

PERSPECTIVES ON MATHEMATICS EDUCATION FOR YOUNG CHILDREN

BY

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DISSERTATION

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Abstract

Through interviews with 32 participants, this study cast a wide net in its attempt to understand four professional groups' views on mathematics education for young children. These groups included (a) mathematicians, (b) teacher educators, (c) cognitive/developmental psychologists who study young children's mathematics learning, and (d) preschool and kindergarten teachers.

Both similarities and differences were found across groups. All participants have optimistic views on children's mathematical ability. Newly suggested in this study, by psychologists, is to connect how children construct mathematical knowledge with pedagogical strategies.

There was substantial agreement on the range of content areas that should be taught to young children. These included (a) number sense and operations, (b) algebraic thinking, (c) geometry, (d) measurement, and (e) data analysis. Mathematicians suggested an ideal sequence for teaching mathematical content areas.

All four groups addressed three substantial ideas related to pedagogy: (a) instilling a positive attitude about math, (b) using diverse teaching strategies, and (c) finding various ways to support teachers. Teacher educators were particularly concerned that current teacher education programs do not prepare teachers well enough to teach math effectively because of the limited number of required courses related to teaching math.

Implications for practice and suggestions for further research are presented. A critical goal for future research should be to get these four groups talking with each other.

To my Family

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Chapter 1

Introduction

From birth, a newborn's environment is filled with mathematical opportunities (Geist, 2004). As infants become toddlers and enter school, they engage in activities where they can have mathematical experiences. In free play, children sort, count, classify, add, and subtract. Mathematics is everywhere and integrated into the child's daily life. When children are standing in a line (ordinal numbers) or buying lunch (counting money), they continually encounter mathematical opportunities. The world of children is full of mathematical opportunities (Ginsburg, 2006).

As well as being exposed to mathematical opportunities, children develop an impressive understanding of various aspects of mathematics. Research suggests that children develop remarkable mathematical competence early in their lives. According to Gelman and Gallistel (1978), 2-, 3-, and 4-year-olds can recognize numbers of items under four. Canfield and Smith (1996) found that even infants have ability to notice abstract number information. They reported that five-month-old infants used visual expectation to show the ability to distinguish three pictures presented in one location from two pictures in another location. This indicated that infants as young as five months can count up to three. Starkey (1992) also found that young children have the ability to reason numerically. Children actively construct mathematical knowledge through daily experiences and have the ability to understand this knowledge intuitively (Baroody, 2000).

The problem is that even though children are capable of doing mathematics as a natural part of their daily routine, teachers often fail to recognize or take advantage of these teachable moments. Research has shown that many early childhood teachers do not create mathematically

rich environments in their classrooms (Copple, 2004; Ginsburg, Inoue, & Seo, 1999; Graham, Nash, & Paul, 1997) and mathematics-related teacher-child interactions seldom occur in the classroom (Sarama, DiBiase, Clements, & Spitler, 2004). There is substantial work on the development of mathematical thinking among young children from a cognitive developmental perspective, but this complex understanding of mathematical thinking has rarely been integrated into educational approaches. The research explains much about how children construct mathematical knowledge, but it tells little about pedagogical strategies. Most early childhood teachers “do not know what to do about mathematics for the young children with whom they work” (Clements, Sarama, & DiBiase, 2004, p. x) even though they recognize the importance of mathematics education for young children.

In order to help teachers understand what students are expected to learn in mathematics and how these mathematical concepts should be taught, NCTM’s Standards, State’s Learning Standards, and the Common Core State Standards are seen as guidelines for teachers. Given the development of these standards, one might conjecture that there is substantial agreement on the knowledge needed for teaching children mathematics. A close look at the standard, however, suggests a lack of consensus about what teachers actually need to know to teach math. States have their own varied standards, making it difficult to reach agreement on the best way to teach math to young children.

With the development of the Common Core State Standards and states’ effort to adopt them, there has been some improvement. Although the implementation of the common core standards has been recently completed (the end of the 2011-2012 school year), a new assessment system based on the common core standards has not been developed yet (scheduled to be in place by the 2013-2014 school year). The standards themselves fail to provide specific

information, such as the organization, composition, and characteristics of content knowledge for teaching as well as pedagogical strategies for teachers. Teachers remain unclear about what and how to teach mathematics to young children. Given this contemporary reality, systematic examination of what early childhood mathematics is and how it is best taught is needed.

In order to teach children mathematics effectively, it is believed that teachers need to have high level of pedagogical content knowledge. Shulman (1986) first addressed Pedagogical Content Knowledge (PCK) in teaching. PCK is a knowledge base combining a thorough understanding of *what* to teach with both *who* is taught and *how* to teach them. Shulman's notion of PCK not only brought to light questions about what teachers need to understand about the subjects they teach, but also promoted empirical applications of PCK in the field of mathematics education.

Ball (1988) interviewed pre-service elementary teachers to examine how much mathematical PCK they had. Of the 19 teachers Ball interviewed, only five were able to adequately explain why the mathematics procedures they knew worked. Building on Ball's work, Ma (1999) conducted a comparative study of American and Chinese teachers. Ma found that the Chinese teachers showed more comprehensive knowledge of mathematics compared to their U.S. peers. Recently, Hill, Rowan and Ball (2005) found a strong relationship between elementary teachers' mathematical PCK and students' academic outcomes. In early childhood education, McCray (2008) reported a similar finding that strength in mathematical PCK is positively related not only to teaching practices but also to students' mathematical outcomes.

Hill, Ball, and Schilling (2008) proposed a model they called Mathematical Knowledge for Teaching (MKT) that shows what teachers need for high quality math teaching. MKT contains three component of PCK as well as some other elements. The difference between MKT

and PCK is that MKT subdivides subject matter knowledge, one subset of PCK, into (a) Common Content Knowledge (CCK), (b) Specialized Content Knowledge (SCK), and (c) knowledge of mathematical horizon. Another difference is that MKT adds one more strand, knowledge of curriculum, to Knowledge of Content and Student (KCS) and Knowledge of Content and Teaching (KCT), two other subsets of PCK.

This research begins with the assumption that a deep understanding of three major elements that combine to generate PCK for mathematics—a well-explored *what* and *how* of mathematics teaching when applied to students (*who*)—allows teachers to teach effectively. If mathematics content and teaching strategies are examined as well as the learning needs and proclivities of young children, teachers can provide more effective mathematics teaching, increasing children's mathematical achievement.

How, then, to help teachers to develop their PCK in mathematics? As stated earlier, PCK is the overlap of three areas: math content knowledge, the purview of mathematicians; pedagogical knowledge, the purview of teacher educators; and knowledge of children's cognition, the purview of cognitive/developmental psychologists. The perspectives of these three professional groups, combined with teachers' views, can begin to shed light on how best to teach mathematics to young children. Figure 1 presents a diagram of the construct of mathematical PCK by incorporating people who have specialty in each component of PCK.

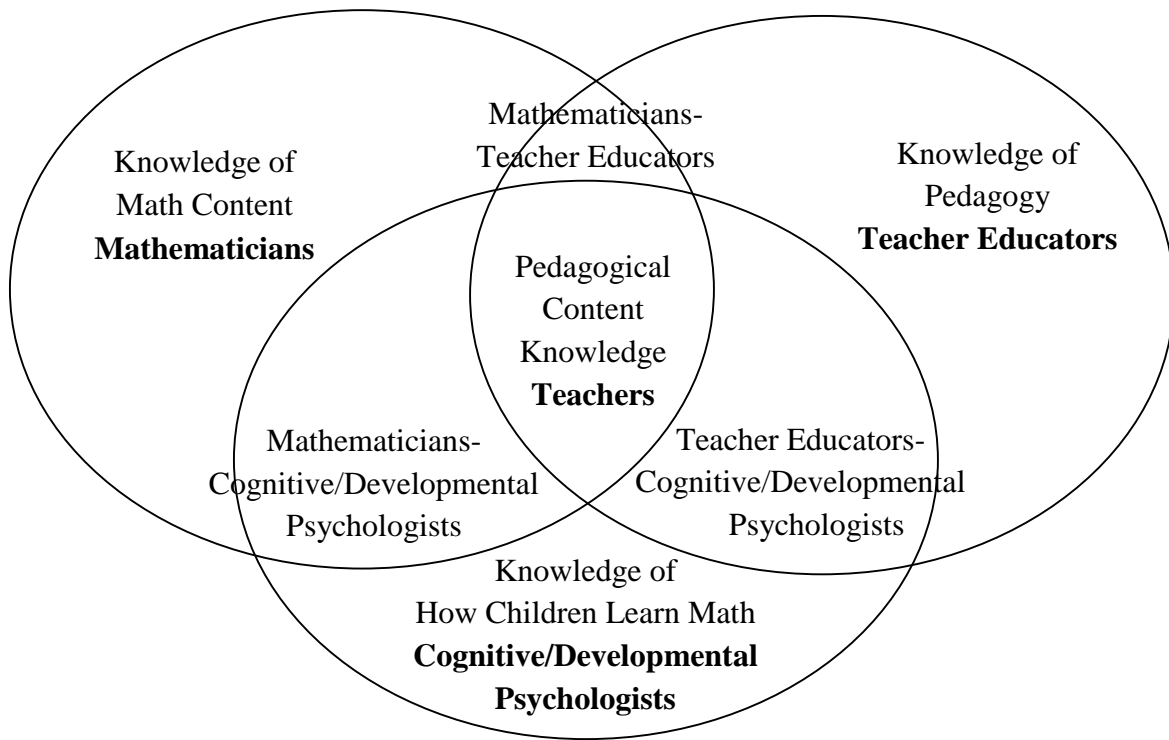


Figure 1. The construct of mathematical PCK

Purpose of the Study

The purpose of this study is to examine the views of four professional groups on mathematics education for young children. These groups are (a) mathematicians, (b) teacher educators, (c) cognitive/developmental psychologists who study young children's math learning, and (d) early childhood teachers. Using structured and semi-structured interviews (described in detail in chapter 3), the study addresses the following questions:

1. What are the views on mathematics education for young children of the following groups: mathematicians, teacher educators, cognitive/developmental psychologists who study young children's math learning, and teachers?
 - a) How do young children learn mathematics?
 - b) What mathematics content should be taught to young children?
 - c) How should this content be taught?
2. How are the perceptions of these groups similar and different?

3. How do the views of these groups reflect current learning standards (i.e., NCTM's Standards, State Learning Standards, and Common Core State Standards)?

Significance of the Study

Mathematical experiences and knowledge in early childhood are important because they affect children's later mathematical competence and achievement. The mathematics learned at an early age forms the basis for all higher mathematics. It also contains rudimentary versions of concepts that are crucial in more advanced forms of the discipline, so getting it right in the first place takes on great importance.

This study is important because although the National Council Teachers of Mathematics (NCTM) and the Council of Chief State School Officers (CCSSO) in cooperation with the National Governors Association Center for Best Practices (NGA Center) have provided standards to help guide the math curriculum, states are providing inconsistent math programs resulting in a math curriculum that is "a mile wide and an inch deep" (Schmidt, McKnight, & Raizen, 1997, p. 2, as cited in NCTM, 2006, p. 3).

The views of early childhood professionals, particularly teacher educators and cognitive/developmental psychologists who study young children's math learning, and to a lesser extent, teachers, have dominated discussions of math education for young children. These views have been challenged as limiting the understanding of mathematics education, particularly math content knowledge (what to teach). The views of mathematicians themselves have not been adequately explored. Wu (2006) noted that the mathematics and education communities have not been on speaking terms in the figurative sense for at least three decades. He was alarmed by the harm that this communication gap and lack of collaboration has caused mathematics education, and he alerted the mathematics community to the urgent need for active participation in education. I hope the process of comparing and contrasting ideas among these four professional

groups can help to map what teachers must know to effectively teach mathematics to young children.

Chapter 2

Literature Review

This chapter reviews research on Pedagogical Content Knowledge (PCK), the theoretical framework for the study, and early childhood mathematics education. First, I review the literature on the definition of Pedagogical Content Knowledge (PCK) and, then, its empirical application to mathematics education. Next, I examine research on early childhood mathematics education. I look at the history of early childhood mathematics education and the research classroom mathematics education practices in early childhood classrooms. The last section reviews the standards for early childhood mathematics education.

Pedagogical Content Knowledge (PCK)

Defining PCK. In 1986, Shulman introduced the term *Pedagogical Content Knowledge* (PCK) to establish the importance for teaching of a specific kind of subject matter knowledge. The key distinction between general subject matter knowledge and PCK is that PCK is a knowledge base that combines a thorough understanding of *what* to teach (general subject matter knowledge) with an understanding of both *who* is being taught and *how* to teach them. Shulman (1987) described PCK as “special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (p. 8). PCK is the “knowledge of subject matter for teaching which consists of an understanding of how to represent specific subject matter topics and issues appropriate to the diverse abilities and interest of learners” (Shulman & Grosman, 1988, p. 9). Shulman (1986) argued that teachers with strong PCK can teach effectively, resulting in optimal achievement for students.

Fennema and Franke (1992) proposed a conceptual model to help understand PCK (see Figure 2). In this model, content knowledge, pedagogical knowledge, and knowledge of learners’

cognition are combined together to produce PCK, which makes highly effective teaching practices possible. Fennema and Franke asserted that the components of the teachers' knowledge should not be seen as separate entities. They insisted that it is necessary to acknowledge and examine the integrated nature of this knowledge. A distinctive characteristic of Fennema and Franke's model is that the components are dynamic and interactive. The model also illustrates the interaction between the teachers' knowledge and affective factors, such as beliefs, which are of particular concern to Fennema and Franke.

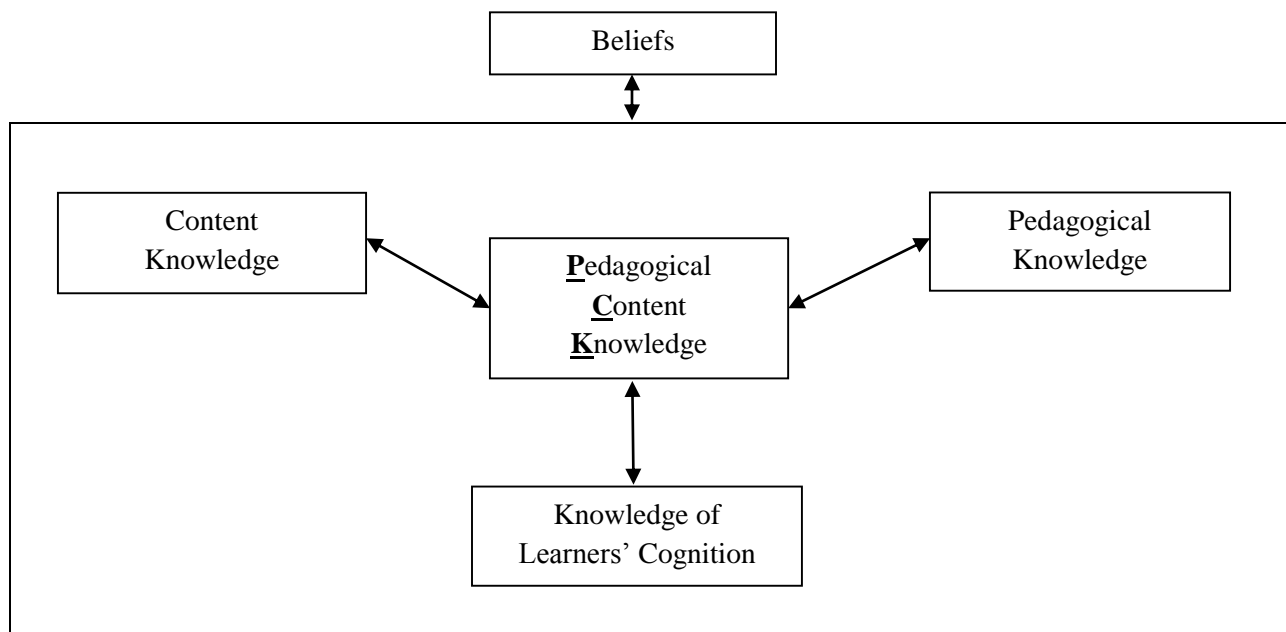


Figure 2. Fennema & Franke's PCK model (1992)

Cochran, DeRuiter, and King (1993) proposed a modification of PCK based on a constructivist perspective. They argued that PCK is not static knowledge, rather it is dynamic. To emphasize this dynamic nature, they used the term “*pedagogical content knowing (PCKg)*,” which they described as “a teacher’s understanding of four components of pedagogy, subject matter content, student characteristics and the environmental context of learning” (p. 266).

Recently, McCray (2008) provided a revised version of the pedagogical content knowledge model (see Figure 3). In her model, McCray used a Venn diagram to describe relationships between different components of teacher knowledge. Her model focuses on interactions between each of the three possible knowledge pairings; *what* content is for *whom*, *how* to teach given *who* the learners are, and *how* best to teach *what* the specific content is. This modified model stresses that in order to achieve PCK one must consider any one of the PCK components in light of the other two. McCray's model suggests directions for a more detailed analysis of how pedagogical content knowledge is constructed.

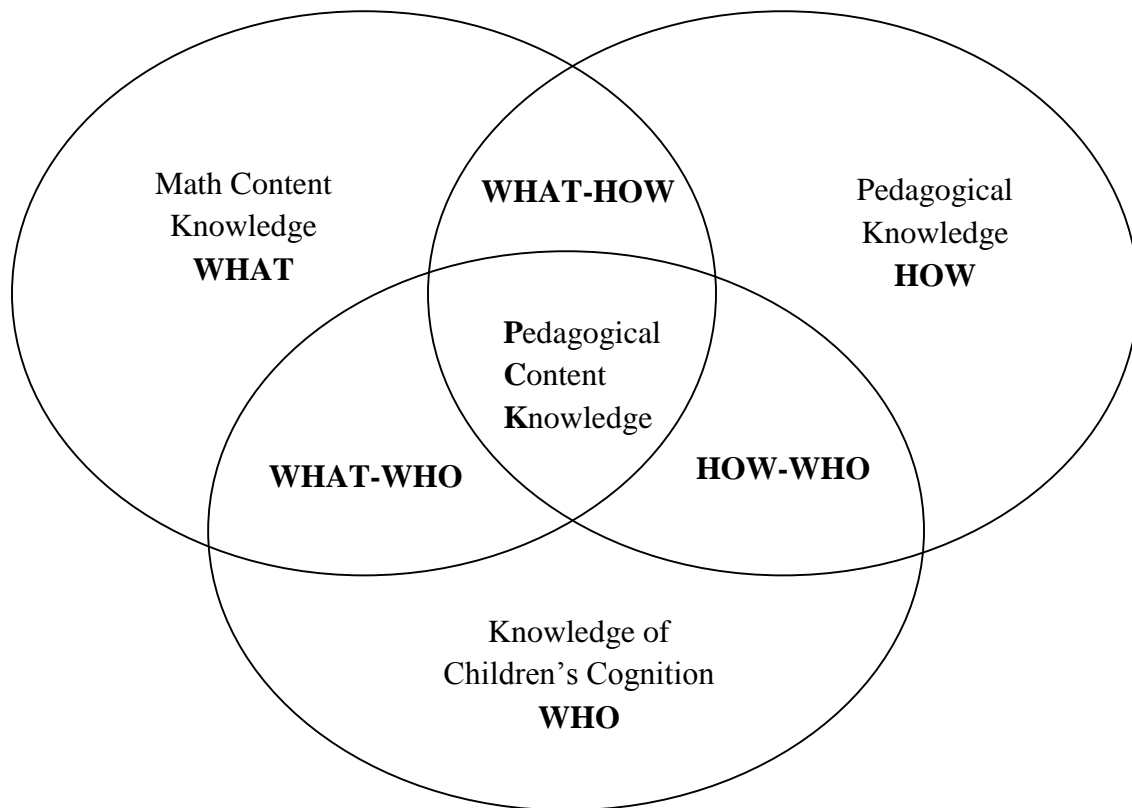


Figure 3. McCray's revised PCK model (2008)

PCK in mathematics. Shulman's ideas about PCK not only addressed the fundamental questions about what teachers need to understand about the subjects they teach but also promoted

empirical applications of PCK in the field of mathematics education. Ball (1988) highlighted prospective elementary school teachers' inability to explain procedures they used to solve mathematical problems. In order to examine how much teachers really know about mathematics as well as how much mathematical PCK teachers have, Ball developed a set of interview questions. The interview questions were built on a likely teaching scenario in which a student asks a question or presents an idea, and the teachers were asked to respond to it. Of the 19 teachers Ball interviewed, only five were able to provide adequate explanations. Ball's findings reflect the current situation in early childhood mathematics education.

Building on Ball's work, Ma (1999) compared the answers of American teachers on a mathematics interview to those of Chinese teachers. The results demonstrated that American teachers generally could not come up with illustrative teaching examples for mathematics problems and could not explain to students why the procedures they prescribed worked. The Chinese teachers demonstrated a significantly "more comprehensive knowledge of the mathematics taught in elementary school" (p. xx). A "profound understanding of fundamental mathematics" (PUFM) (p. xxiv), present among the Chinese teachers, was notably absent among American teachers. Even though Ma's study did not attempt to tie teachers' PCK to students' academic achievement, the finding that Chinese elementary school teachers possessed a markedly higher level of elementary school mathematical PCK than their U.S. peers suggests that teacher knowledge may be part of the reason that American students have performed comparatively poorly on tests administered internationally.

Recently, Ball and colleagues have looked at the relationship between elementary teachers' mathematical PCK and students' academic outcomes. Hill, Rowan, and Ball (2005)

found a positive relation between teachers' mathematical PCK and first- and third-grade students' gains in mathematics achievement scores.

Similarly, in early childhood education, McCray (2008) found that strength in mathematical PCK is positively related not only to teaching practices but also to students' mathematical outcomes. The results from a teacher interview that assesses PCK in preschool mathematics suggest two underlying constructs: Elaborative PCK and Evaluative PCK. Elaborative PCK is knowledge that supports the teachers' efforts to scaffold and elaborate on the children's self-directed activities, while Evaluative PCK is knowledge that allows teachers to evaluate the children's understandings relative to teacher-imposed learning goals. Analysis using hierarchical linear modeling found significant relationships between Elaborative PCK and gains in children's learning. This is the strong evidence that teachers' PCK affects student learning.

Early Childhood Mathematics

In this section, I first present how early childhood mathematics education has evolved historically. I then describe the current state of mathematics education in early childhood classrooms. Finally, I explore three kinds of learning standards for early childhood mathematics, which are recommended as guidelines for teachers.

History of early childhood mathematics education. Early childhood mathematics education has evolved in different ways across different times. At times it was limited to teaching numbers and simple arithmetic operations. At other times, it included learning about various shapes and patterns. Some programs placed more emphasis on direct teaching while others let children learn mathematics incidentally. Revisiting the history of early childhood mathematics education allows one to understand the big picture of the field, shedding light on how to improve mathematics education for young children.

Early optimistic views. Mathematics education for young children has a long history and can be traced back to a founding figure of early childhood education, Freidrich Froebel, who founded the first kindergarten, in Germany in 1837. Froebel's ultimate goal for educating children was to instill an understanding of the mathematically generated logic underlying creation (Clements & Sarama, 2009).

To facilitate children's learning and development, Froebel developed curriculum based on the *Gifts* (small manipulative materials for children to handle in prescribed ways, promoting learning about color, shape, counting, measuring, contrasting and comparison) and the *Occupations* (objects designed to teach specific skills such as paper weaving, paper folding, paper cutting, sewing, drawing, painting, and clay modeling). Through manipulations of the *gifts* and *occupations*, children had the opportunity to analyze and synthesize various geometric forms. For example, triangles, well known to children as parts of faces or other pictures, were used to teach concepts in plane geometry. Children covered the faces of cubes with square tiles and peeled them away to show parts, properties, and congruence. Many blocks and tiles were in the carefully planned shapes that fit in a grid in different ways. Shapes, rings, and slats were used on the grid, arranged and rearranged into shifting, symmetric patterns or geometric borders. Structured activities providing exercises in basic arithmetic, geometry, and the beginning of reading followed.

Froebel regarded mathematics as an essential element of the kindergarten curriculum and believed that kindergarten's universal and alternative language of geometric form could cultivate children's innate ability to observe, reason, express and create. According to Froebel, learning the sacred language of geometry in youth would provide a common ground for all people, and

advance each individual and society in general into a realm of fundamental unity (Brosterman, 1997, pp. 12-13).

As in Froebel's kindergarten, programs developed by Maria Montessori also included a strong emphasis on concrete materials as means of introducing a general principle or concept. Montessori believed that children gained information about the world through their senses, thus she thought that training the senses could make children more intelligent. She developed a series of sensory materials that can help children gain knowledge of the basic academic skills. Her educational materials were considered to be deeply mathematical in nature because most of the activities with the materials dealt with mathematical concepts such as comparisons, explorations, and identification of patterns, variables, sameness, and differences. Through individual work with the manipulative materials, children learned about shape, comparison of size and quantity, as well as the operations of addition, subtraction, multiplication, and division. For instance, red and blue rods, based on a unit of the decimal system, were used to introduce arithmetic. Children compared sizes and found multiples of smaller sizes. The units were then given the number names, and children learned to trace sand paper numerals before writing them. Using golden beads, which were strung on wires organized into units of ten, then rows of ten, squares of hundreds, and cubes of thousands, allowed the children to do more elaborate mathematical operations.

The distinctiveness of Froebel's and Montessori's ideas is that they viewed the development of young children as a process of unfolding and encouraged children to realize their full potential (Saracho & Spodek, 2008). Based on the careful observations of children, Froebel and Montessori advanced the notion that young children are capable of complex mathematical

thought and enjoy using mathematics to explore and understand the world around them (Balfanz, 1999, p.3).

Widespread pessimistic views, with some exceptions. Over the course of the twentieth century, researchers came to have pessimistic views about the nature of young children's mathematical competence that focused on what children cannot do. Thorndike (1922) concluded that young children were so mathematically inept that "little is gained by doing arithmetic before grade 2" (p. 198). He supported the social theorists' assumption that "young children started school with no prior mathematical knowledge or experience and that limited instruction on the first 10 numbers, simple addition and subtraction and recognition of basic shapes was sufficient for the early grades" (Balfanz, 1999, p. 8). Thorndike viewed mathematics as a "hierarchy of mental habits or connections" (p. 52) that must be carefully sequenced, explicitly taught, and practiced with much repetition in order for learning to occur. He contended that the purpose of mathematics education is helping children acquire quick and accurate arithmetic skills and that these skills are best learned in a drill and practice manner. Since Thorndike's pedagogy emphasized mechanical practice rather than logical thinking, one can infer a denial of the ability of children to reason about mathematical concepts.

Different from Thorndike, who advocated mathematics as a mental discipline, Dewey advocated a more utilitarian view of mathematics. Dewey took the view that mathematics is suitable not for memorization (drill and practice) but for inquiry and insisted that mathematics should be taught with an emphasis on processes such as problem solving, reasoning, and connecting to the real world. According to Dewey, traditional mathematics education just funnels the knowledge into children through books or direct instruction without connecting the knowledge to their realities (Dewey & McClellan, 1895). Dewey asserted that mathematics can

be best learned when it reflects children's interests and needs and that solving practical problems in daily experiences requiring mathematical concepts or skills is a more efficient way to learn mathematics than pre-planned instruction (Dewey, 1938). He also insisted that teaching children things that are impractical wastes their time and stifles their interest. According to Dewey, the present experience should always be the focus to avoid the pitfall of creating a disconnection between what is being taught and the experience of the children. As proponents of positive educative experience, Dewey and McClellan (1895) wrote, "It is a cardinal precept of the newer school of education that the beginning of instruction shall be made with the experience learners already have; that this experience and the capacities that have been developed during its course provide the starting point for all further learning" (p. 74). Even though Dewey didn't directly mention the mathematical competence of children, his view that children possess some mathematical knowledge can be inferred from his pedagogy, which emphasized the children's present situation (current knowledge) as the basis for lessons.

In the 1960's, a new trend that reshaped mathematics education for young children appeared. Jean Piaget's work contributed enormously to understanding of the development of mathematical reasoning. Piaget's research resulted in many important findings regarding geometric reasoning and numerical reasoning. For the former, Piaget maintained that children are born without knowledge of space, or even of permanent objects (Piaget & Inhelder, 1967). According to his topological primacy theory, children move through stages of egocentric spatial constructions (e.g., objects within reach or objects in front of the child, including topological relations of connectedness, enclosure, and continuity) to allocentric constructions (e.g. objects far away, including having relationships to one another). In his experiment known as the *Three Mountain Problem*, Piaget supported this theory. In the experiment, a child sat in front of a

mountain that has a cross visible only from the child's side. In addition, there was a doll on the other side of the mountain. According to Piaget, if preoperational children are asked to say what the doll can see, their response would reflect what can be seen from their perspective only. Piaget also found that children first are aware of topological features (i.e., inside and outside, open and closed) before they are aware of projective features of objects (i.e., the similarity of shapes and projections) or Euclidean geometric features of objects (i.e., area and shape congruence).

Regarding numerical reasoning, Piaget insisted that understanding number concepts means understanding number conservation, which is the ability to state quickly and assuredly that the numerosity of a collection had not changed after a change in the relationship among its elements. He ascertained that children do not acquire a notion of quantity and then conserve it and that children can discover true quantification only when they become capable of conservation (Piaget, 1965). According to Piaget, children's number conservation develops in three stages. Stage 1 is the "gross quantity stage" in which children make global perceptual judgments without one-to-one correspondence. Stage 2 is the "intensive quantity stage" where children can make one-to-one correspondences perceptually. In stage 3, children construct the notion of unit and numerical correspondence so that they understand that inverse changes in length and density can compensate each other and thus that changes are reversible. Through the experiment of having young children less than six years of age view two equally spaced rows of equivalent objects with the same numbers of objects in each row (e.g., five blue buttons for the first row and five red buttons for the second row), Piaget (1965) supported his number conservation stage theory. He found that children tend to say that the two rows have unequal numbers of objects if the spaces in one row are increased and contend that the longer row of objects has more objects in it. Also, to Piaget, counting was viewed as ineffectual with "no

connection between the acquired ability to count and the actual operations of which the child is capable” (Piaget & Szeminisk, 1952, p. 61). He asserted that “children are capable of meaningful counting only upon reaching the level of reversible operations” (p. 184). Thus, Piaget didn’t pay attention to children’s counting, which he considered a rote skill, rather he emphasized classification, seriation, and one-to-one correspondence activities that support children’s logical foundation for acquiring number concepts.

Piaget’s work drew interest to young children’s mathematical thinking and knowledge. Most of his work, however, stressed children’s lack of logical (mathematical) reasoning, which resulted in limiting the expectations about what young children can learn and be taught.

Highly optimistic views. In the last quarter of the 20th century, new perspectives on children’s mathematical competency emerged. Researchers started to take optimistic views about children’s mathematical competency and to focus on what they can do. Through her *Magic Experiment*, Gelman (1969) showed numerical and arithmetic competencies in young children who were not operational in Piaget’s assessment. In her experiment, two plates were shown: one with two mice designated the loser and the other with three mice designated the winner. After a series of identification tasks, she surreptitiously altered the winner with changing the spatial arrangement and altering the identity of the item or size of the collection. The results showed that children as young as three and sometimes two and half years seem to know that transformations involving displacements do not change the numerical value of a display, an early form of conservation of number (Gelman & Gallistel, 1978). Also, findings suggested that children preferred to make decisions based on equivalence or nonequivalence of verbal numbers, rather than one-to-one correspondences emphasized by Piaget. Even though Gelman admitted children’s failing to solve mathematical tasks dealing with large numbers, she sought the

explanations based not on children's lack of logical thinking but rather in their limited memory capabilities. According to Gelman, children's numerical abilities are classified into two categories: *number-abstraction* (the ability to represent a certain amount in an array or collection of entity) and *number-reasoning* (the ability to reason and think through about the number) (Gelman & Bailargeon, 1983). She asserted that number-abstraction develops prior to number-reasoning, and children's counting emerged at the beginning of number-abstraction and is closely related to the development of number-reasoning.

Even for infants, researchers provided evidence of children's initial mathematical competence. In her *Puppet Experiment*, Wynn (1998) investigated infants' ability to individuate and enumerate physical actions. In the experiment, infants aged five- to eight-months saw a puppet that jumped two hops repeatedly, and then they saw the same display except the puppet switched between two hops and three. The infants looked longer at the novel number of jumps, indicating that the infants had counted how many jumps occurred in each sequence. Wynn concluded that "infants are able to identify individual jumps and enumerate them" (p. 9). In later refinements of her experiment, she confirmed the findings that these very young children can discriminate the number of actions of the puppet (Wynn, Bloom, & Chiang, 2002).

Other researchers also reported that babies in the first 6 months of life can discriminate one object from two, and two objects from three (Starkey, Spelke, & Gelman, 1990). In their experiment, infants saw a sequence of pictures containing two circles that varied in their attributes, such as size, density, brightness or color. At the beginning, the differences between the pictures kept the infants' attention so that they continued to look at each picture in turn. The infants, however, came to habituate to the displays so that they looked at the screen less, their eyes wandered more often, and their breathing became more relaxed. Once the infants

habituated, they saw a new collection of three circular regions that were similar in attribute to those they had previously seen. The researchers found that the infants focused intently on this new collection and their breathing was more rapid. The infants' renewed interest to the display with a different number of objects provided evidence that they are sensitive to numbers (Starkey, Spelke, & Gelman, 1990).

Recent intermediate views. Researchers in the past ten years have included that the optimistic views exaggerate children's abilities and have called for more balanced views of children's mathematical competence. Among these researchers, children's initial competence is generally accepted, but they assert that children have informal mathematical knowledge and that their math competency develops by age (Baroody, Benson, & Lai, 2003; Mix, Huttenlocher, & Levine, 2002). Using the *mental models view*, Mix, Huttenlocher, and Levine (2002) showed that how children represent numbers evolves. In this view, children's representation of numbers gradually develops through three steps: inexact perception-based representation, exact nonverbal representation, and number-word representation. In the first step, *pretransition 1*, children represent all quantities nonverbally and inexactly using one or more perceptual cues, such as area, contour length (total perimeter), density, and length. In the second step, *transition 1*, children develop the ability to mentally represent the intuitive numbers nonverbally and exactly. In the last step, *transition 2*, children are able to do exact verbally based representation of any number by the aid of transitions 1 and the development of counting (Mix et al., 2002).

Paralleling the increasing evidence for these middle ground views, the National Council Teachers of Mathematics (NCTM) revised their standards in 2000. NCTM produced five content standards and five process standards for mathematics education for young children. The five strands of the content standards are (a) number sense and operations, (b) algebra, (c) geometry,

(d) measurements, and (e) data analysis and probability. The five strands of the process are (a) problem solving, (b) connections, (c) reasoning, (d) representation, and (e) communication (NCTM, 2000).

Based on NCTM's Standards, researchers, within the past 10 years or so, have developed research-based mathematics curricula for young children. The primary principle of these curricula lies in the belief in the power of young children's mathematical thinking and the power of combined teaching strategies to bring forth and develop each child's potential. A representative program is *Pre-K Mathematics Curriculum* (Klein & Starkey, 2002), which comprises 29 small-group classroom activities and 18 home activities. "The activities are designed to be sensitive to the developmental needs of individual children. Suggestions are provided for scaffolding children who experience difficulty" (Klein, Starkey, Clements, & Sarama, 2007, p. 5). The content of the program covers six areas (which mostly overlap with NCTM's Standards): (a) number and operations, (b) space, (c) geometry, (d) patterns, (e) measurement and data, and (f) logical reasoning.

Another representative program is the *High/Scope Curriculum* (Hohmann & Weikart, 2002), which has been updated and renamed *Number Plus*. Overcoming the criticism that the earlier version is limited in scope and content, *Number Plus* provides mathematics at a more challenging level. As can be inferred from the title, the curriculum focuses mostly on number sense and operations, but it also provides activities in shape, space, measurement, algebra (mostly patterns), and data analysis. The hallmark of *Numbers Plus* is that children's mathematical learning is sequenced within activities. Each curriculum kit includes (a) 120 activities divided by content, (b) a teacher's manual, and (c) parents' booklets. It also contains professional development activities.

Big Math for Little Kids (BMLK) is the culmination of four years of work extending from 1998-2002 to provide children with a developmentally appropriate and research-based curriculum that promotes children's mathematical competence. BMLK is based on 4 design principles: (a) Young children are ready and able to learn mathematics; (b) children need adult guidance to reach their full mathematics potential—playing is not enough; (c) low-income children benefit from rich math learning experiences; and (d) young children are capable of learning from a comprehensive and developmentally appropriate curriculum (Morgenlander & Manlapig, 2006). BMLK provides teachers with many different opportunities to help children learn “Big Math” concepts. First, the curriculum offers teachers a sequenced, extensive, and in-depth coverage of various mathematical concepts. The curriculum also presents opportunities for math learning to directly connect from the classroom to the home (Balfanz, Ginsburg, & Greenes, 2003). The BMLK curriculum covers six units: number, shape, patterns and logic, measurement, number operations, and space, and each of these math concepts is first introduced in the pre-K curriculum and then further developed in the kindergarten curriculum (Greenes, Ginsburg, & Balfanz, 2004).

The Measurement-based Approach (Sophian, 2004) was developed for teaching children mathematics in the Head Start program. Based on the work of the Russian psychologist Davydov (1975) and in collaboration with classroom teachers, Sophian developed an intervention program that includes weekly classroom projects with various supplementary activities and weekly home activities for parents to do with their children. The program emphasizes the concept of unit and the early understanding of number, measurement, and geometric shapes. Sophian noted that the intervention positively affected the children's mathematics achievement and that the program elevated teachers and parents' expectations about children's potential for learning mathematics.

Storytelling Sagas (Casey, 2004) is a set of six storybooks that can be used as supplementary teaching materials. Each of the books, combined with oral storytelling and hands-on activity, focuses on a different content area such as space, pattern, and measurement and so on. The books also include vivid visuals and spatial reasoning components. The embedded assumption of *Storytelling Sagas* is that language plays an important role in children's active learning of mathematics.

Another representative mathematics education curriculum for young children is *Building Blocks*, which is a nationwide project designed to help children to develop solid mathematical content knowledge and higher-order thinking. Based on theory and research on early childhood mathematics teaching and learning, *Building Blocks*' basic approach is to find mathematics in, and develop mathematics from, children's activity (Clements & Sarama, 2004). *Building Blocks* was designed to help children extend and mathematize their everyday activities. *Building Blocks* was developed in five phases (Sarama & Clements, 2002). It began with the identification of a significant domain of mathematics. In the second phase, an explicit model of children's knowledge, including hypothesized learning trajectories, was built. Then, researchers created an initial design for software and activities. In the fourth phase, components of the software were tested using clinical interviews and observations. In the last phase, researchers continued to evaluate the prototype to improve it in a more complete form. Consistent with both NCTM's *Principles and Standards for School Mathematics* (PSSM) and an extensive review of the research, *Building Blocks* incorporates math objects, including numbers and shapes, and math actions, such as counting or transforming shapes through manipulatives and objects, computer activities, books, and more.

The *Number Worlds* curriculum (Griffin, 2007b) deals with basic number concepts from preschool through the sixth grade. The curriculum helps children navigate the world of real quantities that exist in space and time, the world of counting numbers, and the world of formal symbols (Griffin, 2007a, p. 375) by teaching fundamental concepts for learning and promoting connections among them. Through hands-on games and activities that “capture children’s emotions and imagination as well as their minds” (p. 379), the curriculum helps children develop mathematical competence.

RightStart Mathematics is a hands-on and visual program for grades K-4 developed by Joan Cotter. The distinctive characteristic of this program is that it does not emphasize counting as the starting point for mathematics. Instead it uses the visualization of quantities and provides strategies (visual pictures) for learning the facts. Understanding and problem solving are emphasized throughout the curriculum. Practice is provided through Math Card Games. Place value is made easier by using number names that are an exact match with the mathematical value: For example, 12 is called “ten 2” (or “1-ten 2”) and 23 is “2-ten 3.” It has daily lesson plans for teachers in a lesson book, and worksheets for children in another book. Lesson books are labeled Levels A through E corresponding to grades K-4. Level A includes reproducible worksheets in the lesson book while Levels B through E have separate worksheet books. The child uses the AL Abacus, a specially designed abacus to help visualize quantities, and various other manipulatives and aids, such as card games, colored tiles, geoboards, and thousand cubes. The basic manipulative set includes AL Abacus, place value cards, base 10 picture cards, six special card decks, fraction charts, a drawing board geometry set, geoboards, color cubes, colored tiles, a calculator, a geared clock, Math Balance, tangrams, centimeter cubes, a 4-in-1 ruler, a folding meter stick, and an angle measurer.

Current ECE classroom mathematics education practices. Research has shown that many teachers of young children do not create mathematically rich environments in their classrooms (Copple, 2004; Ginsburg, Inoue, & Seo, 1999; Graham, Nash, & Paul, 1997). While studies that measure mathematical activities in early childhood classroom are rare (Sarama, DiBiase, Clements, & Spitler, 2004), those that do exist suggest that mathematics related teacher-child interactions in the classroom are rare.

Ginsburg, Inoue, and Seo (1999) observed 30 African American and Latino preschoolers from low SES homes as they engaged in free play at a day care center and found that 44% of free play involves mathematical activity. The mathematical engagements were patterns and shapes, dynamic exploration of the processes of change or transformation, relations-magnitude evaluation or comparison, classification sorting grouping or categorizing, enumeration and quantification or numerical judgment. They concluded that these children appeared to have the intellectual ability to engage in advanced mathematical exploration and activities necessary to succeed in school. Nevertheless, Ginsburg et al. reported that “the teachers . . . do little to promote children’s everyday mathematical activities” and that there was “little evidence in our observations of any adult involvement – explicit teaching or indirect assistance – in children’s mathematical explorations” (p. 97). The authors argued for not only a more varied and rich curriculum for preschoolers but also for teacher’s active involvement in mathematical activities.

In another observational study of two child care centers serving professional and university-affiliated families (Graham, Nash, & Paul, 1997), six half-hour classroom observations were conducted over a three week period of time. Observers focused on classroom activities involving counting, shapes, patterns, sets, adding, subtracting, money, time, and so on. During the 12 hours of observations, only 12 instances of mathematical discussion were

observed, six at each center. The authors remarked that of the 12 instances of mathematical discussion in the 12 hours of observation, only two instances were sustained for longer than 60 seconds. The results confirmed the fact that mathematical activities in the preschool classroom are rare.

A recent observational study tells a similar story. Rudd, Lambert, Satterwhite, and Zaier (2008) observed eleven teachers from six classrooms in the Child Development Center (CDC) located at the Southwest University. They investigated the types and frequency of mathematical language used in preschool classrooms. The results indicated that utterances pertaining to spatial relations exceeded any other type of mathematical concepts, and there were a paucity of utterances involving higher-level mathematical concepts. In addition, they found differences in the kinds of mathematical utterances used by teachers based on teacher characteristics (teaching experience, educational background), as well as differences in math-mediated language between the classes of varying age groups. Rudd et al. pointed out the need for employing higher-level mathematical concepts and planned activities around mathematical concepts.

Standards for early childhood mathematics. In this section, standards for early childhood mathematics education are reviewed. Currently, three sets of standards are widely used in the field of early childhood mathematics education. They include NCTM's Standards, State's Learning Standards, and the Common Core State Standards. Table 1 summarizes these three standards. The Illinois Early Learning and Development Standards, the updated version of the Illinois Early Learning Standards, are also introduced.

