DEFORMATIONS OF THE HILBERT SCHEME OF POINTS
ON A DEL PEZZO SURFACE

BY

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DISSERTATION

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Abstract

The Hilbert scheme of n points in a smooth del Pezzo surface S parameterizes zero-dimensional subschemes with length n on S. We construct a flat family of deformations of $\mathrm{Hilb}^n S$ which can be conceptually understood as the family of Hilbert schemes of points on a family of noncommutative deformations of S. Further we show that each deformed $\mathrm{Hilb}^n S$ carries a generically symplectic holomorphic Poisson structure. Moreover, the generic deformation of $\mathrm{Hilb}^n S$ has a $(k+2)$-dimensional moduli space, where the del Pezzo surface is the blow up of projective plane at k sufficiently general points; and each of the fibers is of the form that we construct. Our work generalizes results of Nevins-Stafford constructing deformations of the Hilbert scheme of points on the plane, and of Hitchin studying those deformations from the viewpoint of Poisson geometry.

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Chapter 1

Introduction

1.1 Overview

Given a space X, the configuration space that parameterizes the collections of n points on X is given by the n-th symmetric product of the space X. In algebraic geometry, X is usually a smooth projective variety. When the dimension of X is greater than 1, this classical configuration space can be very singular in the algebraic sense. One modification for this configuration space is to keep track of the information of how these points collide. The information for 2 colliding points not only contains the position of the point, but also a tangent direction. This idea is highly successful at least when X is a surface.

To be specific, one may define the space $\mathrm{Hilb}^n X$ that parameterizes zero-dimensional subschemes with length n on X. Grothendieck's foundational work in [14] ensures that this configuration space can be realized as a reasonable and nice model in algebraic geometry. $\mathrm{Hilb}^n X$ is actually a projective variety, which is frequently not the case if one parameterizes some more general objects. Although $\mathrm{Hilb}^n X$ may still have singularities when the dimension of X and n are large numbers, when X is a smooth surface, $\mathrm{Hilb}^n X$ is smooth and irreducible by the milestone work [6] by Fogarty.

The question that I address in the thesis is how can we construct deformations for $\mathrm{Hilb}^n X$. This question is wildly unknown when the dimension of X is greater than or equal to 3. We first of all focus on the case that X is a surface S. The surface case has been investigated in much literature —

Bogomolov, Bottacin [4], Fantechi [5], Goto [11], Hitchin [16], Nevins, Stafford [25], etc. Given a flat family of irreducible smooth surfaces, there is a flat family of Hilbert schemes of points on each surface. One natural question is whether all deformations of $\mathrm{Hilb}^n S$ are induced by those of the surface. When the surface S is of general type, it has been shown in [5] that the deformation of $\mathrm{Hilb}^n S$ is always induced by the deformation of S. The case that S has trivial canonical divisor (K3 surface case) has been studied by Beauville [2] and Fujiki [8] in order to construct more examples of higher-dimensional symplectic manifolds; in their case, the Hilbert scheme has a deformation which is not induced from S: the coarse moduli space for marked K3 surfaces has dimension 20, while that for the Hilbert scheme has dimension 21.

By the classification result of algebraic surfaces, the remaining case includes the projective space $\mathbf{P}^2$, del Pezzo surfaces, ruled surfaces, Abelian surfaces, Enriques surfaces, elliptic surfaces and hyperelliptic surfaces. The most interesting case is when the surface itself is rigid, which means S has no deformation itself. In [16], Hitchin showed that in this case every deformation of the Hilbert scheme is obtained from a holomorphic Poisson structure on the Hilbert scheme that is induced from a holomorphic Poisson structure on S. Each Poisson structure induces a Kodaira-Spencer class which can be integrated to a one-parameter family of deformations. In particular, in the $\mathbf{P}^2$ case, where the Poisson structure is determined by its vanishing locus which is a cubic curve, Hitchin shows that the generic deformation has a two-dimensional local moduli space, which is parametrized by the modulus of the cubic curve and a degree 3 line bundle on the curve. This rephrases the results in [25] by Nevins and Stafford. In their work, the deformation is induced from the deformation of $\mathbf{P}^2$ as a noncommutative surface (the Hilbert scheme of points on such a noncommutative surface is commutative!). They construct the deformation as the moduli space of graded right modules (quotient by right bounded ones) with rank 1, trivial c_1 and $\chi = 1 - n$ over a Skylanin algebra.

At the end we focus on the Hilbert schemes of del Pezzo surfaces. By Hitchin's approach from Poisson deformation or general deformation theory, there is an upper bound estimate for the generic dimension which is $11 - d$, i.e., 2 plus the number of blown up points. However, we cannot get more than 2 explicit parameters by this approach. That is the reason why we try the method with inspiration from non-commutative geometry to construct the deformations from non-commutative deformations. From this non-commutative style approach, the first obstacle is that in general we do not know how to make the non-commutative construction on the ring level. However, as in [1] and [29], we can talk about the bounded derived category of sheaves on a non-commutative del Pezzo surface. These categories are a deformed version of the derived category of sheaves on commutative del Pezzo surfaces, and each of them is generated by an exceptional collection and has a semiorthogonal decomposition, hence we can use the Beilinson type spectral sequence to reconstruct objects there. On the other hand $\mathrm{Hilb}^n S$ parameterizes those stable objects with some fixed invariants, and we may construct $\mathrm{Hilb}^n S$ via geometric invariant theory.

Now since the deformation space is not constructed as a Hilbert scheme of modules over rings, we can establish few properties for the deformed spaces outside of the commutative case. However, by considering these spaces in a family $\mathfrak{M}_{\mathfrak{B}} \to \mathfrak{B}$ (where $\mathfrak{B}$ contains the parameter of the datum of non-commutative del Pezzo surfaces including an elliptic curve E, two degree 3 line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ (embedding and translation) and k blowing-up points on E), we show that this morphism is proper and smooth, in other words we get a deformation and each fiber shares the same good properties as the $\mathrm{Hilb}^n S$ in commutative case.

In the next few sections of this chapter, we review some classical results about del Pezzo surfaces and the Hilbert scheme of points on surfaces. In Chapter 2, we introduce the derived category of sheaves on del Pezzo surfaces and non-commutative del Pezzo surfaces from [1]. In addition, we give a description of the Hilbert scheme of a commutative del Pezzo surface via GIT.

Theorem 1.1.1 (Proposition 2.5.5)**.** *$Hilb^n S \simeq MK^{ss}(n) \mathbin{/\!/} (G/\mathbb{C}^\times)$. Here we write $MK^{ss}(n)$ for the moduli space of framed semistable K-complexes with type $(n, 2n+1, n+1, \ldots, n+1, n)$ (see*

Definition 2.4.1). G is a product of general linear groups and $G/\mathbb{C}^\times$ acts freely on $MK^{ss}(n)$.

K-complexes are defined in Section 2.4. In Chapter 3, we set up the construction of deformations spaces. To be specific, given a family datum (μ_A, p_i) of non-commutative del Pezzo surfaces, the family of their Hilbert scheme of n points is constructed as $\mathcal{M}^s_{\mu_A,p_i}(n) \to \mathrm{Spec}A$. We then prove some properties of the morphism $\mathcal{M}^s_{\mu_A,p_i}(n) \to \mathrm{Spec}A$ and get our main technical result.

