RENORMALIZATION ANALYSIS OF CONTINUUM MODELS ON FRACTAL DOMAINS

BY

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THESIS

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ABSTRACT

This thesis builds on the recently begun extension of continuum thermomechanics to fractal media which are specified by a fractional mass scaling law of the resolution length scale *R*. The focus is on *pre-fractal media* (i.e., those with lower and upper cut-offs) through renormalization analysis into continuum models, in which the fractal dimension *D* is also the order of fractional integrals employed to state global balance laws. While the original formulation was based on a Riesz measure—and thus more suited to isotropic media - the new model is based on a product measure capable of describing local material anisotropy. Other choices of calculus on fractals are discussed while the product measure shows great simplicity. This formulation allows one to grasp the anisotropy of fractal dimensions on a mesoscale and the ensuing lack of symmetry of the Cauchy stress. Two continuum models of fractal media are formulated: classical continua and micropolar continua, according to symmetric and asymmetric Cauchy stress. Finally, the reciprocity, uniqueness and variational theorems are proved for development of approximate numerical solutions.

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CHAPTER 1: INTRODUCTION

1.1 Motivation

Fractals date back to research by Hausdorff and Besicovich on monster sets over a hundred years ago, and then to the seminal work of Mandelbrot [1]. He was then followed by physicists and mathematicians. The first category was primarily comprise of condensed matter physicists who focused on the effects of fractal geometries of materials on bulk responses [2]. A number of specialized models have also been developed to particular problems like wave scattering at fractals [3], computational mechanics [4], fracture mechanics [5-7], or geomechanics [8]. While in recent years mathematicians began to look at partial differential equations starting with Laplace's or heat equation on fractal sets [9-10], an analogue of continuum physics and mechanics still needs to be developed, which motivates the present thesis research.

A new step relying on renormalization analysis was taken by Tarasov [11-12], who developed continuum-type equations of conservation of mass, momentum and energy for fractal porous media, and on that basis studied several fluid mechanics and wave motion problems. In principle, one can then map a mechanics problem of a fractal onto a problem in the Euclidean space in which this fractal is embedded, while having to deal with coefficients explicitly involving fractal dimension D and resolution length R. As it turns out, D is also the order of fractional integrals employed to state global balance laws. This has very interesting ramifications for formulating continuum-type mechanics of fractal media, which needs to be further explored. The great promise stems from the fact that much of the framework of continuum mechanics/physics may be generalized and partial differential equations may still be employed [13-14]. Prior research has already involved an extension to continuum thermomechanics and fracture mechanics, a generalization of extremum and variational principles, and turbulent flows in fractal porous media [15-18].

Whereas the original formulation of Tarasov was based on the Riesz measure— and thus more suited to isotropic media— the model proposed in this thesis is based on a product measure that grasps the anisotropy of fractal geometry (i.e., different fractal dimensions in different directions) on mesoscale, which, in turn, leads to asymmetry of the Cauchy stress. This leads to a framework of micropolar mechanics of fractal media to be examined in this thesis.

1.2 Thesis outline

In the subsequent chapters, we conduct the study in the following sequence:

(a) In Chapter 2 we formulate local fractional integrals to reflect materials' fractal mass scaling and study calculus formulas of fractional integrals and derivatives.

(b) Chapter 3 extends continuum models to fractal media. We study two continuum models: classical continua and micropolar continua, due to symmetric or asymmetric Cauchy stress.

(c) Chapter 4 conducts mathematical analysis of formulated partial differential equations on fractal media. We prove the solution uniqueness and their variational structures.

CHAPTER 2: FRACTAL PRODUCT MEASURE AND CALCULUS

2.1 Mass power law and fractal product measure

By a fractal solid we understand a medium B having a fractal geometric structure. The mass of the medium m obeys a power law with respect to the length scale of measurement R (or resolution)

$$m(R) = kR^{D}, \ \ D < 3, \tag{2.1}$$

where D is the fractal dimension of mass, and k is a proportionally constant. We note that in practice a fractional power law relation (2.1) is widely recognized and can be determined in experiments by a log-log plot of m and R [19]. Now, following Tarasov [11], the fractional integral is employed to represent mass in a three-dimensional region W

$$m(W) = \int_{W} \rho(\mathbf{r}) dV_{D} = \int_{W} \rho(\mathbf{r}) c_{3}(D,r) dV_{3}. \tag{2.2}$$

Here the first and the second equality involve fractional integrals and conventional integrals, respectively. The coefficient $c_3(D,r)$ provides a transformation between the two. Using Riesz fractional integrals $c_3(D,r)$ reads the form

$$c_{3}(D,r) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} r^{D-3}, \quad r = \sqrt{\sum_{i=1}^{3}\left(x_{i}\right)^{2}} \ . \tag{2.3}$$

Note that $c_3(D,r)$ above solely depends on the scalar distance r, which in turn confines the formulations to isotropic fractals. However, in general the medium exhibits different fractal dimensions along different directions – it is anisotropic! A practical example is given in Carpinteri [6], where the porous concrete structure is investigated and they suggested representing the specimen as a Sierpinski carpet in cross-section and in the longitudinal axis a Cantor set.

This consideration leads us to replace (2.1) by a more general power law relation with respect to each spatial coordinate

$$m(x_1, x_2, x_3) \sim x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \tag{2.4}$$

In order to account for such anisotropies the fractional integral representing mass distribution is specified via a product measure

$$m(x_1, x_2, x_3) = \iiint \rho(x_1, x_2, x_3) d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3). \tag{2.5}$$

Here the length measurement $d\mu_k\left(x_k\right)$ in each coordinate is provided by

$$d\mu_k\left(x_k\right) = c_1^{(k)}\left(\alpha_k, x_k\right) dx_k, \quad k = 1, 2, 3. \tag{2.6}$$

Generally, the fractal dimension is not necessarily the sum of each projected fractal dimension, while as noted by Falconer [20], “Many fractals encountered in practice are not actually products, but are product-like.” It follows that the volume coefficient c_3 is given by

$$c_3 = c_1^{(1)} c_1^{(2)} c_1^{(3)} = \prod_{i=1}^{3} c_1^{(i)}. \tag{2.7}$$

For the surface coefficient c_2 we typically consider a cubic volume element, whose each surface element is specified by the corresponding normal vector (along axis i, j, or k, see Fig. 2.1). Therefore, the coefficient $c_2^{(k)}$ associated with surface $S_d^{(k)}$ is shown to be:

$$c_2^{(k)} = c_1^{(i)} c_1^{(j)} = \frac{c_3}{c_1^{(k)}}, \quad i \neq j \text{ and } i, j \neq k. \tag{2.8}$$

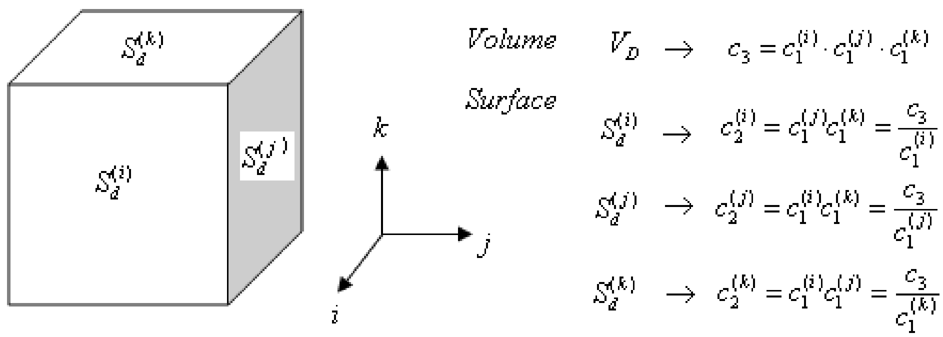

Fig. 2.1: Constructing coefficients c_2 and c_3 via product measures.

