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ASYMPTOTIC FORMULAE FOR CERTAIN ARITHMETIC FUNCTIONS
PRODUCED BY FRACTIONAL LINEAR TRANSFORMATIONS

BY

PAUL SPIEGELHALTER

DISSERTATION

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Doctoral Committee:

Professor Bruce C Berndt, Chair
Professor Alexandru Zaharescu, Director of Research
Professor A J Hildebrand
Professor Florin Boca

Abstract

K.T. Atanassov introduced the two arithmetic functions

$$I(n) = \prod_{\nu=1}^k p_{\nu}^{1/\alpha_{\nu}} \quad \text{and} \quad R(n) = \prod_{\nu=1}^k p_{\nu}^{\alpha_{\nu}-1}$$

called the irrational factor and the strong restrictive factor, respectively. A variety of authors have studied the properties of these arithmetic functions. We consider weighted combinations $I(n)^{\alpha}R(n)^{\beta}$ and characterize pairs (α, β) in order to measure how close n is to being k -power full or k -power free.

We then generalize these functions to a class of arithmetic functions defined in terms of fractional linear transformations arising from certain 2×2 matrices, establish asymptotic formulae for averages of these functions, and explore certain maps that arise from considering the leading terms of these averages.

We further generalize to a larger class of maps by introducing real moments, which allow us to explore new properties of these arithmetic functions. We additionally study the influence of the eigenvalues of a matrix on the associated arithmetic function, and obtain results on the local density of eigenvalues through their connection to a particular surface.

Finally, we present a further generalization involving arithmetic functions defined by certain complex-valued fractional linear transformations, explore some of the properties of these new functions, and present a few open problems.

To Mom, Dad, and Michael.

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Notation

We write $f(x) = O(g(x))$, or $f(x) \ll g(x)$, if there is a constant C such that $|f(x)| \leq Cg(x)$ for relevant values of x . We write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$ as x tends to some limit, and $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$.

We use the notation $a|b$ to mean that a divides b , and $p^\nu || b$ to mean that $p^\nu | b$ but $p^{\nu+1} \nmid b$. Generally s denotes a complex number, with real part $\Re(s) = \sigma$ and imaginary part $\Im(s) = t$, so that $s = \sigma + it$. The Riemann zeta function $\zeta(s)$, the Möbius function $\mu(n)$, and the Euler phi function $\phi(n)$ are defined as normal.

Other symbols are defined on the following pages:

$\mathcal{A}(Q, x)$	p.42
$F_{\alpha, \beta}(s)$	p.10
$F_A(s)$	p.16
$F_{A, \lambda}(s)$	p.23
$\mathcal{F}_Q$	p.20
$\mathbb{G}$	p.24
$h(n)$	p.16
$I(n)$	p.1
$M_{\alpha, \beta}(x)$	p.10
$M_A(x)$	p.14
$M_{A, \lambda}(x)$	p.24
$\Psi_{Q, x}$	p.43
$R(n)$	p.1
$S_{A, \lambda}(x)$	p.24
$\theta(x, \lambda)$	p.27

Chapter 1

Introduction and history

In [3] and [4], Atanassov introduced the two arithmetic functions

$$I(n) = \prod_{\nu=1}^k p_{\nu}^{1/\alpha_{\nu}} \quad \text{and} \quad R(n) = \prod_{\nu=1}^k p_{\nu}^{\alpha_{\nu}-1},$$

where $n = \prod_{\nu=1}^k p_{\nu}^{\alpha_{\nu}}$ is the prime factorization of n , and called them the irrational factor and the strong restrictive factor, respectively. These functions are multiplicative, and they satisfy the inequality

$$I(n)R(n)^2 \geq n \tag{1.1}$$

with equality if and only if n is square-free. In [2], it was noted that $I(n)$ roughly measures how far away n is from being k -power free or k -power full; if $S(n)$ denotes the square-free part of n , and if n is k -power free, then $n^{1/(k-1)} \leq S(n) \leq I(n)^{k-1}$. If n is k -power full, then $I(n) \leq S(n)^{1/k}$.

In [16], L. Panaitopol showed that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} < e^2.$$

He further proved that the arithmetic function

$$G(n) = \prod_{\nu=1}^n I(\nu)^{1/\nu}$$

satisfies the inequalities

$$\frac{n}{e^7} < G(n) < n,$$

for each $n \geq 1$. E. Alkan, A.H. Ledoan, and A. Zaharescu [2] proved that the sequence $\{G(n)/n\}_{n \geq 1}$ is convergent, and established the asymptotic formula

$$G(n) = c_1 n + O(\sqrt{n})$$

for some positive absolute constant c_1 . They also showed that $I(n)$ is very regular on average, and proved that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) I(n) = c_2 x^2 + O\left(x^{3/2} e^{-c_3 (\log x)^{3/5} (\log \log x)^{-1/5}}\right)$$

for some positive absolute constants c_2 and c_3 , and that

$$\sum_{n \leq x} I(n) = 3c_2 x^2 + O\left(x^{3/2} (\log x)^{9/4}\right).$$

Improvements to the error term in the last asymptotic formula have recently been obtained by J.-M. De Koninck and I. Kátai [12]. In [15], the following result was proved: for $0 < \lambda < 1$ and $\epsilon > 0$, there is a positive constant c_λ depending only on λ such that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) R(n)^\lambda = c_\lambda x^{\frac{1+\lambda}{2}} + E(\lambda, x),$$

where

$$E(\lambda, x) = \begin{cases} O_{\lambda, \epsilon} \left(x^{\frac{1}{2} - \epsilon}\right) & \text{if } 0 < \lambda < 1/4, \\ O_{\lambda, \epsilon} \left(x^{\frac{1}{3} + \frac{2\lambda}{3} + \epsilon}\right) & \text{if } 1/4 \leq \lambda < 1. \end{cases}$$

Our goal is, first, to extend the work of previous authors by producing results on weighted combinations of $I(n)$ and $R(n)$ from which previous results can be recovered as special cases. We continue by recognizing that $I(n)$ and $R(n)$ are particular examples of arithmetic functions defined in terms of fractional linear transformations of integers by 2×2 matrices and prove several results in this more general framework. Finally, we further extend these functions by considering fractional linear transformations of complex numbers.

Chapter 2

Preliminary results

2.1 Perron's formula

The analytic behavior of a convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is closely related to the behavior of the sequence $f(n)$. In particular, one can recover information about averages of $f(n)$ using the following methods.

Consider the integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < 1, \\ 1/2 & \text{if } y = 1, \\ 1 & \text{if } y > 1, \end{cases} \quad (2.1)$$

where $c > 0$. One can isolate the coefficients of $F(s)$ by setting $y = x/n$ in the above formula, multiplying by $f(n)$, and summing over n . Formally, one obtains Perron's formula

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds, \quad (2.2)$$

for a non-integer $x > 0$ and some value of c within the abscissa of convergence of F . If x is an integer then the formula takes a similar form, with the exception that the $n = x$ term is $\frac{1}{2}f(n)$ rather than $f(n)$. For a full proof of Perron's formula, see Chapter 17 of [7].

From this, one can arrive at

$$\sum_{n \leq x} f(n)(x^k - n^k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{s+k} \left(\frac{1}{s} - \frac{1}{s+k} \right) ds$$

for a positive integer k (see [19]). We use a version with $k = 1$, namely

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) f(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^{s+1}}{s(s+1)} ds. \quad (2.3)$$

This is especially useful if the function $\frac{F(s)}{s(s+1)}$ has a pole, in which case one can apply Cauchy's residue theorem. We note that the left-hand side of (2.3) is equal to the “double average”

$$\frac{1}{x} \sum_{\substack{j \leq x \\ n \leq j}} f(n).$$

2.2 Behavior of $\zeta(\sigma + it)$ near the critical strip

The Riemann zeta function $\zeta(s)$, defined for $\Re(s) > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

can be extended to a meromorphic function with a simple pole of residue 1 at $s = 1$. The function has “trivial” zeros at $s = -2, -4, -6, \dots$, and famously has an infinite number of “non-trivial” zeros located in the critical strip $0 < \Re(s) < 1$. With $s = \sigma + it$, the number of such zeros with $0 \leq t \leq T$ (here we write $s = \sigma + it$) is asymptotic to $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$. B. Riemann's famous 1860 conjecture is that all of the non-trivial zeros of $\zeta(s)$ lie on the line $\sigma = 1/2$.

In 1899 C.-J. de la Vallée Poussin showed that for some positive constant c_0 the region

$$\sigma > 1 - \frac{c_0}{\log t}$$

for $t \geq 2$ is free of zeros of $\zeta(s)$. This can be used to give a bound of

$$\frac{1}{\zeta(s)} \ll \log t$$

in this region (see Chapter 13 of [7]).

J. Littlewood expanded the zero-free region in 1922 to

$$\sigma > 1 - \frac{c_0 \log \log t}{\log t}.$$

I.M. Vinogradov [22] and N.M. Korobov [13] further improved this result by establishing the larger zero-free

region

$$\sigma \geq 1 - c_0(\log t)^{-2/3}(\log \log t)^{-1/3} \quad (2.4)$$

for $t \geq t_0$ and improving the previous estimate to

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3}(\log \log t)^{1/3}$$

(see also Chapters 2 and 5 of [23]).

Further estimates for $\zeta(s)$ near the critical strip can be found in [21], §3.11 and §5.1. In particular,

$$|\zeta(\sigma + it)| = \begin{cases} O(t^{(1-\sigma)/2}), & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1 \\ O(\log t), & \text{if } 1 \leq \sigma \leq 2 \\ O(1), & \text{if } \sigma \geq 2. \end{cases} \quad (2.5)$$

Chapter 3

Weighted combinations

In order to more fully develop the ideas presented by previous authors, we begin by studying weighted combinations of $I(n)$ and $R(n)$.

Definition 3.0.1. *We say that a pair of real numbers (α, β) is a strong Atanassov pair if*

$$I(n)^\alpha R(n)^\beta \geq n \tag{3.1}$$

for every natural number n .

Thus by (1.1), the pair $(1, 2)$ is a strong Atanassov pair. Clearly, if $\alpha > 1$ and $\beta > 2$ then also (α, β) is a strong Atanassov pair.

Definition 3.0.2. *We say that a pair of real numbers (α, β) is a weak Atanassov pair if the inequality*

$$I(n)^\alpha R(n)^\beta \geq n$$

is true on average, in the sense that

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) (I(n)^\alpha R(n)^\beta - n) > 0$$

for all sufficiently large x .

Evidently, a strong Atanassov pair is also a weak Atanassov pair. Are there any weak Atanassov pairs that are not strong Atanassov pairs? We will see that the answer is yes. Our goal is to completely characterize the strong and weak Atanassov pairs. We show that the following theorem holds.

Theorem 3.0.1. *A pair (α, β) is a strong Atanassov pair if and only if it lies in the region*

$$R = \bigcap_{k=1}^{\infty} L_k$$

where

$$L_1 = \left\{ (\alpha, \beta) : \beta \geq 0, \alpha \geq 1 \right\}$$

and

$$L_k = \left\{ (\alpha, \beta) : \beta \geq -\frac{1}{k(k-1)}\alpha + \frac{k}{k-1} \right\}$$

for $k \geq 2$.

This is a convex region with a piecewise linear boundary. Figure 3.1 shows a plot of the boundary lines of each of the regions L_k for $k \leq 40$.

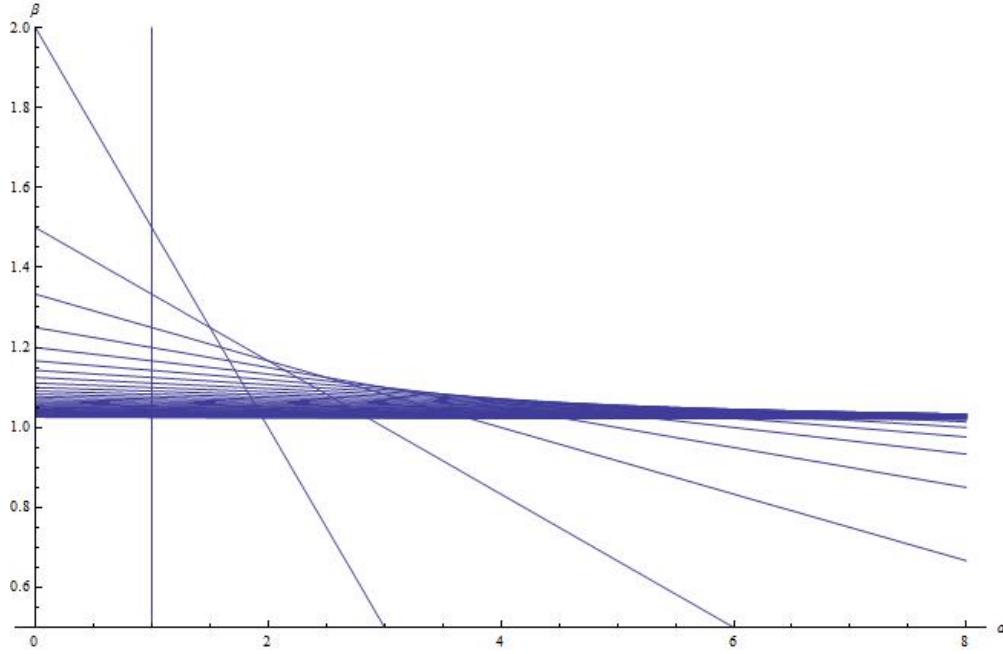


Figure 3.1: Plot of the lines ∂L_k , with $k \leq 40$

Theorem 3.0.1 gives us a more precise notion of $I(n)$ being a measure of how close an integer n is to being k -power full or k -power free; if n is “predominantly” a k -th power, then the pairs (α, β) for which $I(n)^\alpha R(n)^\beta$ is close to n will be those pairs that are close to the boundary line ∂L_k of the region L_k .

Figure 3.2 shows the region R along with the lines ∂L_2 , ∂L_3 , and ∂L_4 . The point $P = (5, 8/5)$ is closer to ∂L_3 and ∂L_4 than to ∂L_2 , so we expect that $|I(n)^5 R(n)^{8/5} - n|$ will be smaller for those n that are cubes or 4-th powers than for square n . On the other hand, P lies inside the region L_3 , but outside the region L_4 , so the inequality (3.1) holds for all n that are cubes, but not all n that are divisible by a 4-th power.

Moreover, we show the following theorem.

Theorem 3.0.2. *Suppose $\alpha, \beta > 0$ are such that (α, β) is not a strong Atanassov pair. Further suppose that*

$\beta < 3\alpha/2$. Then

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) I(n)^\alpha R(n)^\beta = \frac{\zeta(3\alpha/2 - \beta + 2)}{6\zeta(2)\zeta(3\alpha - 2\beta + 4)} K_{\alpha,\beta}(\alpha + 1) x^{\alpha+1} \\ + O_{\alpha,\beta} \left(x^{1/2+\alpha} \exp\{-c_2(\log x)^{3/5}(\log \log x)^{-1/5}\} \right)$$

where the strictly positive constant $K_{\alpha,\beta}(\alpha + 1)$ is defined in (3.2) in Section 3.

Remark 3.0.3. One can easily derive similar asymptotic formulas for $\beta \geq 3\alpha/2$; if $\beta > 3\alpha/2$ then the asymptotic formula will have two main terms.

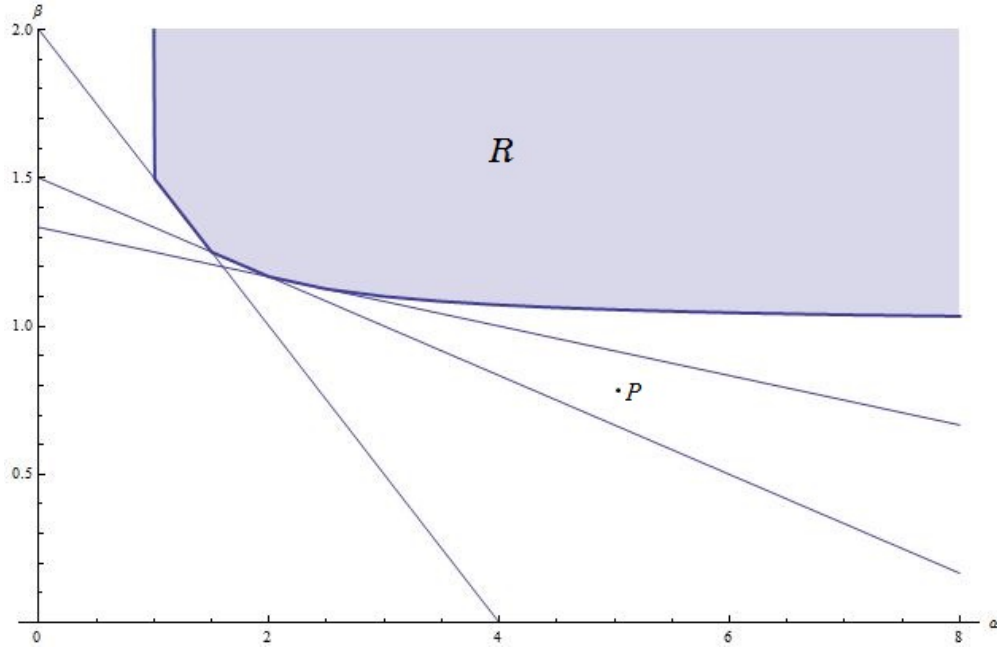


Figure 3.2: The region R

Since

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) n \sim \frac{x^2}{6},$$

we have the following corollary. Let β_0 be defined by the unique solution $\beta = \beta_0$ of the equation

$$\frac{\zeta(7/2 - \beta)}{\zeta(2)\zeta(7 - 2\beta)} K_{1,\beta}(2) = 1.$$

Corollary 3.0.4. A pair (α, β) that is not a strong Atanassov pair is a weak Atanassov pair if and only if

it lies in the region

$$R_w = \{(\alpha, \beta) : \alpha > 1, \beta > 0\} \cup \{(\alpha, \beta) : \alpha = 1, \beta > \beta_0\}$$

Numerical calculation shows that $\beta_0 = 1.341\dots$. Thus for example $(1, 7/5)$ is a weak Atanassov pair that is not a strong Atanassov pair. The point P in Figure 3.2 is in R_w , so $(5, 8/5)$ is also a weak Atanassov pair that is not a strong Atanassov pair.

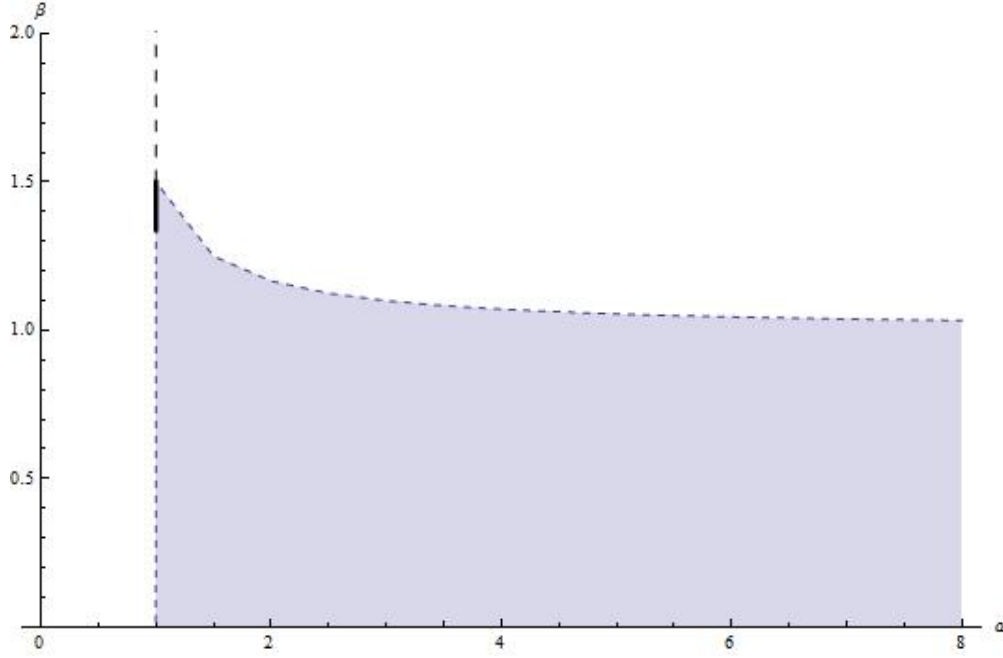


Figure 3.3: The region R_w

3.1 Distribution of strong Atanassov pairs

Since the function $I(n)^\alpha R(n)^\beta$ is multiplicative, we examine its behavior on integers of the form p^k . We have that for each fixed k , the desired inequality

$$I(p^k)^\alpha R(p^k)^\beta \geq p^k$$

is equivalent to

$$\alpha/k - \beta(k-1) \geq k.$$

Clearly the inequality holds for $n = p^k$ in the region

$$L_k = \left\{ \beta \geq -\frac{1}{k(k-1)}\alpha + \frac{k}{k-1} \right\}$$

and so holds for any k in the union of all such regions L_k .

