LENGTH FUNCTIONS IN FLAT METRICS

BY

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DISSERTATION

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Abstract

This dissertation is concerned with equivalence relations on homotopy classes of curves coming from various spaces of flat metrics on a genus $g \geq 2$ surface. We prove an analog of a result of Randol (building on work of Horowitz) for subfamilies of flat metrics coming from q-differentials. In addition we also describe how these equivalence relations are related to each other.

Acknowledgements

I would like to thank Chris Leininger for very useful math and all other discussions, for suggestions and comments while working on this thesis, as well as tremendous guidance and support.

I want to thank my family for all they have done for me.

Finally, I want to thank all my great friends for their moral support, for making my life fun and enjoyable and for putting up with me all this time.

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Chapter 1

Introduction

1.1 History

This thesis was motivated by the work of Horowitz [11] in 1972 and Randol [19] in 1980. In [11], Horowitz studies representations of a free group F into $SL(2, K)$, where K is a commutative ring with 1 and characteristic 0. He gives an example to show that, except for the free group on one generator, there are arbitrarily many distinct elements $u_1, \ldots, u_n \in F$, such that no two elements are conjugate, or inverses of each other, and yet all have equal characters for representations of F into $SL(2, K)$. In [19], Randol applies Horowitz's examples to representations of free subgroups of the fundamental group of a closed oriented surface S into $SL(2, \mathbb{R})$. As a consequence he shows that for every $n > 0$ there exist n distinct homotopy classes of curves $\gamma_1, \ldots, \gamma_n$ on S such that for every hyperbolic metric m, $l_m(\gamma_i) = l_m(\gamma_j)$, for all i, j, where $l_m(\gamma_k)$ denotes the length of the geodesic representative of γ_k in the metric m.

This strange phenomenon has prompted further study of this behavior for curves on surfaces. In [2] Anderson provides a broad discussion and references to this as well as generalizations to other settings. For example, in 2000 Masters [14] proved that the same statement holds for hyperbolic metrics on 3–manifolds; in 1998 Sandler [20] generalized the construction to representations in $SU(2, 1)$; and work of Kapovich, Levitt, Schupp, and Shpilrain [12] in 2007 proved a generalization in the context of free groups acting on $\mathbb{R}$–trees. In [13], Leininger studied a generalization to the space of metrics coming from quadratic differentials, and relates this to the case of hyperbolic metrics (2003). This is the starting point for this dissertation.

1.2 Main results

In [13] Leininger asks whether other families of metrics exhibit similar behaviour to the family of hyperbolic metrics.

Question 1.2.1. *[13] Do there exist pairs of distinct homotopy classes of curves γ and γ' which have the same length with respect to every metric in a given family of path metrics.*

For an arbitrary family of metrics one expects the answer to be no. In fact, as noted in [19] for most metrics m, $l_m(\gamma) = l_m(\gamma') \Rightarrow \gamma = \gamma'$. Hence, we can see that the set of all hyperbolic metrics on S is non–generic in this respect.

Here we study Question 1.2.1 for the metrics coming from q-differentials on S, for all $q \geq 1$. More precisely, let $Flat(S)$ denote the set of non-positively curved Euclidean cone metrics on S and $Flat(S, q)$ those that come from q-differentials. (See Chapter 2 for more details.) Let $\mathcal{C}(S)$ denote the set of homotopy classes of nontrivial closed curves on S. For every $q \in \mathbb{Z}_+$, define an equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_q \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$, $\forall m \in Flat(S, q)$.

In [13] Leininger answers Question 1.2.1 in the affirmative for metrics in $Flat(S, 2)$. In fact, writing $\gamma \equiv_h \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$, $\forall m$ hyperbolic, he proves:

Theorem 1.2.2. *[13] For every γ, $\gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_2 \gamma'$*

Consequently there are arbitrary large $\equiv_2$ - equivalence classes. Here we resolve Question 1.2.1 for all families $Flat(S, q)$, $q \geq 1$, proving $\equiv_q$ is nontrivial. In fact there are arbitrary large $\equiv_q$ - classes of curves.

Theorem 3.3.1. *For every $q_0, k \in \mathbb{Z}_+$ there are k distinct homotopy classes of curves $\gamma_1, \ldots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j$, $\forall i, j$ and $\forall q \leq q_0$. Thus $\forall q \in \mathbb{Z}_+$, the relation $\equiv_q$ is non-trivial.*

However, in the limit this phenomenon disappears. To describe this, define $\gamma \equiv_\infty \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$, $\forall m \in Flat(S, q)$, $\forall q \in \mathbb{Z}_+$.

Theorem 3.2.1. *The equivalence relation $\equiv_\infty$ is trivial.*

Therefore, in general if $q \neq q'$, then $\equiv_q$ and $\equiv_{q'}$ are different equivalence relations. However we have:

Theorem 3.1.1. *For every* $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \;\Leftrightarrow\; \gamma \equiv_2 \gamma'$

In [13] it is also shown that the implication in Theorem 1.2.2 can not be reversed. As a consequence of our construction we see that a similar statement is true for any $q \in \mathbb{Z}_+$.

Theorem 3.3.5. *For every* $q \in \mathbb{Z}_+$, $\exists \gamma, \gamma' \in \mathcal{C}(S)$ *so that* $\gamma \equiv_q \gamma'$ *but* $\gamma \not\equiv_h \gamma'$.

The outline of the dissertation is as follows. In Chapter 2 we define Euclidean cone and flat metrics, give standard definitions and state known theorems and sketch their proofs. In Chapter 3, Section 3.1 contains the proof of Theorem 3.1.1. In Section 3.2 we show that for every metric $m \in Flat(S)$ there is a sequence of metrics in $\bigcup_{q \in \mathbb{Z}_+} Flat(S, q)$ that converge to m and using this we prove that the equivalence relation $\equiv_\infty$ is trivial [Theorem 3.2.1]. Finally, in Section 3.3 we describe constructions of curves reflecting properties of the metrics in $Flat(S, q)$ and we prove Theorem 3.3.1.

Chapter 2

Flat surfaces

Let S denote a closed oriented surface of genus at least 2.

2.1 Euclidean Cone Metrics

A metric m on S is called a *Euclidean cone metric* if it satisfies the following properties:

(i) m is a geodesic metric (not necessaraly uniquely geodesic): the distance between 2 points is the length of a geodesic path between them.

(ii) There is a finite set $X \subset S$ such that m on $S \backslash X$ is Euclidean, that is, locally isometric to $\mathbb{R}^2$ with the Euclidean metric.

(iii) $(\forall x \in X)(\exists \epsilon > 0) B_\epsilon(x)$ is isometric to some cone. More precisely $B_\epsilon(x)$ is isometric to the metric space obtained by gluing together some (finite) number of sectors of ϵ -balls about 0 in $\mathbb{R}^2$. Each x therefore has a well defined cone angle $c(x) \in \mathbb{R}_+$ which is the sum of the angles of the sectors used in construction. See Figure 2.1. For any $x \in S \backslash X$ we define $c(x) = 2\pi$.

See Figure 2.2 for an example of a Euclidan cone metric on a genus 2 surface.

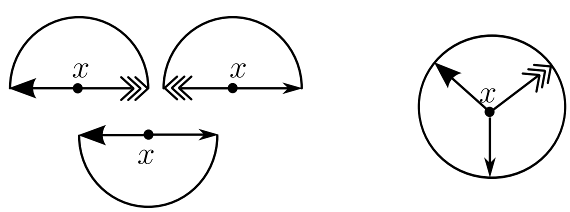

Figure 2.1: An example of a cone angle $c(x) = 3\pi$.

The *holonomy* homomorphism associated to m at any point $x \in S \backslash X$ is a homomorphism

$$\rho_x : \pi_1(S \backslash X, x) \to O(T_x(S \backslash X))$$

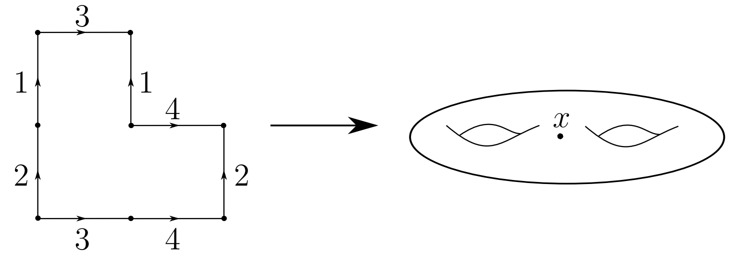

Figure 2.2: An example of a surface with Euclidean cone metric obtained by gluing sides of a polygon in $\mathbb{R}^2$ by translations as indicated. The result is a surface of genus 2 with a single cone point x with $c(x) = 6\pi$.

where $O(T_x(S\backslash X))$ is a group of orthogonal transformations of the tangent space of $S\backslash X$ at x. This is obtained by parallel translating a vector in $T_x(S\backslash X)$ along a loop in $S\backslash X$ based at x. Since our surface is oriented, the image is a subgroup of $SO(T_x(S\backslash X))$ - the group of rotations.

An orientation preserving isometry $\phi : T_x(S\backslash X) \to \mathbb{R}^2$ determines an isomorphism $SO(T_x(S\backslash X)) \to SO(2)$ independent of the choice of isometry ϕ. We therefore view the holonomy homomorphism as a homomorphism to $SO(2)$. For a Euclidean cone metric m define $Hol = Hol(m) < SO(2)$ to be the image of the holonomy homomorphism.

We will construct Euclidean cone surfaces by gluing sides of polygons by maps $\{\rho_i \circ \tau_i\}_{i=1}^k$, which are compositions of translations τ_i and rotations ρ_i. Given $\gamma \in \pi_1(S\backslash X, x)$, $\rho_x(\gamma)$ is given by the composition of the rotations for the side gluings of the sides of the polygons crossed by γ. Therefore $Hol \leq \langle \rho_1, \rho_2, ...\rho_k \rangle$. See Figure 2.3.

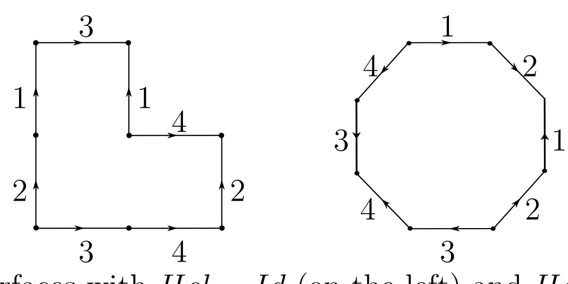

Figure 2.3: Genus 2 surfaces with $Hol = Id$ (on the left) and $Hol = \langle \rho_{\frac{\pi}{2}} \rangle$ (on the right).

