\mathfrak{m}$.

Definition 2.2.11. A local ring is called a **Gorenstein ring** if it has finite injective dimension as a module over itself.

We will make use of the following structure theorem of complete local rings (that is, local rings where $A \cong \widehat{A}$) due to Cohen [Coh46].

Theorem 2.2.12 ([Mat89, Theorem 29.4])**.** *Let A be a complete local ring. Then there is a surjection $V[[X_1, \ldots, X_n]] \to A$, where V is either a field or a complete discrete valuation ring, and $V[[X_1, \ldots, X_n]]$ denotes the power series ring over V in n variables.*

We will also use the following structure theorem of exact sequences due to Buchsbaum and Eisenbud.

Theorem 2.2.13 ([BE73])**.** *Let A be a ring and let*

$$\mathbf{F}_\bullet : \qquad 0 \longrightarrow F_h \xrightarrow{d_h} F_{h-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

be a complex of free A-modules. For $r \geq 1$ and $k \geq 1$, define $I_r(d_k)$ to be the ideal generated by the $r \times r$ minors of d_k. Then $\mathbf{F}_\bullet$ is exact if and only if

1. *The **rank** of the map d_k (that is, the largest r such that $I_r(d_k)$ is nonzero) is equal to*

$$r_k = \sum_{i=k}^{h} (-1)^i \operatorname{rank} F_i;$$

2. *The grade of each ideal $I_{r_k}(d_k)$ is at least k.*

2.3 Double Complexes and the Koszul Complex

Definition 2.3.1. Let A be a ring. A **double complex $\mathbf{D}_{\bullet\bullet}$** is a doubly-indexed set of A-modules D_{ij}, together with two sets of boundary maps,

$$\delta_{ij} : D_{ij} \to D_{i-1,j}$$

and

$$\epsilon_{ij} : D_{ij} \to D_{i,j-1},$$

such that the following diagram commutes:

$$\begin{tikzcd} D_{ij} \arrow[r, "\delta_{ij}"] \arrow[d, "\epsilon_{ij}"'] & D_{i-1,j} \arrow[d, "\epsilon_{i-1,j}"] \\ D_{i,j-1} \arrow[r, "\delta_{i,j-1}"] & D_{i-1,j-1} \end{tikzcd}$$

We will refer to the differentials δ_{ij} as **horizontal differentials** and ϵ_{ij} as **vertical differentials**.

The simplest example of a double complex is one formed by tensoring two complexes. Given complexes $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ with differentials d_i^F and d_i^G, respectively, we can form a double complex $D_{\bullet\bullet}$ by:

$$D_{ij} = F_i \otimes G_j,$$

with differentials

$$\delta_{ij} = d_i^F \otimes \mathrm{id}_{G_j} : F_i \otimes G_j \to F_{i-1} \otimes G_j$$

and

$$\epsilon_{ij} = \mathrm{id}_{F_i} \otimes d_j^G : F_i \otimes G_j \to F_i \otimes G_{j-1},$$

where id_{F_i} and id_{G_j} represent the identity maps on F_i and G_j, respectively.

We will often write $(\mathbf{F} \otimes \mathbf{G})_{\bullet\bullet}$ for the double complex formed by tensoring $\mathbf{F}_\bullet$ by $\mathbf{G}_\bullet$, although we will always explicitly say that we mean this, since there does not seem to be a standard notation.

Definition 2.3.2. Let $\mathbf{D}_{\bullet\bullet}$ be a double complex with horizontal differentials δ_{ij} and vertical differentials ϵ_{ij}. We define the **total complex** of $\mathbf{D}_{\bullet\bullet}$, written $\mathrm{Tot}(\mathbf{D}_{\bullet\bullet})$, as follows:

$$F_n = \bigoplus_{i+j=n} D_{ij}$$

with differentials

$$d_n = \sum_{i+j=n} \delta_{ij} + (-1)^i \epsilon_{ij}.$$

Definition 2.3.3. Let $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ be two complexes. We define the tensor complex by

$$\mathbf{F}_\bullet \otimes \mathbf{G}_\bullet = \mathrm{Tot}((\mathbf{F} \otimes \mathbf{G})_{\bullet\bullet});$$

that is, the total complex of the double complex formed by tensoring $\mathbf{F}_\bullet$ by $\mathbf{G}_\bullet$.

We now define the Koszul complex.

Definition 2.3.4. Let A be a ring and $x \in A$. We write $\mathbf{K}_\bullet(x)$ for the **Koszul complex** of A generated by x, defined by

$$\cdots \longrightarrow K_2(x) \longrightarrow K_1(x) \xrightarrow{x} K_0(x) \longrightarrow \cdots,$$

where $K_0(x) = A$, $K_1(x) = A$, and all other $K_i(x) = 0$; and the map x is given by multiplication by x.

Definition 2.3.5. Let A be a ring, M a finitely generated A-module, and $x_1, \dots, x_n \in A$. We define the **Koszul complex $\mathbf{K}_\bullet$** by

$$\mathbf{K}_\bullet(x_1, \dots, x_n) = \mathbf{K}_\bullet(x_1) \otimes \cdots \otimes \mathbf{K}_\bullet(x_n),$$

and

$$\mathbf{K}_\bullet(x_1, \dots, x_n; M) = \mathbf{K}_\bullet(x_1, \dots, x_n) \otimes M.$$

We write $H_i(x_1, \dots, x_n)$ (respectively, $H_i(x_1, \dots, x_n; M)$) for the ith homology of the complex $\mathbf{K}_\bullet(x_1, \dots, x_n)$ (respectively, $\mathbf{K}_\bullet(x_1, \dots, x_n; M)$).

One can detect regular sequences using Koszul complexes as follows:

Theorem 2.3.6 ([Mat89, Theorem 16.8])**.** *Let $(A, \mathfrak{m})$ be a local ring, M a finitely generated A-module, and let $x_1, \dots, x_n \in \mathfrak{m}$. The following are equivalent:*

1. *$x_1, \dots, x_n$ form an M-sequence.*
2. *$H_i(\underline{x}; M) = 0$ for all $i > 0$.*
3. *$H_1(\underline{x}; M) = 0$.*

Koszul complexes are also connected with multiplicity in the following way:

Theorem 2.3.7 ([Ser65, Theorem 1 of Chapter IV])**.** *Let A be a local ring, M a finitely generated A-module, and let $x_!, \dots, x_r \in A$ be a system of parameters for M. Then for each $i \geq 0$, $H_i(\underline{x}; M)$ has finite length, and*

$$e(\underline{x}; M) = \sum_{i=0}^{r} \ell(H_i(\underline{x}; M)).$$

2.4 Spectral Sequences

More details on the contents of this section can be found in [Wei94].

Definition 2.4.1. Let A be a ring. A **homology spectral sequence** of A-modules starting at an integer a consists of the following data:

1. A family $\{E^r_{ij}\}$ of A-modules, defined for all integers i, j, and $r \geq a$.
2. Maps
$$d^r_{ij} : E^r_{ij} \to E^r_{i-r,j+r-1}$$
that are differentials in the sense that $d^r d^r = 0$. This means that "lines of slope $-(r+1)/r$" in the lattice E^r_{**} form complexes.
3. Isomorphims between E^{r+1}_{ij} and the homology of E^r_{**} at the spot E^r_{ij}:
$$E^{r+1}_{ij} \cong \ker(d^r_{ij}) / \operatorname{im}(d^r_{i+r,j-r+1}).$$

The **total degree** of E^r_{ij} is $n = i + j$. A spectral sequence is **bounded** if there are only finitely many nonzero terms of each total degree. A spectral sequence is **regular** if for each i and j, the differentials d^r_{ij} are zero for sufficiently large r.

We note that a bounded spectral sequence is regular.

Definition 2.4.2. Let A be a ring, and let E^r_{ij} be a homology spectral sequence of A-modules. There is a nested family of submodules of E^a_{ij}

$$0 = B^a_{ij} \subseteq \cdots \subseteq B^r_{ij} \subseteq B^{r+1}_{ij} \subseteq \cdots \subseteq Z^{r+1}_{ij} \subseteq Z^r_{ij} \subseteq \cdots \subseteq Z^a_{ij} = E^a_{ij},$$

where each $E^r_{ij} \cong Z^r_{ij}/B^r_{ij}$. We define

$$B^\infty_{ij} = \bigcup_{r=a}^{\infty} B^r_{ij}$$

and

$$Z^\infty_{ij} = \bigcap_{r=a}^{\infty} Z^r_{ij},$$

and then define

$$E^\infty_{ij} = Z^\infty_{ij}/B^\infty_{ij}.$$

We note that if a spectral sequence is bounded, then for sufficiently large r, $E^r_{ij} = E^\infty_{ij}$.

We now will define convergence of spectral sequences.

Definition 2.4.3. Let A be a ring and let $\{E^r_{ij}\}$ be a homology spectral sequence of A-modules. We say that the spectral sequence **weakly converges** to H_*, where H_* is a set of A-modules H_n, if there is a filtration of each H_n:

$$\cdots \subseteq F_i H_n \subseteq F_{i+1} H_n \subseteq \cdots H_n,$$

and isomorphisms

$$E_{ij}^{\infty} \cong F_i H_{i+j} / F_{i-1} H_{i+j}$$

for all i and j.

We say that the spectral sequence **approaches** H_* if it weakly converges to H_* and for all n,

$$H_n = \bigcup_i F_i H_n \text{ and } \bigcap_i F_i H_n = 0.$$

We say that the spectral sequence **converges** to H_* if it approaches H_*, it is regular, and

$$H_n = \varprojlim_i H_n / F_i H_n.$$

If the spectral sequence converges to H_*, we write

$$E_{ij}^a \Rightarrow H_{i+j}.$$

The main example of a spectral sequence we need is induced from a double complex.

Theorem 2.4.4 ([Wei94, Section 5.6])**.** *Let A be a ring, and let $\mathbf{D}_{\bullet\bullet}$ be a double complex. Then there are two spectral sequences, ${}^{I}E_{ij}^2$ and ${}^{II}E_{ij}^2$, both converging to the homology of* $\operatorname{Tot}(\mathbf{D}_{\bullet\bullet})$*, defined as follows:*

$${}^{I}E_{ij}^2 = H_i^h H_j^v(\mathbf{D}_{\bullet\bullet})$$

and

$${}^{II}E_{ij}^2 = H_i^v H_j^h(\mathbf{D}_{\bullet\bullet}),$$

*where H^v indicates taking vertical homology, and H^h indicates taking horizontal homology. We say ${}^{I}E_{ij}^2$ is induced by **filtration by columns**, and ${}^{II}E_{ij}^2$ is induced by **filtration by rows**.*

2.5 Local Cohomology and Local Duality

More details on the contents of this section can be found in [Har67] and [ILL+07].

Definition 2.5.1. Let A be a ring, let $I \subseteq A$ be an ideal, and let M be an A-module. The ith **local cohomology** of M with respect to the ideal I is defined as

$$H_I^i(M) = \varinjlim_t \operatorname{Ext}^i(A/I^t, M).$$

The directed system $\operatorname{Ext}^i(A/I^t, M)$ is induced by the surjections $A/I^{t+1} \to A/I^t$ for all $t > 0$.

