GAUSSIAN-LIKE VON NEUMANN ALGEBRAS AND NONCOMMUTATIVE BROWNIAN MOTION

BY

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DISSERTATION

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Abstract

The q-Gaussian von Neumann algebras were first defined and studied by Bożejko and Speicher in connection with noncommutative brownian motion. The main results of the present work is to establish that the q-Gaussian von Neumann algebras have the weak* completely contractive approximation property for all $-1 < q < 1$ and any number of generators, and they are strongly solid for all $-1 < q < 1$ and any finite number of generators.

To My Parents, Thomas and Susan Avsec, for their many years of loving support.

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Chapter 1

Introduction

In [2], Bożejko and Speicher gave the following definition of "generalized brownian motion"

Definition 1.0.1. An *n-dimensional generalized Brownian motion* is a triple $(\mathcal{A}, \rho, (a_I^1, \cdots, a_I^n)_{I \in \mathcal{R}}$ where $\mathcal{A}$ is a $*$-algebra, ρ is a state, and $I \to (a_I^1, \ldots, a_I^n)$ is a finitely additive mapping such that

1. pyramidally ordered moments factorize.
2. the moments $\rho(\hat{c}_{I_1+t}^{k(1)} \cdots \hat{c}_{I_n+t}^{k(n)})$ are independent of $t \in \mathbb{R}$ for all $n \in \mathbb{N}$, $k(j) \in \{1, \ldots, m\}$ and $I_j \in \mathcal{R}$ where $\hat{c}$ stands for c or c^* and $I + t = \{s + t | s \in I\}$.
3. for all $n \in \mathbb{N}$, $k(j) \in \{1, \ldots, m\}$, $I \in \mathcal{R}$

$$\rho(\hat{c}_I^{k(1)} \cdots \hat{c}_I^{k(n)}) = \begin{cases} 0, & \text{if n is odd} \\ \lambda(I)^{\frac{n}{2}} \rho(\hat{c}_{[0,1)}^{k(1)} \cdots \hat{c}_{[0,1)}^{k(n)}) & \text{if n is even} \end{cases}$$

Here $\mathcal{R}$ is the semiring of all half-open intervals of $\mathbb{R}$ and "pyramidally ordered moments factorize" means that

$$\rho(a_1 \cdots a_m b_m \cdots b_1 = \rho(a_1 b_1) \cdots \rho(a_m b_m)$$

for $a_j, b_j \in \mathcal{A}_{I_j}$ and $I_j < I_k$ for $j < k$.

Their primary example of such a brownian motion was q-brownian motion which is generated by operators $a(f)$ for $f \in L^2(\mathbb{R}, d\lambda)$ satisfying the q-commutation relation

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g \rangle 1.$$

In [2], Bożejko and Speicher went on to prove that these commutation relations could be realized by the annhilation operator on a Fock space symmetrized in the following way. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{F}(\mathcal{H}) := \oplus_{n \geq 0} \mathcal{H}^{\otimes n}$ be the algebraic Fock space over $\mathcal{H}$. Here $\mathcal{H}^{\otimes 0} \simeq \mathbb{C}$ is spanned by a vector Ω called the *vacuum vector.* Define a sesquilinear form $\langle , \rangle_q$ by

$$\langle h_1 \otimes \cdots \otimes h_m, k_1 \otimes \cdots \otimes k \rangle_q = \delta_{mn} \sum_{\sigma \in S_n} q^{\iota(\sigma)} \prod_{j=1}^{n} \langle h_j, k_{\sigma(j)} \rangle_{\mathcal{H}}.$$

where $\iota(\sigma)$ denotes the number of inversions of the permutation σ. Bożejko and Speicher also proved that this sesquilinear form was positive definite so that $a_q(f)$ was representable on a Hilbert space $\mathcal{F}_q(\mathcal{H})$ and that $a_q(f)^* = a_q^*(f)$.

The q-Gaussian variables $s_q(h) = a_q(h) + a_q^*(h)$ were introduced by Bożejko and Speicher in [2] as an interpolation between classical Gaussian variables in the case q = 1, fermionic variables in the case q = -1, and Voiculescu's free Gaussians in the case q = 0. These variables can be defined functorially from a real Hilbert space H as being generated by self-adjoint elements $s_q(h)$, $h \in H$, which satisfy the moment formula

$$\tau(s_q(h_1) \dots s_q(h_n)) = \begin{cases} 0 & \text{if n is odd} \\ \sum_{\rho \in P_2(n)} q^{\iota(\rho)} \prod_{\{j,k\}} \langle h_j, h_k \rangle & \text{if n is even} \end{cases}.$$

1.1 q-Gaussian von Neumann Algebras

Later, the von Neumann algebras generated by the q-Gaussian variables, denoted by $\Gamma_q(H)$, were shown to have properties similar to those of the free group factors. Note that the free group factors $L\mathbb{F}_n$ are isomorphic to $\Gamma_0(H)$ where $n = \dim(H)$ by a famous result of Voiculescu in [31]. Indeed, for a certain range of q and $\dim(H)$, Bożejko, Kümmerer, and Speicher in [1] established that the q-Gaussian algebras are factors. Ricard proved in [25] that $\Gamma_q(H)$ is a factor for all $-1 < q < 1$ and all $\dim(H) \geq 2$. That they do not have property Γ was established by Sniady in [29] for a certain range of q and large dimension. Nou proved that the q-Gaussian algebras are non-injective for all $-1 < q < 1$ and $\dim(H) \geq 2$ in [15].

Shlyakhtenko proved in [26] that the q-Gaussian algebras are solid in the sense of Ozawa for $|q| < \sqrt{2} - 1$ using estimates on non-microstates free entropy dimension. In [27], Shlyakhtenko further proved that the q-Gaussian algebras do have Cartan subalgebras for a small range of q. Recently, Dabrowski improved these estimates in [8] to prove that an n-tuple of q-Gaussian variables have microstate free entropy dimension n for $|q|n < 1$ and $q^2 n \leq 0.0169$. This result implies the present results for this range of q and n, though we use radically different techniques.

In their landmark papers [17] and [18], Ozawa and Popa introduced the notion of strong solidity. A type II_1 von Neumann algebra $\mathcal{M}$ is *strongly solid* if for any diffuse injective von Neumann subalgebra $\mathcal{A} \subset \mathcal{M}$, the normalizer of $\mathcal{A}$, $\mathcal{N}_{\mathcal{M}}(\mathcal{A}) := \{u \in U(\mathcal{M}) | u^* A u = A\}$, generates an injective von Neumann algebra. In [17], Ozawa and Popa prove that the free group factors have this property. Their proof relies on two properties: that the free group factors admit a particular approximation property (the weak* completely contractive approximation property) and that the free group admits a proper 1-cocycle in a representation which is weakly contained in the left regular representation. We establish analogous results for the q-Gaussian von Neumann algebras.

1.1.1 Approximation Properties

Approximation properties of C*-algebras and von Neumann algebras have provided a fundamental tool for many landmark results and applications of operator algebras. The strongest approximation property is the (weak*) completely positive approximation property which was shown by Effros and Choi in [3] to be equivalent to nuclearity in the C*-algebra setting and shown by Connes in [4] to be equivalent to injectivity in the von Neumann algebra setting. These are equivalent to amenability in the group setting. A less restrictive property is the (weak*) completely bounded approximation property, which Haagerup first introduced in [12] and [9]. This property has become very important following the seminal work of Ozawa and Popa and the recent follow up paper of Ozawa ([16]). It is equivalent to weak amenability of groups.

Our first main result is the following:

Theorem A. *$\Gamma_q(H)$ has the weak*CCAP for all $q \in (-1, 1)$ and $\dim(H) \geq 1$.*

1.1.2 Bimodules

The q-Gaussian algebras admit an "s-malleable" deformation in the sense of Popa (see, for example, [21] or [22]). However, we show that the bimodule associated to this deformation is not weakly contained in the coarse bimodule. The situation is similar in spirit to that of [28] but lacks the structure of an underlying group. In this paper, we find a subbimodule which is weakly contained in the coarse bimodule. Combining this and an adjustment to Popa's estimate for s-malleable deformations (see Lemma 2.1 in [23]), we were able to establish our second main result which is proved in Chapter 4.

Theorem B. *For all $-1 < q < 1$ and all $\dim(H) < \infty$, $\Gamma_q(H)$ is strongly solid.*

Chapter 2

Preliminaries

2.1 Preliminaries

We use standard notation and results from von Neumann algebra theory (see e.g. [30]) and operator space theory (see e.g. [11] or [19]).

2.1.1 The CBAP

A von Neumann algebra $\mathcal{M}$ has the *weak* completely bounded approximation property* (w*CBAP) if there exists a net of completely bounded, finite-rank maps $\varphi_\alpha : \mathcal{M} \to \mathcal{M}$ such that $\varphi_\alpha \to Id$ in the point-weak* topology and such that $\|\varphi_\alpha\|_{cb} \leq C$ for all α. The minimal such constant is called the *Cowling-Haagerup constant* and is denoted by $\Lambda_{cb}(\mathcal{M})$. $\mathcal{M}$ has the *w*CCAP* if $\Lambda_{cb}(\mathcal{M}) = 1$.

Cowling and Haagerup ([7]) proved that for a discrete group Γ, $\Lambda_{cb}(\Gamma) = \Lambda_{cb}(L\Gamma)$. Since the free groups have Cowling-Haagerup constant 1 ([12]), it was known that the free group factors had the w*CCAP. The equivalent definition for a C*-algebra would simply require that the net of finite-rank maps converge to the identity in the point-norm topology.

2.1.2 Operators Spaces

We use the row and column operator space structures on an abstract Hilbert space which are given by

$$H_c = B(H, \mathbb{C}) \qquad \text{and} \qquad H_r = B(\mathbb{C}, \bar{H})$$

respectively. We have that $(H_c)^* = \bar{H}_r$ and $(H_r)^* = \bar{H}_c$. See Section 3.4 of [11] for detailed proofs. We shall also utilize the Haagerup tensor product. Given two operator spaces E and F, let $E \otimes F$ be their algebraic tensor product. Let $x = (x_{ij}) \in \mathbb{M}_n(E \otimes F)$. Define

$$\|x\|_{h,n} = \inf_{r \geq 1} \{ \|y\|_{\mathbb{M}_{n,r}(E)} \|z\|_{\mathbb{M}_{r,n}(F)} | x_{ij} = \sum_k y_{ik} z_{kj} \}$$

The operator space tensor product defined by these norms is called the Haagerup tensor product and is denoted by $E \otimes_h F$. The following two facts about the Haagerup tensor product will be very useful.

Remark 2.1.1. For the row and column Hilbertian operator spaces, we have from Corollaries 5.8 and 5.10 of [19] and Proposition 9.3.4 of [11]

1. $H_c \otimes_h K_c = (H \otimes_2 K)_c$

2. $H_c \otimes_h \bar{K}_r \simeq \mathcal{K}(K, H)$

3. $H_r \otimes_h \bar{K}_c \simeq S_1(K, H)$

where the last two complete isometries are given by

$$(\xi \otimes_h \eta)(k) = \langle \eta, k \rangle \xi$$

and

$$(\xi \otimes_h \eta)(k) = \langle \eta, k \rangle \xi$$

where $\xi \in H$, $\eta \in \bar{K}$, and $k \in K$.

Lemma 2.1.2. *The multiplication map*

$$m : L^2_r(\mathcal{M}) \otimes_h \overline{L^2_c(\mathcal{M})} \to L^1(\mathcal{M})$$

defined by $m(a \otimes_h b) = ab^*$ is completely contractive for any finite von Neumann algebra $\mathcal{M}$.

Proof. We start by observing that $\overline{L^1(\mathcal{M})}^* \simeq \mathcal{M}$ completely isometrically for any von Neumann algebra. See page 139 of [19].

Let

$$\pi : \mathcal{M}^{op} \to B(\overline{L^2(\mathcal{M})})$$

be the usual right representation defined by $\pi(x)\hat{a} = \widehat{xa}$ where $\hat{a}$ denotes the image in $L^2(\mathcal{M})$ of $a \in \mathcal{M}$. We claim that $\pi = m^*$. Let $x, a, b \in \mathcal{M}$. Let $\widehat{a} \in L^2_r(\mathcal{M})$ and $\hat{b} \in L^2_c(\mathcal{M})$ be the canonical images of a and b

in the row and column spaces respectively, and let $\{e_i\}$ be an orthonormal basis of $L^2(\mathcal{M})$. We have

$$\begin{aligned}
\langle \pi(x), \hat{a} \otimes_h \hat{b} \rangle &= Tr\left(\pi(x)^*\left(\hat{a} \otimes_h \hat{b}\right)\right) \\
&= \sum_i \langle e_i, \pi(x)^*\left(\widehat{a} \otimes_h \hat{b}\right) e_i \rangle \\
&= \sum_i \langle \pi(x) e_i, \langle \hat{b}, e_i \rangle \hat{a} \rangle \\
&= \sum_i \langle \pi(x) \langle e_i, \hat{b} \rangle e_i, \hat{a} \rangle \\
&= \langle \pi(x)\widehat{b}, \hat{a} \rangle = \langle \widehat{xb}, \hat{a} \rangle = \tau(b^* x^* a) = \tau(x^* a b^*) = \langle x, m(\hat{a} \otimes_h \hat{b}) \rangle
\end{aligned}$$

Therefore, $m^* = \pi$ since $\mathcal{M}$ is dense in $L^2(\mathcal{M})$ and so $\|m\|_{cb} = \|\pi\|_{cb} = 1$ since π is a $*$-homomorphism. $\square$

Remark 2.1.3. Note that for every von Neumann algebra $\mathcal{M}$ there is a quotient map

$$q : S_1(L^2(\mathcal{M})) \to L^1(\mathcal{M})$$

which is the predual of the canonical representation

$$\pi : \mathcal{M} \to B(L^2(\mathcal{M})).$$

The map m above is simply this quotient map once we identify $S_1(L^2(\mathcal{M}))$ with $L^2_r(\mathcal{M}) \otimes_h \overline{L^2_c(\mathcal{M})}$.

