# SPARSE COLOR-CRITICAL GRAPHS AND RAINBOW MATCHINGS IN EDGE-COLORED GRAPHS

BY

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DISSERTATION

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# Abstract

This thesis focuses on extremal problems about coloring graphs and on finding rainbow matchings in edge-colored graphs.

A graph $G$ is *$k$-critical* if $G$ is not $(k-1)$-colorable, but every proper subgraph of $G$ is $(k-1)$-colorable. Let $f_k(n)$ denote the minimum number of edges in an $n$-vertex $k$-critical graph. We present a lower bound, $f_k(n) \geq F(k,n)$, that is sharp whenever $n = 1 \pmod{k-1}$. It establishes the asymptotics of $f_k(n)$ for every fixed $k$, and confirms a conjecture by Gallai. We also present several applications of the proof, including a polynomial-time algorithm for $(k-1)$-coloring a graph $G$ that satisfies $|E(G[W])| < F(k,|W|)$ for all $W \subseteq V(G)$, several results about 3-coloring planar graphs, and a lower bound on the minimum number of edges in an $n$-vertex $k$-critical hypergraph.

We also present a lower bound on the number of edges in graphs that cannot be 1-defectively 2-colored. The bound solves an open problem in vertex Ramsey theory.

A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. The *color degree* of a vertex $v$ is the number of different colors on edges incident to $v$. We show that if $G$ is a $n$-vertex graph with minimum color degree $k$, then $G$ contains a rainbow matching of size at least $k/2$ always and size at least $k$ if $n \geq 4.25k^2$.

# Acknowledgements

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# Chapter 1

# Introduction

## 1.1 Notation and Conventions

We define $[k] = \{1, \ldots, k\}$. The *degree of a vertex* $v$, denoted $d(v)$, is the number of edges that contain $v$. The *maximum degree* of a graph $G$, denoted $\Delta(G)$, is the maximum degree of the vertices of $G$. For a graph or hypergraph $G$ and a set of vertices $S \subseteq V(G)$, the induced graph of $G$ on $S$, denoted $G[S]$, is the subgraph such that $V(G[S]) = S$ and $E(G[S]) = \{e \in E(G) : e \subseteq S\}$. For a set of vertices $S$ in a graph $G$, the graph $G - S$ is the induced subgraph on the vertex set $V(G) \setminus S$.

A *vertex-coloring* of a graph $G$, or graph coloring, is a function $\phi : V(G) \to [k]$. For a vertex $v \in V(G)$, the *color* on $v$ is $f(v)$. An *edge-coloring* is a function $\phi : V(G) \to [k]$. Let $A \subseteq V(G)$ and $B \subseteq V(G)$ such that $V(G) = A \cup B$. If $\phi_A$ is a coloring of $G[A]$ and $\phi_B$ is a coloring of $G[B]$ such that $\phi_A(v) = \phi_B(v)$ for each $v \in A \cap B$, then $\phi_A \cup \phi_B$ is a coloring $\phi$ of $G$ where $\phi(u) = \phi_A(u)$ when $u \in A$ and $\phi(w) = \phi_B(w)$ when $w \in B$.

A *proper coloring* of a graph $G$ is a graph coloring such that $\phi(u) \neq \phi(v)$ if $uv \in E(G)$. If there exists a proper coloring on $G$ with $k$ colors, then we say that $G$ is *$k$-colorable*. The *chromatic number* of a graph $G$, denoted $\chi(G)$, is the smallest $k$ such that $G$ is $k$-colorable. A graph $G$ is *$k$-chromatic* if $\chi(G) = k$. A graph $G$ is *$k$-critical* if $G$ is not $(k-1)$-colorable, but every proper subgraph of $G$ is $(k-1)$-colorable. Every $k$-critical graph is $k$-chromatic. Every graph with chromatic number at least $k$ contains a $k$-critical graph as a subgraph.

A *proper $k$-coloring* of a hypergraph $G$ is a function $f : V(G) \to [k]$ such that for each

$e \in E(G)$, the vertices in $e$ do not all map to the same color. The definitions of chromatic number and $k$-critical generalize in the natural way.

For a coloring $f$ of a graph $G$, the set of vertices $\{v \in V(G) : f(v) = i\}$ is called the *$i^{th}$ color class of $f$*. A *$(d_1, \ldots, d_k)$-improper coloring* of a graph $G$, or just $(d_1, \ldots, d_k)$-coloring, is a coloring $f$ of $G$ using at most $k$ colors such that for all $i$, the $i^{th}$ color class of $f$ induces a graph with maximum degree at most $d_i$. This is also called a $(d_1, \ldots, d_k)$-relaxed coloring. If $d_i = d$ for $1 \leq i \leq k$, then this is also known as $d$-defectively $k$-coloring. A graph $G$ is *improperly $(d_1, \ldots, d_k)$-colorable*, or just $(d_1, \ldots, d_k)$-colorable, if there is a $(d_1, \ldots, d_k)$-coloring of $G$. We define a graph $G$ to be *$(d_1, \ldots, d_k)$-critical* if $G$ is not $(d_1, \ldots, d_k)$-colorable, but every proper subgraph $H$ of $G$ is $(d_1, \ldots, d_k)$-colorable.

A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. In the literature, a rainbow subgraph has also been called "totally multicolored," "polychromatic," or "heterochromatic." The *color degree* of a vertex $v$ in an edge-colored graph, $d^c(v)$, is the number of distinct colors that appear on the edges incident to $v$. The *minimum color degree* of a graph $G$ with edge-coloring $\phi$, $\delta^c(G, \phi)$, is the minimum color degree of the vertices in $G$ under the edge-coloring $\phi$. A *$t$-factor* of a graph $G$ is a subgraph $H$ of $G$ such that $V(G) = V(H)$ and $d_H(v) = t$ for all $v \in V(H)$.

A graph $G$ is *connected* if there exists a path with ends $x$ and $y$ for every pair of vertices $x$ and $y$ in $G$. A *separating set* in a connected graph $G$ is a subset of vertices $S$ such that $G - S$ is disconnected. If $G$ is a graph such that there exists a pair of vertices in $G$ that do not form an edge, then the connectivity of $G$, denoted $\kappa(G)$, is the smallest size of a separating set in $G$. A graph $G$ is *$k$-connected* if the connectivity of $G$ is at least $k$.

For a graph $G$ and distinct vertices $u, v \in V(G)$, the operation of *merging* $u$ and $v$ is to create a new graph $G'$, where $V(G') = V(G) - u - v + w$ and $N_{G'}(w) = (N_G(u) \cup N_G(v)) - \{u, v\}$. If $G$ is a graph and $v \in V(G)$, then the operation of *splitting* $v$ into vertices $u_1, \ldots u_t$ is to create a new graph $G'$, where merging vertices $u_1, \ldots u_t$ in $G'$ into one vertex $v$ creates

$G$. Splitting and merging are inverse operation of each other.

## 1.2 Sparse Critical Graphs

The problem of graph coloring has long been associated with resource assignment. In this setting, each color is a type of resource and each vertex needs exactly one type of resource.

A simple application of proper colorings of graphs is the problem of assigning radio frequencies to a network of radio towers. Suppose a radio broadcasting company owns a collection of radio stations and towers on which those stations broadcast. The company also pays a subscription fee for a set of frequencies on which their towers can broadcast. If two towers are too close to each other, as when both are located in Miami, then to avoid interference with each other's transmission they must broadcast on different frequencies. If two towers are sufficiently apart, as when one is in Atlanta and the other is in Portland, then they are allowed to use the same frequency. The problem of assigning frequencies to each tower can then be modeled as a proper coloring problem: each vertex represents a tower that requires a frequency, each frequency is represented by a color, and an edge between two vertices represents towers that are close enough to experience interference.

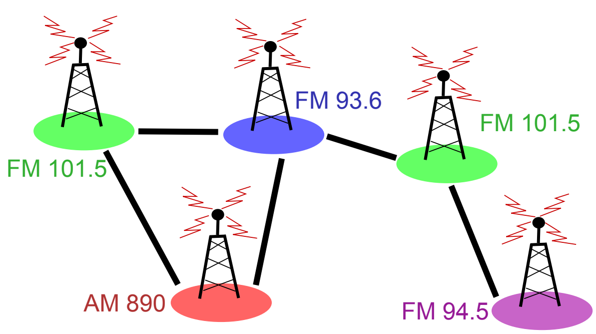


Figure 1.1: An example application of properly coloring a graph.

The problem of graph coloring dates back to 1852, when mathematicians considered the problem of coloring the counties of England on a map. Interest in the problem gained momentum when applications to computational efficiency were discovered. One such application involves scheduling several tasks with conflicting interest for register access. This problem can be modeled via graph coloring. Each task is a vertex. If two tasks need a register that cannot be shared, then the two tasks form an edge. Each color represents a dedicated time slot when the associated tasks can run. In this model, a proper graph coloring with few colors corresponds to a schedule in which all tasks can be completed in a short time-span.

Proper graph coloring continues to be studied today because of the complexity of the problem. Determining whether a graph is $k$-colorable when $k \geq 3$ was in the original list of 21 NP-complete problems by Karp [48]. Furthermore, even reasonable approximations of the fewest number of colors necessary is an NP-complete problem [82].

There is very little one can say about the structure of $k$-chromatic graphs beyond the fact that they contain a $k$-critical subgraph. All $k$-critical graphs are $(k-1)$-edge connected; not all $k$-chromatic graphs are. All $k$-critical graphs are 2-connected; not all $k$-chromatic graphs are. The general structure of a $k$-chromatic graph is best described as a $k$-critical graph plus extraneous stuff. Proofs of many statements in graph coloring begin with a redaction to the case of $k$-critical graphs. Structural results about $k$-critical graphs are then applied. Hence, applications to the large and prominent research area of graph coloring place a strong motivation for characterizing critical graphs.

Chapter 2 is directed towards describing the structure of $k$-critical graphs. The main theorem is a lower bound on the number of edges in a $k$-critical graph with $n$ vertices. Informally, this is a statement claiming that critical graphs are not "sparse" with edges. The theorem confirms a conjecture by Gallai, and partially solves a conjecture by Ore. Chapter 2 also includes an algorithm for quickly coloring sparse graphs. Section 2.2 contains an important lemma, which is interesting on its own. For example, the lemma is used again in Section 3.3.

The original intent of critical graphs was to use them as a tool for answering questions in graph coloring problems. Chapter 3 is dedicated to presenting some applications of the results in Chapter 2. Section 3.1 presents short proofs of several results relating to 3-coloring planar graphs, several of which are new. Section 3.2 includes a construction of graphs with high chromatic number but low chromatic number on all small subgraphs. The results of Chapter 2 improve the previous constructions. Section 3.3 presents a short proof of a result on coloring graphs with low Ore-degree. Section 3.4 presents a similar result for hypergraphs.

Recall the application of graph coloring to assigning frequencies to radio towers. Recently, Havet and Sereni [41] described a variation of this problem:

> In this paper, we investigate the following problem proposed by Alcatel, a satellite building company. A satellite sends information to receivers on earth, each of which is listening on a frequency. Technically it is impossible to focus the signal sent by the satellite exactly on receiver. So part of the signal is spread in an area around it creating noise for the other receivers displayed in this area and listening on the same frequency. A receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too big, i.e. does not exceed a certain threshold $T$.

In this problem, each vertex is allowed to have up to $T$ neighbors operating with the same color. This is a generalization of the traditional problem: if $T = 0$, then the problem reduces to finding a proper coloring. The analogue of this generalized problem is improperly coloring a graph. Using the notation of improper colorings, Alcatel is interested in a $(T, \ldots, T)$-coloring of their satellite receivers.

The interest in structural results about $(T, \ldots, T)$-critical graphs comes from the same set of applications on $k$-critical graphs. Esperet, Montassier, Ochem, and Pinlou [30] proved that if there exists a $(0, 1)$-critical planar graph with girth at least $g$ and maximum degree at most $d$, then determining if a planar graph with girth $g$ and maximum degree $d$ is $(0, 1)$-colorable is NP-complete if a planar graph with girth $g$ and maximum degree $d$ is $(0,1)$-colorable is NP-complete. Because of the similarity in the problems of improper coloring to proper coloring, the theorems about improper colorings we strive to produce are similar to those theorems regarding proper coloring. The main objective of Chapter 4 is a lower bound on the number of edges in $n$-vertex $(1,1)$-critical graphs. This result solves an open problem in vertex-Ramsey theory.

## 1.3 Rainbow Matchings

The main message of our work on critical graphs is that graphs that do not admit simple colorings contain a large and complex subgraph. We now present the converse: large graphs with complex colorings must contain simple substructures. The field of anti-Ramsey theory studies, for fixed graphs $G$ and $H$, the minimum number of colors on the edges of $G$ such that $G$ is forced to contain a rainbow subgraph isomorphic to $H$. Local anti-Ramsey theory, for fixed graphs $G$ and $H$, determines the smallest minimum color degree of $G$ such that $G$ is forced to contain a rainbow subgraph isomorphic to $H$.

Anti-Ramsey theory and local anti-Ramsey theory are essentially solved when $\chi(H-e) \geq 3$ for all $e \in E(H)$. This is because many extremal graphs in anti-Ramsey theory can be constructed by turning an extremal graph in Turán theory into a rainbow subgraph, with all of the non-edges collected into a single color class. However, Turán theory is still largely open when $\chi(H) = 2$. The active directions in these fields involve examining the answers when $H$ is bipartite. Chapter 5 specifically investigates local anti-Ramsey values when $H$ is a matching.

The choice of a matching for $H$ lends deeper significance to the results presented here. Ryser's conjecture on the size of the largest transversal in a Latin Square is equivalent to finding the largest rainbow matching in $G = K_{n,n}$ with an edge-coloring where each color class forms a 1-factor. There exist several results in design theory about finding large rainbow matchings in graphs where each color class forms a $t$-factor, for some $t$. As such, the results in Chapter 5 relate to and augment existing results in design theory.

Additionally, there has been a significant amount of research in the relationship of rainbow subgraphs to minimum color degree in the theory of rainbow subgraphs, as studied by computer scientists. There exists an extensive amount of literature on the computational complexity of finding rainbow subgraphs for fixed subgraph characteristics. The majority of the results determine that the problem is computationally difficult (many of the problems are some variation of NP). This led to an interest in finding easily verifiable conditions that are sufficient for the existence of a rainbow subgraph of given type. The minimum color degree is one such easily verifiable condition. Our results solve two open problems about the sufficiency of a large minimum color degree for a rainbow matching of given size.

# Chapter 2

# Sparse Critical Graphs

Even though $k$-critical graphs have more structure than $k$-chromatic graphs, there is still much unknown about them when $k \geq 4$. The only 1-critical graph is $K_1$, and the only 2-critical graph is $K_2$. The set of 3-critical graphs are the cycles with odd length. See Figure 2.1 for several graphs that demonstrate the variety of 4-critical graphs.

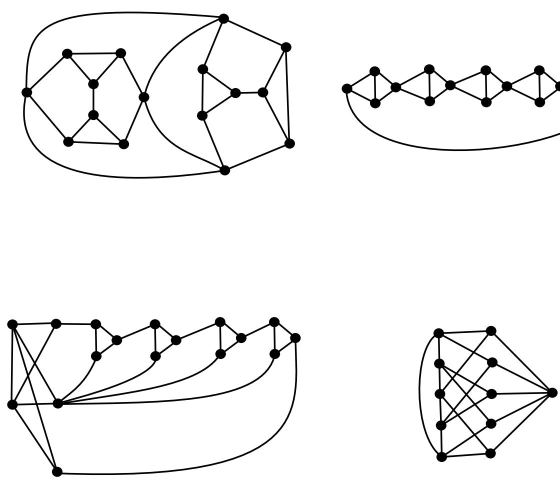

Figure 2.1: Several 4-critical graphs.

Proofs of many statements in graph coloring begin with a redaction to the case of $k$-critical graphs. Structural results about $k$-critical graphs are then applied. Hence, applications to the large and prominent research area of graph coloring place a strong motivation for characterizing critical graphs. Dirac [25] first defined critical graphs in 1951, when he used them to describe the space of $k$-colorings on a graph and the affects on that space from deleting a vertex. In 1957 Dirac [27] asked for $f_k(n)$, the fewest edges in a $k$-critical graph with $n$ vertices. Since then, determining $f_k(n)$ has shown up on several lists of open problems in graph theory [44, 45, 77].

It is known that $k$-critical graphs exist if and only if $n = k$ or $n \geq k+2$, and so we define that to be the domain of $f_k(n)$. Gallai determined exact values when $k+2 \leq n \leq 2k-2$ [33]: $f_k(n) = \frac{1}{2}\left((k-1)n + (n-k)(2k-n)\right) - 1$. Until our recent results, exact values of $f_k(n)$ were known for only a few other values of $n$ for each $k$.

Bounds on $f_k(n)$ fall into two categories: Dirac-type bounds and Gallai-type bounds. Note that by edge-connectivity (and therefore minimum degree), $f_k(n) \geq \frac{k-1}{2}n$. This is tight if and only if $n = k$. Dirac-type bounds prove that $f_k(n) \geq \frac{k-1}{2}n + h(k)$ for some function $h$ under some restrictions on $n$. Gallai-type bounds prove that $f_k(n) \geq h'(k)n$, for some function $h'(k) > \frac{k-1}{2}$ under some restrictions on $n$.

Dirac-type bounds earned their name because Dirac was the first to explicitly work towards this type of result. However, the earliest Dirac-type bound belongs to Brooks [21], who proved that if $G$ is $k$-critical and not $K_k$, then $\Delta(G) \geq k$, which implies that $f_k(n) \geq \frac{k-1}{2}n + \frac{1}{2}$ when $n \neq k$. Brooks' result is sharp when $k = 4$ and $n = 7$. Dirac strengthened this result [27] by showing that $f_k(n) \geq \frac{k-1}{2}n + \frac{k-3}{2}$ when $n \neq k$. Dirac's result is sharp when $n = 2k-1$. Much later, Kostochka and Stiebitz [52] would give the best Dirac-type bound known today, which is $f_k(n) \geq \frac{k-1}{2}n + k - 3$ when $n \notin \{k, 2k-1\}$. This result is sharp when $n \in \{2k, 3k-2\}$.

Gallai [33] started Gallai-type bounds by proving that $f_k(n) \geq (\frac{k-1}{2} + \frac{k-3}{2(k^2-3)})n$ when $n \neq k$. Krivelevich [57] proved the stronger Gallai-type bound $f_k(n) \geq (\frac{k-1}{2} + \frac{k-3}{2(k^2-2k-1)})n$ when $n \neq k$. Then Kostochka and Stiebitz [52] proved that $f_k(n) \geq (\frac{k-1}{2} + \frac{k-3}{k^2+6k-11-6/(k-2)})n$ for $k \geq 6$ when $n \neq k$,.

While Gallai-type bounds are stronger than Dirac-type bounds when $n >> k$, there are no pairs $(n, k)$ for which any of the above Gallai-type bounds are sharp. In fact, for these bounds, $\lim_{k\to\infty}\left(h'(k) - \frac{k-1}{2}\right) = 0$, while the best constructions imply that for the

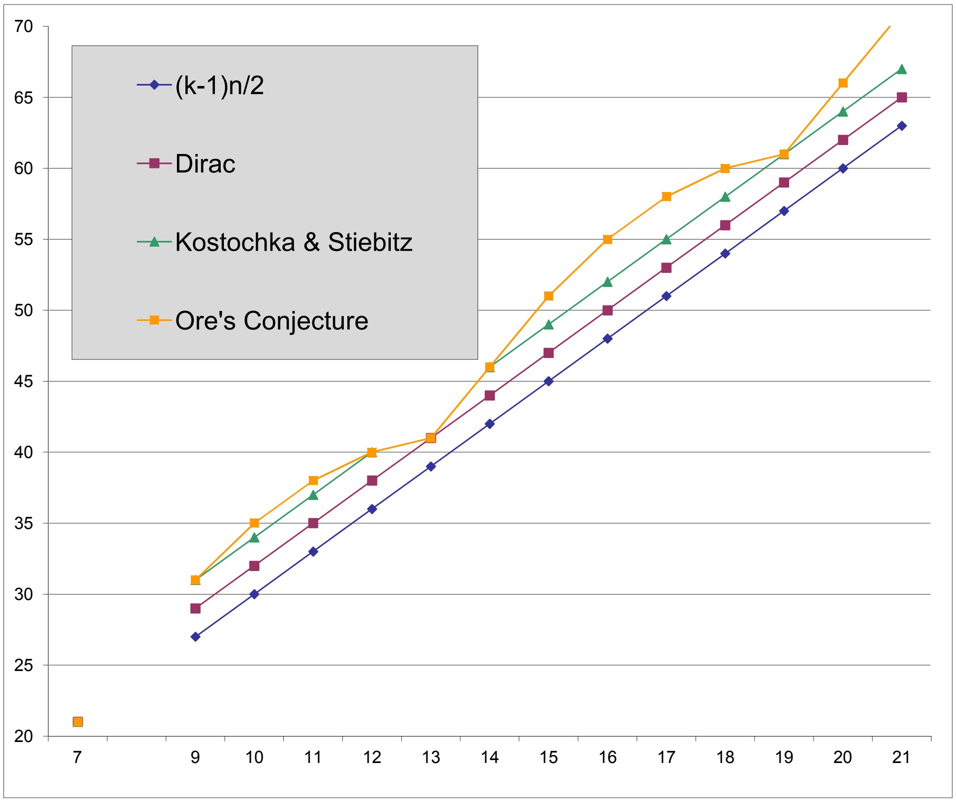


Figure 2.2: Comparison of Dirac-type bounds when $k = 7$.

best bound $\lim_{k\to\infty}\left(h'(k) - \frac{k-1}{2}\right) = \frac{1}{2}$. In 1963, Gallai [33] was able to prove several results in this direction, and conjectured that it is true under specific conditions. However, this 50-year-old conjecture remains open.

**Conjecture 2.1 (Gallai's Conjecture [33])** *If $n = 1 + t(k-1)$ for $t \in \mathbb{N}$, then*

$$f_k(n) = t\left(\binom{k}{2} - 1\right) + 1 = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}.$$

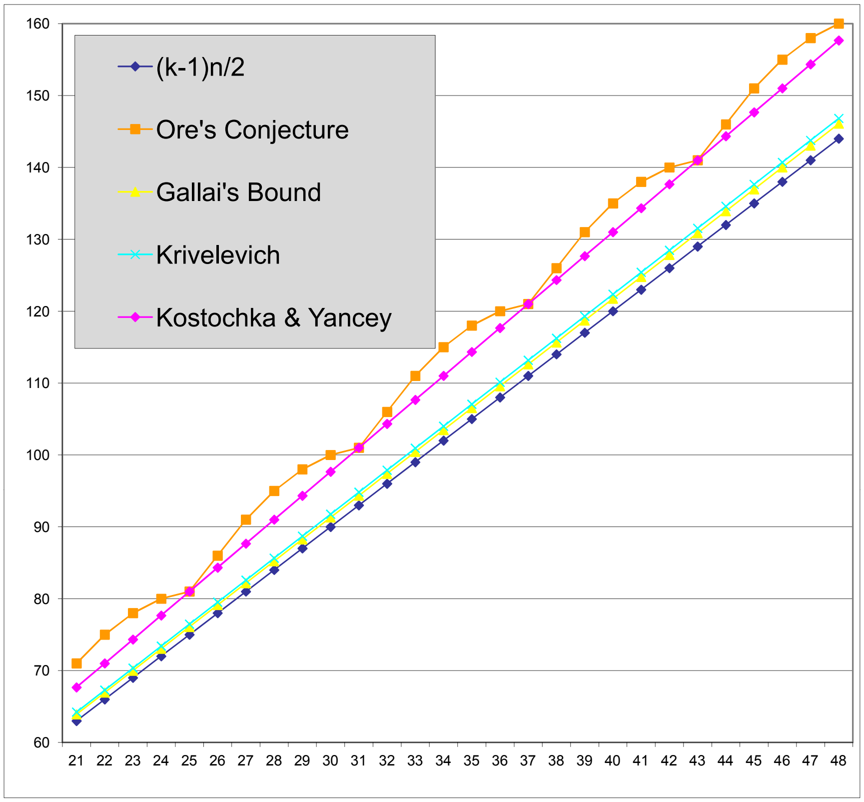


Figure 2.3: Comparison of Gallai-type bounds when $k = 7$.

The bound in Gallai's conjecture is matched by a specific construction, proving that the conjecture is sharp if true. In 1967, Gallai's conjecture was generalized by Ore to a conjecture on all $n$. Ore's conjecture is based on the construction in Gallai's conjecture, with varying initial conditions. Similar to Gallai's conjecture, there are known constructions for which Ore's conjecture is sharp.

**Conjecture 2.2 (Ore's Conjecture [63])** *If $n = k$ or $n \geq k+2$, then*

$$f_k(n) = \frac{(k+1)(k-2)}{2(k-1)}n + \epsilon(n,k),$$

*where $\epsilon(n+k-1,k) = \epsilon(n,k)$.*

Note that combining results from Gallai, Dirac, and Kostochka and Stiebitz, exact values for $f_k(n)$ are known when $k+2 \leq n \leq 2k$, and hence exact values for $\epsilon(n,k)$ are known in all cases.

For a fixed $k$, the function $\epsilon(n,k)$ is quadratic over $2k \leq n \leq 3k-2$. This suggests a rather beautiful nature to the conjectured values of $f_k(n)$, especially since the values of $f_k(n)$ have not been explicitly determined over one continuous period of $\epsilon(n,k)$, but rather pieced together from several periods. Another beautiful feature of the conjectured function for $f_k(n)$ is that for all $k$, $f_k(1) = 1$. Although this is considered outside the domain of $f_k(n)$ because we only consider simple graphs, one could consider one vertex and a loop a $k$-critical graph under certain relaxations of the definition.

There have been multiple constructions of sparse $k$-critical graphs. Hajós' Construction [39] is the most popular. We present a construction below, which is similar to Hajós' Construction. It has been altered to match our needs for further results.

**Construction 2.3** *An* Ore-composition *of graphs $G_1$ and $G_2$, called $O(G_1, G_2)$, is the graph obtained as follows:*

*(i) delete some edge $xy$ from $G_1$;*

*(ii) split some vertex $z$ of $G_2$ into two vertices $z_1$ and $z_2$ of positive degree;*

*(iii) glue $x$ with $z_1$ and $y$ with $z_2$.*

We attached Ore's name to it because we will use Construction 2.3 to make further progress on proving Ore's Conjecture. Note that Ore's Construction [63] is different than Ore-composition.

Ore observed that if $G_1$ and $G_2$ are $k$-critical and $G_2$ is not $k$-critical after $z$ has been split, then $O(G_1, G_2)$ also is $k$-critical. If $|N(z_1) \cap V(G_2)| \leq k-2$, then $G_2$ is not $k$-critical after $z$ has been split, and so all pairs of $k$-critical graphs have some valid split. Ore's Conjecture was specifically that if $f_k(|V(G_1)|) = |E(G_1)|$ and $G = O(G_1, K_k)$, then $f_k(|V(G)|) = |E(G)|$. By this construction with $G_2 = K_k$,

$$\epsilon(n+k-1, k) \leq \epsilon(n, k). \tag{2.1}$$

Conjecture 2.2 can be re-stated as that equality always holds in (2.1).

Recently, Kostochka and Yancey [55] proved the best possible Gallai-type bound. It proves that Conjecture 2.1 is true. It also proves a lower bound on $\epsilon(n, k)$, which implies that the recursive version of Ore's Conjecture, that (2.1) is equal in all cases, can be false for at most a finite number of values of $n$ for each $k$. It also proves that Ore's Conjecture is true when $k = 4$. It is also the first result to give exact values of $f_k(n)$ for infinitely many $n$ for each $k$. Figures 2.2 and 2.3 gives a comparison of Theorem 2.5 to Ore's conjecture and previous Gallai-type bounds.

**Definition 2.4** *Let* $F(k, n) = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}$.

**Theorem 2.5 ([55])** *If* $k \geq 4$ *and* $n \geq k$, $n \neq k+1$, *then*

$$f_k(n) \geq \lceil F(k, n) \rceil .$$

We define *k-Ore graphs*, $O_k$, recursively: $G \in O_k$ if $G = K_k$ or if $G = O(G_1, G_2)$ and $G_1, G_2 \in O_k$. It is easy to see that if $G \in O_k$, then $|V(G)| \equiv 1 (\bmod\, k-1)$ and $|E(G)| = F(k, |V(G)|)$. Kostochka and Yancey [55] also proved a Brooks-type extension of

Theorem 2.5: if $G$ is $k$-critical and $|E(G)| = F(k, |V(G)|)$, then $G \in O_k$.

**Theorem 2.6 ([56])** *Let $k \geq 4$ and let $G$ be a $k$-critical graph. If $G \notin O_k$, then $|E(G)| \geq \frac{(k^2-k-2)|V(G)|-\alpha_k}{2(k-1)}$, where $\alpha_k = \max\{2k-6, k^2-5k+2\}$. Furthermore, if $k \geq 4$ and $n \geq k$, with $n \neq k+1$ and $n \not\equiv 1 (\operatorname{mod} k-1)$, then*

$$f_k(n) \geq F(k, n) + z_k,$$

*where $z_4 = \frac{1}{3}$, $z_5 = \frac{3}{4}$, and $z_k = 1$ otherwise.*

The value of $z_k$ (and equivalently $\alpha_k$) in Theorem 2.6 is best possible. The family of graphs $O_k$ is defined using the Ore-composition, and graphs that can be produced by Ore-composition are characterized by having connectivity equal to two. On the other hand, for $k \geq 4$, there exists an infinite family of 3-connected $k$-critical graphs with $|E(G)| = \frac{(k^2-k-2)|V(G)|-\alpha_k}{2(k-1)}$. The idea of this construction and the examples for $k \in \{4, 5\}$ are due to Toft [76].

Kostochka and Yancey [55, 56] determined several consequences of Theorems 2.5 and 2.6. The proofs of the theorems exclusively use constructive arguments. With some additional work, they provide the core of a polynomial-time algorithm to properly color graphs that are sufficiently sparse. They also determine exact values of $f_k(n)$ and confirm Ore's Conjecture under several different sufficient conditions.

**Corollary 2.7 ([55])** *Let $G$ be a graph. If $k \geq 4$ and $|E(W)| < F(k, |V(W)|)$ for every subgraph $W \subseteq G$, then $G$ can be properly $(k-1)$-colored in $O(k^{3.5}n^{6.5}\log(n))$ time.*

**Corollary 2.8 ([56])** *Conjecture 2.2 holds and the value of $f_k(n)$ is known if*

- *$k \leq 5$,*

- *$k = 6$ and $n \equiv 0 (\operatorname{mod} 5)$,*

- $k = 6$ *and* $n \equiv 2 (\mathrm{mod}\, 5)$,

- $k = 7$ *and* $n \equiv 2 (\mathrm{mod}\, 6)$, *or*

- $n \equiv 1 (\mathrm{mod}\, k - 1)$.

In Section 2.1 we introduce notation and prove some basic results. In Section 2.2 we give an important lemma relating to list coloring, which is interesting on its own merits. In Section 2.3 we determine several structural properties of $k$-Ore graphs. The proofs of Theorems 2.5 and 2.6 are merged into one argument, which is given in three stages: Section 2.4 demonstrates that extra-sparse subgraphs have freedom in their coloring, Section 2.5 contains a collection of inductive arguments on specific structures, and Section 2.6 concludes by using discharging to show that any graph that satisfies the demonstrated properties cannot be sparse. Constructions demonstrating the sharpness of these theorems are given in Section 2.7, which includes a proof of Corollary 2.8. In Section 2.8 we outline the algorithm described in Corollary 2.7.

## 2.1 Preliminaries

**Definition 2.9** *For $R \subseteq V(G)$, define* the $k$-potential of $R$ *to be*

$$\rho_{k,G}(R) = (k-2)(k+1)|R| - 2(k-1)|E(G[R])|. \tag{2.2}$$

*When there is no chance for confusion, we will use $\rho_k(R)$. Let $P_k(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho_k, (R)$. We will also use the related parameter $\widetilde{P}_k(G)$, which is the minimum of $\rho_k(W)$ over all $W \subset V(G)$ such that $2 \leq |W| \leq |V(G)| - 1$.*

**Fact 2.10** *For the $k$-potential defined by (2.2), we have*

1. $\rho_{k,K_k}(V(K_k)) = k(k-3)$,

*2.* $\rho_{k,K_1}(V(K_1)) = (k-2)(k+1)$,

*3.* $\rho_{k,K_2}(V(K_2)) = 2(k^2-2k-1)$,

*4.* $\rho_{k,K_{k-1}}(V(K_{k-1})) = 2(k-2)(k-1)$.

We restate Theorems 2.5 and 2.6 in the following equivalent manner.

**Theorem 2.11** *Let $\alpha_k = \max\{2k-6, k^2-5k+2\}$. If $k \geq 4$ and $G$ is a $k$-critical graph, then $\rho_k(V(G)) \leq k(k-3)$. If $G \notin O_k$ then $\rho_k(V(G)) \leq \alpha_k$.*

Now, we prove some basic results.

**Definition 2.12** *For a graph $G$, a set $R \subset V(G)$ and a proper $(k-1)$-coloring $\phi$ of $G[R]$, the graph $Y(G,R,\phi)$ is constructed as follows. First, for $1 \leq i \leq k-1$, let $R_i'$ denote the set of vertices in $V(G)-R$ adjacent to at least one vertex $v \in R$ with $\phi(v)=i$. Second, let $X = \{x_1, \dots, x_{k-1}\}$ be a set of new vertices disjoint from $V(G)$. Now, let $Y(G,R,\phi)$, or just $Y$, be the graph with vertex set $V(G)-R+X$, such that $Y[V(G)-R] = G-R$ and $N(x_i) = R_i' \cup (\{x_1,\dots,x_{k-1}\} - x_i)$ for $1 \leq i \leq k-1$.*

**Claim 2.13** *Let $\chi(G) \geq k$. If $R \subset V(G)$ and $\phi$ is a proper $(k-1)$-coloring of $G[R]$, then $\chi(Y(G,R,\phi)) \geq k$.*

**Proof.** Let $G' = Y(G,R,\phi)$. Let $\phi' : V(G') \to C$ be a proper $(k-1)$-coloring. By construction of $G'$, the colors of all $x_i$ in $\phi'$ are distinct. By changing the names of the colors, we may assume that $\phi'(x_i) = i$ for $1 \leq i \leq k-1$. By construction of $G'$, we have $\phi'(u) \neq i$ for $u \in R_i'$. Therefore $\phi|_R \cup \phi'|_{V(G)-R}$ is a proper $(k-1)$-coloring of $G$, which contradicts the assumption $\chi(G) \geq k$.. $\square$

**Claim 2.14** *Let $R \subset V(G)$, $\phi$ be a proper $(k-1)$-coloring of $G[R]$, and let $G' = Y(G, R, \phi)$. Let $W \subseteq V(G')$. If $W \cap X = \{x_{i_1}, \ldots, x_{i_q}\}$ and $R|_W$ denotes the set of vertices $v \in R_*$ such that $\phi(v) \in \{i_1, \ldots, i_q\}$, then*

$$\rho_{k,G}(W-X+R) = \rho_{k,G'}(W)-\rho_{k,G'}(W\cap X)+\rho_{k,G}(R)-2(k-1)|E_G(W-X, R-R|_W)|. \quad (2.3)$$

**Proof.** Since $\rho_{k,G}(U)$ is a linear combination of the numbers of vertices and edges in $G[U]$, it is enough to check that every vertex and edge of $G[W - X + R]$ has a net affect of one appearance in the RHS of (2.3) and the every other vertex or edge has a net affect of zero appearances.

The weight of every vertex and edge of $G'[W\cap X]$ appears once positively by $\rho_{k,G'}(W)$ and once negatively by $-\rho_{k,G'}(W\cap X)$, and therefore has zero net effect. Edges in $E_G(W-X, R_W)$ are counted once positively by $\rho_{k,G'}(W)$, and edges in $E_G(W-X, R-R_W)$ are counted once by $-2(k-1)|E_G(W-X, R-R|_W)|$. Hence, edges in $E(W-X, R)$ are counted once. The weight of every vertex and edge of $G'[W-X]$ is counted once, as is every vertex and edge in $G[R]$. Note that $G'[W-X] = G[W-X]$. Because the vertices and edges in $G[W-X+R]$ are the union of vertices and edges of $G[R]$, $G[W-X]$ and $E(W-X, R-R_W)$, we have that the edges and vertices of $G[W-X+R]$ are counted exactly once.

□

## 2.2 Orientations and List Colorings

A *digraph*, or directed graph, is a set of vertices $V$ and arcs $E$, where each arc is an ordered pair of vertices. The *out-neighbors* of a vertex $u$, denoted $N^+(u)$, is the set $\{w \in V(G) : uw \in E(G)\}$. The *out-degree* of $u$, denoted $d^+(u)$, is $|N^+(u)|$. The *in-neighbors* of a vertex $u$, denoted $N^-(u)$, is the set $\{w \in V(G) \ : \ wu \in E(G)\}$. The *neighbors* of a vertex $u$, denoted $N(u)$, is $N^+(u) \cup N^-(u)$. A digraph $G$ is an *orientation* of a multigraph $G'$ if $V(G) = V(G')$ and there is a bijection between the arcs $uw, wu \in E(G)$ and edges $uw \in E(G')$.

Recall that a *list assignment* for a graph $G$ is a mapping $L$ of $V(G)$ into the family of finite subsets of $\mathbf{N}$. For a given list $L$, a graph $G$ is *$L$-colorable*, if there exists a coloring $f \ : \ V(G) \to \mathbf{N}$ such that $f(v) \in L(v)$ for every $v \in V(G)$ and $f(v) \neq f(u)$ for every $uv \in E(G)$.

We consider loopless digraphs. A *kernel* in a digraph $D$ is an independent set $F$ of vertices such that each vertex in $V(D) - F$ has an out-neighbor in $F$. A digraph $D$ is *kernel-perfect* if for every $A \subseteq V(D)$, the induced subdigraph $D[A]$ has a kernel. It is known that kernel-perfect orientations form a useful tool for list colorings. The following fact is well known but we include its proof for completeness.

**Lemma 2.15 (Bondy, Boppana, Siegel (see [5]))** *If $D$ is a kernel-perfect digraph and $L$ is a list assignment such that*

$$|L(v)| \geq 1 + d^+(v) \qquad \textit{for every } v \in V(D), \tag{2.4}$$

*then $D$ is $L$-colorable.*

**Proof.** We use induction on $|V(D)|$. When $D$ has only one vertex, the statement is trivial. Suppose the statement holds for all pairs $(D', L')$ satisfying (2.4) with $|V(D')| \leq n - 1$. Let $|V(D)| = n$, and let $(D, L)$ satisfy (2.4). Let $v \in V(D)$, and let $\alpha$ be a color present in $L(v)$. Let $V_\alpha = \{x \in V(D) : \alpha \in L(x)\}$. Since $D$ is kernel-perfect, $D[V_\alpha]$ has a kernel $K$. Color all vertices of $K$ with $\alpha$ and consider $(D', L')$, where $D' = D - K$ and $L'(y) = L(y) - \alpha$ for all $y \in V(D')$. Since the outdegree of every $x \in V_\alpha - K$ decreased by at least 1, $(D', L')$ satisfies (2.4), and so by the induction hypothesis $D'$ has an $L'$-coloring. Together with coloring of

$K$ by $\alpha$, this yields an $L$-coloring of $D$, as claimed. $\square$

Richardson's Theorem [65] states that every orientation of a bipartite multigraph is kernel-perfect. We prove a somewhat stronger result.

**Lemma 2.16** *Let $A$ be an independent set in a graph $G$, and let $B = V(G) - A$. Let $D$ be the digraph obtained from $G$ by replacing each edge in $G[B]$ by two opposite arcs and by giving an arbitrary orientation to the edges joining $A$ and $B$. The digraph $D$ is kernel-perfect.*

**Proof.** Let $D$ be a counterexample with the fewest vertices. If every $b \in B$ has an outneighbor in $A$, then $A$ is a kernel. Otherwise, some $b \in B$ has no outneighbors in $A$. This yields $N(b) = N^-(b)$. We consider $D' = D - b - N^-(b)$. By the minimality of $D$, the digraph $D'$ has a kernel $K$. Now $K + b$ is a kernel of $D$. $\square$

For a graph $G$ and disjoint vertex subsets $A$ and $B$, let $G(A, B)$ denote the bipartite graph with partite sets $A$ and $B$ whose edges are all edges of $G$ with one endpoint in $A$ and the other in $B$. The main result of this section is the following.