In a local ring, local cohomology is connected to depth and dimension.

Theorem 2.5.2 ([ILL$^+$07, Theorems 9.1, 9.3])**.** *Let $(A, \mathfrak{m})$ be a local ring. Then*

$$\operatorname{depth} A = \inf \left\{ i \,\middle|\, H^i_{\mathfrak{m}}(A) \neq 0 \right\}$$

and

$$\dim A = \sup \left\{ i \,\middle|\, H^i_{\mathfrak{m}}(A) \neq 0 \right\}.$$

Definition 2.5.3. Let $(A, \mathfrak{m}, k)$ be a local ring. We define the **injective hull** of k, denoted $E_A(k)$, to be the unique (up to isomorphism) module E containing k such that

1. E is injective; and
2. For any submodule $N \subseteq E$, $N \cap k = k$.

Definition 2.5.4. Let $(A, \mathfrak{m}, k)$ be a local ring, and let M be an A-module. We define the **Matlis Dual** of M as

$$M^\vee = \operatorname{Hom}(M, E_A(k)).$$

Theorem 2.5.5 ([Mat89, Theorem 18.6])**.** *Let $(A, \mathfrak{m}, k)$ be a local ring, and let $E = E_A(k)$ be the injective hull of k.*

1. *If M is an A-module and $0 \neq x \in M$, then there exists $\phi \in M^\vee$ such that $\phi(x) \neq 0$. In other words, the canonical map $\theta : M \to M^{\vee\vee}$ defined by $\theta(x)(\phi) = \phi(x)$ for $x \in M$ and $\phi \in M^\vee$ is injective.*
2. *If M is an A-module of finite length, then $\ell(M) = \ell(M^\vee)$, and the canonical map $M \to M^{\vee\vee}$ is an isomorphism.*
3. *Let $\hat{A}$ be the completion of A; then E is also an $\hat{A}$-module, and is an injective hull of k as an $\hat{A}$-module.*
4. $\operatorname{Hom}_A(E, E) = \operatorname{Hom}_{\hat{A}}(E, E) = \hat{A}$. *In other words, each endomorphism of the A-module E is multiplication by a unique element of $\hat{A}$.*
5. *E is artinian as an A-module, and also as an $\hat{A}$-module. Assume now that A is complete, and write $\mathcal{N}$ (respectively $\mathcal{A}$) for the category of noetherian (respectively, artinian) A-modules. Then if $M \in \mathcal{N}$, we have $M^\vee \in \mathcal{A}$ and $M \cong M^{\vee\vee}$; if $M \in \mathcal{A}$, we have $M^\vee \in \mathcal{N}$ and $M \cong M^{\vee\vee}$.*

The following theorem is known as Groethendieck's Local Duality Theorem.

Theorem 2.5.6 ([ILL$^+$07, Theorem 11.29])**.** *Let $(A, \mathfrak{m}, k)$ be a local Gorenstein ring of dimension d, and let M be a finitely generated A-module. Then*

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{d-i}_A(M, A)^\vee.$$

Definition 2.5.7. Let $(A, \mathfrak{m})$ be a local ring. A finitely generated A-module ω is called a **canonical module** for A if

$$\hat{\omega} \cong H^d_{\mathfrak{m}}(A)^\vee.$$

By Cohen's structure theorem (Theorem 2.2.12), every complete local ring has a canonical module. In particular, if R is a regular local ring surjecting onto A, then we claim

$$\omega = \operatorname{Ext}_R^{\dim R - \dim A}(A, R)$$

is a canonical module for A. By local duality over R, we have

$$\omega \cong \operatorname{Ext}_R^{\dim R - \dim A}(A, R)^{\vee\vee} \cong H^{\dim A}_{\mathfrak{m}}(A)^\vee,$$

as required.

2.6 Characteristic p

This section describes features of rings of characteristic p. More information and details can be found in [PS73].

Definition 2.6.1. Let $p > 0$ be a prime number. We say that a ring A has characteristic p if there is an injective ring homomorphism $\mathbb{Z}/p\mathbb{Z} \to A$. In this case, the map $f : A \to A$ defined by $x \mapsto x^p$ is a ring homomorphism, which is called the **Frobenius** homomorphism. We write f^n for the nth iterate of the Frobenius; that is, the map $x \mapsto x^{p^n}$.

Definition 2.6.2. Let A be a ring of characteristic p. We write ${}^f\!A$ for bi-algebra A with the action on the left by the Frobenius, and the action on the right by the identity. That is, if $a \in A$ and $x \in {}^f\!A$,

$$a \cdot x = a^p x \text{ and } x \cdot a = xa.$$

Definition 2.6.3. The **Frobenius functor** is a functor F from the category of A-modules to itself defined by

$$F(M) = M \otimes {}^f\!A.$$

Example 2.6.4. Let A be a ring and I an ideal generated by $x_1, \ldots, x_t$. Then

$$F(A/I) = A/I^{[p]},$$

where

$$I^{[p]} = (x_1^p, \ldots, x_t^p).$$

Example 2.6.5. Let

$$\mathbf{L}_\bullet : \qquad \cdots \longrightarrow L_k \xrightarrow{\phi_k} L_{k-1} \longrightarrow \cdots L_1 \xrightarrow{\phi_1} L_0 \longrightarrow 0$$

be a complex of free A-modules. Applying the Frobenius functor to $\mathbf{L}_\bullet$ gives an induced complex

$$F(\mathbf{L}_\bullet) : \qquad \cdots \longrightarrow L_k \xrightarrow{\phi_k^{[p]}} L_{k-1} \longrightarrow \cdots L_1 \xrightarrow{\phi_1^{[p]}} L_0 \longrightarrow 0,$$

where, if ϕ is a map of free A-modules corresponding to a matrix (a_{ij}), then $\phi^{[p]}$ is defined to be the map corresponding to the matrix (a_{ij}^p).

Theorem 2.6.6 ([PS73, Proposition 1.4])**.** *The Frobenius functor commutes with localization. That is, if* $\mathfrak{p} \in \operatorname{Spec} A$*, then*

$$F(-) \otimes_A A_\mathfrak{p} = F(- \otimes_A A_\mathfrak{p}).$$

Theorem 2.6.7 ([PS73, Proposition 1.5 and Theorem 1.7])**.** *If M is a finitely generated A-module, then* $\operatorname{Supp} F(M) = \operatorname{Supp} M$*. If, moreover, M has finite projective dimension, then $F(M)$ also has finite projective dimension,* $\operatorname{Tor}_i(M, {}^f\!A) = 0$ *for all $i \geq 1$, and* $\operatorname{pd} F(M) = \operatorname{pd} M$*.*

Kunz [Kun69] investigated how the length of a module grows when the Frobenius is applied. He proved:

Theorem 2.6.8. *Let A be a local ring of dimension d, and let M be an A-module of finite length. Then there is some constant $L(M)$ such that, for all $n \geq 0$,*

$$1 \leq \frac{\ell(F^n(M))}{p^{nd}} \leq L(M).$$

In 1983, Dutta [Dut83] introduced the following **asymptotic intersection multiplicity**:

Definition 2.6.9. Let A be a local ring, and let M and N be a finitely-generated A-modules such that $\ell(M \otimes N) < \infty$, and suppose that $\operatorname{pd} M < \infty$. We define

$$\chi_\infty(M, N) = \lim_{n\to\infty} \frac{\chi(F^n(M), N)}{p^{n \cdot \operatorname{codim} M}}.$$

Dutta showed that if $\dim N \leq \operatorname{codim} M$ (i.e., the Dimension Inequality holds for M and N), then $\chi_\infty(M, N)$ exists. Seibert [Sei89, Proposition 1] generalized this result as follows:

Lemma 2.6.10 (Seibert)**.** *Writing $d = \dim A$, we let $L_\bullet$ be a complex of finitely generated free modules with homologies of finite length. For each $i \geq 0$, the limit*

$$\lim_{n\to\infty} \frac{\ell(H_i(F^n(L_\bullet)))}{p^{nd}}$$

exists and is rational, and if $H_0(L_\bullet) \neq 0$*, then*

$$\lim_{n\to\infty} \frac{\ell(H_0(F^n(L_\bullet)))}{p^{nd}} \geq 1.$$

We introduce the following notion of **asymptotic Hilbert-Samuel multiplicity**:

Definition 2.6.11. Let M be a finitely-generated A-module and let $\underline{x} = x_1, \ldots, x_r$ be a system of parameters on M. We define

$$e_\infty(\underline{x}; M) = \lim_{n\to\infty} \frac{e(\underline{x}; F^n(M))}{p^{n\cdot \operatorname{codim} M}}.$$

We discuss various applications and properties of the above definition in Chapter 4.

3 Main Results: The Grade Conjecture

3.1 History

We will begin with one result contained in Serre's intersection theorem (Theorem 1.1.1):

Theorem 3.1.1 (Serre)**.** *Let A be a regular local ring, and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$. Then*

$$\dim M + \dim N \leq \dim A.$$

Recall that if A is regular local, then $\dim A = \operatorname{depth} A$. Therefore, using the Auslander-Buchsbaum Formula (Theorem 2.2.7) in the setting of Serre's theorem, we have

$$\dim M + \dim N \leq \operatorname{depth} M + \operatorname{pd} M.$$

Since $\dim M \leq \operatorname{depth} M$ for any finitely generated module M, we conclude that

$$\dim N \leq \operatorname{pd} M.$$

Recall that all modules over a regular local ring have finite projective dimension; in 1973, Peskine and Szpiro used the above formulation to generalize Serre's theorem, replacing the condition that A be regular local with the much weaker condition that $\operatorname{pd} M < \infty$:

The Intersection Theorem (Peskine-Szpiro, Hochster, Roberts)**.** *Let A be a local ring, and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$. Assume that* $\operatorname{pd} M < \infty$*. Then*

$$\dim N \leq \operatorname{pd} M.$$

Peskine and Szpiro [PS73] proved the Intersection Theorem in the following cases:

1. A has characteristic $p > 0$.
2. A is essentially of finite type over a field of characteristic 0.
3. A is ind-étale over a ring which is essentially of finite type over a field of characteristic 0.

Recall that A is ind-étale when $A = \varinjlim B_\alpha$, where:

1. For each α, B_α is local and essentially finite type over a field.
2. $B_\alpha \to B_{\alpha'}$ is the localization of an étale map.

In 1974, Hochster [Hoc74] proved the existence of Big Cohen-Macaulay modules in equicharacteristic, and used it to prove the Intersection Theorem for any equicharacteristic local ring. In 1987, Roberts [Rob87] used the theory of localized Chern characters of Baum, Fulton, and MacPherson [BFM75] to prove the Intersection Theorem in mixed characteristic.

