ROOT DISTRIBUTION OF POLYNOMIALS AND DISTANCE SUMS ON THE UNIT CIRCLE

BY

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DISSERTATION

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Abstract

Our first topic is the study of self-inversive polynomials. We establish sufficient conditions for self-inversive polynomials to have all zeros on the unit circle. We examine how such polynomials are used to generate further such polynomials. We also analyze the distribution of zeros of certain families of self-inversive polynomials and evaluate their discriminants. Our second topic is the sum of distances between points on the unit circle. We consider the sums over the vertices of the regular N-gon with some stretching factors. We also consider the N vertices of a regular N-gon with charges on the unit circle and obtain the maximal sum of squared distances from a point on the unit circle to the charged N-gon. For a certain set of charges, we study the minimal polynomial of the maximum value and its generating function.

To my parents

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Chapter 1

Introduction

A polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is self-inversive if $a_k = \zeta\ \overline{a_{n-k}}$ for $0 \le k \le n$ and $|\zeta| = 1$. It is easy to see that if a polynomial has all its zeros on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ then it is self-inversive. In fact, A. Cohn proved that a polynomial $P(z)$ has all zeros on the unit circle if and only if it is self-inversive and its derivative $P'(z)$ has all its zeros in the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$.

In Chapter 3, we will establish sufficient conditions for self-inversive polynomials to have all zeros on the unit circle. We will also examine how such polynomials may be used to generate further such polynomials. Chapter 4 analyzes the distribution of zeros of certain families of self-inversive polynomials and evaluates their discriminants.

A central problem in the study of root distribution is the distribution of the roots of $P'(z)$ relative to those of $P(z)$. One approach is to consider for an arbitrary $\alpha \in \mathbb{C}$ how the distances from α to the roots of $P(z)$ compare on average with distances to the roots of $P'(z)$. As the Gauss-Lucas theorem shows that the roots of $P'(z)$ are dominated by the roots of $P(z)$ in the sense that the convex hull of the roots of $P(z)$ contains that of the roots of $P'(z)$, a theorem of De Bruijn-Springer [5] shows a similar result about the distances. In particular, it asserts that on average the distances to the roots of $P(z)$ exceeds the distances to the roots of $P'(z)$. Now S. M. Malamud [18] has obtained very general results of the De Brujin-Springer type by generalizing the concept of vector majorization to apply to complex vectors. In fact, his method also yields majorization results for complex vectors of which the Gauss-Lucas theorem is a special case. Many of these results were also proved independently by R. Pereira [22]. This is a motivation for the inclusion of problems involving sums of distances between points. Another motivation is the problem maximizing the discriminant of a monic polynomial all of whose roots lies on the unit circle. This is the same as maximizing the product of all mutual $\binom{n}{2}$ distances and of course the answer is that they form a regular n-gon. But what about the sums (or sums of powers) of mutual distances?

For the sum of λth powers of such distances, the problem has a simple solution for $\lambda = 2$: the center of mass must be at the origin. For $0 < \lambda < 2$, the problems is non-degenerate. For $0 < \lambda \le 1$, there is a relatively elementary solution based on convexity [8], but for $1 < \lambda < 2$, the problem is very difficult and was only solved recently using, among other techniques, the theory of orthogonal polynomials [4]. For an appropriate normalization of the $\lambda \to 0$ case, this reduces to the discriminant maximization problem.

In this thesis, we examine perturbations in the radial direction (henceforth called stretchings) of the vertices of the regular N-gon and try to determine their effect on the distance sums. For stretching parameters $\{q^{a_1}, \cdots, q^{a_N}\}$, we associate the vector $\{a_1, \cdots, a_N\}$. Then to a great extent whether the stretching $a = \{a_1, \cdots, a_N\}$ gives a smaller distance sum than $b = \{b_1, \cdots, b_N\}$ depends upon whether or not $a \prec b$ in the sense of vector majorization, but the precise behavior is somewhat more subtle.

In Section 5.1, we consider the sums over the vertices of the regular N-gon with some stretching factors. In Section 5.2, we consider the N vertices $\{p_j\}_{j=1}^N$of a regular N-gon with charges $\{c_j\}_{j=1}^N$ on the unit circle and we obtain the maximum of $\sum_{j=1}^N c_j|e^{i\theta} - p_j|^2$. For a certain set of charges, we study the minimal polynomial of the maximum value and its generating function.

Chapter 2

Preliminaries

2.1 Hypergeometric Functions

In this section, we give the definition of the hypergeometric function and its important properties and transformation formulas which we will use later.

Definition 2.1. *The hypergeometric function* ${}_2F_1(a,b;c;x)$ *is defined by the series*

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

for $|x| < 1$*, and by analytic continuation elsewhere where* $(a)_n$ *denotes the rising factorial defined by*

$$(a)_n = a(a+1)\cdots(a+n-1) \ \ \textit{for} \ \ n > 0, \quad (a)_0 = 1.$$

When $x = 1$, we have the following theorem.

Theorem 2.2 (Gauss)**.** *For* $\Re(c-a-b) > 0$*, we have*

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The corollary below is the special case where one of the upper parameters is a negative integer, thereby making the ${}_2F_1$ a finite sum.

Corollary 2.3 (Chu-Vandermonde)**.**

$${}_2F_1\left(\begin{matrix} -n, a \\ c \end{matrix}; 1\right) = \frac{(c-a)_n}{(c)_n}.$$

Next, we have linear and quadratic transformations.

Theorem 2.4.
$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right) \quad \textit{(Pfaff)}, \tag{2.1} $$
$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; x\right) \quad \textit{(Euler)}. \tag{2.2} $$

Theorem 2.5. *For all x where the two series converge,*
$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix}; x\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a/2, (1+a)/2-b \\ a-b+1 \end{matrix}; \frac{-4x}{(1-x)^2}\right), \tag{2.3} $$
$$ {}_2F_1\left(\begin{matrix} a, b \\ 2b \end{matrix}; \frac{4x}{(1+x)^2}\right) = (1+x)^{2a} {}_2F_1\left(\begin{matrix} a, a+\frac{1}{2}-b \\ b+\frac{1}{2} \end{matrix}; x^2\right). \tag{2.4} $$

2.2 Classical Orthogonal Polynomials

In this section, we define Jacobi polynomials, ultraspherical polynomials, and Chebyshev polynomials and give some of their properties.

Definition 2.6. *The Jacobi polynomial of degree n is defined by*
$$ P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right). $$

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$ for $\alpha > -1$ and $\beta > -1$. We can represent the Jacobi polynomials in terms of hypergeometric functions:
$$ \begin{aligned} P_n^{(\alpha,\beta)}(x) &= P_n^{(\beta,\alpha)}(-x) \\ &= (-1)^n \frac{(\beta+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}; \frac{1+x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, -n-\beta \\ \alpha+1 \end{matrix}; \frac{x-1}{x+1}\right) \\ &= \frac{(n+\alpha+\beta+1)_n}{n!} \left(\frac{x-1}{2}\right)^n {}_2F_1\left(\begin{matrix} -n, -n-\alpha \\ -\alpha-\beta-2n \end{matrix}; \frac{2}{1-x}\right). \end{aligned} \tag{2.5} $$

We will use the last hypergeometric function representation later.

The next theorem gives us the discriminant of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

Theorem 2.7 (Theorem 8.5.3, [1]). *The discriminant of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is*

$$2^{n(n-1)}\prod_{j=1}^{n}\frac{j^j(\alpha+j)^{j-1}(\beta+j)^{j-1}}{(\alpha+\beta+n+j)^{n+j-2}}.$$

Define the Klein's symbol by

$$E(u)=\begin{cases}0, & \text{if } u\le 0,\\ [u], & \text{if } u>0, \quad u \text{ non-integral},\\ u-1, & \text{if } u=1,2,3,\cdots.\end{cases}$$

The following theorem gives us the zero-distribution of the general Jacobi polynomials.

Theorem 2.8 (Stieltjes, [27]). *Let α, β be arbitrary real values, and set*

$$\begin{aligned} X=X(\alpha,\beta) &= E\left(\frac{1}{2}(|2n+\alpha+\beta+1|-|\alpha|-|\beta|+1)\right),\\ Y=Y(\alpha,\beta) &= E\left(\frac{1}{2}(-|2n+\alpha+\beta+1|+|\alpha|-|\beta|+1)\right),\\ Z=Z(\alpha,\beta) &= E\left(\frac{1}{2}(-|2n+\alpha+\beta+1|-|\alpha|+|\beta|+1)\right).\end{aligned}$$

Then the numbers of the zeros of the general Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ in $-1<x<1$, $-\infty<x<-1$, $1<x<\infty$, respectively, are

$$\begin{aligned} N_1=N_1(\alpha,\beta) &= \begin{cases}2[(X+1)/2] & \textit{if } (-1)^n\binom{n+\alpha}{n}\binom{n+\beta}{n}>0,\\ 2[X/2]+1 & \textit{if } (-1)^n\binom{n+\alpha}{n}\binom{n+\beta}{n}<0,\end{cases}\\ N_2=N_2(\alpha,\beta) &= \begin{cases}2[(Y+1)/2] & \textit{if } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n}>0,\\ 2[Y/2]+1 & \textit{if } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n}<0,\end{cases}\\ N_3=N_3(\alpha,\beta) &= \begin{cases}2[(Z+1)/2] & \textit{if } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n}>0,\\ 2[Z/2]+1 & \textit{if } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n}<0.\end{cases}\end{aligned}$$

Some special cases of Jacobi polynomials are:

1. the ultraspherical polynomials, for $\alpha=\beta$,

2. the Chebyshev polynomials of the first kind, for $\alpha=\beta=-1/2$,

3. the Chebyshev polynomials of the second kind, for $\alpha = \beta = 1/2$.

Definition 2.9. *The ultraspherical polynomials $C_n^\lambda(x)$ are defined by*

$$C_n^\lambda(x) := \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x). \tag{2.6}$$

They are orthogonal with respect to the weight function $(1-x^2)^{\lambda-1/2}$ when $\lambda > -1/2$ and their generating function is

$$(1-2xz+z^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) z^n.$$

There is another representation of $C_n^\lambda(x)$ [1, (2.6.4)]:

$$C_n^\lambda(x) = \frac{(\lambda)_n}{n!}(2x)^n {}_2F_1\left(\begin{matrix} -n/2, (1-n)/2 \\ 1-n-\lambda \end{matrix}; \frac{1}{x^2}\right). \tag{2.7}$$

Definition 2.10. *The Chebyshev polynomials of the first and second kinds, denoted respectively by $T_n(x)$ and $U_n(x)$, are defined by*

$$P_n^{(-1/2,-1/2)}(x) = \frac{(2n)!}{2^{2n}(n!)^2} T_n(x) = \frac{(2n)!}{2^{2n}(n!)^2} \cos n\theta$$

and

$$P_n^{(1/2,1/2)}(x) = \frac{(2n+2)!}{2^{2n+1}[(n+1)!]^2} U_n(x) = \frac{(2n+2)!}{2^{2n+1}[(n+1)!]^2} \frac{\sin(n+1)\theta}{\sin\theta}, \tag{2.8}$$

where $x = \cos\theta$.

We also have explicit formulas for the Chebyshev polynomials:

$$\begin{aligned} T_n(x) &= \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2} &&= n\sum_{k=0}^{n} (-2)^k \frac{(n+k+1)!}{(n-k)!(2k+1)!}(1-x)^k \quad (n>0), \\ U_n(x) &= \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}} &&= \sum_{k=0}^{n} (-2)^k \frac{(n+k-1)!}{(n-k)!(2k)!}(1-x)^k \quad (n>0). \end{aligned} \tag{2.9}$$

The three-term recurrence relations are given by

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x) \quad \text{and} \quad 2xU_n(x) = U_{n+1}(x) + U_{n-1}(x). \tag{2.10}$$

The Chebyshev polynomials of the first and second kinds are related by the following equations:

$$\begin{aligned}
T_n(x) &= \frac{1}{2}(U_n(x) - U_{n-2}(x)), \\
T_{n+1}(x) &= xT_n(x) - (1-x^2)U_{n-1}(x), \\
T_n(x) &= U_n(x) - xU_{n-1}(x).
\end{aligned}$$

If we differentiate the Chebyshev polynomials in their trigonometric forms, it is easy to find their derivatives:

$$\begin{aligned}
\frac{dT_n(x)}{dx} &= nU_{n-1}(x), \\
\frac{dU_n(x)}{dx} &= \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}.
\end{aligned}$$

Both the Chebyshev polynomials of the first and second kinds have extrema at the end points of the interval $[-1, 1]$, given by:

$$T_n(1) = 1, \quad T_n(-1) = (-1)^n, \quad U_n(1) = n+1, \quad \text{and} \quad U_n(-1) = (-1)^n(n+1).$$

The following theorem helps us to find the distribution of zeros of families of polynomials in next chapter.

Theorem 2.11 (Theorem 5.4.1, [1]). *Suppose $\{P_n(x)\}$ is a sequence of orthogonal polynomials with respect to the distribution $d\alpha(x)$ on the interval $[a, b]$. Then $P_n(x)$ has n simple zeros in $[a, b]$.*

Chapter 3

Self Inversive Polynomials with All Zeros on the Unit Circle

Definition 3.1. *A polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree n is self-inversive if*

$$P(z) = \zeta P^*(z)$$

for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, where

$$P^*(z) := z^n \overline{P\left(\frac{1}{\bar{z}}\right)} \quad \textit{for all} \quad z \neq 0.$$

The zeros of a self-inversive polynomial either lie on the unit circle or occur in pairs conjugate to the unit circle [24, Chap.7].

We will show that certain self-inversive polynomials have all their zeros on the unit circle $U = \{x \in \mathbb{C} : |x| = 1\}$ and examine how such polynomials may be used to generate further such polynomials. We call them *U-polynomials.*

Our initial method is to take a polynomial with all zeros in the closed disc $\{z \in \mathbb{C} : |z| \leq 1\}$ and symmetrize it(Section 3.1).

Another procedure is to start with a self-inversive real polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, so $A = a_0 = a_n$ for some A, and increase $|A|$ to a sufficient extent. Recently there have been a number of results in this direction. Lakatos [11] proved that all zeros of the self-reciprocal polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ of degree $n \geq 2$ with real coefficients $a_k \in \mathbb{R}$ (i.e. $a_n \neq 0$ and $a_k = a_{n-k}$ for all $k = 1, \cdots, [n/2]$) are on U if

$$|a_n| \geq \sum_{k=1}^{n-1} |a_k - a_n|.$$

Schinzel [23] generalized this result for self-inversive polynomials: all zeros of the self-inversive polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ are on U if

$$|a_n| \geq \inf_{c,d \in \mathbb{C}, |d|=1} \sum_{k=0}^{n} |c a_k - d^{n-k} a_n|.$$

which can be deduced from our theorem (Theorem 3.5). For more results in this direction, see [12, 13, 14, 15, 17].

Next, if we start with a single self-inversive polynomial $P(x)$ of even degree that has all its zeros on U, the following two procedures will lead to further U-polynomials.

Procedure 1. Define

$$P(x,s) = \frac{P(x)+P(-x)}{2} + s\frac{P(x)-P(-x)}{2}$$

and choose $|s|$ sufficiently close to 1. We can get the value of s from the set of the roots of the discriminant of $P(x,s)$ with respect to x.

Procedure 2. Take any self-inversive factor of the numerator of

$$Q(x) = \frac{d}{dx}\left(x^{-n/2}P(x)\right)$$

where n is the degree of $P(x)$.

Procedure 2 is proved in [7] (Lemma 4.2.1).

In Section 3.2, we will show that in certain situations **Procedure 1** and **Procedure 2** are closely related. One may also generate further such polynomials starting with two of them. One could simply multiply them together. A more sophisticated approach would be to take their Schur-Szegö convolution product. The methods could of course be combined with **Procedure 1** and **Procedure 2**. In Section 3.3, we shall show how this approach leads to 2 variable sums for a polynomial $p(t)$ of a sort that do not seem to have been studied previously. These have the form

$$\sum_{k=1}^{m+1} \frac{c(k)t^{m+1-k}p(t)^k}{1-kx}$$

where the $c(k)$ are constants. Note that if the terms are combined, the resulting denominator is a polynomial generator for Stirling numbers of the first kind.

3.1 Symmetrization

A. Cohn's theorem helps us determine whether a polynomial has all zeros on the unit circle.

Theorem 3.2 (Theorem 7.1.3, [24])**.** *Suppose that the polynomial $P(z)$ of degree n is self-inversive and let τ be the number of zeros of $P(z)$ on the unit circle and ν be the number of zeros of $P'(z)$ in the closed disc*

$\{z \in \mathbb{C} : |z| \leq 1\}$. *Then*

$$\tau = 2(\nu + 1) - n.$$

A. Cohn's theorem is the case $\nu = n - 1$.

Corollary 3.3 (A. Cohn)**.** *For a self-inversive polynomial $P(z)$, all zeros of $P(z)$ are on the unit circle if and only if all zeros of $P'(z)$ lies in $\{z : |z| \leq 1\}$.*

To use A. Cohn's theorem, we need to analyze the zero-distribution of the derivative $P'(z)$. For example, see [26]. In this section, we will give another method to determine whether all zeros are on the unit circle.