**Lemma 2.17** *Let $G$ be a $k$-critical graph. Let disjoint vertex subsets $A$ and $B$ be such that*
*(a) at least one of $A$ and $B$ is independent;*
*(b) $d_G(a) = k - 1$ for every $a \in A$;*
*(c) $d_G(b) = k$ for every $b \in B$.*
*Under these conditions, (i) $\delta(G(A, B)) \leq 2$ and*
*(ii) either some $a \in A$ has at most one neighbor in $B$ or some $b \in B$ has at most three neighbors in $A$.*

**Proof.** Suppose $G$ is a counterexample. If $A \cup B = \emptyset$, then both statements are trivial. Otherwise, since $G$ is $k$-critical, there exists a proper $(k-1)$-coloring $f$ of $G - A - B$. Fix any such $f$. We will show that this proper coloring extends to all of $G$, by list coloring the graph induced on $A \cup B$ in a way that avoids conflict with $f$.

For every $x \in A \cup B$, let $L(x)$ be the set of colors in $[k-1]$ not used in $f$ on neighbors of $x$. Letting $G' = G[A \cup B]$, we have

$$\text{for every } a \in A,\ |L(a)| \geq d_{G'}(a), \text{ and for every } b \in B,\ |L(b)| \geq d_{G'}(b) - 1. \tag{2.5}$$

CASE (i): $\delta(G(A,B)) \geq 3$. Let $G''$ be obtained from $G(A,B)$ by splitting each $b \in B$ into $\lceil d_{G(A,B)}(b)/3 \rceil$ vertices of degree at most 3. In particular, a vertex $b$ of degree 3 in $G(A,B)$ is not split. The graph $G''$ is bipartite with partite sets $A$ and $B'$, where $B'$ is obtained from $B$. By the hypothesis, the degree of each $a \in A$ in $G''$ is at least 3. By the splitting procedure, the degree of each vertex $b \in B'$ is at most 3. It follows that if $S \subseteq A$, then $|N_{G''}(S)| \geq |S|$. By Hall's Theorem, $G''$ has a matching $M$ covering $A$. We construct a digraph $D$ from $G'$ as follows:

(1) replace each edge of $G[B]$ or in $G[A]$ (whichever is nonempty) with two opposite arcs,

(2) orient every edge of $G(A,B)$ corresponding to an edge in $M$ toward $A$,

(3) orient all other edges of $G(A,B)$ toward $B$.

By Lemma 2.16, $D$ is kernel-perfect. Moreover, by (2.5), for every $a \in A$, $d^+_D(a) = d_{G'}(a) - 1 \leq |L(a)| - 1$, and for every $b \in B$,

$$d^+_D(b) \leq d_{G'}(b) - \lfloor \frac{2}{3} d_{G(A,B)}(b) \rfloor \leq (|L(b)| + 1) - 2 = |L(b)| - 1.$$

Thus by Lemma 2.15, $G'$ has $L$-coloring $f'$. This means that $f \cup F'$ is a proper $(k-1)$-coloring of $G$, which is a contradiction. This proves (i).

CASE (ii): Each $a \in A$ has at least two neighbors in $B$ and each $b \in B$ has at least four neighbors in $A$. Obtain $G''$ by splitting each $b \in B$ into $\lceil d_{G(A,B)}(b)/2 \rceil$ vertices of degree at most 2. Similarly to Case 1, the graph $G''$ is bipartite with partite sets $A$ and $B'$, where $B'$

is obtained from $B$. The degree of each $a \in A$ in $G''$ is at least 2, and the degree of each vertex $b \in B'$ is at most 2. Again by Hall's Theorem, $G''$ has a matching $M$ covering $A$. We construct a digraph $D$ from $G''$ according to Rules (1)–(3) in Case 1. Again, by Lemma 2.16, $D$ is kernel-perfect, and by (2.5), for every $a \in A$, $d_D^+(a) = d_{G'}(a) - 1 \leq |L(a)| - 1$. For every $b \in B$, since $d_{G(A,B)}(b) \geq 4$, by (2.5),

$$d_D^+(b) \leq d_{G''}(b) - \lfloor \frac{1}{2} d_{G(A,B)}(b) \rfloor \leq (|L(b)| + 1) - 2 = |L(b)| - 1.$$

This proves (ii) $\square$

**Corollary 2.18** *Let $G$ be a $k$-critical graph. Let disjoint vertex subsets $A$ and $B$ be such that*

*(a) $A$ or $B$ is independent;*

*(b) $d_G(a) = k - 1$ for every $a \in A$;*

*(c) $d_G(b) = k$ for every $b \in B$;*

*(d) $|A| + |B| \geq 3$.*

*Under these conditions, (i) $|E(G(A,B))| \leq 2(|A| + |B|) - 4$ and (ii) $|E(G(A,B))| \leq |A| + 3|B| - 3$.*

**Proof.** First we prove (i) by induction on $|A|+|B|$. If $|A|+|B| = 3$, then since $G(A,B)$ is bipartite, it has at most $2 = 2 \cdot 3 - 4$ edges. Suppose now that $|A| + |B| = m \geq 4$ and that the corollary holds for $3 \leq |A|+|B| \leq m-1$. By Lemma 2.17(i), $G(A,B)$ has a vertex $v$ of degree at most 2. By the minimality of $m$, $G(A,B) - v$ has at most $2(m-1) - 4$ edges. Now $|E(G(A,B))| \leq 2 + 2(m-1) - 4 = 2m - 4$, as claimed.

The base case $|A| + |B| = 3$ for (ii) is slightly more complicated. If $|A| = 3$, then $|E(G(A,B))| = 0 = |A| + 3|B| - 3$. If $|B| \geq 1$, then $|A| + 3|B| \geq 5$ and $|E(G(A,B))| \leq 2 = 5 - 3 \leq |A| + 3|B| - 3$. The proof of the induction step is very similar to the previous paragraph, using Lemma 2.17(ii). $\square$

## 2.3 Ore Graphs

The next fact is a restatement of the definition of $k$-Ore graphs.

**Fact 2.19** *Fix $k$ with $k \geq 3$. Every $k$-Ore graph $G \neq K_k$ has a separating set $\{x, y\}$ and two vertex subsets $A$ and $B$ such that*
*(i) $A \cap B = \{x, y\}$, $A \cup B = V(G)$, and no edge of $G$ connects $A - x - y$ with $B - x - y$,*
*(ii) the graph $\widetilde{G}(x, y)$ obtained from $G[A]$ by adding edge $xy$ is a $k$-Ore graph, and*
*(iii) the graph $\widehat{G}(x, y)$ obtained from $G[B]$ by merging $x$ with $y$ into a new vertex $w$ is a $k$-Ore graph.*

In this case, $G$ is an Ore-composition of $\widetilde{G}(x, y)$ and $\widehat{G}(x, y)$, and we will say that $\widetilde{G}(x, y)$ and $\widehat{G}(x, y)$ are *$x, y$-children* (or simply *children*) of $G$. Moreover, $\widetilde{G}(x, y)$ will be always the *first child* and $\widehat{G}(x, y)$ will be the *second child*.

**Claim 2.20** *For every $k$-Ore graph $G$ and every nonempty $R \subsetneq V(G)$, we have $\rho_{k,G}(R) \geq (k + 1)(k - 2)$.*

**Proof.** Let $G$ be a smallest counterexample to the claim. If $G = K_k$, then the statement immediately follows from Fact 2.10. For $G \neq K_k$, let a separating set $\{x, y\}$ and two vertex subsets $A$ and $B$ be as in Fact 2.19. Let $R$ have the smallest size among nonempty proper subsets of $V(G)$ with connected $G[R]$ and $\rho_{k,G}(R) < (k+1)(k-2)$. If $\rho_{k,G}(R') < (k+1)(k-2)$ and $G[R']$ is disconnected, then the vertex set of some component of $G[R']$ also has potential less than $(k+1)(k-2)$. So, such $R$ exists. Since $\rho_{k,G}(R) < (k+1)(k-2)$ and $R$ is non-empty, $|R| \geq k$.

CASE 1: $\{x, y\} \cap R = \emptyset$. Since $G[R]$ is connected, $R$ is a non-empty proper subset either of $V(\widetilde{G}(x, y))$ or of $V(\widehat{G}(x, y))$. By the minimality of $G$, the potential of $R$ is at least $(k + 1)(k - 2)$, a contradiction.

CASE 2: $\{x, y\} \cap R = \{x\}$. The set $R \cap A$ induces a non-empty connected subgraph of $G$. By the minimality of $|R|$, $\rho_{k,G}(R \cap A) \geq (k+1)(k-2)$. Similarly, $\rho_{k,G}(R \cap B) \geq (k+1)(k-2)$. Now

$$\rho_{k,G}(R) = \rho_{k,G}(R \cap A) + \rho_{k,G}(R \cap B) - \rho_{k,G}(K_1) \geq (k + 1)(k - 2),$$

a contradiction.

CASE 3: $\{x, y\} \subseteq R$. If $A \subset R$, then $\rho_{k,\widehat{G}(x,y)}((R - A) + x * y) = \rho_{k,G}(R)$. By the minimality of $G$, this is at least $(k + 1)(k - 2)$, a contradiction. Similarly, if $B \subseteq R$, then $\rho_{k,\widetilde{G}(x,y)}(R \cap A) = \rho_{k,G}(R)$, a contradiction again. So, suppose $A - R \neq \emptyset$ and $B - R \neq \emptyset$. Now $\rho_{k,\widetilde{G}(x,y)}(R \cap A) \geq (k+1)(k-2)$. Since $xy$ is an edge in $\widetilde{G}(x, y)$ but not in $G$, this yields $\rho_{k,G}(R \cap A) \geq (k + 1)(k - 2) + 2(k - 1)$. Similarly, $\rho_{k,\widehat{G}(x,y)}((R - A) + x * y) \geq (k + 1)(k - 2)$ and thus $\rho_{k,G}(R \cap B) \geq 2(k + 1)(k - 2)$. Now

$$\rho_{k,G}(R) = \rho_{k,G}(R \cap A) + \rho_{k,G}(R \cap B) - 2\rho_{k,G}(K_1) \geq (k + 1)(k - 2) + 2(k - 1),$$

a contradiction. $\square$

**Lemma 2.21** *Let $G$ be a $k$-Ore graph. Let $uv$ be an edge in $G$ such that $\rho_{k,G-uv}(W) > (k+1)(k-2)$ for every $W \subseteq V(G-uv)$ with $2 \leq |W| \leq |V(G)|-1$. For each $w \in V(G)-u-v$, there is a proper $(k - 1)$-coloring $\phi_w$ of $G - uv$ such that $\phi_w(w) \neq \phi_w(u) = \phi_w(v)$.*

**Proof.** We use induction on $|V(G)|$. For $G = K_k$, the statement is evident. Otherwise, let $x, y, A, B, \widetilde{G}(x, y)$ and $\widehat{G}(x, y)$ be as in Fact 2.19. Now $\rho_{k,G}(A) = k(k - 3) + 2(k - 1) = (k + 1)(k - 2)$. Thus by the definition of $uv$, $\{u, v\} \subset A$.

CASE A: $w \in A$. By the induction assumption, there exists a proper $(k-1)$-coloring $\phi'_w$ of $\widetilde{G}(x,y) - uv$ such that $\phi'_w(w) \neq \phi'_w(u) = \phi'_w(v)$. Since $\phi'_w(x) \neq \phi'_w(y)$, this coloring extends to a proper $(k-1)$-coloring of $G$.

CASE B: $w \in B - x - y$. Let $\phi'$ be any proper $(k-1)$-coloring of $\widetilde{G}(x,y) - uv$. Since $\widetilde{G}(x,y)$ is $k$-critical, $\phi'(u) = \phi'(v)$.

CASE B1: $\phi'(u) = \phi'(x)$. Let $G_0 = G[B] + xw$. Since $u, v \notin V(G_0)$, under conditions of the lemma, $\rho_{k,G_0}(W) > (k+1)(k-2) - 2(k-1) = k(k-3)$ for every nonempty $W \subseteq V(G_0)$. Thus $G_0$ has a proper $(k-1)$-coloring $\phi''$. Since it is also a proper $(k-1)$-coloring of $G[B]$, we have $\phi''(x) \neq \phi''(y)$. Renaming the colors in $\phi''$ so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, we obtain a proper $(k-1)$-coloring $\phi = \phi' \cup \phi''$ with $\phi(u) = \phi(x) \neq \phi(w)$.

CASE B2: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k \geq 5$. Take any proper $(k-1)$-coloring $\phi''$ of $G[B]$ such that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$. If $\phi''(w) \in \{\phi''(x), \phi''(y)\}$, then $\phi = \phi' \cup \phi''$ is what we need. Otherwise, since $k-1 \geq 4$, we can change the names of colors in $\phi''$ so that $\phi''(w) \neq \phi'(u)$ and again take $\phi = \phi' \cup \phi''$.

CASE B3: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k = 4$. Let $G_0$ be obtained from $G[B]$ by adding a new vertex $z$ adjacent to $x, y$ and $w$. If $G_0$ has a proper 3-coloring $\phi''$, then $z$ has the color distinct from $\phi''(x)$ and $\phi''(y)$, and thus $\phi''(w) \in \{\phi''(x), \phi''(y)\}$. In this case renaming the names of colors in $\phi''$ so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, yields a proper $(k-1)$-coloring of $G$ as required. Therefore, $G_0$ contains a 4-critical subgraph $G_1$. Since $G_1$ is not a subgraph of $G$, we have $z \in V(G_1)$. Since $\delta(G_1) \geq 3$, we have $\{x, y, w\} \subset V(G_1)$. Let $W = V(G_1)$. Since $\rho_{4,G_0}(W) \leq 4$, we have $\rho_{4,G}(W - z) \leq 12$ and $\{x, y, w\} \subset W - z$. Now

$$\rho_{4,G}(A \cup W - z) \leq \rho_{4,G}(A) + \rho_{4,G}(W - z) - 2\rho_4(K_1) \leq 10 + 12 - 20 = 2,$$

a contradiction to Claim 2.20. $\square$

A set $A$ of vertices in a $k$-Ore graph $G$ is *standard* if

(a) $\rho_k(A) = (k+1)(k-2)$ and

(b) $G$ has a separating set $\{x, y\}$ such that the graph $\widetilde{G}(x, y)$ (defined in Fact 2.19) has vertex set $A$.

**Lemma 2.22** *Let $G$ be a $k$-Ore graph. If $W \subset V(H)$ with $|W| \geq 2$, and $\rho_k(W) = (k+1)(k-2)$, then $W$ is the union of standard sets and $G[W]$ is connected.*

**Proof.** The second part of the statement follows directly from Claim 2.20. We prove the first part by induction on the order of $G$. The graph $K_k$ simply does not have sets $W$ with $|W| \geq 2$ and $\rho_k(W) = (k+1)(k-2)$. Let $G$ be a smallest $k$-Ore graph containing $W \subset V(G)$ with $|W| \geq 2$ and $\rho_k(W) = (k+1)(k-2)$ that is not the union of standard sets.

Suppose first that $W$ contains a standard set $A$ and that separating set $\{x, y\}$ separates $A - x - y$ from $V(G) - A$. Let $\widehat{G}$ be the second child of $G$ at $\{x, y\}$; that is, $\widehat{G}$ is obtained from $G - (A - x - y)$ by merging $x$ and $y$ into a new vertex $w$. Let $W'' = W - A + w$. Now $\widehat{G}$ is a $k$-Ore graph and $W''$ is a non-empty proper subset of $V(\widehat{G})$. Furthermore, $\rho_{k,\widehat{G}}(W'') = \rho_{k,G}(W) - \rho_{k,G}(A) + (k+1)(k-2) = \rho_{k,G}(W) = (k+1)(k-2)$. If $|W''| = 1$, then $W = A$ is a standard set. Otherwise, by the minimality of $G$, the set $W''$ is the union of standard sets in $\widehat{G}$. This makes $W$ the union of standard sets in $G$. Thus we may assume that

$$W \text{ does not contain any standard set.} \tag{2.6}$$

Since $G \neq K_k$, let $x, y, w, A, B, \widetilde{G}(x, y)$ and $\widehat{G}(x, y)$ be as in Fact 2.19.

CASE 1: $W \cap \{x, y\} = \emptyset$. By the minimality of $G$, the set $W$ is the union of standard sets either in $\widetilde{G}(x, y)$ or in $\widehat{G}(x, y)$. In either case, $W$ is then the union of standard sets in $G$, a contradiction.

CASE 2: $W \cap \{x, y\} = \{x\}$. Let $W_1 = W \cap A$ and $W_2 = W \cap B$. Note that $W_2' = W_2 - x + w$ is a non-empty subset of $V(\widehat{G}(x, y))$ and that $W_1$ is a non-empty proper subset of $V(\widetilde{G}(x, y))$.

So, $\rho_{k,\widetilde{G}(x,y)}(W_1) \geq (k+1)(k-2)$. If equality holds, then $W_1$ either consists only of $x$ or is the union of standard sets (which also are standard in $G$). Now

$$\rho_{k,\widehat{G}(x,y)}(W_2') = \rho_{k,G}(W_2) = \rho_{k,G}(W) - \rho_{k,G}(W_1) + \rho_k(K_1) = 2(k+1)(k-2) - \rho_{k,G}(W_1).$$

By Claim 2.20, we conclude that $\rho_{k,G}(W_1) = (k+1)(k-2)$. By (2.6), $W_1$ is not the union of standard sets. So, $W_1 = \{x\}$. Observe that

$$y \text{ has no neighbors in } W, \tag{2.7}$$

since otherwise $\rho_{k,\widehat{G}(x,y)}(W_2') \leq \rho_{k,G}(W_2) - 2(k-1) = k^2 - 3k$, a contradiction to Claim 2.20. If $W_2' \neq V(\widehat{G}(x,y))$, then $W_2'$ is the union of standard sets in $\widehat{G}(x,y)$, a contradiction to (2.6). If $W_2' = V(\widehat{G}(x,y))$, then $y$ does have a neighbor in $W_2$, a contradiction to (2.7).

CASE 3: $W \cap \{x,y\} = \{x,y\}$. Again, let $W_1 = W \cap A$ and $W_2 = W \cap B$. Since $\{x,y\} \subset W_1$, and (2.6) yields $W_1 \neq V(\widetilde{G}(x,y))$, we have $\rho_{k,\widetilde{G}(x,y)}(W_1) \geq (k+1)(k-2)$. Moreover, by the minimality of $G$, if $\rho_{k,\widetilde{G}(x,y)}(W_1) = (k+1)(k-2)$, then $W_1$ is the union of standard sets in $\widetilde{G}(x,y)$. Let $W_2' = W_2 - x - y + w$. If $W_2' \neq V(\widehat{G}(x,y))$, then $\rho_{k,\widehat{G}(x,y)}(W_2') \geq (k+1)(k-2)$, so

$$\begin{aligned}
\rho_{k,G}(W) &= \rho_{k,G}(W_1) + \rho_{k,G}(W_2) - 2(k+1)(k-2) \\
&\geq \Big(\rho_{k,\widetilde{G}(x,y)}(W_1) + 2k - 2\Big) + \Big(\rho_{k,\widehat{G}(x,y)}(W_2') + (k+1)(k-2)\Big) - 2(k+1)(k-2) \\
&\geq k^2 + k - 4,
\end{aligned}$$

a contradiction to the choice of $W$. So, $W_2' = V(\widehat{G}(x,y))$ and $\rho_{k,\widehat{G}(x,y)}(W_2') = k^2 - 3k$. If $\rho_{k,\widetilde{G}(x,y)}(W_1) \geq (k+1)(k-2) + 2$, then again

$$\rho_{k,G}(W) \geq (\rho_{k,\widetilde{G}(x,y)}(W_1) + 2k - 2) + (\rho_{k,\widehat{G}(x,y)}(W_2') + (k+1)(k-2)) - 2(k+1)(k-2) \geq k^2 - k,$$

a contradiction. Thus $\rho_{k,\widetilde{G}(x,y)}(W_1) = (k+1)(k-2)$, and $W_1$ is the union of standard sets in $\widetilde{G}(x,y)$. A standard set in $\widetilde{G}(x,y)$ not containing $\{x,y\}$ is also a standard set in $G$. By (2.6), $W_1$ is standard in $\widetilde{G}(x,y)$ by itself. Now $W$, which equals $W_1 \cup W_2$, is standard in $G$, as claimed. $\square$

**Claim 2.23** *Let $G$ be a $k$-Ore graph. Let $u$ be a vertex in $G$ such that $\rho_{k,G}(W) > (k+1)(k-2)$ for every $W \subseteq V(G) - u$ with $|W| \geq 2$. There exists $S \subseteq V(G) - u$ such that (1) $G[S] \cong K_{k-1}$, (2) $d_G(v) = k-1$ for all $v \in S$, and (3) $(N(S) - S)$ is an independent set.*

**Proof.** We use induction on $|V(G)|$. For $G = K_k$, the statement is evident. Otherwise, let $x, y, w, A, B, \widetilde{G}(x,y)$ and $\widehat{G}(x,y)$ be as in Fact 2.19. Now $\rho_{k,G}(A) = (k+1)(k-2)$, and hence $\{u\} \subset A$.

By assumption, if there exists $W \subseteq V(\widehat{G}(x,y))$ such that $\rho_{k,\widehat{G}(x,y)}(W) \leq (k+1)(k-2)$, then $w \in W$. By the induction hypothesis, there exists $S \subseteq V(\widehat{G}(x,y)) - w$ such that (1) $\widehat{G}(x,y)[S] \cong K_{k-1}$, (2) $d_{\widehat{G}(x,y)}(v) = k-1$ for all $v \in S$, and (3) $(N(S) - S)$ is an independent set in $\widehat{G}(x,y)$. By construction, (1) $S \subseteq V(G) - u$, (2) $G[S] \cong K_{k-1}$, and (3) $(N_G(S) - S)$ is an independent set in $G$. Furthermore, $N_G(x) \cap N_G(y) \cap (B - x - y) = \emptyset$, and $(N_G(x) \cup N_G(y)) \cap (B - x - y) = N_{\widehat{G}(x,y)}(w)$, and $E_G(A - x - y, B - x - y) = \emptyset$, so $d_G(v) = k-1$ for all $v \in S$. $\square$

## 2.4 Sets with Small Potential

A graph $H$ is *smaller than* graph $G$, if either $|E(G)| > |E(H)|$, or $|E(G)| = |E(H)|$ and $G$ has fewer pairs of vertices with the same closed neighborhood. If $|V(G)| \geq |V(H)|$, $\rho_{k,G}(V(G)) \leq \rho_{k,H}(V(H))$, and equality does not hold in both cases, then $H$ is smaller than $G$.

Let $G$ be the smallest graph with respect to our order that contradicts Theorem 2.11. Clearly, we may assume that $G$ is not $k$-Ore. Let $n = |V(G)|$. This implies that

$$\text{if } H \text{ is smaller than } G \text{ and } P_k(H) > k(k-3), \text{ then } H \text{ is } (k-1)\text{-colorable.} \tag{2.8}$$

Recall Definition 2.9: $\rho_{k,G}(R) = (k-2)(k+1)|R| - 2(k-1)|E(G[R])|$ for $R \subseteq V(G)$. For each $v \in V(G)$,

$$\rho_k(V(G)-v) = \rho_k(V(G)) - (k^2-k-2) + 2(k-1)d(v) > \begin{cases} \alpha_k + k^2 - 3k + 4 & \text{if } d(v) = k-1, \\ \alpha_k + k^2 - k + 2 & \text{if } d(v) \geq k \end{cases} \tag{2.9}$$

**Claim 2.24** *There is no nonempty $R \subsetneq V(G)$ with $\rho_k(R) \leq \alpha_k + 2k - 2$.*

**Proof.** Let $R$ have the smallest potential among nonempty proper subsets of $V(G)$. Let $\rho_k(R) = m$. Since $m \leq \alpha_k + 2k - 2 < k^2 - k - 2$, we have $|R| \geq k$. Since $G$ is $k$-critical, $G[R]$ has a proper $(k-1)$-coloring $\phi$. Let $G' = Y(G, R, \phi)$. By Claim 2.13, $\chi(G') \geq k$. Then $G'$ contains a $k$-critical graph $G''$. By the minimality of $G$, $\rho_{k,G'}(V(G'')) \leq k^2 - 3k$. Since $G$ is $k$-critical, $V(G'') \cap X \neq \emptyset$. Let $Z = V(G'') - X + R$. Since every non-empty subset of $X$ has potential at least $k^2 - k - 2$, by (2.3),

$$\rho_{k,G}(Z) \leq \rho_{k,G'}(V(G'')) - \rho_{k,G'}(V(G'') \cap X) + m \tag{2.10}$$

$$\leq k^2 - 3k - (k^2 - k - 2) + m = m - 2k + 2.$$

Since $Z \supset R$, it is nonempty. By the minimality of the potential of $R$, we have $Z = V(G)$. Now $\rho_k(V(G)) \leq (\alpha_k + 2k - 2) - 2k + 2 = \alpha_k$, a contradiction. $\square$

**Claim 2.25** *$G$ is 3-connected.*

**Proof.** Suppose that $G$ has a separating set $\{x, y\}$ and two vertex subsets $A$ and $B$ such that $A \cap B = \{x, y\}$, $A \cup B = V(G)$, and no edge of $G$ connects $A - x - y$ with $B - x - y$. Since $G$ is $k$-critical, there are proper $(k-1)$-colorings $\phi_A$ and $\phi_B$ of $G[A]$ and $G[B]$, respectively. Since $G$ has no proper $(k-1)$-coloring, by symmetry, we may assume that $\phi_A(x) = \phi_A(y)$ and $\phi_B(x) \neq \phi_B(y)$. Moreover, since $G$ is $k$-critical, for each edge $e \in E(G[A])$, the graph $G[A] - e$ has a proper $(k-1)$-coloring $\phi_{A,e}$ such that $\phi_{A,e}(x) \neq \phi_{A,e}(y)$. Similarly, for each edge $e \in E(G[B])$, the graph $G[B] - e$ has a proper $(k-1)$-coloring $\phi_{B,e}$ such that $\phi_{B,e}(x) = \phi_{B,e}(y)$. It follows that the graph $\widetilde{G}$ obtained from $G[A]$ by inserting edge $xy$ and the graph $\widehat{G}$ obtained from $G[B]$ by merging $x$ and $y$ are $k$-critical. Both graphs are smaller, so by minimality, $\rho_{k,\widetilde{G}}(V(\widetilde{G}) \leq k^2 - 3k$ and $\rho_{k,\widehat{G}}(V(\widehat{G})) \leq k^2 - 3k$. Moreover, by the minimality of $G$, if $\widetilde{G}$ (respectively, $\widehat{G}$) is not a $k$-Ore graph, then the potential of its vertex set is at most $\alpha_k$. Now

$$\begin{aligned}
\rho_k(V(G)) &\leq (\rho_k(V(\widetilde{G})) + 2(k-1)) + (\rho_k(V(\widehat{G})) + (k+1)(k-2)) - 2 \cdot (k+1)(k-2) \\
&\leq \rho_k(V(\widetilde{G})) + \rho_k(V(\widehat{G})) - k^2 + 3k.
\end{aligned}$$

If at least one of $\widetilde{G}$ and $\widehat{G}$ is not a $k$-Ore graph, then the last expression is at most $(k^2 - 3k) + \alpha_k - (k^2 - 3k) = \alpha_k$, as claimed. If both $\widetilde{G}$ and $\widehat{G}$ are $k$-Ore graphs, then $G$ also is a $k$-Ore graph, which is a contradiction. $\square$

**Claim 2.26** *For $R \subsetneq V(G)$ with $|R| \geq 2$ and $x, y \in R$, the graph $G[R] + xy$ is $(k-1)$-colorable.*

**Proof.** Let $R$ be a smallest subset of vertices such that $2 \leq |R| < n$ and for some $xy \notin E(G)$, the graph $H$, where $H = G[R] + xy$, is not $(k-1)$-colorable. By Claim 2.24, $\rho_{k,H}(R) = -(2k-2) + \rho_{k,G}(R) > \alpha_k$. So, by the minimality of $G$, the graph $H$ contains a $k$-Ore subgraph $H_1$. By the minimality of $R$, the set $V(H_1)$ is all of $R$. If $H_1 \neq H$, then $H$ has at least one extra edge, so $\rho_{k,H}(R) \le \rho_{k,H_1}(R) - 2k + 2 \le k^2 - 5k + 2$. Since $\alpha_k \ge k^2 - 5k + 2$, we get

$$\rho_{k,G}(R) = (2k-2) + \rho_{k,H}(R) \le k^2 - 3k \le \alpha_k + 2k - 2,$$

a contradiction to Claim 2.24. So, $H$ is a $k$-Ore graph by itself. Let $R_*$ be the set of vertices in $R$ that have a neighbor outside of $R$. By Claim 2.25, $|R_*| \ge 3$. We want to prove that

$$G[R] \text{ has a proper } (k-1)\text{-coloring } \psi \text{ such that } R_* \text{ is not monochromatic.} \tag{2.11}$$

Let $\{u, v, w\} \subseteq R_*$.

CASE 1: $\{x, y\} \subset R_*$. Let $w \in R_* - x - y$. If there exists a subset $R' \subsetneq R$ such that $\{x, y\} \not\subset R'$ and $\rho_k(R) = (k+1)(k-2)$, then by Claim 2.22, there is a standard set $A$ in $H$. There then exists a pair of vertices $\{a, b\} \subset A$ such that $G[A] + ab$ is not $(k-1)$-colorable, which contradicts the minimality of $R$. By Lemma 2.21, there is a proper $(k-1)$-coloring $\phi_w$ of $H - xy$ such that $\phi_w(w) \ne \phi_w(x) = \phi_w(y)$. For $\psi = \phi_w$, (2.11) holds.

CASE 2: $\{x, y\} \not\subset R_*$. Let $H_0 = G[R] + uv$. If $H_0$ has a proper $(k-1)$-coloring, then (2.11) holds. If not, then there is a $k$-Ore subgraph $H_0'$ of $H_0$ that contains the edge $uv$. By the minimality of $R$, $H_0' = H_0$. The rest follows similarly to Case 1. This proves (2.11).

Let $\psi$ satisfy (2.11). Let $G' = Y(G, R, \psi)$. By Claim 2.13, $G'$ is not $(k-1)$-colorable. Hence it contains a $k$-critical subgraph $G''$. Let $W = V(G'')$. By the minimality of $G$, $\rho_k(W) \le k(k-3)$. Since $G$ is $k$-critical, $W \cap X \ne \emptyset$. Since every nonempty subset of $X$ has potential at least $(k-2)(k+1)$,

$$\rho_{k,G}(W - X + R) \le \rho_{k,G'}(W) - \rho_{k,G'}(X \cap W) + \rho_{k,G}(R) \le \rho_{k,G}(R) - 2k + 2 \le k^2 - 3k. \tag{2.12}$$

Since $W - X + R \supset R$, we have $|W - X + R| \ge 2$. Since $\alpha_k + (2k-2) \ge k^2 - 3k$, by Claim 2.24, $W - X + R = V(G)$. If $|W \cap X| \ge 2$, then $\rho_k(X \cap W) \ge 2(k-1)(k-2)$, and by (2.12),

$$\rho_{k,G}(W - X + R) \leq k^2 - 3k - 2(k-1)(k-2) + (k-2)(k+1) = 2k - 6 \leq \alpha_k,$$

a contradiction. So $|X \cap W| = 1$. Because $R_*$ is not monochromatic and $|X \cap W| = 1$, there is a vertex $z \in R_* - W$. Now by (2.3), instead of (2.12), we have

$$\rho_{k,G}(W - X + R) \leq k^2 - 3k - (k+1)(k-2) + (k-2)(k+1) - 2k + 2 = k^2 - 5k + 2 \leq \alpha_k,$$

a contradiction. $\square$

**Claim 2.27** *If $X$ is a $(k-1)$-clique in $G$, $u, v \in X$, $N(u) - X = \{a\}$, and $N(v) - X = \{b\}$, then $a = b$.*

**Proof.** Assume $a \neq b$. Let $G' = G - u - v + ab$ if $ab \notin E(G)$ and $G' = G - u - v$ otherwise. By Claim 2.26, $G'$ is $(k-1)$-colorable. A proper $(k-1)$-coloring of $G'$ can easily be extended to a proper $(k-1)$-coloring of $G$. $\square$

**Claim 2.28** *$G$ does not contain $K_k - e$.*

**Proof.** If $G[R] = K_k - e$, then $R \neq V(G)$, but adding the missing edge to $G[R]$ creates a $k$-chromatic graph on $R$, a contradiction to Claim 2.26. $\square$

**Definition 2.29** *A* cluster *is a maximal set $R \subseteq V(G)$ such that $d_G(x) = k-1$ for $x \in R$ and $N[x] = N[y]$ for $x, y \in R$.*

**Claim 2.30** *If $X$ is a $(k-1)$-clique, then there is a unique cluster $T$ inside of $X$ (possibly $T = \emptyset$), and $|T| \leq k-3$.*

**Proof.** Two clusters inside of $X$ would contradict Claim 2.27. If $|T| \geq k-2$, then by Claim 2.27, there is a vertex $y \in V(G) - X$ adjacent to at least $k-2$ vertices in $X$, a contradiction to Claim 2.28. $\square$

**Claim 2.31** *For every partition $(A, B)$ of $V(G)$ with $2 \leq |A| \leq n-2$, $|E_G(A, B)| \geq k$.*

**Proof.** Let $A_*$ (respectively, $B_*$) be the set of vertices in $A$ (respectively, $B$) that have neighbors in $B$ (respectively, $A$). Since $G$ is 3-connected, $|A_*| \geq 3$ and $|B_*| \geq 3$. By Claim 2.26, $G[A]$ therefore has a proper $(k-1)$-coloring $\phi_A$ such that $A_*$ is not monochromatic, and $G[B]$ has a proper $(k-1)$-coloring $\phi_B$ such that $B_*$ is not monochromatic. Gallai and Toft (see [75, p. 157]) independently proved that if $|E_G(A, B)| = k-1$, then either $A_*$ is monochromatic in every proper $(k-1)$-coloring of $G[A]$ or $B_*$ is monochromatic in every proper $(k-1)$-coloring of $G[B]$. Thus, $|E_G(A, B)| \geq k$. $\square$

**Lemma 2.32** *Let $z$ and $s$ be integers such that $z \geq s \geq 2$. Let $R_* = \{u_1, \ldots, u_s\}$ be a vertex set, and let $w : R_* \to \{1, 2, \ldots\}$ be a positive integral valued weight function on $R_*$ such that $w(u_1) + \ldots + w(u_s) = z$ and $w(u_1) \geq \ldots \geq w(u_s)$. For each $1 \leq i \leq z/2$, there exists a graph $F$ with $V(F) = R_*$ and $|E(F)| \leq i$ such that for each $1 \leq j \leq s$, $d_F(u_j) \leq w(u_j)$, and for every independent set $M$ in $F$ with $|M| \geq 2$,*

$$\textstyle\sum_{u \in R_* - M} w(u) \geq i. \tag{2.13}$$

*Moreover, if $s \geq 3$ and $2i < z$, then at least one of the three statements below holds:*

*(1) such $F$ with property (2.13) can be chosen as a graph with $i-1$ edges, or*

*(2) a hypergraph $H$ with property (2.13) can be chosen with $i-1$ edges of size 2 and one edge of size 3, or*

*(3) $s = i+1$ and $w(u_2) = \ldots = w(u_s) = 1$.*

*The weight arrangement is $i$-*special *if (3) holds.*

**Proof.** CASE 1: $w(u_2) + \ldots + w(u_s) \le i-1$. Let $E(F) = \{u_1u_j \,:\, 2 \le j \le s\}$. If $M$ is any independent set with $|M| \ge 2$, then $u_1 \notin M$, and $u_1$ witnesses that (2.13) holds. To prove the "Moreover" part in this case, observe that $F$ has at most $i-1$ edges.

CASE 2: $w(u_2)+\ldots+w(u_s) \ge i$. Choose the largest $j$ such that $w(u_{j+1})+\ldots+w(u_s) \ge i$. Let $a = i - w(u_{j+2}) + \ldots + w(u_s)$. Since $2i \le r$, we also have $w(u_1) + \ldots + w(u_{j+1}) \ge i + a$. By the choice of $j$ and the ordering of the vertices, $0 \le a \le w(u_{j+1}) \le w(u_1)$. We draw $a$ edges connecting $u_1$ with $u_{j+1}$ and $i-a$ edges connecting $\{u_{j+2}, \ldots, u_s\}$ with $\{u_1, \ldots, u_{j+1}\}$ so that for each $\ell$, the degree of $u_\ell$ in the resulting multigraph $F$ is at most $w(u_\ell)$. Let $M$ be any nonempty independent set in $F$. By the definition of $F$, since $M$ is independent,

$$\sum_{u \in R_* - M} w(u) \ge \sum_{u \in R_* - M} d_F(u) \ge \frac{1}{2} \sum_{u \in R_*} d_F(u) = i,$$

as claimed. Note that in this case, (2.13) holds for *every* independent set $M$, even if $|M| = 1$.

To prove the "Moreover" part in this case, observe that since $2i < z$, for some $1 \le \ell \le s$ the degree in $F$ of $u_\ell$ is less than $w(u_\ell)$. If $F - u_\ell$ has an edge $e$, then we can enlarge $e$ to $e+u_l$ and still keep (2.13). Otherwise, $\ell = 1$ and $u_1$ is adjacent to each of $u_2, \ldots, u_s$. If after replacing multiple edges with single edges we have fewer than $i$ edges, then we are done again. So, the remaining case is $w(u_2) = \ldots = w(u_s) = 1$ and so $s = i+1$. Therefore we satisfy (3). $\square$

**Lemma 2.33** *If $R \subsetneq V(G)$ and $2 \le |R| \le n-2$, then $\rho(R) \ge 2(k-1)(k-2)$. Moreover, if $\rho(R) = 2(k-1)(k-2)$, then $G[R] = K_{k-1}$.*

**Proof.** Assume that the claim does not hold. Let $i$ be the smallest integer such that there exists $R \subsetneq V(G)$ with $2 \leq |R| \leq n-2$, $G[R] \neq K_{k-1}$, and

$$\alpha_k + 2i(k-1) < \rho_k(R) \leq \alpha_k + 2(i+1)(k-1). \tag{2.14}$$

By Claim 2.26, $i \geq 1$. Since $\alpha_k + (k+1)(k-1) \geq k^2 - 5k + 2 + (k+1)(k-1) > 2(k-1)(k-2)$, we have $i \leq \frac{k}{2}$. By the integrality of $i$, if $k$ is odd, then $i \leq \frac{k-1}{2}$. Moreover, if $k = 4$, then $\alpha_k = \max\{2 \cdot 4 - 6, 4^2 - 5 \cdot 4 + 2\} = 2$ and so $y_4 + 4(4-1) = 14 > 12 = 2(4-1)(4-2)$. Thus

$$i \leq \frac{k}{2}\text{ , with strict inequality if } k \text{ is odd or } k = 4. \tag{2.15}$$

Let $R$ be the smallest set among $R \subsetneq V(G)$ with $2 \leq |R| \leq n-2$, $\rho(R) \leq 2(k-1)(k-2)$, and $G[R] \neq K_{k-1}$ for which (2.14) holds. Let $m = \rho_{k,G}(R)$. Since $G[R] \neq K_{k-1}$ and for $2 \leq s \leq k-2$, $\rho_k(K_s) > 2(k-1)(k-2)$, we have $|R| \geq k$.

For $u \in R$, let $w(u) = |N(u) \cap (V(G) - R)|$. Let $R_* = \{u \in R : w(u) \geq 1\}$. By Claim 2.31, $\sum_{u \in R_*} w(u) = |E_G(R, V(G) - R)| \geq k$. Let $R_* = \{u_1, \ldots, u_s\}$ and $w(u_1) \geq \ldots \geq w(u_s)$. By Claim 2.25, $s \geq 3$. By Lemma 2.32, we can add to $G[R_*]$ a set $E_0$ of at most $i$ edges such that for every independent subset $M$ of $R_*$ in $G \cup E_0$ with $|M| \geq 2$, (2.13) holds. Let $H = G[R] \cup E_0$ and $X' = X \cap W$.