This suggests that many other results true for regular local rings can be relaxed to arbitrary local rings if one adds the condition that one (or both) modules have finite projective dimension. In particular, Peskine and Szpiro posed the following:

Conjecture 3.1.2 (The Dimension Inequality). *Let A be a local ring, and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$. Assume that* $\operatorname{pd} M < \infty$. *Then*

$$\dim M + \dim N \leq \dim A.$$

Peskine and Szpiro also proved:

Lemma 3.1.3 ([PS73, Lemma 4.8]). *Let A be a local ring, and let M be a finitely generated A-module. Then*

$$\operatorname{depth} A \leq \operatorname{grade} M + \dim M \leq \dim A.$$

When A is Cohen-Macaulay, this immediately implies:

Corollary 3.1.4. *Suppose A is Cohen-Macaulay. Then*

$$\operatorname{grade} M + \dim M = \dim A$$

for all finitely generated modules M.

The above result prompted them to introduce the following conjecture:

The Grade Conjecture. *Let A be a local ring, and let M be a finitely generated A-module with finite projective dimension. Then*

$$\operatorname{grade} M + \dim M = \dim A.$$

This is sometimes also referred to as the Codimension Conjecture.

When combined with the Dimension Inequality (Conjecture 3.1.2), this suggests the following statement:

The Strong Intersection Conjecture. *Let A be a local ring, and let M and N be finitely generated A-modules with $\ell(M \otimes N) < \infty$. Assume that* $\operatorname{pd} M < \infty$. *Then*

$$\dim N \leq \operatorname{grade} M.$$

Peskine and Szpiro [PS73] proved the following relationship between the above conjectures.

Proposition 3.1.5. *The Strong Intersection Conjecture is equivalent to the Grade Conjecture plus the Dimension Inequality.*

The Dimension Inequality (and hence the Strong Intersection Conjecture) is unknown in almost any case when A is not regular. The Grade Conjecture, as asserted in Corollary 3.1.4, is true when A is Cohen-Macaulay, but unknown in most other cases. We will show that the Grade Conjecture holds when A is a complete, equidimensional local ring.

3.2 The Equidimensional Case

When M is perfect, the Grade Conjecture follows immediately from the Intersection Theorem:

Corollary 3.2.1 (Peskine-Szpiro)**.** *Let A be a local ring, M a finitely generated A-module of finite projective dimension, and suppose that M is perfect. Then*

$$\operatorname{grade} M + \dim M = \dim A.$$

Proof. Let $x_1, \ldots, x_r$ be a system of parameters for M. By the Intersection Theorem (and since M is perfect),

$$\dim A/\underline{x} \leq \operatorname{pd} M = \operatorname{grade} M.$$

Using $\dim A/\underline{x} \geq \dim A - \dim M$ gives $\dim A \leq \operatorname{grade} M + \dim M$; and the reverse inequality follows from Lemma 3.1.3. □

Next, we have some basic results about the annihilator of modules with finite projective dimension.

Lemma 3.2.2. *Let A be a local ring. Suppose that there exists a finitely generated A-module M of finite projective dimension such that* $\dim M = 0$*. Then M is perfect and A is Cohen-Macaulay.*

Proof. Since $\dim M = 0$, it follows that $I = \operatorname{ann} M$ is $\mathfrak{m}$-primary, so $\operatorname{grade} M = \operatorname{depth} A$. By Auslander-Buchsbaum,

$$\operatorname{pd} M + \operatorname{depth} M = \operatorname{depth} A,$$

and so $\operatorname{depth} M = 0$ implies $\operatorname{pd} M = \operatorname{grade} M$, i.e., M is perfect; and by Corollary 3.2.1, A is Cohen-Macaulay. □

Lemma 3.2.3. *Let A be a local ring and M a finitely generated A-module of finite projective dimension. Then* $\operatorname{ht} \operatorname{ann} M = \operatorname{grade} M$*.*

Proof. Let $x_1, \dots, x_g$ be a maximal A-sequence contained in $I = \operatorname{ann} M$ (so $g = \operatorname{grade} M$). Then

$$\operatorname{Hom}(M, A/\underline{x}) = \operatorname{Ext}^g(M, A) \neq 0.$$

Choose a minimal prime $\mathfrak{p}$ in $\operatorname{Supp} M \cap \operatorname{Ass}(A/\underline{x})$. Then

$$\operatorname{Hom}(M_{\mathfrak{p}}, A_{\mathfrak{p}}/\underline{x}A_{\mathfrak{p}}) \neq 0,$$

so $\operatorname{grade} M_{\mathfrak{p}} = \operatorname{grade} M = g$ and $\operatorname{ht} \mathfrak{p} \geq \operatorname{ht} I$.

Since $\mathfrak{p} \in \operatorname{Ass} A/\underline{x}$, $\operatorname{depth} A_{\mathfrak{p}} = g$; by Auslander-Buchsbaum,

$$\operatorname{depth} M_{\mathfrak{p}} + \operatorname{pd} M_{\mathfrak{p}} = g;$$

and hence $\operatorname{pd} M_{\mathfrak{p}} = g = \operatorname{grade} M_{\mathfrak{p}}$, i.e., $M_{\mathfrak{p}}$ is perfect over $A_{\mathfrak{p}}$. Corollary 3.2.1 then implies that

$$\operatorname{grade} M_{\mathfrak{p}} + \dim M_{\mathfrak{p}} = \dim A_{\mathfrak{p}}.$$

Now we choose a prime $\mathfrak{q}$ with $I \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, minimal over I, with $\dim A_p/\mathfrak{q}A_p = \dim M_{\mathfrak{p}}$. We also note that $\dim A_{\mathfrak{q}} \geq \operatorname{depth} A_{\mathfrak{q}} \geq g$.

We now have

$$\begin{aligned}
\dim A_{\mathfrak{p}} &\geq \dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}} + \dim A_q \\
&= \dim M_{\mathfrak{p}} + \dim A_{\mathfrak{q}} \\
&\geq \dim M_{\mathfrak{p}} + g \\
&= \dim A_{\mathfrak{p}},
\end{aligned}$$

which implies that the above inequalities are equalities; in particular, $\dim A_{\mathfrak{q}} = g$, which means that $\operatorname{ht} I \leq \operatorname{ht} \mathfrak{q} = g = \operatorname{grade} M$. Since the reverse inequality is always true, we have equality. □

From this, we get a special case of the Grade Conjecture. As stated earlier, the equicharacteristic case of this result was proved by Foxby [Fox79].

Theorem 3.2.4. *Suppose that A is a complete equidimensional local ring, and let M be a finitely generated A-module of finite projective dimension. Then*

$$\operatorname{grade} M + \dim M = \dim A.$$

Proof. We will show that $\operatorname{grade} M + \dim M \geq \dim A$; equality then holds because the reverse inequality is just Lemma 3.1.3.

We choose a prime $\mathfrak{p} \in \operatorname{Supp} M$ with $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \operatorname{ann} M$, and then choose a minimal prime $\mathfrak{q} \subseteq \mathfrak{p}$ with $\operatorname{ht} \mathfrak{p}/\mathfrak{q} = \operatorname{ht} \mathfrak{p}$. We then have

$$\begin{aligned} \dim A &= \dim A/\mathfrak{q} \\ &= \dim A/\mathfrak{p} + \operatorname{ht} \mathfrak{p}/\mathfrak{q} \\ &= \dim A/\mathfrak{p} + \operatorname{ht} \mathfrak{p} \\ &= \dim A/\mathfrak{p} + \operatorname{grade} M \\ &\leq \dim M + \operatorname{grade} M \end{aligned}$$

where the first equality follows because A is equidimensional; the second because A is complete, and hence catenary; and the fourth from Lemma 3.2.3. Finally, the inequality holds since $\dim A/\mathfrak{p} \leq \dim M$ always. □

Next, we prove that the Grade Conjecture is connected with the dimension of a particular Ext module. First, we prove a result that will be used several times in the next chapter.

Lemma 3.2.5. *Let A be a local ring, M a finitely generated A-module of finite projective dimension, and let $d = \dim A$ and $r = \dim M$. Then*

$$\dim \operatorname{Ext}^{d-r}(M, A) = r$$

if and only if there is a prime $\mathfrak{p} \in \operatorname{Ass} M$ with $\dim A/\mathfrak{p} = r$ and $\operatorname{ht} \mathfrak{p} = d - r$.

Proof. If $\dim \operatorname{Ext}^{d-r}(M, A) = r$, then there is some prime

$$\mathfrak{p} \in \operatorname{Supp} \operatorname{Ext}^{d-r}(M, A) \subseteq \operatorname{Supp} M$$

with $\dim A/\mathfrak{p} = r$. Such a prime is necessarily minimal over $\operatorname{ann} M$, so it is in $\operatorname{Ass} M$. Furthermore, $\operatorname{pd} M_{\mathfrak{p}} \geq d - r$ (since $\operatorname{Ext}^{d-r}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, A_{\mathfrak{p}}) \neq 0$), so

$$\operatorname{ht} \mathfrak{p} = \dim A_{\mathfrak{p}} \geq \operatorname{depth} A_{\mathfrak{p}} \geq \operatorname{pd} M_{\mathfrak{p}} \geq d - r$$

by Auslander-Buchsbaum. Since $\operatorname{ht} \mathfrak{p}$ certainly cannot be larger, $\operatorname{ht} \mathfrak{p} = d - r$.

Conversely, suppose that $\mathfrak{p} \in \operatorname{Ass} M$ with $\dim A/\mathfrak{p} = r$ and $\operatorname{ht} \mathfrak{p} = d - r$. Then $M_{\mathfrak{p}}$ is of finite length and finite projective dimension, so by Lemma 3.2.2, $M_{\mathfrak{p}}$ is perfect of projective dimension $d - r$ over $A_{\mathfrak{p}}$, and hence

$$\operatorname{Ext}^{d-r}(M, A)_{\mathfrak{p}} \neq 0,$$

so $\dim \operatorname{Ext}^{d-r}(M, A) = r$ (the dimension clearly cannot be larger than r). □

This is connected with the Grade Conjecture as follows:

Proposition 3.2.6. *Let A be a local ring, M a finitely generated A-module of finite projective dimension, and assume that $\operatorname{grade} M + \dim M = \dim A$ (i.e.,* *the Grade Conjecture holds for M). Let $d = \dim A$ and $r = \dim M$. Then*

$$\dim \operatorname{Ext}^{d-r}(M, A) = r.$$

Proof. Let $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim A/\mathfrak{p} = r$. Then $\ell(M_\mathfrak{p}) < \infty$, so by Lemma 3.2.2, $M_\mathfrak{p}$ is perfect over the Cohen-Macaulay ring $A_\mathfrak{p}$. We then have

$$d - r = \operatorname{grade} M \leq \operatorname{grade} M_\mathfrak{p} = \dim A_\mathfrak{p} = \operatorname{ht} \mathfrak{p},$$

so we are done by Lemma 3.2.5. □

3.3 Reduction to Characteristic p

Peskine and Szpiro [PS73] introduced some techniques to reduce many homological statements from equicharacteristic zero to characteristic p. Hochster [Hoc75] generalized these techniques into a "metatheorem" that is the standard tool for this type of reduction. Here we will show how to reduce the Grade Conjecture, as well as some auxiliary questions, from equicharacteristic zero to characteristic p.