2.1.3 The q-Fock Space and q-Gaussian Algebras

Let H be a real Hilbert space, $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, and $\mathcal{F}(H) = \oplus_{n \geq 0} H_{\mathbb{C}}^{\otimes n}$ be the algebraic Fock space over $H_{\mathbb{C}}$. Here $H_{\mathbb{C}}^{\otimes 0}$ is understood to be a one-dimensional space spanned by a unit vector Ω. In [2], Bożejko and Speicher defined the following sesquilinear form on $\mathcal{F}(H)$.

$$\langle h_1 \otimes \ldots \otimes h_n, k_1 \otimes \ldots \otimes k_m \rangle = \delta_{m,n} \sum_{\sigma \in S_n} q^{\iota(\sigma)} \prod_j \langle h_j, k_{\sigma(j)} \rangle$$

where S_n denotes the symmetric group on n characters, $\iota(\sigma)$ denotes the number of inversions of $\sigma \in S_n$, and $-1 \leq q \leq 1$. By the main result in [2], this form is nonnegative definite, in fact, strictly positive definite if $-1 < q < 1$ for each n and thus defines an inner product. Denote by $\mathcal{F}_q(H)$ the Hilbert space completion of $\mathcal{F}(H)$ with respect to this inner product. Now for $h \in H$,$h_1, \ldots, h_n \in H_{\mathbb{C}}$, define

$$l_q(h)h_1 \otimes \ldots \otimes h_n = h \otimes h_1 \otimes \ldots \otimes h_n$$

to be the left creation operator, and its adjoint the left annihilation operator

$$l_q^*(h)h_1 \otimes \cdots \otimes h_n = \sum_{j=1}^{n} q^{j-1}\langle h, h_j\rangle h_1 \otimes \cdots \otimes \hat{h}_j \otimes \cdots \otimes h_n$$

where $\hat{h}_j$ indicates that h_j is omitted from the tensor. By [2], $l_q(h) \in B(F_q(H))$ for $-1 \leq q < 1$, and $l_q(h)$ is closable for $q = 1$. Let $s_q(h) = l_q(h) + l_q(h)^*$. We define the q-Gaussian von Neumann algebra to be

$$\Gamma_q(H) := \{s_q(h) | h \in H\}''$$

for $-1 \leq q < 1$, and

$$\Gamma_1(H) = \{e^{is_1(h)} | h \in H\}''.$$

Furthermore, in [1], it is proved in Proposition 2.3 that the vector Ω is cyclic and separating, and defines a finite trace $\tau(x) = \langle \Omega, x\Omega\rangle$. Therefore, $\Gamma_q(H)$ is a finite von Neumann algebra in standard form. The following two results can also be found in [1] (Proposition 2.7 and Theorem 2.11 respectively).

Theorem 2.1.4. *For each $\xi \in \mathcal{F}(H)$ there exists a unique element $W(\xi) \in \Gamma_q(H)$ such that $W(\xi)\Omega = \xi$. $W(\xi)$ is called the* Wick product *of ξ.*

Theorem 2.1.5. *Let $u : H \to K$ be a contractive map between real Hilbert spaces. There exists a trace-preserving unital completely positive (cput) map $\Gamma_q(u) : \Gamma_q(H) \to \Gamma_q(K)$ such that*

1. *$\Gamma_q(u)$ is a *-automorphism if u is an orthogonal tranformation.*
2. *$\Gamma_q(u)$ is a *-embedding if u is an inclusion.*
3. *$\Gamma_q(u)$ is a conditional expectation if u is a projection.*

In this sense, Γ_q can be seen as a functor between the category of real Hilbert spaces with contractions and the category of II_1 factors with completely positive maps (see [31] for the free case).

2.1.4 Wick products

The following two maps were introduced in [1] and studied in a very general setting in [14]. Define a map

$$U_{n,k} : H_c^{\otimes n-k} \otimes_h \bar{H}_r^{\otimes k} \to B(\mathcal{F}_q(H))$$

by

$$U_{n,k}\left((h_1 \otimes \ldots \otimes h_{n-k}) \otimes_h (\bar{h}_{n-k+1} \otimes \ldots \otimes \bar{h}_n)\right) \qquad = l_q(h_1)\ldots l_q(h_{n-k}) l_q^*(\bar{h}_{n-k+1})\ldots l_q^*(\bar{h}_n)$$

and

$$R_{n,k} : \bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k} \to H^{\otimes n}$$

by

$$R_{n,k}(h_1 \otimes \cdots \otimes h_{n-k} \otimes_h h_{n-k+1} \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes h_n$$

It is shown in [15] that

$$\begin{aligned} & R^*_{n,k}\left(h_1 \otimes \ldots \otimes h_n\right) \\ &= \sum_{x \in S_n/S_{n-k}\times S_k} q^{\iota(x)} (h_{\sigma_x(1)} \otimes \ldots \otimes h_{\sigma_x(n-k)}) \otimes_h (h_{\sigma_x(n-k+1)} \otimes \ldots \otimes h_{\sigma_x(n)}) \end{aligned}$$

where $x \in S_n/S_{n-k} \times S_k$ are right cosets, $\sigma_x \in x$ is the representative of x with the fewest inversions, and $\iota(x) = \iota(\sigma_x)$. The following theorem is Theorem 1 in [14]

Theorem 2.1.6. *Let $\xi \in H^{\otimes n}$.*

$$W(\xi) = \sum_{k=0}^{n} U_{n,k} R^*_{n,k}(\xi)$$

for $\xi = h_1 \otimes \ldots \otimes h_n$.

Observation 2.1.7. We may associate to any right coset $x \in S_n/S_{n-k} \times S_k$ a subset $A \subset \{1, \ldots, n\}$ such that $|A| = k$ in the following way. For any permutation $\sigma \in x$, $\sigma(j) \in A^c$ for $1 \leq j \leq n-k$ and $\sigma(j) \in A$ for $n-k+1 \leq j \leq n$. Suppose $A^c = (\alpha_1, \ldots, \alpha_{n-k})$ and $A = (\beta_1, \ldots, \beta_k)$. Then

$$\sigma_x(1, \ldots, n) = (\alpha_1, \ldots, \alpha_{n-k}, \beta_1, \ldots, \beta_k)$$

From now on, we may replace a *right coset* $x \in S_n/S_{n-k} \times S_k$ with its corresponding *subset* of cardinality k where convenient.

2.1.5 The q-Ornstein-Uhlenbeck Semigroup and its Markov Dilation

Let $u_t : H \to H$ be the map $h \mapsto e^{-t}h$ for $t \geq 0$. u_t is clearly a contraction, and so by Theorem 2.1.5, we have a trace-preserving, completely positive, unital (*cput*) map $T_t = \Gamma_q(u_t)$. Since $u_s \circ u_t = u_{s+t}$, $T_s \circ T_t = T_{s+t}$ by functorality, and $T_0 = Id$. Therefore, T_t is a cput semigroup. We shall denote by N its (positive) generator, which is called the number operator.

Definition 2.1.8. We say a cput semigroup T_t on a (semi-)finite von Neumann algebra $\mathcal{M}$ admits a *Markov dilation* if there is a larger (semi-)finite von Neumann algebra $\widetilde{\mathcal{M}}$ with increasing filtration $\widetilde{\mathcal{M}} = \vee_{t\geq 0}\widetilde{\mathcal{M}}_t$ $\widetilde{\mathcal{M}}_t \subset \widetilde{\mathcal{M}}_s$ when $t < s$ together with a sequence of $*$-homomorphisms $\varphi_t : \mathcal{M} \to \widetilde{\mathcal{M}}_t$ such that $E_s \circ \varphi_t(x) = \varphi_s \circ T_{t-s}(x)$ for all $t > s$, $x \in \mathcal{M}$, where E_s denotes the conditional expectation onto $\vee_{s\geq t\geq 0}\mathcal{M}_t$.

In this case, our Markov dilation is special in that $\tilde{\mathcal{M}}_t = \tilde{\mathcal{M}}$ for all $t > 0$ and the φ_t are of the form $\alpha_t \circ \pi$ where α_t is an automorphism group of $\widetilde{\mathcal{M}}$, π is a canonical inclusion of $\mathcal{M} \subset \widetilde{\mathcal{M}}$. In particular, let $\widetilde{\mathcal{M}} = \Gamma_q(H \oplus H)$ and let $R_t : H \oplus H \to H \oplus H$ be the rotation

$$R_t = \begin{pmatrix} e^{-t}Id & -\sqrt{1-e^{-2t}}Id \\ \sqrt{1-e^{-2t}}Id & e^{-t}Id \end{pmatrix}$$

Let $\alpha_t = \Gamma_q(R_t)$. By Theorem 2.1.5, this can be extended to a group of *-automorphisms of $\Gamma_q(H \oplus H)$. Let $P_1 : H \oplus H \to H$ be the projection onto the first coordinate and $E_1 = \Gamma_q(P_1)$. Then $T_t(x) = E_1 \circ \alpha_t(x)$ and $\Gamma_q(H) \subset \Gamma_q(H \oplus H)$ by including H in the first coordinate. This identity is not difficult to show.

Definition 2.1.9. A von Neumann subalgebra $P \subset \mathcal{M}$ is rigid with respect to a continuous family of completely positive maps $\theta_t : \mathcal{M} \to \mathcal{M}$ if $\theta_t \to Id$ as $t \to 0$ uniformly on the unit ball of $L^2(P)$.

We have the following theorem follows immediately from Theorem 5.4 of [20] regarding rigid subalgebras of $\Gamma_q(H)$ when $\dim(H) < \infty$.

Theorem 2.1.10. *Let $B \subset \Gamma_q(H)$ be a von Neumann subalgebra. Then TFAE*

1. *B is rigid with respect to α_t.*
2. *B is rigid with respect to T_t.*
3. *B is atomic.*

Proof. 3) $\Rightarrow$ 2): T_t is compact on $L^2(\Gamma_q(H)) \simeq \mathcal{F}_q(H)$ since $H^{\otimes n}$ is a finite dimensional eigenspace of T_t with eigenvalue e^{-nt}. Let $B \subset \Gamma_q(H)$ be a diffuse subalgebra. Suppose B is rigid with respect to T_t. Then there exists t_0 such that
$$\|T_t(u) - u\|_2 \leq \frac{1}{2}$$
for all $t < t_0$ and $u \in \mathcal{U}(B)$. Let $A \subset B$ be a maximal abelian subalgebra of B. A is diffuse since B is diffuse, so there exists a unitary $v \in A$ such that $\tau(v^m) = 0$. The sequence $\hat{v}_m \in L^2(\mathcal{M})$ converges weakly

to 0. Since T_t is compact, we get that $\|T_t(\hat{v}_m)\|_2 \to 0$ for any fixed t. Therefore

$$\|T_t(\hat{v}_m) - \hat{v}_m\|_2 = \|\hat{v}_m\|_2 = 1$$

contradicting that $\|T_t(u) - u\|_2 \leq \frac{1}{2}$ for all $t < t_0$ and $u \in \mathcal{U}(B)$.

2) $\Rightarrow$ 3): Suppose B is Type I. Since $\Gamma_q(H)$ is finite, $B = \oplus_{\alpha \in I} M_{n_\alpha}$ for some countable index set I since $\Gamma_q(H)$ has a separable predual. We have projections e_α such that $e_\alpha B e_\alpha = M_{n_\alpha}$ and $\sum_{\alpha \in I} \tau(e_\alpha) = 1$. For any $\varepsilon > 0$, there is a finite set $F \subset I$ such that $\sum_{\alpha \in F} \tau(e_\alpha) > 1 - \varepsilon$. Let $\iota_F : B \to L^2(B)$ be the map $x \mapsto \sum_{\alpha \in F} e_\alpha x e_\alpha$. A simple estimate shows that $\|\iota_{F^c}\| \leq \sqrt{\varepsilon}$. Therefore

$$\lim_{t \to 0} \sup_{\|x\|_\infty \leq 1} \|T_t(\iota_F(x))\|_2 = 0$$

and so T_t converges uniformly on $(B)_1$.

1) $\Rightarrow$ 2): For any $x \in L^2(\Gamma_q(H))$ we have

$$\begin{aligned} \|\alpha_t(x) - x\|_2^2 &= 2\langle x, x\rangle - \langle \alpha_t(x), x\rangle - \langle x, \alpha_t(x)\rangle \\ &= 2(\langle x, x\rangle - \langle x, T_t(x)\rangle) \\ &\leq 2\|x\|\|T_t(x) - x\|_2 \end{aligned}$$

and so if T_t converges uniformly, α_t converges uniformly.

2) $\Rightarrow$ 1): Similarly,

$$\begin{aligned} \|T_t(x) - x\|_2 &= \|E_{\Gamma_q(H)}\left(\alpha_t(x) - x\right)\|_2 \\ &\leq \|\alpha_t(x) - x\|_2 \end{aligned}$$

so if α_t converges uniformly, T_t converges uniformly. □

2.1.6 Central Limit Theorem

From now on, we shall drop the subscript q and simply assume that q is a fixed parameter between -1 and 1. We shall need the following two results. The first is Proposition 2 in [2].

Theorem 2.1.11. *Let $h_1, \dots, h_{2n}$ be vectors in H. Then*

$$\tau(s(h_1)\dots s(h_{2n})) = \sum_{\sigma\in P_2(2n)} q^{\iota(\sigma)} \prod_{\{i,j\}\in\sigma} \langle h_i, h_j\rangle$$

where $P_2(2n)$ denotes the pair partitions of the set $\{1,\dots,2n\}$ and $\iota(\sigma)$ denotes the number of crossings of the partition σ.