Theorem 1.1.2 (Theorem 3.4.3)**.** *Let A be a noetherian ring such that* $\mathrm{Spec}A$ *is a smooth curve over $\mathbb{C}$ and (μ_A, p_i) be a flat family of noncommutative del Pezzo surfaces with one fiber being the commutative S (see Definition 2.3.1). Then $\mathcal{M}^s_{\mu_A,p_i}(n) \to \mathrm{Spec}A$ is a smooth family of deformations of $Hilb^n S$.*

The main part of the proof is on the smoothness, which is contained in the section 3.4. To show the smoothness, we translate the problem of whether the Jacobian matrix has full rank to the exactness of global sections of a complex of sheaves on an elliptic curve. As a byproduct, when the degree of the del Pezzo surface is 8, 2 or 1, $\mathrm{Hilb}^n S$ is described as the moduli space parameterizing some specific tuples of vectors bundles on an elliptic curve and maps between them. In Chapter 4, we construct the natural Poisson structure on the deformed Hilbert schemes of del Pezzo surfaces. Then by the results in [16], we may describe the Kodaira-Spencer classes tangent to each direction of the deformation of $\mathrm{Hilb}^n S$, and compute the generic dimension of versal deformation base space of $\mathrm{Hilb}^n S$ showing that the base space of the family that we construct has the same dimension.

Theorem 1.1.3 (Proposition 4.1.2)**.** *Suppose the degree of the del Pezzo surface is 8, 2 or 1, and let $(\mu_{\mathbb{C}}, p_i)$ be a data of noncommutative del Pezzo surface (see Definition 2.2.1). Then $\mathcal{M}^s_{\mu,p_i}(n)$ admits a natural Poisson structure which is generically symplectic. The generic deformation of $Hilb^n S$ has a $(2+k)$-dimensional space of moduli and each of them is of the form $\mathcal{M}^s_{\mu_A,p_i}(n)$ depending on a smooth elliptic curve E, k blown up points on it and two degree 3 line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ (accurately speaking, the difference of them $\mathcal{L}_2^{-1} \otimes \mathcal{L}_1$).*

1.2 del Pezzo surfaces

In this section, we review the basic facts of del Pezzo surfaces. We always assume that the scheme/variety is over the complex number field $\mathbb{C}$.

Definition 1.2.1. *A* Fano variety *is a complete variety whose anticanonical bundle is ample. A* del Pezzo *surface is a two-dimensional Fano variety. In the thesis, when we talk about a del Pezzo surface, we always assume it is smooth.*

Let K be the canonical bundle of a del Pezzo surface. The *degree* of the surface is defined to be the intersection number K^2. We denote del Pezzo surface with degree n by D_n.

Example 1.2.2. *Here are examples of del Pezzo surfaces.*

1. *The projective plane* $\boldsymbol{P}^2$ *has canonical sheaf* $O(-3)$. *Up to isomorphism, it is the unique del Pezzo surface with degree* 9.

2. *The Hirzebruch surface* Σ_0, *which is isomorphic to* $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, *has canonical sheaf* $O(-2,-2)$. *The Hirzebruch* Σ_1 *is isomorphic to* $\boldsymbol{P}^2$ *blown up at a point. Fix a section* C_0 *lifted from* $\boldsymbol{P}^1$ *and a fiber* f, Σ_1 *has canonical divisor* $-2C_0 - 3f$. *Both surfaces have degree* 8.

3. *The smooth cubic surface* D_3 *in* $\boldsymbol{P}^3$ *has canonical divisor* $O_{D_3}(-1)$. *It has degree* 3 *and is isomorphic to* $\boldsymbol{P}^2$ *blown up at six points in general position.*

4. *The complete intersection of two quadrics in* $\boldsymbol{P}^4$ *has canonical divisor* $O_{D_4}(-1)$. *It has degree* 4 *and is isomorphic to* $\boldsymbol{P}^2$ *blown up at five points in general position.*

By the classification result, except for $\mathbf{P}^1 \times \mathbf{P}^1$, del Pezzo surfaces can be realized as $\mathbf{P}^2$ blown-up at $9-n$ points. To be a del Pezzo surface, the blown-up points are on general positions: no three lie on a line, no six lie on a conic and no eight of them lie on a cubic having a node at one of them.

Let X be a normal projective variety; we denote by $\mathrm{Pic}(X)$ the Picard group. We consider $\mathrm{Pic}_{\mathbb{Q}}(X)(=\mathrm{Pic}(X)\otimes\mathbb{Q})$ to be the $\mathbb{Q}$-Cartier divisors. We denote by $N^1(X)_{\mathbb{Q}}$ the $\mathbb{Q}$-Néron-Severì space, i.e., the quotient of $\mathrm{Pic}_{\mathbb{Q}}(X)$ by the numerically trivial $\mathbb{Q}$-divisors. We denote by $N_1(X)_{\mathbb{Q}}$ the quotient group of $\mathbb{Q}$-1-cycles on X by the subgroup of numerically trivial $\mathbb{R}$-1-cycles. A divisor $D \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is *nef* if $(D \cdot C) \geq 0$ for every irreducible curve $C \subset X$.

The Picard group of $\mathbf{P}^1 \times \mathbf{P}^1$ is generated by the classes $[\mathcal{O}(1,0)]$ and $[\mathcal{O}(0,1)]$. In the other general cases, let ℓ_0 be a curve on S which is the proper transform of a projective line on $\mathbf{P}^2$ not through any blown-up point; let ℓ_i, $1 \leq i \leq 9-n$, be the other exceptional lines. Then the Picard group of D_n is generated by the classes $[\mathcal{O}(\ell_0)]$, $[\mathcal{O}(\ell_1)]$, $\ldots$ $[\mathcal{O}(\ell_{9-n})]$. We may also denote ℓ_i as the generator $[\mathcal{O}(\ell_i)]$ of $\mathrm{Pic}(S)$ when it causes no confusion. The intersection numbers of the pairs are

$$(\ell_0)^2 = 1,\ \ell_0 \cdot \ell_i = 0,\ \ell_i \cdot \ell_j = -\delta_{ij}, \text{ for any } 1 \leq i, j \leq 9-n.$$

The class of the canonical divisor K is $-3\ell_0 + \ell_1 + \cdots + \ell_{9-n}$.

Notation 1.2.3. *We consider the following cones.*

- *Nef*$(X) \subset N^1(X)_{\mathbb{R}}$*: the closed cone of nef divisors;*
- *Eff*$(X) \subset N^1(X)_{\mathbb{R}}$*: the closed cone of effective divisors;*
- *NE*$(X) \subset N_1(X)_{\mathbb{R}}$*: the closed cone of effective 1-cycles.*

A reference for these terminologies is [20]. By Kleiman's criterion $\mathrm{Nef}(X)$ and $\mathrm{NE}(X)$ are dual to each other. When X is a smooth surface, $\mathrm{Eff}(X) \subset N^1(X)_{\mathbb{R}}$ and $\mathrm{NE}(X) \subset N_1(X)_{\mathbb{R}}$ are identified together.