Because $|E_0| \leq i < k \leq |E_G(R, V(G) - R)|$, $H \prec G$. So if $P_k(H) > k(k-3)$, then $H$ is $(k-1)$-colorable.

CASE 1: $H$ has a proper $(k-1)$-coloring $\phi$. Let $F = Y(G, R, \phi)$. By Claim 2.13, $F$ is not $(k-1)$-colorable. Thus $F$ contains a $k$-critical subgraph $F'$. Let $W = V(F')$. Since $G$ is $k$-critical by itself, $W \cap X \neq \emptyset$. By the choice of $i$, since

$$\rho_{k,G}(W - X + R) \leq k(k-3) - (k-2)(k+1) + m \leq m - 2(k-1), \tag{2.16}$$

$|W - X + R| \geq n-1$. Moreover, if $|W - X + R| = n-1$, then by (2.9), $k^2 - 3k - (k-2)(k+$

$1) + m \geq \alpha_k + k^2 - 3k + 4$ and so $m \geq \alpha_k + k^2 - k + 2 > 2(k-1)(k-2)$, a contradiction to $m \leq 2(k-1)(k-2)$. So $W - X + R = V(G)$.

Suppose $X' = \{x_j\}$. It follows that $W = V(F) - X + x_j$. Let $R_j = \{u \in R_* : \phi(u) = c_j\}$. If $|R_j| = 1$, then $F \cong G[W - x_j \cup R_j]$, a contradiction to the fact that $G$ itself is $k$-critical. If $|R_j| \geq 2$, then by the construction of $H$, at least $i$ edges connect the vertices in $R_* - R_j$ with $V(G) - R$. Adjusting (2.16) to account for these edges and using (2.14), we have

$$\rho_{k,G}(W - \{x_j\} + R) \leq k(k-3) - (k-2)(k+1) - 2i(k-1) + m = m - 2(i+1)(k-1) \leq \alpha_k$$

which is a contradiction. So assume $|X'| \geq 2$.

Suppose $F'$ is not a $k$-Ore graph, and so $\rho_{k,F}(W) \leq \alpha_k$. Since every $2 \leq |X'| \leq k-1$ has potential at least $2(k-1)(k-2)$, by (2.3), $\rho_{k,G}(W - X + R) \leq \alpha_k - 2(k-1)(k-2) + m \leq \alpha_k$, a contradiction. So assume $F'$ is a $k$-Ore graph.

If $X' \neq X$ or $F' \neq F$, then instead of (2.16), we would have

$$\rho_{k,G}(W - X + R) \leq -2(k-1) + k^2 - 3k - 2(k-1)(k-2) + 2(k-1)(k-2) = k^2 - 5k + 2 \leq \alpha_k,$$

a contradiction. So, $H = F = F'$.

Since $|R| \leq n-2$, we have $F \neq K_k$. Let the separating set $\{x, y\}$, vertex $w$, vertex subsets $A$ and $B$, and graphs $\widetilde{F}(x, y)$ and $\widehat{F}(x, y)$ be as in Fact 2.19. Since $F[X']$ is a clique and $E_F(A - x - y, B - x - y) = \emptyset$, either $X' \subseteq A$ or $X' \subseteq B$. Since $xy \notin E(F)$ we may assume that either $X' \subset A - y$ or $X' \subset B - y$. Suppose first that $X' \subset A - y$. The graph $\widehat{F} - w$ is a subgraph of $G$, namely, it is $G[B - x - y]$, and

$$d_{\widehat{F}-w}(v) = d_G(v) \text{ for every } v \in B - x - y. \tag{2.17}$$

If $\widehat{F} - w$ has a vertex subset $S$ with $|S| \geq 2$ of potential at most $(k+1)(k-2)$, then by

Lemma 2.22, $S$ is the union of standard sets and $F[S]$ is connected. But in each standard set $S'$, there are two vertices $u$ and $u'$ such that $F[S'] + uu'$ is not $(k-1)$-colorable. This contradicts Claim 2.26. Thus $\rho_{k,\widehat{F}}(S) > (k+1)(k-2)$ for every $S \subseteq V(\widehat{F}) - w$ with $|S| \geq 2$. Then by Claim 2.23, there exists an $S \subseteq V(\widehat{F}) - w = B - x - y$ such that $\widehat{F}[S] \cong K_{k-1}$, and $d_{\widehat{F}}(v) = k-1$ for all $v \in S$. By (2.17), this contradicts Claim 2.30.

Thus $X' \subset B - y$. Similarly to (2.17), the graph $\widetilde{F} - x$ is a subgraph of $G$, namely, it is $G[A - x]$, and

$$d_{\widetilde{F}-x}(v) = d_G(v) \text{ for every } v \in A - x - y. \tag{2.18}$$

As in the previous paragraph, $\rho_{k,\widetilde{F}}(S) > (k+1)(k-2)$ for every $S \subseteq V(\widetilde{F}) - x$ with $|S| \geq 2$. So again by Claim 2.23, there exists an $S \subseteq V(\widetilde{F}) - x = A - x$ such that $\widetilde{F}[S] \cong K_{k-1}$, and $d_{\widetilde{F}}(v) = k-1$ for all $v \in S$. But $|S - y| \geq k-2$, which together with (2.18) contradicts Claim 2.30.

CASE 2: The set of weights $\{w(u_1), \ldots, w(u_s)\}$ is $i$-special: $s = i+1$ and $w(u_2) = \ldots = w(u_s) = 1$. This means that many (at least $i+1$) edges connect $u_1$ with $Q = V(G) - R$ and each of the vertices $u_2, \ldots, u_{i+1}$ is connected to $Q$ by exactly one edge. For $j = 2, \ldots, i+1$, let $q_j$ be the vertex in $Q$ such that $u_j q_j \in E(G)$. Recall that since $G$ is 3-connected, $i \geq 2$. Let $E_0 = \{u_1 u_j : 2 \leq j \leq i\}$ and $H_0 = G[R] \cup E_0$. Since $|E_0| = i - 1$, by (2.14), $\rho_{k,H_0}(R') > \alpha_k + 2k - 2 \geq k^2 - 3k$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by induction, $H_0$ has a proper $(k-1)$-coloring $\phi$. If $\phi(u_{i+1}) \neq \phi(u_1)$, then for every monochromatic subset $M$ of $R_*$ in $G \cup E_0$ with $|M| \geq 2$, (2.13) holds. Then we can repeat the argument of Case 1.

Thus suppose $\phi(u_{i+1}) = \phi(u_1)$. Let $G_0$ be obtained from $G[V(G) - (R - u_1)]$ by adding edge $u_1 q_{i+1}$. By Claim 2.26, $G_0$ has a proper $(k-1)$-coloring $\phi'$. Since $i \leq \frac{k}{2}$, we can rename the colors in $\phi'$ so that $\phi'(u_1) = \phi(u_1) = \phi(u_{i+1})$ and $\phi(\{u_2, \ldots, u_i\}) \cap \phi'(\{q_2, \ldots, q_i\}) = \emptyset$. Then $\phi \cup \phi'$ is a proper $(k-1)$-coloring of $G$, a contradiction.

CASE 3: The set of weights $\{w(u_1), \ldots, w(u_s)\}$ is not $i$-special and $2i < |E(R, V(G) - R)|$.

If Statement (1) of the "Moreover" part of Lemma 2.32 holds, then we take this set $E_0$ of $i-1$ edges and let $H_0 = G[R] + E_0$. In this case by (2.14), $\rho_{k,H_0}(R') > \alpha_k + 2k - 2 \geq k^2 - 3k$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by induction, $H_0$ has a proper $(k-1)$-coloring $\phi$, and we simply repeat the argument of Case 1.

Suppose now that Statement (2) holds: *there is a hypergraph $F$ with $i-1$ 2-edges and a 3-edge $e_0 = \{u, v, w\}$ such that $d_F(u_j) \leq w(u_j)$ for all $j = 1, \ldots, s$ and (2.13) holds.* Let $H_1$ be obtained from $G[R]$ by adding the set of edges $E(F) - e_0$ and edge $uv$. If $H_1$ has a proper $(k-1)$-coloring, then we repeat the proof of Case 1. So suppose not. It follows that $H_1$ has a $k$-critical subgraph $H_1'$. Let $R' = V(H_1')$. Since $\rho_{k,H_1}(R') \leq k^2 - 3k \leq \alpha_k + 2k - 2$, we have $\rho_{k,G}(R') \leq \alpha_k + 2(i+1)(k-1)$. By the minimality of $R$, we have $R' = R$. Furthermore, if $H_1'$ is not a $k$-Ore graph, then $\rho_{k,H_1}(R) \leq \alpha_k$ and so $\rho_{k,G}(R) \leq \alpha_k + 2i(k-1)$, a contradiction to (2.14). So, $H_1'$ is a $k$-Ore graph. If $H_1' \neq H_1$, then it has the same vertex set and at least one fewer edge, in which case,

$$\rho_{k,G}(R) \leq \rho_{k,H_1'}(R) + 2i(k-1) \leq \rho_{k,H_1}(R) + 2(i-1)(k-1) \leq k^2 - 3k + 2(i-1)(k-1) \leq \alpha_k + 2i(k-1),$$

a contradiction to (2.14). So, $H_1$ is a $k$-Ore graph and $\rho_{k,G}(R) = k^2 - 3k + 2i(k-1)$. Again by (2.14) and the minimality of $|R|$, for every $W \subset R$ with $|W| \geq 2$, we have

$$\rho_{k,H_1-uv}(W) > \alpha_k + 2(i+1)(k-1) - 2(i-1)(k-1) = \alpha_k + 4k - 4 \geq (k+1)(k-2).$$

Thus by Lemma 2.21, $H_1 - uv$ has a proper $(k-1)$-coloring $\phi$ with $\phi(w) \neq \phi(u)$. With this coloring, we simply repeat the argument of Case 1.

CASE 4: The set of weights $\{w(u_1), \ldots, w(u_s)\}$ is not $i$-special and $2i \geq |E(R, V(G)-R)|$. By Claim 2.31, $|E(R, V(G)-R)| \geq k$. So by (2.15), in order to have $2i \geq |E(R, V(G)-R)|$, we need $i = \frac{k}{2}$ and $k \geq 6$. For $k \geq 6$, we know that $\alpha_k = k^2 - 5k + 2$. By Lemma 2.32 for $i-1$

instead of $i$, we can add to $G[R_*]$ a set $E_1$ of at most $i-1$ edges such that for every independent subset $M$ of $R_*$ in $G \cup E_1$ with $|M| \geq 2$, (2.13) holds. Let $H_1 = G[R] \cup E_1$. By (2.14), $\rho_{k,H_1}(R') > \alpha_k + 2k - 2 = k^2 - 3k$ for every $R' \subseteq R$ with $|R'| \geq 2$. So, by minimality of $G$, $H_1$ has a proper $(k-1)$-coloring $\phi$. Then we try to repeat the argument of Case 1 with $H_1$ in place of $H$. It does not work only in the case of $X \cap W = \{x_j\}$. But in this case, since $F'$ is $k$-critical, $d_{F'}(x_j) \geq k-1$. This means that $|E_G(R_j, V(G)-R)| \geq k-1$. Also, by construction, at least $i-1$ edges connect $R - R_j$ with $V(G) - R$. Since $k \geq 6$ and $2i = k$, it follows that $|E(R, V(G)-R)| \geq (k-1) + \frac{k}{2} - 1 \geq k+1$, a contradiction to the conditions of the case. □

## 2.5 Reducible Configurations

We will now quickly refresh the reader on relevant statements. A graph $H$ is *smaller than* graph $G$, if either $|E(G)| > |E(H)|$, or $|E(G)| = |E(H)|$ and $G$ has fewer pairs of vertices with the same closed neighborhood. The graph $G$ is the smallest graph with respect to our order that contradicts Theorem 2.11.

We have proven the following statements:

Claim 2.23: If $H \in O_k$ and $u \in V(H)$ such that $\rho_{k,H}(W) > (k+1)(k-2)$ for every $W \subseteq V(H) - u$ with $|W| \geq 2$, then there exists $S \subseteq V(G) - u$ such that $G[S] \cong K_{k-1}$ and $d_G(v) = k-1$ for all $v \in S$.

Claim 2.26: For $R \subsetneq V(G)$ with $|R| \geq 2$ and $x, y \in R$, the graph $G[R] + xy$ is $(k-1)$-colorable.

Claim 2.28: $G$ does not contain $K_k - e$.

Claim 2.30: If $X$ is a $(k-1)$-clique in $G$, then there is a unique cluster $T$ inside of $X$, and $|T| \leq k-3$.

Lemma 2.33: If $R \subsetneq V(G)$ and $2 \leq |R| \leq n-2$, then $\rho(R) \geq 2(k-1)(k-2)$ with equality only when $G[R] = K_{k-1}$.

**Claim 2.34** *If $v$ is not in $(k-1)$-clique $X$, then $|N(v) \cap X| \leq \frac{k-1}{2}$. Furthermore, if $T$ is a cluster in a $(k-1)$-clique $X$, then $|T| \leq \frac{k-1}{2}$.*

**Proof.** If $|N(v) \cap X| \geq \lceil k/2 \rceil$, then $\rho_k(X+v) \leq 2(k-2)(k-1)-2$. Since $n \geq k+2$, this is a contradiction to Lemma 2.33.

Let $\{v\} = N(T) - X$. Then $|N(v) \cap X| \geq |T|$. $\square$

**Claim 2.35** *Let $T$ be a cluster in $G$ and $t = |T| \geq 2$.*
*(a) If $N(T) \cup T$ does not contain $K_{k-1}$, then $d_G(v) \geq k-1+t$ for every $v \in N(T) - T$;*
*(b) If $N(T) \cup T$ contains a $K_{k-1}$ with vertex set $X$, then $d_G(v) \geq k-1+t$ for every $v \in X - T$.*

**Proof.** Let $v \in N(T) - T$ such that $k \leq d(v) \leq k-2+t$ and if $N(T) \cup T$ contains a $K_{k-1}$ with vertex set $X$, then $v \in X$. By definition, $|T \cup N(T)| = k$. By this and Claim 2.28, $T$ is contained in at most one $(k-1)$-clique, and so

$$N(T) \cup T - v \text{ does not contain } K_{k-1}. \tag{2.19}$$

By the choice of $v$, $|N(v) - T| \leq k-2$. Let $u \in T$ and $G' = G - v + u'$, where $N[u'] = N[u]$. If $G'$ has a proper $(k-1)$-coloring $\phi'$, then there is a proper $(k-1)$-coloring $\phi$ of $G$ as follows: set $\phi|_{V(G)-T-v} = \phi'|_{V(G')-T-u'}$, $\phi(v) \in C - \phi'(N(v) - T)$, and then color $T$ using colors in $\phi'(T \cup u') - \phi(v)$. This is a contradiction, so there is no proper $(k-1)$-coloring of $G'$. Thus $G'$ contains a $k$-critical subgraph $G''$. Let $W = V(G'')$. By minimality of $G$, it follows that $\rho_{k,G'}(W) \leq k(k-3)$.

Since $G''$ is not a subgraph of $G$, $u' \in W$. By symmetry, it follows that $T \subset W$. But then

$$\rho_{k,G}(W - u') \leq k(k-3) - (k-2)(k+1) + 2(k-1)(k-1) = 2(k-2)(k-1).$$

This implies that either $G[W-u']$ is a $K_{k-1}$ or $W-u' = V(G)-v$ If the former holds, then we have a contradiction to (2.19). If the latter holds, then by (2.9) and $d(v) \geq k$,

$$\rho_k(V(G)) \leq \rho_{k,G}(W-u')+(k+1)(k-2)-2k(k-1) \leq 2(k-2)(k-1)-k^2+k-2 = k^2-5k+2 \leq \alpha_k,$$

a contradiction. $\square$

**Claim 2.36** *If $xy \in E(G)$, $N[x] \neq N[y]$, $x$ is in a cluster of size $s$, $y$ is in a cluster of size $t$, and $s \geq t$, then $x$ is in a $(k-1)$-clique. Furthermore, $t = 1$.*

**Proof.** Assume that $x$ is not in a $(k-1)$-clique. Let $G' = G - y + x'$, where $N[x'] = N[x]$. If $G'$ has a proper $(k-1)$-coloring $\phi'$, then we extend it to a proper $(k-1)$-coloring $\phi$ of $G$ as follows: define $\phi|_{V(G)-x-y} = \phi'|_{V(G')-x-x'}$, then choose $\phi(y) \in C - (\phi'(N(y)-x))$, and $\phi(x) \in \{\phi'(x), \phi'(x')\} - \{\phi(y)\}$.

So, $\chi(G') \geq k$ and $G'$ contains a $k$-critical subgraph $G''$. Let $W = V(G'')$. Since $G''$ is not a subgraph of $G$, so $x' \in W$. Then $\rho_{k,G}(W-x') \leq k(k-3)-(k-2)(k+1)+2(k-1)(k-1) = 2(k-2)(k-1)$. If $|W-x'| \leq n-2$, this contradicts Lemma 2.33. Otherwise, $W-x' = V(G)-y$ and $G'$ itself is a $k$-critical graph. Since $G'$ is smaller than $G$, by the minimality of $G$, either $\rho_{k,G'}(V(G')) \leq \alpha_k$ or $G'$ is a $k$-Ore graph. The former is a contradiction to (2.9), so suppose $G'$ is a $k$-Ore graph.

Since $n > k$, we have $G' \neq K_k$. Let the separating set $\{x, y\}$, vertex $w$, vertex subsets $A$ and $B$, and graphs $\widetilde{G}'(x,y)$ and $\widehat{G}'(x,y)$ be as in Fact 2.19. If $x' \notin A$, then $\widetilde{G}'(x,y)$ is obtained from a proper subgraph of $G$ by adding an edge. This contradicts Claim 2.26. So, $x' \in A$. By symmetry, $x \in A$. By the same claim, $\rho_{k,G'}(W) > (k+1)(k-2)$ for every $W \subseteq V(G') - x'$ with $|W| \geq 2$. In particular, $\rho_{k,\widehat{G}'(x,y)}(W) > (k+1)(k-2)$ for every $W \subseteq V(\widehat{G}') - w$ with $|W| \geq 2$. Then by Claim 2.23, there exists a $S \subseteq V(\widehat{G}'(x,y)) - w$ such that $\widehat{G}'(x,y)[S] \cong K_{k-1}$, and $d_{\widehat{G}'(x,y)}(v) = k-1$ for all $v \in S$. By Claim 2.30, vertex $y$ in $G$

is adjacent to at most $k-3$ vertices in $S$. So $S$ contains one cluster $T$ and $|T| \geq 2$. Then by Claim 2.35(b), the degree of each vertex in $S-T$ in $G$ is at least $k+1$. This is impossible, since each of them has in $G$ at most one extra neighbor (and it is $y$, if exists) in comparison with $\widehat{G}'(x,y)$. This proves that $x$ is in a $K_{k-1}$.

The second part follows from the fact that each vertex of degree $k-1$ contained in a copy $X$ of $K_{k-1}$ has exactly neighbor outside of $X$. $\square$

## 2.6 Discharging

We will now quickly refresh the reader on relevant statements. A graph $H$ is *smaller than* graph $G$, if either $|E(G)| > |E(H)|$, or $|E(G)| = |E(H)|$ and $G$ has fewer pairs of vertices with the same closed neighborhood. The graph $G$ is the smallest graph with respect to our order that contradicts Theorem 2.11.

We have proven the following statements:

Corollary 2.18: Let disjoint vertex subsets $A$ and $B$ of $G$ be such that $A$ or $B$ is independent, $d_G(a) = k-1$ for every $a \in A$, $d_G(b) = k$ for every $b \in B$, and $|A| + |B| \geq 3$. Under these conditions, (i) $|E(G(A,B))| \leq 2(|A|+|B|) - 4$ and (ii) $|E(G(A,B))| \leq |A| + 3|B| - 3$.

Claim 2.23: If $H \in O_k$ and $u \in V(H)$ such that $\rho_{k,H}(W) > (k+1)(k-2)$ for every $W \subseteq V(H) - u$ with $|W| \geq 2$, then there exists $S \subseteq V(G) - u$ such that $G[S] \cong K_{k-1}$ and $d_G(v) = k-1$ for all $v \in S$.

Claim 2.27: If $X$ is a $(k-1)$-clique in $G$, $u, v \in X$, $N(u) - X = \{a\}$, and $N(v) - X = \{b\}$, then $a = b$.

Claim 2.30: If $X$ is a $(k-1)$-clique in $G$, then there is a unique cluster $T$ inside of $X$, and $|T| \leq k-3$.

Lemma 2.33: If $R \subsetneq V(G)$ and $2 \leq |R| \leq n-2$, then $\rho(R) \geq 2(k-1)(k-2)$ with equality only when $G[R] = K_{k-1}$.

### 2.6.1 Case $k = 4$

**Claim 2.37** *Each vertex with degree* 3 *has at most* 1 *neighbor with degree* 3*.*

**Proof.** Let $x$ be such that $N(x) = \{a, b, c\}$ and $d(a) = 3$. By Claim 2.36, $G[\{x, b, c\}]$ is a $K_3$. So by Claims 2.30 and 2.34, $d(b), d(c) \geq 4$. $\square$

We will now use discharging to show that $|E(G)| \geq \frac{5}{3}n$, which will finish the proof to the case $k = 4$, since $y_4 = 4^2 - 5 \cdot 4 + 6 = 2 > 0$. Each vertex begins with charge equal to its degree. If $d(v) \geq 4$, then $v$ gives charge $\frac{1}{6}$ to each neighbor. Note that $v$ will be left with charge at least $\frac{5}{6}d(v) \geq \frac{10}{3}$. By Claim 2.37, each vertex of degree 3 will end with charge at least $3 + \frac{2}{6} = \frac{10}{3}$. $\square$

### 2.6.2 Case $k = 5$

**Claim 2.38** *If $k = 5$, then each cluster has only one vertex.*

**Proof.** Assume $N[x] = N[y]$ and $d(x) = d(y) = 4$. Let $N(x) = \{y, a, b, c\}$. Since $G$ does not contain $K_5 - e$, $|E(G[\{a, b, c\}])| \leq 1$. By Claim 2.35 we can rename the vertices in $\{a, b, c\}$ so that $ab \notin E(G)$ and $d(c) \geq 5$.

We obtain $G'$ from $G$ by deleting $x$ and $y$ and merging $a$ and $b$ into a vertex $u$. If $G'$ is 4-colorable, then so is $G$. This is because a proper 4-coloring of $G'$ will have at most 2 colors on $N[x] - \{x, y\}$ and therefore could be extended greedily to $x$ and $y$.

So $G'$ contains a $k$-critical subgraph $G''$. Let $W' = V(G'')$; then $\rho_{5,G'}(W') \leq 10$. Since $G''$ is not a subgraph of $G$, $u \in W'$. Let $W = W' - u + a + b + x + y$. But then $\rho_{5,G}(W) \leq 10 + 54 - 40 = 24$. Because $ab \notin E(G)$, we have that $G[W]$ is not a $K_4$. By Lemma

2.33, $|W| \geq n-1$. Therefore $|W \cap \{b,c\}| \geq 1$, and we did not account for at least two of the edges in $E(\{b,c\},\{x,y\})$, so $\rho_{5,G}(W) \leq 10+54-56=8$. By (2.9), this implies $W=V(G)$. But then we may use all four edges of $E(\{b,c\},\{x,y\})$, so $\rho_{5,G}(W) \leq 10+54-72<0$. $\square$

**Claim 2.39** *If $k=5$, then each $K_4$-subgraph of $G$ contains at most one vertex with degree 4. Furthermore, if $d(x)=d(y)=4$ and $xy \in E(G)$, then each of $x$ and $y$ is in a $K_4$.*

**Proof.** The first statement follows from Claims 2.30 and 2.38. The second statement follows from Claims 2.36 and 2.38. $\square$

**Definition 2.40** *We define $H \subseteq V(G)$ to be the set of vertices of degree 5 not in a $K_4$, and $L \subseteq V(G)$ to be the set of vertices of degree 4 not in a $K_4$. Set $\ell=|L|$, $h=|H|$ and $e_0=|E(L,H)|$.*

**Claim 2.41** $e_0 \leq 3h+\ell$.

**Proof.** This is trivial if $h+\ell \leq 2$ and follows from Corollary 2.18(ii) and Claim 2.39 for $h+\ell \geq 3$. $\square$

We will now use discharging to show that $|E(G)| \geq \frac{9}{4}n$, which will finish the proof to the case $k=5$. We will do discharging in two stages. Let every vertex $v \in V(G)$ have initial charge $d(v)$. The first half of discharging has one rule:

**Rule R1:** Each vertex in $V(G)-H$ with degree at least 5 gives charge 1/6 to each neighbor.

**Claim 2.42** *After the first round of discharging, each vertex in $V(G)-H-L$ has charge at least 4.5.*

**Proof.** Let $v \in V(G) - H - L$. If $d(v) = 4$, then $v$ receives $1/6$ from at least 3 neighbors and gives no charge. If $d(v) = 5$, then $v$ gives $1/6$ to 5 neighbors, but receives $1/6$ from at least 2 neighbors. If $d(v) \geq 6$, then $v$ is left with charge at least $5d(v)/6 \geq 4.5$. $\square$

For the second round of discharging, all charge in $H \cup L$ is taken up and distributed evenly among the vertices in $H \cup L$.

**Claim 2.43** *After the first round of discharging, the sum of the charges on the vertices in $H \cup L$ is at least $4.5|H \cup L|$.*

**Proof.** By Rule R1, vertices in $L$ receive from outside of $H \cup L$ the charge at least $\frac{1}{6}(4\ell - |E(H,L)|)$. By Claim 2.41, $|E(H,L)| \leq 3h + \ell$. So, the total charge on $H \cup L$ is at least

$$5h + 4\ell + \frac{1}{6}(4\ell - (3h + \ell)) = 4.5(h + \ell),$$

as claimed. $\square$

Combining Claims 2.42 and 2.43, the average degree of the vertices in $G$ is at least 4.5, a contradiction.

### 2.6.3 Case $k \geq 6$

**Claim 2.44** *Suppose $k \geq 6$, $X$ is a $(k-1)$-clique, and $v \in X$ has degree $k-1$. Then $X$ contains at least $(k-1)/2$ vertices with degree at least $k+1$.*

**Proof.** Let $\{u\} = N(v) - X$. Assume that $X$ contains at least $k/2$ vertices with degree at most $k$. By Claim 2.34, $|N(u) \cap X| < k/2$, so there exists a $w \in X$ such that $xw \notin E(G)$ and $d(w) \leq k$. By Claim 2.27, $d(w) = k$, so assume $N(w) - X = \{a, b\}$. Let $G'$ be obtained from $G - v$ by adding edges $ua$ and $ub$.

Suppose $G'$ has a proper $(k-1)$-coloring $f$. If $f(u)$ is not used on $X-w-v$, then we recolor $w$ with $f(u)$. So, $v$ will have at least two neighbors of color $f(u)$, and we can extend the proper $(k-1)$-coloring to $G$.

Thus $G'$ is not $(k-1)$-colorable and so contains a $k$-critical subgraph $G''$. Let $W = V(G'')$. Then $\rho_{k,G'}(W) \leq k(k-3)$ and so $\rho_{k,G}(W) \leq k(k-3)+2(k-1)(2) = k^2+k-4 < 2(k-2)(k-1)$. If $W \neq V(G')$ then this is a contradiction, since in this case $|W| \leq |V(G')|-1 \leq n-2$. So, $W = V(G')$.

If $G''$ is not a $k$-Ore graph, then by the minimality of $G$, $\rho_{k,G''}(W) \leq \alpha_k$, and so

$$\rho_{k,G}(V(G)) \leq \rho_{k,G'}(W) + (k-2)(k+1)(1) - 2(k-1)(k-3) < \alpha_k$$

when $k \geq 6$. Similar calculation works when $G'' \neq G'$. So, our case is that $G'$ is a $k$-Ore graph. Since $G'-ua-ub$ is a subgraph of $G$, $\rho_{k,G'}(U) > (k+1)(k-2)$ for every $U \subseteq V(G')-u$ with $|U| \geq 2$. Then by Claim 2.23, there exists a $S \subseteq V(G')-u$ such that $G'[S] \cong K_{k-1}$, and $d_{G'(v)} = k-1$ for all $v \in S$. But for every $z \in S-a-b$, $d_G(z) = d_{G'}(z)$. Since $k-1 \geq 5$, this contradicts Claim 2.34. $\square$

**Claim 2.45** *If $k = 6$ and a cluster $T$ is contained in a 5-clique $X$, then $|T| = 1$.*

**Proof.** By Claim 2.34, assume that $T = \{v_1, v_2\}$. Let $N(v_1) - X = \{y\}$ and $\{u, u', u''\} = X - T$. By Claim 2.44, $d(u), d(u'), d(u'') \geq 7$. Obtain $G'$ from $G-T$ by merging $u$ and $y$ into a new vertex $w$.

Suppose that $G'$ has a proper 5-coloring. We can extend this coloring to a proper coloring on $G$ by greedily assigning colors to $T$, because only 3 different colors appear on the set $\{u, u', u'', y\}$. So we may assume that $\chi(G') \geq 6$. Then $G'$ contains a 6-critical subgraph $G''$. Let $W = V(G'')$, and so $\rho_{6,G'}(W) \leq 18$. Since $G''$ is not a subgraph of $G$, $w \in W$. Let $t = |\{u', u''\} \cap W|$.

*Case 1:* $t = 0$. We have $\rho_{6,G}(W - w + y + X) \leq 18 + 28(5) - 10(12) = 38$. By Lemma 2.33, $|W - w + y + X| \geq n - 1$. We did not account for edges in $E(\{u', u''\}, V(G) - X)$, and each of $u', u''$ has at least 3 neighbors outside of $X$. Thus $\rho_{6,G}(W - w + y + X) \leq 38 - 10 \cdot 4 < 0$.

*Case 2:* $t = 1$. We have $\rho_{6,G}(W - w + y + u + C) \leq 18 + 28(3) - 10(7) = 32$. By Lemma 2.33, $|W - w + y + u + C| \geq n - 1$, so $W - w + y + u + C$ is either $V(G) - u'$ or $V(G) - u''$. But this contradicts (2.9).

*Case 3:* $t = 2$. We have $\rho_{6,G}(W - w + y + u + C) \leq 18 + 28(3) - 10(9) = 12$. By Lemma 2.33 and (2.9), $W - w + y + u + C = V(G)$. If $G''$ is not $k$-Ore or if $G'' \neq G'$, then $\rho_{6,G}(W - w + y + u + C) \leq 2$, which is a contradiction.

Since $G'' - w$ is a subgraph of $G$, we have $\rho_{k,G'}(U) > (k+1)(k-2)$ for every $U \subseteq V(G') - w$ with $|U| \geq 2$. By Claim 2.23, there exists a $S \subseteq V(G') - w$ such that $G'[S] \cong K_{k-1}$, and $d_{G'(v)} = k - 1$ for all $v \in S$. By Claim 2.34, there exists a $z \in S$ such that $d_G(z) \neq d_{G'}(z)$. This means that $zy, zu \in E(G)$, and so $G'' \neq G'$ because $G'$ has a multi-edge, which is a contradiction. $\square$

**Definition 2.46** *We partition $V(G)$ into four classes: $L_0$, $L_1$, $H_0$, and $H_1$. Let $H_0$ be the set of vertices with degree $k$, $H_1$ be the set of vertices with degree at least $k + 1$, and $H = H_0 \cup H_1$. Let*

$$L = \{u \in V(G) : d(u) = k - 1\},$$

$$L_0 = \{u \in L : N(u) \subseteq H\},$$

*and*

$$L_1 = L - L_0.$$

*Set $\ell = |L_0|$, $h = |H_0|$ and $e_0 = |E(L_0, H_0)|$.*

**Claim 2.47** *$e_0 \leq 2(\ell + h)$.*

**Proof.** This is trivial if $h+\ell \leq 2$ and follows from Corollary 2.18(i) for $h+\ell \geq 3$. $\square$

Let every vertex $v \in V(G)$ have initial charge $d(v)$. We first do a half-discharging with two rules:

**Rule R1:** Each vertex in $H_1$ keeps for itself charge $k-2/(k-1)$ and distributes the rest equally among its neighbors of degree $k-1$.

**Rule R2:** If a $K_{k-1}$-subgraph $C$ contains $s$ $(k-1)$-vertices adjacent to a $(k-1)$-vertex $x$ outside of $C$ and not in a $K_{k-1}$, then each of these $s$ vertices gives charge $\frac{k-3}{s(k-1)}$ to $x$.

**Claim 2.48** *Each vertex in $H_1$ donates at least $\frac{1}{k-1}$ charge to each neighbor of degree $k-1$.*

**Proof.** If $v \in H_1$, then $v$ donates at least $\frac{d(v)-k+2/(k-1)}{d(v)}$ to each neighbor. Note that this function increases as $d(v)$ increases, so the charge is minimized when $d(v)=k+1$. But then each vertex gets charge at least $(1+2/(k-1))/(k+1) = 1/(k-1)$. $\square$

**Claim 2.49** *Each vertex in $L_1$ has charge at least $k-2/(k-1)$.*

**Proof.** Let $v \in L_1$ be in a cluster $C$ of size $t$.

*Case 1:* $v$ is in a $(k-1)$-clique $X$ and $t \geq 2$. By Claim 2.45, this case only applies when $k \geq 7$.

By Claim 2.35 each vertex in $X-C$ has degree at least $k-1+t \geq k+1$, and therefore $X-C \subseteq H_1$. Furthermore, each vertex in $X-C$ has at least $k-2-t$ neighbors with degree at least $k$. Therefore each vertex $u \in (X-C)$ donates charge at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+2+t}$ to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u)=k-1+t$. It follows that $u$ gives to $v$ charge at least $\frac{t-1+2/(k-1)}{2t+1}$.

So, $v$ has charge at least $k-1+(k-1-t)(\frac{t-1+2/(k-1)}{2t+1}) - \frac{k-3}{t(k-1)}$, which we claim is at least $k-2/(k-1)$. Let

$$g_1(t) = (k-1-t)((t-1)(k-1)+2) - (2t+1)(k-3)(1+\frac{1}{t}).$$

We claim that $g_1(t) \geq 0$, which is equivalent to $v$ having charge at least $k-2/(k-1)$. Let

$$\widetilde{g}_1(t) = (k-1-t)((t-1)(k-1)+2) - (2t+1)(k-3)(3/2).$$

Note that $\widetilde{g}_1(t) \leq g_1(t)$ when $t \geq 2$, so we need to show that $\widetilde{g}_1(t) \geq 0$ on the appropriate domain. $\widetilde{g}_1(t)$ is quadratic with a negative coefficient at $t^2$, so it suffices to check its values at the boundaries. They are

$$\widetilde{g}_1(2) = (k-3)(k-6.5)$$

and

$$\begin{aligned} 4\widetilde{g}_1(\frac{k-1}{2}) &= (k-1)\left((k-3)(k-1)+4\right) - 6k(k-3) \\ &= k^3 - 11k^2 + 29k - 7 \\ &= (k-7)(k^2-4k+1). \end{aligned}$$

Each of these values is non-negative when $k \geq 7$.

*Case 2:* $t \geq 2$ and $v$ is not in a $(k-1)$-clique. By Claim 2.35, each neighbor of $v$ outside of $C$ has degree at least $k-1+t \geq k+1$ and is in $H_1$. Therefore $v$ has charge at least $k-1+(k-t)(\frac{t-1+2/(k-1)}{k-1+t})$. We define

$$\begin{aligned} g_2(t) &= (k-t)(t-1+\frac{2}{k-1}) - \frac{k-3}{k-1}(k-1+t) \\ &= t(k-t) - 2(1-\frac{2}{k-1})(k-1) \\ &= t(k-t) - 2(k-3). \end{aligned}$$

Note that $g_2(t) \geq 0$ is equivalent to $v$ having charge at least $k - 2/(k-1)$. The function $g_2(t)$ is quadratic with a negative coefficient at $t^2$, so it suffices to check its values at the boundaries. They are

$$g_2(2) = 2(k-2) - 2(k-3) = 2$$

and

$$g_2(k-3) = (k-3)(3) - 2(k-3) = k-3.$$

Each of these values is positive.

*Case 3:* $t = 1$. If $v$ is not in a $(k-1)$-clique $X$, then by Claim 2.36 the vertex adjacent to $v$ with degree $k-1$ is in a $(k-1)$-clique and cluster of size at least 2. In this case $v$ will receive charge $(k-3)/(k-1)$ in total from that cluster. Therefore we may assume that $v$ is in a $(k-1)$-clique $X$.

By Claim 2.44, there exists a $Y \subset X$ such that $|Y| \geq \frac{k-1}{2}$ and every vertex in $Y$ has degree at least $k+1$. Furthermore, each vertex in $Y$ has at least $k-3$ neighbors with degree at least $k$. Therefore each vertex $u \in Y$ donates at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+3}$ charge to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k+1$. It follows that $u$ gives to $v$ charge at least $\frac{1+2/(k-1)}{4}$, and $v$ has charge at least

$$k - 1 + \frac{k-1}{2}\left(\frac{1+2/(k-1)}{4}\right) = k + \frac{k-7}{8},$$

which is at least $k - 2/(k-1)$ when $k \geq 6$. $\square$

We then observe that after that half-discharging,

a) the charge of each vertex in $H_1 \cup L_1$ is at least $k - 2/(k-1)$;

b) the charges of vertices in $H_0$ did not decrease;

c) along every edge from $H_1$ to $L_0$ the charge at least $1/(k-1)$ is sent.

Thus by Claim 2.47, the total charge $F$ of the vertices in $H_0 \cup L_0$ is at least

$$kh + (k-1)\ell + \frac{1}{k-1}\left(\ell(k-1) - e(G_0)\right) \geq k(h+\ell) - \frac{1}{k-1}2(h+\ell) = (h+\ell)\left(k - \frac{2}{k-1}\right),$$

and so by a), the total charge of all the vertices of $G$ is at least $n\left(k - \frac{2}{k-1}\right)$, a contradiction.
$\square$

## 2.7 Sharpness

We will construct sparse 3-connected $k$-critical graphs. As it was pointed out in the introduction, Construction 2.50 and the infinite series of 3-connected sparse 4- and 5-critical graphs are due to Toft [76]. Because $k$-Ore graphs have connectivity 2, this will imply that the bound in Theorem 2.11 on non-$k$-Ore graphs is tight.

**Construction 2.50 (Toft [76])** *Let $u, v, w$ be distinct vertices in a $k$-critical graph $G$ such that $uv \in E(G)$. Suppose also that $\phi(w) = \phi(u) = \phi(v)$ for all proper $(k-1)$-colorings $\phi$ of $G - uv$. Let $S_1 \cup S_2 \cup S_3$ be a partition of the vertex set $X$ of a copy of $K_{k-1}$ such that each $S_i$ is nonempty. We construct $G'$ as $V(G') = V(G) \cup V(X)$ and $E(G') = E(G) \cup E(X) \cup E'$, where*

$$E' = \{ua : a \in S_1\} \cup \{vb : b \in S_2\} \cup \{wc : c \in S_3\}.$$

**Claim 2.51** *If $G$ is a 3-connected $k$-critical graph and $G'$ is created using $G$ and Construction 2.50, then $G'$ is a 3-connected $k$-critical graph.*

**Proof.** We will use the names and definitions from Construction 2.50.

If there is a proper $(k-1)$-coloring $\phi$ of $G'$, then all $k-1$ colors must appear on $X$, so $\phi(u)$ appears on a vertex in $S_2$ or $S_3$. Now either $\phi(v) \neq \phi(u)$ or $\phi(w) \neq \phi(u)$, which contradicts the assumptions of Construction 2.50. So $\chi(G') \geq k$.

By way of contradiction, suppose there exists $f \in E(G')$ such that $\chi(G' - f) \geq k$. If $f \in E(G)$, then let $\phi_1$ be a proper $(k-1)$-coloring of $G - f$. Because $e \in E(G) - f$, we have $\phi_1(u) \neq \phi_1(v)$, and so $\phi_1$ extends easily to $G' - f$. If $f \subset X$, then a proper $(k-1)$-coloring of $G - e$ can be extended to $G' - f$, because $X$ can be colored with $k - 2$ colors, while $N(X) = \{u, v, w\}$ is colored with 1 color. If $f \in E'$, then a proper $(k-1)$-coloring of $G - e$ extends to $G' - f$, because the unique color on $\{u, v, w\}$ can be given to $f \cap X$. Therefore $G'$ is $k$-critical.