In fact, we can obtain any Euclidean cone surface by gluing sides of a single *generalized Euclidean polygon* immersed in $\mathbb{R}^2$. To explain, consider a triangulation of S by Euclidean triangles for which the vertex set is precisely the set of cone points. Such a triangulation exists by Theorem 4.4. in [15], for example (this is actually a Δ–complex structure as in [9] instead of a proper triangulation, but the distinction is unimportant). We get a dual graph of the triangulation constructed by defining a vertex for each triangle and an edge for each pair of triangles that share an edge. See Figure 2.4. This graph has a maximal tree. Now by cutting along the edges of the triangles whose dual edges do not belong in the maximal tree we get a simply connected surface that is a union of Euclidean triangles which isometrically immerses in the plane. This is the generalized Euclidean polygon, which we denote P. The surface S can be reconstructed from P by gluing pairs of edges. See Figure 2.5. If we glue P by translations and rotations $\{\rho_i \circ \tau_i\}_{i=1}^{k}$, then $Hol = \langle \rho_1, \rho_2, \ldots, \rho_k \rangle$.

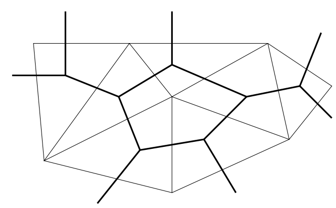

Figure 2.4: The dual graph of a triangulation.

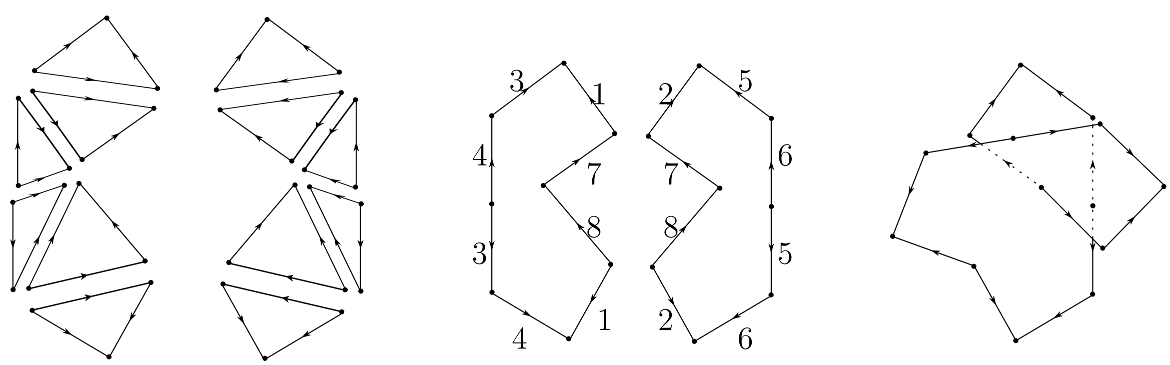

Figure 2.5: A genus 2 surface obtained by gluing triangles (on the left), two polygons (in the middle) and by gluing an immersed polygon (on the right).

Another important tool for us is the following well known fact:

Proposition 2.1.1. (Gauss-Bonnet formula) *Let R be a closed surface of genus $g \geq 0$ equipped with a Euclidean cone metric m. Then*

$$2\pi\chi(R) = \sum_{x \in X} (2\pi - c(x))$$

where X is the set of cone points.

Proof. Pick a triangulation of the surface R with triangles $\{\Delta_i\}_{i=1}^{F}$, so that cone points are the vertices of some of the triangles $\{\Delta_i\}$. Let the number of vertices of this triangulation be V, number of edges E and number of faces F. By Euler's theorem

$$V - E + F = \chi(R).$$

Each triangle has 3 edges, 2 edges are identified in pairs, therefore we get

$$3F = 2E.$$

Now we have

$$2V - 2E + 2F = 2\chi(R)$$

and therefore

$$2V - F = 2\chi(R).$$

This gives

$$2\pi V - \pi F = 2\pi\chi(R). \tag{1}$$

Denote $\alpha_i, \beta_i, \gamma_i$ the angles of Δ_i. Since all the triangles are Euclidean we have

$$\pi F = \sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i).$$

On the other hand after gluing the triangles the sum of all angles in the triangles becomes the sum of angles around each vertex. Let $\mathcal{U}$ denote the set of all vertices and $X \subset \mathcal{U}$ the set of all cone points. We then get:

$$\sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i) = \sum_{x \in X} c(x) + \sum_{x \in \mathcal{U} \setminus X} 2\pi.$$

This gives us

$$\pi F = \sum_{x \in X} (c(x) - 2\pi) + \sum_{x \in \mathcal{U}} 2\pi = \sum_{x \in X} (c(x) - 2\pi) + 2\pi V. \tag{2}$$

From (1) and (2) we get our formula

$$2\pi\chi(R) = \sum_{x \in X} (2\pi - c(x)).$$

□

Note: A similar proof for compact surfaces with geometric structure of constant curvature can be found in [17].

2.2 Flat Metrics

A Euclidean cone metric is called NPC (non-positively curved) if it is locally CAT(0). By the Gromov Link Condition this is equivalent to $c(x) \geq 2\pi$ for all $x \in S$ (See [5]). For example the surface in Figure 2.2 has one cone point and its angle is 6π, and so is NPC.

Define

$$Flat(S) = \{m \,|\, m \text{ is NPC Euclidean cone metric on } S\}.$$

By a metric on S we mean a metric inducing the given topology.

We are interested in the following class of metrics: For any $q \in \mathbb{Z}_+$ define

$$Flat(S,q) = \{m \in Flat(S) \,|\, Hol(m) \subset \langle \rho_{\frac{2\pi}{q}} \rangle\},$$

where ρ_θ is a rotation by angle θ.

Alternatively, $Flat(S,q)$ is the space of metrics coming from the set $\mathcal{Q}_q = \{q$–differentials on $S\}$. To explain this, suppose we are given a complex structure and a holomorphic q–differential φ (see Chapter II of [7], for example). We can pick a small disk neighborhood U of any point $p_0 \in S$ with $\varphi(p_0) \neq 0$, containing no zeros of φ, and define preferred coordinates ζ for φ by

$$\zeta(p) = \int_{p_0}^{p} \sqrt[q]{\varphi}.$$

In these coordinates $\varphi = d\zeta^q$. Preferred coordinates give an atlas of charts on $S \backslash \{zeros(\varphi)\}$ to $\mathbb{C}$ with transition functions of the form $T(z) = e^{\frac{2\pi ik}{q}} z + w$ for some $k \in \mathbb{Z}_+$, $w \in \mathbb{C}$. Pulling back the Euclidean metric we get Euclidean metric on $S \backslash \{zeros(\varphi)\}$. The completion of this metric is obtained by adding back in $\{zeros(\varphi)\}$ and at a zero of order k we have a cone angle $2\pi + \frac{2\pi k}{q}$. Therefore, the metric lies in $Flat(S,q)$. Conversely, take any metric in $Flat(S,q)$, choose local coordinates away from the singularities which are local isometries and so that the transition functions are translations and rotations by integer multiples of $\frac{2\pi}{q}$. Since these are holomorphic transformations preserving dz^q, this determines a complex structure and dz^q determines a holomorphic q-differential, and this extends over the singularities (compare with [16] and [21], for example, for the case $q = 2$).

To give some idea of how "big" $Flat(S,q)$ is, we calculate the dimension. By the Riemann–Roch Theorem, the real dimension of the space of q–differentials is

$$dim(\mathcal{Q}_q) = 2(2q-1)(g-1).$$

Since $\mathcal{Q}_q$ is a vector bundle over the Teichmüller space, and since every two q–differentials that differ by some rotation define the same metric on S, we get the real dimension of $Flat(S,q)$:

$$\begin{aligned} dim(Flat(S,q)) &= dim(\mathcal{Q}_q) + dim(\mathcal{T}(S)) - 1 \\ &= 2(2q-1)(g-1) + 6g - 7. \end{aligned}$$

For every $q \in \mathbb{Z}_+$, $Flat(S,q) \subset Flat(S)$ and $\lim_{q\to\infty} dim(Flat(S,q)) = \infty$. Thus,

$$dim(Flat(S)) = \infty.$$

2.3 Closed curves

Given a metric m and a homotopy class of curve $\gamma \in \mathcal{C}(S)$ we define the length function

$$l_m(\gamma) = \inf\{l_m(c) \,|\, c \in \gamma\}.$$

For every curve $\gamma \in \mathcal{C}(S)$ there is a geodesic (that is, a length minimizing) representative on S due to the Arzela–Ascoli theorem. Therefore $l_m(\gamma)$ is the length of its m-geodesic representative.

Proposition 2.3.1. *For m in $Flat(S)$, a curve $\gamma \in \mathcal{C}(S)$ is an m-geodesic if and only if γ is a closed Euclidean geodesics or a concatination of Euclidean segments between cone points such that angles between consecutive segments are $\geq \pi$ on each side of the curve γ. (See Figure 2.6.)*

Proof. Assume γ is a geodesic. Away from cone points m is Euclidean, thus in the complement of the cone points geodesics are straight Euclidian segments. If γ enters a cone point x and exits at an angle less than π we can find a path shorter than γ in the neighborhood of the cone point. Therefore all geodesics have to make an angle greater than equal to π on both sides around a cone point.

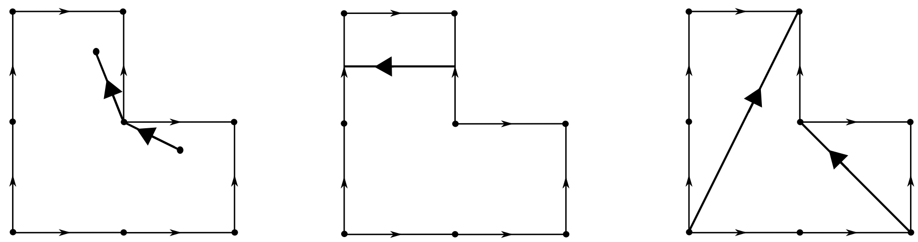

Figure 2.6: A geodesic through a cone point (on the left), a closed geodesic with no cone points (in the middle) and a closed self-intersecting geodesic containing a cone point (on the right). See Figure 2.2 for the gluings.