Table 1

Learning Standards for Mathematics

NCTM's Standards (2000)		Illinois Learning Standards (1997)		Common Core State Standards (2010)	
Principals	<ul style="list-style-type: none"> - Equity - Curriculum - Teaching - Learning - Assessment - Technology 	Skills	<ul style="list-style-type: none"> - Applications of learning - Solving problems - Communications - Applying technology - Working on teams - Making connections 	Standards	Practice standards
		Goals	<ul style="list-style-type: none"> - Number Sense - Estimation and measurement - Algebra and analytical methods - Geometry - Data analysis and probability 		Content standards
Standards	Content standards	Standards	Specific to each grade level		
	Process standards	Benchmarks	Specific to each grade level		
Grade band	Pre-K to grade 2 Grades 3–5 Grades 6–8 Grades 9–12.	Grade level	Preschool Kindergarten Grade 1 Grade 2 Grade 3 Grade 4 Grade 5 Grade 6 Grade 7 Grade 8 Grade 9 Grade 10 Grade 11 Grade 12	Grade level/ band	Kindergarten Grade 1 Grade 2 Grade 3 Grade 4 Grade 5 Grade 6 Grade 7 Grade 8 Grades 9-12

NCTM's Standards. The National Council of Teachers of Mathematics (NCTM)'s effort to improve math education began with the release of *Curriculum and Evaluation Standards for School Mathematics* in 1989. Because it was the first time a professional organization established goals for teachers and policymakers in a school discipline, the document is considered a historically significant step to provide focus, coherence, and new ideas to math education.

In 1991, NCTM published *Professional Standards for Teaching Mathematics* which described the elements of effective mathematics teaching. In 1995, NCTM issued *Assessment Standards for School Mathematic*, which sets objectives against which assessment practices can be measured. On the basis of these three standards, NCTM launched “the Standards 2000 project” in 1997. The aim of the project was to update the standards document while periodically examining, evaluating, and revising these standards to keep them relevant. A draft version of the new document was finished in 1998, and NCTM substantially revised the document based on the feedback from many different sources in response to the draft. The resulting publication was *Principles and Standards for School Mathematics*, which appeared in 2000. It has been considered a guideline to help realize visions of high-quality mathematics education (Bredekamp, 2004).

The distinctive characteristic of *Principles and Standards for School Mathematics* is that, including pre-kindergarteners for the first time, it acknowledges the importance of mathematics education in the early years. *Principles and Standards for School Mathematics* consist of two major parts. The first, the principles for school mathematics, deals with the broad issues of equity, curriculum, teaching, learning, assessment, and technology (NCTM, 2000). The first principle asserts that excellence in mathematics education requires equity, that is, the need to have high expectations and to provide strong support for all students. According to the second principle, a curriculum must be more than a collection of activities. In other words, it should be coherent, focused on important mathematic concepts, and well articulated across the grade levels. The third principle, teaching, insists that mathematics teaching requires an understanding of what students know (prior knowledge) and what they need to know (new knowledge) and then challenging and supporting them to learn it well. The fourth principle emphasizes the importance

of learning mathematics by building new knowledge from prior experience and knowledge. The fifth principle emphasizes the appropriate usage of assessment, which not only supports mathematical learning but also provides useful information to both teachers and students. The last principle notes that technology can be a helpful tool to enhance good teaching and students' learning.

Following these principles, the standards for school mathematics describe comprehensive goals for mathematics instruction. NCTM produced five content standards and five process standards for mathematics education, and these standards describe the mathematical understanding, knowledge, and skills that students should acquire from pre-kindergarten through grade 12. The standards are treated in greater detail in four grade-band chapters; pre-kindergarten through grade 2, grades 3-5, grades 6-8, and grades 9-12. The five strands of the content standards are number sense and operations, algebra, geometry, measurements, and data analysis and probability (NCTM, 2000).

For each of the five content standards, a set of expectations specific to each grade-band is included. Number sense and operations consist of counting, comparing, ordering, grouping, and adding to and taking away a quantity. The emphasis of algebraic thinking in early childhood involves finding patterns. Patterns are a way for children to recognize order and to organize their environment (NCTM, 2000). "Geometry can be used to understand and to present objects, directions and locations in our world, and the relationship between them" (Clements, 2004, p. 39). The subtopics of geometry include shape, location direction and coordinates, visualization and spatial reasoning, and transformations and symmetry. Measurement, one of the most widely used applications, determines how much of an attribute an object possesses, such as length, weight, and capacity. Measuring involves the use of tools such as rulers and also includes

nonstandard ways of measuring such as paper clips. Data analysis uses information to classify, organize, and answer questions.

The content standards are connected by process standards that are related to all content areas. The five strands of the process are problem solving, connections, reasoning, representation, and communication (NCTM, 2000). For each of the five process standards, the examples demonstrate what the standard should look like in each grade-band and what the teacher's role should be to achieve the standards. All of the standards apply to all grade-bands, but each grade-band put a relatively different emphasis on the particular standards.

Although NCTM has provided standards to help guide the mathematic curriculum, states are providing inconsistent mathematical programs resulting in a math curriculum that is “a mile wide and an inch deep” (Schmidt, McKnight, & Raizen, 1997, p. 2, as cited in NCTM, 2006, p. 3). In order to assist teachers in identifying the most critical content when implementing the standards, NCTM issued a document called *Curriculum Focal Points* in 2006. The document established which mathematical topics are imperative to cover in pre-kindergarten through grade eight. *Curriculum Focal Points* enumerates the primary mathematical concepts for each grade level. The document asserted that focal points should be addressed in contexts and should emphasize the process standards. In early childhood, the focal points are number sense and operations, geometry, and measurement (NCTM, 2006). Number sense in early childhood centers on developing an understanding of the meaning of whole numbers, including one-to-one correspondence, counting, cardinality and comparison. Geometry in early childhood focuses on recognizing spatial relationships and identifying shapes. Measurement in early childhood concentrates on using terms such as *more* or *less* to identify and compare measurable attributes (NCTM, 2006).

Illinois Learning Standards. Even though NCTM's Standards are recommended as a guideline for mathematics curricula, states are providing their own standards, and Illinois is no exception. Current learning standards for mathematics in Illinois were adopted in 1997. They define in detail what students need to know and to be able to do as a result of their schooling. *The Illinois Learning Standards for Mathematics* was developed by Illinois teachers based on their work and experience to match the classroom realities in Illinois public schools. The assumption conveyed in the standards is that mathematics is much more than a collection of concepts and skills, rather it is a way of approaching new challenges through investigating, reasoning, visualizing, and problem solving with the goal of communicating the relationships observed and problems solved to others. The standards clearly denote the skills: applications of learning, solving problems, communications, applying technology, working on teams and making connections, which students need to accomplish as a result of engaging in mathematics learning experiences.

The distinctive feature of the Illinois Learning Standards for Mathematics is the hierarchy in the standards. Five general goals are defined that cut across the school mathematics curriculum. The first goal is to demonstrate and apply a knowledge and sense of numbers, including numeration and operations (addition, subtraction, multiplication, division), patterns, ratios, and proportions. The second goal is to estimate, make, and use measurements of objects, quantities, and relationships and to determine acceptable levels of accuracy. The third is to use algebraic and analytical methods to identify and describe patterns and relationships in data, solve problems, and predict results. The fourth goal is to use geometric methods to analyze, categorize, and draw conclusions about points, lines, planes, and space. The last one is to collect, organize, and analyze data using statistical methods; predict results; and interpret uncertainty using

concepts of probability. All of these general goals apply to all grade levels. Under these five general goals, three-to-four learning standards are listed, and each standard has several benchmarks under it. These standards and benchmarks are specific to each grade level. The Illinois Learning Standards *for Mathematics* also provides descriptors (simple examples) to help understanding of the goals, standards, and benchmarks more clearly.

Common Core State Standards. In order to provide a common understanding of what students are expected to learn, the Council of Chief State School Officers (CCSSO) and the National Governors Association Center for Best Practices (NGA Center) developed *The Common Core State Standards*. When they decided to develop the standards, they incorporated the most effective models across the U.S. as well as from other countries. The standards were developed in collaboration with teachers, school administrators, and experts. CCSSO and NGA Center received feedback on the initial draft from a range of groups, such as teachers, postsecondary educators, civil rights groups, English language learners, and students with disabilities. Following the initial round of feedback, the draft standards were opened for public comment. Based on this feedback, the organization finalized the standards and presented the final Common Core State Standards documents in 2010.

The most distinctive characteristic of the document is that it aims to provide consistent standards along with appropriate benchmarks for all students regardless of where they live, that is, nationally agreed upon learning standards. The standards define the knowledge and skills students should have within their K-12 education. It includes standards for mathematical practice as well as for mathematical content. The first standard for mathematical practice is to make sense of problems and persevere in solving them. The second is to reason abstractly and quantitatively. The third standard is to construct viable arguments and critique the reasoning of others. The

fourth is to model with mathematics, applying mathematical knowledge to solve problems in daily life. The fifth is to use appropriate tools strategically when solving a mathematical problem. The sixth standard is to communicate precisely to others when explaining reasoning. The seventh is to look for and make use of patterns or structure. The last is to look for and express regularity in repeated reasoning. Along with these eight standards for mathematical practice, standards for mathematical content based on the following mathematical topics are established: (a) counting and cardinality, (b) operations and algebraic thinking, (c) number and operations in base 10, (d) measurement and data, (e) geometry, (f) number and operations in fractions, (g) ratios and proportional relationships, (h) number system, (i) expressions and equations, (j) statistics and probability, and (k) algebra. These content standards are grade-specific, and the mathematical topics dealt with vary for each grade level. For kindergarten, only the first five content areas are listed.

Because the Common Core State Standards aim to disseminate the nationally agreed upon learning standards, CCSSO and NGA Center encourage states to adopt the standards. Currently, 48 states, two territories (U.S. Virgin Islands and Northern Mariana Islands), and the District of Columbia have adopted these standards and are in the process of replacing their own state learning standards with the Common Core State Standards.

In spite of its lofty ambition, flaws in the Common Core State Standards have been pointed out, and researchers have started to question its effectiveness. The standards were completed quickly, in approximately one year, in response to President Obama's educational reform agenda—making all high school graduates college- and career-ready. The short time allotted for the development process raised several concerns about the development, content, and use of the 500 pages of standards and supporting documents (Mathis, 2010).

Facing the threat of reduced Title I federal funds as well as the application deadline for *Race to the Top* funds, states had to adopt the standards hurriedly. The federal pressure on states to adopt the standards within two months prevented a thoughtful and comprehensive review. Many researchers claimed that standards of this scale, complexity, and importance should be field-tested and revised for validity, focus, and effects before being implemented (Mathis, 2010, p. 16).

Second, different from other subject-matter standards developed by area specialists working in universities and schools, the level of input from school-based practitioners in the Common Core State Standards appears to have been minimal (Mathis, 2010). Of the 65 people in “Achieve” work groups, who took charge in developing the standards, there was only one K-12 educator (p. 5). Practitioners and subject matter experts complained that they were excluded from the development process, which may have produced a tendentious result.

Lastly, the major educational professional organizations, such as the American Association of School Administrators, National Association of State Boards of Education, National Education Association, American Federation of Teachers, and National School Boards Association, voiced serious concerns about the standards and conditioned their support on the provision of adequate resources and professional development as well as on active involvement by practitioners (Mathis, 2010, p. 11).

Math and English teacher associations worried about the content of the standards. The National Council of Teachers of Mathematics (NCTM) reported that the curriculum is not properly articulated from one grade to the next (Mathis, 2010). NCTM also objected to the lack of focus on mathematical understanding and to the short-changing of technology, statistics, and data analysis (Usiskin, 2011). According to NCTM, fractions get too much attention. The

National Council of Teachers of English (NCTE) also noted that the standards are too narrow and inappropriately prescriptive and that grade-to-grade articulation is deficient. They especially criticized the standards' concentration on lower-order rote learning. Overall, both associations worried that the standards are inadequate and fall short of the mark.

Illinois Early Learning and Development Standards. Replacing the Illinois Learning Standards for Mathematics with the Common Core State Standards was completed at the end of the 2011-2012 school year in Illinois. The Common Core State Standards did not include pre-kindergarten in their target grade level/band. *The Illinois Early Learning and Development Standards*, previously the Illinois Early Learning Standards, were introduced in draft form in January, 2013. These standards target children from three to kindergarten.

The composition of the Illinois Early Learning and Development Standards is almost the same as the Illinois Learning Standards. Under the five general goals, three-to-four learning standards are stated. Each standard includes several benchmarks and example performance descriptors. The standards are aligned to the Illinois Kindergarten Standards and the Common Core State Standards for Kindergarten. Even though the standards represent an alignment with the K-12 standards, they appear to be a developmentally appropriate set of goals and objectives for young children.

Early childhood professionals from public and private schools, Head Start, colleges, and community based early care and learning programs will review and critique the revised draft document. Because the standards are still a draft, the recommendations from early childhood professionals will be considered making revisions of the standards.

Summary

The history of mathematics education for young children demonstrates that opportunities to learn mathematics have been provided to children from the beginning even though the focus of programs varied. Educators who misunderstood children's interest in and ability to engage in mathematical thinking, however, frequently weakened these programs (Saracho & Spodek, 2008). The past century has seen a good deal of debate about the origins and early development of mathematical knowledge between researchers with optimistic views of children's mathematical competence and those with pessimistic views. Fortunately, the recent trend in mathematics education for young children moves away from this tedious disputation and seeks the best way to help children construct knowledge of mathematics in the everyday environment.

In spite of this effort, most teachers still have difficulty providing effective mathematics teaching for children. NCTM's Standards, State's Learning Standards, and the Common Core State Standards are recommended as a guideline for teachers, but these standards seem to fail in advancing this goal. Instead, a deep understanding of the three major elements that combine to generate PCK for mathematics; well-explored *what* and *how* of mathematics teaching as applied to young children (*who*) provides a possible viable solution for teachers to effectively teach mathematics to young children.

Chapter 3

Methodology

This chapter begins with a description of the research methods I utilized. I describe the participants, instruments of the study, and data collection processes. Then I move on to data analysis and discussing the strength of the data. Finally, I discuss the challenges and difficulties I faced in the course of the study.

Research Methods

This study investigated four professional groups' views on mathematics education for young children. These groups included (a) mathematicians, (b) teacher educators, (c) cognitive/developmental psychologists who study young children's mathematics learning, and (d) preschool and kindergarten teachers. I selected these four groups because they are specialists in each component of mathematical PCK, which is the knowledge base needed for teachers to effectively teach young children mathematics. The general methodology was structured and semi-structured interviews.

Krathwohl (1998) referred to the interview as “a desire to tap an internal process” (p. 286). “It boils down to picking the brains of a handful of people who know more about a subject than you do. You try to find out how to reach the people you want to reach in your study, what sort of questions you should be asking them, what sort of language you should be using” (Pisani, 2008, p. 49). Spradley (1979) defined interviews as a “series of friendly conversations into which the researcher slowly introduces new elements to assist informants to respond as informants” (p. 58). Interviewing, done well, is an efficient tool to obtain insiders' perceptions, feelings, beliefs, and stories. Interviewing also allows the researcher to effectively collect large amounts of data regarding the research topic (Fontana & Frey, 2000; Kvale, 1996).

Interviews helped me gather in-depth knowledge about four professional groups' views on how best to teach mathematics to young children. As Figure 4 shows, my study had three stages. First, a pilot study was conducted with four participants to improve my interview questions. Based on the feedback on my interview questions from the pilot participants, I revised the interview questions. The second stage was interviewing 32 participants to identify their general ideas about mathematics education for young. The initial findings from the first round of interviews gave me some insights about which directions I should pursue in further interviews. The third stage was interviewing 12 participants from the original 32 to explore their ideas in depth. The in-depth interviews with the participants expanded the scope of my inquiry and clarified my findings. They allowed me to uncover different layers of thoughts about how best to teach mathematics to young children.

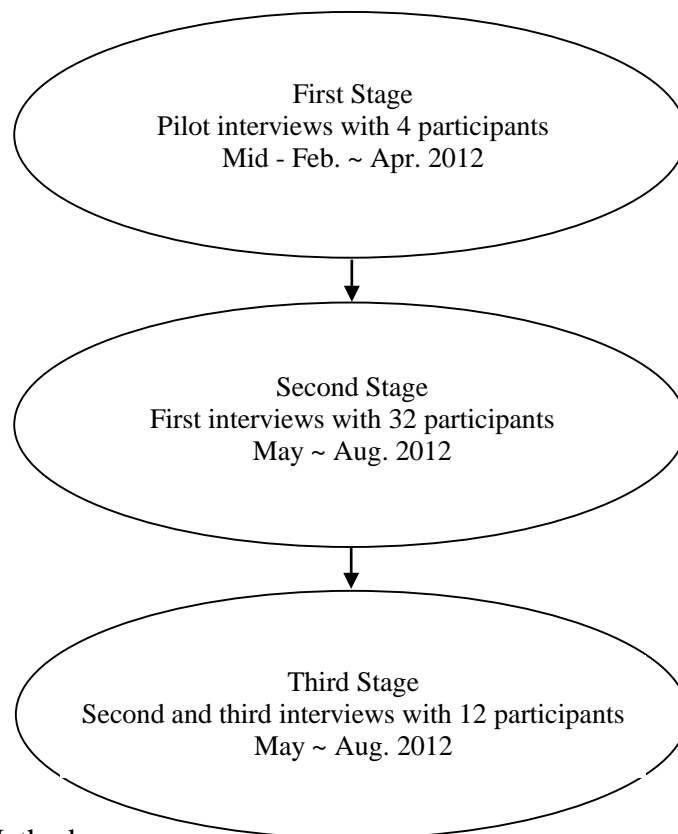


Figure 4. Research Methods

In this research all interviews were audio-recorded and transcribed. All interviews except one were conducted face-to-face in a place chosen by the subject. I conducted one set of interviews over the phone with one subject who said she was too busy to meet in person.

Participants. For the first interviews, I picked eight participants from each of the four groups for a total of 32. The mathematicians are all faculty members at colleges and universities. The teacher educators are all faculty members at universities. The psychologists all study young children's math learning and are faculty members at universities. The teachers at kindergartens and preschools were recruited. All research participants were recruited in the state of Illinois.

From these eight participants, I targeted three, for a total of 12, to continue with the additional interviews based on (a) their desire to continue and (b) their experiences. The mathematicians are all faculty members in math departments at colleges and universities. Two of the teacher educators are in college of education while the third is in a math department. The psychologists all study young children's math learning. One cognitive psychologist is in a college of education; the other two, a cognitive psychologist and a developmental psychologist are in psychology departments. Teachers included one public kindergarten teacher, one private kindergarten teacher, and one private preschool teacher. The following table summarizes each participant's demographics as well as the interview dates.

Table 2

Participant Information

	Mathematician 1	Mathematician 2	Mathematician 3
<u>Job title</u>	Professor in math department	Professor in math department	Assistant professor in math department
<u>Teaching grade level</u>	Undergraduate, Graduate	Undergraduate, Graduate	Undergraduate
<u>Educational background</u>	PhD	PhD	Master's
<u>Working experience</u>	45 years	22 years	20 years
<u>Gender</u>	Male	Male	Female
<u>Age</u>	73	50	42
<u>Ethnicity</u>	European American	European American	European American
<u>Interview 1</u>	May 17, 2012	May 18, 2012	May 21, 2012
<u>Interview 2</u>	June 3, 2012	June 2, 2012	May 28, 2012
<u>Interview 3</u>	June 8, 2012	June 7, 2012	June 4, 2012

	Teacher Educator 1	Teacher Educator 2	Teacher Educator 3
<u>Job title</u>	Clinical assistant professor in college of education	Instructor in math department	Director of online teaching program in college of education
<u>Teaching grade level</u>	Undergraduate (some master's students)	Undergraduate	Undergraduate
<u>Educational background</u>	Master's	Master's	PhD
<u>Working experience</u>	10 years in high school 5 years at university	26 years in K-12 7 years at university	27 years
<u>Gender</u>	Male	Female	Male
<u>Age</u>	36	62	50
<u>Ethnicity</u>	European American	European American	European American
<u>Interview 1</u>	May 15, 2012	May 16, 2012	May 17, 2012
<u>Interview 2</u>	May 29, 2012	May 30, 2012	June 4, 2012
<u>Interview 3</u>	June 5, 2012	June 14, 2012	June 14, 2012

Table 2 (*continued*)

	Psychologist 1	Psychologist 2	Psychologist 3
<u>Job title</u>	Professor emeritus in college of education	Professor emeritus in psychology department	Associate professor in psychology department
<u>Teaching grade level</u>	Undergraduate	Undergraduate	Undergraduate
<u>Educational background</u>	PhD	PhD	PhD
<u>Working experience</u>	34 years	38 years	6 years
<u>Gender</u>	Male	Male	Female
<u>Age</u>	65	65	34
<u>Ethnicity</u>	European American	European American	European American
<u>Interview 1</u>	May 18, 2012	May 22, 2012	July 31, 2012
<u>Interview 2</u>	June 11, 2012	June 5, 2012	August 7, 2012
<u>Interview 3</u>	August 2, 2012	June 19, 2012	August 10, 2012

	Teacher 1	Teacher 2	Teachers 3
<u>Job title</u>	Head teacher	Head teacher	Head teacher
<u>Teaching grade level</u>	Kindergarten (public)	Kindergarten (private)	Preschool (private)
<u>Educational background</u>	Bachelor's	Master's	Bachelor's
<u>Working experience</u>	42 years	22 years	34 years
<u>Gender</u>	Female	Female	Female
<u>Age</u>	63	60	57
<u>Ethnicity</u>	European American	European American	Puerto Rican
<u>Interview 1</u>	May 1, 2012	May 3, 2012	May 13, 2012
<u>Interview 2</u>	May 8, 2012	June 13, 2012	May 18, 2012
<u>Interview 3</u>	May 16, 2012	June 20, 2012	June 8, 2012

Instruments. The instruments in this study were the interview protocols for each group. The interviews had two parts: (a) general information questions and (b) interview questions. Before developing interview questions, I read James Spradley's (1979) *The Ethnographic Interview*. I developed and revised my questions based on his suggestions. The interview questions were piloted from mid-February to April 2012 with four participants (one mathematician, one teacher educator, one cognitive psychologist, and one teacher). I received

feedback on my interview questions from the pilot participants and finalized the interview questions based on those comments.

The feedback included (a) adding questions to get general background information for participants, (b) making questions more specific, (c) starting with a question on a topic that participants are very familiar with, and (d) excluding questions about learning standards (especially for mathematicians because they are not familiar with the standards).

From the pilot interviews, I also made some tentative observations. First, mathematicians see a big picture of what math is and what purpose doing math serves. They emphasize the logical thinking embedded in math rather than mathematical concepts themselves. The second is that teacher educators worry about the number of courses that pre-service teachers are required to take for certification. They think that taking 2 to 3 courses is not enough to prepare pre-service teachers to teach math well. The third observation is that cognitive psychologists have a deep understanding of children's mathematical thinking, but this knowledge tends to be quite domain-specific (e.g., number sense and operations or geometry). Also, they have few opportunities to share their knowledge with teachers. Finally, teachers do not have much difficulty in providing math activities, but they struggle with implementing those activities. These initial findings gave me some insight about which directions I should pursue in further interviews.

I conducted three sets of interviews, each lasting 30 minutes. The first interviews were more structured; the next two, less so. In the first round of interviews, I asked four predetermined questions (see Appendix C2, C3, C4, and C5) with possible probes depending on the answers. The questions varied by group. The interview questions focused on what participants think about teaching math to young children: (a) what is children's mathematical understanding? (b) What

mathematical concepts should be taught to young children, and (c) which teaching strategies promote children's mathematical competency?

Data collection. The pilot interviews were conducted from mid-February to April, 2012, and the first round of interviews and the in-depth interviews were conducted from May to August, 2012. The first round of interviews aimed to identify (a) mathematicians', (b) teacher educators', (c) cognitive/developmental psychologists', and (d) teachers' general ideas about mathematics education for young children. Later interviews were intended to explore what the participants thought about math education for young children in depth. The participant recruiting process for each stage was the same.

To recruit mathematicians, teacher educators, and cognitive/developmental psychologists, I did the following: From Google and from a review of literature as well as from other faculty members' referrals, I obtained contact information for people interested in math education for young children. I emailed them a description of my research and asked them whether they were interested in participating. People who were interested contacted me directly.

To recruit kindergarten teachers, I worked through the Office of School-University Research Relations (OSURR) to identify schools and teachers in the Champaign County area. OSURR contacted 6 elementary schools in Urbana and 11 elementary schools in Champaign and distributed the summary of my study to the principals. In case I could not find teachers in Urbana-Champaign, I made a list of 41 elementary schools in 14 other school districts in Champaign County, which I gave to OSURR. Once I got permission from OSURR, I sent an e-mail outlining my research to the principals of these schools. The principals forwarded the e-mail to teachers. Teachers contacted me directly via e-mail if they wanted to participate.

To recruit preschool teachers, I contacted directors of 23 preschools directly. Because most preschools are private, I did not have to go through OSURR. I shared a brief research proposal and an example of consent letters. I described how I would maintain confidentiality. With the directors' approval, I left a flyer outlining my research for the teachers to read and consider. Teachers contacted me directly via e-mail if they wanted to participate.

After recruiting participants, I conducted three sets of interviews with the participants from May to August, 2012. The purpose of the first set of interviews was to identify the participants' general ideas about mathematics education for young children and to find participants for the second and third interviews.

I conducted the second and third sets of interviews after closely analyzing the first set of interviews. As Krathwohl (1998) notes, data from early interviews can evolve into questions for later interviews. The second and third interviews were tailored based on analysis of the participants' responses in the first interview. Before beginning the second and third interviews, the participants and I spoke informally about their responses on the first interview to remind them of what they had talked about. I also made fieldnotes during the interviews. After the interviews I made an interview log that included my reflections and was organized by interview date. As Graue and Walsh (1998) asserted, I found writing fieldnotes to be a very important component of my data collection.

As described earlier, for the first interviews, I picked eight participants from each of the four groups. From these, I targeted three from each group for additional interviews. Selecting participants for the second and third interviews was a sensitive task because not all participants were equally good informants. Selection criteria included demographics, program affiliation (in the case of teachers, whether they had been involved in one or more programs), a willingness to

share additional insights about their experience, and strong opinions about what a good math education for young children is. I attempted to have a broad and representative range of interviewees. Table 3 summarizes the contact information resources, the criteria for including/excluding the participants, and the research site for each professional group.

Table 3

Participant Contact Information Resources

Participants	Contact Information Resources	Criteria for the Participants	Site
Mathematicians	<ul style="list-style-type: none"> - Google - Literature review - Other faculty members' referrals 	- Faculty members who teach math courses for undergraduates and graduates	The state of Illinois
Teacher Educators		- Faculty members who teach math education courses for undergraduates	
Cognitive/ Developmental Psychologists		- Cognitive/developmental psychologists who study young children's math learning	
Teachers	<ul style="list-style-type: none"> - Google - The yellow pages - Previous researchers - Other teachers' referrals - Parents' referrals - My working experience - OSURR 	- Preschool or kindergarten teachers who are interested in teaching math	

All subjects signed an informed consent form and were provided with a copy. With their consent, I audio-recorded all interviews, and audio files were coded with a unique ID number. Both handwritten notes and audio-taped interviews were transcribed using pseudonyms. An identifying key that linked the subjects' names to their pseudonyms was maintained in a file separate from the data collection file on a password-locked computer. In audio-recording files, all personal information such as names and school names were eliminated.

Data analysis. According to Marshall and Rossman (1999), data analysis is the process of bringing order, structure, and interpretation to the mass of collected data (p. 150). The interviews were transcribed and organized as described below.

To interpret and analyze the data, I used both top-down and bottom-up codes. First, the transcription was categorized and coded by themes. I created themes based on the research questions (first level) and participants' responses (second level). Because initial first-level coding was based on the PCK model, three top-down categories had already been selected. These three categories were (a) views related to children's mathematical understanding, (b) content-related views (what to teach), and (c) pedagogy-related views (how to teach).

After coding the data into these three main categories, I generated additional themes emerging from the data that did not fit the initial categories. Based on these, I developed the second level of coding. At this point in the analysis, I followed Creswell's (2002) suggestions: I organized and documented data according to the type of information source, and then read through the data "to obtain a general sense of the information and to reflect on its overall meaning" (p. 191). During analysis, I moved back and forth between the raw and the analyzed data. Through this process, I broke down the data to examine details and then attempted to bring these details to form a larger picture.

Data warrant. To ensure the credibility of the data, I continued to return to my research questions and the purpose of my study. I also frequently revisited the original data to maintain a sense of the larger context as I answered the research questions. To further establish credibility, I did member checks by frequently sharing what I had learned with the interviewees and by referring to the interviewees' previous interview responses when conducting the second and third sets of interviews. To check the interview transcription for accuracy, two editors read all

transcriptions thoroughly. If I was unsure what someone was saying on the recording, I asked my editors to listen to the recording with me.

Challenges and Difficulties

My initial difficulty was recruiting participants, especially groups other than teachers. Even though I started the recruiting process in mid-February when I started my pilot study, the response rate to my recruiting e-mail, from university faculty, then in the middle of the spring semester, was low. Fortunately, my actual study began in May when spring semester finished. Some had agreed to participate during the semester, and others asked me to contact them again after the semester ended. In the middle of May, I resent reminder e-mails to these people. By doing so, I was able to recruit enough participants.

Achieving a diverse sample of participants was another challenge. Although I tried to interview non-European-Americans, most participants were European Americans. The age range of participants was limited. Many participants were middle-aged or older. Perhaps senior faculty members have more time during the summer than junior faculty.

In designing interview questions, I had to consider the fact that participants may tend to give “expected” answers instead of expressing their frank opinions about optimal math education for young children. In interview questions, I used the phrase “math education” instead of “optimal math education,” even though I was seeking participants’ opinions about what good math education is. Before conducting the interviews, I also mentioned that the purpose of the research was to hear different perspectives and their sources, not to evaluate responses.

In conducting interviews, developing rapport with the participants was a difficult task for me. In interviews, establishing a trusting relationship with the participant is important because such a relationship allows both the researcher and the participants to share opinions openly. The

relationship also allows the interviewer to gain a deeper and more meaningful understanding of the participant's responses. To build rapport with the participants, I briefly explained my background and why I became interested in this topic. I shared my experiences with math education. I also tried to assure the participants that my study did not seek to evaluate their thoughts or opinions, but to discover and understand their perspectives. The participants showed a strong interest in my research, and most of them participated very actively. Spradley (1980) emphasizes expressing interest in and ignorance of what informants address as a way of encouraging active involvement in interviews. I attempted to show interest in my participants' stories and to let them know that they were presenting perspectives that I had limited knowledge of. My goal was to encourage them to elaborate.

The fieldwork was in many ways an ongoing decision-making process that exposed and continually evolved beyond my own subjectivity. The data I wanted to collect, and the ways I collected it, were necessarily constrained by my own perspectives. Generating early and detailed plans, however, helped me to identify possible sources of subjectivity and possible bias. Monitoring the data collection processes and organizing data after each fieldwork session also provided me with an opportunity to improve my research (Glesne & Peshkin, 1992). Discussing my research with my colleagues helped me think more clearly about the issues that arose during the study. Getting feedback from other people allowed me to consider different points of view related to the topic during the analytic process and to increase the validity of my research (Wolcott, 1990).

Chapter 4

Mathematicians: Math Education for Young Children

To learn what mathematicians think about math education for young children, I started out with eight mathematicians. The tentative findings of the first round interviews with the eight mathematicians also gave me some insight about which directions I should pursue in further interviews with the three mathematicians. All mathematicians talked about the aspects of mathematics that appealed to them and explained how these characteristics of math helped them like math. They also talked about the people who influenced the development of their careers. Even though they study higher mathematics, which people often think too abstract and alienated from the real world, mathematicians do not ignore the connection between mathematics and daily lives.

From these eight, I selected three based on their willingness to continue and their strong opinions about what good math education for young children is. They are all faculty members in math departments at their respective colleges and universities. I interviewed all mathematicians face to face in their offices.

I found the first mathematician most interesting. He tries to decipher the world with math concepts. For example, when he explained his academic interest, which is number theory, he went back to his birthday. He was born on a prime number day in a prime number month, and if you take the product of those two numbers and add 1900, that is the year he was born. He also said that when being asked why he is interested in number theory, he always mentions that his grandmother decided to have a prime number of children. His grandparents chose the prime number 17, and his mother was the 15th of 17 children.

The second mathematician is a bit unusual in that he didn't decide to be a professional

mathematician until he was 26. He was in graduate school in math, but was using his PhD studies in math as a way to get TA support while he was studying for tests to get into medical school. He, however, told me that he had always liked math, even before he decided to become a professional mathematician. He is very interested in math education and frequently interacts with faculty members in math education.

The third mathematician is an assistant professor who worked as a secondary school teacher. She has three daughters, and the youngest was attending kindergarten during the interviews. She provided multiple layers of perspectives, as a mathematician, a teacher, and a parent.

This chapter has four sections. In the first, I explore mathematicians' thoughts on children's mathematical understanding. I then look at what mathematics content mathematicians think should be taught to young children. In the third section, I examine what mathematicians think about how to teach mathematics to young children. The final section examines how mathematicians define math and math learning.

How Children Learn Mathematics

All three mathematicians agreed that children have math ability, but they pointed out huge individual difference in children's mathematical competence.

I think there's going to be a lot of children who are ready for math and you do them a great disservice if you don't feed that. But at the same time, there are children who are probably not ready and somehow they sometimes get the message that they can't do it because they weren't ready when the other ones were. So I think the answer is going to be individually dependent. I think there are a lot of students who are very ready and you almost hurt them, if you don't allow them to run because they can get bored. But the ones who aren't ready, it doesn't mean they won't be ready but they are just not ready yet. (M2, May 18, 2012)

I do believe that everybody has at least a certain degree of capability with mathematics. Obviously not everyone is going to get a PhD in math and so forth, but I do think that

everybody has the ability to understand the basics and to go to a certain level, and we need to start promoting that better. (M3, May 21, 2012)

I think children are all capable of doing math, but it varies from child to child. They might have the ability to think about mathematics quite early, but somehow you have to draw that out and make it interesting for them and want them to think about these things. So it all depends on the child and their interest. (M1, May 17, 2012)

First of all I want to be careful about the word “competent” because I think they’re all competent and capable of learning. But especially if you’re talking about it at the young age preschool, kindergarten, where they’re just entering sort of the formal education area, they’re coming with very different backgrounds. (M3, May 28, 2012)

Holding generally optimistic views about children’s ability in math, the mathematicians criticized ideas, wide spread in our society, that underestimate children’ ability in math. One mathematician explicitly condemned the idea that only children with special math talent can do math.

Here we put mathematics in a separate category that you have to have special talent to learn mathematics. I mean it is true that people are not created equal with the abilities they’re born with. But on the other hand, I think, especially for my associates in Japan, it’s considered a subject just like history. In other words, you don’t necessarily have to have a special ability, and people are not categorized into mathematically oriented or not mathematically oriented. In Korea, I think it’s closer to Japan than the U.S. in that respect. I think part of the fact that students don’t want to engage in these activities or have negative attitudes, probably is cultural. It comes from the culture around them. They’ve gained these negative attitudes from teachers, other students, or parents, or just the culture in general. It has to be realized that it’s a cultural problem. So how do you get around this? How do you remedy this? That’s a tough question. (M1, June 8, 2012)

Another mathematician criticized the idea that children can learn math concepts only when concrete objects are provided. She asserted that children have ability to understand abstract math concepts.

I guess I wouldn’t say that you shouldn’t do concrete first and move into abstract, even though most teachers think that way and limit children’s ability. I just think you have to be open to all the possibilities and go in with what the child can provide. For example, I have three daughters. The youngest is in kindergarten, and from time to time all of them have asked me for help with math. Usually, I don’t start out with manipulatives or concrete objects to explain. However, if my words or the questions I’m using don’t help them see it, then I’ll go get the concrete and see if that makes a difference. So,

sometimes they can do without. They can use a frame of reference that's in their mind as opposed to right in front of them in the world, and sometimes that frame of reference is too hard for them to create, and then we create it out of manipulatives. So, I think they can do some of both. (M3, May 21, 2012)

Mathematics Content (What to Teach)

The mathematicians mentioned a wide range of content areas, such as number sense and operations (comparing), algebraic thinking in terms of finding patterns, geometry, measurement, data analysis and probability.

Number is really basic to everything, so we should teach it early. Another thing I think important is inequalities because we use them all the time. You might go into a store and you see two items but they're priced differently and it's not clear which is cheapest. You want the one that's cheapest. You might have to do some calculations or solve another problem or subsidiary problem to find which item is cheapest. I would probably add inequalities in numbers. (M1, June 8, 2012)

I think it's nice to introduce patterns to children because that's what mathematicians do—they look for patterns. And you can find patterns in addition obviously and subtracting, but you can also find patterns in algebra and geometry. You can find patterns in data analysis, which is how we end up with something that looks like the normal curve. Patterns. That's where the number e (an irrational number well known for its patterns – the first few digits are 2.7182818284590452353602874713527– called Euler's number after Leonhard Euler) comes from. That's a lot of what math is – recognizing patterns. (M3, June 4, 2012)

Here's an example from one of my former PhD students, who was teaching a remedial class. He said on an exam, "If you could get a fourteen-inch pizza or two seven inch pizzas at the same price, which would you choose?" He said the majority of the class said two seven inch pizzas. You can give problems like that to show the importance of working with numbers and finding geometric areas. You make your teaching interesting by giving some sort of examples like that. And you could cook up examples just in terms of measurement and length or something as well. Like, traveling distances by different routes along rectangles or circles or something. They have to use some geometrical knowledge to solve a problem. You're going to walk to a store by this route or this route. Which is the most time saving route? Maybe we could tie this up with some geometry where they have to calculate the perimeter or half the perimeter of a rectangle or something like this. (M1, June 8, 2012)

The thing is you can really find math anywhere. So, in certain situations, it's natural to talk about probability. In other situations, it's natural to talk about measurement. (M3, May 28, 2012)

The mathematicians, however, said that the number of content areas introduced to students is not important, rather how thoroughly students understand each content area is important.

I mean, spread too many concepts, and you don't learn any of them particularly well. So, all this rush for more and more concepts, it's like teaching poetry to the person who is still trying to sound out the words, it doesn't work. Poetry's a way higher order concept. You really have to read smoothly. And not be sounding out the words, half the time getting it not quite right. And so I have students who basically are doing algebra at the speed and comprehension of sounding out the words. And they say correctly, "If you give me enough time, I can solve the problem." Just like that student who sounds out the words; if you give them enough time, they will read the sentence, but they can't read well enough to do poetry. It's not going to happen. (M2, June 2, 2012)

The mathematicians also emphasized the importance of sequence in introducing content areas in mathematics. They agreed that number should always come first because it is the basis of other mathematical concepts. Number concepts, counting, ordering, and the four basic operations should be presented in order.

Having a basic number concept, that the number 5 represents 5 objects or something like that and counting, order—things go in a certain order. That I think would have to come first. And then the four basic operations—addition and subtraction, really everything else comes from those two because multiplication is repeated addition, you just keep adding over and over again and division is repeated subtracted, you keep subtracting over and over again. So when you divide by 7, you keep subtracting 7 until you can't subtract 7 anymore. So if they can have a strong sense of addition and subtraction, then that will allow them to have a strong sense of multiplication and division. And everything comes from there. That's the basis for algebra, that's the basis for everything you do past that. I think the basic operations have to be pretty near the beginning, but after that I think you can go a lot of different directions based on interest. (M3, June 4, 2012)

Following number, the agreement was that algebraic thinking, geometry, and measurement should come in order. Data analysis should come later because it needs more sophistication and maturity.

As I said, algebra, geometry and measurements depend so much on number, so those should come after number. Data analysis requires more sophistication and maturity. I think it is something that comes much later. Organizing, it is often very difficult and takes a sort of expertise. My epidemiologist daughter is always telling me how a lot of

the data that she receives is not organized properly. That's at a more sophisticated level, but it's very important in any level. Again, this takes maturity and experience. It's very important. But I would say this is certainly much later in grade school than the other four activities, all of which I think are important. (M1, June 8, 2012)

One mathematician said that measurement is relatively less important than other content areas. He also confirmed the idea that data analysis should be introduced later.

Number, algebra and geometry certainly require much more time and effort than measurement. I think measurement is important because it's useful and it's a verification so to speak that number, algebra, geometry are important. So I agree that the first three are the most important. And measurement is important, but it shouldn't require a core of the curriculum. And I think data analysis—really in order to do something meaningful, it has to be later on when the student is more mature. (M2, June 7, 2012)

Pedagogy (How to Teach)

With respect to how to teach mathematics to young children, the mathematicians ranged widely. Their thoughts fit into three categories: (a) various ways to help students develop a positive attitude toward math, (b) teaching strategies, and (c) support for teachers.

A positive attitude. When asked how they got interested in the field of mathematics, all the mathematicians replied that they always liked math and had a positive attitude toward it.

It's just something I always liked and was relatively good at. I remember that when I was in grade school, we used to take exams which qualified us and said where we're at. In mathematics, they said I was at a college level even in grade school. So I knew it was something I was good at. (M1, May 17, 2012)

When I was younger, I was always good at mathematics. I liked working with numbers, and I used to compute batting averages. So, I always enjoyed mathematics, and I was relatively good at it. (M1, May 17, 2012)

Describing their attitudes toward math, they often mentioned teachers back in schools who continuously motivated them and fostered their interest in math.

I liked my high school math teacher very much, and she was challenging. Out of our graduating class, particularly those students who had her, there was a high percentage of students who went in to math and math-related careers. I don't think everyone liked her, but we certainly liked her and we learned a lot. She taught for understanding, not just for memorization and being able to do well on standardized tests and things like that. She

really helped us understand mathematics, which really gave us a lot of opportunities. (M3, May 21, 2012)

When I went to college, I decided I really liked mathematics better. I had a wonderful mathematics teacher for many of my classes in this small liberal arts college. He was a wonderful teacher, and he was the kind of person I sort of admired and I'd like to be. His explanations were always very clear, so I never had any trouble doing the homework problems. I can't remember any incidents where I went home and said to myself, "I didn't understand this." So his explanations were very clear, so I never had any difficulty understanding the material. (M1, June 3, 2012)

When I was in college, I had a professor with a very sparkling personality. He was writing a new research book, so he was giving a course at a graduate level about this new emerging subject as he was working through it to write this book. And he, in particular, had open questions because it was a new subject, so that was very interesting to me. The other thing was that the subject turned out to be very interesting, fortuitous timing. He was very enthusiastic when I would go to him with questions, or maybe the fact that I was paying particular attention I was the only student who was actually paying close attention to his course after the first few weeks, but he was extremely encouraging. And there was another visiting faculty, and he was just brilliant and very positive, and he spent a lot of hours just talking with me about math, and he and the professor whom I mentioned earlier were sharing a house, so this visiting faculty would talk to me about the ideas of the class as well, so it made me feel very much like an adult to be talking to these two very serious senior mathematicians. And they treated me like I was a serious student, and they were both very colorful, full-of-stories kind of people. So it was just a lot of fun for me, and the course was at where my skills had just reached a point, they were right at the edge of what I knew, but I understood everything that they were talking about, I was able to work it all out. (M2, June 2, 2012)

The mathematicians also talked about the aspects of mathematics that appealed to them and explained how these characteristics of math helped them like math.

I've always found math interesting, even when it was difficult. And I find the challenge and being able to reach that final answer and the solution to the problem, I find that interesting and satisfying. I'm a person who likes puzzles, whether they're logic puzzles or even reading mysteries, trying to solve that puzzle. I like the neatness of it, the logic, everything can come to a sort of a nice conclusion. It's somewhat open-ended, and you have some opportunities to do different things, and I like that aspect. But ultimately there's one goal or one solution or one right answer. It's really broad, so there's a wide range of things you can study in it. So I always can find something new that's interesting in it. And it's everywhere. And it's really what makes so much of our world keep going—new technology, economy, all kinds of things are so dependent on understanding mathematics that I find it to be really powerful. (M3, May 21, 2012)

When I was a child, we had to learn English and spelling. The teacher would give us rules for spelling and then start to give us all the counter examples. Like *i* before *e* except after *c* and then give you a whole list of common words like *seize* that don't obey those rules. Or you would take a science class, and they would tell you something that's true, but then you would learn more science and find out that that was overly simplified and it was actually not true. Math was the only thing that when you learn certain things they remain fixed and true. So that was the first appeal of math. I didn't like people changing the rules. They would say, "Learn this." And I would learn it, and a year later that wouldn't apply. It was very frustrating. Or it'd be subject to the teacher's interpretation. Is this a good way to write a paragraph? This teacher loves it. The next teacher doesn't like it. I hate that. (M2, May 18, 2012)

In addition to sharing their experiences, the mathematicians directly suggested several ways to help students feel good about math. First, they discussed teacher's or parents' modeling as important for passing on a positive attitude toward math. One mathematician talked about how a teacher's negative attitude toward math can detrimentally affect students' attitude to math.