Suppose there exists a set $S$ such that $|S| < 3$ and nonempty $A$ and $B$ such that $E(A, B) = \emptyset$ and $A \cup B \cup S = V(G')$. Because critical graphs are 2-connected, $|S| = 2$. Because $X$ is a clique, without loss of generality $X \subseteq A \cup S$. By construction, there is no set of size 2 such that $X = A \cup S$, so $S$ also separates $G - e$. Because $\kappa(G) \geq 3$, $e$ has an endpoint in each component of $G - S - e$. Now the components of $G' - S$ are connected via paths through $X$. $\square$

The assumptions in Construction 2.50 are strong. Most edges $uv$ in $k$-critical graphs can not be matched with a third vertex $w$, and some $k$-critical graphs do not have any edge-vertex pairs $(uv, w)$ that satisfy the assumptions. We will construct for each $k$ an infinite family of sparse graphs with high connectivity, $\mathbb{G}_k$, that do satisfy the assumptions.

The family is generated for each $k$ by finding a small 3-connected $k$-critical graph $G'_k$ such that $\rho_k(G'_k) = \alpha_k$. We will describe a subgraph $H'_k \leq G'_k$ with two vertices, $u$ and $w$, such that $\phi'(u) = \phi'(w)$ in every proper $(k-1)$-coloring $\phi'$ of $H'_k$. Construction 2.50 can then be applied to $G'_k$, using any edge $uv$ that is not in $H'_k$ and $u, v, w$ are distinct vertices. Because Construction 2.50 does not decrease the degree of $u$, this process can be iterated indefinitely to populate $\mathbb{G}_k$.

Note that Construction 2.50 adds the same number of vertices and edges as Ore's composition with $G_2 = K_k$. Therefore $\rho_k(G) = \alpha_k$ for every graph $G \in \mathbb{G}_k$. Furthermore, $G$ is also $k$-critical and 3-connected, and therefore $G$ is not $k$-Ore. This implies the sharpness of Theorem 2.11.

All that is left is to find suitable graphs for $G'_k$ and $H'_k$. Figure 2.4 illustrates $G'_4$ and $G'_5$. We will need a second construction for larger $k$.

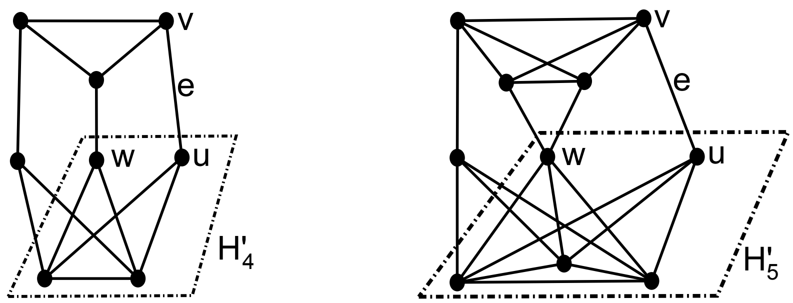


Figure 2.4: Graphs $G'_4$ and $G'_5$, with appropriate substructures labeled for constructing $\mathbb{G}_4$ and $\mathbb{G}_5$.

**Construction 2.52** *Fix a $t$ such that $1 \leq t < k/2$. Let*

$$V(H_{k,t}) = \{u_1, u_2, \ldots, u_{k-1}, v_1, v_2, \ldots, v_{k-1}, w\}$$

*and*

$$E(H_{k,t}) = \{u_iu_j : 1 \leq i < j \leq k-1\} \cup \{v_iv_j : 1 \leq i < j \leq k-1\} \cup \{u_iv_j : i, j \leq t\}$$

$$\cup \{wu_i : i > t\} \cup \{wv_i : i > t\}.$$

Note that $H_{k,1}$ is a $k$-Ore graph, $H_{k,t}$ is $k$-critical, $\kappa(H_{k,t}) = t+1$, $|V(H_{k,t})| = 2k-1$, and $|E(H_{k,t})| = k(k-1) - 2t + t^2$. Moreover, $\rho_k(H_{k,2}) = \alpha_k$. For $k \geq 6$, we choose $G'_k = H_{k,2}$. We will next find $H'_k$ for $k \geq 6$, which will complete the argument.

**Claim 2.53** *If $H'_k = H_{k,2} - \{u_1v_1, u_1v_2\}$, then $\phi'(u_1) = \phi'(w)$ in every proper $(k-1)$-coloring $\phi'$ of $H'_k$.*

**Proof.** Let $\phi'$ be a proper $(k-1)$-coloring of $H'_k$. Note that all $k-1$ colors appear on $\{u_1, u_2, \ldots, u_{k-1}\}$ and appear again on $\{v_1, v_2, \ldots, v_{k-1}\}$. Thus $\phi'(w)$ appears on a vertex $a \in \{u_1, u_2\}$ and again on a vertex $b \in \{v_1, v_2\}$. So $ab \notin E(G)$, which implies that $a = u_1$. $\square$

Next, we repeat the earlier statement of sharpness in Theorem 2.11.

**Corollary 2.8** *If $n \equiv 1 \pmod{k-1}$ and $n \geq k$, then $f_k(n) = F(k, n)$.*

*If one of the following holds:*

1. $k = 4$,

2. $k = 5$,

3. $k = 6$ *and* $n \equiv 0 (\mathrm{mod}\, 5)$,

4. $k = 6$ *and* $n \equiv 2 (\mathrm{mod}\, 5)$, *or*

5. $k = 7$ *and* $n \equiv 2 (\mathrm{mod}\, 6)$,

*then*

$$f_k(n) = \lceil F(k, n) + z_k \rceil .$$

**Proof.** By (2.1), if we construct an $n_0$-vertex $k$-critical graph for which our lower bound on $f_k(n_0)$ is exact, then the bound on $f_k(n)$ is exact for every $n$ of the form $n_0 + s(k-1)$. So, we only need to construct

1. a $k$-critical $k$-vertex graph with $\binom{k}{2}$ edges,

2. a 4-critical 6-vertex graph with 10 edges,

3. a 4-critical 8-vertex graph with 13 edges,

4. a 5-critical 7-vertex graph with 16 edges,

5. a 5-critical 8-vertex graph with 18 edges,

6. a 5-critical 10-vertex graph with 22 edges,

7. a 6-critical 10-vertex graph with 28 edges,

8. a 6-critical 12-vertex graph with 33 edges, and

9. a 7-critical 14-vertex graph with 46 edges.

Case (a) follows from $K_k$. The other eight cases are presented in Figure 2.5. If $G$ is a $(k-1)$-critical graph $H$ plus a vertex $v$ whose neighborhood is $V(H)$, then $G$ is $k$-critical. Cases (b), (d), (e), and (g) are created using this construction on odd cycles and $k$-Ore graphs.

Let $H$ be a $k$-critical graph, and let $uv \in E(H)$. If $H' = H - uv + w$, where $N_{H'}(w) = N_H(u) - v$, then in every proper $(k-1)$-coloring $\phi$ of $H'$, $\phi(u) = \phi(v) = \phi(w)$. Moreover, if any edge of $H'$ is deleted, then there is a proper $(k-1)$-coloring of $H'$ where $|\{\phi(u), \phi(v), \phi(w)\}| \geq 2$. Let $S_1 \cup S_2 \cup S_3$ be a partition of the vertex set $X$ of a copy of $K_{k-1}$ such that each $S_i$ is nonempty. If $G$ is a graph where $V(G) = V(H') \cup V(X)$ and $E(G) = E(H') \cup E(X) \cup E'$, where

$$E' = \{ua : a \in S_1\} \cup \{vb : b \in S_2\} \cup \{wc : c \in S_3\},$$

then $G$ is $k$-critical. The proof that $G$ is $k$-critical follows exactly the proof of Claim 2.51. Cases (c), (f), (h), and (i) are created using this construction with $H = K_k$. $\square$

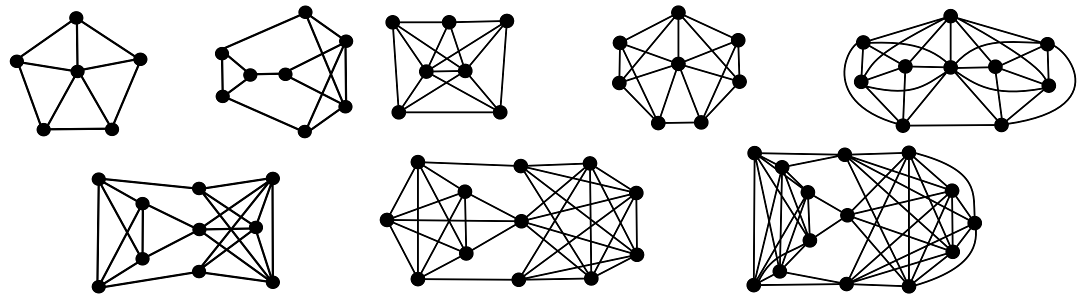

Figure 2.5: Minimal $k$-critical graphs.

## 2.8 Algorithm

Recall that $\rho_{k,G}(W) = (k+1)(k-2)|W| - 2(k-1)|E(G[W])|$ and that $P_k(G)$ is the minimum of $\rho_{k,G}(W)$ over all nonempty $W \subseteq V(G)$. We will also use the related parameter $\widetilde{P}_k(G)$ which is the minimum of $\rho_{k,G}(W)$ over all $W \subset V(G)$ with $2 \leq |W| \leq |V(G)| - 1$.

### 2.8.1 Procedure R1

The input of the procedure $R1_k(G)$ is a graph $G$. The output is one of the following five:

(S1) a nonempty set $R \subseteq V(G)$ with $\rho_{k,G}(R) \leq k(k-3)$, or

(S2) conclusion that $k(k-3) < \widetilde{P}_k(G) < (k+1)(k-2)$ and a nonempty set $R \subsetneq V(G)$ with $\rho_{k,G}(R) = \widetilde{P}_k(G)$, or

(S3) conclusion that $\widetilde{P}_k(G) < 2(k-1)(k-2)$, and a set $R \subset V(G)$ with $2 \leq |R| \leq n-1$ and $\rho_{k,G}(R) = \widetilde{P}_k(G)$, or

(S4) conclusion that $\widetilde{P}_k(G) = 2(k-1)(k-2)$, and a set $R \subset V(G)$ with $k \leq |R| \leq n-1$ and $\rho_{k,G}(R) = 2(k-1)(k-2)$, or

(S5) conclusion that $\widetilde{P}_k(G) \geq 2(k-1)(k-2)$ and that every set $R \subseteq V(G)$ with $\rho_{k,G}(R) = 2(k-1)(k-2)$ has size $k-1$ and induces $K_{k-1}$.

First we calculate $\rho_k(V(G))$, and if it is at most $k(k-3)$, then we are done. Suppose

$$(k+1)(k-2)|V(G)| - 2(k-1)|E(G)| \geq 1 + k(k-3). \tag{2.20}$$

Consider the auxiliary network $H = H(G)$ with vertex set $V \cup E \cup \{s, t\}$ and the set of arcs $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{sv : v \in V\}$, $A_2 = \{et : e \in E\}$, and $A_3 = \{ve : v \in V, e \in E, v \in e\}$. The capacity $c$ of each $sv \in A_1$ is $(k+1)(k-2)$, of each $et \in A_2$ is $2(k-1)$, and of each $ve \in A_3$ is $\infty$.

Since the capacity of the cut $(\{s\}, V(H) - s)$ is finite, $H$ has a maximum flow $f$. Let $M(f)$ denote the value of $f$, and let $(S, T)$ be the minimum cut in it. By definition, $s \in S$ and $t \in T$. Let $S_V = S \cap V$, $S_E = S \cap E$, $T_V = T \cap V$, and $T_E = T \cap E$.

Since $c(ve) = \infty$ for every $v \in e$,

$$\text{no edge of } H \text{ connects } S_V \text{ to } T_E. \tag{2.21}$$

It follows that if $e = vu$ in $G$ and $e \in T_E$, then $v, u \in T_V$. On the other hand, if $e = vu$ in $G$, $v, u \in T_V$ and $e \in S_E$, then moving $e$ from $S_E$ to $T_E$ would decrease the capacity of the cut by $2(k-1)$, a contradiction. So, we get

**Claim 2.54** $T_E = E(G[T_V])$.

By the claim,

$$M(f) = \min_{W \subseteq V} \Big\{(k+1)(k-2)|W| + 2(k-1)(|E| - |E(G[W])|\Big\} = 2(k-1)|E| + \min\Big\{P_k(G), 0\Big\}. \tag{2.22}$$

So, if $M(f) < 2(k-1)|E|$, then $P_k(G) < 0$ and any minimum cut gives us a set with small potential. Otherwise, consider for every $e_0 \in E$ and every vertex $v_0$ not incident to $e_0$, the network $H_{e_0,v_0}$ that has the same vertices and edges and differs from $H$ in the following:

(i) the capacity of the edge $e_0t$ is not $2(k-1)$ but $2(k-1)+2(k-1)(k-2)=2(k-1)^2$;

(ii) for every $v \in V(G)-v_0$, the capacity of the edge $sv$ is $(k+1)(k-2)-\frac{1}{2n}$;

(iii) the capacity of the edge $sv_0$ is $(k+1)(k-2)-\frac{1}{2n}+2(k-1)(k-2)+1$.

Then for every $e_0 \in E$ and $v_0 \in V(G)$, the capacity of the cut $(V(H_{e_0,v_0})-t,t)$ is $2(k-1)|E|+2(k-1)(k-2)$. Since this is finite, $H_{e_0,v_0}$ has a maximum flow $f_{e_0,v_0}$. As above, let $M(f_{e_0,v_0})$ denote the value of $f_{e_0,v_0}$, and let $(S,T)$ be the minimum cut in it. By definition, $s \in S$ and $t \in T$. Let $S_V = S \cap V$, $S_E = S \cap E$, $T_V = T \cap V$, and $T_E = T \cap E$. By the same argument as above, (2.21) and Claim 2.54 hold. Let $M_k(G)$ denote the minimum value over $M(f_{e_0,v_0})$.

By (2.20), for every $e_0 \in E$ and $v_0 \in V(G)$, the capacity of the cut $(s, V(H_{e_0,v_0})-s)$ is at least

$$\left((k+1)(k-2)-\frac{1}{2n}\right)n+2(k-1)(k-2)+1 \geq 2(k-1)|E|+2(k-1)(k-2)+\frac{1}{2}.$$

If the potential of some nonempty $W \neq V$ is less than $(k+1)(k-2)$, then $G[W]$ contains some edge $e_0$ and there is $v_0 \in V-W$. So, in the network $H_{e_0,v_0}$, the capacity of the cut $(\{s\}\cup(V-W)\cup(E-E(G[W])), W\cup E(G[W])\cup\{t\})$ is

$$\left((k+1)(k-2)-\frac{1}{2n}\right)|W|+2(k-1)(|E|-|E(G[W])|)=2(k-1)|E|+\rho_{k,G}(W)-\frac{|W|}{2n}.$$

On the other hand, for every nonempty $W \neq V$, every edge $e_0$ and every $v_0 \in V$, the capacity of the cut $(\{s\}\cup(V-W)\cup(E-E(G[W])), W\cup E(G[W])\cup\{t\})$ is at least

$$\left((k+1)(k-2)-\frac{1}{2n}\right)|W|+2(k-1)(|E|-|E(G[W])|)>2(k-1)|E|+\rho_{k,G}(W)-\frac{1}{2}.$$

Thus if $M_k(G) \leq k(k-3)+2(k-1)|E|$, then (S1) holds and if $k(k-3)+2(k-1)|E| < M_k(G) <$

$(k+1)(k-2)+2(k-1)|E|$, then (S2) holds. Note that if a nonempty $W$ is independent, then $E(G[W]) = \emptyset$, and the capacity of the cut $(\{s\} \cup (V-W) \cup (E-E(G[W])), W \cup E(G[W]) \cup \{t\})$ is at least

$$2(k-1)|E| + 2(k-1)(k-2) + (k+1)(k-2).$$

Thus, if

$$(k+1)(k-2) + 2(k-1)|E| \leq M_k(G) < 2(k-1)(k-2) - 1 + 2(k-1)|E|,$$

then (S3) holds.

Similarly, if

$$2(k-1)(k-2) - 1 + 2(k-1)|E| \leq M_k(G) < 2(k-1)(k-2) + 2(k-1)|E| - \frac{k-1}{2n},$$

then there exists $W \subset V$ with $k \leq |W| \leq n-1$ with potential $2(k-1)(k-2)$. Then (S4) holds. Finally, if $M_k(G) \geq 2(k-1)(k-2) + 2(k-1)|E| - \frac{k-1}{2n}$, then (S5) holds.

Since the complexity of the max-flow problem is at most $Cn^2\sqrt{|E|}$ and $|E| \leq kn$, the procedure takes time at most $Ck^{1.5}n^{4.5}$.

### 2.8.2 Outline of the algorithm

We consider the outline for $k \geq 7$. For $k \leq 6$, everything is quite similar and easier.

Let the input be an $n$-vertex $e$-edge graph $G$. The algorithm will be recursive. The output will be either a proper coloring of $G$ with $k-1$ colors or return a nonempty $R \subseteq V(G)$ with $\rho_{k,G}(R) \leq k(k-3)$. The algorithm runs through 7 steps, which are listed below. If a step is triggered, then a recursive call is made on a smaller graph $G'$. Some steps will then require a second recursive call on another graph $G''$.

The algorithm does not make the recursive call if $|E(G')| \leq k^2/2$. In this case, $G'$ is either $(k-2)$-degenerate or $K_k$ minus a matching, and so is easily $(k-1)$-colorable in time $O(k|V(G')|^2)$. This also holds for $G''$.

After all calls have been made, the algorithm will return a proper coloring or a subgraph with low potential, skipping the other steps.

1) We check whether $G$ is disconnected or has a cut-vertex or has a vertex of degree at most $k-2$. In the case of any "yes", we consider smaller graphs (and at the end will reconstruct the coloring).

2) We run $R1_k(G)$ and consider possible outcomes. If the outcome is (S1), we are done.

3) Suppose the outcome is (S2). The algorithm makes a recursive call on $G' = G[R]$, which returns a proper $(k-1)$-coloring $\phi$. Let $G''$ be the graph $Y(G, R, \phi)$ described in Definition 2.12. The proof of Claim 2.26 yields that $P_k(G'') \geq k(k-3)$, and thus the recursive call will return with a proper coloring. Let $\phi'$ be the proper coloring returned. It is straightforward to combine the colorings $\phi$ and $\phi'$ into a proper $(k-1)$-coloring of $G$.

4) Suppose the outcome is (S3) or (S4). We choose $i$ using (2.6) and add $i$ edges to $G[R]$ as in the proof of Claim 2.33. Denote the new graph $G'$. The algorithm makes a recursive call on $G' = G[R]$, which returns a proper $(k-1)$-coloring $\phi$. Let $G''$ be the graph $Y(G, R, \phi)$ described in Definition 2.12. The proof of Claim 2.33 yields that $P_k(G'') \geq k(k-3)$, and thus the recursive call will return with a proper coloring. Let $\phi'$ be the coloring returned. It is straightforward to combine the colorings $\phi$ and $\phi'$ into a proper $(k-1)$-coloring of $G$.

5) So, the only remaining possibility is (S5). For every $(k-1)$-vertex $v \in V(G)$, check whether there is a $(k-1)$-clique $K(v)$ containing $v$ (since (S5) holds, such a clique is unique, if exists). We certainly can do this in $O(kn^2)$ time. Let $a_v$ denote the neighbor of $v$ not in

$K(v)$ and $T_v$ denote the set of $(k-1)$-vertices in $K(v)$. Then for every pair $(v, K(v))$ such that $d(v) = k-1$ and $K(v)$ exists, do the following:

(5.1) If there is $w \in T_v - v$ with $a_w \neq a_v$, then consider the graph $G' = G - v - w + a_v a_w$. By Claim 2.27, $P_k(G') > k(k-3)$. So, the algorithm will return with a proper $(k-1)$-coloring of $G'$, which we then extend to $G$.

(5.2) Suppose that $|T_v| \geq 2$ and $K(v) - T_v$ contains a vertex $x$ of degree at most $k-2+|T_v|$. Let $G' = G - x + v'$, where the closed neighborhood of $v'$ is the same as of $v$. By Claim 2.35, $P_k(G') > k(k-3)$, so the algorithm returns a proper $(k-1)$-coloring of $G'$, which is then extended to $G$ as in the proof of Claim 2.35.

(5.3) Suppose that $T_v = \{v\}$ and $K(v)$ contains at least $k/2 - 1$ vertices of degree $k$. Since (S5) holds, there is $x \in K(v) - v$ of degree at most $k$ not adjacent to $a_v$. Let $x_1$ and $x_2$ be the neighbors of $x$ outside of $K_v$. Let $G'$ be obtained from $G - v$ by adding edges $a_v x_1$ and $a_v x_2$. By the proof of Claim 2.44, $P_k(G') > k(k-3)$, so the algorithm finds a proper $(k-1)$-coloring of $G'$, which is then extended to $G$ as in the proof of Claim 2.44.

6) Let $C_v$ denote the *cluster of* $v$, i.e. the set of vertices that have the same closed neighborhood as $v$. We certainly can find $C_v$ for every $(k-1)$-vertex $v \in V(G)$ in $O(kn^2)$ time. Then for every pair $(v, C_v)$ such that $d(v) = k-1$, do the following:

(6.1) Suppose that $|C_v| \geq 2$ and $N(v) - C_v$ contains a vertex $x$ of degree at most $k-2+|T_v|$. Then do the same as in (5.2).

(6.2) Suppose that $N(v) - C_v$ contains a $(k-1)$-vertex $w$ and that $|C_w| \leq |C_v|$. If $v$ is not in a $(k-1)$-clique, then consider $G' = G - w + v'$, where the $v'$ is a new vertex whose closed neighborhood is the same as that of $v$. By the proof of Claim 2.36, $P_k(G') > k(k-3)$, and so we find a proper $(k-1)$-coloring of $G'$ and then extend it to $G$ as in the proof of Claim 2.36.

7) Let $L_0$, $H_0$, and $e_0$ be as defined in Definition 2.46. If $e_0 \geq 2(|L_0| + |H_0|)$, then iteratively remove vertices in $L_0$ with at most two neighbors in $H_0$ and vertices in $H_0$ with at most two neighbors in $L_0$. Let $H$ be the graph that remains, and $G' = G - V(H)$. Clearly $P_k(G') > k(k-3)$, so the recursive call returns a proper coloring of $G'$. Give each vertex $v \in V(H)$ a list of colors $L(v) = \{c_1, \ldots, c_{k-1}\}$, then remove from that list the colors on $N(v) \cap V(G')$. Orient the edges of $H$ as in Case 1 of the proof of Lemma 2.17, then extend the coloring of $G'$ to a proper coloring of $G$ by list coloring $H$ using the system described in the proof to Lemma 2.15.

### 2.8.3 Analysis of correctness and running time

The proof of Theorem 2.5 consists in proving that at least one of the situations in steps 1 through 7 described above must happen. Moreover, the main theorem proves that $G', G'' \prec G$ by a partial order with finite descending chains, and therefore the algorithm will terminate. We claim that the algorithm makes at most $O(k^2 n^2 \log(n))$ recursive calls, and each call only takes $O(k^{1.5} n^{4.5})$ time, so the algorithm runs in $O(k^{3.5} n^{6.5} \log(n))$ time. Let $e$ denote the number of edges in $G$. Note that if $\rho_k(V(G)) \geq 0$, then $e \leq nk$.

If a call of the recursive algorithm terminates on Step 2, we will refer to this as 'Type 1,' a call terminating on Step 1, 3, 4, 5.1, 5.3, 6.1, or 7 is 'Type 2,' and a call terminating on Step 5.2 or 6.2 is 'Type 3.' If a call is made on a Type 1, then the whole algorithm stops.

If a Type 3 happens, then the algorithm makes one recursive call with a graph with the same number of edges and strictly more pairs of vertices with the same closed neighborhood. The proof of Claim 2.34 shows that the number of pairs of vertices with the same closed neighborhood is bounded by $kn$. We have that at least one out of every $kn$ consecutive recursive calls is Type 1 or 2.

Consider an instance of a Type 2 call with input graph $H$. If $H'$ is the graph in the first recursive call and $H''$ is the graph in the second call (if necessary), then $|E(H')|, |E(H'')| < |E(H)|$ and $|E(H)| \geq |E(H')| + |E(H'')| - k^2/2$. Let $g_k(e,i)$ denote the number of Type 2 recursive calls made on graphs with $i$ edges. Note that if $i \leq k^2/2$ then $g_k(e,i) = 0$ and $g_k(e,e) = 1$. By tracing calls up through their parent calls, it follows that

$$e \geq i + (g_k(e,i) - 1)\left(i - k^2/2\right)$$

when $i > k^2/2$. Therefore

$$g_k(e,i) < \frac{e}{(i - k^2/2)}.$$

The total number of calls that our algorithm makes is at most

$$kn \sum_{i=k^2/2+1}^{e} g_k(e,i) < kne\log(e).$$

Because $e \leq nk$, we have that the total number of calls is $O(k^2 n^2 \log(n))$.

A call may run algorithm R1 once, which will take $O(k^{1.5}n^{4.5})$ time. Constructing the appropriate graphs for recursion in Steps 3, 4, 5, and 6 will take $O(kn^2)$ time. Combining colorings in Steps 1, 3, 4, 5, and 6 will take $O(n)$ time. Properly coloring a degenerate graph will take $O(kn^2)$ time, which happens at most twice. The only thing left to consider is Step 7. Iteratively removing vertices will take $O(n^2)$ time. Splitting the vertices and orienting the edges using network flows will take $O(n^{2.5}k^{0.5})$ time. Finding a kernel will take $O(n^2)$ time, which happens at most $n$ times. Therefore each instance of the algorithm takes $O(k^{1.5}n^{4.5})$ time.

# Chapter 3

# Applications of Theorem 2.5

## 3.1 3-colorable Planar Graphs

An *embedding* $\sigma$ of a graph $G$ in a surface $\Sigma$ is an injective mapping of $V(G)$ to a point set $P$ in $\Sigma$ and $E(G)$ to non-self-intersecting curves in $\Sigma$ such that
(a) for all $v \in V$ and $e \in E$, $\sigma(v)$ is never an interior point of $\sigma(e)$,
(b) $\sigma(v)$ is an endpoint of $\sigma(e)$ if and only if $v$ is a vertex of $e$, and
(c) for all $e, h \in E$, $\sigma(h)$ and $\sigma(e)$ can intersect only in vertices of $P$.
A graph is *planar* if it has an embedding in the plane. A graph with its embedding in the [projective] plane is a *[projective] plane* graph. A cycle in a graph embedded in $\Sigma$ is *contractible* if it splits $\Sigma$ into two surfaces where one of them is homeomorphic to a disk.

We restate Theorem 2.5 for 4-critical graphs:

**Theorem 3.1** *If $G$ is a 4-critical $n$-vertex graph, then*

$$|E(G)| \geq \frac{5n-2}{3}.$$

One of the most famous results of graph theory is that planar graphs are 4-colorable [7, 8]. An entire area of research centers on variations of this result, one prominent direction is showing that planar graphs with additional restrictions are 3-colorable. The most famous among those results is Grötzsch's Theorem [36] that triangle-free planar graphs are 3-colorable. The original proof of Grötzsch's Theorem is somewhat sophisticated. There were subsequent simpler proofs (see, e.g. [74] and references therein). Using Theorem 3.1, we can give a very short proof of Grötzsch's Theorem.

**Theorem 3.2 ([36])** *Every triangle-free planar graph is 3-colorable.*

**Proof.** Let $G$ be a plane graph with fewest elements (vertices and edges) for which the theorem does not hold. Such a graph must be 4-critical. Let $G$ have $n$ vertices, $m$ edges, and $f$ faces.

CASE 1: $G$ has no 4-faces. Then $5f \leq 2m$ and so $f \leq 2m/5$. By this and Euler's Formula $n - m + f = 2$, we have $n - 3m/5 \geq 2$, i.e., $m \leq \frac{5n-10}{3}$, a contradiction to Theorem 3.1.

CASE 2: $G$ has a 4-face $(x, y, z, u)$. Since $G$ has no triangles, $xz, yu \notin E(G)$. If the graph $G_{xz}$ obtained from $G$ by merging $x$ and $z$ has no triangles, then by the minimality of $G$, it is 3-colorable, and so $G$ also is 3-colorable. Thus $G$ has an $x, z$-path $< x, v, w, z >$ of length 3. Since $G$ itself has no triangles, $\{y, u\} \cap \{v, w\} = \emptyset$, and there are no edges joining $\{y, u\}$ and $\{v, w\}$. Now $G$ has no $y, u$-path of length 3, since such a path must cross the path $< x, v, w, z >$. Thus the graph $G_{yu}$ obtained from $G$ by merging $y$ and $u$ has no triangles, and so, by the minimality of $G$, is 3-colorable. Now $G$ also is 3-colorable, a contradiction. $\square$

This section presents short proofs of some other theorems on proper 3-coloring of graphs close to planar. Most of these results are strengthenings of Grötzsch's Theorem. If $G$ is a triangle-free planar graph, then the following are true:

1. If $H$ is a graph such that $G = H - h$ for some edge $h$, then $\chi(H) \leq 3$. [3, 43]

2. If $H$ is a graph such that $G = H - v$ for some vertex $v$ of degree 3, then $\chi(H) \leq 3$. [43]

3. If $F$ is a face of $G$ of length at most 5, then each proper 3-coloring of $F$ can be extended to a proper 3-coloring of $G$. [37]

4. Each coloring of two non-adjacent vertices can be extended to a proper 3-coloring of $G$. [4]

Another way to strengthen Grötzsch's Theorem is to allow a small number of triangles. If $G$ is planar and has at most three triangles, then $\chi(G) \leq 3$ [2, 10, 38]. The original proof of this result by Grünbaum [38] was incorrect, and a correct proof was provided by Aksenov [2]. A simpler proof was given by Borodin [10].

Youngs [81] constructed triangle-free graphs in the projective plane that are not 3-colorable. Thomassen [73] showed that if $G$ is embedded in the projective plane without contractible cycles of length at most 4 then $G$ is 3-colorable.

Borodin, Kostochka, Lidický, and Yancey [19] constructed a collection of short proofs of statements that either match or are stronger than each of the above statements. We will present those claims and prove them below.

Without restriction on triangles, Steinberg conjectured [69] that every planar graph without 4- and 5-cycles is 3-colorable. Erdős suggested to relax the conjecture and asked for the smallest $k$ such that every planar graph without cycles of length 4 to $k$ is 3-colorable. The best known bound for $k$ is 7 [11].

We will also present a statement of Borodin, Kostochka, Lidický, and Yancey [19] that is related to Steinberg's conjecture.

### 3.1.1 Short proofs

The following lemma is a well-known tool to reduce the number of 4-faces.

**Lemma 3.3** *Let $G$ be a plane graph and $F = v_0v_1v_2v_3$ be a 4-face in $G$ such that $v_0v_2, v_1v_3 \notin E(G)$. Let $G_i$ be obtained from $G$ by identifying $v_i$ and $v_{i+2}$ where $i \in \{0, 1\}$. If the number of triangles increases in both $G_0$ and $G_1$ then there exists a triangle $v_iv_{i+1}z$ for some $z \in V(G)$*

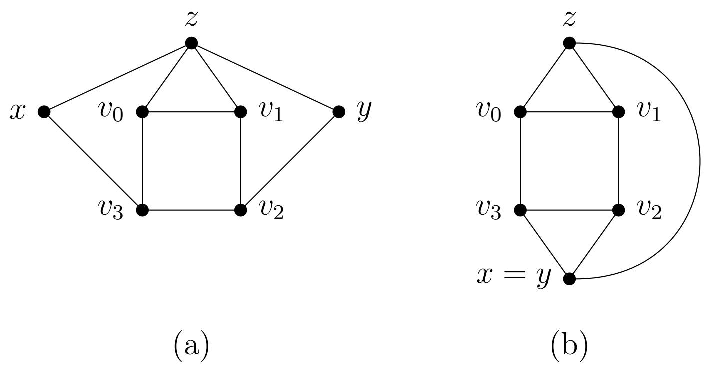


Figure 3.1: Triangle adjacent to a 4-face in Lemma 3.3.

*and $i \in \{0, 1, 2, 3\}$. Moreover, $G$ contains vertices $x$ and $y$ not in $F$ such that $v_{i+1}zxv_{i-1}$ and $v_izyv_{i+2}$ are paths in $G$. Indices are modulo 4. See Figure 3.1.*

**Proof.** This is a restatement of the procedure used in the proof of Theorem 3.2. $\square$

**Theorem 3.4** *If $G$ is a triangle-free planar graph, and $H$ is a graph such that $G = H - h$ for some edge $h$ of $H$, then $H$ is 3-colorable.*

**Proof.** Let $H$ be a smallest counterexample and $G$ be a triangle-free plane graph such that $G = H - h$ for some edge $h$. Let $h = uv$. Let $H$ have $n$ vertices and $m$ edges and $G$ have $f$ faces. Note that $G$ has $n$ vertices and $m - 1$ edges. By minimality, $H$ is 4-critical, so Theorem 3.1 implies $m \geq \frac{5n-2}{3}$.

CASE 1: $G$ has at most one 4-face. This yields $5f - 1 \leq 2(m - 1)$ and hence $f \leq (2m - 1)/5$. By this and Euler's Formula $n - (m - 1) + f = 2$ applied to $G$, we have $5n - 3m + 1 \geq 5$, i.e., $m \leq \frac{5n-4}{3}$. This contradicts Theorem 3.1.

CASE 2: Every 4-face of $G$ contains both $u$ and $v$, and there are at least two such 4-faces $F_x$ and $F_y$. Let $F_x = ux_1vx_2$ and $F_y = uy_1vy_2$. If there exists $z \in \{x_1, x_2\} \cap \{y_1, y_2\}$ then $z$ has degree two in $G$ which contradicts the 4-criticality of $G$.

Let $G'$ be obtained from $G$ by merging $x_1$ and $x_2$ into a new vertex $x$. If $G'$ is not triangle-free, then there is a path $P$ of vertices $x_1q_1q_2x_2$ in $G$, where $q_1, q_2 \notin F_x$. Since $P$ must cross $uy_1v$ and $uy_2v$, we may assume that $y_1 = q_1$ and $y_2 = q_2$. However, $y_1y_2 \notin E(G)$. This contradicts the existence of $P$. Hence $G'$ is triangle-free. Let $H' = G' + h$. By the minimality of $H$, there is a proper 3-coloring $\varphi$ of $H'$. This contradicts that $H$ is not 4-colorable, since $\varphi$ can be extended to $H$ by letting $\varphi(x_1) = \varphi(x_2) = \varphi(x)$.

CASE 3: $G$ has a 4-face $F$ with vertices $v_0v_1v_2v_3$ in cyclic order, where $h$ is neither $v_0v_2$ nor $v_1v_3$. Since $G$ is triangle-free, neither $v_0v_2$ nor $v_1v_3$ are edges of $G$. Lemma 3.3 implies that either $v_0$ can be merged with $v_2$ or $v_1$ can be merged with $v_3$ without creating a triangle. Without loss of generality assume that $G'$, obtained by from $G$ identification of $v_0$ and $v_2$ to a new vertex $v$, is triangle-free. Let $H' = G' + h$. By the minimality of $H$, there is a proper 3-coloring $\varphi$ of $H'$. The proper 3-coloring $\varphi$ can be extended to $H$ by letting $\varphi(v_0) = \varphi(v_2) = \varphi(v)$, which contradicts the 4-criticality of $H$. $\square$

**Theorem 3.5** *If $G$ is a triangle-free planar graph, and $H$ is a graph such that $G = H - v$ for some vertex $v$ of degree 4, then $H$ is 3-colorable.*

**Proof.** Let $H$ be a smallest counterexample, and let $G$ be a triangle-free plane graph such that $G = H - v$ for some vertex $v$ of degree 4. Let $H$ have $n$ vertices and $m$ edges, and let $G$ have $f$ faces. Note that $G$ has $n-1$ vertices and $m-4$ edges. By minimality, $H$ is 4-critical, so Theorem 3.1 implies $m \geq \frac{5n-2}{3}$.

CASE 1: $G$ has no 4-faces. Here $5f \leq 2(m-4)$, and hence $f \leq 2(m-4)/5$. By this and Euler's Formula $(n-1) - (m-4) + f = 2$ applied to $G$, we have $5n - 3m - 8 \geq -5$, i.e., $m \leq \frac{5n-3}{3}$. This contradicts Theorem 3.1.

CASE 2: $G$ has a 4-face $F$ with vertices $v_0v_1v_2v_3$ in the cyclic order. Since $G$ is triangle-free, neither $v_0v_2$ nor $v_1v_3$ is an edge of $G$, and Lemma 3.3 applies. Without loss of generality, assume that $G_0$ obtained from $G$ by merging $v_0$ and $v_2$ is triangle-free.

By the minimality of $H$, the graph obtained from $H$ by merging $v_0$ and $v_2$ is 3-colorable, since it satisfies the hypothesis. Now $H$ is 3-colorable, a contradiction. $\square$

**Theorem 3.6** *If $G$ is a triangle-free planar graph, and $F$ is a face of $G$ of length at most 5, then each proper 3-coloring of $F$ can be extended to a proper 3-coloring of $G$.*

**Proof.** Let the proper 3-coloring of $F$ be $\varphi$.

CASE 1: $F$ is a 4-face where $v_0v_1v_2v_3$ are its vertices in cyclic order.

CASE 1.1: Suppose $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) = \varphi(v_3)$. Let $G'$ be obtained from $G$ by adding a vertex $v$ adjacent to $v_0, v_1, v_2$ and $v_3$. Since $G'$ satisfies the assumptions of Theorem 3.5, there is a proper 3-coloring $\varrho$ of $G'$. In any such proper 3-coloring, $\varrho(v_0) = \varrho(v_2)$ and $\varrho(v_1) = \varrho(v_3)$. Hence by renaming the colors in $\varrho$ we obtain an extension of $\varphi$ to a proper 3-coloring of $G$.

By symmetry, the other subcase is the following.

CASE 1.2: $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) \neq \varphi(v_3)$. Let $G'$ be obtained from $G$ by adding the edge $v_1v_3$. Since $G'$ satisfies the assumptions of Theorem 3.4, there is a proper 3-coloring $\varrho$ of $G'$. In any such proper 3-coloring, $\varrho(v_1) \neq \varrho(v_3)$, and hence $\varrho(v_0) = \varrho(v_2)$. By renaming the colors in $\varrho$ we obtain an extension of $\varphi$ to a proper 3-coloring of $G$.

CASE 2: $F$ is a 5-face where $v_0v_1v_2v_3v_4$ are its vertices in cyclic order. Observe that up to symmetry there is just one proper coloring of $F$. So without loss of generality assume that $\varphi(v_0) = \varphi(v_2)$ and $\varphi(v_1) = \varphi(v_3)$.

Let $G'$ be obtained from $G$ by adding a vertex $v$ adjacent to $v_0, v_1, v_2$ and $v_3$. Since $G'$ satisfies the assumptions of Theorem 3.5, there is a proper 3-coloring $\varrho$ of $G'$. Note that in any such proper 3-coloring $\varrho(v_0) = \varrho(v_2)$ and $\varrho(v_1) = \varrho(v_3)$. Hence by renaming the colors in $\varrho$ we can extend $\varphi$ to a proper 3-coloring of $G$. $\square$

**Theorem 3.7** *If $G$ is a triangle-free planar graph, then each coloring of two non-adjacent vertices can be extended to a proper 3-coloring of $G$.*

**Proof.** Let $G$ be a smallest counterexample, and let $u, v \in V(G)$ be the two non-adjacent vertices colored by $\varphi$. If $\varphi(u) \neq \varphi(v)$, then the result follows from Theorem 3.4 by considering graph obtained from $G$ by adding the edge $uv$. Hence assume that $\varphi(u) = \varphi(v)$.