Conversely, if any pair (α, β) lies outside of this region, then it lies outside of L_{k_0} for some k_0 , say. Hence there are integers of the form p^{k_0} that do not satisfy the desired inequality.

This completes the proof of Theorem 3.0.1, showing that the strong Atanassov pairs are precisely those pairs lying in the region R .

3.2 Distribution of weak Atanassov pairs

For the proof of Theorem 3.0.2 we work with the weighted sum

$$M_{\alpha,\beta}(x) = \sum_{n \leq x} \left(1 - \frac{n}{x}\right) I(n)^\alpha R(n)^\beta.$$

Since both the functions $I(n)$ and $R(n)$ are multiplicative, and since for each n we have $I(n) \leq n$ and $R(n) \leq n$, the associated Dirichlet series has an Euler product representation which converges for $\Re(s) > 1 + \alpha + \beta$.

This Euler product can be written as follows:

$$\begin{aligned} F_{\alpha,\beta}(s) &:= \sum_{n=1}^{\infty} \frac{I(n)^\alpha R(n)^\beta}{n^s} \\ &= \prod_p \left(1 + \frac{1}{p^{s-\alpha}} + \frac{1}{p^{2s-\alpha/2-\beta}} + \dots + \frac{1}{p^{ks-\alpha/k-\beta(k-1)}} + \dots \right) \\ &= \frac{\zeta(s-\alpha)\zeta(2s-\alpha/2-\beta)}{\zeta(2(s-\alpha))\zeta(2(2s-\alpha/2-\beta))} K_{\alpha,\beta}(s), \end{aligned}$$

where

$$K_{\alpha,\beta}(s) = \prod_p (1 + A_{\alpha,\beta,p}(s)) \tag{3.2}$$

and

$$A_{\alpha,\beta,p}(s) = \frac{1}{\left(1 + \frac{p^\alpha}{p^{s+1}}\right) \left(1 + \frac{p^{\alpha/2+\beta}}{p^{2(s+1)}}\right)} \left(-\frac{p^{3\alpha/2+\beta}}{p^{3(s+1)}} + \sum_{k \geq 3} \frac{p^{\alpha/k+\beta(k-1)}}{p^{k(s+1)}} \right).$$

Now, given $\epsilon > 0$, we have

$$\frac{1}{\left(1 + \frac{p^\alpha}{p^{s+1}}\right) \left(1 + \frac{p^{\alpha/2+\beta}}{p^{2(s+1)}}\right)} \ll_\epsilon 1$$

for $\Re(s) > \max\{\alpha - 1 + \epsilon, \beta + \alpha/2 - 1 + \epsilon\}$. So

$$\begin{aligned} |A_{\alpha,\beta,p}(s)| &\ll_{\epsilon} \left| -\frac{p^{3\alpha/2+\beta}}{p^{3(s+1)}} + \sum_{k \geq 3} \frac{p^{\alpha/k+\beta(k-1)}}{p^{k(s+1)}} \right| \\ &\ll_{\epsilon} \frac{p^{3\alpha/2+\beta}}{p^{3(\sigma+1)}} + \frac{p^{\alpha/3+2\beta}}{p^{3(\sigma+1)}} \frac{1}{1 - \frac{p^{\beta}}{p^{\sigma+1}}}. \end{aligned}$$

Provided that $\Re(s)$ is in the given range, the above expression is

$$\ll_{\epsilon} \frac{1}{p^{3(\sigma+1) - \max\{3\alpha/2+\beta, \alpha/3+2\beta\}}},$$

hence $\sum_p |A_{\alpha,\beta,p}(s)| < \infty$ in any half plane of the form

$$\sigma \geq \sigma_0 > \max\{\alpha/2 + \beta/3 - 2/3, \alpha/9 + 2\beta/3 - 2/3, \alpha - 1 + \epsilon, \beta + \alpha/2 - 1 + \epsilon\}.$$

It follows that the product $K_{\alpha,\beta}(s)$ is uniformly bounded on the half-plane $\Re(s) > \sigma_0$ (see §14.2, p. 15 of [20]). Since $K_{\alpha,\beta}(s)$ is analytic in this half-plane, the Dirichlet series $F_{\alpha,\beta}(s)$ has a meromorphic continuation to this region. Furthermore, since $\zeta(s - \alpha)$ has a simple pole at $s = 1 + \alpha$ and $\zeta(2s - \alpha/2 - \beta)$ a simple pole at $s = 1/2 + \alpha/4 + \beta/2$, and since the functions $\zeta(2s - 2\alpha)$ and $\zeta(4s - \alpha - 2\beta)$ have no zeros on the half planes $\Re(s) > 1/2 + \alpha$ and $\Re(s) > 1/4 + \alpha/4 + \beta/2$, respectively, it follows that $F_{\alpha,\beta}(s)$ has an analytic continuation to the half-plane $\Re(s) > \max\{1/2 + \alpha, 1/4 + \alpha/4 + \beta/2\}$ with the exception of a simple pole at $s = 1 + \alpha$ and (possibly) at $s = 1/2 + \alpha/4 + \beta/2$, with the existence of the latter pole depending on whether $\beta > 3\alpha/2$.

3.3 The main term

To continue, we utilize the variant of Perron's formula given by (2.3) and write

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) I(n)^{\alpha} R(n)^{\beta} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s \zeta(s - \alpha) \zeta(2s - \alpha/2 - \beta) K_{\alpha,\beta}(s)}{\zeta(2(s - \alpha)) \zeta(2(2s - \alpha/2 - \beta)) s(s+1)} ds,$$

where $1 + \alpha < c < 2 + \alpha$. For the sake of brevity we specialize to the case $\alpha = 1$; the methods by which we obtain the results for $\alpha \neq 1$ will be clear.

We apply the Vinogradov-Korobov zero-free region for $\zeta(s)$ given by (2.4) in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3} (\log \log t)^{1/3}.$$

Fix $0 < U < T < x^2$, let $\nu = 3/2$ and let

$$\eta = \nu - c_0(\log U)^{-2/3}(\log \log U)^{-1/3}.$$

Deform the path of integration into the union of the line segments

$$\left\{ \begin{array}{ll} \gamma_1, \gamma_9 : s = c + it & \text{if } |t| \geq T, \\ \gamma_2, \gamma_8 : s = \sigma \pm iT & \text{if } \nu \leq \sigma \leq c, \\ \gamma_3, \gamma_7 : s = \nu + it & \text{if } U \leq |t| \leq T, \\ \gamma_4, \gamma_6 : s = \sigma \pm iU & \text{if } \eta \leq \sigma \leq \nu, \\ \gamma_5 : s = \eta + it & \text{if } |t| \leq U. \end{array} \right.$$

The integrand is analytic on and within this modified contour. Hence by the residue theorem

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) I(n)^\alpha R(n)^\beta = \frac{\zeta(7/2 - \beta)}{6\zeta(2)\zeta(7 - 2\beta)} K_{1,\beta}(2)x^2 + \sum_{k=1}^9 J_k,$$

the main contribution being due to the residue of the simple pole at the point $s = 2$.

3.4 Upper bounds for the error integrals

In order to estimate the integral along our modified contour we will make use of the bounds given by (2.5).

On the line segments on which $s = c + it$, $|t| \geq T$, we have $\zeta(s - 1) = O(\log t)$, $\zeta(2s - 1/2 - \beta) = O(1)$, $1/\zeta(2s - 2) = O(\log t)$, and $1/\zeta(4s - 1 - \beta) = O(\log t)$, so

$$\begin{aligned} |J_1|, |J_9| &\ll \int_T^\infty |F_{\alpha,\beta}(c + it)| \frac{|x^{c+it}|}{|c + it||c + 1 + it|} dt \\ &\ll x^c \int_T^\infty \frac{\log^3 t}{t^2} dt \\ &\ll \frac{x^c}{T^{9/10}}. \end{aligned}$$

On the line segments on which $s = \sigma + iT$, $\nu \leq \sigma \leq c$, we have $\zeta(s - 1) = O(T^{1-\sigma/2} \log T)$, $\zeta(2s - 1/2 - \beta) =$

$O(\log T)$, $1/\zeta(2s-2) = O(\log T)$, and $1/\zeta(4s-1-\beta) = O(\log T)$, so

$$\begin{aligned} |J_2|, |J_8| &\ll \int_{\nu}^c |F_{\alpha,\beta}(\sigma + iT)| \frac{|x^{\sigma+iT}|}{|\sigma + iT||\sigma + 1 + iT|} d\sigma \\ &\ll \frac{\log^3 T}{T} \int_{\nu}^c \left(\frac{x}{T^{1/2}}\right)^{\sigma} d\sigma \\ &\ll \frac{\log^3 T}{T} \max \left\{ \left(\frac{x}{T^{1/2}}\right)^{\nu}, \left(\frac{x}{T^{1/2}}\right)^c \right\}. \end{aligned}$$

On the line segments on which $s = c + it$, $U \leq |t| \leq T$, we have $\zeta(s-1) = O(t^{1/8-\beta/4} \log t)$, $\zeta(2s-1/2-\beta) = O(\log t)$, $1/\zeta(2s-2) = O(\log t)$, and $1/\zeta(4s-1-\beta) = O(\log t)$, so

$$\begin{aligned} |J_3|, |J_7| &\ll \int_U^T |F_{\alpha,\beta}(\nu + it)| \frac{|x^{\nu+it}|}{|\nu + it||\nu + 1 + it|} dt \\ &\ll x^{\nu} \int_U^T \frac{t^{1/8} \log^4 t}{t^2} dt \\ &\ll \frac{x^{\nu}}{U^{3/4}}. \end{aligned}$$

On the line segments on which $s = \sigma + iU$, $\eta \leq \sigma \leq \nu$, we have $\zeta(s-1) = O(U^{1-\sigma/2} \log U)$, $\zeta(2s-1/2-\beta) = O(U^{3/4-\beta/2-\sigma} \log U)$, $1/\zeta(2s-2) = O(\log U)$, and $1/\zeta(4s-1-\beta) = O(\log U)$, so

$$\begin{aligned} |J_4|, |J_6| &\ll \int_{\eta}^{\nu} |F_{\alpha,\beta}(\sigma + iU)| \frac{|x^{\sigma+iU}|}{|\sigma + iU||\sigma + 1 + iU|} d\sigma \\ &\ll \frac{\log^4 U}{U^{1/4+\beta/2}} \int_{\eta}^{\nu} \left(\frac{x}{U^{3/2}}\right)^{\sigma} d\sigma \\ &\ll \frac{\log^4 U}{U^{1/4+\beta/2}} \max \left\{ \left(\frac{x}{U^{3/2}}\right)^{\eta}, \left(\frac{x}{U^{3/2}}\right)^{\nu} \right\}. \end{aligned}$$

On the line segments on which $s = \eta + it$, $|t| \leq U$, we have $\zeta(s-1) = O((|t|+1)^{1-\eta/2} \log(|t|+1))$ and $\zeta(2s-1/2-\beta) = O((|t|+1)^{5/8+\beta/4-\eta} \log(|t|+1))$. Since $\beta < 3/2$ this becomes $\zeta(2s-1/2-\beta) = O((|t|+1)^{1-\eta} \log(|t|+1))$. Also, $1/\zeta(2s-2) = O(\log(|t|+1))$, and $1/\zeta(4s-1-\beta) = O(\log(|t|+1))$, so

$$\begin{aligned} |J_5| &\ll \int_{-U}^U |F_{\alpha,\beta}(\eta + it)| \frac{|x^{\eta+it}|}{|\eta + it||\eta + 1 + it|} dt \\ &\ll x^{\eta} \int_{-U}^U (|t|+1)^{-3\eta/2} \log^4(|t|+1) dt. \end{aligned}$$

Since $\eta > 1$, say, for U sufficiently large, the above integral converges, hence $|J_5| \ll x^{\eta}$.

We collect all estimates, and take $T = x^2$ and $U = \exp\{c_2(\log x)^{3/5}(\log \log x)^{-1/5}\}$ to obtain the desired result. This completes the proof of Theorem 3.0.2.

Chapter 4

A first generalization

One can generalize the theory of the functions $I(n)$ and $R(n)$, and the previously established results, to a much wider context. In particular, one can establish functions $f_A(n)$, where A belongs to a particular class of 2×2 matrices. In this case $I(n)$ and $R(n)$ correspond to functions $F_A(n)$ for particular choices of A . In this chapter we begin our study of these generalized functions and explore some of their properties.

Consider for a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and determinant ± 1 the fractional linear transformation Az given by

$$Az = \frac{az + b}{cz + d}.$$

For each positive integer n , define

$$f_A(n) = \prod_{p^\alpha || n} p^{\frac{a\alpha + b}{c\alpha + d}}.$$

As an example, the function $I(n)$ is equal to $f_{A_0}(n)$ for

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We shall consider weighted averages of the functions $f_A(n)$. Let

$$M_A(x) = \frac{1}{x} \sum_{n \leq x} \left(1 - \frac{n}{x}\right) f_A(n).$$

Consider the subset $\mathcal{A}$ of 2×2 matrices with integer-valued entries given by

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = -1, a, b, d \geq 0, c \geq 1 \right\}.$$

Define for each positive rational number r

$$E_r = \{A \in \mathcal{A} : M_A(x) \asymp x^r \text{ as } x \rightarrow \infty\}.$$

Note that if $r_1 \neq r_2$, then $E_{r_1} \cap E_{r_2} = \emptyset$. We will prove that each E_r with $r > 0$ consists of exactly one element.

For each matrix A in $\mathcal{A}$ we define the *associated series* $(A_n)_{n \in \mathbb{N}}$ by

$$A_n = An = \frac{an + b}{cn + d}.$$

As we shall see, the associated series plays an important role in our computations. Clearly, if $A \in \mathcal{A}$ then A_n is monotone decreasing and has the finite limit $A_\infty := a/c$.

We have the following result.

Theorem 4.0.1. *Given $A \in \mathcal{A}$, there are positive real-valued constants K_A and c such that*

$$M_A(x) = K_A x^{A_1} + O_A \left(x^{A_1 - 1/2} \exp\{-c(\log x)^{3/5} (\log \log x)^{-1/5}\} \right).$$

We remark that under the Riemann hypothesis, for a restricted class of matrices one has an asymptotic formula for the error term in Theorem 4.0.1 of the form

$$M_A(x) - K_A x^{A_1} \sim \tilde{K}_A x^{\frac{1}{2}(A_2 - 1)} \quad (4.1)$$

for a real-valued constant $\tilde{K}_A$. This naturally leads one to consider the maps $\psi_j : \mathcal{A} \rightarrow \mathbb{Q}_+$ for $j = 1, 2$ given by

$$\psi_j(A) = A_j. \quad (4.2)$$

Since, as mentioned above, each E_r consists of exactly one element, it follows that there is a well-defined map $s : \mathbb{Q}_+ \rightarrow \mathbb{Q}_+$ given by

$$s(r) = \psi_2 \circ \psi_1^{-1}(r). \quad (4.3)$$

The map $s(r)$ tells us how accurately the main term $K_A x^{A_1}$ approximates $M_A(x)$ in (4.1), in the sense that it gives the exact order of magnitude of the error $M_A(x) - K_A x^{A_1}$.

Although it can be shown that this map is nowhere continuous, one can obtain asymptotic formulas for the average value of $s(r)$, with r in various ranges. For example, define the height function for each rational

$r = p/q$ with $q \geq 1$ and $(p, q) = 1$ by

$$h(r) := \max\{|p|, |q|\}.$$

We have the following result.

Theorem 4.0.2. *For any $\delta > 0$,*

$$\sum_{\substack{r \in \mathbb{Q}_+ \cap [0,1] \\ h(r) \leq X}} s(r) = \frac{3}{2\pi^2} X^2 + O_\delta(X^{11/6+\delta}).$$

4.1 Asymptotics of the average

Consider the Dirichlet series

$$F_A(s) = \sum_{n=1}^{\infty} \frac{f_A(n)}{n^s}.$$

We will take advantage of the meromorphic continuation of $F_A(s)$ in the case where $\det A = -1$.

Proof of Theorem 4.0.1. We prove the result with

$$K_A = \frac{1}{(1 + A_1)(2 + A_1)\zeta(2)} T_A(1 + A_1).$$

If $\det A = -1$ then $p^{A\alpha} \leq p^{A_1}$ for all $\alpha \geq 1$, so $f_A(n) \leq n^{A_1}$, hence $F_A(s)$ converges in the half plane $\Re(s) = \sigma > 1 + A_1$. Moreover, $F_A(s)$ has an Euler product in that region. Write

$$F_A(s) = \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} \prod_p (1 + g_p(s)),$$

where

$$g_p(s) = \left(1 + \frac{p^{A_1}}{p^s}\right)^{-1} \sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}}.$$

Note that if $\det(A) = -1$, then $A_1 - A_2 = \frac{1}{(c+d)(2c+d)} \leq \frac{1}{2}$. Take $\epsilon > 0$. For $\sigma \geq A_1 + \epsilon$ we have

$$\left(1 + \frac{p^{A_1}}{p^s}\right)^{-1} \ll_{\epsilon} 1.$$

Also, for $\sigma \geq \frac{1}{2}(1 + A_2 + \epsilon)$ we have

$$\sum_{k=2}^{\infty} \frac{p^{A_k}}{p^{ks}} \ll \frac{p^{A_2}}{p^{2s}} \ll_{\epsilon} \frac{1}{p^{1+\epsilon}}$$

Thus for $\sigma \geq \max\{A_1 + \epsilon, \frac{1}{2}(1 + A_2 + \epsilon)\}$ the sum $\sum_p |g_p(s)|$ converges, hence

$$T_A(s) = \prod_p (1 + g_p(s))$$

is analytic for $\sigma > \sigma_0 = \max\{A_1, \frac{1}{2}(1 + A_2)\}$, so $F_A(s)$ is meromorphic there, with a pole at $s = 1 + A_1$.

4.2 The main term

To continue, we utilize a variant of Perron's formula and write

$$\sum_{n \leq x} \left(1 - \frac{n}{x}\right) f_A(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} T_A(s) \frac{x^s}{s(s+1)} ds$$

where $1 + A_1 < c \leq 5/4 + A_1$.

We apply the Vinogradov-Korobov zero-free region for $\zeta(s)$ given by (2.4) in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3} (\log \log t)^{1/3}.$$

Fix $0 < U < T \leq x$, let $\nu = 1/2 + A_1$ and set

$$\eta = \nu - c_0 (\log U)^{-2/3} (\log \log U)^{-1/3}.$$

Deform the path of integration into the union of the line segments

$$\left\{ \begin{array}{ll} \gamma_1, \gamma_9 : s = c + it & \text{if } |t| \geq T, \\ \gamma_2, \gamma_8 : s = \sigma \pm iT & \text{if } \nu \leq \sigma \leq c, \\ \gamma_3, \gamma_7 : s = \nu + it & \text{if } U \leq |t| \leq T, \\ \gamma_4, \gamma_6 : s = \sigma \pm iU & \text{if } \eta \leq \sigma \leq \nu, \\ \gamma_5 : s = \eta + it & \text{if } |t| \leq U. \end{array} \right.$$

The integrand is analytic on and within this modified contour, hence by the residue theorem

$$xM_A(x) = \frac{1}{(1 + A_1)(2 + A_1)\zeta(2)} T_A(1 + A_1) x^{1+A_1} + \sum_{k=1}^9 J_k,$$

with the main terms coming from the residue at the simple pole at $s = 1 + A_1$.