Theorem 2.1.12. *Let $s_j(h) = s(h\otimes e_j)$ for some orthonormal basis $\{e_j\}_j \subset \ell^2_N(\mathbb{R})$ and $h\in H$ for an arbitrary Hilbert space H. Consider the operator*

$$u_N(h) = N^{-\frac{1}{2}}\sum_{j=1}^{N} s_j(h).$$

Then

$$\tau(u_N(h_1)\cdots u_N(h_m)) = \tau(s(h_1)\cdots s(h_m)).$$

for all $h_1,\dots,h_m \in H$.

Proof. Observe that the map $\iota_N : H \to H\otimes \ell^2_{\mathbb{R}}(N)$ where

$$\iota_N(h) = N^{-\frac{1}{2}}\sum_{j=1}^{N} h\otimes e_j$$

is an isometric embedding. We apply Theorem 2.1.5 and the theorem follows immediately. □

Fix a free ultrafilter $\mathcal{U}$ on the natural numbers. For a sequence of Banach spaces $\{X_n\}$, we may define the ultraproduct $X_{\mathcal{U}}$ by

$$X_{\mathcal{U}} = \prod\nolimits_{\mathcal{U}} X_n := \prod\nolimits_n X_n/I_{\mathcal{U}}$$

where

$$I_{\mathcal{U}} = \{(x_n) : \lim_{n,\mathcal{U}} \|x_n\|_{X_n} = 0\}.$$

Define

$$u_{\mathcal{U}}(h) = (u_N(h))^{\bullet} \in \prod\nolimits_{\mathcal{U}} L^p(\Gamma_q(\ell^2_N(\mathbb{R})\otimes H))$$

where the notation $(x_N)^{\bullet}$ is used for the equivalence class of the sequence (x_N) in the von Neumann algebra

$$\tilde{\mathcal{N}}_{\mathcal{U}} := \left(\prod_{N,\mathcal{U}} \overline{L^1(\Gamma_q(H\otimes \ell^2_N(\mathbb{R})))}\right)^{*}.$$

However, $\tilde{\mathcal{N}_{\mathcal{U}}}$ is not in general finite, but it contains a canonical finite subalgebra $\mathcal{N}_{\mathcal{U}}$ which is obtained as the image of bounded sequences in the Hilbert space $\prod_{\mathcal{U}} L^2\left(\Gamma_q(H\otimes\ell^2_N(\mathbb{R}))\right)$, obtained by the GNS construction for the trace

$$\tau_{\mathcal{U}}=\lim_{N,\mathcal{U}}\tau_N,$$

where τ_N is the vacuum trace associated to $\Gamma_q(H\otimes\ell^2_N(\mathbb{R}))$. Thus there is a canonical inclusion

$$L^p(\mathcal{N}_{\mathcal{U}})\subseteq\prod\nolimits_{\mathcal{U}} L^p(\Gamma_q(H\otimes\ell^2_N(\mathbb{R}))).$$

By Theorem 2.1.12, we have an injective *-homomorphism $\pi_{\mathcal{U}}:\Gamma_q(H)\to\mathcal{N}_{\mathcal{U}}$ such that $\pi_{\mathcal{U}}(s(h))=(u_N(h))^\bullet$. See [19] Section 9.10 or [24] for more information regarding ultraproducts of finite von Neumann algebras.

Let $u:H\to K$ be any contraction on real Hilbert spaces. Note that

$$u\otimes Id_N : H\otimes\ell^2_N(\mathbb{R})t\ K\otimes\ell^2_N(\mathbb{R})$$

is also a contraction. By Theorem 2.1.5, there is a completely positive map

$$\Gamma_q(u\otimes Id_N):\Gamma_q(H\otimes\ell^2_N(\mathbb{R}))\to\Gamma_q(K\otimes\ell^2_N(\mathbb{R})).$$

Therefore we may define a completely positive map

$$\Gamma_q^{\mathcal{U}}(u):\mathcal{N}_{\mathcal{U}}\to\mathcal{N}_{\mathcal{U}}.$$

such that $\Gamma_q^{\mathcal{U}}(u)\left((u_N(h))^\bullet\right)=(u_N(u(h)))^\bullet$. In particular, if u is an isometry, $\Gamma_q^{\mathcal{U}}(u)$ is a $*$-homomorphism. The following lemma shall be crucial.

Lemma 2.1.13. *Let $E_{\mathcal{U}}:\mathcal{N}_{\mathcal{U}}\to\Gamma_q(H)$ be the unique conditional expectation. Then*

1. *For any contraction $u:H\to K$, $\Gamma_q^{\mathcal{U}}(u)\circ\pi_{\mathcal{U}}=\pi_{\mathcal{U}}\circ\Gamma_q(u)$.*

2. $$E_{\mathcal{U}}\left(N^{-\frac{m}{2}}\sum_{\substack{j_1\neq\dots\neq j_m\\ 1\leq j_k\leq N}} s_{j_1}(h_1)\dots s_{j_m}(h_m)\right)^\bullet=W(h_1\otimes\cdots\otimes h_m)$$

(Here the indices are taken to be pair-wise not equal.)

Proof. 1) is obvious. However, note that 1) implies $\pi_{\mathcal{U}}$. For 2), let

$$y_N = N^{-\frac{m}{2}} \sum_{\substack{j_1 \neq \cdots \neq j_m \\ 1 \le j_k \le N}} s_{j_1}(h_1) \cdots s_{j_m}(h_m)$$

Let $y = E_{\mathcal{U}}(y_N)^{\bullet}$. Using i) and that the indices j_k are all different, we know that

$$T_t(y) = E_{\mathcal{U}}(T_t^{\mathcal{U}}(y_N)^{\bullet}) = e^{-mt} E_{\mathcal{U}}(y_N) = e^{-mt} y$$

Therefore, we have

$$\tau(s(\bar{f}_{m'}) \cdots s(\bar{f}_1) y) = \tau(P_m(s(\bar{f}_{m'}) \cdots s(\bar{f}_1)) y)$$

where $P_m : \Gamma_q(H) \to \Gamma_q(H)$ denotes the projection defined by

$$P_m(W(\xi)) = \begin{cases} W(\xi) & \text{if } \xi \in H^{\otimes m} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we must only check the case where $m' = m$. Using Theorem 2.1.11, we obtain

$$\begin{aligned} &\tau(s(\bar{f}_m) \cdots s(\bar{f}_1) y) = \tau_{\mathcal{U}}(s_{\mathcal{U}}(\bar{f}_m) \cdots s_{\mathcal{U}}(\bar{f}_1) y_{\mathcal{U}}) \\ &= \lim_N N^{-m} \sum_{k_1, \ldots, k_m} \sum_{j_1 \neq \cdots \neq j_m} \tau(s_{k_m}(\bar{f}_m) \cdots s_{k_1}(\bar{f}_1) s_{j_1}(h_1) \cdots s_{j_m}(h_m)) \\ &= \lim_N N^{-m} \sum_{k_1, \ldots, k_{m'}} \sum_{j_1 \neq \cdots \neq j_m} \sum_{\sigma \in P_2(2m)} q^{\iota(\sigma)} \prod_{\{\alpha, \beta\} \in \sigma} \langle \bar{f}_\alpha \otimes e_{k_\alpha}, h_\beta \otimes e_{j_\beta} \rangle \end{aligned}$$

Now simply counting the number of possible indices which make the inner product $\langle \bar{f}_\alpha \otimes e_{k_\alpha}, h_\beta \otimes e_{j_\beta} \rangle$ non-zero, we get

$$\begin{aligned} &\lim_N N^{-m} \sum_{k_1, \ldots, k_m} \sum_{j_1 \neq \cdots j_m} \sum_{\sigma \in P_2(2m)} q^{\iota(\sigma)} \prod_{\{\alpha, \beta\} \in \sigma} \langle \bar{f}_\alpha \otimes e_{k_\alpha}, h_\beta \otimes e_{j_\beta} \rangle \\ &= \lim_N N^{-m} \prod_{j=0}^{m-1} (N - j) \sum_{\sigma \in P_2(2m)} q^{\iota(\sigma)} \prod_{\{\alpha, \beta\} \in \sigma} \langle \bar{f}_\alpha, h_\beta \rangle \\ &= \langle f_1 \otimes \cdots \otimes f_m, h_1 \otimes \cdots \otimes h_m \rangle \end{aligned}$$

It is easy to see that

$$f_1 \otimes \cdots \otimes f_m = P_m(s(f_1) \cdots s(f_m) \Omega)$$

and so we get

$$\langle W(f_1 \otimes \cdots \otimes f_m), y\rangle = \langle f_1 \otimes \cdots \otimes f_m, h_1 \otimes \cdots \otimes h_m\rangle$$

Hence $y\Omega = h_1 \otimes \cdots \otimes h_m$ as required. □

2.1.7 Bimodules

Bimodules over von Neumann algebras were first defined and studied by Connes in his unpublished notes [5] and were use by Connes and Jones in [6] in order to define property (T) for von Neumann algebras. Specifically, for von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, an $\mathcal{M}$-$\mathcal{N}$-bimodule is a *-representation of $\mathcal{M} \otimes_{bin} \mathcal{N}^{op}$ on a Hilbert space $\mathcal{H}$. See [10] for the definition of the bin tensor norm. A simple and important example of a bimodule is the *coarse* bimodule $L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$ where $\mathcal{M}$ acts on the left on $L^2(\mathcal{M})$ and $\mathcal{N}$ acts on the right on $L^2(\mathcal{N})$. Just as for group representations, Connes and Jones gave the following definition of weak containment for these bimodules.

Definition 2.1.14. Let $\mathcal{H}$ and $\mathcal{K}$ be $\mathcal{M}$-$\mathcal{N}$-bimodules. $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$, denoted by $\mathcal{H} \prec \mathcal{K}$, if for all ε, $\xi \in \mathcal{H}$, $F \subset \mathcal{M}$ finite, and $E \subset \mathcal{N}$ finite, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$|\langle \xi, x\xi y\rangle - \sum_{j=1}^{n} \langle \eta_j, x\eta_j y\rangle| < \varepsilon$$

for all $x \in F$ and $y \in E$.

For two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, denote the state space of the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ by $S(\mathcal{A} \odot \mathcal{A}) = S(\mathcal{A} \otimes_{max} \mathcal{B})$. In [10], Effros and Lance show that if $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, $f \in S(\mathcal{M} \odot \mathcal{N})$ is such that $(x, y) \mapsto f(x \otimes y)$ is weak* continuous in each variable if and only if the maps

$$T_f(x)(y) := f(x \otimes y)$$

defines completely positive map $T_f : \mathcal{M} \to \mathcal{N}_*$ and $T_f^* : \mathcal{N} \to \mathcal{M}_*$ defines a completely positive map. We may define an element of $\varphi_\xi \in S(\mathcal{M} \odot \mathcal{N})$ from $\xi \in \mathcal{H}$ such that $\|\xi\|_{\mathcal{H}} = 1$ simply by $\varphi_\xi(x \otimes y) = \langle \xi, x\xi y\rangle$. These definitions give the following proposition.

Lemma 2.1.15. *Let $\mathcal{M}$ and $\mathcal{N}$ be finite von Neumann algebras with separable predual and $\mathcal{H}$ be an $\mathcal{M}$-$\mathcal{N}$-bimodule. Then for the following,* (2) *and* (3) *are equivalent and* (1) *implies* (2) *and* (3).

1. *$T_\xi := T_{\varphi_\xi}$ extends to an element of $S_2(L^2(\mathcal{M}), L^2(\mathcal{N}))$.*

2. *For $\xi \in \mathcal{H}$ such that $\|\xi\|_{\mathcal{H}} = 1$, $\varphi_\xi \in S(\mathcal{M} \odot \mathcal{N})$ is continuous with respect to the minimal tensor norm.*

3. $\mathcal{H} \prec L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$.

Proof. Note that $T_\xi(y)(x) = \varphi_\xi(x \otimes y) = \langle \xi, x\xi y\rangle$. Also note that the coarse bimodule $L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$ is isomorphic to $S_2(L^2(\mathcal{M}), L^2(\mathcal{N}))$ by identifying simple tensors $\xi \otimes \eta \in L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$ with rank one operators. The bimodule structure comes from pre-composing with operators from $\mathcal{M}$ and composing with operators from $\mathcal{N}$. Therefore we have

$$L^2(\mathcal{M}) \otimes L^2(\mathcal{N}) \simeq L^2(\mathcal{M}\bar{\otimes}\mathcal{N}^{op}) \simeq S_2(L^2(\mathcal{M}), L^2(\mathcal{N})).$$

Let

$$\pi : \mathcal{M} \otimes_{bin} \mathcal{N} \to B(L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$$

denote the representation described above which defines the bimodule structure.

(1) $\Rightarrow$ (2): Using these identifications, T_ξ corresponds to $\zeta \in L^2(\mathcal{M}\bar{\otimes}\mathcal{N}^{op})$ by

$$\tau_{\mathcal{N}}(T_\xi(x)y) = \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}(\pi(x \otimes y)\zeta).$$

Since $\mathcal{M}$ and $\mathcal{N}$ are finite, we have that $\zeta \in L^1(\mathcal{M}\bar{\otimes}\mathcal{N}^{op})$. Using the Kaplansky density theorem, $L^1(\mathcal{M}\bar{\otimes}\mathcal{N})$ embeds isometrically in $(\mathcal{M} \otimes_{min} \mathcal{N})^*$ since $\mathcal{M} \otimes_{min} \mathcal{N}$ is weak* dense in $\mathcal{M}\bar{\otimes}\mathcal{N}$. Therefore we have that $\|\varphi_\xi\|_{min^*} \leq \|T_\xi\|_{HS}$.

