A LYAPUNOV-BASED SMALL-GAIN THEOREM FOR
INTERCONNECTED SWITCHED SYSTEMS

BY

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THESIS

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ABSTRACT

Stability of an interconnected system consisting of two switched systems is investigated in the scenario where in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). We have shown that, providing the switching signals neither switch too frequently nor activate non-ISS subsystems for too long, a small-gain theorem can be used to conclude global asymptotic stability (GAS) of the interconnected system. For each switched system, with the constraints on the switching signal being modeled by an auxiliary timer, a correspondent hybrid system is defined to enable the construction of hybrid ISS Lyapunov functions. Apart from justifying the ISS property of their corresponding switched systems, these hybrid ISS Lyapunov functions are then combined to establish a Lyapunov-type small-gain condition which guarantees that the interconnected system is globally asymptotically stable.

To my parents, for their love and support

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CHAPTER 1

INTRODUCTION

The study of interconnected systems plays a significant role in the development of stability theory of dynamic systems, as it allows one to investigate the stability property of a complex system by analyzing its less complicated components. In this context, the small-gain theorems have proved to be important tools in the analysis of feedback connections of multiple systems, which appear frequently in the control literature. A comprehensive summarization of classical small-gain theorems involving input-output gains of linear systems can be found in [1]. This technique was then generalized to nonlinear feedback systems in [2] and [3] within the input-output context. The notion of input-to-state stability (ISS) proposed by Sontag [4] was naturally adopted and extended in [5] to establish a general nonlinear small-gain theorem which guaranteed both external and internal stabilities. Instead of analyzing the behavior of solution trajectories, Jiang et al. [6] have developed a Lyapunov-type nonlinear small-gain theorem based on the construction of ISS Lyapunov functions. A variety of nonlinear small-gain theorems were summarized in [7, Section 10.6].

In this thesis, we explore the stability property of interconnected switched systems. The study of switched systems has attracted a lot of attention in recent years (see, e.g., [8] and references therein). It is well known that, in general, a switched system does not necessarily inherit the stability properties of its subsystems. For example, in [8, Part II] it is shown that a switched system consisting of two asymptotically stable subsystems may not be stable. In the linear system context, it was proved in [9] that such a switched system can achieve asymptotic stability providing the switching signal satisfies a certain dwell-time condition. This approach was then generalized to the nonlinear system context and to the concept of average dwell-time condition in [10]. In [11], a similar result was developed for a linear switched system with both stable and unstable subsystems by restricting the fraction of time in which the unstable subsystems are active. The development of stability property inheritance in switched systems was extended to the ISS context by Vu et al. [12] and to the IOSS (input/output-to-state stability) context by Müller and Liberzon [13], both for nonlinear switched systems. Furthermore, in [13] the IOSS property of a switched nonlinear system was studied also for the general case where some of the subsystems are not input/output-to-state stable.

In this thesis, a sufficient condition is formulated to guarantee the global asymptotic stability (GAS) of an interconnected system consisting of two switched systems. We have considered a very general scenario: in both switched systems there may exist some subsystems that are not input-to-state stable (non-ISS). It is proved that, providing the switching signal neither switches too frequently (average dwell-time constraint) nor activates non-ISS subsystems for too long (time-ratio constraint), a small-gain theorem can be established by introducing an auxiliary timer and adopting hybrid system techniques. In particular, for each switched system, a hybrid system is defined such that their solutions are correspondent and the constraints on the switching signal are modeled by the auxiliary timer. An ISS Lyapunov function for each hybrid system is then constructed to show that any complete solution to the hybrid system, and therefore any solution to the switched system, is ISS. (Although the result that a switched system with non-ISS subsystems is ISS under certain average dwell-time condition and time-ratio condition has already been proved in [13], the Lyapunov-type formulation in this thesis exhibits an improvement in the sense that it not only generates an ISS Lyapunov function which is used later in the study of the interconnected system, but provides means for robustness analysis as well.) With these two ISS Lyapunov functions, a small-gain condition is then established to prove the GAS property of the interconnected switched system.

Hybrid systems are dynamic systems that possess both continuous-time and discrete-time features. Trajectory-based small-gain theorems for interconnected hybrid systems were first presented in [14] and [15], while Lyapunov-based formulations were introduced in [16]. The concept of ISS Lyapunov function was extended to hybrid systems in [17]. In our analysis of hybrid systems, we have adopted the modeling framework proposed by Goebel et al. [18], which proved to be general and natural from the viewpoint of Lyapunov stability theory. In the hybrid system context, a detailed study of small-gain theorems based on the construction of ISS Lyapunov functions using this modeling framework can be found in [19], [20] and [21]. Comparing to [21], our result in modifying the ISS Lyapunov function to guarantee its decrease along solutions is more general in the sense that it applies to the situation where the original ISS Lyapunov functions are increasing not only at the jumps but also during some of the flows. Based on the idea of restricting non-ISS subsystems' activation time proportion proposed in [11] and [13], an aforementioned auxiliary timer is introduced in the construction of the hybrid system to manage the non-ISS flows.

This thesis is structured as follows. In Chapter 2, we introduce some mathematical preliminaries. Our main result—the small-gain theorem for interconnected switched systems with both ISS and non-ISS subsystems—is presented and interpreted in Chapter 3, followed by a corollary discussing relaxations in the assumptions to conclude GAS when all subsystems are ISS. A detailed proof, prefaced by an introduction to hybrid systems, is provided in Chapter 4. Chapter 5 concludes the thesis with a short summary and an outlook on future research.

CHAPTER 2

PRELIMINARIES

Consider a family of dynamic systems

$$\dot{x} = f_p(x, u), \qquad p \in \mathcal{P}, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input and $\mathcal{P}$ the *index set* (which can in principle be arbitrary). For all $p \in \mathcal{P}$, f_p is locally Lipschitz and $f_p(0, 0) = 0$. A *switched system*

$$\dot{x} = f_\sigma(x, u) \tag{2.2}$$

is generated by the family (2.1) and a *switching signal* $\sigma\colon \mathbb{R}_{\geq 0} \to \mathcal{P}$ which specifies the index of the active system at time t. In this thesis, it is assumed that the switching signal σ is piecewise constant, right-continuous and has no accumulation point. Thus it has at most one switch at any time instant and finitely many switches in any finite time interval. Let ψ_i ($i \in \mathbb{Z}_{>0}$) denote the time when the i-th switch occurs and let $\Psi := \{\psi_i : i \in \mathbb{Z}_{>0}\}$. A function u is an admissible input to the switched system (2.2) if it is measurable and locally essentially bounded.