The following theorem (a generalization of Problem 1 in [24, p. 232]) shows that symmetrizing a polynomial with all zeros in the closed disc $\{z \in \mathbb{C} : |z| \leq 1\}$ will provide a U-polynomial.

Theorem 3.4. *Let $Q(z)$ be a polynomial of degree n with all zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$. Then, for each ζ on the unit circle and for all nonnegative integers N,*

$$z^N Q(z) + \zeta Q^*(z)$$

has all its zeros on the unit circle.

Proof. Let $z_1, z_2, \cdots, z_n$ be zeros of $Q(z)$. Then $|z_k| \leq 1$ for $k = 1, \cdots, n$, and we have

$$Q(z) = a_n \prod_{k=1}^{n} (z - z_k) \text{ and } Q^*(z) = \overline{a_n} \prod_{k=1}^{n} (1 - \overline{z_k} z).$$

If $|z_k| = 1$ for all $k = 0, \cdots, n$, then

$$Q^*(z) = \frac{\overline{a_n}}{a_n \prod_{k=1}^{n} (-z_k)} Q(z).$$

Hence

$$z^N Q(z) + \zeta Q^*(z) = \left(z^N + \frac{\zeta \overline{a_n}}{a_n \prod_{k=1}^{n} (-z_k)} \right) Q(z)$$

has all zeros on the unit circle since

$$\left| \frac{\zeta \overline{a_n}}{a_n \prod_{k=1}^{n} (-z_k)} \right| = 1.$$

Now, suppose that we have at least one z_k strictly inside the unit circle. Let $x^N Q(x) + \zeta Q^*(x) = 0$. We will show that $|x| = 1$. If $Q^*(x) = 0$, $|x| \geq 1$ and $x^N Q(x) = 0$. Thus, x is on the unit circle. If $Q^*(x) \neq 0$, consider the following Blaschke product:

$$B(z)=\frac{Q(z)}{Q^*(z)}=\frac{a_n}{\overline{a_n}}\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}.$$

We have $|B(z)|<1$ for $|z|<1$, $|B(z)|=1$ for $|z|=1$, and $|B(z)|>1$ for $|z|>1$. Since $x^NQ(x)+\zeta Q^*(x)=0$,

$$B(x)=\frac{Q(x)}{Q^*(x)}=-\frac{\zeta}{x^N}.$$

If $|x|<1$, $1>|B(x)|=|x|^{-N}>1$. And, if $|x|>1$, $1<|B(x)|=|x|^{-N}<1$. Therefore, $|x|=1$. $\square$

Note that any self-inversive polynomial can be represented as the form of $x^NQ(x)+\zeta Q^*(x)$ for some polynomial $Q(z)$. One of these representations is the following:

From Definition 3.1, for the self-inversive polynomial $P(z)=a_0+a_1z+\cdots+a_nz^n$, we can show that $a_k=\zeta\overline{a_{n-k}}$ for $0\le k\le n$. If we let

$$Q(z)=\begin{cases}\sum\limits_{k=0}^{N-1}a_{N+k}z^k & \text{if } n=2N-1,\\ \frac{a_N}{2}+\sum\limits_{k=1}^{N}a_{N+k}z^k & \text{if } n=2N,\end{cases}\tag{3.1}$$

then, for $n=2N-1$,

$$Q^*(z)=\sum_{k=0}^{N-1}\overline{a_{n-k}}z^k=\zeta^{-1}\sum_{k=0}^{N-1}a_kz^k$$

and therefore

$$z^NQ(z)+\zeta Q^*(z)=z^N\sum_{k=0}^{N-1}a_{N+k}z^k+\sum_{k=0}^{N-1}a_kz^k=P(z).$$

Similarly, for $n=2N$, we have that $z^NQ(z)+\zeta Q^*(z)=P(z)$.

Then we have the theorem on the sufficient condition for a self-inversive polynomial to have all zeros on the unit circle.

Theorem 3.5. *Let $P(z)=z^NQ(z)+\zeta Q^*(z)$ be a self-inversive polynomial. If $Q(z)$ has all zeros in the closed unit disc $\{z\in\mathbb{C}:|z|\le 1\}$, then $P(z)$ is a U-polynomial.*

In [3], Chen proved a sufficient and necessary condition for a self-inversive polynomial to have all its zeros on the unit circle.

Theorem 3.6 (Theorem 1, [3])**.** *A necessary and sufficient condition for all the zeros of $p(z)=\sum_{k=0}^{n}a_kz^k$ with complex coefficients to lie on the unit circle is that there is a polynomial $q_{n-l}(z)$ with all its zeros in or* *on the unit circle such that*

$$p_n(z) = z^l q_{n-l}(z) + e^{i\theta} q^*_{n-l}(z)$$

for some nonnegative integer l and real θ.

It is the same as our result which is proved independently. The sufficient condition is Theorem 3.5. The necessary condition is easy to prove by setting $Q(z) = \frac{1}{2}P(z)$. We will now give applications of our theorem.

From known results on the maximum moduli of the zeros, we obtain some corollaries. We begin with the theorem of Eneström-Kakeya.

Theorem 3.7 (Eneström-Kakeya, [19])**.** *If all a_k are real and if $a_0 \geq a_1 \geq \cdots \geq a_n > 0$, then $\sum_{k=0}^{n} a_k z^k \neq 0$ for $|z| < 1$.*

Our theorem together with the Eneström-Kakeya Theorem provides a very large class of self-inversive polynomials having roots on the unit circle with decreasing symmetric coefficients.

Corollary 3.8. *(i) For $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$,*

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_{N-1} z^{N-1} + a_N z^N + a_N z^{N+1} + a_{N-1} z^{N+2} + \cdots + a_0 z^{2N+1}$$

is a U-polynomial.

(ii) For $a_0 \geq a_1 \geq a_2 \geq \cdots \geq \frac{a_N}{2} \geq 0$,

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_{N-1} z^{N-1} + a_N z^N + a_{N-1} z^{N+1} + \cdots + a_0 z^{2N}$$

is a U-polynomial.

Proof. Set $Q(z) = a_N + a_{N-1} z + a_{N-2} z^2 + \cdots + a_0 z^N$. By the Eneström-Kakeya theorem, all zeros of $Q(z)$ are in $\{z \in \mathbb{C} : |z| \leq 1\}$. □

Example 3.9. *From this corollary, the following families of polynomials have all their zeros on the unit circle:*

$$\sum_{k=0}^{n} \frac{z^k}{\binom{n}{k}^{\alpha}} \qquad \text{for } \alpha > 0, \qquad \text{and}$$

$$q^N + q^{N-1} z + q^{N-2} z^2 + \cdots + q z^{N-1} + z^N + z^{N+1} + q z^{N+2} + q^2 z^{N+3} + \cdots + q^N z^{N+1} \qquad \text{for } q \geq 1.$$

Also, we can show that self-inversive polynomials with slow increasing coefficients have all their zeros on the unit circle.

Corollary 3.10. *If* $0 \le a_N - a_{N-1} \le a_{N-1} - a_{N-2} \le \cdots \le a_2 - a_1 \le a_1 - a_0 \le a_0$*, then*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{N-1} z^{N-1} + a_N z^N + a_N z^{N+1} + a_{N-1} z^{N+2} + \cdots + a_0 z^{2N+1}$$

is a U*-polynomial.*

Proof. Let $Q(z) = (a_N - a_{N-1}) + (a_{N-1} - a_{N-2})z + \cdots + (a_1 - a_0)z^N$. Then it has all its zeros in the closed disc by the Eneström-Kakeya theorem. And it is easy to check $(z-1)P(z) = z^{N+2}Q(z) - Q^*(z)$. Therefore $P(z)$ has all its zeros on the unit circle. □

Corollary 3.11. *Let* $P(z) = z^N Q(z) + \zeta Q^*(z)$ *be a self-inversive polynomial with* $Q(z) = \sum_{k=0}^{N} a_k z^k$*. If* $\sum_{k=0}^{N-1} |a_k| \le |a_N|$*, then all zeros of* $P(z)$ *are on the unit circle.*

Proof. From the Gershgorin circle theorem [9], we know that if z_0 is a root of $Q(z)$ then either $|z_0| \le 1$ or $|z_0| \le \sum_{k=0}^{N-1} |a_k|/|a_N|$. □

Corollary 3.12. *Let* $P(z) = z^N Q(z) + \zeta Q^*(z)$ *be a self-inversive polynomial with* $Q(z) = \sum_{k=0}^{N} a_k z^k$*. If* $|a_0| \le |a_1|$ *and* $2|a_k| \le |a_{k+1}|$ *for* $1 \le k \le N-1$*, then all zeros of* $P(z)$ *are on the unit circle.*

Proof. Kojima [19, p. 107, Exercise 6] proved that all zeros of $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ lie in the circle $|z| \le r$, $r = \max\left(|a_0|/|a_1|, 2|a_k/a_{k+1}|\right)$ for $k = 1, 3, \cdots, n-1$. □

Corollary 3.13. *Let* $P(z) = z^N Q(z) + \zeta Q^*(z)$ *be a self-inversive polynomial with* $Q(z) = \sum_{k=0}^{N} a_k z^k$*. If* $|a_0| + \sum_{k=0}^{N-1} |a_{k+1} - a_k| \le |a_N|$*, then all zeros of* $P(z)$ *are on the unit circle.*

Proof. From the result of Montel-Marty [19, p. 107, Exercise 8], we have that all zeros of the polynomial $f(z) = a_0 + a_1 z + \cdots + z^n$ lie in the circle $|z| \le \max\left(L, L^{1/(n+1)}\right)$ where L is the length of the polygonal line joining in succession the points $0, a_0, a_1, \cdots, a_{n-1}, 1$; i.e. $L = |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}|$. □

Last, we deduce the result of Schinzel from our theorem.

Corollary 3.14 (Schinzel, [23])**.** *All zeros of the self-inversive polynomial* $P(z) = \sum_{k=0}^{n} a_k z^k$ *are on the unit circle if*

$$|a_n| \ge \inf_{c,d \in \mathbb{C}, |d|=1} \sum_{k=0}^{n} |c a_k - d^{n-k} a_n|.$$

Proof. Let $d = 1$ and $P(z) = \zeta P^*(z)$ for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Then

$$\zeta \overline{a_{n-k}} = a_k \quad \text{for} \quad k = 0, 1, \cdots, n. \tag{3.2}$$

Set $Q(z)$ as in (3.1). For $n = 2N-1$, we have the following inequality:

$$\begin{aligned}
& |a_N| + \sum_{k=1}^{N-1} |a_{N+k} - a_{N+k-1}| \\
= \;& \frac{1}{2}\left(|a_N| + |\zeta\overline{a_N}| + \sum_{k=1}^{N-1} |a_{N+k} - a_{N+k-1}| + \sum_{k=1}^{N-1} |\zeta\overline{a_{N+k}} - \zeta\overline{a_{N+k-1}}|\right) \qquad \text{since} \quad |\zeta| = 1 \\
= \;& \frac{1}{2}\left(|a_N| + |a_{N-1}| + \sum_{k=1}^{N-1} |a_{N+k} - a_{N+k-1}| + \sum_{k=1}^{N-1} |a_k - a_{k-1}|\right) \qquad \text{by (3.2)} \\
\leq \;& \frac{1}{2}|\sum_{k=1}^{n} |a_k - a_{k-1}| \\
\leq \;& \frac{1}{2}\left(\sum_{k=1}^{n} \left|a_k - \frac{a_n}{c}\right| + \sum_{k=1}^{n} \left|a_{k-1} - \frac{a_n}{c}\right|\right) \\
= \;& \sum_{k=0}^{n} \left|a_k - \frac{a_n}{c}\right| - \frac{1}{2}\left(\left|a_0 - \frac{a_n}{c}\right| + \left|n_0 - \frac{a_n}{c}\right|\right) \\
\leq \;& \left|\frac{a_n}{c}\right| - \frac{1}{2}\left(\left|\zeta\overline{a_n} - \frac{a_n}{c}\right| + \left|a_n - \frac{a_n}{c}\right|\right) \\
\leq \;& |a_n|
\end{aligned}$$

for any nonzero $c \in \mathbb{C}$. Similarly, we have the same inequality for the case when $n = 2N$. By Corollary 3.13, $P(z)$ has all zeros on the unit circle. Since $|d| = 1$, $P(dz)$ has all its zeros on the unit circle. □

3.2 Procedure 1 and Procedure 2

We begin with $P_n(t) = \sum_{k=0}^{2n} t^k$ which clearly has all zeros on the unit circle. Its generating function is

$$\sum_{n=0}^{\infty} P_n(t)x^n = \frac{1+tx}{(1-x)(1-t^2x)}.$$

Theorem 3.15. *For $m = 1, 2, \cdots$, let*

$$\sum_{n=0}^{\infty} P_n^{(m)}(t)x^n = \frac{1+tx}{(1-x)^m(1-t^2x)^m}.$$

Then $P_n^{(m)}(t)$ is a U-polynomial. Also, $P_{n-1}^{(m+1)}(t)$ is the result of applying **Procedure 2** *to $P_n^{(m)}(t)$.*

Proof. Replace x by x/t to get

$$\sum_{n=0}^{\infty} t^{-n} P_n^{(m)}(t)x^n = \frac{1+x}{(1-\frac{x}{t})^m(1-tx)^m}.$$

By taking the derivative with respect to t, we have

$$\sum_{n=1}^{\infty} \frac{d}{dt}\left(t^{-n}P_n^{(m)}(t)\right)x^n = \frac{mx(1+x)(1-\frac{1}{t^2})}{(1-\frac{x}{t})^{m+1}(1-tx)^{m+1}}.$$

Replacing x by xt and multiplying by $\frac{t}{x}$ will give us

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{d}{dt}\left(t^{-n}P_n^{(m)}(t)\right)t^{n+1}x^{n-1} &= \frac{m(1+xt)(t^2-1)}{(1-x)^{m+1}(1-t^2x)^{m+1}} \\ &= m(t^2-1)\sum_{n=0}^{\infty} P_n^{(m+1)}(t)x^n.\end{aligned}$$

Hence, it follows that

$$\frac{d}{dt}\left(t^{-n}P_n^{(m)}(t)\right)t^{n+1} = m(t^2-1)P_{n-1}^{(m+1)}.$$

i.e. $P_{n-1}^{(m+1)}(t)$ is the result of applying **Procedure 2** to $P_n^{(m)}(t)$. Therefore, $P_n^{(m)}(t)$ has all zeros on the unit circle. □

Now, the generating function for the polynomials defined in this setting by **Procedure 1**, namely $H_n(s,t) = 1 + t^2 + t^4 + \cdots + t^{2n} + s(t + t^3 + \cdots + t^{2n-1})$, is

$$\sum_{n=0}^{\infty} H_n(s,t)x^n = \frac{1+stx}{(1-x)(1-t^2x)}.$$

Theorem 3.16. *For $m = 1, 2, \cdots$, let*

$$\sum_{n=0}^{\infty} H_n^{(m)}(s,t)x^n = \frac{1+stx}{(1-x)^m(1-t^2x)^m}.$$

Then the generating function of $H_n^{(m)}(1+\frac{2m-1}{n}, -t)$ is

$$\sum_{n=1}^{\infty} H_n^{(m)}\left(1+\frac{2m-1}{n}, -t\right)x^{n-1} = m(1-t)^2\sum_{n=0}^{\infty} P_n^{(m+1)}(t)x^n$$

and hence exactly the result of **Procedure 2**.