I believe the teacher should be enthusiastic about teaching math because that really conveys something to the student. There's a classic story from one student this was in first grade. The teacher said, "We're not having fun anymore. It's time to teach math." That was part of the day, no more fun today, we're teaching math. So immediately the teacher is conveying that this is not fun but we have to do it. So to me, that sends the wrong message to the student. (M2, June 7, 2012)

Another mathematician described our society's tacit agreement that it's okay to hate math.

One of the big problems in the United States is that most teachers hate math, and they pass on that attitude. If they run out of time on one thing that day, they'll leave out the math. Or if they're not comfortable with it, they'll probably spend less time on it. And the same thing with parents. Parents give an implicit okay to their child to be bad at math because they'll say things like, "Well don't worry about it because I was always bad at math too." That's essentially telling the child that it's okay. We have that issue today with our whole society. You could go to a social gathering, a party, and if someone asks me what I do and they find out I'm a math teacher, I don't know how often, but it's a lot, people will tell me, "Oh I was horrible in math. I hated math. I was an F student in math." or whatever, and they say it proudly quite often, like it's something to be proud of. Nobody would ever say, "I never learned to read. I'm so proud of that." People are almost proud of being ignorant in math, and I think that's a problem we have in our society. We have to change that attitude, and we have to—not that anyone should be ashamed of not being good at math—but we have to believe that it's okay and it's not geeky and it's not nerdy to be good at math. (M3, May 21, 2012)

I think one thing we need to do is to get the message out that they share their attitudes with their children. So if parents are talking badly about math at home, their child is going to feel that's okay. They're going to possibly adopt that attitude themselves, and that's going to make it a very uphill battle for them. I think a lot of parents share their dislike for math with their kids. And it doesn't lead to positive things usually. (M3, June 4, 2012)

Second, they recommended early exposure to math concepts as a way of helping kids be interested in math. Early exposure will help children later remember things easily even though they don't understand the concepts at the time they were introduced to those concepts.

I started talking with them about numbers and counting. That's the first thing to do, counting. I have a granddaughter who will be one in June. She was here a couple weeks ago, and I started to tell her about counting, and I don't think it was way too early. I told her about one and two and things like this. Anyway I start early even though they might not—so in other words, I think you can start talking about numbers and counting even when they don't really understand it yet because when they can understand it, they probably will remember, “Oh yes, this is what this guy was saying earlier.” In fact, I often find I'll attend a lecture, and I won't understand very much of it, but later I'll hear it again, so things start to click. So, even when you don't understand something, it might be valuable. So I think you can start very early and not worry about whether the child understands something or not. You're implanting the ideas there, and it might take a while for the ideas to develop. I don't think you can start too early. (M1, May 17, 2012)

I always think you should start early. Even just counting. Some students don't even get that until it's later than everyone else. Kids can count by the time they're 2 or 3 with understanding of what they're doing. Certainly adding and subtracting, we don't have to call it that or write a plus sign, but we can talk about having some of these and adding more of those to the pile, and we can even do that with subtracting, and you can even do that with borrowing if you have representations of things. And fractions, half of an apple, a third of an apple. I think those are all things that can be introduced on a certain level that makes sense to them so that later on when they come up in a more formal kind of way, there's already a basis, there's already a sense of what that means. And I guess that's part of what I mean, when I talk about number sense. A lot of people don't have number sense with fractions. But if you go back to a visual representation of what a fraction is, most people get that. They just don't ever make the connection between that and the way we write a fraction. And the numbers and how we add and subtract. So I think all of those things—you can start a good foundation that will lead to better understanding later and some children can do even more than that. (M3, May 21, 2012)

Third, providing interesting problems and examples was mentioned. Mathematicians said that by doing so, children can see math as interesting and enjoyable.

I think you have to provide interesting examples and interesting problems. I have an eight-year-old granddaughter. She's in our house every day, and I try to tell her interesting ways of looking at mathematics and doing calculations. So now she's just involved in addition, subtraction, and various things and multiplication. I try to give her problems that are interesting. So I gave her recently some numbers to add, and I cooked them up so that it was a very nice answer at the end. She was so surprised that it came out to such a beautiful, nice number. So I think you have to try to make problems interesting to them and if you can kind of make them connect to things in their life. If they're interested in basketball, you might say well here's this team that scored so many points in this quarter. What are the different ways you can score these points? So you can connect mathematics to some other thing they're interested in. (M1, May 17, 2012)

One mathematician, however, was concerned about the risk of forcing students to be interested in math. Arousing students' interest in a natural manner, such as asking interesting questions or providing interesting examples, is beneficial, but forcing students to like math or putting pressure on students is detrimental.

It doesn't work if they put lots of pressure and say, "Won't you be interested? Won't you be interested? Won't you be interested?" If anything, it tends to make them not interested. And I do think there's a danger when we do any subject when they're not interested and they're not ready, and you're going to try to sort of force them to be interested. You can try a couple different angles, but if they're just not interested or they're not ready, you only just make them feel bad, if you put pressure on them to like it. So now, they have to make a decision, and they put their foot down and tell you, "I don't like it." Now that they've said, "I don't like this, I don't like math," they have to fight you all the time. "No you're going to love it. If you just let me do this, you're going to love it." I think you have to say, "I respect the fact that you're not interested now, and I hope that you'll find me interesting later." And then they don't go away. (M2, May 18, 2012)

He continued,

This is what happens in math. Those that are ready they take it, they love it, they're happy with it. The other ones, you basically rape them. You force them to learn math, and they're not ready. They don't like it. They just do it because they want to please you. "Okay I'll make you happy. I'll do these. I don't know what the hell I'm doing, but I'll do it to make you happy." Or you have the other ones who say, "You can't make me like this," and draw a line in the sand, and you've got to work really hard to correct that world view. But even the ones who are doing it are really hard to correct because after a while it's like "Yeah I know how to do it. I can memorize it. And you're happy when I do it. I'm just doing this to make you happy." I think math is something like that. You can get somewhat better if you don't push on people too hard and let them if they want to learn it. (M2, May 18, 2012)

We tend to think that education is going to be different. “So I’m not interested.” “Oh no no, you will be interested and you’re going to love it if I just force this onto you.” And so now the poor child, to defend themselves, either has to say, “You’re right, I’m wrong, I will love it. Okay just go ahead. Give it to me.” And they go ahead and go through the motions to make you happy. And they’re not happy, and they figure I don’t like this subject, and I’ll just go through it with the minimum I need to survive. (M2, June 2, 2012)

Fourth, introducing math concepts with some connection to real life is necessary for students to be able to see math as real and meaningful.

Maybe introduce the subject with some experience from real life which might get their attention more than just saying these are numbers and what do you do with them. So in other words, you can say, “The answer to this problem that you might encounter going to a store or something . . .” You need to talk about addition or subtraction or whatever it is later on. Maybe later on, it will be in algebra, so you might say, “You went to the post office and the postmaster said put on 45 cents of stamps. But you don’t have a 45 cent stamp. You have these kinds of stamps. Can you do it? Then if you can do it, how do you do it?” So that’s just one thing that popped into my mind, but there could be lots of problems that they would encounter in a store or post office or the playground. (M1, May 17, 2012)

For some, you have to make connections with the real world. If you make these connections—I mean some students have told me, “Ah, it’s much easier” if they make connections with something they know about. So it might be something about the sports they’re playing or the musical instrument they’re playing. So that’s one way you can. (M1, June 3, 2012)

Math isn’t a collection of rules disconnected from reality. It’s a collection of reflections of reality that we use to predict and measure. If you want to know that you have enough carpet, that’s prediction. And somehow the kids aren’t realizing that the power of math is to save money, save lives, save resources, save time. It’s not as much a test, it’s to make their lives easier. So maybe if you can get that concept through, you would find an easier time getting them to do some of the other things. (M2, June 7, 2012)

Lastly, the importance of self-motivation for learning was mentioned. One mathematician shared a story about his son’s self-motivation. This story clearly shows how self-motivation can be a driving force to engage students in mathematical investigation.

This weekend, my son came home with a Rubix cube from school. Some kids were playing with the Rubix cube, and he got interested. He never showed interest in it before. By Saturday, he solved the Rubix cube. He spent hours until his fingers and eyes were

sore. He was complaining, but he solved it in two days. In the sense that he can do the whole thing, no matter what, when you give him about two minutes now—by day three, he could. If I had tried to convince him to do that Rubix cube, I could have talked forever, and he would have never had any interest in it. I don't know what to do about that, but that kind of focus! He actually did something. The thing I was most amazed about was how tenaciously he worked to solve it. He likes to be the best at things. The main thing was that people at school were interested, and so he wanted to impress them, and so that made him work fiendishly hard. It's a funny thing – I would've loved to see that kind of work ethic from time to time. I've never seen it before. I know from having kids before, I may not see it again for years. (M2, June 7, 2012)

Teaching strategies. The mathematicians addressed various teaching strategies for teaching young children. First, they discussed the importance of assessment. They said that knowing where each student is at is critical because through this process, a teacher can get a good sense of each student's weakness and provide individualized help.

I had some very good teachers, especially in high school, and some of what I do is what they do. One of them was diligent about always looking at students' homework, and she got a good sense of what were my weaknesses versus what were Johnny's weaknesses. And she could use that to really help us individually. So I like to really look at student work and get a sense of where each of them is because they are usually all in very different places. And what this person consistently makes as a mistake, this person never makes as a mistake and vice versa. So that's one thing that I've picked up on is just really trying to see their work and that's where the process comes in too. I don't just look at the answer. (M3, May 28, 2012)

One mathematician pointed out the huge difference between the efficient way (a test) and the best way to assess children's mathematical understanding. She asserted that informal evaluation, which focuses on assessing children's mathematical thinking process, is necessary to get the best picture of where children are..

You know *efficient*—you give everyone a paper test. They write their answer, you grade it quickly, and you have a grade you can put in the grade book. I don't think that gives you the best picture. So I think talking to kids about their math while they're doing a problem is really the best way to have some sense of what they understand and what they don't. The difficulty with that, especially for public schools where you have this accountability issue with parents and so forth, is that it's really difficult to put a grade on that. Is that a 1, 2, or 3? Is that an A, B, or C? So I think the best way you can guide them is through some oral and some written communication with the student to see what

they understand and what they don't. It's just difficult to turn that into any kind of thing you can put in your grade book. (M3, May 28, 2012)

Second, the mathematicians suggested approaching students in different ways depending on their difference. Teachers should assess how students learn and adjust and fine-tune their teaching strategies to best fit each student.

Not everyone is created with the same intellectual abilities. With many, it's just a matter of time before they learn. Really, I don't think it's a matter of IQ or intelligence. Also, some students learn by hearing better than they do by seeing and conversely. So for example for myself, I'm not very good at discussing mathematics with people. And as far as doing research, I have to go home and sit in my chair and think about these things before. But others think very well on their feet in discussions. I'm not one of those. (M1, June 3, 2012)

In some sense, there are differences in my students, and I have to approach them in different ways. Some need more direction. So, in other words, I have to say, "Read this paper and try to prove this theorem or generalize this theorem." For some, I can just say, "If you read this paper, you should find lots of questions to ask." So one of the huge major differences between students is that—and probably you can extrapolate this all the way down to grade school—some people are very good at asking questions. And they will come up with all sorts of questions and find things to write on. Others are not good at asking questions at all. So you have to lead them—even after you sort of bring them into a stage of asking questions—in that direction. It still takes a long time for them to learn to do this. I guess that's my first suggestion is to try to help students learn to ask questions. (M1, June 3, 2012)

Well, it can be anything from having visual students as opposed to students who are more auditory, to having students who need tactile, concrete things to show them how it works. And you're going to have some of each, so the more ways you can show them the same thing, allows for more students to hopefully understand it. So you can show fractions by taking a pie or an apple and cutting it into pieces. Some students need that, and they need to be able to touch that, and they need to be able to manipulate the pieces. So you could even take a piece of paper and just draw a circle and cut the circle into parts. So when you ask them for three eighths, they pull out three of the pieces that come from an eight cut slice. And they need that to make the connections about adding fractions and subtracting fractions and things like that. Someone else, all you have to do is tell them, "What if you had a pie and you cut it into eight pieces?" They can see that in their mind. They don't need to touch it, to move it around. So I think the more ways you can offer that, the more likely you are to reach all of the students. (M3, May 28, 2012)

I think having some respect for learning styles is important. So I generally see two types. One is the student who understands everything immediately, and they have a good

understanding. They don't need a lot of written practice over and over again, but they do have to understand it and if they don't understand it they will get frustrated and drop it. The other student is what would be considered just a good student all around. They're probably good in most of their classes because they know how to do things like take notes and memorize formulas and things like that. So your job with them is to make sure they're also understanding it, not just memorizing just to get the A or to do the minimum to get whatever grade they're hoping for. So I think you have to have that in mind in order to bring out the best math in your students as well. (M3, May 21, 2012)

Third, mathematicians said that math is not a collection of isolated facts. Teachers should avoid rushing to teach facts or pushing students to memorize. Rather, they should encourage students' reasoning and critical thinking.

So my teachers, who I think were good math teachers, did not emphasize memorization. I can't remember much about . . . from 8th grade onward, but I can remember my teachers in mathematics. They did not memorize or require memorization. There were problems where you were required to reason. (M1, May 17, 2012)

I strongly believe that people should understand what they're doing. If you perform some sort of algebraic or manipulations using some formulas from calculus, you don't really retain that unless you understand it. Once you understand it, then you can retain them in your mind. If you're just told to memorize it, you evaluate this integral by this formula, you just don't remember things as long. (M1, May 17, 2012)

One mathematician elaborated on why rote memorization is problematic and deep understanding of topics or procedures is important.

And most of them relied on memorizing and remembering a procedure but had no understanding of the procedure and therefore didn't always know when to apply it. And even though some of the questions they asked were very basic and probably they could have just figured it out by thinking about it a little bit, they didn't ever do that. They never chose to just think about what the problem meant and finding an answer. They always tried to think of a procedure that they think might work, and often times they picked the wrong procedure or they couldn't remember which procedure might work. So there's a place for procedure, but I think it comes along the way. For example, if you can get a student to understand what's happening in a certain procedure. For example, if you want to find the middle of two numbers on a number line. So you want to know what's half way between these two. You could draw a number line and count. You could do that all day long if you wanted. You understand what it means to find the middle. If you do enough practice you might be able to understand, "Oh what I really need to do is add them up and divide by two." I'm averaging them and averaging is adding up. And so the procedure is important, but it should follow the understanding. So the understanding of what you're doing should come first, and then we talk about, "Now how can I make this

easier on myself? What's an algorithm that will always fit this situation that I can use over and over again?" Then that algorithm has some meaning, and they're not as likely to forget it, and they certainly won't confuse when to use it because it fits that particular situation. (M3, May 28, 2012)

Memorization, however, is necessary for some basic concepts.

In recent times, there's been an emphasis away from memorization. You can go too far in that direction on the memorization, but I think an understanding of the concepts is important. But on the other hand, you can save a lot of time when you memorize basic facts. I don't see anything wrong with some stress on memorizing multiplication tables or something like that. I think memorization is important. (M1, May 17, 2012)

Although we don't like to teach this way, maybe they have to memorize. Sometimes, that's the only way they can do it. (M1, June 3, 2012)

Fourth, mathematicians said that encouraging students to come up with diverse ways to solve the problem is important because it encourages different kinds of reasoning.

Often telling them easier ways to do things, so you might first say, "Well, here's one way of doing this problem." And then you might ask, "Is there an easier way of doing it, so you don't have to do so much work?" In other words, you might think of another way. You should always encourage children—there isn't any right or wrong way of thinking about problems. There are different kinds of reasoning that may lead you to the correct answer. And sometimes, the reasoning might be good, but you might make a little mistake along the way and get an incorrect answer. But still the reasoning might be very good. So you should always encourage good thinking and good reasoning, even though maybe they get the wrong answer at the end. Maybe some little thing just needs to be corrected. But I think you should correct students' answer, when they got wrong answer. They shouldn't go on thinking that it's correct. But maybe you should go through their reasoning with them, as to how they get their answer. In other words, there might be very good things about their reasoning. And the next child might have entirely different kind of reasoning, but be just as good. And even though something might be longer reasoning, that still might be good because it might involve more creativity. And the long reasoning might be better in another problem later. There isn't any right way or wrong way. Students should be encouraged to think differently, and not worry about it if someone or a friend next to them thinks different. (M1, May 17, 2012)

Often there are lots of ways of doing or working a problem. Even at the elementary level, there's ways of thinking about a problem that you would have never have thought of. One wants to encourage different ways of thinking and creativity. So a student might have a wrong answer but have a very clever approach, and made a slight miscalculation or maybe one thing the student just forgot about in his or her analysis. That's the most important thing—how a student thinks. If you ask questions and have the student tell

you why these things are true or why he or she believes this or reasons this, that's much better. (M1, June 8, 2012)

I think you should take a kid anywhere they can go with understanding. If you're just teaching them process and facts that they don't have the tools for understanding yet, you should wait until they do. Some students have very valid ways of thinking through problems, but they're not recognized because they're not thought of as the standard method of doing the problem. If you can allow them to get through that and accept that, I think then you can introduce the standard way of doing it as an alternative way of looking at the problem. And they might appreciate it more than just being told, "No, you can't do it your way. You have to do it my way." You can present it as both ideas work, and sometimes you might choose your method and there may be other opportunities where you might choose my method for whatever reason. So yes, I think they should be allowed to do it in different ways. (M3, May 21, 2012)

When encouraging students to come up with different ways of solving problems, not criticizing students is very important because it may hurt students' motivation to investigate mathematical problems.

Pedagogically, I never criticize students in class because that just makes them feel bad and decreases their desire to learn the material. I can't say I've always been perfect. I can remember a couple incidents where I probably wasn't as kind as I should have been, and these bothered me afterwards. I mean, we all make mistakes, and I've certainly made them. Generally, never criticize students. And if someone asks a good question—always encourage questions. I don't mind if students interrupt me while I'm lecturing. If there's something they don't understand, it's good to stop and re-explain it and probably explain it in another way. So I encourage questions at all times during a fifty minute lecture period. So encouraging questions, complementing them if they ask good questions, and if they don't understand something trying to explain it in other way and with another vehicle and with another example maybe. So when I answer questions, I assume he doesn't understand something and try to give a completely different explanation from what I had said. (M1, June 3, 2012)

The only thing I can say is that you have to give them the sorts of problems that motivate and encourage them and when they do something correct, praise them. It's sort of an individual situation. What works for one child might not work for another. At any rate, no criticism, keep praising (M1, June 8, 2012)

They added that by asking students questions and having conversations with them, teachers can help students elaborate their thoughts and lead them to good reasoning.

I want to know how they're getting the answer, what they are thinking about to get it. If I know what they're thinking, I can help direct it a different way and say, "Can you see

where your thinking is not correct there? What can we do to fix that?" (M3, May 28, 2012)

I think it is important that you want to know what they're thinking. I was volunteering in my daughter's school. And since the teacher knew that I know math, she had a student come work with me who had been absent. And she wanted me to explain a concept to him. And the interesting thing with him was that he could go straight to the answer, but he wasn't following the process that she [the teacher] wanted. So I asked him what he was thinking to get to that answer. So I think you have to know what they're thinking and you have to use that to try to help them understand further things. So if they always think of a problem in a certain way, for example if they want to add 11 and 12, some kids will think to add 10 and 10 first and then add 1 and 2 and find the sum that way. Some of them have to think of it in terms of columns and going through the procedure that the teacher tells them. So if you know the way they're thinking about it, you can use that to help them later with more difficult concepts. (M3, May 21, 2012)

When they [students]'re actually trying things on their own, I think you have to be present and you have to be walking around and answering questions and watching what they're doing. And if you have a parent or an assistant who can help you, the more people the better to interact as they're thinking through the process. And I think the idea of asking them what they did is good. With my daughter, some of the homework papers that come home, often the very last question is, "Explain how you did this. Explain what you did." And I think that's a good thing to ask of younger kids as well as older so that they think about why think came up with that answer. (M3, May 28, 2012)

Fifth, the mathematicians talked about the values of collaborative learning among students. They said that people view things differently, so collaborating or discussing something with others helps students think about problems in a new way.

I think people view things in different ways. So if you are collaborating or discussing something with some people, often it's just the way they say it or something which will light up—put a light bulb in your head. Often if I'm discussing something with my students, they will say something which will lead me to "Oh yes, this leads to a problem or an idea." So collaborating or even just discussing problems or mathematics with your peers can be very instructive. (M1, June 8, 2012)

I encourage my students to work together and to talk to each other at various points during classes when we're learning concepts because talking about something can help you think about it and hearing what someone else says about it can help you think about it in a new way. And I don't want me during the instruction to be the only person they hear talk about it because that's just one view. I think they ask each other good questions sometimes that they might not want to ask me because they'll think that they appear stupid or silly. And I try to make sure they don't feel that way. But sometimes they're more comfortable asking someone else who is also learning rather than asking someone

who is supposed to be the expert. I think student collaboration is very good in mathematics actually. It's very helpful. (M3, June 4, 2012)

It's an opportunity for the stronger students to gain confidence in their strengths. I don't know how many times, in particular, women, you'll see a strong student in the classroom who doesn't have confidence in their abilities. By working in a smaller group, they become aware of the fact that they have something to offer, which is awesome. But it also helps the weaker students because they see that there are plenty of questions that the smarter guys actually can't answer. So it's okay to be confused. There's this real positiveness to seeing that the hotdog that comes out of there and always seems like they're bored and just has the answers, when you start working collaborative, you've discovered they had a lot of the answers, but they don't always have the answer. And when they're lost, they're just as lost as you are. It just takes them a different spot to get lost, and I think that really is useful. (M2, June 7, 2012)

They also provided some tips when grouping kids for collaborative learning. Even though their ideas about how to group students varied, there were some common implications.

The strong and the average. The average will probably realize the benefit more than the strong student. For the weaker students, I think there's a dilemma. You could pair them with students of their own abilities, in which case it probably wouldn't help too much. You could pair them with students of better ability, but on the other hand, that might have the opposite effect of what you intend. They'll see this student is smarter than they are, and they could never learn as fast as he or she. So I think pairing up with different abilities is dangerous, in my own opinion. You're better off if you want to encourage someone not to just put them with better students but just to encourage them where they're at. And when they make an advance or get a problem correct or have a good idea, compliment them. (M1, June 8, 2012)

I think it really depends on the students and some of it is not just where their ability is, but it's also personality and things like that. Because some students no matter how hard you try will not interact a lot with someone else. You actually may be better off having someone who is extroverted with someone who is introverted even if they're about the same ability because it brings the introvert out a little bit to talk about what they normally wouldn't talk about, and I think sometimes it allows the other person to be a better listener. Ability should be taken into account because you don't want the blind leading the blind. Some students will explain with great confidence, and they'll explain it completely wrong. And that's not helpful to the other student. So you do have to take ability into account, but I think there are other things to look at as well in terms of personality and willingness to share, willingness to talk about ideas. And I think you can look at a student's weakness too. If you've been working with the student a while, you should be pretty aware of what kinds of mistakes that student is likely to make versus the types of mistakes another student is likely to make. (M3, June 4, 2012)

If you have four people working in a group, peers, it works better if they're slightly homogeneous. If you take a thirty five year old and an eighteen year old and a mother of four and somebody else and you put them in collaborative working, it does not work as well. But when you have a slightly homogeneous group, it works pretty well. But they also, they have different ways of expressing how something works. You learn a lot just by helping them, looking at how their processes go, and what's working and what doesn't. (M2, June 2, 2012)

Now this is tricky business. I've found that in the past, over many years, engineering the groups is a mistake. They know it; they feel it. So what I prefer is I come in the room, and I make up some jingle, and I count around the room sort of at random, and then I have them break up into groups based on that. So "I love math," I need to break you into four groups. Or "I love math a lot," and so then you'll be "I," "love," "math," "a lot." All the "I's" are here and the "love's" are – and it works way better, because sometimes they're in good groups, sometimes they're in bad groups. By the end of the year, all their group grades are essentially the same, and it amounts to attendance because all the good students are mixed in with the bad ones at the same rate and what have you. But some days, you get a group that doesn't work well. But when you try and engineer it, they sense it. They know that "I must be the weak student, I'm always put with this strong student," "I must be the difficult student, I'm always put with the" – I mean, it's sort of like when I went to school and they really didn't tell us that we were in graded classrooms, within three weeks we all knew I was in the middle group. You all know. "Oh, he's in that classroom? Oh, he's an idiot, so that must be the bottom classroom." "Oh, he's in that, he's really smart, he must, that must be the top." You just figure it out – and your groups, they all know that. If you're mixing groups every day – and so when you try to engineer them – now what I would do is I would engineer some for continuity. So like these two are labs, and those labs would stay fixed for three or four weeks, and I would set those. So there was a total randomness, and there was also an engineered part. When we did away with the labs due to budget cuts, I sometimes experimented with having, when I met five days a week, the Friday one was an engineered one, and the other four were random. If you engineer them, and then two people don't show up, then that group has two people. Right? If you're totally random, there's never any of that that happens. And if someone can't stand to work with somebody, they have the equal chance of being with them as anybody else. (M2, June 7, 2012)

So when I group the students, I don't consider the students' capabilities. I prefer to pretend that every student's capable of learning the material and they're all going to work hard and therefore, there's no reason for me to engineer the sections. And I tell them that on the first day. I say, "Look, I know that you're all at different levels in ability, but I believe you all will try hard and you can learn it, so I don't need to differentiate, because I trust that, when you're in a group, you'll be consciously trying to work and you won't be trying to hurt the other students and that you will get it. It might take you a little longer or a little less time than someone else, because if you get done sooner, that's something for you to practice and teach." But I don't find that – I found that my weaker students rose up and my stronger students – because after a few weeks, they all know. Get in a group, "Okay, who's the stronger one? We all know who it is." Or two of them, whatever, and if you don't meddle, they find their own way. We are a

community, collaborative preachers. They're supportive of everybody. (M2, June 7, 2012)

Lastly, mathematicians said that integrating math with other subject areas is beneficial since it helps students understand math well and make connections better.

I think any time you can tie math with a different subject is positive. It gives the child a new way to connect the things that they've learned. Basically, the more connections our brain makes with something, the more likely we are to remember, recall, and understand it. So I do think that is a positive. I think if you only teach it in that way and don't have any time where just look at math by itself, you're losing something. Math in itself can be a really beautiful subject for people. It's very logical, it follows the rules, and it can be looked at in its own way like that. But then you can also connect it to other things and see that it's an addition, it's practical. (M3, June 4, 2012)

One mathematician suggested several ideas that teachers can make good use of when integrating math with other subject areas.

In fact, about a month or so ago, we had a professor in number theory from Harvard come, and he's also a composer. So he told a lot about the ways music and mathematics interact. So this is something I didn't know much about at all, but if a student is interested in music, there are lots of sources whereby the student can relate the mathematics she or he is learning with music. Not being a musician, I can't give you the sources, but this came to mind because I was just fascinated by this fellow's lecture because I just didn't know any of this before. (M1, June 3, 2012)

He continued,

Since I'm not involved with elementary education, I haven't really thought much about how I would integrate anything at the elementary level. So I like to talk about it at a more advanced level if that's all right. Actually one of the ways I like to integrate is by using history because mathematicians have an interesting history, especially how they have interacted with various things in society. Often in my lectures, I will bring out historical details to interest students so that they know that this is an ongoing activity that has had a lot of effect over the years. I think I mentioned last time in calculus there are lots of applications one could discuss. Although I don't have time to discuss in detail every particular application, I often will mention the applications which hopefully they'll get in their courses or sometime in their careers. Often, there's just not time to go into details of the applications, but one should at least mention that these applications exist. As I mentioned last time for combinatorics, you can get lots of applications by reading, for example Fred Roberts' book on combinatorics. There are just hundreds of applications to real life. (M1, June 8, 2012)

Support for teachers. The mathematicians agreed that support from teacher education programs is needed to help teachers prepare themselves to teach math more effectively and become confident about teaching math.

Hopefully we would catch that beforehand, and I think that's what college and universities need to strive to do—is create a level of mastery in math and help teachers obtain it and don't just say, “You have to obtain it or you're out.” But help them reach that. (M3, May 28, 2012)

I think sometimes you have to go back to the math itself and you have to help teachers become more confident about it. We teach mathematics for elementary school teachers here. We're getting a new textbook and we're going to change some of the things we do with the class. We used to require a mastery test at the end that would test basic mathematical concepts. So you have to be able to add fractions and you have to be able to do some higher degree multiplication without a calculator, the kinds of things that they'll be expected to teach their students. And if you don't get a 90% or above, you don't pass the class. In the past, it was done at the end of the semester. We're thinking of moving it to the beginning and then remediating with the students who don't pass it, who don't get the 90%. So at that point, they will need to go back and strengthen those skills. They can stay in the class, and we can help them with that. But hopefully that's going to be a time of building their confidence in their understanding of the mathematics. We can go back and help fill in some of the gaps that they've had through the years that have made them feel not confident about their abilities in mathematics. And then our hope is that as we work our way through the class and also talk about not only how you do those things but how you teach those things, then they'll feel more confident about that and grow in their own understanding. So I sometimes think you have to just go back and say, “Okay if you're not okay with fractions, then we have to get you comfortable with fractions. We have to explain it to you in a new way so that you can learn fractions and feel good about them. We can't just brush over it and say, ‘Well just show them how to do this and you'll be fine. Or just do what the textbook says and you'll be fine.’” We want them to have a feel for it themselves. (M3, May 28, 2012)

When talking about support from teacher education programs, the mathematicians often mentioned the importance of strengthening teachers' mathematical content knowledge. They believe that there should be more content-related courses because teachers need to develop a thorough understanding of the content areas.

In general, I think they need more math content than they get right now. I know we teach a math class for elementary education teachers here. I've never taught it before. I'm scheduled to teach it in the fall for the first time. It's similar to a class that I teach now, and based on the similarities, I would say it's not enough content. (M3, May 21, 2012)

They also emphasized thinking about the big picture in math by making connections across the content areas. The content areas in math are not isolated but closely interconnected.

I think the content—if you don't really understand the subject or where it fits in the big picture then you're going to have a difficult time convincing the student that it's important, that it's relevant, and you don't have as many tools for explaining to them either. If they struggle and you're struggling with it, it becomes a very difficult thing to do. (M3, May 21, 2012)

I think you have to have a knowledge base beyond what you're teaching. So if you're teaching Algebra 1 all the time that doesn't mean you shouldn't understand Calculus. Because so much of Algebra 1 leads into Calculus, and if you know where it's going, you can help your students know where it's going, which will hopefully help them to see the relevance. So I think it's important to always be stretched beyond what you're teaching your students and not just know enough to teach them, because it provides a bigger picture and a perspective on where this fits in with everything else and why it's important. And hopefully you can address that with your students—you can tell them, "What is this used for? Why is it important?" You can't guarantee that every student will use every concept you teach them, but you can tell them how it's used. And they may choose to go down that path. (M3, May 21, 2012)

When I was a TA in my graduate program, I was teaching a calculus class. One of my students wanted to be a high school math teacher. And he at various times in the semester got frustrated with the calculus and wondered why he needed to know this anyway because he wasn't going to teach calculus. He was going to teach Algebra 1 or Algebra 2. And what he doesn't realize or didn't at that time, hopefully he came to realize that, is that the more math you take beyond Algebra 1, Algebra 2 and the things what you might be wanting to teach, the more you understand about the Algebra 1 and the Algebra 2. So when you gain new experience in math, it helps you with the old experiences in math, to have a deeper understanding of them, to see how they fit. And when one of his high school students ask him why do you have to learn Algebra 1, he can have a better view of all of the different ways that Algebra 1 can be used and he also himself has more tools in his toolbox to use in reference when explaining to his students and trying to help them understand it as well as trying to help them understand what the use of it is, why it's important to learn it. There are connections, sometimes you see the connections later on. So you can talk about some topic here and some topic over here that seem to not be related, but it all comes together for a new concept. (M3, May 28, 2012)

One mathematician recommended three courses that would be helpful for teachers to broaden their mathematical knowledge and teaching repertoire.

When I advise students who are in math education . . . some of my advisees are in math education. They ask me what courses to take which would help them. I think the best math courses would be—I can think of three of them: the most commonly taken is a course in geometry. So we have two courses in geometry which people in math education frequently take. (M1, May 17, 2012)

But I also think number theory is a good course to take because it doesn't require any background, but it might require maturity and sophistication to think about some problems. So the ideas in number theory and concepts you can teach in primary school very easily because they just involve properties of divisibility and factoring and things of this type. So you can actually give children very interesting problems in number theory to challenge them at an early age. And number theory gives you interesting properties of numbers. (M1, May 17, 2012)

And the third course I suggest is combinatorics where you find different ways of counting and counting different things. So in other words, if you ask a person to count certain kinds of patterns, it might be very difficult unless you have—you can shorten the process by using combinatorial reasoning. These are again problems you can give to children at a very early age. You can say, "You got six objects. How many ways can you choose two objects from six objects?" So students—they will attack it by choosing these two, these two, these two. And then they say, "Oh, it's becoming more complicated. Is there a shorter way?" And then you can show them a shorter way. (M1, May 17, 2012)

Second, mathematicians said that collaboration in teaching can be a good support for teachers. Like children learn from each other, teachers can also learn teaching strategies from or share ideas with other teachers.

The other thing I think is to attempt to pair someone who's not comfortable with math with a coordinating teacher or supervising teacher who is. And then sometimes when you hear someone teaching it in a way you've never seen before or saying things in a way that helps you understand it better, you automatically feel more comfortable yourself and you gain a little confidence that way. So I think that could be a positive—talking to pre-service teachers about what do they feel are their strengths and weaknesses and trying to find a complement to that in a supervising teacher. (M3, May 28, 2012)

And you can possibly fill in some gaps the same way two teachers would who, one's comfortable with Algebra and one's comfortable with Geometry. But they work together to fill in the gaps. (M3, June 4, 2012)

Well I think the obvious merit is the idea of strengths and weaknesses and everyone has them. And when you're working with someone, you can hopefully fill in each other's gaps. And when one person is particularly excited about a topic, then they can take the lead on that. There may be a different topic where the other person takes the lead. And I

also think that sharing ideas and materials is positive as well because getting something from someone else that has been successful—even something as simple as a worksheet that really seemed to connected with the students well—can get a person who is uncomfortable with that topic off on the right foot and possibly be able to help them build on top of that. So I think that the more teachers share with each other in terms of ideas, materials, and even sometimes sharing a classroom I think all those are really beneficial. (M3, June 4, 2012)

One mathematician shared a conversation he had with his son to show the merits of collaboration.

For one thing, none of us are particularly smart. But collectively we're stellarly genius. My son, true story: This morning we were driving back from Indiana, so we're in the car and he said, he's in seventh grade, "You know, cars are really amazing." I said, "Yeah." He said, "Just to think of how to make one, I don't know if I could even think how to make this door." And he talked about the curve of the door and the hole for the handle, and I said it takes thousands of people to design a car. And it took us hundreds and hundreds of man hours probably to design that door. And then there's the process of building it, and the process, it takes thousands of people, and engineers and scientists to think and then improve upon the previous person's to make that door. And I agree with him completely. The process is fantastically amazing. And even more amazing is that they change them every four or five years. What looks like cosmetic changes to us are whole machines being changed, and pieces re-fitting. It's a lot of cool stuff. (M2, June 7, 2012)

He continued,

As teachers, I don't feel it's really so different. I'm going to be teaching math for twenty- one years in the fall, and I taught [a course] two years ago and I have a collection of worksheets and more recent is a book, it's pretty good. But I will go through the worksheets of the people who taught it the previous year to get ideas, to see what worked and didn't work. We kept notes of when I did it before. I had another person use the worksheets together, and he kept notes of how some were too long, too short, successful, whatever. And then I met with the TAs each week and talked about their impressions of how various things worked and didn't work. So I think there's the students' feedback, but there's also the instructors using your materials or similar materials, what was successful and not successful. I've got little stories of things that people found that worked in the classroom, little light bulbs that come on, and I kept track of those things, and then I can incorporate those positive things. So I was able to improve by working with the instructors and working with the students and taking what I'd learned from the two. But it wasn't one smart person, it was taking, collecting from about thirty people every year. (M2, June 7, 2012)

Third, mathematicians talked about the importance of collaboration with parents.

Parents are eager to help teachers but often they do not know how. If parents were more involved, they could support teachers and schools more, and, as a result, we would be in a lot better shape in general.

I think it's important to give the parents as much support as possible because if you're not doing math a lot then you forget it. If you're not using it very much, you tend to lose it. There are parents out there who would like to help their children but they can't because they've forgotten the math that they once know. So I think the schools providing support—I know that some schools send home a family math unit and it has the answers to all of homework problems. It does have a letter explaining some of the concepts that they're going to be talking about and then the back page has all the solutions to the homework sets for that unit. And so they may be able to get some understanding from the letter part of it. I'm sure, for parents, that can be really helpful if they tuck that away somewhere and if a question comes up that they are unsure about, they can look back and help their child. I think that's a positive thing. I think just attitude is so much of it. Parents are the answer to all the education problems, in my opinion. If we could get more involved parents and parents who support education more and support teachers more and support schools more, then we would be in a lot better shape in general. And math is certainly no different and it's maybe more so. (M3, June 4, 2012)

They provided some ideas that can help parents get involved in their children's math learning and promote their mathematical investigation.

The first thing to do is to always ask them what they learned. So I do this with my granddaughter. I usually have dinner with her every evening. That's a starting point. If you see where they're at and what they've learned then maybe you could ask a more specific question. Different children—some of them are more forthcoming, and others are very silent, so you have to maybe draw out what one child learned more than others. I like looking at the book from which they're learning and then seeing the examples and the problems they're working. Because then I can see these and I can often think of other kinds of problems at the same level. To me, this is very helpful, looking at their books and their lessons because then—of course I have more experience than most parents because this is my job—but then maybe I can see, oh my son is interested in basketball and I can give a problem related to this. I think making sure the child brings his or her book or homework back so you can see it. This sort of leads to other problems to conjure up that will pique their interest. (M1, June 8, 2012)

The past week one kindergarten in town did a math-games family night. So they had prizes and snacks and things like that and all kinds of math games going on. I think that was a good event. Before we lived in this area, my older daughter, her school had a

family math night, same kind of thing. So those are good, positive things that schools can do to get parents get involved. (M3, June 4, 2012)

Defining Mathematics

Mathematicians saw a large picture of what math is and what purpose doing math serves. They said that math is not a collection of facts or concepts and that doing math is a process of finding logic. They emphasized logical thinking embedded in math rather than mathematical concepts themselves.

In my opinion, the major learning goal of doing math in early childhood is to raise children's mathematical potential to be good at math later in their lives. I think doing math is a kind of logical training, so developing logical thinking capability is the core of math education. Whether your answer is right or wrong is not that important, rather what you are learning from the process of doing math is more important. (M1, May 17, 2012)

One mathematician explained what math is and what doing math means by comparing it to writing a composition.

I think that learning math is similar to creating a writing composition. Imagine that you are writing an essay. Once the topic is decided, first thing you do is to think about strategies—how you are going to persuade readers to agree with your opinion. How to approach the topic or perspective to analyze it vary, but you have a big idea [a topic sentence] and several supporting sentences to back it up. You show your logic in your essay, and if it is reasonable, your readers agree with your assertion. When you solve a math problem, the process is the same. The difference is there is always the right answer for the math problem. But, still how to get to the right answer can be different. By asking children how they get their answers, we can conjecture about their logic to solve the problem. We need to encourage children to create their own story. I think verbally expressing their logic is very important. Even in the case that their answer is wrong, children can learn many things. I think correcting children and giving the right answer at the spot is not good. It is even harmful to children. When their answer is wrong, there is always a flaw in their logic. By asking questions about how they get to their answer, we can help children to find the flaw in their logic so that they can self-correct their irrational logic. (M2, May 18, 2012)

Summary

The mathematicians believed that all children have some math ability and are ready to learn math. They criticized pessimistic views about the nature of young children's mathematical competence that are widespread in our society in terms of underestimating children's ability in math.

They mentioned a wide range of content areas that should be taught to young children: (a) number sense and operations (comparing), (b) algebraic thinking in terms of finding patterns, (c) geometry, (d) measurement, and (e) data analysis. The number of content areas introduced to students is not as important as how thoroughly students understand each content area. They also emphasized the importance of sequencing the teaching of the content areas in mathematics.

Three major ideas were addressed concerning pedagogy. First, as a way of helping children develop a positive attitude toward math, they made five suggestions: (a) teachers' or parents' sharing a positive attitude to math with children, (b) early exposure to math concepts to help children get interested in math, (c) providing interesting problems and examples so children see math as meaningful and enjoyable, and avoiding forcing children to like math or putting pressure on children, (d) introducing math concepts in ways that related to daily life, and (e) encouraging self-motivation for learning.

Second, they suggested following teaching strategies: (a) emphasizing ongoing assessment, (b) providing individualized teaching by approaching students in different ways, (c) avoiding rushing to teach facts or pushing memorization, (d) encouraging students to come up with different ways to solve the problem, (e) encouraging collaborative learning among students, and (f) integrating math with other subject areas.

Third, as ways of supporting teachers, (a) support from teacher education programs, (b) collaboration with other teachers, and (c) collaboration with parents were mentioned.

The mathematicians saw a big picture of what math is and what purpose doing math serves. For them, math is not a collection of facts or concepts but rather a process of finding logic. They emphasized the logical thinking embedded in math rather than mathematical concepts themselves.

Chapter 5

Teacher Educators: Math Education for Young Children

In order to understand how teacher educators think about math education for young children, I began with eight teacher educators. From the first round of interviews with the eight teacher educators, I made some observations. Teacher educators worry about the disparity between what pre-service teachers have learned in grade school and high school and what they are going to teach as teachers. They think that what teachers know in terms of math content is often outdated and needs to be updated. They also believe that teachers persist in their own ways of solving problems which they learned in grade school and high school, require students to follow those methods, which results in narrow thinking and reasoning. These initial findings inspired me to decide directions for the second and third interviews with the three teacher educators.

I targeted three from the original eight to continue with the second and third interviews based on their desire to continue as well as their experiences that contribute to the study. Two were in college of education teaching pedagogy courses and one in a math department teaching content courses. They teach one or two courses a year to undergraduates studying to be certified to teach in the primary grades. I interviewed all teacher educators in their offices.

The first teacher educator has a son in kindergarten. He also has teaching experience as a secondary math teacher before starting to teach at university. In addition, he has a plan to apply for the ICTM (Illinois Council of Teachers of Mathematics) Board of Directors, which encourages an active interest in all areas of mathematics education, enhances the teaching and learning of mathematics and the role of mathematics in other disciplines, and provides opportunities for exchange of views regarding the teaching and learning of mathematics. His

different experiences made him a good person to interview.

The second teacher educator also has a range of teaching experiences. She taught 12 years in parochial schools (K-12), which, she noted, do not require certification. For five years of those 12, she taught in an international school in Tokyo. She came back to the U.S. and got her teaching certificate. She taught in the public schools for 14 years. She has been at the university for seven years. Her teaching experience in Japan intrigued me because Japanese students rank highly on international math exams. Even though she didn't teach in Japanese schools, I hoped that she might be able to compare math education in Japan and the U.S. Unfortunately, she did not do this.