Following Morse [9], we say that a switching signal σ satisfies the *dwell-time condition* if there exists a constant $\tau_d \in \mathbb{R}_{>0}$, called the *dwell-time*, such that for all consecutive switching instants $\psi_i, \psi_{i+1} \in \Psi$,

$$\psi_{i+1} - \psi_i \geq \tau_d. \tag{2.3}$$

A generalized concept was introduced by Hespanha and Morse [10]: a switching signal σ is said to satisfy the *average dwell-time condition* if there exists a constant $\tau_a \in \mathbb{R}_{>0}$, called the *average dwell-time*, and a constant $N_0 \in \mathbb{Z}_{\geq 0}$ such that

$$N(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a} \qquad \forall\, t_2 \geq t_1 \geq 0, \tag{2.4}$$

where $N(t_2, t_1)$ denotes the number of switchings in the time interval $(t_1, t_2]$. Note that the dwell-time condition can be interpreted as a special case of the average dwell-time condition with $N_0 = 1$ and $\tau_a = \tau_d$.

For two vectors x_1, x_2, (x_1, x_2) is used to denote their concatenation, that is, $(x_1, x_2) := (x_1^\top, x_2^\top)^\top$.

For a vector $x \in \mathbb{R}^n$, we use $|x|$ to denote its Euclidean norm. For a compact set $A \subset \mathbb{R}^n$, we use $|x|_A$ to denote the Euclidean distance from a vector x to A. For a function $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|u\|_t$ is used to denote its essential supremum (Euclidean) norm on the interval $[0, t]$.

A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing and positive definite. α is of *class* $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{x\to\infty} \alpha(x) = \infty$. In particular, this implies that α is globally invertible. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for all fixed t and $\beta(x, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(x, t) = 0$ for all fixed x.

As introduced by Sontag [4], a dynamic system from family (2.1) is called *input-to-state stable* (ISS) if there exist functions $\gamma \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$ such that for all initial states $x(0) \in \mathbb{R}^n$ and all inputs $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^m$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}. \tag{2.5}$$

The definition of input-to-state stability (ISS) also applies to switched systems. Note that for an autonomous dynamic system (i.e. $u \equiv 0$), the ISS property (2.5) is equivalent to the notion of *global asymptotic stability* (GAS) [22, Proposition 2.5].

CHAPTER 3

MAIN RESULT

3.1 Interconnected Switched System with Both ISS and Non-ISS Subsystems

Consider two switched systems

$$\begin{aligned} \dot{x}_1 &= f_{1,\sigma_1}(x_1, u_1), \\ \dot{x}_2 &= f_{2,\sigma_2}(x_2, u_2), \end{aligned} \tag{3.1}$$

where, for $i \in \{1, 2\}$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $\sigma_i \in \mathcal{P}_i$.[1] Suppose these two switched systems fulfill the same assumptions as those imposed on (2.2) in Chapter 2. If $m_1 = n_2$ and $m_2 = n_1$, an *interconnected switched system* with the state $(x_1, x_2) \in \mathbb{R}^{n_1+n_2}$ can be constructed by letting $u_1 = x_2$ and $u_2 = x_1$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_{1,\sigma_1}(x_1, x_2) \\ f_{2,\sigma_2}(x_2, x_1) \end{bmatrix}. \tag{3.2}$$

Suppose that in the interconnected switched system (3.2), both switched systems may contain ISS as well as non-ISS subsystems. For $i \in \{1, 2\}$, let $\mathcal{P}_{s,i}$ and $\mathcal{P}_{u,i}$ denote the subsets of $\mathcal{P}_i$ containing the indexes of ISS and non-ISS subsystems, respectively. Then $(\mathcal{P}_{s,i}, \mathcal{P}_{u,i})$ forms a partition of $\mathcal{P}_i$ (i.e., $\mathcal{P}_{s,i} \cup \mathcal{P}_{u,i} = \mathcal{P}_i, \mathcal{P}_{s,i} \cap \mathcal{P}_{u,i} = \varnothing$). Following Müller and Liberzon [13], we define $T_{s,i}(t_2, t_1)$ as the activation time of ISS subsystems on the time interval $(t_1, t_2]$ (i.e., $\sigma_i \in \mathcal{P}_{s,i}$) and $T_{u,i}(t_2, t_1)$ that for non-ISS subsystems. Then $T_{s,i}(t_2, t_1) + T_{u,i}(t_2, t_1) = t_2 - t_1$.

The three constraints in the following assumption are commonly used in the context of

[1] We use f_{i,σ_i} instead of f_{σ_i} to avoid confusion in case the two index sets $\mathcal{P}_1, \mathcal{P}_2$ contain common elements.

switched systems.

Assumption 1. For all $i, j \in \{1, 2\}$ such that $i \neq j$, the following constraints are satisfied:

UNIFORM ISS LYAPUNOV-TYPE CONSTRAINT There exists a family of positive definite $\mathcal{C}^1$ functions $V_{i,p} \colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ $(p \in \mathcal{P}_i)$ such that the following conditions hold:

1. $\exists\ \alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i \in \mathcal{P}_i$,

$$\alpha_{1,i}(|x_i|) \leq V_{i,p_i}(x_i) \leq \alpha_{2,i}(|x_i|). \tag{3.3}$$

2. $\exists\ \phi_i \in \mathcal{K}_\infty, \lambda_{s,i}, \lambda_{u,i} \in \mathbb{R}_{>0}$ such that for all $x_i \in \mathbb{R}^{n_i}, x_j \in \mathbb{R}^{n_j}$ and all $p_s \in \mathcal{P}_{s,i}, p_u \in \mathcal{P}_{u,i}$,

$$|x_i| \geq \phi_i(|x_j|) \ \Rightarrow \ \begin{cases} \dfrac{\partial V_{i,p_s}(x_i)}{\partial x_i} \cdot f_{i,p_s}(x_i, x_j) \leq -\lambda_{s,i} V_{i,p_s}(x_i), \\ \dfrac{\partial V_{i,p_u}(x_i)}{\partial x_i} \cdot f_{i,p_u}(x_i, x_j) \leq \lambda_{u,i} V_{i,p_u}(x_i). \end{cases} \tag{3.4}$$

3. $\exists\ \mu_i \in \mathbb{R}_{\geq 1}$ such that for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i, q_i \in \mathcal{P}_i$,

$$V_{i,p_i}(x_i) \leq \mu_i V_{i,q_i}(x_i). \tag{3.5}$$

TIME-RATIO CONSTRAINT There exists $\rho_i \in [0, 1)$ and $T_{0,i} \in \mathbb{R}_{\geq 0}$ such that the activation time of non-ISS subsystems satisfies

$$T_{u,i}(t_2, t_1) \leq T_{0,i} + \rho_i(t_2 - t_1) \qquad \forall\ t_2 \geq t_1 \geq 0. \tag{3.6}$$

AVERAGE DWELL-TIME CONSTRAINT The switching signal σ_i satisfies the average dwell-time condition (2.4) with constants $\tau_{a,i} \in \mathbb{R}_{>0}$ and $N_{0,i} \in \mathbb{Z}_{\geq 0}$.

Remark 1. The constraints in Assumption 1 apply to each switched system separately. Moreover, the Uniform ISS Lyapunov-Type Constraint is a constraint on the subsystems' dynamics, while the Time-Ratio Constraint and the Average Dwell-Time Constraint are constraints on the switching signals.