Proof. Let

$$\sum_{n=0}^{\infty} h_n^{(m)}(t)x^n = \frac{1}{(1-x)^m(1-t^2x)^m}.$$

Then $H_n^{(m)}(s,t) = h_n^{(m)}(t) + st\ h_{n-1}^{(m)}(t)$ for $n = 1, 2, \cdots$. By differentiating with respect to x, we obtain

$$\sum_{n=0}^{\infty} n\ h_n^{(m)}(t)x^n = \frac{m(1+t^2-2t^2x)}{(1-x)^{m+1}(1-t^2x)^{m+1}}.$$

Now, when $s = 1 + \dfrac{2m-1}{n}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} H_n^{(m)}\left(1+\frac{2m-1}{n}, -t\right)x^{n-1} &= \sum_{n=1}^{\infty} n\left\{h_n^{(m)}(-t) - \left(1+\frac{2m-1}{n}\right)t\ h_{n-1}^{(m)}(-t)\right\}x^{n-1} \\
&= \frac{m(1-tx)(1+t^2-2t^2x)}{(1-x)^{m+1}(1-t^2x)^{m+1}} - \frac{2mt}{(1-x)^m(1-t^2x)^m} \\
&= m(1-t)^2\frac{1+tx}{(1-x)^{m+1}(1-t^2x)^{m+1}} \\
&= m(1-t)^2\sum_{n=0}^{\infty} P_n^{(m+1)}(t)x^n.
\end{aligned}$$

□

Thus, there is a strong connection between **Procedure 1** and **Procedure 2** in this case. The connection is somewhat weaker for polynomials such as

$$4 + 3t + 2t^2 + t^3 + 2t^4 + 3t^5 + 4t^6$$

which, by Theorem 3.5, have all their zeros on the unit circle. Here the generating function is

$$\begin{aligned}
L(t,x) &= \frac{1-4t^2x^2+2t^2x^3+2t^4x^3-t^4x^4}{(1-x)^2(1-tx)(1-t^2x)^2} \\
&= \sum_{n=0}^{\infty} L_n(t)x^n \\
&= 1 + (2+t+2t^2)x + (3+2t+t^2+2t^3+3t^4)x^2 + \cdots.
\end{aligned}$$

In this case, examples show that **Procedure 1** and **Procedure 2** give different results. **Procedure 2** applied to

$$L_4(t) = 5 + 4t + 3t^2 + 2t^3 + t^4 + 2t^5 + 3t^6 + 4t^7 + 5t^8$$

gives, by removing the $2(t^2-1)$ from the numerator,

$$p_1(t) = 10 + 6t + 13t^2 + 7t^3 + 13t^4 + 6t^5 + 10t^t.$$

For **Procedure 1**, we evaluate the discriminant with respect to t of

$$5 + 3t^2 + t^4 + 3t^2 + 5t^8 + s(4t + 2t^3 + 2t^5 + 4t^7),$$

which is

$$-6400(-17 + 12s)(17 + 12s)(178605 + 47777s^2 + 4985s^4 + 3200s^6)^2.$$

The result of **Procedure 1** with $s = -17/12$ and the $(1-t)^2/6$ factor removed is

$$p_2(t) = 30 + 26t + 40t^2 + 37t^3 + 40t^4 + 26t^5 + 30t^6.$$

Hence, $p_1(t) \neq p_2(t)$. However, $p_1(t)$ and $p_2(t)$ both have roots near $-.139 \pm .990i$ that are remarkably close, and for $L_n(t)$ for n large, the zeros of the two different polynomials produced by **Procedure 1** and **Procedure 2** appear to be remarkably close.

3.3 Combined Procedure

In our use of **Procedure 1** or **Procedure 2**, we always reduce the degree of the polynomial by 2. We now consider **Procedure 2** in the case in which the polynomial $P(x)$ is first multiplied by a self-inversive quadratic polynomial before introducing the factor $t^{-\frac{\text{degree}}{2}}$ and differentiating. Thus, we will generate a sequence of polynomials of the same degree.

Example 3.17. *Start with* $P_3^{(1)}(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + t^6$. *Do* **Procedure 2** *with premultiplication by* $(1 + at + t^2)$. *Let the resulting sequences of sixth degree polynomials be* $Q_n^{(3)}(t)$ *with* $Q_0^{(3)}(t) = P_3^{(1)}(t)$.

$$\begin{aligned}\frac{d}{dt}\left(t^{-4}P_3^{(1)}(t)(1 + at + t^2)\right) &= \frac{d}{dt}\left(t^{-3}P_3^{(l)}(t)(t^{-1} + a + t)\right) \\ &= t^{-3}P_3^{(l)}(t)(-t^{-2} + 1) + t^{-4}(t^2 - 1)P_2^{(2)}(t)(t^{-1} + a + t).\end{aligned}$$

Multiplying by $\dfrac{t^5}{t^2 - 1}$ *gives us*

$$Q_1^{(3)}(t) = \frac{t^5}{t^2 - 1}\frac{d}{dt}\left(t^{-4}P_3^{(1)}(t)(1 + at + t^2)\right) = \left\{P_3^{(1)}(t) + P_2^{(2)}(t)(1 + at + t^2)\right\}.$$

Similarly, we have the following:

$$
\begin{aligned}
\frac{t^5}{t^2-1}\frac{d}{dt}\left(t^{-4}P_2^{(2)}(t)(1+at+t^2)^2\right) &= 2(1+at+t^2)\left\{P_2^{(2)}(t)+P_1^{(3)}(t)(1+at+t^2)\right\}, \\
\frac{t^5}{t^2-1}\frac{d}{dt}\left(t^{-4}P_1^{(3)}(t)(1+at+t^2)^3\right) &= 3(1+at+t^2)^2\left\{P_1^{(3)}(t)+P_0^{(4)}(t)(1+at+t^2)\right\}, \\
\frac{t^5}{t^2-1}\frac{d}{dt}\left(t^{-4}P_0^{(4)}(t)(1+at+t^2)^4\right) &= 4(1+at+t^2)^3.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q_2^{(3)}(t) &= \frac{t^5}{t^2-1}\frac{d}{dt}Q_1^{(3)}(t) \\
&= \frac{t^5}{t^2-1}\frac{d}{dt}\left\{P_3^{(1)}(t)+P_2^{(2)}(t)(1+at+t^2)\right\}(1+at+t^2) \\
&= \frac{t^5}{t^2-1}\frac{d}{dt}\left\{P_3^{(1)}(t)(1+at+t^2)+P_2^{(2)}(t)(1+at+t^2)^2\right\} \\
&= \left\{P_3^{(1)}(t)+P_2^{(2)}(t)(1+at+t^2)\right\}+2(1+at+t^2)\left\{P_2^{(2)}(t)+P_1^{(3)}(t)(1+at+t^2)\right\} \\
&= P_3^{(1)}(t)+(1+2)P_2^{(2)}(t)(1+at+t^2)+2P_1^{(3)}(t)(1+at+t^2)^2
\end{aligned}
$$

and

$$
\begin{aligned}
Q_3^{(3)}(t) &= \frac{t^5}{t^2-1}\frac{d}{dt}Q_2^{(3)}(t) \\
&= \frac{t^5}{t^2-1}\frac{d}{dt}\left\{P_3^{(1)}(t)+3P_2^{(2)}(t)(1+at+t^2)+2P_1^{(3)}(t)(1+at+t^2)^2\right\}(1+at+t^2) \\
&= P_3^{(1)}(t)+7P_2^{(2)}(t)(1+at+t^2)+12P_1^{(3)}(t)(1+at+t^2)^2+6(1+at+t^2)^3.
\end{aligned}
$$

If we consider the generating function

$$\sum_{n=0}^{\infty} Q_n^{(3)}(t)x^n,$$

then

$$Q_n^{(3)}(t) = a_{n,1}P_3^{(1)}(t)+a_{n,2}P_2^{(2)}(t)(1+at+t^2)+a_{n,3}P_1^{(3)}(t)(1+at+t^2)^2+a_{n,4}(1+at+t^2)^3$$

where $a_{n,1}$, $a_{n,2}$, $a_{n,3}$, and $a_{n,4}$ can be founded by the recurrence relations:

$$a_{n,1}=1,\ \ a_{n,2}=a_{n-1,1}+2a_{n-1,2},\ \ a_{n,3}=2a_{n-1,2}+3a_{n-1,3},\ \ \textit{and}\ a_{n,4}=3a_{n-1,3}+4a_{n-1,4}$$

so that $a_{n,1}=1$, $a_{n,2}=1\cdot 2^n-1$, $a_{n,3}=1\cdot 3^n-2\cdot 2^n+1$, and $a_{n,4}=1\cdot 4^n-3\cdot 3^n+3\cdot 2^n-1$. We note *that there are binomial coefficients involved. Therefore, we obtain the generating function:*

$$\begin{aligned}\sum_{n=0}^{\infty} Q_n^{(3)}(t)x^n &= \frac{1}{1-x}\left\{P_3^{(1)}(t) - P_2^{(2)}(t)(1+at+t^2) + P_1^{(3)}(t)(1+at+t^2)^2 - (1+at+t^2)^3\right\}\\ &\quad+\frac{1}{1-2x}\left\{P_2^{(2)}(t)(1+at+t^2) - 2P_1^{(3)}(t)(1+at+t^2)^2 + 3(1+at+t^2)^3\right\}\\ &\quad+\frac{1}{1-3x}\left\{P_1^{(3)}(t)(1+at+t^2)^2 - 3(1+at+t^2)^3\right\} + \frac{1}{1-4x}(1+at+t^2)^3\\ &= \frac{(-1+2a+a^2-a^3)t^3}{1-x} + \frac{(-2-2a+3a^2)t^2(1+at+t^2)}{1-2x}\\ &\quad+\frac{(1-3a)t(1+at+t^2)^2}{1-3x} + \frac{(1+at+t^2)^3}{1-4x}.\end{aligned}$$

With similar computation, we can get the generating function for the combined procedure of the polynomial $P_4^{(1)}(t) = 1+t+\cdots+t^8$.

$$\begin{aligned}\sum_{n=0}^{\infty} Q_n^{(4)}(t)x^n &= \frac{(1+2a-3a^2-a^3+a^4)t^3}{1-x} + \frac{(-2+6a+3a^2-4a^3)t^3(1+at+t^2)}{1-2x}\\ &\quad+\frac{(-3-3a+6a^2)t^2(1+at+t^2)^2}{1-3x} + \frac{(1-4a)t(1+at+t^2)^3}{1-4x} + \frac{(1+at+t^2)^4}{1-5x}.\end{aligned}$$

In general, we have

$$\sum_{n=0}^{\infty} Q_n^{(l)}x^n = \sum_{k=0}^{l}\frac{1}{1-(k+1)x}\sum_{i=k}^{l} P_{l-i}^{i+1}(t)(1+at+t^2)^i(-1)^{i-k}\binom{i}{k}. \tag{3.3}$$

Conjecture 3.18. *Let*

$$C^{(l)}(a,x,t) = \sum_{k=0}^{l}\frac{(-1)^k t^{l-k}(1+at+t^2)^k \frac{d^k R_l(a)}{da^k}/k}{1-(1+k)x},$$

where

$$R_l(a) = \sum_{j=0}^{l} a^{l-k}(-1)^{l-\lceil j/2\rceil}\binom{l-\lceil j/2\rceil}{\lfloor j/2\rfloor} = \frac{(-t)^l}{\alpha-\beta}\left(\alpha^{l+1}-\alpha^l-\beta^{l+1}+\beta^l\right)$$

with

$$\alpha = \frac{a+\sqrt{a^2-4}}{2} \qquad \textit{and} \qquad \beta = \frac{a-\sqrt{a^2-4}}{2}.$$

For $-2 \le a \le 2$, the polynomials generated by $C^{(l)}(a,x,t)$ have all their zeros on the unit circle.

We claim that $C^{(l)}(a,x,t)$ is the generating function of $Q_n^{(l)}(t)$, i.e.,

$$C(a,x,t) = \sum_{n=0}^{\infty} Q_n^{(l)}x^n.$$

If we could verify the equation

$$\sum_{i=0}^{l} P_{l-i}^{(i+1)}(t)(1+at+t^2)^i(-1)^i = t^l \sum_{j=0}^{l} a^{l-k}(-1)^{l-\lceil j/2\rceil}\binom{l-\lceil j/2\rceil}{\lfloor j/2\rfloor},$$

then it would follow by induction that

$$\sum_{i=k}^{l} P_{l-i}^{(i+1)}(t)(1+at+t^2)^i(-1)^{i-k}\binom{i}{k} = t^{l-k}\frac{d^kR(a)}{da^k}\frac{(-1)^k}{k}, \qquad \text{for} \quad k>0.$$

Hence, from (3.3), we can show that $C^{(l)}(a,x,t)$ would be the generating function of $Q_n^{(l)}(t)$ and this would prove Conjecture 3.18.

3.4 Related Generating Functions

The following is a generalization of Theorem 3.15.

Theorem 3.19. *For any real number $p>0$, the polynomials generated by*

$$K(p,t,x) := \frac{1+tx}{(1-x)^p(1-t^2x)^p}$$

are U-polynomials.

Proof. By using the binomial theorem, we can expand $K(p,t,x)$ as follow:

$$\begin{aligned}
K(p,t,x) &= (1+tx)(1-x)^{-p}(1-t^2x)^{-p} \\
&= (1+tx)\sum_{j=0}^{\infty}\binom{-p}{j}(-x)^j\sum_{k=0}^{\infty}\binom{-p}{k}(-t^2x)^k \\
&= (1+tx)\sum_{n=0}^{\infty}(-x)^n\sum_{k=0}^{n}\binom{-p}{n-k}\binom{-p}{k}t^{2k} \\
&= (1+tx)\sum_{n=0}^{\infty}\frac{(p)_n}{n!}x^n\sum_{k=0}^{n}\frac{(-n)_k(p)_k}{k!(1-p-n)_k}t^{2k} \\
&= (1+tx)\sum_{n=0}^{\infty}\frac{(p)_n}{n!}x^n {}_2F_1\left(\begin{matrix}-n,p\\1-p-n\end{matrix};t^2\right).
\end{aligned}$$

Hence, the polynomial generated by $K(p,t,x)$ is

$$\frac{(p)_n}{n!}{}_2F_1\left(\begin{matrix}-n,p\\1-p-n\end{matrix};t^2\right)+\frac{(p)_{n-1}}{(n-1)!}t\,{}_2F_1\left(\begin{matrix}-n+1,p\\2-p-n\end{matrix};t^2\right)$$

which is

$$
\begin{aligned}
& \frac{(p)_n}{n!}(1+t^2)^n {}_2F_1\left(\begin{matrix}-n/2,(1-n)/2\\1-p-n\end{matrix};\left(\tfrac{2t}{1+t^2}\right)^2\right)\\
& +\frac{(p)_{n-1}}{(n-1)!}t(1+t^2)^{n-1}{}_2F_1\left(\begin{matrix}(1-n)/2,(2-n)/2\\2-p-n\end{matrix};\left(\tfrac{2t}{1+t^2}\right)^2\right) \qquad \text{by (2.3) and (2.1)}\\
= \; & t^n\left\{C_n^p\left(\frac{1+t^2}{2t}\right)+C_{n-1}^p\left(\frac{1+t^2}{2t}\right)\right\} \qquad \text{by (2.7)}\\
= \; & t^n\left\{\frac{(2p)_n}{(p+1/2)_n}P_n^{(p-1/2,p-1/2)}\left(\frac{1+t^2}{2t}\right)+\frac{(2p)_{n-1}}{(p+1/2)_{n-1}}P_{n-1}^{(p-1/2,p-1/2)}\left(\frac{1+t^2}{2t}\right)\right\} \qquad \text{by (2.6)}\\
= \; & t^n\frac{(2p)_{n-1}(-1)^n}{(p+1/2)_n}\left\{(2p+n-1)P_n^{(p-1/2,p-1/2)}\left(-\frac{1+t^2}{2t}\right)-(p+n-1/2)P_{n-1}^{(p-1/2,p-1/2)}\left(-\frac{1+t^2}{2t}\right)\right\},
\end{aligned}
$$

where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial. Recall an identity for Jacobi polynomials((6.4.21), [1]):

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=(n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(x)-(n+\beta)P_{n-1}^{(\alpha+1,\beta)}(x).$$

From this, we have that the polynomial generated by $K(p,t,x)$ is

$$2\frac{(2p)_{n-1}}{(p+1/2)_{n-1}}t^nP_n^{(p-1/2,p-3/2)}\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right).$$

For $p>0$, the Jacobi polynomial has all zeros in the interval $[-1,1]$. Hence, $P_n^{(p-1/2,p-3/2)}(\frac{1}{2}(t+1/t))$ has all zeros on the unit circle because $\frac{1}{2}(t+1/2)\in[-1,2]\Leftrightarrow|t|=1$. Therefore, they have all zeros on the unit circle. $\square$

The following conjectures are related to Theorem 3.19.

Conjecture 3.20. *If $p>r\geq 1$, then the polynomials generated by*

$$\frac{1+tx}{(1-x)^p(1-t^2x)^r}$$

have all zeros t satisfying $|t|\geq 1$. If $r>p\geq 1$, all zeros t satisfy $|t|\leq 1$.

Conjecture 3.21. *For any integer $p\geq 1$, the function $K(p,t,x)^2$ generates U-polynomials.*

Chapter 4

The Root Distribution of Families of Polynomials

In this chapter, we analyze the root distributions of certain families of polynomials. We also evaluate their discriminants and related resultants. We connect them with classical orthogonal polynomials such as Jacobi polynomials, and Chebyshev polynomials of the first and second kind.