(3) $\Rightarrow$ (2): If $\mathcal{H} \prec L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$, using the definition, we have elements $\eta_1, \ldots, \eta_n \in L^2(\mathcal{M}) \odot L^2(\mathcal{N})$ such that

$$|\varphi_\xi(x \otimes y) - \sum_{j=1}^{n} \varphi_{\eta_j}(x \otimes y)| < \varepsilon$$

for all $x \in E$ finite, $y \in F$ finite, $\varepsilon > 0$. Since $\eta_j \in L^2(\mathcal{M}) \odot L^2(\mathcal{N})$,

$$\varphi_{\eta_j}(x \otimes y) = \langle \eta_j, \pi(x \otimes y)\eta_j\rangle.$$

Hence $\varphi_{\eta_j} \in B(L^2(\mathcal{M})\otimes L^2(\mathcal{N})_*$. Therefore φ_ξ is a cluster point of elements of the form $\sum_{j=1}^{n} \varphi_{\eta_j}$. According to [10], this implies that φ_ξ is min-continuous.

(2) $\Rightarrow$ (3): Suppose φ_ξ is continuous with respect to the min-norm. We observe $\varphi_\xi \in (\mathcal{M} \otimes_{min} \mathcal{N})^*$ if it lifts to $B(L^2(\mathcal{M}) \otimes L^2(\mathcal{N}))^*$. Therefore, by [10], we may write φ_ξ as the limit of elements $\varphi_\eta \in B(L^2(\mathcal{M}) \otimes L^2(\mathcal{N})_*$. This implies directly that $\mathcal{H} \prec L^2(\mathcal{M}) \otimes L^2(\mathcal{N})$. □

Chapter 3

CCAP for the q-Gaussian algebras

3.1 Proof of the w*CCAP

In this section, we shall prove that $\Gamma_q(H)$ has the w*CCAP. This is a crucial property to proving strong solidity given the result of Ozawa and Popa ([17] Theorem 3.5) that every amenable subalgebra of a von Neumann algebra with the w*CCAP is weakly compact. This result is made significantly easier by using Theorem 2.1.5 and Theorem 1 from [15]. We begin by recalling Nou's result from Theorem 1 of [15].

Theorem 3.1.1. *Let K be a complex Hilbert space. Then for all $n \geq 0$ and for all $\xi \in B(K) \otimes_{\min} H^{\otimes n}$ we have*

$$\begin{aligned}\max_{0\leq k\leq n} \|(Id \otimes R_{n,k}^*(\xi)\| &\leq \|(Id \otimes W)(\xi)\|_{\min} \\ &\leq C_q(n+1) \max_{0\leq k\leq n} \|(Id \otimes R_{n,k}^*)(\xi)\|\end{aligned}$$

Let X_n denote the ℓ^∞ direct sum of the spaces

$$H_c^{n-k} \otimes_h \bar{H}_r^k$$

for k ranging from 0 to n. This theorem means the map $\Phi_n : H^{\otimes n} \to \Gamma_q(H)$ defined by $\Phi_n(\xi) = W(\xi)$ has cb-norm less than $C_q(n+1)$. Here the operator space structure on $H^{\otimes n}$ is realized by demanding

$$(R_{n,k}^*) : H^{\otimes n} \to X_n$$

be a completely isometric embedding. From here, we shall denote this operator space by $H_{Nou}^{\otimes n}$. In other words, the following diagram commutes

$$\begin{tikzcd} H_{Nou}^{\otimes n} \arrow[d, "(R_{n,k}^*)"'] \arrow[dr, "\Phi_n"] & \\ X_n \arrow[r, "(U_{n,k})"'] & \Gamma_q(H) \end{tikzcd}$$

Nou proves that $\|U_{n,k}\|_{cb} \leq C_q$ and so $\|\Phi_n\|_{cb} \leq C_q(n+1)$. Recall that $\overline{L^1(\mathcal{M})}^*$ is completely isometric to $\mathcal{M}$ for a finite von Neumann algebra where the duality is with respect to the trace, i.e.

$$\langle x, y\rangle = \tau(x^* y)$$

which is consistent with the multiplication map from Lemma 2.1.2. Let $P_n : \Gamma_q(H) \to \Gamma_q(H)$ be the projection defined by

$$P_n(W(\xi)) = \begin{cases} W(\xi) & \text{if } \xi \in H^{\otimes n} \\ 0 & \text{otherwise} \end{cases}$$

Our goal shall be to show that $\|P_n\|_{cb} < p(n)$ for some polynomial p. Once we have established this fact, we see that

$$\|\sum_{n=0}^N P_n\|_{cb} \leq \sum_{n=0}^N \|P_n\|_{cb} < p_1(n)$$

for some polynomial p_1 of higher degree. We may then compose these projections with the operators T_t which has for which $\|T_t|_{F_n}\|_{cb} = e^{-t}$ where $F_n = \{W(\xi)|\xi \in H^{\otimes n}$. This exponential decay will balance the growth of the norms of the projections, as in [12]. Let us define a map

$$\Psi_n : (H_{Nou}^{\otimes n})^* \to \overline{L^1(\Gamma_q(H))}$$

such that $\Psi_n(\xi) = W(\xi)$. Note that $(X_n)^*$ is the ℓ_1 direct sum of the spaces

$$\bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k}$$

as k ranges from 0 to n. Similarly to [15], we define the map

$$\overline{\Psi}_n : (X_n)^* \to \overline{L^1(\Gamma_q(H))}$$

such that

$$\overline{\Psi}_n(h_1 \otimes \cdots \otimes h_{n-k} \otimes_h h_{n-k+1} \otimes \cdots \otimes h_n)_k = W(h_1 \otimes \cdots \otimes h_n).$$

Thus, since

$$R_{n,k} : \bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k} \to (H_{Nou}^{\otimes n})^*$$

is defined by

$$R_{n,k}(h_1 \otimes \cdots \otimes h_{n-k} \otimes_h h_{n-k+1} \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes h_n,$$

the following diagram commutes.

$$\begin{tikzcd}
(X_n)^* \arrow[d, "(R_{n,k})"'] \arrow[dr, "\overline{\Psi}_n"] & \\
(H_{Nou}^{\otimes n})^* \arrow[r, "\Psi_n"'] & \overline{L^1(\Gamma_q(H))}
\end{tikzcd}$$

From these definitions, we have the following lemma.

Lemma 3.1.2. *For P_n, Φ_n, and Ψ_n as defined above,*

$$P_n = \Phi_n \circ \Psi_n^*$$

Proof. Let $\xi \in H^{\otimes m}$ and $\eta \in \left(H_{Nou}^{\otimes n}\right)$. We have

$$\langle \eta, \Psi_n^*(W(\xi)) \rangle = \langle \Psi_n(\eta), W(\xi) \rangle = \langle W(\eta), W(\xi) \rangle = \langle \eta, \xi \rangle$$

Therefore,

$$\Psi_n^*(W(\xi)) = \begin{cases} \xi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Since $\Phi_n(\xi) = W(\xi)$, this completes the proof. $\square$

Therefore, $\|P_n\|_{cb} \leq \|\Phi_n\|_{cb} \|\Psi_n\|_{cb}$, and so we must show that $\|\Psi_n\|_{cb} < p(n)$ for some polynomial p. Since $R_{n,k}^*$ is a quotient map by definition, we must show that $\|\bar{\Psi}_n\|_{cb} < p(n)$. With this in mind, we have the following proposition.

Proposition 3.1.3. *Let $\beta_{n,k} : \bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k} \to \overline{L^1(\Gamma_q(H))}$ be defined by*

$$\beta_{n,k}(h_1 \otimes \cdots \otimes h_{n-k} \otimes_h h_{n-k+1} \otimes \cdots \otimes h_n) = W(h_1 \otimes \cdots \otimes h_n).$$

so that $\Psi_n = (\beta_{n,k})$. *We have* $\|\beta_{n,k}\|_{cb} \leq C_q$.

The proof of this proposition will require several lemmas. However, note that it follows directly from this proposition that $\|P_n\|_{cb} < C_q n^2$.

Lemma 3.1.4. *Let* $v_{n,k} : \bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k} \to \overline{L^1(\Gamma_q(H))}$ *be the map*

$$v_{n,k}((h_1 \otimes \ldots \otimes h_{n-k}) \otimes_h (h_{n-k+1} \otimes \ldots \otimes h_n))$$
$$W(h_1 \otimes \ldots \otimes h_{n-k}) W(h_{n-k+1} \otimes \ldots \otimes h_n)$$

Then $\|v_{n,k}\|_{cb} = 1$.

Proof. $v_{n,k} = m \circ (W \otimes W)$ where W is the Wick word and $\bar{m} : \overline{L_r^2(\Gamma_q(H))} \otimes L_c^2(\Gamma_q(H)) \to \overline{L^1(\Gamma_q(H))}$ is the canonical multiplication map from Lemma 2.1.2, which is completely contractive. Note that $W : F_q(H) \to L^2(\Gamma_q(H))$ a unitary. $\square$

Lemma 3.1.5. *Define* $w_{n,k}^j : \bar{H}_r^{\otimes n-k} \otimes_h H_c^{\otimes k} \to \overline{L^1(\Gamma_q(H))}$ *by*

$$w_{n,k}^j = v_{n-2j,k-j} \circ (Id_{n-k-j} \otimes m_j \otimes Id_{k-j}) \circ (R_{n-k,j}^* \otimes R_{k,k-j}^*)$$

where Id_k *is the identity on* $H^{\otimes k}$ *and* $m_j : \bar{H}_r^{\otimes j} \otimes H_c^{\otimes j} \to \mathbb{C}$ *is simply a duality bracket pairing.*

Then $\|w_{n,k}^j\|_{cb}$ *is bounded by a constant depending only on* q.

Notation 3.1.6. For an n-tensor $\xi = h_1 \otimes \cdots \otimes h_n$ and a subset $A = \{\iota_1, \ldots, \iota_k\} \subset \{1, \ldots, n\}$, denote by ξ_A the tensor $h_{\iota_1} \otimes \cdots \otimes h_{\iota_k}$.

Remark 3.1.7. As the maps $w_{n,k}^j$ are crucial to our argument, we further describe the image of an element of $H_r^{\otimes n-k} \otimes_h H_c^{\otimes k}$.

$$w_{n,k}^j(\xi_{n-k} \otimes_h \xi^k) = \sum_{\substack{A \subset \{1,\ldots n-k\} \\ |A|=j}} \sum_{\substack{B \subset \{n-k+1,\ldots n\} \\ |B|=j}} q^{\iota(A)+\iota(B)} \langle \xi_{n-k,A}, \xi_B^k \rangle_q W(\xi_{n-k,A^c}) W\left(\xi_{B^c}^k\right)$$

$$= \sum_{\substack{A \subset \{1,\ldots,n-k\} \\ |A|=j}} \sum_{\substack{B \subset \{n-k+1,\ldots,n\} \\ |B|=j}} \sum_{\sigma \in S_j} q^{\iota(A)+\iota(B)+\iota(\sigma)} \prod_{s=1}^{j} \langle h_{a_s}, h_{b_{\sigma(s)}} \rangle W(\xi_{n-k,A^c}) W(\xi_{B^c}^k)$$

where a_s is the sth element of A and likewise for b_s. $\iota(A)$ and $\iota(B)$ are the number of inversions of the corresponding right cosets from Observation 2.1.7.

Proof of Lemma 3.1.5. From Lemma 3.1.4 we know that $\|v_{n-2j,k-j}\|_{cb} \leq 1$ and it is clear that the middle term in the composition defining $w^j_{n,k}$ is completely contractive. It is shown in [15] that $\|R_{n,k}\|_{cb} \leq C_q$ where $C_q = \prod_{j\geq 1}(1-q^j)^{-1}$. Therefore, it is clear that $\|w^j_{n,k}\|_{cb} \leq C_q^2$. □

Lemma 3.1.8. *Let $\xi = h_1 \otimes \cdots \otimes h_n$, $\eta = k_1 \otimes \cdots \otimes k_m$, and $\theta = f_1 \otimes \cdots \otimes f_\ell$. We have*

$$\tau(W(\xi)W(\eta)W(\theta)) = \sum_{A,B,C} q^{\iota(A)+\iota(B)+\iota(C)} \langle \xi_A, \eta_B\rangle \langle \xi_{A^c}, \theta_C\rangle \langle \eta_{B^c}, \theta_{C^c}\rangle$$

where $A \subset \{1,\dots,n\}$, $B \subset \{1,\dots,m\}$, and $C \subset \{1,\dots,\ell\}$, such that $|A| = |B|$, $|A^c| = |C|$, and $|B^c| = |C^c|$ and the sum ranges over all such subsets.

Proof. We note first that $\ell + m + n$ must be even. Let α, β, and γ be multi-indices of lengths n, m, and ℓ respectively such that they are each pairwise not equal (i.e. $\alpha_j \neq \alpha_k$ for $j \neq k$). Let $x_\alpha = s_{\alpha_1}(h_1)\cdots s_{\alpha_n}$, $y_\beta = s_{\beta_1}(k_1)\cdots s_{\beta_m}(k_m)$, and $z_\gamma = s_{\gamma_1}(f_1)\cdots s_{\gamma_\ell}(f_\ell)$. We apply Lemma 2.1.13 and Theorem 2.1.11.

$$\begin{aligned}
&\tau(W(\xi)W(\eta)W(\theta)) = \lim_N N^{(n+m+\ell)/2} \sum_{\alpha,\beta,\gamma} \tau(x_\alpha y_\beta z_\gamma) \\
&= \lim_N N^{(n+m+\ell)/2} \sum_{\alpha,\beta,\gamma} \sum_{\sigma\in P_2(n+m+\ell)} q^{\iota(\sigma)} \prod_{\{\alpha_r,\beta_s\}\in\sigma} \langle h_r, k_2\rangle \prod_{\{\alpha_r,\gamma_s\}\in\sigma} \langle h_r, f_s\rangle \prod_{\{\beta_r,\gamma_s\}\in\sigma} \langle k_r, f_s\rangle.
\end{aligned}$$

Identifying A, B, and C as the subsets of multi-indices α, β, and γ so that $\{\alpha_j, \beta_k\} \in \sigma$ if and only if $\alpha_j \in A$ and $\beta_k \in B$ and $\{\alpha_j, \gamma_k\} \in \sigma$ if and only if $\alpha_j \in A^c$ and $\gamma_k \in C$, from Observation 2.1.7, we get the result. □

Notation 3.1.9. We shall denote by $P_{1,2}(m)$ the set of all partitions of $\{1,\dots,m\}$ whose parts are no larger than two and set

$$P^k_{1,2}(m) = \{\sigma \in P_{1,2} | \{i,j\} \in \sigma \Rightarrow i \in \{1,\dots,m-k\} \text{ and } j \in \{m-k+1,\dots m\}\}.$$

For $\sigma \in P_{1,2}(m)$, $\iota(\sigma)$ will denote the number of "crossings" of σ, i.e.

$$\iota(\sigma) = |\{\{i,j\},\{k,\ell\} \in \sigma : i < k < j < \ell\}| + |\{\{i,j\},\{k\} \in \sigma : i < k < j\}|.$$

We shall denote by $P^{j,k}_{1,2}(n)$ the subset of $P^k_{1,2}(n)$ with exactly j pairs.