The expressions of length coefficients $c_1^{(k)}$ depend on forms of specific fractional integrals. We adopt a modified Riemann-Liouville fractional integral recently formulated by Jumarie [21-22]. It follows that

$$c_1^{(k)} = \alpha_k (l_k - x_k)^{\alpha_k - 1}, \ k = 1, 2, 3. \tag{2.9}$$

where l_k is the total length (integral interval) along axis x_k. Let us examine it in two special cases:

1. Uniform mass: The mass is distributed uniformly in a cubic region W with a power law relation (2.4). Denoting the mass density by ρ_0 and the cubic length by l, we obtain

$$m(W) = \rho_0 l^{\alpha_1} l^{\alpha_2} l^{\alpha_3} = \rho_0 l^{\alpha_1+\alpha_2+\alpha_3} = \rho_0 l^D. \tag{2.10}$$

Which is consistent with the mass power law (2.1).

2. Point mass: The distribution of mass is concentrated at one point, so that the mass density is denoted by the Dirac function $\rho(x_1, x_2, x_3) = m_0 \delta(x_1)\delta(x_2)\delta(x_3)$. The fractional integral representing mass becomes

$$m(W) = \alpha_1\alpha_2\alpha_3 l^{\alpha_1-1} l^{\alpha_2-1} l^{\alpha_3-1} m_0 = \alpha_1\alpha_2\alpha_3 l^{D-3} m_0. \tag{2.11}$$

When $D \rightarrow 3\,(\alpha_1, \alpha_2, \alpha_3 \rightarrow 1)$, $m(W) \rightarrow m_0$ and the conventional concept of point mass is recovered [23]. Note that using the Riesz fractional integral will always give zero (0^{D-3}) except when $D=3$ (if let $0^0=1$), which on the other hand shows a non-smooth transition of the mass with respect to its fractal dimension. This also supports our choice of the non-Riesz type expressions for $c_1^{(k)}$ in (2.9).

Note that the above expression $c_1^{(k)}$ shows a length dimension and thus the mass m will involve a nonusual physical dimension following from the fractional integral (2.5). This is understandable since in mathematics a fractal curve only exhibits finite measure with respect to a fractal dimensional length unit [1]. While practically we prefer to adopt usual dimensions of physical quantities. An alternate way to address this issue is to nondimensionize coefficients $c_1^{(k)}$. Here we suggest replacing $(l_k - x_k)$ by $(l_k - x_k)/l_0$ in (2.9) (l_0 is a characteristic scale, e.g. the mean porous size).

2.2 Fractional calculus and some integral theorems

At this point we recall two basic integral theorems extensively employed in continuum mechanics: the Gauss theorem which relates a certain volume integral to the integral over its bounding surface, and the Reynold's transport theorem concerning the rate of change of any volume integral for a continuous medium. In the following, we derive their fractional generalizations and, moreover, introduce a definition of fractal derivatives, which together provide a stepping-stone to construct a continuum mechanics in the setting of fractals.

The derivation of a fractional Gauss theorem is analogous to Tarasov's [12] dimensional regularization, albeit formulated in the framework of product measures discussed above. First, let us recall the surface integral in a fractal medium:

$$S_d\left[\overline{f}\right] := \int_{S_d} \overline{f}\cdot\hat{n}dS_d = \int_{S_d} f_k n_k dS_d \, . \tag{2.12}$$

Here $\overline{f} = f_k\mathbf{e}_k$ is any vector field and $\hat{n} = n_k\mathbf{e}_k$ is the unit normal vector of the surface. The Einstein's summation convention is assumed. In order to compute (2.12), we relate the integral element $\hat{n}dS_d$ to its conventional forms $\hat{n}dS_2$ via fractal surface coefficients $c_2^{(i)}, c_2^{(j)}, c_2^{(k)}$. Note that, by definition, any infinitesimal surface element dS_d in the integrand can be regarded as a plane (aligned in an arbitrary direction with normal vector $\hat{n}$). Since the coefficients $c_2^{(i)}$'s are built on coordinate planes Ox_jx_k's, we consider their projections onto each coordinate plane. The projected planes $n_i dS_d$ can then be specified by coefficients $c_2^{(i)}$'s and this totally provides a representation of the integral element $\hat{n}dS_d$ (see Fig. 2.2). Thus, we have:

$$\int_{S_d} \overline{f}\cdot\hat{n}dS_d = \int_{S_2} f_k c_2^{(k)} n_k dS_2 \, . \tag{2.13}$$

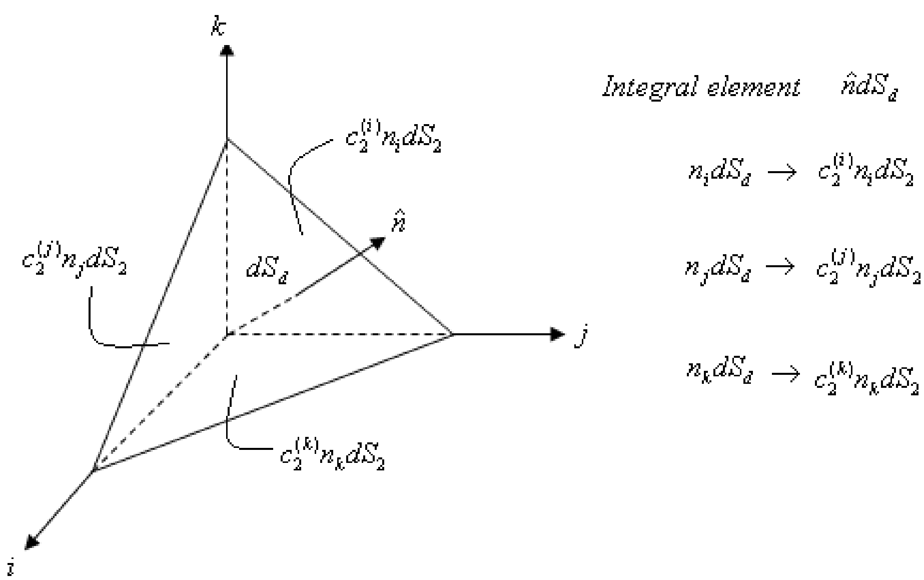

Fig. 2.2: A representation of the fractional integral element $\hat{n}dS_d$ under product measures.

Now, following the conventional Gauss theorem, we get

$$\int_{\partial W} f_k c_2^{(k)} n_k dS_2 = \int_W \left(f_k c_2^{(k)} \right)_{,k} dV_3 \,. \tag{2.14}$$

Note that from the expression (2.8) $c_2^{(k)}$ is independent of the variable x_k. And we write (2.14) in the fractional form

$$\int_{\partial W} f_k n_k dS_d = \int_W \left(f_k c_2^{(k)} \right)_{,k} c_3^{-1} dV_D = \int_W f_{k,k} c_2^{(k)} c_3^{-1} dV_D = \int_W \frac{f_{k,k}}{c_1^{(k)}} dV_D := \int_W \nabla_k^D f_k dV_D \,. \tag{2.15}$$

This equation is a fractional generalization of the Gauss theorem. Here and after we use the notation of fractal derivative ∇_k^D with respect to the coordinate x_k

$$\nabla_k^D := \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} (\cdot) \,. \tag{2.16}$$

The definition of ∇_k^D is similar to Tarasov's [12] ($\nabla_k^D = c_3^{-1} \left(c_2 \cdot \right)_{,k}$). But our form is simplified for product measures. We now examine three properties of the operator ∇_k^D.