4.1 The U-Polynomials $G(n, d, x)$

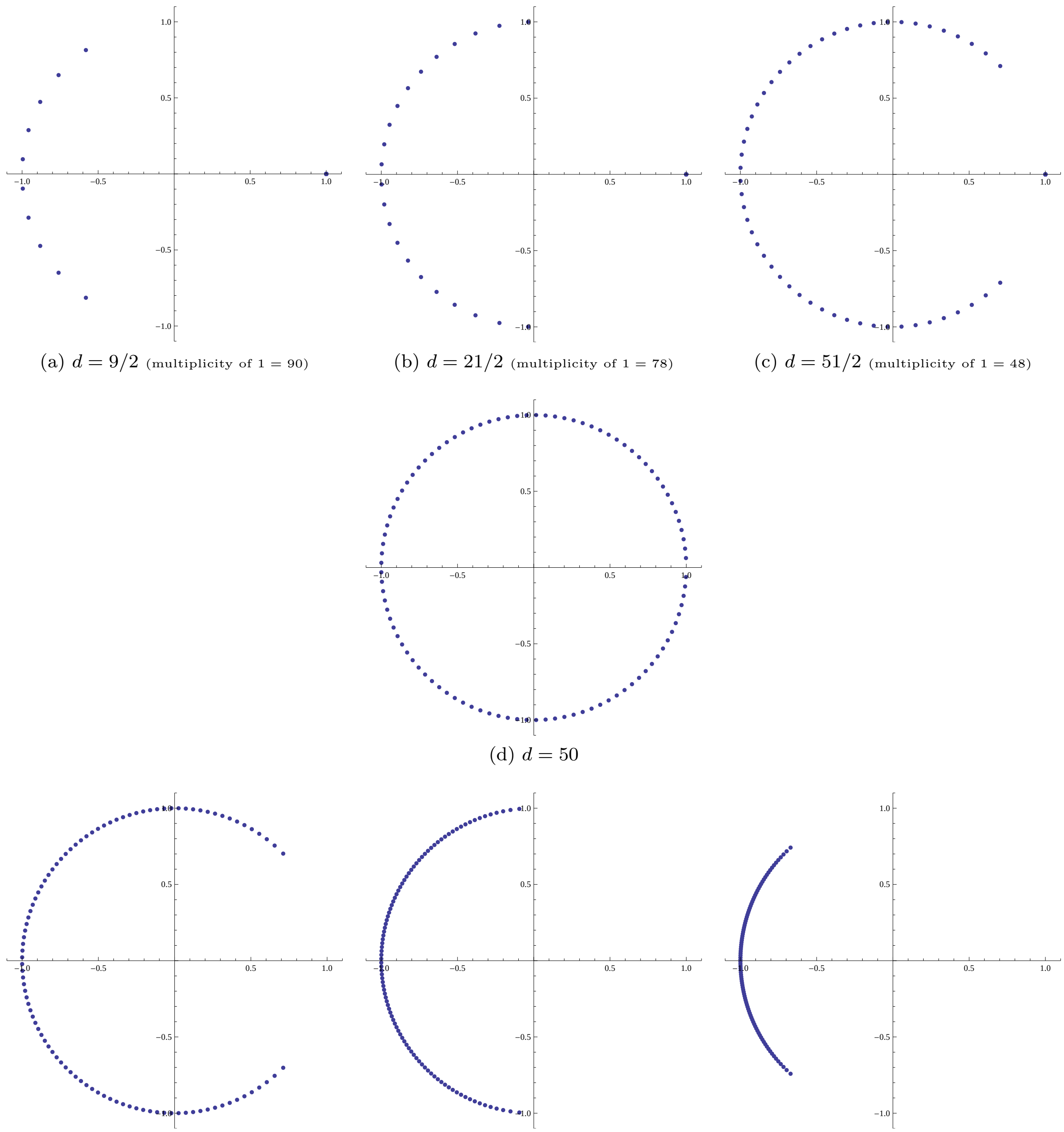
Two extreme U-polynomials are $u_1(x) = (x-1)^{2n}$ and $u_2(x) = (x+1)^{2n}$. We can imagine all roots of $u_1(x)$ separating away from each other and away from their initial value of 1 with half moving along the unit circle above the x-axis and half below until all roots again get closer together and finally coincide at $x = -1$. This gives a transition from u_1 to u_2. We shall show there is a simply described family of U-polynomials whose coefficients depend on a single parameter d such that as d goes from $-1/2$ to ∞ the polynomial goes from a constant multiple of u_1 to a constant multiple of u_2 with the value $\sum_{k=0}^{2n} z^k$ when $d = n$. See Figure 4.1. In general, the coefficients of these polynomials, after the highest power of $(x-1)$ has been factored out, increase in magnitude as the central term is approached, so considerations of the sort used in Corollary 3.8 are not applicable. Our proof is based on properties of Jacobi polynomials. One might expect that the roots are most evenly spread apart on the unit circle when $d = n$. However, a very natural measure of spread (see the discussion after Theorem 4.6) indicates that the spread is maximal for values of d asymptotic to $\sqrt{n/2}$.

For $d \geq -\frac{1}{2}$, let

$$G(n, d, x) := \sum_{j=-n}^{n} \frac{\Gamma(d+1-j)\Gamma(d+1+j)}{(n-j)!(n+j)!} x^{(n+j)}.$$

Note that as d goes from $-1/2$ to ∞, the $G(n, d, x)$ goes from $\dfrac{\pi}{(2n)!}u_1(x)$ to $\dfrac{1}{(2n)!}u_2(x)$ with $G(n, n, x) = \sum\limits_{k=0}^{2n} z^k$.

Theorem 4.1. *If $d > n - \frac{3}{2}$, $G(n, d, x)$ is a U-polynomial.*



(a) $d = 9/2$ (multiplicity of 1 = 90)

(b) $d = 21/2$ (multiplicity of 1 = 78)

(c) $d = 51/2$ (multiplicity of 1 = 48)

(d) $d = 50$

(e) $d = 100$

(f) $d = 300$

(g) $d = 1000$

Figure 4.1: The root distribution of $G(50, d, x)$

Proof. We can rewrite $G(n,d,x)$ as follows:

$$\begin{aligned}
G(n,d,x) &= \sum_{j=-n}^{n} \frac{\Gamma(d+1-j)\Gamma(d+1+j)}{(n-j)!(n+j)!} x^{(n+j)} \\
&= \sum_{j=0}^{2n} \frac{\Gamma(d+1-j+n)\Gamma(d+1+j-n)}{(2n-j)!j!} x^{j} \\
&= \frac{\Gamma(d+1-n)\Gamma(d+1+n)}{(2n)!} {}_2F_1\left(\begin{matrix} -2n, d+1-n \\ -d-n \end{matrix}; x\right) \\
&= \frac{\Gamma(d+1-n)\Gamma(d+1+n)}{(2n)!} (1-x)^{2n} {}_2F_1\left(\begin{matrix} -n, -d-1/2 \\ -d-n \end{matrix}; \frac{-4x}{(1-x)^2}\right) \qquad \text{by (2.3)} \\
&= \frac{\Gamma(d+1-n)\Gamma(d+1)}{(1/2)_n} (-x)^n P_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right) \qquad \text{by (2.5).}
\end{aligned}$$

If $-n+d+\frac{1}{2} > -1$, $P_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}(x+\frac{1}{x})\right)$ has all zeros in the interval $[-1,1]$. Since $(x+1/x) \in [-2,2] \Leftrightarrow |x| = 1$, $G(n,d,x)$ has all zeros on the unit circle. □

Lemma 4.2. *For $k = 0, 1, \cdots, n-1$,*

$$G(n, k-\frac{1}{2}, x) = \frac{(2k)!\Gamma(k+\frac{1}{2}-n)}{(2n)!\Gamma(n+\frac{1}{2}-k)} (1-x)^{2(n-k)} G(k, n-\frac{1}{2}, x).$$

Proof. By Euler's transform (2.2), we have that

$${}_2F_1\left(\begin{matrix} -2n, k+\frac{1}{2}-n \\ -k+\frac{1}{2}-n \end{matrix}; x\right) = (1-x)^{2(n-k)} {}_2F_1\left(\begin{matrix} n-k+\frac{1}{2}, -2k \\ -k+\frac{1}{2}-n \end{matrix}; x\right).$$

Hence, we obtain

$$\begin{aligned}
G(n, k-\frac{1}{2}, x) &= \frac{\Gamma(k+\frac{1}{2}-n)\Gamma(k+\frac{1}{2}+n)}{(2n)!} {}_2F_1\left(\begin{matrix} -2n, k+\frac{1}{2}-n \\ -k+\frac{1}{2}-n \end{matrix}; x\right) \\
&= \frac{\Gamma(k+\frac{1}{2}-n)\Gamma(k+\frac{1}{2}+n)}{(2n)!} (1-x)^{2(n-k)} {}_2F_1\left(\begin{matrix} n-k+\frac{1}{2}, -2k \\ -k+\frac{1}{2}-n \end{matrix}; x\right) \\
&= \frac{\Gamma(k+\frac{1}{2}-n)\Gamma(k+\frac{1}{2}+n)}{(2n)!} (1-x)^{2(n-k)} \frac{(2k)!}{\Gamma(n+\frac{1}{2}-k)\Gamma(n+\frac{1}{2}+k)} G(k, n-1/2, x) \\
&= \frac{(2k)!\Gamma(k+\frac{1}{2}-n)}{(2n)!\Gamma(n+\frac{1}{2}-k)} (1-x)^{2(n-k)} G(k, n-1/2, x).
\end{aligned}$$

□

Theorem 4.3. *$G(n, k-\frac{1}{2}, x)$ is a U-polynomial for $k = 0, 1, \cdots$.*

Proof. If $k - 1/2 > n - 3/2$, the result follows from the previous theorem. Let $k \leq n-1$. By Lemma 4.2, we have

$$G(n,k-\frac{1}{2},x)=\frac{(2k)!\Gamma(k+\frac{1}{2}-n)}{(2n)!\Gamma(n+\frac{1}{2}-k)}(1-x)^{2(n-k)}G(k,n-\frac{1}{2},x).$$

Since $n-1/2>k-3/2$, $G(k,n-\frac{1}{2},x)$ has all its zeros on the unit circle by Theorem 4.1. $\square$

We have analyzed the root distribution of $G(n,d,x)$ for $d>n-\frac{3}{2}$ or $d=k-\frac{1}{2}$ for $k=0,1,\cdots,n-1$. There remains the case $d<n-\frac{3}{2}$ with $d+1/2\notin\mathbb{Z}^+$. Since we have shown that

$$G(n,d,x)=\frac{\Gamma(d+1-n)\Gamma(d+1)}{(1/2)_n}(-x)^nP_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)$$

in the proof of Theorem 4.1, we can use the root distribution of the general Jacobi polynomials.

Theorem 4.4. *Let $n-3/2-k<d<n-1/2-k$ for $k\in\mathbb{Z}^+$. If $1\le k\le n-1$, the number of zeros of $G(n,d,x)$ on the unit circle is 2(n-k).*

Proof. In the Stieltjes' Theorem (Theorem 2.8), for

$$G(n,d,x)=\frac{\Gamma(d+1-n)\Gamma(d+1)}{(1/2)_n}(-x)^nP_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right),$$

we have

$$\begin{aligned} X &= E\left(\frac{1}{2}(|2n-n+d+1/2-1/2+1|-|-n+d+1/2|-|-1/2|+1)\right)\\ &= E(d+1)\\ &= \lceil d\rceil \end{aligned}$$

since $n-3/2-k<d<n-1/2-k$ for $k=1,2,\cdots,n-1$. Here,

$$\begin{aligned} &(-1)^n\binom{n+\alpha}{n}\binom{n+\beta}{n}=(-1)^n\binom{d+1/2}{n}\binom{n-1/2}{n}>0\\ \Leftrightarrow\ &(-1)^n(d+1/2)\cdots(d-n+3/2+k)(d-n+1/2+k)\cdots(d-n+1/2+1)>0\\ \Leftrightarrow\ &(-1)^{n+k}>0. \end{aligned}$$

If $(-1)^{n+k}>0$, $n-k$ is even. For $n-k-3/2<d\le n-k-1$, $X=n-k-1$ so that $N_1=2\left[\frac{n-k-1+1}{2}\right]=n-k$. And, for $n-k-1<d<n-k-1/2$, $X=n-k$ so that $N_1=2\left[\frac{n-k+1}{2}\right]=n-k$. Similarly, if $(-1)^{n+k}<0$, we can get $N_1=n-k$. Therefore, there are $(n-k)$ zeros of $P_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)$ in the interval $[-1,1]$ which implies that there are $2(n-k)$ zeros of $G(n,d,x)$ on the unit circle. $\square$

In the next theorem, we can obtain the discriminant of $G(n,d,x)$ when it has all zeros on the unit circle. Recall that the discriminant of the polynomial $f(x)$ of degree n is defined by

$$\Delta_x f(x) = (-1)^{n(n-1)/2} a_n^{2n-2} \prod_{j<k} (r_j - r_k)$$

where a_n is the leading coefficient of $f(x)$ and $\{r_j\}$ is the set of roots of $f(x)$.

Theorem 4.5. *The discriminant of $G(n,d,x)$ for $d > n - 3/2$ is*

$$\begin{aligned}\Delta_x G(n,d,x) &= 2^{n(n-1)} \left\{ \frac{\Gamma(d+1-n)\Gamma(d+1)}{(2n)!} \right\}^{4n-2} \left(\prod_{j=1}^{2n} j^j \right) \\ &\quad \times \prod_{j=1}^{n} (d+j)^{2(n-j)} (2d-2n+2j+1)^{2j-1}.\end{aligned}$$

Proof. Since $d > n - 3/2$, the set of zeros of $G(n,d,x)$ is $\{\alpha_j, \overline{\alpha_j}\}_{j=1}^n$ with $|\alpha_j| = 1$. So, the discriminant of $G(n,d,x)$ is

$$\begin{aligned}\Delta_x G(n,d,x) &= (-1)^{2n(2n-1)/2} \left\{ \frac{\Gamma(d+1-n)\Gamma(d+1+n)}{(2n)!} \right\}^{4n-2} \prod_{j=1}^{n} (\alpha_j - \overline{\alpha_j})^2 \\ &\quad \times \left(\prod_{j<k} (\alpha_j - \alpha_k)(\alpha_j - \overline{\alpha_k})(\overline{\alpha_j} - \alpha_k)(\overline{\alpha_j} - \overline{\alpha_k}) \right)^2.\end{aligned}$$

First, we compute the first product $\prod_{j=1}^{n} (\alpha_j - \overline{\alpha_j})^2$:

$$\begin{aligned}\prod_{j=1}^{n} (\alpha_j - \overline{\alpha_j})^2 &= \prod_{j=1}^{n} (2i\Im\alpha_j)^2 \\ &= (-1)^n \prod_{j=1}^{n} 4\left\{1 - (\Re\alpha_j)^2\right\} \quad \text{since} \quad |\alpha_j| = 1 \\ &= (-1)^n \prod_{j=1}^{n} (2 - 2\Re\alpha_j) \prod_{j=1}^{n} (2 + 2\Re\alpha_j).\end{aligned}$$

Since

$$\begin{aligned}G(n,d,x) &= \frac{\Gamma(d+1-n)\Gamma(d+1+n)}{(2n)!} {}_2F_1\left(\begin{matrix} -2n, d+1-n \\ -d-n \end{matrix} ; x \right) \\ &= \frac{\Gamma(d+1-n)\Gamma(d+1+n)}{(2n)!} \prod_{j=1}^{n} (z - \alpha_j)(z - \overline{\alpha_j}),\end{aligned}$$

by using the Chu-Vandermonde theorem (Theorem 2.3) and (2.3), we get

$$\prod_{j=1}^{n}(2-2\Re\alpha_j) = {}_2F_1\left(\begin{matrix}-2n, d+1-n\\ -d-n\end{matrix};1\right) = \frac{(-2d-1)_{2n}}{(-d-n)_{2n}}$$

and

$$\begin{aligned}\prod_{j=1}^{n}(2+2\Re\alpha_j) &= {}_2F_1\left(\begin{matrix}-2n, d+1-n\\ -d-n\end{matrix};-1\right)\\ &= 2^{2n}{}_2F_1\left(\begin{matrix}-n, -d-1/2\\ -d-n\end{matrix};1\right) = 2^{2n}\frac{(-n+1/2)_n}{(-d-n)_n}.\end{aligned}$$

Hence,

$$\prod_{j=1}^{n}(\alpha_j-\overline{\alpha_j})^2 = \frac{2^{4n}(1/2)_n(-d-1/2)_n}{\{(d+1)_n\}^2}.$$

For the second product, from

$$(\alpha_j-\alpha_k)(\alpha_j-\overline{\alpha_k})(\overline{\alpha_j}-\alpha_k)(\overline{\alpha_j}-\overline{\alpha_k}) = 4(\Re\alpha_j-\Re\alpha_k)^2,$$

we get

$$\prod_{j<k}(\alpha_j-\alpha_k)(\alpha_j-\overline{\alpha_k})(\overline{\alpha_j}-\alpha_k)(\overline{\alpha_j}-\overline{\alpha_k}) = 2^{n(n-1)}\prod_{j<k}(\Re\alpha_j-\Re\alpha_k)^2.$$

Since

$$G(n,d,x) = \frac{\Gamma(d+1-n)\Gamma(d+1)}{(1/2)_n}(-x)^n P_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right),$$

$\{\Re\alpha_j\}_{j=1}^{n}$ is the set of zeros of $P_n^{(-n+d+\frac{1}{2},-\frac{1}{2})}(x)$. Recall Theorem 2.7: the discriminant of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is

$$2^{n(n-1)}\prod_{j=1}^{n}\frac{j^j(\alpha+j)^{j-1}(\beta+j)^{j-1}}{(\alpha+\beta+n+j)^{n+j-2}}.$$

Thus

$$\prod_{j<k}(\alpha_j-\alpha_k)(\alpha_j-\overline{\alpha_k})(\overline{\alpha_j}-\alpha_k)(\overline{\alpha_j}-\overline{\alpha_k}) = 2^{2n(n-1)}\prod_{j=1}^{n}\frac{j^j(j-1/2)^{j-1}(d-n+j+1/2)^{j-1}}{(d+j)^{n+j-2}}.$$

Therefore, the discriminant of $G(n,d,x)$ is

$$\begin{aligned}\Delta_x G(n,d,x) &= 2^{n(n-1)}\left\{\frac{\Gamma(d+1-n)\Gamma(d+1)}{(2n)!}\right\}^{4n-2}\left(\prod_{j=1}^{2n} j^j\right)\\ &\quad\times\prod_{j=1}^{n}(d+j)^{2(n-j)}(2d-2n+2j+1)^{2j-1}.\end{aligned}$$

$\square$

Lastly, consider sums of squared distances between zeros of $G(n,d,x)$.