Our main goal is to prove:

Theorem 3.3.1. *Suppose that the Grade Conjecture holds over every local ring of characteristic p. Then it also holds for every local ring of equicharacteristic zero.*

In this section, if $f_1, \ldots, f_N \in \mathbb{Z}[X_1, \ldots, X_m]$, and $x_1, \ldots, x_m$ are elements of some ring R, then we say that $\underline{x}$ is a solution of $f_1, \ldots, f_N$ if

$$f_1(\underline{x}) = 0, \ldots, f_N(\underline{x}) = 0.$$

Theorem 3.3.2 (Hochster)**.** *Let*

$$f_1, \ldots, f_N \in \mathbb{Z}[X_1, \ldots, X_d, W_1, \ldots, W_t],$$

where X_i and W_i are indeterminates. If $f_1, \ldots, f_N$ have a solution $(\underline{x}, \underline{w})$ in a local ring A of equicharacteristic zero, with $x_1, \ldots, x_d$ forming a system of parameters for A, then $f_1, \ldots, f_N$ have a solution $(\underline{x}', \underline{w}')$ in a local ring A' of characteristic p, with $x'_1, \ldots, x'_d$ forming a system of parameters for A'.

We will use a more generalized version [Kur94]:

Theorem 3.3.3 (Kurano)**.** *Let*

$$f_1, \ldots, f_N \in \mathbb{Z}[Y_1, \ldots, Y_n, X_1, \ldots, X_d, G_1, \ldots, G_l, W_1, \ldots, W_t],$$

where X_i, Y_i, G_i, and W_i are indeterminates. If $f_1, \ldots, f_N$ have a solution $(\underline{y}, \underline{x}, \underline{g}, \underline{w})$ in a regular local ring R of equicharacteristic zero, with $y_1, \ldots, y_n$

forming a regular system of parameters for R, and $x_1, \ldots, x_d$ forming a system of parameters for $R/\underline{g}$, then $f_1, \ldots, f_N$ have a solution $(\underline{y}', \underline{x}', \underline{g}', \underline{w}')$ in a regular local ring R' of characteristic p, with $y_1', \ldots, y_n'$ forming a regular system of parameters for R', and $x_1', \ldots, x_d'$ forming a system of parameters for $R'/\underline{g}'$.

The goal, then, is to find equations that preserve the particular properties that we're interested in. The main result we need, Lemma 3.3.10, asserts that we can preserve the dimension of homology modules. First, we need some preliminary results.

Lemma 3.3.4. *Let R be a regular local ring, and let $I = (g_1, \ldots, g_l)$ be a proper ideal with $h = \operatorname{ht} I$. Then there are polynomials $f_1, \ldots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates*

1. $Y_1, \ldots, Y_n$
2. $G_1, \ldots, G_l$
3. $W_1, \ldots, W_t$ *(for some sufficiently large t)*

such that

1. *There are $\underline{y}$ and $\underline{w}$ in R such that $(\underline{y}, \underline{g}, \underline{w})$ is a solution of $f_1, \ldots, f_N$, and $y_1, \ldots, y_n$ forms a regular system of parameters for R.*
2. *If $(\underline{y}', \underline{g}', \underline{w}')$ is a solution of $f_1, \ldots, f_N$ in a regular local ring R' with $y_1', \ldots, y_n'$ forming a regular system of parameters for R', then*

$$\operatorname{ht}(g_1', \ldots, g_l') \geq h.$$

Proof. First we suppose that $h = n$; i.e., I is $\mathfrak{m}$-primary, so $\sqrt{I} = \mathfrak{m}$. Thus, for each y_i, there is some integer $e_i \geq 0$ such that $y_i^{e_i} \in I$, so we can write

$$y_i^{e_i} = \sum_{j=1}^{l} c_{ij} g_j.$$

We use extra indeterminates C_{ij} and set $f_1, \ldots, f_N$ to be the polynomials

$$Y_i^{e_i} - \sum_{j=1}^{l} C_{ij} G_j$$

for $1 \leq i \leq n$, which preserves the fact that I is $\mathfrak{m}$-primary.

Now we do the general case; that is, $h < n$. We write $d = n - h$, and let $x_1, \ldots, x_d$ be a system of parameters for R/I, and choose polynomials to preserve the fact that $(g_1, \ldots, g_l, x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary, as above.

If $g_1', \ldots, g_l', x_1', \ldots, x_d'$ are part of a solution to $f_1, \ldots, f_N$ in a regular local ring R', then, as before, $(g_1', \ldots, g_l', x_1', \ldots, x_d')$ has height n. Removing the d elements $x_1', \ldots, x_d'$, we conclude that $(g_1', \ldots, g_l')$ has height at least $n - d = h$. □

Lemma 3.3.5. *Let R be a regular local ring, and suppose that*

$$R^a \xrightarrow{\phi} R^b \xrightarrow{\psi} R^c$$

is an exact sequence of free R modules. We choose matrices (u_{ij}) and (v_{ij}) that represent the maps ϕ and ψ. Then there are polynomials $f_1, \dots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates

1. $Y_1, \dots, Y_n$
2. U_{ij} *and* V_{ij}*, corresponding to the matrices* u_{ij} *and* v_{ij}
3. $W_1, \dots, W_t$ *(for some sufficiently large t)*

such that

1. *There are $\underline{y}$ and $\underline{w}$ in R such that $(\underline{y}, (u_{ij}), (v_{ij}), \underline{w})$ is a solution of $f_1, \dots, f_N$, and $y_1, \dots, y_n$ forms a regular system of parameters for R.*
2. *If $(\underline{y}', (u'_{ij}), (v'_{ij}), \underline{w}')$ is a solution of $f_1, \dots, f_N$ in a regular local ring R' with $y'_1, \dots, y'_n$ forming a regular system of parameters for R', then, letting ϕ' and ψ' be maps corresponding to the matrices (u'_{ij}) and (v'_{ij}),*

$$(R')^a \xrightarrow{\phi} (R')^b \xrightarrow{\psi} (R')^c$$

   is an exact sequence.

Proof. Since R is regular local, we can extend the maps $R^a \to R^b \to R^c$ to an exact sequence of free R-modules

$$0 \longrightarrow F_h \xrightarrow{\delta_h} F_{h-1} \longrightarrow \cdots \longrightarrow F_3 \xrightarrow{\delta_3} R^a \xrightarrow{\phi} R^b \xrightarrow{\psi} R^c,$$

so to prove the lemma, we will construct polynomials to preserve the exactness of this sequence.

First, we choose matrices (d^k_{ij}) to represent the maps δ_k, and choose corresponding variables D^k_{ij}. From each pairwise composition $\delta_{k+1}\delta_k = 0$, we get a system of polynomial equations on the D^k_{ij} that preserves the property that $\mathbf{F}_\bullet$ is a complex.

To preserve the property that $\mathbf{F}_\bullet$ is acyclic, we use the Buchsbaum-Eisenbud criteria for exactness (Theorem 2.2.13). To ensure that the rank of each δ_k is

$$r_k = \sum_{i=k}^{h} (-1)^i \operatorname{rank} F_i,$$

we add polynomials so that each r_{k+1} minor of δ_k vanishes. To ensure that the grade of each $I_{r_k}(\delta_k)$ is at least k, recall that since R is regular local, the grade of an ideal coincides with its height. By Lemma 3.3.4, we can find polynomials so that the height of each $I_{r_k}(\delta_k)$ does not decrease. □

Lemma 3.3.6. *Let $(R, \mathfrak{m})$ be a regular local ring, and let $x \in \mathfrak{m}$. Then there are polynomials $f_1, \ldots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates*

1. $Y_1, \ldots, Y_n$
2. X
3. $W_1, \ldots, W_t$ *(for some sufficiently large t)*

such that

1. *There are $\underline{y}$ and $\underline{w}$ in R such that $(\underline{y}, x, \underline{w})$ is a solution of $f_1, \ldots, f_N$, and $y_1, \ldots, y_n$ forms a regular system of parameters for R.*
2. *If $(\underline{y}', x', \underline{w}')$ is a solution of $f_1, \ldots, f_N$ in a regular local ring $(R', \mathfrak{m}')$ with $y'_1, \ldots, y'_n$ forming a regular system of parameters for R', then $x' \in \mathfrak{m}'$.*

Proof. We write $x = \sum_{i=1}^{n} c_i y_i$, and use extra indeterminates C_i and the one polynomial

$$X - \sum_{i=1}^{n} C_i Y_i.$$

□

Lemma 3.3.7. *Let R be a regular local ring, and let $I = (g_1, \ldots, g_l)$ be an ideal such that $g_1, \ldots, g_l$ minimally generate I. Then there are polynomials $f_1, \ldots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates*

1. $Y_1, \ldots, Y_n$
2. $G_1, \ldots, G_l$
3. $W_1, \ldots, W_t$ *(for some sufficiently large t)*

such that

1. *There are $\underline{y}$ and $\underline{w}$ in R such that $(\underline{y}, \underline{g}, \underline{w})$ is a solution of $f_1, \ldots, f_N$, and $y_1, \ldots, y_n$ forms a regular system of parameters for R.*
2. *If $(\underline{y}', \underline{g}', \underline{w}')$ is a solution of $f_1, \ldots, f_N$ in a regular local ring R' with $y'_1, \ldots, y'_n$ forming a regular system of parameters for R', then $g'_1, \ldots, g'_l$ minimally generate the ideal $I' = (g'_1, \ldots, g'_l)$.*

Proof. Let

$$R^a \xrightarrow{\phi} R^l \xrightarrow{(g_1, \ldots, g_l)} R$$

be the end of a minimal finite free resolution for R/I. By Lemma 3.3.5, we can find polynomials to preserve the exactness of this sequence. We can also preserve the fact this is a minimal resolution; that is, that each g_i and each entry of a matrix corresponding to ϕ is in the maximal ideal $\mathfrak{m}$; by using Lemma 3.3.6 for each entry of each matrix. □

Lemma 3.3.8. *Let R be a regular local ring, and let $I = (g_1, \dots, g_l)$ be a proper ideal with $h = \operatorname{ht} I$. Then there are polynomials $f_1, \dots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates*

1. $Y_1, \dots, Y_n$
2. $G_1, \dots, G_l$
3. $W_1, \dots, W_t$ *(for some sufficiently large t)*

such that

1. *There are $\underline{y}$ and $\underline{w}$ in R such that $(\underline{y}, \underline{g}, \underline{w})$ is a solution of $f_1, \dots, f_N$, and $y_1, \dots, y_n$ forms a regular system of parameters for R.*
2. *If $(\underline{y}', \underline{g}', \underline{w}')$ is a solution of $f_1, \dots, f_N$ in a regular local ring R' with $y'_1, \dots, y'_n$ forming a regular system of parameters for R', then*

$$\operatorname{ht}(g'_1, \dots, g'_l) = h.$$

Proof. First, we note that $\operatorname{ht} I = \operatorname{ht} \sqrt{I}$, so by adding polynomial equations from the containments

$$\sqrt{I}^e \subseteq I \subseteq \sqrt{I},$$

we can reduce to the case where $I = \sqrt{I}$.