Example 3.1.10. Let

$$\sigma = \{\{1\},\{2,5\},\{3\},\{4,7\},\{6\},\{8\}\} \in P^{2,4}_{1,2}(8).$$

We may represent σ using the following figure.

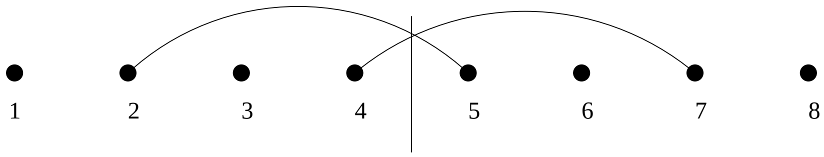

Figure 3.1: σ, $\iota(\sigma) = 3$

We can see that $\iota(\sigma) = 3$ since $\{2, 5\}$ crosses the singleton $\{3\}$, $\{4, 7\}$ crosses the singleton $\{6\}$, and the pairs $\{2, 5\}$ and $\{4, 7\}$ cross.

Before we state the key proposition, we shall need to study two "color" operators. In the case of y_j, j tensors are given an arbitrary new color, whereas for z_j, j tensors are given a new color in decreasing order of colors.

Definition 3.1.11. Let H be a real Hilbert space and $\{e_\ell\}_{\ell=0}^{j}$ an orthonormal basis of $\ell^2_{j+1}(\mathbb{R})$, we define

$$y_j : H^{\otimes n} \to (H \otimes \ell^2_{j+1}(\mathbb{R}))^{\otimes n}$$

by

$$y_j(h_1 \otimes \cdots \otimes h_n) = \sum_{\substack{A \subset \{1,\cdots,n\} \\ |A|=j}} \sum_{\substack{f:A\to\{1,\cdots,j\} \\ f|_A \text{ a bijection} \\ f(A^c)=\{0\}}} (h_1 \otimes e_{f(1)}) \otimes \cdots \otimes (h_n \otimes e_{f(n)}).$$

Also, we define

$$z_j : H^{\otimes n} \to (H \otimes \ell^2_{j+1}(\mathbb{R}))^{\otimes n}$$

by

$$z_j(h_1 \otimes \cdots \otimes h_n) = \sum_{\substack{A \subset \{1,\ldots,n\} \\ |A|=j}} (h_1 \otimes e_{f_A(1)}) \otimes \ldots \otimes (h_n \otimes e_{f_A(n)})$$

where

$$f_A(\ell) = \begin{cases} 0 & \text{if } \ell \notin A \\ j-k+1 & \text{if } \ell \text{ is the kth largest element of } A \end{cases}.$$

Notation 3.1.12. For $\{e_\ell\}_{\ell=0}^{j}$ and orthonormal basis of $\ell^2_{j+1}(\mathbb{R})$, let

$$E_j : \Gamma_q(H \otimes \ell^2_{j+1}(\mathbb{R})) \to \Gamma_q(H \otimes \ell^2_j(\mathbb{R}))$$

be the conditional expectation given by $E_j = \Gamma_q(P_j)$, where P_j is the projection such that $P_j(e_j) = 0$ and

$P_j(e_\ell) = e_\ell$ for $\ell \neq j$.

Definition 3.1.13. Let $\sigma_\emptyset \in P_{1,2}^k(n)$ be the singleton partition, and let $\sigma_j \in P_{1,2}^{j,k}(n)$ and $\sigma_{j-1} \in P_{1,2}^{j-1,k}(n)$ be such that $\sigma_j \setminus \sigma_{j-1} = \{\ell_1, \ell_2\}$ where if $\{k_1, k_2\} \in \sigma_{j-1}$, then $\ell_1 < k_1$. We define a new function ι' on $P_{1,2}^k(n)$ recursively by

1. $\iota'(\sigma_\emptyset) = 0$.

2. $\iota'(\sigma_j) = \iota'(\sigma_{j-1}) +$

$$|\{\{m\} \in \sigma_{j-1} : \ell_1 < m < \ell_2\}| + 2|\{\{k_1, k_2\} \in \sigma_{j-1} : \ell_1 < k_1 < k_2 < \ell_2\}|$$

Example 3.1.14. Let

$$\sigma = \{\{1,6\}, \{2,5\}, \{3\}, \{4\}, \{7\}, \{8\}\}$$

shown in Figure 3.2.

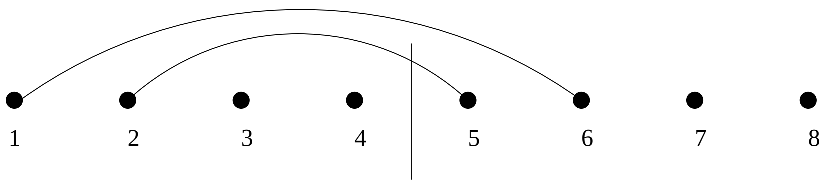

Figure 3.2: $\iota'(\sigma) = 6$, $\iota(\sigma) = 4$

We can see that $\iota'(\sigma) = 6$ since $\{1,6\}$ "contains" $\{2,5\}$, $\{3\}$, and $\{4\}$, and $\{2,5\}$ "contains" $\{3\}$ and $\{4\}$.

We now have the following lemma

Lemma 3.1.15. *For $w_{n,k}^j$, y_j, and z_j as above, we have*

$$q^{\binom{j}{2}} w_{n,k}^j(\xi_{n-k} \otimes \xi^k) = E_{\mathcal{U}}\left(E_1 \cdots E_j\left(z_j(u_N(\xi_{n-k}))y_j(u_N(\xi^k))\right)\right)$$

Proof. For the left hand side, we have that

$$
\begin{aligned}
&q^{\binom{j}{2}} w^j_{n,k}(\xi_{n-k} \otimes \xi^k) \\
&= q^{\binom{j}{2}} \sum_{\substack{A \subset \{1,\ldots,n-k\} \\ |A|=j}} \sum_{\substack{B \subset \{1,\ldots,k\} \\ |B|=j}} q^{\iota(A)+\iota(B)} \langle \xi_{n-k,A}, \xi^k_B \rangle_q W(\xi_{n-k,A^c}) W(\xi^k_{B^c}) \\
&= \sum_{\substack{A \subset \{1,\ldots,n-k\} \\ |A|=j}} \sum_{\substack{B \subset \{1,\ldots,k\} \\ |B|=j}} \sum_{\sigma \in S_j} q^{\iota(A)+\iota(B)+\iota(\sigma)+\binom{j}{2}} \prod_{s=1}^{j} \langle h_{a_s}, h_{b_{\sigma(s)}} \rangle W(\xi_{n-k,A^c}) W(\xi^k_{B^c}) \\
&= \sum_{\rho \in P^{j,k}_{1,2}(n)} q^{\iota(A)+\iota(B)+\iota(\sigma)+\binom{j}{2}} \prod_{\{\ell_1,\ell_2\} \in \rho} \langle h_{\ell_1}, h_{\ell_2} \rangle W(\xi_{n-k,\rho}) W(\xi^k_\rho),
\end{aligned}
$$

where ξ_ρ denotes ξ with the pairs of ρ removed. For the right hand side, we have that

$$
\begin{aligned}
&E_{\mathcal{U}} \left(E_1 \cdots E_j \left(z_j(u_N(\xi_{n-k})) y_j(u_N(\xi^k)) \right) \right) \\
&= E_{\mathcal{U}} \left(\sum_{\substack{A \subset \{1,\ldots,n-k\} \\ |A|=j}} \sum_{\substack{B \subset \{1,\ldots,k\} \\ |B|=j}} \sum_{\substack{g: B \to \{1,\ldots,j\} \\ g \text{ a bijection}}} E_1 \cdots E_j \left(u_N(\xi^A_{n-k}) u_N(\xi^{k,(B,g)}) \right) \right),
\end{aligned}
$$

where ξ^A_{n-k} is corresponding tensor in image of z_j in $(H \otimes \ell^2_{j+1}(\mathbb{R}))^{\otimes n-k}$ and similarly for $\xi^{k,(B,g)}$. We examine

$$
E_1 \cdots E_j \left(u_N(\xi^A_{n-k}) u_N(\xi^{k,(B,g)}) \right)
$$

for fixed A, B, and g. For this term and fixed N, we have

$$
\begin{aligned}
&E_1 \cdots E_j \left(u_N(\xi^A_{n-k}) u_N(\xi^{k,(B,g)}) \right) \\
&= q^{\ell_2-\ell_1-1} \langle h_{\ell_1} \otimes e_{\alpha_{\ell_1}}, h_{\ell_2} \otimes e_{\beta_{\ell_2}} \rangle E_1 \cdots E_{j-1} \left(N^{-1} u_N(\xi^{A \setminus \{\ell_1\}}_{n-k}) u_N(\xi^{k,(B \setminus \{\ell_2\},g')}) \right) \\
&= q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\} \in \rho} \langle h_{\ell_1} \otimes e_{\alpha_{\ell_1}}, h_{\ell_2} \otimes e_{\beta_{\ell_2}} \rangle N^{-j} u_N(\xi_{n-k,\rho}) u_N(\xi^k_\rho),
\end{aligned}
$$

where $\rho \in P^{j,k}_{1,2}(n)$ denotes the partition whose pairs are given by $\{\ell_1, \ell_2\} \in \rho \Rightarrow f_A(\ell_1) = g(\ell_2) \neq 0$. The factor N^{-j} comes from the fact that we are shortening the tensors so we must compensate for the factor of N in the formula for $u_N(\xi)$. To see that $\iota'(\rho)$ is the appropriate power of q, after applying E_j we have a term $q^{\ell_2-\ell_1-1}$. However, we have removed h_{ℓ_1} and h_{ℓ_2} from the tensor, so we must compensate for the remaining pairs in ρ which cross $\{\ell_2\}$. Letting $\rho = \rho_j$ and the partition with the same pairs of ρ except

$\{\ell_1, \ell_2\}$ be ℓ_2, we get,

$$\iota'(\rho_j) - \iota'(\rho_{j-1}) = \ell_2 - \ell_1 - 1 - |\{\{k_1, k_2\} \in \rho_2 : k_1 < \ell_2 < k_2\}|$$
$$= |\{\{m\} \in \rho_j : \ell_1 < m < \ell_2\}| + 2|\{\{k_1, k_2\} \in \rho_j : \ell_1 < k_1 < k_2 < \ell_2\}|$$

where the last equality follows by simply observing that those are the elements remaining after removing the elements of $\{\{k_1, k_2\} \in \rho_2 : k_1 < \ell_2 < k_2\}$. Let's reconsider Example 3.1.14.

Example 3.1.16. Recall we used

$$\sigma = \{\{1, 6\}, \{2, 5\}, \{3\}, \{4\}, \{7\}, \{8\}\}$$

After applying y_2 and z_2, we have colored the indices which appear in the two pairs of σ as shown in Figure 3.3.

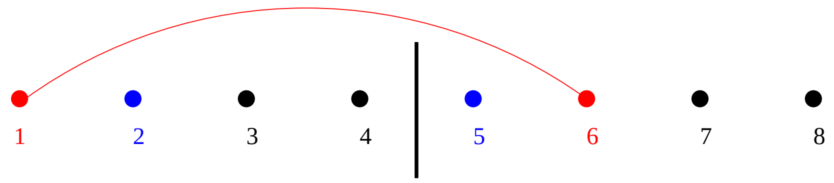

Figure 3.3: $z_j \otimes y_j$

We then apply the conditional expectation E_2, and the result is shown in Figure 3.4.

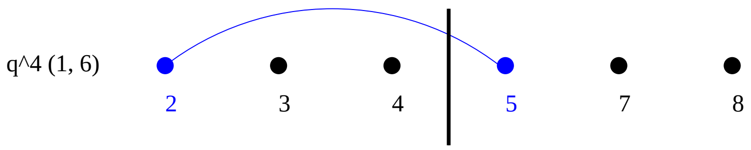

Figure 3.4: E_2

Finally, we apply E_1, and the result is shown in Figure 3.5.

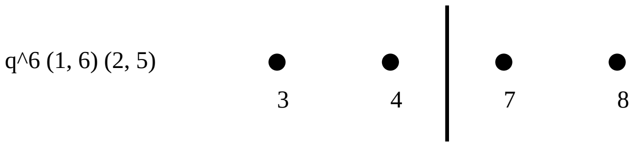

Figure 3.5: E_1

This coincides with $\iota'(\sigma)$ as shown in Example 3.1.14.