Conversely, because of the non-positive curvature, to see that paths satisfying this conditions are geodesics we just need to show that they locally minimize the length. For this we only need to check the cone points. Let x be a cone point on γ and denote each ray of γ coming out of x inside a small ball B around x containing no other cone point, with γ_- and γ_+. Construct two different straight line rays starting at x and making angles $\frac{\pi}{2}$ with γ_- on either side, and do the same for γ_+. See Figure 2.7. Notice that these rays define two non-intersecting neighborhoods of γ_- and γ_+ since angles on each side of γ at x are greater than or equal to π. Now define a projection inside B in the following way. Every point in the region bounded by the two neighborhoods of γ_- and γ_+ project orthogonally onto γ, and every other point maps to x. This projection is distance non-increasing since orthogonal projection and projection to a point do not increase distances. Therefore γ is a local geodesic and that completes the proof. $\square$

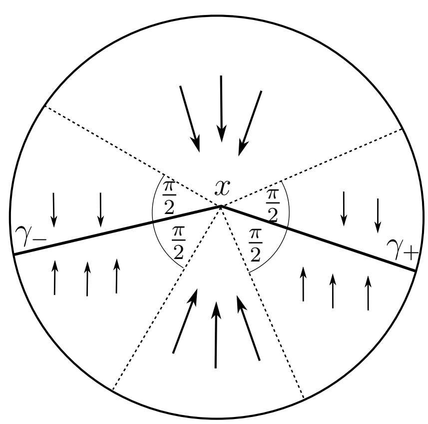

Figure 2.7: Projection onto γ.

If an m-geodesic representative of a curve in $\mathcal{C}(S)$ is not unique in its homotopy class then the set of geodesic representatives foliates a cylinder in S, and each geodesic representative has the same length in m. (See Figure 2.8.)

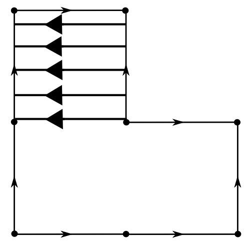

Figure 2.8: A cylindar on a genus 2 surface foliated by closed geodesics. See Figure 2.2 for the gluings.

For two curves $\alpha, \beta \in \mathcal{C}(S)$ we can define the geometric intersection number, $i(\alpha, \beta) =$ minimum number of double points of intersection of any two representatives of α and β. For hyperbolic metrics geodesic representatives realize geometric intersection number.

Define $\mathcal{S}(S) \subset \mathcal{C}(S)$ to be the set of homotopy classes of simple closed curves on S.

2.4 Measured foliations

A *measured foliation* $\mathcal{F}$ is a foliation with singularities such that every point in $S \backslash X$ (where X is a finite set of singular points) has a chart $\varphi : U \to \mathbb{R}^2$, such that $\varphi^{-1}(y = const)$ consists of leaves of $\mathcal{F}|_U$ and the transition functions are $\varphi_{ij}(x, y) = (\phi_{ij}(x, y), \pm y + c)$. In these charts the transverse measure is given by $|dy|$. Around each point $x \in X$ (called the singular set of $\mathcal{F}$) there is a chart $\varphi : U \to \mathbb{R}^2$ carrying $U \cap \mathcal{F}$ to W_k, a k-prong singularity for some $k \geq 3$ (see Figure 2.9). See [8] for more details.

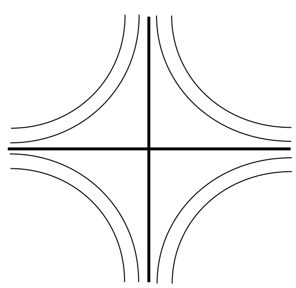

Figure 2.9: W_4, a k-prong singularity for k=4.

For $\gamma \in \mathcal{C}(S)$, $\int_\gamma \mathcal{F}$ is a total variation of the y-coordinate of $p \in \gamma$ as p traverses γ with respect to any chart. Define

$$I(\mathcal{F}, \alpha) = \inf_{\alpha_0 \in \alpha} \int_{\alpha_0} \mathcal{F}.$$

The measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are called *measure equivalent* (also called *Schwartz equivalent*) if $I(\mathcal{F}_1, \alpha) = I(\mathcal{F}_2, \alpha)$, for every $\alpha \in \mathcal{S}(S)$. Denote $\mathcal{MF}$ the set of measure equivalence classes of measured foliations.

We say that two foliations are Whitehead equivalent if one foliation may be transformed to the other by isotopies and elementary deformations called Whitehead moves as in Figure 2.10. As proved in [8] measured foliations are Schwartz equivalent if and only if they are Whitehead equivalent. Thus, $\mathcal{MF}$ can also be considered as the set of Whitehead equivalence classes.

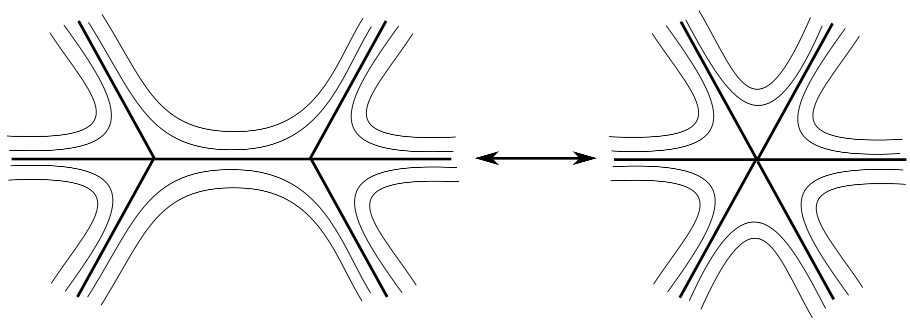

Figure 2.10: A Whitehead move.

Thurston constructed an embedding $\mathbb{R}_+ \times \mathcal{S}(S) \hookrightarrow \mathcal{MF}(S)$, so that for $\beta \in \mathcal{S}(S)$ and $(1, \beta) \mapsto \mathcal{F}_\beta$,

$$t\, i(\beta, \alpha) = I(t\, \mathcal{F}_\beta, \alpha),$$

where $\alpha \in \mathcal{C}(S)$, $t \in \mathbb{R}_+$, and $t\, \mathcal{F}_\beta$ is $\mathcal{F}_\beta$ with measure scaled by t. We use this to identify $\mathcal{S}(S)\,(= \{1\} \times \mathcal{S}(S))$ with its image in $\mathcal{MF}$, writing $\beta = \mathcal{F}_\beta \in \mathcal{MF}$.

A consequence of Thurston's [8], [18] and Bonahon's [4] work on measured laminations, closed curves and intersection number is the following:

Theorem 2.4.1. *Given any separating simple closed curve $\beta \in \mathcal{S}(S)$, there is a sequence $\{(t_n, \beta_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathcal{S}(S)$ where each β_n is non-separating, so that for all $\alpha \in \mathcal{C}(S)$, $t_n i(\beta_n, \alpha) \to i(\beta, \alpha)$ as $n \to \infty$.*

Sketch of the proof. Let τ be a non-separating simple closed curve on S that intersects β twice. Define a sequence of simple closed curves $\beta_n = T_\beta^n(\tau)$, where T_β is a Dehn twist around β. See Figure 2.11. It follows that β_n is non-separating.

For each simple closed curve σ, we have the following formula (see [8]):

$$\left|i(T_\beta^n(\tau),\sigma) - n\, i(\tau,\beta)i(\beta,\sigma)\right| \leq i(\tau,\sigma).$$

Since $i(\tau,\beta)=2$, by multiplying both sides by $\frac{1}{2n}$ we get

$$\left|\frac{1}{2n}i(\beta_n,\sigma) - \frac{1}{2n}\,n\,2\,i(\beta,\sigma)\right| \leq \frac{1}{2n}i(\tau,\sigma).$$

Now let $n\to\infty$. It follows that

$$\lim_{n\to\infty}\left|\frac{1}{2n}i(\beta_n,\sigma) - i(\beta,\sigma)\right| \leq \lim_{n\to\infty}\frac{1}{2n}i(\tau,\sigma) = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{2n}\,i(\beta_n,\sigma) = i(\beta,\sigma), \quad \forall\sigma\in\mathcal{S}(S).$$

Identifying simple closed curves with their embedding in the space of measured foliations we see that $\frac{1}{2n}\beta_n \to \beta$ in $\mathcal{MF}(S)$.

Now chose $t_n = \frac{1}{2n}$. By continuity of Bonahon's intersection number i, for every curve $\alpha\in\mathcal{C}(S)$, we have $\lim\limits_{n\to\infty} t_n i(\alpha,\beta_n) = i(\alpha,\beta)$. □

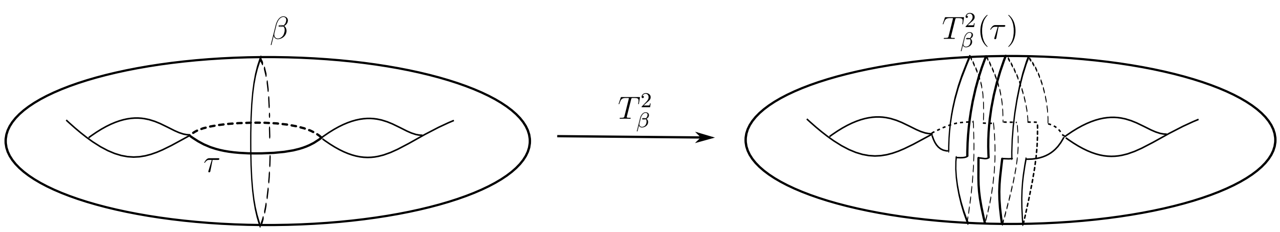

Figure 2.11: A Dehn twist around β.

We say that γ and $\gamma'\in\mathcal{C}(S)$ are simple intersection equivalent, $\gamma\equiv_{si}\gamma'$, if $i(\gamma,\alpha)=i(\gamma',\alpha)$ for every $\alpha\in\mathcal{S}(S)$.

We will also need the next fact.