Lemma 1.2.4. *When* $7 \geq n \geq 3$, *Eff*(D_n) *is a convex polyhedral cone with all extremal rays spanned by* ℓ_a, $\ell_0 - \ell_a - \ell_b$, *and* $2\ell_0 - \ell_a - \ell_b - \ell_c - \ell_d - \ell_e$. *Here* a, b, c, d, e *are distinct and greater than* 0, *whenever these classes exist.*
When $n = 2$, *there are* 7 *more extremal rays spanned by* $-K - \ell_a$.
When $n = 1$, *there are* 4 *more sets of extremal rays:*

- $-K - \ell_a + \ell_b$; $-K + \ell_0 - \ell_a - \ell_b - \ell_c$; $-2K - \ell_0 + \ell_a + \ell_b$; $-2K - \ell_a$.

Proof. Each divisor class listed above has self-intersection number -1. Each divisor can be represented by the strict transform of certain curve on $\mathbf{P}^2$ across some blown up points with certain multiplicities. By Lemma 1.22 in [18], each of them spans an extremal ray for $\mathrm{Eff}(D_n)$.

To see that all the extremal rays are listed in the lemma, let C be a nonsingular irreducible curve other than the (-1)-curves listed in the lemma. Since $C \cdot \ell_i \in \mathbb{Z}_{\geq 0}$, for $0 \leq i \leq 9-n$, the divisor class of C can be written as

$$a_0\ell_0 - a_1\ell_1 - a_2\ell_2 - \cdots - a_{9-n}\ell_{9-n},$$

where all a_i's are non-negative integers and in addition that $a_0 > 0$. By the adjunction formula (page 361 in [15]), we have

$$C \cdot C \geq -1.$$

By Corollary 1.21 in [18], if the rank of the Picard group is not 1 and C spans an extremal ray, then $C \cdot C$ is either -1 or 0. Since $C \cdot L \geq 0$ for any L listed in the lemma, we have the following constrains for the pre-extremal class $a_0\ell_0 - a_1\ell_1 - a_2\ell_2 - \cdots - a_{9-n}\ell_{9-n}$.

$$\begin{aligned} a_i &\geq 0; \\ a_0^2 - (a_1^2 + \cdots + a_{9-n}^2) &= 0 \text{ or } -1; \\ a_0 &\geq a_i + a_j; \\ 2a_0 &\geq a_i + a_j + a_k + a_l + a_m; \\ &\text{other constrains when } n \leq 2. \end{aligned}$$

By an elementary argument and a case-by-case checking, the self-intersection number of C must be 0 and $[C]$ is spanned by the exceptional divisors. The details are in the appendix. □

Example 1.2.5. *The total number of* (-1)*-curves is obtained by computing the extremal rays in Lemma 1.2.4.* D_7 *has 3* (-1)*-curves.* D_6 *has 6* (-1)*-curves.* D_5 *has 10* (-1)*-curves.* D_4 *has 16* (-1)*-curves.* D_3 *has 27* (-1)*-curves.* D_2 *has 56* (-1)*-curves.* D_1 *has 240* (-1)*-curves.*

The following example from [24] says that the effective cone can be terrible when we blow up more than 8 points on $\mathbf{P}^2$.

Example 1.2.6. *Consider a pencil of cubic curves; the base locus has nine points. Let* D_0 *be* $\boldsymbol{P}^2$ *blown up at the nine points. For any point on* D_0*, there is a unique cubic curve in the pencil passing through it. This induces a map by mapping each point on* D_0 *to the pencil* $\boldsymbol{P}^1$*. The map is actually a morphism with fibers of cubic curves. Fix one of the nine exceptional lines as the zero section, the other sections on cubic curve fibers generate an infinite group of actions on* X *along the fibers.* X *has infinitely many* (-1)*-curves. Again by Lemma 1.22 in [18], all of the curves span an extremal ray of* $Eff(X)$*. The details are in [24].*

1.3 Hilbert scheme of points

Hilbert scheme

Let X be a quasi-projective scheme over $\mathbb{C}$, H be an ample divisor. For any closed subscheme $Z \subset X$, which is proper over $\mathbb{C}$, the Hilbert polynomial of Z is defined in the usual sense:

$$P_Z(n) := \chi(O_Z \otimes O_X(nH)).$$

Given a polynomial P, the Hilbert scheme functor is defined as following.

$$\begin{aligned} \mathrm{Hilb}^P(X) : \mathrm{Schemes}^{op} &\to \mathrm{Sets} \\ S &\mapsto \{\mathscr{Z} \subset S \times X | \mathscr{Z} \text{ proper and flat}/S, P_{\mathscr{Z}_s} = P \text{ for any fiber } s \in S\}. \end{aligned}$$

$\mathrm{Hilb}^P(X)$ is a well-defined contravariant functor: for any morphism $f : T \to S$, the family $\mathscr{Z}_T := (f \times id_X)^{-1}(\mathscr{Z})$ is flat and proper over T.

Theorem 1.3.1 (Grothendieck [14]). *The functor $Hilb^P(X)$ is represented by a quasiprojective scheme. If X is projective, then $Hilb^P(X)$ is also projective.*

In particular, fixing the constant Hilbert polynomial $P \equiv n$, the scheme $\mathrm{Hilb}^n(X)$ parameterizes all zero-dimensional subschemes of length n.

Example 1.3.2. *We consider the zero-dimensional subscheme support at one point on a surface. We may assume that the surface is $Spec\mathbb{C}[x,y]$ and the subscheme is supported at $(0,0)$. These objects are parameterized by the subscheme $Hilb^n_0(\mathbb{C}^2)$.*

1. $n = 1$: the ideal is $\langle x, y\rangle$.

2. $n = 2$: $\langle x^2, ax + by, y^2\rangle$. (a,b) is a nonzero pair of complex numbers. Two subschemes are isomorphic to each other when and only when $(a : b)$'s are the same. The information of a zero-dimensional subscheme with length 2 contains the information of a point and the tangent direction $ax + by = 0$.

3. $n = 3$: $\langle ax + by + cx^2 + dxy + ey^2, x^3, x^2y, xy^2, y^3\rangle$ and $\langle x^2, xy, y^2\rangle$. (a,b) is a nonzero pair of complex numbers.