The third teacher educator is currently teaching a distance education course. His academic interest is in the use of technology in math teaching. He sees a heavy technology focus in schools now because of new learning technologies. He belongs to many professional organizations such as The American Society for Curriculum Developers, The National Council for Teachers of Mathematics, AERA, The International Society for Technology Education, The Illinois Council of Teachers of Mathematics, The Illinois Science Teachers Association, and The National Science Teachers Association. He belongs to these organizations in order to receive the journals and to stay aware of what is going on in the field. I was impressed by his zest for the field of math education.

This chapter has four sections. In the first, I look at the teacher educators' views on children's mathematical understanding. The second section deals with what mathematical content they believe should be taught to young children. In the third section, I delve into their ideas on how to teach mathematics to young children. In the final section, I explore the teacher educators' thoughts about current teacher education programs and their effectiveness in

preparing students to teach math.

How Children Learn Mathematics

The teacher educators generally agreed that children have some math ability and are ready to learn math. They also talked about the signs of a child's readiness to learn mathematics.

Eventually they get to an age, as their curiosity grows, as they start to be able to quantify objects. And they start with their oral counting which isn't very meaningful to them yet. They'll count to ten, but really not know what the number ten means. As they start being able to, some being taught, some just making sense of their world, start placing a meaning for the different numbers. That number is a quantity, and we can use it to count and to mark a set of things. Once they start getting that "quantity readiness," there's awareness that I can look at a pile of things and that there is a number that represents that pile, often called the "cardinality principle." Once that readiness starts happening, there's a lot you can start doing. (TE1, May 15, 2012)

My son is in kindergarten right now. He's capable of quite a bit, but it's not about giving him just abstract problems to apply rules to. He'll make sense out of things, and I'll ask him questions. I'll take out some of his toys and say, I'll give him 8 toys and say, "Ok Mike, you have 8 toys, and let's suppose you really wanted 15. How many more would I have to give you to get 15?" And he can't just in his head just calculate the right answer, but if he has the concrete things there, he starts getting strategies. He can think about how he's going to keep track and move things in and is capable of doing that. So I believe kids in kindergarten are capable of doing operations, especially addition and subtraction and comparison-type situations. (TE1, May 15, 2012)

I believe that they, even my son, and again I'm using him as my example, can understand number relationships. For example, when I've seen him do this as well, again I've worked with him and some of his classmates. For example, we were looking at number relationships to ten. Between me and him, we had ten cookies, and we were thinking of all the ways we could split them up. So I had him think about all the different ways we could do that. Like he could have one and I could have nine, and he was making a list, trying to figure it out. You know, "If I had five, dad would have five. If dad had six, I would have four." And again, it's not all mentally done. Often they're starting with manipulatives, but they're making sense. They're dividing the ten things up and seeing how different facts make ten. Getting the idea of different ways of partitioning a number, of breaking down a number. And so young students are very capable of working on those number relationships, looking for patterns, doing addition/subtraction. Multiplication can build on addition. Some kindergartners, starting to stretch a little bit there. I think that's often more. I think developmentally for most kids would be first grade, second grade. With that said, I won't say kindergartners can't because there are many who can. Because once they get the idea, it's just a shortcut for repeated addition. Again it's not that they memorize their multiplication table, but they can make sense out of it. Once a kid realizes that "two times six" is just saying, "I have two piles of six cookies." Well they can realize

what that is. They can use a counting strategy or a doubling strategy to do that. So, with preschool and kindergarten, I think that there's quite a bit of mathematical knowledge that they can work on in those types of ways. "More than, less than" comparisons is a big one. (TE1, May 15, 2012)

And before you know it, you've got kids who are actually thinking in terms of quantities and values and putting things together and grouping things. So, I think we can make the kids numerically literate – I think you can make everyone numerically literate in kindergarten. Then if you also consider logic to be a branch of mathematics, which it is, you can clearly start teaching little ones logic. Okay, wait a minute, today is Monday, what does that mean? Tomorrow is Tuesday. That's logic! Well, if tomorrow is Tuesday, what will three days from now be? Mathematics is the language of the universe! So, kids are speaking mathematics in kindergarten. Even if they can't communicate, they can still put sticks together on the floor and let me know that they understood that this many fingers was this many sticks on the floor or something. (TE2, May 16, 2012)

My colleague Jane has three sons, and even the youngest learns very quickly what a third is because they know what's fair. So I think it would be pretty tough to give the five-year-old a sixth and give the other two equal parts of the remaining five-sixths and get away with it. Young kids have abilities, certainly in counting and the ability to determine what's fair numerically. It's something that young children can do. In terms of kindergarten, I saw a presentation last year from Angela Andrews about the number line, and you really can introduce at the early levels empty boxes that can be replaced by numerals to help kids along the way towards counting. I think kids are doing math all the time; it's just bringing it out. They notice curves, they notice shapes. They notice regular and irregular, they see patterns; all of those things are the foundations for math. The skills don't matter so much. (TE3, May 17, 2012)

One of the teacher educators, however, pointed out that the views about children's mathematical capabilities differ based on how one defines math.

I agree – people have two views of mathematics, maybe more than two. But you could look at it as this. People who say that kids can't do math at that age, they think of math as kind of a set of algorithms and rules that are to be followed, and these kids aren't ready to follow these rules. And the other view is that math is a way of making sense of our world. That's a language that we use to describe what we see, and students much younger than preschool are already making sense of their world. They know that it's dessert time, and they see two cookies in front of them and their sister has three, younger than preschool, they start to do comparison right there and know that something isn't fair or that there's different quantities. I mean they won't be able to distinguish "She has one more than me." yet, but ideas of more or less quantity come very early as kids experience and make sense out of their world. (TE1, May 15, 2012)

They also mentioned the importance of understanding children's mathematical cognition when teaching math to children.

I teach what we refer to here as the content courses. I'm actually supposed to be looking at the mathematics that the students [pre-service teachers] will be teaching, but the assumption I make is that they know how to do elementary level mathematics. So that's not what I'm interested in. What I'm interested in is that they understand how children learn and absorb the concepts of mathematics. So it's sometimes a struggle because they think I'm just going to ask them to do math, and they already know how to do it. So in many cases, they're not really as open to thinking about how children understand mathematics because if the future teachers can't approach mathematics from the point of view of a child who doesn't understand it, they're never really going to be good at helping children understand it. (TE2, May 16, 2012)

When I first started teaching, I remember one of the professional developers who came in and spoke to our group. She said if all you do is to cover content then there's nothing that couldn't be replaced by a computer. You could've just as easily seen a video. It might have even been better because you could stop it and start it as opposed to listening to a person explaining things on a board. So teachers have to do much more than that today, and, in particular, have a variety of approaches. All teachers, all the best teachers at least, have to teach students, and not just teach math. That is, you have to know who your students are, and that human interaction is as important as mathematics knowledge, I would say at all levels, but most especially at earlier levels where you're getting students their first experience with mathematics, you want them to be excited by it and feel successful, so you want to know who they are and have some understanding of how they can best learn. (TE 3, May 17, 2012)

There has been a cognitive revolution in understanding that there is a process by which children come to understand things: learning trajectories and zones of proximal development and things like that, a lot of talk. But underlining it, for the purposes of teachers, is trying to find those spaces that are going to be the most effective for the student, so you can take your lead from where the students are in terms of their mathematical understanding. (TE 3, June 14, 2012)

Mathematics Content (What to Teach)

While the teacher educators did not list specific mathematical content areas they thought should be taught to young children, they did talk about what teachers should pay attention to when teaching content as well as the importance of understanding concepts. They equated math content with the concepts, and asserted that teachers should help students figure out what each concept exactly means rather than knowing just facts.

I think the most difficult one for elementary teachers is the concept of the concept. Like, what is addition? And they think addition is adding two plus three and getting five. No. That's rote memorization. My elementary teachers want to just think that you memorize your addition facts and then you know about addition. So, memorization of the quote-unquote "facts" has nothing to do with the concept. So it's very difficult for me to get elementary teachers to care about the concept. Is multiplication repeated addition? No! If you want to call it a short hand for addition, but that isn't the purpose of multiplication. Where is the repeated addition in four squared? What are you adding? Are you adding four twice? No! So, to me, engaging students in the content so that they take the time to understand what it means to add, subtract, multiply and divide, and why actually the only operations are addition and multiplication – the field operations are just addition and multiplication. We teach four, but there are really only two. They need to engage in that, because it's going to help them understand why students don't understand. When I say "understand the content" I do not mean "understand the algorithms." Yes, the basic concept, not "I know how to add." I don't care. I mean, I know how to drive a car, but I know nothing about the car. So content is number one, but we have to be careful about what we mean by content. (TE2, May 30, 2012)

If you look at the book which I am using for courses for elementary teachers, you'll see that it is not a book that looks like a math textbook. It's all about, "What is the core content topic that you're trying to get through to the students." For instance, if we're going to do fractions, we talk about "What is the definition of a fraction? What's the difference between a fraction and a ratio?" And we decide that the concept that we want students to understand most clearly about fractions is that they're a part of a whole. So when we talk about fractions, we are thinking about a proper fraction, but we don't really use those particular terms anymore. And then we go through and we talk about "How do you get the kids to understand which of these could be called fractions?" because the assumption is that if something is in fourths, four pieces have to be the same size. What teachers often think is that they don't actually realize that's something that they should dwell with the kids on. I think the key is for them to understand the major ideas. (TE3, June 14, 2012)

As well as emphasizing the importance of teaching students the concepts well enough that they makes sense to the students, they stressed connections between each content area and looking at the big picture.

They should know more than merely the topics that they teach at every level. I think all of us who teach want to understand more broadly than merely the topics that we teach our students. (TE3, June 4, 2012)

That is a genuine danger, for elementary school teachers; we teach tricks rather than mathematical knowledge. So we'll teach FOIL (First, Outer, Inner, Last) rather than teach larger, more general ideas. And so, students never get the big picture. They never see a whole. (TE3, June 14, 2012)

Along with the connections between each content area in mathematics, they talked about the order of introducing content areas. There is a fixed sequence to teaching topics, and this order is very important, but their comments regarding teaching concepts following the sequence only covered number-related topics.

With regard to the sequence, I think the people who thought through learning arithmetic (adding, subtracting, multiplying and maybe or maybe not long division algorithms), as long as they're taught with meaning. I think the order is as important to me as the meaning. (TE3, June 14, 2012)

I think there should be continuity to the order that things are presented. I think that kids should always have a sense of "this built on what we just learned." In so many elementary textbooks, you're rolling along and you do something and you turn and it says "chapter four," and you start something brand new, and they don't really try to make an effort to connect what you did in chapter three to what you're doing in chapter four. That never made any sense to me. I think of mathematics as kind of like unfolding for the kids, so I always try to make sure that, just like a novel – there needs to be a little bit of foreshadowing, so if you know you're going to move into shapes or something, then you start bringing shapes into your counting. So, "Let's count all the edges, let's count all the corners, let's count all the faces." You know, "How can we stand this up, and if we stand this up, which one do I think is the base?" "What if somebody puts this down? Then this is the base." So when you move to the geometry chapter, wherever it is, you can say, "Remember when we were counting all those edges and faces? How many faces did it have? How many sides did it have?" So, I don't think you could do the geometry shapes before the kids knew how to count the faces and the edges. (TE2, June 14, 2012)

She continued,

So I would say that there would be some sequencing – I do think we should do counting first because counting just makes them familiar with the symbols which are their numbers which are totally abstract. See, a lot of teachers don't realize that. The number three is a totally abstract squiggle, and until you make something concrete out of it for every child, it's just a squiggle. It's just like learning the alphabet. Until you put a letter – until you put a sound to those shapes, they mean nothing. The same is true with numbers. So, clearly you need to be able to count before you actually start putting numbers to the things that you're counting. (TE2, June 14, 2012)

Pedagogy (How to Teach)

The teacher educators proposed several ideas about how to teach math to young children. Three major themes emerged: (a) the importance of a positive attitude about math, (b) teaching strategies, and (c) support for teachers.

A positive attitude. They all said that having a positive attitude about and passion for math is very important for optimizing math learning.

The things that optimize learning are the enthusiasm of the students, which a teacher can facilitate, the circumstances around which they can explore and bring their learning to the next level, which a teacher can be aware of and try to facilitate, and the opportunities that students have to become engaged, which again, a teacher and a community can help make happen. (TE3, June 4, 2012)

In order to help students perceive math as meaningful and enjoyable, they said that not pushing students to do math is crucial because it may harm student's interest in math.

I've seen children frustrated. I saw with my own daughter that I would kind of push more than she was interested in; I think that's not good. Some students you might have to wait out. Which is, they might not be interested now, but they will be later. My own feeling is that there are basic things that students should know. I think you have to take your lead from the child. Some pushing. I guess it depends on how hard and in what ways they push back. But, except that case, pushing kids so hard may result in loss of interest in math. (TE3, June 14, 2012)

The teacher educators also thought that teachers should focus on students who have had very negative experiences with math and try to give them better experiences.

As math teachers, we get upset when people don't think that math is important, that they dismiss it and will admit to being no good at math where they won't admit to being illiterate. They'd be ashamed to be illiterate, but they're not ashamed of being innumerate. And I think we've said enough; to emphasize that math is important is beating a dead horse. We know that math is important. But we should look to those who've had very negative experiences in math and try to give them better. We owe them more than that. We owe them an opportunity to find some joy in it, and math teachers and mathematicians, quite frankly, bear some responsibility for this because it's the academy and its elitism around mathematics. (TE3, June 4, 2012)

On top of students' enthusiasm for math, they stressed the importance of teachers' passion for math – their attitudes toward math get passed on to their students.

At the elementary level, it's important that they [students] not be frightened by mathematics, that they feel empowered by it. Many elementary teachers feel much stronger about reading than they do about mathematics. So they tend to avoid . . . well they don't avoid, but they often think mathematics is just skills, so they don't have much enthusiasm for it, so that's a problem. (TE3, May 17, 2012)

But you [teachers] have to be a regular user and you have to enjoy it. You wouldn't want to learn driving from someone who hates driving or is frightened of the road. In the same way, you want to learn math from someone who enjoys it and finds it engaging. (TE3, June 14, 2012)

One teacher educator noted the importance of a teacher's passion for math by connecting this passion with a thorough understanding of the subject.

I think you need to have a passion for mathematics. I think that you really need to understand it as just a beautiful system of organization that works not only with numbers, but with nature and concepts of science and with things like electricity. I mean, whoever thought that we would be able to control electricity to the point where it would be the major means of communication for everyone. All of our devices now are basically electronic devices. Computer science is electronic mathematics. If you don't embrace mathematics as a huge, powerful tool, then you're going to resort to just having the kids memorize. It isn't about having the kids do well on a test. If you think that you can be judged by your students' grades in your class, you're not a good teacher. If you look for the questions that your students ask that show you their understanding – I would say the most important thing is that you have a passion for and an understanding of the subject. (TE2, May 16, 2012)

Teaching strategies. When asked how best to teach math to young children, teacher educators had various ideas to offer. First, they said that assessing where students are is very important. They said that rather than focusing on test assessment for math, the focus should be on overall comprehension.

You need to work hard at assessing whether or not your children are learning the concepts and not just how to do the problem. You're paying attention to keeping the door open. The door only opens through understanding. You can fake it just so long, and then it stops. So knowing whether your students are really understanding it or whether they're faking it – it's important for teachers to be able to assess that. (TE2, May 16, 2012)

I think you would need to know where your students are at. At the early levels, especially, students are in different places with regard to that zone of proximal development. What's the next place they need to go? I think it could be very broad in high school, and I'm sure even more so in elementary school or kindergarten because you have kids who are all over the place, in terms of what they might be ready for and what they might be challenged by, and what they're interested in. (TE3, May 17, 2012)

They added that using multiple forms of assessment is a more effective way of judging students' understanding of math than a single form is.

I think one key principle of assessment is that you really need multiple forms of assessment to really paint a large picture of student's understanding. There are many types of assessment, but it's important to utilize different means of assessment. As a teacher, the first line of assessment, which is very important, is formative assessment. During class, listening to your students or watching them work, that is a huge form of assessment. If the teacher's at the board the whole time and doing all the talking and writing, then they don't have a chance to see what students are doing or listen to students. That type of assessment is really important to know, "Are my students getting it? How are they making sense of it? What's happening?" So first in lessons, that's the first key thing, but also with assessment, it's important to be able to collect student work individually, whether that's a form of a problem that a student's working on to give feedback. I still believe there's a role for some quizzes and more regular summative types of things to check measure. It's one measurement to see how they are they doing with our benchmarks. (TE1, May 29, 2012)

One teacher educator elaborated on why one needs many kinds of assessment tools.

According to him, using multiple forms of assessment give more detailed pictures of how students are thinking when engaging with math.

Some of them are formative: interviewing, listening, watching. And there's some more traditional assessment: written work, quizzes, tests. There are different forms of assessment that teachers use, sometimes checklists, rubrics. The two usual ones you think about are your basic formative and summative assessment, again both needed. I'm suggesting that if you're assessing only on traditional, turned in quiz/test/homework type stuff for kids, all you know as a teacher is what they know at the moment they took the test. And you don't know the way they're thinking about it. You often don't know what's really in their thinking that's giving them problem or what's making them successful. When you structure your class in a way that you can constantly listen, observe, watch, and interact, that's really what informs of how students are thinking in the moment. You can ask the right question or pose that next problem to help them get over a misconception in their head or rethink something. They really just work hand in hand. (TE1, May 29, 2012)

The second suggestion was using differentiated instruction. They asserted that students' mathematical competence varies especially at a young age, which makes it necessary to approach math teaching in accordance with students' individual differences.

They look at me like I'm crazy when I say, "These things are important – if you don't ever want the door on mathematics to close for your students, then you have to keep making it clear where the path is." They want to just teach them one way to add, one way to subtract, one way to multiply, one way to divide, and be over with it. I say to them, "But clearly that doesn't work for all students in this country, or everyone would be able to go to college and get a degree in mathematics." So if you're in charge of that door, and you don't do everything you can to keep that door open, then quite frankly I don't want you in the classroom. (TE2, May 16, 2012)

Sometimes you have to get rid of some of your prejudices in terms of "what's the best way to teach math," because the best way to teach math is the way that works for each individual child. (TE2, May 30, 2012)

There's a lot of fashionable discussion of differentiated instruction. But it isn't that it's a big problem: it's a problem of education in general, confronted as elementary teachers are with perhaps 25 to 30 different students, each with different strengths and challenges. So meeting the needs of all of them is always the challenge. Each of students has different dispositions, different levels of attention that they may be willing to give to your mathematics topic on that day. I can say that I would subscribe to what Richard Feynman, the physicist, said when he was asked: "After your years of teaching, what have you learned about the best ways to teach?" And he said, "Different things for different students." One person may be excited about this aspect, and another person may be about this. One person may find this box-folding very exciting and interesting, and another would find it very boring. So I think individualized teaching based on each student's interest and learning style is necessary. It can reach a lot of students. (TE3, June 4, 2012)

As a teacher, that's what differentiated instruction is about. That's obviously the big buzzword for a while now in education. In our class, if we have students who are ready for abstraction. Then if we're doing things and they are not being challenged, then as a teacher, part of our job is to think about what's the question that I can give to some of those students or ask them to think about or do, or make the numbers larger or give a different type of standard for them to work on and try. It's easier said than done, but I think that's exactly the notion of what differentiated instruction is. At different times, certain students will understand more quickly, and just trying to keep everyone adequately challenged. (TE1, May 29, 2012)

The discussion about using differentiated instruction to teach math to different types of learners led to discussions of different ways of learning. Teacher educators said that these ways should be considered when a teacher is trying to reach out to students.

So, with dispositions and attitudes, that means a couple things. That definitely means a math teacher needs to be patient. A lot of students will learn math at different paces and have different struggles as they try and make connections, so teachers need to be patient and realize that since students construct understanding in different ways. A good teacher has to be flexible and realize that some students may get it with these manipulatives; some students may get it with a real-life context; some students may need more direct instruction; some students may get it when they talk with other students. So you have to realize that students will understand in different ways. (TE1, May 15, 2012)

Third, they said that thinking processes and making connections should be the focus in math education. Teachers should avoid having students memorize facts or merely knowing procedures, and instead should encourage students to reason and think critically.

There is a famous interview with a five year-old. The interviewer is asking him some questions, and he says, "Do you want me to get the answer using math, or should I think?" I think a lot of times, we teach math as though it's just a set of rules that aren't related to thinking, and that's a big mistake. (TE2, May 30, 2012)

If our view of math is just getting the right answer, but there's a lot more to math than that. That's definitely a part of it. There is a right answer a lot of the time with most questions, so that's important, but we do focus a lot on the process. The idea of allowing students to actually gain what we call a "relational understanding" or a "conceptual understanding," which an understanding that sets kids up for success later on, is very important. (TE1, May 15, 2012)

If we just give kids a quick rule and they can follow the rule for a little bit, and it even looks like they're doing ok, but there's not really any understanding behind it. If students haven't had a chance to internalize to make sense of it through some real world interesting scenarios, or a chance with manipulatives, then often those rules are quickly forgotten. Short-term success may be easier if we're just going for a right answer. I can get my class to get a right answer just by telling them exactly what to do. For many kids, that's a short-term solution because then it falls away because there's not a relational understanding, a long-term impact. So we're trying to get our pre-service teachers to think about what it means for us to have a conceptual understanding of these math concepts. What are different ways of solving it? What are ways of posing questions to students to allow them to use their prior knowledge to make sense out of it, to use manipulatives, to model things directly? (TE1, May 15, 2012)

It's much easier for a math teacher to, "Ok, I'm on page seven. Today I teach this rule, and the kids do these seven problems, and I'll see if they can do that." And I'll be the first to admit it's much easier for a teacher to do that, but again I think, not just in our program but nationally, we're being challenged, and they've been doing that for a while, to think much deeper than that. The NCTM standards provide a goal that when problems have a right answer, that's important, but along the way we want to develop We hear the words "problem-solvers" a lot, students who have reasoning abilities, students who can communicate their ideas and communicate mathematically, students who can use multiple representations to understand things in different ways, and students who can make connections between different topics. And those process goals are a great tool for pre-service teachers and in-service teachers to start thinking about how we develop those in our classroom. (TE1, May 15, 2012)

By sharing their thoughts on the definition of doing math, they confirmed the idea of focusing on process.

When I hear the words "doing math," that often is a phrase that you see in methods books and math terminology that actually has specific meaning. For example, when we think about a teacher just telling student, "Ok, to find out if two fractions are equal, you multiply the numerator by the denominator of the second one, and then do it with the other one, and if those numbers come out to be the same, then the two fractions are equal. Now here's a worksheet, now you try it out and go through it." And then students say, "Oh this is easy. Multiply these two, now multiply these two, ok." That's not really "doing math" because for most students, they're applying a procedure a teacher told them, and they're even good at it, they're getting the right answers, but it doesn't really make sense. The teacher could have told them incorrectly and said, "Multiply these two and these two, and then compare," and the students could be following a completely wrong rule, and it's just mechanical. So it's not really doing math because doing math, part of that means understanding the math and not just applying a teacher-told rule, that often again you have no way of knowing if you're right or not without the teacher specifically telling you if you're right or not. (TE1, May 29, 2012)

He continued,

So "doing mathematics" has to involve students' reasoning, students' being able to explain and express their reasoning behind an answer. Often at times, there are times for practice, to do exercises and practice procedures, but at the same time, "doing math" more to me means when students have a problem, the teacher is not looking for one specific way to do it, even if there is a right answer. Even if you're asking the same type of question, "Ok, here's two fractions. $\frac{2}{3}$ and $\frac{7}{8}$. Which one's bigger? Why? How do you know?" To even start there, students have to share some reasoning and try to think and try and choose what type of fractions. They might draw something, they may choose and try to reason something out. But that process of trying to make sense out of it, that's what doing mathematics is. (TE1, May 29, 2012)

Doing math to me is more that aspect of allowing students to reason through situations where they're not just applying a specific set of mechanical rules. But like I said, there is a role for that more reason side, and that's important, and there is a role for practice. I do think that after students have been given a chance to make sense of fractions and do different things, I do see value in still giving opportunities to apply whatever method they're using and become efficient. If they have different methods, they need to get efficient and see different types of problems and to practice. So it's a part of doing mathematics, but often I think we leave out the important part, which is focused on the student reasoning. (TE1, May 29, 2012)

I forget who said it first, but they said what the goal of mathematics was to get from this. [drawing a question mark and an arrow that leads to an exclamation mark] And they say that this is the "a-ha!" Do I have a question, and can I find a way to an answer? And I see the mathematical process as one in which you are also going get to more questions. So, you think of yourself as engaging in something that will get richer all the time. I think this is the process that we should promote in teaching math. (TE3, June 14, 2012)

The teacher educators suggested several approaches to facilitating students' reasoning and critical thinking. These included creating optimal environments, asking questions that promote students' thinking, and not correcting but analyzing student's errors.

The best you can do is to create optimal environments for letting the process go forward. The optimal environment is something you might approach, but you're never going to reach. You can approach asymptotically as it were. (TE3, June 4, 2012)

I was leaving the school and they had the little "goodbye" thing, and they were asking the students, "What did you like best about Mrs. Lyon?" And one little girl said, "I liked the fact that she never answered a single one of my questions." And I'm like, "What do you mean? I never answered your questions?" She said, "Whenever I asked you a question, you never answered my question right away but you always asked me a question back. By the time you left my desk, I knew the answer, and I thought I came up with it all by myself!" She continued, "One time, I was at home working with my dad." I said to him, "Well, just ask me some questions," and he was like, "Well, what kind of questions am I supposed to ask you?" "I don't know, but I always get it whenever Mrs. Lyon asks me questions." So, to me asking questions is a huge assessment tool. You ask students a question and listen to what they answer, and that informs you for your next question facilitating students' critical thinking. (TE2, May 30, 2012)

There's a couple different ways that you can analyze what your students are doing, and one of them is to closely look at their work and break it down into what they're doing and why they're doing it, especially for the mistakes that they make. At what point in my career did I not correct the students, but instead analyze the students' mistakes? I don't

know when that happened, but I realized when I started analyzing the students' mistakes rather than correcting their papers, I became a much better teacher. So I thought it was important for me to give that to my future teachers. So, you put the problem at the table and say, "Okay, first off, is the student correct or incorrect?" And they're like, "Well, if the student's correct then you don't look at it." And I'm like, "No. You always look at it, because you not only have to tell me if the student is correct or not, but you have to tell me what they did." And so, they will go and the answer's correct. But they'll tell me they did it wrong. And I'm like, "How do you know they did it wrong?" "Well, because they didn't borrow," or they didn't do this or they didn't do that," but, in fact, when we all come back together, I will have tables that say, "They did it in a really interesting way." You know, we figured out what the student was thinking, and they did it on three different problems and so they clearly understand it, and this is how they multiply. And if they hadn't been forced to actually analyze the student work, they never would have thought, "There is another way to do this." (TE2, June 14, 2012)

Fourth, they discussed the value of student-invented strategies to solve problems. They said that encouraging students to come up with their own problem-solving methods promotes different kinds of reasoning, and through this process children easily make sense of the problems.

I think a good math teacher knows that there are some efficient ways of solving problems, but I believe first sometimes we'll give students the opportunity to problem-solve themselves, to scaffold and allow students to think about possible solutions and make sense out of it. And the teacher then can present other efficient solutions or ways to think about it, but usually that's done after students get the big idea or have had a chance to know what the question's really all about. Maybe they already do know and can figure out the answer and have a way of a solution that they can present to the class. Again, a good math teacher realizes that students come into class with many ideas and wants to hear from the students. They value what the students have to say. They value students' ideas and strategies, and they realize that while there may be a traditional way of doing it, there are many right strategies and many right thinking patterns to do it. And so, I think a good teacher is always looking for students and giving opportunities to see, to have them try some things and make sense of it, have students share some of their ideas, and students can learn from each other. That said, I mean, often the teacher will still need to show the efficient way or standard way. That can be helpful. But I believe that's just a way; it's another way. It's not THE way. (TE1, May 15, 2012)

We encourage student-invented strategies, that before we tell them exactly how to do it, to be able to pose a situation or problem and allow students to reason, to draw, to think numerically themselves and to share some solutions. Sometimes through that, students amaze us with what they think. Sometimes, they get stuck, and that's ok. Then through that, we can offer and show some more traditional ways, at times, of thinking about it, and they can practice and go through. But an important part there is the making meaning,

allowing students opportunity to make sense of mathematical ideas, and then we can help them by giving some algorithmic solutions and things like that, which is important. (TE1, May 15, 2012)

By introducing activities that he is doing with pre-service teachers in class, one teacher educator explained in detail why encouraging multiple ways of solving problems is helpful.

One thing we're learning about in our textbook is called "student-invented strategies," meaning that, for example, when we're working with multiplication, students will often make sense of math in different ways that make sense to them. And so in class with our students, we look at the basic facts of multiplication and thought about, "OK, 8×7 ." At this point, many of us have this memorized as college students or adults, we're at what's called "recall phase," but for students who aren't at recall phase, they have to somehow learn to know what 8×7 means and how to get that. There are many strategies to do that, so in class first sometimes I'll challenge our students to think. "Ok, pretend you don't know 8×7 . Pretend you don't have that memorized as 56. What are different ways that students could make sense out of it?" And in class, we'll brainstorm ways, and students in groups will think about it, and they'll come up with ways that obviously we could at 8 groups of 7 or 7 groups of 8. But we also look at things like what prior knowledge might a student have. For example, if a student knows 8×8 , they bring up ideas. Well they know 8×8 , then how can we get 8×7 based of knowing 8×8 . Or maybe they don't know 8×7 but they do know 4×7 . So in groups, they have a chance to think about all the different ways and share in class the different ways that students try and make sense out of it. And the book also has different types of strategies, but it's encouraging them to think about that there's more than one way to think about it and make sense of it, and think about different ways that could work. The book labels certain strategies, common strategies by students, and so you can think about what they would do to find this out using a "doubles strategy" or using "number line strategies" – things like that. (TE1, May 29, 2012)

He continued,

One big thing we're trying to get teachers to realize is there's more than one right way. But often there is a traditional way which is great, but often it's important for students to make sense out of things and so we'd like to brainstorm and look at the text and think about different ways students may make sense out of a problem. There often is a traditional strategy for some math problems, and that's good, and that's helpful, but one thing we promote in this class is, "That's just A strategy." And often it's a nice, efficient strategy, so it has good uses, but sometimes it's good to allow young children to try to make sense of it themselves and try to develop strategies themselves. Some will, some won't. Sometimes students really do surprise you. And then we can share the traditional strategy if that didn't come up, as a common strategy, but I think the notion is not to push right away, "Everyone here should do it exactly like this. Follow me. Now you do it this way." Sometimes there's a place for that after students have had a chance to make sense out of things and try some strategies. And we can say, "Here's the traditional way. Let's try this way together." But still it's the context of that knowing that math, especially at the

elementary age, there are just lots of ways to reason through problems. (TE1, May 29, 2012)

He also told a story of a mother who shared her concerns about home-schooling her daughter.

We need to give our students a chance to make sense out of things. I keep going back to that, but it's really true, that don't force your method on the kids right away. I actually shared this email with a parent just the other day. I got an email from a home-schooling parent, and it said "I'm realizing that my daughter is doing her calculations, and I ask her how she's doing it, and she's doing it in ways that are different from mine. I'll try and show her my way, and she wants to keep doing her own ways. Is this ok or not? Should I be telling her to do it the other way?" So my advice to her as a parent was that "It's a great opportunity here to discuss with your daughter that there are many ways to a right answer in mathematics and that people do think about math differently to make sense out of it." So I encouraged her, "Do share your strategies with your daughter. When she's doing it one way, help your daughter think about her method, but then share with her, 'That's interesting, but here's how I think about as well.' But don't force that method on your daughter. Instead of forcing the strategy on your daughter, allow her to see it though especially in homeschool because they don't get to see other kids' strategies. But to focus on that aspect; if it's making sense to your daughter and her reasoning is working, well then have her share that with you. Praise reasoning, praise the ideas, and share other ways as well. Some she may internalize first, some she may not, and her method ends up falling short as she gets to a harder problem, then you have the need for something new. Then there's a need, a time, to think about it." (TE1, June 5, 2012)

Another teacher educator agreed with the idea that teachers should encourage students to come up with their own ways to solve the problems.

Student teachers tend to stick to the standardized form always. The way they were taught – they just want to do it the way they were taught. It works for them; it'll work for their students. And part of my job is to convince them that clearly that's not true because there are as many ways to do a math problem as there are to eat an apple. There are a lot of ways to eat an apple. (TE2, May 16, 2012)

She continued,

What upsets me particularly in my elementary classes is that they want the student to do it their way. It's my way, or it's wrong. And I really don't like that. Especially when we have so many students who come in and out of our classroom all the time, and to assume that every child was taught to do something in the same way as whatever series of textbooks we might be using in our district is just wrong as far as I'm concerned. If the student shows you that they understand what they're doing, and they are consistently correct in their work, let them be. And if they run into difficulty and they ask you, "What did I do wrong?" you don't show them your method and say, "Well, then do it this way, because this way's easier." No. You take the time as a teacher to figure out what they were doing, and then you remember next time they're doing a problem, "We're going to

use your way of thinking about it.” So, the right way to do it is the way that makes sense to the child. (TE2, May 16, 2012)

You do have to teach certain things like how to find a common denominator, so that when they get to rational expressions, they at least have a shot at how to figure out and how to proceed. I guess there are lots of things, but I usually teach them, because I’ve built up to it. They come in at the beginning of class, and I give the kids a problem, and they can’t solve it. And then you step back and say, “Well, what do you think we need to know that we don’t know?” And as soon as they come up with what it is they want to do, like, “Well, we want to put these two expressions together, but in order to do that, we’d have to find a common denominator and we don’t really know how to do that.” And then you say, “Well, would you like me to show you some methods?” And they’re like, “Yeah”! And then you show them, and they pick up whichever one they like, because there are several usually, several ways of doing things. And you show them all, and you say, you know, “Pick one you like, one that makes sense for you.” (TE2, May 30, 2012)

The third one shared a story about his daughter.

I remember that she taught herself, when she was in first grade, how to add and subtract left to right when she had an addition problem. So let’s see, 23 and 42 [writing] . . . so she could do 65 starting with the ten, and she could be pretty reliable. She didn’t make mistakes, she got the right answer, but she had to do the corrections. Say if there was a carry of 5 and an 8, two, six [writing] . . . suppose if she was adding something like this, she would go 8 and then 2, nine . . . so she would make the marks down here in the answer rather than up here. But it worked for her, and her teachers let her do that so I was very pleased. Later on, they said that she should do it right to left with regrouping, but they encouraged her when she was finding her own way to do things, and I think that is very important. So I think her elementary school teachers were very good in that regard, letting students have their own invented ways to get to do things where it is possible. (TE3, May 17, 2012)

He insisted on the importance of not only encouraging students to solve the problems on their own but also helping students realize there are multiple ways of solving problems.

The standard algorithms are fine, but I think it’s important that students realize that there are multiple ways to get to the same place. It’s not like there are multiple right answers, but there are multiple paths to getting to the right answer, some more efficient than others. It helps develop their reasoning skills and the notion that they’re creating their own math, creating their own structures for understanding. (TE3, May 17, 2012)

I think their focus should be on the love of learning itself, on the desire to explore new things. Have you read the book by Liping Ma on knowing and teaching elementary mathematics? There is a story near the end of it where she’s interviewing a Chinese teacher, and he’s talking about his students. He’s been teaching for a long time, and he says he’s learned so much from them. He was doing a problem where they were given

this kind of kite structure and were trying to calculate the area. He had been doing it for years, and one of the students showed him a way to create a box around it to calculate the area much more efficiently. And it was after years of his doing it that the student came up with it, just kind of on his own, thinking about, “Oh yeah if I can get this one and I can get that one, it can all come out very easily,” and that came from the students. Whereas the method he had been doing for a long time was saying, “Well here if you’ve got this side then you can calculate this next one and you can calculate the area.” I’m not remembering the problem precisely, just the general point of the story that the teacher was open to learning from the student, and to be open to being amazed at the discoveries that kids would have and the things that they would show him. And I think that’s always true, and so I would think that’s what would be most important for every teacher to have and to convey: the sense that there are things out there that we need to discover, that there are new things, that it’s not just information that I have as a teacher that I give to you and then you have it, it’s that we’re building knowledge together. I know what the order of operations are, why we have to do that, and I know what the standard algorithms are. But within that framework, it’s like a sonnet. A sonnet is a poem with fourteen lines, an octet, an iambic pentameter, and so many . . . there are rules, but within that the options are so many. It’s so difficult to do one well, but looking at Shakespeare, you’re finding new things all the time. Math is like that. Instead of rules, there are restrictions – those are constant. But within that structure, and I think particularly fundamental mathematics, you’ve got to always be looking for the really creative things like, “Wow, you can add left to right, and it works.” Why one way over the other? I don’t know. You know, maybe the only reason we do it right to left is because then you make the mess up here and the answer is cleaner. But if it works as well, why not? So that’s what I would want teachers to be, the kind of teachers I would want for my daughter, which is teachers who would allow her to explore and express some enthusiasm for the topic. (TE3, May 17, 2012)

There was a study a few years back that found that students in the first grade who had invented algorithms were much more successful in third grade. So it was this process of trying to understand it on their own, trying to invent a way to make something work, that led to more success down the road. That makes sense to me. I thought my daughter could understand more because she had multiple ways of getting at the same problem, so I think there are things that students need to know and that we should help them get there. But we shouldn’t worry too much about how quickly. Speed is not the problem. (TE3, June 14, 2012)

Fifth, the teacher educators talked about the values of collaborative math learning among students.

From a teacher’s perspective, if you get your cooperative groups to work well together so that they answer each other’s questions, that gives you a much broader scope of time to walk around and observe what’s happening in the groups and whether or not the students are understanding. (TE2, May 30, 2012)

I think studies have been done. I think there were four different pedagogies that they looked at, and peer learning was the most effective. It's a study that goes way back. It was the Cost Effectiveness of Four Approaches for Improving Mathematics Instruction and Reading Performance, 1) Computer Assisted Instruction, 2) Reducing Class Size and 3) Increasing the Length of the School Day, and 4) Peer Tutoring. They found that peer tutoring was the most effective, and that's not surprising. I guess what was surprising to me was that it wasn't reported all that widely, but I think it's a common experience that being able to explain something helps you learn it and understand it well. It's as true for kids as it is for adults. All of us need to practice our communication skills, so the advantages for kids are many. It's difficult; some kids are shy and withdrawn, but we all need to be able to share. (TE3, June 14, 2012)

One teacher educator provided some ideas that can maximize the effectiveness of collaborative learning.

I'm a firm believer that if you can explain your understanding to someone else, then you really do understand it. It's not that you can just do the problem, but you actually have to internalize "how is it that I know how to do this problem, because I have to answer the question for someone else." So you have to encourage your groups not just to do each other's work, but to explain to each other, you know, how it is that you understand it. (TE2, June 14, 2012)

They talked about how best to group the kids for collaborative learning. Though there was no consistency among views, their ideas suggested implications for grouping kids.

When I get ready to put the kids into cooperative groups to start a new task, I never make any assumption about who I think is going to succeed and who I think is going to stumble. I generally try to do it just randomly. Unless, we're at a point in the year where I know that putting two kids together would be a disaster because they would fight or they wouldn't get along or whatever. I watch and see if the way in which the groups are organized – then I can determine who is going to do well on these tasks and who is not. So, that's why I like to change my groups a lot. The kids don't understand why I'm changing them. But then the next day, if they come in, I will have made assumptions that this group really got it, so I'll totally split that group up and put each of those four kids in a group that didn't get it. So the first thing that I'm doing is I'm making sure that at least every group has a good shot at it, because I've attempted on my second day of grouping to put somebody in the group who gets it. And then look at who is very vocal but didn't get it, because sometimes the kids who get it are the ones who are not vocal. So, it's really hard to describe. But, the ones that you really need to be most concerned about are the ones who are not getting it and are not vocal. So you have to place them very carefully to make sure that somehow or another, their needs are going to be taken care of. You put them with the person who is always, "Do you understand this? Are you with us?" So, I look at it not only from the concept of ability but also from the concept of personality. (TE2, June 14, 2012)

I think that would be one of the things that would take the most time: figuring out which groups work best. It seems to make sense to say, okay, we'll put a struggling kid with an energetic kid, but you never know which personalities are going to work. Attendance patterns would factor into that too. You want to have students who are very sharp, but is it reliable for being there in a group project? All the factors that would go into if you were working on a team and some of the difficulties you would have negotiating that team. Different people will take different roles and will have different strengths; maybe it's not just a quiet kid and an outgoing kid, but a student who feels more comfortable with being artistic and will say, "I'm not good at math, I have nothing to offer," creating the problems that ensure that kids do have something to offer that builds on their separate strengths. (TE3, June 14, 2012)

While they admitted that there were some possible drawbacks of collaborative learning, they also provided ideas on how best to overcome those drawbacks.

The only downside I ever see to cooperative groups is if you determine that you're going to give a group grade, and then someone in the group takes that to mean, "I don't have to do anything and I'm still going to get a good grade." And there are kids who will be free riders. So you have to find ways of making sure that if there is a free rider in the group, and you are going to give a group grade, that the group understands that they need to put some pressure on this person so that this person understands their responsibility in the group. And there will be times when the group does have someone in the group who is not working, and they need to come and tell you that. And you say, "I promise you your grade will not suffer." And then you take that child aside and you say, "You know? I'm not going to grade you on the group rubric. I'm going to grade you on your rubric and I'm watching you. So, if I don't see you doing your part, you are not getting the group grade." (TE2, June 14, 2012)

Lastly, teacher educators agreed that integrating math with other subjects could be an effective way of teaching. They, however, doubted the efficiency of that method of teaching in the current test-oriented school environment.

Mathematics is the language of the universe. There you go. It doesn't matter what you want to study. There is mathematics in it. And if you don't exploit the rest of the universe and use it to teach mathematics, then you're wasting a wonderful opportunity to convince students of the power of mathematics. (TE2, June 14, 2012)

I think integrating other content is effective. I think it's important, I think it's correct. It's probably not efficient. By that I mean that the experience of individual teachers has taught that if students are going to be assessed on tests, the thing to do is to prepare them for those tests. And usually those tests don't have anything to do with math, science, art or those other things. So, we measure efficiency against the speed for which, for example, they would learn the algorithm for adding fractions or long division. It's probably more

effective to teach students in the context of other things, but it's probably less efficient as measured by performance on the upcoming assessment that doesn't include them. I think it's more effective to have a broad view of mathematics. I don't think it's efficient. (TE3, June 14, 2012)

Support for teachers. The educators all said that in order to teach math effectively and meaningfully, support for teachers is needed. One way to provide support for teachers suggested was collaboration among teachers. They asserted that collaboration in teaching is not merely beneficial but essential because it enables teachers to develop professionally within a community of professionals.

I guess it starts with the assumption or belief that even the best pre-service program doesn't fully prepare people to go teach the rest of their lives. It gives them a foundation to start with. But once they go out get their first job, they need avenues to continue to grow and learn. I can't cite research, but I know that it exists. But looking at what's involved in that, a big part of that is not being an individual, alone, isolated in the classroom, but having discussion with other professionals. (TE1, June 5, 2012)

I think it's extremely important that the students understand that they are going to work within a community called the faculty. And that you are going to – in most cases – take students from one teacher and pass them on to the other. And if the teachers who had the students before you and the teachers who will get the students after you are not all part of the same conversation, then the student suffers. So, I'm a huge believer that the purpose of being on a faculty and building camaraderie on a faculty so that everyone knows what their place on the passage of this student through the building is. And also, like for instance in my classes when I put my kids together on the four at a table, and they come from all different schools, the way they approach the problem is different. And they learn from how each of them was taught, and just that alone brings out the point that not everyone teaches it the same way. And you shouldn't actually teach anything just one way; you should teach it as many ways as you need to in order to reach all of the students in your class. And if you're never exposed to those other methods through collaboration, how will you even know they exist? You don't have to reinvent the wheel. There will be members on every faculty who have been there for decades, and they're just a wealth of information, and if you don't take the time to collaborate with the older members of the faculty, you're not doing your students justice. (TE2, June 14, 2012)

I do think this working in groups, collaborating with others on mathematics programs and having to reason with each other and learn is ok sometimes. If I wasn't wrong, that we can learn together is a great start because where a lot of fear comes in math is that I'm afraid that I'm wrong, and that's when we think of teaching, teachers who are not confident, often their classrooms are very procedural because they'll really feel like, "I know one way to do this, so I'm going to teach my students this one way to do it because

then if we all do it the same way, I'll know if it's right or now, and I'll feel ok,” where if we're not confident in being able to think about if a student tries a different method or has a different way of thinking about it, that I could reason with the student to see if that's a correct way to think about or not or just to be ok with that there are more than one ways to think about a problem. So in the groups, they experience that, they often will think about things differently, and they have a chance to share that and see that. It's the beginning of a culture where that's ok. (TE1, June 5, 2012)

One teacher educator shared his past experience as a teacher and explained how he benefited from collaboration with other teachers.