Theorem 2.4.2. *[13] For every γ, $\gamma'\in\mathcal{C}(S)$, $\gamma\equiv_h\gamma' \Rightarrow \gamma\equiv_{si}\gamma' \Leftrightarrow \gamma\equiv_2\gamma'$.*

We will use the implication $\gamma \equiv_{si} \gamma' \Rightarrow \gamma \equiv_2 \gamma'$. We sketch the proof here:

Sketch of the proof. Assume $\gamma \equiv_{si} \gamma'$. We know that $\overline{\mathcal{S}(S) \times \mathbb{R}_+} = \mathcal{MF}(S)$ (see [8]). Therefore if $i(\gamma, \alpha) = i(\gamma', \alpha)$ for every $\alpha \in \mathcal{S}(S)$ then $i(\gamma, \mathcal{F}) = i(\gamma', \mathcal{F})$ for every measured foliation $\mathcal{F} \in \mathcal{MF}$, by continuity of i. Let $m \in Flat(S, 2)$ and let $\nu^\theta \in \mathcal{MF}(S)$ be a straight line foliation of (S, m) in the direction $\theta \in [0, \pi]$. From [6] we know

$$l_m(\gamma) = \frac{1}{2} \int_0^\pi i(\nu^\theta, \gamma) d\theta,$$

$$l_m(\gamma') = \frac{1}{2} \int_0^\pi i(\nu^\theta, \gamma') d\theta.$$

Therefore, since we have $i(\gamma, \nu^\theta) = i(\gamma', \nu^\theta)$, for every $\theta \in [0, \pi]$, it follows $l_m(\gamma) = l_m(\gamma')$. The metric $m \in Flat(S, 2)$ was arbitrary, so $\gamma \equiv_2 \gamma'$. $\square$

Chapter 3

Equivalent curves on flat surfaces

3.1 Relations $\equiv_1$ and $\equiv_2$

We now turn to the proof of

Theorem 3.1.1. *For every* $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \;\Leftrightarrow\; \gamma \equiv_2 \gamma'$

Proof. Given γ, $\gamma' \in \mathcal{C}(S)$, we have

$$\gamma \equiv_{si} \gamma' \;\Rightarrow\; \gamma \equiv_2 \gamma' \;\Rightarrow\; \gamma \equiv_1 \gamma'$$

by Theorem 2.4.2 and the fact $Flat(S,1) \subseteq Flat(S,2)$. We want to prove

$$\gamma \equiv_1 \gamma' \;\Rightarrow\; \gamma \equiv_{si} \gamma'.$$

We claim that if $\gamma \equiv_1 \gamma'$, then $i(\alpha, \gamma) = i(\alpha, \gamma')$ for every non-separating curve α. Then if β is a separating curve, by Theorem 2.4.1 there is a sequence of non-separating curves β_n and positive real numbers t_n such that $t_n i(\gamma, \beta_n) \to i(\gamma, \beta)$ and $t_n i(\gamma', \beta_n) \to i(\gamma', \beta)$. Since $t_n i(\beta_n, \gamma) = t_n i(\beta_n, \gamma')$ by the claim, we will have $i(\beta, \gamma) = i(\beta, \gamma')$ for every separating simple closed curve β, hence every $\beta \in \mathcal{S}(S)$, and thus $\gamma \equiv_{si} \gamma'$.

For any $g \geq 2$, we can construct a closed genus g surface by gluing the arcs in the boundary of a cylinder so that in the resulting surface the core curve is non-separating. Moreover, this construction can be carried out on a Euclidean cylinder so that the resulting Euclidean cone metric has trivial holonomy. See Figure 3.1.

Suppose S has genus g and let α be a non-separating curve on S. Let X_g^ϵ be the surface of genus g obtained by gluing a rectangle as in Figure 3.1 with horizontal side length 1 and vertical side length $\epsilon > 0$. Let α_g be the core nonseparating curve of the cylinder See Figure 3.2. We assume that the obvious affine map from the 1×1 square to the $1 \times \epsilon$ rectangle descends to a homeomorphism $f_\epsilon : X_g^1 \to X_g^\epsilon$. Choose homeomorphisms $f : S \to X_g^1$ so that

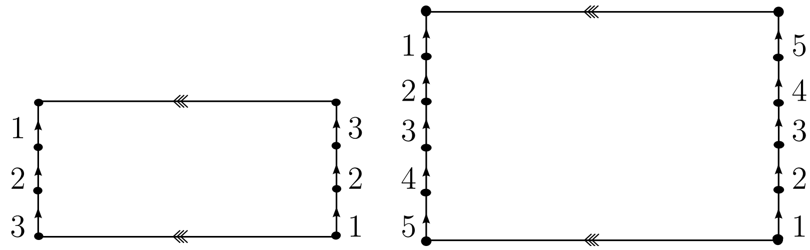

Figure 3.1: Examples of genus 2 and 3 surfaces. Gluing top sides to the bottom sides results in a cylinder, the rest of the gluing produces the closed surfaces.

α is sent to α_g and let $h_\epsilon = f_\epsilon \circ f : S \to X_g^\epsilon$. Write $m_\alpha^\epsilon \in Flat(S, 1)$ to denote the metric obtained by pulling back via h_ϵ.

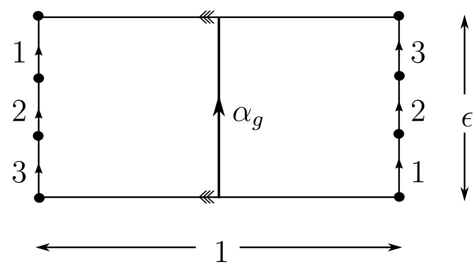

Figure 3.2: The surface X_2^ϵ and curve α_g.

Let γ be a curve on S. The geodesic representative of γ in m_α^ϵ is sent by h_ϵ to a union of straight lines running from one side of the rectangle to another and along the vertical boundary curves as in Figure 3.3.

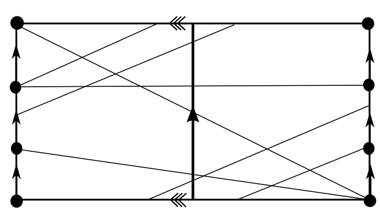

Figure 3.3: The geodesic representative of γ in m_α^ϵ. See Figure 3.2 for the gluings.

Thus we have

$$l_{m_\alpha^\epsilon}(\gamma) \geq i(\alpha, \gamma) \cdot 1 \tag{1}$$

since each geodesic segment that crosses the cylinder contributes 1 to intersection number and at least 1 to the length.

The curve γ is homotopic to a curve $\bar{\gamma}$ which is sent by h_ϵ to a union of straight line segments parallel to the side of length 1 of the rectangle and some segments of the vertical ϵ-length sides as in Figure 3.4.

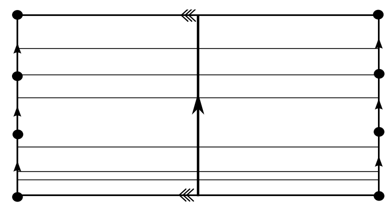

Figure 3.4: The representative $\bar{\gamma}$ in m_α^ϵ. See Figure 3.2 for the gluings.

From this we get

$$l_{m_\alpha^\epsilon}(\gamma) \leq length(\bar{\gamma}) \leq i(\alpha, \gamma) \cdot 1 + n_\gamma \epsilon \tag{2}$$

where n_γ is the number of vertical segments of $\bar{\gamma}$.

Now suppose $\gamma \equiv_1 \gamma'$. Then for every $\epsilon > 0$ we have $l_{m_\alpha^\epsilon}(\gamma) = l_{m_\alpha^\epsilon}(\gamma')$. By (1) and (2)

$$l_{m_\alpha^\epsilon}(\gamma) - n_\gamma \epsilon \leq i(\alpha, \gamma) \leq l_{m_\alpha^\epsilon}(\gamma)$$

and

$$l_{m_\alpha^\epsilon}(\gamma') - n_{\gamma'} \epsilon \leq i(\alpha, \gamma') \leq l_{m_\alpha^\epsilon}(\gamma').$$

Letting $\epsilon \to 0$ we get $i(\alpha, \gamma) = i(\alpha, \gamma')$, proving the claim.

Hence $\gamma \equiv_{si} \gamma'$, completing the proof. $\square$

Corollary 3.1.2. $\forall q \in \mathbb{Z}_+, \quad \gamma \equiv_q \gamma' \;\Rightarrow\; \gamma \equiv_2 \gamma'$.

Proof. Since $Flat(S, 1) \subset Flat(S, q)$ for every q, we have

$$\gamma \equiv_q \gamma' \Rightarrow \gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_2 \gamma'.$$

$\square$

3.2 The equivalence relation $\equiv_\infty$

Next we will prove

Theorem 3.2.1. *The equivalence relation $\equiv_\infty$ is trivial.*

To prove this we first prove a weaker statement. Define another equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_{\mathbb{R}} \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every $m \in Flat(S)$.

Theorem 3.2.2. *The equivalence relation $\equiv_{\mathbb{R}}$ is trivial.*

Proof. To prove this theorem we will show that for every $\gamma, \gamma' \in \mathcal{C}(S)$ there is $m \in Flat(S)$ so that $l_m(\gamma) \neq l_m(\gamma')$.

Let γ and $\gamma' \in \mathcal{C}(S)$. Pick a hyperbolic Riemannian metric g on S. If the geodesic representatives of γ and γ' have the same length in g, let $\epsilon > 0$ be small enough so that γ' does not intersect ϵ-ball around a point $x \in \gamma$. Let $\varphi : S \to \mathbb{R}$ be a smooth function so that $\varphi \equiv 0$ on $S \backslash B_\epsilon(x)$, $\varphi(x) = 1$ and $\varphi(S) = [0, 1]$. For $\delta > 0$ consider the Riemannian metric $g_\delta = g(1 - \delta\varphi)$. The metric g_δ is negatively curved for δ sufficiently small and the g_δ-length of γ is smaller than the g-length, while the length of γ' is not changed. Thus, for any two curves γ and $\gamma' \in \mathcal{C}(S)$ there is a negatively curved Riemannian metric m' such that $l_{m'}(\gamma) \neq l_{m'}(\gamma')$. In fact there are negatively curved metrics where there are no closed curves with the same length (see [19], [1], [3] for more general result).

Pick a geodesic triangulation of the surface S so that the geodesic representatives of γ and γ' are unions of edges of triangles. Being a locally $CAT(k)$ space, $k < 0$, implies also being locally $CAT(0)$. Now build a Euclidean cone metric m on S by replacing the triangles with Euclidean triangles with the same length sides. By the $CAT(0)$ property the angles of the Euclidean triangles are larger than the angles in the triangles they replaced. The vertices of the triangles are the only possible cone points. There are finitely many of them, and their angles are therefore $\geq 2\pi$. Hence $m \in Flat(S)$.