4.3 Upper bounds for the error integrals

In order to estimate the integral along our modified contour we will make use of the bounds given by (2.5).

On the line segments on which $s = c + it$, $|t| \geq T$, we have $\zeta(s - A_1) \ll \log t$ and $1/\zeta(2s - 2A_1) \ll \log t$, so

$$\begin{aligned} |J_1|, |J_9| &\ll \int_T^\infty (\log t)^2 \frac{x^c}{|(c + it)(c + 1 + it)|} dt \\ &\ll \frac{x^c (\log T)^2}{T}. \end{aligned}$$

On the line segments on which $s = \sigma + iT$, $\nu \leq \sigma \leq c$, we have $1/\zeta(2s - 2A_1) \ll \log T$, $\zeta(s - A_1) \ll T^{(1-\sigma+A_1)/2}$ for $\nu \leq \sigma \leq 1 + A_1$, and $\zeta(s - A_1) \ll \log T$ for $1 + A_1 \leq \sigma \leq c$. So

$$\begin{aligned} |J_2|, |J_8| &\ll \int_\nu^{1+A_1} T^{\frac{1}{2}(1-\sigma+A_1)} \log T \frac{x^\sigma}{T^2} d\sigma + \int_{1+A_1}^c (\log T)^2 \frac{x^\sigma}{T^2} d\sigma \\ &\ll T^{\frac{1}{2}(1+A_1)} \log T \max \left\{ \left(\frac{x}{\sqrt{T}} \right)^\nu, \left(\frac{x}{\sqrt{T}} \right)^{1+A_1} \right\} + \frac{(\log T)^2}{T^2} x^c. \end{aligned}$$

On the line segments on which $s = \nu + it$, $U \leq |t| \leq T$, we have $\zeta(s - A_1) \ll t^{(1-\nu+A_1)/2}$ and $1/\zeta(2s - 2A_1) \ll \log t$, so

$$\begin{aligned} |J_3|, |J_7| &\ll \int_U^T (\log t) t^{\frac{1}{2}(1-\nu+A_1)} \frac{x^\nu}{|(\nu + it)(\nu + 1 + it)|} dt \\ &\ll \frac{\log T}{U^{3/4}} x^\nu. \end{aligned}$$

On the line segments on which $s = \sigma + iU$, $\eta \leq \sigma \leq \nu$, we have $\zeta(s - A_1) \ll U^{(1-\sigma+A_1)/2}$ and $1/\zeta(2s - 2A_1) \ll \log U$, so

$$\begin{aligned} |J_4|, |J_6| &\ll \int_\eta^\nu (\log U) U^{\frac{1}{2}(1-\sigma+A_1)} \frac{x^\sigma}{U^2} d\sigma \\ &\ll U^{\frac{1}{2}(1+A_1)-2} \log U \max \left\{ \left(\frac{x}{\sqrt{U}} \right)^\nu, \left(\frac{x}{\sqrt{U}} \right)^\eta \right\}. \end{aligned}$$

On the line segment on which $s = \eta + it$, $|t| \leq U$, we have $\zeta(s - A_1) \ll (|t| + 1)^{(1-\eta+A_1)/2}$ and $1/\zeta(2s - 2A_1) \ll$

$\log(|t| + 1)$, so

$$\begin{aligned} |J_5| &\ll \int_{-U}^U (|t| + 1)^{1-\eta+A_1} \log(|t| + 1) \frac{x^\eta}{|\eta + it||\eta + 1 + it|} dt \\ &\ll x^\eta \int_{-U}^U (|t| + 1)^{\frac{1}{2}(1-\eta+A_1)-2} \log(|t| + 1) dt. \end{aligned}$$

Since $\frac{1}{2}(1 - \eta + A_1) - 2 \leq -\frac{3}{2}$ for U sufficiently large, the above integral converges, hence $|J_5| \ll x^\eta$.

We collect all estimates, and take $T = x^2$ and $U = \exp\{c_2(\log x)^{3/5}(\log \log x)^{-1/5}\}$ to obtain the desired result. $\square$

One could instead factor

$$\left(1 + \frac{p^{A_1}}{p^s} + \frac{p^{A_2}}{p^{2s}} + \frac{p^{A_3}}{p^{3s}} + \cdots\right) = \left(1 + \frac{p^{A_1}}{p^s}\right) \left(1 + \frac{p^{A_2}}{p^{2s}}\right) (1 + g_p(s))$$

with

$$g_p(s) = \left(1 + \frac{p^{A_1}}{p^s}\right)^{-1} \left(1 + \frac{p^{A_2}}{p^{2s}}\right)^{-1} \left(-\frac{p^{A_1+A_2}}{p^{3s}} + \sum_{k=3}^{\infty} \frac{p^{A_k}}{p^{ks}}\right)$$

so that

$$F_A(s) = \frac{\zeta(s - A_1)}{\zeta(2s - 2A_1)} \frac{\zeta(2s - A_2)}{\zeta(4s - 2A_2)} \prod_p (1 + g_p(s)).$$

Under the Riemann hypothesis, we find a second order term of the form $\tilde{K}_A x^{A_2}$ in the asymptotic formula for $F_A(s)$ provided that $\frac{1}{4} + A_1 < \frac{1}{2}(1 + A_2)$, that is, provided that

$$a + b < \frac{c + d}{2} - \frac{1}{2c + d}.$$

This occurs for matrices A in $\mathcal{A}$ with restrictions on c and d . One can see that A_1 will lie in the interval $(0, 1/2)$.

4.4 Mapping through matrices

We now return to the two maps ψ_1 and ψ_2 defined in (4.2).

Lemma 4.4.1. *The map ψ_1 is bijective.*

Proof. For $\frac{p}{q} \in \mathbb{Q}_+$, consider the set of matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$ such that $\psi_1(A) = \frac{p}{q}$. We note that any such quadruple (a, b, c, d) is constrained by $c \geq 0, d \geq 0$,

$$ad - bc = -1 \tag{4.4}$$

and

$$c + d = \frac{q}{p}(a + b). \tag{4.5}$$

(Note that (4.4) implies that p cannot be zero.) By (4.5) we have

$$c = \frac{q}{p}(a + b) - d.$$

Inserting this into (4.4) gives us

$$ad - b(a + b)\frac{q}{p} + bd = -1,$$

so

$$(a + b)(pd - qb) = -p.$$

Write $a + b = \pm n$ for some positive integer $n|p$. By (4.5) we have $c + d = \frac{q}{p}(\pm n) \in \mathbb{Z}$ so $p|n$, hence $p = n$.

There are two cases: If $a + b = -p$ then $c + d = -q$. This contradicts the assumptions that $q \geq 1$ and c and d are non-negative. On the other hand, if $a + b = p$, then $c + d = q$, so (4.4) gives us

$$a(q - c) - bc = -1$$

so

$$pc = 1 + aq. \tag{4.6}$$

So c is uniquely determined by $cp \equiv 1 \pmod{q}$ and $1 \leq c < q$. Then d is uniquely determined by $d = q - c$, and a and b by $a = \frac{1 - pc}{q}$ and $b = p - a$. $\square$

In the case where $p/q \in (0, 1]$, we identify p/q as an element of $\mathcal{F}_Q$, the Farey fractions of order Q , with $Q \geq q$. If we consider the “minimal” set of Farey fractions $\mathcal{F}_q$ containing p/q , then elementary properties of Farey fractions (see for example Chapter 3 of [10]) give that the adjacent Farey fractions $p'/q' < p/q < p''/q''$

satisfy $q' = \bar{p}$, $p' = \bar{q}$, $p'' = p - \bar{q}$ and $q'' = q - \bar{p}$. Here $\bar{p}$ is the unique integer $1 \leq \bar{p} < q$ satisfying $p\bar{p} \equiv 1 \pmod{q}$ and $\bar{q}$ is the unique integer $1 \leq \bar{q} < p$ satisfying $q\bar{q} \equiv 1 \pmod{p}$. We can write

$$\psi_1(p/q) = \begin{pmatrix} \bar{q} & p - \bar{q} \\ \bar{p} & q - \bar{p} \end{pmatrix}.$$

That is, the matrix $\psi_1(p/q)$ is comprised of the “parent” Farey fractions in $\mathcal{F}_{q-1}$.

Additionally, we can write the function $s(p/q)$ from (4.3) uniquely as

$$s(p/q) = \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)} \quad (4.7)$$

4.5 Proof of Theorem 4.0.2

To prove Theorem 4.0.2, we will use the following result (see Lemma 2.3 of [6]).

Lemma 4.5.1. *Assume that $q \geq 1$ and h are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q , and $f : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$ is a C^1 function. Then for any integer $T \geq 1$ and any $\delta > 0$*

$$\sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ ab \equiv h \pmod{q} \\ \gcd(a,b)=1}} f(a,b) = \frac{\phi(q)}{q^2} \iint_{\mathcal{I} \times \mathcal{J}} f(x,y) dx dy + \mathcal{E},$$

with

$$\begin{aligned} \mathcal{E} &\ll_{\delta} T^2 \|f\|_{\infty} q^{1/2+\delta} \gcd(h,q)^{1/2} \\ &\quad + T \|\nabla f\|_{\infty} q^{3/2+\delta} \gcd(h,q)^{1/2} + \frac{\|\nabla f\|_{\infty} |\mathcal{I}| |\mathcal{J}|}{T}, \end{aligned}$$

where $\|f\|_{\infty}$ and $\|\nabla f\|_{\infty}$ denote the sup-norm of f and respectively $|\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}|$ on $\mathcal{I} \times \mathcal{J}$.

Let $Q = \lfloor X \rfloor$. Since $r \in \mathcal{F}_Q$ we have

$$\sum_{\substack{r \in \mathbb{Q}_+ \cap [0,1] \\ h(r) \leq X}} s(r) = \sum_{\substack{1 \leq q \leq Q \\ 1 \leq p < q \\ (p,q)=1}} s(p/q).$$

We use (4.7) and Lemma 4.5.1 with $T = q^{\frac{1}{6} - \frac{\delta}{3}}$ to deduce that the right-hand sum is equal to

$$\begin{aligned} \sum_{1 \leq q \leq Q} \sum_{\substack{1 \leq p < q \\ 1 \leq \bar{p} < q \\ p\bar{p} \equiv 1 \pmod{q} \\ (p, q) = 1}} \frac{\bar{p}p - 1 + pq}{q(\bar{p} + q)} &= \sum_{1 \leq q \leq Q} \frac{\phi(q)}{q^2} \iint_{[1, q]^2} \frac{vu - 1 + uq}{q(v + q)} du dv + \mathcal{E} \\ &= \sum_{1 \leq q \leq Q} \phi(q) \iint_{[1/q, 1]^2} \frac{xy - \frac{1}{q^2} + x}{y + 1} dx dy + \mathcal{E}, \end{aligned}$$

where $\mathcal{E} \ll_{\delta} q^{5/6 + \delta}$. The integral is equal to

$$\frac{1}{2} \left(1 - \frac{1}{q^2}\right) \left(1 - \frac{1}{q}\right) - \frac{q-1}{q^3} \left(\log 2 - \log \left(1 + \frac{1}{q}\right)\right) = \frac{1}{2} + O\left(\frac{1}{q}\right),$$

so

$$\sum_{\substack{r \in \mathbb{Q}_+ \cap [0, 1] \\ h(r) \leq X}} s(r) = \frac{1}{2} \sum_{1 \leq q \leq Q} \phi(q) + O\left(\sum_{1 \leq q \leq Q} \frac{\phi(q)}{q}\right) + O\left(\sum_{1 \leq q \leq Q} q^{5/6 + \delta}\right).$$

One can use the methods of Section 2 to estimate the sums over $\phi(q)$, or use partial summation along with standard estimates (see for example [23] or Chapter 18 of [10]). This gives the main term of our theorem; the first error term above is $O(X)$, and the second is $O_{\delta}(X^{11/6 + \delta})$.

Chapter 5

A second generalization: λ -moments

We complement the results of the previous chapter by considering real-valued λ -moments of these arithmetic functions $f_A(n)$, defined over a larger class of 2×2 matrices. This method offers greater flexibility, since the λ -weighted moments of the Dirichlet series associated with $f_A(n)$ may have meromorphic continuation to a region in which the original Dirichlet series has an essential singularity. As we shall see, this expands previous results to a more general framework, and leads to new results within this more general setting.

Consider the subset $\mathcal{A}$ of 2×2 matrices with integer-valued entries given by

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det A = 1, c \geq 0, d > 0 \right\}.$$

For each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$ consider the fractional linear transformation

$$Az = \frac{az + b}{cz + d}.$$

For each $\lambda > 0$ we examine the λ -moment

$$(f_A(n))^\lambda = \prod_{p^\alpha || n} p^{\lambda A\alpha}.$$

A key tool in our study of the λ -moment is the Dirichlet series

$$F_{A,\lambda}(s) = \sum_{n=1}^{\infty} \frac{(f_A(n))^\lambda}{n^s}.$$

We say that the pair (A, λ) is *good* if there exists a half-plane where $F_{A,\lambda}$ has meromorphic continuation

with at least one pole. Consider the space $\mathbb{G}$ in $\mathcal{A} \times \mathbb{R}_+$ of pairs (A, λ) that are good. Information about $\mathbb{G}$ leads to information about λ -moments

$$S_{A,\lambda}(x) = \sum_{n \leq x} (f_A(n))^\lambda$$

and more precise estimates about the weighted λ -moments such as

$$M_{A,\lambda}(x) = \sum_{n \leq x} \left(1 - \frac{n}{x}\right) (f_A(n))^\lambda.$$

Asymptotic formulas for such moments are given in the following section.

Theorem 5.0.2. *Suppose λ is a positive real number. Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathcal{A}$, necessary and sufficient conditions for the pair (A, λ) to be in $\mathbb{G}$ are*

- if $A = R_k = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$, $k = 1, 2, 3, \dots$ and $0 < \lambda < 1/k$,
- if $A \neq R_k$, $b \geq -1$ and $\lambda \in (0, \infty)$,
- if $A \neq R_k$, $b < -1$ and $0 < \lambda < -\frac{dc}{bc+1}$.

For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathcal{A}$ define $h(A) = \max\{|a|, |b|, |c|, |d|\}$. For any positive integer Q , define

$$g_Q(\lambda) = \frac{\#\{A \in \mathcal{A} : h(A) \leq Q \text{ and } (A, \lambda) \in \mathbb{G}\}}{\#\{A \in \mathcal{A} : h(A) \leq Q\}}.$$

Figure 1 shows the behavior of $g_Q(\lambda)$ for values of Q up to 75. We will prove the following:

Theorem 5.0.3. *The functions g_Q converge uniformly on compact subintervals of $(0, \infty)$ to g as $Q \rightarrow \infty$, where*

$$g(\lambda) = \begin{cases} 1 - \frac{\lambda}{4} & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{2} + \frac{1}{4\lambda} & \text{if } \lambda \geq 1. \end{cases}$$

5.1 A general region of meromorphic continuation

To prove Theorem 5.0.2 we begin with the following lemma:

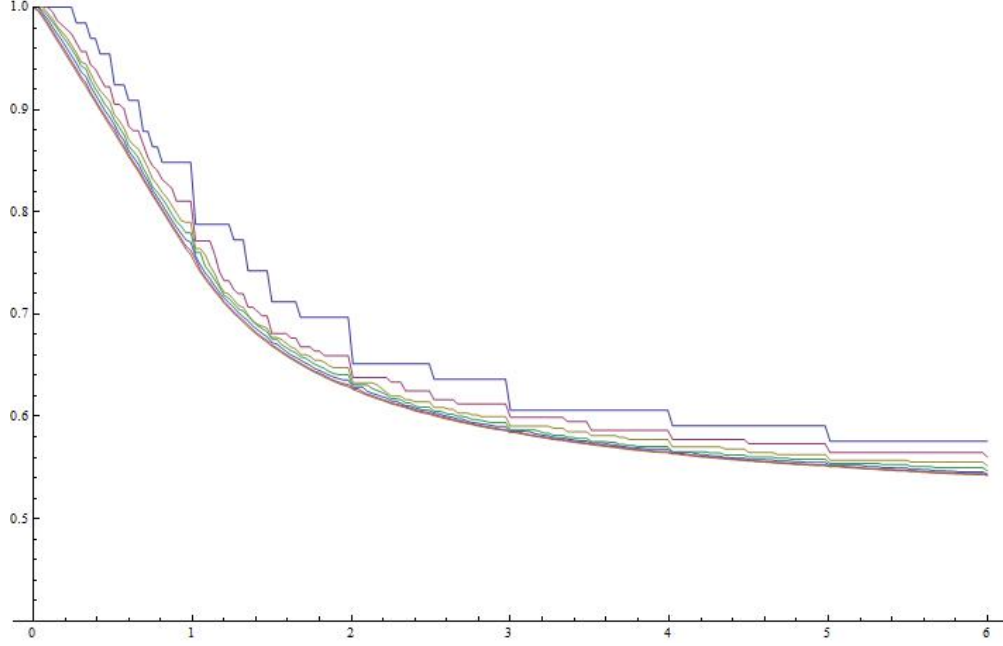


Figure 5.1: Plot of the function $g_Q(\lambda)$

Lemma 5.1.1. *Suppose $F(s)$ is a Dirichlet series with Euler product*

$$F(s) = \prod_{p \text{ prime}} \left(1 + \frac{p^{c_1}}{p^s} + \frac{p^{c_2}}{p^{2s}} + \frac{p^{c_3}}{p^{3s}} + \dots \right),$$

where $c_1, c_2, \dots$ are real numbers independent of p . Assume there exists a finite set of natural numbers $\mathcal{N} = \{n_1, n_2, \dots, n_l\}$ such that for all $1 \leq j, k \leq l$ we have $\frac{1}{n_j} (1 + c_{n_j}) = \frac{1}{n_k} (1 + c_{n_k})$ and such that $\frac{1}{n_j} (1 + c_{n_j}) > \sup_{n \notin \mathcal{N}} \left\{ \frac{1}{n} (1 + c_n) \right\}$. Then $F(s)$ satisfies

$$F(s) = G(s) \prod_{j=1}^l \zeta(n_j s - c_{n_j})$$

where $G(s)$ is analytic in the half plane

$$\Re(s) > \sup_{\substack{m \in \mathcal{N} \\ n \notin \mathcal{N}}} \left\{ \frac{1}{n} (1 + c_n), \frac{1}{m} (1/2 + c_m) \right\}$$

and is bounded in any closed half plane contained in this region.