<u>1. It is the "inverse" operator of fractional integrals.</u> Since for any function $f(x)$ we have

$$\nabla_x^D \int f(x) d\mu^D(x) = \frac{1}{c_1(x)} \frac{d}{dx} \int f(x) c_1(x) dx = \frac{1}{c_1(x)} \left[f(x) c_1(x) \right] = f(x) \tag{2.17}$$

and

$$\int \nabla_x^D f(x) d\mu^D(x) = \int \left[\frac{1}{c_1(x)} \frac{df(x)}{dx} \right] c_1(x) dx = \int \frac{df(x)}{dx} dx = f(x) \,. \tag{2.18}$$

For this reason we name ∇_k^D a "fractal derivative" (so as to distinguish it from the fractional derivatives already in existence).

<u>2. The rule of "term-by-term" differentiation is satisfied</u>

$$\nabla_k^D \left(AB \right) = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} \left(AB \right) = \frac{1}{c_1^{(k)}} \frac{\partial \left(A \right)}{\partial x_k} B + \frac{1}{c_1^{(k)}} \frac{\partial \left(B \right)}{\partial x_k} A = B \nabla_k^D \left(A \right) + A \nabla_k^D \left(B \right) , \tag{2.19}$$

whereby note that this is invalid in Tarasov's [12] notation.

<u>3. Its operation on any constant is zero.</u>

$$\nabla_k^D (C) = \frac{1}{c_1^{(k)}} \frac{\partial (C)}{\partial x_k} = 0 . \tag{2.20}$$

Here we recall that the usual fractional derivative (Riemann-Liouville) of a constant does not equal zero neither in fractional calculus [24], nor in Tarasov [12] formulation.

This fractional calculus can be generalized to vector calculus in fractal space and it is found that the four fundamental identities of the conventional vector calculus still holds [25], a great promise from the product measure. As to the fractional generalization of Reynold's transport theorem, we follow the line of conventional continuum mechanics distinguishing between the reference and deformed configurations

$$\begin{aligned}
\frac{d}{dt}\int_{W_t} P dV_D &= \frac{d}{dt}\int_{W_0} PJ dV_D^0 = \int_{W_0} \frac{d}{dt}(PJ) dV_D^0 = \int_{W_0} \left(\frac{d}{dt} P \cdot J + P \cdot \frac{d}{dt} J \right) dV_D^0 \\
&= \int_{W_0} \left(\frac{d}{dt} P \cdot J + P \cdot v_{k,k} J \right) dV_D^0 = \int_{W_0} \left(\frac{d}{dt} P + P \cdot v_{k,k} \right) J dV_D^0 \\
&= \int_{W_t} \left(\frac{d}{dt} P + P \cdot v_{k,k} \right) dV_D = \int_{W_t} \left(\frac{\partial}{\partial t} P + P_{,k} v_k + P \cdot v_{k,k} \right) dV_D \\
&= \int_{W_t} \left(\frac{\partial}{\partial t} P + (P v_k)_{,k} \right) dV_D .
\end{aligned} \tag{2.21}$$

Here P is any quantity accompanied by a moving material system W_t, $\mathbf{v} = v_k \mathbf{e}_k$ is the velocity field, and J is the Jacobian of the transformation from the current configuration x_k to the referential configuration X_K. Note that the result is identical to its conventional representation. The fractal material time derivative is thus the same

$$\left(\frac{d}{dt} \right)_D P = \frac{d}{dt} P = \frac{\partial}{\partial t} P + P_{,k} v_k . \tag{2.22}$$

While we note that the alternate form of fractional Reynold's transport theorem which involves surface integrals is different from the conventional and rather complicated. This is because the fractal volume coefficient c_3 depends on all coordinates x_k's (not like $c_2^{(k)}$

that is independent of x_k when deriving fractional Gauss theorem). Continuing on (2.22), the formulation follows as

$$\begin{aligned}
\frac{d}{dt}\int_{W_t} P dV_D &= \int_{W_t}\left(\frac{\partial}{\partial t}P + \left(Pv_k\right)_{,k}\right)dV_D = \int_{W_t}\frac{\partial}{\partial t}P dV_D + \int_{W_t}\left(Pv_k\right)_{,k} c_3 dV_3 \\
&= \int_{W_t}\frac{\partial}{\partial t}P dV_D + \int_{W_t}\left(\int\left(Pv_k\right)_{,k} c_3 dx_k\right)_{,k} dV_3 = \int_{W_t}\frac{\partial}{\partial t}P dV_D + \int_{\partial W_t}\left(\int\left(Pv_k\right)_{,k} c_3 dx_k\right) n_k dS_2 \\
&= \int_{W_t}\frac{\partial}{\partial t}P dV_D + \int_{\partial W_t}\left(Pv_k c_3 - \int Pv_k c_{3,k} dx_k\right)\left(c_2^{(k)}\right)^{-1} n_k dS_d \\
&= \int_{W_t}\frac{\partial}{\partial t}P dV_D + \int_{\partial W_t} P c_1^{(k)} v_k n_k dS_d - \int_{\partial W_t}\left(\int P c_{1,k}^{(k)} v_k dx_k\right) n_k dS_d .
\end{aligned}$$

(2.23)

2.3 Discussions of calculus on fractals

The above formulations provide one choice of calculus on fractals, i.e. through fractional product integrals (2.5) to reflect the mass scaling law (2.4) of fractal media. The advantage is that it is connected with conventional calculus through coefficients $c_1 \sim c_3$ and therefore well suited for development of continuum mechanics and partial differential equations on fractal media as we shall see in the next chapter. Besides, the product formulation allows a decoupling of coordinate variables, which profoundly simplifies the Gauss theorem (2.15) and many results thereafter. Now we investigate other choices of calculus on fractals to complement the proposed formulation.