We now apply $E_{\mathcal{U}}$ to an element such as

$$q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\}\in\rho} \langle h_{\ell_1} \otimes e_{\alpha_{\ell_1}}, h_{\ell_2} \otimes e_{\beta_{\ell_2}}\rangle N^{-j} u_N(\xi_{n-k,\rho}) u_N(\xi_\rho^k).$$

From Theorem 2.1.13 we get

$$\begin{aligned}
&E_{\mathcal{U}} q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\}\in\rho} \langle h_{\ell_1} \otimes e_{\alpha_{\ell_1}}, h_{\ell_2} \otimes e_{\beta_{\ell_2}}\rangle N^{-j} u_N(\xi_{n-k,\rho}) u_N(\xi_\rho^k) \\
&= q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\}\in\rho} \langle h_{\ell_1}, h_{\ell_2}\rangle E_{\mathcal{U}} u_N(\xi_{n-k,\rho}) u_N(\xi_\rho^k) \\
&= q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\}\in\rho} \langle h_{\ell_1}, h_{\ell_2}\rangle W(\xi_{n-k,\rho}) W(\xi_\rho^k)
\end{aligned}$$

Here the factor N^{-j} is offset since for each pair $\{\ell_1, \ell_2\} \in \rho$, $\alpha_{\ell_1} = \beta_{\ell_2}$ in order for the inner product to be non-zero. For each of the j partitions, there are N possibilities for indices which match. This gives a factor of N^j. From Theorem 2.1.13, we get that

$$E_{\mathcal{U}} u_N(\xi) u_N(\eta) = W(\xi) W(\eta)$$

Now since summing over all possible A, B, and g such that $|A| = |B| = j$ is the same as summing over all elements of $P_{1,2}^{j,k}(n)$, we see that the right hand side is equal to

$$\sum_{\rho\in P_{1,2}^{j,k}(n)} q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\}\in\rho} \langle h_{\ell_1}, h_{\ell_2}\rangle W(\xi_{n-k,\rho}) W(\xi_\rho^k)$$

Therefore, to prove the lemma, we must only check that $\iota'(\rho) = \iota(A) + \iota(B) + \iota(\sigma) + \binom{j}{2}$. Recall that A and B are subsets of $\{1, \ldots, n-k\}$ and $\{n-k+1, \ldots, n\}$ respectively, and that $|A| = |B| = j$. Recall also that $\iota(A)$ and $\iota(B)$ are given by associating A and B to cosets in $S_{n-k}/S_{n-k+j} \times S_j$ and $S_k/S_j \times S_{k-j}$ respectively as in Observation 2.1.7. The permutation $\sigma \in S_j$ identifies how to pair elements of A with elements of B so that we may associate these three data with an element of $P_{1,2}^{j,k}(n)$. Therefore, for $j = 0$, we have

$$\iota'(\rho_0) = \iota(\emptyset) + \iota(\emptyset) + \iota(\sigma_\emptyset) + \binom{0}{2} = 0$$

where ρ_0 is the element of $P_{1,2}^k$ containing no pairs.

Now let ρ_j and ρ_{j-1} be such that $\rho_j \setminus \rho_{j-1} = \{\ell_1, \ell_2\}$ where if $\{k_1, k_2\} \in \rho_{j-1}$, $\ell_1 < k_1$. Let $c_1 = \iota(\rho_j) - \iota(\rho_{j-1})$, and $c_2 = \iota'(\rho) - \iota'(\rho_{j-1})$. By our inductive hypothesis, we assume

$$\iota'(\rho_{j-1}) = \iota(A_{j-1}) + \iota(B_{j-1}) + \iota(\sigma_{j-1}) + \binom{j-1}{2},$$

where A_{j-1}, B_{j-1}, and σ_{j-1} are associated to ρ_{j-1} as described above. Let A_j, B_j, and σ_j be associated to ρ_j similarly. Then we have

$$\begin{aligned}
&\iota(A_j) + \iota(B_j) - \iota(A_{j-1}) - \iota(B_{j-1}) \\
&= |\{\{m\} \in \rho_j : \ell_1 < m < \ell_2\}| + |\{\{k_1, k_2\} \in \rho_j : \ell_1 < k_1 < \ell_2 < k_2\}| \\
&= c_1 - (j - 1 - \frac{c_2 - c_1}{2}),
\end{aligned}$$

since ℓ_1 and ℓ_2 must cross all of the singletons $\{m\}$ such that $\ell_1 < m < \ell_2$. However we subtract the term $j - 1 - \frac{c_2-c_1}{2}$ since these are the pairs $\{k_1, k_2\}$ such that $k_1 < \ell_2 < k_2$ which had to cross the singleton ℓ_2 in $\iota(B_{j-1})$. Since $\frac{c_2-c_1}{2}$ is the number of pairs $\{k_1, k_2\} \in \rho_j$ such that $\ell_1 < k_1 < k_2 < \ell_2$, and there are $j-1$ pairs in ρ_{j-1}, we get that there are $j - 1 - \frac{c_2-c_1}{2}$ such pairs. For the permutations, we get

$$\iota(\sigma_j) - \iota(\sigma_{j-1}) = \frac{c_2 - c_1}{2}$$

since we must multiply the element $(1, \frac{c_2-c_1}{2}) \in S_j$ onto σ_{j-1} (where σ_{j-1} is viewed as an element of the subgroup $S_1 \times S_{j-1}$). Since ι is multiplicative on S_j, we get the equality above. Therefore

$$\begin{aligned}
&\iota(A_j) + \iota(B_j) + \iota(\sigma_j) + \binom{j}{2} - \iota'(\rho_{j-1}) \\
&= c_1 - \left(j - 1 - \left(\frac{c_2 - c_1}{2}\right)\right) + \frac{c_2 - c_1}{2} + \binom{j}{2} - \binom{j-1}{2} \\
&= c_1 + c_2 - c_1 - (j-1) + j - 1 = c_2
\end{aligned}$$

which finishes the proof. □

Now it is time for an example which clarifies this inductive step.

Example 3.1.17. Let

$$\rho_2 = \{\{1\}, \{2, 5\}, \{3\}, \{4, 7\}, \{6\}, \{8\}\},$$

and

$$\rho_3 = \{\{1, 6\}, \{2, 5\}, \{3\}, \{4, 7\}, \{8\}\}.$$

Then for ρ_2, $A_2 = \{2, 4\}$, $B_2 = \{5, 7\}$, and $\sigma_2 = 1$, as shown in Figure 3.6.

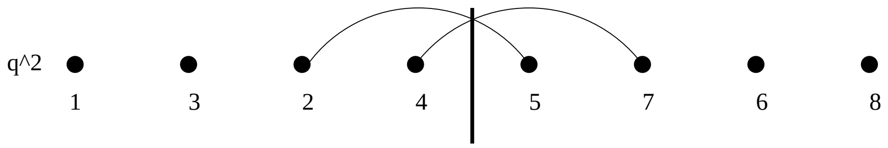

Figure 3.6: $\iota(A_2) = 1$, $\iota(B_2) = 1$, $\iota(\sigma_2) = 0$

For ρ_3, $A_3 = \{1, 2, 4\}$, $B_3 = \{5, 6, 7\}$, so after arranging A and B as seen in Figure 3.7.

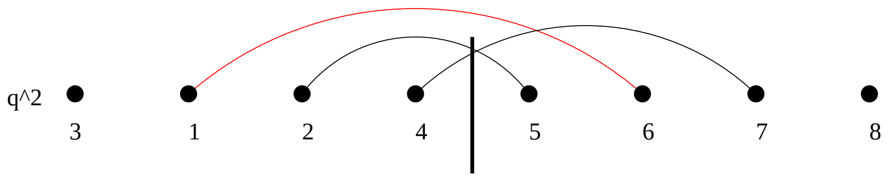

Figure 3.7: $\iota(A_3) = 2$, $\iota(B_3) = 0$

Now we must apply the transposition $\sigma_3 = (5, 6)$, and the result can be seen in Figure 3.8.

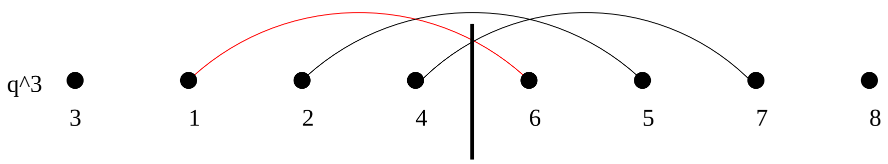

Figure 3.8: $\iota(\sigma_3) = 1$

Adding $\binom{2}{2} = 1$ to $\iota(A_2) + \iota(B_2) + \iota(1)$, we get $\iota'(\rho_2) = 3$, and adding $\binom{3}{2} = 3$ to $\iota(A_3) + \iota(B_3) + \iota(\sigma_3)$, we get $\iota'(\sigma_3) = 6$.

Proposition 3.1.18. *We have for $\beta_{n,k}$ and $w_{n,k}^j$ as above*

$$\beta_{n,k} = \sum_{j=0}^{k \vee n-k} (-1)^j q^{\binom{j}{2}} w_{n,k}^j$$

ere we are defining $w_{n,k}^0 := v_{n,k}$.

Proof. Let $\xi = h_1 \otimes \cdots \otimes h_n$, and let ξ_{n-k} and ξ^k be as before. From Lemma 3.1.8, we get that

$$W(\xi_k)W(\xi^{n-k}) = \sum_{\sigma \in P_{1,2}^k(n)} q^{\iota(\sigma)} \prod_{\{i,j\} \in \sigma} \langle h_i, h_j \rangle W(\xi)_\sigma$$

We can see this by examining $\langle W(\eta), W(\xi_k)W(\xi^{n-k}\rangle = \tau(W(\eta)^*W(\xi_k)W(\xi^{n-k}))$ with the formula from Lemma 3.1.8. Recall from Lemma 2.1.13 that

$$W(h_1 \otimes \ldots \otimes h_n) = E_{\mathcal{U}}((N^{-n/2} \sum_{j_1 \neq \cdots \neq j_n} s_{j_1}(h_1) \cdots s_{j_n}(h_n))^\bullet)$$

For $\xi = h_1 \otimes \cdots \otimes h_n$, let

$$u_N(\xi) = N^{-\frac{n}{2}} \sum_{\alpha} s_{\alpha_1}(h_1) \cdots s_{\alpha_n}(h_n)$$

From Lemma 3.1.15, we have that

$$\begin{aligned}
\sum_{j=0}^{k \vee n-k} (-1)^j q^{\binom{j}{2}} w^j_{n,k}(\xi_{n-k} \otimes \xi^k) &= \sum_{j=0}^{k \vee n-k} (-1)^j E_{\mathcal{U}} E_1 \cdots E_j \left(z_j(u_N(\xi_{n-k})) y_j(u_N(\xi^k)) \right) \\
&= \sum_{j=0}^{k \vee n-k} (-1)^j \sum_{\rho \in P^{j,k}_{1,2}(n)} q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\} \in \rho} \langle h_{\ell_1}, h_{\ell_2} \rangle E_{\mathcal{U}} \left(u_N(\xi_{n-k,\rho}) u_N(\xi^k_\rho) \right)
\end{aligned}$$

Now from Theorem 2.1.13 and Lemma 3.1.8, we get that

$$\begin{aligned}
&\sum_{j=0}^{k \vee n-k} (-1)^j \sum_{\rho \in P^{j,k}_{1,2}(n)} q^{\iota'(\rho)} \prod_{\{\ell_1,\ell_2\} \in \rho} \langle h_{\ell_1}, h_{\ell_2} \rangle \sum_{j'=0}^{k-j \vee n-k-j} \sum_{\sigma \in P^{j',k-j}_{1,2}} q^{\iota(\sigma)} \prod_{\{\ell_1,\ell_2\} \in \sigma} \langle h_{\ell_1}, h_{\ell_2} \rangle W((\xi_\rho)_\sigma) \\
&= \sum_{j=0}^{k \vee n-k} (-1)^j \sum_{j'=0}^{k-j \vee n-k-j} \sum_{\rho \in P^{j,k}_{1,2}(n)} \sum_{\sigma \in P^{j',k-j}_{1,2}(n-2j)} q^{\iota'(\rho)+\iota(\sigma)} \prod_{\{\ell_1,\ell_2\} \in \rho \cup \sigma} \langle h_{\ell_1}, h_{\ell_2} \rangle W(\xi_{\rho \cup \sigma}) \\
&= \sum_{j=0}^{k \vee n-k} (-1)^j \sum_{m=j}^{k \vee n-k} \sum_{\rho \in P^{j,k}_{1,2}(n)} \sum_{\sigma \in P^{m-j,k-j}_{1,2}(n-2j)} q^{\iota'(\rho)+\iota(\sigma)} \prod_{\{\ell_1,\ell_2\} \in \rho \cup \sigma} \langle h_{\ell_1}, h_{\ell_2} \rangle W(\xi_{\rho \cup \sigma}) \\
&= \sum_{m=0}^{k \vee n-k} \sum_{j=0}^{m} (-1)^j \sum_{\rho \in P^{j,k}_{1,2}(n)} \sum_{\sigma \in P^{m-j,k-j}_{1,2}(n-2j)} q^{\iota'(\rho)+\iota(\sigma)} \prod_{\{\ell_1,\ell_2\} \in \rho \cup \sigma} \langle h_{\ell_1}, h_{\ell_2} \rangle W(\xi_{\rho \cup \sigma}) \\
&= \sum_{m=0}^{k \vee n-k} \sum_{\pi \in P^{m,k}_{1,2}(n)} \sum_{j=0}^{m} (-1)^j \sum_{\substack{\rho \in P^{j,k}_{1,2}(n) \\ \sigma \in P^{m-j,k}_{1,2}(n-2j) \\ \rho \cup \sigma = \pi}} q^{\iota'(\rho)+\iota'(\sigma)} \prod_{\{\ell_1,\ell_2\} \in \pi} \langle h_{\ell_1}, h_{\ell_2} \rangle W(\xi_\pi),
\end{aligned}$$

where $\rho \cup \sigma \in P^k_{1,2}(n)$ has pairs from both ρ and σ. From here, it is clear that for $m = 0$, we simply have $W(\xi)$. Therefore, the following claim finishes the proof.