The curves γ and γ' in the new metric become concatinations of Euclidean segments (sides of triangles). Since the sum of angles of a cone point on one side of the curve in the metric m is greater than or equal to the sum of the angles in the metric m' we get that the angle around every cone point on each side of γ and γ' is $\geq \pi$. Therefore γ and γ' are also geodesic in the metric m.

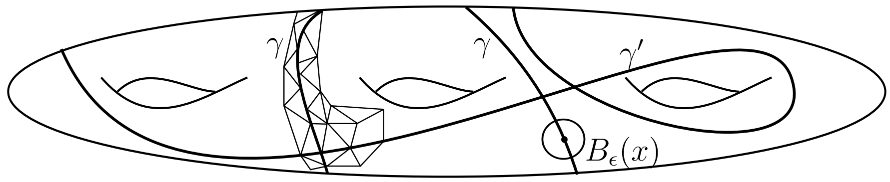

Figure 3.5: Part of a triangulation of a genus 3 surface with curves γ and γ' as unions of edges of the triangles.

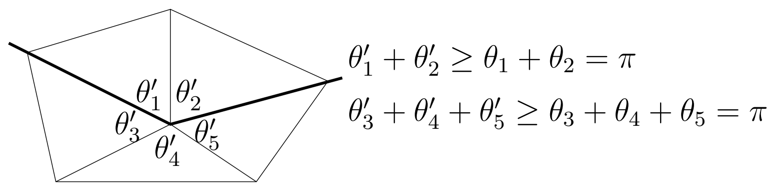

Figure 3.6: Euclidean triangles and a geodesic segment containing edges of the triangles.

We have

$$l_m(\gamma) = l_{m'}(\gamma) \neq l_{m'}(\gamma') = l_m(\gamma'),$$

and we proved our theorem. □

Theorem 3.2.3. $\overline{\bigcup_{q \in \mathbb{Z}_+} Flat(S,q)} = Flat(S)$. *More precisely, for every* $m \in Flat(S)$, $\exists m_n \in \bigcup_{q \in \mathbb{Z}_+} Flat(S,q)$, *so that* $id : (S, m_n) \to (S, m)$ *is* K_n*-bilipschitz and* $K_n \to 1$ *as* $n \to \infty$.

Proof. Let $m \in Flat(S)$. Take a triangulation of S by Euclidean triangles with all vertices being cone points. Using this triangulation, view S as obtained from a generalized Euclidean polygon P isometrically immersed in $\mathbb{R}^2$ by gluing the edges in pairs by isometries as in Section 2.1. Orient the edges s_i using the boundary orientation coming from the immersion into $\mathbb{R}^2$. Let z_i be a complex number that represents the edge s_i and θ_i the argument of z_i. We have $z_1 + z_2 + ... + z_{2n} = 0$. Assume, by rotating if necessary, that $\theta_1 = 0$ and by relabeling if necessary that z_{2i-1} and z_{2i} correspond to the edges that are glued together, $\forall i = 1, \dots, n$. We have $|z_1| = |z_2|$, $|z_3| = |z_4|, \dots, |z_{2n-1}| = |z_{2n}|$. See Figure 3.7.

If $z_{2i-1} = -z_{2i}$, $\forall i = 1, \dots, n$ then $Hol(S) = \{Id\}$ and therefore $m \in Flat(S, 1)$ and we set $m_n = m$ for all n.

Therefore we assume that there is some i so that $z_{2i-1} \neq -z_{2i}$. After changing indices if necessary we may assume $z_{2n-1} \neq -z_{2n}$. Fix $0 < \epsilon < \dfrac{|z_{2n-1} + z_{2n}|}{2}$. We will construct an

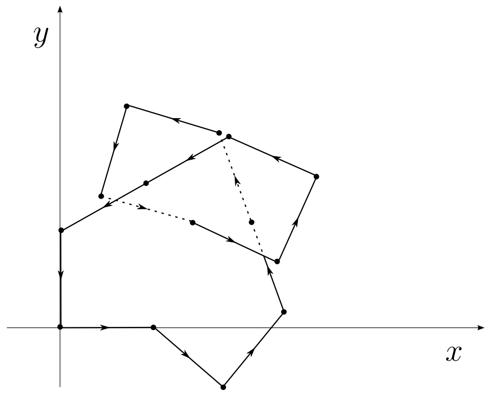

Figure 3.7: An immersed polygon in $\mathbb{R}^2$ with one edge on the x-axis.

isometrically immersed polygon P_ϵ so that the angles the sides of P_ϵ make with the real axis are all in $\mathbb{Q}\pi$, and $P_\epsilon \to P$ as $\epsilon \to 0$.

Denote $\tau = -z_1 - \ldots - z_{2n-2}$. Since $\tau = z_{2n-1} + z_{2n}$ it follows that $|\tau| > 2\epsilon$. Let $\widetilde{z}_1 = z_1$. For $2 \le j \le 2n-4$ choose $\widetilde{z}_j$ so that $|z_j| = |\widetilde{z}_j|$, $\widetilde{\theta}_j \in \mathbb{Q}\pi$ and $|z_j - \widetilde{z}_j| < \frac{\epsilon}{2n}$. Now, choose $\widetilde{z}_{2n-3}$, $\widetilde{z}_{2n-2}$ so that $|\widetilde{z}_{2n-3}| = |\widetilde{z}_{2n-2}|$, $\widetilde{\theta}_{2n-3}$, $\widetilde{\theta}_{2n-2} \in \mathbb{Q}\pi$, $|z_i - \widetilde{z}_i| < \frac{\epsilon}{2n}$ for $i = 2n-3, 2n-2$, and also so that the angle of $\widetilde{z}_1 + \ldots + \widetilde{z}_{2n-3} + \widetilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. We arrange this in the following way: First construct z_i', $i = 2n-3, 2n-2$, so that their arguments are in $\mathbb{Q}\pi$ and $|z_i'| = |z_i|$, $|z_i' - z_i| < \frac{\epsilon}{4n}$ for $i = 2n-3$, $2n-2$. Then construct $\widetilde{z}_i$, $i = 2n-3$, $2n-2$, so that $Arg(z_i') = Arg(\widetilde{z}_i)$, $|z_i' - \widetilde{z}_i| < \frac{\epsilon}{4n}$ and the angle of $\widetilde{z}_1 + \ldots + \widetilde{z}_{2n-3} + \widetilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. See Figure 3.8.

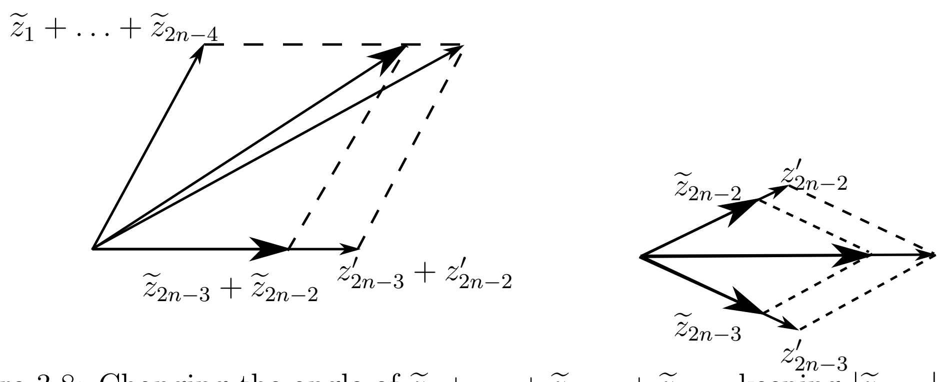

Figure 3.8: Changing the angle of $\widetilde{z}_1 + \ldots + \widetilde{z}_{2n-3} + \widetilde{z}_{2n-2}$ keeping $|\widetilde{z}_{2n-3}| = |\widetilde{z}_{2n-2}|$.

Denote $\widetilde{\tau} = -\widetilde{z}_1 - \ldots - \widetilde{z}_{2n-2}$. By the triangle inequality $|\tau - \widetilde{\tau}| \le \sum_{j=1}^{2n-2} |z_j - \widetilde{z}_j| < \epsilon$. Observe that $\widetilde{\tau} \neq 0$, because if $\widetilde{\tau} = 0$ then $2\epsilon < |\tau| < \epsilon$, which is a contradiction.

Since $\tau \neq 0$, the point z_{2n-1} is on the perpendicular bisector l of the vector τ. Let $\widetilde{l}$ be the perpendicular bisector of the vector $\widetilde{\tau}$. Let $\delta = 2\text{dist}(z_{2n-1}, \widetilde{l})$. Note that as $\epsilon \to 0$, $\delta \to 0$. Now choose $\widetilde{z}_{2n-1} \in \widetilde{l} \cap B(z_{2n-1}, \delta)$ so that the argument of $\widetilde{z}_{2n-1}$ is in $\mathbb{Q}\pi$. See Figure 3.9. Set $\widetilde{z}_{2n} = \widetilde{\tau} - \widetilde{z}_{2n-1}$. Since $\widetilde{z}_{2n-1}$ is on the perpendicular bisector of $\widetilde{\tau}$ then $|\widetilde{z}_{2n-1}| = |\widetilde{z}_{2n}|$ and since $Arg(\widetilde{\tau}) = \frac{1}{2}(\widetilde{\theta}_{2n-1} + \widetilde{\theta}_{2n})$, it follows that $\widetilde{\theta}_{2n} \in \mathbb{Q}\pi$.

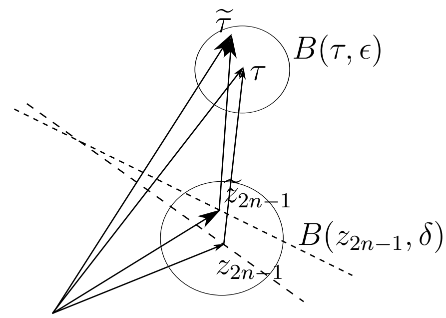

Figure 3.9: Constructing the vector $\widetilde{z}_{2n-1}$.

We have $\widetilde{z}_1 + \ldots + \widetilde{z}_{2n} = 0$ and $\widetilde{\theta}_i \in \mathbb{Q}\pi$ for every i. Therefore for $\epsilon > 0$ sufficiently small $\{\widetilde{z}_i\}_{i=1}^{2n}$ determines an isometrically immersed polygon P_ϵ with all sides having angles being rational multiples of π.