To begin with, we define a mapping $P^{\alpha} : L \to m(L)$ that maps the length L to its mass m in fractal media with fractal dimension α $(0 < \alpha \le 1)$. The mass scaling law (2.4) requires the *fractality* property of P^{α}

$$P^{\alpha}\left(bL\right) = b^{\alpha} P^{\alpha}(L), \qquad 0 < b \le 1 \tag{2.24}$$

Note that the proposed fractional integral (2.5) is one way to reflect this property. Now in an analogue of developing integrals on the real line, we decompose the fractal media into pieces and "combine" them together to recover the whole. But the fractality property does not allow a direct Riemann sum of each piece. To illustrate this, considering a fractal with length L and fractal dimension α $(0<\alpha<1)$, it follows that

$$P^{\alpha}\left(\frac{L}{2}\right)+P^{\alpha}\left(\frac{L}{2}\right)=\frac{P^{\alpha}(L)}{2^{\alpha}}+\frac{P^{\alpha}(L)}{2^{\alpha}}\neq P^{\alpha}(L) \qquad (2.25)$$

We define an operator Λ^{α} on P^{α} satisfying the *combination* property:

$$P^{\alpha}\left(L\right)=\Lambda^{\alpha}\left(P^{\alpha}(l_1),P^{\alpha}(l_2),\ldots,P^{\alpha}(l_n)\right), \quad l_i>0,\sum_{i=1}^{n}l_i=L \qquad (2.26)$$

Let $m=P^{\alpha}(L)$, $b_i=l_i/L$. Following the fractality property (2.24), we have

$$m=\Lambda^{\alpha}\left(b_1^{\alpha}m,b_2^{\alpha}m,\cdots,b_n^{\alpha}m\right), \quad 0<b_i\le 1,\sum_{i=1}^{n}b_i=1 \qquad (2.27)$$

A straightforward choice of Λ^{α} is an analogue of the p-norm in L^p space:

$$\Lambda^{\alpha}\left(p_1,p_2,\ldots p_n\right)=\left(p_1^{1/\alpha}+p_2^{1/\alpha}+\ldots+p_n^{1/\alpha}\right)^{\alpha}=\left(\sum_{i=1}^{n}p_i^{1/\alpha}\right)^{\alpha} \qquad (2.28)$$

In the limit $n\to\infty$, (2.28) induces another choice of P^{α}:

$$P^{\alpha}(L)=m=\left(\int_L\left[\rho(x)\right]^{1/\alpha}dx\right)^{\alpha} \qquad (2.29)$$

where m is the mass of fractal media with length L and fractal dimension α $(0<\alpha\le 1)$, and $\rho(x)$ is the local mass density. (2.29) is consistent with the fractality property (2.24). A generalization to 3D fractals follows similarly through product formulations. While we note that (2.29) cannot be transformed to conventional linear integrals through coefficients $c_1\sim c_3$ and the corresponding Gauss theorem is much more complicated.

The combination operator (2.28) suggests one way to build up global forms based on established local formulations. To this end, we note that the proposed product measure is suitable for local properties of fractal media. The global formulation requires a nonlinear assembly of local forms through (2.28). To write it formally:

$$P^{\alpha} = \left[\int \left(dP^{\alpha} \right)^{1/\alpha} \right]^{\alpha} \tag{2.30}$$

It is challenging to obtain analytical forms of global formulations. While we note that the discrete form can be easier formulated in finite element implementations, where the assembly of elements is replaced by (2.30). In the following we shall discuss continuum mechanics based on the proposed local fractional integral (2.5). The assembly procedure and finite element implementations are not pursued further beyond this point.

CHAPTER 3: CONTINUUM MODELS OF FRACTAL MEDIA

In Chapter 2 we have discussed fractional integrals under product measures and thereby generalized some basic integral theorems. Now we proceed to develop a framework of continuum mechanics in fractal setting. We will formulate the field equations analogous to those in continuum mechanics but based on fractional integrals.

3.1 Classical continuum models

Note that the notions of continuum mechanics rely on geometry configurations of the body. We shall first examine some physical concepts and definitions on account of the fractal geometry.

Let us recall the formula of fractal mass (2.2) which expresses the mass power law via fractional integrals. From a homogenization standpoint this allows an interpretation of the fractal (intrinsically discontinuous) medium as a continuum and a 'fractal metric' embedded in the equivalent 'homogenized' continuum model, saying that

$$dl_D = c_1 dx, \quad dS_d = c_2 dS_2, \quad dV_D = c_3 dV_3. \tag{3.1}$$

Here dl_D, dS_d, dV_D represent the line, surface, volume element in the fractal body and dx, dS_2, dV_3 denote those in the homogenized model, see Fig. 3.1. The coefficients $c_1,\ c_2,\ c_3$ provide the relation between the two.

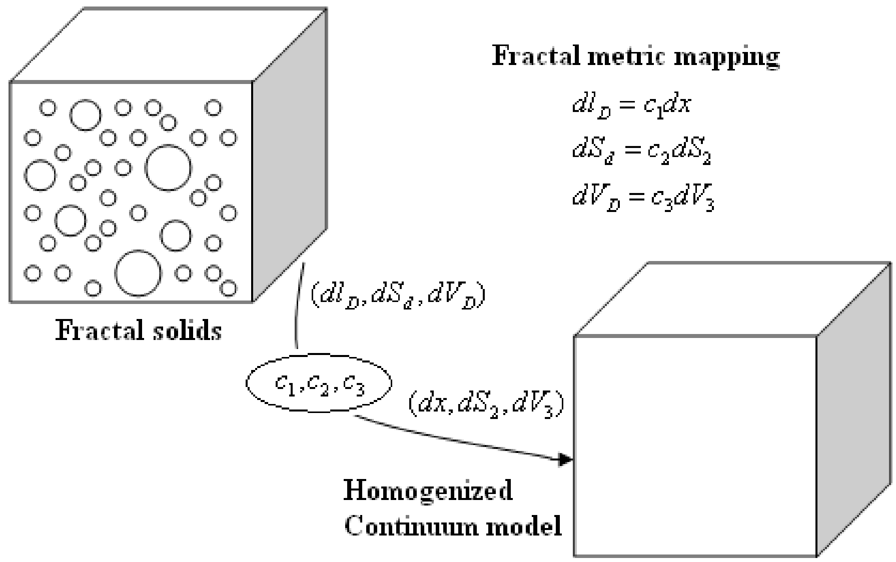

Fig. 3.1: An illustration of the homogenization process from geometry configurations.

The definitions of stress and strain must be modified accordingly. The Cauchy stress is now specified to express the surface force F_k^S via fractional integrals

$$F_k^S = \int_{\partial W} \sigma_{kl} n_l dS_d = \int_{\partial W} \sigma_{kl} n_l c_2^{(l)} dS_2 \, . \tag{3.2}$$

As to the configuration of strain, we recommend to replace all the spatial derivatives $\partial / \partial x_k$ with fractal derivatives ∇_k^D introduced in Section 3. This can be understood by observing from (3.1) that

$$\nabla_k^D = \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} = \frac{\partial}{\partial l_D^{(k)}} \, . \tag{3.3}$$

For small deformation, the expression of strain in fractal solids thus gives

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_j^D u_i + \nabla_i^D u_j\right) = \frac{1}{2}\left(\frac{1}{c_1^{(j)}} u_{i,j} + \frac{1}{c_1^{(i)}} u_{j,i}\right). \tag{3.4}$$

Note that the stress-strain pairs must be conjugate from the viewpoint of energy. We shall examine the consistency of these definitions later when deriving wave equations in the next section. Now, let us consider the balance law of linear momentum in fractal solids.