I began my teaching with a vision to try and teach meaningfully, and it was hard. It was hard, and often I felt under-resourced, under-timed. It was difficult, but luckily for me, I was in a department that did have a community where there were two female teachers who had some seniority, and both of them were big NCTM believers. They had a vision of us being a standards-based department, promoting communication, promoting reasoning, using projects and activities to teacher. So I walked into a department where they would share resources. They had a vision in our meetings where they would talk about it and share some of what they were doing. And they didn't always have success either, but they would be open, and that was really big as a young teacher, to see other teachers who were trying to have a student-centered classroom and supporting me. I taught there for 10 years, and there were ups and downs, successes and failures, but I learned a lot along the way. And it was important that I had some mentors in the beginning to help me to not give up and to support and to help give a vision and support of that vision in teaching. (TE1, June 5, 2012)

He also described a current movement in schools that seeks to promote collaboration among teachers.

In the U.S. right now, a big push has been PLCs – professional learning communities – as well as at the elementary level, a new onset of, I don't know what the elementary term is, I think it's “elementary specialists,” often like a coach, a math specialist in the elementary schools who can work with and coach other teachers so that they're not alone, but they're collaborating with a specialist who doesn't have their own classroom all the time, but their job is to work with teachers. That's been a relatively newer, commonly accepted practice in the US that's growing, and I think it has a lot of merit. Just reflecting on practice, being exposed to new ideas, having other people's perspectives on our classroom and our teaching. That's a wonderful avenue for growth, and so we like our teachers to have that mindset, that it's okay to share ideas, that I can learn from others, that there's more than one way to think about this. So I think that's a mindset of a teacher that we'd like to promote and have them leave here expecting, wanting it. I hope they want to work with other teachers, to share ideas in that way. (TE1, June 5, 2012)

Another, however, brought up the closed culture that often exists among teachers and addressed the difficulty of working as a community.

That is something that not many teachers have the opportunity to do, but which really needs to be emphasized, that you aren't teaching alone. As a school teacher you often shut the door, and it's you and your students. I think teachers need to support each other and work collaboratively, and that is sometimes a challenge. (TE3, June 4, 2012)

The more I see it, the more I believe it's essential; the more I observe teachers, and the more I reflect on my own teaching. One of the great problems is that teachers act individually and solve problems on their own that could more easily or readily be worked out with the community. It's not the kind of knowledge that one can read in a book and acquire, or a research article for that matter. There may be some tricks that you could get from a workshop or reading, but really, what's needed are communities that meet and talk about ideas and solve problems together. But there is rarely enough time for teachers for that. It hasn't been part of the structure that we have in American schools for that collaboration, so it's a problem. (TE3, June 14, 2012)

Second, they discussed the importance of collaboration with parents. They used their own experiences doing activities with their own children as examples of how parents could become more involved. They also suggested possible activities parents could do with their children in order to facilitate math learning.

I think attempting to engage parents in the child's learning of mathematics is beneficial. And if the adult who you can engage is a parent, that's the best of all possible worlds. (TE2, June 14, 2012)

We read tons of books to our kids, and so I intentionally chose books that had to do with numbers and counting when they were as young as two. I don't know exactly when, but even while they were in their twos, they were interested in books. We had basic counting books, and they could eventually count with us. Different books had some interesting takes on just comparing again and thinking about "how many more." And intentionally that way, getting them to think about number. And then again more as a parent more than a teacher, with young kids just the idea of. "All right, we're eating today, let's see how many goldfish you have." And just counting with them, using amusing everyday experiences to have them realize that number is a powerful tool to describe what's happening around them. (TE1, May 15, 2012)

Family math night and things like that, and having children take home problems. It's kind of fun if children take home problems that they can work on with their parents. For instance, here is an algebra problem. If you have three toothpicks that make a triangle, and then let's say, if I add two more . . . well let's say there are tables that I'm making out

of toothpicks. This is one, and now I have two. And now if I do a third one . . . so I have one, two, three, four, five, six, seven . . . so now if I keep doing this, I have seven toothpicks to make three tables. How many toothpicks would it take for me to make ten tables? I think that that would be a fun thing for a parent and a kid to work on, the pattern. You could go crazy with that and say, okay fine, this is one, and then if I add this . . . this is two now, this whole thing. And then . . . [laughs] I could keep going and keep going and say okay, how many toothpicks when I had a side that was ten. And I think those are fun things to work on together, depending on the child and what his parents are up for. (TE3, June 14, 2012)

One, however, emphasized that when facilitating parental involvement, teachers should not deny children who don't have parents providing support for them. She offered some ideas for how to remain inclusive to children without support.

But if the teacher goes into it with the assumption that the parents are going to be participants in the child's learning in any grade level at any semester, then some of those children are going to be denied because there are no parents there. Or if the parents are there, they don't have the time to do it. Or, in some cases, they don't have the skills to do it. So, the only time that you can actually count on the parents doing a really good job with the kids in ways that you would like them to do it, is when you bring the parents into your classroom. You help to train them in what it is you want to do. And you will see, early on, which students never have anyone come with them, and then you can elicit another parent and say, "When we do these activities, can I put Susie with you?" or "Can I put Jay with you?" or you can enlist other people from the community to come in and take on the role as the adult. (TE2, June 14, 2012)

She continued,

Because I think it's wonderful for the kids to see adults doing something that they're doing, so if they say, "When will I ever use this?" and they see adults all around them who are using it all the time with them, then that question doesn't come up. I'm going to be doing this with my children when my children are in school. This is how I'm going to engage with them. These are the activities we're going to do that are fun and we learn from them, and it's time I get to spend with these people in my life that I really like. But unfortunately, the vast majority of schools in this country do not have parents who are engaged the way they should be engaged. And once again, the teacher can't fix that. So you have to be careful in making sure that you don't isolate some of your kids because you expect parental involvement. (TE2, June 14, 2012)

Teacher Education Programs

Of the three teacher educators I interviewed, two were critical of teacher education programs. The first concern they expressed was that the few math courses in the programs were

insufficient to prepare teachers to teach math well.

I feel, especially in the United States' education system, one difficult thing is that our elementary teachers aren't specialists, which can be a strength because kids have the same teacher for a major part of the day, but it's also a weakness because for many of our elementary teachers, math is often the subject where they don't have self-confidence, or their own depth of knowledge in that area isn't extremely strong. That impacts the choices they make as a teacher. If you want students to reason and communicate about math, but if you're a teacher and you don't have confidence in your ability to really understand things deeply, it won't happen because you're like, "I know how to do the problem like this, so I want to make sure my students do the problem exactly like this because otherwise I'm not sure about other ways or if it's right or not." So the math knowledge is very important. They take a few math courses in the mathematics department and few methods courses math in college of education in which we try and look at some things as well... and again I've heard various things about the quality of those classes and am not sure taking just a few courses is enough to make pre-service teachers well prepared to teach math or have confidence in math and teaching it. (TE1, May 29, 2012)

I don't like our program here because we only require the students to take one course, and that is certainly not sufficient. I know, in this country, that we are generalists, and therefore they have to teach everything, and we can't ask them to take three courses in everything, but I think mathematics is important enough. I believe it is one of the major gateways to barring students in this country from going as far as they can in education, and I think we should have our teachers take at least three courses in mathematics. (TE2, May 16, 2012)

They also talked about the right timing to start taking math or math methods courses. One teacher educator said that pre-service teachers should not start taking math or math method courses as freshman. She seemed to think that freshmen are not mature enough to understand the importance of being a teacher in the front of the classroom. They need to postpone taking math or math method courses until they are ready to see themselves as teachers.

I think they should not be taking it as freshmen. They come in as freshmen and say, "I want to be a teacher" and so the first couple of years they go through courses that actually show them what it means to be a teacher. And then they can decide if that's what they really want. I'm trying to get them to understand what it's going to be like to teach mathematics in a classroom, but they're not even in college of education yet. So there are things about the program I would change, but for me the most important thing is for them to understand the importance of being the professional in the front of the room who is

supposed to bring children to an understanding of mathematics. And they can't do that unless they themselves develop a much richer understanding. (TE2, May 16, 2012)

Second, the teacher educators stressed the importance of on-going teacher education for in-service teachers because once pre-service teachers become in-service teachers, it is more difficult to improve their ability to teach math.

When they leave pre-service instruction and get certified, it's just the beginning, and it takes more than just a pre-service program. It takes continued induction and new teacher support. (TE1, June 5, 2012)

Sometimes young teachers who just graduated from college naturally get an idea and a vision, so they go out, and they are just ready to change the world. They go into their class and try and teach the lesson, and it just totally bombs, and the students are confused. They say, "Wow, I'm not a good teacher," or "It's just easier if I tell them what to do." They need a mentor who's there to help them think through it so that their confidence doesn't go. Let's think through why, let's change it, let's adapt it so that next time what would we do, and let's keep the vision. Let's not just give up. How can we provide that support? That's hard, especially when you leave the college and become in-service teachers. (TE 1, June 5, 2012)

Summary

The teacher educators generally agreed that children have some math ability and are ready to learn math. They talked about some signs of readiness to learn math that children show. They pointed out, however, that the views toward children's capability for doing math can be different based on how you define math. They mentioned the importance of understanding children's mathematical cognition.

They didn't specifically address which content areas should be taught to young children. They, however, equated math content with the concepts, and emphasized thorough understanding of these concepts. They also discussed the connections between each content area and stressed looking at the big picture in math. They agreed that there is a fixed sequence to teach topics and this order is very important.

They proposed three big ideas about how to teach mathematics to young. First, they talked about the importance of a positive attitude toward math. They asserted that to optimize math learning, both students' passion and teachers' passion for math are needed.

Second, they suggested the following ways to improve teaching strategies in math: (a) knowing where students are through multiple forms of assessment, (b) providing differentiated instruction based on students' individual needs, (c) facilitating students' reasoning and critical thinking, (d) valuing student-invented strategies to solve problems, (e) encouraging collaborative learning among students, and (f) integrating math with other subject areas.

Third, as ways of supporting teachers, (a) collaboration with other teachers and (b) collaboration with parents were seen as critical.

Most teacher educators said that current teacher education programs are not satisfactory. They said that taking a few courses is insufficient to prepare teachers to teach math well and that these courses should be introduced to pre-service teachers not in freshman or sophomore year but later. They also mentioned that the target of teacher education should also embrace in-service teachers.

Chapter 6

Psychologists: Math Education for Young Children

I initially interviewed eight psychologists. From these, I selected three who were willing to share additional thoughts on math education for young children. I made some observations from the first round interviews with the eight psychologists. Psychologists have a deep understanding of children's mathematical thinking, but this knowledge often comes from a controlled laboratory research or one-to-one teaching. They rarely attempt to apply this knowledge to classrooms or integrate it into educational approaches. These initial observations gave me some ideas about what I should delve into more deeply during the second and third interviews with the three psychologists.

Of the three psychologists I interviewed in Stage 3, two were cognitive psychologists and one developmental. One of the cognitive psychologists is in a college of education, and the other cognitive and the developmental psychologists are in psychology departments. I interviewed the first psychologist in his office, the second, in his office (first interview) and at a coffee shop (second and third interviews). I interviewed the third, who said she was too busy to meet in person, by phone.

The first psychologist I interviewed is a cognitive psychologist whose research focuses on the teaching and learning of basic counting, number, and arithmetic concepts, as well as children with learning difficulties. His main responsibility has been research, but he also has done some teaching related to mathematics education for young children. I selected him because of his experiences as a researcher and an educator.

The second psychologist is a developmental psychologist. His research interest concerns children's innate perceptual abilities related to math. He has three children of his own, and when

his oldest son was in preschool (in the early 80s), he went into his class and taught his son and classmates Logo, a computer program created in 1967 for constructivist teaching by Daniel G. Bobrow, Wally Feurzeig, Seymour Papert and Cynthia Solomon—in Logo children move a turtle and, as they do so, learn different math concepts, for example, estimating how many units the turtle needs to go forward to reach the other side of the room. I selected this psychologist because of his developmental perspectives. His volunteering experience in preschool also intrigued me.

The third psychologist studies cognitive development with a primary focus on how children think, learn, and solve problems in mathematics. Her work encompasses several interrelated areas such as numerical representation, symbolic reasoning, concept construction, skill acquisition, and problem solving. I selected her because of her research interests, and because, unlike the first two, she was a woman, and a junior faculty member.

This chapter has four sections. In the first, I look at psychologists' views of children's mathematical understanding. The second section focuses on what mathematical content should be taught to young children. I then look at the psychologists' ideas of how to teach mathematics to young children. In the last section, I explore their perspectives which integrate all three areas discussed in the previous sections.

How Children Learn Mathematics

All the psychologists saw children as capable of doing math. They agreed that children have a more sophisticated understanding of math than most people think, and that all children are ready to learn math.

Young children are capable of learning a great deal. For example, they form some basic understanding of numbers and which number's larger, and they have some understanding of the operations of addition and subtraction. They can even construct a concept of infinity if led in that direction. I asked one little kindergarten girl, just out of curiosity,

what she thought the biggest number was, and she thought for a moment and said, “A million!” I asked, “Well, what number do you think comes after a million?” Well, that really surprised her. She thought for a moment and said, “A million and one?” and I said, “What number do you think comes after a million and one?” And she thinks for a moment and says, “A million and two!” And I asked again and she said, “Oh wait, there is no largest number.” So, that is the level of abstraction that kids can get to. Now this girl probably was not typical. I mean she had a lot of informal experiences probably provided by her family. But it illustrates what kids are capable of doing if given the opportunity. (PSY1, May 18, 2012)

I think mathematics is something that kids can do as a very early age but obviously it needs to be tailored to their interests and their abilities, just as anything does. (PSY2, May 22, 2012)

Well again, certainly in my own, partly because I was interested in it, but the idea of counting, making size estimates and doing some of the things that I said, I mean they [children] are able to do that without much difficulty. A lot of the things that we think of as math, I think they're able to do. It's just then to begin to connect that to more formal ways of doing it. (PSY2, May 22, 2012)

I think that young children have a pretty sophisticated understanding of math, more than a typical person would think. There is controversy about whether it's innate or not, but it looks like they have an understanding of at least two basic numerical processes. So they can represent small, exact numbers – one, two and three, and they can perform simple arithmetic calculations on those numbers. They also have a representational system for representing large approximate numbers. So they can tell you if you show them ten dots and twenty dots, they can tell you which one is larger. By kindergarten, most have an understanding of cardinality and can count, and they understand that when you count, the final number that you say is the total number of objects in the set. Those are the three main areas. They have knowledge of small exact numbers, they have knowledge of large approximate numbers, and they are early understanders of integer counting. (PSY3, July 31)

I think that they're always learning math from the time that they're born, so I think they're always ready. I have a twenty-two month old, and she's already interested in numbers, she tries to count. She's right now only counting one-two-one-two, that's how she counts, and I'm not sure if you're familiar with Karen Lin's work, but typically when they're learning the integers, children start by being one-knowers, so they can think they understand one and only one. If you ask them for any more than one, they just give you a pile. They don't distinguish two from three from four, it's just one versus all, everything that's there. Then they become two-knowers. So they know one and two, and then everything that's bigger they just sort of . . . that's all bigger. That's where my daughter is now. She knows two but nothing bigger, then they become three-knowers where they know one, two, and three, and everything bigger is sort of up there. And then it looks like that when they become four-knowers, that's sort of a major shift, and then they can understand any number in their count list. Now that's a controversial, a little bit, but it's

either right when they learn four or very soon after that they seem to understand the integers. So I would say I think children are always ready. (PSY3, July 31)

One psychologist summarized five key signs of a child's readiness to learn math, which are prerequisites for mathematical learning.

Several key signs are, can the child recognize small collections of one, two, and three and be able to label them with a number word? That turns out to be a very critical skill because it allows kids to do so many other things. It's the basis for doing so much else. For example, if a child can recognize a collection of three items and say, "That's three," it's much more likely that when teachers try to teach them how to count, it's going to make sense to them. So if they can see that's three, and the teacher goes, "Let's count! One, two, threeee," and emphasizes the last term because that has special significance, it indicates the total. Well, a child, who can see that's three already, will have a better chance of understanding why the teacher is emphasizing that last term or why the teacher repeats the last term. One, two, three. See, three. It'll make much more sense to them. (PSY1, May 18, 2012)

Subitizing also allows kids to see operations on number and begin to construct an understanding of addition and subtraction. So they can see that that's three, but if one item is taken away, now they see it's two. And then this also allows them to start building up an understanding of the ordinal meaning of number, which is three is more than two; you can see that three is more than two, two is more than one. So this ability to recognize very small collections is very important skill. And what we see among kids who are really struggling in Kindergarten and first grade, they don't have that skill. (PSY1, May 18, 2012)

Another important skill would be the ability to read numerals. The one digit numerals. Ability to read the numerals is the bridge between the informal knowledge of number and formal knowledge of number and arithmetic. My post-doc did an analysis where he found that the ability to read numerals and relate them to specific quantities – that is, recognize that four represents four things – those two skills together mediate informal and formal knowledge. Basically if a child has those two skills then there's a bridge between formal and informal knowledge. So reading numerals is very important. (PSY1, May 18, 2012)

Another skill that I would say is highly predictive. One would be the ability to compare numbers, recognizing that seven not only follows six in sequence, but it represents more items than six. So if you ask a child which is more, seven or six? They can tell you seven is more because it comes later in the count sequence so it's a bigger number. And that's a very important skill, especially if you're trying to understand formal arithmetic. It doesn't make sense to see an expression "six plus one equals what?" or an equation "six plus one equals seven" if you don't understand those ordinal relations. (PSY1, May 18, 2012)

And the other skill that I would say is very important is the ability to add one. How much is seven and one more? But kids will know, for example, that seven comes after eight for

quite a while. But if you ask them how much seven and one more is, they have to count, or count out blocks or not get the answer at all, then they don't see the connection between adding one and the number after knowledge. But once they understand that adding one is like just going to the next number in the count sequence, voila! Then they can add one with any number for which they know the count sequence. And this is the very important bridge between concrete addition and mental addition in various forms, including the ability to estimate, the ability to count on. For example, seven plus three – a child might think, “well, if seven plus one more is eight, then seven and three more must be three more beyond seven! So that's eight, nine, ten.” Also, being able to add one allows kids to do all sorts of other reasoning strategies such as, if four and four is eight, well then four plus five, five is one more than four so it's one more than four plus four, and so it's, eight plus one is nine. So I would say those four skills are pretty important. And there's research to suggest that they are. (PSY1, May 18, 2012)

One area where views overlapped was that children have their own unique mathematical competence. This was interesting because they did not see differences in mathematical competency as a difference in ability. Rather, they considered it an issue of timing and opportunity.

The fact that you believe in individual differences doesn't mean that teaching doesn't count or curriculum doesn't count. But I think you can't start off with a belief that everybody's going learn this by a certain time period. They're going learn different amounts. (PSY2, May 22, 2012)

It's not an ability that you either have or don't have. We're talking about building knowledge, and, therefore, it's a matter of how much practice you made. Some kids will do it once or twice and understand the idea. Other kids may need to do it ten times or twenty times. So it's kind of a timing issue. (PSY2, June 5, 2012)

I think there are ability differences but it's probably more helpful with children to think of it as timing because you have no way of really knowing accurately what their full capacity is. I think it's probably too low to. And there'll be indications of that, how quickly they pick things up and so on. I just worry, in talking to teachers, that's probably a better way to talk about it so they don't prematurely start making estimates of intelligence and deciding some kids just can't do certain things. (PSY2, June 19, 2012)

Keep it in mind that there are absolutely huge individual differences almost from the beginning. I would say that the key basis for individual difference is experience – the opportunity to form informal knowledge. Some kids have rich opportunities to do so. Their parents are talking about numbers all the time, they're engaging the child and looking at the world in terms of numbers and thinking mathematically. They encourage the kids to think mathematically and solve problems, and other kids get almost nothing. And so what you see beginning at around two and a half or so are kids who are beginning

to recognize one, two, and three or at least one and two. And other kids who can't tell the difference, I mean, they can see there's a different connection, but they can't label them. Without labeling them, it's going to be real hard to remember things and do arithmetic in your head and other things. So there is just huge difference in terms of experience, and that can account for much of the individual differences we see. There may be an innate component that permits some kids to find patterns more easily, to make more connections, to be more interested in math, but my guess is that's probably more secondary. (PSY1, May 18, 2012)

Most of the variance is due to opportunity, rather than ability, certainly at the elementary level. People who have more opportunity are going to do better in terms of elementary achievement than those who don't. That's not to say some of the variance isn't accounted for by ability, but I would say that would be less than what's accounted for by opportunity and motivation and other factors. (PSY1, August 2, 2012)

A least some of the gap comes from the nature of children's environments at home. So there's a lot of evidence to suggest that children from low income environments are behind, even prior to the start of formal schooling. It's related to not only the kinds of activities the parents do with the children involving number and counting, but also a recent study suggests that the kinds of vocabulary words that parents use early on, so their type of math talk in their language affects it. So I think the environment plays a big role. (PSY3, July 31, 2012)

They put understanding children's mathematical cognition as an important factor for teaching math effectively.

I think that teachers, especially in elementary school, have to take developmental psychology classes to understand broader issues of cognitive development and how children's language skills at certain levels of development might affect how they're going to learn math, and children's understanding of symbols just generally, not just math symbols, and how their symbolic development is going to affect the way they understand what you're trying to teach them in mathematics. (PSY3, August 7, 2012)

Certainly, understanding how children think is a lot more important than anything. I don't know how you could do early childhood education if you don't have some knowledge about children's cognition. You have to be able to be a good evaluator of an individual child, what motivates them, what interests them, so that you can be flexible with the strategies you use, the particular examples you draw on. I think good teachers do that all the time. You know, it's a Vygotsky thing: you know what their zone of proximal development is, and you know what their particular interests and their particular . . . the things that get them excited and the things that lose their interest. (PSY2, June 5, 2012)

Mathematics Content (What to Teach)

The three psychologists laid out a wide spectrum of mathematical content areas which should be taught to young children. This included counting and operations, geometry, and measurement. While discussing these content areas, they often mentioned children's cognitive ability related to each content area. From their point of view, children's cognitive ability provides a good basis for understanding what children need to learn because it shows that children are capable of understanding the concepts in each content area.

Ultimately, they're going to learn words and concepts that are going to go beyond perception, but you can build on that if they make the discriminations effectively. That can help with the concept of numbers and placing words, different sets of things, and the idea of sizes getting bigger when you're measuring. I think they can understand that pretty well. Even things like areas, they can estimate this pie is bigger than this pie, that kind of thing, so the things that eventually they're going to learn in geometry, they can see all those differences. They can see that some angles are bigger than other angles, for instance, understand the idea of angles and, even the difference between polygons and circles things like that, having angles versus not having angles. Again, those are all perceptual discriminations. It seems like when we start with older kids, we too often go right to the cognitive stuff. Try to convince them that we can turn a polygon into a circle by increasing the number of angles. I mean, that's too abstract but if you show it to them, they can understand it. They can understand that a stop sign is more like a circle than a square is, for instance. So the idea that as you're making your angles bigger and bigger, you're increasing the number of sides and you're moving toward ultimately a circle, just as an example. (PSY2, June 5, 2012)

Counting and operations. The psychologists placed counting and operations as the content areas that should be taught to young children. For counting, psychologists addressed (a) a child's ability to understand a small exact system, (b) a child's ability to understand a large approximate system, and (c) a child's perceptual ability to visually quantify the objects. For basic operations, specifically for division, psychologists mentioned children's ability to understand equal partitioning, which is an informal analogy for division.

I think counting and cardinality is a key skill that they need to know before and during kindergarten, before they're going to move on to grade school. I think getting a better understanding of the base ten system is important, especially for children in the United

States. For the numbers eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, because that is one difference that we have from children who are in East Asian countries where their language makes the base ten system more transparent instead of saying eleven, twelve, thirteen, they say ten-one, ten-two, ten-three. So I think that is an important area, working with numbers from eleven to nineteen, so they're going to be gaining a better understanding of place value. (PSY3, August 10, 2012)

If you think of that as a perceptual ability, kids at five have memory for pictures that's actually better than adults, and we've known that forever. Their verbal memory is poorer than adults', but their visual memory is actually better. So to me, that would be how you would start it. You would do stuff that's highly perceptually based. For example, you can make use of that perceptual ability in teaching how to count. Counting is a very sensory/motor kind of thing, so there's no reason they can't start counting, dealing with that at a very young age. (PSY2, May 22, 2012)

Right now there's some debate about how children learn counting. Is it building on the small exact system? Or is it building from the large approximate system? Or is it a combination math between those two systems? I think that would be something very helpful to know because then we could know whether we should be focusing on activities to draw out or promote linking between that small exact system to counting or the large approximate system in counting. Based on the current evidence, it's probably the case that doing activities that help draw on children's existing knowledge. So I think that's an effective teaching strategy across the board, like starting where kids are and then getting them to the next thing. Since there's a lot of evidence that they do have this understanding of small exact and large approximate, you might start with activities related to those and then bridge them to the next thing you would want to teach them, which is probably the integers and counting. And then after you have a sense of that, then you can move on to – well they even have an intuitive idea of some basic addition and subtraction, say, again, when you're introducing addition and subtraction, then maybe start by introducing it in the context of those core systems – the small exact and the large approximate and then build on it to the integers. (PSY3, August 10, 2012)

I think all those basic things, see the focus on things like adding and subtracting, to me those are the next things you should do. And I think at that age, they could begin doing those. You could do real simple stuff with that. You have three dolls and you take one away, how many do you have now? You have two dolls and we're going to put another one in, how many do you have? So they certainly could have the rudiments of addition and subtraction from a pretty early age. (PSY2, May 22, 2012)

How do you introduce division to children? Well, it goes back to the big idea of equal partitioning, and an informal analogy for equal partitioning is fair sharing. So I've got twelve apples and I want to share them fairly among four people, what's the size of each person's share? I divided it up one at a time to each person and find out there's three apples in each pile. That's basically your informal basis for understanding division and children clearly understand this concept. (PSY1, August 2, 2012)

Geometry. Another content area thought important was geometry. Among several subtopics in geometry including shape, location direction and coordinates, visualization and spatial reasoning, and transformations and symmetry, the psychologists focused on shape, angle, and distance, where children have perceptual ability to distinguish the differences.

Understanding shapes, I imagine, is part of the geometry standard, and I think that's important. (PSY3, August 8, 2012)

You should try to build on competencies children already have. In this case, the competencies are perceptual. They don't have words for them, so they don't really have ways to think about them. So you're giving them a way to think about things they already know about. To me, that's the key thing. It's not that you have to drill them, but you really are giving them new content just as you are when you're teaching them the letters of the alphabet or teaching them words for things. So I think that when you teach some things, you get them to say that these things are different and then you give them names for things, and you get them to see that there are differences in size and then you give them a way to measure the difference in size, and then you ask them to show you that they know those things. Again, not on a written test, I would do it more on . . . so if you've got a group of kids that are around, you hand out different shapes and then each of them has to tell you about its shape, and tell you as many things as they can about it. So they're being told, they have to produce and show comprehension and memory of those words. The words in this case represent basic mathematical concepts, so you could ask the little kid who has the circle, "How's the circle different from all the other ones the kids have?" They should be able to talk about that, they should be able to say it doesn't have any corners on it. If you take your finger and just keep moving around it never bumps into anything. (PSY2, May 22, 2012)

I think they become able to understand the abstract ideas, but I just think from what we know of children's cognitive development, they start off more perceptually and eventually they develop cognitive understanding of things out of their interaction with the world. So drawing from their perceptual abilities, allowing them to manipulate things, just like the example I gave from vision about a square versus an octagon and a circle, if you just have them feel the objects and say which of those two is more like a circle? They begin to get the idea that as you add more to the sides that you're moving out toward the circle concept and that the circle is an object that doesn't have angles in it—it's a rounded object without angles. Which is an abstract kind of thing or, especially the distinction between a circle and non-circle? It sounds abstract, but I think getting to it through, again, the perceptual mode of exploration, it becomes easier. (PSY2, June 5, 2012)

For example, how do you present an angle to a child? What is an angle? And my students by and large would tell me how they would explain it, and I'd say, "Well I'm a child who is six years old. How am I going to understand what that means? What does

that even mean?” And quite often, their definitions were just partial definitions and not totally correct. The distance between the two lines, the two rays. Well, is that the distance here? The distance here? The distance here? No, it’s none of those. So what would be a useful analogy for helping young children understand what an angle is . . . well, a very nice analogy would be, the amount of turn you do. So if you do a one-eighty, you go this far. Halfway around. You do a three-sixty, you go all the way around. So an angle, informally, is simple a measure of how much turning you’re doing. (PSY1, August 2, 2012)

Measurement. They also talked about measurement. The reason why they recommended measurement as a content area that should be taught to young children was because, according to them, children have the perceptual ability to understand sub-topics in measurement.

Weight is another perceptual thing. So the idea that things weigh different amounts depending on how heavy they feel in your hands – they could start there. I think there are plenty of things we could teach them. The idea of using a ruler to decide which of these things are longer, so you look at them and for them to learn, then it's hard to tell with those two, but if you use a special measuring stick, you can put it next to them, and it can tell you. (PSY2, May 22, 2012)

I think they [children] are able to estimate sizes and distances. They clearly have the idea that we can do that, and that’s useful. They do a lot of, “Which is bigger? How is it bigger? How could we tell which is bigger, which is heavier?” They could start learning things like – this is getting into some Piagetian conservation but even at a younger age – the idea that a ball of clay and a balloon the same size look the same but they weigh different things. Weight means how much they push down in your hand. So I think that idea that things have different mass, things have different length, collections have different numbers of units, and all the basic building blocks of math, I think they could do. You could do little measurement exercises where the kids are measured against the wall. (PSY2, May 22, 2012)

When listing content areas that should be taught to young children, they emphasized the importance of understanding of the concepts in each content area.

The most important thing is to understand the mathematics. If you understand the mathematics, you can always figure out a way to solve a problem. Well, not always, but you stand a much better chance. So if you understand the mathematics, you can always reinvent the formal algorithm, for example. (PSY1, June 11, 2012)

Students should certainly know the math that comes before and after what they are learning as well as some notion about where it’s going or how people use it, more than a

superficial understanding of what it is. For example, if you learn division, you ought to know more than just go through the steps. You have to think about what that really means. It is an understanding at in-depth concept level, and not just the ability to perform the skill. (PSY3, August 7, 2012).

I think the focus should be building the concepts first. Some research suggests that if you teach children a procedure for getting the correct answer, then they won't pay attention to the concept. But if you teach them the concepts first then sometimes they can generate their own procedures for getting the right answer, and even if they don't generate their own procedures, they can understand the procedure when you teach them and then will be grounded in the conceptual understanding. So, I guess my bias would be to focus on concepts rather than procedures. (PSY3, August 10, 2012)

One psychologist said that when teaching children the content areas, following a fixed sequence, which helps children understand the concepts, is important. She, however, lamented the fact that a fixed sequence has not been clarified yet.

I don't think that the science has progressed enough for us to know, but I think there probably is an ideal organizational structure and sequence in teaching mathematical topics. I just don't think we know what that is yet. (PSY3, August 10, 2012)

Pedagogy (How to Teach)

The psychologists had much to say about teaching math. They talked about (a) teachers' roles in teaching math, (b) the importance of a positive attitude toward math, (c) teaching strategies, and (d) support for teachers.

Teachers' roles. Psychologists generally agreed that teachers should take an active role in teaching math to young children. They asserted that teachers should help children understand how their experiences can be interpreted in a mathematical way. They, however, also noted the importance of students' learning via self-discovery. One psychologist introduced four approaches to teaching math and recommended one intermediate approach as the best one to promote children's mathematical thinking and learning.

I think there are four different approaches to teaching math. On one extreme, it's the teacher knows all, it's the teacher's duty to impart this wisdom to children who know nothing. At the other extreme, it's basically unstructured discovery learning. The teacher

is just a participant, just one of the kids, and basically there's very little direction on the part of the teacher. The teacher stands back and lets the kids discover the stuff for themselves. Some of that may be helpful but chances are kids are not going to rediscover all the mathematics they need to survive in the world with just that approach. There are two intermediate positions. The math wars are typically defined as direct instruction and drill versus unstructured discovery. But I would say most people don't fall in those two camps. Most fall in the two intermediate camps. In the approach that's closer to direct instruction, the teacher sees their role as helping kids learn strategies and may be open to kids inventing their own strategy, but at the end of the day, the teacher wants the students to know the formal algorithm, the right way. So while there's some patience for informal procedures, and the teacher may encourage some of that, ultimately they want kids to know the right way of doing things. So in this approach, what they might do with manipulatives, for example, they want kids to understand the procedure, so they might model the procedure with manipulatives to help explain why the formal algorithm works. So in a real sense, in this view, teachers believe that understanding can be imposed on children. Sadly, my experience as a parent and teacher does not corroborate that view. The other intermediate position, somewhat closer to the unstructured discovery, is a structured discovery type of position where the teacher serves as a guide and directs learning by choosing games and activities, by encouraging kids to question each other, by raising questions when kids come up with a strategy or an answer, and basically always encouraging reflection, discussion and some kind of consensus on the part of the students. And so the role here is significantly different than any of the other three approaches. Here the teacher might pose a problem, and then encourage kids to use what they know to find a solution. So here with manipulatives, a teacher might say, "Here's a problem. And here are these manipulatives and these manipulatives and these manipulatives. They might help you solve the problem, but I'm not going to tell you how." Then basically leave it up to the kids to use what they learned previously to come up with their own solution. And then from there, you might work towards helping kids understand the formal algorithm as one of several possibilities. For example, the invert-multiply procedure, the formal invert-multiply procedure for fractions is not the most efficient strategy, it's just taught. It's not necessarily the most efficient way of doing it. So basically, here you might encourage them to learn a formal algorithm as one of several options and then evaluate what they think is the most useful option to them. I think this approach is most efficient to help children build mathematical knowledge. (PSY1, June 11, 2012)

The third psychologist confirmed the efficiency of this intermediate approach.

I think that the evidence suggests that guided discovery is better than just discovery or very direct instruction. So I would say that it's best to do what I was suggesting early, where the teachers, based on their knowledge of the concepts, can set up very structured, organized activities that guide children to struggle with and reach discoveries on their own. But, it's not just sort of throwing them out there and having them discover, you really have to, have to have a guided activity that sort of leads them to what you want them to discover. (PSY3, August 10, 2012)

A positive attitude. All three considered a positive attitude toward math as indispensable for effective math learning. If students are not comfortable with math and do not have self-confidence about their math abilities, they often avoid learning math which reduces their opportunities to learn math. They discussed possible reasons for students losing their confidence in math abilities and coming to have a negative attitude toward math.

If you think that mathematics is something that's innate and can't be learned, you're not going to try really hard. If you've been told again and again that you're not good at mathematics, then you start giving up and you don't try. So if you think that mathematics should be something that can be done easily, and you run into a difficult problem, you're probably not going to have the patience to keep working at the problem. So that's basically an affective aspect of competence that I think is really important. The beliefs, the attitudes, that would support the learning and use of mathematics. So you need to get beliefs, which is an affective component. (PSY1, May 18, 2012)

The other key thing is that we, as a society, tend to think of math as an innate ability, like you're either good at math or you're bad at math. And I think that's extremely harmful to people's confidence because if they start doing poorly in math then, "Oh, I guess I'm just not good at math." And then it escalates because if they're not good at math, then they're not going to be working on math, and if they're not working on math then they're not going to get any better at math. I think that we really need to try as a society to understand better that a lot of people's success in math has to do with hard work. It's almost like the opposite here. If you have to work hard in math, that must mean you're not good at it. If they were struggling, then they thought, "Oh well, I'm just not good at math." And that they just kept that instead of thinking, "Oh well, I can work harder and gain a better understanding." If they had a better understanding, then they could improve their mathematical skill through effort then I think that would help them become more confident because then it would be in their control. (PSY3, July 31, 2012)

The problem is that teachers often come up with the pedagogies of hundred years ago or something. But to get the average teacher, the reason they can't come up with a new pedagogy is the whole style of math teachers going back to very authoritarian days. It's like here's what you have to learn, and you have to learn it without reference. You have to learn all these formulas, so it's a very intimidating style and it's a style that says, again, you have to come to this highly abstract subject matter that only the smartest people in the world are good at, which makes everybody else feel inferior. Well you don't have to be the smartest person in the world to teach or learn math, but you have to be somebody who's comfortable with the math, not somebody who's afraid of it. (PSY2, May 22, 2012)

In order to help students feel comfortable with math and have a positive attitude toward it, the psychologists suggested introducing math concepts in a natural way by making

connections between math and students' daily lives. In particular, they emphasized helping students recognize that math is a part of their real world in order to make math more meaningful to them.

I'm viewing math as just a development of our everyday knowledge about the world, not math as this highly abstract, very theoretical kind of thing with its own symbolic language and all that. That's how math developed. Math is a marketplace. How did we come to an agreement between how many goats I'm going to sell you, how many coins you're going to give me, and how're we going to remember that? So the writing part of it becomes important. How're we going to represent how many goats you have? Are we going to draw, you know, you gave me fifty goats today, are we going to draw fifty lines? Is there some better way to do that? You could have kids do this, almost like go through the history of math and see the value of what was developed for the same reason people discovered them in the first place. This is just one example, but what I want to say is that knowing the fact that math is closely related to our daily lives and our lives are full of math is very important. (PSY2, May 22, 2012)

For the little kids, I really think what you're teaching is the kind of everyday utility of math. So helping kids get that kind of basic appreciation of how math gets used in a variety of everyday things is important. I think kids are going to learn math well when they're in real life situations that demand that. So if you have them understand our daily lives are full of math, then they will learn math better. (PSY2, June 5, 2012)

I would approach it in terms of, there's math involved in everyday activities and that's where they're going to start. They're not going to start from a math textbook written by a mathematician. They're going to start from the children's' experience, everyday experience, and kind of inform them how those experiences can be talked about in a mathematical way. So I think if they view that more as their job, they're not mathematicians doing high level problems but they're teaching the rudiments of math, often providing labels and stuff that kids are already doing, I think that'll help them. And I think in the process of taking the math methods courses, their own knowledge of math will actually get a lot deeper because often math is taught much more as just routines to solve things and, in some ways when you're teaching the beginnings of things, it's sort of forcing you to think in a much more theoretical way about what is this? What is this all about? So I think they'll be pleasantly surprised. And to realize themselves all the ways they are in fact using math and are completely capable of doing that. You know, it's only by the standards of advanced math that they feel incapable. But they already know and are doing all the elementary math, pleased to find that out. (PSY2, June 19, 2012)

If you want the essence of mathematics, it's solving problems. That's what algorithms are. They just formalized ways of solving problems, a problem that we constantly run into in everyday life. So I think it's important for teachers to help students to apply mathematical concepts into their daily lives. How're you going to make math meaningful if you don't connect it to their daily life? How're you going to help motivate them to

learn it if you don't say this is an important tool for doing these things in your life? So, relating math to your everyday life, your everyday situations, needs, and interests is important. (PSY1, June 11, 2012)

With young children, there should always be a connection to real things. There should be time spent learning ways to talk about quantity, to develop number sense, to realize that ten is more than two. I can remember someone talking to me about how a few asked people to close their eyes and to visualize a cat, and an image occurs. But if you ask people to close their eyes and imagine five, most people think of the numeral or the symbol, not five things. You could say that it's the same if a picture of a cat comes to mind: you don't see alphabet C-A-T. Even if we did something abstract like blue, often times you would think of something that is blue such as the sky or a shirt, whereas with numbers, we have separated them far from what they mean. People don't think of five things, rather they just think of that five. And so, it has no more meaning than the two. I mean, parents like to teach their kids to count really early so the kids can say those words. But unless you count things, it's meaningless. So I think always, always, always connect, connect, connect. (PSY3, July 31, 2012).

Also mentioned was the importance of teachers' passion for math because teachers' attitudes get passed on to their students.

Simply because a child isn't interested in mathematics doesn't mean we aren't going to teach him math. That's just craziness. Interest is something that can be learned. So if the teacher is excited about learning mathematics, quite often their kids can be too. But, it is hard to find a teacher who is interested in math, unfortunately. (PSY1, August 2, 2012)

Teaching strategies. The psychologists proposed various strategies for teaching math to young children. First, knowing where your students are in their mathematical competence is important. In order to determine each student's level of mathematical competency, one psychologist recommended using hypothetical learning trajectories.

Hypothetical learning trajectories, HLTs, have become very popular in the last ten years or so. What a hypothetical learning trajectory does is to help a teacher understand how knowledge unfolds, how the skills and concepts combine, and how you build more advanced knowledge. And what it permits you to do is to try to figure out where the child is on the trajectory so you can know what the child already knows. Once you figure out what the child already knows, then you can figure out what's the next step that he needs to know. And you monitor the progress. Once a child knows step B, then you can think about doing step C and D. Most of the problems I have seen are that teachers don't take into account any kind of developmental progression. As a result, they just teach stuff that doesn't connect with what kids already know. As a result, kids don't have the flightiest idea what the teacher's talking about, and they have a couple of choices, either they can

play the game and memorize stuff by rote, at least until the test and then forget most of it, or they can say, I'm not going to bother. (PSY1, May 18, 2012)

The first thing they need to do is to be familiar with how kids develop mathematically. So once they can locate where a child is on a learning trajectory, then they know what they can build on and what they need to do to take the next step in the learning trajectory. (PSY1, June 11, 2012)

He also suggested several ways to evaluate students' mathematical understanding such as using a scoring rubric, performance assessment and open-ended assessment.

I'm sorry I forget the name of it...but basically it's a structured task where you ask kids to solve a problem and then you evaluate using a scoring rubric. That can be very useful, and that's one of the key devices we use to assess children's mathematical understanding. For example, the exam involves showing how you can use base ten blocks to model the following decimal multiplication model. And then there is a scoring rubric that we use to analyze different students' efforts, and basically we use that scoring rubric to evaluate the student's performance on the exam. So using scoring rubrics with a specific task that they have to complete is a very useful technique. (PSY1, June 11, 2012)

If you want to assess something more specific, then...basically we use the scoring rubric. So I looked at, did they represent the ones, tenths and hundredths in a systematic and reasonable fashion, and then there's probably seven or eight criteria that I looked at, and each was weighted, each had a certain number of points. (PSY1, August 2, 2012)

One, I think, particularly important way of doing it is performance assessment, where you give a student a problem and actually observe how they solve it. And typically what you need is a scoring rubric. The observation doesn't always have to be direct. For example, students can either use the manipulatives and show me or can write it out and say this is how I do it step by step. Most of them prefer to write it out. But I would say a good number of them did come up and demonstrate it and explain how they were doing it. (PSY1, August 2, 2012)

I think open-ended assessment is a good way of gauging flexibility and creativity, that sort of thing. (PSY1, August 2, 2012)

The third psychologist suggested interviewing students as another way to assess their mathematical understanding.