CASE 1: $G$ has at most two 4-faces. Let $H$ be a graph obtained from $G$ by identification of $u$ and $v$. Any proper 3-coloring of $H$ yields a proper 3-coloring of $G$ where $u$ and $v$ are colored the same. By this and the minimality of $G$, we conclude that $H$ is 4-critical. Let $G$ have $m$ edges, $n+1$ vertices and $f$ faces.

Since $G$ is planar, $5f - 2 \leq 2m$. By this and Euler's formula,

$$2m + 2 + 5(n+1) - 5m \geq 10,$$

and hence $m \leq (5n-3)/3$, a contradiction to Theorem 3.1.

CASE 2: $G$ has at least three 4-faces. Let $F$ be a 4-face with vertices $v_0v_1v_2v_3$ in cyclic order. Since $G$ is triangle-free, neither $v_0v_2$ nor $v_1v_3$ are edges of $G$. Hence Lemma 3.3 applies.

Without loss of generality, let $G_0$ from Lemma 3.3 be triangle-free. By the minimality of $G$, $G_0$ has a proper 3-coloring $\varphi$ where $\varphi(u) = \varphi(v)$ unless $uv \in E(G_0)$. Since $uv \notin E(G)$, without loss of generality $v_0 = u$ and $v_2v \in E(G)$. Moreover, the same cannot happen to $G_1$ from Lemma 3.3, hence $G_1$ contains a triangle. Thus $G$ contains a path $v_1q_1q_2v_3$ where $q_1, q_2 \notin F$, and $G$ also contains a 5-cycle $C = uv_1q_1q_2v_3$ (see Figure 3.2). By Theorem 3.6, $C$ is a 5-face. Hence $u$ has degree 2 and is incident with only one 4-face.

Then there is a 4-face not incident to $u$, and we may repeat the argument. By this symmetric argument, $v$ also has degree 2 and is incident with exactly one 4-face and 5-face.

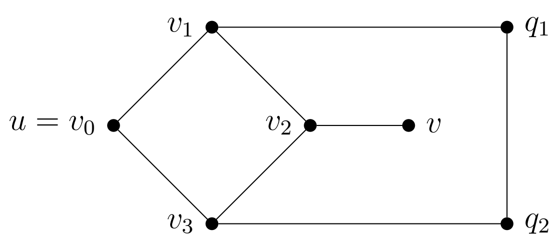


Figure 3.2: Configuration from Theorem 3.7.

However, $G$ has at least one more 4-face where merging of vertices does not result in the edge $uv$, a contradiction to the minimality of $G$. $\square$

**Theorem 3.8** *Let $G$ be a graph embedded in the projective plane such that the embedding has at most two contractible cycles of length 4 or one contractible cycle of length three such that all other cycles of length at most 4 are non-contractible. Under these conditions, $G$ is 3-colorable.*

**Proof.** Let $G$ be a minimal counterexample with $m$ edges, $n$ vertices and $f$ faces. By minimality, $G$ is 4-critical, and $G$ has at most two 4-faces or one 3-face. From embedding, $5f - 2 \leq 2m$. By Euler's formula, $2m + 2 + 5n - 5m \geq 5$. Hence $m \leq (5n - 3)/3$, a contradiction to Theorem 3.1. $\square$

Borodin used in his proof of Theorem 3.9 a technique called *portionwise coloring*. We avoid it and build the proof on the previous results arising from Theorem 3.1.

**Theorem 3.9** *If $G$ is a planar graph containing at most three triangles, then $G$ is 3-colorable.*

**Proof.** Let $G$ be a smallest counterexample. By minimality, $G$ is 4-critical and every triangle is a face. By Theorem 3.6 for every separating 4-cycle and 5-cycle $C$, both the interior and exterior of $C$ contain triangles.

CASE 1: $G$ has no 4-faces. Then $5f-6 \leq 2m$ and by Euler's Formula $3m+6+5n-5m \geq 10$, i.e., $m \leq \frac{5n-4}{3}$. This contradicts Theorem 3.1.

CASE 2: $G$ has a 4-face $F = v_0v_1v_2v_3$ such that $v_0v_2 \in E(G)$. By the minimality, $v_0v_1v_2$ and $v_0v_3v_2$ are both 3-faces and hence $G$ has 4 vertices, 5 edges and it is 3-colorable.

CASE 3: For every 4-face $F = v_0v_1v_2v_3$, neither $v_0v_2$ nor $v_1v_3$ are edges of $G$. By Lemma 3.3, there exist paths $v_0zyv_2$ and $v_1zxv_3$.

CASE 3.1: $G$ contains a 3-prism with one of its 4-cycles being a 4-face. We may assume that this face is our $F$ and $x = y$, see Figure 3.1(b). Theorem 3.6 implies that one of $zv_0v_3x$, $zv_1v_2x$ is a 4-face. Without loss of generality, assume that $zv_1v_2x$ is a 4-face. Let $G_0$ be obtained from $G$ by merging $v_0$ and $v_2$ to a new vertex $v$. Since $G_0$ is not 3-colorable, it contains a 4-critical subgraph $G_0'$. Note that $G_0'$ contains triangle $xvz$ that is not in $G$, but $v_0v_1z$ is not in $G_0'$ since $d(v_1) = 2$ in $G_0$. By the minimality of $G$, there exists another triangle $T$ that is in $G_0$ but not in $G$. By planarity, $x \in T$. Hence there is a vertex $w_1 \neq v_3$ such that $v_0$ and $x$ are neighbors of $w_1$.

By considering merging $v_1$ and $v_3$ and by symmetry, we may assume that there is a vertex $w_2 \neq v_0$ such that $v_3$ and $z$ are neighbors of $w_2$. By planarity we conclude that $w_1 = w_2$. This contradicts the fact that $G$ has at most three triangles. Therefore $G$ is 3-prism-free.

CASE 3.2: $G$ contains no 3-prism with one of its 4-cycles being a 4-face. Then $x \neq y$, see Figure 3.1(a). If $v_0x \in E(G)$, then $G - v_0$ is triangle-free, and Theorem 3.5 gives a proper 3-coloring of $G$, a contradiction. Similarly, $v_1y \notin E(G)$.

Suppose that $zv_0v_3x$ is a 4-face. Let $G'$ be obtained from $G - v_0$ by adding edge $xv_1$. If the number of triangles in $G'$ is at most three, then $G'$ has a proper 3-coloring $\varphi$, by the minimality of $G$. Let $\varrho$ be a proper 3-coloring of $G$ such that $\varrho(v) = \varphi(v)$ if $v \in V(G')$ and $\varrho(v_0) = \varphi(x)$. Since the neighbors of $v_0$ in $G$ are neighbors of $x$ in $G'$, $\varrho$ is a proper 3-coloring, a contradiction. Therefore $G'$ has at least four triangles and hence $G$ contains a vertex $t \neq z$ adjacent to $v_1$ and $x$. Since $v_1y \notin E(G)$, the only possibility is $t = v_2$. Having edge $xv_2$

results in a 3-prism being a subgraph of $G$, which is already excluded. Hence $zv_0v_3x$ is not a face, and by symmetry $zv_1v_2y$ is not a face either.

Since neither $zv_0v_3x$ nor $zv_1v_2y$ is a face, each of them contains a triangle in its interior. Since we know the location of all three triangles, Theorem 3.6 implies that $zyv_2v_3x$ is a 5-face. It also implies that the common neighbors of $z$ and $v_3$ are exactly $v_0$ and $x$, and the common neighbors of $z$ and $v_2$ are exactly $v_1$ and $y$. Without loss of generality, let $zyv_2v_3x$ be the outer face of $G$.

Let $H_1$ be obtained from the 4-cycle $zv_0v_3x$ and its interior by adding edge $zv_3$. The edge $zv_3$ is in only two triangles, and there is only one triangle in the interior of the 4-cycle. Hence by the minimality of $G$, there exists a proper 3-coloring $\varphi_1$ of $H_1$.

Let $H_2$ be obtained from the 4-cycle $zv_1v_2y$ and its interior by adding edge $zv_2$. By the same argument as for $H_1$, there is a proper 3-coloring of $\varphi_2$ of $H_2$.

If we rename the colors in $\varphi_2$ so that $\varphi_1(z) = \varphi_2(z)$, $\varphi_1(v_0) = \varphi_2(v_2)$ and $\varphi_1(v_3) = \varphi_2(v_1)$, then $\varphi_1 \cup \varphi_2$ is a proper 3-coloring of $G$, a contradiction. $\square$

**Theorem 3.10** *If $G$ is a 4-chromatic projective planar graph where every vertex is in at most one triangle, then $G$ contains a cycle of length 4,5 or 6.*

**Proof.** Let $G$ be a 4-chromatic projective plane graph where every vertex is in at most one triangle and let $G$ be 4-,5- and 6-cycle free. Hence $G$ contains a 4-critical subgraph $G'$. Let $G'$ have $m$ edges, $n$ vertices and $f$ faces. Since $G'$ is also 4-,5- and 6-cycle-free and every vertex is in at most one triangle, we get $f \leq \frac{n}{3} + \frac{2m-n}{7}$. By Euler's formula, $7n + 6m - 3n + 21n - 21m \geq 21$. Hence $m \leq 5n/3 - 21/15$, a contradiction to Theorem 3.1. $\square$

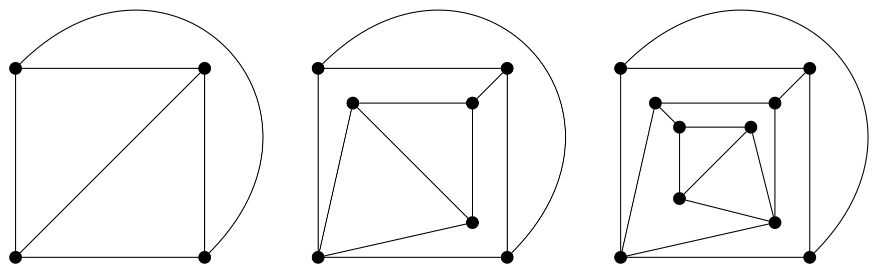

Figure 3.3: First three 4-critical graphs from the family described by Thomas and Walls [72].

### 3.1.2 Tightness

This section shows examples where Theorems 3.4, 3.5, 3.6, 3.7, 3.9, and 3.8 are tight.

Theorem 3.4 is best possible because there exists an infinite family [72] of 4-critical graphs that become triangle-free and planar after removal of just two edges. See Figure 3.3. Moreover, the same family shows also the tightness of Theorem 3.9, since the construction has exactly four triangles.

Aksenov [2] showed that every plane graph with one 6-face $F$ and all other faces being 4-faces has no proper 3-coloring in which the colors of vertices of $F$ form the sequence $(1, 2, 3, 1, 2, 3)$. This implies that Theorem 3.6 is best possible. It also implies that Theorems 3.5 and 3.7 are best possible. See Figure 3.4 for constructions where coloring of three vertices or an extra vertex of degree 5 force a coloring $(1, 2, 3, 1, 2, 3)$ of a 6-cycle.

Theorem 3.8 is best possible because there exist embeddings of $K_4$ in the projective plane with three 4-faces or with two 3-faces and one 6-face.

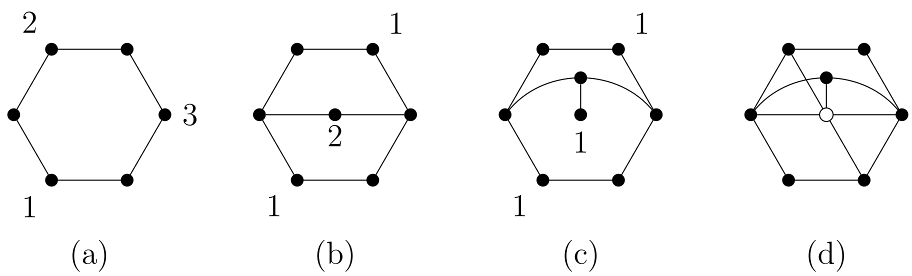


Figure 3.4: Coloring of three vertices by colors 1, 2 and 3 in (a), (b) and (c) or an extra vertex of degree 5 in (d) forces a coloring of the 6-cycle by a sequence $(1, 2, 3, 1, 2, 3)$ in cyclic order.

## 3.2 Local vs. Global Graph Properties

Krivelevich [57] presented several nice applications of his lower bounds on $f_k(n)$ and related graph parameters to questions of existence of complicated graphs whose small subgraphs are simple. We indicate here how to improve two of his bounds using Theorem 2.5.

Let $f(\sqrt{n}, 3, n)$ denote the maximum chromatic number over $n$-vertex graphs in which every $\sqrt{n}$-vertex subgraph has chromatic number at most 3. Krivelevich proved that for every fixed $\epsilon > 0$ and sufficiently large $n$,

$$f(\sqrt{n}, 3, n) \geq n^{6/31-\epsilon}. \tag{3.1}$$

He used his result that every 4-critical $t$-vertex graph with odd girth at least 7 has at least $31t/19$ edges. If instead of this result, we use our bound on $f_4(n)$, then repeating almost word by word Krivelevich's proof of his Theorem 4 (choosing $p = n^{-0.8-\epsilon'}$), we get that for every fixed $\epsilon$ and sufficiently large $n$,

$$f(\sqrt{n}, 3, n) \geq n^{1/5-\epsilon}. \tag{3.2}$$

Another result of Krivelevich is:

**Theorem 3.11 ([57])** *There exists $C > 0$ such that for every $s \geq 5$ there exists a graph $G_s$ with at least $C\left(\frac{s}{\ln s}\right)^{\frac{33}{14}}$ vertices and independence number less than $s$ such that the independence number of each 20-vertex subgraph at least 5.*

He used the fact that for every $m \leq 20$ and every $m$-vertex 5-critical graph $H$,

$$\frac{|E(H)| - 1}{m - 2} \geq \frac{\lceil 17m/8 \rceil - 1}{m - 2} \geq \frac{33}{14}.$$

From Theorem 2.5 we instead get

$$\frac{|E(H)| - 1}{m - 2} \geq \frac{\left\lceil \frac{9m-5}{4} \right\rceil - 1}{m - 2} \geq \frac{43}{18}.$$

Then repeating the argument in [57] we can replace $\frac{33}{14}$ in the statement of Theorem 3.11 with $\frac{43}{18}$.

## 3.3 Ore-degrees

The *Ore-degree*, $\Theta(G)$, of a graph $G$ is the maximum of $d(x) + d(y)$ over all edges $xy$ of $G$. Let $\mathcal{G}_t = \{G \,:\, \Theta(G) \leq t\}$. It is easy to prove (see, e.g. [49]) that $\chi(G) \leq 1 + \lfloor t/2 \rfloor$ for every $G \in \mathcal{G}_t$. Clearly $\Theta(K_{d+1}) = 2d$ and $\chi(K_{d+1}) = d + 1$. The graph $O_5$ in Fig 3.5 is the only 9-vertex 5-critical graph with $\Theta$ at most 9. We have $\Theta(O_5) = 9$ and $\chi(O_5) = 5$.

A natural question is to describe the graphs in $\mathcal{G}_{2d+1}$ with chromatic number $d + 1$. Kierstead and Kostochka [49] proved that for $d \geq 6$ each such graph contains $K_{d+1}$. Then Rabern [64] extended the result to $d = 5$. Each $(d + 1)$-chromatic graph $G$ contains a $(d + 1)$-critical subgraph $G'$. Since $\delta(G') \geq d$ and $\Theta(G') \leq \Theta(G) \leq 2d + 1$,

$$\Delta(G') \leq d + 1, \text{ and vertices of degree } d + 1 \text{ form an independent set.} \tag{3.3}$$

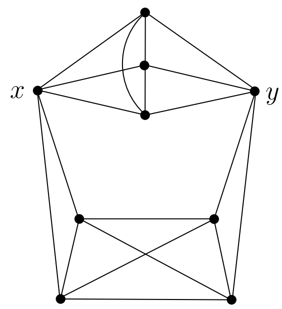


Figure 3.5: The graph $O_5$.

Thus the results in [49] and [64] mentioned above could be stated in the following form.

**Theorem 3.12 ([49, 64])** *Let $d \geq 5$. Then the only $(d+1)$-critical graph $G'$ satisfying (3.3) is $K_{d+1}$.*

The case $d = 4$ was settled by Kostochka, Rabern, and Stiebitz [51]:

**Theorem 3.13 ([51])** *Let $d = 4$. Then the only 5-critical graphs $G'$ satisfying (3.3) are $K_5$ and $O_5$.*

Theorem 2.5 and Corollary 2.18 yield simpler proofs of Theorems 3.12 and 3.13. The key observation is the following.

**Lemma 3.14** *Let $d \geq 4$ and $G'$ be a $(d+1)$-critical graph satisfying (3.3). If $G'$ has $n$ vertices of which $h > 0$ vertices have degree $d+1$, then*

$$h \geq \left\lceil \frac{(d-2)n - (d+1)(d-2)}{d} \right\rceil \tag{3.4}$$

*and*

$$h \leq \left\lfloor \frac{n-3}{d-1} \right\rfloor . \tag{3.5}$$

**Proof.** By definition, $2e(G') = dn + h$. So, by Theorem 2.5 with $k = d+1$,

$$dn + h \geq (d + 1 - \frac{2}{d})n - \frac{(d+1)(d-2)}{d},$$

which yields (3.4).

Let $B$ be the set of vertices of degree $d+1$ in $G'$ and $A = V(G') - B$. By (3.3), $e(G'(A, B)) = h(d+1)$. So, by Corollary 2.18(ii) with $k = d+1$,

$$h(d+1) \leq 3h + (n-h) - 3 = 2h + n - 3,$$

which yields (3.5). $\square$

Another ingredient is the following old observation by Dirac.

**Lemma 3.15 (Dirac [26])** *Let $k \geq 3$. There are no $k$-critical graphs with $k+1$ vertices, and the only $k$-critical graph (call it $D_k$) with $k+2$ vertices is obtained from the 5-cycle by adding $k-3$ all-adjacent vertices.*

Suppose $G'$ with $n$ vertices of which $h$ vertices have degree $d+1$ is a counterexample to Theorems 3.12 or 3.13. Since the graph $D_{d+1}$ from Lemma 3.15 has a vertex of degree $d+2$, $n \geq d+4$. So since $d \geq 4$, by (3.4),

$$h \geq \left\lceil \frac{(d-2)(d+4) - (d+1)(d-2)}{d} \right\rceil = \left\lceil \frac{3(d-2)}{d} \right\rceil \geq 2.$$

On the other hand, if $n \leq 2d$, then by (3.5),

$$h \leq \left\lfloor \frac{2d-3}{d-1} \right\rfloor = 1.$$

Thus $n \geq 2d+1$.

Combining (3.4) and (3.5) together, we get

$$\frac{(d-2)n-(d+1)(d-2)}{d} \leq \frac{n-3}{d-1}.$$

Solving with respect to $n$, we obtain

$$n \leq \left\lfloor \frac{(d+1)(d-1)(d-2)-3d}{d^2-4d+2} \right\rfloor. \tag{3.6}$$

For $d \geq 5$, the RHS of (3.6) is less than $2d+1$, a contradiction to $n \geq 2d+1$. This proves Theorem 3.12.

Suppose $d = 4$. Then (3.6) yields $n \leq 9$. So, in this case, $n = 9$. By (3.4) and (3.5), we get $h = 2$. Let $B = \{b_1, b_2\}$ be the set of vertices of degree 5 in $G'$. By a theorem of Stiebitz [70], $G' - B$ has at least two components. Since $|B| = 2$ and $\delta(G') = 4$, each such component has at least 3 vertices. Since $|V(G') - B| = 7$, we may assume that $G' - B$ has exactly two components, $C_1$ and $C_2$, and that $|V(C_1)| = 3$. Again because $\delta(G') = 4$, $C_1 = K_3$ and all vertices of $C_1$ are adjacent to both vertices in $B$. So, if we color both $b_1$ and $b_2$ with the same color, this can extended to a proper 4-coloring of $G' - V(C_2)$. Thus to have $G'$ 5-chromatic, we need $\chi(C_2) \geq 4$ which yields $C_2 = K_4$. Since $\delta(G') = 4$, $e(V(C_2), B) = 4$. So, since each of $b_1$ and $b_2$ has degree 5 and 3 neighbors in $C_1$, each of them has exactly two neighbors in $C_2$. This proves Theorem 3.13. $\square$

## 3.4 Hypergraphs

A *proper $k$-coloring* of a hypergraph $G$ is a function $f : V(G) \to [k]$ such that for each $e \in E(G)$, the vertices in $e$ do not all map to the same color. The definitions of chromatic number and $k$-critical generalize in the natural way.

There are several generalizations of $f_k(n)$. One generalization is to extend the result to hypergraphs. Critical hypergraphs create a more complex problem: even the family of 3-critical hypergraphs is not known. We present several constructions of critical hypergraphs. An example of a 3-critical hypergraph constructed using Constructions 3.16 and 3.17 is shown in Figure 3.6.

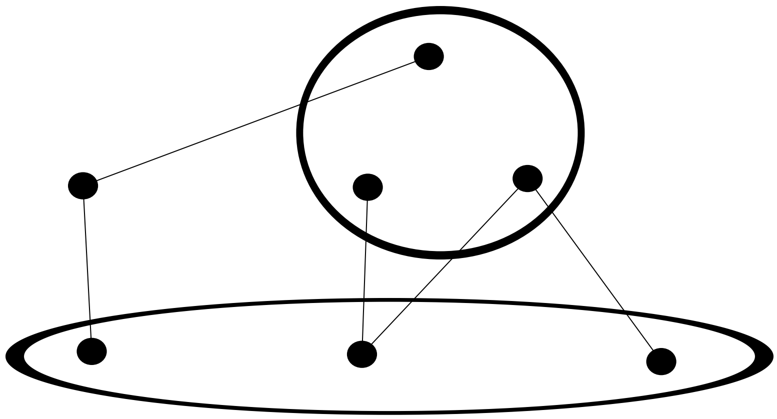

Figure 3.6: An example of a 3-critical hypergraph that is not an odd cycle.

**Construction 3.16** *Let $G_1$ and $G_2$ be hypergraphs. Let $e_1 \in E(G_1)$, $v_1 \in e_1$, $e_2 \in E(G_2)$, and $v_2 \in e_2$. Let $G$ be comprised of a copy of $(G_1 - e_1)$ and a copy of $(G_2 - e_2)$ with $v_1$ and $v_2$ merged into a vertex $v_*$, and then add edge $e = (e_1 \cup e_2) - v_*$. See Figure 3.7.*

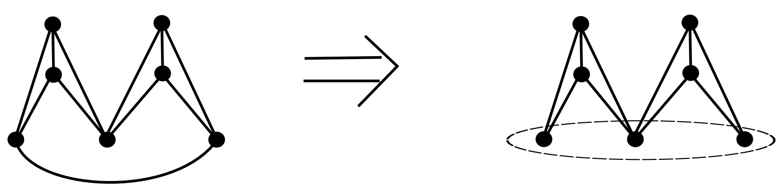

Figure 3.7: Construction 3.16.

**Construction 3.17** *Let $G'$ be a hypergraph, $e' \in E(G')$, and $u \in e'$. If there exists a vertex $v$ such that $\phi(v) = \phi(u)$ in every proper $(k-1)$-coloring $\phi$ of $(G' - e')$, then let $e = e' \cup \{v\}$ and $G = G' - e' + e$. See Figure 3.8.*

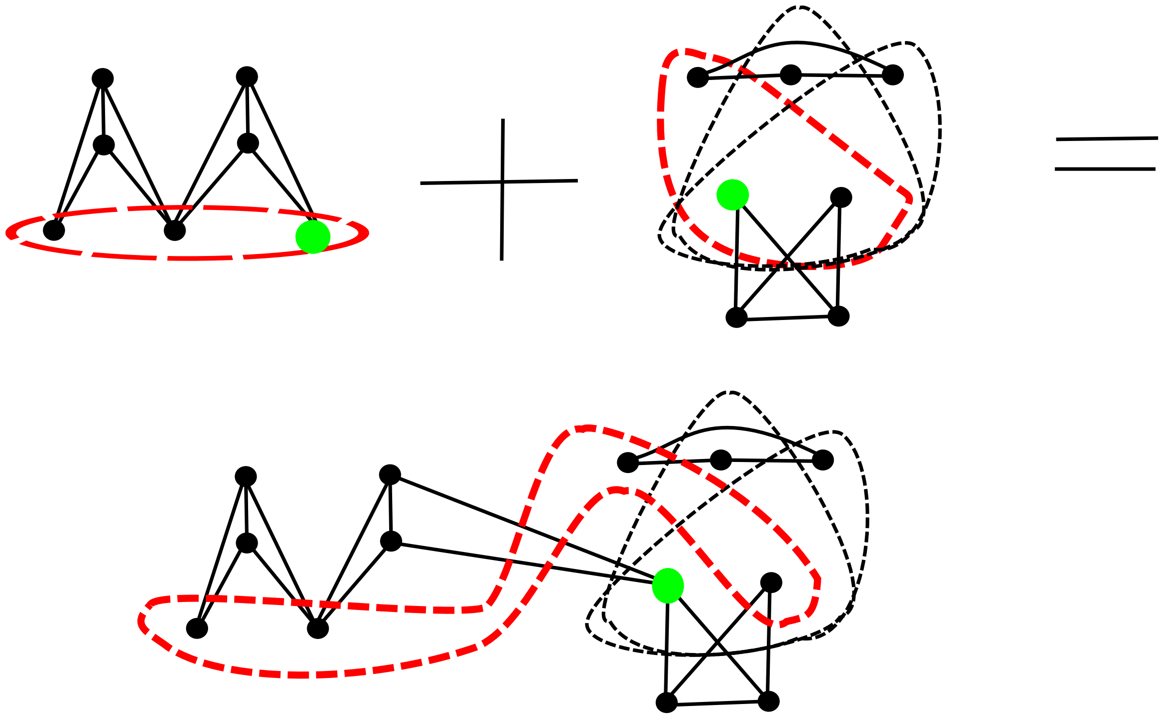

Figure 3.8: Construction 3.17.

**Construction 3.18** *Let $G_1$ and $G_2$ be hypergraphs. Let $\{u_1, \ldots u_t\} = e \in E(G_1)$ and $v \in V(G_2)$. Let $G_2'$ be $G_2$ with $v$ split into $t$ vertices $v_1, \ldots v_t$, each with non-zero degree in $G_2'$. Let $G$ be obtained from $(G_1 - e)$ and $G_2'$ with $u_i$ and $v_i$ merged for $1 \leq i \leq t$. See Figure 3.9.*

If $G_1$, $G_2$, and $G'$ are $k$-critical hypergraphs, then Constructions 3.16 and 3.17 produce new $k$-critical hypergraphs. Construction 3.18 is a generalization of Construction 2.3 in Chapter 2. If $G_2'$ is not $k$-critical, then Construction 3.18 produces a new $k$-critical graph. We will present the proof of Construction 3.16. The proof of Construction 3.17 is clear, and the proof of Construction 3.18 follows exactly the proof of Construction 2.3.

**Proof of Construction 3.16** Let $G$, $G_1$, $G_2$, $e_1$, $e_2$, $v_1$, and $v_2$ be as in the statement of the construction. First, we will show that $\chi(G) \geq k$. Suppose $\phi$ is a proper $(k-1)$-coloring of $G$. The vertices in $e$ are not monochromatic, so there must be some vertex $u \in e$ where

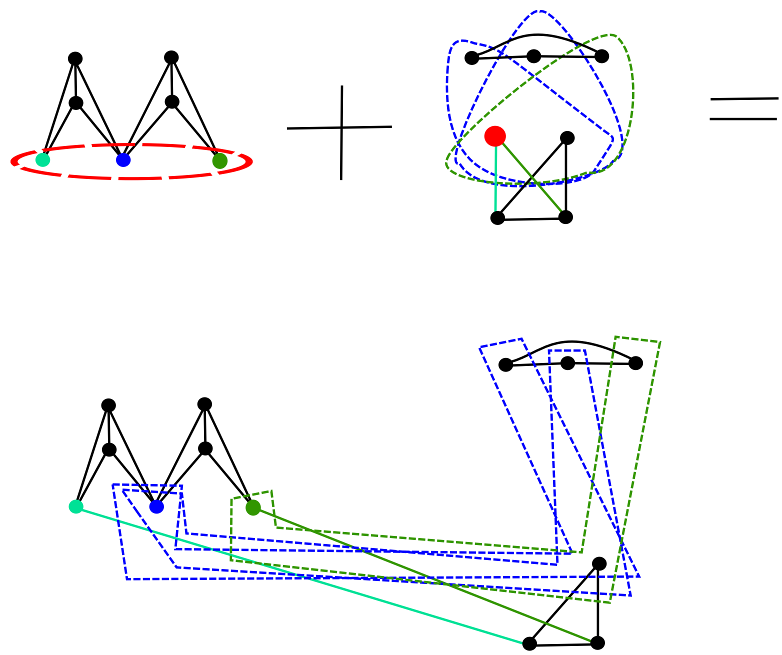

Figure 3.9: Construction 3.18.

$\phi(u) \neq \phi(v_*)$. Without loss of generality, we may assume that $v_*$ is in the copy of $G_1 - e_1$. Note that $\phi$ induces a proper $(k-1)$-coloring of $G_1 - e_1$. Because $\phi(u) \neq \phi(v_*)$, we have that $\phi$ is a proper $(k-1)$-coloring of $G_1$, which is a contradiction.

Let $f \in E(G)$. We will show that $G - f$ is $(k-1)$-colorable. Without loss of generality, assume that $f \in E(G_1)$. There exists a proper $(k-1)$-coloring $\phi_1$ of $G_1 - f$. Note that there is a vertex $u_1 \in e_1$ such that $\phi_1(u) \neq \phi_1(v_1)$. Let $\phi_2$ be a proper $(k-1)$-coloring of $G_2 - e_2$. By symmetry, we may assume $\phi_1(v_1) = \phi_2(v_2)$. Let $u_2 \in e_2 - v_2$. By criticality of $G_2$, $\phi_2(u_2) = \phi_2(v_2)$. Now, $\phi_1 \cup \phi_2$ is a proper $(k-1)$-coloring of $G - e$. This is also a proper $(k-1)$-coloring of $G$, because $\phi_1(u_1) \neq \phi_1(v_1) = \phi_2(v_2) = \phi_2(u_2)$. $\square$

It is easier to properly color larger edges. We formalize this with a statement below.

**Claim 3.19** *Let $G$ be a hypergraph and $e \in E(G)$. If $e' \subseteq e$ and $G' = G - e + e'$, then $\chi(G') \geq \chi(G)$.*

**Proof.** A proper coloring of $G'$ is also a proper coloring of $G$. $\square$

A consequence of this statement is that if $H$ is a hypergraph, and $G$ is a graph where each edge of $H$ is shrunk to have size 2, then $\chi(G) \geq \chi(H)$. Therefore, it follows intuitively that critical hypergraphs would have as many edges as critical graphs. We will prove this rigorously and also characterize the extremal hypergraphs.

**Theorem 3.20** *If $k \geq 3$, and $G$ is a $k$-critical hypergraph, then* $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$.

*For $k \geq 4$, let $G$ be $k$-critical with $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. If there exists $e \in E(G)$ such that $|e| \geq 3$, then $G$ can be constructed from smaller $k$-critical hypergraphs using Construction 3.16, 3.17, or 3.18.*

By Theorem 2.6, we know all of the extremally sparse $k$-critical graphs, so this statement characterizes all of the extremally sparse $k$-critical hypergraphs when $k \geq 4$. The bound $|E(G)| \geq |V(G)|$ for 3-critical hypergraphs was already known.

Let $f_{k,t}(n)$ denote the fewest edges in an $n$-vertex, $k$-critical, $t$-uniform hypergraph. Let $\alpha(k,t) = \lim_{n\to\infty} f_{k,t}(n)/n$. The best bounds known on $\alpha(k,t)$ for small $k$ are by Abbott and Hare [1]: $\max\{1,(k-1)/t\} \leq \alpha(k,t) < k-2$ when $t \geq 3$ and $k \geq 4$. Abbott and Hare gave slightly stronger bounds under specific conditions (such as $k = t+1$) to determine that $\alpha(4,3) \in [10/9, 35/19]$. Theorem 3.20 improves on these bounds for small $k$ as a simple corollary.

**Corollary 3.21** *Let $k \geq 3$. For all $t$, $\alpha(k,t) \geq \frac{k}{2} - \frac{1}{k-1}$.*

*Moreover, $\alpha(4,3) \geq 5/3$.*

For large $k$, Kostochka and Stiebitz [53] proved that $\alpha(k,t) \geq k - 3k^{\frac{2}{3}}$ when $t \geq 3$, which shows that $\lim_{k\to\infty} \alpha(k,t)/k = 1$. Corollary 3.21 is the strongest known result for $k \leq 215$.

We will prove Theorem 3.20 in two stages. In the first stage we will prove the lower bound on the number of edges in $k$-critical hypergraphs, which will be described by Lemma

3.26. In the second stage we will prove that all extremally sparse $k$-critical hypergraphs can be constructed via Construction 3.16, 3.17, or 3.18, which will be described by Lemma 3.29.

We generalize the definition of potential to hypergraphs in the natural way. Note that we now consider $\rho_3(W) = 4|W| - 4|E(G[W])|$. We can do this because the statement

$$\text{if } G \text{ is } k\text{-critical, then } \rho_k(V(G)) \leq k(k-3) \tag{3.7}$$

is still true when $k = 3$.

We also use the following generalizations:

**Definition 3.22** *For a hypergraph $G$ and a set $R \subset V(G)$, let*

$$R_* = \{u \in R \ : \ N(u) \not\subseteq R\}.$$

**Construction 3.23** *For a hypergraph $G$, a set $R \subset V(G)$, and a proper $(k-1)$-coloring $\phi$ of $G[R]$, the hypergraph $Y(G, R, \phi)$ is constructed as follows. Let $t$ be the number of colors used on $R_*$. We may renumber the colors so that the colors used on $R_*$ are $1, \ldots, t$. Let $Y(G, R, \phi) = G - R + X$, where $X = \{x_1, \ldots, x_t\}$ is a set of new vertices which form a clique. Then for each edge $e \in E(G)$ such that $e \not\subseteq R$:*
*(i) if $e \cap R = \emptyset$, then it stays where it was;*
*(ii) if $\{\phi(v) \ : \ v \in R \cap e\} = \{i\}$ for some $i$, then we add to $Y(G, R, \phi)$ edge $e - R + x_i$;*
*(iii) if $|\{\phi(v) \ : \ v \in R \cap e\}| \geq 2$, then we add nothing to $Y(G, R, \phi)$.*

**Claim 3.24** *Let $H$ be a hypergraph such that $\chi(H) \geq k$. If $R \subset V(H)$ and there exists a proper $(k-1)$-coloring $\phi$ of $H[R]$, then $\chi(Y(H, R, \phi)) \geq k$.*

**Proof.** Let $H' = Y(H, R, \phi)$. Suppose $H'$ has a proper $(k-1)$-coloring $\phi' : V(H') \to C$. By construction of $H'$, the colors of all $x_i$ in $\phi'$ are distinct. By changing the names of the colors, we may assume that $\phi'(x_i) = c_i$ for $1 \le i \le k-1$. By construction of $H'$, $\phi|_R \cup \phi'|_{V(H)-R}$ is a proper coloring of $H$, a contradiction. $\square$

**Definition 3.25** *For a hypergraph $H$, let*

$$e_i(H) = |\{e \in E(H) \ : \ |e| = i\}|.$$

*Let $G$ and $H$ be hypergraphs. Let $i$ be the largest integer such that $e_i(G) \neq e_i(H)$. If such an $i$ exists and $e_i(H) < e_i(G)$, then $H$ is* smaller than *$G$.*

Note that if $G'$ is a subgraph of $G$, then $G'$ is smaller than $G$.

### 3.4.1 Sparse

**Lemma 3.26** *If $k \ge 3$ and $H$ be $k$-critical hypergraph, then $\rho_k(V(H)) \le k(k-3)$.*

Let $H$ be a minimal hypergraph with respect to our relation that contradicts Lemma 3.26.

**Claim 3.27** *There is no nonempty $R \subsetneq V(H)$ with $|R| \ge 2$ and $\rho_{k,H}(R) < (k-2)(k+1)$.*

**Proof.** Let $R$ have the smallest potential among nonempty proper subsets of $V(H)$. Let $\rho_k(R) = m$. Since $H$ is $k$-critical, $H[R]$ has a proper coloring $\phi$. Let $H' = Y(H, R, \phi)$. By Claim 3.24, $\chi(H') \ge k$. Then $H'$ contains a $k$-critical hypergraph $H''$.

We claim that $H'$ is smaller than $H$, and therefore $H''$ is smaller than $H$. Edges added using rules (i), (ii), and (iii) to $H'$ are in 1-to-1 correspondence with edges in $H$ that are at least as large. All edges added inside of $X$ have size 2. If $H[R]$ contains an edge with size at least 3, then it follows that $H'$ is smaller than $H$. So assume every edge in $H[R]$ has size

2. $\rho_{k,H}(R) < (k-2)(k+1) \leq 2(k-2)(k-1)$, so $|R| \geq k$. Because $\rho_{k,H}(R) < \rho_{k,H'}(X)$ and $|R| > |X|$, it follows that $e_2(H[R]) > e_2(H'[X])$. Therefore $H'$ is smaller than $H$.

By the minimality of $H$, $\rho_k(V(H'')) \leq k(k-3)$. Since $H$ is $k$-critical, $V(H'') \cap X \neq \emptyset$. Let $Z = V(H'') - X + R$. So, since every nonempty subset of $X$ has potential at least $(k-2)(k+1)$,

$$\begin{aligned}\rho_{k,H}(Z) &\leq \rho_{k,H'}(V(H'')) - \rho_{k,H'}(V(H'') \cap X) + m \\ &\leq k(k-3) - (k+1)(k-2) + m \\ &= m - 2k + 2.\end{aligned}$$

Since $Z \supset R$, we have that $Z$ is nonempty. By the minimality of the potential of $R$, it follows that $Z = V(H)$. Now

$$\rho_{k,H}(V(H)) \leq m - 2k + 2 < (k-2)(k+1) - 2k + 2 = k(k-3),$$

a contradiction again. $\square$

**Corollary 3.28** $P_k(H) \geq k(k-3) + 1$.

**Proof.** This follows from Claim 3.27, the assumption $\rho_{k,H}(V(H)) \geq k(k-3) + 1$, and the fact $\rho_k(K_1) = (k-2)(k+1)$. $\square$

Let $e$ be a maximum-sized edge in $H$. If $|e| = 2$ then the claim follows from Theorem 2.5, so assume $|e| \geq 3$. Let $e = \{v_1, v_2, \dots v_t\}$.

Let $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_2, \dots v_t\}$. Let $H_1 = H - e + e_1$ and $H_2 = H - e + e_2$. By Claim 3.19, $\chi(H_1), \chi(H_2) \geq \chi(H) = k$. So there exist $k$-critical hypergraphs $H'$ and $H''$ such that $e_1 \in H' \subseteq H_1$ and $e_2 \in H'' \subseteq H_2$. Let $W' = V(H')$ and $W'' = V(H'')$. By minimality of $H$,

$$\rho_{k,H}(W'), \rho_{k,H}(W'') \leq k(k-3) + 2(k-1) = (k-2)(k+1).$$

*Case 1:* $W' = V(H)$. Here $e \subseteq W'$ and $\rho_{k,H}(V(H)) = \rho_{k,H'}(V(H')) \leq k(k-3)$, a contradiction.

*Case 2:* $W' \cap W'' \neq V(H)$. Because $v_2 \in W' \cap W''$, we have that $\rho_{k,H}(W' \cap W'') \geq (k-2)(k+1)$. By submodularity,

$$\begin{aligned} \rho_{k,H}(W' \cup W'') &\leq \rho_{k,H}(W') + \rho_{k,H}(W'') - \rho_{k,H}(W' \cap W'') \\ &\leq (k-2)(k+1) + (k-2)(k+1) - (k-2)(k+1) \\ &= (k-2)(k+1). \end{aligned}$$

Note that $e \subseteq W' \cup W''$, which was not accounted for previously. So $\rho_{k,H}(W \cup W'') \leq (k-2)(k+1) - 2(k-1) = k(k-3)$. But this contradicts Corollary 3.28. $\square$

### 3.4.2 Construction

**Lemma 3.29** *If $H$ is $k$-critical, $\rho_k(V(H)) = k(k-3)$, $k \geq 4$, and there exists an $e \in E(H)$ such that $|e| \geq 3$, then $H$ can be constructed from smaller $k$-critical hypergraphs using Construction 3.16, 3.17, or 3.18.*

Let $H$ be a minimal hypergraph with respect to our relation that contradicts Lemma 3.29.