This gives

$$\frac{d}{dt}\int_W \rho \mathbf{v} dV_D = \mathbf{F}^B + \mathbf{F}^S , \tag{3.5}$$

where $\mathbf{v} = v_k \mathbf{e}_k$ denotes the velocity vector, and $\mathbf{F}^B$, $\mathbf{F}^S$ are the body and surface forces, respectively. Writing the equation (3.5) in indicial notation and expressing forces in terms of fractional integrals, we obtain

$$\frac{d}{dt}\int_W \rho v_k dV_D = \int_W f_k dV_D + \int_{\partial W} \sigma_{kl} n_l dS_d . \tag{3.6}$$

On observation of fractional Gauss' theorem (2.15) and Reynold's transport theorem (2.23), this gives

$$\int_W \rho \left(\frac{d}{dt}\right)_D v_k dV_D = \int_W \left(f_k + \nabla_l^D \sigma_{kl} \right) dV_D . \tag{3.7}$$

Here the operators of fractal derivative ∇_k^D and material derivative $\left(\frac{d}{dt}\right)_D$ are employed, which are specified in (2.16) and (2.22), respectively. Note that the region W is arbitrary. On account of (3.7), we obtain the balance equation in local form

$$\rho \left(\frac{d}{dt}\right)_D v_k = f_k + \nabla_l^D \sigma_{kl} . \tag{3.8}$$

The specification of constitutive equations involves more arguments in physics. We recommend keeping the relations of stress and strain while modifying their definitions in fractal setting. This is understood in that the fractal geometry solely influences our configurations of some physical quantities (like stress and strain) while it takes no effect on physical laws (like the conservation principles, and constitutive relations that are inherently due to material properties). We note that this justification is verified in [26] where the scale effects of material strength are discussed by the fractal argument of stress definitions and confirmed in experiments of both brittle and plastic materials.

Now, we consider a specific example: linear elastic solids with small deformation. The constitutive equations take linear forms as usual

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} , \tag{3.9}$$

where λ and μ are material parameters (Lame constants), σ_{ij} and ε_{ij} are fractal stress and strain defined in (3.2) and (3.4), respectively.

Under small displacements, the linearization of stress equations (3.8) gives

$$\rho\frac{\partial^2 u_k}{\partial t^2} = f_k + \nabla_l^D\sigma_{kl} , \tag{3.10}$$

where $\mathbf{u} = u_k\mathbf{e}_k$ is the displacement field. Note that (3.4), (3.9) and (3.10) constitute a complete set of equations describing the problem (excluding boundary conditions).

3.2 Micropolar continuum models

Analogous to the classical continuum mechanics, first we specify the surface force $\mathbf{T}^s$ in terms of the Cauchy stress tensor $\boldsymbol{\sigma}$ via fractional integrals

$$T_k^S = \int_{\partial W}\sigma_{lk}n_l dS_d \ . \tag{3.11}$$

The conservation of linear and angular momentum in fractal media can be written as

$$\frac{d}{dt}\int_W \rho v_k dV_D = \int_W X_k dV_D + \int_{\partial W}\sigma_{lk}n_l dS_d , \tag{3.12}$$

and

$$\frac{d}{dt}\int_W \rho e_{ijk}x_j v_k dV_D = \int_W e_{ijk}x_j X_k dV_D + \int_{\partial W} e_{ijk}x_j\sigma_{lk}n_l dS_d . \tag{3.13}$$

Here v_k denotes the velocity and X_k is the body force density; e_{ijk} is the permutation tensor. On account of the fractional Gauss theorem (2.15) and Reynold's transport theorem (2.23), we obtain the balance equations of linear and angular momentum in local form:

$$\rho\left(\frac{d}{dt}\right)_D v_k = X_k + \nabla_l^D\sigma_{lk} \tag{3.14}$$

and

$$e_{ijk} \frac{\sigma_{jk}}{c_1^{(j)}} = 0. \tag{3.15}$$

In general, $c_1^{(j)} \neq c_1^{(k)}$ meaning that the medium exhibits anisotropic fractal dimensions, thus making the Cauchy stress tensor asymmetric—$\sigma_{jk} \neq \sigma_{kj}$. This can be physically understood by noting that fractal media display a heterogeneous fine structure at arbitrarily small scales, also note [27]—this is incorporated into our formulations by coefficients c_1, c_2, c_3 as functions of anisotropic fractal dimensions. By contrast, in classical continuum mechanics material microstructures are ignored, thus leading to a symmetric Cauchy stress. The micropolar continuum model, which treats its microstructures as rigid bodies instead of continuous points [28,29], captures the asymmetry of Cauchy stress in a simplest possible way, and thereby furnishes a good candidate to model fractal media.

Focusing now on physical fractals (so-called pre-fractals), we consider a body that obeys a fractal mass power law (2.4) between the lower and upper cutoffs. The choice of the continuum approximation is specified by the resolution R. Choosing the upper cut-off, we arrive at the *fractal representative volume element* (RVE) involves a region up to the upper cutoff L, which is mapped onto a homogenized continuum element in the whole body. The micropolar point homogenizes the very fine microstructures into a rigid body (with 6 degrees of freedom) at the lower cutoff l. The two-level homogenization processes are illustrated in Fig. 3.2.

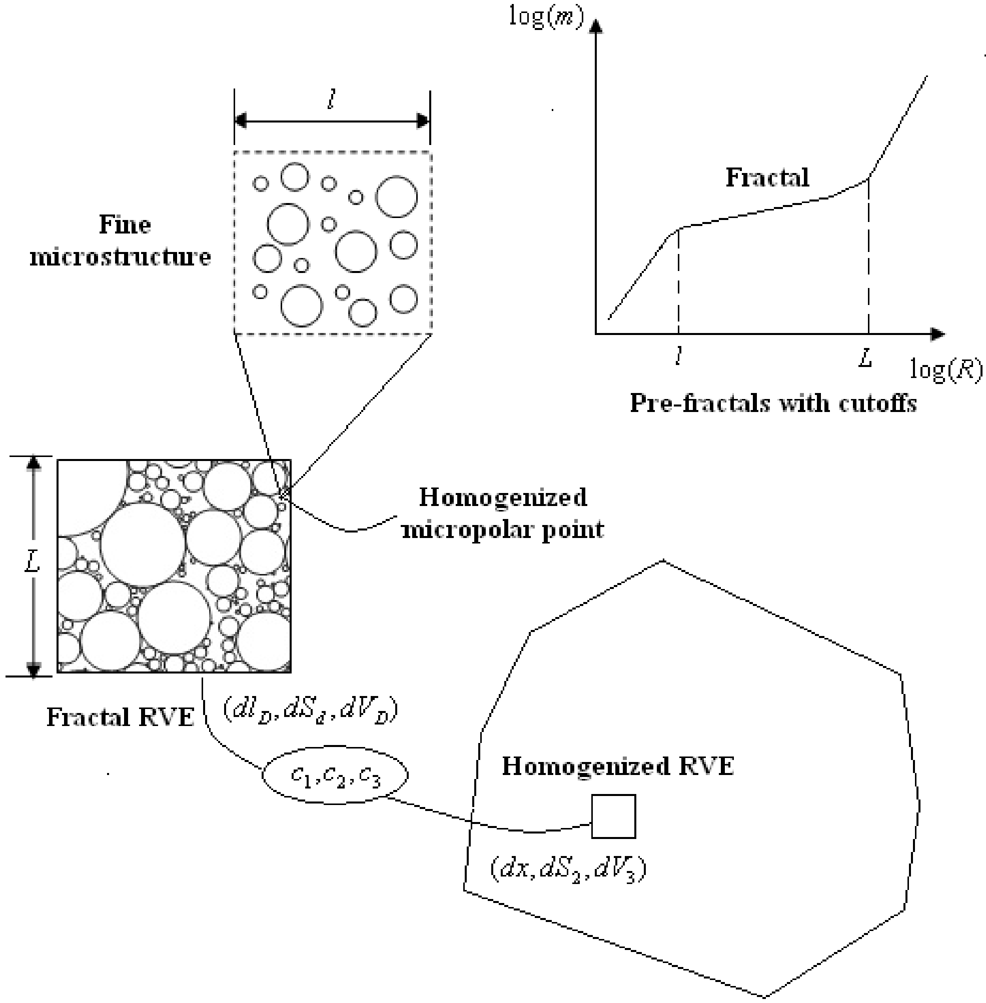

Fig. 3.2: Illustration of the two-level homogenization processes: fractal effects are present between the resolutions *l* and *L* in a fractal RVE.