Remark 2. The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is “Lyapunov-type” in the sense that it constrains not only the ISS subsystems, but the non-ISS subsystems as well. The existence of functions V_{i,p_s} satisfying (3.4) for $p_s \in \mathcal{P}_{s,i}$ follows from the fact that these subsystems are ISS [23], while the existence of functions V_{i,p_u} satisfying (3.4) for $p_u \in \mathcal{P}_{u,i}$ is equivalent to the forward completeness property of non-ISS subsystems [24].

Remark 3. The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is “uniform” since for switched system i, it is satisfied by ISS Lyapunov-type functions $V_{i,p}$ for all subsystems, with fixed functions $\alpha_{1,i}, \alpha_{2,i}, \phi_i$ and constants $\lambda_{s,i}, \lambda_{u,i}, \mu_i$. This uniformity can be concluded automatically for some particular types of index sets. For example, (3.3) is guaranteed if $\mathcal{P}$ is finite and all subsystems are ISS [12, Remark 1]. Besides, for positive definite functions $V_{i,p}$, the existence of the uniform ratio bound μ_i in (3.5) is a sufficient condition for the existence of the uniform comparison functions $\alpha_{1,i}, \alpha_{2,i}$ in (3.3).

Remark 4. The idea of restricting the fraction of time during which non-ISS subsystems are active in the Time-Ratio Constraint is essentially introduced by Zhai et al. [11] and Müller and Liberzon [13].

Our main result is stated as the following theorem.

Theorem 1. *Consider an interconnected switched system (3.2). Suppose that Assumption 1 holds with constants satisfying*

$$\lambda_{s,i} > \frac{\ln(\mu_i)}{\tau_{a,i}} + \rho_i(\lambda_{s,i} + \lambda_{u,i}) =: \gamma_i \qquad \forall\, i \in \{1, 2\}. \tag{3.7}$$

For all $i, j \in \{1, 2\}$ such that $i \neq j$, let

$$\Gamma_i := N_{0,i} \ln(\mu_i) + T_{0,i}(\lambda_{s,i} + \lambda_{u,i}), \tag{3.8}$$

and let $\chi_i \in \mathcal{K}_\infty$ be defined as

$$\chi_i(r) = \alpha_{2,i}(\phi_i(\alpha_{1,j}^{-1}(r))) \exp(\Gamma_i). \tag{3.9}$$

Then the interconnected switched system is globally asymptotically stable if the following *small-gain condition is satisfied:*

$$\chi_1(\chi_2(r)) < r \qquad \forall\ r \in \mathbb{R}_{>0}. \tag{3.10}$$

3.2 Interconnected Switched System with Only ISS Subsystems

In the stability analysis of the interconnected switched system (3.2), a less complicated scenario arises when all the subsystems are ISS for both of the switched systems (i.e., $\forall\ i \in \{1,2\},\ \mathcal{P}_{s,i} = \mathcal{P}_i, \mathcal{P}_{u,i} = \varnothing$). In this case, the global asymptotic stability of the interconnected switched system can be established under less restrictive assumptions, as stated in the following assumption and corollary:

Assumption 2. For all $i, j \in \{1,2\}$ such that $i \neq j$, the following constraints are satisfied:

Uniform ISS Lyapunov Constraint There exists a family of positive definite $\mathcal{C}^1$ functions $V_{i,p}\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ $(p \in \mathcal{P}_i)$ such that the following conditions hold:

1. $\exists\ \alpha_{1,i}, \alpha_{2,i} \in \mathcal{K}_\infty$ such that (3.3) holds for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i \in \mathcal{P}_i$.
2. $\exists\ \phi_i \in \mathcal{K}_\infty, \lambda_i \in \mathbb{R}_{>0}$ such that for all $x_i \in \mathbb{R}^{n_i}, x_j \in \mathbb{R}^{n_j}$ and all $p_i \in \mathcal{P}_i$,

$$|x_i| \geq \phi_i(|x_j|) \ \Rightarrow\ \frac{\partial V_{i,p_i}(x_i)}{\partial x_i} \cdot f_{i,p_i}(x_i, x_j) \leq -\lambda_i V_{i,p_i}(x_i). \tag{3.11}$$

3. $\exists\ \mu_i \in \mathbb{R}_{\geq 1}$ such that (3.5) holds for all $x_i \in \mathbb{R}^{n_i}$ and all $p_i, q_i \in \mathcal{P}_i$.

Average Dwell-Time Constraint The switching signal σ_i satisfies the average dwell-time condition (2.4) with constants $\tau_{a,i} \in \mathbb{R}_{>0}$ and $N_{0,i} \in \mathbb{Z}_{\geq 0}$.

Remark 5. If (3.5) is satisfied with $\mu_i = 1$, then the Uniform ISS Lyapunov Constraint is equivalent to the existence of a common ISS Lyapunov function for the subsystems of the switched system i, which guarantees the ISS of that switched system under arbitrary switching [25].

Corollary 1. *Consider an interconnected switched system (3.2). Suppose that Assumption 2 holds with constants satisfying:*

$$\lambda_i > \frac{\ln(\mu_i)}{\tau_{a,i}}. \tag{3.12}$$

For all $i, j \in \{1, 2\}$ such that $i \neq j$, let $\chi_i \in \mathcal{K}_\infty$ be defined as

$$\chi_i(r) = \alpha_{2,i}(\phi_i(\alpha_{1,j}^{-1}(r))) \exp(N_{0,i} \ln(\mu_i)). \tag{3.13}$$

Then the interconnected switched system is globally asymptotically stable if the small-gain condition (3.10) is satisfied.

Remark 6. For an interconnected switched system (3.2) in which only one of the two switched systems contains non-ISS subsystems, the global asymptotic stability can be established if Assumption 1 is satisfied by this switched system, while Assumption 2 is satisfied by the other, and the small-gain condition (3.10) is satisfied with the functions χ_1, χ_2 defined according to (3.9) and (3.13), respectively.

Remark 7. Suppose that in one of the two switched systems, instead of the average dwell-time condition (2.4), the switching signal satisfies the dwell-time condition (2.3) with dwell-time $\tau_{d,i}$. Then the same result holds if the average dwell-time $\tau_{a,i}$ in (3.7) (or (3.12), in case this switched system consists of only ISS subsystems) is substituted by $\tau_{d,i}$ and the constant $N_{0,i}$ in (3.9) (or (3.13)) is equal to one.

CHAPTER 4

PROOF OF THE MAIN RESULT

In this chapter, a detailed proof of Theorem 1 is presented. We start by introducing some preliminaries for hybrid systems in Section 4.1. In Section 4.2, a correspondent interconnected hybrid system is constructed for the interconnected switched system under Assumption 1. An ISS Lyapunov function for each hybrid system is defined in Section 4.3. In Section 4.4, the ISS property of each hybrid system is proved.[1] Section 4.5 concludes the proof of Theorem 1 by showing that the interconnected hybrid system, and therefore the interconnected switched system, is GAS.