Theorem 4.6. *For fixed n, let R_d be the set of zeros of $G(n,d,x)$. Then, for $d > n-\frac{3}{2}$ or $d-1/2 = 0, 1, \cdots$, the sum of squared distances between zeros of $G(n,d,x)$ and zeros of $G(n,d+k,x)$ is*

$$\sum_{r\in R_d}\sum_{s\in R_{d+k}}|r-s|^2 = 8n^2\frac{(2d+1+k)(2n-1)}{(d+1+k)(d+n)}.$$

Proof. For $d > n-\frac{3}{2}$ or $d-1/2 = 0, 1, \cdots$, all zeros of $G(n,d,x)$ are on the unit circle by Theorem 4.1 and Theorem 4.3. Since $G(n,d,x)$ is self-inversive, the set of zeros of $G(n,d,x)$ is $R_d = \{\alpha_{d,j}, \overline{\alpha_{d,j}}\}_{j=1}^n$ where $|\alpha_{d,j}| = 1$. Hence, we have

$$\begin{aligned}\sum_{r\in R_d}\sum_{s\in R_{d+k}}|r-s|^2 &= 2\sum_{i=1}^{n}\sum_{j=1}^{n}\left(|\alpha_{d,i}-\alpha_{d+k,j}|^2+|\alpha_{d,i}-\overline{\alpha_{d+k,j}}|^2\right)\\ &= 8\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\Re\alpha_{d,i}\Re\alpha_{d+k,j})\\ &= 8n^2-2\left(\sum_{i=1}^{n}2\Re\alpha_{d,i}\right)\left(\sum_{j=1}^{n}2\Re\alpha_{d+k,j}\right)\\ &= 8n^2-2\left(\sum_{i=1}^{n}(\alpha_{d,i}+\overline{\alpha_{d,i}})\right)\left(\sum_{j=1}^{n}(\alpha_{d+k,j}+\overline{\alpha_{d+k,j}})\right).\end{aligned}$$

Since

$$\sum_{i=1}^{n}(\alpha_{d,i}+\overline{\alpha_{d,i}}) = -\frac{2n(d+1-n)}{d+n},$$

the sum of squared distances between zeros of $G(n,d,x)$ and zeros of $G(n,d+k,x)$ is

$$\sum_{r\in R_d}\sum_{s\in R_{d+k}}|r-s|^2 = 8n^2-8n^2\frac{(d+1-n)(d+k+1-n)}{(d+n)(d+k+n)} = 8n^2\frac{(2d+1+k)(2n-1)}{(d+1+k)(d+n)}.$$

$\square$

We get the maximal sum when d is asymptotic to $(-1+\sqrt{2n-1})/2$. We expect that the zeros are most evenly spread apart on the unit circle when $d=n$ so that the maximal sum occurs for this case. But, for $d \le n-3/2$, $G(n,d,z)$ has all zeros on the unit circle when d is $k-1/2$ where k is nonnegative integer. As k increases from 0 to $n-1$, $2k$ zeros are moving close to $z=-1$ with $2(n-k)$ zeros at $z=1$ so that we could not get the maximal sum from the case when zeros are well distributed on the unit circle.

4.2 A Family of Salem Polynomials

A self-inversive polynomial is called self-reciprocal if it has only real coefficients. Then the zeros of self-reciprocal polynomials are symmetric with respect to the unit circle and the real line. A monic reciprocal polynomial with integer coefficients having exactly one zero of multiplicity 1 outside the unit circle is called a *Salem polynomial* [2]. Hence a Salem polynomial provides a *Salem number* which is a real algebraic integer $\alpha > 1$ of degree ≥ 4, all of whose conjugates, apart form α and α^{-1}, lie on the unit circle. In this section, we discuss a family of Salem polynomials. Consider the polynomials with increasing odd coefficients

$$R_n(x) = \sum_{k=0}^{n} (2k+1)x^k.$$

Let

$$S_n(x) = R_n(x^2) + x^{2n+1} R_n(1/x^2).$$

Then $S_n(x)$ are the self-reciprocal polynomial of degree $2n+1$ with all odd coefficients. For example,

$$S_6(x) = 1 + 13x + 3x^2 + 11x^3 + 5x^4 + 9x^5 + 7x^6 + 7x^7 + 9x^8 + 5x^9 + 11x^{10} + 3x^{11} + 13x^{12} + x^{13}.$$

Since $S_n(-1)$=0 for all n, replace $S_n(x)$ by $S_n(x)/(x+1)$. It is easy to show that

$$S_n(x) = \frac{x^{2n+4} - 1 + 2(n+1)x(x^{2n+2}-1) + (2n+3)x^2(x^{2n}-1)}{(x^2-1)(x+1)^2} \tag{4.1}$$

with generating function

$$\frac{1-(1-x)^2 t - x^2 t^2}{(1-t)^2(1-tx^2)^2} = \sum_{n=0}^{\infty} S_n(x) t^n.$$

Theorem 4.7. *$S_n(x)$ has all zeros on the unit circle except exactly two. i.e. $S_n(x)$ is a Salem polynomial.*

Proof. From (4.1), we can write $S_n(x)$ as the sum of hypergeometric functions as follows:

$$\begin{aligned}
(1+x^2)S_n(x) &= \frac{x^{2n+4}-1+2(n+1)x(x^{2n+2}-1)+(2n+3)x^2(x^{2n}-1)}{(x^2-1)} \\
&= \sum_{l=0}^{n+1} x^{2l} + (2n+2)x\sum_{l=0}^{n} x^{2l} + (2n+3)x^2\sum_{l=0}^{n-1} x^{2l} \\
&= {}_2F_1\left(\begin{matrix}-n-1,1\\-n-1\end{matrix};x^2\right) + (2n+2)x\,{}_2F_1\left(\begin{matrix}-n,1\\-n\end{matrix};x^2\right) + (2n+3)x^2\,{}_2F_1\left(\begin{matrix}-n+1,1\\-n+1\end{matrix};x^2\right) \\
&= (1+x)^{2n+2}\,{}_2F_1\left(\begin{matrix}-n-1,-n-3/2\\-2n-3\end{matrix};\frac{4x}{(1+x)^2}\right) \\
&\quad +(2n+2)x(1+x)^{2n}\,{}_2F_1\left(\begin{matrix}-n,-n-1/2\\-2n-1\end{matrix};\frac{4x}{(1+x)^2}\right) \\
&\quad +(2n+3)x^2(1+x)^{2n-2}\,{}_2F_1\left(\begin{matrix}-n+1,-n+1/2\\-2n+1\end{matrix};\frac{4x}{(1+x)^2}\right) \qquad \text{by (2.4).}
\end{aligned}$$

By the change of variable $x \mapsto \dfrac{x+\sqrt{x^2-4}}{2}$, we have

$$\begin{aligned}
\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{-n} S_n\left(\frac{x+\sqrt{x^2-4}}{2}\right) &= (x+2)^n\,{}_2F_1\left(\begin{matrix}-n-1,-n-3/2\\-2n-3\end{matrix};\frac{2}{1-(-x/2)}\right) \\
&\quad +(2n+2)(x+2)^{n-1}\,{}_2F_1\left(\begin{matrix}-n,-n-1/2\\-2n-1\end{matrix};\frac{2}{1-(-x/2)}\right) \\
&\quad +(2n+3)(x+2)^{n-2}\,{}_2F_1\left(\begin{matrix}-n+1,-n+1/2\\-2n+1\end{matrix};\frac{2}{1-(-x/2)}\right).
\end{aligned}$$

By using (2.5), (2.8), and $U_n(-x) = (-1)^n U_n(x)$, we can represent these hypergeometric functions as Chebyshev polynomials of the second kind:

$$(x+2)^n\,{}_2F_1\left(\begin{matrix}-n,-n-1/2\\-2n-1\end{matrix};\frac{2}{1-(-x/2)}\right) = (-1)^n U_n\left(-\frac{x}{2}\right) = U_n\left(\frac{x}{2}\right)$$

so that

$$\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{-n} S_n\left(\frac{x+\sqrt{x^2-4}}{2}\right) = \frac{1}{x+2}\left\{U_{n+1}\left(\frac{x}{2}\right) + (2n+2)U_n\left(\frac{x}{2}\right) + (2n+3)U_{n-1}\left(\frac{x}{2}\right)\right\}.$$

Lastly, use the recurrence relation of the Chebyshev polynomials of the second kind (2.10) to get

$$S_n\left(\frac{x+\sqrt{x^2-4}}{2}\right) = \left(\frac{x+\sqrt{x^2-4}}{2}\right)^n \frac{(\frac{x}{2}+n+1)U_n\left(\frac{x}{2}\right) + (n+1)U_{n-1}\left(\frac{x}{2}\right)}{\frac{x}{2}+1}. \tag{4.2}$$

Since $(-1+n+1)U_n\left(\frac{-2}{2}\right)+(n+1)U_{n-1}\left(\frac{-2}{2}\right)=0$,

$$\frac{(\frac{x}{2}+n+1)U_n\left(\frac{x}{2}\right)+(n+1)U_{n-1}\left(\frac{x}{2}\right)}{\frac{x}{2}+1}$$

is a polynomial of degree n, and $(\frac{x}{2}+n+1)U_n\left(\frac{x}{2}\right)+(n+1)U_{n-1}\left(\frac{x}{2}\right)$ has only real roots since

$$\frac{(\frac{x}{2}+n+1)U_n\left(\frac{x}{2}\right)+(n+1)U_{n-1}\left(\frac{x}{2}\right)}{\frac{x}{2}+1}=\det(A-xI)$$

where

$$A=\begin{pmatrix} -(2n+2) & \sqrt{2n+2} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2n+2} & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & -1 & & & 0 \\ 0 & 0 & -1 & 0 & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & & & \ddots & 0 & -1 \\ 0 & 0 & 0 & & & -1 & 0 \end{pmatrix}$$

which is real symmetric.

For $x=1+u$ with $u>0$ and $n\geq 2$, from (2.9), we have

$$\begin{aligned} & U_{n+1}(1+u)+2(n+1)U_n(1+u)+(2n+3)U_{n-1}(1+u) \\ = \quad & \sum_{k=0}^{n+1}(2u)^k\frac{(n+2+k)!}{(n+1-k)!(2k+1)!}+2(n+1)\sum_{k=0}^{n}(2u)^k\frac{(n+1+k)!}{(n-k)!(2k+1)!} \\ & +(2n+3)\sum_{k=0}^{n-1}(2u)^k\frac{(n+k)!}{(n-1-k)!(2k+1)!}. \end{aligned}$$

For $0\leq k\leq n-1$, the coefficient of u^k is

$$\frac{2^{k+1}(n+k)!}{(n+1-k)!(2k+1)!}\left\{k^2-n(2n+3)k+2(n+1)^3\right\}>0.$$

The coefficient of u^n is $2^{n+3}(n+1)>0$ and the coefficient of u^{n+1} is $2^{n+1}>0$. Hence,

$$U_{n+1}(1+u)+2(n+1)U_n(1+u)+(2n+3)U_{n-1}(1+u)$$

has all positive coefficients. By Decartes's rule of signs, there is no root >1. Similarly, for $x=-1-u$ with

$u > 0$,

$$\begin{aligned} & U_{n+1}(-1-u) + 2(n+1)U_n(-1-u) + (2n+3)U_{n-1}(-1-u) \\ = \; & (-1)^{n+1}\sum_{k=0}^{n+1}(2u)^k\frac{(n+2+k)!}{(n+1-k)!(2k+1)!} + 2(n+1)(-1)^n\sum_{k=0}^{n}(2u)^k\frac{(n+1+k)!}{(n-k)!(2k+1)!} \\ & +(-1)^{n-1}(2n+3)\sum_{k=0}^{n-1}(2u)^k\frac{(n+k)!}{(n-1-k)!(2k+1)!}. \end{aligned}$$

For $0 \le k \le n-1$, the coefficient of u^k is

$$(-1)^n\frac{2^k(n+k)!}{(n+1-k)!(2k+1)!}\frac{k}{2n-1}\left\{k-(n+2+\frac{4}{2n-1})\right\}.$$

The coefficient of u^n is 0 and the coefficient of u^{n+1} is $(-1)^{n+1}2^{n+1} > 0$. By Decartes's rule of signs, there is exactly one root < -1. Therefore, $U_{n+1}(x/2) + 2(n+1)U_n(x/2) + (2n+3)U_{n-1}(x/2)$ has all its zeros in the interval $[-2, 2]$ except one. From (4.2), we can conclude that $S_n(x)$ has all its zeros on the unit circle except exactly two. □

We found above a family of self-reciprocal polynomials of odd degree with increasing positive coefficients a_{2k} that have all zeros on the unit circle except two. We have the following conjecture for polynomials with those properties.

Conjecture 4.8. *Let* $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. *Suppose* $0 < a_0 < a_1 < \cdots < a_n$. *Then*

$$P(x^2) + x^{2n+1}P(1/x^2) = a_0 + a_nx + a_1x^2 + a_{n-1}x^3 + a_2x^4 + \cdots + a_1x^{2n-1} + a_nx^{2n} + a_0x^{2n+1}$$

has all zeros on the unit circle except two.

If we consider $P(x^2)+x^{2n-N}P(1/x^2)$ for any nonnegative integer N, by Theorem 3.5 and the Eneström-Kakeya Theorem (Theorem 3.7), we will have all zeros on the unit circle. However, for the case $N = -1$, there seem to be two exceptions.

4.3 The Procedure 1 Polynomials $v_{2n}(p, x)$

Consider the following example:

$$v_8(p, x) = 1 + px + x^2 + px^3 + x^4 + px^5 + x^6 + px^7 + x^8.$$

By studying its discriminant

$$\Delta_x v_8(p,x) = (5-4p)(5+4p)(400+204p^2+93p^4+32p^6)^2,$$

we can see that for $|p| \le 5/4$ all its zeros are on the unit circle and for $|p| > 5/4$ there is exactly one pair not on the unit circle. These root distributions are true in general for

$$v_{2n}(p,x) := (1+x^2+\cdots+x^{2n}) + px(1+x^2+\cdots+x^{2n-2}).$$

Theorem 4.9. *$v_{2n}(p,x)$ has all zeros on the unit circle for $|p| \le \dfrac{n+1}{n}$ and has exactly one pair not on the unit circle for $|p| > \dfrac{n+1}{n}$.*

Proof. First, $v_{2n}(p,x)$ can be written as a linear combination of two successive Chebyshev polynomials of the second kind:

$$\begin{aligned}
v_{2n}(p,x) &= (1+x^2+\cdots+x^{2n}) + px(1+x^2+\cdots+x^{2n-2}) \\
&= \frac{x^{2(n+1)}-1}{x^2-1} + px\frac{x^{2n}-1}{x^2-1} \\
&= x^n\left(\frac{x^{n+1}-x^{-(n+1)}}{x-x^{-1}} + p\frac{x^n-x^{-n}}{x-x^{-1}}\right) \\
&= x^n\left\{U_n\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right) + pU_{n-1}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)\right\}.
\end{aligned}$$

Let

$$u_n(p,x) = U_n(x) + pU_{n-1}(x).$$

Consider $u_n(1+x)$ for $x>0$. By (2.9), we can analyze its coefficients:

$$u_n(p,1+x) = \sum_{k=0}^{n}(2x)^k\frac{(n+k+1)!}{(n-k)!(2k+1)!} + p\sum_{k=0}^{n-1}(2x)^k\frac{(n+k)!}{(n-k-1)!(2k+1)!}.$$

For $0 \le k \le n-1$, the coefficient of x^k is

$$\frac{2^k(n+k)!}{(n-k)!(2k+1)!}\left\{(n+k+1)+p(n-k)\right\}$$

which is positive for $p > -\dfrac{n+1}{n}$ and negative for $p < -\dfrac{n+1}{n}$. And the coefficient of x^n is $2^n > 0$. By Decartes's rule of signs, $u_n(p,x)$ has no root >1 for $p \ge -\dfrac{n+1}{n}$ and exactly one root >1 for $p < -\dfrac{n+1}{n}$. Similarly, by considering $u_n(p,-1-x)$ for $x>0$, we can show that $u_n(p,x)$ has no root <-1 for $p \le \dfrac{n+1}{n}$

and has exactly one root < -1 for $p > \dfrac{n+1}{n}$. Therefore, $v_{2n}(p,x)$ has all zeros on the unit circle for $|p| \le \dfrac{n+1}{n}$ and all zeros on the unit circle except one pair for $|p| > \dfrac{n+1}{n}$ since $|z| = 1 \iff -1 \le x \le 1$ when $z = \frac{1}{2}(x + \frac{1}{x})$. $\square$

By Corollary 3.1 in [6], the discriminant of $v_{2n}(p,x)$ with respect to x is

$$\Delta_x(v_{2n}(p,x)) = (-1)^n \left\{(n+1)^2 - n^2p^2\right\} a_{n-1}(p)^2,$$

where

$$a_{n-1}(p) = 2^{n(n-1)}(-1)^n \frac{(2n+1)^n p^n}{(n+1)^2 - n^2p^2} \left[U_n\left(-\frac{n+1+np^2}{(2n+1)p}\right) + pU_{n-1}\left(-\frac{n+1+np^2}{(2n+1)p}\right)\right]$$

is an even polynomial in k of degree $2n-2$ with positive integer coefficients. By Theorem 4 in [6],

$$\Delta_x(u_n) = 2^{n(n-1)} a_{n-1}(p).$$

Hence, $\Delta_x v_{2n}/(\Delta_x u_n)^2$ has only simple factors, namely

$$(-1)^n \frac{(n+1)^2 - n^2p^2}{2^{2n(n-1)}}.$$

Now, we consider similar properties for Chebyshev polynomials of the first kind. Let

$$t_n(p,x) = T_n(x) + pT_{n-1}(x).$$

We want to find a sequence of polynomials $Q_n(x)$ such that for

$$s_{2n}(p,x) = Q_n(x^2) + pxQ_{n-1}(x^2),$$

we have the following two properties:

1. All zeros of $Q_n(x)$ are on the unit circle and
2. $\Delta_x s_{2n}(p,x)/(\Delta_x t_n(p,x))^2$ has only simple factors.