Proof. We restrict ourself to the case $l = 1$; the remaining cases are similar. We factor

$$\prod_p \left(1 + \frac{p^{c_1}}{p^s} + \frac{p^{c_2}}{p^{2s}} + \frac{p^{c_3}}{p^{3s}} + \dots \right) = \prod_p \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right) \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

and note that

$$\prod_p \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right) = \frac{\zeta(n_1 s - c_{n_1})}{\zeta(2n_1 s - 2c_{n_1})}.$$

Write $s = \sigma + it$. The function $1/\zeta(2n_1 s - 2c_{n_1})$ is analytic for $\sigma > \frac{1}{n_1} \left(\frac{1}{2} + c_{n_1} \right)$. Also, for any fixed $\epsilon > 1/2$ and $\sigma \geq \frac{1}{n_1} (\epsilon + c_{n_1})$ we have that

$$\left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \ll_{\epsilon} 1,$$

and for $\sigma > \epsilon + \sup_{n \neq n_1} \left\{ \frac{1}{n} (1 + c_n) \right\}$ we have

$$\sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \ll \sup_{n \neq n_1} \frac{p^{c_n}}{p^{n\sigma}} \ll_{\epsilon} \frac{1}{p^{1+\epsilon}}.$$

So

$$\sum_p \left| \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right|$$

converges in any half plane of the form

$$\sigma \geq \sigma_0 > \epsilon + \sup_{n \neq n_1} \left\{ \frac{c_{n_1}}{n_1}, \frac{1}{n} (1 + c_n) \right\}.$$

It follows that the product

$$\prod_p \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

is uniformly bounded on the half-plane $\Re(s) > \sigma_0$ (see §14.2, p. 15 of [20]). Hence

$$G(s) = \frac{1}{\zeta(2n_1 s - 2c_{n_1})} \prod_p \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

is uniformly bounded on $\Re(s) \geq \max \left\{ \sigma_0, \frac{1}{n_1} \left(\frac{1}{2} + c_{n_1} \right) \right\}$. Since $G(s)$ is analytic in this half-plane, the Dirichlet series $F(s)$ has a meromorphic continuation to this region, where it satisfies $F(s) = \zeta(n_1 s -$

$c_{n_1})G(s)$. □

5.2 Asymptotics of the λ -moments

We now give the proof of Theorem 5.0.2. Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$, if $\alpha \geq 1$, then $\frac{a\alpha+b}{c\alpha+d} < (|a| + |b|)\alpha$. So $f_{A,\lambda}(n) \leq n^{|a|+|b|}$, and hence the Dirichlet series $F_{A,\lambda}(s)$ is analytic in the region $\Re(s) > 1 + |a| + |b|$. In this region, F has an Euler product

$$F_{A,\lambda}(s) = \prod_p \left(1 + \frac{p^{\lambda A_1}}{p^s} + \frac{p^{\lambda A_2}}{p^{2s}} + \frac{p^{\lambda A_3}}{p^{3s}} + \cdots \right).$$

Let

$$\theta^{(1)} = \sup_{n \geq 1} \{\theta_n(\lambda)\}, \tag{5.1}$$

where $\theta_n = \theta_n(\lambda) = \frac{1}{n}(1 + \lambda A n)$. If the supremum in (5.1) is attained, then by employing Lemma 5.1.1 one can show that $(A, \lambda) \in \mathbb{G}$. Next, we identify this supremum by considering the function

$$\theta(x, \lambda) = \frac{1}{x} \left(1 + \lambda \frac{ax + b}{cx + d} \right)$$

where x is positive and real-valued.

If $c = 0$, then

$$\begin{aligned} \theta(x, \lambda) &= \frac{1}{x} \left(1 + \lambda \frac{ax + b}{d} \right) \\ &= \frac{\lambda a}{d} + \frac{1}{x} \left(1 + \frac{\lambda b}{d} \right). \end{aligned}$$

If $c \neq 0$, then

$$\frac{ax + b}{cx + d} = \frac{a}{c} - \frac{\det A}{c(cx + d)},$$

and upon writing

$$\frac{1}{x(cx + d)} = \frac{1/d}{x} - \frac{c/d}{cx + d}$$

we find that

$$\theta(x, \lambda) = \frac{1}{x} \left(1 + \frac{\lambda b}{d} \right) + \frac{1}{cx + d} \frac{\lambda \det A}{d}.$$

Since $cx + d > 0$ for all positive x , the expression $\theta(x, \lambda)$ will be a decreasing function of x if the coefficients of $\frac{1}{x}$ and $\frac{1}{cx + d}$ are positive. If $\det A = 1$ with $a \geq 1$ and $b \geq 0$, then $\theta(x, \lambda)$ is a decreasing function of x for any $\lambda > 0$, so $\sup_{n \geq 2} \{\theta_n\} < \theta_1$, and $(A, \lambda) \in \mathbb{G}$ for any λ in $(0, \infty)$. If $b \leq 0$, then $1 + \lambda \frac{b}{d}$ is positive provided that $\lambda < -d/b$.

The partial derivative

$$\frac{\partial}{\partial x} \theta(x, \lambda) = - \frac{(1 + \lambda \frac{b}{d})(cx + d)^2 + \frac{c\lambda}{d} x^2}{x^2(cx + d)^2}$$

is negative for large enough x provided that

$$0 < c^2 \left(1 + \lambda \frac{b}{d} \right) + \frac{c\lambda}{d}. \quad (5.2)$$

If $b = -1$ and $c = 1$ then (5.3) gives that $(A, \lambda) \in \mathbb{G}$ for any λ in $(0, \infty)$ for such matrices A . More generally, if $b \neq -1$ then $bc \neq -1$, and so (5.2) is equivalent to

$$\lambda < - \frac{dc}{bc + 1}. \quad (5.3)$$

We see that $\theta(x, \lambda)$ has a maximum provided that $\lambda > 0$ is in this range, and hence so does $\theta_n(\lambda)$. This completes the proof of Theorem 5.0.2.

If we take x_0 to be the value of x for which $\theta(x, \lambda)$ is maximal, then $\theta^{(1)}$ is equal to θ_{n_1} , where $n_1 = \lfloor x_0 \rfloor$ or $n_1 = \lceil x_0 \rceil$. We remark that if λ is such that the above maximum is attained at both $\lfloor x_0 \rfloor$ and $\lceil x_0 \rceil$, where x_0 is not an integer, then $F_{A, \lambda}(s)$ has a double pole at $s = \theta^{(1)}$. Furthermore, we note that for a given matrix A , the set of such exceptional λ is at most countable.

One can obtain an asymptotic formula for $M_{A, \lambda}(x)$ using the techniques of the previous chapter, which can be summarized as follows:

Write $F_{A, \lambda}(s)$ in the form given by Lemma 5.1.1 and use a variant of Perron's formula, namely

$$\sum_{n \leq x} \left(1 - \frac{n}{x} \right) f_{A, \lambda}(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_{A, \lambda}(s) \frac{x^s}{s(s+1)} ds,$$

where $\sigma_0 < c < \sigma_0 + \delta$ for some $\delta > 0$.

We apply the Vinogradov-Korobov zero-free region for $\zeta(s)$ given by (2.4) in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3} (\log \log t)^{1/3}.$$

Fix U and T to be chosen later, with $0 < U < T < x^2$, and let $\nu = \frac{1}{n_1} \left(\frac{1}{2} + \lambda A_{n_1} \right)$ and

$$\eta = \nu - c_0 (\log U)^{-2/3} (\log \log U)^{-1/3}.$$

Deform the path of integration into the union of the line segments

$$\left\{ \begin{array}{ll} \gamma_1, \gamma_9 : s = c + it & \text{if } |t| \geq T, \\ \gamma_2, \gamma_8 : s = \sigma \pm iT & \text{if } \nu \leq \sigma \leq c, \\ \gamma_3, \gamma_7 : s = \nu + it & \text{if } U \leq |t| \leq T, \\ \gamma_4, \gamma_6 : s = \sigma \pm iU & \text{if } \eta \leq \sigma \leq \nu, \\ \gamma_5 : s = \eta + it & \text{if } |t| \leq U. \end{array} \right.$$

The integrand is analytic on and within this modified contour, and by the residue theorem,

$$M_{A,\lambda}(x) = K_1 x^{\theta^{(1)}} + \sum_{k=1}^9 J_k,$$

the main contribution being due to the residue of the simple pole at the point $s = \theta^{(1)}$.

In order to estimate the integral along our modified contour we will make use of the bounds given by (2.5).

Upon estimating $|J_i|$, $i = 1, 2, \dots, 9$, and selecting U and T so as to optimize the error terms, we see that if $(A, \lambda) \in \mathbb{G}$ and the pole of $F_{A,\lambda}(s)$ at $\theta^{(1)}$ is simple, then

$$M_{A,\lambda}(x) = K_1 x^{\theta^{(1)}} + R_{A,\lambda}(x),$$

where

$$R_{A,\lambda}(x) \ll_{A,\lambda} \max \left\{ x^{\theta^{(2)}}, x^{\frac{1}{n_1} \left(\frac{1}{2} + \lambda A_{n_1} \right)} \exp \{ -c (\log x)^{3/5} (\log \log x)^{-1/5} \} \right\}.$$

Notice that in the case where $x^{\theta^{(2)}}$ is of a larger order than

$$x^{\frac{1}{n_1} \left(\frac{1}{2} + \lambda A_{n_1} \right)} \exp \{ -c (\log x)^{3/5} (\log \log x)^{-1/5} \},$$

one obtains a secondary term in the asymptotic formula for $M_{A,\lambda}(x)$ of the form $K_2 x^{\theta^{(2)}}$.

We remark that given a matrix A there may possibly be a finite or countable set of λ for which $F_{A,\lambda}(s)$ has a double pole at $s = \theta^{(1)}$. In these rare cases $M_{A,\lambda}(x)$ has a different order of magnitude. More precisely,

$$M_{A,\lambda}(x) \sim K' x^{\theta^{(1)}} \log x$$

as $x \rightarrow \infty$, where K' is a positive constant that depends only on A and λ .

5.3 Examples

One can examine the different cases represented in Theorem 5.0.2 for various matrices A by plotting the functions $\theta(x, \lambda)$ for $x \geq 1$ and different ranges of λ (depending on A). Recall that if we can identify the value $x = x_0$ for which $\theta(x, \lambda)$ is maximal in these ranges, then the Dirichlet series $F_{A,\lambda}(s)$ will have a pole at $s = \theta^{(1)} = \frac{1}{k} \left(1 + \lambda \frac{ak+b}{ck+d} \right)$, where either $k = \lfloor x_0 \rfloor$ or $k = \lceil x_0 \rceil$.

If

$$A = A^{(1)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

the matrix associated with the restrictive factor function $R(n)$, then we can see in Figure 5.2 that, for a few chosen values of λ with $0 < \lambda < 1$, the maximal value of $\theta(x, \lambda)$ for $x \geq 1$ occurs at $x = 1$. Similar behavior holds for any matrix of the form

$$A = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

and $0 < \lambda < 1/k$.

If

$$A = A^{(2)} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

then we can see in Figure 5.3 that, for values of λ larger than 0, the maximal value of $\theta(x, \lambda)$ for $x \geq 1$ occurs at $x = 1$. Similar behavior holds for any matrix in $\mathcal{A}$ with $b \geq 1$ for any positive λ .

Matrices in $\mathcal{A}$ for which the maximal value of $\theta(x, \lambda)$ occurs at some value x_0 strictly larger than 1 appear as a subset of those represented by the third case in Theorem 5.0.2. For example, the matrix

$$A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

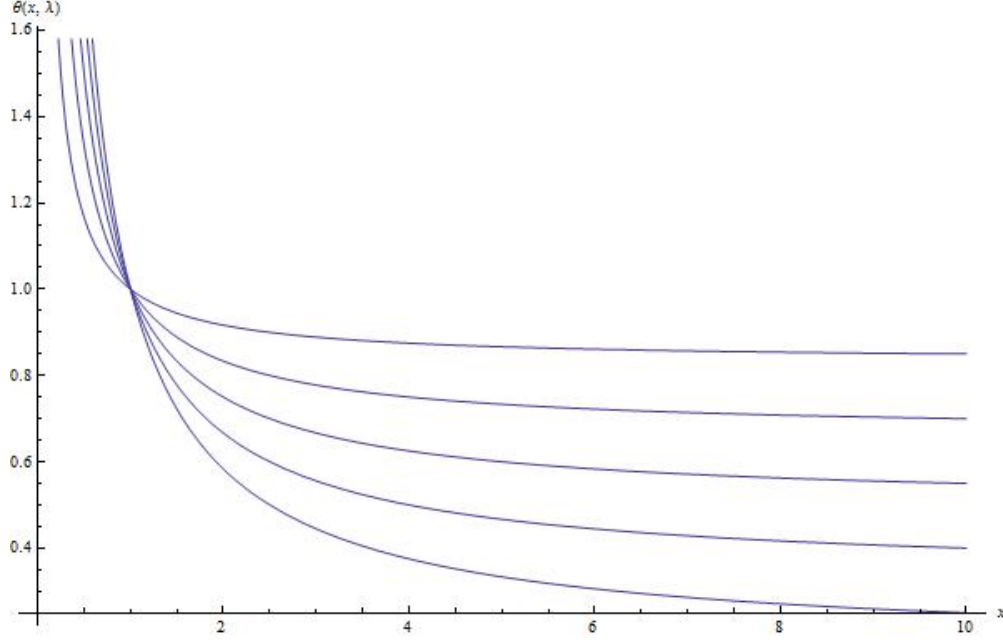


Figure 5.2: Plot of the function $\theta(x, \lambda)$ for $A^{(1)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ with $\lambda = 1/6, 1/3, 1/2, 2/3, 5/6$

will yield a pair $(A^{(3)}, \lambda)$ in $\mathbb{G}$ provided that $0 < \lambda < 2/3$. Figure 5.4 shows plots of $\theta(x, \lambda)$ for $(A^{(3)})$ for several values of λ in this range. Note also that the curve $\theta(x, 2/3)$ appears in Figure 5.4 at the bottom. This curve does not appear to attain its maximum, which is not surprising given that we expect such a maximum to be attained only if λ is in the *open* interval $(0, 2/3)$.

One can see in Figure 5.5 the behavior of $\theta(x, \lambda)$ for $A^{(3)}$ with λ in the interval $(0, 2/3)$. While $\theta(x, \lambda)$ attains its maximum at $x = 1$ for small values of λ , this behavior changes dramatically at values of λ close to $2/3$.

Figure 5.6 shows behavior of $\theta(x, \lambda)$ for $A^{(3)}$ with λ in a much more restricted range. The curves $\theta(x, 0.6)$, $\theta(x, 0.625)$, and $\theta(x, 0.65)$ attain their maxima at $x = 1$, $x = 2.58114$, and $x = 10.7009$ respectively. The bottom-most curves $\theta(x, 0.675)$ and $\theta(x, 0.7)$, which do not attain their maximum, are shown for comparison.

Figure 5.7 shows the behavior of $\theta(x, \lambda)$ for $A^{(3)}$ in the entire interval $0.6 < \lambda < 0.7$. If we consider only those λ in a smaller interval contained in the upper range of the interval $(0, 2/3)$, as in Figures 5.8 and 5.9, this behavior becomes more pronounced. The curves $\theta(x, 0.645)$, $\theta(x, 0.65)$, and $\theta(x, 0.655)$ shown in Figures 5.8 attain their maxima at $x = 7.60576$, $x = 10.7009$ (as before), and $x = 16.4315$ respectively.

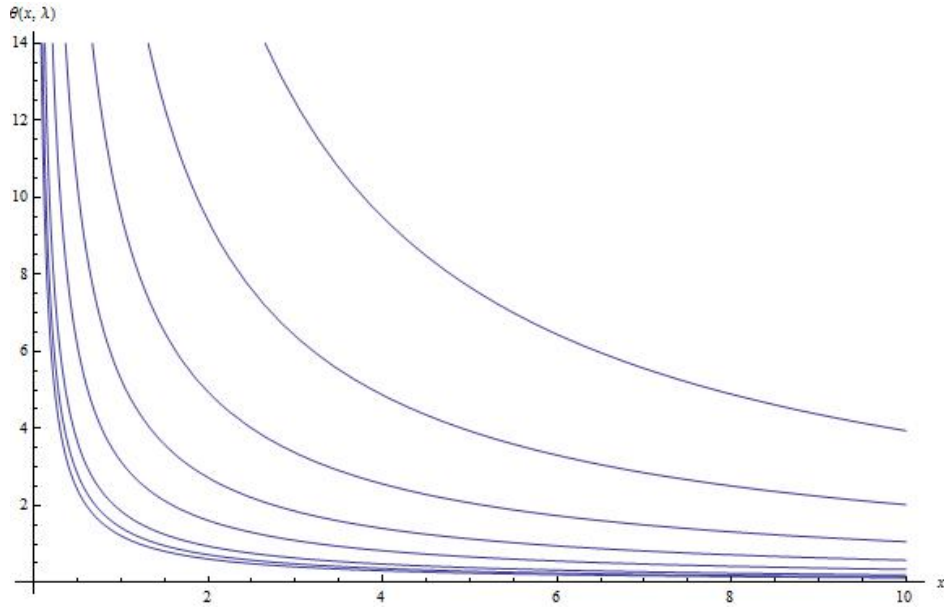


Figure 5.3: Plot of the function $\theta(x, \lambda)$ for $A^{(2)} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ with $\lambda = 1/2, 1, 2, 5, 10, 20, 40, 80$

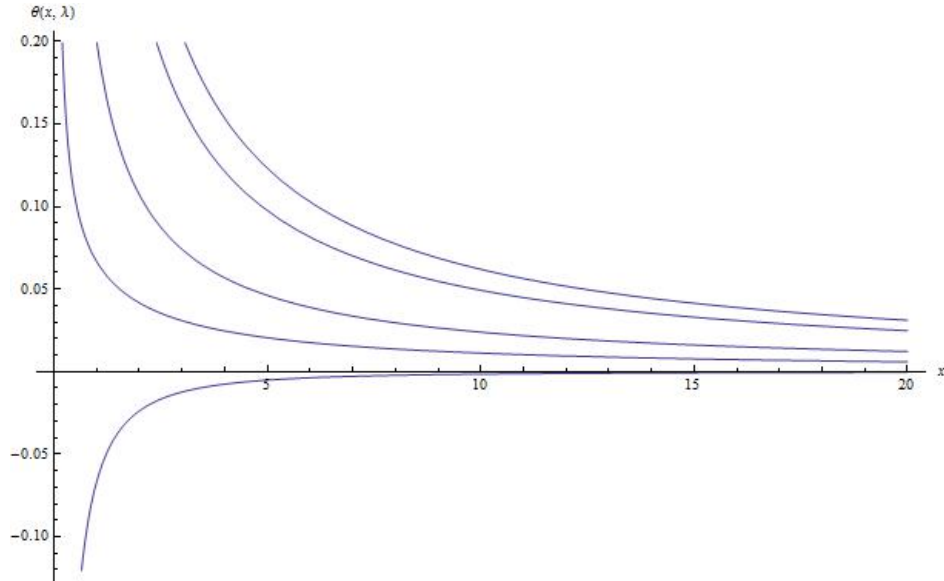


Figure 5.4: Plot of the function $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $\lambda = 1/4, 1/3, 1/2, 7/12, 2/3$

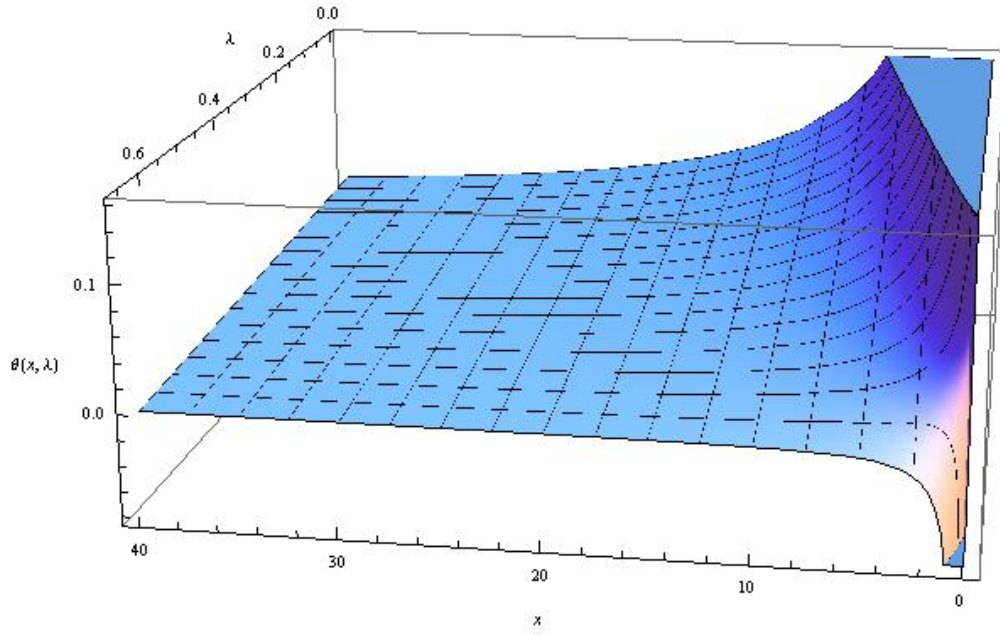


Figure 5.5: The surface $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $0 < \lambda < 2/3$