We now choose $x_1, \dots, x_h \in I$ to be a maximal regular sequence in I that forms part of a minimal generating set for I. (To do so, choose $x_{i+1} \in I$ to avoid both $\mathfrak{m}I$ and all associated primes of $R/(x_1, \dots, x_i)$.)

We let S be the complement of the union of the minimal primes containing I and let $A = S^{-1}R$. We have (since I is radical)

$$\sqrt{(x_1, \dots, x_h)A} = IA,$$

so there is some $u \in S$ such that for some integer $e \geq 0$,

$$uI^e \subseteq (x_1, \dots, x_h), \tag{3.1}$$

and so we add polynomials to ensure that equation (3.1) holds.

We now claim that either u is a unit or $(x_1, \dots, x_h, u)$ is minimally generated by all $h + 1$ elements. If neither holds, then $u \in \mathfrak{m}$, and one of the x_i, say, x_1, can be written in terms of the others, as well as u:

$$x_1 = \sum_{i=2}^{h} b_i x_i + bu. \tag{3.2}$$

Now, we observe that for each minimal prime $\mathfrak{p}$ of I, each $x_i \in \mathfrak{p}$ but $u \notin \mathfrak{p}$, which implies that $b \in \mathfrak{p}$; in other words, $b \in \sqrt{I} = I$. Now, reading equation

(3.2) mod $\mathfrak{m}I$, we have (since $b \in I$ and $u \in m$)

$$x_1 \equiv \sum_{i=2}^{h} \overline{b_i} x_i \mod \mathfrak{m}I.$$

But this contradicts the fact that $x_1, \ldots, x_h$ are part of a minimal generating set for I. Therefore either

1. u is a unit; or

2. $(x_1, \ldots, x_h, u)$ is minimally generated by all $h+1$ elements.

In case (1), then we write $uv = 1$ and add the polynomial $UV - 1$; in case (2), we use Lemma 3.3.7 to get polynomials that preserve the minimal number of generators of the ideal $(x_1, \ldots, x_h, u)$. Note that in either case, it will ensure that

$$u \notin (x_1, \ldots, x_h). \tag{3.3}$$

We then use Lemma 3.3.4 to add polynomials to ensure that the heights of $(g_1, \ldots, g_l)$ and $(x_1, \ldots, x_h)$ do not decrease. Finally, we add conditions to show that each $x_i \in (g_1, \ldots, g_l)$.

To show that the height is preserved, we suppose that $\underline{g}', \underline{x}', u'$ are part of a solution to $f_1, \ldots, f_N$ in a regular local ring R'. By Lemma 3.3.4, we have that

$$\operatorname{ht}(g_1', \ldots, g_l') \geq h \text{ and } \operatorname{ht}(x_1', \ldots, x_h') \geq h.$$

The second condition implies that $x_1', \ldots, x_h'$ is a regular sequence, and since it's contained in $I' = (g_1', \ldots, g_l')$, we just have to ensure that there are no zerodivisors on $(x_1', \ldots, x_h')$ contained in I'. (Recall again that since R' is regular local, the height of an ideal equals its grade.)

Suppose, instead, that $r' \in I'$ is a nonzerodivisor on $(x_1', \ldots, x_h')$. By the polynomials (3.1), we have

$$u'(r')^e \in (x_1', \ldots, x_h').$$

Since r' is a nonzerodivisor on $(x_1', \ldots, x_h')$, so is $(r')^e$, and hence $u' \in (x_1', \ldots, x_h')$. But this contradicts the choice of u' by equation (3.3).

Thus the height of I' exactly equals h, as desired. $\square$

Lemma 3.3.9. *Let R be a regular local ring, and let M be a finitely generated R-module with*

$$R^a \xrightarrow{\phi} R^b \longrightarrow M \longrightarrow 0.$$

an exact sequence. We choose a matrix u_{ij} that represets the map ϕ. Then the there are polynomials $f_1, \ldots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates

1. $Y_1, \ldots, Y_n$

2. U_{ij}*, corresponding to the matrix* u_{ij}

3. $W_1, \ldots, W_t$ *(for some sufficiently large* t*)*

such that

1. There are $\underline{y}$ *and* $\underline{w}$ *in* R *such that* $(\underline{y}, (u_{ij}), \underline{w})$ *is a solution of* $f_1, \ldots, f_N$*, and* $y_1, \ldots, y_n$ *forms a regular system of parameters for* R*.*

2. If $(\underline{y}', u'_{ij}, \underline{w}')$ *is a solution of* $f_1, \ldots, f_N$ *in a regular local ring* R' *with* $y'_1, \ldots, y'_n$ *forming a regular system of parameters for* R'*, then, letting* ϕ' *be a map corresponding to the matrix* u'_{ij} *and* $M' = \operatorname{coker} \phi'$*, we have*

$$\dim M' = \dim M.$$

Proof. If $a < b$, then $\dim M = \dim R$, so we don't need any polynomials; thus, we can assume without loss that $b \leq a$.

Let $I = I_b(\phi)$ be the ideal of maximal minors of ϕ. Recall that

$$\sqrt{\operatorname{ann} M} = \sqrt{I},$$

so

$$\dim M = \dim R - \operatorname{ht} \operatorname{ann} M = \dim R - \operatorname{ht} I.$$

By Lemma 3.3.8, we can find polynomials to preserve the height of I, whose generators we can write as polynomials in the u_{ij}, and this will preserve the dimension of M as well. □

Now we come to the main lemma, that we can preserve the dimension of homology modules in the following sense.

Lemma 3.3.10. *Let* R *be a regular local ring, and let* $A = R/(g_1, \ldots, g_l)$ *be a quotient such that* $g_1, \ldots g_l$ *minimally generate the ideal* $(g_1, \ldots, g_l)$*. Suppose that*

$$A^a \xrightarrow{\phi} A^b \xrightarrow{\psi} A^c$$

is a complex of free A *modules with homology of dimension* h *(where we set* $h = -\infty$ *if the complex is exact). We choose matrices* $(\overline{u}_{ij})$ *and* $(\overline{v}_{ij})$ *that represent the maps* ϕ *and* ψ*, with* $u_{ij}, v_{ij} \in R$*. Then there are polynomials* $f_1, \ldots, f_N$ *with coefficients in* $\mathbb{Z}$ *in indeterminates*

1. $Y_1, \ldots, Y_n$

2. $G_1, \ldots, G_l$

3. U_{ij} *and* V_{ij}*, corresponding to the matrices* u_{ij} *and* v_{ij}

4. $W_1, \ldots, W_t$ *(for some sufficiently large* t*)*

such that

1. *There are* $\underline{y}$ *and* $\underline{w}$ *in* R *such that* $(\underline{y}, \underline{g}, (u_{ij}), (v_{ij}), \underline{w})$ *is a solution of* $f_1, \ldots, f_N$*, and* $y_1, \ldots, y_n$ *forms a regular system of parameters for* R*.*

2. *If* $(\underline{y}', \underline{g}', (u'_{ij}), (v'_{ij}), \underline{w}')$ *is a solution of* $f_1, \ldots, f_N$ *in a regular local ring* R' *with* $y'_1, \ldots, y'_n$ *forming a regular system of parameters for* R'*, then, setting* $A' = R'/\underline{g}'$*, and letting* ϕ' *and* ψ' *be maps corresponding to the matrices* $(\overline{u}'_{ij})$ *and* $(\overline{v}'_{ij})$*,*

$$(A')^a \xrightarrow{\phi'} (A')^b \xrightarrow{\psi'} (A')^c$$

is a complex of free A' *modules with homology of dimension* h*.*

Proof. In the course of this proof, we will use the following notational conventions.

If $\mathbf{F}_\bullet$ is a complex, then F_i will denote the ith module of the complex and $f_i : F_i \to F_{i-1}$ will denote the ith differential (in general, for the differential, we will use the lowercase of whichever uppercase letter denotes the complex).

If $\epsilon : \mathbf{F}_\bullet \to \mathbf{G}_\bullet$ is a chain map of complexes, then $\epsilon_i : F_i \to G_i$ will denote the ith map.

Step 1: Construct free complexes over R, and some equations, that capture the hypotheses of the lemma.

Let $K_\phi, I_\phi, C_\phi; K_\psi, I_\psi, C_\psi$ be the kernel, image, and cokernel of ϕ and ψ, respectively, and let $H = K_\psi / I_\phi$ be the homology at A^b. Let $\mathbf{K}_{\phi\bullet}, \mathbf{I}_{\phi\bullet}, \mathbf{C}_{\phi\bullet}, \mathbf{K}_{\psi\bullet}, \mathbf{I}_{\psi\bullet}, \mathbf{C}_{\psi\bullet}$ be minimal free resolutions for each over R. This gives us exact sequences of acyclic complexes of free R-modules

$$0 \longrightarrow \mathbf{K}_{\phi\bullet} \longrightarrow \mathbf{F}^{\mathbf{0}}_\bullet \xrightarrow{\alpha_\phi} \mathbf{I}_{\phi\bullet} \longrightarrow 0 \tag{3.4}$$

$$0 \longrightarrow \mathbf{I}_{\phi\bullet} \xrightarrow{\beta_\phi} \mathbf{F}^{\mathbf{1}}_\bullet \longrightarrow \mathbf{C}_{\phi\bullet} \longrightarrow 0 \tag{3.5}$$

$$0 \longrightarrow \mathbf{K}_{\psi\bullet} \xrightarrow{\iota_\psi} \mathbf{F}^{\mathbf{2}}_\bullet \xrightarrow{\alpha_\psi} \mathbf{I}_{\psi\bullet} \longrightarrow 0 \tag{3.6}$$

$$0 \longrightarrow \mathbf{I}_{\psi\bullet} \xrightarrow{\beta_\psi} \mathbf{F}^{\mathbf{3}}_\bullet \longrightarrow \mathbf{C}_{\psi\bullet} \longrightarrow 0 \tag{3.7}$$

where $\mathbf{F}^{\mathbf{0}}_\bullet$ is a finite free resolution of A^a over R, $\mathbf{F}^{\mathbf{1}}_\bullet$ and $\mathbf{F}^{\mathbf{2}}_\bullet$ of A^b, and $\mathbf{F}^{\mathbf{3}}_\bullet$ of A^c; and where $H_0(\beta_\phi \alpha_\phi) = \phi$ and $H_0(\beta_\psi \alpha_\psi) = \psi$.