For sufficiently small ϵ there is a bilipschitz homeomorphism $f_\epsilon : P \to P_\epsilon$, which is linear on edges. For example, when $\epsilon > 0$ is sufficiently small, we can take a triangulation of P_ϵ with the same combinatorics as the one used to construct P. Map each triangle of P linearly to the corresponding triangle of P_ϵ. That way we get a map f_ϵ which is linear, and thus bilipschitz, on each triangle. It follows that f_ϵ is bilipschitz on P and as $\epsilon \to 0$ bilipschitz constant $K(f_\epsilon) \to 1$. The surface S is obtained by gluing the polygon P, so $S = P/_\sim$ where $t \sim \varphi(t)$ and $\varphi : \cup s_i \to \cup s_i$ is the gluing map. Define $\varphi_\epsilon = f_\epsilon \varphi f_\epsilon^{-1} : \cup f_\epsilon(s_i) \to \cup f_\epsilon(s_i)$. The map f_ϵ descends to $\bar{f}_\epsilon : P/_\sim \to P_\epsilon/_{\sim_\epsilon}$ where $t \sim_\epsilon \varphi_\epsilon(t)$. Define a Euclidean cone metric m_ϵ on S by pulling back the metric on $P_\epsilon/_{\sim_\epsilon}$ by $\bar{f}_\epsilon$. The holonomy of m_ϵ is generated by rotations through angles in $\mathbb{Q}\pi$. Therefore there is $q_\epsilon \in \mathbb{Z}_+$ so that $m_\epsilon \in Flat(S, q_\epsilon)$ and $m_\epsilon \to m$ as $\epsilon \to 0$. □

Theorem 3.2.1 now follows easily from the previous theorem.

Proof of Theorem 3.2.1. Suppose that we are given curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_\infty \gamma'$. By Theorem 3.2.3 it follows that $\forall m \in Flat(S)$, there is a sequence of metrics $\{m_n \in Flat(S, q_n)\}_{n=1}^\infty$ such that $id : (S, m_n) \to (S, m)$ is K_n-bilipschitz and $K_n \to 1$ as $n \to \infty$.

Thus $\lim_{n\to\infty} l_{m_n}(\gamma) = l_m(\gamma)$ and $\lim_{n\to\infty} l_{m_n}(\gamma') = l_m(\gamma')$. Since $l_{m_n}(\gamma) = l_{m_n}(\gamma')$ by assumption then $l_m(\gamma) = l_m(\gamma')$. That implies $\gamma \equiv_{\mathbb{R}} \gamma'$ which is a contradiction by Theorem 3.2.2. $\square$

Using the same idea as in Theorem 3.2.3 we can prove a stronger statement.

Theorem 3.2.4. *For every infinite sequence of distinct positive integers* $\{q_i\}_{i=1}^{\infty}$,

$$\overline{\bigcup_{i=1}^{\infty} Flat(S, q_i)} = Flat(S).$$

Proof. The proof of this theorem follows the proof of Theorem 3.2.3 if we approximate angles θ_j with angles $\widetilde{\theta}_j \in \mathbb{Z}\frac{2\pi}{q_l}$ for appropriately chosen $q_l \gg 0$. $\square$

In contrast to Theorem 1.2.2 we observe the following.

Corollary 3.2.5. *For all but finitely many* $q \in \mathbb{Z}_+$ *there exist curves* $\gamma, \gamma' \in \mathcal{C}(S)$ *so that* $\gamma \equiv_h \gamma'$ *but* $\gamma \not\equiv_q \gamma'$.

Proof. Assume that there exist infinitely many $q_i \in \mathbb{Z}_+$ so that $\forall \gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_{q_i} \gamma'$. Then, since the relation $\equiv_h$ is non-trivial, there are two distinct curves γ and γ' so that $\gamma \equiv_{q_i} \gamma'$ for some infinite sequence $\{q_i\}_{i=1}^{\infty}$ of positive integers, which is a contradiction to Theorem 3.2.4. $\square$

3.3 Curves in q-differential metrics and relations $\equiv_q$

In this section we prove:

Theorem 3.3.1. *For every $q_0, k \in \mathbb{Z}_+$ there are k distinct homotopy classes of curves $\gamma_1, \ldots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j$, $\forall i, j$ and $\forall q \leq q_0$. Thus $\forall q \in \mathbb{Z}_+$, the relation $\equiv_q$ is non-trivial.*

The proof uses the following two lemmas.

Lemma 3.3.2. *For any $q_0 \in \mathbb{Z}_+$, there is a curve $\gamma \in \mathcal{C}(S)$ such that for every $m \in Flat(S, q)$, $q \leq q_0$ the geodesic representative γ_m of γ contains a cone point and $[\gamma] \neq 0$ in $H_1(S)$.*

Proof. Fix a hyperbolic metric on S and identify the universal cover as $\mathbb{H}^2 \to S$. Fix $q_0 \in \mathbb{Z}_+$, and take $q_0 + 1$ geodesics $\alpha_1, \ldots, \alpha_{q_0+1}$ in $\mathbb{H}^2$ meeting at 0. See Figure 3.10.

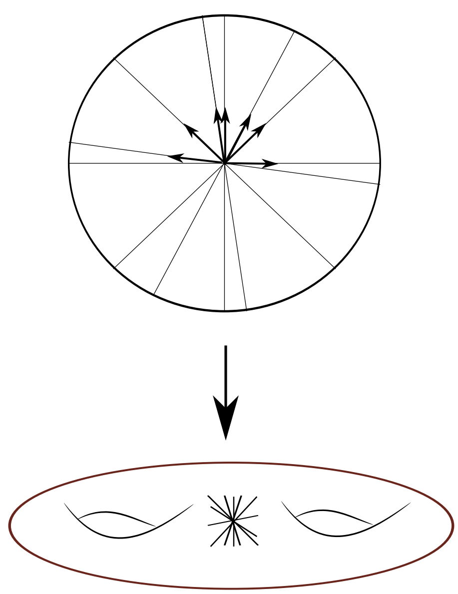

Figure 3.10: Projecting $\alpha_1, \ldots, \alpha_{q_0+1}$ into the surface S.

By ergodicity of the geodesic flow, there is a dense geodesic on S [10]. Given $\epsilon > 0$, we can therefore construct an ϵ-dense closed geodesic.

From this it follows that we can construct a closed geodesic γ on S with lifts $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{q_0+1}$ having tangent vectors within ϵ distance from the tangent vectors of $\alpha_1, \ldots, \alpha_{q_0+1}$ at 0. For ϵ sufficiently small the end points of $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{q_0+1}$ pairwise link in the same pattern as

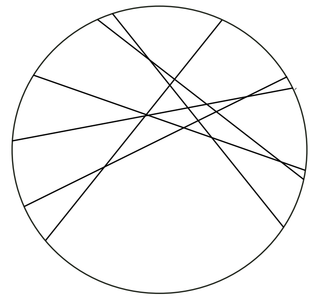

Figure 3.11: Lifts of γ_m to $\mathbb{H}^2$.

$\alpha_1, \ldots, \alpha_{q_0+1}$. In particular, every pair $\widetilde{\alpha}_i, \widetilde{\alpha}_j$ intersects. For every $m \in Flat(S,q)$ and m-geodesic representative γ_m of γ in m there are lifts $(\widetilde{\gamma}_1)_m, \ldots, (\widetilde{\gamma}_{q_0+1})_m$ which are quasi-geodesics in $\mathbb{H}^2$, with the same endpoints as $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{q_0+1}$, and hence with endpoints that all pairwise link. See Figure 3.11.

If $[\gamma] = 0$ in $H_1(S)$, we replace γ with a different curve as follows. Let δ be any curve with $[\delta] \neq 0$ in $H_1(S)$ that intersects γ. Construct a new curve which runs n times around γ, then at the intersection point switches and runs once around δ. If n is large enough, this constructs a curve which maintains the property of having $q_0 + 1$ lifts with endpoints that pairwise link. This new curve is homologous to δ, and so we replace γ with this curve.

Now assume that there is $m \in Flat(S,q)$, $q \leq q_0$ so that γ_m does not go through a cone point. Then there is an isometrically immersed Euclidean cylinder $S^1 \times [0,a] \to S$, for some $a > 0$, such that the image of $S^1 \times \{\frac{a}{2}\}$ is the geodesic γ_m and the image of $S^1 \times [0,a]$ does not contain any cone points. It follows that γ_m has only finitely many transverse self intersecting points. Because the holonomy group of m is in $\langle \rho_{\frac{2\pi}{q}} \rangle$, the angles of intersections are integer multiples of $\frac{2\pi}{q}$.

Lifts of γ_m have lifted cylinders which are strips isometric to $\mathbb{R} \times [0,a]$ in the universal cover. We consider the $q_0 + 1$ lifts $(\widetilde{\gamma}_1)_m, \ldots, (\widetilde{\gamma}_{q_0+1})_m$ as above and fix one of the lifts $(\widetilde{\gamma}_1)_m$ and a strip about it. Let $P \in (\widetilde{\gamma}_1)_m \cap (\widetilde{\gamma}_2)_m$ be the point of intersection of $(\widetilde{\gamma}_2)_m$ with $(\widetilde{\gamma}_1)_m$. For every $j \geq 3$ parallel transport the tangent vector to $\widetilde{\gamma}_j$ at the point of intersection $(\widetilde{\gamma}_1)_m \cap (\widetilde{\gamma}_j)_m$ inside the strip to P. The point P is not a cone point and there are no cone points inside the strip. If $(\widetilde{\gamma}_1)_m$, $(\widetilde{\gamma}_i)_m$, $(\widetilde{\gamma}_j)_m$ do not intersect in a common point then they form a triangle in $\widetilde{S}$. Since $\widetilde{S}$ is a $CAT(0)$ space the angle sum of a triangle is less than π, and hence $(\widetilde{\gamma}_i)_m$ and $(\widetilde{\gamma}_j)_m$ cannot make the same angle with $(\widetilde{\gamma}_1)_m$. Hence no two tangent vectors get transported to the same vector. See Figure 3.12.

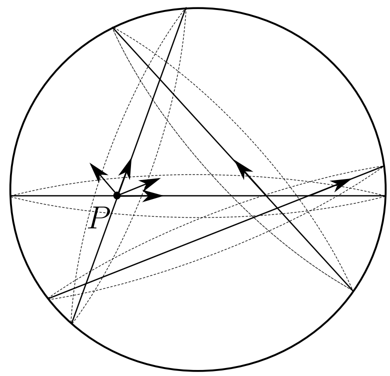

Figure 3.12: Strips in the universal cover.