There is a $\mathbb{C}^\times\times\mathbb{C}^\times$ action on $\mathrm{Hilb}^n_0(\mathbb{C}^2)$: (a,b) maps each ideal $\langle p_1(x,y), \dots, p_k(x,y)\rangle$ to $\langle p_1(ax,by), \dots, p_k(ax,by)\rangle$. There are finitely many fixed ideals by the action. Each fixed ideal can be written in the form $\langle x^{\alpha_0}, x^{\alpha_1}y^{\beta_k}, \dots, x^{\alpha_k}y^{\beta_1}, y^{\beta_0}\rangle$, where $\alpha_0 > \alpha_1 > \cdots > \alpha_k > 0$ and $\beta_0 > \cdots > \beta_k > 0$. There is a one-to-one correspondence between the partitions λ of n and fixed ideals of the $\mathbb{C}^\times \times \mathbb{C}^\times$ action.

$$\lambda = (\lambda_1, \dots, \lambda_k) \leftrightarrow \langle x^{\lambda_1}, x^{\lambda_2}y^1, \dots, x^{\lambda_k}y^{k-1}, y^k\rangle.$$

Example 1.3.3 (Monad resolution for quotients). *A free resolution for the quotient $\mathbb{C}[x,y]/(x,y)$ is simply given by:*

$$\mathbb{C}[x,y] \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} \mathbb{C}[x,y]^{\oplus 2} \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} \mathbb{C}[x,y] \to \mathbb{C}[x,y]/(x,y).$$

A free resolution for the quotients $Q_{a,b} = \mathbb{C}[x,y]/(x^2, ax+by, y^2)$ *can be given by:*

$$\mathbb{C}[x,y]^{\oplus 2} \xrightarrow{\begin{bmatrix} y & a \\ 0 & y \\ -x & b \\ 0 & -x \end{bmatrix}} \mathbb{C}[x,y]^{\oplus 4} \xrightarrow{\begin{bmatrix} x & -b & y & a \\ 0 & x & 0 & y \end{bmatrix}} \mathbb{C}[x,y]^{\oplus 2} \to Q_{a,b}.$$

Here the last map can be written as $\begin{bmatrix} x & b \end{bmatrix}$ *or* $\begin{bmatrix} y & -a \end{bmatrix}$ *whenever a or b is non-zero respectively.*
A free resolution for the quotient $Q_{(2,1)} = \mathbb{C}[x,y]/(x^2, xy, y^2)$ *is given by:*

$$\mathbb{C}[x,y]^{\oplus 3} \to \mathbb{C}[x,y]^{\oplus 6} \xrightarrow{\begin{bmatrix} x & 0 & -1 & y & 0 & 0 \\ 0 & x & 0 & 0 & y & -1 \\ 0 & 0 & x & 0 & 0 & y \end{bmatrix}} \mathbb{C}[x,y]^{\oplus 3} \xrightarrow{\begin{bmatrix} x & y & 1 \end{bmatrix}} Q_{(2,1)}.$$

A free resolution for the quotient $Q_\lambda = \mathbb{C}[x,y]/\langle x^{\lambda_1}, x^{\lambda_2}y^1, \ldots, x^{\lambda_k}y^{k-1}, hy^k\rangle$ *can be written as:*

$$\mathbb{C}[x,y]^{\oplus|\lambda|} \to \mathbb{C}[x,y]^{\oplus 2|\lambda|} \xrightarrow{M_2} \mathbb{C}[x,y]^{\oplus|\lambda|} \xrightarrow{M_3} Q_\lambda.$$

M_2 *is*

$$\begin{bmatrix} xId_{\lambda_s^*} & -I(\lambda_s^*, \lambda_{s-1}^*) & 0 & 0 & J_y(\lambda_s^*) & 0 & 0 & 0 \\ 0 & \cdots & -I(\lambda_3^*, \lambda_2^*) & 0 & 0 & \cdot\cdot & 0 & 0 \\ 0 & 0 & xId_{\lambda_2^*} & -I(\lambda_2^*, \lambda_1^*) & 0 & 0 & \cdot\cdot & 0 \\ 0 & 0 & 0 & xId_{\lambda_1^*} & 0 & 0 & 0 & J_y(\lambda_1^*) \end{bmatrix}.$$

Here $s = \lambda_1$ *and* $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_s^*)$ *is the dual partition of* λ*.* $J_y(c)$ *is a c by c Jordan block.*

$I(c,d)$ is a c by d matrix.

$$J_y(c) = \begin{bmatrix} y & -1 & 0 & \cdot\cdot & 0 \\ 0 & y & -1 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & 0 & \cdot\cdot & y \end{bmatrix};\ I(c,d) = \begin{bmatrix} 0 & \cdot\cdot 0 & -1 & 0 & \cdot\cdot & 0 \\ 0 & \cdot\cdot 0 & 0 & -1 & \cdot\cdot & 0 \\ \cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & \cdot\cdot 0 & 0 & 0 & \cdot\cdot & -1 \end{bmatrix}.$$

M_3 is written as:

$$(x^{\lambda_1-1}y^{\lambda_s^*-1}, x^{\lambda_1-1}y^{\lambda_s^*-2}, \dots x^{\lambda_1-1}, x^{\lambda_1-2}y^{\lambda_{s-1}^*-1}, x^{\lambda_1-2}y^{\lambda_{s-1}^*-2}, \dots x^{\lambda_1-2}, \dots, xy^{\lambda_2^*-1}, \dots, x, y^{\lambda_1^*-1}, y^{\lambda_1^*-2}, \dots, 1)$$

1.3.1 Geometric properties of Hilbert scheme

We review some classical results for $\mathrm{Hilb}^n X$. When a scheme represents a functor, there is an intrinsic description of the Zariski tangent space at closed points.

Theorem 1.3.4 (Grothendieck [14])**.** *Let $[Z] \in Hilb^n(X)$ be a closed point representing a subscheme $Z \subset X$. Then there is a canonical isomorphism*

$$T_{[Z]}Hilb^n(X) \simeq Hom_{O_X}(\mathcal{I}_Z, O_Z).$$

Sketch of proof. A tangent vector at $[Z]$: Spec $\mathbb{C} \to \mathrm{Hilb}^n X$ corresponds to a morphism τ : Spec $\mathbb{C}[t]/(t^2) \to \mathrm{Hilb}^n(X)$ such that Spec $\mathbb{C} \to$ Spec $\mathbb{C}[t]/(t^2) \to \mathrm{Hilb}^n(X)$ is $[Z]$. By the definition of $\mathrm{Hilb}^n X$ as a functor, τ corresponds an ideal $\tilde{I}$ on $X\times$ Spec $\mathbb{C}[t]/(t^2)$ which is flat over Spec $\mathbb{C}[t]/(t^2)$ and restricts to $\mathcal{I}_Z$ on the closed point on Spec $\mathbb{C}[t]/(t^2)$. By the flatness of $\tilde{I}$, there is a short exact sequence:

$$0 \to \mathcal{I}_Z \xrightarrow{\cdot t} \tilde{I} \to \mathcal{I}_Z \to 0$$

on Spec $\mathbb{C}[t]/(t^2)$. The inclusion $\tilde{I} \subset O_X[t]/(t^2)$ embeds the the quotient $\tilde{I}/\mathcal{I}_Z$ $(\simeq I_Z)$ into $O_X[t]/(tI_Z, t^2)$

$= \mathcal{O}_X \oplus t\mathcal{O}_Z$. The second factor $\rho : \mathcal{I}_Z \to t\mathcal{O}_Z$ gives the map in $\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$. Conversely, such a ρ defines an inclusion from $\mathcal{I}_Z$ to $\mathcal{O}_X[t]/(tI_Z, t^2)$. It lifts to an ideal sheaf $\tilde{I}$ in $\mathcal{O}[t]$ and hence determines a tangent vector. $\square$

Theorem 1.3.5 (Fogarty [6])**.** *Let X be a smooth connected quasi-projective surface. Then the Hilbert scheme $Hilb^n(X)$ is connected and smooth of dimension $2n$.*

Sketch of proof. The connectedness holds in a more general case by Hartshorne's connectedness theorem: the Hilbert scheme that parameterizes subschemes of $\mathbf{P}^r$ with fixed Hilbert polynomial is connected. $\mathrm{Hilb}^n X$ always contains a connected $2n$-dimensional configuration space of unordered n-tuples of pairwise disjoint points. Adopting the connectedness theorem by Hartshorne, $\mathrm{Hilb}^n X$ is connected.