Let

$$Q_n(t) = \frac{(t+1+\sqrt{t^2+t+1})^n + (t+1-\sqrt{t^2+t+1})^n}{2}.$$

Then we can easily see that

$$Q_n(t^2) = t^n T_n(t + 1/t)$$

with generating function

$$\sum_{n=0}^{\infty} Q_n(t)x^n = \frac{1 + t - tx}{1 - 2(t+1)x + tx^2}.$$

From $Q_n(t^2) = t^n T_n(t + 1/t)$, we have the first property: $Q_n(x)$ has all roots on the unit circle since Chebyshev polynomials have all zeros on the interval $[-1, 1]$ and $-1 \le t + 1/t \le 1 \Rightarrow |t| = 1$. And we have more information on the distribution of roots: the real parts of the roots are $\le -1/2$ since

$$Q_n(e^{i\theta}) = e^{in\theta/2} T_n(2\cos\theta) = 0 \Rightarrow -1 \le 2\cos\frac{\theta}{2} \le 1 \Rightarrow Re(x) \le -1/2.$$

Let $s_{2n}(p, x) := Q_n(x^2) + pxQ_{n-1}(x^2)$. i.e. $s_{2n}(p, x) = x^n t_n(p, x + 1/x)$. We evaluate the discriminant of $t_n(p, x)$ which is not done in either [6] or [10].

Theorem 4.10. *The discriminant of $t_n(p, x)$ with respect to x is*

$$\Delta_x(t_n(p, x)) = 2^{(n-1)(n-2)} \frac{p^n(2n-1)^n}{(1+p)(1-p)} \left\{ T_n\left(-\frac{p^2(n-1)+n}{p(2n-1)}\right) + pT_{n-1}\left(-\frac{p^2(n-1)+n}{p(2n-1)}\right) \right\},$$

and the discriminant of $s_{2n}(p, x)$ with respect to x is

$$\Delta_x(s_{2n}(p, x)) = (-1)^n s_{2n}(p, 1) s_{2n}(p, -1) \left(\Delta_x(t_n(p, x))\right)^2$$

where

$$s_{2n}(p, 1) = \frac{1}{2}\left[\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} + p\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}\right]$$

and

$$s_{2n}(p, -1) = \frac{1}{2}\left[\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} - p\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}\right].$$

To prove Theorem 4.10, we need some properties of resultants and discriminants. Let

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \\ g(x) &= b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m \end{aligned}$$

be two given polynomials. And suppose that the zeros of f and g are $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_m$, respectively.

Then we define the resultant of f and g by

$$Res(f,g) = a_0^m b_0^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j).$$

The most important properties are

$$Res(f,g) = a_0^m \prod_{i=1}^{n} g(\alpha_i), \tag{4.3}$$

$$Res(f,g) = (-1)^{nm} Res(g,f), \tag{4.4}$$

$$Res(f,pq) = Res(f,p)Res(f,q) \tag{4.5}$$

where p and q are polynomials, and

$$Res(g,f) = b_0^{n-l} Res(g,r) \tag{4.6}$$

if we can write $f(x) = q(x)g(x) + r(x)$ with polynomials q and r, and $\deg r = l$.

We also need the resultant of two homogeneous polynomials

$$\begin{aligned} F(x,y) &= a_0x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n, \\ G(x,y) &= b_0x^m + b_1x^{m-1}y + \cdots + b_{m-1}xy^{m-1} + b_my^m. \end{aligned}$$

Then their resultant with respect to x and y, denoted by $Res_{x,y}(F,G)$, is defined by the Sylvester determinant:

$$Res_{x,y}(F,G) = \begin{vmatrix} f_0 & f_1 & \cdots & \cdots & f_n & & & & \\ & f_0 & f_1 & \cdots & \cdots & f_n & & & \\ & & \ddots & \ddots & & & & \ddots & \\ & & & f_0 & f_1 & \cdots & \cdots & & f_n \\ g_0 & g_1 & \cdots & \cdots & g_n & & & & \\ & g_0 & g_1 & \cdots & \cdots & g_n & & & \\ & & \ddots & \ddots & & & & \ddots & \\ & & & g_0 & g_1 & \cdots & \cdots & & g_n \end{vmatrix}$$

The basic properties are analogous to those of $Res(f,g)$. We will use the following chain rule for resultants of homogeneous polynomials.

Lemma 4.11 (McKay and Wang, [21])**.** *Let $F(x,y)$, $G(x,y)$, $H(u,v)$, and $K(u,v)$ be homogeneous polyno-*

mials of degree at least 1, and suppose that $\deg H = \deg K$. *Then*

$$Res_{u,v}(F(H,K), G(H,K)) = [Res_{x,y}(F,G)]^{\deg H} [Res_{u,v}(H,K)]^{\deg F \deg G}.$$

Proof of Theorem 4.10. Since the discriminant $\Delta_x f(x)$ of a polynomial $f = \sum_{k=0} a_k x^k$ can be given in terms of a resultant of f and f' by

$$\Delta_x f(x) = \frac{(-1)^{n(n-1)/2}}{a_n} Res(f, f'),$$

begin by considering

$$\begin{aligned}
(1-x^2)\frac{d}{dx}t_n(p,x) &= (1-x^2)\{nU_{n-1}(x) + p(n-1)U_{n-2}(x)\} \\
&= -\{nx + p(n-1)\}\, t_n(p,x) + \left\{p(2n-1)x + p^2(n-1) + n\right\} T_{n-1}(x).
\end{aligned}$$

This with (4.5) and (4.6) gives

$$\begin{aligned}
& Res_x\left(t_n(p,x), (1-x^2)\frac{d}{dx}t_n(p,x)\right) \\
= \;& 2^{n-1} Res_x\left(t_n(p,x), \left\{p(2n-1)x + p^2(n-1) + n\right\} T_{n-1}(x)\right) \\
= \;& 2^{n-1} Res_x\left(t_n(p,x), p(2n-1)x + p^2(n-1) + n\right) Res_x\left(t_n(p,x), T_{n-1}(x)\right).
\end{aligned} \tag{4.7}$$

The first resultant on the right side above can be computed by using (4.3),

$$Res_x\left(t_n(p,x), p(2n-1)x + p^2(n-1) + n\right) = (-1)^n p(2n-1) t_n\left(p, -\frac{p^2(n-1)+n}{p(2n-1)}\right).$$

In [10], Theorem 3.1 shows that

$$Res_x(T_n(x) + kT_{n-1}(x), T_{n-1}(x) + hT_{n-2}(x)) = (-1)^{n(n-1)/2} 2^{n^2-3n-3} h^n \left[T_n\left(\frac{1+hk}{2h}\right) - kT_{n-1}\left(\frac{1+hk}{2h}\right)\right]$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind. So, the second resultant on the right-hand side of (4.7) is

$$Res_x\left(t_n(p,x), T_{n-1}(x)\right) = 2^{(n-1)(n-2)}(-1)^{n(n-1)/2}.$$

On the other hand, we have

$$\begin{aligned}
Res_x\left(t_n(p,x),(1-x^2)\frac{d}{dx}t_n(p,x)\right) &= Res_x\left(t_n(p,x),\frac{d}{dx}t_n(p,x)\right)Res_x\left(t_n(p,x),(1-x^2)\right) \qquad \text{by (4.3)}\\
&= \left\{(-1)^{\frac{n(n-1)}{2}}2^{n-1}\Delta_x\left(t_n(p,x)\right)\right\}\left\{(-1)^{2n}(-1)^2 t_n(p,1)t_n(p,-1)\right\}\\
&= (-1)^{\frac{n(n+1)}{2}}2^{n-1}(1+p)(1-p)\Delta_x\left(t_n(p,x)\right).
\end{aligned}$$

Therefore,

$$\Delta_x(t_n(p,x)) = 2^{(n-1)(n-2)}\frac{p^n(2n-1)^n}{(1+p)(1-p)}\left\{T_n\left(-\frac{p^2(n-1)+n}{p(2n-1)}\right)+pT_{n-1}\left(-\frac{p^2(n-1)+n}{p(2n-1)}\right)\right\}.$$

For the discriminant of $s_{2n}(p,x)$, consider $\dfrac{d}{dx}s_{2n}(p,x)$ as before;

$$xs_{2n}'(p,x) = ns_{2n}(p,x)+x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)(x^2-1).$$

From (4.5) and (4.6), we can get

$$\begin{aligned}
Res_x\left(s_{2n}(p,x),x\frac{d}{dx}s_{2n}(p,x)\right) &= Res_x\left(s_{2n}(p,x),ns_{2n}(p,x)+x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)(x^2-1)\right)\\
&= Res_x\left(s_{2n}(p,x),x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)(x^2-1)\right)\\
&= Res_x\left(s_{2n}(p,x),(x^2-1)\right)Res_x\left(s_{2n}(p,x),x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)\right)\\
&= s_{2n}(p,1)s_{2n}(p,-1)Res_x\left(s_{2n}(p,x),x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)\right).
\end{aligned}$$

To examine $Res_x\left(s_{2n}(p,x),x^{n-1}\frac{d}{dx}t_n\left(p,x+1/x\right)\right)$, set

$$f(w) := t_n(w) = f_0w^n+f_1w^{n-1}+\cdots+f_n$$

and

$$g(w) := f'(w) = g_0w^{n-1}+g_1w^{n-2}+\cdots+g_{n-1}.$$

Then define the homogeneous functions F and G by

$$z^nf\left(\frac{z^2+1}{z}\right) = f_0(z^2+1)^n+f_1(z^2+1)^{n-1}z+\cdots+f_nz^n = F(z^2+1,z)$$

and

$$z^{n-1}g\left(\frac{z^2+1}{z}\right) = g_0(z^2+1)^{n-1} + g_1(z^2+1)^{n-2}z + \cdots + g_{n-1}z^{n-1} = G(z^2+1,z).$$

And let $H(z,u) = z^2+u^2$ and $K(z,u) = zu$. Then, by Lemma 4.11, we have

$$\begin{aligned}
Res_z\left(F(z^2+1,z), G(z^2+1,z),\right) &= Res_{z,u}\left(F(H,K), G(H,K)\right) \\
&= \left[Res_{x,y}\left(F(x,y), G(x,y)\right)\right]^2 \left[Res_{z,u}\left(H(z,u), K(z,u)\right)\right]^{n(n-1)} \\
&= \left[Res_x\left(F(x,1), G(x,1)\right)\right]^2 \left[Res_z\left(H(z,1), K(z,1)\right)\right]^{n(n-1)} \\
&= \left[(-1)^n 2^{n-1}\Delta_x(t_n(p,x))\right]^2 (-1)^{n(n-1)} \\
&= 2^{2(n-1)}\left[\Delta_x(t_n(p,x))\right]^2.
\end{aligned}$$

On the other hand,

$$Res_z\left(F(z^2+1,z), G(z^2+1,z)\right) = Res_z\left(s_{2n}(p,z), z^{n-1}\frac{d}{dx}t_n\left(p, \frac{z^2+1}{z}\right)\right).$$

Hence,

$$Res_x\left(s_{2n}(p,x), x\frac{d}{dx}s_{2n}(p,x)\right) = s_{2n}(p,1)s_{2n}(p,-1)2^{2(n-1)}\left(\Delta_x(t_n(p,x))\right)^2.$$

Since

$$\begin{aligned}
Res_x\left(s_{2n}(p,x), x\frac{d}{dx}s_{2n}(p,x)\right) &= Res_x\left(s_{2n}(p,x), \frac{d}{dx}s_{2n}(p,x)\right) Res_x\left(s_{2n}(p,x), x\right) \\
&= (-1)^n 2^{(n-1)}\Delta_x(s_{2n}(p,x)),
\end{aligned}$$

we can represent the determinant of $s_{2n}(p,x)$ in terms of the determinant of $t_n(p,x)$:

$$\Delta_x(s_{2n}(p,x)) = (-1)^n s_{2n}(1)s_{2n}(-1)\left(\Delta_x(t_n(p,x))\right)^2.$$

Here,

$$s_{2n}(p,1) = \frac{1}{2}\left[\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} + p\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}\right]$$

and

$$s_{2n}(p,-1) = \frac{1}{2}\left[\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\} - p\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}\right].$$

□

Chapter 5

Distance Sums for Points on the Unit Circle

5.1 Sums for the Vertices of the Stretched N-gon

Let $\{p_1, \cdots, p_N\}$ be the vertices of a regular N-gon with center at 0, i.e. $p_j = e^{i\frac{2\pi j}{N}}$ for $j = 1, \cdots, N$. Define stretching factors $\{q^{a_1}, q^{a_2}, \cdots, q^{a_N}\}$ where $q > 0$ is a real number and the a_j are nonnegative integers such that $a_1 + a_2 + \cdots + a_N = N$. The *stretched N-gon* is the N-gon with vertices $\{q^{a_1}p_1, q^{a_2}p_2, \cdots, q^{a_N}p_N\}$. This also can be considered as the q-analogue of a regular N-gon. For the stretched N-gon and $\lambda > 0$, let

$$S_\lambda(q, \{a_j\}_{j=1}^N) := \sum_{1 \le j < k \le N} |q^{a_j}p_j - q^{a_k}p_k|^\lambda.$$

We consider the question of when the smallest sums occur and when the largest sums occur in a neighborhood of $q = 1$. Naturally, we can conjecture that the smallest sum occurs for $\{a_j\}_{j=1}^N = \{1, 1, \cdots, 1\}$ and the largest sum occurs for $\{a_j\}_{j=1}^N = \{N, 0, \cdots, 0\}$.

If we consider this question for the regular simplex, then we have the complete answer from the theory of majorization. We begin with the definition of majorization and a theorem.

Definition 5.1. *Let $a = (a_j)_1^m$ and $b = (b_j)_1^m$ be two real sequences. Let $\hat{a}$ and $\hat{b}$ be these sequences, reordered to be decreasing. If*

$$\sum_{j=1}^n \hat{a}_j \le \sum_{j=1}^n \hat{b}_j, \quad n = 1, 2, \cdots, m \qquad \text{and} \qquad \sum_{j=1}^m \hat{a}_j = \sum_{j=1}^m \hat{b}_j,$$

then a is majorized by b. This is denoted by $a \prec b$.

Theorem 5.2 (4.B.1 [20])**.** *Let $a, b \in \mathbb{R}^m$. The inequality*

$$\sum_{j=1}^m f(a_j) \le \sum_{j=1}^m f(b_j)$$

holds for all continuous convex function $f : \mathbb{R} \to \mathbb{R}$ if and only if $a \prec b$.

From this theorem, we have the complete answer for sums of squared distances between vertices on the stretched regular simplex in $\mathbb{R}^N$.