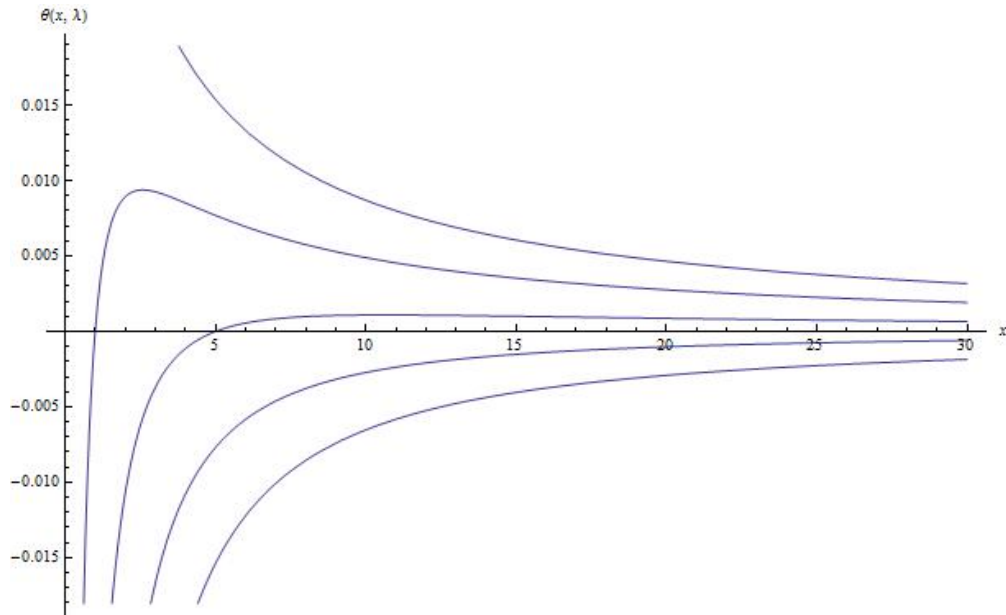


Figure 5.6: Plot of the function $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $\lambda = 0.6, 0.625, 0.65, 0.675, 0.7$

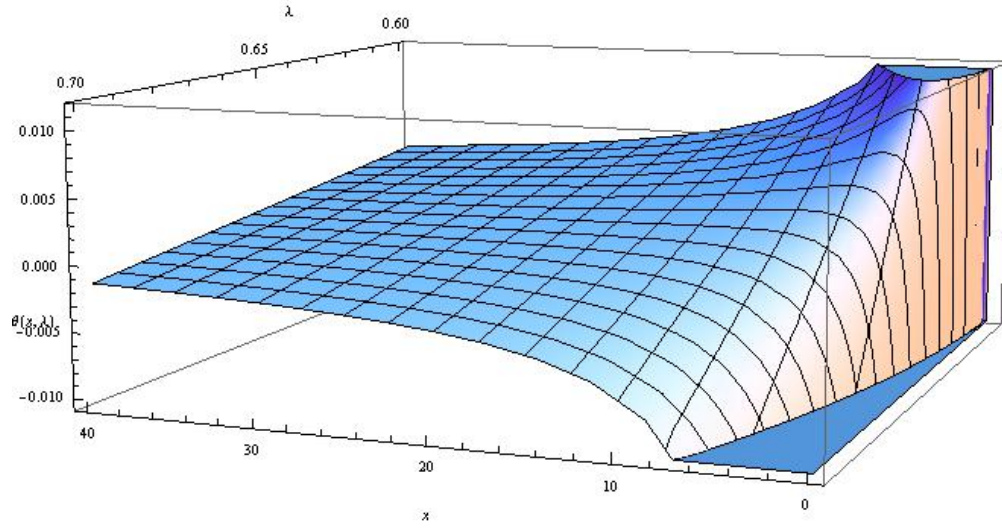


Figure 5.7: The surface $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $0.6 < \lambda < 0.7$

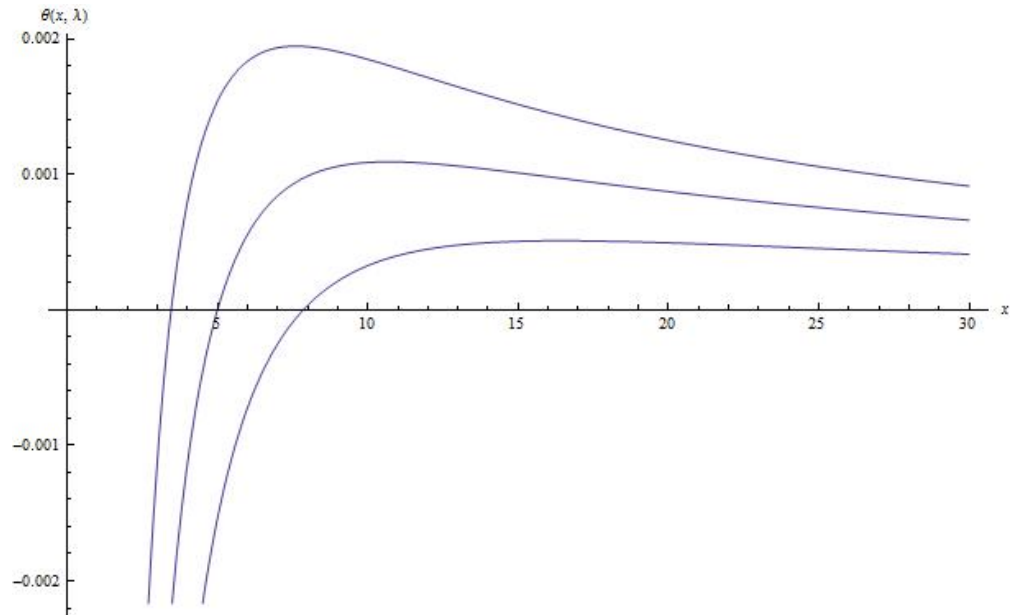


Figure 5.8: Plot of the function $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $\lambda = 0.645, 0.65, 0.655$

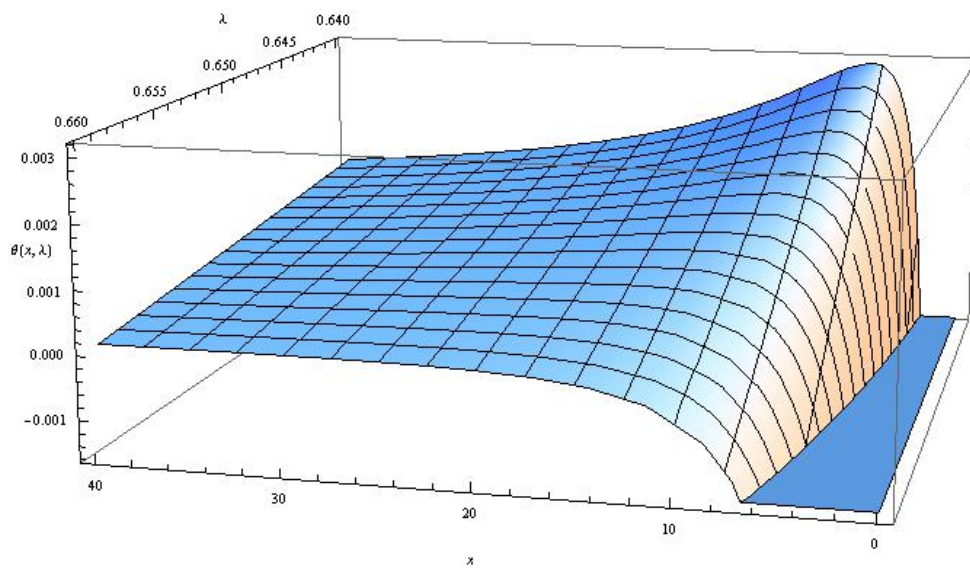


Figure 5.9: The surface $\theta(x, \lambda)$ for $A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ with $0.645 < \lambda < 0.655$

5.4 Further remarks on $M_{A,\lambda}(x)$ and $S_{A,\lambda}(x)$

For asymptotic formulas for the sums $S_{A,\lambda}(x)$ we use the following form of Perron's formula (see [20], Sections 9.42 and 9.44; and [21], Section 3.12):

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} A(s) \frac{x^s}{s} ds + R(x, c, T),$$

where $A(s)$ is the Dirichlet series associated with $a(n)$ and

$$|R(x, c, T)| \leq \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|a(n)|}{n^c |\log x/n|}.$$

It is more natural to consider this sum $S_{A,\lambda}(x)$ instead of $M_{A,\lambda}(x)$, but we note that the variant of Perron's formula used for $M_{A,\lambda}(x)$ has an extra factor of $\frac{1}{s+1}$ in the integrand, which makes estimations easier.

In order to estimate $S_{A,\lambda}(x)$, we apply the above version of Perron's formula with $a(n) = f_{A,\lambda}(n)$ and $A(s) = F_{A,\lambda}(s)$. Upon shifting the path of integration and replacing it with a rectangular path with vertices $c - iT$, $c + iT$, $\nu + iT$, and $\nu - iT$, one can apply the residue theorem as before to obtain

$$\sum_{n \leq x} (f_A(n))^\lambda = K'' x^{\theta^{(1)}} + \sum_{k=1}^3 J_k,$$

where K'' is a positive constant depending only on A and λ . Upon estimating the remaining terms J_k and $R(x, c, T)$ and selecting T so as to optimize the resulting error terms, one finds that

$$\sum_{n \leq x} (f_A(n))^\lambda = K'' x^{\theta^{(1)}} + R'_{A,\lambda}(x),$$

where

$$R'_{A,\lambda}(x) \ll_{A,\lambda} \max \left\{ x^{\theta^{(2)}}, x^{\frac{1}{n_1}(\frac{1}{2} + \lambda A_{n_1})} (\log x)^r \right\}$$

and r is a positive constant that depends only on A and λ .

We also remark that even if (A, λ) is not in $\mathbb{G}$, one can still use Perron's formula to find nontrivial upper and lower bounds for $M_{A,\lambda}(x)$ of the form

$$x^C \ll_{A,\lambda} M_{A,\lambda}(x) \ll_{A,\lambda,\epsilon} x^{C+\epsilon}$$

for some positive constant C depending on A and λ .

5.5 Proof of Theorem 5.0.3: λ -densities of matrices in $\mathbb{G}$

We begin with the following simple result.

Lemma 5.5.1. *For any $0 < \alpha < \beta$ and any $\epsilon > 0$ we have*

$$\#\{\alpha q \leq m \leq \beta q : (m, q) = 1\} = (\beta - \alpha)\phi(q) + O_\epsilon(q^\epsilon).$$

Proof. We have

$$\begin{aligned} \#\{\alpha q \leq m \leq \beta q : (m, q) = 1\} &= \sum_{\substack{\alpha q \leq m \leq \beta q \\ d|m \\ d|q}} \sum_{d|q} 1 \\ &= \sum_{d|q} \mu(d) \sum_{\substack{\alpha q \leq m \leq \beta q \\ d|m}} 1 \\ &= \sum_{d|q} \mu(d) \left(\left\lfloor \beta \frac{q}{d} \right\rfloor - \left\lfloor \alpha \frac{q}{d} \right\rfloor \right). \end{aligned}$$

Since $|\sum_{d|q} \mu(d)| \leq \sum_{d|q} 1 \ll_\epsilon q^\epsilon$,

$$\begin{aligned} \#\{\alpha q \leq m \leq \beta q : (m, q) = 1\} &= (\beta - \alpha)q \sum_{d|q} \frac{\mu(d)}{d} + O_\epsilon(q^\epsilon) \\ &= (\beta - \alpha)\phi(q) + O_\epsilon(q^\epsilon). \end{aligned}$$

□

We now turn our attention to the proof of Theorem 5.0.3.

Note that if $c \geq 0$ and $b < 0$, then the relation $ad - bc = 1$ implies that $a \leq 0$. Replace b with $-b$ and a with $-a$. The given conditions become $bc - ad = 1$, $a, b, c, d > 0$, and

$$0 < \lambda < \frac{dc}{bc - 1}. \tag{5.4}$$

Let $N_{\mathcal{A},a}(\lambda) = \#\{A \in \mathcal{A} : h(A) = c, (\lambda, A) \in \mathbb{G}\}$, and define $N_{\mathcal{A},b}(\lambda)$, $N_{\mathcal{A},c}(\lambda)$, and $N_{\mathcal{A},d}(\lambda)$ similarly for the cases where $h(A) = b$, $h(A) = c$, and $h(A) = d$, so that

$$\#\{A \in \mathcal{A} : h(A) \leq Q, (\lambda, A) \in \mathbb{G}\} = N_{\mathcal{A},a}(\lambda) + N_{\mathcal{A},b}(\lambda) + N_{\mathcal{A},c}(\lambda) + N_{\mathcal{A},d}(\lambda) + O(Q).$$

If $h(A) = c$, then $bc - ad = 1$ implies $ad \equiv -1 \pmod{c}$, hence $a \equiv -\bar{d} \pmod{c}$. Since also $1 \leq a < c$, we have $a = c - \bar{d}$, where $\bar{d}$ is the unique inverse of $d \pmod{c}$ satisfying $1 \leq \bar{d} \leq c$. So

$$b = \frac{1 + ad}{c} = \frac{1 + cd - d\bar{d}}{c}.$$

Inserting this into equation (5.4), we obtain

$$\bar{d} > c \frac{\lambda - 1}{\lambda}.$$

We note that there are only $O(1)$ matrices A with $h(A) = 1$. Hence

$$\begin{aligned} N_{\mathcal{A},c}(\lambda) &= \sum_{2 \leq q \leq Q} \# \left\{ 1 \leq d \leq q : (d, q) = 1, \bar{d} > q \frac{\lambda - 1}{\lambda} \right\} + O(1) \\ &= \sum_{2 \leq q \leq Q} \# \left\{ 1 \leq m \leq q : (m, q) = 1, m > q \frac{\lambda - 1}{\lambda} \right\} + O(1) \\ &= \sum_{2 \leq q \leq Q} \# \left\{ \max \left\{ 1, q \frac{\lambda - 1}{\lambda} \right\} \leq m \leq q : (m, q) = 1 \right\} + O(1) \\ &= \begin{cases} \sum_{2 \leq q \leq Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{\lambda} \sum_{2 \leq q \leq Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \geq 1. \end{cases} \end{aligned}$$

Using the well-known estimate

$$\sum_{n \leq X} \phi(n) = \frac{1}{2\zeta(2)} X^2 + O(X \log X)$$

(see for example [23] or Chapter 18 of [10]), we see that

$$N_{\mathcal{A},c}(\lambda) = \begin{cases} \frac{1}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{2\lambda\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \geq 1. \end{cases}$$

If $h(A) = b$, then $bc - ad = 1$ implies $ad \equiv -1 \pmod{b}$, hence $a \equiv -\bar{d} \pmod{b}$. Since also $1 \leq a < b$, we have $a = b - \bar{d}$, where $\bar{d}$ is the unique inverse of $d \pmod{b}$ satisfying $1 \leq \bar{d} \leq b$. So

$$c = \frac{1 + ad}{b} = \frac{1 + bd - d\bar{d}}{b}.$$

Inserting this into equation (5.4), we obtain

$$d > b\lambda + \frac{1}{\bar{d}-1}.$$

This can only hold if $\lambda < 1$, hence for $\lambda \geq 1$, we have $\#\{A \in \mathcal{A} : h(A) = b, (\lambda, A) \in \mathbb{G}\} = 0$. When $0 < \lambda < 1$, we note that $\bar{d} = 1$ only when $d = 1$ since $1 \leq d < b$. We find that $d > b\lambda$ for all but a bounded number of integers b . Also, we note that again $h(A) = 1$ for only $O(1)$ of these matrices. Hence for $0 < \lambda < 1$,

$$\begin{aligned} N_{\mathcal{A},b}(\lambda) &= \sum_{2 \leq q \leq Q} \# \left\{ 1 < d \leq q : (d, q) = 1, d > b\lambda + \frac{1}{\bar{d}-1} \right\} + O(1) \\ &= \sum_{2 \leq q \leq Q} \left(\# \left\{ 1 < d \leq q : (d, q) = 1, d > b\lambda + \frac{1}{\bar{d}-1} \right\} + O(1) \right) + O(1) \\ &= \sum_{2 \leq q \leq Q} \# \{ \lambda q \leq m \leq q : (m, q) = 1 \} + O(Q) \\ &= (1 - \lambda) \sum_{2 \leq q \leq Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) \\ &= \frac{1 - \lambda}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}). \end{aligned}$$

If $h(A) = a$, then $bc - ad = 1$ implies $bc \equiv 1 \pmod{a}$, hence $b \equiv \bar{c} \pmod{a}$. Since also $1 \leq b < a$, we have $b = \bar{c}$, where $\bar{c}$ is the unique inverse of $c \pmod{a}$ satisfying $1 \leq \bar{c} \leq a$. So

$$d = \frac{bc - 1}{a} = \frac{c\bar{c} - 1}{a}.$$

Inserting this into equation (5.4), we see that $a\lambda < c$. This can only hold if $\lambda < 1$, hence for $\lambda \geq 1$, we have $\#\{A \in \mathcal{A} : h(A) = a, (\lambda, A) \in \mathbb{G}\} = 0$. Furthermore, if $bc = 1$ then $ad = 2$, so there are $O(1)$ such matrices. When $0 < \lambda < 1$,

$$\begin{aligned} N_{\mathcal{A},a}(\lambda) &= \sum_{2 \leq q \leq Q} \# \{ 1 \leq c \leq q : (c, q) = 1, c > a\lambda \} + O(1) \\ &= \sum_{2 \leq q \leq Q} \# \{ \lambda q \leq m \leq q : (m, q) = 1 \} + O(1) \\ &= (1 - \lambda) \sum_{2 \leq q \leq Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) \\ &= \frac{1 - \lambda}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}). \end{aligned}$$

If $h(A) = d$, then $bc - ad = 1$ implies $bc \equiv 1 \pmod{d}$, hence $b \equiv \bar{c} \pmod{d}$. Since also $1 \leq b < c$, we

have $b = \bar{c}$, where $\bar{c}$ is the unique inverse of $c \bmod d$ satisfying $1 \leq \bar{c} \leq d$. So

$$d = \frac{bc - 1}{d} = \frac{c\bar{c} - 1}{d}.$$

Inserting this into equation (5.4), we find

$$\bar{c} < \frac{d}{\lambda} + \frac{1}{c}.$$

So $\bar{c} < \frac{d}{\lambda}$ for all but a bounded number of integers c . Since again there are $O(1)$ matrices with $bc = 1$, we have

$$\begin{aligned} N_{\mathcal{A},d}(\lambda) &= \sum_{2 \leq q \leq Q} \# \left\{ 1 \leq c \leq q : (c, q) = 1, \bar{c} < \frac{d}{\lambda} + \frac{1}{c} \right\} + O(1) \\ &= \sum_{2 \leq q \leq Q} \left(\# \left\{ 1 \leq c \leq q : (c, q) = 1, \bar{c} < \frac{d}{\lambda} \right\} + O(1) \right) + O(1) \\ &= \sum_{2 \leq q \leq Q} \# \left\{ 1 \leq m \leq \min\{q, \frac{q}{\lambda}\} : (m, q) = 1 \right\} + O(Q) \\ &= \begin{cases} \sum_{2 \leq q \leq Q} \phi(q) + O_{\epsilon}(Q^{1+\epsilon}) & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{\lambda} \sum_{2 \leq q \leq Q} \phi(q) + O_{\epsilon}(Q^{1+\epsilon}) & \text{if } \lambda \geq 1. \end{cases} \\ &= \begin{cases} \frac{1}{2\zeta(2)} Q^2 + O_{\epsilon}(Q^{1+\epsilon}) & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{2\lambda\zeta(2)} Q^2 + O_{\epsilon}(Q^{1+\epsilon}) & \text{if } \lambda \geq 1. \end{cases} \end{aligned}$$

Combining the above four cases, and the fact that if $\det A = -1$ then $(A, \lambda) \in \mathbb{G}$ for all $\lambda > 0$, one obtains the desired result after a short calculation.