By the holonomy condition all angles of intersections are integer multiples of $\frac{2\pi}{q}$, which is greater than or equal to $\frac{2\pi}{q_0}$. On the other hand we have q_0+1 vectors based at P, no two of them equal, therefore there is a pair of vectors with the angle between them less than $\frac{2\pi}{q_0}$. It follows that for some j, $(\widetilde{\gamma}_1)_m$ and $(\widetilde{\gamma}_j)_m$ make an angle which is not an integer multiple of $\frac{2\pi}{q_0}$. This is a contradiction and thus proves that γ_m has to contain at least one cone point for every $m \in Flat(S,q)$, $q \leq q_0$. $\square$

Lemma 3.3.3. *Let γ be the curve constructed in Lemma 3.3.2, and let m be a metric in $Flat(S,q)$. Given two lifts $\widetilde{\gamma}$, $\widetilde{\gamma}_0$ with linking endpoints and stabilizers generated by h and g, respectively, let $w \in \widetilde{\gamma} \cap \widetilde{\gamma}_0$. Then the m-geodesic from $h^{(q+2)}(w) \in \widetilde{\gamma}$ to any point along $\widetilde{\gamma}_0$ must run along a positive length segment of $\widetilde{\gamma}$. Moreover, this geodesic meets $\widetilde{\gamma}_0$ on $[g^{-(q+2)}(w), g^{(q+2)}(w)] \subset \widetilde{\gamma}_0$.*

Proof. It could happen that $l_m(\widetilde{\gamma}_0 \cap \widetilde{\gamma}) > 0$ as in Figure 3.13. We first claim that $l_m(\widetilde{\gamma}_0 \cap \widetilde{\gamma}) < l_m(\gamma)$. To show this let v, u be cone points at each end of $\widetilde{\gamma}_0 \cap \widetilde{\gamma}$. If $l_m(\widetilde{\gamma}_0 \cap \widetilde{\gamma}) \geq l_m(\gamma)$ then since the translation length of g and h is equal to $l_m(\gamma)$, we have $g(v) = h(v)$ or $g^{-1}(v) = h(v)$ which is a contradiction as $\pi_1(S)$ acts freely on $\widetilde{S}$.

Now let $v_1, \ldots, v_n$ be all the different cone points along $[h^{(q+1)}w, h^{(q+2)}(w)) \subset \widetilde{\gamma}$ and let θ_i be the angles $\widetilde{\gamma}$ makes at v_i all on one side of $\widetilde{\gamma}$, and let θ_i' be all the angles at v_i on the other side. Observe that there is some i so that $\theta_i > \pi$. Otherwise there is a small neighborhood of one side of $\widetilde{\gamma}$ with no cone points and we can find a geodesic homotopic to γ with no cone points by doing a straight line homotopy as shown in Figure 3.14. Similarly, there is j so that $\theta_j' > \pi$.

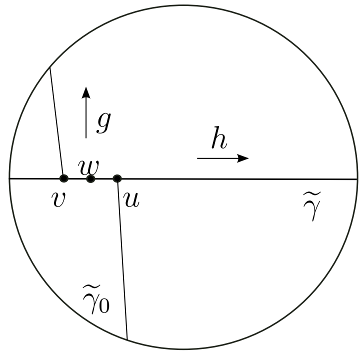

Figure 3.13: Two lifts of γ to the universal cover.

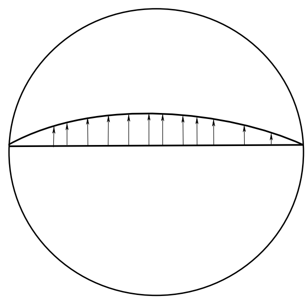

Figure 3.14: Homotopy.

Let $v = v_i$ with $\theta_i > \pi$ and $v' = v_j$ with $\theta'_j > \pi$. Let σ be the continuation of the geodesic segment $[h^{(q+2)}(w), v] \subset \widetilde{\gamma}$ to a geodesic ray starting at $h^{(q+2)}(w)$ and at every cone point past v having angle π on the right. See Figure 3.15. We define σ' similarly as the continuation of $[h^{(q+2)}(w), v'] \subset \widetilde{\gamma}$ with angle π on the left.

Claim: The rays σ and σ' do not intersect $\widetilde{\gamma}_0$.

Proof. Assume that $\sigma \cap \widetilde{\gamma}_0 \neq \emptyset$. Then $\widetilde{\gamma}$, $\widetilde{\gamma}_0$ and σ form a triangle in $\widetilde{S}$. Denote the angles of the triangle by β_1, β_2 and β_3. See Figure 3.16.

Take two copies of the given triangle and glue them together to get a sphere R with an induced Euclidean cone metric. See Figure 3.16. The Euler characteristic of R is 2, and by the Gauss-Bonnet formula (see Proposition 2.1.1)

$$-2\pi\chi(R) = \sum_{x \in X} (c(x) - 2\pi),$$

where X is the set of all cone points of R. Three cone points come from the triangle vertices

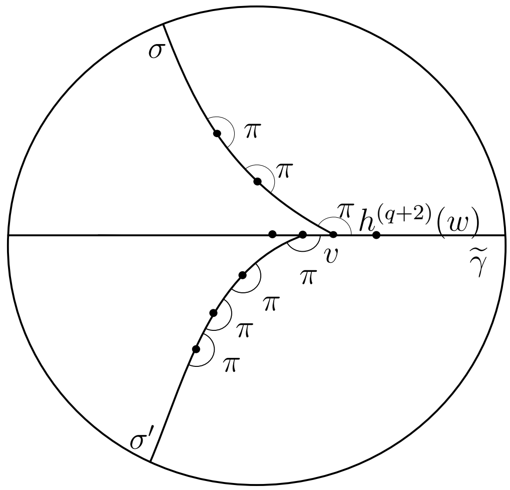

Figure 3.15: Geodesic rays as continuations of the geodesic segment γ.

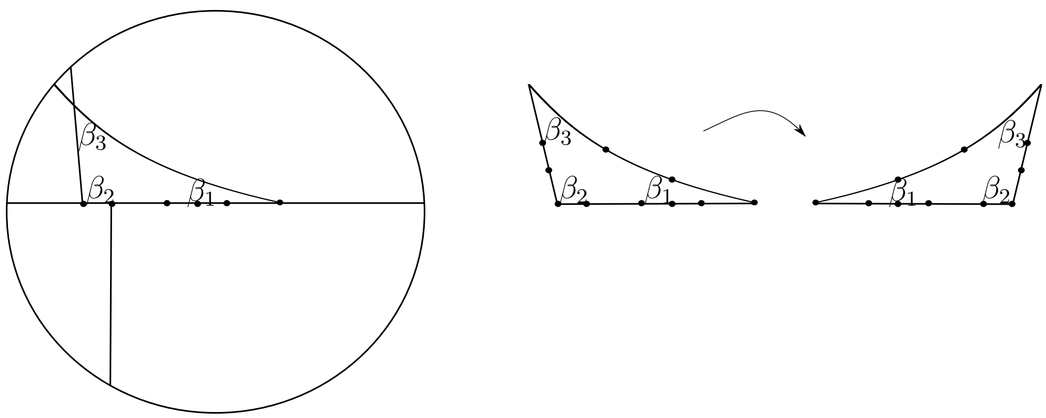

Figure 3.16: Gluing two triangles to get a sphere.

and they have cone angles $2\beta_l$, $l = 1, 2, 3$. The rest come from cone points along the sides or the inside of the triangles and have cone angles $> 2\pi$. In particular we have cone points $h^{-1}(v_j), \ldots, h^{-q}(v_j)$ with cone angles $2\theta_j$, $j = 1, \ldots, n$. Therefore we get the following inequality:

$$-4\pi = \sum_{x \in X}(c(x) - 2\pi) \geq \sum_{i=1}^{q}\sum_{j=1}^{n}(2\theta_j - 2\pi) + (2(\beta_1 + \beta_2 + \beta_3) - 6\pi).$$

It follows that

$$\pi - (\beta_1 + \beta_2 + \beta_3) \geq \sum_{i=1}^{q}\sum_{j=1}^{n}(\theta_j - \pi).$$

The holonomy for every metric in $Flat(S, q)$ is a rotation through an angle of the form $\frac{2\pi}{q}k$,

$k \in \mathbb{Z}$. Therefore we get $\sum_{j=1}^{n}(\theta_j - \pi) = \frac{2\pi}{q}k$ for some integer $k > 0$, and so $\sum_{j=1}^{n}(\theta_j - \pi) \geq \frac{2\pi}{q}$.

It follows that $\sum_{i=1}^{q}\sum_{j=1}^{n}(\theta_j - \pi) = q\sum_{j=1}^{n}(\theta_j - \pi) \geq q\frac{2\pi}{q} = 2\pi$, and thus:

$$\pi - (\beta_1 + \beta_2 + \beta_3) \geq 2\pi$$

which is a contradiction. Therefore $\sigma \cap \widetilde{\gamma}_0 = \emptyset$. The same argument shows $\sigma' \cap \widetilde{\gamma}_0 = \emptyset$. This proves the claim.

Now we'll show that every geodesic from $h^{(q+2)}(w)$ to $\widetilde{\gamma}_0$ has to share a positive length geodesic segment with $\widetilde{\gamma}$.

There is a unique geodesic between any 2 distinct points in $\widetilde{S}$, since it is a $CAT(0)$ space. Let ζ be a geodesic from $h^{(q+2)}(w)$ to some point on $\widetilde{\gamma}_0$. Let $\sigma_0 = \sigma - \widetilde{\gamma}$ and $\sigma_0' = \sigma' - \widetilde{\gamma}$. Assume ζ does not share a positive length geodesic segment of $\widetilde{\gamma}$. By the previous claim ζ must intersect one of σ_0 and σ_0'. See Figure 3.17. Without loss of generality assume that $A \in \zeta \cap \sigma_0$ is such a point. The path from $h^{(q+2)}(w)$ to v following the geodesic segments on $\widetilde{\gamma}$ and then from v to A along σ_0 is a geodesic by construction since all cone angles are $\geq \pi$ on both sides of the path. Since the initial arc of ζ to A provides a different geodesic, this contradicts the uniqueness of geodesics in $\widetilde{S}$ and we can see that ζ cannot cross either of σ_0 or σ_0' and hence must share a positive length segment with $\widetilde{\gamma}$.