**Claim 3.30** *There is no nonempty $R \subsetneq V(H)$ with $|R| \geq 2$ and $\rho_{k,H}(R) \leq (k-2)(k+1)$.*

**Proof.** Let $R$, $\phi$, $H'$ and $H''$ be the same as in the proof of Claim 3.27. If possible, pick $\phi$ such that there are at least two colors appearing on the vertices of $R_*$. By Lemma 3.26, $\rho_k(V(H'')) \leq k(k-3)$. Let $Z = V(H'') - X + R$.

$$\begin{aligned}\rho_{k,H}(Z) &\leq \rho_{k,H'}(V(H'')) - \rho_{k,H'}(V(H'')\cap X) + \rho_{k,H}(R)\\ &\leq k(k-3) + \rho_{k,H}(R) - \rho_{k,H'}(V(H'')\cap X)\end{aligned}$$

Since $H$ is $k$-critical, $V(H'')\cap X \neq \emptyset$. If $\emptyset \neq W \subseteq X$, then $\rho_{k,H'}(W) \geq (k-2)(k+1) > k(k-3)$. By the minimality of the potential of $R$, we have $Z = V(H)$. If $|V(H'')\cap X| \neq 1$, then $\rho_{k,H'}(V(H'')\cap X) > (k-2)(k+1)$ (this is where $k \geq 4$ is applied). It follows that $\rho_{k,H}(V(H)) < \rho_{k,H}(R) - 2(k-1)$, which contradicts the assumptions that $\rho_k(V(H)) = k(k-3)$ and $\rho_{k,H}(R) \leq (k-2)(k+1)$. Therefore $|V(H'')\cap X| = 1$.

Without loss of generality, assume $V(H'')\cap X = \{x_1\}$. Let $R_1 = \{u \in R_* \ :\ \phi(u) = c_1\}$.

*Case A:* There exists an edge with an endpoint in $R - R_1$ and another endpoint in $V(H) - R$. This edge was not accounted for above, so $\rho_{k,H}(V(H)) \leq k(k-3) - 2(k-1)$, which contradicts the assumptions.

*Case B:* There are no edges with an endpoint in $R - R_1$ and another endpoint in $V(H) - R$. Note that this implies that $R_1 = R_*$. By our choice of $\phi$, every proper coloring of $H[R]$ is monochromatic on $R_1$. Let $e'$ be an edge with $e' = R_1$. Then $H$ can be constructed by Construction 3.18, where $G_1 = H[R] + e'$ and $G_2$ is $H - (R - R_1)$ with all vertices of $R_1$ are merged into one vertex. $\square$

Let $e = \{v_1, v_2, \ldots v_t\}$, $e_1 = \{v_1, v_2\}$, and $e_2 = \{v_2, \ldots v_t\}$. Let $H_1 = H - e + e_1$ and $H_2 = H - e + e_2$. By Claim 3.19, $\chi(H_1), \chi(H_2) \geq \chi(H) = k$. So there exist $k$-critical graphs $e_1 \in H' \subseteq H_1$ and $e_2 \in H'' \subseteq H_2$. Let $W' = V(H')$ and $W'' = V(H'')$. By Lemma 3.26,

$$\rho_{k,H}(W'), \rho_{k,H}(W'') \leq k(k-3) + 2(k-1) = (k-2)(k+1).$$

*Case 1:* $W' = V(H)$ (or $W'' = V(H)$, symmetrically). Here $H$ can be constructed from $H'$ by iteratively applying Construction 3.17.

*Case 2:* $W' \cap W'' \neq V(H)$. Because $v_2 \in W' \cap W''$, we have $\rho_{k,H}(W' \cap W'') \geq (k-2)(k+1)$. By submodularity,

$$\begin{aligned}\rho_{k,H}(W' \cup W'') &\leq \rho_{k,H}(W') + \rho_{k,H}(W'') - \rho_{k,H}(W' \cap W'') \\ &\leq (k-2)(k+1) + (k-2)(k+1) - (k-2)(k+1) \\ &= (k-2)(k+1).\end{aligned}$$

Note that $e \subseteq W' \cup W''$, which was not accounted for previously. So $\rho_{k,H}(W \cup W'') \leq (k-2)(k+1) - 2(k-1) = k(k-3)$.

Because of Claim 3.30, $W' \cup W'' = V(H)$ and all of the inequalities are equalities. Specifically, this implies that $\rho_{k,H}(W' \cap W'') = (k-2)(k+1)$ (and therefore $|W' \cap W''| = 1$), and there are no edges with an endpoint in $W' - W''$ and another endpoint in $W'' - W'$. Thus $H$ can be constructed from $H'$, $H''$, and Construction 3.16. $\square$

# Chapter 4

# (1,1)-Critical Graphs

Recall the application of graph coloring to assigning frequencies to radio towers. If two towers are too close to each other, such as when if both are located in Miami, then to avoid interference with each other's transmission they must broadcast on different frequencies. If two towers are sufficiently apart, such as when if one is in San Diego and the other is in Portland, then they are allowed to use the same frequency. The problem of assigning frequencies to each tower can then be modeled as a proper coloring problem: each vertex represents a tower that requires a frequency, each frequency is represented by a color, and an edge joining two vertices represents towers that are close enough to experience interference. Recently, Havet and Sereni [41] described a variation of this problem:

> In this paper, we investigate the following problem proposed by Alcatel, a satellite building company. A satellite sends information to receivers on earth, each of which is listening on a frequency. Technically it is impossible to focus the signal sent by the satellite exactly on receiver. So part of the signal is spread in an area around it creating noise for the other receivers displayed in this area and listening on the same frequency. A receiver is able to distinguish the signal directed to it from the extraneous noises it picks up if the sum of the noises does not become too big, i.e. does not exceed a certain threshold $T$.

For a coloring $f$ of a graph $G$, the set of vertices $\{v \in V(G) : f(v) = i\}$ is called the $i^{th}$ *color class of* $f$. A $(d_1, \ldots, d_k)$-*improper coloring* of a graph $G$, or just $(d_1, \ldots, d_k)$-coloring, is a coloring $f$ of $G$ using at most $k$ colors such that for all $i$, the $i^{th}$ color class of $f$ induces a graph with maximum degree at most $d_i$. In the problem described in the quotation, each vertex is allowed to have up to $T$ neighbors broadcasting with the same color. This is a generalization of the traditional problem: if $T = 0$, then the problem reduces to finding a proper coloring. Alcatel is interested in a $(T, \dots, T)$-coloring of their satellite receivers.

A graph $G$ is *improperly* $(d_1, \dots, d_k)$*-colorable*, or just $(d_1, \dots, d_k)$-colorable, if there is a $(d_1, \dots, d_k)$-coloring of $G$. We define a graph $G$ to be $(d_1, \dots, d_k)$*-critical* if $G$ is not $(d_1, \dots, d_k)$-colorable, but every proper subgraph $H$ of $G$ is $(d_1, \dots, d_k)$-colorable.

The interest in improper colorings predates the application described by Havet and Sereni. Publications on this topic extend back to the 1960's, and many of them involve generalizing famous results in proper colorings. For example, Lovász [62] showed that every graph $G$ is $(d_1, \dots, d_k)$-colorable whenever $(d_1 + 1) + \dots + (d_k + 1) \geq \Delta(G) + 1$; this is a generalization of Brooks' Theorem.

The most famous result in proper coloring, and arguably in all of graph theory, is the Four Color Theorem [7, 8], which states that every planar graph is 4-colorable. Cowen, Cowen, and Woodall [23] proved that every planar graph is $(2, 2, 2)$-colorable. For each $i$ and $j$, there exist an infinite family of planar $(1, i, j)$-critical graphs.

Borodin and Kostochka [18] constructed $(j, k)$-critical graphs. Those graphs are planar, and those graphs are outerplanar when $j = 1$. Without loss of generality, assume $j \leq k$. Let $G_{j,k}$ be $X$, with $V(X) = \{x_1, x_2, x_3\}$ and $X$ is isomorphic to $K_{1,2}$, plus $(1, k)$-critical graphs $C_{a,b}$ for $1 \leq a \leq 3$ and $1 \leq b \leq k + 1$, and each vertex in $C_{a,b}$ is adjacent to $x_a$. Note that in each $(1, j, k)$-coloring of $G_{i,j}$, there is a vertex from the second and third color classes in each $C_{a,b}$. Now each vertex of $X$ is in the first color class, because it has at least $k+1$ neighbors in the second and third color classes. We have that the first color class has maximum degree at least 2, which is a contradiction. Therefore $G_{j,k}$ is a $(1, j, k)$-critical planar graph. Because Borodin and Kostochka constructed an infinite family of outerplanar $(1, k)$-critical graphs, this creates an infinite number of $(1, j, k)$-critical planar graphs.

The girth of a graph $G$, written $g(G)$, is the length of a shortest cycle in $G$. If $G$ does not have a cycle, then we consider $g(G)$ to be infinite. The maximum average degree of a graph $G$, $\mathrm{mad}(G)$, is $\max\left\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right\}$. The maximum average degree is related to the girth of a planar graph, as they both bound how sparse the graph is. Specifically, $\mathrm{mad}(G) \leq \frac{2g(G)}{g(G)-2}$ for a planar graph $G$. Grötzsch [36] proved that planar graphs with girth at least 4 are 3-colorable. The literature on the relationship of girth in planar graphs to improper colorability is extensive.

Borodin et al. [15] proved that every planar graph $G$ with $g(G) \geq 5$ is $(2, 13)$- and $(3, 7)$-colorable. In a follow-up paper [16], they showed that every planar graph $G$ with $g(G) \geq 6$ is $(1, 5)$-colorable and with $g(G) \geq 7$ is $(1, 2)$-colorable. Borodin et al. [14] constructed for each $\ell$ a planar $(0, \ell)$-critical graph with girth 6 and proved that every planar graph $G$ with $g(G) \geq 7$ is $(0, 8)$-colorable. They also showed that if $g(G) \geq 8$, then $G$ is $(0, 4)$-colorable. Borodin and Kostochka [18] proved that if $\ell \geq 2j + 2$ and $\mathrm{mad}(G) \leq 2\left(2 - \frac{\ell+2}{(j+2)(\ell+1)}\right)$, then $G$ is $(j, \ell)$-colorable. This result implies that for a planar graph $G$

a) if $g(G) \geq 5$, then $G$ is $(2, 6)$-colorable,

b) if $g(G) \geq 6$, then $G$ is $(1, 4)$-colorable,

c) if $g(G) \geq 7$, then $G$ is $(0, 4)$-colorable, and

d) if $g(G) \geq 8$, then $G$ is $(0, 2)$-colorable.

Their result on maximum average degree is known to be sharp, but the best relationship between girth and improper colorability for planar graphs remains open.

Glebov and Zambalaeva [35] proved that every planar graph $G$ with girth at least 16 is $(0, 1)$-colorable. This was strengthened by Borodin and Ivanova [12]: they proved that every graph $G$ with $\mathrm{mad}(G) < \frac{7}{3}$ is $(0, 1)$-colorable, which implies that every planar graph $G$ with $g(G) \geq 14$ is $(0, 1)$-colorable. Borodin and Kostochka [17] proved that every graph $G$ with $\mathrm{mad}(G) < \frac{12}{5}$ is $(0, 1)$-colorable, which implies that every planar graph $G$ with $g(G) \geq 12$

is $(0,1)$-colorable. Esperet, Montassier, Ochem, and Pinlou [30] have constructed a planar $(0,1)$-critical graph with girth 9.

Our main contribution to this area is slightly stronger than a statement about the maximum average degree. We show that any graph that is not $(1,1)$-colorable has a $(1,1)$-critical subgraph that is not sparse. We prove the following result:

**Theorem 4.1 ([20])** *If $G$ is a $(1,1)$-critical graph, then $5|E(G)| > 7|V(G)|$.*

**Corollary 4.2 ([20])** *Every planar graph $G$ with $g(G) \geq 7$ is $(1,1)$-colorable.*

Theorem 4.1 and Corollary 4.2 improve the previous-best conditions for $(1,1)$-colorability. The best results had been from applying a result of Havet and Sereni [42], who showed that a graph $G$ with $\text{mad}(G) < \frac{4k+4}{k+2}$ is $(k,k)$-colorable (they were working under the more general setting of *improper choosability*). Theorem 4.1 is also strong enough to improve a different type of result from Borodin and Ivanova [13]. They showed that every graph $G$ with $g(G) \geq 7$ and $\text{mad}(G) < \frac{14}{5}$ can be partitioned into two strong linear forests. A strong linear forest is a graph where each component contains at most 2 edges; our result shows that each component contains at most 1 edge.

There is a different generalization of improper colorability called *vertex Ramsey*. We say that $G \xrightarrow{v} (H_1, \ldots, H_k)$ if for every coloring $\phi : V(G) \to [k]$ there exists $i$ such that $H_i$ is a subgraph of the $i^{th}$ color class. It is clear that a graph is $(d_1, \ldots, d_k)$-colorable if and only if $G \not\xrightarrow{v} (K_{1,d_1+1}, \ldots, K_{1,d_k+1})$. We are interested in the same problems in this generalized setting: what affect does maximum degree, planarity, girth, and maximum average degree have on the vertex Ramsey problem?

Let $m_{cr}(H_1, \ldots, H_k) = \inf\{\text{mad}(F) \ : \ F \xrightarrow{v} (H_1, \ldots, H_k)\}$. Borodin and Kostochka's results [17, 18] directly state exact values for $m_{cr}(K_{1,j+1}, K_{1,\ell+1})$ if $\ell \geq 2j+2$ or $j = 0$.

Kurek and Ruciński [58] proved that

$$\sum_{i=1}^{k} \max_{H_i' \leq H_i} \delta(H_i') \leq m_{cr}(H_1, \ldots, H_k) \leq 2 \sum_{i=1}^{k} \max_{H_i' \leq H_i} \delta(H_i').$$

As a corollary, it follows that $m_{cr}(K_s, \ldots, K_s) = k(s-1)$. However, $m_{cr}(H_1, \ldots, H_k)$ is still unknown in general.

In the same paper, Kurek and Ruciński showed that $8/3 \leq m_{cr}(K_{1,2}, K_{1,2}) \leq 14/5$. Ruciński offered 400,000 PLZ cash prize for the exact value of $m_{cr}(K_{1,2}, K_{1,2})$. Theorem 4.1 proves that $m_{cr}(K_{1,2}, K_{1,2}) = 14/5$.

We will first convert the problem to an equivalent problem on graphs with "special" vertices. In Section 4.1 we will demonstrate the sharpness of the result. In Section 4.2 we will prove the theorem.

For a fixed graph $G$ and partition $V(G) = V_s \cup V_r$, we define a *good coloring* to be a map $\phi : V(G) \to \{a, b\}$ such that

- if $v \in V_s$, then $|\{u \in N(v) \ : \ \phi(u) = \phi(v)\}| = 0$, and
- if $v \in V_r$, then $|\{u \in N(v) \ : \ \phi(u) = \phi(v)\}| \leq 1$.

We refer to the vertices in $V_s$ as *super vertices* and the vertices in $V_r$ as *regular vertices*. We say that $H$ is *contained* in $G$ if $V_s(H) \subseteq V_s(G)$ and $H \subseteq G$. A graph $G$ is *good* if it has a good coloring, and *critical* if it is not good but every graph contained by $G$ is good. For a subset of vertices $S \subseteq V(G)$, we define the *potential* of $S$ to be

$$\rho_G(S) = 7|S \cap V_r| + 3|S \cap V_s| - 5|E(G[S])|.$$

We will prove:

**Theorem 4.3** *If $G$ is critical, then $\rho_G(V(G)) \leq -1$.*

By setting $V_s = \emptyset$, Theorem 4.3 implies Theorem 4.1. We will show that they are equivalent. Given $G$, $V_r$, and $V_s$, we create a new graph $\pi : (G, V_r, V_s) \to G'$ as follows: for each vertex $u \in V_s$, add a copy of $K_4 - e$, called $F_u$, and merge a vertex of degree 3 in $F_u$ to $u$. Let $\phi$ be a $(1,1)$-coloring of $G'$. Because every $(1,1)$-coloring of a path on three vertices contains both colors, each $u \in V_s$ has no neighbors in $V(G') - F_u$ colored $\phi(u)$. Therefore $\phi$ is a good coloring of $G$, and Theorem 4.1 implies Theorem 4.3.

## 4.1 Sharpness

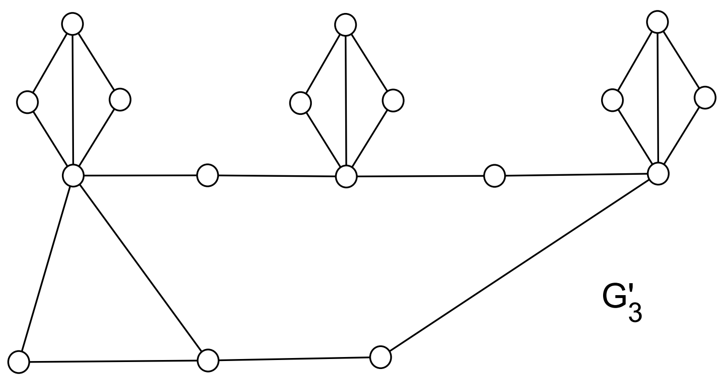


Figure 4.1: An example of a sparse $(1,1)$-critical graph.

To show that Theorem 4.3 is sharp, we will construct a sequence $\{G_i\}_{i=1}^{\infty}$ of graphs such that each $G_i$ is critical and $\rho(G_i) = -1$. We define

$$V_r(G_i) = \{x_{2j} \,:\, 1 \leq j < i\} \cup \{u_1, u_2, u_3\},$$

$$V_s(G_i) = \{x_{2j-1} \,:\, 1 \leq j \leq i\},$$

and

$$E(G_i) = \{x_j x_{j+1} \,:\, 1 \leq j \leq 2i-2\} \cup \{u_1u_2, u_1x_1, u_2u_3, u_2x_1, u_3x_{2i-1}\}.$$

It is easy to calculate that $\rho_{G_i}(V(G_i)) = 7(i+2) + 3(i) - 5(2i+3) = -1$.

See Figure 4.2 for a picture of $G_3$. See Figure 4.1 for the corresponding $(1,1)$-critical graph.

We will now show that $G_i$ is critical. If $\phi$ is a good coloring of $G_i$, then $\phi(x_{2j-1}) = \phi(x_{2j+1})$ for $1 \leq j < i$. Moreover, $\phi(x_1) = \phi(x_{2i-1})$. Because there are only two colors, it follows that $\phi(u_1) = \phi(u_2) = \phi(u_3)$. This is a contradiction, because $\Delta(G_i[\{u_1, u_2, u_3\}]) = 2$. Therefore $G_i$ is not good.

It is easy to see that if an edge is removed from $G_i[\{x_1, \ldots, x_{2i-1}\}]$ or a vertex is changed from special to regular, then there exists a good coloring of $G_i$ such that $\phi(x_1) \neq \phi(x_{2i-1})$. Also, if an edge in $G_i[\{u_1, u_2, u_3\}]$ is removed, then $\Delta(G_i[\{u_1, u_2, u_3\}]) = 1$. If an edge in $E(\{x_1, \ldots, x_{2j-1}\}, \{u_1, u_2, u_3\})$ is removed, then there exists a good coloring where $\{u_1, u_2, u_3\}$ is not monochromatic. Therefore $G_i$ is critical.

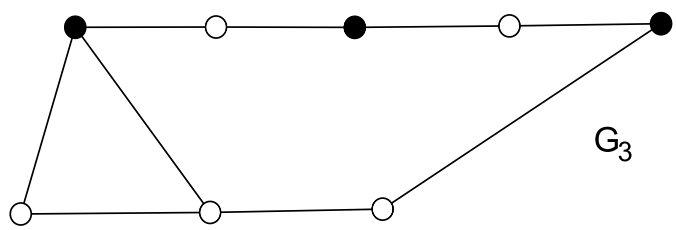


Figure 4.2: An example of a critical graph. The super vertices are solid black; the regular vertices are hollow. Note that $\pi(G_3) = G_3'$.

## 4.2 Proof of Theorem 4.1

An *unimportant vertex* is a regular vertex with degree 2 contained in a triangle, a *semi-important vertex* is a regular vertex with degree 2 not contained in a triangle; and all other vertices are *important*. We denote these sets by $V_0$, $V_1$, and $V_2$, respectively. We say $H$ *precedes* $G$ if

1. $|V_2(H)| < |V_2(G)|$,
2. $|V_2(H)| = |V_2(G)|$ and $|V_1(H)| < |V_1(G)|$,
3. $|V_2(H)| = |V_2(G)|$, $|V_1(H)| = |V_1(G)|$, and $|V_0(H)| + 3|V_s(H)| < |V_0(G)| + 3|V_s(H)|$, or
4. $|V_2(H)| = |V_2(G)|$, $|V_1(H)| = |V_1(G)|$, $|V_0(H)| + 3|V_s(H)| = |V_0(G)| + 3|V_s(H)|$, and

$$\sum_{u \in V(H)} d(u)^2 > \sum_{w \in V(G)} d(w)^2.$$

As shorthand, we write "$H$ precedes $G$" as $H \prec G$. By the definition, if $H$ is contained in $G$, then $H \prec G$.

By induction, assume that $G$ is an earliest (under $\prec$) critical graph with $\rho_G(V(G)) \geq 0$. Because $G$ is critical, $G$ is 2-edge-connected, and therefore $\delta(G) \geq 2$. It follows that $V_0$ and $V_1$ partition the vertices of degree 2 and $V_2$ is the set of vertices with degree at least 3.

**Lemma 4.4** *If $xy \in E(G)$, then $\{x, y\} \cap V_2 \neq \emptyset$.*

Let $N(x) = \{y, u\}$ and $N(y) = \{x, v\}$. By criticality, construct a good coloring $\phi$ of $G - x - y$. We may then extend $\phi$ to a good coloring of $G$ by setting $\phi(x) \neq \phi(u)$ and $\phi(y) \neq \phi(v)$, which is a contradiction.

### 4.2.1 Sets with low potential

A set of vertices $\{v_1, v_2, v_3\}$ is called a *clump* if $G[\{v_1, v_2, v_3\}] \cong K_3$, $v_1 \in V_s$, $v_2 \in V_r$, and $v_3 \in V_0$. Note that every nonempty subgraph of a clump has potential at least 2. For a set of vertices $T \subset V(G)$, let $T^* = \{u \in T \ : \ N(u) \not\subseteq T\}$.

**Definition 4.5** *Fix $\emptyset \neq T \subsetneq V(G)$ and a good coloring $\phi_T$ of $G[T]$. Let $T_a = \{w \in V(G) - T \ : \ \exists u \in T, \phi_T(u) = a, wu \in E(G)\}$ and $T_b = \{x \in V(G) - T \ : \ \exists y \in T, \phi_T(y) =$*

$b, xy \in E(G)\}$. *Define* $Y(T,\phi_T)$ *to be the graph constructed from* $G$ *with* $T$ *removed, super vertex* $v_1$ *and regular vertices* $v_2$ *and* $v_3$ *added, and add edges*

$$v_1v_2 + v_1v_3 + v_2v_3 + \{v_1w \ : \ w \in T_a\} + \{v_2x \ : \ x \in T_b\}.$$

**Remark 4.6** *By construction,* $Y(T,\phi_T)$ *is not good (or else there would be a good coloring of* $G$*). Moreover,* $Y(T,\phi_T)$ *contains a critical subgraph.*

**Remark 4.7** *If* $W \subseteq V(Y(T,\phi_T))$*, then*

$$\begin{aligned}\rho_G(W - \{v_1,v_2,v_3\} + T) &= \rho_{Y(T,\phi_T)}(W) - \rho_{Y(T,\phi_T)}(W \cap \{v_1,v_2,v_3\}) \\ &\quad + \rho_G(T) - 5|E(\{v_1,v_2,v_3\}, V(Y(T,\phi_T)) - W)|.\end{aligned}$$

**Proposition 4.8** *If* $\emptyset \neq T \subsetneq V(G)$*, and* $\phi_T$ *is a good coloring of* $G[T]$*, then* $\rho_G(T) \geq 3$ *or* $Y(T,\phi_T) \not\prec G$*.*

**Proof.** By way of contradiction, let $T$ be a largest proper subset of $V(G)$ such that $\rho_G(T) \leq 2$. Let $G'$ be a critical subgraph of $Y(T,\phi_T)$. Because $G' \prec Y(T,\phi_T) \prec G$, by minimality $\rho_{G'}(V(G')) \leq -1$. Critical graphs do not have proper subgraphs that are critical, so $V(G') \cap \{v_1,v_2,v_3\} \neq 0$. Therefore $\rho_{Y(T,\phi_T)}(V(G') \cap \{v_1,v_2,v_3\}) \geq 2$. Hence $\rho_G(V(G') - \{v_1,v_2,v_3\} + T) \leq -1$. Because $T$ was maximal and $\rho_G(V(G)) \geq 0$, this is a contradiction. $\square$

The above proposition states that every set with potential at most 2 must precede a clump. We will show that this is not possible.

**Lemma 4.9** *If* $\emptyset \neq T \subsetneq V(G)$ *and* $\rho_G(T) \leq 2$*, then* $T$ *is a clump.*

**Proof.** Let $T$ have the smallest potential among all non-empty proper subsets of $V(G)$.

First, we will show that $T^* \subseteq V_2$. If $u \in T^* - V_2$, then $d(u) = 2$ and $|N(u) \cap T| \leq 1$ by definition of $T^*$. Furthermore, $\rho_G(\{u\}) = 7$. Hence $\rho_G(T-u) = \rho_G(T)-2$, which contradicts minimality.

Suppose there exists a good coloring $\phi_T$ of $T$ such that $\phi_T(u) = \phi_T(v)$ for all $u, v \in T^*$. Let $G' = G-T+w$, with $w$ a super vertex and $N(w) = N(T)-T$. If $G'$ has a good coloring $\phi'$, then by symmetry we may assume $\phi'(w) = \phi_T(u)$ for all $u \in T^*$. This is a contradiction, because then $\phi' \cup \phi_T$ is a good coloring of $G$. Therefore $G'$ contains a critical subgraph $G''$ and $w \in G''$. Because each vertex of $T^*$ is important, $G'' \prec G' \prec G$, and so by induction $\rho_{G''}(V(G'')) \leq -1$. Hence $\rho_G(V(G'') - w + T) \leq \rho_G(T) - 4$, which contradicts minimality.

Therefore every good coloring of $G[T]$ has at least two colors in $T^*$. Moreover, $|T^*| \geq 2$ and $T$ has at least two important vertices.

If $\rho_G(T) \leq 2$, then $Y(T, \phi_T) \not\prec G$. By the definition of a clump, it follows that $|V_2 \cap T| = 2$, $|V_1 \cap T| = 0$, and $|V_0 \cap T| + 3|V_s \cap T| \leq 4$. Moreover, $T^* = V_2 \cap T$. If $V_s \cap T = \emptyset$, then there exists a good coloring of $T$ where $T^*$ is monochromatic and all other vertices of $T$ are colored the other color. This contradicts an above statement, so $V_s \cap T \neq \emptyset$. It follows that $T$ is a clump. $\square$

**Proposition 4.10** *If $T$ is a separating set in $G$, then $T$ is not a clump.*

**Proof.** By way of contradiction, suppose $V(G) - T$ partitions into sets $U_1$ and $U_2$ such that $E(U_1, U_2) = \emptyset$. By criticality of $G$, there exist good colorings $\phi_1$ of $G[U_1 \cup T]$ and $\phi_2$ of $G[U_2 \cup T]$. By construction of clumps, up to symmetry of the colors, there is a unique good coloring of $T$, and we may therefore assume $\phi_1|_T = \phi_2|_T$. Moreover, no vertex of $T$ has a neighbor with the same color outside of $T$ in either $\phi_1$ or $\phi_2$. Therefore $\phi_1 \cup \phi_2$ is a good coloring of $G$, which is a contradiction. $\square$

A consequence of Lemma 4.9 is that every nonempty proper subset of vertices of $G$ has potential at least 2. Among other things, this forbids two special vertices from being adjacent. Also, every subset of vertices of $G$ has non-negative potential. We will now characterize each proper subset of vertices with potential 3 as being one super vertex or at least $|V(G)|-1$ in size. This will unfold over a series of smaller statements.

**Proposition 4.11** *If $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$, then*

*(a) $G[T]$ is connected,*

*(b) no vertex outside of $T$ has more than one neighbor in $T$,*

*(c) $T^* \subseteq V_2$,*

*(d) there is no good coloring of $G[T]$ such that $T^*$ is monochromatic, and*

*(e) for each vertex in $T^*$ there is a neighbor in $(V_1 \cup V_2) - T$, and the neighbors are distinct.*

**Proof.** If $G[T]$ is not connected, then one of the components has potential at most $\lfloor 3/2 \rfloor = 1$, which is a contradiction. If $u \notin T$ and $|N(u) \cap T| \geq 2$, then $\rho_G(T + u) \leq 0$, which is a contradiction.

If $u \in T^* - V_2$, then then $d(u) = 2$ and $|N(u) \cap T| \leq 1$ by definition of $T^*$. Furthermore, $\rho_G(\{u\}) = 7$. Hence $\rho_G(T - u) = \rho_G(T) - 2 = 1$, which is a contradiction because $T - u$ is a non-empty proper subset of vertices.

Suppose there exists a good coloring $\phi_T$ of $T$ such that $\phi_T(u) = \phi_T(v)$ for all $u, v \in T^*$. Let $G' = G - T + w$, with $w$ a super vertex and $N(w) = N(T) - T$. If $G'$ has a good coloring $\phi'$, then by symmetry we may assume $\phi'(w) = \phi_T(u)$ for all $u \in T^*$. This is a contradiction, because now $\phi' \cup \phi_T$ is a good coloring of $G$. Therefore $G'$ contains a critical subgraph $G''$, and $w \in G''$. Because each vertex of $T^*$ is important, $G'' \prec G' \prec G$, and so by minimality $\rho_{V(G'')}(V(G'')) \leq -1$. But then $\rho_G(V(G'') - w + T) \leq \rho_G(T) - 4 = -1$, which is a contradiction.

Let $x \in T^*$ and $y \in N(x) - T$. $V_0, V_1$ and $V_2$ partition $V(G)$. If $y \in V_0$, then $y$ forms a triangle with $x$ and a vertex $z$. By part (b), $z$ is not in $T$ and $z$ is not a neighbor to any other vertex in $T$. By Lemma 4.4, $z \in V_2$. $\square$

**Proposition 4.12** *Let $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$. Suppose $u, v \in T^*$ are not in the same clump. Let $Z(T, u, v) = G[T] + z$, where $w$ is a regular vertex with $N(z) = \{u, v\}$. Under these assumptions, $Z(T, u, v)$ has a good coloring.*

**Proof.** By Proposition 4.11(e), there are at least two vertices in $V(G) - (T \cup V_0)$. Each vertex in $T$ is no more important in $Z(T, u, v)$ than in $G$, and $d_{G'}(z) = 2$, so $z \notin V_2(Z(T, u, v))$. Therefore $Z(T, u, v) \prec G$. By the assumption that $u$ and $v$ are not in the same clump and Lemma 4.9, there is no set with potential at most 2 which contains $\{u, v\}$. Because the addition of one regular vertex and two edges decreases the potential by 3, for every $W \subset Z(T, u, v)$, we have $\rho_{Z(T,u,v)}(W) \geq 0$. By minimality, there are no critical graphs contained in $Z(T, u, v)$, and therefore $Z(T, u, v)$ is good. $\square$

**Proposition 4.13** *If $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$, then $|T^*| \geq 3$.*

**Proof.** By criticality, there exists a good coloring of $G[T]$, which is not monochromatic. Therefore $|T^*| \geq 2$, and by way of contradiction we may assume $T^* = \{t_1, t_2\}$. Because clumps are not separating sets, $t_1$ and $t_2$ are not in the same clump. Let $G' = Z(T, t_1, t_2)$.

Let $\phi_T$ be a good coloring of $G'$. Without loss of generality, let $\phi_T(t_1) = \phi_T(u) = a$ and $\phi_T(t_2) = b$. Let $G''$ be $G - (T - T^*)$ with edge $t_1t_2$ added if it does not exist and $t_2$ become a special vertex if it is not one already. Suppose $G''$ has a good coloring $\phi''$, and without loss of generality, assume $\phi''(t_1) = a$. By construction, $\phi''(t_1) \neq \phi''(t_2)$, so $\phi''(t_2) = \phi_T(t_2)$.

Moreover, $t_1$ has no neighbors of the same color in $T$ and $t_2$ has no neighbors with the same color in $V(G) - T$. Therefore $\phi_T \cup \phi''$ is a good coloring of $G$, which is a contradiction. So $G'' \not\prec G$ or there exists a critical graph $G^\circ$ contained in $G''$.

Suppose $G'' \not\prec G$. Every vertex in $T^*$ is important in $G$, so $V_2 \cap T = T^*$, $V_1 \cap T = \emptyset$, and $|V_0(T)| + 3|V_s(T)| \leq 3$. If $V_s \cap T = \emptyset$, then there exists a good coloring of $T$ where $T^*$ is monochromatic and all other vertices of $T$ are colored the other color. This contradicts an above statement, so $V_s \cap T \neq \emptyset$. But this is impossible because no graph on two vertices has potential 3.

Thus, we may assume that there exists a critical graph $G^\circ$ contained in $G''$. By minimality, $\rho_{G''}(V(G^\circ)) \leq -1$. Because critical graphs do not contain critical graphs, $t_2 \in V(G^\circ)$. Note that every graph on two vertices with potential at least 2 has potential at least 5. Hence $\rho_G(V(G^\circ) - T^* + T) \leq -1 - 5 + 3 = -3$, which is a contradiction. $\square$

**Proposition 4.14** *Let $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$. For every good coloring $\phi_T$ of $G[T]$, each vertex in $T^* \cap V_r$ has a neighbor with each color.*

**Proof.** By way of contradiction, let $\phi_T$ be a good coloring of $G[T]$ such that $u \in T^*$ has no neighbors colored $a$. We may assume that $\phi_T(u) = a$. By the above, there exists a vertex $w \in N(u) - V_0 - T$. If $w$ is a special vertex, then $\rho_G(T \cup w) = 1$, which is a contradiction.

Construct $G'$ from $Y(T, \phi_T)$ by deleting the edge from $w$ to $\{v_1, v_2, v_3\}$ and turning $w$ into a special vertex.

Suppose there exists a good coloring $\phi'$ of $G'$. Without loss of generality, we may assume $\phi'(v_1) = a$ and $\phi'(v_2) = b$. We claim that $\phi_T \cup \phi'$ is a good coloring of $G$, which is a contradiction. By construction, each edge in $E(T, V(G) - T)$ has different colors on its endpoints except for possibly $wu$. Because $w$ and $u$ have no other neighbors with the same color and both vertices are regular, $\phi_T \cup \phi'$ is a good coloring even if $\phi_T(u) = \phi'(w)$. Therefore

$G'$ is not good and contains a critical graph $G''$. By induction, $G' \not\prec G$ or $\rho_{G'}(V(G'')) \leq -1$.

Because $|T^*| \geq 3$, $T$ contains at least 3 important vertices. Because $w \notin V_0$, either $|V_2(G)| > |V_2(G')|$, or $|V_2(G)| = |V_2(G')|$ and $|V_1(G)| > |V_1(G')|$. Therefore $\rho_{G'}(V(G'')) \leq -1$.

If $w \in V(G'')$, then $\rho_G(V(G'') - \{v_1, v_2, v_3\} + T) \leq -1 + 4 + 3 - 5 \leq 1$. So $V(G'') - \{v_1, v_2, v_3\} + T = V(G)$. Now we did not account for at least two edges in $E(T, V(G) - T - w)$, and so $\rho_G(V(G'') - \{v_1, v_2, v_3\} + T) \leq 1 - 10$, which is a contradiction.

We may assume $w \notin V(G'')$. Because critical graphs do not contain other critical graphs, $V(G'') \cap \{v_1, v_2, v_3\} \neq \emptyset$. Therefore $\rho_{G'}(V(G'') \cap \{v_1, v_2, v_3\}) \geq 2$, and thus $\rho_G(V(G'') - \{v_1, v_2, v_3\} + T) \leq -1 - 2 + 3 \leq 0$. Now $V(G'') - \{v_1, v_2, v_3\} + T$ is either empty or $V(G)$, but this is a contradiction because it contains $T$ and does not contain $w$. □

**Proposition 4.15** *Let $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$. If $u, v \in T^*$ are not in the same clump, then there exists a good coloring $\phi_T$ of $G[T]$ such that $\phi_T(u) = \phi_T(v)$.*

**Proof.** If $\phi_T$ be a good coloring of $Z(T, u, v)$ then $\phi_T$ is a good coloring of $G[T]$. Furthermore, $\phi_T(u) \neq \phi_T(z)$ and $\phi_T(v) \neq \phi_T(z)$. Because there are only two colors, $\phi_T(u) = \phi_T(v)$. □

**Lemma 4.16** *There are no sets $T \subseteq V(G)$ such that $2 \leq |T| \leq |V(G)| - 2$ and $\rho_G(T) = 3$.*

**Proof.** By way of contradiction, let $T$ be the smallest set with potential 3 and at least two vertices. Let $\phi_1$ be a good coloring of $G[T]$. Let $U_a = \{u \in T^* : \phi_1(u) = a\}$ and $U_b = T^* - U_a$. By symmetry, we may assume $|U_a| \geq |U_b|$. Because $T^*$ is not monochromatic, $U_b \neq \emptyset$. Because $|T^*| \geq 3$, we have $|U_a| \geq 2$.

Let $\{u_1, u_2\} \subseteq U_a$, and let $v \in U_b$. Each clump has only two important vertices, and clumps may not overlap without their union having negative potential. By symmetry, we may then assume that $\{u_1, v\}$ is not contained in a clump. There is a second good coloring $\phi_2$ of $G[T]$ such that $\phi_2(v_1) = \phi_2(u)$. Let $W_\Delta = \{u \in T^* \ : \ \phi_1(u) \neq \phi_2(u)\}$ and $W_\theta = T^* - W_\Delta$. By symmetry, we may assume $|W_\Delta| \leq |W_\theta|$. By construction, $W_\Delta \neq \emptyset$.

*Case 1:* $W_\Delta \subseteq U_a$ ($W_\Delta \subseteq U_b$ follows symmetrically).

Let $G'$ be constructed from $Y(T, \phi_T)$ by deleting the edges $E(N(W_\Delta) - T, \{v_1, v_2, v_3\})$, adding a special vertex $v_0$, and adding edges so that $N(v_0) = N(W_\Delta) - T$. Suppose there exists a good coloring $\phi'$ of $G'$. Without loss of generality, we may assume $\phi'(v_1) = a$ and $\phi'(v_2) = b$. Then either $\phi_1 \cup \phi'$ or $\phi_2 \cup \phi'$ is a good coloring of $G$. This is a contradiction, so $G'$ is not good and contains a critical graph $H'$. By induction, $G' \not\prec G$ or $\rho_{G'}(V(H')) \leq -1$.

Suppose $\rho_{G'}(V(H')) \leq -1$. If $v_0 \in V(H')$, then $\rho_G(V(H') - v_0 + T) \leq -1$, which is a contradiction. We may assume $v_0 \notin V(H')$. By criticality, $H'$ is not contained in $G$, so $V(H') \cap \{v_1, v_2, v_3\}) \neq \emptyset$. Therefore $\rho_{G'}(V(H') \cap \{v_1, v_2, v_3\}) \geq 2$, and thus $\rho_G(V(H') - \{v_1, v_2, v_3\} + T) \leq -1 - 2 + 3 \leq 0$. So $V(H') - \{v_1, v_2, v_3\} + T = V(G)$. Now we did not account for at least one edge in $E(W_\Delta, V(G) - T)$, and so $\rho_G(V(H') - \{v_1, v_2, v_3\} + T) \leq -5$, which is a contradiction. Therefore $G' \not\prec G$.