Let $\mathbf{L}_\bullet$ be the minimal free resolution of A over R; we will write $\mathbf{L}^n_\bullet$ for the nth direct sum of $\mathbf{L}_\bullet$. Since $\mathbf{F}^{\mathbf{0}}_\bullet$ and $\mathbf{L}^a_\bullet$ are both free resolutions of A^a over R (and likewise for $\mathbf{F}^{\mathbf{1}}_\bullet$, and so on), we have short exact sequences of complexes

$$0 \longrightarrow \mathbf{L}^a_\bullet \xrightarrow{\epsilon^0} \mathbf{F}^{\mathbf{0}}_\bullet \longrightarrow \mathbf{E}^{\mathbf{0}}_\bullet \longrightarrow 0 \tag{3.8}$$

$$0 \longrightarrow \mathbf{E}^{\mathbf{1}}_\bullet \longrightarrow \mathbf{F}^{\mathbf{1}}_\bullet \xrightarrow{\epsilon^1} \mathbf{L}^b_\bullet \longrightarrow 0 \tag{3.9}$$

$$0 \longrightarrow \mathbf{E}^{\mathbf{2}}_\bullet \longrightarrow \mathbf{F}^{\mathbf{2}}_\bullet \xrightarrow{\epsilon^2} \mathbf{L}^b_\bullet \longrightarrow 0 \tag{3.10}$$

$$0 \longrightarrow \mathbf{L}^c_\bullet \xrightarrow{\epsilon^3} \mathbf{F}^{\mathbf{3}}_\bullet \longrightarrow \mathbf{E}^{\mathbf{3}}_\bullet \longrightarrow 0 \tag{3.11}$$

where each $\mathbf{E^i_\bullet}$, is a split exact sequence of free R-modules, and where each $H_0(\epsilon^i)$ is the identity map.

Now we observe that $H_0(\epsilon^1\beta_\phi\alpha_\phi\epsilon^0) = \phi$; so, writing $\phi^R : R^a \to R^b$ as the map corresponding to the "lifted" matrix (u_{ij}), we have

$$\epsilon^1_0\beta_{\phi 0}\alpha_{\phi 0}\epsilon^0_0 - \phi^R$$

is a map from R^a to R^b that is zero on A. Therefore, there exists some map $\delta_\phi : L^a_0 \to L^b_1$ such that

$$\epsilon^1_0\beta_{\phi 0}\alpha_{\phi 0}\epsilon^0_0 - \phi^R = l^b_1\delta_\phi. \tag{3.12}$$

(Recall that l^b_1 denotes the 1st differential of the complex $\mathbf{L^b_\bullet}$.)

Similarly, we have

$$H_0(\epsilon^3)\psi H_0(\epsilon^2) = \psi = H_0(\beta_\psi\alpha_\psi);$$

so writing $\psi^R : R^b \to R^c$ as the map corresponding to the lifted matrix (v_{ij}), we have

$$\epsilon^3_0\psi^R\epsilon^2_0 - \beta_{\psi 0}\alpha_{\psi 0}$$

is a map from R^a to R^b that is zero on A. Therefore, there exists some map $\delta_\psi : F^2_0 \to F^3_1$ such that

$$\epsilon^3_0\psi^R\epsilon^2_0 - \beta_{\psi 0}\alpha_{\psi 0} = f^3_1\delta_\psi. \tag{3.13}$$

(Recall that f^3_1 denotes the 1st differential of the complex $\mathbf{F^3_\bullet}$.)

Finally, we lift the inclusion $I_\phi \to K_\psi$ to a chain map $\eta : \mathbf{I}_{\phi\bullet} \to \mathbf{K}_{\psi\bullet}$. We can construct the tail of a finite free resolution of the homology H by

$$I_{\phi 0} \oplus K_{\psi 1} \overset{\eta_0 \oplus k_{\psi 1}}{\longrightarrow} K_{\psi 0} \;\longrightarrow\; H \;\longrightarrow\; 0.$$

(Recall that $k_{\psi 1}$ is the 1st differential of the complex $\mathbf{K}_{\psi\bullet}$.)

Now we note that $H_0(\epsilon^1\beta_\phi) = H_0(\epsilon^2\iota_\psi\eta)$ (both are just the inclusion $I_\phi \to A^b$), so we have

$$\epsilon^1_0\beta_{\phi 0} - \epsilon^2_0\iota_{\psi 0}\eta_0$$

is a map from $I_{\phi 0}$ to R^b that is zero on A. Therefore, there exists some map $\delta_\eta : I_{\phi 0} \to L^b_1$ such that

$$\epsilon^1_0\beta_{\phi 0} - \epsilon^2_0\iota_{\psi 0}\eta_0 = l^b_1\delta_\eta. \tag{3.14}$$

We now construct polynomials $f_1, \ldots, f_N$ that preserve the fact that

- $\mathbf{K}_{\phi\bullet}, \mathbf{I}_{\phi\bullet}, \mathbf{C}_{\phi\bullet}, \mathbf{K}_{\psi\bullet}, \mathbf{I}_{\psi\bullet}, \mathbf{C}_{\psi\bullet}, \mathbf{F^0_\bullet}, \mathbf{F^1_\bullet}, \mathbf{F^2_\bullet}, \mathbf{F^3_\bullet}, \mathbf{L_\bullet}$ are acyclic complexes (Lemma 3.3.5);

- $\mathbf{E^0_\bullet}, \mathbf{E^1_\bullet}, \mathbf{E^2_\bullet}, \mathbf{E^3_\bullet}$ are split exact complexes (Lemma 3.3.5 also);

- Equations (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) are exact sequences of complexes (Lemmas 3.3.5 and equations so that the differentials commute with the maps between the complexes);
- Equations (3.12), (3.13), (3.14) hold;
- $\mathbf{L}_\bullet$ is a minimal free resolution for A, and the last differential uses the variables $G_1, \dots, G_l$ corresponding to $g_1, \dots, g_l$ (Lemmas 3.3.7 and 3.3.6);
- $\eta : \mathbf{I}_{\phi\bullet} \to \mathbf{K}_{\psi\bullet}$ is a chain map (equations so that the differentials commute with η), and $h = \dim \operatorname{coker}(\eta_0 \oplus k_{\psi 1})$ (Lemma 3.3.9).

Step 2: Show that solutions of the complexes and equations in Step 1 actually preserve the data we want.

Let R' be a regular local ring with solution $(\underline{y}', \underline{g}', (u'_{ij}), (v'_{ij}), \underline{w}')$ to the equations $f_1, \dots, f_N$, such that $y'_1, \dots, y'_n$ forms an R'-sequence. Let $A' = R'/(g'_1, \dots, g'_l)$, and let ϕ' and ψ' be the maps corresponding to the matrices $(\overline{u}'_{ij})$ and $(\overline{v}'_{ij})$, respectively. By construction, we have

- Acyclic complexes of free R'-modules:

$$\mathbf{K}'_{\phi\bullet}, \mathbf{I}'_{\phi\bullet}, \mathbf{C}'_{\phi\bullet}, \mathbf{K}'_{\psi\bullet}, \mathbf{I}'_{\psi\bullet}, \mathbf{C}'_{\psi\bullet}, \mathbf{F}^{\mathbf{0}\prime}_\bullet, \mathbf{F}^{\mathbf{1}\prime}_\bullet, \mathbf{F}^{\mathbf{2}\prime}_\bullet, \mathbf{F}^{\mathbf{3}\prime}_\bullet, \mathbf{L}'_\bullet;$$

- Split exact complexes of free R'-modules: $\mathbf{E}^{\mathbf{0}\prime}_\bullet, \mathbf{E}^{\mathbf{1}\prime}_\bullet, \mathbf{E}^{\mathbf{2}\prime}_\bullet, \mathbf{E}^{\mathbf{3}\prime}_\bullet$;
- "Prime" versions of equations (3.4), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11) are exact sequences of complexes;
- "Prime" versions of equations (3.12), (3.13), (3.14) hold;
- $\mathbf{L}'_\bullet$ is a minimal free resolution for A';
- $\eta' : \mathbf{I}'_{\phi\bullet} \to \mathbf{K}'_{\psi\bullet}$ is a chain map, and $h = \dim \operatorname{coker}(\eta'_0 \oplus k'_{\psi 1})$.

First we note that, since each complex is either acyclic or split exact, each short exact sequence of complexes gives rise to a short exact sequence on H_0; e.g., (3.4') gives rise to

$$0 \longrightarrow H_0(\mathbf{K}'_{\phi\bullet}) \longrightarrow H_0(\mathbf{F}^{\mathbf{0}\prime}_\bullet) \xrightarrow{H_0(\alpha'_\phi)} H_0(\mathbf{I}'_{\phi\bullet}) \longrightarrow 0.$$

Also, since each complex $\mathbf{E}^{\mathbf{i}\prime}_\bullet$ is split exact, we have that each $H_0(\mathbf{E}^{\mathbf{i}\prime}_\bullet) = 0$.

Now, taking H_0 of the exact sequences (3.4'), (3.5'), (3.8'), and (3.9'), and using the fact that $\mathbf{E}^{\mathbf{0}\prime}_\bullet$ and $\mathbf{E}^{\mathbf{1}\prime}_\bullet$ are split exact, we have a composite map

$$\begin{tikzcd}
H_0(\mathbf{L}^{a\prime}_\bullet) \arrow[r, "H_0(\epsilon^{0\prime})"] \arrow[d, equal] & H_0(\mathbf{F}^{\mathbf{0}\prime}_\bullet) \arrow[r, "H_0(\alpha'_\phi)"] & H_0(\mathbf{I}'_\bullet) \arrow[r, "H_0(\beta'_\phi)"] & H_0(\mathbf{F}^{\mathbf{1}\prime}_\bullet) \arrow[r, "H_0(\epsilon^{1\prime})"] & H_0(\mathbf{L}^{b\prime}_\bullet) \arrow[d, equal] \\
(A')^a & & & & (A')^b
\end{tikzcd}$$

By equation (3.12'), this composition equals ϕ', and hence (since $H_0(\alpha'_\phi\epsilon^{0'})$ is surjective)

$$\operatorname{im}(\phi') = \operatorname{im}(H_0(\epsilon^{1'}\beta'_\phi)). \tag{3.15}$$

Next, taking H_0 of the exact sequences (3.6'), (3.7'), (3.10'), and (3.11'), and using the fact that $\mathbf{E}_\bullet^{\mathbf{2}'}$ and $\mathbf{E}_\bullet^{\mathbf{3}'}$ are split exact, we have a composite map

$$H_0(\mathbf{F}_\bullet^{\mathbf{2}}) \xrightarrow{H_0(\epsilon^{2'})} H_0(\mathbf{L}_\bullet^b) \xrightarrow{\psi'} H_0(\mathbf{L}_\bullet^c) \xrightarrow{H_0(\epsilon^{3'})} H_0(\mathbf{F}_\bullet^{\mathbf{3}})$$

By equation (3.13'), this composition equals $H_0(\beta_\psi\alpha_\psi)$. Now consider the combination of exact sequences (3.6') and (3.7'):

$$0 \longrightarrow \mathbf{K}'_{\psi\bullet} \xrightarrow{\iota_\psi{}'} \mathbf{F}_\bullet^{\mathbf{2}'} \xrightarrow{\beta'_\psi\alpha'_\psi} \mathbf{F}_\bullet^{\mathbf{3}'} \longrightarrow \mathbf{C}'_{\psi\bullet} \longrightarrow 0;$$

we have, because $H_0(\epsilon^{3'})$ is an isomorphism,

$$\operatorname{im}(H_0(\iota'_\psi)) = \ker(H_0(\beta'_\psi\alpha'_\psi)) = \ker(\psi' H_0(\epsilon^{2'}))$$