Claim 3.1.19. For $m \geq 1$,

$$\sum_{j=0}^{m} (-1)^j \sum_{\substack{\rho \in P^{j,k}_{1,2}(n) \\ \sigma \in P^{m-j,k}_{1,2}(n-2j) \\ \rho \cup \sigma = \pi}} q^{\iota'(\rho)+\iota'(\sigma)} = 0$$

Proof of Claim. We proceed by induction. For $m = 1$, fix $\pi \in P_{1,2}^{1,k}(n)$. We have

$$q^{\iota'(\pi)} - q^{\iota(\pi)} = 0$$

since $\iota'(\pi) = \iota(\pi)$ for $\pi \in P_{1,2}^{1,k}(n)$. Let $\pi_m \in P_{1,2}^{m,k}(n)$, and let $\pi_{m-1} \in P_{1,2}^{m-1,k}(n)$ be such that $\pi_m \setminus \pi_{m-1} = \{\ell_1, \ell_2\}$ where $\ell_1 < \ell_1'$ for all pairs $\{\ell_1', \ell_2'\} \in \pi_{m-1}$. By our inductive hypothesis, we have that $S_{\pi_{m-1}} = 0$. However, we have

$$S_{\pi_m} = q^{c_1} S_{\pi_{m-1}} + q^{c_2} S_{\pi_{m-1}} = 0$$

where $c_1 = \iota(\pi_m) - \iota(\pi_{m-1})$ and $c_2 = \iota'(\pi_m) - \iota'(\pi_{m-1})$. The first term comes from $\{\ell_1, \ell_2\} \in \sigma$, and the second term similarly comes from $\{\ell_1, \ell_2\} \in \rho$. This finishes the proof of the claim and the proposition. $\square$

of Proposition 3.1.3. Since $\|w_{n,k}^j\|_{cb} \leq C_q$ by Lemma 3.1.5, we get from Proposition 3.1.18 that

$$\|\beta_{n,k}\|_{cb} \leq \sum_{j=0}^{k \vee n-k} |q|^{\binom{j}{2}} \|w_{n,k}^j\|_{cb} < C_q$$

$\square$

We are now ready to prove Theorem A.

Theorem A. *For all* $-1 < q < 1$ *and all* $\dim(H) \geq 2$,

1. $\Gamma_q(H)$ *has the weak* completely contractive approximation property.*
2. $\mathcal{A}_q(H)$ *has the completely contractive approximation property.*

Proof. For (1), we follow Haagerup's standard argument from [12] except the cb-norm of the projections onto words of length n are bounded by cn^2 instead of cn. From second quantization, we know that $E = \Gamma_q(P)$ for any projection P is a conditional expectation and $T_t = \Gamma_q(e^{-t} Id)$ is a ucp semigroup. Furthermore $\|T_t|_{F_n}\| = e^{-nt}$ where F_n is the subspace spanned by the Wick words of degree n. We now estimate the cb-norm for $P_n : \Gamma_q(H) \to F_n$. From above we know that

$$\|P_n\|_{cb} \leq C_q(n+1) \sum_{j=0}^{n} \|\beta_{n,k}\|_{cb}$$

From Proposition 3.1.18, we get

$$\|\beta_{n,k}\|_{cb} \leq \sum_{j=0}^{k\vee n-k} |q|^{\binom{j}{2}} \|w^j_{n,k}\| \leq C_q$$

So we conclude that $\|P_n\|_{cb} \leq C_q n$. Let $P_{\leq n}$ be the projection onto $\oplus_{k\leq n} F_k$. Finally, we define the following net of maps.

$$U_\alpha = T_{t_\alpha} \circ P_{\leq n_\alpha} \circ E_{k_\alpha}$$

Where $E_{k_\alpha} = \Gamma_q(P_{k_\alpha})$ where P_k is a sequence of projections of rank k whose union is the identity, and $e^{-t_\alpha n_\alpha} C n_\alpha^2 \leq 1$ for all α, but $n_\alpha \to \infty$ and $t_\alpha \to 0$. Clearly U_α is finite rank, completely contractive, and converges to the identity in the point-weak* topology.

For (2), observe that what we have shown is that there exists functions $f_\alpha : \mathbb{N} \to \mathbb{R}$ where $\alpha = (t, m, \varepsilon)$ such that

$$\sum_{n=0}^{m} e^{-tn} n^2 \leq 1 + \varepsilon$$

which satisfies the following conditions:

1. The pointwise limit of f_α is 1.

2. f_α has finite support for each n.

3. $\|f_\alpha(N)\|_{cb} \leq 1 + \varepsilon_\alpha$.

Recall that N is the number operator, which generates the semigroup T_t. Since $\Gamma_q(H)$ is faithfully represented on $\mathcal{F}_q(H)$, we have that the Wick words linearly generate $\mathcal{A}_q(H)$. For $\xi \in H^{\otimes n}$, we have that

$$f_\alpha(N)W(\xi) = f_\alpha(n)W(\xi).$$

Therefore, $f_\alpha(N)$ converges to the identity in the point-norm topology and $\|f_\alpha\|_{cb} \leq 1$ for appropriately chosen α. □

Remark 3.1.20. We observe that $T_t : \Gamma_q(H) \to \mathcal{A}_q(H)$ since for $x \in \Gamma_q(H)$, we have that

$$T_t(x) = \sum_{n\geq 0} e^{-nt} P_n(x).$$

Clearly $P_n(x) \in \mathcal{A}_q(H)$ for H finite dimensional since the range of P_n is spanned by Wick words. For H infinite dimensional, we get that $P_n(x) \in \mathcal{A}_0(H)$ for all n and $x \in \Gamma_0(H)$. Recall from [15] that

$\|W(\xi)\|_\infty \leq C_q \|W(x)\|_2$. Therefore for all $\varepsilon > 0$,

$$\|T_t(x) - \sum_{n=0}^{M} e^{-nt} P_n(x)\| \leq \| \sum_{n>M+1} e^{-nt} P_n(x)\| < \varepsilon \|x\|$$

for M such that $C \sum_{n>M} e^{-nt} n^2 < \varepsilon$. Therefore, $F_M(x) := \sum_{n=0}^{n} e^{-nt} P_n(x)$ converges in norm to $T_t(x)$. Hence $T_t(x) \in \mathcal{A}_q(H)$. It is obvious that $\mathcal{A}_q(H)$ has the CCAP since we may then simply apply the projections P_n to an element of the form $T_t(x)$.

Chapter 4

Strong Solidity of the q-Gaussian algebras

4.1 Weak Containment

In this section, we shall show that while $L^2_0(\Gamma_q(H \oplus H))$ is not obviously weakly contained in the coarse bimodule, there is a subbimodule of $L^2_0(\Gamma_q(H \oplus H))$ which is weakly contained in the coarse bimodule. Define the following subspaces of $L^2_0(\Gamma_q(H \oplus H))$.

$$\begin{aligned} F_m &= \{W(h_1 \otimes \ldots \otimes h_n) | \exists \iota_1 \ldots \iota_m \in \{1, \ldots, n\}, h_{\iota_k} \in 0 \oplus H\}^{\|\cdot\|_2} \\ E_m &= \oplus_{k=0}^m F_k \end{aligned}$$

Note that F_m and E_m are $\Gamma_q(H)$-$\Gamma_q(H)$-bimodules simply by the action restricted from $L^2_0(\Gamma_q(H \oplus H))$. The main result of this section is the following.

Proposition 4.1.1. *Let $m > -\frac{\log(d)}{2\log(|q|)}$ where $d = \dim(H)$. Then $E_{m-1}^\perp \prec L^2(\Gamma_q(H)) \bar{\otimes} L^2(\Gamma_q(H))$.*

It will turn out that this sub-bimodule, $E_{m-1}^\perp$, will be "large enough" to replace $L^2_0(\Gamma_q(H \oplus H))$ in the proof of strong solidity from [13] and [17]. Throughout this section, we shall denote $\Gamma_q(H)$ by $\mathcal{M}$, $\Gamma_q(H \oplus H)$ by $\widetilde{\mathcal{M}}$, $(h, 0) \in H \oplus H$ by simply h, and $(0, h)$ by $\tilde{h}$. Define $\Phi_{\xi,\eta} : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ by $\Phi_{\xi,\eta}(x) = E_{\mathcal{M}}(W(\xi)^* x W(\eta))$ for $\xi, \eta \in F_k$.

Lemma 4.1.2. *If $\xi, \eta \in F_k$, then $\Phi_{\xi,\eta} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is Schatten p-class for $p > -\frac{\log(d)}{k\log(|q|)}$. In particular $\Phi_{\xi,\eta}$ is Hilbert-Schmidt for $k \geq -\frac{\log(d)}{2\log(|q|)}$.*

We shall first need two additional lemmas.

Lemma 4.1.3. *Let $H = \oplus_{j \geq 0} H_j$ be a graded Hilbert space and $A = [A_{ij}] : H \to H$ be an operator such that*

1. *$A_{ij} = 0$ if $|i - j| \geq L$ for some $L > 0$.*
2. *There exists j_0 such that*

$$\|A_{ij}\| \leq C r^{jk}$$

for all $j \geq j_0$ and for some constants $0 < r < 1$, k and C independent of i and j.

3. $\dim(H_j) = d^j$.

Then $A \in S_p(H)$ for $p > -\frac{\log(d)}{k\log(r)}$.

Proof. Let $K_1 = \oplus_{j=0}^{j_0-1} H_j$ and $K_2 = \oplus_{j \geq j_0} H_j$, then

$$\begin{aligned}\|A\|_{S_p} \leq \|A : K_1 \to K_1\|_{S_p} + \|A : K_1 \to K_2\|_{S_p} \\ + \|A^* : K_1 \to K_2\|_{S_p} + \|A : K_2 \to K_2\|_{S_p}\end{aligned}$$

Since K_1 is finite dimensional, we may control the first three norms simply by a constant depending on the dimension of K_1 and the norm of A, so we only must estimate

$$\|A : K_2 \to K_2\|_{S_p}.$$

For $\|A : K_2 \to K_2\|_{S_p}$, we have that

$$\begin{aligned}\|A\|_{S_p} &\leq C + \sum_{\ell=-L}^{L} \sum_{j \geq L} \|A_{j,j+\ell}\|_{S_p} \\ &\leq C + \sum_{\ell=-L}^{L} ((\sum_{j=j_0}^{\infty} d^j \|A_{j+\ell,j}\|^p)^{\frac{1}{p}}) \\ &\leq C + (\sum_{j=j_0}^{\infty} (2L) d^j C^p r^{jkp})^{\frac{1}{p}}\end{aligned}$$

The sum converges if and only if $dr^{kp} < 1$, which is equivalent to $p > -\frac{\log(d)}{k\log(r)}$. $\square$

Lemma 4.1.4. *Let $H_k = H^{\otimes k} \otimes e \otimes H^{n-k}$ where e is a unit vector in an ambient Hilbert space which is orthogonal to H. Then*

$$\|l^*(e) : H_k \to H^{\otimes n}\| \leq C_q |q|^k.$$

Proof. For $\xi, \eta \in H_k$, we have that

$$\begin{aligned}\langle l^*(e)\xi_1 \otimes e \otimes \xi_2, l^*(e)\eta_1 \otimes e \otimes \eta_2\rangle &= q^{2k}\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2\rangle \\ &= q^{2k}\langle e \otimes \xi_1 \otimes \xi_2, e \otimes \eta_1 \otimes \eta_2\rangle\end{aligned}$$

since $\xi_1, \eta_1 \in H^{\otimes k}$. It is straightforward to check that $\|\xi\|_{H_k} \geq C_q \|e \otimes \xi_1 \otimes \xi_2\|$, and so we get that $\|\ell^*(e)\|_{H_k(e)} \leq C_q |q|^k$. $\square$

Proof of Lemma 4.1.2. Let $x = W(\theta)$ for $\theta = f_1 \otimes \cdots \otimes f_j$, $\xi = h_1 \otimes \cdots \otimes h_n$ and $\eta = k_1 \otimes \cdots \otimes k_m$. Assume $j > m + n$. Clearly $\Phi_{\xi,\eta}(x) \in \oplus_{\ell=j-m-n}^{j+m+n} H^{\otimes \ell}$. We shall estimate $\|\Phi_{\xi,\eta} : H^{\otimes j} \to H^{\otimes j+\ell}\|$ for all $|\ell| \leq m + n$.