Chapter 6

Eigenvalues

Given a matrix A that gives rise to a function $f_A(n)$, it is natural to ask how A influences the behavior of this associated arithmetic function. We have seen how certain values of the fractional linear transformation $\alpha \mapsto A\alpha = \frac{a\alpha+b}{c\alpha+d}$, with $\alpha \in \mathbb{N}$, affect the average values of the function f_A . Do the eigenvalues of A also influence the behavior of $f_A(n)$? To answer this question, we establish results on the local density of eigenvalues through their natural connection to a particular surface.

For each given matrix A and a positive real number x , we define the weighted average

$$M_A(x) = \sum_{1 \leq n \leq x} \left(1 - \frac{n}{x}\right) f_A(n).$$

We also consider λ_A^+ and λ_A^- , the positive and negative real eigenvalues of A , respectively. Thus, λ_A^+ and λ_A^- are solutions of the quadratic equation

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0,$$

with

$$\lambda_A^+ = \frac{a+d + \sqrt{(a+d)^2 + 4}}{2} \tag{6.1}$$

and

$$\lambda_A^- = \frac{a+d - \sqrt{(a+d)^2 + 4}}{2}. \tag{6.2}$$

Furthermore, λ_A^+ and λ_A^- satisfy the inequalities $\lambda_A^- < 0 < \lambda_A^+$ and the identity $\lambda_A^+ \lambda_A^- = -1$.

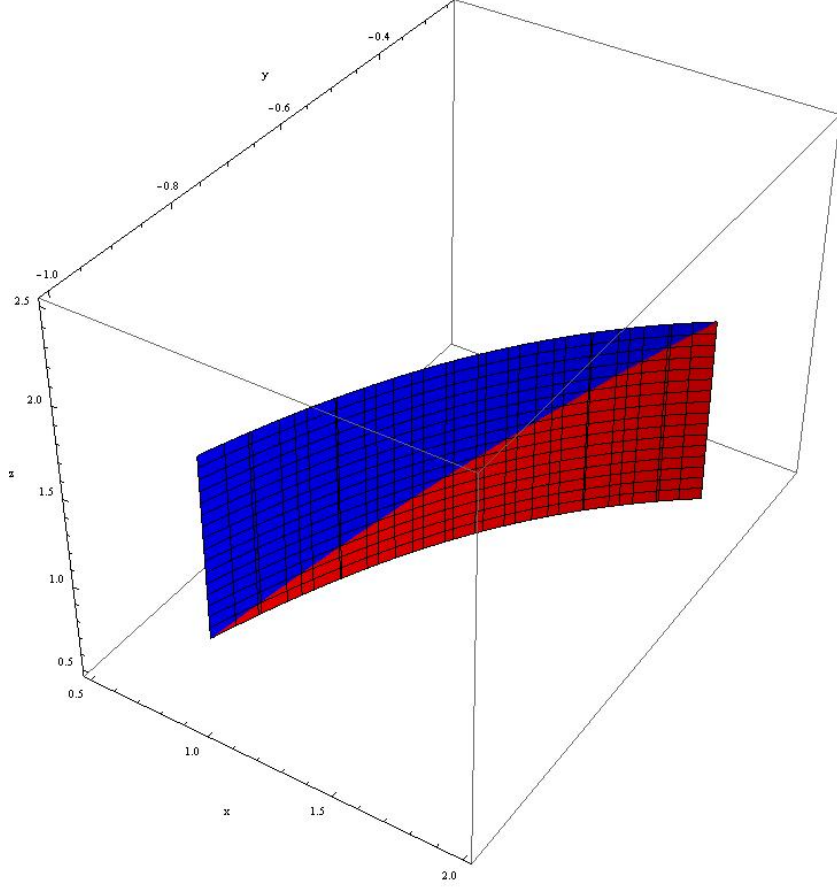


Figure 6.1: The surface $\mathcal{S}$.

6.1 A connection between $\mathcal{A}(Q, x)$ and the surface $\mathcal{S}$

For a large Q and a much larger x , we consider the subset $\mathcal{A}(Q, x)$ of 2×2 matrices with integer entries given by

$$\mathcal{A}(Q, x) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : 1 \leq a, b, c, d \leq Q, ad - bc = -1, \left(\frac{\lambda_A^+}{Q}, Q\lambda_A^-, \frac{\log M_A(x)}{\log x} \right) \in \mathcal{S} \right\},$$

where the surface $\mathcal{S}$ is given by

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : 1 < x, z < 2, xy = -1\}.$$

(See Figure 6.)

The map

$$\Psi_{Q,x} : \mathcal{A}(Q, x) \longrightarrow \mathcal{S}$$

defined by

$$\Psi_{Q,x}(A) = \left(\frac{\lambda_A^+}{Q}, Q\lambda_A^-, \frac{\log M_A(x)}{\log x} \right)$$

associates to each matrix $A \in \mathcal{A}(Q, x)$ a unique point on $\mathcal{S}$. Given such a matrix A , the normalized eigenvalues λ_A^+ and λ_A^- give the first and second coordinates of the corresponding point, with λ_A^+ divided by Q and λ_A^- multiplied by Q . We note that λ_A^+ is close to $a + d$, which can be $2Q$ at most. It follows that $\lambda_A^+/Q < 2$, with very few exceptions.

For the sake of simplicity, we restrict our attention to the case when λ_A^+/Q is in the interval $(1, 2)$ and leave to the reader to make the adaptation to the case when λ_A^+/Q is in the interval $(0, 1)$, as the two cases are similar.

In the third coordinate of such a point on $\mathcal{S}$, we observe that for any A with positive entries, $f_A(n) \geq 1$ for all n . It follows that $M_A(x) > x/2$. Hence,

$$\frac{\log M_A(x)}{\log x} > 1 - \frac{\log 2}{\log x}.$$

Finally, for simplicity's sake, we consider only the case when z is in the interval $(1, 2)$. In like manner, one can study the case when z is in the interval $(2, \infty)$.

6.2 Properties of $\Psi_{Q,x}$

In order to investigate the influence of the eigenvalues λ_A^+ and λ_A^- of A on the behavior of the associated arithmetic function $f_A(n)$, we examine the joint distribution of λ_A^+ , λ_A^- , and $(\log M_A(x))/\log x$. That is, we examine the image of $\Psi_{Q,x}$ on $\mathcal{S}$. More precisely, for a given point $(\alpha, -1/\alpha, \beta)$ on $\mathcal{S}$ we consider, for each small $\delta > 0$, the neighborhood $\mathcal{V}_{\alpha,\beta,\delta}$ of $(\alpha, -1/\alpha, \beta)$ in $\mathcal{S}$ given by

$$\mathcal{V}_{\alpha,\beta,\delta} = \{(x, y, z) \in \mathcal{S} : |x - \alpha| < \delta, |z - \beta| < \delta\}.$$

We would like to estimate the number of matrices A in $\mathcal{A}(Q, x)$ for which $\Psi_{Q,x}(A)$ lies in $\mathcal{V}_{\alpha,\beta,\delta}$. We expect the number of such matrices to grow like a constant times $\delta^2 Q^2$ as Q and x tend to infinity, with x much

larger than Q , while $\delta > 0$ is kept fixed. This leads us to consider the limit of the ratio

$$\frac{\#\{\Psi_{Q,x}^{-1}(\mathcal{V}_{\alpha,\beta,\delta})\}}{\delta^2 Q^2} = \frac{\#\{A \in \mathcal{A}(Q,x) : \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha,\beta,\delta}\}}{\delta^2 Q^2},$$

as x approaches infinity and then Q approaches infinity. Lastly, we take the limit of this expression as $\delta \rightarrow 0^+$.

Our main result can be summarized as follows.

Theorem 6.2.1. *Fix a point $(\alpha, -1/\alpha, \beta) \in \mathcal{S}$, where α and β are real numbers such that $1 < \alpha, \beta < 2$.*

Then we have

$$\lim_{\delta \rightarrow 0} \lim_{Q \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{\#\{A \in \mathcal{A}(Q,x) : \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha,\beta,\delta}\}}{\delta^2 Q^2} = \begin{cases} \frac{24}{\pi^2} \left(\frac{\beta - \alpha}{\beta - 1} \right), & \text{if } \beta \geq \alpha; \\ 0, & \text{if } \beta < \alpha. \end{cases}$$

Thus, the images via $\Psi_{Q,x}$ of almost all matrices A lie on the part of the surface $\mathcal{S}$ where $z \geq x$, depicted in blue in Figure 6. If we fix two points $P_1 = (\alpha_1, -1/\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, -1/\alpha_2, \beta_2)$ on that portion of the surface $\mathcal{S}$ and compare the local densities of the points in $\Psi_{Q,x}(\mathcal{A}(Q,x))$ around P_1 and P_2 , respectively, as a direct consequence of our theorem we deduce the following:

Corollary 6.2.2. *Let α_j and β_j be real numbers such that $1 < \alpha_j < \beta_j < 2$ for $j \in \{1, 2\}$. Then we have*

$$\lim_{\delta \rightarrow 0} \lim_{Q \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{\#\{A \in \mathcal{A}(Q,x) : \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha_1,\beta_1,\delta}\}}{\#\{A \in \mathcal{A}(Q,x) : \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha_2,\beta_2,\delta}\}} = \frac{(\beta_1 - \alpha_1)(\beta_2 - 1)}{(\beta_2 - \alpha_2)(\beta_1 - 1)}.$$

6.3 The subset $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$

We begin the proof of Theorem 6.2.1 by fixing α and β in the interval $(1, 2)$ and $\delta > 0$ small enough so that α and β belong to the interval $(1 + \delta, 2 - \delta)$. We also consider the set of matrices

$$\mathcal{D}_{\alpha,\beta,\delta,Q,x} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}(Q,x) : 1 \leq a, b, c \leq d \leq Q, ad - bc = -1, \right. \\ \left. (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, (\beta - 1 - \delta)d < b < (\beta - 1 + \delta)d \right\}.$$

The cardinality of $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ is given by

$$\begin{aligned}
\#\mathcal{D}_{\alpha,\beta,\delta,Q,x} &= \sum_{1 \leq d \leq Q} \sum_{\substack{1 \leq c \leq d \\ \gcd(c,d)=1}} \#\{(a,b) : 1 \leq a, b \leq d, ad - bc = -1, \\
&\quad (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, \\
&\quad (\beta - 1 - \delta)d < b < (\beta - 1 + \delta)d\} \\
&= \sum_{1 \leq d \leq Q} \sum_{\substack{1 \leq c \leq d \\ \gcd(c,d)=1 \\ (\alpha - \delta)Q \leq d + (c\bar{c} - 1)/d \leq (\alpha + \delta)Q \\ (\beta - 1 - \delta)d < \bar{c} < (\beta - 1 + \delta)d}} 1. \tag{6.3}
\end{aligned}$$

As in Chapter 4, $\bar{c}$ is used to denote the unique multiplicative inverse of c modulo d in the interval $[1, d]$. The second step in (6.3) follows from the fact that the conditions $1 \leq b \leq d$ and $ad - bc = -1$ imply that b is equal to $\bar{c}$. Hence, a is uniquely determined and is given by $a = (bc - 1)/d$. Furthermore, the contribution of the terms in (6.3) for which $d < (\alpha - \delta)Q/2$ is zero. Indeed, since $a \leq d$, we see that if $d < (\alpha - \delta)Q/2$, then $a + d < (\alpha - \delta)Q$.

Setting $q = d$, $x = c$ and $y = \bar{c}$, we obtain $\#\mathcal{D}_{\alpha,\delta,Q}$ in the form

$$\#\mathcal{D}_{\alpha,\beta,\delta,Q,x} = \sum_{(\alpha - \delta)Q/2 \leq q \leq Q} \#\{(x, y) \in \Omega_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2 : xy \equiv 1 \pmod{q}\}, \tag{6.4}$$

where

$$\begin{aligned}
\Omega_{\alpha,\beta,\delta,Q,q} &= \{(u, v) \in \mathbb{R}^2 : 1 \leq u, v \leq q, (\alpha - \delta)qQ - q^2 \leq uv \leq (\alpha + \delta)qQ - q^2, \\
&\quad (\beta - 1 - \delta)q \leq v \leq (\beta - 1 + \delta)q\}. \tag{6.5}
\end{aligned}$$

We estimate the summands in (6.4) by using Lemma 4.5.1. We break the region $\Omega_{\alpha,\beta,\delta,Q,q}$ into squares of side length $L = [Q^\eta]$ for some $0 < \eta < 1$, and denote by I_j those squares lying entirely within $\Omega_{\alpha,\beta,\delta,Q,q}$, and B_i those squares which intersect both $\Omega_{\alpha,\beta,\delta,Q,q}$ and its complement in $\mathbb{R}^2$, where $1 \leq j \leq n$ and $1 \leq i \leq m$ for some natural numbers n and m . We have

$$\begin{aligned}
\#\{(u, v) \in \Omega_{\alpha,\beta,\delta,Q,q} : ab \equiv 1 \pmod{q}\} &= \sum_{1 \leq j \leq n} \#\{(u, v) \in I_j : ab \equiv 1 \pmod{q}\} \\
&\quad + \sum_{1 \leq i \leq m} \#\{(u, v) \in B_i \cap \Omega_{\alpha,\beta,\delta,Q,q} : \\
&\quad ab \equiv 1 \pmod{q}\}.
\end{aligned}$$

By Lemma 4.5.1, each of the summands on the right-hand side above is equal to

$$\frac{\phi(q)}{q^2}L^2 + O_\epsilon(q^{1/2+\epsilon}).$$

If we take Ω' to be the subset of $\Omega_{\alpha,\beta,\delta,Q,q}$ formed by removing from $\Omega_{\alpha,\beta,\delta,Q,q}$ a neighborhood of the boundary of $\Omega_{\alpha,\beta,\delta,Q,q}$ of width $L\sqrt{2}$, we find that $\Omega' \subset \bigcup I_j \subset \Omega_{\alpha,\beta,\delta,Q,q}$ and

$$\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) - \text{Area}(\Omega') = O(qL).$$

Hence,

$$\text{Area}\left(\bigcup I_j\right) = \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL).$$

Since

$$\begin{aligned} \text{Area}\left(\bigcup I_j\right) &= \sum_{1 \leq j \leq n} \#\{(u, v) \in I_j : ab \equiv 1 \pmod{q}\} \\ &= n \frac{\phi(q)}{q^2} L^2 + O_\epsilon(nq^{1/2+\epsilon}), \end{aligned}$$

we have

$$nL^2 = \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL),$$

and in particular

$$n = O\left(\frac{Q^2}{L^2}\right).$$

Thus,

$$\begin{aligned} \sum_{1 \leq j \leq n} \#\{(u, v) \in I_j : ab \equiv 1 \pmod{q}\} &= n \frac{\phi(q)}{q^2} L^2 + O_\epsilon(nq^{1/2+\epsilon}) \\ &= \frac{\phi(q)}{q^2} \left(\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL) \right) + O_\epsilon\left(\frac{Q^2}{L^2} q^{1/2+\epsilon}\right) \\ &= \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(L) + O_\epsilon\left(\frac{Q^{5/2+\epsilon}}{L^2}\right). \end{aligned}$$

Similarly, we find that $m = O(Q/L)$ and

$$\begin{aligned}
0 &\leq \sum_{1 \leq i \leq m} \#\{(u, v) \in B_i \cap \Omega_{\alpha, \beta, \delta, Q, q} : ab \equiv 1 \pmod{q}\} \\
&\leq \sum_{1 \leq i \leq m} \#\{(u, v) \in B_i : ab \equiv 1 \pmod{q}\} \\
&= m \frac{\phi(q)}{q^2} L^2 + O_\epsilon(mq^{1/2+\epsilon}) \\
&= O(L) + O_\epsilon\left(\frac{Q^{3/2+\epsilon}}{L}\right).
\end{aligned}$$

Taking $\eta = 5/6$, we have

$$\#\{(u, v) \in \Omega_{\alpha, \beta, \delta, Q, q} : ab \equiv 1 \pmod{q}\} = \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha, \beta, \delta, Q, q}) + O_\epsilon(Q^{5/6+\epsilon}).$$

Thus,

$$\#\mathcal{D}_{\alpha, \beta, \delta, Q, x} = M + E, \tag{6.6}$$

where

$$M = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Omega_{\alpha, \beta, \delta, Q, q}), \tag{6.7}$$

and

$$E = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} E_{\alpha, \beta, \delta, Q, q} = O_\epsilon(Q^{11/6+\epsilon}). \tag{6.8}$$

6.4 The main term from $\mathcal{D}_{\alpha, \beta, \delta, Q, x}$

To examine the main term M in (6.7), we recall from the definition of the set $\Omega_{\alpha, \beta, \delta, Q, q}$ in (6.5) that

$$(\alpha - \delta)qQ - q^2 \leq uv \leq (\alpha + \delta)qQ - q^2.$$

We first point out that when $\alpha > \beta$ and δ is small enough, all the areas $\text{Area}(\Omega_{\alpha, \beta, \delta, Q, q})$ are zero for all values of q . Indeed, if $\alpha > \beta$ and $(u, v) \in \text{Area}(\Omega_{\alpha, \beta, \delta, Q, q})$, then

$$(\alpha - 1 - \delta)q^2 \leq (\alpha - \delta)qQ - q^2 \leq uv \leq qv \leq (\beta - 1 + \delta)q^2.$$

This shows that for $\delta > 0$ small enough, all of the sets $\text{Area}(\Omega_{\alpha, \beta, \delta, Q, q})$ are empty. In what follows we will restrict to the case $\alpha < \beta$. From the position of the hyperbolas $uv = (\alpha - \delta)qQ - q^2$ and $uv = (\alpha + \delta)qQ - q^2$,

the horizontal lines $v = (p - 1 - \delta)q$ and $v = (p - 1 + \delta)q$, and their points of intersection with the boundary of the square $[1, q] \times [1, q]$, we find that

$$\Omega_{\alpha, \beta, \delta, Q, q} = \mathcal{L} \cap ([1, q] \times [1, q]),$$

where $\mathcal{L}$ is the “parallelogram shaped” region that lies between the hyperbolas and horizontal lines.

It is easy to see that if $q < (\alpha - \delta)Q/(\beta + \delta)$, then $\mathcal{L}$ lies completely outside the square $[1, q] \times [1, q]$. Furthermore, one can verify that if $(\alpha - \delta)Q/(\alpha + \delta) \leq q \leq (\alpha + \delta)Q/(\beta - \delta)$, then $\mathcal{L}$ intersects the square $[1, q] \times [1, q]$ but does not lie entirely inside it. This implies that $\mathcal{L}$ lies close enough to the boundary of the square $[1, q] \times [1, q]$ that the total contribution of these values of q to the main term M is negligible. Hence, we are left with the sum

$$\sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\mathcal{L}). \quad (6.9)$$

Here, $\text{Area}(\mathcal{L})$ is asymptotic to the area of the parallelogram. That is, if δ is small enough, then we have

$$\begin{aligned} \text{Area}(\mathcal{L}) &\sim 2\delta q \left(\frac{(\alpha + \delta)qQ - q^2}{(\beta - 1)q} - \frac{(\alpha - \delta)qQ - q^2}{(\beta - 1)q} \right) \\ &= 2\delta q \left(\frac{2\delta Q}{\beta - 1} \right) \\ &= \frac{4\delta^2 q Q}{\beta - 1}, \end{aligned} \quad (6.10)$$

as $Q \rightarrow \infty$. Inserting (6.10) into (6.9), we obtain

$$M \sim \frac{4\delta^2 Q}{\beta - 1} \sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q}. \quad (6.11)$$

We estimate the summation in (6.11) by employing the following result from [5].