Since $H_0(\epsilon^{2'})$ is an isomorphism, we conclude that

$$\operatorname{im}(H_0(\epsilon^{2'}\iota'_\psi)) = \ker(\psi'). \tag{3.16}$$

Thus, letting H' be the homology of the complex

$$A'^a \xrightarrow{\phi'} A'^b \xrightarrow{\psi'} A'^c,$$

equations (3.15) and (3.16) imply that

$$H' = \frac{\operatorname{im}(H_0(\epsilon^{2'}\iota'_\psi))}{\operatorname{im}(H_0(\epsilon^{1'}\beta'_\phi))}.$$

By equation (3.14'), we have $H_0(\epsilon^{1'}\beta'_\phi) = H_0(\epsilon^{2'}\iota'_\psi\eta')$, so

$$H' = \frac{\operatorname{im}(H_0(\epsilon^{2'}\iota'_\psi))}{\operatorname{im}(H_0(\epsilon^{2'}\iota'_\psi\eta'))}.$$

Since $H_0(\epsilon^{2'}\iota'_\psi)$ is injective, we conclude that

$$H' = \operatorname{coker} H_0(\eta').$$

But this equals $\operatorname{coker}(\eta'_0 \oplus k'_{\psi 1})$, which has dimension h, as desired. □

This allows us to preserve many properties of modules. In particular:

Lemma 3.3.11. *Let R be a regular local ring, let $A = R/(g_1, \ldots, g_l)$ be a quotient, and let M be an A-module of finite projective dimension. We let*

$d = \dim A, r = \dim M, g = \operatorname{grade}_A M$, and $e_i = \dim \operatorname{Ext}^i_A(M, A)$. Let

$$F_\bullet : \qquad 0 \longrightarrow A^{b_h} \xrightarrow{\phi^h} A^{b_{h-1}} \longrightarrow \cdots \xrightarrow{\phi^1} A^{b_0}$$

be a minimal free resolution for M. We choose matrices $(\overline{u}^k_{ij})$ that represent the maps ϕ^k, for $1 \le k \le h$. Then there are polynomials $f_1, \ldots, f_N$ with coefficients in $\mathbb{Z}$ in indeterminates

1. *$Y_1, \ldots, Y_n$*
2. *$X_1, \ldots, X_d$*
3. *$G_1, \ldots, G_l$*
4. *U^k_{ij}, corresponding to the matrices u^k_{ij}*
5. *$W_1, \ldots, W_t$ (for some sufficiently large t)*

such that

1. *There are $\underline{y}, \underline{x}, \underline{w}$ in R such that $(\underline{y}, \underline{x}, \underline{g}, (u^k_{ij}), \underline{w})$ is a solution to $f_1, \ldots, f_N$, with $y_1, \ldots, y_n$ a regular system of parameters for R and $x_1, \ldots, x_d$ a system of parameters for A.*
2. *If $(\underline{y}', \underline{x}', \underline{g}', ({r'}^k_{ij}), \underline{w}')$ is a solution of $f_1, \ldots, f_N$ in a regular local ring R' with $y'_1, \ldots, y'_n$ forming a regular system of parameters for R' and $x'_1, \ldots, x'_d$ is a system of parameters for $A' = R'/\underline{g}'$, then letting ${\phi'}^k$ be maps corresponding to the matrices $(\overline{u'}^k_{ij})$, we have*

$$F'_\bullet : \qquad 0 \longrightarrow (A')^{b_h} \xrightarrow{{\phi'}^h} (A')^{b_{h-1}} \longrightarrow \cdots \xrightarrow{{\phi'}^1} (A')^{b_0}$$

is exact; and, setting $M' = \operatorname{coker}({\phi'}^1)$, we have $r = \dim M', g = \operatorname{grade}_{A'} M'$, and $e_i = \dim \operatorname{Ext}^i_{A'}(M', A')$.

Proof. For each pair of maps in the free resolution for M,

$$A^{b_{k+1}} \xrightarrow{\phi^{k+1}} A^{b_k} \xrightarrow{\phi^k} A^{b_{k-1}},$$

we choose equations as in Lemma 3.3.10, using the same variables U^k_{ij} for each map ϕ^k. This preserves the projective resolution of M, as well as $\dim M$.

Now taking $\operatorname{Hom}(-, A)$ of the free resolution for M, for each pair of maps

$$A^{b_{k-1}} \xrightarrow{\phi^{k*}} A^{b_k} \xrightarrow{\phi^{k+1*}} A^{b_{k+1}},$$

we choose equations as in Lemma 3.3.10, using the same variables for the dual of a map as we used for the map itself. This preserves $\dim \operatorname{Ext}^k_A(M, A)$ (including whether $\operatorname{Ext}^k_A(M, A) = 0$), and so it also preserves $\operatorname{grade} M$. Combining these sets of equations, we can preserve all the numerical data, as desired. $\square$

We can now reduce the Grade Conjecture to characteristic p as follows.

Theorem 3.3.1. *Suppose that the Grade Conjecture holds over every local ring of characteristic p. Then it also holds for every local ring of equicharacteristic zero.*

Proof. We let A be an equicharacteristic local ring and let M be a finitely generated A-module of finite projective dimension, and we will show that

$$\operatorname{grade} M + \dim M = \dim A.$$

Without loss of generality, we can replace A and M by $\hat{A}$ and $\hat{M}$, since completion does not change grade, dimension, or projective dimension. By the Cohen Structure Theorem (Theorem 2.2.12), we can choose a regular local ring R of which A is a quotient.

By Lemma 3.3.11, we can find a local ring A' of characteristic p and a finitely generated A'-module M' such that

1. $\dim A = \dim A'$;
2. M' has finite projective dimension over A';
3. $\dim_{A'} M' = \dim_A M$ and $\operatorname{grade}_{A'} M' = \operatorname{grade}_A M$.

Since A' has characteristic p, M' satisfies the Grade Conjecture, and so M does as well. $\square$

4 Main Results: Asymptotic Intersection Multiplicity

Throughout this chapter, unless otherwise specified, we suppose A is a complete local ring of characteristic p.

4.1 Asymptotic Intersection Multiplicity

To begin, we recall the definition of e_∞:

Definition 2.6.11. Let M be a finitely-generated A-module and let $\underline{x} = x_1, \ldots, x_r$ be a system of parameters on M. We define

$$e_\infty(\underline{x}; M) = \lim_{n\to\infty} \frac{e(\underline{x}; F^n(M))}{p^{n\cdot \operatorname{codim} M}}.$$

We will relate the positivity of e_∞ to the dimension of $\operatorname{Ext}^{\operatorname{codim} M}(M, A)$ using Lemma 3.2.5 as follows:

Theorem 4.1.1. *Suppose* $\operatorname{pd} M < \infty$*, and set* $d = \dim A$ *and* $r = \dim M$*. Then*

$$\dim \operatorname{Ext}^{d-r}(M, A) = r$$

if and only if $e_\infty(\underline{x}; M) > 0$ *for some (= every) system of parameters* $x_1, \ldots, x_r$ *for* M *that is part of a system of parameters for* A*.*

Proof. We apply Theorem 2.1.7 to $\underline{x}$ and $F^n(M)$, which says that

$$e(\underline{x}; F^n(M)) = \sum_{\mathfrak{p}} e(\underline{x}; A/\mathfrak{p})\ell(F^n(M)_{\mathfrak{p}}),$$

where the sum is taken over all primes $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim A/\mathfrak{p} = r$. Now, we recall that the Frobenius functor commutes with localization (Theorem 2.6.6), i.e.,

$$\ell(F^n_A(M)_{\mathfrak{p}}) = \ell(F^n_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})).$$

Dividing by $p^{n(d-r)}$ and taking the limit as $n \to \infty$, we have

$$e_\infty(\underline{x}; F^n(M)) = \sum_{\mathfrak{p}} e(\underline{x}; A/\mathfrak{p}) \lim_{n\to\infty} \frac{\ell(F^n_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}))}{p^{n(d-r)}}.$$

For each prime $\mathfrak{p}$, consider

$$\lim_{n\to\infty} \frac{\ell(F^n_{A_\mathfrak{p}}(M_\mathfrak{p}))}{p^{n(d-r)}};$$

by Lemma 2.6.10, this is positive if and only if $\dim A_\mathfrak{p} = d - r$ (i.e., $\operatorname{ht}\mathfrak{p} = d - r$). Therefore, $e_\infty(\underline{x}; M) > 0$ if and only if there is *some* prime $\mathfrak{p}$ with $\dim A/\mathfrak{p} = r$ and $\operatorname{ht}\mathfrak{p} = d - r$; which, by Lemma 3.2.5, happens if and only if $\dim \operatorname{Ext}^{d-r}(M, A) = r$. $\square$

Remark 4.1.2. If the Dimension Inequality is true, then it implies that any system of parameters for M is part of a system of parameters for A, so the last assumption in the above theorem would be superfluous.

In order to relate e_∞ to χ_∞, we need a lemma about the Euler characteristic of a spectral sequence.

Lemma 4.1.3. *Let A be a ring, and let $\{E^r_{ij}\}$ be a homology spectral sequence of A-modules for $r \geq a$ converging to H_*. Assume that*

1. *There are only finitely many nonzero E^a_{ij}; and*
2. *Each E^a_{ij} has finite length.*

Then

1. *For every $r \geq a$, there are only finitely many nonzero E^r_{ij}, and each E^r_{ij} has finite length; and*
2. *There are only finitely many nonzero H_n, and each H_n has finite length.*

Moreover, for $r \geq a$, let

$$\chi(E^r_{ij}) = \sum_{i,j} (-1)^{i+j} \ell(E^r_{ij})$$

and

$$\chi(H_*) = \sum_n (-1)^n \ell(H_n).$$

*(These are known as **Euler characteristics**.) Then for all $r \geq a$,*

$$\chi(E^r_{ij}) = \chi(H_*).$$

Proof. Since E^{r+1}_{ij} is a subquotient of E^r_{ij}, it follows by induction on r that there are only finitely many nonzero entries, and each has finite length. Also, the spectral sequence is bounded, so eventually it stabilizes, so there are only finitely many nonzero E^∞_{ij}, and each has finite length. Therefore, since each H_n is filtered by subquotients that equal the E^∞_{ij}, there are only finitely many of them, and each has finite length.

Next, we claim that for all $r \geq a$,

$$\chi(E^r_{ij}) = \chi(E^{r+1}_{ij}).$$

To see this, recall that E^{r+1}_{ij} is the homology of the complex of "lines of slope $-(r+1)/r$" in the lattice E^r_{**}. By Lemma 2.1.12, it follows that, for each pair of integers a and b,

$$\sum_t (-1)^t \ell(E^r_{a-tr,b+t(r-1)}) = \sum_t (-1)^t \ell(E^{r+1}_{a-tr,b+t(r-1)}).$$

For fixed a and b, the points $(a - tr, b + t(r - 1))$ form a line; for each such line L, we choose a single representative point (a_L, b_L). Taking an alternate sum over all lines L, we get

$$\sum_{L,t} (-1)^{a_L+b_L+t} \ell(E^r_{a_L-tr,b_L+t(r-1)}) = \sum_{L,t} (-1)^{a_+b_L+t} \ell(E^{r+1}_{a_L-tr,b_L+t(r-1)}).$$

We now make a change of variables $i = a_L - tr$ and $j = b_L + t(r - 1)$. Since we've chosen a unique representative (a_L, b_L) for each line L, summing over all lines L and all integers t is equivalent to summing over all integers i and j:

$$\chi(E^r_{ij}) = \sum_{i,j} (-1)^{i+j} \ell(E^r_{ij}) = \sum_{i,j} (-1)^{i+j} \ell(E^{r+1}_{ij}) = \chi(E^{r+1})_{ij}),$$

as claimed.