To determine the inertia tensor at any micropolar point, we consider a rigid particle *p* having volume element *P*. Its angular momentum gives

$$\sigma_A = \int_P (\mathbf{x} - \mathbf{x}_A) \times \mathbf{v}(\mathbf{x}, t) \rho(\mathbf{x}) dV_D \quad (3.16)$$

Since *p* is a rigid body, following [23] $\mathbf{v}(\mathbf{x}, t)$ is a helicoidal vector field, i.e.

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}_A, t) + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_A) \quad (3.17)$$

where $\boldsymbol{\omega}$ is the rotational velocity vector. Substituting (3.17) into (3.16) we obtain

$$\sigma_A = \int_P (\mathbf{x} - \mathbf{x}_A) \times \mathbf{v}(\mathbf{x}_A, t)\rho(\mathbf{x}) dV_D + \int_P (\mathbf{x} - \mathbf{x}_A) \times \left[\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_A) \right] \rho(\mathbf{x}) dV_D \tag{3.18}$$

The first term above gives angular momentum associated with translational motion, while the second term refers to rotational motion. It follows that the mapping

$$J_A : \boldsymbol{\omega} \mapsto \int_P (\mathbf{x} - \mathbf{x}_A) \times \left[\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{x}_A) \right] \rho(\mathbf{x}) dV_D \tag{3.19}$$

is a linear operator representing the inertial tensor of P with respect to point A. If A is the origin A=O fixed in P, we have

$$\begin{aligned} J_O(\mathbf{u}) \cdot \mathbf{v} &= \int_P \rho(\mathbf{x}) \left[\mathbf{x} \times (\mathbf{u} \times \mathbf{x}) \right] \cdot \mathbf{v} dV_D \\ &= \int_P \rho(\mathbf{x}) \left[|\mathbf{x}|^2 \mathbf{u} - (\mathbf{x} \cdot \mathbf{u}) \mathbf{x} \right] \cdot \mathbf{v} dV_D \\ &= \int_P \rho(\mathbf{x}) \left[|\mathbf{x}|^2 \mathbf{u} \cdot \mathbf{v} - (\mathbf{x} \cdot \mathbf{u})(\mathbf{x} \cdot \mathbf{v}) \right] dV_D = J_O(\mathbf{v}) \cdot \mathbf{u} \end{aligned} \tag{3.20}$$

This shows that the mapping $(\mathbf{u}, \mathbf{v}) \mapsto J_o(\mathbf{u}) \cdot \mathbf{v}$ has a bilinear symmetric form, from which we obtain each component of the inertial tensor I_{ij} as $I_{ij} = (J_O)_{ij} = J_O(\mathbf{e}_i) \cdot \mathbf{e}_j$ or, effectively,

$$I_{ii} = (J_O)_{ii} = \int_P \rho \left[|\mathbf{x}|^2 - x_i^2 \right] dV_D, \quad I_{ij} = (J_O)_{ij} = \int_P \rho x_i x_j \, dV_D \, , i \neq j \tag{3.21}$$

In the development of micropolar continuum mechanics, we introduce a couple-stress tensor $\boldsymbol{\mu}$ and a rotation vector $\boldsymbol{\varphi}$ augmenting, respectively, the Cauchy stress tensor $\boldsymbol{\tau}$ (thus denoted so as to distinguish it from the symmetric $\boldsymbol{\sigma}$) and the deformation vector $\mathbf{u}$. The surface force and surface couple in the fractal setting can be specified by fractional integrals of $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$, respectively, as

$$T_k^S = \int_{\partial W} \tau_{lk} n_l dS_d, \quad M_k^S = \int_{\partial W} \mu_{lk} n_l dS_d. \tag{3.22}$$

Now, proceeding in a fashion similar as before, we arrive at the balance equations of linear and angular momentum

$$\rho\left(\frac{d}{dt}\right)_D v_i = X_i + \nabla_j^D \tau_{ji}, \tag{3.23}$$

$$I_{ij}\left(\frac{d}{dt}\right)_D w_j = Y_i + \nabla_j^D \mu_{ji} + e_{ijk}\frac{\tau_{jk}}{c_1^{(j)}}. \tag{3.24}$$

In the above, X_i is the external body force density, Y_i is the body force couple, while $v_i\ (=\dot{u}_i)$ and $w_i\ (=\dot{\varphi}_i)$ are deformation and rotation velocities, respectively.

Let us now consider the conservation of energy. It has the following form

$$\frac{d}{dt}\int_W (u+k)\,dV_D = \int_W (X_i v_i + Y_i w_i)\,dV_D + \int_{\partial W}(t_i v_i + m_i w_i)\,dS_d \tag{3.25}$$

where $k = (1/2)\left(\rho v_i v_i + I_{ij} w_i w_j\right)$ is the kinetic energy density and u denotes the internal energy density. (Note here that, just like in conventional continuum mechanics, the balance equations of linear momentum (3.23) and angular momentum (3.24) can be consistently derived from the invariance of energy (3.25) with respect to rigid body translations ($v_i \to v_i + b_i,\ w_i \to w_i$ and rotations ($v_i \to v_i + e_{ijk} x_j \omega_k,\ w_i \to w_i + \omega_i$), respectively.) Next, we want to obtain the expression for the rate of change of internal energy, and so we start with

$$\begin{aligned}&\int_W \left[\left(\frac{d}{dt}\right)_D u + \rho v_i \left(\frac{d}{dt}\right)_D v_i + I_{ij} w_i \left(\frac{d}{dt}\right)_D w_j\right] dV_D \\ &= \int_W \left[\left(X_i v_i + Y_i w_i\right) + \nabla_j^D\left(\tau_{ji} v_i + \mu_{ji} w_i\right)\right] dV_D,\end{aligned} \tag{3.26}$$

which yields the local form

$$\left(\frac{d}{dt}\right)_D u + \rho v_i \left(\frac{d}{dt}\right)_D v_i + I_{ij} w_i \left(\frac{d}{dt}\right)_D w_j \;=\left(X_i v_i + Y_i w_i\right) + \nabla_j^D\left(\tau_{ji} v_i + \mu_{ji} w_i\right). \tag{3.27}$$

In view of (3.23) and (3.24), and noting the "term by term" rule of ∇_j^D, we find

$$\left(\frac{d}{dt}\right)_D u = \tau_{ji}\left(\nabla_j^D v_i - e_{kji}\frac{w_k}{c_1^{(j)}}\right) + \mu_{ji}\nabla_j^D w_i. \tag{3.28}$$

Here and after we consider small deformations, where we have $\left(d/dt\right)_D u = \dot{u}$. It is now convenient to define the strain tensor γ_{ji} and the curvature tensor κ_{ji} in fractal media as

$$\gamma_{ji} = \nabla_j^D u_i - e_{kji} \frac{\varphi_k}{c_1^{(j)}}, \quad \kappa_{ji} = \nabla_j^D \varphi_i. \tag{3.29}$$

so that the energy balance (3.28) can be written as

$$\dot{u} = \tau_{ji} \dot{\gamma}_{ji} + \mu_{ji} \dot{\kappa}_{ji} \tag{3.30}$$

Assuming *u* to be a state function of γ_{ji} and κ_{ji} only, leads to

$$\tau_{ji} = \frac{\partial u}{\partial \gamma_{ji}}, \qquad \mu_{ji} = \frac{\partial u}{\partial \kappa_{ji}} \tag{3.31}$$

which shows that, in the fractal setting, (τ_{ji}, γ_{ji}) and (μ_{ji}, κ_{ji}) are still conjugate pairs.