Because $G' \not\prec G$, we know that $V_2 \cap T = T^*$, $|T^*| = 3$, $V_1 \cap T = \emptyset$, and $|V_0 \cap T| + 3|V_s \cap T| \leq 7$. By Lemma 4.4, if $u \in V_0 \cap T$, then $N(u) \subset T^*$. By counting potential, it follows that

$$3 = 21 - 4|V_s \cap T| - 5|E(G[T^*])| - 3|V_0 \cap T|. \tag{4.1}$$

Every vertex on a shortest path between two vertices of $T^*$ is not in $V_0$, so $G[T^*]$ is connected. This implies that $|E(G[T^*])| \geq 2$.

If $|V_s \cap T| \geq 2$, then $\rho_G(T) \leq 3$. Because clumps cannot be separating sets and $T$ is a minimal set with potential 3, $T = T^*$. Hence there is a good coloring of $G[T]$ such that the regular vertex has no neighbors with the same color, which contradicts Proposition 4.14.

With all of these restrictions, the only valid solution to (4.1) is $|V_s \cap T| = 0$, $|E(G[T^*])| = 3$, and $|V_0 \cap T| = 1$. Hence $G[T] \cong K_4 - e$, and there is a good coloring of $G[T]$ such that a regular vertex in $T^*$ has no neighbors with the same color. This contradicts Proposition 4.14.

*Case 2:* $|W_\Delta| \geq 2$. This implies that $|T^*| \geq 4$.

Let $G''$ be constructed from $Y(T, \phi_T)$ by deleting the edges $E(N(W_\Delta) - T, \{v_1, v_2, v_3\})$, adding a clump with vertex set $\{v_1', v_2', v_3'\}$, and adding edges so that $N(v_1') = N(W_\Delta \cap U_a) - T$ and $N(v_2') = N(W_\Delta \cap U_b) - T$. Suppose there exists a good coloring $\phi''$ of $G''$. Without loss of generality, we may assume $\phi''(v_1) = a$ and $\phi''(v_2) = b$. Now either $\phi_1 \cup \phi''$ or $\phi_2 \cup \phi''$ is a good coloring of $G$. This is a contradiction, so $G''$ is not good and contains a critical graph $H''$. By minimality, $G'' \not\prec G$ or $\rho_{G''}(V(H'')) \leq -1$.

Suppose $\rho_{G''}(V(H'')) \leq -1$. $H''$ must include at least one of the vertices in the two clumps added to $G$ to make $G''$, by symmetry assume that $V(H'') \cap \{v_1, v_2, v_3\} \neq \emptyset$. Therefore $\rho_{V(G'')}(V(H'') \cap \{v_1, v_2, v_3\}) \geq 2$. Let $S = V(H'') - \{v_1, v_2, v_3, v_1', v_2', v_3'\} + T$. Thus $\rho_G(S) \leq -1 - 2 + 3 \leq 0$. So $S = V(G)$. We did not account for at least one edge in $E(W_\Delta, V(G) - T)$, and so $\rho_G(S) \leq -5$, which is a contradiction. Therefore $G'' \not\prec G$.

Because $G'' \not\prec G$, we know that $V_2 \cap T = T^*$, $|T^*| = 4$, $V_1 \cap T = \emptyset$, and $|V_0 \cap T| + 3|V_s \cap T| \leq 8$. By Lemma 4.4, if $u \in V_0 \cap T$, then $N(u) \subset T^*$. Moreover, $N(u)$ is an edge of $G[T^*]$. By counting potential, it follows that

$$3 = 21 - 4|V_s \cap T| - 5|E(G[T^*])| - 3|V_0 \cap T|. \tag{4.2}$$

Every vertex on a shortest path between two vertices of $T^*$ is not in $V_0$, so $G[T^*]$ is connected. This implies that $|E(G[T^*])| \geq 3$. If $|E(G[T^*])| \geq 5$, then $\rho_G(T^*) \leq 3$. Similarly to Case 1, we have $T = T^*$, the only graph with four vertices and five edges is $K_4 - e$, which is a contradiction. So $|E(G[T^*])| \in \{3, 4\}$.

With all of these restrictions, the only valid solution to (4.2) is $|V_s \cap T| = 1$, $|E(G[T^*])| = 3$, and $|V_0 \cap T| = 2$. Because of Proposition 4.14, every regular vertex that is a leaf in $G[T^*]$ is also adjacent to a vertex in $V_0 \cap T$.

*Case 2.A:* $G[T^*] \cong K_{1,3}$. Because $K_{1,3}$ has three leaves and there are only two unimportant vertices in $T$, then the special vertex in $T^*$ is one of the leaves of $G[T^*]$ and the neighborhoods of the two unimportant vertices of $T$ is the other two leaves with the center of the star. Thus there is a good coloring of $G[T]$ where the center of the star is color $a$, and all other vertices are color $b$. This contradicts Proposition 4.14.

*Case 2.B:* $G[T^*] \cong P_4$. We may assume that this path is $(t_1, t_2, t_3, t_4)$. By symmetry, we may assume that the special vertex is either $t_1$ or $t_2$. Let $x_1$ and $x_2$ be the unimportant vertices, and let $y$ be the special vertex of $T$. By symmetry, we may assume $N(x_1) = \{t_3, t_4\}$. Now there is a good coloring $\phi_T$ of $G[T]$ where $\phi_T(t_2) = \phi_T(t_4) = a$, $\phi_T(t_1) = \phi_T(t_3) = b$, and $\phi_T(y) = \phi_T(x_1) \neq \phi_T(x_2)$. In this coloring, at least one of regular vertices of $T^*$ will not have a neighbor colored with the same color. This contradicts Proposition 4.14. $\square$

### 4.2.2 Reducible configurations

**Lemma 4.17** *Let each of $u$ and $v$ be either a special vertex or in a clump. If $u$ and $v$ are not in the same clump, then the distance between $u$ and $v$ is at least* 3.

**Proof.** Let $P$ be a shortest path between $u$ and $v$. Let $S$ be the set of vertices composed of the union of $P$, $u$, $v$, and vertices in any clump containing $u$ or $v$. If $P$ contains at most 3 vertices, then $S$ has potential at most 3. $\square$

**Lemma 4.18** *Let $v \in V(G)$ be a regular vertex with degree 3. Then every neighbor of $v$ is in $V_2$.*

**Proof.** By way of contradiction, let $N(v) = \{x, y, z\}$ and $v$ be a regular vertex.

*Case A:* $v$ is in a clump. Without loss of generality, let $x$ be outside of that clump, $z$ be the special vertex of the clump, and $N(y) = \{v, z\}$.

*Case A.1* $d(x) \geq 3$. Let $G' = G - y + y'$, where $N(y') = \{x, z\}$. Note that $v$ is important in $G$, but $v$ and $y'$ are at most semi-important in $G'$. Also, $x$ is important in $G$. Therefore $G' \prec G$. If there exists $T \subseteq V(G')$ such that $\rho_{G'}(T) \leq -1$, then $y' \in T$. This implies that $\{x, z\} \subseteq T$. However, $\rho_G(T - y' + y) = \rho_{G'}(T)$, which is a contradiction. Therefore we may find a good coloring $\phi' : V(G') \to \{a, b\}$.

Without loss of generality, assume that $\phi'(z) = a$. Because $z$ is a special vertex, $\phi'(v) = \phi'(y') = b$. From this, we deduce that $\phi'(x) = a$. We may generate a good coloring $\phi$ of $G$ by setting $\phi|_{V(G)-y} = \phi'|_{V(G')-y'}$ and $\phi(y) = b$.

*Case A.2:* $N(x) = \{v, u\}$. By Lemma 4.4, $u$ is important. Because $u$ is distance 2 to $v$ and $v$ is in a clump, we have that $u$ is a regular vertex and not in a clump. Let $G'$ be $G$ with $v$, $x$, and $y$ deleted, and $u$ turned into a special vertex. Because $v$ and $u$ are important in $G$, it follows that $G' \prec G$.

If there exists $T \subset V(G')$ such that $\rho_{G'}(T) \leq -1$, then $u \in T$. It follows that $\rho_G(T) \leq 3$, which is a contradiction because $T \cap \{x, y, v\} = \emptyset$ and $|T| \geq 2$. Therefore $G'$ has a good coloring $\phi'$. We create a good coloring $\phi$ of $G$ by setting $\phi|_{G-x-y-v} = \phi'|_{G'}$, $\phi(x) = \phi'(z)$, and $\phi(v) = \phi(y) \neq \phi'(z)$.

*Case B:* $x \in V_0$ (symmetrically, $y \in V_0$ or $z \in V_0$). Without loss of generality, assume $N(x) = \{v, y\}$. By Lemma 4.4, $y \in V_2$. Because $v$ is not in a clump, we have $y \in V_r$. Let $G'$ be $G$ with $v$ and $x$ deleted and $y$ is changed into a special vertex. By construction, $G' \prec G$.

If $T \subset V(G')$ with $\rho_{G'}(T) \leq -1$, then $y \in T$. Now, $\rho_G(T + x + v) \leq -1 + 4 - 1 = 2$. Since $v$ was not in a clump, this implies that $T + x + v = V(G)$. We did not account for the edge $vz$, and so $\rho_G(T+x+v) \le 2-5 < 0$, a contradiction.

Let $\phi'$ be a good coloring of $G'$. If $w \in N_{G'}(y)$, then $\phi'(w) \neq \phi'(y)$. Let $\phi$ be a coloring of $G$ where $\phi|_{V(G)-x-v} = \phi'|_{V(G')}$, $\phi(x) = \phi'(z)$, and $\phi(v) \neq \phi'(z)$. Either $\phi(y) = \phi(v)$ or $\phi(y) = \phi(x)$, but not both. Therefore $\phi$ is a good coloring.

For cases 1-3, assume $v$ is not in a clump and $v$ has no neighbors of degree 2 in a triangle.

*Case 1:* $v$ is adjacent to exactly two neighbors of degree 2. Let $N(x) = \{v, x'\}$ and $N(y) = \{v, y'\}$. By Lemma 4.4, $x'$ and $y'$ are important. Without loss of generality, assume $d(x') \ge d(y')$.

*Case 1.1:* $x' \neq y'$. Let $G' = G - y + y^\circ$, where $N(y^\circ) = \{v, x'\}$. We claim that $G'$ precedes $G$. Because $x'$ and $y'$ are important in $G$, they cannot be more important in $G'$. Suppose $y^\circ$ is more important than $y$. Then $y$ was in a triangle, which contradicts the assumption. Therefore $G'$ precedes $G$ by the condition on the degrees.

If $T$ is such that $\rho_{G'}(T) \le -1$, then $y^\circ \in T$ and also $v \in T$. Thus $\rho_G(T - y^\circ) \le -1+3 = 2$, which contradicts the assumption that $v$ was not in a clump.

Therefore there exists a good coloring $\phi'$ of $G'$. Without loss of generality, let $\phi'(x') = a$. Let $\phi$ be a coloring of $G$ where $\phi|_{V(G)-x-y-v} = \phi'|_{V(G')-x-y^\circ-v}$.

- If $\phi'(z) = b$, then color $\phi(x) = b$, $\phi(v) = a$, and $\phi(y) \neq \phi(y')$.
- If $\phi'(z) = a$ and $\phi'(v) = b$, then either $\phi'(x) = a$ or $\phi'(y^\circ) = a$. Furthermore, for all $u \in N_{G'}(x') - \{x, y^\circ\}$, we have $\phi'(u) = b$. Color $\phi(x) = a$, $\phi(v) = b$, and $\phi(y) \neq \phi(y')$.
- If $\phi'(z) = \phi'(v) = \phi'(y') = a$, then color $\phi(x) = \phi(y) = b$ and $\phi'(v) = a$.
- If $\phi'(z) = \phi'(v) = a$ and $\phi'(y') = b$, then color $\phi(x) = \phi(v) = b$ and $\phi(y) = a$.

The above assumptions exhaust all possibilities for $\phi'$. Moreover, each provides a good coloring of $G$.

*Case 1.2:* $x' = y'$. Let $G' = G - y + y^\circ$, where $N(y^\circ) = \{v, z\}$. Because $y \in V_1$ and $y^\circ \in V_0$, $G'$ precedes $G$.

If $T$ is such that $\rho_{G'}(T) \le -1$, then $y^\circ \in T$ and also $\{v, z\} \subseteq T$. Thus $\rho_G(T - y^\circ) \le -1 + 3 = 2$, which contradicts the assumption that $v$ was not in a clump. Therefore $G'$ has a good coloring.

Let $\phi' : V(G') \to \{a, b\}$ be a good coloring. Without loss of generality, let $\phi'(z) = a$. We have that $\phi'(x) = a$ or $a \in \{\phi'(v), \phi'(y^\circ)\}$. If neither of these statements are true then $\phi'(v) = \phi'(x) = \phi'(y^\circ) = b$, and so $v$ is adjacent to two vertices with the same color in $\phi'$, which is a contradiction.

Let $\phi$ be a coloring on $G$ where $\phi|_{V(G)-x-y-v} = \phi'|_{V(G')-x-y^\circ-v}$ and

- if $\phi'(x) = a$, then $\phi(x) = a$, $\phi(y) \neq \phi(x')$, and $\phi(v) = b$, or
- otherwise if $a \in \{\phi'(v), \phi'(y^\circ)\}$, then $\phi(x) = \phi(y) \neq \phi'(x')$ and $\phi(v) = \phi'(x')$.

Under these conditions, $\phi$ is a good coloring.

*Case 2:* $v$ is adjacent to exactly one vertex of degree 2. Let $N(x) = \{v, x'\}$. Let $G' = G - x + x^\circ$, where $N(x^\circ) = \{y, z\}$. Because $x^\circ, v \notin V_2(G')$, it follows that $G' \prec G$.

If $T$ is such that $\rho_{G'}(T) \le -1$ then $x^\circ \in T$ ans also $\{y, z, v\} \subseteq T$. Thus, $\rho_G(T - x^\circ) \le -1+3$. This contradicts the assumption that $v$ was not in a clump. Therefore $G'$ has a good coloring.

Let $\phi'$ be a good coloring of $G'$. Without loss of generality, let $\phi'(y) = a$.

If $\phi'(z) = a$, then create a coloring $\phi|_{V(G)-x-v} = \phi'|_{V(G')-x^\circ-v}$, $\phi(v) = b$, and $\phi(x) \neq \phi'(x')$. This is a good coloring of $G$, which is a contradiction. Thus, we may assume $\phi'(z) = b$. Because $\phi'$ is a good coloring, it follows that $\phi'(v) \neq \phi'(x^\circ)$, or else $y$ or $z$ will have two neighbors with the same color. Therefore all other neighbors of $y$ have color $b$ and all other neighbors of $z$ have color $a$. We color $G$ with coloring $\phi|_{V(G)-x-v} = \phi'|_{V(G')-x^\circ-v}$, $\phi(x) \neq \phi'(x')$, and $\phi(v) = \phi'(x')$. Note that $\phi(v)$ may be the same as $\phi(y)$ or $\phi(z)$, but it will not be the same as both. Hence, $\phi$ is a good coloring of $G$.

*Case 3:* $v$ is adjacent to three vertices of degree 2. Let $G' = G - v$. Let $\phi'$ be a good coloring of $G'$. In $G'$, $x$, $y$, and $z$ all have degree one. Without loss of generality, we may assume each of them has no neighbors with the same color as themselves. Extend the coloring on $G'$ to one on $G$ by coloring $v$ the color that appears the least in the list $(\phi'(x), \phi'(y), \phi'(z))$. Because some color appears at most once and that vertex has no other neighbors with the same color, this is a good coloring of $G$. $\square$

One consequence of this is that the important regular vertex in a clump has degree at least 4.

**Lemma 4.19** *Let $v$ be a special vertex. Then $d(v) \geq 3$.*

**Proof.** Let $v$ be a special vertex and $N(v) = \{x, y\}$. Without loss of generality, assume $d(x) \geq d(y)$.

*Case 1:* $xy \notin E(G)$ and $N(x) \cap N(y) = \{v\}$. Let $G' = G - v - x - y + z$, where $z$ is a regular vertex and $N(z) = (N(y) \cup N(x)) - v$. If $u \in V(G') - z$ is important in $G'$, then $u$ is important in $G$. If $z$ is important in $G'$, then $d_G(x) \geq \lceil d(z)/2 \rceil + 1 \geq 3$, so $x$ is important in $G$. Therefore $|V_2(G')| < |V_2(G)|$, and so $G' \prec G$.

If $T \subset V(G')$ such that $\rho_{G'}(T) \leq -1$, then $z \in T$. Thus, $\rho_G(T - z + x + y + v) = \rho_{G'}(T)$, a contradiction.

Therefore $G'$ has a good coloring, $\phi'$. We can create a good coloring $\phi$ of $G$ by setting $\phi|_{G-v-x-y} = \phi'|_{G'-z}$, $\phi(x) = \phi(y) = \phi'(z)$, and $\phi(v) \neq \phi'(z)$.

*Case 2:* $xy \in E(G)$ or there exists a $w$ such that $w \in (N(x) \cap N(y)) - v$. Let $G' = G - v + z$, where $z$ is a special vertex and $N(z) = (N(y) \cup N(x)) - v$. If $u \in V(G') - z$ is important in $G'$, then $u$ is important in $G$. If $u \in V(G') - z$ is semi-important in $G'$, then $u$ is important or semi-important in $G$. Because $G$ is connected, if $y \in V_0$, then $x \in V_2$. Therefore, either $|V_2(G')| < |V_2(G)|$, or $|V_2(G')| = |V_2(G)|$ and $|V_1(G')| < |V_1(G)|$. In either case, $G' \prec G$.

If $T \subset V(G')$ such that $\rho_{G'}(T) \leq -1$, then $z \in T$. Now $\rho_G(T - z + x + y + v) \leq -1$, a contradiction.

Therefore $G'$ has a good coloring, $\phi'$. We can create a good coloring of $\phi$ by setting $\phi|_{V(G)-v-x-y} = \phi'|_{V(G')-z}$, $\phi(x) = \phi(y) = \phi'(z)$, and $\phi(v) \neq \phi'(z)$. $\square$

**Lemma 4.20** *If $v \in V(G)$ such that $N(v) = \{u_1, u_2, u_3, u_4\}$, $v$ and each $u_i$ are regular vertices, and $N(u_i) = \{x_i, v\}$ for all $i$, then $x_i$ is a super vertex or in a clump for all $i$.*

**Proof.** Without loss of generality, let $x$, $u_i$ and $x_j$ be as above, and assume that $x_1$ is not a super vertex or in a clump. By Lemma 4.4, each $x_j$ is important. Let $G' = G - \{v, u_1, u_2, u_3, u_4\}$, and $x_1$ is changed to a special vertex. By construction, $G' \prec G$.

Suppose that $G'$ has a good coloring $\phi'$. We construct a good coloring $\phi$ of $G$ as follows:

- Set $\phi|_{V(G)-\{v,u_1,u_2,u_3,u_4\}} = \phi'|_{V(G')}$.
- Set $\phi(u_i) \neq \phi'(x_i)$ for $i \in \{2, 3, 4\}$.
- Set $\phi(v)$ equal to the color that appears the least in the list $(\phi(u_2), \phi(u_3), \phi(u_4))$.
- Set $\phi(u_1) \neq \phi(v)$.

Under these conditions, $\phi$ is a good coloring of $G$. We conclude that $G'$ has no good coloring, and that there exists $T \subseteq V(G')$ such that $\rho_{G'}(T) \leq -1$. For this to be possible, $x_1 \in T$. Thus $\rho_G(T) \leq -1 + 4 = 3$. Because $\{v, u_1, u_2, u_3, u_4\} \cap T = \varnothing$, this is a contradiction. $\square$

### 4.2.3 Discharging

By assumption on $G$, we have

$$\sum_{v\in V_r}(5d(v)-14)+\sum_{v\in V_s}(5d(v)-6)\le 0. \tag{4.3}$$

The initial charge of each vertex $v$ of $G$ is $\mu(v)=5d(v)-14$ if $v\in V_r$ and $\mu(v)=5d(v)-6$ if $v\in V_s$. The final charge $\mu^*(v)$ is determined by applying the following rules:

**R1.** Let $x$ be a special vertex. Every vertex adjacent to $x$ with degree 2 gets charge 2.5 from $x$ and charge 1.5 from its other neighbor.

**R2.** Let $x$ be a regular important vertex in a clump. Every vertex adjacent to $x$ with degree 2 that is not in the same clump as $x$ gets charge 2.5 from $x$ and charge 1.5 from its other neighbor.

**R3.** Every regular vertex with degree 2 not adjacent to a special vertex or a clump gets 2 from each of its neighbors.

By Lemmas 4.17, there is no conflict with Rules 1 and 2. By Lemma 4.4, there is no conflict with Rule 3.

**Lemma 4.21** *For every $v\in V(G)$, we have $\mu^*(v)\ge 0$. Moreover, if $\mu^*(v)=0$, then either*

*a) $v$ is regular and $d(v)=2$, or*

*b) $v$ is regular, $d(v)=4$, and exactly three neighbors of $v$ have degree 2.*

**Proof.** Each special vertex has degree at least 3, and therefore for all $v\in V_s$, we have $\mu^*(v)\ge 2.5d(v)-6>0$.

Every vertex with degree 2 receives 4 charge and has final charge 0.

If $v$ is a regular vertex and $d(v)=3$, then $v$ is not adjacent to a vertex with degree 2 and hence $\mu^*(v)=\mu(v)=1$.

If $v$ is a regular vertex with $d(v)\ge 5$, then $\mu^*(v)\ge 3d(v)-14\ge 1$.

If $v$ is a regular vertex not in a clump with $d(v)=4$, then $v$ has at most three neighbors with degree 2. Now $\mu^*(v)\ge 3d(v)+2-14=0$, with equality if and only if $v$ has exactly three neighbors with degree 2. If $v$ is the important regular vertex in a clump and $d(v) = 4$, then $\mu^*(v) \geq 3d(v) + 2.5 - 14 = 0.5$. $\square$

By the above lemma, in order (4.3) to hold, we need $\mu^*(v) = 0$ for every $v \in V(G)$. By the same lemma, $G$ has only regular vertices of degree 2 and 4, and each vertex with degree 4 has at most one neighbor of degree 4. By Lemma 4.4, each vertex with degree 2 has no neighbors of degree 2. In such a graph $G$, if we color all vertices with degree 4 with color $a$ and all vertices with degree 2 color $b$, then we get a good coloring of $G$, a contradiction.

# Chapter 5

# Rainbow Matchings

The motivation to study the existence of rainbow matchings comes from three distinct sources: an equivalent problem in design theory, its relationship to anti-Ramsey theory, and the overall interest in rainbow subgraphs.

One of the fundamental problems in design theory involves finding transversals in Latin Squares. A *Latin Square* with length $n$ is an $n$ by $n$ matrix with entries in $[n]$ such that each row and each column form a permutation of $[n]$. A *transversal* of a Latin Square $L$ is a set of coordinates $(a_i, b_i)_{1 \leq i \leq k}$ such that $1 \leq a_i, b_i \leq n$ and there is no repetition of values in the sets $\{a_i : 1 \leq i \leq k\}$, $\{b_i : 1 \leq i \leq k\}$, and $\{L(a_i, b_i) : 1 \leq i \leq k\}$.

Ryser [67] conjectured that if $n$ is odd, then Latin Squares with length $n$ contains a transversal with size $n$. It is known that if $n$ is even, then there exist Latin Squares with length $n$ with no transversal with size $n$. It is also conjectured that every Latin Square with length $n$ has a transversal of size $n-1$. The best lower bound on the size of a largest transversal of a Latin Square with length $n$ remains open. After a sequence of successively better results, the best known bound today comes from Hatami and Shor [40], which states that in Latin Squares with length $n$ a transversal of size $n - O(log^3(n))$ exists.

It turns out that searching for this substructure in a matrix has an analogous problem in graph coloring. Let $L$ be a Latin Square with length $n$ and transversal $T = (a_i, b_i)_{1 \leq i \leq k}$. Consider an edge-coloring $\phi$ of the graph $G = K_{n,n}$ with parts $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ such that $\phi(x_i, y_j) = L(i, j)$. Each coordinate pair $(a_i, b_i) \in T$ maps to an edge of $G$. Let $M$ denote the subgraph of $G$ that is the union of the edges that are images of elements of $T$.

No value is repeated in the sets $\{a_i : 1 \leq i \leq k\}$ and $\{b_i : 1 \leq i \leq k\}$, so $\Delta(M) = 1$. No value is repeated in the set $\{L(a_i, b_i) : 1 \leq i \leq k\}$, so $M$ is a rainbow subgraph of $G$. By removing isolated vertices, there is a correspondence from $T$ to $M$ that relates transversals in $L$ to rainbow matchings in $G$. Ryser's conjecture is thus equivalent to finding a largest rainbow matching in an edge-colored $K_{n,n}$ where each color class forms a 1-factor.

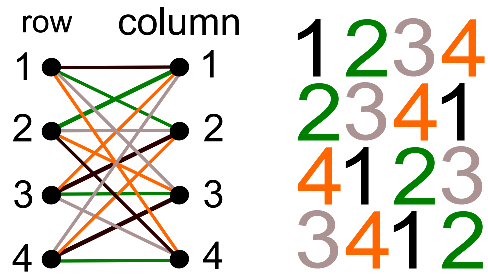


Figure 5.1: An example that demonstrates the correspondence between Latin Squares and edge-colored graphs.

Design theorists saw this connection and used it to generalize the problem to orthogonal matchings. For a graph $G$ whose edges are partitioned into $k$ $t$-factors, an *orthogonal matching* is a matching with exactly one edge from each $t$-factor. By labeling each $t$-factor as a color class, an orthogonal matching directly translates into a rainbow matching.

The largest matching in a graph with $n$ vertices is at most $n/2$. This implies that Ryser's conjecture is infeasible in this generalized setting without additional constraints on the order of the graph. One of the earliest results in orthogonal matchings showed that the smaller of the two bounds is almost possible in all cases. Anstee and Caccetta [6] showed that every $k$-regular graph with $n$ vertices where each color class is a 1-factor has a rainbow matching of size

$$\min\{\frac{n}{2} - \frac{3}{2}(\frac{n}{2})^{2/3}, k - \frac{3}{2}k^{2/3}\}. \tag{5.1}$$

This result provides the intuition that the size of the graph and the number of colors are the main issues that may prevent large rainbow matchings from existing.

Anstee and Caccetta were also able to provide stronger results if the color classes are large. They proved in the same paper that if each color class is a $t$-factor, and $t \geq 3$, then $G$ has a rainbow matching containing an edge from each color class. If each color class is $t$-regular and there are $k$ colors, then $G$ has at least $tk+1$ vertices; so the assumption on large color classes is stronger than an assumption on the size of the graph. Furthermore, Anstee and Caccetta showed that if each color class is a 2-factor, then the graph has a rainbow matching with size $k - k^{2/3}$. Stong [71] improves this bound for large graphs, showing that a rainbow matching containing an edge from each color class exists if each of the $k$ color classes forms a 2-factor and $n \geq 3k - 2$.

Anstee and Caccetta [6] used the fact that if each color class is regular and spanning, there are $k$ colors, and the graph has at least $4k - 3$ vertices, then a greedy algorithm will produce a rainbow matching of size $k$. This statement is easy to prove (their paper only devotes one paragraph to it), and immediately implies their result when the color classes are $t$-factors for $t \geq 4$. A greedy algorithm will not work if the color classes are not regular or not spanning.

A second motivation for studying rainbow matchings is anti-Ramsey theory. For a fixed graph $H$ and parameter $n$, the *anti-Ramsey number*, $AR(n, H)$, is the largest $\ell$ such that there exists an edge-coloring of $K_n$ using exactly $\ell$ colors which contains no rainbow $H$. Solutions to the anti-Ramsey problem are known for trees [46], matchings [31], and complete graphs [68, 22] (see [32] for a more complete survey).

Erdős, Simonovits, and Sós made a deep connection[28] between anti-Ramsey theory and Turán theory. For a fixed graph $H$ and parameter $n$, the *Turán number* is the largest $\ell$ such that there exists a graph with $n$ vertices, $\ell$ edges, and does not contain $H$ as a subgraph. Let $X(H) = min\{\chi(H - e) \ : \ e \in E(H)\}$. In their work, Erdős, Simonovits, and Sós demonstrated that $AR(n, H) = (1 - 1/(X(H) - 1) + o(1))n^2$ by comparing anti-Ramsey numbers to Turán numbers, whose asymptotic values were well known at the time for graphs with chromatic number at least 3. In fact, the connection runs deeper than just the numerical values. Many extremal graphs in anti-Ramsey theory can be constructed by turning an extremal graph in Turán theory into a rainbow subgraph, with all of the non-edges collected into a single color class. The open problems in Turán theory for bipartite graphs left much interest in anti-Ramsey theory for bipartite graphs.

Several variations of anti-Ramsey theory have been introduced. Rödl and Tuza proved there exist graphs $G$ with arbitrarily large girth such that every proper edge coloring of $G$ contains a rainbow cycle [66]. Erdős and Tuza asked for which graphs $H$ there is a $d$ such that there is a rainbow copy of $H$ in any edge-coloring of $K_n$ with exactly $|E(H)|$ colors such that each color class forms a spanning subgraph with minimum degree $d$. They found positive results for trees, forests, $C_4$, and $K_3$ and found negative results for several infinite families of graphs [29]. Another variation on anti-Ramsey theory is to determine for fixed graphs $G$ and $H$ the largest $\ell$ such that there exists an edge-coloring of $G$ using exactly $\ell$ colors which contains no rainbow $H$. This reduces to the traditional anti-Ramsey problem when $G = K_n$. Local anti-Ramsey [9] seeks for a fixed graph $H$ to find the maximum $k$ such that there is an edge coloring of $K_n$ with minimum color degree $k$ that contains no rainbow copy of $H$.

The third motivation to study rainbow matchings is the overall interest in rainbow subgraphs. Computer scientists have spent an extensive amount of effort to quickly find and prove the existence of rainbow subgraphs in vertex and edge-colored graphs. Typical results involve determining the computational complexity of partitioning the graph into monochromatic or rainbow subgraphs of specified type (paths, trees, cycles, etc). Other results involve determining the existence of large rainbow subgraphs (paths, trees, Hamiltonian cycles, etc) given certain assumptions. Adding conditions on colors seems to complicate problems in interesting ways. For example, finding the largest matching in a graph is a polynomial problem, while finding the largest rainbow matching is NP-complete [34]. See [47] for a survey of 90 such results.

Let $r(G, \phi)$ be the size of the largest rainbow matching in a graph $G$ with edge-coloring $\phi$. The relationship of the size of the largest rainbow matching to the minimum color degree was first investigated by Kano and Li [79], who showed that $r(G, \phi) \geq \left\lceil \frac{5\delta^c(G,\phi)-3}{12} \right\rceil$. This bound is tight for $K_4$ with color classes that form a 1-factor. However, it is tight for only finitely many other graphs. They conjectured that complete graphs with color classes that form a 1-factor would be the extremal cases.

**Conjecture 5.1 ([79])** *If* $\delta^c(G, \phi) \geq k \geq 4$*, then* $r(G, \phi) \geq \lceil \frac{k}{2} \rceil$.

LeSaulnier, Stocker, Wenger, and West [59] proved that $r(G, \phi) \geq \lfloor \frac{k}{2} \rfloor$. They also proved Conjecture 5.1 in full under several additional conditions, which include if $G$ is large relative to $k$. Hence the conjecture remained open only if $k$ is odd and for only finitely many graphs for each $k$.

Wang [78] showed that $r(G, \phi) \geq 3\delta^c(G, \phi)/5$ when $|V(G)| \geq 1.6k$, demonstrating that Conjecture 5.1 is sharp only for graphs with few vertices relative to $k$. Similar to orthogonal matchings, Wang conjectured that a rainbow matching of size $\delta^c(G, \phi)$ should be possible when the graph is large enough.

**Question 5.2 ([78])** *Is there a function* $f$ *such that if* $|V(G)| \geq f(\delta^c(G, \phi))$ *and* $\phi$ *is a proper coloring, then* $r(G, \phi) \geq \delta^c(G, \phi)$*?*

This would be best possible for graphs with color classes that form a 1-factor. Independently of our work, Diemunsch et al. [24] showed that $f$ exists, and $f(k) \leq 98k/23$. However, the value of the smallest function $f$ such that this is true remains open. Currently, the largest known properly edge-colored graph with no rainbow matching of size $\delta^c(G, \phi)$ has only $2\delta^c(G, \phi)$ vertices, and is constructed using a Latin Square with length $n$ and no transversal of size $n$. It is possible that completely solving the problem of the best function $f$ would solve Ryser's conjecture as a special case.

Lo and Tan [60] later proved that $f(k) \leq 4k-3$ and that $f(k) = 4k-3$ when $k \leq 3$. Lo and Tan have continued to work on the problem and have a proof that $f(k) \leq 11k/3 - 2$, which is the best upper bound on $f$ today. They currently have no plans to submit the better bound for publication [61], because they continue to work on even stronger results. It is worth noting that Diemunsch et al. used the assumption that $\phi$ is a proper coloring, while Lo and Tan's result and Lo's unpublished result do not assume that $\phi$ is a proper coloring.

Our first contribution towards finding rainbow matchings has no condition on the size of the graph.

**Theorem 5.3 ([54])** *Let $\phi$ be an edge coloring of $G$. If $G$ is not $K_4$, then $G$ contains a rainbow matching of size at least $\left\lceil \frac{\delta^c(G,\phi)}{2} \right\rceil$.*

Our second contribution towards finding rainbow matchings adds an assumption on the size of the graph.

**Theorem 5.4 ([50])** *Let $G$ be an $n$-vertex graph and $\phi$ be an edge-coloring of $G$. If $n > 4.25\left(\delta^c(G,\phi)\right)^2$, then $G$ contains a rainbow matching with at least $\delta^c(G,\phi)$ edges.*

If $G$ is colored with $k$ colors and each color class is a $t$-factor, then $\delta^c(G,\phi) \geq k$. Theorems 5.3 and 5.4 give new results on orthogonal matchings by relaxing the global assumption that each color class is regular and spanning to a local assumption on the minimum color degree. This relaxation eliminates the possibility of using certain methods, like the greedy algorithm described by Anstee and Caccetta for large graphs, but our conclusions are almost as strong.

Theorems 5.3 and 5.4 give new results in local anti-Ramsey theory when $H$ is a matching with the generalization that possibly $G \neq K_n$, which is a generalization used in other areas of anti-Ramsey theory.

Theorem 5.3 confirms Conjecture 5.1 as true. It is the best possible result without additional assumptions on the size of the graph. Theorem 5.4 confirms that the function in Question 5.2 exists, and that a similar function also exists for non-proper colorings. The bound $r(G, \phi) \geq \delta^c(G, \phi)$ is best possible. However, the bound on $n$ is not sharp. The best bounds on $n$ today are Lo and Tan's unpublished bound that $n \geq 11k/3 - 2$ suffices, and the best lower bound uses the same construction for properly-colored graphs that shows $f(k) \geq 2k + 1$.

We will present the proof of Theorem 5.3 in Section 5.1. We will present the proof of Theorem 5.4 in Section 5.2.

## 5.1 Proof of Theorem 5.3

By way of contradiction, let $G$ with edge coloring $f$ be a counterexample to Theorem 5.3 with the fewest edges. Let $k = \delta^c(G, f)$, $r := r(G, f)$, and $n := |V(G)|$. By [59], we may assume that $k$ is odd and $r = \frac{k-1}{2}$. Woolbright and Fu [80] proved the case when $G$ is a properly colored complete graph, so assume $n > k + 1$.

**Claim 5.5** *The edges of each color class of $f$ form a forest of stars.*

**Proof.** Let $F$ be a color class of $f$. If an edge $e \in F$ connects two vertices of degree at least two in $F$, then the color degrees of all vertices in $G$ and $G - e$ are the same, and any rainbow matching in $G - e$ is a rainbow matching in $G$. This contradiction to the minimality of $G$ yields the claim. □

Most of the results and notation in this section come from [59].

Let $M$ be a maximum rainbow matching in $G$, with edge set $\{e_j \,:\, 1 \leq j \leq r\}$, where $e_j = u_j v_j$. Let $H = G - V(M)$. Let $E_j$ denote the set of edges connecting $V(H)$ with $\{u_j, v_j\}$. Let $E'$ be the set of edges connecting $V(H)$ with $V(M)$, i.e., $E' = \bigcup_{j=1}^{r} E_j$. Define $p = |V(H)| = n - 2r = n - (k - 1)$. Because $n \geq k + 2$, it follows that $p \geq 3$. Label the vertices of $H$ as $\{w_1, w_2, \ldots, w_p\}$.

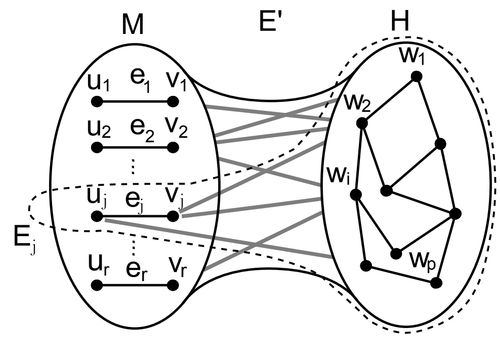


Figure 5.2: An example of $G$ with notation.

Without loss of generality, we will assume that edge $e_i$ is colored $i$ for $i = 1, \ldots, r$. A *free color* is a color not used on any of the edges of $M$. A *free edge* is an edge colored with a free color. If a free edge is contained in $H$, then $M$ is not a maximum rainbow matching, so this is not the case.

**Definition 5.6** *Let $\pi : V(M) \to [k-1]$ be the ordering with*

$$\pi(u_1) < \pi(v_1) < \pi(u_2) < \pi(v_2) < \ldots < \pi(u_{\frac{k-1}{2}}) < \pi(v_{\frac{k-1}{2}}).$$

*A free edge $wx$ colored $\alpha$ is important if $x \in V(M), w \in V(H)$, and $\pi(x) = \min_y\{\pi(y) : wy \in E', wy \text{ is colored } \alpha\}$. All other free edges in $E'$ are unimportant.*

The motivation for this definition is that for each $w \in V(H)$ and each free color $\alpha$ used on an edge incident with $w$, there is exactly one $\alpha$-colored important edge incident with $w$.

**Lemma 5.7 ([59])** *For any $1 \leq j \leq r$, if there are three vertices in $V(H)$ incident with important edges in $E_j$, then only one such vertex can be adjacent to two important edges.*

*Configuration A* in the set $E_j$ is a set $A_j$ of important edges such that (a) it contains all $p$ edges connecting $v_j$ with $H$ and one edge, say $u_jw$, incident with $u_j$; (b) the color of $u_jw$ (say $\alpha$) is also the color of every edge in $A_j$ apart from the edge $v_jw$ (which is different).

In this case, $\alpha$ will be called the *main color* for $E_j$. Note that in our definition we are assuming that $v_i$ is the vertex with $p$ important edges and not $u_i$. This assumption will be used for the rest of the paper.

**Corollary 5.8 ([59])** *If $p \geq 4$, then there are at most $p+1$ important edges in $E_j$ for each $j$. Furthermore, if $E_j$ has $p+1$ important edges, then $E_j$ contains Configuration A.*

Define *Configuration B* to be the set of four edges $B_j = \{wu_j, w'u_j, wv_j, w'v_j\} \subseteq E_j$ such that $w, w' \in V(H)$, all four edges are important, $f(wu_j) = f(w'v_j)$ and $f(wv_j) = f(w'u_j)$.

In this case $f(wu_j)$ and $f(wv_j)$ will be called the *major colors* for $E_j$.

**Corollary 5.9** *If $p = 3$, then there are at most $p+1 = 4$ important edges in $E_j$ for each $j$. Furthermore, if $E_j$ has 4 important edges, then $E_j$ contains either Configuration A or Configuration B.*

**Proof.** If each of $w_1, w_2, w_3$ is incident with an important edge, then the proof of Corollary 5.8 goes through and implies that $E_j$ contains Configuration A. If only two of them, say $w_1$ and $w_2$, are incident with important edges, then, in order to have four such edges, the set of important edges in $E_j$ must be $\{w_1u_j, w_2u_j, w_1v_j, w_2v_j\}$. Since $M$ is a maximum rainbow matching, $f(w_1u_j) = f(w_2v_j)$ and $f(w_1v_j) = f(w_2u_j)$. $\square$

While Configuration A can occur in graphs of any order, Configuration B only occurs when $p = 3$. Let $J_A$ denote the set of indices $j$ such that $E_j$ contains Configuration A. Let $J_B$ denote the set of indices $j$ such that $E_j$ contains Configuration B. By definition, $J_A \cap J_B = \emptyset$. Define $a = |J_A|$ and $b = |J_B|$. The values of $a$ and $b$ will depend on $G$, $f$, and the choice of $M$.