Lemma 6.4.1 (Lemma 2.3 from [5]). *Suppose that a and b are two real numbers such that $0 < a < b$, $q \in \mathbb{N}^*$ and f is a piecewise C^1 function defined on $[a, b]$. Then we have*

$$\sum_{a < q \leq b} \frac{\phi(q)}{q} f(q) = \frac{1}{\zeta(2)} \int_a^b f(x) dx + O \left(\log b \left(\|f\|_\infty + \int_a^b |f'(x)| dx \right) \right).$$

Applying Lemma 6.4.1, we find that

$$\sum_{(\alpha+\delta)Q/(\beta-\delta) \leq q \leq Q} \frac{\phi(q)}{q} = \frac{1}{\zeta(2)} \int_{(\alpha+\delta)Q/(\beta-\delta)}^Q dt + O(\log Q). \quad (6.12)$$

Then inserting (6.12) into (6.11), we find that

$$\frac{M}{\delta^2 Q^2} \rightarrow \frac{4}{(\beta-1)\zeta(2)} \left(1 - \frac{\alpha}{\beta}\right), \quad (6.13)$$

as $Q \rightarrow \infty$ first and then followed by $\delta \rightarrow 0$.

6.5 The subset $\mathcal{C}_{\alpha,\beta,\delta,Q,x}$

Next, we consider the set of matrices

$$\mathcal{C}_{\alpha,\beta,\delta,Q,x} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}(Q, x) : 1 \leq a, b, d \leq c \leq Q, ad - bc = -1, \right. \\ \left. (\alpha - \delta)Q \leq a + d \leq (\alpha + \delta)Q, (\beta - 1 - \delta)c \leq a \leq (\beta - 1 + \delta)c \right\}.$$

Estimating the cardinality of $\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ in a similar fashion to that in (6.3), we write

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = \sum_{1 \leq c \leq Q} \sum_{\substack{1 \leq d \leq c \\ \gcd(c,d)=1 \\ (\alpha-\delta)Q \leq c-\bar{d}+d \leq (\alpha+\delta)Q \\ (\beta-\delta)c \leq c-\bar{d} \leq (\beta-1+\delta)c}} 1. \quad (6.14)$$

The equality in (6.14) follows by noticing that the conditions $1 \leq a \leq c$ and $ad - bc = -1$ imply that a is to equal $c - \bar{d}$, where $\bar{d}$ is the multiplicative inverse of d modulo c in the interval $[1, c]$. Furthermore, note that in (6.14) the terms for which $c < (\alpha - \delta)Q/2$ have no contribution to the sum. Indeed, the inequality $(\alpha - \delta)Q \leq c - \bar{d} + d$ implies $(\alpha - \delta)Q < 2q$. Hence, setting $q = c$, $x = d$ and $y = \bar{d}$, we obtain $\#\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ in the form

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} \#\{(x, y) \in \Gamma_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2 : xy \equiv 1 \pmod{q}\}, \quad (6.15)$$

where

$$\begin{aligned}\Gamma_{\alpha,\beta,\delta,Q,q} &= \{(u, v) \in \mathbb{R}^2 : 1 \leq u, v \leq q, \\ &\quad (\alpha - \delta)Q - q \leq u - v \leq (\alpha + \delta)Q - q, \\ &\quad (2 - \beta - \delta)q \leq v \leq (2 - \beta + \delta)q\}.\end{aligned}\tag{6.16}$$

Applying Lemma 4.5.1 as before, we obtain

$$\#\{(x, y) \in \Gamma_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2 : xy \equiv 1 \pmod{q}\} = \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha,\beta,\delta,Q,q}) + E'_{\alpha,\beta,\delta,Q,q},\tag{6.17}$$

where

$$E'_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{5/6+\epsilon}).\tag{6.18}$$

Then inserting (6.17) and (6.18) into (6.15), we find that

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = M' + E',\tag{6.19}$$

where

$$M' = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha,\beta,\delta,Q,q})\tag{6.20}$$

and

$$E' = \sum_{(\alpha-\delta)Q/2 \leq q \leq Q} E'_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{11/6+\epsilon}).\tag{6.21}$$

6.6 The main term from $\mathcal{C}_{\alpha,\beta,\delta,Q,x}$

From the definition of the set $\Gamma_{\alpha,\beta,\delta,Q,q}$ in (6.16), we see that

$$\Gamma_{\alpha,\beta,\delta,Q,q} = \mathcal{M} \cap ([1, q] \times [1, q]),$$

where $\mathcal{M}$ is the parallelogram that lies between the slant lines $v = u + q - (\alpha + \delta)Q$ and $v = u + q - (\alpha - \delta)Q$ and the horizontal lines $v = (2 - \beta - \delta)q$ and $v = (2 - \beta + \delta)q$. First, we observe that if $\alpha > \beta$, then for δ small enough all parallelograms $\mathcal{M}$ lie outside the square $[1, q] \times [1, q]$. In this situation, the sets $\Gamma_{\alpha,\beta,\delta,Q,q}$ are empty, so the main term M' is zero.

In what follows, we consider the case when $\alpha < \beta$. If $q < (\alpha - \delta)Q/(\beta + \delta)$, then the parallelograms $\mathcal{M}$

still lie outside the square $[1, q] \times [1, q]$. Hence, we may restrict to the interval $[(\alpha - \delta)Q/(\beta + \delta), Q]$.

Next, if q belongs to the interval $[(\alpha - \delta)Q/(\beta + \delta), (\alpha + \delta)Q/(\beta - \delta)]$, then $\mathcal{M}$ intersects the square $[1, q] \times [1, q]$ but is not entirely contained in it. This implies that $\mathcal{M}$ lies close to the boundary of the square $[1, q] \times [1, q]$, so that all those values of q satisfying this property have negligible contribution to the main term M' .

Thus we may restrict the summation over q to the interval $[(\alpha + \delta)Q/(\beta - \delta), Q]$. For all such values of q , we see that $\mathcal{M}$ is entirely contained in the square $[1, q] \times [1, q]$ and its area is equal to exactly $4\delta^2 qQ$. Hence, the main term in (6.20) is given by

$$M' = \sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q^2} \text{Area}(\Gamma_{\alpha, \beta, \delta, Q, q}) \quad (6.22)$$

$$= 4\delta^2 Q \sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q}. \quad (6.23)$$

Using Lemma 6.4.1, we find that

$$\sum_{(\alpha + \delta)Q/(\beta - \delta) \leq q \leq Q} \frac{\phi(q)}{q} = \frac{Q}{\zeta(2)} \left(1 - \frac{\alpha + \delta}{\beta - \delta} \right) + O(\log q). \quad (6.24)$$

Then inserting (6.24) into (6.22), we see that

$$\frac{M'}{\delta^2 Q^2} \rightarrow \frac{4}{\zeta(2)} \left(1 - \frac{\alpha + \delta}{\beta - \delta} \right), \quad (6.25)$$

as $Q \rightarrow \infty$ first and then followed by $\delta \rightarrow 0$.

6.7 Proof of Theorem 6.2.1: Local density results

On combining the above estimates for $\#\mathcal{D}_{\alpha, \beta, \delta, Q, x}$ and $\#\mathcal{C}_{\alpha, \beta, \delta, Q, x}$ when β is larger than α and recalling that both quantities are zero when β is less than α , we deduce that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{Q \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{\#\mathcal{D}_{\alpha, \beta, \delta, Q, x} + \#\mathcal{C}_{\alpha, \beta, \delta, Q, x}}{\delta^2 Q^2} &= \begin{cases} \frac{4 \left(1 - \frac{\alpha}{\beta} \right)}{(\beta - 1)\zeta(2)} + \frac{4 \left(1 - \frac{\alpha}{\beta} \right)}{\zeta(2)}, & \text{if } \alpha \leq \beta; \\ 0, & \text{if } \alpha < \beta; \end{cases} \\ &= \begin{cases} \frac{4}{\zeta(2)} \left(\frac{\beta - \alpha}{\beta - 1} \right), & \text{if } \alpha \leq \beta; \\ 0, & \text{if } \alpha > \beta. \end{cases} \end{aligned} \quad (6.26)$$

Recall that, by Theorem 4.0.1,

$$M_A(x) = K_A x^{1+(a+b)/(c+d)} + O_A(x^{1/2+(a+b)/(c+d)} \exp\{-c'(\log x)^{3/5}(\log \log x)^{-1/5}\}).$$

Theorem 4.0.1 shows us that

$$\frac{\log M_A(x)}{\log x} \sim 1 + \frac{a+b}{c+d},$$

as $x \rightarrow \infty$. Since

$$\frac{a+b}{c+d} = \frac{a}{c} - \frac{\det(A)}{c(c+d)} = \frac{b}{d} + \frac{\det(A)}{d(c+d)},$$

when $d > c$ we see that

$$\left| \frac{\log M_A(x)}{\log x} - \frac{b}{d} \right| = O\left(\frac{1}{d^2}\right),$$

as $x \rightarrow \infty$. When $c > d$, we have

$$\left| \frac{\log M_A(x)}{\log x} - \frac{a}{c} \right| = O\left(\frac{1}{c^2}\right),$$

as $x \rightarrow \infty$.

We partition $\mathcal{A}(Q, x)$ into two subsets, according to whether $1 \leq \max(c, d) \leq \sqrt{Q}$ or $\max(c, d) > \sqrt{Q}$. There are at most $O(Q^{3/2})$ matrices of the first type, and for the second type we have $O(1/d^2) = O(1/Q)$ and $O(1/c^2) = O(1/Q)$ when $d > c$ and $c > d$, respectively, as $Q \rightarrow \infty$.

We note that the δ in our definitions of $\mathcal{D}_{\alpha, \beta, \delta, Q, x}$ and $\mathcal{C}_{\alpha, \beta, \delta, Q, x}$ should be replaced by an expression of the form $\delta + \delta_E(Q)$, where the function $\delta_E(Q) = O(1/Q)$, but in what follows we let Q tend to infinity before letting δ tend to zero, so in our case we may replace one by the other.

Since $1 + (a+b)/(c+d) < \beta + \delta < 2$, we find that $a < c$, and similarly $b \leq d$. So the conditions $a, b \leq d$ and $a, b \leq c$ in $\mathcal{D}_{\alpha, \beta, \delta, Q, x}$ and $\mathcal{C}_{\alpha, \beta, \delta, Q, x}$ are satisfied. Thus,

$$\lim_{x \rightarrow \infty} \left| \frac{\#\mathcal{D}_{\alpha, \beta, \delta, Q, x} + \#\mathcal{C}_{\alpha, \beta, \delta, Q, x}}{\delta^2 Q^2} - \frac{\#\{A \in \mathcal{A}(Q, x) : \Psi_{Q, x}(A) \in \mathcal{V}_{\alpha, \beta, \delta}\}}{\delta^2 Q^2} \right| = O\left(\frac{1}{\delta^2 \sqrt{Q}}\right)$$

as $Q \rightarrow \infty$. This, in combination with (6.26), completes the proof of the theorem.

Chapter 7

f_A -type maps on the complex plane

So far we have considered only the action via fractional linear transformation of 2×2 matrices on real numbers. This type of action can be easily extended to fractional linear transformations of complex variables z . It is natural to consider how the functions $f_A(n)$ and $f_{A,\lambda}(n)$ behave in this more general setting.

For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and determinant ± 1 , consider the set of complex numbers $z = \tau + i\gamma$ with the property that $|z + \frac{d}{ck}| > \epsilon$ for all positive integers k , provided that $c \neq 0$. Further, consider the fractional linear transformations Az and Akz , where

$$Akz = \frac{akz + b}{ckz + d}.$$

Define $f_A : \mathbb{N} \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$f_A(n, z) = \prod_{p^\alpha \parallel n} p^{A\alpha z}.$$

How does this function behave on average?

In keeping with the methods we have developed, we wish to discover information about regions in the complex plane in which the Dirichlet series

$$F_A(s, z) = \sum_{n=1}^{\infty} \frac{f_A(n, z)}{n^s}$$

has a meromorphic continuation with at least one pole. Let us suppose that $|z| < R$ for some large R . If $c = 0$ then

$$|A\alpha z| = \left| \frac{a\alpha z + b}{d} \right| \leq \alpha|az| + |b| \leq (aR + |b|)\alpha.$$

Note that in particular $|z| > \epsilon$. Then for $c \geq 1$,

$$|A\alpha z| \leq \frac{|a\alpha z + b|}{\epsilon} \leq \frac{|a|\alpha R + |b|}{\epsilon} \leq \left(\frac{|a|R + |b|}{\epsilon} \right) \alpha.$$

Hence $|f_A(n, z)| \leq n^{(|a|R + |b|)/\epsilon}$, and so $F_A(s, z)$ is analytic in the region $\Re(s) > 1 + (|a|R + |b|)/\epsilon$ and so has an Euler product in that region given by

$$F_A(s, z) = \prod_p \left(1 + \frac{p^{Az}}{p^s} + \frac{p^{A2z}}{p^{2s}} + \frac{p^{A3z}}{p^{3s}} + \dots \right).$$

7.1 Regions of meromorphic continuation

We wish to factor out terms of the form $\left(1 + \frac{p^{Aks}}{p^{ks}}\right)$ in the product form of the function $F_A(s, z)$, in which case we can write

$$F_A(s, z) = \zeta(ks - Akz)G_A(s, z)$$

for some function $G_A(s, z)$ that is not necessarily analytic. This gives a singularity at $s = \frac{1}{k}(1 + Akz)$. If we can show that this singularity is a pole rather than an essential singularity, and further identify the value of k such that this pole is the furthest to the right of all such poles (if such a k exists), then we can guarantee that there is a half plane that includes the pole at $s = \frac{1}{k}(1 + Akz)$ and on which the function $G_A(s, z)$ is analytic.

Consider for $x \geq 1$ the function

$$\theta^*(x, z) := \frac{1}{x} (1 + \Re(Axz))$$

which is comparable to the function $\theta(x, \lambda)$ from Chapter 5.

If $c = 0$ then

$$\theta^*(x, z) = \frac{1}{x} \left(1 + \frac{b}{d} \right) + \frac{a\tau}{d}.$$

This function is decreasing for $x \geq 1$ provided that $b > -d$. If we take a to be non-negative, then the condition $b > -d$ is equivalent to $b \geq 0$ if $\det A = 1$ and to $b \geq 2$ if $\det A = -1$. Note that in order to apply this to matrices of the form

$$A = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}$$

and to the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

associated to the function $R(n)$ in particular, one must introduce λ -moments as in Chapter 5.

If $c \neq 0$, write

$$\begin{aligned} Akz &= \frac{akz + b}{ckz + d} \cdot \frac{ck\bar{z} + d}{ck\bar{z} + d} \\ &= \frac{ac|z|^2k^2 + (adz + bc\bar{z})k + bd}{c^2|z|^2k^2 + 2cd\tau k + d^2} \end{aligned}$$

so that

$$\Re(Akz) = \frac{ac|z|^2k^2 + (ad + bc)\tau k + bd}{c^2|z|^2k^2 + 2cd\tau k + d^2}.$$

A short Mathematica calculation shows that

$$\frac{\partial}{\partial x}\theta^*(x, z) = -\frac{1}{x^2} + \frac{(bc + ad)\tau + 2acx|z|^2}{d^2x + 2cd\tau x^2 + c^2x^3|z|^2} - \frac{(bd + (bc + ad)\tau x + acx^2|z|^2)(d^2 + 4cd\tau x + 3c^2x^2|z|^2)}{(d^2x + 2cd\tau x^2 + c^2x^3|z|^2)^2}.$$

Note that, as x tends to infinity,

$$\frac{(bc + ad)\tau + 2acx|z|^2}{d^2x + 2cd\tau x^2 + c^2x^3|z|^2} \sim \frac{2a}{cx^2}$$

and

$$\frac{(bd + (bc + ad)\tau x + acx^2|z|^2)(d^2 + 4cd\tau x + 3c^2x^2|z|^2)}{(d^2x + 2cd\tau x^2 + c^2x^3|z|^2)^2} \sim \frac{3a}{cx^2}$$

hence

$$\frac{\partial}{\partial x}\theta^*(x, z) \sim -\frac{1}{x^2} \left(1 + \frac{a}{c}\right).$$

So this derivative is eventually negative provided that $-a < c$. For other cases, one could introduce λ -moments as in Chapter 5.

This shows that there do exist matrices A and complex numbers z with non-zero imaginary parts for which $F_A(s, z)$ has a meromorphic continuation with at least one pole. In such cases, we identify for fixed z the values of x for which

$$\sup_{x \geq 1} \theta^*(x, z)$$

is attained at x_j for $1 \leq j \leq M$. We further consider $\lfloor x_j \rfloor$ and $\lceil x_j \rceil$ for each $j = 1, 2, \dots, M$, and take as k

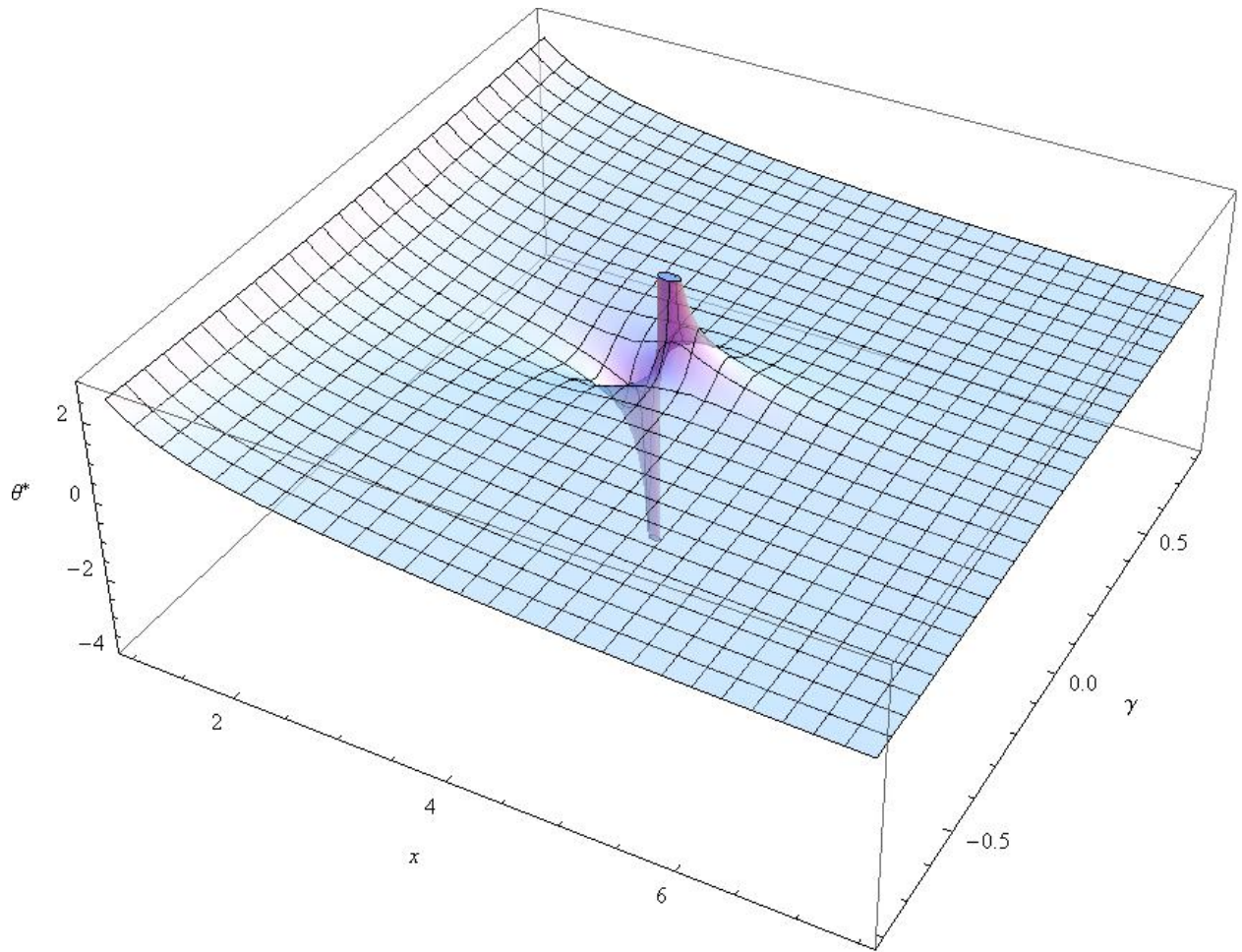


Figure 7.1: The surface $\theta^*(x, -1/2 + i\gamma)$ for $A^{(4)}$

whichever of these gives the larger value of

$$\frac{1}{k} (1 + \Re(Akz)),$$

In some cases more than one of the values $\lfloor x_j \rfloor$ and $\lceil x_j \rceil$ maximize the above quantity simultaneously. In this situation we obtain up to $2M$ different values of k and we obtain either a pole of $F_A(s, z)$ of order larger than 1, or multiple poles with the same real part and different imaginary parts.