Theorem 5.3. *Let $\{v_j\}_1^N$ be the vertices of a regular $N-1$ dimensional simplex in $\mathbb{R}^N$ where v_j is the jth row of the identity matrix I_N. Consider the stretched regular simplex $\{q^{a_1}v_1, q^{a_2}v_2, \cdots, q^{a_N}v_N\}$ with the stretching parameters $a \in \mathbb{R}^N$ and $q \in \mathbb{R}$. If $a \prec b$ for $a, b \in \mathbb{R}^N$, then*

$$\sum_{j<k} |q^{a_j}v_j - q^{a_k}v_k|^2 \leq \sum_{j<k} |q^{b_j}v_j - q^{b_k}v_k|^2.$$

Proof. For $a \in \mathbb{R}^N$, the sum of squared distances between the vertices of a stretched regular simplex is

$$\begin{aligned}\sum_{j<k} |q^{a_j}v_j - q^{a_k}v_k|^2 &= \sum_{j<k} \left(q^{2a_j} + q^{2a_k}\right) \\ &= (N-1)\sum_{j=1}^{N} q^{2a_j}.\end{aligned}$$

Since $f(x) = q^{2x}$ is convex on $\mathbb{R}$ for $q \in \mathbb{R}$, by Theorem 5.2,

$$\sum_{j<k} |q^{a_j}v_j - q^{a_k}v_k|^2 \leq \sum_{j<k} |q^{b_j}v_j - q^{b_k}v_k|^2 \qquad \text{if } \ a \prec b.$$

$\square$

In addition, if we have $\sum_{j=1}^{N} a_j = N$, then the stretching factor $\{N, 0, \cdots, 0\}$ gives the largest sum and the stretching factor $\{1, 1, \cdots, 1\}$ gives the smallest sum. However, for the regular N-gon, majorization cannot give the full answer. The main reason is lack of full symmetry in the mutual distances. The theory of majorization is invariant under permutations. However, the sum of distances between vertices on the stretched regular N-gon is affected by permutations of the stretching factors. Consider the following example.

Example 5.4. *Consider the vertices of the square $\{1, i, -1, -i\}$ on the unit circle. Then we have the following inequalities for the stretching factor $\{2, 11/10, 9/10, 0\} \succ \{2, 1, 1, 0\}$:*

$$\begin{aligned}S_2(q, \{2, 9/10, 11/10, 0\}) \geq S_2(q, \{2, 1, 1, 0\}) &\geq S_2(q, \{2, 11/10, 9/10, 0\}) \\ &\geq S_2(q, \{2, 11/10, 0, 9/10\}) \geq S_2(q, \{2, 1, 0, 1\}).\end{aligned}$$

The sum of squared distances $S_2(q, \{a_j\}_i^N)$ on the stretched N-gon is

$$S_2(q, \{a_j\}_1^N) = N \sum_{j=1}^{N} q^{2a_j} - \left| \sum_{j=1}^{N} q^{a_j} e^{i\frac{2\pi j}{N}} \right|^2 .$$

The first sum can be handled with majorization theory since $f(x) = q^{2x}$ is convex on $\mathbb{R}$. However, for the second sum, we need to consider the permutations. If q is close to 1 and N is large, then the first sum is much larger than the second sum. Hence, we can expect that majorization still can give the solution for some special cases provided that the majorization is sufficiently large. We shall give specific results rather than a quantification of majorization in this context.

Our results are as follows. In Section 5.1.1, for $\lambda = 2$, we will find the smallest and the largest sum for some stretching factors. For example, when $N = 2^3$, we consider $A_0 = \{1, \cdots, 1\}$, $A_1 = \{2, 0, 2, 0, 2, 0, 2, 0\}$, $A_2 = \{2^2, 0, 0, 0, 2^2, 0, 0, 0\}$, and $A_3 = \{2^3, 0, \cdots, 0\}$ for $\{a_j\}_{j=1}^{2^3}$. Then we obtain $S_2(q, A_0) \leq S_2(q, A_1) \leq S_2(q, A_2) \leq S_2(q, A_3)$. And, for a fixed $\{a_j\}_{j=1}^N$, the closer the center of mass of $\{q^{a'_1}p_1, q^{a'_2}p_2, \cdots, q^{a'_N}p_N\}$ is from the origin where $\{a'_j\}$ is a permutation of $\{a_j\}$, the larger the sum is. From this, we have $S_2(q, \{4, 4, 0, \cdots, 0\}) \leq S_2(q, \{4, 0, 4, 0, \cdots, 0\}) \leq S_2(q, \{4, 0, 0, 4, 0, \cdots, 0\}) \leq S_2(q, A_2)$. Next, we prove that $S_2(q, A_1) < S_2(q, \{4, 4, 0, \cdots, 0\})$. Hence we can list the stretching factors in the order of larger sums. In Section 5.1.2, we will prove that for any $\lambda > 0$, the difference between sums of any two stretching factors $\{a_j\}$ and $\{a'_j\}$ is $\mathcal{O}((q-1)^2)$ as $|q-1| \to 0$.

5.1.1 The Sums for Some Special Stretching Factors

In this section, we will consider the case when λ is 2, i.e. $S_2(q, \{a_j\}_{j=1}^N)$.

Lemma 5.5. *Let $\{a_j\}_{j=1}^N$ be fixed. For any permutation σ of $\{j\}_{j=1}^N$,*

$$S_2(q, \{a_{\sigma(j)}\}_{j=1}^N) \leq N \sum_{j=1}^{N} q^{2a_j} .$$

Also the center of mass of $\{q^{a_j}p_j\}_{j=1}^N$ is at the origin, i.e. $\sum\limits_{j=1}^{N} q^{a_j}p_j = 0$, if and only if

$$S_2(q, \{a_j\}_{j=1}^N) = N \sum_{j=1}^{N} q^{2a_j} .$$

Proof. Use $|p-q|^2 = |p|^2 + |q|^2 - 2\langle p, q\rangle$, where $p, q \in \mathbb{R}^2$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^2$, to get

$$\begin{aligned}
S_2(q, \{a_j\}_{j=1}^N) &= \frac{1}{2}\sum_{j=1}^N \sum_{k=1}^N \left| q^{a_j} e^{i\frac{2\pi j}{N}} - q^{a_k} e^{i\frac{2\pi k}{N}} \right|^2 \\
&= \frac{1}{2}\sum_{j=1}^N \sum_{k=1}^N \left(\left| q^{a_j} e^{i\frac{2\pi j}{N}} \right|^2 + \left| q^{a_k} e^{i\frac{2\pi k}{N}} \right|^2 - 2\langle q^{a_j} e^{i\frac{2\pi j}{N}}, q^{a_k} e^{i\frac{2\pi k}{N}} \rangle \right) \\
&= N\sum_{j=1}^N q^{2a_j} - \sum_{j=1}^N \sum_{k=1}^N \langle q^{a_j} e^{i\frac{2\pi j}{N}}, q^{a_k} e^{i\frac{2\pi k}{N}} \rangle \\
&= N\sum_{j=1}^N q^{2a_j} - \langle \sum_{j=1}^N q^{a_j} e^{i\frac{2\pi j}{N}}, \sum_{k=1}^N q^{a_k} e^{i\frac{2\pi k}{N}} \rangle \\
&= N\sum_{j=1}^N q^{2a_j} - \left| \sum_{j=1}^N q^{a_j} e^{i\frac{2\pi j}{N}} \right|^2 . \qquad (5.1)
\end{aligned}$$

Hence, for a permutation σ,

$$\begin{aligned}
S_2(q, \{a_{\sigma(j)}\}_{j=1}^N) &= N\sum_{j=1}^N q^{2a_{\sigma(j)}} - \left| \sum_{j=1}^N q^{a_{\sigma(j)}} e^{i\frac{2\pi j}{N}} \right|^2 \\
&= N\sum_{j=1}^N q^{2a_j} - \left| \sum_{j=1}^N q^{a_{\sigma(j)}} e^{i\frac{2\pi j}{N}} \right|^2 \\
&\leq N\sum_{j=1}^N q^{2a_j}.
\end{aligned}$$

Also the center of mass of $\{q^{a_j} p_j\}_{j=1}^N$ is at the origin, i.e. $\sum_{j=1}^N q^{a_j} p_j = 0$, if and only if

$$S_2(q, \{a_j\}_{j=1}^N) = N\sum_{j=1}^N q^{2a_j}. \qquad (5.2)$$

□

Let $N = b^n$ for some integer $b \geq 2$. For $l = 0, \cdots, n$, we consider the following sequences $A_l^b = \{a_{l,j}\}_{j=1}^{b^n}$ where

$$a_{l,j} = \begin{cases} b^l, & \text{if } j \equiv 1 \mod b^l, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.6. *For $q > 0$,*

$$S_2(q, A_0^b) \leq S_2(q, A_1^b) \leq S_2(q, A_2^b) \leq \cdots \leq S_2(q, A_n^b).$$

Proof. Since the center of mass is at the origin for $l = 0, 1, \cdots, n-1$, by (5.2), we have

$$S_2(q, A_l^b) = N \sum_{j=1}^{N} q^{2a_j} = N \left\{ \frac{N}{b^l} q^{2 \cdot b^l} + \left(N - \frac{N}{b^l} q^{2 \cdot 0} \right) \right\} = \frac{N^2}{b^l} \left(q^{2b^l} + b^l - 1 \right). \tag{5.3}$$

And, for $l = n$, from (5.1),

$$S_2(q, A_n^b) = (N-1) q^{2N} + 2q^N + (N^2 - N - 1).$$

Hence, for $l = 0, \cdots, n-2$, the difference between $S_2(q, A_{l+1}^b)$ and $S_2(q, A_l^b)$ is

$$\begin{aligned} S_2(q, A_{l+1}^b) - S_2(q, A_l^b) &= \frac{N^2}{b^{l+1}} \left(q^{2b^{l+l}} + b^{l+1} - 1 \right) - \frac{N^2}{b^l} \left(q^{2b^l} + b^l - 1 \right) \\ &= \frac{N^2}{b^{l+1}} \left(q^{2b^{l+1}} - b q^{2b^l} + b - 1 \right) \\ &= \frac{N^2}{b^{l+1}} \left(q^{2b^l} - 1 \right) \left(q^{2b^l(b-1)} + q^{2b^l(b-2)} + \cdots + q^{2b^l} + 1 - b \right) \geq 0 \end{aligned}$$

and the last difference is

$$\begin{aligned} S_2(q, A_n^b) - S_2(q, A_{n-1}^b) &= (N-1) q^{2N} + 4q^N + 2(N^2 - N - 1) - \frac{N^2}{b^{n-1}} \left(q^{2b^{n-1}} + b^{n-1} - 1 \right) \\ &= (b^n - 1) q^{2b^n} + 2q^{b^n} - b^{n+1} q^{2b^{n-1}} + (b-1) b^n - 1 \geq 0. \end{aligned}$$

$\square$

Theorem 5.7. *For $l = 0, \cdots, n-2$,*

$$S_2(q, A_l^b) < S_2(q, \{a_j\}_{j=1}^N)$$

where $a_1 = \cdots = a_{b^{n-l-1}} = b^{l+1}$ and $a_{b^{n-l-1}+1} = \cdots = a_N = 0$.

Proof. By Lemma 5.5 and (5.3),

$$\begin{aligned}
&S_2(q,\{a_j\}_{j=1}^N) - S_2(q, A_l^b) \\
&= \frac{N^2}{b^{l+1}}\left(q^{2b^{l+1}} + b^{l+1} - 1\right) - \left|\sum_{j=1}^{N} q^{a_j} e^{i\frac{2\pi j}{N}}\right|^2 - \frac{N^2}{b^l}\left(q^{2b^l} + b^l - 1\right) \\
&= \frac{N^2}{b^{l+1}}\left(q^{2b^l} - 1\right)\left(q^{2b^l(b-1)} + q^{2b^l(b-2)} + \cdots + q^{2b^l} + 1 - b\right) - \left|q^{2b^l}\sum_{j=1}^{b^{n-l-1}} e^{i\frac{2\pi j}{N}} + \sum_{j=b^{n-l-1}+1}^{N} e^{i\frac{2\pi j}{N}}\right|^2 \\
&= \frac{N^2}{b^{l+1}}\left(q^{2b^l} - 1\right)\left(q^{2b^l(b-1)} + q^{2b^l(b-2)} + \cdots + q^{2b^l} + 1 - b\right) - \left|\frac{e^{i\frac{2\pi}{N}}}{1 - e^{i\frac{2\pi}{N}}}\left(q^{2b^l} - 1\right)\left(1 - e^{i\frac{2\pi}{b^{l+1}}}\right)\right|^2 \\
&= \left(q^{2b^l} - 1\right)^2\left\{\frac{N^2}{b^{l+1}}\left(q^{2b^l(b-2)} + 2q^{2b^l(b-3)} + \cdots + (b-2)q^{2b^l} + (b-1)\right) - \left(\frac{\sin\frac{\pi}{b^{l+1}}}{\sin\frac{\pi}{N}}\right)^2\right\}.
\end{aligned}$$

Since $0 \le \dfrac{\pi}{b^{l+1}} \le \dfrac{\pi}{N} \le \dfrac{\pi}{2}$,

$$\left(\frac{\sin\frac{\pi}{b^{l+1}}}{\sin\frac{\pi}{N}}\right)^2 \le \left(\frac{\frac{\pi}{b^{l+1}}}{\frac{2}{N}}\right)^2.$$

From this with $q^{2b^l(b-2)} + 2q^{2b^l(b-3)} + \cdots + (b-2)q^{2b^l} + (b-1) \ge b-1 \ge 1$, it follows that

$$\begin{aligned}
S_2(q,\{a_j\}_{j=1}^N) - S_2(q, A_l^b) &\ge \left(q^{2b^l} - 1\right)^2\left\{\frac{N^2}{b^{l+1}} - \left(\frac{\frac{\pi}{b^{l+1}}}{\frac{2}{N}}\right)^2\right\} \\
&= \left(q^{2b^l} - 1\right)^2\left\{\left(\frac{N}{b^l}\right)^2 - \left(\frac{\pi}{2b}\frac{N}{b^l}\right)^2\right\} > 0.
\end{aligned}$$

□

Theorem 5.6 and Theorem 5.7 show that if $a \prec a'$ then $S_2(q,a) \le S_2(q,a')$ when we consider the stretching factors a and a' consisting of b^l or 0.

5.1.2 $S_\lambda(q,\{a_j\}_{j=1}^N) - S_\lambda(q,\{a'_j\}_{j=1}^N) = \mathcal{O}((q-1)^2)$

Theorem 5.8. *For any $\lambda > 0$ and any pair $\{a_j\}_{j=1}^N$ and $\{a'_j\}_{j=1}^N$ where $a_1+\cdots+a_N = N$ and $a'_1+\cdots+a'_N = N$, we have*

$$S_\lambda(q,\{a_j\}_{j=1}^N) - S_\lambda(q,\{a'_j\}_{j=1}^N) = \mathcal{O}((q-1)^2).$$

Moreover,

$$\begin{aligned} & S_\lambda(q, \{a_j\}_{j=1}^N) \\ = \quad & 2^{\lambda/2} \sum_{1\le j<k\le N} \left|\sin \frac{\pi(j-k)}{N}\right|^\lambda + \left\{\lambda 2^{\lambda/2} N \sum_{l=1}^{n-1} \left(\sin \frac{\pi l}{N}\right)^\lambda\right\}(q-1) + \mathcal{O}((q-1)^2) \end{aligned}$$

where the constant term and the coefficient of $(q-1)$ *are independent of the choice of* $\{a_j\}_{j=1}^N$.

To prove the theorem, we need the following proposition.