4.1 Preliminaries for Hybrid Systems

Following Goebel et al. [18, Chapter 2], a *hybrid system* with inputs can be modeled as

$$\begin{cases} \dot{z} \in F(z,u), & z \in C, \\ z^+ \in G(z,u), & z \in D, \end{cases} \tag{4.1}$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input, $C \subset \mathbb{R}^n$ the *flow set*, $D \subset \mathbb{R}^n$ the *jump set*, $F\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ the *flow map* and $G\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ the *jump map*.[2] (In this model, if $z \in C \cap D$, then there are two possible cases: either the state does not jump and $\dot{z} \in F(z,u)$, or the state jumps to a state $z^+ \in G(z,u)$ and continues from there.) Similarly to the assumptions imposed on the switched system (2.2), in this thesis we assume that a solution to the hybrid system (4.1) jumps at most once at any time instant and finitely many times in any finite time interval. $\mathcal{H} = (C, F, D, G)$ is called the *data* of the hybrid system.

[1]This section is not directly related to the proof of Theorem 1 but considered as an independent result.
[2]We use "$\rightrightarrows$" to denote a set-valued mapping.

The solutions of the hybrid system are defined on the so-called hybrid time domain. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J} ([\theta_j, \theta_{j+1}], j) \tag{4.2}$$

for some finite sequence of times $0 = \theta_0 < \theta_1 < \cdots < \theta_{J+1}$. E is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. A *hybrid arc* is a function $z\colon \operatorname{dom} z \to \mathbb{R}^n$ defined on a hybrid time domain such that for each fixed $j \in \mathbb{Z}_{\geq 0}$, $z(\cdot, j)$ is locally absolutely continuous on $\{t : (t, j) \in \operatorname{dom} z\} =: \Theta_j^z$. A *hybrid input* is a function $u\colon \operatorname{dom} u \to \mathbb{R}^m$ defined on a hybrid time domain such that for each fixed $j \in \mathbb{Z}_{\geq 0}$, $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, j) \in \operatorname{dom} u\}$. A hybrid arc $z\colon \operatorname{dom} z \to \mathbb{R}^n$ is a solution to a hybrid system $\mathcal{H} = (C, F, D, G)$ with hybrid input $u\colon \operatorname{dom} u \to \mathbb{R}^m$ if the following conditions hold:

1. $\operatorname{dom} z = \operatorname{dom} u$.
2. For all $j \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \Theta_j^z$, $z(t, j) \in C$ and $\dot{z}(t, j) \in F(z(t, j), u(t, j))$.[3]
3. For all $(t, j) \in \operatorname{dom} z$ such that $(t, j + 1) \in \operatorname{dom} z$, $z(t, j) \in D$ and $z(t, j + 1) \in G(z(t, j), u(t, j))$.

With proper assumptions on the data $\mathcal{H}$, one can establish the local existence of solutions to the hybrid system, which may not be necessarily unique (see, e.g., [18, Proposition 2.10]). A solution to a hybrid system is *complete* if its domain is unbounded.[4]

Following Cai and Teel [17], for a function defined on a hybrid time domain $z\colon \operatorname{dom} z \to \mathbb{R}^n$, the essential supremum (Euclidean) norm up to hybrid time (t, j) is denoted by $\|z\|_{(t,j)}$ and defined as

$$\|z\|_{(t,j)} = \max \left\{ \operatorname*{ess\,sup}_{(s,k) \in \operatorname{dom} z \setminus J(z),\ s \leq t,\ k \leq j} |u(s, k)|, \sup_{(s,k) \in J(z),\ s \leq t,\ k \leq j} |u(s, k)| \right\},$$

[3]Here $z(t, j)$ represents the state of the system at time t and after j jumps.

[4]In this thesis, since we assume that a solution to the hybrid system (4.1) jumps at most once at any time instant and finitely many times in any finite time interval, a complete solution is a solution with infinite time horizon.

where $J(z)$ is the set of all $(s,k) \in \operatorname{dom} z$ such that $(s,k+1) \in \operatorname{dom} z$. $\|z\|_{A,(t,j)}$ is used to denote the essential supremum Euclidean distance to a compact set A up to hybrid time (t,j).

4.2 A Correspondent Hybrid System

Consider the following interconnected hybrid system with state (z_1, z_2): for $i, j \in \{1,2\}$ such that $i \neq j$, $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i) \in \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i] =: Z_i$ and the system dynamics are

$$\begin{cases} \dot{z}_i \in F_i(z_i, z_j), & z_i \in C_i, \\ z_i^+ \in G_i(z_i), & z_i \in D_i, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} F_i(z_i, z_j) &:= \begin{cases} \begin{bmatrix} \{f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j)\} \\ \{0\} \\ [0, \gamma_i] \end{bmatrix}, & \text{if } \widetilde{\sigma}_i \in \mathcal{P}_{s,i}, \\ \begin{bmatrix} \{f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j)\} \\ \{0\} \\ \{\gamma_i - (\lambda_{s,i} + \lambda_{u,i})\} \end{bmatrix}, & \text{if } \widetilde{\sigma}_i \in \mathcal{P}_{u,i}, \end{cases} \\ C_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i], \\ G_i(z_i) &:= \{\widetilde{x}_i\} \times (\mathcal{P}_i \backslash \{\widetilde{\sigma}_i\}) \times \{\tau_i - \ln(\mu_i)\}, \\ D_i &:= \mathbb{R}^{n_i} \times \mathcal{P}_i \times [\ln(\mu_i), \Gamma_i], \end{aligned}$$

and γ_i, Γ_i are constants defined in (3.7) and (3.8). We will show that the following proposition holds.

Proposition 1. *For each solution (x_1, x_2) to the interconnected switched system (3.2) with the switching signals σ_1, σ_2, if σ_1, σ_2 satisfy the Time-Ratio Constraint and Average Dwell-Time Constraint in Assumption 1, there is a complete solution (z_1, z_2) ($z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)$, $i \in \{1,2\}$) to the interconnected hybrid system (4.3) such that*

$$\widetilde{x}_i(t,j) = x_i(t) \qquad \forall\ (t,j) \in \operatorname{dom} z_i, \forall\ i \in \{1,2\}.$$

Proof. Suppose (x_1, x_2) is a solution to the interconnected switched system (3.2) with the switching signals σ_1, σ_2. For all $i \in \{1,2\}$, let $\Psi_i = \{\psi_{i,k} : k \in \mathbb{Z}_{>0}\}$ be the set of the switching times of σ_i and define $\psi_{i,0} = 0$. For all $T \in \mathbb{R}_{\geq 0}$, let $K_T := \max\{k \in \mathbb{Z}_{\geq 0} : \psi_{i,k} \leq T\}$ and

$$E_T := \left(\bigcup_{k=0}^{K_T - 1} ([\psi_{i,k}, \psi_{i,k+1}], k) \right) \cup ([\psi_{K_T}, T], K_T), \tag{4.4}$$

then E_T is a compact hybrid time domain. Consider the hybrid arc $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)$ defined such that for all $T \in \mathbb{R}_{\geq 0}$,