By the theorem 1.3.4 of Grothendieck, the smoothness can be proved by showing that

$$Hom_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \simeq \mathbb{C}^{2n},$$

for any 0-dimensional subscheme Z with length n on X. This reduced to the case that Z is supported on a point. We may assume that Z is support on $(0, 0)$ and X is $\mathbb{C}^2$. Under the $\mathbb{C}^* \times \mathbb{C}^*$ action, any such ideal contracts to an I_λ. We only need show the smoothness at all I_λ's. This can be proved by an induction argument on the shape of Young tableaux. The details are left to the reader. Another approach is to apply $\mathrm{Hom}(-, \mathcal{O}_Z)$ on the short exact sequence:

$$0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0.$$

Then since $\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \simeq \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)^* = \mathbb{C}^n$, and $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = 0$, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathbb{C}^{2n}$. $\square$

The geometry of $\mathrm{Hilb}^n X$ can be terrible in the higher dimensional cases.

Example 1.3.6. *Consider the three dimensional affine space $\mathbb{C}^3$. Let Z be the subscheme given by the ideal $I_Z = \langle x^2, y^2, z^2, xy, yz, zx\rangle$. Then*

$$Hom_{O_{\mathbb{C}^3}}(\mathcal{I}_Z, O_Z) \simeq Hom_{\mathbb{C}}(\ span_{\mathbb{C}}\{x^2, y^2, z^2, xy, yz, xz\},\ span_{\mathbb{C}}\{x, y, z\}) \simeq \mathbb{C}^{18}.$$

Since $Hilb^4\mathbb{C}^3$ is connected and has dimension 12 at a general point, $Hilb^4\mathbb{C}^3$ is not smooth.

Remark 1.3.7 (Vakil [28])**.** *Murphy's Law: every singularity type of finite type over $\mathbb{Z}$ appears on some component of some Hilbert scheme.*

1.3.2 Hilbert-Chow morphism

Definition 1.3.8. *Let X be a smooth quasi-projective surface, $Sym^n(X)$ be the n-th symmetric product of X. The Hilbert-Chow map is defined as:*

$$\begin{aligned} HC : Hilb^n(X) &\to Sym^n(X) \\ [Z] &\mapsto \sum_{x \in X} length(O_{Z,x}) \cdot x \end{aligned}$$

at set-theoretical level.

Proposition 1.3.9 ([17] Example 4.3.6)**.** *The Hilbert-Chow map defined as above is a morphism.*

Notation 1.3.10. *We denote the 'big diagonal' in Sym^nX as Δ: in other words,*

$$\Delta := \{(x_1, x_2, \ldots, x_n) | x_i = x_j, \text{for some } i \neq j\} \subset Sym^nX.$$

We denote the exceptional locus of the Hilbert-Chow morphism as B. As a set,

$$B = \{[Z] \in Hilb^nX | \exists x \in X : length(O_{Z,x}) \geq 2\}.$$

1.3.3 Göttsche's Formula

Let X be a smooth irreducible projective surface. As a manifold, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is smooth compact with real dimension $4n$. The j-th Betti number $b_j(\mathrm{Hilb}^n(X))$ can be computed in terms of $b_j(X)$ by the Göttsche's formula.

Theorem 1.3.11 (Göttsche's Formula [12])**.** *Adopt the notations as above. Let $P_t(Hilb^nX)$ be the Poincaré polynomial of $Hilb^nX$. The Göttsche's formula is given by:*

$$\sum_{n=0}^{\infty} P_t(Hilb^n X)q^n = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}}.$$

Here t and q are formal parameters.

Remark 1.3.12. *1. When $b_1(X)$ is zero, the Hilbert scheme $Hilb^nX$ has zero odd homology groups. 2. (Reference [7]). The Betti numbers of the Hilbert schemes stabilize as n increases. In particular, $b_2(Hilb^nX)$ is $b_2(X)+1$ when $n \geq 1$. The $N^1(Hilb^nX)$ is generated by the pull-back divisors from Sym^nX and the exceptional divisor $B/2$.*

Recall that $\mathrm{Hilb}^n_0(\mathbb{C}^2)$ ($\subset \mathrm{Hilb}^n(\mathbb{C}^2)$) parameterizes the subschemes supported on $(0,0)$. Its Poincaré polynomial is

$$\sum_{n=0}^{\infty} P_t(Hilb^n_0(\mathbb{C}^2))q^n = \prod_{m=1}^{\infty}(1-t^{2m-2}q^m)^{-1}.$$

To see this, recall that we have the $\mathbb{C}^\times \times \mathbb{C}^\times$ action whose fixed points are labeled by the partitions of n. We may choose a sub group $H_{a,b} = \{(t^a, t^b) | t \in C^\times\}$ in $\mathbb{C}^\times \times \mathbb{C}^\times$ such that the fixed point set of H is the same as that of $\mathbb{C}^\times \times \mathbb{C}^\times$. Here (a,b) are positive integers depending on n, for example, we may choose (a,b) such that the ratio $\frac{a}{b}$ is not $\frac{k}{m}$ for any $0 \leq k, m \leq n$. There is a stratification on $\mathrm{Hilb}^n_0(\mathbb{C}^2)$:

$$Hilb^n_0(\mathbb{C}^2) := \coprod_{\lambda \vdash n} Hilb^{\lambda,(a,b)}_0(\mathbb{C}^2),$$

where $\mathrm{Hilb}^{\lambda,(a,b)}_0(\mathbb{C}^2) = \{[Z] \in \mathrm{Hilb}^n_0(\mathbb{C}^2) | \, lim_{H \ni g \to 0} g \cdot I_Z = I_\lambda\}$. The $\overline{Hilb^{\lambda,(a,b)}_0(\mathbb{C}^2)}$ can be the representatives for the homology classes of $\mathrm{Hilb}^n_0(\mathbb{C}^2)$.

Example 1.3.13. *$Hilb^3_0(\mathbb{C}^2)$. The fixed ideals of the $\mathbb{C}^\times \times \mathbb{C}^\times$ are $I_{(3)} = \langle x^3, y\rangle$, $I_{(2,1)} = \langle (x^2, xy, y^2)$ and $I_{(1,1,1)} = (x, y^3)$. Let the subgroup be (t, t^4), then each ideal $\langle a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2\rangle$ contracts to a fixed ideal according to the values of a_1 and a_3:*

$$\begin{aligned} a_1 \neq 0, &\qquad \langle x, y^3\rangle; \\ a_1 = 0 \text{ and } a_3 \neq 0, &\qquad \langle x^2, xy, y^2\rangle; \\ a_1 = a_3 = 0, &\qquad \langle x^3, y\rangle. \end{aligned}$$

Chapter 2

Hilbert scheme of points on a del Pezzo surface

2.1 Exceptional sheaves on del Pezzo surfaces

2.1.1 Exceptional objects

Let $\mathbf{T}$ be the bounded derived category of coherent sheaves on a smooth projective variety (more generally, a triangulated category linear over $\mathbb{C}$ and for any two objects A, $B \in \mathbf{T}$ the space $\oplus_i Hom^i(A, B)$ is finite-dimensional). The references for the following concepts are [1], [9], [19] and [23].