We choose to keep the form of constitutive relations while modifying the definitions of stress and strain to the fractal setting. This is consistent with [26], where scale effects of material strength and stress (i) are discussed from the standpoint of fractal geometry rather than mechanical laws, and (ii) are confirmed by experiments. Thus, focusing on elastic materials, we have

$$\tau_{ij} = C_{ijkl}^{(1)} \gamma_{kl} + C_{ijkl}^{(3)} \kappa_{kl}, \quad \mu_{ij} = C_{ijkl}^{(3)} \gamma_{kl} + C_{ijkl}^{(2)} \kappa_{kl}. \tag{3.32}$$

Equations (3.23), (3.24), (3.29), and (3.32) constitute a complete set of equations describing the initial-boundary value problems in fractal media.

CHAPTER 4: ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we consider some theoretical issues related to the analysis of these equations. First, we prove the uniqueness theorem following [30], where the uniqueness was proved without any definiteness assumptions on the material moduli. First a reciprocity relation is established involving two elastic processes at different instants, on which the uniqueness theorem is subsequently built. We also establish variational principles starting from balance equations. The consistency verifies our entire formulation. These results are useful in theoretical developments, such as uniqueness, stability, and approximate solutions.

4.1 Reciprocity and uniqueness theorem

To establish the reciprocity relation, we consider two external loading systems $L^{(\alpha)}=\left\{\mathbf{X}^{(\alpha)},\mathbf{Y}^{(\alpha)},\mathbf{t}^{(\alpha)},\mathbf{m}^{(\alpha)}\right\}$, resulting in $S^{(\alpha)}=\left\{\mathbf{u}^{(\alpha)},\boldsymbol{\varphi}^{(\alpha)},\boldsymbol{\gamma}^{(\alpha)},\boldsymbol{\kappa}^{(\alpha)},\boldsymbol{\tau}^{(\alpha)},\boldsymbol{\mu}^{(\alpha)}\right\}$ on the same material body ($\alpha=1,2$). The reciprocity shows

Theorem 1. (Reciprocity relation) Let

$$
\begin{aligned}
E_{12}(r,s) = & \int_{\partial W}\left[t_i^{(1)}(\mathbf{x},r)u_i^{(2)}(\mathbf{x},s)+m_i^{(1)}(\mathbf{x},r)\varphi_i^{(2)}(\mathbf{x},s)\right]dS_d \\
& + \int_{W}\left[X_i^{(1)}(\mathbf{x},r)u_i^{(2)}(\mathbf{x},s)+Y_i^{(1)}(\mathbf{x},r)\varphi_i^{(2)}(\mathbf{x},s)\right]dV_D \\
& - \int_{W}\left[\rho\ddot{u}_i^{(1)}(\mathbf{x},r)u_i^{(2)}(\mathbf{x},s)+I_{ij}\ddot{\varphi}_j^{(1)}(\mathbf{x},r)\varphi_i^{(2)}(\mathbf{x},s)\right]dV_D.
\end{aligned}
\tag{4.1}
$$

Then

$$E_{12}(r,s)=E_{21}(s,r) \tag{4.2}$$

Proof. Let

$$J_{\alpha\beta}(r,s)=\tau_{ij}^{(\alpha)}(r)\gamma_{ij}^{(\beta)}(s)+\mu_{ij}^{(\alpha)}(r)\kappa_{ij}^{(\beta)}(s) \quad (\alpha,\beta=1,2). \tag{4.3}$$

Substituting constitutive equations (3.32) into (4.3) we have

$$\begin{aligned}J_{\alpha\beta}(r,s)=C_{ijkl}^{(1)}\gamma_{kl}^{(\alpha)}(r)\gamma_{ij}^{(\beta)}(s)+C_{ijkl}^{(2)}\kappa_{kl}^{(\alpha)}(r)\kappa_{ij}^{(\beta)}(s)\\+C_{ijkl}^{(3)}\left[\kappa_{kl}^{(\alpha)}(r)\gamma_{ij}^{(\beta)}(s)+\gamma_{kl}^{(\alpha)}(r)\kappa_{ij}^{(\beta)}(s)\right].\end{aligned}$$

Note that the constitutive coefficients $C_{ijkl}^{(m)}$ satisfy symmetry relations $C_{ijkl}^{(m)}=C_{klij}^{(m)}$ ($m=1\sim 3$). It follows that $J_{\alpha\beta}(r,s)=J_{\beta\alpha}(s,r)$. On the other hand, on account of the "term by term" property of the operator ∇_j^D and in view of (3.23), (3.24) and (3.29), we have

$$\begin{aligned}J_{\alpha\beta}(r,s)=\nabla_j^D\left[\tau_{ji}^{(\alpha)}(r)u_i^{(\beta)}(s)+\mu_{ji}^{(\alpha)}(r)\varphi_i^{(\beta)}(s)\right]+X_i^{(\alpha)}(r)u_i^{(\beta)}(s)+Y_i^{(\alpha)}(r)\varphi_i^{(\beta)}(s)\\-\left[\rho\ddot{u}_i^{(\alpha)}(r)u_i^{(\beta)}(s)+I_{ij}\ddot{\varphi}_j^{(\alpha)}(r)\varphi_i^{(\beta)}(s)\right].\end{aligned}$$

Using the fractional Gauss theorem and (3.22) we find $\int_W J_{\alpha\beta}(r,s)\,dV_D=E_{\alpha\beta}(r,s)$, which implies (4.2). □

As a consequence we have:

Corollary. Let

$$P(r,s)=\int_W\left[X_i(r)u_i(s)+Y_i(r)\varphi_i(s)\right]dV_D+\int_{\partial W}\left[t_i(r)u_i(s)+m_i(r)\varphi_i(s)\right]dS_d\,. \tag{4.4}$$

Then

$$\begin{aligned}\frac{d}{dt}\int_W\left(\rho u_iu_i+I_{ij}\varphi_i\varphi_j\right)dV_D=\int_0^t\left[P(t-s,t+s)-P(t+s,t-s)\right]ds\\+\int_W\left\{\rho\left[\dot{u}_i(2t)u_i(0)+\dot{u}_i(0)u_i(2t)\right]+I_{ij}\left[\dot{\varphi}_i(2t)\varphi_j(0)+\dot{\varphi}_i(0)\varphi_j(2t)\right]\right\}dV_D.\end{aligned} \tag{4.5}$$

Proof. From (4.2) we have

$$\int_0^t E_{11}(t+s,t-s)\,ds=\int_0^t E_{11}(t-s,t+s)\,ds. \tag{4.6}$$

In view of (4.1) and (4.4) we find

$$\begin{aligned}\int_0^t E_{11}(t+s,t-s)\,ds=\int_0^t P(t+s,t-s)ds\\-\int_0^t\int_W\left[\rho\ddot{u}_i(t+s)u_i(t-s)+I_{ij}\ddot{\varphi}_j(t+s)\varphi_i(t-s)\right]dV_D ds,\end{aligned} \tag{4.7}$$

and

$$\int_0^t E_{11}(t-s,t+s)\,ds = \int_0^t P(t-s,t+s)ds \\ -\int_0^t \int_W \left[\rho\ddot{u}_i\left(t-s\right)u_i\left(t+s\right)+I_{ij}\ddot{\varphi}_j\left(t-s\right)\varphi_i\left(t+s\right)\right]dV_D ds, \tag{4.8}$$

Note that by "integration in part"

$$\int_0^t \ddot{f}(t+s)g(t-s)\,ds = \dot{f}(2t)g(0)-\dot{f}(t)g(t)+\int_0^t \dot{f}(t+s)\dot{g}(t-s)\,ds, \\ \int_0^t \ddot{g}(t-s)f(t+s)\,ds = \dot{g}(t)f(t)-\dot{g}(0)f(2t)+\int_0^t \dot{g}(t-s)\dot{f}(t+s)\,ds. \tag{4.9}$$

Combining (4.6) ~ (4.9) we obtain (4.5).