- $\operatorname{dom} z_i \cap ([0,T] \times \{0, 1, \ldots, K_T\}) = E_T$;
- $\forall\ (t,k) \in E_T$, $\widetilde{x}_i(t,k) = x_i(t)$ and $\widetilde{\sigma}_i(t,k) = \sigma_i(\psi_{i,k})$;
- $\forall\ (t,k) \in E_T$,

$$\tau_i(t,k) = \begin{cases} \Gamma_i, & \text{if } k = 0, \\ \min\{\Gamma_i, \bar{\tau}_{s,i}(t,k)\}, & \text{if } k > 0, \sigma_i(\psi_{i,k}) \in \mathcal{P}_{s,i}, \\ \bar{\tau}_{u,i}(t,k), & \text{if } k > 0, \sigma_i(\psi_{i,k}) \in \mathcal{P}_{u,i}, \end{cases}$$

where

$$\bar{\tau}_{s,i}(t,k) := \tau_i(\psi_{i,k}, k-1) - \ln(\mu_i) + \gamma_i(t - \psi_{i,k}),$$
$$\bar{\tau}_{u,i}(t,k) := \bar{\tau}_{s,i}(t,k) - (\lambda_{s,i} + \lambda_{u,i})(t - \psi_{i,k}).$$

We will show that, if the switching signals σ_1, σ_2 satisfy the Time-Ratio Constraint and Average Dwell-Time Constraint in Assumption 1, (z_1, z_2) is a *complete* solution to the interconnected hybrid system (4.3).

Indeed, by construction, (z_1, z_2) satisfies the dynamic equations (4.3) and thus is a solution to the hybrid system. For all $i \in \{1,2\}$, z_i is complete if $\operatorname{dom} z_i$ has infinite time horizon and $\forall\ (t,j) \in \operatorname{dom} z_i$, $z_i(t,j) \in Z_i$, which amount to showing the following three properties:

1. $\widetilde{x}_i \in \mathbb{R}^{n_i}$ is equivalent to the fact that x_i has no finite escape time, which is guaranteed by the Uniform ISS Lyapuno-Type Constraint in Assumption 1. By construction, this

property also implies that $\operatorname{dom} z_i$ is unbounded.

2. $\widetilde{\sigma}_i \in \mathcal{P}_i$ is guaranteed by construction.

3. $\tau_i \le \Gamma_i$ is guaranteed by construction, and $\tau_i \ge 0$ is shown in the following: For all $(t,k) \in \operatorname{dom} z_i$, let $(t_0,k_0) := \arg\max_{(s,l)\in \operatorname{dom} z_i}\{s+l \le t+k : \tau_i(s,l) = \Gamma_i\}$. Then according to the Time-Ratio Constraint and Average Dwell-Time Constraint in Assumption 1 and the definitions of γ_i in (3.7) and Γ_i in (3.8), we have

$$\begin{aligned}
\tau_i(t,k) &= \tau_i(t_0,k_0) - N(t,t_0)\ln(\mu_i) + T_{s,i}(t,t_0)\gamma_i + T_{u,i}(t,t_0)(\gamma_i - (\lambda_{s,i}+\lambda_{u,i})) \\
&\ge \Gamma_i - (N_{0,i} + (t-t_0)/\tau_{a,i})\ln(\mu_i) + (t-t_0)\gamma_i - (T_{0,i} + \rho_i(t-t_0))(\lambda_{s,i}+\lambda_{u,i}) \\
&= \Gamma_i - N_{0,i}\ln(\mu_i) - T_{0,i}(\lambda_{s,i}+\lambda_{u,i}) + (\gamma_i - \ln(\mu_i)/\tau_{a,i} - \rho_i(\lambda_{s,i}+\lambda_{u,i}))(t-t_0) \\
&= 0.
\end{aligned}$$

Therefore, the hybrid arc (z_1,z_2) constructed above is a complete solution to the interconnected hybrid system (4.3).[5] □

4.3 ISS Lyapunov Functions for the Hybrid Systems

Consider the interconnected hybrid system (4.3). For all $i \in \{1,2\}$, define a function $V_i \colon Z_i \to \mathbb{R}_{\ge 0}$ as

$$V_i(z_i) = V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i). \tag{4.5}$$

For all $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i) \in Z_i$, since $V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)$ is $\mathcal{C}^1$ with respect to $\widetilde{x}_i$, $V_i(z_i)$ is continuously differentiable with respect to $\widetilde{x}_i$ and τ_i. We will show that, for all $i \in \{1,2\}$, V_i satisfies the following ISS Lyapunov conditions.

Proposition 2. *For all $i,j \in \{1,2\}$ such that $i \ne j$, the following conditions hold:*

[5] By Goebel et al. [18, Proposition 2.10], for a hybrid system with local existence of solutions, a solution is complete if it has no finite escape time and does not jump out of the union of the jump set and the closure of the flow set. Unfortunately, we cannot apply this result since in the hybrid system (4.3), the local existence of solutions is not satisfies everywhere. In particular, at $z_i = (\widetilde{x}_i, \widetilde{\sigma}_i, 0)$ where $\widetilde{\sigma}_i \in \mathcal{P}_{u,i}$, the condition (VC) in [18, Proposition 2.10] does not hold. However, the hybrid arcs we constructed will not arrive at such points.

1. $\exists\ \underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ *such that*

$$\underline{\alpha}_i(|z_i|_{A_i}) \leq V_i(z_i) \leq \overline{\alpha}_i(|z_i|_{A_i}) \qquad \forall\ z_i \in Z_i, \tag{4.6}$$

where

$$A_i := 0^{n_i} \times \mathcal{P}_i \times [0, \Gamma_i]. \tag{4.7}$$

2. $\exists\ \lambda_i \in \mathbb{R}_{>0}$ *such that*

$$|z_i|_{A_i} \geq \phi_i(|z_j|_{A_j}) \ \Rightarrow\ \frac{\partial V_i(z_i)}{\partial z_i} \cdot v_i \leq -\lambda_i V_i(z_i) \qquad \forall\ v_i \in F_i(z_i, z_j), \forall\ z_i \in C_i, z_j \in Z_j. \tag{4.8}$$

3. We have

$$V_i(z_i^+) \leq V_i(z_i) \qquad \forall\ z_i^+ \in G_i(z_i), \forall\ z_i \in D_i. \tag{4.9}$$

Proof. For all $i, j \in \{1, 2\}$ such that $i \neq j$,

1. Let $\underline{\alpha}_i(r) := \alpha_{1,i}(r)$, $\overline{\alpha}_i(r) := \alpha_{2,i}(r)\exp(\Gamma_i)$, then (4.6) is satisfied according to (3.3).