Definition 2.1.1. *An object E in $\boldsymbol{T}$ is called* exceptional *if*

$$Hom^i(E, E) = 0, \ for \ \ i \neq 0; \ Hom^0(E, E) = \mathbb{C}.$$

An ordered collection of exceptional objects $\{E_0, \ldots, E_m\}$ is called an exceptional collection *if*

$$Hom^{\bullet}(E_i, E_j) = 0, for \ i > j.$$

An exceptional collection of two objects is called an exceptional pair.

Definition 2.1.2. *Let $\mathcal{E} = \{E_0, \ldots, E_n\}$ be an exceptional collection. We call $\mathcal{E}$* strong*, if $Hom^k(E_i, E_j)$ $= 0$, for all i and j and $k \neq 0$. We call $\mathcal{E}$* full*, if $\mathcal{E}$ generates $\boldsymbol{T}$ under homological shifts, cones and direct summands.*

Let $S = D_{9-k}$ be a smooth del Pezzo surface (exclude $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$) with k $(1 \leq k \leq 9)$ points in general position blown up on $\mathbf{P}^2$. We denote $\pi : S \to \mathbf{P}^2$ as the projection map, and $\mathcal{T}_{\mathbf{P}^2}$ as the tangent sheaf on $\mathbf{P}^2$. Let $\mathrm{D}^b(S)$ be the the bounded derived category of coherent sheaves on S.

Proposition 2.1.3 (Theorem 2.5 in [1]). *$D^b(S)$ has a full strong exceptional collection:*

$$\{E_0, E_1, \dots, E_k, E_{k+1}, E_{k+2}\} = \{\mathcal{O}(\ell_0), \mathcal{O}(\ell_1 + \ell_0), \dots, \mathcal{O}(\ell_k + \ell_0), \pi^*\mathcal{T}, \mathcal{O}(2\ell_0)\}.$$

Proof. 'Strong exceptional' is due to the calculation by using Riemann-Roch and Serre duality for line bundles and the short exact sequence: $0 \to \mathcal{O} \to \mathcal{O}(\ell_0)^{\oplus 3} \to \pi^*\mathcal{T} \to 0$. For example, $\mathrm{Hom}(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}) = 0$, $\mathrm{Ext}^2(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}) \simeq (\mathrm{Hom}\ (\mathcal{O}, \mathcal{O}(-2\ell_0+2\ell_1+\dots+\ell_k))^* = 0$, $\chi(\mathcal{O}(-\ell_1-\ell_0))$ $= (-\ell_1-\ell_0)\cdot(-\ell_1-\ell_0+(3\ell_0-\ell_1-\dots-\ell_k))/2 + 1 = -1$ implies $\mathrm{Ext}^1(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}) = \mathbb{C}$; in the same way $\mathrm{Hom}(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}(\ell_0)) = 0$, $\mathrm{Ext}^2(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}(\ell_0)) \simeq (\mathrm{Hom}\ (\mathcal{O}(\ell_0), \mathcal{O}(-2\ell_0+2\ell_1+\dots+\ell_k))^* = 0$, $\chi(\mathcal{O}(-\ell_1)) = (-\ell_1)\cdot(-\ell_1+(3\ell_0-\ell_1-\dots-\ell_k))/2 + 1 = 0$ implies $\mathrm{Ext}^1(\mathcal{O}(\ell_1+\ell_0), \mathcal{O}) = 0$. Now apply $\mathrm{Hom}(\mathcal{O}(l_0+l_1), -)$ to the short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(\ell_0)^{\oplus 3} \to \pi^*\mathcal{T} \to 0$, we get Ext^i $(\mathcal{O}(\ell_0+\ell_1), \pi^*\mathcal{T}) = \mathbb{C}$ for $i = 0$, and $= 0$ for $i \neq 0$. The other relations can be checked in a similar way. 'Full' is due to Theorem 2.5 in [1]. □

A zero-dimensional subscheme of S is equivalently described by an ideal sheaf, in other words, a rank 1 torsion-free sheaf with trivial first Chern class. We compute $\mathrm{Ext}^i(E_j, -)$'s of these sheaves at first.

Lemma 2.1.4. Let $\mathcal{F}$ be a rank 1, torsion-free sheaf with trivial first Chern class on S. Write $n = 1 - \chi(\mathcal{F})$. Then we have

$$dimExt^i(E_j, \mathcal{F}) = \begin{cases} n, \text{when } i = 1, j = 0, k+2; \\ n+1, \text{when } i = 1, 1 \le j \le k; \\ 2n+1, \text{when } i = 1 j = k+1; \\ 0, \text{otherwise.} \end{cases}$$

Proof. Consider the natural map $\mathcal{F} \to \mathcal{F}^{**}$. Since $\mathcal{F}$ is torsion-free and S is smooth, the map is injective and the cokernel is supported on a finite set. Since $c_1(\mathcal{F})$ is trivial and $\mathrm{Pic}^0(S)$ is trivial,

$\mathcal{F}^{**} \simeq \mathcal{O}_S$. Apply $\mathrm{Ext}^i(E_j, -)$ to the short exact sequence: $0 \to \mathcal{F} \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$. Since all E_j's are locally free and Z has dimension 0, $\mathrm{Ext}^i(E_j, \mathcal{O}_Z) = 0$ for $i = 1, 2$.

$\mathrm{Ext}^0(E_j, \mathcal{O}_S) = 0$: when $j \neq k+1$, E_j is a line bundle and the divisor is effective. When $j = k+1$, E_{k+1} is the cokernel of $\mathcal{O} \to \mathcal{O}(\ell_0)^{\oplus 3}$. $\mathrm{Ext}^2(E_j, \mathcal{O}_S) = 0$: when $j \neq k+1$, E_j is a line bundle and the divisor is not greater than the anti-canonical divisor $3\ell_0 - \ell_1 - \cdots - \ell_k$. When $j = k+1$, $\pi^*\mathcal{T}$ is the kernel of $\mathcal{O}(2\ell_0)^{\oplus 3} \to \mathcal{O}(3\ell_0)$. By Serre duality, $\mathrm{Ext}^2(E_j, \mathcal{F}) = 0$.