Since the spectral sequence eventually stabilizes, it remains to show that

$$\chi(E^\infty_{ij}) = \chi(H_*).$$

Indeed, let F_iH_n be the filtration associated with H_*. Since the spectral sequences converges to H_*, we have

$$\begin{aligned} \chi(H_*) &= \sum_n (-1)^n \ell(H_n) = \sum_{n,i} (-1)^n \ell(F_{i+1}H_n/F_iH_n) \\ &= \sum_{n,i} (-1)^n \ell(E^\infty_{i,n-i}) = \chi(E^\infty_{ij}), \end{aligned}$$

as desired. □

We will now relate e_∞ to χ_∞ for a suitably chosen system of parameters:

Proposition 4.1.4. *Suppose* $\operatorname{pd} M < \infty$ *and* $\dim M < \dim A$. *Then there is a system of parameters* $x_1, \ldots, x_r$ *for* M*, that is part of a system of parameters for* A*, such that*

$$e(\underline{x}; M) = \chi(M, A/\underline{x}).$$

Proof. We first claim that we can choose a system of parameters $x_1, \ldots, x_r$ for M that is part of a system of parameters for A, such that the higher Koszul homologies have finite length, i.e.,

$$\ell(H_i(\underline{x};A)) < \infty \text{ for all } i \geq 1.$$

To do so, choose $x_{i+1} \in \mathfrak{m}$

1. a parameter on $M/(x_1,\dots,x_i)M$
2. a parameter on $A/(x_1,\dots,x_i)A$
3. not in any prime $\mathfrak{p} \neq \mathfrak{m}$ in $\operatorname{Ass}(A/(x_1,\dots,x_i)A)$

Then, if we localize the Koszul complex $K_\bullet(\underline{x};A)$ at any nonmaximal prime, all the x_i will be either units or nonzerodivisors mod $x_1,\dots,x_{i-1}$, so the higher homologies will vanish, and hence they all have finite length.

Now, let $D_{\bullet\bullet}$ be the double complex from tensoring $K_\bullet(\underline{x};A)$ with a free resolution $L_\bullet$ for M. By Theorem 2.4.4, there are two associated spectral sequences, both converging to $H_*(\operatorname{Tot}(D_{\bullet\bullet}))$:

1. ${}^I E^2_{st} = H_t(\underline{x};M)$ when $s=0$, and zero otherwise
2. ${}^{II} E^2_{st} = \operatorname{Tor}_s(M, H_t(\underline{x};A))$

Since all the above E^2 terms have finite length, and there are only finitely many of them (they vanish for $s > \operatorname{pd} M$ and $t > r$), Lemma 4.1.3 implies that the Euler characteristics of the two E^2 terms are equal, so we have

$$\sum_t (-1)^t \ell(H_t(\underline{x};M)) = \sum_{s,t} (-1)^{s+t} \ell(\operatorname{Tor}_s(M, H_t(\underline{x};A))).$$

By Theorem 2.3.7, the left-hand side is $e(\underline{x};M)$; and the right side is just

$$\sum_t (-1)^t \chi(M, H_t(\underline{x};A)).$$

For $t \geq 1$, the $H_t(\underline{x};A)$ have finite length, by the choice of $\underline{x}$ above; since $\dim M < \dim A$, we have $\chi(M, H_t(\underline{x};A)) = 0$ for $t \geq 1$. Thus

$$e(\underline{x};M) = \chi(M, H_0(\underline{x};A)) = \chi(M, A/\underline{x}),$$

as desired. □

Note that the choice of $x_1,\dots,x_r$ in the proof of Proposition 4.1.4 depends only on $\operatorname{Supp} M$, so the same system of parameters can be used for $F^n(M)$ for all n. We can therefore use Proposition 4.1.4 to translate Theorem 4.1.1 to χ_∞:

Theorem 4.1.5. *Suppose* $\operatorname{pd} M < \infty$, *and set* $d = \dim A$ *and* $r = \dim M$. *Then there is a system of parameters* $x_1,\dots,x_r$ *for* M, *that is part of a system of parameters for* A, *such that*

$$\chi_\infty(M, A/\underline{x}) > 0 \textit{ if and only if } \dim \operatorname{Ext}^{d-r}(M,A) = r.$$

For the proof of the next theorem, we need the following theorem due to Dutta [Dut96]:

Theorem 4.1.6 (Dutta)**.** *Let $F_\bullet$ be a complex of finitely-generated free modules with homologies of finite length. Let N be a finitely generated module. Let W_{jn} denote the jth homology of $\operatorname{Hom}(F^n(F_\bullet), N)$, and write*

$$\tilde{N} = \operatorname{Hom}(H^d_{\mathfrak{m}}(N), E),$$

where $E = E(k)$ is the injective hull of k.

We have the following:

1. *If $\dim N < \dim A$, then $\lim \ell(W_{jn})/p^{nd} = 0$.*
2. *If $\dim N = \dim A$, and*
   (a) *$j < d$, then $\lim \ell(W_{jn})/p^{nd} = 0$;*
   (b) *$j = d$, then $\lim \ell(W_{jn})/p^{nd} = \lim \ell(F^n(H_0(F_\bullet)) \otimes \tilde{N})/p^{nd}$, which is positive;*
   (c) *$j > d$, then $\lim \ell(W_{jn})/p^{nd} = \lim \ell(H_{j-d}(F^n(F_\bullet)) \otimes \tilde{N})/p^{nd}$.*

We will now establish a special case of asymptotic positivity:

Theorem 4.1.7. *Let $d = \dim A$ and $r = \dim M$, and suppose that $\operatorname{pd} M = d - r$. Then any system of parameters $x_1, \ldots, x_r$ for M is part of a system of parameters for A, and*

$$\chi_\infty(M, A/\underline{x}) > 0.$$

Proof. We write $B = A/\underline{x}$. By the Intersection Theorem,

$$\dim B \leq \operatorname{pd} M = d - r,$$

and so we can choose $y_1, \ldots, y_{d-r}$ such that $\ell(B/\underline{y}B) < \infty$. Since $d = \dim A$, it follows that $x_1, \ldots, x_r, y_1, \ldots, y_{d-r}$ is a system of parameters for A, proving the first claim.

Now we let $L_\bullet$ be a free resolution for M over A, and let $F_\bullet = \operatorname{Hom}(L_\bullet, B)$. We will apply Theorem 4.1.6 to $F_\bullet$ over the ring B, with $N = B$. We note that

$$W_{jn} = \operatorname{Tor}_{d-r-j}(F^n(M), B),$$

since the length of the complex $L_\bullet$ is $\operatorname{pd} M = d - r$ (we are flipping the indices for $F_\bullet$; that is, $F_i = \operatorname{Hom}(L_{d-r-i}, B)$). By Theorem 4.1.6,

$$\lim_{n\to\infty} \frac{\ell(W_{jn})}{p^{n(d-r)}} = 0,$$

for all $j < d - r$, which means that

$$\chi_\infty(M, B) = \lim_{n\to\infty} \frac{\ell(F^n(M) \otimes B)}{p^{n(d-r)}} > 0$$

by Lemma 2.6.10, proving the theorem. $\square$

4.2 Connection With The Grade Conjecture

We will now prove a connection between the Grade Conjecture and χ_∞. Recall the following Proposition from Chapter 3.

Proposition 3.2.6. *Let A be a local ring (of any characteristic), M a finitely generated A-module of finite projective dimension, and assume that* $\operatorname{grade} M + \dim M = \dim A$ *(i.e., the Grade Conjecture holds for M). Let $d = \dim A$ and $r = \dim M$. Then* $\dim \operatorname{Ext}^{d-r}(M, A) = \dim M$.

By Theorem 4.1.5, this implies:

Theorem 4.2.1. *Let A be a local ring in characteristic p, M a finitely generated A-module of finite projective dimension, and assume that the Grade Conjecture holds for M. Then there is a system of parameters $x_1, \ldots, x_r$ for M, that is part of a system of parameters for A, such that*

$$\chi_\infty(M, A/\underline{x}) > 0.$$

We also can use χ_∞ to show special cases for which $\dim \operatorname{Ext}^{d-r}(M, A) = r$.

Theorem 4.2.2. *Let A be a local ring in characteristic p, M a finitely generated A-module of finite projective dimension and let $d = \dim A$ and $r = \dim M$. If* $\operatorname{pd} M = d - r$, *then* $\dim \operatorname{Ext}^{d-r}(M, A) = r$.

Proof. Choose $x_1, \ldots, x_r$ as in Proposition 4.1.4. We then have

$$e_\infty(\underline{x}; M) = \chi_\infty(M, A/\underline{x}).$$

Since $\operatorname{pd} M = d - r$, Theorem 4.1.7 implies that $\chi_\infty(M, a/\underline{x}) > 0$, so $e_\infty(\underline{x}; M) > 0$ as well. Theorem 4.1.1 then implies that $\operatorname{Ext}^{d-r}(M, A) = r$, as desired. $\square$

The statement of Theorem 4.2.2 can be translated to equicharacteristic zero using the techniques of Section 3.3.

Theorem 4.2.3. *Let A be a local ring of equal characteristic, M a finitely generated A-module of finite projective dimension and let $d = \dim A$ and $r = \dim M$. If* $\operatorname{pd} M = d - r$, *then* $\dim \operatorname{Ext}^{d-r}(M, A) = r$.

Proof. Suppose the hypotheses of the theorem hold in a ring A of equicharacteristic zero. By Lemma 3.3.11, we can find a local ring A' of characteristic p and a finitely generated A'-module M' such that

1. $\dim A' = d$;
2. $\dim_{A'} M' = rM$.
3. $\operatorname{pd}_{A'} M' = d - r$.

4. $\dim_{A'} \operatorname{Ext}^{d-r}_{A'}(M', A') = \dim_A \operatorname{Ext}^{d-r}_A(M, A)$.

By Theorem 4.2.2, the conclusion of the theorem is true of A'; that is,

$$\dim_{A'} \operatorname{Ext}^{d-r}_{A'}(M', A') = r,$$

which implies the same conclusion over the ring A as well. $\square$

Corollary 4.2.4. *Let A be a local ring of equal characteristic, M a finitely generated A-module of finite projective dimension, and let $d = \dim A$. If $\dim M = 1$, then $\dim \operatorname{Ext}^{d-1}(M, A) = 1$.*

Proof. By the Intersection Theorem, either $\operatorname{pd} M = d - 1$ or $\operatorname{pd} M = d$.

In the former case, the result follows from Theorem 4.2.3. In the latter case, A is Cohen-Macaulay, so the grade conjecture holds (Corollary 3.1.4) and the result follows from Proposition 3.2.6. $\square$

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