2. Let $\lambda_i := \lambda_{s,i} - \gamma_i$, then $\lambda_i > 0$ by (3.7). For all $z_i \in C_i, z_j \in Z_j$ and all $v_i \in F_i(z_i, z_j)$, since $V_i(z_i)$ is continuously differentiable with respect to $\widetilde{x}_i$ and τ_i, the inner product in (4.8) is well defined. Then, according to (3.4), $|z_i|_{A_i} \geq \phi_i(|z_j|_{A_j})$ implies

   (a) if $\widetilde{\sigma}_i \in \mathcal{P}_{s,i}$,

$$\begin{aligned}
\frac{\partial V_i(z_i)}{\partial z_i} \cdot v_i &\leq \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \widetilde{x}_i} \cdot f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j) + \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \tau_i} \cdot \gamma_i \\
&\leq \frac{\partial V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)}{\partial \widetilde{x}_i} \cdot \exp(\tau_i) f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j) + V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i)\gamma_i \\
&\leq -(\lambda_{s,i} - \gamma_i) V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)\exp(\tau_i) \\
&= -\lambda_i V_i(z_i).
\end{aligned}$$

(b) if $\widetilde{\sigma}_i \in \mathcal{P}_{u,i}$,

$$\begin{aligned}
\frac{\partial V_i(z_i)}{\partial z_i} \cdot v_i &= \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \widetilde{x}_i} \cdot f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j) + \frac{\partial V_i(\widetilde{x}_i, \widetilde{\sigma}_i, \tau_i)}{\partial \tau_i} \cdot (\gamma_i - (\lambda_{s,i} + \lambda_{u,i})) \\
&= \frac{\partial V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i)}{\partial \widetilde{x}_i} \cdot \exp(\tau_i) f_{i,\widetilde{\sigma}_i}(\widetilde{x}_i, \widetilde{x}_j) + V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i) \exp(\tau_i)(\gamma_i - (\lambda_{s,i} + \lambda_{u,i})) \\
&\leq (\lambda_{u,i} + \gamma_i - (\lambda_{s,i} + \lambda_{u,i})) V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i) \exp(\tau_i) \\
&= -\lambda_i V_i(z_i).
\end{aligned}$$

Thus (4.8) is satisfied.

3. For all $z_i \in D_i$ and all $z_i^+ \in G_i(z_i)$, according to (3.5),

$$\begin{aligned}
V_i(z_i^+) &= V_{i,\widetilde{\sigma}_i^+}(\widetilde{x}_i^+) \exp(\tau_i^+) \\
&\leq \mu_i V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i) \exp(\tau_i - \ln(\mu_i)) \\
&= V_{i,\widetilde{\sigma}_i}(\widetilde{x}_i) \exp(\tau_i) \\
&= V_i(z_i).
\end{aligned}$$

Thus (4.9) is satisfied.

□

4.4 Digression on ISS of the Switched Systems

In this section, we will show that, for all $i \in \{1, 2\}$, the ISS Lyapunov function V_i defined in (4.5) can be conveniently used to prove the ISS property of the switched system i. While not directly related to the proof of Theorem 1, this result is presented here to show the advantage of our method in comparison with that of [13].

For all $i \in \{1, 2\}$, let $\alpha_i \in \mathcal{K}_\infty, \beta_i \in \mathcal{KL}$ be defined as

$$\begin{cases}
\alpha_i(r) = \underline{\alpha}_i^{-1}(\overline{\alpha}_i(\phi_i(r))), \\
\beta_i(r,t) = \underline{\alpha}_i^{-1}(\overline{\alpha}_i(r) \exp(-\lambda_i t)),
\end{cases}$$

then we have the following proposition:

Proposition 3. *Suppose (z_1, z_2) is a complete solution to the interconnected hybrid system (4.3), then*

$$|z_i(t,k)|_{A_i} \leq \beta_i(|z_i(0,0)|_{A_i}, t) + \alpha_i(\|z_j\|_{A_j,(t,k)}) \qquad \forall\ (t,k) \in \operatorname{dom} z_i, \forall\ i,j \in \{1,2\} : i \neq j, \tag{4.10}$$

where A_1, A_2 are sets defined in (4.7).

Proof. For all $i, j \in \{1,2\}$ such that $i \neq j$, according to (4.8) and (4.9), for all $(t,k), (t_0,k_0) \in \operatorname{dom} z_i$ such that $t + k \geq t_0 + k_0$,

$$|z_i(s,l)|_{A_i} \geq \phi_i(\|z_j\|_{A_j,(s,l)}) \qquad \forall\ (s,l) \in \operatorname{dom} z_i \cap ([t_0,t] \times \{k_0, k_0+1, \ldots, k\})$$

implies

$$V_i(z_i(t,k)) \leq V_i(z_i(t_0,k_0)) \exp(-\lambda_i(t - t_0)). \tag{4.11}$$

From here, our proof follows similar argument to the proof of [17, Proposition 2.7]. Consider the following cases:

1. $|z_i(t,k)|_{A_i} \leq \phi_i(\|z_j\|_{A_j,(t,k)})$. Then according to (4.6),

$$\begin{aligned}
|z_i(t,k)|_{A_i} &\leq \phi_i(\|z_j\|_{A_j,(t,k)}) \\
&\leq \underline{\alpha}_i^{-1}(\overline{\alpha}_i(\phi_i(\|z_j\|_{A_j,(t,k)}))) \\
&= \alpha_i(\|z_j\|_{A_j,(t,k)}) \\
&\leq \beta_i(|z_i(0,0)|_{A_i}, t) + \alpha_i(\|z_j\|_{A_j,(t,k)}).
\end{aligned}$$

2. $\forall\ (s,l) \in \operatorname{dom} z_i \cap ([0,t] \times \{0, 1, \ldots, k\})$, $|z_i(s,l)|_{A_i} > \phi_i(\|z_j\|_{A_j,(s,l)})$. Then by (4.11) we have

$$V_i(z_i(t,k)) \leq V_i(z_i(0,0)) \exp(-\lambda_i t).$$

According to (4.6),

$$\begin{aligned}|z_i(t,k)|_{A_i} &\le \underline{\alpha}_i^{-1}(\overline{\alpha}_i(|z_i(0,0)|_{A_i})\exp(-\lambda_i t))\\ &= \beta_i(|z_i(0,0)|_{A_i},t)\\ &\le \beta_i(|z_i(0,0)|_{A_i},t)+\alpha_i(\|z_j\|_{A_j,(t,k)}).\end{aligned}$$

3. $\exists\ (t_0,k_0)\in \operatorname{dom} z_i\cap([0,t]\times\{0,1,\dots,k\})$ such that $|z_i(t_0,k_0)|_{A_i}=\phi_i(\|z_j\|_{A_j,(t_0,k_0)})$ and $\forall\ (s,l)\in \operatorname{dom} z_i\cap([t_0,t]\times\{k_0,k_0+1,\dots,k\})\backslash\{(t_0,k_0)\},\ |z_i(s,l)|_{A_i}>\phi_i(\|z_j\|_{A_j,(s,l)})$. Then by (4.11) we have

$$V_i(z_i(t,k))\le V_i(z_i(t_0,k_0))\exp(-\lambda_i(t-t_0)).$$

According to (4.6),

$$\begin{aligned}|z_i(t,k)|_{A_i} &\le \underline{\alpha}_i^{-1}(\overline{\alpha}_i(|z_i(t_0,k_0)|_{A_i})\exp(-\lambda_i(t-t_0)))\\ &\le \underline{\alpha}_i^{-1}(\overline{\alpha}_i(|z_i(t_0,k_0)|_{A_i}))\\ &= \alpha_i(\|z_j\|_{A_j,(t_0,k_0)})\\ &\le \beta_i(|z_i(0,0)|_{A_i},t)+\alpha_i(\|z_j\|_{A_j,(t,k)}).\end{aligned}$$