7.2 Examples

Consider the matrices

$$A^{(4)} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad A^{(3)} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}.$$

The matrix $A^{(4)}$ satisfies $-a < c$ so we expect $\sup_{x \geq 1} \theta^*(x, z)$ to be attained at a finite (though not necessarily unique) value of x . The function $\theta^*(x, z)$ has poles at points (x, z) for which $xz = -2$. If we fix $\tau = -1/2$, say, one can graph θ^* as a function of x and γ as shown in Figure 7.1.

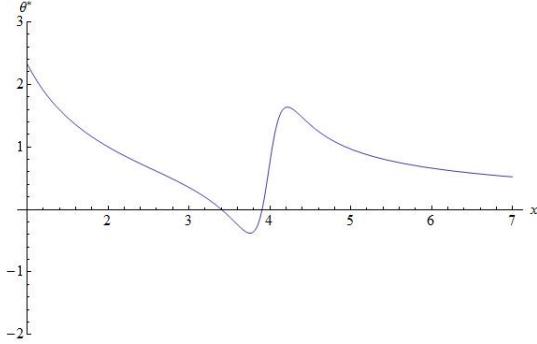


Figure 7.2: $\theta^*(x, -1/2 + 0.03i)$ for $A^{(4)}$

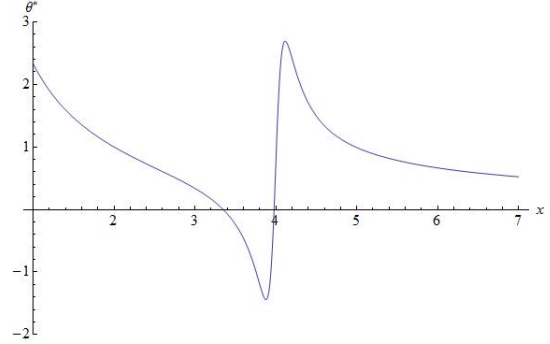


Figure 7.3: $\theta^*(x, -1/2 + 0.015i)$ for $A^{(4)}$

When $\tau = -1/2$ the function has a pole at $(x, \gamma) = (4, 0)$. For $|\gamma|$ much larger than 0 the function is maximal at $x = 1$, but as γ approaches 0 the function grows in absolute value near $x = 4$. This behavior is shown in Figures 7.2 and 7.3. The function is maximal at $x = 1$ when $\gamma = 0.03$ and at $x = 4$ when $\gamma = 0.015$. We note that $|\gamma|$ is bounded away from 0 when $\tau = -1/2$ by assumption. If we fix $\gamma > 0$ and allow τ to vary, we can examine the behavior of $\theta^*(x, z)$ with no interference from possible poles. Figure 7.4 shows the behavior of $\theta^*(x, \tau + i0.015)$ for $A^{(4)}$.

The matrix $A^{(3)}$ does not satisfy $-a < c$, so the function $\theta^*(x, z)$ may or may not attain its maximal value for a finite x . The function $\theta^*(x, z)$ has poles at points (x, z) for which $xz = -3/2$. If we fix $\tau = -1/2$, say, one can graph θ^* as a function of x and γ as shown in Figure 7.5.

When $\tau = -1/2$ the function has a pole at $(x, \gamma) = (3, 0)$. As in the previous example the function grows in absolute value near $x = 3$ as γ approaches 0. For $|\gamma|$ much larger than 0 the function increases monotonically to 0 as $x \rightarrow \infty$. This behavior is shown in Figures 7.6 and 7.7.

The function is maximal at $x \approx 3$ when $\gamma = 0.02$, but while it has a local maximum near $x = 3$ when $\gamma = 0.09$, that maximal value is less than the limit of the function as $x \rightarrow \infty$. We note again that $|\gamma|$ is bounded away from 0 when $\tau = -1/2$ by assumption. If we fix $\gamma > 0$ and allow τ to vary, we can examine the behavior of $\theta^*(x, z)$ with no interference from possible poles. Figure 7.8 shows the behavior of $\theta^*(x, \tau + i0.015)$ for $A^{(3)}$.

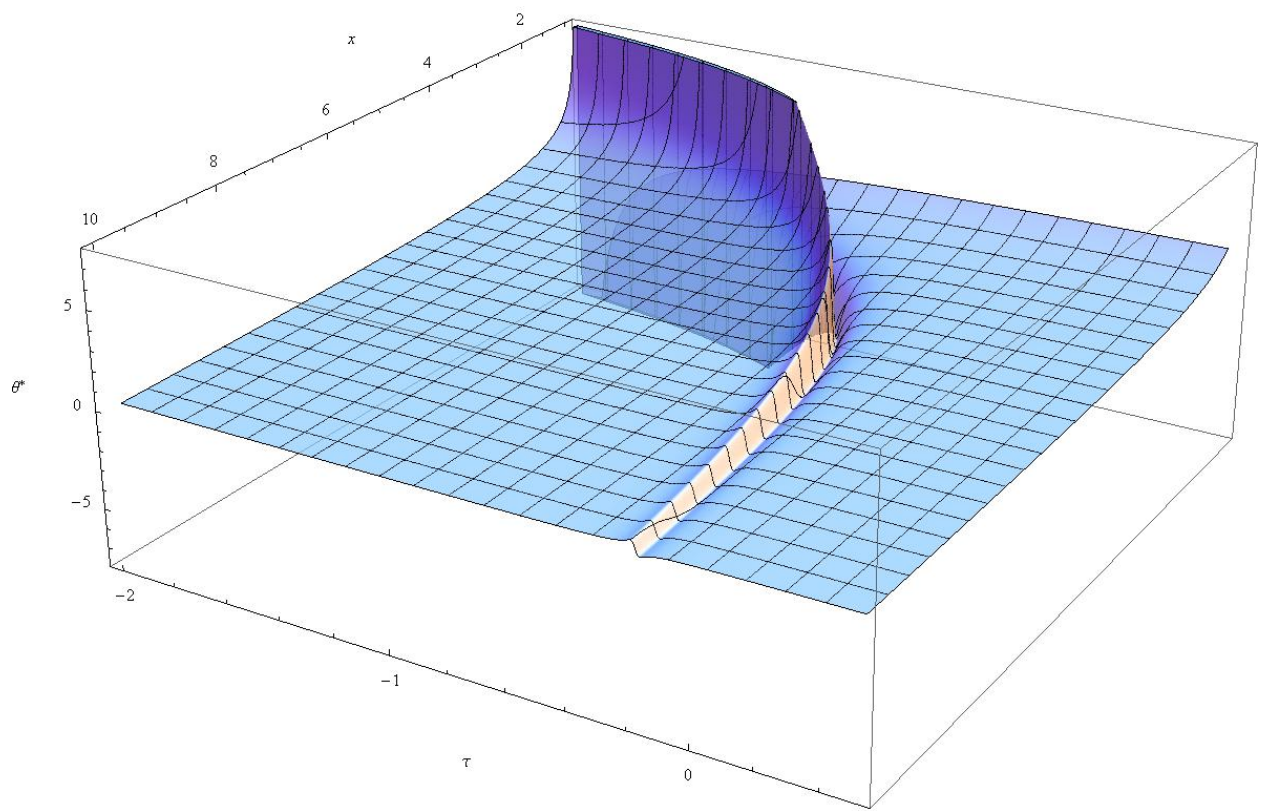


Figure 7.4: The surface $\theta^*(x, \tau + i0.015)$ for $A^{(4)}$

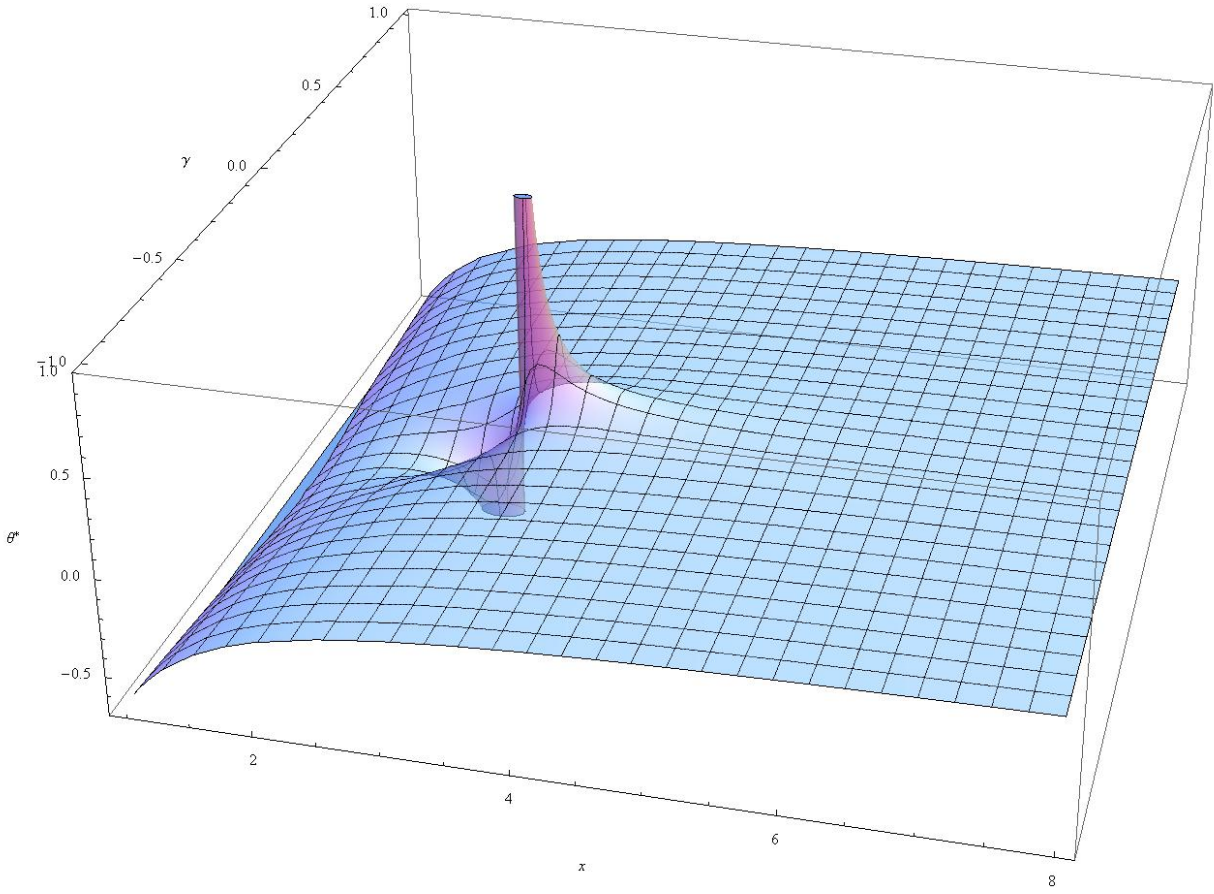


Figure 7.5: The surface $\theta^*(x, -1/2 + i\gamma)$ for $A^{(3)}$

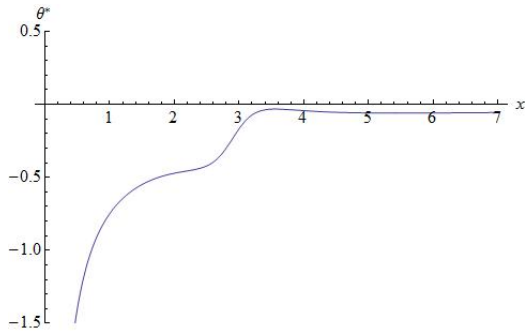


Figure 7.6: $\theta^*(x, -1/2 + 0.09i)$ for $A^{(3)}$

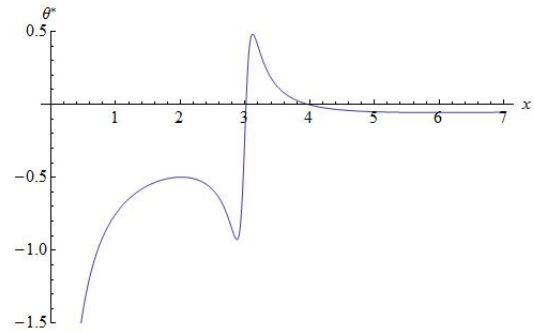


Figure 7.7: $\theta^*(x, -1/2 + 0.02i)$ for $A^{(3)}$

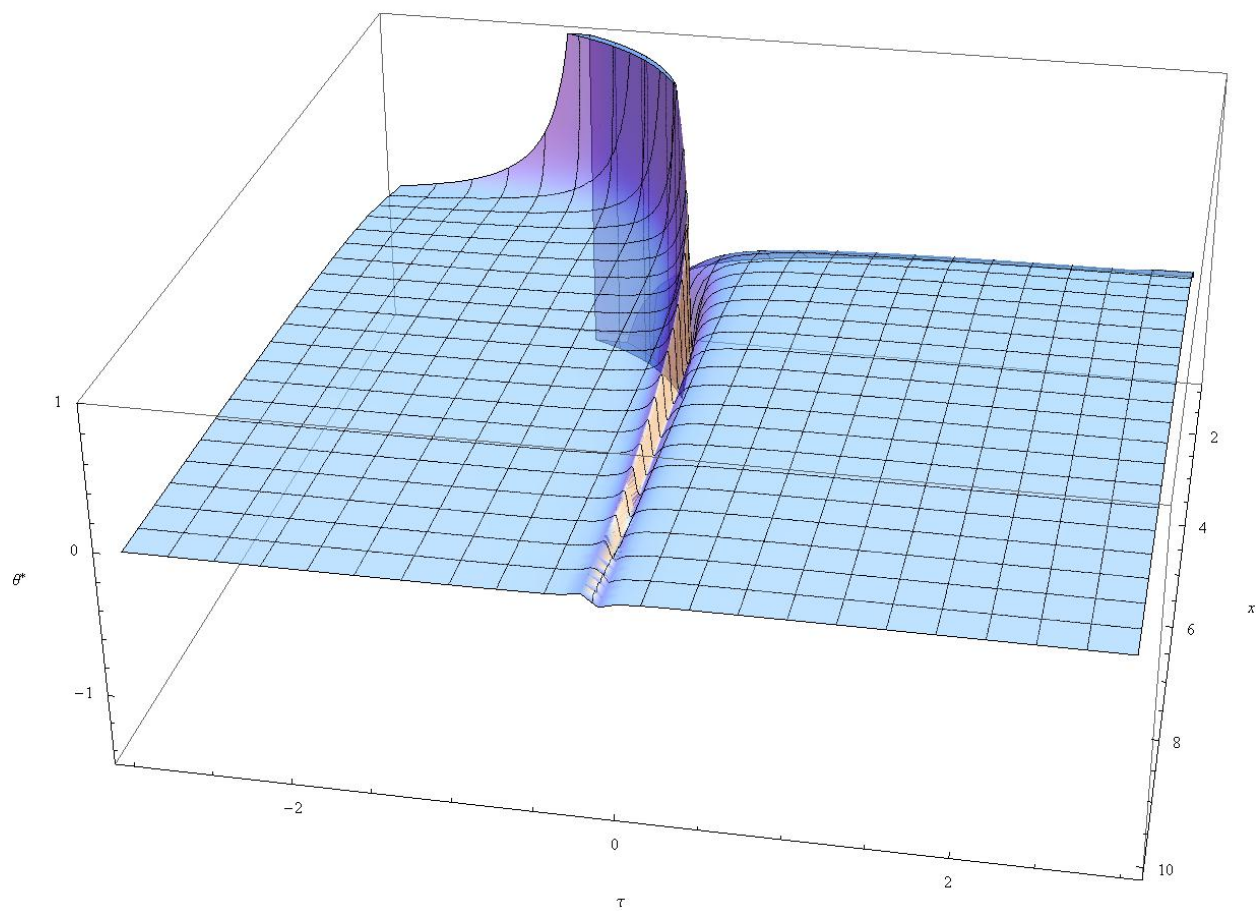


Figure 7.8: The surface $\theta^*(x, \tau + i0.02)$ for $A^{(3)}$

7.3 Open questions: The regions $R_A(k, z)$

Note that if we identify k for a particular A and z , then also this value of k will give us a pole of $F_A(s, \omega)$, where $|z - \omega| < \delta$ for some small $\delta > 0$.

Let

$$R_A(k, z) = \left\{ \omega \in \mathbb{C} : F_A(s, \omega) \text{ has rightmost pole at } s = \frac{1}{k} (1 + Akz) \right\}.$$

Since the function $\frac{1}{x} (1 + \Re(Axz))$ is continuous in both x and z for $x \geq 1$ and z not purely real, there should be some regions $R_A(k)$ and $R_A(j)$ that have a nonempty intersection (although the interior of this intersection will be empty). Values of ω in this intersection correspond to double poles of the function F_A if $k = j + 1$.

If we take all values of k for which $F_A(s, z)$ has a meromorphic continuation for *some* complex number z , then those values of z necessarily lie in the union of regions

$$R_A = \bigcup_k R_A(k).$$

How many of these regions $R_A(k)$ touch each other? Is the entire region R_A connected? Is it possible that R_A is composed of two connected domains that don't intersect? Of five nonintersecting connected domains?

7.4 Open questions: Fixed points

We say that a complex number z_0 is a fixed point of the transformation $z \rightarrow Az$ if

$$Az_0 = \frac{az_0 + b}{cz_0 + d} = z_0. \tag{7.1}$$

We follow the convention that $A_\infty = a/c$ if $c \neq 0$ and $A_\infty = \infty$ if $c = 0$.

There is a clear relationship between the fixed points of the map induced by a particular matrix A and the notion of $f_A(n, z)$ being a measure of how close n is to being k -power full or k -power free. Given a matrix A and a non-infinite fixed point z_0 , for a given k the map

$$n \mapsto f_A(n, z_0/k)^{k/z_0}$$

is the identity map provided that n is a k -th power. It is natural to explore the properties of this relationship.

If $c = 0$ then ∞ is a fixed point. Note that in this case

$$Az_0 = \frac{a}{d}z_0 + \frac{b}{d}$$

so also $z_0 = \frac{b}{d-a}$ is a fixed point. If $\det A = 1$ then $a = d$ so we take $z_0 = \infty$, and if $\det A = -1$ then $a = -d$ and $z_0 = -b/2$. On the other hand, if $c \neq 0$ then by multiplying equation (7.1) by $cz_0 + d$ and solving the resulting quadratic equation we find that a fixed point z_0 must satisfy

$$z_0 = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

This is non-real when $(a-d)^2 + 4bc < 0$. A short calculation shows that this is equivalent to $\text{tr}(A)^2 < 4 \det A$.

This is impossible when $\det A = -1$, and is equivalent to $|\text{tr}(A)| < 2$ when $\det A = 1$.

An exposition on these fixed points can be found in [1]. Given a 2×2 matrix with real entries, a transformation for which $\text{tr}(A)^2 < 4$ is said to be *elliptic*. A non-identity transformation for which $\text{tr}(A)^2 = 4$ is said to be *parabolic* (in which case the transformation has a single real fixed point), and a transformation for which $\text{tr}(A)^2 > 4$ is said to be *hyperbolic* (in which case the transformation has two real fixed points).

Does each k have an ‘optimal’ choice of matrix A ? How does the choice of a matrix A affect the properties of the map

$$n \mapsto f_A(n, z_0/k)^{z_0/k}$$

when the transformation associated to A is elliptic, parabolic, or hyperbolic?

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