Proposition 5.9. *Let* f *be a function and* $\{a_j\}_{j=1}^N$ *be a sequence. If* $f(-m) = f(m)$ *and* $f(N-m) = f(m)$, *then*

$$\sum_{j=1}^N \sum_{k=1}^N f(j-k)(a_j + a_k) = 2\sum_{l=0}^{N-1} f(l) \sum_{j=1}^N a_j.$$

Proof. Here either $j - k = l$ or $j - k = l - N$ or $j = k$. Call these terms of the *1st*, *2nd*, and *3rd kind*. The sum over terms of the *3rd kind* is clearly $2f(0)\sum_{j=1}^N a_j$. For the terms of the *1st kind*, we have the summand

$$f(l)(a_{k+l} + a_k) \text{ where } j = k + l \text{ so } 1 \le k \le N - l.$$

For the terms of the *2nd kind*, we have the summand

$$f(l-N)(a_{k+l-N} + a_k) \text{ where } j = k + l - N \text{ so } N + 1 - l \le k \le N.$$

Since $f(-m) = f(m)$ and $f(N-m) = f(m)$, $f(l-N) = f(N-l) = f(l)$ so that the terms of *1st kind* are

$$\{a_{l+1}, a_{l+2}, \cdots, a_N\} \text{ and } \{a_1, a_2, \cdots, a_{N-l}\} \tag{5.4}$$

while those of the *2nd kind* are

$$\{a_1, a_2, \cdots, a_l\} \text{ and } \{a_{N+1-l}, a_{N+2-l}, \cdots, a_N\}. \tag{5.5}$$

Since each a_j occurs exactly twice in (5.4) and (5.5), we obtain

$$\sum_{j=1}^N \sum_{k=1}^N f(j-k)(a_j + a_k) = 2\sum_{l=0}^{N-1} f(l) \sum_{j=1}^N a_j.$$

□

Proof of Theorem 5.8. For any $\{a_j\}_{j=1}^N$, we have

$$\begin{aligned} S_\lambda(q,\{a_j\}_{j=1}^N) &= \sum_{1\le j<k\le N} |q^{a_j}v_j - q^{a_k}v_k|^\lambda \\ &= \sum_{1\le j<k\le N} \left\{q^{2a_j}+q^{2a_k}-2q^{a_j+a_k}\cos\frac{2\pi(j-k)}{N}\right\}^{\lambda/2} \end{aligned}$$

so that, for $q=1$, it follows that

$$S_\lambda(1,\{a_j\}_{j=1}^N) = \sum_{1\le j<k\le N}\left\{2-2\cos\frac{2\pi(j-k)}{N}\right\}^{\lambda/2} = 2^{\lambda/2}\sum_{1\le j<k\le N}\left|\sin\frac{\pi(j-k)}{N}\right|^\lambda.$$

Since the derivative of $S_\lambda(1,\{a_j\}_{j=1}^N)$ with respect to q is

$$\begin{aligned} \frac{\partial S_\lambda}{\partial q}(q,\{a_j\}_{j=1}^N) &= \frac{\lambda}{2}\sum_{1\le j<k\le N}\left\{q^{2a_j}+q^{2a_k}-2q^{a_j+a_k}\cos\frac{2\pi(j-k)}{N}\right\}^{\lambda/2-1} \\ &\quad\times\left\{2a_jq^{2j-1}+2a_kq^{2k-1}-2(a_j+a_k)q^{a_j+a_k}\cos\frac{2\pi(j-k)}{N}\right\}, \end{aligned}$$

$$\begin{aligned} \frac{\partial S_\lambda}{\partial q}(1,\{a_j\}_{j=1}^N) &= \lambda 2^{\lambda/2-1}\sum_{1\le j<k\le N}\left\{1-\cos\frac{2\pi(j-k)}{N}\right\}^{\lambda/2}(a_j+a_k) \\ &= \lambda 2^{\lambda-2}\sum_{1\le j,k\le N}\left|\sin\frac{\pi(j-k)}{N}\right|^\lambda (a_j+a_k). \end{aligned}$$

If we let $f(l)=\left|\sin\frac{\pi l}{N}\right|^\lambda$, then it satisfies $f(-l)=f(l)$ and $f(N-l)=f(l)$. By Proposition 5.9, we obtain

$$\frac{\partial S_\lambda}{\partial q}(1,\{a_j\}_{j=1}^N) = \lambda 2^{\lambda-2}2\sum_{l=0}^{N-1}\left|\sin\frac{\pi l}{N}\right|^\lambda\sum_{k=1}^N a_k = \lambda 2^{\lambda-1}N\sum_{l=1}^{N-1}\left(\sin\frac{\pi l}{N}\right)^\lambda.$$

Hence, for any $\{a_j\}_{j=1}^N$ such that $a_1+\cdots+a_N=N$, the expansion of $S_\lambda(q,\{a_j\}_{j=1}^N)$ about $q=1$ is

$$S_\lambda(q,\{a_j\}_{j=1}^N) = 2^{\lambda/2}\sum_{1\le j<k\le N}\left|\sin\frac{\pi(j-k)}{N}\right|^\lambda + \left\{\lambda 2^{\lambda-1}N\sum_{l=0}^{N-1}\left(\sin\frac{\pi l}{N}\right)^\lambda\right\}(q-1)+\mathcal{O}((q-1)^2)$$

and therefore, for any pair $\{a_j\}_{j=1}^N$ and $\{a'_j\}_{j=1}^N$ such that $a_1+\cdots+a_N=N$ and $a'_1+\cdots+a'_N=N$,

$$S_\lambda(q,\{a_j\}_{j=1}^N) - S_\lambda(q,\{a'_j\}_{j=1}^N) = \mathcal{O}((q-1)^2).$$

□

Remark 5.10. *For $\lambda = 2$, we have the simpler expression*

$$S_2(q, \{a_j\}_{j=1}^N) = N^2 + 2N^2(q-1) + \mathcal{O}((q-1)^2).$$

for any $\{a_j\}_{j=1}^N$ where $a_1 + \cdots + a_N = N$.

5.2 Squared Distance Sum

In [25], Stolarsky studied the sum of the distances to N points on a sphere with the following question: if $p_1, \cdots, p_N$ be points on the unit sphere S^{m-1}, for what $p \in S^{m-1}$ is

$$\sum_{j=1}^N |p - p_j|^\lambda, \quad 0 < \lambda < 2$$

maximal? When is the sum minimal? By introducing the concept of uniform power maxima, he determined the maximum or minimum for a regular m-dimensional octahedron, cube, simplex, and a regular N-gon on the unit circle. We generalize the latter as follows. Let $\{p_j\}_{j=1}^N$ be the N vertices of a regular N-gon with vertex charges $\{c_j\}_{j=1}^N$ on the unit circle. We will obtain the maximum of $\sum_{j=1}^N c_j |e^{i\theta} - p_j|^2$. For a certain set of charges, we study the minimal polynomial of the maximum value and its generating function.

5.2.1 On the Set of Charged Vertices on the Regular N-gon

Consider $\{p_1, \cdots, p_N\}$, the vertices of a regular N-gon on the unit circle, i.e. $p_j = e^{i\frac{2\pi}{N}j}$ for $j = 1, \cdots, N$. Let c_j be a charge on the p_j. We define

$$S(N, \{c_j\}_{j=1}^N, e^{i\theta}) := \sum_{j=1}^N c_j |e^{i\theta} - e^{\frac{2\pi i}{N}j}|^2$$

to be the sum of squared distances from a point on the unit circle to charged vertices on the regular N-gon. Then it is easy to show that

$$S(N, \{c_j\}_{j=1}^N, e^{i\theta}) = \sum_{j=1}^N c_j - 2\cos\theta \sum_{j=1}^N c_j \cos\frac{2\pi}{N}j - 2\sin\theta \sum_{j=1}^N c_j \sin\frac{2\pi}{N}j.$$

By considering $\frac{\partial S}{\partial \theta} = 0$, we have that the maximal sum of squared distances is

$$\begin{aligned} M(N, \{c_j\}_{j=1}^N) &:= \max_\theta S(N, \{c_j\}_{j=1}^N, e^{i\theta}) \\ &= 2\left(\sum_{j=1}^N c_j + \left|\sum_{j=1}^N c_j e^{i\frac{2\pi}{N}j}\right|\right) \end{aligned}$$

where

$$\sin\theta = \pm\frac{\sum_{j=1}^N c_j \sin\frac{2\pi}{N}j}{\left|\sum_{j=1}^N c_j e^{i\frac{2\pi}{N}j}\right|} \text{ and } \cos\theta = \pm\frac{\sum_{j=1}^N c_j \cos\frac{2\pi}{N}j}{\left|\sum_{j=1}^N c_j e^{i\frac{2\pi}{N}j}\right|}.$$

The case when $N = 2n$ and $c_j = 1$ for $1 \le j \le n$, $c_j = -1$ for $n+1 \le j \le 2n$

In this case, it is easy to see that

$$\sum_{j=1}^N c_j = 0 \text{ and } \sum_{j=1}^N c_j e^{i\frac{2\pi}{N}j} = \frac{2i}{e^{i\frac{\pi}{N}}\sin\frac{\pi}{N}}.$$

Thus, the maximum squared distance sum is

$$M(2n, \{c_j\}_{j=1}^{2n}) = 4\csc\frac{\pi}{2n} \quad \text{where} \quad \theta = i\frac{(3n+1)\pi}{2n}.$$

If n is odd, the maximal value $M(2n, \{c_j\}_{j=1}^{2n})$ is the root of an irreducible monic polynomial of degree $\phi(n)/2$ since $2\sin\left(2\pi\frac{k}{l}\right)$ is an algebraic integer of degree $\phi(l)/2$ or $\phi(l)$ according as l is or is not a multiple of 4 by Lehmer's Theorem [16]. Now, we determine the minimal polynomial of this maximal value.

Lemma 5.11. *Let $n = 2m+1$ and*

$$P_m(t) = \begin{cases} \dfrac{(-1)^k 4^m \cos((m+\frac{1}{2})\sec^{-1}\frac{t}{4})}{\cos^m(\sec^{-1}\frac{t}{4})\cos(\frac{1}{2}\sec^{-1}\frac{t}{4})}, & m = 2k-1, \\ \dfrac{(-1)^k 4^m \sin((m+\frac{1}{2})\sec^{-1}\frac{t}{4})}{\cos^m(\sec^{-1}\frac{t}{4})\sin(\frac{1}{2}\sec^{-1}\frac{t}{4})}, & m = 2k. \end{cases}$$

then $P_m\left(M(2(2m+1), \{c_j\}_{j=1}^{2(2m+1)})\right) = 0$ for $m \ge 1$, $\Delta_t P_m(t) = 2^{3m(m-1)}(2m+1)^{m-1}$, and

$$P(x,t) = \frac{(1+tx)(1-8t-t^2x^2)}{1+(64-2t^2)x^2+t^4x^4} = \sum_{m=0}^\infty P_m(t)x^m \quad \textit{where} \quad \deg P_m(t) = m.$$

Proof. The degree of $P_m(t)$ is m and its leading coefficient is 1. Now, consider the case when $m = 2k-1$,

$k \geq 1$. For $j = 0, \cdots, m-1$,

$$\begin{aligned} P_m\left(4\csc\frac{4(j-k+1)+1}{2m+1}\frac{\pi}{2}\right) &= P_m\left(4\sec\frac{2\pi j}{2m+1}\right) \\ &= \frac{(-1)^k 4^m \cos((m+\frac{1}{2})\frac{(2j+1)\pi}{2m+1})}{\cos^m \frac{(2j+1)\pi}{2m+1}\cos(\frac{1}{2}\frac{(2j+1)\pi}{2m+1})} = 0. \end{aligned}$$

Similarly, when $m = 2k$ and $k \geq 1$, $P_m\left(4\csc\frac{4(j-k)+1}{2m+1}\frac{\pi}{2}\right) = 0$ for $j = 0, \cdots, m-1$. Therefore,

$$P_m(t) = \prod_{j=0}^{m-1}\left(t - 4\csc\frac{4(j-[m/2])+1}{2m+1}\frac{\pi}{2}\right).$$

In particular, $P_m\left(M(2(2m+1), \{c_j\}_{j=1}^{2(2m+1)})\right) = P_m\left(4\csc\frac{\pi}{2(2m+1)}\right) = 0$ for $m \geq 1$.

Next, we compute $\Delta_t P_m(t)$ by using the following formula for the discriminant:

$$\Delta_x f(x) = \frac{(-1)^{n(n-1)/2}}{a_n} Res(f, f')$$

where the degree of f is n and a_n is its leading coefficient. When m is even, $m = 2k$, the derivatives of $P_m(t)$ at its roots are, for $j = 1, \cdots, m$,

$$P_m'(4\sec\frac{2\pi j}{2m+1}) = \frac{(-1)^{k+j}2^{2m-3}(2m+1)}{\cos^{m-2}\left(\frac{2\pi j}{2m+1}\right)\sin\left(\frac{2\pi j}{2m+1}\right)\sin\left(\frac{\pi j}{2m+1}\right)}.$$

Thus, we get the discriminant of $P_m(t)$, namely

$$\Delta_t P_m(t) = (-1)^{m(m-1)/2}\prod_{j=1}^{m} P_m'\left(4\sec\frac{2\pi j}{2m+1}\right) = 2^{m(2m-3)}(2m+1)^{m-1},$$

by using

$$\prod_{k=1}^{N-1}\sin\frac{\pi k}{N} = 2^{1-N}N \qquad \text{and} \qquad \prod_{k=1}^{N-1}\cos\frac{\pi k}{N} = 2^{1-N}\sin\frac{N\pi}{N}.$$

Similarly, for the case when m is odd, we will obtain $\Delta_t P_m(t) = 2^{3m(m-1)}(2m+1)^{m-1}$.

By considering $P_m(4\sec\theta)$, we have the following recurrence relation:

$$P_m(t) = (2t^2 - 64)P_{m-2}(t) - t^4 P_{m-4}(t), \quad m \geq 4$$

with the initial conditions

$$P_0(t) = 1, \quad P_1(t) = t - 8, \quad P_2(t) = t^2 - 8t - 64, \quad \text{and} \quad P_3(t) = t^3 - 16t^2 - 64t + 512.$$

From this recurrence relation, we obtain the generating function of $P_m(t)$:

$$\begin{aligned} P(x,t) &= \sum_{m=0}^{\infty} P_m(t)x^m \\ &= \sum_{m=0}^{3} P_m(t)x^m + \sum_{m=4}^{\infty} \left\{(2t^2-64)P_{m-2}(t) - t^4 P_{m-4}(t)\right\} x^m \\ &= (1+tx)(1-8x-t^2x^2) + \left\{(2t^2-64)x^2 - t^4x^4\right\} P(x,t). \end{aligned}$$

Hence, $P(x,t) = \dfrac{(1+tx)(1-8t-t^2x^2)}{1+(64-2t^2)x^2+t^4x^4}$. □

By Lemma 5.11, the minimal polynomial $\Psi_m(t)$ of $M(2n, \{c_j\}_{j=1}^{2(2m+1)})$ is a divisor of $P_m(t)$. Often $P_m(t)$ is reducible. However, when $2m+1$ is prime, $P_m(t)$ is irreducible so that the minimal polynomial is $P_m(t)$. In fact, $P_m(t)$ has the same factorization pattern as $\dfrac{t^{2m+1}-1}{t-1}$ as follows: let

$$P_m^l(t) = \prod_{\substack{j=0 \\ (j,2m+1)=l}}^{m-1} \left(t - 4\csc \frac{4(j-[m/2])+1}{2m+1}\frac{\pi}{2}\right).$$

Here, all $P_m^l(t)$ are irreducible since $2\sin\left(\frac{2\pi l}{2(2m+1)}\right)$ is an algebraic integer of degree $\phi\left((2m+1)/l\right)/2$. Then

$$P_m(t) = \prod_{l|(2m+1)} P_m^l(t) \quad \text{and} \quad \Psi_m(t) = P_m^1(t).$$

5.2.2 With the Charged Point on the Unit Circle

We consider the sum of squared distances from a charged point to charged vertices on the regular N-gon as follows:

$$S(N, \{c_j\}_{j=1}^N, me^{i\theta}) := \sum_{j=1}^{N} c_j |me^{i\theta} - e^{\frac{2\pi i}{N}j}|^2.$$

Then

$$\begin{aligned} M(N, \{c_j\}_{j=1}^N, m) &:= \max_\theta S(N, \{c_j\}_{j=1}^N, me^{i\theta}) \\ &= (m^2+1)\sum_{j=1}^{N} c_j + 2m\left|\sum_{j=1}^{N} c_j e^{i\frac{2\pi}{N}j}\right|. \end{aligned}$$

The case when $N=2n+1$ and $c_{2n+1}=0$, $c_{n+1}=-1$, and $c_j=1$ for the other j's

Since

$$\sum_{j=1}^{N} c_j = (2n-2) \text{ and } \sum_{j=1}^{N} c_j e^{i\frac{2\pi}{2n+1}j} = -2e^{i\frac{\pi}{2n+1}} - 1,$$

we have

$$M(2n+1,\{c_j\}_{j=1}^{2n+1},m) = 2(m^2+1)(n-1) + 2m\sqrt{5+4\cos\frac{\pi}{2n+1}}.$$

Let $\Psi_{2n+1}(x)$ be the minimal polynomial of $4\cos\frac{\pi}{2n+1}$. i.e.

$$\Psi_{2n+1}(x) = \prod_{\substack{j=1\\(j,2n+1)=1}}^{n} \left(x - 4\cos\frac{j\pi}{2n+1}\right).$$

Then the degree of $\Psi_{2n+1}(x)$ is $\phi(2n+1)/2$. Define

$$H_n(p,y) := (-p)^{\phi(2n+1)/2}\Psi_{2n+1}(-5-y/p).$$

It is a polynomial of p and y, $\deg_p H_n(p,y) = \deg_y H_n(p,y) = \phi(2n+1)/2$, and

$$H_n\left(-4m^2, \left(M(2n+1,\{c_j\}_{j=1}^{2n+1},m) - 2(m^2+1)(n-1)\right)^2\right) = 0.$$

Finally, $H_n(p,y)$ has positive coefficients since

$$H_n(p,y) = p^{\phi(2n+1)/2} \prod_{\substack{j=1\\(j,2n+1)=1}}^{n} \left(y/p + 5 + 4\cos\frac{j\pi}{2n+1}\right).$$

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