A color $\alpha$ is *basic* for $E_j$ if either $j \in J_A$ and $\alpha$ is the main color for $E_j$ or $j \in J_B$ and $\alpha$ is a major color for $E_j$.

**Claim 5.10** *The basic colors for distinct $E_j$ are distinct.*

**Proof.** The edges of a basic color for $E_j$ are incident with at least $p-1$ vertices in $H$. So if some color $\alpha$ was basic for $E_j$ and $E_{j'}$, then some $w \in V(H)$ would be incident with two edges of color $\alpha$, and so one of them would be unimportant, a contradiction. $\square$

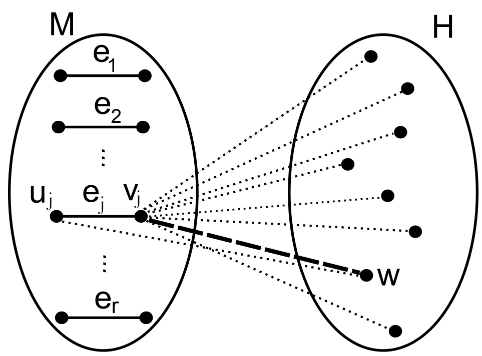


Figure 5.3: Configuration A.

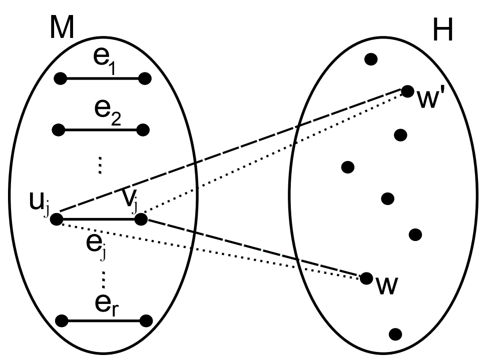


Figure 5.4: Configuration B.

**Definition 5.11** *Let $d_I(e_j)$ denote the number of important edges that are incident with $u_j$ or $v_j$. Let $d_I(w_j)$ be the number of important edges incident with $w_j$.*

There are only $r = \frac{k-1}{2}$ non-free colors, and each vertex is incident with at least $k$ distinct colors, therefore

$$\text{each vertex in } H \text{ is incident with at least } \tfrac{k+1}{2} \text{ important edges.} \tag{5.2}$$

The number of important edges coming out of $V(M)$ equals the number of important edges coming out of $H$, which gives the inequality

$$\sum_{j=1}^{\frac{k-1}{2}} d_I(e_j) = \sum_{i=1}^{p} d_I(w_i) \geq p\ \frac{k+1}{2} = pr + p. \tag{5.3}$$

Since $d_I(e_j) \leq p+1$ for each $j$, in order to satisfy (5.3),

$$\text{there are at least } p \text{ distinct values of } j \text{ such that } d_I(e_j) = p+1, \text{ i.e. } a+b \geq p \geq 3. \tag{5.4}$$

**Lemma 5.12** *Let $i$ be such that all of the free edges in $E_i$ are important. Let $\pi$ be the ordering of $V(M)$ described in the Definition 5.6. If $j < i$ is fixed, then in the ordering $\pi'$ of $V(M)$, where*

$$\pi'(u_1) < \pi'(v_1) < \pi'(u_2) < \pi'(v_2) < \ldots < \pi'(u_{j-1}) < \pi'(v_{j-1}) < \pi'(u_i) < \pi'(v_i) <$$

$$< \pi'(u_j) < \pi'(v_j) < \ldots < \pi'(u_{i-1}) < \pi'(v_{i-1}) < \pi'(u_{i+1}) < \pi'(v_{i+1}) < \ldots \pi'(u_{\frac{k-1}{2}}) < \pi'(v_{\frac{k-1}{2}}),$$

*the set of edges that are important is the same for $\pi$ and $\pi'$.*

**Proof** The only change from $\pi$ to $\pi'$ is that $u_i$ and $v_i$ come earlier. Thus, we will consider the effect of moving one pair of vertices to another spot in the ordering. Note that the number of important edges is not affected by the order of the vertices, only the selection of the set of important edges. Thus, for every edge that is changed from important to unimportant, there must be an edge that changes from unimportant to important. Therefore, since the relative order among all other vertices does not change, it suffices to show that if the status of the edges incident with $u_i$ and $v_i$ does not change, which will imply that the set of important edges in the whole graph does not change.

Let $e$ be an edge incident with $u_i \in V(M)$ and $w \in V(H)$ (the case when $e$ is incident with $v_i$ is symmetric). Since $e$ is already important by the hypothesis, it cannot *change* into an important edge. By the definition of an important edge, $e$ can turn from important to unimportant if and only if $u_i$ is the earliest edge with its color incident with $w$ and then moved after another edge with the same color. And since $u_i$ is being moved earlier by hypothesis, it cannot change into an unimportant edge. $\square$

Because $M$ is a maximum rainbow matching, if $E_j$ contains Configuration A or B, then $E_j$ contains exactly $p+1$ free edges. That is, if $j \in J_A \cup J_B$, then every free edge of $E_j$ is important.

**Definition 5.13** *A* special vertex *$v$ is a vertex with $d(v) = n-1$ and $d^c_G(v) = k$ such that one color appears on $n-k = p-1$ distinct edges incident with $v$ (each other color appears exactly once). For a special vertex $v$, the color that appears $n-k$ times is called the* main color *of $v$.*

If a color is on $n-k$ different edges incident with $v$, then $v$ is special. This proves that if $j \in J_A$ then $v_j$ is a special vertex.

We call an edge $xy$ a *main edge* if $x$ is special and $xy$ is colored with the main color of $x$. Let $M$ have the most main edges among all rainbow matchings in $G$ with $r$ edges. This implies that

$$\text{if } i \in J_A \text{ then } u_i \text{ is special and } i \text{ is the main color of } u_i. \tag{5.5}$$

This is because $v_i$ is special and its main color is free, and therefore not $i$. Since $e_i$ could be replaced by one of the main edges of $v_i$, the choice of $M$ shows that $e_i$ is already a main edge. This shows that $e_i$ is a main edge of $u_i$.

### 5.1.1 Case $a > 0$

We will use a fixed index $i \in J_A$. By Lemma 5.12 and the remark immediately after, we may assume $i = 1$. Consider edges $u_1u_j$ for $j \in J_A \cup J_B$. These edges exist for $j \neq 1$ because $u_1$ is a special vertex.

*Case 1: $f(u_1u_j)$ is the main color of $v_1$, or the main color of $v_j$ (if $j \in J_A$).* Without loss of generality, we will assume that $u_1u_j$ is the main color of $v_1$. By the definition of Configuration A, the main color of $v_1$ is free and it is on an edge that is incident with $u_1$ and a vertex in $H$. Thus there are two different edges incident with $u_1$ with the main color of $v_1$, but only the main color may be repeated at special vertex $u_1$. This creates a contradiction.

*Case 2: $f(u_1u_j)$ is 1, $j$, or free; and neither the main color of $v_j$ nor a major color for $E_j$.* In this case, a larger rainbow matching can be obtained by replacing $e_1$ and $e_j$ with three edges: $u_1u_j$, a main edge of $v_1$ (we have $p - 1$ choices for such an edge), and either a main edge of $v_j$ (if $i \in J_A$) or a major edge of $E_j$ (if $j \in J_B$).

*Case 3: $j \in J_B$ and $f(u_1u_j)$ is a major color of $E_j$.* If $f(u_1v_j)$ is not a major color of $E_j$, then we may swap $u_j$ and $v_j$ ( because Configuration B is symmetrical) and get Case 2. So suppose each of $f(u_1u_j)$ and $f(u_1v_j)$ is a major color of $E_j$. Then each of $u_j$ and $v_j$ has a free color repeated on edges incident with it. Configuration B only occurs only when $p = 3$, so a vertex is special when a color is repeated $n - k = p - 1 = 2$ times. Therefore both $u_j$ and $v_j$ are special with free main colors. This implies that $e_j$ is not a main edge. But this is a contradiction because $M$ could have contained more main edges by replacing edge $e_j$ with a main edge of $v_j$.

*Case 4: $f(u_1u_j) = h$, where $2 \leq h \leq r$, and $j \in J_A$.* Consider an important edge $e \in E_h$. It cannot be colored with the main color of $v_1$ or $v_j$, or else some vertex in $H$ will be incident with two important edges with the same color, which is a contradiction. We will attempt to replace $e_h$, $e_1$, and $e_j$ with edges $e$, $v_1w_s$, $u_1u_j$, and $v_jw_t$ for some $s \neq t$ that give the main colors of $v_1$ and $v_j$. The only way for this to not be possible is if $p = 3$, and the two main edges of $v_1$ and $v_j$ form a $C_4$ that is incident with $e$. But in this case, the important edge that is incident with $v_1$ and is not a main edge of $v_1$ is incident with the important edge of $v_j$ that is not a main edge, and they must have different colors. Then we can replace $e_h$, $e_1$, and $e_j$ with edges $e$, $v_1w_s$, $u_1u_j$, and $v_jw_t$ for some $s \neq t$ that give the main color of $v_1$ or $v_j$, and a free color that is not the main color of either $v_1$ or $v_j$ and not the color of $e$. Therefore $E_h$ has no important edges.

*Case 5: $f(u_1u_j) = h$, where $2 \leq h \leq r$, and $j \in J_B$.* Since $p = 3$, $V(H) = \{w_1, w_2, w_3\}$. Without loss of generality, assume that the major edges of $E_j$ are incident with $w_1$ and $w_2$.

Consider an important edge $e \in E_h$. It cannot have the main color of $v_1$, or have a major color of $E_j$ and be incident with $w_1$ or $w_2$. Suppose first that the edges with the main color of $v_1$ incident with $v_1$ go to $w_1$ and $w_2$. If $f(e)$ is a major color of $E_j$ and is incident with $w_3$, (without loss of generality, assume that $f(e) = f(v_jw_1)$), then replace $e_1$, $e_j$, and $e_h$ with $u_1u_j$, $e$, $v_jw_2$, and $v_1w_1$. This will also work if $e$ has any other free color and is incident with $w_3$. If $e$ is incident with $w_1$ and $f(e) \neq v_1w_3$, then replace $e_1$, $e_j$, and $e_h$ with $u_1u_j$, $e$, $v_1w_3$, and $v_jw_2$. This works symmetrically if $e$ is incident with $w_2$. This leaves only the case when $f(e) = f(v_1w_3)$ and $e$ is incident with $w_1$ or $w_2$. By the minimality of $G$, only two such edges may exist.

Suppose now that the edges with the main color of $v_1$ go to $w_1$ and $w_3$ ($w_2$ and $w_3$ is a symmetric situation). If $e$ is incident with $w_1$, then replace $e_1$, $e_j$, and $e_h$ with $u_1u_j$, $e$, $v_1w_3$, and $v_jw_2$. If $e$ is incident with $w_2$, then replace $e_1$, $e_j$, and $e_h$ with $u_1u_j$, $e$, $v_1w_3$, and $v_jw_1$. This leaves only the case when $e$ is incident with $w_3$. Since $G$ is a simple graph, only two such edges may exist.

Cases 1, 2, and 3 all led to contradictions. The vertex $u_1$ is special with main color 1. Therefore, there must be $a - 1$ instances of Case 4 and $b$ instances of Case 5. This creates $a - 1$ values of $i$ where $E_i$ has no important edges and $b$ other values of $i$ where $E_i$ has at most 2 important edges. By definition, for all $i \notin J_A \cup J_B$, the set $E_i$ has at most $p$ important edges.

$$
\begin{aligned}
\sum_{i=1}^{r} d_I(e_i) &= \sum_{i \in J_A \cup J_B} d_I(e_i) + \sum_{i \notin J_A \cup J_B} d_I(e_i) \\
&\leq (p+1)(a+b) + \Big((a+b-1)2 + p(r-(a+b)-(a+b-1))\Big) \\
&= pr + (a+b) - (p-2)(a+b-1).
\end{aligned}
$$

Recall that by (5.4), $a+b \geq 3$. Thus, since $p \geq 3$ and $a \geq 1$,

$$
\sum_{i=1}^{r} d_I(e_i) < pr + p, \tag{5.6}
$$

a contradiction to (5.3).

### 5.1.2 Case $a = 0$

If $a = 0$, then $b \geq 3$ by (5.4). This also implies that $p = 3$ and $V(H) = \{w_1, w_2, w_3\}$.

We will partition $J_B$ into three sets: $J_B^1$ will be the set of indices $i$ such that the free edges of $E_i$ are incident with $w_1$ and $w_2$; $J_B^2$ will be the set of indices $i$ such that the free edges of $E_i$ are incident with $w_1$ and $w_3$; and $J_B^3$ will be the set of indices $i$ such that the free edges of $E_i$ are incident with $w_2$ and $w_3$. We define $b_1 = |J_B^1|$, $b_2 = |J_B^2|$, and $b_3 = |J_B^3|$, so that $b_1 + b_2 + b_3 = b$.

We will analyze separately the cases when at least two of the values $b_1$, $b_2$, and $b_3$ are positive and when at least two of the values are zero. The vertices $w_1$, $w_2$, and $w_3$ can be reordered, so that $b_1$ is the smallest positive value of the three. Then $0 < b_1 \leq b_2 + b_3$ if two of the values are positive, and $b_3 = b_2 = 0$ and $b_1 = b$ if two of the values are zero.

In both cases, $b_1 > 0$. We will use a fixed index $i \in J_B^1$. By Lemma 5.12 and the remark immediately after, we may assume $i = 1$. We will show that

$$\text{there are } b-1 \text{ values for } j \text{ such that } E_j \text{ has 2 or fewer important edges.} \tag{5.7}$$

If (5.7) holds, then it generates a contradiction to (5.3) exactly as in (5.6).

**Subcase A:** $a = 0$ and $1 \leq b_1 \leq b_2 + b_3$. Since $k = n-2$, $d^c(u_1) \geq n-2$. Thus the number of distinct colors on the edges connecting $u_1$ with $\bigcup_{i \in J_B^2 \cup J_B^3} \{u_i, v_i\}$ is at least $2(b_3 + b_2) - 1 \geq b - 1$.

*Case A.i: $i \in J_B^3$, and the edge $u_1u_i$ exists.* This is symmetric to the case when $i \in J_B^2$.

If $f(u_1u_i) = f(v_1w_1)$ (Case $f(u_1u_i) = f(v_iw_3)$ is symmetric), then we replace edges $e_1$ and $e_i$ in $M$ with edges $v_1w_2$, $v_iw_3$, and $u_1u_i$. If $f(u_1u_i)$ is equal to a free color other than $f(v_1w_1)$ or $f(v_iw_3)$, then replace edges $e_1$ and $e_i$ in $M$ with edges $v_1w_1$, $v_iw_3$, and $u_1u_i$.

It follows that $f(u_1u_i)$ is not free. We will consider what important edges may be in $E_h$ for $f(u_1u_i) = h$. Suppose $e \in E_h$ is an important edge. First, assume that $e$ is incident with $w_2$. Since $w_2$ is incident with at most one important edge of each color, $f(e) \neq f(u_1w_2)$ and $f(e) \neq f(u_iw_2)$. So, since $f(u_1w_2) = f(v_1w_1)$ and $f(u_iw_2) = f(v_iw_3)$, we can replace edges $e_1$, $e_i$, and $e_h$ in $M$ with edges $u_1u_i$, $e$, $v_1w_1$ and $v_iw_3$. Thus $e$ is not incident with $w_2$. Second, assume that $e$ is incident with $w_3$. Since $w_3$ is incident with at most one important edge of color $f(e)$, we have $f(e) \neq f(u_iw_3) = f(v_iw_2)$. If also $f(e) \neq f(v_1w_1)$, then we replace in $M$ edges $e_1$, $e_i$, and $e_h$ with $u_1u_i$, $e$, $v_1w_1$ and $v_iw_2$. Finally, assume that $f(e) = f(v_1w_1)$. Again, since $w_3$ is incident with at most one important edge of color $f(v_1w_1)$, only one edge incident with $w_3$ in $E_h$ can be important. So, altogether $E_h$ has at most two important edges.

*Case A.ii: $i \in J_B^2 \cup J_B^3$, and the edge $u_1v_i$ exists.* By the symmetry of Configuration B, the proof is exactly the same as in case A.

This implies that there are $b-1$ values for $j$ such that $E_j$ has 2 or fewer important edges. Thus (5.7) holds.

**Subcase B:** $a = 0$ and $b_3 = b_2 = 0$.

Let $i \in J_B^1 = J_B$. Suppose that edge $w_3v_i$ exists. Since $E_i$ has Configuration B, edge $w_3v_i$ cannot be free. Let $f(w_3v_i) = h$. Suppose $e$ is a free edge in $E_h$. Assume first that $e$ is incident with $w_1$. Since $w_1$ is incident with at most one important edge of color $f(e)$, $f(e) \neq f(v_iw_1) = f(u_iw_2)$. So we can replace edges $e_i$ and $e_h$ in $M$ with edges $v_iw_3$, $u_iw_2$, and $e$, a contradiction. Hence $e$ is not incident with $w_1$ and similarly is not incident with $w_2$. Thus all important edges in $E_h$ are incident with $w_3$. It follows that $E_h$ has at most two such edges.

Similarly to the start of subcase A, since $d^c(w_3) \geq k = n - 2$, at least $b_1 - 1 = b - 1$ distinct colors were used on the edges in the set $\{w_3v_i \ : \ i \in J_B^1\}$. This implies that there are $b - 1$ values for $j$ such that $E_j$ has 2 or fewer important edges. So (5.7) holds again.

## 5.2 Proof of Theorem 5.4

Let $(G, \phi)$ be a counterexample to our theorem with the fewest edges in $G$. For brevity, let $k := \delta^c(G, \phi)$. Since $(G, \phi)$ is a counterexample, $n := |V(G)| > 4.25k^2$. The theorem is trivial for $k = 1$, and it is easy to see that if $\delta^c(G) = 2$ and $(G, \phi)$ does not have a rainbow matching of size 2, then $|V(G)| \leq 4$. Therefore $k \geq 3$.

**Claim 5.14** *Each color class in $(G, \phi)$ forms a star forest.*

**Proof.** Suppose that the edges of color $\alpha$ do not form a star forest. Then there exists an edge $uv$ of color $\alpha$ such that an edge $ux$ and an edge $vy$ also are colored with $\alpha$ (possibly, $x = y$). Then the graph $G' = G - uv$ has fewer edges than $G$, but $\delta^c(G', \phi) = k$. By the minimality of $G$, $r(G', \phi) \geq k$. But then $r(G, \phi) \geq k$, a contradiction. $\square$

We will denote the set of maximal monochromatic stars of size at least 2 by $\mathcal{S}$. Let $E_0 \subseteq E(G)$ be the set of edges not incident to another edge of the same color, i.e. the maximal monochromatic stars of size 1.

**Claim 5.15** *For every edge $v_1v_2 \in E(G)$, there is an $i \in \{1,2\}$, such that $d^c(v_i) = k$ and $v_1v_2$ is the only edge of its color at $v_i$.*

**Proof.** Otherwise, we can delete the edge and consider the smaller graph. $\square$

**Claim 5.16** *All leaves $v \in V(G)$ of stars in $\mathcal{S}$ have $d^c(v) = k$.*

**Proof.** This follows immediately from Claim 5.15. $\square$

For the sake of exposition, we will now direct all edges of our graph $G$. With an abuse of notation, we will still call the resulting directed graph $G$. In every star in $\mathcal{S}$, we will direct the edges away from the center. All edges in $E_0$ will be directed in a way such that the sequence of color outdegrees in $G$, $d_0^{c+} \geq d_1^{c+} \geq \ldots \geq d_n^{c+}$ is lexicographically maximized. Note that by Claim 5.14,

$$\text{the set of edges towards } v \text{ forms a rainbow star, and so } d^-(v) \leq d^c(v). \tag{5.8}$$

Let $C$ be the set of vertices with non-zero outdegree and $L := V \setminus C$. Let $\mathcal{S}^* \subseteq \mathcal{S}$ be the set of maximal monochromatic stars with at least two vertices in $L$, and let $E_0^* \subseteq E_0 \cup \mathcal{S}$ be the set of maximal monochromatic stars with exactly one vertex in $L$. For a color $\alpha$, let $E_H[\alpha]$ be the set of edges colored $\alpha$ in a graph $H$. If there is no confusion, we will denote it by $E[\alpha]$.

**Claim 5.17** *For every $v \in V(G)$ with $d^c(v) \geq k+1$, $d^-(v) = 0$. In particular, $d^-(v) \leq k$ for every $v \in V(G)$. Moreover, for all $w \in L$, $d(w) = k$.*

**Proof.** Suppose that $d^c(v) \geq k+1$, and let $w_i v$ be the edges directed towards $v$. By Claim 5.15 and (5.8), $d^c(w_i) = k$ and $w_i v \in E_0$ for all $i$. Then $d^{c+}(w_i) \leq d^c(w_i) = k$. Reversing all edges $w_i v$ would increase the color outdegree of $v$ with a final value larger than $k$ while decreasing the color outdegree of each $w_i$, which was at most $k$. Hence the sequence of color outdegrees would lexicographically increase, a contradiction to the choice of the orientation of $G$.

By the definition of $L$, if $w \in L$, then $d^+(w) = 0$. So in this case by the previous paragraph, $k \leq d^c(w) \leq d^-(w) \leq k$, which proves the second statement. □

**Claim 5.18** *No color class in $(G, \phi)$ has more than $2k-2$ components.*

**Proof** Otherwise, remove the edges of a color class $\alpha$ with at least $2k-1$ components, and use induction to find a rainbow matching with $k-1$ edges in the remaining graph. This matching can be incident to at most $2k-2$ of the components of $\alpha$, so there is at least one component of $\alpha$ not incident to the matching, and we can pick any edge in this component to extend the matching to a rainbow matching on $k$ edges. □

We consider three cases. If $n > 4.25k^2$, then at least one of the three cases will apply. The first two cases will use greedy algorithms.

**Case A:** $|\mathcal{S}^*| + \frac{1}{2}|E_0^*| \geq 2.5k^2$. For every $S \in \mathcal{S}^*$, assign a weight of $w_1(e) = 1/|S \cap L|$ to each of the edges of $S$ incident to $L$. Assign a weight of $w_1(e) = 1/2$ to every edge $e \in E_0^*$. Edges in $G[C]$ receive zero weight. Let $G_0 \subset G$ be the subgraph of edges with positive weight. For every set of edges $E' \subseteq E(G)$, let $w_1(E')$ be the sum of the weights of the edges in $E'$. For every vertex, let $w_1(v) = \sum_{a \in N^+(v)} w_1(va) + \sum_{b \in N^-(v)} w_1(bv)$. Note that $G_0$ is bipartite with partite sets $C$ and $L$ and that $w_1(e) \leq 1/2$ for every edge $e \in E(G)$.

Furthermore,

$$\frac{1}{2}\sum_{v\in V(G)} w_1(v) = \sum_{e\in E(G)} w_1(e) = |\mathcal{S}^*| + \frac{1}{2}|E_0^*| \geq 2.5k^2.$$

**Claim 5.19** *For every $v \in V(G)$, $w_1(v) \leq 2(k-1)$.*

**Proof.** Suppose $(G,\phi)$ has a vertex $v$ with $w_1(v) > 2(k-1)$. Let $G' = G - v$. Then $\delta^c(G',\phi) \geq k-1$ and $|V(G')| = n-1 > 4.25(k-1)^2$. By the minimality of $(G,\phi)$, the colored graph $(G',\phi)$ has a rainbow matching $M$ of size $k-1$. At most $k-1$ of the stars $v$ is incident to have colors appearing in $M$; each of them contributes a weight of at most 1 to $w_1(v)$. As $w_1(v) > 2(k-1)$, there are at least $2k-1$ edges incident to $v$ with colors not appearing in $M$. At least one of these edges is not incident to $M$. Thus $(G,\phi)$ has a rainbow matching of size $k$, a contradiction. $\square$

We propose an algorithm that will find a rainbow matching of size at least $k$. For $i = 1, 2, \ldots$, at Step $i$:

0. If $G_{i-1}$ has no edges or $i-1 = k$, then stop.
1. If a vertex $v$ of maximum weight has $w_1(v) > 2(k-i)$ in $G_{i-1}$, then set $G_i = G_{i-1} - v$ and go to Step $i+1$.
2. If the largest color class $E[\alpha]$ of $G_{i-1}$ has at least $2(k-i)+1$ components, then set $G_i = G_{i-1} - E[\alpha]$ and go to Step $i+1$.
3. If $w_1(v) \leq 2(k-i)$ for all $v \in V(G_{i-1})$ and every color class has at most $2(k-i)$ components, then set $G_i = G_{i-1} - x - y - E[\phi(xy)]$ for some edge $xy \in E(G_{i-1})$.

We will refer to these as options (1), (2), and (3) for Step $i$. We call the difference in the total weight of the remaining edges between $G_{i-1}$ and $G_i$ the *weight of Step* $i$ or $W_1(i)$. When both options (1) and (2) are possible, we will pick option (1). Option (3) is only used when neither of options (1) and (2) are possible.

Let $G_r$ be the last graph created by the algorithm, i.e., $r = k$ or $G_r$ has no edges. We will first show by reversed induction on $i$ that

$$G_i \text{ has a rainbow matching of size at least } r - i. \tag{5.9}$$

This trivially holds for $i = r$. Suppose (5.9) holds for some $i$, and $M_i$ is a rainbow matching of size $r - i$ in $G_i$. If we used Option (1) in Step $i$, then there is some edge $e \in E(G_{i-1})$ incident with $v$ that is not incident with $M_i$ and whose color does not appear on the edges of $M_i$, similarly to the proof of Claim 5.19. If we used Option (2) in Step $i$, then there is some component of $E_{G_{i-1}}[\alpha]$ that is not incident with $M_i$, and we let $e$ be an edge of that component. If we used Option (3) in Step $i$, then let $e = xy$. In each scenario, $M_i + e$ is a rainbow matching of size $r - i + 1$ in $G_{i-1}$. This proves the induction step and thus (5.9). So, if $r = k$, then we are done.

Assume $r < k$. Then the algorithm stopped because $E(G_{r+1}) = \emptyset$. This means that

$$\sum_{i=1}^{r} W_1(i) = \sum_{e \in E(G)} w_1(e) \geq 2.5k^2. \tag{5.10}$$

We will show that this is not the case. Suppose that at Step $i$, we perform Option (3). By the bipartite nature of $G_0$, we may assume that $y \in L$. By Claim 5.17, $w_1(y) - w_1(xy) \leq \frac{k-1}{2}$. Because Options (1) and (2) were not performed at Step $i$, $w_1(x) + w_1(E_{G_{i-1}}[\phi(xy)]) \leq 4(k - i)$. Therefore the weight of Step $i$ is at most $\frac{k-1}{2} + 4(k - i) < 4.5k - 4i$.

By Claims 5.18 and 5.19, Option (3) is performed at Step 1. If $W_1(i) < 4.5k - 4i$ for all $i$, then $\sum_{i=1}^{r} W_1(i) < \sum_{i=1}^{r} 4.5k - 4i = 4.5kr - 2r(r+1) \leq 2.5k(k-1)$, a contradiction to (5.10). Let $i$ be the first time that $W_1(i) \geq 4.5k - 4i$, and $j < i$ be the last time Option (3) is performed prior to $i$. By the choice of $i$, $W_1(a) < 4.5k - 4a$ when $a \leq j$. Because Option (1) and (2) were not chosen at Step $j$, $W_1(i') \leq 2(k-j)$ for each Step $i'$ such that $i' > j$ and Option (1) or (2) is used. Note that by choice of $i$ and $j$, this bound applies for all steps between $j+1$ and $i$. Furthermore, by the choice of $i$, $2(k-j) > 4.5k - 4i' - 1$ for $i' > i$. It follows that $W_1(b) \leq 2(k-j)$ for each $b > j$, and so

$$\sum_{a=1}^{r} W_1(a) \leq \sum_{a=1}^{j}(4.5k - 4a) + 2(k-j)(r-j) \leq 4.5kj - 2j(j+1) + 2(k-j)(k-1-j)$$

$$= k(0.5j + 2k - 2) < 2.5k^2,$$

a contradiction to (5.10).

**Case B:** $|C| \geq 1.75k^2$. We will use a different weighting: for every vertex $v \in C$ and outgoing edge $vw$, if $vw \in E_0$, we let $w_2(vw) = 1/d^{c+}(v)$, where $d^{c+}(v)$ is the color outdegree of $v$, and if $vw$ is in a star $S \in \mathcal{S}$, then we let $w_2(vw) = 1/(d^{c+}(v)\|S\|)$. For a vertex $v \in V(G)$, let $w^+(v)$ and $w^-(v)$ denote the accumulated weights of the outgoing and incoming edges, respectively, and $w_2(v) = w^+(v) + w^-(v)$. By definition, $w^+(v) = 1$ for each $v \in C$. Then

$$\sum_{e \in E(G)} w_2(e) = \sum_{v \in V(G)} w^-(v) = \sum_{v \in V(G)} w^+(v) = |C| \geq 1.75k^2.$$

**Claim 5.20** *Let $uv$ be a directed edge in $G$ and $e$ an edge incident to $u$ that is not $uv$. Then $w_2(e) \leq 1/2$.*

**Proof.** The result is easy if $e$ is in a monochromatic star with size at least 2, so assume $e \in E_0$. If $e$ is directed away from $u$, then $d^{c+}(u) \geq 2$ and the claim follows. Suppose now that $e$ is directed towards $u$, say $e = wu$, and $w_2(e) = 1$. Then $d^{c+}(w) = 1$, and reversing $e$ we obtain the orientation of $G$ where the color outdegree of $w$ decreases from 1 to 0, and the color outdegree of $u$ increases from $d^{c+}(u) \geq 1$ to $d^{c+}(u) + 1$. The new orientation has a lexicographically larger outdegree sequence, which is a contradiction. $\square$

**Claim 5.21** *For every color $\alpha$, we have $w_2(E[\alpha]) \leq 1.5(k-1)$.*

**Proof.** Otherwise, remove the edges of a color class $E[\alpha]$ with $w_2(E[\alpha]) > 1.5(k-1)$, and use induction to find a rainbow matching with $k-1$ edges in the remaining graph. For every directed edge $vw \in M$, $v$ can be incident to a component of $E[\alpha]$ of weight at most $1/2$, and $w$ can be incident to a component of $E[\alpha]$ of weight at most 1, so there is at least one component of $E[\alpha]$ not incident to the vertices of $M$, and we can pick any edge in this component to extend $M$ to a rainbow matching of $k$ edges. $\square$

We will use the following greedy algorithm: start from $G$, and at Step $i$, choose a color $\alpha$ with the minimum value $w_2(E[\alpha]) > 0$, and pick any edge $e_i \in E[\alpha]$ of that color, and put it in the matching $M$, and then delete all edges of $G$ that are either incident to $e_i$ or have the same color as $e_i$. Without loss of generality, we may assume that edge $e_i$ has color $i$. If we can repeat the process $k$ times, we have found our desired rainbow matching, so assume that we run out of edges after $r < k$ steps, and call the matching we receive $M$. Let $h \leq k-1$ be the first step after which only edges with colors present in $M$ remain in $G_h$. Let $\beta$ be a color not used in $M$ such that the last edges in $E[\beta]$ were deleted at Step $h$. Such $\beta$ exists, since $G$ has at least $k$ colors on its edges.

By Claim 5.20, one step can reduce the weight $w_2(E[\beta])$ by at most 1.5. It follows that $w_2(E[\beta])$ at Step $i \leq h$ is at most $1.5(h-i+1)$. As we always pick the color with the smallest weight, the color $i \leq h$ also had weight at most $1.5(h-i+1)$ when we deleted it in Step $i$. Every color $i > h$ which appears in $M$ has weight at most $1.5(k-1)$ by Claim 5.21. Thus, the total weight of colors in $M$ at the moment of their deletion is at most $1.5\sum_{i=1}^{h} i + 1.5(k-1)(k-1-h)$.

**Claim 5.22** *For each vertex $v$, $w_2(v) \leq (k+1)/2$.*

**Proof.** Suppose there are two edges, $e_1$ and $e_2$, incident with $v$ such that $w_2(e_1) = w_2(e_2) = 1$. By Claim 5.20, both edges are directed towards $v$ and are in $E_0$. Consider the orientation of $G$ where the directions of $e_1$ and $e_2$ have been reversed. Then the outdegree of $v$ has been increased by 2, while the outdegree of two other vertices changed from 1 to 0. This creates a lexicographically larger outdegree sequence, a contradiction.

By Claim 5.17, if $d^c(v) \geq k+1$, then $w_2(v) = 1$. If $d^c(v) = k$, then by the above $w_2(v) \leq 1 + (k-1)/2$. $\square$

If an edge $e$ has a color $\beta$ not in $M$ or has color $i \leq h$ but was deleted at Step $j$ with $j < i$, then $e$ is incident to the edges $\{e_1, \dots e_h\}$. By Claim 5.22, the total weight of such edges is at most $2h(k+1)/2$.

However, this is a contradiction because it implies

$$|C| \leq h(k+1) + \frac{3}{2}\sum_{i=1}^{h} i + \frac{3}{2}(k-1)(k-1-h) = \frac{3k^2}{2} - 3k + \frac{3}{2} + \frac{3h^2}{4} - \frac{hk}{2} + \frac{13h}{4} < 1.75k^2.$$

**Case C:** $|L| > |\mathcal{S}^*| + 0.5|E_0^*|$. We will introduce yet another weighting, now of vertices in $L$. For every star $S \in \mathcal{S}^*$, add a weight of $1/|L \cap V(S)|$ to every vertex in $L \cap V(S)$. For every edge $e \in E_0^*$, add a weight of $1/2$ to the vertex in $L \cap e$. For every $v \in L$, let $w_3(v)$ be the resulting weight of $v$.

Since $\sum_{v \in L} w_3(v) = |\mathcal{S}^*| + 0.5|E_0^*| < |L|$, there is a vertex $v \in L$ with $w_3(v) < 1$. Let $S_1, S_2, \dots S_k$ be the $k$ maximal monochromatic stars incident to $v$ ordered so that $|L \cap V(S_i)| \leq |L \cap V(S_j)|$ for $1 \leq i < j \leq k$ (where $S_1 \in E_0$ is allowed). Since $v \notin C$, all these stars have different centers and different colors. Now we greedily construct a rainbow matching $M$ of size $k$, using one edge from each $S_i$ as follows. Start from including into $M$ the edge in $S_1$ containing $v$. Assume that for $\ell \geq 2$, we have picked a matching $M$ containing one edge from each $S_i$ for $1 \leq i \leq \ell - 1$. Since $w_3(v) < 1$, we know that $|L \cap V(S_\ell)| > \ell$ for $\ell \geq 2$. As every edge in $M$ contains at most one vertex in $L$, we can extend the matching with an edge from the center of $S_\ell$ to an unused vertex in $L \cap V(S_\ell)$.

To finish the proof, let us check that at least one of the above cases holds. If Cases 1 and 2 do not hold, then $|C| < 1.75k^2$ and $|\mathcal{S}^*| + 0.5|E_0^*| < 2.5k^2$. Then, since $n > 4.25k^2$, $|L| > 4.25k^2 - 1.75k^2 = 2.5k^2$, and we have Case 3. $\square$

# Chapter 6

# Concluding Remarks and Open Problems

## 6.1 Critical Graphs

Exact values of $f_k(n)$ still remains open for most pairs $(n, k)$. It would be an impressive result to completely solve Conjecture 2.2. The techniques in this paper, specifically the potential function, make heavy use of the linearity of the bound. It is non-trivial to attempt to account for the non-linear nature of $\epsilon(n, k)$.

As there exist problems on coloring graphs with specific structure, there are analogues of $f_k(n)$ for the family of graphs that are $k$-critical with some additional constraints. Specifically, there is an interest in describing the fewest edges in a $k$-critical $n$-vertex, $K_s$-free graph. This is a natural direction given the breadth of graph coloring problems for $K_s$-free graphs (such as Grötzsch's Theorem that every triangle-free planar graph is 3-colorable). Other constraints commonly considered involve girth, planarity, and forbidding cycles of specific length.

As with the result on $(1,1)$-critical graphs, there is also significant motivation to find similar results on alternate versions of graph coloring. The most common alternate version of graph coloring asked about by colleagues is list-coloring. Unfortunately, it appears that the arguments used in this paper do not apply to list-coloring. The opening lemma, where non-sparse subgraphs are replaced with specific constructions, requires a finite fixed set of colors used on the overall graph. The technique clearly does apply to improper colorings of graphs. However, attempts to identify the fewest number of edges in $(2,2)$-critical graphs have stalled over specific, difficult-to-solve gaps. Future work would focus on improper coloring graphs with many colors, to make full use of Corollary 2.18 and possible variations of it.

A common generalization of any problem in graph theory is to restate the problem for hypergraphs. Theorem 3.20 gives a rather conclusive answer for hypergraphs, but there are other questions to answer. Specifically, it is an easy inductive argument to prove that $k$-critical hypergraphs constructed from Constructions 3.16, 3.17, and 3.18 contain at least three edges $e_1, e_2, e_3$ such that $|e_i| = 2$. Most results on $k$-critical hypergraphs are interested in $r$-uniform hypergraphs. From this discussion, it follows that Theorem 3.20 is not sharp for $r$-uniform hypergraphs when $r \geq 3$. Investigating stronger bounds with the added assumption on uniform edges is a promising direction for future research.

A second question left open by Theorem 3.20 is an exact characterization of extremal 3-critical hypergraphs. To demonstrate the sharpness of our characterization of extremal $k$-critical hypergraphs when $k \geq 4$, we provide two constructions of 3-critical hypergraphs $H$ such that $|E(H)| = |V(H)|$.

**Construction 6.1** *Let $T$ be a tree with set of leaves $L$ such that for all $u, v \in T$, $dist(u, v)$ is even. Let $e$ be a hyper edge such that $e = L$. Then $T + e$ is a 3-critical hypergraph.*

**Construction 6.2** *Let $G_1$ and $G_2$ be hypergraphs. Let $xy, yz \in E(G_1)$, $ab, e \in E(G_2)$, $b \in e$, and $a \notin e$. Let $G$ be comprised of a disjoint copy of $(G_1 - xy - yz)$ and $(G_2 - e)$ with $y$ glued to $b$, $z$ glued to $a$, and then add edge $e' = e \cup \{x\}$.*

The graph depicted in Figure 6.1 (a $K_{1,3}$, plus an edge around the leaves) can be constructed using Construction 6.1 or 6.2, but it cannot be constructed from Constructions 3.16, 3.17, and 3.18. This shows that the limitation of Theorem 3.20 to describe all extremal $k$-critical hypergraphs only when $k \geq 4$ is not just in the proof. Instead, there is something more complicated going on.

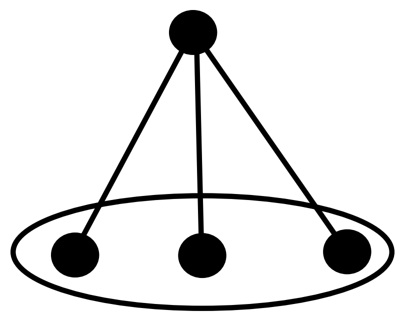

Figure 6.1: An example of a 3-critical hypergraph that cannot be described using Construction 3.16, 3.17, or 3.18.

## 6.2 Rainbow Matchings

An exact answer for $f(k)$ is the most interesting question today in rainbow matching. Although results by the author do not assume proper coloring, the relationship to Ryser's conjecture on Latin Squares makes this assumption a reasonable one.

Other questions in rainbow subgraphs remain open. For example, the length of the longest path given a condition on the minimum color degree has had a sequence of successively stronger results. Given the amount of work done, it would appear that new results would be both interesting to a significantly sized audience and hard to obtain.

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