Thus (4.10) is satisfied. □

Combining Propositions 1 and 3 gives that, for each solution (x_1,x_2) to the interconnected switched system (3.2), if Assumption 1 holds,

$$|x_i(t)|\le\beta_i(|x_i(0)|,t)+\alpha_i(\|x_j\|_t)\qquad\forall\ t\in\mathbb{R}_{\ge0},\forall\ i,j\in\{1,2\}:i\ne j. \tag{4.12}$$

Remark 8. Notice that for all $i,j\in\{1,2\}$ such that $i\ne j$, if we substitute x_j by an admissible input u to the switched system (2.2) in all assumptions and results, all the proofs remain valid. Therefore, we have the following corollary, which has been first proved by Müller and Liberzon [13] using trajectory analysis.

Corollary 2 ([13, Theorem 2]). *Consider a switched system (2.2) where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\sigma \in \mathcal{P}$. Let $\mathcal{P}_s, \mathcal{P}_u$ denote the sets of indexes of the ISS and non-ISS subsystems, respectively, and $T_u(t_2, t_1)$ the length of activation time of non-ISS subsystems in a time interval $(t_1, t_2]$. Suppose the following constraints are satisfied:*

UNIFORM ISS LYAPUNO-TYPE CONSTRAINT *There exists a family of positive definite $\mathcal{C}^1$ functions $V_p \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ ($p \in \mathcal{P}$) such that the following conditions hold:*

1. $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $p \in \mathcal{P}$,

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|). \tag{4.13}$$

2. $\exists\ \phi \in \mathcal{K}_\infty, \lambda_s, \lambda_u \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $p_s \in \mathcal{P}_s, p_u \in \mathcal{P}_u$,

$$|x| \geq \phi(|u|) \ \Rightarrow \ \begin{cases} \dfrac{\partial V_{p_s}(x)}{\partial x} \cdot f_{p_s}(x,u) \leq -\lambda_s V_{p_s}(x) \\ \dfrac{\partial V_{p_u}(x)}{\partial x} \cdot f_{p_u}(x,u) \leq \lambda_u V_{p_u}(x) \end{cases}. \tag{4.14}$$

3. $\exists\ \mu \in \mathbb{R}_{\geq 1}$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$,

$$V_p(x) \leq \mu V_q(x). \tag{4.15}$$

TIME-RATIO CONSTRAINT *The switching signal σ satisfies (3.6) with constants $\rho \in [0, 1)$ and $T_0 \in \mathbb{R}_{\geq 0}$.*

AVERAGE DWELL-TIME CONSTRAINT *The switching signal σ satisfies the average dwell-time condition (2.4) with constants $\tau_{a,i} \in \mathbb{R}_{>0}$ and $N_{0,i} \in \mathbb{Z}_{\geq 0}$.*

Then the switched system is input-to-state stable if

$$\lambda_s > \frac{\ln(\mu)}{\tau_a} + \rho(\lambda_s + \lambda_u). \tag{4.16}$$

4.5 GAS of the Interconnected Switched System

As shown by Jiang et al. [6], by (3.10) there exists a $\mathcal{K}_\infty$ function δ such that

$$\chi_1^{-1}(r) > \delta(r) > \chi_2(r) \qquad \forall\, r \in \mathbb{R}_{>0}, \tag{4.17}$$

and δ is $\mathcal{C}^1$ on $\mathbb{R}_{>0}$.

Let $z = (z_1, z_2) \in Z_1 \times Z_2 =: Z$ be the state of the interconnected hybrid system (4.3) and define a function $V\colon Z \to \mathbb{R}_{\geq 0}$ as

$$V(z) = \max\{\delta(V_1(z_1)), V_2(z_2)\}. \tag{4.18}$$

Since for all $i \in \{1, 2\}$, V_i is continuously differentiable with respect to $\widetilde{x}_i$ and τ_i, and δ is $\mathcal{K}_\infty$ and $\mathcal{C}^1$ and $(0, \infty)$, V is locally Lipschitz and thus absolutely continuous and almost everywhere differentiable with respect to $\widetilde{x}_1, \widetilde{x}_2, \tau_1$ and τ_2 (Rademacher's theorem [26]). We will show that, according to Proposition 2, V satisfies the following Lyapunov conditions:

1. $\exists\ \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|z|_A) \leq V(z) \leq \overline{\alpha}(|z|_A) \qquad \forall\, z \in Z, \tag{4.19}$$

with $A := A_1 \times A_2$, where A_1, A_2 are sets defined in (4.7).

Indeed, let

$$\begin{cases} \underline{\alpha}(r) := \min\{\delta(\underline{\alpha}_1(r/\sqrt{2})), \underline{\alpha}_2(r/\sqrt{2})\}, \\ \overline{\alpha}(r) := \max\{\delta(\overline{\alpha}_1(r)\exp(\Gamma_1)), \overline{\alpha}_2(r)\exp(\Gamma_2)\}, \end{cases}$$

then (4.19) is satisfied according to (4.6). In particular, for all $z \in Z$,

$$\begin{aligned}
\underline{\alpha}(|z|_A) &= \min\{\delta(\underline{\alpha}_1(|z|_A/\sqrt{2})), \underline{\alpha}_2(|z|_A/\sqrt{2})\} \\
&\leq \min\left\{ \max\{\delta(\underline{\alpha}_1(|z_1|_{A_1})), \delta(\underline{\alpha}_1(|z_2|_{A_2}))\}, \max\{\underline{\alpha}_2(|z_1|_{A_1}), \underline{\alpha}_2(|z_2|_{A_2})\}\right\} \\
&\leq \max\{\delta(\underline{\alpha}_1(|z_1|_{A_1})), \underline{\alpha}_2(|z_2|_{A_2})\} \\
&\leq \max\{\delta(V_1(z_1)), V_2(z_2)\} \\
&= V(z).
\end{aligned}$$

2. There exists a positive definite function $h \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\frac{\partial V(z)}{\partial z} \cdot v \leq -h(V(z)) \qquad \forall\, v \in F(z), \forall\, z \in (C_1 \times C_2)\backslash\Omega, \tag{4.20}$$

where

$$F(z) := \begin{bmatrix} F_1(z_1, z_2) \\ F_2(z_2, z_1) \end{bmatrix},$$

and Ω is a set of Lebesgue-measure zero which contains all points at which V is not differentiable.