By the results of [2] and [1], $x \to xW(\eta)$ is bounded by a constant which depends only on η. Since $x = W(\theta)$, $xW(\eta) \in \oplus_{\ell=-m}^{m} (H \oplus H)^{\otimes j+\ell} \cap F_k$. We may decompose

$$
\begin{aligned}
H^{\otimes j} W(\eta) &= \oplus_{j_1=j-m}^{j+m} \oplus_{j_2,\ldots,j_{k+1}} H_1^{\otimes j_1} \otimes h'_{\iota_1} \otimes H_1^{\otimes j_2} \otimes \cdots \otimes H_1^{\otimes j_k} \otimes h'_k \otimes H_1^{\otimes j_{k+1}} \\
&:= H_{j_1,\ldots,j_{k+1}}(\eta),
\end{aligned}
$$

where $H_1 = H \oplus 0$ and $(h'_{\iota_1}, \ldots, h'_{\iota_k})$ are the tensor components of η which come from $0 \oplus H$. We now use the formula from Theorem 2.1.6 to estimate the operator norm of

$$\sum_A q^A E_{\mathcal{M}} \ell(h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_r}) \ell^*(h_{\beta_1} \otimes \cdots \otimes h_{\beta_{n-r}})$$

when applied to $H_{j_1,\ldots,j_{k+1}}(\eta)$ where $\ell(h_1 \otimes \cdots \otimes h_r) = \ell(h_1) \cdots \ell(h_n)$ and similarly for ℓ^*. Again we are associating a subset $A = \{\alpha_1, \ldots, \alpha_r\} \subset \{1, \cdots, n\}$ to a coset x as described in Observation 2.1.7. Note that since $\xi \in F_k$, there are k indices, $\iota_1, \ldots, \iota_k$, such that $h_{\iota_s} \in H_2$. Thanks to the conditional expectation, the vectors h_{ι_s} can only act as annihilation operators. By Lemma 4 of [2], we know that $\|\ell(h)\| = \|\ell^*(h)\| = C_q \|h\|_H$, where $C_q = \max\{1, \frac{1}{\sqrt{1-q}}\}$ if $h \in H_1$. However, for $h \in H_2$, we may use Theorem 2.1.13 to write

$$\|\ell^*(h) : H_{j_1,\ldots,j_r}(\eta) \to \oplus_s H_{j_1,\ldots,j_{s-1}+j_s,\ldots,j_{r-1}}(\eta')\| \leq |q|^{j_1} \|h\|_H C_q$$

as a linear combination of operators like that in Lemma 4.1.4, since all of the lengths must be greater than or equal to j_1. Therefore,

$$
\begin{aligned}
\|E_{\mathcal{M}} W(\xi) : H_{j_1,\ldots,j_{k+1}} \to \oplus_{\ell=-(m+n)}^{m+n} H_1^{j+\ell}\| &\leq |q|^{(j-m-n)k} n C_q^n C_\eta \prod_s \|h_s\|_H \\
&\leq |q|^{jk} C(\xi, \eta)
\end{aligned}
$$

since $j_1 \geq j - m - n$ since $j \geq (m + n)$. Now we are in a position to exploit Lemma 4.1.3, simply by setting $r = |q|$ and $H_j = H^{\otimes j}$. Therefore $\|\Phi_{\xi,\eta}\|_{S_p} \leq C(\xi, \eta)(\sum_{j \geq m+n} d^j |q|^{jkp})^{\frac{1}{p}}$.

$\square$

Proof of Proposition 4.1.1. Let $\xi \in E_{m-1}^{\perp}$. With the natural $\mathcal{M}$-$\mathcal{M}$-bimodule structure on $E_{m-1}^{\perp}$, we can see that

$$\langle \xi, x\xi y\rangle = \tau(W(\xi)^* x W(\xi) y) = \tau(E_{\mathcal{M}}(W(\xi)^* x W(\xi) y)) = \tau(\Phi_{\xi,\xi}(x) y)$$

for $x, y \in \mathcal{M}$. Note that $\Phi_{\xi,\xi}$ coincides with T_{φ_ξ} in Lemma 2.1.15. Thus since $\Phi_{\xi,\xi}$ is Hilbert-Schmidt for all ξ, $E_{m-1}^{\perp} \prec L^2(\mathcal{M}) \otimes L^2(\mathcal{M})$ by Lemma 2.1.15. $\square$

4.2 Strong Solidity

As shown in the previous section, $L_0^2(\Gamma_q(H \oplus H))$ is not necessarily weakly contained in the coarse correspondence for $q^2 \dim(H) \geq 1$. However, the submodule $E_k^{\perp}$ is weakly contained in the coarse bimodule for sufficiently large k. This requires us to modify Popa's s-malleable deformation ([23] Lemma 2.1) estimate slightly to suit our new situation. What we need to know is that the image of $\Gamma_q(H)$ under the automorphism group α_t has a "large enough" intersection with $E_k^{\perp}$. This is the purpose of the following proposition.

Proposition 4.2.1. *For a fixed $k \geq 1$, there exists a constant depending only on k, C_k such that*

$$\|(\alpha_{t^k} - id)(x)\|_2 \leq C_k \|E_{k-1}^{\perp}\alpha_t(x)\|_2$$

for $x \in \oplus_{m\geq k} H^{\otimes m} \subset \mathcal{F}_q(H)$ and $t < 2^{-k}$.

Proof. Let $x = h_1 \otimes \cdots \otimes h_n$, $y = k_1 \otimes \cdots \otimes k_n$. Note that $\alpha_{t^k} - id$ and $E_{k-1}^{\perp}\alpha_t$ are both tensor length-preserving operators, so it suffices to prove this estimate on $H^{\otimes n}$ for $n \geq k$. We calculate

$$\begin{aligned}
\langle E_{k-1}^{\perp}\alpha_t(x), E_{k-1}^{\perp}\alpha_t(y)\rangle &= \sum_{m=k}^{n} \langle F_m \alpha_t(x), F_m \alpha_t(y)\rangle \\
&= \sum_{m=k}^{n} \sum_{A,B\subset\{1,\ldots,n\}} e^{-2t(n-m)}(1-e^{-2t})^m \langle x_{A^c} \otimes x_A, y_{B^c} \otimes y_B\rangle_q
\end{aligned}$$

where $x_{A^c} \otimes x_A$ denotes that the indices belonging to A^c come from $H \oplus 0$ and the indices belonging to A come from $0 \oplus H$. Now we expand the q-inner product to get

$$\begin{aligned}
&\sum_{m=k}^{n} \sum_{A,B\subset\{1,\ldots,n\}} e^{-2t(n-m)}(1-e^{-2t})^m \langle x_{A^c} \otimes x_A, y_{B^c} \otimes y_B\rangle \\
&= \sum_{m=k}^{n} e^{-2t(n-m)}(1-e^{-2t})^m \sum_{A} \sum_{x\in S_n/S_A\times S_{A^c}} \sum_{\sigma\in x} q^{\iota(\sigma)} \prod_j \langle h_j, k_{\sigma(j)}\rangle
\end{aligned}$$

since for each $A, B \subset \{1, \ldots, n\}$ only those permutations $\sigma \in S_n$ which map A to B contribute to the inner product. This is equivalent to summing over the right cosets in $S_n/S_A \times S_{A^c}$. However, since we are summing over all permutations in all of the cosets, for each fixed A, we are summing over all the permutations, and so we just get $\langle x, y\rangle_q$ for each fixed A. Therefore we get that

$$\sum_{m=k} \sum_{|A|=k} e^{-2t(n-m)}(1-e^{-2t})^m \langle x, y\rangle_q = \sum_{m=k}^{n} e^{-2t(n-m)}(1-e^{-2t})^m \binom{n}{m} \langle x, y\rangle_q.$$

For $\alpha_{t^k} - id$ we have

$$\begin{aligned}
\langle(\alpha_{t^k} - id)(x), (\alpha_{t^k} - id)(y)\rangle &= \langle \alpha_{t^k}(x), \alpha_{t^k}(y)\rangle - \langle \alpha_{t^k}(x), y\rangle - \langle x, \alpha_{t^k}(y)\rangle + \langle x, y\rangle \\
&= 2(\langle x, y\rangle - \langle x, T_{t^k}(y)\rangle) \\
&= 2(1 - e^{-nt^k})\langle x, y\rangle.
\end{aligned}$$

Therefore, we only have to show that $2(1-e^{-nt^k}) < C\sum_{m=k}^{n}\binom{n}{m}e^{-2(n-m)t}(1-e^{-2t})^m$ for some C independent of n and t. We choose M_k such that $e^{-2nt}\sum_{m=0}^{k-1} C_m n^m t^m < \frac{1}{2}$ for $nt > M_k$. Suppose $nt < M_k$. We have

$$\sum_{m=k}^{n}\binom{n}{m}e^{-2(n-m)t}(1-e^{-2t})^m > \binom{n}{k}e^{-2(n-k)t}(1-e^{-2t})^k > C_k n^k t^k,$$

and

$$2(1-e^{-nt^k}) < C_k nt^k < C_k n^k t^k$$

Now suppose that $nt > M_k$. Then $n > 2^k$ since $t < 2^{-k}$ and so,

$$\begin{aligned}
\sum_{m=k}^{n}\binom{n}{m}e^{-2(n-m)t}(1-e^{-2t})^m &= 1 - \sum_{m=0}^{k-1}\binom{n}{m}e^{-2(n-m)t}(1-e^{-2t})^m \\
&= 1 - e^{-2nt}\sum_{m=0}^{k-1}\binom{n}{m}e^{2mt}(1-e^{-2t})^m \\
&\geq 1 - e^{-2nt}\sum_{m=0}^{k-1} C_m n^m t^m \geq \frac{1}{2}
\end{aligned}$$

However, clearly $2(1-e^{-nt^k}) < 2$ and so we have proved the statement for all n and $t < 2^{-k}$. $\square$

Now we may prove Theorem B, following the proof of Theorem 3.5 in [13]. There are a number of modifications since we are using a proper sub-bimodule of $L_0^2(\Gamma_q(H \oplus H))$.

Theorem B. *For all $-1 < q < 1$ and all $\dim(H) < \infty$, $\Gamma_q(H)$ is strongly solid.*

Proof. Let $P \subset \Gamma_q(H)$ be a diffuse, amenable subalgebra. We want to prove that $\mathcal{N}_{\Gamma_q(H)}(P)''$ is also amenable. P is not rigid with respect to the deformation α_t (Lemma 2.1.10), and P is weakly compact inside of $\Gamma_q(H)$. Since $P \subset \Gamma_q(H)$ is weakly compact, there is a net of elements $(\eta_n) \in L^2(P \otimes \bar{P})$ which satisfy

1. $\lim_n \|\eta_n - (v \otimes \bar{v})\eta_n\|_2 = 0$, $\forall v \in \mathcal{U}(P)$,

2. $\lim_n \|\eta_n - Ad(u \otimes \bar{u})\eta_n\|_2 = 0$, $\forall u \in \mathcal{N}_{\Gamma_q(H)}(P)$ and,

3. $\langle(1 \otimes \bar{x})\eta_n, \eta_n\rangle = \tau(x) = \langle\eta_n, (x \otimes 1)\eta_n\rangle$.

Following [13], let $\mathcal{G}$ denote $\mathcal{N}_{\Gamma_q(H)}(P)$, and let $z \in \mathcal{Z}(\mathcal{G}' \cap \Gamma_q(H))$ be a non-zero projection. Since α_t does not converge uniformly on $(P)_1$, α_t does not converge uniformly on $(Pz)_1$ and so α_t does not converge uniformly on $\mathcal{U}(Pz)$ either. Therefore there exist $0 < c < 1$, a sequence $(u_k) \in \mathcal{U}(Pz)$, and a sequence $t_k \to 0$ such that $\|\alpha_{t_k}(u_k z) - (E_{m-1} \circ \alpha_{t_k})(u_k z)\|_2 \geq c\|z\|_2$ $\forall k \in \mathbb{N}$, by Proposition 4.2.1. Since $\|\alpha_{t_k}(u_k z)\|_2 = \|z\|_2$, we get

$$\|(E_{m-1}^{\perp} \circ \alpha_{t_k})(u_k z)\|_2 \leq \sqrt{1 - c^2}\|z\|_2 \tag{4.2.1}$$

for all $k \in \mathbb{N}$. Let $P_{\mathcal{H}} = E_{m-1}^{\perp}$. Define for all n and k

$$\begin{aligned}
\eta_n^k &= (\alpha_{t_k} \otimes 1)(\eta_n) \in L^2(\Gamma_q(H \oplus H))\bar{\otimes}L^2(\Gamma_q\bar{(}H)) \\
\xi_n^k &= (P_{\mathcal{H}}^{\perp}\alpha_{t_k} \otimes 1)(\eta_n) \in (L^2(\Gamma_q(H \oplus H)) \ominus \mathcal{H})\bar{\otimes}L^2(\Gamma_q\bar{(}H)) \\
\zeta_n^k &= (P_{\mathcal{H}}\alpha_{t_k} \otimes 1)(\eta_n) \in \mathcal{H}\bar{\otimes}L^2(\Gamma_q\bar{(}H))
\end{aligned}$$

Observe that

$$\|(x \otimes 1)\eta_n^k\|_2^2 = \langle(x \otimes 1)(\alpha_{t_k} \otimes 1)\eta_n, (x \otimes 1)(\alpha_{t_k} \otimes 1)\eta_n\rangle$$

Also, for all $x \in \Gamma_q(H)$ we have

$$\begin{aligned}\|(x \otimes 1)\zeta_n^k\|_2 &= \|(x \otimes 1)(P_{\mathcal{H}} \otimes 1)\eta_n^k\|_2 \\ &= \|(P_{\mathcal{H}} \otimes 1)(x \otimes 1)\eta_n^k\|_2 \\ &\leq \|(x \otimes 1)\eta_n^k]\|_2 \\ &= \|x\|_2\end{aligned}$$

Therefore we have the following claim

Claim 4.2.2. For any k sufficiently large,

$$\lim_n \|(z \otimes 1)\zeta_n^k\|_2 \geq \delta$$

Proof. Assume not. Following Houdayer-Shlyakhtenko, we get that this implies that

$$\lim_n \|(z \otimes 1)\eta_n^k - (E_{m-1}\alpha_{t_k}(u_k)z \otimes \bar{u_k})\xi_n^k\|_2 \leq \delta$$

However,

$$\begin{aligned}\|E_{m-1} \circ \alpha_{t_k}(u_k z)\|_2 &\geq \|E_{m-1} \circ \alpha_{t_k}(u_k)z\|_2 - \|z - \alpha_{t_k}(z)\|_2 \\ &\geq \lim_n \|(E_{m-1} \circ \alpha_{t_k}(u_k)z \otimes \bar{u_k})(\eta_n^k)\|_2 - \delta \\ &\geq \lim_n \|(z \otimes 1)\eta_n^k\| - 2\delta \\ &= \|z\|_2 - 2\delta \geq \sqrt{1 - c^2}\|z\|_2\end{aligned}$$

which contradicts (4.2.1). □

From here, we may follow the remainder of the proof in [13] verbatim in what follows their Claim 3.6 since the bimodule$E_k^{\perp}$ is weakly contained in the coarse bimodule for sufficiently large k. □

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