I think interviewing students is a good assessment tool. I don't especially like paper-and-pencil tests. As students get older they [paper-and-pencil tests] become a real part of their world, but I think that other than being efficient, they [paper-and-pencil tests] are not a good assessment tool to accurately evaluate students' mathematical understanding. You don't know where the difficulties started, but you can certainly observe and ask. I think

the most ridiculous classroom is where everybody sits and nobody watches anybody. (PSY3, August 7, 2012)

Second, psychologists said that differentiated instruction based on students' individual differences is needed. They emphasized planning that attended to each child, especially at an early age.

Some kids learn certain things much faster and other things slower. Obviously, there are strong individual differences, and again they represent permanent limits for the speed of learning, and they're going to play a role at times. I think anything that recognizes diversity among children, however, you think about it and being ready then to follow different trajectories, different speed in your learning plan. To me, nothing is worse for the faster learning kids to be asked to do something a hundred times that they already know how to do. Then you're just boring them to death. Or what I think happens to the slower kids in math, to be asked to go to the next activity and they still didn't understand the one before it. So from the beginning experience in math is lost. It's always going too fast, they never know what's going on. I saw that when I taught statistics. You could just see some kids at a certain point freeze up. It's like, "I don't understand this, and on we're going to the next thing." And, it's lost after that since it's so cumulative. (PSY2, June 5, 2012)

Third, they talked about using concrete objects when teaching math to young children. Children can more easily understand mathematical concepts when they have concrete objects to manipulate.

To me, with little kids, you need to work with the things, you don't just work with the symbols of things. That's another mistake that's been a part of traditional education going back to the, I don't know how it worked in other countries but certainly in Europe where education came from the top-down. So the way you learned in university kept being pushed down lower and lower. In other words, they relied on books, everything had to be in a book and you'd learn from books. To me, with little kids, the last thing you get to is books, although kids are surprisingly interested in books. But I would say very much work with objects, and they're very object oriented. So you do not have to get them, you know all their toys are objects, so you could use toys set up to teach this kind of stuff. (PSY2, May 22, 2012)

I'm seeing kids as thinking much more concretely and perceptually, and that's where the learning ought to be taking place. Then you put labels on that kind of things as the last thing you're doing, so they're not learning to the label, they're exploring and elaborating their own experiential knowledge, and eventually you're putting labels on it. So I think examples are ways of staying on a much more concrete level, and the more that you induce rules, then you tell them rules, and have them deduce how you do about it. Division and multiplication are great examples of that. If you go to the rule-based

deductive way of doing those too quickly, I don't think they ever really understand them, whereas when you do those things with blocks or marbles or something like that or with the areas, it's much easier for them to see what you mean. Two times three means you have two rows of three each and three times four means you have three rows . . . and they see then it's just a faster way of adding each thing instead of saying 3 plus 3 plus 3 plus 3. You say three times four. I think the more that is understood at very concrete level, the more they'll understand that multiplication's, just a kind of a faster way of doing addition. (PSY2, June 19, 2012)

They, however, argued the claim that children can learn math only through the use of concrete objects. One pointed out that this idea, which underestimates children's mathematical ability, results from a misunderstanding of Piaget, and what Piaget really meant. They suggested ways to make good use of manipulatives in teaching math.

This is based on a misunderstanding of Piaget. Piaget talked about the concrete operational stage. What he really may have meant was what is familiar to children, as opposed to something that has to be physically in front of them. Now, can manipulatives be helpful? Yes, but it depends on how they're used. Quite often, what teachers try to do is to impose a manipulative model on children, as a way of helping them understand the rationale for a formal procedure. But imposing a manipulative model is almost as bad as imposing an abstract, written, symbolic procedure on kids. So what I suggest to do is to have students try to devise their own manipulative base model based on an analogy they already understood. For example, if we were studying fractions, multiplication could be very tricky with people. One of the everyday analogies that might be helpful to students would be a "groups of" meaning, so three times four means three groups of four. So when they get to fractions, one-third times four means one-third group of four. And if you have that analogy, no one needs to tell you how to use manipulatives to figure the answer out. You can model it, you can figure out how to model it yourself. (PSY1, June 11, 2012)

Fourth, the psychologists emphasize the need to focus more on the process rather than the end product – focus on "how you get there" rather than "what you end up with." Students, they said, can reach a true understanding while figuring out the problems and elaborating their thinking.

I would recommend that any teacher – it's not simply the focus on what the child answers, the products of their work, but focus on the process by which the child is coming up with the answer. And sometimes, a child can be completely wrong, a completely wrong answer, but if you figure out how a child is coming up with the answer, what the child is thinking may make a great deal of sense. At the very least, you can figure out what the child's misconception is. Now that's going to take a certain amount of training and background to do that, but I think that would be important to do. Again,

knowledge of how kids develop is really important so that you can begin to formulate hypotheses about what the child might be doing, what the process is. So teachers really need to be immersed in a developmental theory of mathematical learning and research on it. That way, they're in a position to begin to figure out, in a fairly reasonable and coherent way, what the child might be thinking or what the child might be doing, where the problem might be, without that you're just guessing wildly. (PSY1, May 18, 2012)

Especially when they're first beginning, you're focusing on the wrong thing. You're focusing on the end product. I think you need to focus more on the process. Do they really understand what they're doing? If they put one block or two blocks together and count one, two, three, that may take longer than just seeing a one and a two and a three in front of them, but if they've gone through that process, they understand what they're doing. They're adding. If you teach sort of the routine, five plus two is, you count to five and now what're you going to do with these other two? Well, now they're not just one and two, now they become six and seven. So that's really the notion of adding, that you're continuing, you're counting beyond five because of the two. And if they go through that process routinely, it takes time to do that. I think that's a lot more important than if they can see a five and a two and put a seven under them. I think there's too much emphasis on just showing they can be a little fast calculating machine without any concern if they have any understanding. That's why kids when they get older, they start having problems in math because they're like little calculating machines. They know a whole series of procedures, but they have no understanding of what they're doing beyond just doing procedure ABC and D. (PSY2, June 5, 2012)

I think the key thing in what I hope won't be lost in this is how you can exploit their [students'] experience and their way of thinking to approach these things. The problem in math always has been that the math teachers don't pay enough attention to the intuitive knowledge and immediately try to get you to use their terminology and their operations. I would agree they should be able to do these things, but the real learning will come in the process of getting it, that's why teaching-to-the-tests kind of thing was so problematic that they immediately think, "Well, we've got to teach them to do little addition and subtraction problems because that's what they're going to do on the test." (PSY2, June 19, 2012)

Fifth, the psychologists said that encouraging students to come up with different ways of solving problems is important. Often, teachers stick to a standard way to solve problems and push kids into following only that way. The psychologists, however, said that standardized processes for problem solving are just one way among several to solve problems, and sometimes they are not even the most efficient way to do so. They noted that students can clearly understand the concepts conveyed in the problems by developing their own strategies to solve problems and by explaining rationale for those strategies.

Sometimes, there is not the correct way of doing things. There may be multiple strategies to solve problems, and then you pick the strategy that is most efficient for a particular situation. Or you adapt a strategy to a particular situation. So the correct way? Well, a lot of people think the correct way of dividing fractions is invert and multiply because it is the formal and standardized way to do that. But that's not necessarily the easiest way of doing things. (PSY1, June 11, 2012)

I think you can kind of do both. You can invite the kid saying, "Here's a good way to do it. You can try to find other ways to do it." I still remember that kind of thing in high school geometry where kids would come up with other proofs, other way of proving theory, and the teacher would mark them wrong because it wasn't the prescribed way. I mean that's ridiculous. So I use strategies as just possible proven solution patterns but not necessarily the only way to do it. (PSY2, June 5, 2012)

There is no correct way to solve problems rather there are multiple strategies to solve them. We should encourage students to think about different ways to approach problems. I am not saying that everybody should invent all problem-solving methods that people have been learning the past several thousand years. What I am trying to say is we need to build on what other people have discovered already, in other words, standardized problem-solving methods. There is no point in asking students to completely re-develop the mathematics of an entire culture. But, they still have to be able to develop their own mathematics, so they have to have some way of making sure that whatever mathematics they have in their own head is expanded. (PSY3, August 7, 2012)

Sixth, they described the merits of collaboration among students when learning math.

They said that collaborative learning can boost children's mathematical understanding because it helps children learn from each other.

By and large, doing small group learning really helps people because now you have not only one teacher but many spread through the classroom. Quite often, someone will understand something and be able to help their peers comprehend it. And another person will understand something different and be able to help the others. I always encourage my students to listen very carefully to the dissenting opinion because too often the group will go down the primrose path and be led by one person because they think they know the answer, and that's not necessarily the case. Sometimes that lone wolf is saying, "Wait, wait, wait, I actually have something to say." Sometimes not. But you're not going to find out if you're not listening carefully, and that's part of teaching people how to work in a group. (PSY1, August 2, 2012)

I think they will observe each other and pick things up from each other and since it's sort of like benefitting from models that have similar levels of competence versus models who've already mastered it. So they're more likely to see the stages of the learning process being demonstrated by the other kids. And I also think it makes it more like a play activity that they'll be enthusiastic about. So they're commenting on each other's successes or bragging about their own successes. I think the adult model of learning is too

much based on – it goes back to the early cathedral schools – too much like behaving the way we would in church rather than how we would in the playground. So I think the more you can simulate the playground type environment where they're boisterous and enthusiastic and jumping around and showing each other things and not just sitting in their passive learner. (PSY2, June 19, 2012)

One psychologist addressed the importance of choosing tasks or activities which can facilitate children's collaborative learning.

I think it's good to have some time to think alone, and then to have the opportunity to discuss with a partner or with a group. I think a lot of times in school, there's a collaborative learning organization that doesn't have interesting enough problems for a group to solve. It's important to choose the task wisely so that it's something that actually requires more than one person to solve the problem, one person working and two people watching. I heard William Glasser say one time, "If the task is to carry four books to the office, you don't need four students to do that. But, if the task is to carry a heavy table to the office, then it is a good structure requiring the thinking and the skills of everyone, not just of a few. (PSY3, August 10, 2012).

Another psychologist provided some tips for improving the effectiveness of group learning.

If you're going to use small group learning, you have to prepare students how to work in a group. Because if you just put kids in a group, what's going to happen? The kid who knows it is just going to tell everyone else and that really doesn't help the other kids. Because they really don't understand what the kid is talking about, just like they don't understand the teacher. So simply putting kids in groups is not going to work very effectively, or as effectively as it could. So you have to help kids learn how to work effectively in a group. Then you have to present problems that are going to be challenging for everyone in the group, even if they use different strategies to figure it out, at least everyone's involved and thinking about them. So there's a real challenge right there that most don't pay sufficient attention too. (PSY1, August 2, 2012)

Group work is a tool, so you have to use it carefully and thoughtfully, and flexibly. Sometimes, you can use heterogeneous grouping if a problem is something that can be looked at different ways and solved using different levels of strategies or sophistication. That makes a lot of sense. Other times, it makes a lot of sense to have homogeneous groups because a problem that's going to be challenging for this group is going to be too challenging for this other group and not challenging at all for these other kids here. So under some circumstances, it makes a great deal of sense to have homogenous grouping, especially when kids are behind in math. It does not make sense to simply integrate them with people who are doing far more advanced stuff that they can't comprehend. That's the problem with education mainstreaming. You see some of these kids, they're just sitting there. They're totally lost because they're just not ready. So it depends on the situation, what you're trying to accomplish with the kids. (PSY1, August 2, 2012)

One, however, pointed out the difficulty of pairing up kids for collaborative learning. Knowing how to group the kids depends on teachers' judgment, which needs continuous practice.

I think ultimately there's going to be some judgment involved. I mean research is just going to give you means from groups of people, and I don't think it's very informative for teachers in terms of how to exercise their judgment. So I think that's something they're going to pick up from their practice and their training and experience with kids. All of them have advantages and disadvantages. If you have a child who likes to help then pairing that child up with someone who's having difficulties may work. But, if the child doesn't like to help, then they can become impatient with the other child. So, again, I think there are other considerations that a teacher who's used to working with kids that age would probably be aware of. There's the old thing about learning by teaching. So I'm not the one that thinks you're somehow disadvantaging the faster learners if you put them in the teacher's role. I think that's a good experience for them. But, you have to remember that teaching requires empathy and sociability, and some kids aren't as good at that. Although you could argue that might be a good experience for one to learn fast, we have to be careful to force them into that role. But, I wouldn't offer any general prescriptions on that. I just think there has to be more judgment involved. (PSY2, June 19, 2012)

Lastly, they talked about teaching math by integrating it with other subject areas, which they believed to be not only economical but also efficient.

I do think that it's good to integrate the math activities when you can, because I think one of the problems, that happens with children in math later, is that they are always saying, "Oh, when am I going to use this?" or "Why is this important?" So it's good to have activities where math is needed. They need to understand the math in order to achieve another goal. That helps them understand the usefulness of the mathematics. (PSY3, August 10, 2012)

Well, it's a good way of teaching all the subjects in an economical way. I was originally trained as a science teacher so there are all sorts of ways of integrating mathematics and science structure so that kids learn an interesting way. For example, Law of Levers, I mean you can simply tell kids what the law of levers is. You can simply teach kids that. They won't understand it very well. But you can get them to explore that, discover that law, and at the same time what you're doing is helping them explore a very important area of mathematics which is inverse proportions. So you learned law of levers, and you learned something about indirect proportional thinking. There's no reason why you should exclude the two, you know, do one topic exclusively. (PSY1, August 2, 2012)

They described subject areas that can be easily integrated with math such as science, social studies, reading, and music.

Well it's relatively easy to do with science, even social studies because there are all kinds of data, and that sort of thing. But even with reading, there'll all sorts of stories that can serve as the basis not only for reading instruction but also mathematics. For example, "The Doorbell Rang" is a classic story where you've got two kids, a brother and a sister, and the mother bakes a dozen cookies, and before the kids are able, the mother wants them to share them fairly. So what you could do instead of just reading the story is you could stop and say "Well, you've got 12 cookies and two kids, how many cookies does each person get? What's their share?" And let them solve it informally then discuss that. Then the doorbell rings and another child shows up, so not they have to share 12 cookies among 3, and then another child shows up, now it's 12 shared among four, then two more kids come up and now it's 12 shared among six, and then 12 shared among twelve. And you could go beyond the story and say, what would happen if I've got 12 cookies here and now we have 24 kids? You get into fractions. So all sorts of opportunities to cross pollinate, basically. A lot of mathematics is communication, communicating carefully, thoughtfully, logically, accurately, precisely. I mean that's, those are language skills too. And music. Music absolutely. Music's a great one. (PSY1, August 2, 2012)

Support for teachers. The psychologists said that in order to help teachers prepare themselves to teach math more effectively, support from the teacher education program is necessary. They emphasized teachers' content knowledge as an important factor and argued that teacher education programs should provide courses that can help teachers broaden their mathematical content knowledge.

They need to understand the mathematics that they're going to teach. That's what we should try to do in math methods course. So, basically the instructor should assume that they understood nothing. What the instructor should try to do is teach them in a way she is hoping they would teach their kids, so modeling a way of teaching. And what the instructor should try to do is help them understand the mathematics, to construct an understanding of the mathematics themselves through solving problems or thinking about questions and that sort of thing. (PSY1, August 2, 2012)

I would agree with that you need content knowledge, certainly of what you're teaching. You need to understand the mathematics of what you're teaching, because how are you going to be able to answer questions from students intelligently? How do you know what the goals of your instruction are if you don't understand the mathematics? And above and beyond that, it's nice to know that what you're doing is the basis for something else, the next steps, and that's why important to know beyond your own grade level, the mathematical content beyond your grade level. Because if you don't have some sense where you're going, I think it may limit how you help kids. (PSY1, August 2, 2012)

I do think that it is necessary for the teachers to understand the concepts themselves. There's a really excellent curriculum, *Everyday Math*¹ for elementary school, but I think that it is a really bad curriculum if the teachers don't understand the math that is going on. So I do think it's necessary for teachers to understand the concepts that they're teaching, otherwise I think it's too hard for them to set up conceptual learning activities for the children because then they don't understand how the concept is being represented in the concrete objects or whatever sort of activity they're providing. But, it is very hard for teachers to build this knowledge themselves, so teacher education program should support teachers. (PSY3, August 10, 2012)

They also mentioned cooperating with parents as a way to help teachers facilitate children's mathematical understanding. They talked about the ways to promote parents' involvement in teaching math.

I think finding ways of involving the parents more would be really helpful if that was part of the charge of the early childhood educators – to not just teach the children directly but to get the parents involved in the process too. Because at that level, the parents know enough that they could get involved in just about everything. I don't think they've been engaged enough, and the idea that some of them are even more uncomfortable with the math than the teachers are. But again, I think they could pick it up. (PSY2, June 19, 2012)

Some teachers use like a newsletter that they send home periodically to explain what they're doing in math and why. I think that can help. On parents' night, having a short interesting lesson can involve the parents in learning something new, can help them understand better why you're teaching the way you're teaching. Some teachers use a math game night and they share . . . basically they play games with the parents and then they talk about what the game is trying to accomplish. What mathematics can be learned from it and that can be fun. (PSY1, August 2, 2012)

One, however, expressed her concern with making parents' involvement compulsory.

I think that involving the parents is important. However, I don't think that it's fair to rely on parents to really teach any content because I think that contributes to disparities between children from low income backgrounds and children from high income backgrounds. So I think it's great to include parents. I think the evidence suggests that, if you use Bronfenbrenner's term, making the *mesosystem* stronger is great, so making parent-teacher connections in classrooms, home connections stronger is great, so having them take something home and explain it to their parents or getting data from their home to analyze, things like that are I think good ways of including and involving the parents,

¹ *Everyday Math* is a research-based and field-tested curriculum for pre-K through 6th grade emphasizing 1) use of concrete and real-life examples, 2) repeated exposures to mathematical concepts and skills, 3) frequent practice of basic computation skills, and 4) use of multiple methods and problem-solving strategies, first edition published in 1988, second edition in 2007, <http://everydaymath.uchicago.edu/about/>

but I don't think that anything should be set up where it relies on the parents to teach the content because some parents don't understand the content, and I think that the children of those parents are at a disadvantage if it's left to the parents. (PSY3, August 10, 2012)

Pedagogical Content Knowledge (PCK)

The psychologists addressed teachers' PCK. Of the three, the first cognitive psychologist used the term, pedagogical content knowledge. The other two did not use the term but emphasized the importance of the three elements involved in PCK. They said that teachers often lack the needed knowledge.

They need to understand mathematics. They need to understand the psychology of mathematical learning. They need to be aware of a variety of pedagogical techniques. Most people think teaching is just telling kids stuff, and then expecting them to understand immediately. Well, it's not that simple. It's a little bit more complicated. (PSY3, August 7, 2012)

They not only need to understand the mathematics and have an accurate understanding of these basic concepts and skills, but also need pedagogical knowledge, which includes the ability to take formal ideas and representations, and translate them into something a young child can comprehend, that connects to what's in the child's head. Because if you don't make the connection, then they're basically going to have to memorize what you're teaching, or not learn it at all, or only learn it partially because they're trying to memorize it. (PSY2, June 19, 2012)

In order to have pedagogical content knowledge, you also have to know how kids develop, how their thinking develops, because pedagogical content knowledge means taking mathematical content and translating it into a form that your students are going to understand. If you don't understand your students, you're not going to be able to do that very well. And, part of knowledge about student learning is what kinds of misconceptions do they come to you with? And, how to deal with that? What are the common misconceptions when something is taught? Why does that occur and how do you get around it? So I think pedagogical content knowledge and how kids develop is very closely intertwined. I have a hard time separating them out. They're very dependent on each other. (PSY1, June 11, 2012)

People often think that anyone can teach mathematics to young children because the concepts dealt with at that level are very easy. But, that's really not the case. Over the years, I've had a number of secondary math ed people take the elementary math course. And they thought this was going to be really easy. Of course they know mathematics, they're straight A students in mathematics right through high school and even college. They thought this was going to be a real breeze. Well, we started asking questions like, "Why do you do this or why does this make sense?" No one in their program had asked them these types of questions. So they certainly knew the definitions. They certainly

knew the procedures, but in terms of conceptual understanding, it was often very limited. And certainly to take something and explain it in terms that a child is going to understand, they didn't have, and most people don't have, that knowledge. (PSY1, August 2, 2012)

When talking about PCK, one psychologist said that there is a sequence for teachers to develop each component of PCK— content knowledge, knowledge of children's cognition, and pedagogical knowledge should come in order.

I think if we're talking about which should come first, I do think that the math content knowledge should be in place first when you're learning how to teach math. So you have the content knowledge first, so that sets you up for understanding next the other two aspects – pedagogical knowledge and knowledge of cognition and development, I think, sort of go hand in hand. But, I would say that, understanding of cognition and development might come next in terms of a timeline, first your understanding of the mathematics, now start getting a sense of where children are developmentally, how do they process information at a particular age about symbols, quantity, and language, and then once you have that yourself as a teacher, both your content knowledge and your knowledge of development and cognition, you can actually start generating hypotheses about what might be effective pedagogical approaches. And that will better help you interpret and read the literature on best practices, because you'll go in with intuitive ideas and ideas . . . You know, not just intuitive, but educated, guesses based on both your knowledge of the content and your knowledge of children's cognition and development with ideas about how to teach, and you can use that as a way to be interested in reading articles about best practices and see if they confirm your hypotheses or go against it. So I think that's how I would set up, if you're talking about like, a strategy for preparing people to teach math, that's sort of the sequence that I would think that would be the best approach. (PSY3, August 10, 2012)

Another psychologist asserted that method courses should help pre-service teachers develop pedagogical content knowledge and not merely focus on pedagogy.

The methods course should try to do three things: First, it should try to help students understand the mathematics that they teach. Second, it should help them understand how kids learn the mathematics, how they think, how they develop, and what are common misconceptions that young children bring to your classroom, because their informal knowledge is incomplete or inaccurate in some ways. And the third thing is pedagogy, what are the latest and best methods for teaching mathematics. We should try to accomplish three things in the methods course because that's not always the case. Most methods courses are focused on the pedagogy, so they don't necessarily focus on the mathematical content, understanding the mathematical content, or how kids develop an understanding of mathematics, how their thinking changes, how they learn skills. (PSY1, August 2, 2012)

Summary

The psychologists agreed that children have a sophisticated understanding of math more than most people think, and that children show signs of readiness to learn math often. When discussing children's ability to do math, they considered the differences in mathematical competence between each child. They also mentioned the importance of understanding children's mathematical cognition.

They discussed a wide range of content areas that teachers should focus on: (a) counting and operations, (b) geometry, and (c) measurement. When talking about these content areas, psychologist often mentioned children's cognitive ability related to each content area.

They outlined four major ideas related to pedagogy. The first is that teachers should actively help children interpret their experiences mathematically and understand the relation between real life experiences and mathematics. They also noted the importance of students learning via self-discovery.

Second, a positive attitude is the first step for effective math learning. They discussed some reasons students lose their confidence in their ability to do math and develop negative attitudes, suggesting several solutions. They emphasized the importance of teachers' exhibiting a passion for math because their attitudes affect students' attitudes.

Third, between the three, they offered seven strategies for teaching math: (a) emphasizing assessment in teaching, (b) planning different teaching trajectories at different speeds for each child based on their individual differences, (c) making good use of concrete objects while teaching, (d) focusing on the problem-solving process, (e) valuing students' informal problem-solving strategies, (f) encouraging collaborative learning among students, and (g) integrating math with other subject areas.

Fourth, all emphasized (a) the need for support from teacher education program for strengthening teachers' mathematical content knowledge, and (b) collaboration with parents.

They agreed that teachers often lack the knowledge necessary to teach math to young children effectively. They suggested a sequence for educating teachers to teach math: first, content knowledge; second, knowledge of children's cognition; and, third, pedagogical knowledge. They criticized methods courses that focused on pedagogy and did not address content knowledge and children's mathematical understanding.

Chapter 7

Teachers: Math Education for Young Children

To discover what teachers think about math education for young children, I initially interviewed eight teachers then chose three to continue with the second and third interviews. From the first round interviews with the eight teachers, I found that teachers, even teachers who do not have confidence in their math ability or who hate math, do not have much difficulty in providing math activities to children. They, however, expressed confusion about the move from the Illinois State learning standards to the Common Core Standards. They did not seem to understand the difference between the Illinois Standards and the Common Core Standards or that the Common Core Standards were coming from the federal level. They appeared to see the Common Core Standards were an updated version of the Illinois Standards. These initial findings pointed me in directions I would pursue in the second and third interviews.

My selection of the three teachers for Stage 3 was based on demographics, program affiliation (whether they had been involved in one or more programs), and their willingness to share additional insights about their experience. I interviewed the first and third teachers in their classrooms, the second in a study room at a public library.

The first teacher is a kindergarten teacher in a public school. She has been teaching kindergarten for 42 years (36 years in her current school, six years elsewhere). I hoped she would be able to describe how notions of good math education have changed with time. She was clearly committed to teaching math. She showed me many math games she created as well as teaching materials she used in her classroom.

The second teacher is a kindergarten teacher at a private school. She has 22 years of teaching experience including five years in a pre-kindergarten. Her first degree was not in

education but in retailing. When her daughters entered school, she went back to school to get a teaching certificate. Her story about her master's program intrigued me. She was not strong in math in high school. Math scared her, and she did not really understand it. She took only one math class for her bachelor's degree. When she had to take a math methods course for her certificate, she was scared. She met with the professor before the semester started and said, "I'm almost done with my Master's certification program, but I haven't taken math methods yet. It really worries me because my math is really poor." The professor said, "Well then, I don't know if you're going to be able to finish the program." His response and attitude made her math phobia even worse. She described, however, how she overcame her fears by taking extra math courses and taking professional development courses that had to do with teaching math. I wanted to hear more of her story.

The third teacher teaches at a private preschool. She has a range of teaching experiences. She had worked at one university laboratory school and two child care centers affiliated with universities before her current school. She also worked in Puerto Rico training teachers in her own classroom in developmentally appropriate practices. She has a bachelor's degree in early education, but she does not have early childhood certification. I thought that her experiences in preschools would add to what I would learn from the two kindergarten teachers.

This chapter has four sections. In the first, I examine teachers' views of children's mathematical understanding. In the second section, teachers discuss what mathematical content which should be taught to young children. I then move on to their ideas about how to teach mathematics to young children. Finally, I look at the challenges that teachers face teaching mathematics to young children.

How Children Learn Mathematics

All three teachers believe that children have some math ability and are ready to learn math.

I think they're capable of more than we think. Children come in with tons more knowledge than they did 30, 40 years ago. They are more into it. They have been exposed to so much more. We just expect more of them. (T3, May 13, 2012)

I think kids are ready right at five, if not a little bit before as long as it's made meaningful to them. The kids that are struggling are the ones that don't like the pencil and paper end of it. They don't want to do the writing, but they would love to know all about numbers. Numbers are easier than alphabet. So yes, I think they're very, very ready at five. Very ready. (T1, May 1, 2012)

There are some children that come in ready to do abstract, either because they just have that ability, or because somebody has worked with them before they came to school. My student teacher says her little boy who is 18 months is already starting to count things. She says it's just because he likes it, not because anybody is drilling him on that. (T2, June 13, 2012)

Teachers often mentioned children's eagerness to learn math.

They want to learn. They want to learn more about numbers and their name than anything else. And they want to read. But they aren't going to be capable of that. But they're a lot more capable of numbers than they are in reading. One of the first things we require is you learn your telephone number and your address. Those are full of numbers. If they don't know their address, address is a big word, what do you do? You go home and look at the numbers that are in front of your house. It's required in our school to have your numbers on your house. Those are your address numbers. They will go home, and they'll say, "I've got one of those, and I've got one of those, and I've got one of those on my house." I said, "Well, you need to learn those names." And so there's an incentive to learning it. (T1, May 1, 2012)

Yes, that [math] is one of the first things they want to learn. They want to learn their name, how to write their name, and they want to play all those math games. They will pick that up faster than they will anything else, because they want to do that math game. They like to do these kinds of things at the beginning of the year. They will see that they can't do it, if they don't know. They will bring that card up, and they will look at that, and they say, "Well, how much is this?" And I said, "Do you know what number it is? What's its name?" If they don't know it, I said, "Well its name is eight. Let's count. One, two, three, four, five, six, seven, eight. Okay, go find me that many fishes." And that'll match. (T1, May 1, 2012)

One teacher mentioned the importance of understanding children's mathematical cognition. She said that teachers should consider how children think mathematically when they plan math activities for children.

I think that the cognition, understanding of what they're thinking guides what you do with the kids. If you don't take the time to understand how they're thinking, I don't know how you can teach. I do think that number one is, you have to understand where that child is. (T2, June 13, 2012)

Mathematics Content (What to Teach)

From the examples of math activities that teachers described using in their classrooms, one can infer the range of mathematical content covered.

Number sense and operations. First, teachers said that they do activities involving number sense and operations. These activities cover many subtopics such as number recognition, counting, comparing, and operations. The following examples are about number recognition.

This is another game that we play later on, and it's for mostly recognition. This is spread out on the carpet, and a frog is put on each one. And they take turns from Team A to Team B, and if they can name that number, then they get the frog to put in their team. Then we count, and by the time we play this, hopefully – this is 20 and up – every kid knows the number. They can choose any one they want. So if you've got a child – but you'll notice there's a lot of reversals on here – so if you've got really a top child, he's going to pick 81 or 73 or 86, where your lower kids will probably try to get the 18, and they'll put the frog in their pile. If you're lucky, and most of the time, nobody has messed up, it's a tie. Those are the best games, always a tie. (T1, May 1, 2012)

These I made myself and these go from 20 to 40. When we start this, I'm the caller of the numbers and you have to get a whole row, top or bottom, where you've got some spatial. We play top row, bottom row, we play four corners, we play four middles, to keep it a little livelier. By the time we get to this, they are the caller. I have taken myself out of the equation of running the group and make it more of a social, and they are the callers. They're using their oral skills to recognize the number, and play teacher and be in charge of the class. This is very popular. (T1, May 1, 2012)

This is a hundreds chart, and they have to put the little number chips in the right order and where they are. One hundred is our basic top. I've been doing some things that they're doing writing now. In fact, we just finished it today. One of my kids wrote to 180, because there were 180 little squares on the paper. I've got another one who was

still writing up to 13. Even at this time of the year. Our required minimum, minimum is 30. (T1, May 1, 2012)

Right now, we have to teach through 30. They have to recognize those numerals out of sequence. They have to be able to write them in sequence. (T2, May 3, 2012)

The following activities involve counting.

Now for the first nine weeks, we do a lot of this in our centers. I have ten boxes back here, but I probably have 30 more games that are built on exactly the same concept. They have a number on the card, and they have to put everything out, and counting. (T1, May 1, 2012)

So first thing in the morning, we do calendar time. I have a child that's the lunch count person, so their job is to call the names for each child whether they're hot lunch or cold lunch, and whether they're chocolate milk or white milk. Then their job is to count how many people are chocolate milk, how many people are white milk, and they have to write those numbers on the board. We're doing math with that. (T2, May 3, 2012)

The activities below are related to comparing.

Lots of opportunities for counting and comparing so who has more, who has less. We incorporate a lot of things that are really considered math games, like do you know the card game War? We call it "Top It." Two children would each have a stack of cards, and they each flip one over. The one who has the higher number takes all the cards then, and you just keep playing like that. So we start at a real rudimentary level with that, just which is more which is less, and number recognition. But then as they develop, you can play that same "Top It" game with flipping over two cards and adding them together, or flipping over two cards and subtracting them. You continue to layer on more math. (T3, May 13, 2012)

The following activities are about operations, often in base 10.

Basic addition through 10. Like, one plus nine, two plus seven, things like that. Real basic through 10. Usually not over 10. (T2, May 3, 2012)

I work a lot with number line, so I explain that if you move to the right on the number line, it's greater than, and the number is getting larger. It's what happens when we add. If we go to the left or backwards on the number line, it's a smaller, lesser than number. So there's a lot you can do using a number line. They have one on their nametags too so they can see the numbers and they can add up or add down. We work on counting forwards and backwards. (T2, May 3, 2012)

We've been doing a lot this week of word problems. We did one out of this little tiny math book we have. Somebody went to the zoo, and there was a snake three feet long and there was a snake that was two feet long. Then there was a third snake that was as large as both of those combined. How long is that snake? (T2, June 13, 2012)

We do in terms of “I had ten cats on Monday but on Tuesday I had fifteen cats. How many cats did I get?” (T3, May 13, 2012)

Algebra. Teachers also mentioned activities related to algebra. These activities often involve finding patterns, which is the emphasis for algebraic thinking in early childhood (NCTM, 2000), but some activities are abstract and require a high-level of algebraic thinking.

They’ve got to figure out where it needs to go. It’s a little harder to find out in order. Most of these have a pattern on it. I’m not sure this one has a pattern. Yellow, green, blue. Green, pink. Yellow, green, blue. Yeah, that one didn’t follow through. Most of them have a pattern. We start in September with maybe apples, bus, apples, bus, apples, bus. That’s a pattern, we start with those, so there’s patterns, that’s math. (T1, May 1, 2012)

Now as far as the algebraic concepts, we do a lot of totally oral stuff. Like, okay, “Tracie is not here today. We all know she always drinks a chocolate milk. How many chocolate milks do we need today?” [We usually get 16 chocolate milks, but Tracie is not here, so we need one less] We do a lot of that. Or we count our box tops when we get box tops in because it’s a fundraiser. So we count them, and we constantly say, “Well, I got 10 here, how many more do I need to add to get to that number?” [$10 + X = \text{that number}$, what is X ?] So we count them, and we’re constantly saying, “Okay, how many papers do I need at this table? If there are five kids that sit here, okay go get the papers that you need for your table.” (T1, May 1, 2012)

Geometry. Another content area in which teachers do activities is geometry. There are many subtopics in geometry such as shape, location direction and coordinates, visualization and spatial reasoning, and transformations and symmetry. Nevertheless, the activities that teachers referred to included only shapes.

We used to teach the six basic shapes for geometry. In the new Illinois Common Core Standards, there are ten shapes. They’re not the same six basic shapes. Diamond is not even a shape anymore. It’s not even considered a shape. And they’re going to three-dimensional. (T1, May 1, 2012)

They have to know certain shapes [squares, circles, triangles, hexagons] and how to form those shapes. Plain shapes, the 2-dimensional shapes [lying in a plane or flat]. (T2, May 3, 2012)

Measurement. Teachers also mentioned measurement activities. Measuring activities involved not only the use of standard tools such as rulers but also nonstandard ways of measuring as well.

For us, currently in our setting, there is a study group, and they're studying growth. The idea of a baby, or when they were born, they weighed less, they were shorter, they wore smaller clothes – all of those key vocabulary words – and where they are now. They're wearing larger clothes, they are taller. They have grown quite a few inches and now there's feet associated with that. That concept of measuring . . . there's so much vocabulary and so many tools to use to demonstrate that or to show that. It's exciting when you see that they get like, "My baby sock was this and it was this length, and my current sock is this much longer," and they understand that concept. (T3, May 13, 2012)

Let's say a group of five students are very interested in measuring their jumps. We didn't measure jumps with a ruler. First, we read a book, *Jump Frog Jump* that talks about the frog jumping and how far — can it get further? There's a literacy element that correlates with that. Then we participated. We actively jumped like frogs from a starting point to another point, and we measured with tape, not a ruler. No numbers, just tape. Each student did that, and then we graphed. We moved the tape from the floor to the graph. We saw who jumped the furthest, which jumps were equal, and which jump was the shortest. Then we named the graph, just like you would see in a book, with the X-axis and the Y-axis, we named them, then we titled the graph, and we talked about the graph. That was very student-directed. That was something that they wanted to do. (T3, May 13, 2012)

We actually do that in the sensory table. Like right now, we have flour with cinnamon in it and nutmeg, we've got some spices mixed into it, so they're getting a sensory bombardment. We have a set of four measuring cups: $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, and 1 cup. So they've got a chance to measure in that. (T2, May 3, 2012)

Data analysis. Finally, teachers said that data analysis is other area in which they do activities.

Then in direct instruction, activities might involve collecting data, so they might have a yes or no survey or a "Which do you like better: chocolate or vanilla ice cream?" Then they actually ask people and tally that information, so that they're practicing the data collection. Then representing that, so maybe using Unifix cubes to say that seven people said chocolate, and eight people said vanilla. Which is more? How many more? (T3, May 13, 2012)

One kindergarten teacher summarized the kinds of math activities that she is doing in her classroom. They included most of the content areas mentioned by other teachers above.

At the end of the year, students are supposed to know six basic shapes and be able to reproduce them. Students are supposed to know numbers minimum up to 30. We [teachers] prefer up to 100. You have to be able to count by 5s, and you have to be able to count by 10s. You have to be able to recognize value, say, 100 is a lot, but two is not. And spatial concepts of first, second, third, last. Top, bottom. We were talking today about directions, which are also part of math. Which way we would look for north, south, east, and west. Things like that. (T1, May 1, 2012)

Pedagogy (How to Teach)

The teachers discussed many ways to teach mathematics to young children. These can be put into three main categories: (a) various ways of helping children develop a positive attitude toward math, (b) teaching strategies, and (c) support for teachers.

A positive attitude. Teachers said that helping students develop a positive attitude toward math is the first step in teaching math to young children. Teachers talked about modeling being excited about math as a way to help kids develop a positive attitude toward it.

First of all, the teacher has to be excited about math. You have to relay that to children. I do that. I tell them, "Once you understand how numbers work, they are fun, they are exciting, and they are in your life every day." In transition from one point to the other, I'll tell them, "I'm doing mental math, you need to be quiet." I'm very dramatic about it. I'll say things like, "Today I have 17 children, so I need to be able to count 170 toes in my line." I'll go through, and I'll skip count by tens, and I'll count their toes. I get them very, very excited about math, and I have never had a child not want to do math activities. (T2, June 20, 2012)

I think a teacher maintaining his or her own wonder about learning and being able to say, "This is really exciting, you figured this out in a new way." So to keep that spark about that this is really not just that this is fun, but it's really interesting and helping the child to understand how they're understanding. Yeah, and also having fun with it. I think math is fun. It can be really dry and boring, but we don't want it to be like that! (T3, June 8, 2012)

Then the students, if they know that I'm really good at teaching fractions, and I have a lot of fun, they feed off of that energy. They're having fun with it because I, as their teacher, am having fun with it. If I'm teaching area and perimeter, and I think I am very bad at teaching area and perimeter. If I look like I'm not having fun, then the students are probably like, "She could care less about this." Right? So we need to give them more credit. They know what our interest is. So teaching at the preschool level, they all know that I love teaching math at their level of understanding, and then the next level where they want to go, some of them, I love to take them to that level. (T3, May 18, 2012)

Continually exposing children to math concepts was also mentioned as a way of helping children feel good about math. Teachers said that teaching math every day can help children feel comfortable with it and eventually come to like math.

You should see some math every day. Every day. I don't know how much, but you're going to see. (T1, May 1, 2012)

Math concept exposure should happen many times throughout the day. It should not just happen during the time we set aside for direct instruction. (T3, May 13, 2012)

By introducing activities that they do as routine, teachers showed how they help students feel familiar with math.

I know every one of the five of us [the number of kindergarten teachers in this school] start the day with math. We start the day with the calendars. That's how we all start. Everyone of us. We start with what day of the week it is, what the weather is, what number goes on the calendar. Now, it's not so hard when it's down to 30 and 31, but when you've got all the numbers all over that metal piece here, and they've got to pick out, okay today is the first day of May, and they're all over here, see here is, here's 31 still on here. The yellow chart up there used to be over here, because we've had all those days. And you see the ladybugs on there? Those are 5s and 10 days, so we have our practice and we're always doing our 5s and 10s. (T1, May 1, 2012)

Now even in the hall, we do math. "How many toes do I have in line?" We have 18 children, 16 are missing, how many more do we need? We're playing with numbers all the time. My kids, when I'm trying to figure something out, I'll go like this, [pointing her index finger to her head] and they'll go, "She's using mental math – we need to be quiet!" So I've got them trained to understand these concepts. Now we're working on using words like "equations." Instead of drawing pictures or using things like that, I'm writing the equation " $3+1=4$ " and making them understand whatever happens on this side has to happen on that side. They have to balance or be equal. It's fun. (T2, May 3, 2012)

Providing math activities that interest and engage students was also mentioned.

Teachers said that it is important that students feel that math activities are enjoyable.

Everybody in our school does a lot of this type of thing. Hands-on, a lot of fun ways to learn. The least fun way, of course, is pencil and paper. Absolutely. On Valentine's Day, I took a hundred Hershey's Kisses and put the numbers on the bottom from 1 to 100. Then they had to find the Hershey's Kisses around the room, and if it said number 65, they had to put Hershey's Kiss number 65 on 65. We only found 93 of them, and so they had to sit, and it took three more days to find all of them. Then when we did all of them,

we divided them up, and we all ate our Hershey's Kisses. We did the same thing on St. Patrick's Day, only we had gold coins. And it actually came out 104 gold coins that we had to find. They were everywhere. What we did with the Easter eggs, but there was nothing in the Easter eggs. And what we did with those – I hid them outside, and we just saw which team got the most. I hid them and they found them, and then we graphed them by colors. So those are all math concepts . . . they're just . . . "funner" than pencil and paper writing. I know what they need, and I know what works with five-year-olds. Anything in a fun, game-type situation works. (T1, May 1, 2012)

Yeah, who wants to learn how to learn how to count to 100? But they can't wait as soon as they hear, they go "vrrrrrrm" [imitating car noise]. Off they go, they jump up and start going, "Yeah!" I went through the 50's with the most boring math books ever invented. Therefore I do not look on math as a favorable thing to learn – and that's why I'm trying to make it as fun and interesting as possible for them. (T1, May 8, 2012)

I think it's kind of dependent on the child and what they're interested in. We should help children feel math interesting and fun. I have found that people who try to push children into something that they're not ready for, even if it is a prescribed "it should be done this way," if you try to push a child somewhere where they're not ready, they're not going to learn it. I don't think it has to be just "boom-boom-boom-boom-boom" with young children. (T2, June 20, 2012)

Teachers also mentioned that introducing math concepts in a natural manner by making connections with students' daily lives is important. They said that making connections with students' daily lives can make math more meaningful and help students recognize that math is part of their real world.

Make the connection with something that's real to them, because math is abstract at a certain point. (T1, May 1, 2012)

We try to make it very meaningful for them because it's part of their real world. If you think, two plus two is four, that's just something like they sing in a song. (T1, May 1, 2012)

You need to have math you can use. That's why this program from Chicago, *Everyday Math* – that's why we really like it. You're going to the grocery store. Which can of peas is the cheapest? Which one is your best value? You need lumber to build a fence. How many board feet do you need to buy? You can memorize all the formulas you want, and if you can't figure out how much lumber to buy, it won't work. You've got to be able to apply it to your world. That's why we think it's a really good math program. (T1, May 8, 2012)

You should be able to make math valuable. You should be able to make it applicable to the real world. You can't just say, "Okay it's from 9:00 to 10:00, we're going to do

math.” It’s got to be cross-curriculum, and it’s got to be something kids can use and not just put in a box. “Okay, well, I don’t have to know how to keep score for bowling. It’s not math time.” Or, “I don’t have to know how much money I need to pay my account. Or add my checkbook up.” They have to have a real reason to do it. I just think you need to make it real. If it’s not real, it’s a bunch of formulas or theories and stuff you’ll never use. I think teachers should help students to apply math to their real life and their real world. (T1, May 8, 2012)

I think the important thing to tell anybody with young children is you need to make it real to them. You need to make it fun to them. You can count how many steps it takes to get from your porch to your house, or from the first level to the second level. You can have them go around and ask friends, “What’s your favorite kind of ice cream?” “I need to make cookies. How many dozen cookies?” You can take it to a higher level. I think things like that are very important. (T2, June 20, 2012)

One teacher’s personal experience emphasized the importance of making math real.

I think that in my high school example, when I lost interest in math, it was the same thing. I didn’t know why it was working. I couldn’t see any practical application. I couldn’t visualize it. Some youth can. They can figure out how that concept really works, but I think especially for young children, keeping it very real so it’s not math time. It doesn’t have to be so isolated, being able to figure out problems that are meaningful and really make a difference to the child. (T3, May 18, 2012)

In addition, teachers said that motivating students to study math is necessary because self-motivation provides a strong driving force to engage students in mathematical investigation.