Indeed, let h be defined as

$$h(r) = \min\{\delta'(\delta^{-1}(r))\lambda_1\delta^{-1}(r), \lambda_2 r\}.$$

As δ is $\mathcal{K}_\infty$ and $\mathcal{C}^1$ on $(0, \infty)$, h is positive definite. For all $z \in Z$ and all $v = (v_1, v_2) \in F(z)$, consider the following three cases:

(a) $\delta(V_1(z_1)) > V_2(z_2)$. Then $V(z) = \delta(V_1(z_1))$ and, according to (4.17),

$$V_1(z_1) \geq \delta^{-1}(V_2(z_2)) > \chi_1(V_2(z_2)). \tag{4.21}$$

By (3.3) and the definition of χ_1 in (3.9), $V_1(z_1) \geq \chi_1(v_2(z_2))$ implies $|z_1|_{A_1} \geq$

$\phi_1(|z_2|_{A_2})$. Thus by (4.8),

$$\begin{aligned}\frac{\partial V(z)}{\partial z} \cdot v &= \delta'(V_1(z_1)) \left(\frac{\partial V_1(z_1)}{\partial z_1} \cdot v_1 \right) \\ &\leq -\delta'(V_1(z_1))\lambda_1 V_1(z_1) \\ &= -\delta'(\delta^{-1}(V(z)))\lambda_1 \delta^{-1}(V(z)) \\ &\leq -h(V(z)).\end{aligned}$$

(b) $\delta(V_1(z_1)) < V_2(z_2)$. Then $V(z) = V_2(z_2)$ and, according to (4.17),

$$V_2(z_2) \geq \delta(V_1(z_1)) > \chi_2(V_1(z_1)). \tag{4.22}$$

By (3.3) and the definition of χ_2 in (3.9), $V_2(z_2) \geq \chi_2(v_1(z_1))$ implies $|z_2|_{A_2} \geq \phi_2(|z_1|_{A_1})$. Thus by (4.8),

$$\begin{aligned}\frac{\partial V(z)}{\partial z} \cdot v &= \frac{\partial V_2(z_2)}{\partial z_2} \cdot v_2 \\ &\leq -\lambda_2 V_2(z_2) \\ &= -\lambda_2 V(z) \\ &\leq -h(V(z)).\end{aligned}$$

(c) $\delta(V_1(z_1)) = V_2(z_2)$. Then $V(z) = \delta(V_1(z_1)) = V_2(z_2)$ and (4.21) and (4.22) are both satisfied. By (4.8) and the fact that δ is $\mathcal{K}_\infty$, the set $\{(z_1, z_2) \in Z : \delta(V_1(z_1)) = V_2(z_2)\}$ has Lebesgue-measure zero.

Thus (4.20) is satisfied.

3. For all z, z^+ such that $z \in D_1 \times Z_2, z^+ \in G_1(z_1) \times \{z_2\}$ or $z \in Z_1 \times D_2, z^+ \in \{z_1\} \times G_2(z_2)$ or $z \in D_1 \times D_2, z^+ \in G_1(z_1) \times G_2(z_2)$,

$$V(z^+) \leq V(z). \tag{4.23}$$

Indeed, for all $z^+ = (z_1^+, z_2^+)$ and all z in the sets above, according to (4.9),

$$\begin{aligned} V(z^+) &= \max\{\delta(V_1(z_1^+)), V_1(z_1), V_2(z_2^+), V_2(z_2)\} \\ &\leq \max\{\delta(V_1(z_1), V_2(z_2)\} \\ &= V(z). \end{aligned}$$

Thus (4.23) is satisfied.

Let $z = (z_1, z_2)$ be a complete solution to the interconnected hybrid system (4.3). By definition, $z(t, k)$ is absolutely continuous in t on $\Theta_k^z = \{t : (t, k) \in \operatorname{dom} z\}$ for all $k \in \mathbb{Z}_{\geq 0}$. According to (4.20) and the fact that V is almost everywhere differentiable with respect to $\widetilde{x}_1, \widetilde{x}_2, \tau_1$ and τ_2, by virtue of [27, Lemma 1], for all $k \in \mathbb{Z}_{\geq 0}$, $V(z(t, k))$ is locally absolutely continuous in t on Θ_k^z and

$$\frac{\mathrm{d}V(z(t,k))}{\mathrm{d}t} \leq -h(V(t,k)) \qquad \text{a.e. on } \Theta_k^z. \tag{4.24}$$

Define $\overline{V}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ as

$$\overline{V}(t) = V(z(t, k_t)) + \sum_{l=0}^{k_t} (V(z(\theta_l, l-1)) - V(z(\theta_l, l))),$$

where $k_t = \max\{k : (t, k) \in \operatorname{dom} z\}$ and $\forall\, l \in \{0, 1, \ldots, k_t\}$, $\theta_l = \min \Theta_l^z$. By definition, $\overline{V}$ is locally absolutely continuous on $\mathbb{R}_{\geq 0}$. According to (4.23) and (4.24), it satisfies

$$\overline{V}(t) \geq V(z(t,k)) \geq 0 \qquad \forall\, (t,k) \in \operatorname{dom} z,$$

and

$$\dot{\overline{V}}(t) \leq -h(\overline{V}(t)) \qquad \text{a.e. on } \mathbb{R}_{\geq 0}.$$

Thus, according to [22, Lemma 4.4], there exists $\beta_V \in \mathcal{KL}$ such that

$$\overline{V}(t) \leq \beta_V(\overline{V}(0), t) \qquad \forall\, t \in \mathbb{R}_{\geq 0},$$

and therefore

$$V(z(t,j)) \leq \beta_V(V(z(0,0)),t) \qquad \forall\ (t,j) \in \operatorname{dom} z.$$

Let $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ and define $\beta \in \mathcal{KL}$ as

$$\beta(r,t) = \underline{\alpha}^{-1}(\beta_V(\overline{\alpha}(r),t)).$$

Then, according to (4.19) and the definition of the set A,

$$|\widetilde{x}(t,j)| \leq \beta(|\widetilde{x}(0,0)|,t) \qquad \forall\ (t,j) \in \operatorname{dom} z.$$

Finally, let $x = (x_1, x_2)$ be a solution to the interconnected switched system (3.2). Then, according to Proposition 1,

$$|x(t)| \leq \beta(|x(0)|,t) \qquad \forall\ t \in \mathbb{R}_{\geq 0},$$

that is, the interconnected switched system (3.2) is globally asymptotically stable.

CHAPTER 5

CONCLUSION AND FUTURE RESEARCH

We have studied the stability property of an interconnected system consisting of two switched systems in the scenario where in both switched systems there may exist some subsystems that are not input-to-state stable. We have proved a small-gain theorem as a sufficient condition that guarantees the global asymptotic stability of the interconnected system via the hybrid system approach and the construction of appropriate ISS Lyapunov functions.

In this thesis, for each switched system, we have categorized its subsystems by their ISS property (i.e., ISS or non-ISS). On the other hand, we have only assumed an upper-bound on the effect of the switches (i.e., (3.5)). This lack of symmetry has drawn our attention and could possibly become a future research topic.

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