Now we have the uniqueness theorem:

Theorem 2. (Uniqueness) Assume that (i) ρ is strictly positive and (ii) I_{ij} is positive definite. Then the initial-boundary value problem of linear micropolar elastodynamics for fractal media has at most one solution.

Proof. Suppose we have two solutions, then their difference $\{\bar{u}_i,\bar{\varphi}_i\}$ is a solution corresponding to zero loads and initial-boundary conditions. From (4.5) we have

$$\int_W \left(\rho\bar{u}_i\bar{u}_i + I_{ij}\bar{\varphi}_i\bar{\varphi}_j\right)dV_D = 0.$$

Adopting the assumptions (i) and (ii), we find $\bar{u}_i = 0$ and $\bar{\varphi}_i = 0$, implying that the two solutions must be equal. $\square$

4.2 Variational principles

As to the variational principles, we consider a body with displacements u_i and rotations φ_i plus virtual motions δu_i and $\delta\varphi_i$. In view of the balance equations (3.23) and (3.24), we have

$$\int_W \left[\left(X_i-\rho\ddot{u}_i\right)\delta u_i+\left(Y_i-I_{ij}\ddot{\varphi}_j\right)\delta\varphi_i\right]dV_D + \int_W\left[\nabla_j^D\tau_{ji}\delta u_i+\left(e_{ijk}\frac{\tau_{jk}}{c_1^{(j)}}+\nabla_j^D\mu_{ji}\right)\delta\varphi_i\right]dV_D = 0.$$

Using the "integration by parts" and the fractional Gauss theorem in the second term above, we obtain

$$\begin{aligned}\int_W \left[\left(X_i - \rho \ddot{u}_i\right)\delta u_i + \left(Y_i - I_{ij}\ddot{\varphi}_j\right)\delta\varphi_i\right] dV_D \\ + \int_{\partial W}\left[t_i \delta u_i + m_i \delta\varphi_i\right] dS_d = \int_W \left[\tau_{ji}\delta\gamma_{ji} + \mu_{ji}\delta\kappa_{ji}\right] dV_D.\end{aligned} \tag{4.10}$$

Note that the right hand side denotes the variance of internal energy δW with respect to virtual motions, so that we set up the virtual work principle

$$\begin{aligned}\int_W \left[\left(X_i - \rho \ddot{u}_i\right)\delta u_i + \left(Y_i - I_{ij}\ddot{\varphi}_j\right)\delta\varphi_i\right] dV_D \\ + \int_{\partial W}\left[t_i \delta u_i + m_i \delta\varphi_i\right] dS_d = \delta W.\end{aligned} \tag{4.11}$$

The equation (4.11) can be written as

$$\delta L - \int_W \left[\rho \ddot{u}_i \delta u_i + I_{ij}\ddot{\varphi}_j \delta\varphi_i\right] dV_D = \delta W, \tag{4.12}$$

where

$$\delta L = \int_W \left[X_i \delta u_i + Y_i \delta\varphi_i\right] dV_D + \int_{\partial W}\left[t_i \delta u_i + m_i \delta\varphi_i\right] dS_d \tag{4.13}$$

refers to the external virtual work. Integrating (4.12) over time interval $[t_1, t_2]$

$$\delta \int_{t_1}^{t_2} W dt = \int_{t_1}^{t_2} \delta L dt - \int_{t_1}^{t_2} dt \int_W \left[\rho \ddot{u}_i \delta u_i + I_{ij}\ddot{\varphi}_j \delta\varphi_i\right] dV_D \tag{4.14}$$

Introducing the variance of kinetic energy,

$$\begin{aligned}\delta K = \int_W \rho \dot{u}_i \delta \dot{u}_i dV_D + \int_W I_{ij}\dot{\varphi}_j \delta\dot{\varphi}_i dV_D = \int_W \rho \frac{\partial}{\partial t}\left(\dot{u}_i \delta u_i\right) dV_D - \int_W \rho \ddot{u}_i \delta u_i dV_D \\ + \int_W I_{ij}\frac{\partial}{\partial t}\left(\dot{\varphi}_j \delta\varphi_i\right) dV_D - \int_W I_{ij}\ddot{\varphi}_j \delta\varphi_i dV_D\end{aligned}$$

and integrating it also over $[t_1, t_2]$, and noting that $\delta u_i, \delta\varphi_i$ vanish at $t = t_1$ and $t = t_2$, we find

$$\delta \int_{t_1}^{t_2} K dt = -\int_{t_1}^{t_2} dt \int_W \left[\rho \ddot{u}_i \delta u_i + I_{ij}\ddot{\varphi}_j \delta\varphi_i\right] dV_D \tag{4.15}$$

In view of (4.14) and (4.15), we finally obtain variational principles generalized to micropolar fractal media

$$\delta\int_{t_1}^{t_2}\left(W-K\right)dt=\int_{t_1}^{t_2}\delta L\,dt \tag{4.16}$$

If the external forces are conservative and derivable from a potential V, this shows

$$\delta\int_{t_1}^{t_2}\left(\Pi-K\right)dt=0 \tag{4.17}$$

where $\Pi = W - V$ denotes the total potential energy.

CHAPTER 5: CONCLUSIONS

Our approach builds on, but modifies, Tarasov's approach in that it admits an arbitrary anisotropic structure. This involves, in the first place, a specification of geometry of continua via 'fractal metric' coefficients, which then allows a construction of continuum mechanics of fractal solids. The anisotropy of fractal geometry on the mesoscale leads to the asymmetry of the Cauchy stress and to the appearance of the couple stress, i.e., to a fractal micropolar continuum. In the situations where the resolution R falls outside the [l, L] interval of Fig. 3.2 or when the surface and volume fractal dimensions (d and D) become conventional integers (2 and 3), all the newly derived equations revert back to the well-known forms of conventional continuum mechanics of non-fractal media.

The proposed methodology broadens the applicability of continuum mechanics/physics to studies of material responses. The highly complex, fractal-type media which have, so far, been the domain of condensed matter physics, geophysics and biophysics, etc. (multiscale polycrystals, cracked materials, polymer clusters, gels, rock systems, percolating networks, nervous systems, pulmonary systems, ...) will become open to studies conventionally reserved for smooth materials. This will allow solutions of initial-boundary value problems of very complex, multiscale materials of both elastic and inelastic type.

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