I remember years ago, I asked my class, “Why do you think you need to learn math?” They said, “So you can go to first grade.” I realized I had to change what they thought about math at that point. They have to understand that you don’t just learn math to go to first grade. I gave them an example once. I got on the wrong bus in Chicago because I didn’t see the number. I said, “Think if you can’t read those big numbers. How will you know which bus to get on? You’ll get lost!” (T2, June 20, 2012)

Teaching strategies. Teachers addressed various strategies for teaching math to young children. First, teachers emphasized the importance of assessment. They said that knowing students’ levels is important because it can help the teacher know what they are interested in, what they are challenged by, and what is the next place they need to go. They suggested several ways to evaluate students’ mathematical understanding, including standardized tests and

informal assessments, such as observation, as well as asking children to elaborate on how they interpret and approach a problem.

In order to help children who need further help from teachers, they need to be identified. We have a math screening in the fall, MAP. And I'm not quite sure what MAP stands for, but it's used a lot of places in Illinois. It also just reiterates if a child is really low in a certain area, and it's not just math, but math concepts too. Like first, second, and third, shapes, most, least, a lot of the concepts, which involves language. Once we pick those out, then we've got them in tutoring, then I try to find a corner here, and a corner there. We play a lot of games, and we play a lot of BINGO games. (T1, May 8, 2012)

To me, a good teacher knows where every child is academically, even if I'm not giving them a test. I know where they are. I know which ones aren't understanding what an "8" or a "9" is. I know which ones can read "1,193." I know which kids understand that's an odd number and that's an even number and what that means. (T2, May 3, 2012)

I'm watching them while I'm doing this to see how they're figuring this out. It is like informal assessment. Some of them got it right, and some of them got it wrong. The ones that got it wrong, I watch their faces. I'm always watching their faces to see how they're feeling about themselves. Some of them come in so "programmed" that they can't be wrong. If they're wrong, they're going to cry, or somebody's going to make fun of them. If they get it wrong, I usually will say something like, "I like the way you're thinking. Your thinking is good, and I like your thinking. You're almost there. How can we help you to do this?" I'll say to the whole class, "If you didn't understand this, tell me what I can do to help you." (T2, June 13, 2012)

I do a lot of observation. I will sit with a group and use white boards, little dry erase boards. I might give them a story or a problem, and I'll watch them doing that too. (T2, June 13, 2012)

I think that it's really important that teachers of young children look at the individual child's development rather than shoot for an average mark. You know what I mean? Like everybody should be able to do this. I think that we need to spend the time really even watching the children at play with things like this. What are they really doing, and what they are not doing? I'm going to go back to the counting example. I think that children need lots of opportunities for actually discovering the one-to-one correspondence as they're counting before they go on to grouping and putting the groups together, which would be the beginning of computation. I think it's really breaking it down. I think the teachers need to be very watchful of what the children are doing. Sometimes it's just watching a child put together a jigsaw puzzle, for example, and what they're really doing in their spatial reasoning to put that together. That really informs if a child has great facility with that or if they're really struggling, if they really can't visualize where this puzzle piece could fit. Then they need lots more – they probably need an easier puzzle first, and they need lots more opportunities to think spatially like that. That is directly related to how they develop in math, in their math concepts. (T3, May 18, 2012)

I think that doing something one-on-one where you really are able to watch how they're counting or how they're adding things together, to try to determine what the confusion might be. Often I'll just have a conference with parents to say, what are you noticing at home . . . for example, if they play counting games with them, things where they have to count on and back. Are they running into those same issues? In public schools, I had access to the special education teachers, so I could use their expertise to try to figure out some things that I couldn't figure out on my own. In this environment, we don't have as easy access to the special educators. (T3, June 8, 2012)

Yeah, I think you should always do an informal assessment before moving to the next stage of teaching anything really, but particularly math. An informal assessment, check for understanding. Are they able to do the addition? We're just using that as an example. It could be measuring. It could be time. But are they able to do the addition at the level before successfully? Could they teach it to me? If they can teach it to me, then they are probably ready for the next level. But if they aren't able to teach it to somebody else, we'd better review a little bit more. (T3, May 18, 2012)

Second, the teachers discussed the importance of making good use of concrete objects when teaching math. They agreed that children have the ability to understand abstract mathematical ideas, but they said that children can learn mathematical concepts more easily when concrete objects are provided. They described a sequence in teaching math, starting from the concrete and moving to the abstract.

I think that some children, even very young children, have the ability to think abstractly about math concepts and perhaps even solve problems, but I still think it's important for them to have the other ways of demonstrating that. I think just because a child can do something doesn't mean that child should do it that way. Just because a five year old is able to do rote math, to do $1+1$ is two, and actually write out the equations, I wouldn't spend very much time with that because developmentally, I believe they still need to be visualizing the comparison. They still need to be doing actual counting. Even if they know six plus four is ten, for them to be able to actually see that to put those together and count them, I think is important in making relationships to other things that they're learning later. I'm my own best example. I think that I learned how to do math at a very young age, addition and subtraction, and put it on paper, but I didn't really understand how computation worked. I could kind of do the trick, but I didn't really understand how it worked. So when I got to later math, I had a gap in my understanding of math patterns, for example. (T3, May 8, 2012)

I think you need the concrete first. Once you have the concrete, then you can move from the objects to the pencil and paper to being able to do it. Say you've got turtles and frogs. Three turtles plus two turtles makes five water animals. If they can see three turtles and two frogs, then they can put them together. Later on, then they can write three plus two, and five, without having to count how many. But I think they need the

concrete first at this age, and you can move. It's primary to do the concrete first. (T1, May 8, 2012)

I think there should be a fixed sequence. Because if I say "three plus two," and a child doesn't know, the first thing they're going to do is they're going to use their fingers. So they've already taken it out of my abstract, and they put it back into the concrete. (T1, May 8, 2012)

I think you've got to teach the concrete first. You would love to say teach thinking skills, and higher-order thinking, and problem-solving. But if you don't understand what you have first, then you can't solve a problem if you don't understand it. If we are talking about how much is three, and how much is two, they've got to know what a three is, they've got to know what a two is. They've got to know the value that you assign to it. Then later on, they will be able to- the higher-order thinking. You can focus that on further – well, if I had 200 plus 300, is it the same idea? Is it the same thought process behind it? What if I had three million and two million, is it the same process? You'd be surprised. They think it's not. And it's exactly the same process because they haven't learned how to think it yet. Higher-order thinking skills are always what we strive for, but I think you've really got to hit the concrete ones first. (T1, May 8, 2012)

Teachers elaborated the reasons why concrete objects should be presented first.

Children learn by doing, touching, seeing. There's a saying that goes . . . I don't know how it goes. It's "If you tell me, I listen. But if I see it and touch it and do it, I can learn it." It's something like that. That's not right at all, but that's what it means. It's that you have to be able to use more than just your sense of hearing to learn. That's what kindergarten is all about. You're going to sit in eighth grade most of the time and use your hearing and seeing, you know. But in kindergarten, you've got to learn by doing. That is the way kindergarten kids learn. So however you said with cognitive, they've got to be on all senses across the board to learn their math. I can't just put up a number and say, "This is a one. Okay, this is a two." It's not going to work. What is a two? What does two mean? Can you show me two things? (T1, May 8, 2012)

Third, teachers shared some thoughts on the definition of "doing math." Math is more than just pencil and paper writing or getting right answers. The teachers wanted the focus to shift to the thinking process and making connections.

I think that the process is very, very important. I think that's where we've fallen down in this country, because the way I was taught – and somewhat the way even my daughters were taught – was very much memorizing how to do it, even to look into when you're doing word problems, to look for those certain words in the problem. Because of that, it's very hard to go out in the world and understand math. I think there are certain things they have to learn, but beyond those certain things, I think – I've had children come up with ideas that might not be my idea, but I can look at them and go, "You're right, that is

another way to do this.” I like that, for them to be able to come up with different ideas to get answered. (T2, June 13, 2012)

I want them to be able to understand their own learning process. I want them to understand how to apply it in other situations. I just think that early childhood educators need to be open to trying lots of different things, exposing the children to lots of different ways, and being open to doing direct instruction in very different ways because the same thing doesn’t work for all children. (T3, May 18, 2012)

I want to hear their thought processes on it. “Well, how did you know how many apples he had left?” Some of them can tell you, and some of them will say, “I don’t know, I just guessed.” And then we’ll go over it logically, and I say, “Well. If . . .” And a lot of times, then I will draw something on the board, but that’s my last thing, because I don’t want to show them a concrete and go from there. (T1, May 8, 2012)

Teachers added that students should be encouraged to explain how they figure out math problems and to elaborate their thoughts.

It’s a very good strategy. If you’ve not worked with children in a math setting before, just reading it isn’t going to help you learn it. You have to be able to practice it. You have to be able to talk to a child one-on-one and say, “How did you do that? How do you know how to get that answer?” Suddenly, it’s clear. It just is clear. I’ve taken what I’ve learned from those with my own students. When they tell me, “I just know the answer,” I don’t let it stop there. I say, “No, you need to explain to me what you’re thinking inside of your head.” I’ve even told them, “That’s called metacognition. Metacognition is a very big word, and it means you can tell me what you’re thinking. You can understand that.” Some of them really get it. They like to use big words. (T2, June 13, 2012)

The teachers also mentioned the importance of trial and error.

One math activity, that happens very regularly and they [students] come to expect that, is when we gather on the carpet for a group discussion. There’s one time of the day that it’s oral math, so we pose a problem out loud. For example we might say, “I have ten pennies,” so there’s nothing visual for these. “I have ten pennies, and I bought two gumballs that each cost 2 pennies. How many pennies do I have left?” And the purpose in that kind of daily activity is for them to listen carefully to what is happening. We’re less concerned about the right answer during that particular activity than giving various children the opportunity to explain how they figured it out. We really want them to listen to one another and find out that it’s like trial and error is a good thing. It doesn’t really matter that the answer wasn’t correct but that you were trying in this particular way. (T3, May 13, 2012)

Teachers said that children can learn from trial and error. In order for children to benefit from trial and error, it is important to create an atmosphere where children do not feel ashamed when they get wrong answers or make mistakes.

The first thing you have to do when you teach them anything is make sure that they're comfortable in the setting so that they're comfortable giving an answer, whether it's right or wrong. If they make a mistake, they need to feel comfortable enough and that the other children are not going to say, "Ha ha" and laugh at them. They have to be comfortable making mistakes. I tell my students all the time that nobody's perfect. I make mistakes, and everybody does, and we learn by making mistakes. (T2, June 13, 2012)

No. We don't really say, "You're wrong." But we say, "Well, let's see if somebody else has another idea." And then we ask them, "Well how did you come up with that up idea?" (T1, May 8, 2012)

Fourth, teachers talked about the merits of collaborative learning among students. They said that children can learn from each other.

I don't consider myself the most important person in here. If I cannot teach it, and their neighbor can, then as long as the learning gets done, that's what's important to me. If I'm struggling and getting frustrated with a certain concept, and the kid beside them or the kid across the hall or any kid can say, "Well, this is how you do it because . . ." and all of the sudden, the light bulb comes on— that's what's important to me is the learning, not how I'm in charge, I get to do it, you have to do it my way. Not all children learn in the same learning styles. That's why I do a lot of cooperative learning. Besides, you've to learn to work together with people in this world. You're going to work with people that are smarter than you and people that are slower than you, and if you can work in a group. I'm finding that groups of three are the max. Any more than three, then you start to get a power struggle going on. (T1, May 16, 2012)

I think collaborative learning is very valuable. I think it's valuable in the lower grades especially, where you're not working for a grade. I find that when they get much older, often times, when they work in collaborative groups, if it's for a grade, there are those children that won't work hard, and they'll let the strong child carry it and do most of the work. At the younger level, they're not at that point yet. They're very eager to work together and to help each other. At least in my class, I'm finding that they're very helpful. We've worked on this all year to get them to the point where we know this is the way we work. If somebody is struggling, you don't make fun of them. You find a way to help them, and you work together to find a way to solve whatever it is we're working on. They're getting really good at that, which makes me very pleased. I think collaboration is fantastic at this level. If you don't start it at this level, then those really bright kids aren't going to necessarily want to work together or to help the other ones as they get older because they think they're so much smarter than everybody else. I think it's good for all of them, the low kids and the high kids to learn to work together. (T2, June 20, 2012)

I think the merits are what we've just been saying, that the children have that opportunity to really explain their problem-solving methods and listen to other children and build upon that. For example, I consider boxes-and-junk construction, when they

have lots of different recycled items, and they create something out of that. That is very much of a problem solving activity that I can relate to math because it's spatial thinking. It's maybe getting things that are very disparate shapes to stick together. That's one of the activities where I hear a lot of back-and-forth conversation, where they're [saying], "I can't get this toilet paper tube to stick to the box. The tape won't hold it on, so what can I do?" They really help one another and then become sort of boxes and junk experts. (T3, June 8, 2012)

Teachers identified some simple rules for encouraging collaborative learning among students.

I always tell my children, if you're helping someone else, don't tell them the answer. Help them to find the answer, whether it's math or literacy skills, or whatever we're working on. I've taught them. You can give them clues, and you can help them, but don't tell them the answer. I've actually gone to doing that lately. I put my really high child at a table with lower kids and have him do things with those kids while I'm over here working with another group. It varies, depending on what I feel like we need. Today, I got dry erase boards out, and one of my higher kids asked me, "Can we work on equations?" I said, "Absolutely!" They just took off and just did it. (T2, June 20, 2012)

They also provided some tips for grouping kids for collaborative learning.

I usually try to pair maybe average-to-good students. Not the top student, because the top student will make the lowest student feel really bad, but I try to pair them as a team. They have to work as a team, and a lot of times, your top student has to learn some social skills of not being so superior and things like that, to work with the lower ones. (T1, May 8, 2012)

It would be different for different activities. Sometimes it's important to have them with like learners, so it might be three or four who are struggling with same concept or developing at a common rate about that particular math concept. That is important when I'm going to be working directly with that small group, when they're independent groups, so that I can be working with this group, and the other groups are independent. I just like to mix them so that they're communicating with a lot of different children and problem solving and being able to have those conversations with a lot of different children. (T3, June 8, 2012)

Lastly, teachers said that activities that integrate math with other subject areas such as music, art, PE, reading, and so on are efficient ways to teach math.

Music is a big part with math. You can do a lot of stuff with music and math. Want to hear one? Want to hear the best one we've got? I'll just play you part of it, but the kids love it. [played "Count to 100" song] Now, don't you think they're going to remember that? Yes. Every one of them can sing that to 100. (T1, May 8, 2012)

I like to do it mostly with music. I don't know if you've ever looked at all the things that are available, but there's so much you can do with just songs and music. There's a man

named Jack Hartman, and he is probably the number one for children. It used to be Raffi in the 70's, but Raffi did a lot with rhyme. Jack Hartman has done a lot with more of math, seasons, concepts, counting by 5s and 10s, skip counting, hopping, and that kind of stuff. He's probably the number one on children's, but there's probably thousands and thousands of them out there. There's a lady named Dr. Jean. She also has a lot of musical things. (T1, May 8, 2012)

It's teaching children the concepts they need for reading, writing, and math through music. At this age, with young kids, you can pound it in their head all you want, but you put it to song, and they've got it. Just like that. (T1, May 16, 2012)

I think that's an excellent thing to do with young children is to try to integrate it, so that you're not just doing one thing or the other. Finding ways to do – if you're doing a unit on insect or eggs or whatever. You're doing all these different things at the same time. I might have them measure with nonstandard measurement. I might have them measure the circumference with Unifix cubes, and then they can decorate the egg. We might have some music that goes along with that, or we might have a poem that goes along with it as well. Especially in kindergarten, we integrate everything just about that we do. You just don't have a separate "it's time to do math" necessarily. We're doing it all day long together, and it helps. (T2, June 20, 2012)

I think that we're integrating a lot of things throughout the day, so the time isn't chopped up into discrete hunks of what we're focusing on. I really believe that. Children learn in such different ways. Often it's a song, that is something. (T3, June 8, 2012)

Support for teachers. In order to help teachers prepare themselves to teach math, support from teacher education programs was mentioned.

I think that it really needs to start in our teacher education programs, that we give learning teachers the tools they need to answer questions and to help them to be continual learners themselves. I know far more now than I did when I first started, because it comes from experience and finding out from other people. It's always, that's what's exciting about being a teacher is that you're always learning at the same time as the children. I think that mentorships need to be always offered. I think that our teacher education programs should really be that solid first place where undergraduates can say, "I don't like math, what can we do to be better and feel better about math so that we can be good teachers?" (T3, June 8, 2012)

One teacher also said that teachers should make an effort to improve their own math skills, which builds on the support from teacher education programs.

I had to go back to college to pick up extra math courses, and they made me take the ones that didn't count because my math was so low. Then I jumped into stats – statistics. They didn't want me to take that, but I said I know I can do it, so I took that. Then every time I had a chance, I would take professional development courses on my own that had

to do with teaching math because there were things out there to help teachers that needed help. I took four or five different courses on my own on teaching math to children in different ways. I really began to enjoy teaching math because it just made sense. Now it's fun. I love it because it's all about patterns and seeing how things go together and work and how it works in real life. Now I really love teaching math, and I teach off and on all day. (T2, May 3, 2012)

Second, collaboration with other teachers was recommended as a way to provide support for teachers.

So why can't math be something where there's teaming done? So if I were not able to teach area and perimeter, I should probably team up with somebody who's really good at teaching area and perimeter. But if I'm very comfortable and feel skilled at teaching fractions – now, these are not preschool levels by the way, neither of these two concepts – then I feel good about it. Do you see what I mean? So therefore, that's my strength, and I should team up with somebody with another strength. (T3, May 18, 2012)

All three teachers said that collaboration with parents is the most powerful support for teachers to facilitate children's mathematical understanding. One teacher described workbooks that she sent home for students to do with their parents.

There are take-home books that the children take home, and I keep track of them. They have these books as their homework books. And here is Book One [shows me *Everyday Math Home Links Workbook*]. They get this sheet like this, and that's the first page, and we can either give them the book – I prefer to tear them out and the parent has to do it with them. Goes home on Friday, comes back on Thursday. They have to do the activities at home. This is number one, and parents write down what they did. I tell them, I said, "Don't write me a book, I'm not going to read it." I said, "Just say, you did it, you enjoyed it, you liked it, you hated it." This is from the parent. "Grace learned how to collect data and sort items" on Lesson 16. "Grace is okay with shapes. Shops at the store for different shapes, helped her remember them. Too easy." Now her last one, she's on Lesson 26. That means she took 27 home today. "Yes, we did it. It took twenty minutes." You can't fault a parent in these busy times, as long as they did something with their kid. This [working with parents] is all pre-programmed for me. I can't take a kid home and beat them up if they don't do this. But you hope – and we talk to the parents at the beginning of the year – we hope that they do this. There are four books. See? Book Two, Book Three, Book Four. And we work right through them. Now the other teachers all send this book home, inclusive at the beginning of each nine weeks, and then the parent is supposed to go ahead and do it on a timely basis. They aren't going to do it on a timely basis. So I separate them out. Either they will all do it the first week and get everything done, and then they've crammed it all in, or they will wait, "Oh no, I've got to send this book back, it's the end of the quarter!" They'll cram it all in at the end of the quarter. I make them do it every week and bring it back. (T1, May 1, 2012)

Teachers said that sending equipment [math games] home works because parents want to help, but they often do not know how. The activities that teachers send home can give parents ideas of how to promote children's mathematical understanding.

We also have a lot of games that are "take-home-able" games. When you can get the parents involved, that's the best help you can get. We're finding that the parents around here, they really want to help. They just don't have a clue how. But they're more than willing. So if you send a game home that you have to roll dice and hop three spaces or something, parents will play with them. Without question, they'll be very helpful on that. (T1, May 8, 2012)

The kids think they're playing a game, and all they're doing is practicing naming numbers, which is a lot better than going "flashcard, flashcard, flashcard, flashcard." We do a lot of games like that. Parents don't even think that, "Oh, we don't have cards at home with numbers on them." Well, you take index cards, and you make them. You take a piece of cardboard, and you make them. If they won't, I send them home and do them. (T1, May 16, 2012)

I send my equipment from school home a lot. We play a lot of number BINGO. We do a lot of game format things, where you send a dice home. The kids are having trouble with a number. I'll say, put that sheet on the floor, and roll the dice. Hop that many times. Say that number, recognition. Clap that number. Hopscotch or penny pitch, where they'll flip a penny and it'll land on a number. You get to take that penny off and put it in your penny bank if you can say, "That's number 36 or number 42." There's a lot of ways that parents can do and have fun. (T1, May 16, 2012)

Yes, but I've already checked with the parents by saying, "Hey, I've got a couple of ideas. Have you got time to play at home and do these with them?" We'll go over them, because a lot of times the parents want to help, but they just don't know how. They don't have any idea how. I'll say, "Well, let's make it fun." Say you've got a boy who wants to play with his Matchbox cars but is struggling in counting. I'll say, "Line up your Matchbox cars. Count them. One, two, three, four, five. Now you're going to roll them. Roll them, one, two, three. Roll them like that." (T1, May 16, 2012)

There's so many ways you can do it. You can do matching. You can do same size, same color. You can do groupings, and all you're doing is playing cards with your kids. Simple as that. Parents say, "Oh, I never thought of doing something like that." They can use dice and cards, and their own toys, and pennies. You want to do some counting, you say, "Hey, it's time to set the table. How many people in our family? We have five in our family. We all need a knife, we all need a fork, and we all need a spoon. Go get how many forks? How many spoons? How many knives? And set the table." I told the parents, "First of all, you've got your table set for supper, and the kids have been counting the whole time. They don't even know it while you're busy cooking." You take a real life situation, and it involves just a little bit of work and a lot of communication.

All of the sudden, you're a teacher. That's what I want. I want the parents to be teachers. (T1, May 16, 2012)

Anyway, I've got children coming in that have nothing. Maybe somebody counted with them, but that's all. As a parent myself, I was a parent before I became a teacher. I don't think I did as good of a job as a parent with my children in terms of what they needed as I would have if I'd been a teacher first. You know what I mean? So I understand what it's like. I don't think parents know what they should be doing with their children before they come to school. Some parents think they have to buy workbooks and sit their kids down and go through workbooks every day. That's not the way to do it either. These kids coming in didn't have anything, but I have other children whose parents have worked with them a lot. Not necessarily drilling them, but just exposing them to numbers, and exposing them to letters for that matter. These kids have had so much more given to them, even culturally. They're way ahead of the kids that came out of nowhere. (T2, May 3, 2012)

That's where we like this *Everyday Math* program. It has those little books with all these ideas that parents can use with their children. Those books have some very, very nice ideas on things. As a parent myself, when I think back to when my daughters were in kindergarten, I hadn't gone to school yet to be a teacher. I didn't think of things. I already had a bachelor's degree, and I was a fairly intelligent person, but as a parent, I would love to have had something like that to give me ideas. (T2, June 20, 2012)

Occasionally, I'll send home things for parents to do with their children. I will tell parents things if their children are weak. Mostly what I'm seeing right now is my children don't need math so much as they need work with reading. Right now, that's the big push is to get children reading. I'm sending home more things about that than I am with math. The only thing I'm really doing right now with math is sending home those *Everyday Math* books. My kids just aren't- it's hard to get parents to do things with children. (T2, June 20, 2012)

In terms of trying to get them to do things with their children in math, it's very hard to do. I might have them do a survey, have your child do a survey. Or when I do the egg drop, they're supposed to measure and figure things out, the dynamics of that. So that involves math and other things as well too. Other than that, if I send home too much stuff, they're not going to do it. Or the parents will do it for them and send it back. I try to do as much of it in class as possible. In the ideal world, we could tell parents and give them ideas of things to do, but it doesn't work. (T2, June 20, 2012)

I think that if we're going to send something home, it needs to be, in order to communicate what kind of learning is going on in the classroom, rather than just practice, practice, practice of a particular skill. I have used that in the past, where I say, this is the activity that we did, these are questions you can ask your child about it, here's another way you can try it. When we have parent-teacher conferences, parents always ask what they can be doing at home to support their children's learning. I think it's most important in math concepts, that they again use math in a very real way. Counting out money to pay for something, playing games that involve numbers and math concepts.

Many, many games do that. It's something that parents can- we have children in the class who are playing chess with their parents, even those kind of things. Lots of games and puzzles. (T3, June 8, 2012)

Challenges in the Classroom

Research findings indicate that teachers do not know what to do about mathematics for young children (Clements, Sarama, & DiBiase, 2004, p. x), but the three teachers I interviewed did not have difficulty providing math activities to children. The kindergarten teachers both use a math program selected by their schools, so they do not feel any pressure to come up with their own math activities.

We have a math program here that we follow, so we don't have to make up a lot. We have a program; it's probably evolved out of the old "Math Their Way" in the 70's... Our math program was developed by the University of Chicago, and it is called *Everyday Math*. We have this program that tells exactly what we do, every day, and if you miss a day, you catch up. You can do anything with it. We start with our shapes and the numbers, and we play a lot of songs and number games, counting in sequence and ordering, those kinds of things. (T1, May 1, 2012)

We use the University of Chicago *Everyday Math* curriculum, so I have those ideas. When I plan math activities for my students, I always look at that book. I also have experiential ideas, things that I've done with my previous students or things with my own children. So, I don't have trouble providing math activities to the children. (T2, May 3, 2012)

The preschool also provides a math program. The preschool teacher, however, seemed to have more flexibility to incorporate other ideas into the curriculum. She said that she uses a math program recommended by her school as one among several and does not exclusively depend on it.

So in this classroom, we use the University of Chicago *Everyday Math* curriculum, but we don't follow it letter-by-letter. We incorporate that and also other things. We're fortunate that no one just handed us a curriculum and said, "Use this," whether the children are ready or not, so we can adjust for that. (T3, May 13, 2012)

She added why she does not solely use only one math program.

I've not had to use one curriculum and step through it in my career. I don't think I'd be able to do that. When I was teaching in other school, we were expected to use a particular

math curriculum. It only made sense to me to still add in other things. I haven't seen a math curriculum that can cover everything and really be useful every single day. Maybe they exist, or maybe I'm just fussy. (T3, June 8, 2012)

The challenge the teachers faced was not so much implementing math activities as adjusting the difficulty level to meet the children's differing levels of mathematical competence.

The biggest challenge is the diversity that comes in. You have some children who can walk in the door and count to a hundred. They pretty much know what they're doing and how they're counting, if you put a number out and say, "Is 89 a lot or a little?" Then you have other kids that come in the door, and they don't know what a one or a two or a three is. I thought I saw children who were not ready with a lot of concepts. The diversity was huge. We have to start out with the diversity. (T1, May 1, 2012)

That's the problem right there, is how do you address all those children in one setting when you've got some down here and some way up here. The ones way up here are getting those abstract concepts, but the ones down here are struggling. (T2, May 3, 2012)

Right now I have some children that could recognize just about any numeral I show them, way up in the hundreds. But then I have some children in my class that don't know what the number eight is still. The difference is huge. So when you do things, you're trying to hit all of them, but when you're working with those children that are still very low, they can't deal with the abstract yet. We have to say things like, "You have ten fingers. Use those fingers." Draw pictures. There are strategies you can teach them to help them do these things. I work on teaching those strategies, especially with my lower children. You scaffold and then you start pulling it back away when they're ready for it. (T2, May 3, 2012)

Initially, the struggle was in meeting diverse learners at their developmental level. So in one class, there would be children that were very, very emergent in their mathematical understanding, and then children who were all along the range, and then children who were very adept and had higher level understanding. I think the challenge was in trying to design activities and an environment – a classroom environment where you could meet all of those diverse developmental level needs. (T3, May 13, 2012)

Summary

The teachers believe that children have some math ability and are ready to learn math. Particularly, they often mentioned children's eagerness to learn math. They also mentioned the importance of understanding children's mathematical cognition because they think that they should consider how children think mathematically when they plan math activities for children.

Teachers mentioned five content areas: (a) number sense and operations, (b) algebra, (c) geometry, (d) measurement, and (e) data analysis.

The teachers addressed three big ideas related to pedagogy. First, as a way of helping children develop a positive attitude to math, teachers made five suggestions: (a) modeling being excited about math, (b) continually presenting math concepts so that children feel comfortable with math and eventually like it, (c) providing math activities that students are interested and engaged in so that children see math activities as fun and enjoyable, (d) connecting math to students' daily lives, and (e) motivating students to study math.

Second, teachers suggested five teaching strategies: (a) continually assessing, (b) starting from concrete and moving to abstract, (c) focusing on thinking and making connections, (d) encouraging collaborative learning among students, and (e) integrating math with other subject areas.

Third, teachers need support from (a) teacher education programs, (b) collaboration with other teachers, and (c) collaboration with parents.

The teachers I interviewed did not have difficulty providing math activities to students. Rather, they struggle with the activity levels required to meet the needs of different learners with different levels of mathematical ability.

Chapter 8

Similarities and Differences across Groups

I initiated this study wanting to understand what mathematicians, teacher educators, psychologists, and teachers had to say about the best way to provide math education for young children. As I examined the responses of each group, I searched for patterns that would help me capture the perspectives of the participants. From chapter four to seven, I summarized each group's responses.

In this chapter, I compare and contrast their responses based on the three categories from the PCK model: (a) children's mathematical understanding, (b) mathematics content for young children, and (c) math pedagogy. I add a fourth category emerging from the data that did not fit the initial top-down categories: group-specific responses.

How Children Learn Mathematics

All four groups agreed that children have some math ability and are ready to learn math. They paid attention to individual differences in this ability. The difference between the groups, however, is that teacher educators and teachers considered individual differences in children's math ability as a challenge in teaching while the mathematicians and the psychologists, the two groups farthest removed from early childhood classrooms, talked about individual differences but did not relate these to the actual teaching of math. Psychologists differed from the other groups in that they attributed variance in math ability to opportunities provided children rather than some sort of innate ability. Teacher educators and psychologists focused more on signs that children are ready to learn math. Teachers pointed out children's eagerness to learn math. Teacher educators, psychologists, and teachers asserted that understanding children's mathematical cognition is important because how children think mathematically should be a foundation for developing and providing math activities for children. Table 4 summarizes ideas

proposed by each group related to children's mathematical understanding. The number refers to the number of people in each group who talked about the idea. The symbol in the parentheses represents the strength of the theme—how often brought up and how strongly emphasized (+++: strong theme; ++: average; +: weak).

Table 4

Views related to children's mathematical understanding

	Mathematicians	Teacher Educators	Psychologists	Teachers
Children's math ability	3 (++)	3 (+++)	3 (+++)	3 (+++)
Individual differences in math ability	3 (+++)	1 (++)	2 (+++)	3 (+++)
Signs of children's readiness to learn math	0	1 (++)	1 (++)	0
Children's eagerness to learn math	0	0	0	1 (++)
Importance of understanding children's mathematical cognition	0	2 (++)	2 (++)	1 (++)

Mathematics Content (What to Teach)

Five content areas, which should be taught to young children, were addressed: (a) number sense and operations, (b) algebra, (c) geometry, (d) measurement, and (e) data analysis.

Mathematicians and teachers mentioned all five areas while psychologists mentioned only three: (a) number sense and operations, (b) geometry, and (c) measurement.

The reason that psychologists did not mention the other two content areas seems due not to the secondary importance of these two areas but to their focus, which can be seen as limited. The specialty of the two cognitive psychologists is children's numerical thinking. The developmental psychologist is interested in the content areas where children have innate perceptual ability, specifically geometry and measurement.

Mathematicians emphasized deep understanding of mathematics content. Teacher educators did not directly mention content areas, but they equated math content with concepts

and emphasized thorough understanding of these concepts. Mathematicians and teacher educators mentioned the connections between each content area and stressed looking at the big picture in math. The difference between the two groups was that mathematicians emphasized the importance of teachers' seeing the big picture in math while teacher educators stressed helping students see the big picture. Mathematicians and teacher educators asserted that there is a fixed sequence to teach topics and that this order is very important. The difference between two groups, however, was that mathematicians talked about the order of introducing all five content areas, which they thought to be important to teach to young children, while teacher educators only talked about the sequence of introducing subtopics in "number sense and operations." It appears that mathematicians see a bigger picture than teacher educators across the math content areas. Psychologists also referred to the sequence in teaching mathematics topics, but they did not present the possible sequence. They just lamented that the sequence is not clarified yet. The interesting thing is that teachers never mentioned the order of introducing math concepts. It is possible that teachers are introducing and teaching math concepts not in order but randomly.

Table 5 summarizes content-related views proposed by each group.

Table 5

Content-related views

	Mathematicians	Teacher Educators	Psychologists	Teachers
Number sense and operations	2 (+++)	0	3 (+++)	3 (+++)
Algebra	1 (++)	0	0	1 (++)
Geometry	2 (++)	0	3 (++)	2 (++)
Measurement	1 (+)	0	1 (++)	2 (++)
Data analysis	2 (++)	0	0	2 (++)
Thorough understanding of content areas	1 (+++)	2 (+++)	2 (+++)	0
Connections between content areas	1 (++)	1 (++)	0	0
Sequence for teaching content areas	3 (+++)	2 (+++)	1 (+)	0

Pedagogy (How to Teach)

Three big ideas emerged about pedagogy. The first one was having a positive attitude about math. All four groups suggested using teachers' and parents' excitement toward math as a model in order to help children develop a positive attitude toward math. Mathematicians and teachers proposed two other ways to help children have a positive attitude: (a) early and continuous exposure to math so that children feel comfortable with math and eventually like it, and (b) providing math activities that interest and engage students. Mathematicians, psychologists, and teachers suggested introducing math concepts by making connections between math and students' daily lives. Mathematicians, teacher educators, and teachers noted that teachers should continually motivate students to do math. Table 6 summarizes what the four groups suggested to help kids develop a positive attitude toward math.

Table 6

Building a positive attitude toward math

	Mathematicians	Teacher Educators	Psychologists	Teachers
Teachers'/parents' modeling	3 (+++)	2 (++)	1 (++)	2 (++)
Early/continuous exposure to math	2 (++)	0	0	3 (+++)
Providing fun/enjoyable math activities	2 (++)	1 (++)	0	2 (++)
Introducing math concepts in a natural manner	2 (+++)	0	3 (+++)	3 (+++)
Motivating students to study math	1 (++)	0	0	1 (++)

The second idea related to pedagogy was teaching strategies. All the groups had similar ideas. They emphasized (a) assessment as part of teaching, (b) thinking processes and making connections, (c) encouraging collaborative learning among students, and (d) integrating math with other subject areas. Mathematicians, teacher educators, and psychologists supported providing differentiated instruction based on students' individual differences, but they did not provide specific ideas of how to do it. Psychologists and teachers recommended making use of

concrete objects in teaching. Both groups agreed that children can understand math concepts more easily when the concrete objects are provided. The difference between two groups, however, was that only teachers mentioned the direction: starting from concrete and moving to abstract. This notion, that children learn math concepts through concrete objects and move on to the next stage where they can understand abstract ideas, indicates the long shadow that Piaget's concrete operational stage has cast on early childhood education. It is interesting that teachers seem more influenced by Piagetian theory than psychologists. Teachers' clear distinction between concrete and abstract by equating abstract with memorized number facts shows that teachers have a narrow view of what constitutes abstract mathematical thinking. Memorized number facts may help someone think abstractly, but they are not abstract thinking.

Mathematicians, teacher educators, and psychologists agreed that student-invented strategies to solve problems should be valued because they encourage different kinds of reasoning. They also noted that students can clearly understand the concepts conveyed in the problems by developing their own strategies to solve problems and by explaining the rationale for those strategies. The interesting thing is that teachers, who can actually provide support for students to come up with their own strategies, did not mention or recognize the importance of these informal problem-solving strategies. Table 7 summarizes teaching strategies suggested by four groups.

Table 7

Teaching strategies

	Mathematicians	Teacher Educators	Psychologists	Teachers
Assessment	1 (++)	3 (+++)	2 (+++)	3 (+++)
Individualized teaching	2 (++)	3 (+++)	1 (++)	0
Using concrete objects	0	0	1 (++)	2 (++)
Focusing on process	2 (+++)	3 (+++)	2 (++)	3 (+++)
Encouraging diverse ways to solve problems	2 (+++)	3 (+++)	3 (+++)	0
Collaborative learning	3 (+++)	2 (++)	3 (+++)	3 (+++)
Integrating math with other subjects	2 (++)	2 (++)	2 (++)	3 (+++)

Various ways to support teachers were mentioned. Mathematicians, psychologists, and teachers mentioned teacher education programs. Curiously, teacher educators, who are most involved in teacher education programs, did not. Perhaps they considered such support as a matter of course. Mathematicians, teacher educators, and teachers proposed collaboration with other teachers. The psychologists appeared more familiar with one-to-one teaching, such as conducting a diagnostic test, than collaborative teaching. All four groups suggested collaboration with parents. Table 8 summarizes what four groups talked about supporting teachers to prepare them to teach math effectively.

Table 8

Support for teachers

	Mathematicians	Teacher Educators	Psychologists	Teachers
Support from teacher education programs	2 (+++)	2 (+++)	2 (+++)	2 (++)
Collaboration with other teachers	2 (++)	3 (+++)	0	1 (++)
Collaboration with parents	2 (++)	3 (+)	3 (+++)	3 (+++)

Group-Specific Views

Each group responded in ways that do not belong to the initial three categories. Mathematicians saw a big picture of what math is and what purpose doing math serves. They

said that math is not a collection of facts or concepts. For them, doing math is a process of finding logic. They emphasized the logical thinking embedded in math rather than mathematical concepts themselves.

Teacher educators agreed that current teacher education programs did not adequately teach math. They said that taking a few courses is not sufficient to prepare teachers to teach math well and that these courses should be introduced to pre-service teachers not as a freshmen but later. They also mentioned that the teacher education should embrace in-service teachers.

Psychologists talked about teachers' roles in teaching math. They asserted that teachers should actively help children interpret their experiences mathematically and understand the relations between math and their experiences. Psychologists also noted the importance of students' discovery learning. They addressed teachers' pedagogical content knowledge (PCK), a knowledge base combining a thorough understanding of what to teach with both how to teach and an understanding of how students learn. They said that teachers need a high level of mathematical PCK to teach math effectively, but often they lack this knowledge. Psychologists gave a sequence for developing each component of PCK: content knowledge, knowledge of children's cognition, and pedagogical knowledge should come in that order. Psychologists also said that method courses should help pre-service teachers develop pedagogical content knowledge and not merely focus on pedagogy.

Finally, teachers talked about challenges that they face in the classrooms when teaching math to children. They did not talk about challenges in providing math activities to students. Rather, they struggle to implement those activities in ways that meet the needs of different learners with different levels of mathematical ability. It is interesting to see the difference between what teachers said and research findings, for example, Clements, Sarama, and DiBiase

(2004, p. x) concluded that teachers do not know what to do about mathematics for the young children. This discrepancy may well be due to *Everyday Math*, used by all three teaches.

Everyday Math provides math activities for each day so teachers do not need to make things up day to day. *Everyday Math*, however, is a double-edged sword. It is a resource for teachers, but it can be also possible that teachers mechanically follow the curriculum without a sense of purpose.

Chapter 9

Discussion and Conclusion

I began this study with the intention of exploring the views of four groups—mathematicians, teacher educators, psychologists who study young children’s math learning, and early childhood teachers—about how to effectively teach mathematics to young children. I had three specific questions: (a) how do children learn mathematics, (b) what mathematics content should be taught to young children, and (c) how should this content be taught to young children. I wanted to know how the views of these groups were similar and different and how they reflected current learning standards (i.e., NCTM’s Standards, the Illinois State Learning Standards, and the Common Core Standards).

How Children Learn Mathematics

All four groups agreed that children have some math ability and are ready to learn math. All paid attention to individual differences in this ability. Teacher educators and psychologists focused more on signs that children are ready to learn math while teachers pointed out children’s eagerness to learn math. Teacher educators, psychologists, and teachers asserted that understanding children’s mathematical cognition is important because how children learn math and think mathematically should be a foundation for developing and providing math activities for children.

Not surprisingly, the psychologists discussed this the most. They went far beyond merely talking about children’s ability in math. They addressed two specific ideas of how to relate how children learn to teaching.

Psychologists suggested using *hypothetical learning trajectories* (HLT) as an assessment tool. HLT helps teachers to figure out where the child is on the learning trajectory so

that teachers can check what the child already knows. Once they understand what the child already knows, then they can figure out what the next step that he needs to know is.

Psychologists recommended utilizing children's innate mathematical ability when planning math instruction for children. They saw little effort to directly use this knowledge when designing math activities for children. The psychologists noted that children's innate mathematical ability is mostly perceptual, and this perceptual ability is a sign of readiness to learn a given content area. They asserted that the activities based on this perceptual ability are a good start for introducing children to a content area. For example, in order to teach children different shapes in geometry, given their perceptual ability to visually distinguish different shapes, teachers can start having children sort out different shapes of manipulatives. Then they can teach names and attributes for each shape.

Mathematics Content (What to Teach)

The groups show substantial agreement on the range of content areas that should be taught to young children. This agreement includes (a) number sense and operations, (b) algebraic thinking, (c) geometry, (d) measurement, and (e) data analysis.

Mathematicians, teacher educators, and psychologists emphasized the importance of the proper sequence for introducing content areas, although psychologists lamented that there was not much research on proper sequencing. Mathematicians emphasized foundational content areas. According to mathematicians, number should always come first because number is the basis of all mathematical concepts. For number, number sense, counting, ordering, and the four basic operations should be presented in order. Following number, mathematicians said that algebraic thinking, geometry, and measurement should be introduced in order. Data analysis should come later because it requires more sophistication and maturity. Mathematicians

distinguished the different levels of importance of the content areas. They saw measurement as relatively less important than the others.

Teacher educators also proposed a sequence for teaching, but it covered only two sub-topics in number and operations: counting and arithmetic. They asserted that counting should be introduced earlier than arithmetic. These findings suggest an ideal sequence for teaching mathematical content areas.

Another insightful idea related to mathematical content is that mathematics provides connections linking each content area within itself. Understanding how mathematical ideas interconnect and build on one another to produce a coherent whole is very important. Content areas that should be taught to young children are often listed separately and fragmentarily, so connections among them are easily neglected. Mathematicians and teacher educators, however, asserted that mathematics must be approached as a unified whole because each content area is interrelated in a significant way. For example, in order to solve problems in algebra, students need to have number sense as well as ability to do basic operations. Much of number sense and operations leads into algebra. Helping students understand these connections is very important.

Mathematicians emphasized students' thorough understanding of the concepts in each content area. They stressed that the number of content areas introduced to students is not important, rather how thoroughly students understand each content area is important. They were not referring to merely knowing facts or skills but referring to deeply understanding the meaning of the concept (e.g., what does *fraction* really mean?) and understanding the logic of each concept. Mathematicians saw math as a system of logic, so it is not surprising that they emphasized understanding the algorithm embedded in math concepts. Teacher educators

confirmed this idea by equating math content with concepts and emphasizing thorough understanding of these concepts.

Pedagogy (How to Teach)

All four groups addressed three big ideas related to pedagogy: (a) instilling a positive attitude toward math, (b) teaching strategies, and (c) various ways to support teachers.

Many ideas were proposed to help children have a positive attitude toward math. These included (a) teachers' and parents' modeling, (b) early and continuous exposure to math (c) providing math activities that interest and engage students, (d) introducing math concepts by making connections with students' daily lives, and (e) motivating students to do math. An interesting finding is that all four groups saw teachers' or parents' excitement about math as an important factor that affected students' attitude toward math. Mathematicians, whose positive attitude toward math definitely played an important role in their choice of careers, all mentioned teachers in their schooling who passed on a positive attitude toward math to them. All groups mentioned the power of teachers' or parents' modeling in constructing students' attitude toward math.