The computation on Ext^1 is an application of the Riemann-Roch formula, for example:

$$
\begin{aligned}
&-\mathrm{ext}^1(\mathcal{O}(\ell_1 + \ell_0), \mathcal{F}) = \chi(\mathcal{F}(-\ell_1 - \ell_0)) \\
=&(1, -\ell_1 - \ell_0, \frac{(-\ell_1 - \ell_0)^2}{2} - n) \cdot (1, (3\ell_0 - \ell_1 - \cdots - \ell_k)/2, 1) \\
=&1 - 2 - n = -1 - n.
\end{aligned}
$$

In the case of $\pi^*\mathcal{T}$, the computation is done by using $\mathcal{O} \to \mathcal{O}(\ell_0)^{\oplus 3} \to \pi^*\mathcal{T}$ and the additive property of χ. □

Definition 2.1.5. *The* left transformation $L_E F$ *of an exceptional pair* (E, F) *is defined as the object that fits into the distinguished triangle:*

$$L_E F \to R^{\cdot}Hom(E, F) \otimes E \to F \to L_E F[1].$$

The cocollection *of* $\{E_0, \ldots, E_{k+2}\}$ *is defined to be*

$$\{E_0^\vee, \ldots, E_{k+2}^\vee\} := \{E_0, L_{E_0}E_1, L_{E_0}L_{E_1}E_2, \ldots, L_{E_0}L_{E_1} \ldots L_{E_{k+1}}E_{k+2}\}.$$

The following definition is from [19].

Definition 2.1.6. *An infinite sequence $\{E_i\}_{i\in\mathbb{Z}}$ of objects of the derived category $D^b(S)$ is called a helix of period n if $\{E_{i+1},\ldots,E_{i+n}\}$ an exceptional collection for all $i\in\mathbb{Z}$ and in addition,*

$$E_i = E_{i+n}\otimes\mathcal{O}_S(K)[3-n].$$

Lemma 2.1.7. Let $\{E_0^\vee,\ldots,E_{k+2}^\vee\}$ be the exceptional collection in Lemma 2.1.3. The collection $\{E_0^\vee,\ldots,E_{k+2}^\vee\}$ is

$$\{\mathcal{O}(\ell_0),\mathcal{O}_{\ell_1}(-1)[-1],\ldots,\mathcal{O}_{\ell_k}(-1)[-k],\mathcal{O}(\ell_1+\cdots+\ell_k)[-k],\mathcal{O}(\ell_1+\cdots+\ell_k-\ell_0)[-k]\}.$$

Proof. $E_0^\vee=\mathcal{O}(\ell_0)$ by definition. When $1\le i\le k$, since $\mathrm{Hom}(\mathcal{O}(\ell_i+\ell_0),\mathcal{O}(\ell_j+\ell_0)[t])=0$ for any t and $i\neq j$, $E_i^\vee=\mathcal{O}_{\ell_i}(-1)[-i]$. As the exceptional collection generates the category, the sequence of objects $\{E_i\}_{s\in\mathbb{Z}}$, which is extended from $\{E_0,\ldots,E_n\}$ by setting $E_{-i}=L_{E_{-i+1}}\ldots L_{E_{n-i-1}}E_{n-i}$, is a helix of period $n+1$ by Proposition 1.12 in [19]. As the property of a helix,

$$E_{k+2}^\vee=E_{-1}=E_{k+2}\otimes\omega_S[-k]=\mathcal{O}(\ell_1+\cdots+\ell_k-\ell_0)[-k].$$

Also we know that $L_{E_{k+2}^\vee}(E_{k+1}^\vee)=L_{E_{k+2}^\vee}(L_{E_0}\ldots L_{E_k}E_{k+1})=E_{k+1}\otimes\omega_S[-k]$. By writing down each step of the left mutations, we know that $L_{E_0}\ldots L_{E_k}E_{k+1}$ is concentrated in degree $-k$. In addition, it fits into the short exact sequence

$$0\to E_{-1}\to E_{-2}\otimes\mathrm{Hom}(E_{-2},E_{k+1}^\vee)\to E_{k+1}^\vee\to 0,$$

in other words $0\to\pi^*\mathcal{T}\otimes\mathcal{O}(-3\ell_0+\ell_1+\cdots+\ell_k)\to\mathcal{O}(\ell_1+\cdots+\ell_k-\ell_0)\otimes\mathrm{Hom}(\mathcal{O}(\ell_1+\cdots+\ell_k-\ell_0),E_{k+1}^\vee[k])\to E_{k+1}^\vee[k]\to 0$. Thus, $E_{k+1}^\vee=R_{E_{-1}}E_{-2}=\mathcal{O}(\ell_1+\cdots+\ell_k)[-k]$. □

Proposition 2.1.8 (Formula 2.24 in [10], Proposition on page 82 in [9])**.** *Write $s_{k+2}=k$, $s_{k+1}=k$ and $s_i=i$ for $0\le i\le k$. Given a torsion-free sheaf $\mathcal{F}$ with invariants $(r,c_1,\chi)=(1,0,1-n)$ on*

S, there is a spectral sequence with $E_1^{p,q} = Ext^{q-s_{-p}}(E_{-p}, \mathcal{F}) \otimes E^{\vee}_{-p}[s_{-p}]$ that converges to $\mathcal{F}$ on $p = -q$ diagonal and 0 on the other part. □

2.2 Non-commutative version of del Pezzo surface

We recollect some concepts and notations of non-commutative deformation of del Pezzo surfaces from [1]. Recall from Section 2.1 that the bounded derived category of coherent sheaves on S has a full strong exceptional collection $\{\mathcal{O}_S(\ell_0), \mathcal{O}_S(\ell_1 + \ell_0), \ldots, \pi^*\mathcal{T}_{\mathbf{P}^2}, \mathcal{O}_S(2\ell_0)\}$. Denote the $\mathbb{C}$-linear morphism spaces between those sheaves by:

$$U := \operatorname{Hom}(\mathcal{O}_S(\ell_0), \pi^*\mathcal{T}); \ V := \operatorname{Hom}(\pi^*\mathcal{T}, \mathcal{O}_S(2\ell_0)); \ W := \operatorname{Hom}(\mathcal{O}_S(\ell_0), \mathcal{O}_S(2\ell_0));$$

$$E_i := \operatorname{Hom}(\mathcal{O}_S(\ell_0), \mathcal{O}_S(\ell_i + \ell_0)); \ F_i := \operatorname{Hom}(\mathcal{O}_S(\ell_i + \ell_0), \pi^*\mathcal{T}); \ G_i := \operatorname{Hom}(\mathcal{O}_S(\ell_i + \ell_0), \mathcal{O}_S(2\ell_0)),$$

and the main composition law by

$$\mu : U \otimes V \to W.$$

Since each E_i, F_i has dimension 1, the composition of them determines a morphism in U up to a scalar, we denote each image line by $p_i \in P(U)$, where $P(U)$ stands for the projectivization of U. In other words,

$$p_i := [F_i \circ E_i].$$

The dimensions of U, V, W, E_i, F_i, G_i for $1 \le i \le k$ are $3, 3, 3, 1, 1, 2$ respectively.

Since the exceptional collection above generates $D^b(S)$, by the philosophy of [3] , the triangulated category is determined by the composition laws of the morphisms between them. Namely, a deformation of this category is just a deformation of the composition laws. Under this procedure, we need some extra requirements on the laws to make the deformation non-degenerate.

Definition 2.2.1. We call $(\mu, p_1, \ldots, p_k) \in \operatorname{Hom}_{\mathbb{C}}(U \otimes V, W) \times P(U) \times \ldots \times P(U)$ a datum of *noncommutative del Pezzo surface* if it satisfies the following conditions.