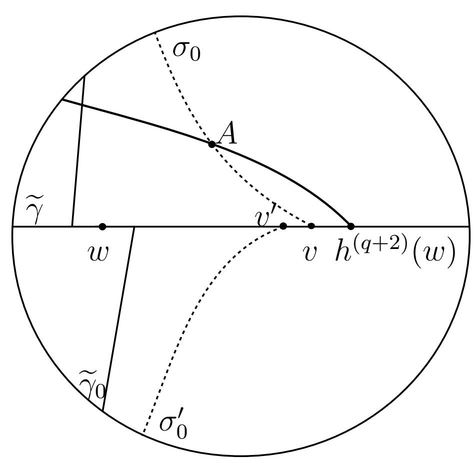

Figure 3.17: A geodesic segment from $\widetilde{\gamma}$ to $\widetilde{\gamma}_1$.

For the same reason the geodesics from $g^{(q+2)}(w)$ or $g^{-(q+2)}(w)$ to $\widetilde{\gamma}$ must share a positive length segment with $\widetilde{\gamma}_0$. Therefore every geodesic from $h^{(q+2)}(w)$ to a point on $\widetilde{\gamma}_0$ meets $\widetilde{\gamma}_0$ on $[g^{-(q+2)}(w), g^{(q+2)}(w)]$. $\square$

Now we can prove the main theorem.

Proof of Theorem 3.3.1. Let $\gamma \in \mathcal{C}(S)$ be as in Lemma 3.3.2. Let $\widetilde{\gamma}$ and $\widetilde{\gamma}_0$ be lifts of γ with endpoints that link. Suppose the stabilizer of $\widetilde{\gamma}$ is generated by h. Let $\widetilde{\gamma}_{-1}$ and $\widetilde{\gamma}_1$ be defined by $\widetilde{\gamma}_i = h^{i(q+2)}(\widetilde{\gamma}_0)$, $i = \pm 1$. See Figure 3.18. Denote h_{-1} and h_1 conjugate elements of $\pi_1(S)$ that generate the stabilizers of $\widetilde{\gamma}_{-1}$ and $\widetilde{\gamma}_1$ respectively.

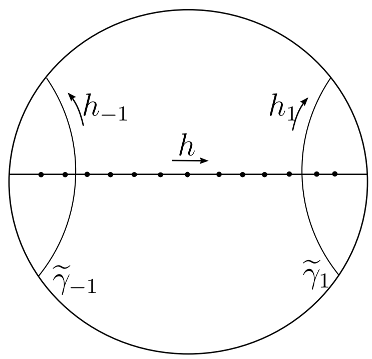

Figure 3.18: Lifts of γ.

Let $F(a, b)$ denote the free group on two generators a and b. Define $\phi : F(a, b) \to \pi_1(S)$ by $a \to h_{-1}^{2(q+2)}$ and $b \to h_1^{2(q+2)}$.

Claim: The ϕ–images of the words

$$w_0 = (ab)^k,\ w_1 = (ab)^{k-1}(ab^{-1}), \ldots, w_{k-1} = (ab)(ab^{-1})^{k-1}$$

represent distinct elements in $\mathcal{C}(S)$ and have the same length in every metric $m \in Flat(S, q)$, $q \in \mathbb{Z}_+$.

Let $m \in Flat(S, q)$. Let $(\widetilde{\gamma}_i)_m$, $(\widetilde{\gamma})_m$ be corresponding lifts of the m-geodesic representatives of γ. We have $(\widetilde{\gamma}_1)_m = h^{2(q+2)}((\widetilde{\gamma}_{-1})_m)$. Let z' be a point on $(\widetilde{\gamma})_m$ so that $(\widetilde{\gamma}_i)_m \cap (\widetilde{\gamma})_m \in [h^{i(q+1)}(z'), h^{i(q+2)}(z')]$, $i = \pm 1$. From Lemma 3.3.3 we know that every geodesic from z' to $(\widetilde{\gamma}_i)_m$, $i = \pm 1$, has to run along positive line segment of $(\widetilde{\gamma})_m$ and meet $(\widetilde{\gamma}_i)_m$ on $[h_i^{-(q+2)}(z'), h_i^{(q+2)}(z')]$.

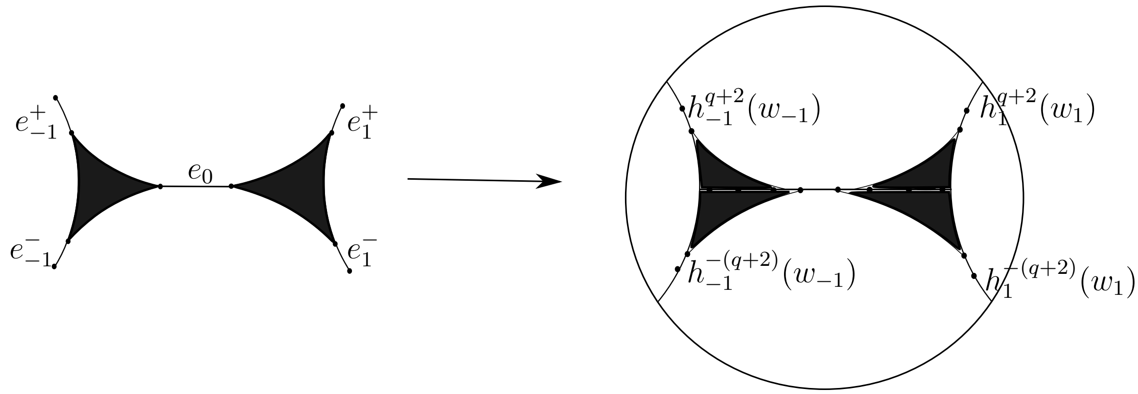

Figure 3.19: The m-convex hull of $(\widetilde{\gamma}_{-1})_m$ and $(\widetilde{\gamma}_1)_m$ on the right, and the 2-complex Γ' on the left.

It follows that the m–convex hull of $(\widetilde{\gamma}_{-1})_m$ and $(\widetilde{\gamma}_1)_m$ consists of $(\widetilde{\gamma}_{-1})_m \cup (\widetilde{\gamma}_1)_m$ together with an arc of $(\widetilde{\gamma})_m$ and two (possibly degenerate) triangles. Choose a point $w_i \in (\widetilde{\gamma})_m \cap (\widetilde{\gamma}_i)_m$ for $i = \pm 1$ and consider the point $h_i^{\pm(q+2)}(w_i)$ along $(\widetilde{\gamma}_i)_m$. Let Γ' be the metric 2-complex with two (possibly degenerate) triangles and five edges determined by $h_i^{\pm(q+2)}(w_i)$ as shown in the Figure 3.19 and view this as mapping into $\widetilde{S}$. Let Γ denote the metric 2-complex obtained by gluing $h_i^{-(q+2)}(w_i)$ to $h_i^{(q+2)}(w_i)$, for $i = \pm 1$. The inclusion $\Gamma' \to \widetilde{S}$ descends to a locally convex, local isometry $f : \Gamma \to S$. Identifying $\pi_1(\Gamma) = F(a, b)$ as shown in Figure 26 we have $f_* = \phi$.

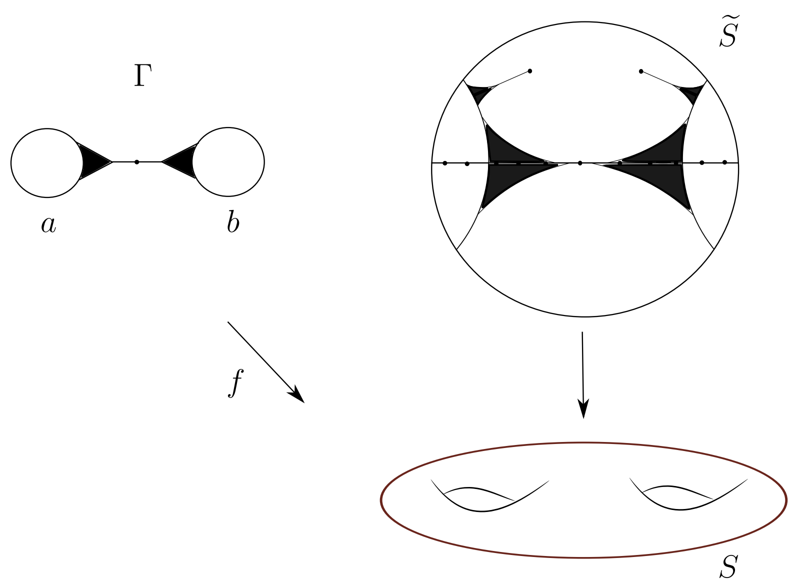

Figure 3.20: Constructing 2-complex Γ from the curve γ.

By the previous construction $[f_*(a)] = [f_*(b)] = \pm 2(q + 2)[\gamma]$ in $H_1(S)$. It follows that

$[f_*(w_j)] = \pm 2(q+2)(2k-2j)[\gamma]$. By construction $[\gamma] \neq 0$, thus $\{f_*(w_j)\}_{j=0}^{k-1}$ are distinct classes of non-null homotopic curves. Since f is a locally convex, local isometry, by measuring lengths on Γ instead of S we can see $l_m(f_*(w_j)) = l_m(f_*(w_i))$ for every $i, j = 0, \ldots, k-1$. $\square$

Corollary 3.3.4. *Let $q_1, q_2 \in \mathbb{Z}_+$. If $q_1 | q_2$ then $\equiv_{q_2} \Rightarrow \equiv_{q_1}$. The reverse implication is not true in general.*

Proof. If $q_1 | q_2$ then $Flat(S, q_1) \subset Flat(S, q_2)$. Thus the first part of the statement follows.

Assume the reverse statement is true for all $q_1 | q_2$. Let $q \in \mathbb{Z}_+$. Then $\equiv_q \Rightarrow \equiv_{q^i}$ for every positive integer i. By Theorem 3.3.1 there are two distinct curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$. By assumption we get $\gamma \equiv_{q^i} \gamma'$, $\forall i$. This is in contradiction to Theorem 3.2.4 if we take our infinite sequence to be $\{q^i\}_{i=1}^{\infty}$. $\square$

Theorem 3.3.5. *For every $q \in \mathbb{Z}_+$, $\exists \gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$ but $\gamma \not\equiv_h \gamma'$.*

Proof. Let γ and γ' be from the construction in Theorem 3.3.1. If $\gamma \equiv_h \gamma'$ then γ and γ' can be oriented so that they represent the same homology class [13]. In our construction γ and γ' have different homology representatives, thus those curves can not be h-equivalent. $\square$

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