All groups except teachers suggested providing differentiated instruction based on students' individual differences. Teachers all said that adjusting the difficulty level of activities to meet the children's differing levels of mathematical competence is a big challenge for them. All groups except teachers also emphasized encouraging diverse ways to solve problems. They discussed the value of student-invented strategies to solve problems compared to traditional ways. Encouraging students to come up with their own problem-solving methods promotes different kinds of reasoning, and through this process students begin to make sense of the problems.

All four groups mentioned ways to support teachers. These included (a) support from teacher education programs, (b) collaboration with other teachers, and (c) collaboration with parents. Teacher educators' concern was that teacher education programs do not prepare teachers well enough to teach math effectively. They saw required courses related to teaching math as too few. They also mentioned that the teacher education should embrace in-service teachers. In-service teachers often lack pedagogical knowledge and strategies for teaching math because they are not provided with enough opportunities to take courses that can expand this knowledge. Once pre-service teachers become in-service teachers, it is more difficult to improve their ability to teach math. In-service teachers need on-going teacher education. Teacher educators pointed out that, in reality, providing the support that teachers need in teacher education programs is a serious challenge.

Mathematicians and psychologists also mentioned that teachers should take more courses related to teaching math. The difference between two groups was that mathematicians emphasized requiring pre-service teachers to take more content courses while psychologists asserted that teachers should expand their knowledge in content, pedagogy, and children's mathematical cognition altogether.

Pedagogical Content Knowledge (PCK)

This study sought to identify exactly what falls into each component of mathematical PCK. The question is *how, then, to help teachers of young children develop their PCK in mathematics by incorporating those findings?*

According to the literature (Ball, 1988; McCray, 2008; Shulman, 1986), deep understanding of three major elements that combine to generate Pedagogical Content Knowledge

for mathematics—knowledge of *what to teach* and *how to teach* combined with knowledge of *how students learn*—is necessary for effective math teaching.

Psychologists argued that the three elements of PCK should be taught in the following sequence: First, teachers should develop math content knowledge, and then they should develop knowledge of how children learn math, and finally pedagogical strategies for teaching math. Psychologists emphasized using this knowledge as an assessment tool. Having assessed where each child is, teachers should develop pedagogy that is effective in that context. Psychologists also said that the latter two steps (assessing how children learn mathematics and then deciding most appropriate pedagogy in that situation) are repeated while conducting math instruction

Implications

Mathematicians, psychologists, and teacher educators are specialists in each component of PCK. Teachers are expected to incorporate these three elements to generate PCK. My assumption was that the synthesized views as well as group-specific views would, together, help teachers clearly know what they need to teach math effectively.

Teachers need special expertise, including a deep and practical knowledge of the content and pedagogy of elementary mathematics and an understanding of children's mathematical thinking in order to teach math effectively (Ball, 1988; McCray, 2008; Shulman, 1986). There is substantial work on the development of mathematical thinking among young children from a cognitive developmental perspective. This complex understanding of mathematical thinking, however, has been rarely integrated into educational approaches. That is, it tells much about how children construct mathematical knowledge, but it tells little about pedagogical strategies. What psychologists suggested in this study, however, is an attempt to apply the understanding of development of children's mathematical thinking into teaching, specifically assessing what

children know and using math activities that incorporate children's innate math ability. Making good use of children's innate mathematical ability in teaching is beneficial because teachers often are challenged to see what aspects of children's mathematical competence should be developed.

The four groups' substantial agreement on the range of content areas that should be taught to young children tells teachers what areas they should focus on while teaching, and what areas they should expand their content knowledge. The view of mathematicians, with whom teachers rarely have contact during teacher preparation, that math content areas should be presented in an appropriate order, provides teachers with some insights into how to organize math activities. Mathematicians' and teacher educators' ideas of emphasizing the connections between content areas can help teachers not to provide math activities fragmentally but to provide them coherently, while thinking about the big picture.

Unfortunately, little research on what an ideal sequence for teaching mathematical content areas is, except for some key topics, and how content areas are interconnected has been done. The findings of this study related to these issues, however, can be a good starting point because they provide some clues. If an ideal sequence for teaching mathematical content areas as well as the interconnections between the content areas can be identified and applied to math education, students can begin to develop a bigger picture of math as well as begin to understand how the content area that they are learning fits in with everything else and why it is important.

Diverse ideas, proposed by the four groups related to teaching strategies, provide teachers with more tools in their toolbox to use when teaching math to young children. They emphasized (a) assessment as part of teaching, (b) thinking processes and making connections, (c) encouraging collaborative learning among students, and (d) integrating math with other subject

areas. The interesting thing was that all groups except teachers suggested providing differentiated instruction based on students' individual differences. There may be many reasons why providing individualized instruction is seen by teachers as difficult, particularly time—individualized instruction takes much time. Unlike the other three groups, teachers are the ones who work with children every day. Teachers recognize the value of differentiated teaching, but they understand that time constrains what they can do. I would argue that the notion of differentiated instruction needs to be reconfigured. If one defines children's individual differences in mathematical competence as their ability to learn and apply a set of skills, differences will stand out. If one, however, defines it as the level of ability to understand the concepts, to formulate different ways to solve problems, and to see the big picture, these differences may be less pronounced.

Another teaching strategy proposed by all groups except teachers was encouraging students to find many ways to solve problems. Possibly, teachers did not mention the value of non-traditional ways to solve problems for practical reasons such as time or classroom organization. Teachers' lack of self-confidence or knowledge in math may be an accurate reason. If teacher do not have confidence in their ability to really understand things deeply, then they are likely not confident about other ways and whether they are correct. Teacher educators pointed out that one problem is that teachers teach only the strategies that they themselves were taught in school.

Teacher educators' concern that pre-service teachers do not take enough courses that can expand their pedagogical knowledge needs serious attention. As cooking is best learned by cooking under the direction of an accomplished cook, pre-service teachers should be provided with enough opportunities to take courses designed with a mixture of didactic and apprenticeship

instruction (Bass, 1999). Ginsburg, Jang, Preston, VanEsselstyn, and Appel (2004) introduced a model course where pre-service teachers learn principles of pedagogy as well as actually practice these in different contexts. Pedagogical knowledge is best learned by experience. Having sufficient opportunities to practice applying their pedagogical knowledge is critical for teacher training.

Most research on teachers' PCK emphasizes the importance of combining content knowledge, pedagogical knowledge, and knowledge of learners' cognition (Ball, 1988; McCray, 2008; Shulman, 1986). This study suggests an order for integrating each element of mathematical PCK and clarifies the process of applying the three elements of PCK to teaching. Even though only one of the four groups, psychologists, talked about the order of integrating the components of PCK, this provides a promising area for further research. Integrating the elements of PCK in an appropriate order may boost teachers' mathematical PCK.

Letting teachers know what falls in each component of PCK, however, does not promise a high level of mathematical PCK. Teachers' Content Knowledge (CK) is a prerequisite for developing PCK. As teacher educators complained, teacher education programs require teachers to take only a few courses (mostly one content course and one or two pedagogy courses) related to teaching math to young children. With this preparation, teachers are unlikely to have sufficient mathematical CK. Increasing the number of required courses could address this problem. Increasing the number of math courses, however, requires decreasing the number of other courses. Teacher education programs do not have many degrees of freedom. The same challenge exists for developing adequate understanding of how children learn math. Courses in teacher education programs tend to focus on how children learn in general, not how they learn in specific domains.

What can be done to help teachers, then? The teachers I interviewed felt comfortable teaching math when they are provided with a math program; they all make a good use of *Everyday Math*. Whether it is a good program or not is still controversial. Mathematicians and scientists from leading universities have expressed opposition to the program and had pointed out serious mathematical shortcomings in it (The Washington Post, 1999), but it is clear that a math program helps teachers have some sense of which activities, among many others, lead to the desired mathematical development. A math program provides content in a certain sequence and pedagogical ideas (usually in the teacher's handbook). In the process of developing *Everyday Math*, the development team claims that how children learn mathematics was definitely considered. In a sense, a math program contains aspects of the three elements of PCK, so it can be good resource for teachers. If teachers fully understand the program, they may be able to develop an adequate level of mathematical PCK. The better the program is, the more helpful for teachers.

Teachers described collaboration with other teachers who are really good at teaching a certain content area as helpful. Teachers, however, often lack necessary mathematical content knowledge. As generalists they are required to teach all core subjects. As a result, they are often “jacks of all trades, masters of none.”

Elementary Mathematics Specialists (EMS) may be one answer. Wu (2009) noted the difficulty of raising the mathematical proficiency of all teachers due to a problem of scale, and suggested producing a smaller corps of EMS as a viable alternative. Ideally EMS would have a deep and broad knowledge of mathematics content, knowledge of how children learn, and expertise in helping others use effective instructional practices. The idea behind EMS is that they are placed in schools to be leaders and to provide on-site, collaborative, professional

development addressing mathematical content, pedagogy, and curriculum in order to enhance instruction and to improve student achievement (AMTE, 2010).

Research has provided evidence that EMS has a significant positive impact on student achievement (Brosna & Erchick, 2009; Campbell & Malkus, 2010; 2011; Kessel, 2009; Meyers & Harris, 2008;). Researchers and professional organizations have recommended making use of EMS in teaching mathematics (Battista, 1994; Conference Board of the Mathematical Sciences, 2001; Learning First Alliance, 1998; Reys & Fennell, 2003). The National Mathematics Advisory Panel (2008) noted, “the use of teachers who have specialized in elementary mathematics teaching could be a practical alternative to increasing all elementary teachers’ content knowledge (a problem of huge scale) by focusing the need for expertise on fewer teachers” (p. 44). Recently, the Association of Mathematics Teacher Educators (AMTE), the Association of State Supervisors of Mathematics (ASSM), the National Council of Supervisors of Mathematics (NCSM), and the National Council of Teachers of Mathematics (NCTM) all recommend the use of (EMS) in pre-kindergarten through 6th grade (2010).

Although only seventeen states (Arizona, California, Georgia, Idaho, Kentucky, Louisiana, Maryland, Michigan, Missouri, North Carolina, Ohio, Oklahoma, Oregon, South Dakota, Texas, Utah, and Virginia) currently offer professional designations for EMS, the number of states providing certification and endorsement for EMS is increasing: Two states are in final stages (Arkansas and Pennsylvania), and 10 states are in process (Alabama, Alaska, Colorado, Illinois, Indiana, New Hampshire, New York, Washington, West Virginia, and Wisconsin) (Rigelman, McGatha, & Goodman, 2012).

Some issues need to be resolved in order to strengthen the expertise of EMS. A close look at the Standards for Elementary Mathematics Specialists (AMTE, 2010) reveals that the

standards address content and pedagogy well but do not adequately address how children learn mathematics. The curricula of graduate/professional development programs for EMS also do not deal adequately with children's mathematical understanding. For example, among six courses offered by University of Nebraska's Primarily Math Specialist Certificate Program, which was the first program to target K-3 teachers, only one part of one course deals with children's mathematical cognition. If this weakness in the standards and curriculum reflecting the standards is remedied, EMS will be able to have a broad knowledge of mathematics content and pedagogy as well as knowledge of children's mathematical understanding.

Learning Standards for Mathematics

One of my research questions was to examine how the views of the four groups reflect current learning standards (i.e., NCTM's Standards, State Learning Standards, and Common Core State Standards). I intended to focus on the Common Core State Standards because they are the latest guideline for teaching math to young children. I decided not to address the standards at length in this report because teachers had only a vague sense of the Common Core State Standards, and only one participant (a mathematician from the original group of eight) from the other three groups mentioned the standards.

The math content the four groups said should be taught to young children overlaps a great deal with NCTM's Standards, the Illinois Learning Standards, and the Common Core State Standards. The teaching strategies the four groups discussed for the most part reflect NCTM's process standards. The skills listed in the Illinois Learning Standards, which incorporated many aspects of NCTM's standards, also resonated with the four groups' views on pedagogy. The fact the four groups' views on how to teach math to young children appear not to reflect the Common Core State Standards as much needs attention. Specifically, the four groups' views on how to

teach math to young children addressed only four among the eight standards for mathematical practice.

I suggest some reasons. First, implementation of the Common Core State Standards was in a transition period during this research. Although most states that adopted the Common Core State Standards require school districts to implement the standards, the majority of these states have not yet required districts to make complementary changes in curriculum and teacher education programs (Center on Education Policy, 2011). This may explain why teachers in this study were not familiar with the Common Core State Standards—what the teachers in this study do in the classroom has changed little since the introduction of Common Core State Standards. As I mentioned in Chapter 7, the teachers assumed that the Common Core was an update of the Illinois Learning Standards, not something new. For now, the teachers are not sure what the Common Core State Standards mean for their daily practice. It may be the case only for these teachers, but it is also possible that many early childhood teachers are perplexed about the new standards. Better promotion of the Common Core State Standards for teachers is needed.

Another possible reason is problems with the Common Core State Standards itself. Flaws in the Common Core State Standards have been pointed out, and researchers have started to question its effectiveness (Mathis, 2010; Usiskin, 2011). The standards were developed in collaboration with teachers, school administrators, and experts, but the level of input from school-based practitioners was minimal. Of the 65 people in “Achieve” work groups, who took charge in developing the standards, there was only one K-12 educator (Mathis, 2010, p. 5). This study indicates that some of the teaching strategies proposed by the groups other than teachers are somewhat idealistic. Given the contemporary realities of schooling, implementing these

strategies will be challenging. Clearly, the involvement of teachers is needed in deciding whether ideas proposed by other groups are feasible.

The standards were completed quickly, in approximately one year, in response to President Obama's educational reform agenda—making all high school graduates college- and career-ready. The short time allotted for the development process raised several concerns about the development, content, and use of the 500 pages of standards and supporting documents (Mathis, 2010). The National Council of Teachers of Mathematics (NCTM) and the National Council of Teachers of English (NCTE) expressed concern that the standards were inadequate and fell short of the mark (Mathis, 2010; Usiskin, 2011).

The content standards of the Common Core State Standards introduce mathematical content areas in an order different from NCTM's Standards and from the order recommended by the mathematicians in this study. Unlike the Illinois Learning Standards, which provide clear descriptors for each standard for teachers, the Common Core State Standards do not.

Recently, Linda Gojak (2013), president of NCTM, argued that, in spite of some flaws in the standards, the Common Core State Standards have the potential to make a difference for students. As mentioned earlier, implementing the Common Core State Standards is still in a transition period. New curriculum and assessment system based on the standards are in the process of development. At this point, it is not clear how the standards are going to evolve and contribute to improving math education for young children. Addressing flaws in the Common Core State Standards can spark discussions of how to improve the standards.

Teaching Mathematics to Young Children Well

My personal experience as a teacher got me interested in how best to teach math to young children. As a kindergarten teacher, I had difficulty teaching math. Although I took three years of

math in high school, I took none in college (it was not required in my program), and I did not like math. I never felt comfortable teaching math. I did take one math-method course in my teacher education program (an elective). I wished I had not; it made the situation worse. It made me feel that I did not have sufficient knowledge to teach math.

I assumed that I was not the only teacher who had a problem teaching math to young children. I initiated this study intending to help teachers like me by identifying what to teach and how to teach it in order to promote children's mathematical thinking and learning.

What I found from mathematicians, teacher educators, psychologists, and teachers provides some clues for effective ways to teach math. Based on those findings, I make some rather general suggestions for math instruction in preschool and kindergarten. The first two suggestions are stronger than those that follow, which remain tentative suggestions at best and are best viewed as questions that will guide further research. As I continue my research, I hope to make these suggestions stronger and more specific.

First, all four groups agreed that the teaching of mathematics to young children should begin with values and attitude. Teachers have to value and have a strong attitude toward math and its critical place in children's learning and development. The four groups proposed many ideas to help children develop a positive attitude toward math.

Second, mathematicians and teacher educators strongly agreed that, for young children, mathematics education should begin with number sense. Young children need to begin their math journey with a strong sense of what numbers are and their role both in mathematics and everyday life.

Third, narrow the curriculum and focus on understanding math as a system of logic—doing less, but going more deeply. There was substantial agreement between the four groups on

the content areas that should be taught to young children, and some agreement on sequence. The content areas, in sequence, include (a) number sense (counting and ordering), (b) the four basic operations, (c) algebraic thinking, (d) geometry, (e) measurement, and (f) data analysis. In the above list, I separate number sense and operations (from earlier lists). I recommend that math teaching in prekindergarten and kindergarten focus more on (a) through (d), which include the basic and core concepts in math. Mathematicians in this study generally agreed that measurement is not that important, that is, not key to understanding mathematics as a system of logic. They also agreed that data analysis should be left until later when children's math knowledge has grown more sophisticated. Many activities in early schooling, however, involve measurement (e) and data analysis (f). Whether these activities are what should be going on in early schooling needs further study and discussion. The focus in the early grades should be on understanding.

Fourth, emphasize collaborative learning. Have children work together to invent many ways to solve problems. In order to encourage collaborative math learning among children, teachers need to think about strategies for grouping children. Clarifying efficient ways to group children can maximize the effectiveness of collaborative math learning and needs further research.

Fifth, continually connect the different content areas and see the big picture in math. Unfortunately, little research on how content areas interconnect and build on one another to produce a coherent whole has been done. This is a promising area for further research.

Sixth, begin with children's innate mathematical ability, which is perceptual, to estimate, to quantify, and to recognize shapes. Focus on what children can see and compare. There is substantial work on the development of mathematical thinking among young children from a cognitive developmental perspective, but this knowledge is rarely connected to pedagogical

strategies. More work needs to be done to connect how children construct mathematical knowledge to the development of teaching strategies.

Seventh, continually assess children's mathematical understanding, for example, by using hypothetical learning trajectories (HLT). Good teaching involves moving children, and to do so, one must know where they are. More research is needed on the development of accurate and useful assessment tools.

Eighth, collaborate with other teachers or EMS. Find people with a deep and practical knowledge of the content and pedagogy and an understanding of children's mathematical learning, and learn from them.

Finally, see the goal of math education as the development of mathematically competent people who are able to reason mathematically, who can communicate their ideas and communicate mathematically, who can use multiple representations to understand things in different ways, and who can make connections between different topics.

Limitations and Future Research Directions

The goal of this study is to improve math teaching for young children by synthesizing the views of mathematicians, teacher educators, psychologists, and teachers. Given that implementing the Common Core State Standards is not completed yet (a new assessment system based on the common core standards will be in place for the 2013-2014 school year) and that some states have started to develop standards for pre-kindergarteners based on the Common Core State Standards (since the Common Core State Standards don't include pre-kindergarteners in their target grade level/band), deliberate discussion involving specialists to improve the standards is required. Involving specialists in deliberate meaning-making can lead the field to find effective methods for math education for young children.

Future research should have more representative numbers of participants. The small number of participants makes it difficult to generalize the findings. In addition, I interviewed only four groups considered as specialists in each component of mathematical PCK. It would be helpful to interview other stakeholders having previously unheard and overlooked voices. Developing a richer understanding of what makes good math education for young children remains a work in progress.

If I have learned one thing in this research, it is this: These four groups should be talking to each other. I see getting this dialog going and maintaining it as central to my future research.

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Appendix A

Summary of the Study for Directors/Principals

UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

February, 2012

Dear Director/Principal,

My name is Youn Joo Jang. I am a doctoral student in Early Childhood Education at the University of Illinois. I am inviting you to participate in research on teaching math to young children.

In this study, I will interview people from four different groups, asking them about their beliefs about teaching math to young children. These groups include (a) mathematicians, (b) teacher educators, (c) cognitive/developmental psychologists who study young children's math learning, and (d) preschool and kindergarten teachers.

I am interested in (a) what these groups believe, (b) the differences and similarities within and between groups, and (c) how these beliefs relate to the current discourse on teaching math to young children.

The importance of this study is this: Although the National Council Teachers of Mathematics (NCTM) has provided standards to help guide the math curriculum, states are providing inconsistent math programs resulting in a math curriculum that is "a mile wide and an inch deep" (Schmidt, McKnight, & Raizen, 1997, p. 2, as cited in NCTM, 2006, p.3).

Of the four groups listed above, teacher educators and cognitive/developmental psychologists have dominated discussions of math education for young children. Not enough is known about what early childhood teachers believe, and the views of mathematicians have not been explored.

Understanding the beliefs of these four groups will provide a foundation for improving math teaching in early schooling.

If you have any questions, please contact me at 1-217-369-0706, yjang9@illinois.edu. or you may contact Dr. Daniel Walsh at 1-217-244-1218, danielw@illinois.edu. If you have any questions about your right as a participant in this study, please contact the University of Illinois Bureau of Educational Research at 1-217-333-3023, info@education.illinois.edu.

(Letter continued)

Sincerely,

Youn Joo Jang

Appendix B

Flyers/ Consent Letters

B1. Flyers/ Consent Letters for Mathematicians

UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

February, 2012

Dear Mathematician,

My name is Youn Joo Jang. I am a doctoral student in Early Childhood Education at the University of Illinois. I am inviting you to participate in research on teaching math to young children.

I am examining the beliefs of mathematicians about teaching math to young children. To understand these beliefs, I am interviewing mathematicians. The interview will take about 30 minutes at a location and a time convenient for you. If you are interested, I may ask you to participate in two further interviews, each about 30 minutes. With your permission, interviews will be audio-taped.

For the first interview, I will provide light refreshments. If you participate in the follow-up interviews, you will receive a small gift certificate worthy of \$20.

This study involves no risks beyond those that exist in daily life. The results of this study will be used primarily for my doctoral dissertation. They may be shared with others through conference presentation and journal articles with your permission. Once transcribed, the audio-tapes will be erased. For any publications or presentations, your identity will be protected by the use of pseudonyms and the removal of other identifying information.

Your participation is completely voluntary. You are free to withdraw from the study at any time, for any reason, without penalty. You are also free to refuse to answer any questions you do not wish to answer. A copy of the research report will be made available to you once the study is completed.

If you are willing to participate, please complete the form below. If you have any questions, please contact me by phone, at 217-369-0706 or by e-mail, at yjang9@illinois.edu, or Dr. Daniel Walsh at 217-244-1218, danielw@illinois.edu. If you have any questions about your right as a participant in this study, please contact the University of Illinois Bureau of Educational Research Office, 217-244-0538, or via email at info@education.illinois.edu.

(Consent letter continued)

Sincerely

Youn Joo Jang

() I **do** agree to participate in this interview.

→For **audiotaping**: () I **do** give permission to **audiotape** this interview.

() I **do not** give permission to **audiotape** this interview.

Name

Signature

Date

B2. Flyers/ Consent Letters for Teacher Educators

UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

February, 2012

Dear Teacher Educator,

My name is Youn Joo Jang. I am a doctoral student in Early Childhood Education at the University of Illinois. I am inviting you to participate in research on teaching math to young children.

I am examining the beliefs of teacher educators about teaching math to young children. To understand these beliefs, I am interviewing teacher educators. The interview will take about 30 minutes at a location and a time convenient for you. If you are interested, I may ask you to participate in two further interviews, each about 30 minutes. With your permission, interviews will be audio-taped.

For the first interview, I will provide light refreshments. If you participate in the follow-up interviews, you will receive a small gift certificate worthy of \$20.

This study involves no risks beyond those that exist in daily life. The results of this study will be used primarily for my doctoral dissertation. They may be shared with others through conference presentation and journal articles with your permission. Once transcribed, the audio-tapes will be erased. For any publications or presentations, your identity will be protected by the use of pseudonyms and the removal of other identifying information.

Your participation is completely voluntary. You are free to withdraw from the study at any time, for any reason, without penalty. You are also free to refuse to answer any questions you do not wish to answer. A copy of the research report will be made available to you once the study is completed.

If you are willing to participate, please complete the form below. If you have any questions, please contact me by phone, at 217-369-0706 or by e-mail, at yjang9@illinois.edu, or Dr. Daniel Walsh at 217-244-1218, danielw@illinois.edu. If you have any questions about your right as a participant in this study, please contact the University of Illinois Bureau of Educational Research Office, 217-244-0538, or via email at info@education.illinois.edu.

Sincerely

Youn Joo Jang

(Consent letter continued)

() I **do** agree to participate in this interview.

→For **audiotaping**: () I **do** give permission to **audiotape** this interview.

() I **do not** give permission to **audiotape** this interview.

Name

Signature

Date

B3. Flyers/ Consent Letters for Psychologists

UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

February, 2012

Dear Psychologist,

My name is Youn Joo Jang. I am a doctoral student in Early Childhood Education at the University of Illinois. I am inviting you to participate in research on teaching math to young children.

I am examining the beliefs of psychologists, who study young childrens' math learning, about teaching math to young children. To understand these beliefs, I am interviewing psychologists. The interview will take about 30 minutes at a location and a time convenient for you. If you are interested, I may ask you to participate in two further interviews, each about 30 minutes. With your permission, interviews will be audio-taped.

For the first interview, I will provide light refreshments. If you participate in the follow-up interviews, you will receive a small gift certificate worthy of \$20.

This study involves no risks beyond those that exist in daily life. The results of this study will be used primarily for my doctoral dissertation. They may be shared with others through conference presentation and journal articles with your permission. Once transcribed, the audio-tapes will be erased. For any publications or presentations, your identity will be protected by the use of pseudonyms and the removal of other identifying information.

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If you are willing to participate, please complete the form below. If you have any questions, please contact me by phone, at 217-369-0706 or by e-mail, at yjang9@illinois.edu, or Dr. Daniel Walsh at 217-244-1218, danielw@illinois.edu. If you have any questions about your right as a participant in this study, please contact the University of Illinois Bureau of Educational Research Office, 217-244-0538, or via email at info@education.illinois.edu.

Sincerely

Youn Joo Jang

(Consent letter continued)

() I **do** agree to participate in this interview.

→For **audiotaping**: () I **do** give permission to **audiotape** this interview.

() I **do not** give permission to **audiotape** this interview.

Name

Signature

Date

B4. Flyers/ Consent Letters for Teachers

UNIVERSITY OF ILLINOIS
AT URBANA-CHAMPAIGN

February, 2012

Dear Teacher,

My name is Youn Joo Jang. I am a doctoral student in Early Childhood Education at the University of Illinois. I am inviting you to participate in research on teaching math to young children.

I am examining the beliefs of early childhood teachers about teaching math to young children. To understand these beliefs, I am interviewing teachers. The interview will take about 30 minutes at a location and a time convenient for you. If you are interested, I may ask you to participate in two further interviews, each about 30 minutes. With your permission, interviews will be audio-taped.

For the first interview, I will provide light refreshments. If you participate in the follow-up interviews, you will receive a small gift certificate worthy of \$20.

This study involves no risks beyond those that exist in daily life. The results of this study will be used primarily for my doctoral dissertation. They may be shared with others through conference presentation and journal articles with your permission. Once transcribed, the audio-tapes will be erased. For any publications or presentations, your identity will be protected by the use of pseudonyms and the removal of other identifying information.

Your participation is completely voluntary. You are free to withdraw from the study at any time, for any reason, without penalty. You are also free to refuse to answer any questions you do not wish to answer. A copy of the research report will be made available to you once the study is completed.

If you are willing to participate, please complete the form below. If you have any questions, please contact me by phone, at 217-369-0706 or by e-mail, at yjang9@illinois.edu, or Dr. Daniel Walsh at 217-244-1218, danielw@illinois.edu. If you have any questions about your right as a participant in this study, please contact the University of Illinois Bureau of Educational Research Office, 217-244-0538, or via email at info@education.illinois.edu.

Sincerely

Youn Joo Jang

(Consent letter continued)

() I **do** agree to participate in this interview.

→For **audiotaping**: () I **do** give permission to **audiotape** this interview.

() I **do not** give permission to **audiotape** this interview.

Name

Signature

Date

Appendix C

Interview Protocols

C1. Interview Questions for Pilot Study

First interview

1. I am really curious about how you got interested in math education for young children. Can you tell me what led you in this field?
2. Can you talk about what you think about young children's capability of learning math?
3. Can you talk about the purpose of math education?
4. What constitute the correct way of doing math? (e.g.: learning specific procedures for solving math problems, thinking process in doing math)
5. Some educators believe that there are three things to teach children math; 1) mathematical content knowledge (what to teach), 2) pedagogical knowledge (how to teach) and 3) understanding of children's cognition (whom to teach). Can you talk about these three?
6. National Council of Teachers of Mathematics (NCTM) produced five content standards for math education for young children; 1) number sense and operations, 2) algebra, 3) geometry, 4) measurements, and 5) data analysis and probability. From your perspective, do these make sense? Can you think of other things needed to teach young children?
7. What do you think that children are able to demonstrate and apply a knowledge and sense of numbers, including numeration and operations (addition, subtraction, multiplication, division), patterns, ratios, and proportion?
8. What do you think that children are capable of using algebraic and analytical methods to identify and describe patterns and relationships in data, to solve problems, and to predict results?
9. What do you think that children have ability to make use of geometric methods to analyze, categorize and draw conclusions about points, lines, planes, and space?
10. What do you think that children are able to estimate, make and use measurements of objects, quantities and relationships, and determine acceptable levels of accuracy?
11. What do you think that children can collect, organize, and analyze data using statistical methods; predict results; and interpret uncertainty using concepts of probability?

12. National Council of Teachers of Mathematics (NCTM) also produced five strands of the process for mathematics education for young children; 1) problem solving, 2) connections, 3) reasoning, 4) representation, and 5) communication. What do you think about these?

13. Suppose that you have new neighbors or relatives who have young children and ask you how to support math development for their children? What is your suggestion?

Second and third interviews

Questions for second and third interviews will depend on what each participant responds to the previous questions because the questions in the second and third interviews will be basically the same as the first interview but more detailed. With the participants' previous answers that the I would analyze, the participants would be led to discussing more about what they said before. This research will use qualitative method so that the orders of the questions can be different and more detailed sub-questions related to big questions can be added or taken out according to each participant's response. However, the big streams of questions written above will not be changed.

C2. Interview Questions for Mathematicians

First Interview

General Information Questions

1. Can you describe your position here at ***? Are you full-time?
2. Do you teach undergrads, grads or both?
3. Can you tell me how long you have been teaching at ***? Have you taught elsewhere?
5. Can you describe your educational and professional background?
6. Do you belong to any professional organizations (e.g., NCTM)?
7. Could you give me some idea of your age; 50 & Up, 40 to 49, 30 to 39, 20 to 29?
8. How would you describe your ethnicity?

Interview Questions (what is math?)

1. (attitude toward math) First of all, I am really curious about how you got interested in the field of mathematics, because many people I know don't like math. Maybe they had bad experiences with math in school. They often say "I hate math, and I am not a math person etc." So I am really interested in what it is about math that appeals to people who like math and do well in it. So, can you talk a little about what aspects of mathematics most appeal to you and why?

Probe: That's really interesting (I didn't know that). Can you tell me who or what has had the greatest influence on the development of your career interests?

Probe: Do you remember any special moments when you were young, in school or out of school, which led you to want to be a mathematician?

2. (teacher knowledge, skills, attitudes) When I was in high school, I had a math teacher who—all he did was make us memorize formulas. He never gave examples to help us understand what we were doing. He didn't explain why the formulas worked. He may have known a lot of math, but he wasn't good at teaching it, at least to me. My sense is that math knowledge is necessary but it is not enough to make someone a good math teacher. What I am trying to understand are the qualities that someone needs to teach math well. So, given your experience in the field of mathematics, can you talk about some of the qualities that a good math teacher needs?

Probe: In the teacher education program at UIUC, people who want to get certified to teach in the primary grades are required to take one math content course in math department (103, theory of arithmetic) and one math methods course in the college of education. One thing I am interested in is exploring the differences between math content courses and methods courses. Do you have any thoughts on these kinds of courses? For example, do you think taking 2 courses is enough to make pre-service teachers well prepared to teach math? Should there more math content or more methods courses?

3. (how and what to teach) When I was teaching math in kindergarten in Korea, I thought I should focus on how children solved problems rather than on getting the correct answer. I wanted to know what they were thinking. But I was never sure that I was doing the right thing. I just didn't have much confidence about teaching math—I was never sure what to teach or how to teach it. How teachers teach math clearly affects how children view what they are doing when they are solving problems. I'm really not sure how to ask this question, but what I am interested in is how best to teach math. Can you talk a little about what teachers need to do when they are

teaching pre-kindergarteners or kindergarteners math? What should their focus be? How can teachers become confident about what they are doing?

Probe: Along with the previous questions, some educators believe that teachers need three things to teach children math effectively; 1) mathematical content knowledge (what to teach), 2) pedagogical knowledge (how to teach) and 3) understanding how children think mathematically. Can you talk about these three and relationships among them?

4. (children's capability for doing math) A big question is what children in prekindergarten and kindergarten are capable of doing in math. Some people think that all children are ready to learn math while others think kids that age aren't ready and we should wait until they are older. When I was teaching kindergarten, I saw children doing some amazing things. For example, when I checked the attendance every morning at circle time, children quickly noticed how many children came to school and how many didn't even though I had more than 30 students in my class. Of course, children's mathematical understanding varies, but I got the impression that they were ready to learn math. Could you talk about children's math ability, about what they are able to do and what they are not able to do? What should we expect from prekindergarten and kindergarten kids?

Probe: One of the goals of schooling has to be the development of mathematically competent people. Can you describe what a mathematically competent young child would be like, and what she needs to know in order to eventually develop into a mathematically competent adult?

Probe: Some people think that children can learn mathematical concepts only when concrete objects are provided. Other people think that children have the ability to understand abstract mathematical ideas. What do you think?

Second and third interviews

Questions for second and third interviews will depend on what each participant responds to the previous questions because the questions in the second and third interviews will be basically the same as the first interview but more detailed. With the participants' previous answers that the I would analyze, the participants would be led to discussing more about what they said before. This research will use qualitative method so that the orders of the questions can be different and more detailed sub-questions related to big questions can be added or taken out according to each participant's response. However, the big streams of questions written above will not be changed.

C3. Interview Questions for Teacher Educators

First Interview

General Information Questions

1. Can you describe your position here at ***? Are you full-time?
2. Do you teach undergrads, grads, or both?
3. Can you tell me how long you have been teaching at ***? Have you taught elsewhere?
4. Can you describe your educational and professional background?
5. Do you belong to any professional organizations (e.g., NCTM)?
6. Could you give me some idea of your age; 50 & Up, 40 to 49, 30 to 39, 20 to 29?
7. How would you describe your ethnicity?

Interview Questions (studying math V.S. teaching math)

1. (teacher preparation) When I recall the time that I was a kindergarten teacher, the struggles that I had when teaching math to children always come to mind. Even though I understood the importance of math education, I had trouble providing good math activities for the children. I wasn't confident about what to do to promote their mathematical thinking. When I was in a teacher education program, I took only one course related to teaching math. I felt really unprepared. So, one of the things that I am really interested in is how teacher education programs best prepare pre-service teachers to teach math. So if you would, could you talk a little about your program? What courses do you teach? What do you think is most important to cover in those courses? Are students in your program getting enough preparation?

2. (teacher knowledge, skills, attitudes) When I was in high school, I had a math teacher who—all he did was make us memorize formulas. He never gave examples to help us understand what we were doing. He didn't explain why the formulas worked. He may have known a lot of math, but he wasn't good at teaching it, at least to me. My sense is that math knowledge is necessary but it is not enough to make someone a good math teacher. What I am trying to understand are the qualities that someone needs to teach math well. So, given your experience in teacher education, can you talk about some of the qualities that a good math teacher needs?

3. (children's capability for doing math) A big question is what children in prekindergarten and kindergarten are capable of doing in math. Some people think that all children are ready to learn math while others think kids that age aren't ready and we should wait until they are older. When I was teaching kindergarten, I saw children doing some amazing things. For example, when I checked the attendance every morning at circle time, children quickly noticed how many children came to school and how many didn't even though I had more than 30 students in my class. Of course, children's mathematical understanding varies, but I got the impression that they were ready for learning math. Could you talk about children's math ability, about what they are able to do and what they are not able to do? What should we expect from prekindergarten and kindergarten kids?

Probe: What can be done to match the math we teach to their abilities?

4. (how and what to teach) When I was teaching math in kindergarten in Korea, I thought I should focus on how children solved problems rather than on getting the correct answer. I wanted

to know what they were thinking. But I was never sure that I was doing the right thing. I just didn't have much confidence about teaching math—I was never sure what to teach or how to teach it. How teachers teach math clearly affects how children view what they are doing when they are solving problems. I'm really not sure how to ask this question, but what I am interested in is how best to teach math. Can you talk a little about what you expect your students to do when they are teaching kids math? What should their focus be? How can they become confident about what they are doing?

Second and third interviews

Questions for second and third interviews will depend on what each participant responds to the previous questions because the questions in the second and third interviews will be basically the same as the first interview but more detailed. With the participants' previous answers that the I would analyze, the participants would be led to discussing more about what they said before. This research will use qualitative method so that the orders of the questions can be different and more detailed sub-questions related to big questions can be added or taken out according to each participant's response. However, the big streams of questions written above will not be changed.

C4. Interview Questions for Psychologists

First Interview

General Information Questions

1. Can you describe your position here at ***?
2. Do you teach courses related to children's math learning?
3. How long you have been teaching at ***? Have you taught elsewhere?
5. Can you describe your educational and professional background?
6. Do you belong to any professional organizations (e.g., AERA, NCTM)?
7. Could you give me some idea of your age; 50 & Up, 40 to 49, 30 to 39, 20 to 29?
8. How would you describe your ethnicity?

Interview Questions (children's mathematical competence – learning trajectory)

1. (children's capability for doing math) A big question is what children in prekindergarten and kindergarten are capable of doing in math. Some people think that all children are ready to learn math while others think kids aren't ready and we should wait until they are older. When I was teaching kindergarten, I saw children doing some amazing things. For example, when I checked the attendance every morning at circle time, children quickly noticed how many children came to school and how many didn't even though I had more than 30 students in my class. Of course, children's mathematical understanding varies, but I got the impression that they were ready, but I wasn't sure what for. I am really interested in what kids are able to do in math. Could you talk about children's math ability; what they are able to do and what they are not able to do? What should we expect from prekindergarten and kindergarten kids?

Probe: How do children show signs of readiness to learn math? What are some signs a teacher might see?

Probe: When I was teaching, some children seemed to learn math more easily than others. Can you talk a little bit about why this might be and what a teacher should do, particularly with kids who struggle?

Probe: One of the goals of schooling has to be the development of mathematically competent people. Can you describe what a mathematically competent young child would be like, and what she needs to know in order to eventually develop into a mathematically competent adult?

2. (how and what to teach based on children's math learning trajectories) When I recall the time that I was a kindergarten teacher, the struggles that I had when teaching math to children always come to mind. Even though I understood the importance of math education, I had trouble providing good math activities for the children. I didn't know what to do or where to go to promote their mathematical thinking. When I was in a teacher education program, I took only one course related to teaching math. I felt really unprepared. I wasn't sure what the sequence of math knowledge and skills should be. So, one of the things that I am really interested in is a general sense of not only what to teach, but how to sequence math teaching. Can you talk a little about this, not only the content, but how the content should be presented? Does the order matter? Are different sequences possible? I know this is a really general question, about scope and sequence, but your answers will give me good insights into this topic.

3. (correct way of doing math) When I was teaching math in kindergarten in Korea, I thought I should focus on how children solved problems rather than on getting the correct answer. I wanted to know what they were thinking. But I was never sure that I was doing the right thing. I just didn't have much confidence about teaching math—I was never sure what to teach or how to teach it. Can you talk a little about what you think teachers should do when they are teaching kids math? What should their focus be?

Probe: Some people think that the purpose of doing math is getting correct answers while others think that it is more important to learn specific procedures for solving math problems. Could you talk about what the definition of doing math is and what constitutes the correct way of doing math?

4. Suppose that you have new neighbors or relatives who have young children and ask you how to support math development for their children. What is your suggestion? Of course, mathematical development varies from child to child. How can parents best prepare their children to learn math in schools taking into account this variability? What resources or teaching materials would you suggest? (e.g., books, educational TV shows, and board games.)

Second and third interviews

Questions for second and third interviews will depend on what each participant responds to the previous questions because the questions in the second and third interviews will be basically the same as the first interview but more detailed. With the participants' previous answers that the I would analyze, the participants would be led to discussing more about what they said before.

This research will use qualitative method so that the orders of the questions can be different and more detailed sub-questions related to big questions can be added or taken out according to each participant's response. However, the big streams of questions written above will not be changed.

C5. Interview Questions for Teachers

First interview

General Information Questions

1. Can you describe your position here at ***? Are you full-time?
2. How old are the children you teach?
3. Can you describe the students you teach (e.g., social class, ethnicity, etc.)?
4. Can you tell me how long you have been teaching in this school? Have you taught elsewhere?
5. Can you describe your educational background?
6. Do you belong to any professional organizations (e.g., AERA, NAEYC, NCTM)?
7. Could you give me some idea of your age; 50 & Up, 40 to 49, 30 to 39, 20 to 29?
8. How would you describe your ethnicity?

Interview Questions (struggles that teachers have when teaching math)

1. (challenges teachers face) When I recall the time that I was a kindergarten teacher, the struggles that I had when teaching math to children always comes to mind. Even though I understood the importance of math education, I had trouble providing good math activities to the children. I didn't know what to do or where to go to promote their mathematical thinking. When I was in a teacher education program, I took only one course related to teaching math. I felt really unprepared. So, one of the things that I am really interested in is the challenges that teachers face when they teach math to young children. So if you would, could you talk a little about the challenges, or the kinds of challenges, that kindergarten teachers face when they teach math to young children?

Probe: When I was teaching, I found algebraic concepts really difficult to teach even though the concepts were included in kindergarten learning standards. It was hard because I often failed to control the difficulty level in my teaching. I tried to teach algebraic concepts in circle time so that every child had an equal opportunity to be exposed to the concepts. But, the children's understanding of algebra varied a lot. My lesson often ended up being boring or too difficult. Can you talk about areas that you find difficult to teach?

2. (kinds of activities/ math content) When I'm supervising student teachers in preschool or primary grade classrooms, I often see teachers do many different kinds of math activities with children. My guess is that these activities vary a lot, and that different teachers do different activities with kids. So what I would really like you to talk about is the kinds of math activities that kindergarten teachers do in their classrooms. Could you give some examples of kinds of things you, or other kindergarten teachers, do with the kids?

Probe: When I am supervising, I am amazed by the many math activities going on in kindergarten classrooms. It seems that each activity is designed for teaching specific mathematical knowledge, such as numbers, shapes, graphing, measuring, adding... lots of things. But it is hard for me to get a good sense of the range of content covered in kindergarten. Can you help me with this—what is your sense of the range of mathematical knowledge covered in kindergarten?

3. (how much time teachers spend on teaching math?) I know that kindergarten teachers are under lots of pressure, for example, learning standards. Most of the standards place a strong

emphasis on early literacy, so one of the things I am wondering about is the amount of time for teaching math. When I was a kindergarten teacher, I couldn't devote much time to teaching math. There were so many other things I was required to do. We were required to prepare for things like parents day, sports day, and to have the walls covered with student projects. I always ran out of time. When I supervise, I am usually in the classroom only for an hour or so. If I were to stay all day in a typical day in a kindergarten, when would I see math activities and how much time would I see allotted to those activities?

4. (children's capability for doing math) A big question is what children in kindergarten are capable of doing in math. Some people think that all children are ready to learn math while others think kids aren't ready and we should wait until they are older. When I was teaching kindergarten, I saw children doing some amazing things. For example, when I checked the attendance every morning at circle time, children quickly noticed how many children came to school and how many didn't even though I had more than 30 students in my class. Of course, children's mathematical understanding varies, but I got the impression that they were ready, but I wasn't sure what for. I am really interested in what kindergarten teachers think kids are able to do in math. Could you talk about children's math ability, about what they are able to do and what they are not able to do? What should we expect from a kindergarten child?

Second and third interviews

Questions for second and third interviews will depend on what each participant responds to the previous questions because the questions in the second and third interviews will be basically the same as the first interview but more detailed. With the participants' previous answers that the

I would analyze, the participants would be led to discussing more about what they said before.

This research will use qualitative method so that the orders of the questions can be different and more detailed sub-questions related to big questions can be added or taken out according to each participant's response. However, the big streams of questions written above will not be changed.