1. For any nonzero $u \in U$, the induced map $\mu_u : V \xrightarrow{u\otimes\bullet} W$ has rank at least 2.
2. Each μ_{p_i} has rank 2: since μ_{p_i} factors as $V \to E_i \bigotimes F_i \bigotimes V \to E_i \bigotimes G_i \to W$, its image has dimension 2.

Further observation shows that the other composition laws of morphisms are determined by μ and p_i's. For example, the composition $E_i \otimes G_i \to W$ identifies G_i as the image of μ_{p_i} in W.

Since μ_u has rank 2 only when $\det\mu_u = 0$ (which is given by a degree 3 equation (or constantly zero) if we write down-to-earth by choosing basis for U, V, W), degenerate points may either form a cubic curve in $P(U)$ or the whole space of U. In the second case, $D^b(S)$ is the bounded derived category of sheaves on the classical commutative surface. The fact that p_i could be any point on $P(U)$ indicates that one can blow-up any point on the surface. In the cubic curve case, the second condition in Definition 2.2.1 says that one can only blow-up points on the degenerate cubic curve E. Namely, the datum (μ, p_i) depends on $(k + 2)$-parameters: as explained in Section 3.2, μ depends on 2 parameters including the modulus of cubic curve and the difference of two degree 3 line bundles; all p_i's are restricted on E. More relations between U, V, W and E are discussed in Section 3.2. Further details on non-commutative deformations of del Pezzo surfaces which we do not use here appear in [1], Section 2.

Remark 2.2.2. Since we are only interested in generic case in this note, when the degenerate curve is a cubic curve, the curve E is assumed to be a smooth elliptic curve.

Notation 2.2.3. *Each μ_{p_i} has a 1-dimensional kernel in V; we denote it by $q_i \in P(V)$. The null spaces of p_i and q_i in U^* and V^* are denoted by Y_{p_i} and Z_{q_i} (or Z_{μ,p_i}) respectively; the dual map of μ, $\mu^* : W^* \to U^* \bigotimes V^*$ has 3-dimensional image in $U^* \bigotimes V^*$, and we denote it by X_μ.*

2.3 Deformation in family

The discussion in Section 2.2 makes sense for any field F over $\mathbb{C}$. We may also talk about it over arbitrary commutative noetherian ring over ground field $\mathbb{C}$.

Definition 2.3.1. *Let A be a commutative noetherian ring, a morphism of A-modules $\mu_A : U_A \otimes V_A \to W_A$ is called* non-degenerate *if for any field F over $\mathbb{C}$ and $A \to F$, the induced morphism $\mu_F : U_F \bigotimes_F V_F \to W_F$ is non-degenerate. An element p_i in U_A is called* degenerate *if $\mu(p_i, V_A)$ is a rank 2 projective submodule of W_A. A* family of deformed noncommutative del Pezzo surfaces *consists of the following datum:*

$$\{\text{a nondegenerate morphism of } A\text{-modules } \mu_A : U_A \otimes V_A \to W_A\text{; degenerate elements } p_1, \dots p_k \text{ in } U_A\},$$

where U_A, V_A, W_A are projective A-modules of rank 3. Similarly as those notations in Section 2.2, we have $Y_{A,p_i} \subset U_A^*$, and $X_{\mu_A} \subset U_A^* \otimes V_A^*$.

For technical reasons, we require that the family satisfies flatness conditions: (μ_A, p_i) is said to be *flat* if the following holds:

1. For any field F and morphism $A \to F$, the canonical maps $Y_{A,p_i} \otimes F \to Y_{p_i \otimes F}$; $X_{\mu_A} \otimes F \to X_{\mu_A \otimes F}$ are isomorphisms.
2. For each i, there is a non-vanishing element q_i in V_A such that $\mu_A(p_i, q_i) = 0$ and the corresponding $Z_{A,q_i} \otimes F \to Z_{q_i \otimes F}$ is an isomorphism; the different choice of q_i gives the same Z_{A,q_i}, we denote it by Z_{μ_A, p_i}.
3. $U_A^*/Y_{A,p_i}$ and $V_A^*/Z_{A,q_i}$, $U_A^* \otimes V_A^*/X_{\mu_A}$ are projective A-modules. (*)

2.4 K-complexes

By a similar strategy to [21] and [25], where the base variety is a projective plane instead of a general del Pezzo surface, we define an analogue of Kronecker complexes for sheaves on a del Pezzo surface.

Definition 2.4.1. *Given a del Pezzo surface S, a* K-complex $\boldsymbol{K}$ *is given by the following data: vector spaces $(H_2, H_1, H_{T_1}, \ldots, H_{T_k}, H_0)$ and maps:*

$$I : \mathcal{O}(-\ell_0) \otimes H_2 \to \mathcal{O} \otimes H_1;$$

$$L_i : H_1 \to H_{T_i}, \textit{for } 1 \le i \le k;$$

$$J : \mathcal{O} \otimes H_1 \to \mathcal{O}(\ell_0) \otimes H_0.$$

Here $\mathcal{O}$ is the structure sheaf on S. $(H_2, H_1, H_{T_1}, \ldots, H_{T_k}, H_0)$ are vector spaces with finite dimensions $(h_2, h_1, h_{T_1}, \ldots, h_{T_k}, h_0)$. To be a K-complex, the data must satisfy the following requirements:

1. $(\oplus_{1 \le i \le k} L_i) \circ I = 0$;
2. $J|_{kerL_i} : \mathcal{O} \otimes kerL_i \to \mathcal{O}(\ell_0) \otimes H_0$ *is zero on* ℓ_i.
3. $J \circ I = 0$.

We call $(h_2, h_1, h_{T_1}, \ldots, h_{T_k}, h_0)$ the type of a K-complex.

Remark 2.4.2. *We have the following two remarks.*

1. *I is equivalently described as a map $I_\bullet$ in $Hom(H_2, H_1) \otimes Hom(\mathcal{O}(-\ell_0), \mathcal{O})$, respectively $J_\bullet$.*

2. *A subcomplex $\widetilde{\boldsymbol{K}}$ of $\boldsymbol{K}$ is the data $(\widetilde{I}, \widetilde{J}, \widetilde{L}_1, \ldots, \widetilde{L}_k; \widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_1}, \ldots, \widetilde{H}_{T_k}, \widetilde{H}_0)$, where each $\widetilde{H}_i$ is a subspace of H_i such that the morphisms are compatible with them, i.e., $I\ (\mathcal{O}(-\ell_0) \otimes \widetilde{H}_2) \subset \mathcal{O} \otimes \widetilde{H}_1$ and so on. The morphisms of $\widetilde{\boldsymbol{K}}$ are just the restriction of the original ones. We can talk about morphisms between two K-complexes. Such a complex is just a collection of maps $(f_2, f_1, f_{T_i}$'s$, f_0)$ between vector spaces $(H_2, H_1, H_{T_i}$'s$, H_0)$ which commutes with $(I, J, L_i$'s$)$.*

It is not hard to check that the kernel and cokernel of these maps are still K-complexes. K-complexes form an Abelian category.

Proposition 2.4.3. *Suppose $\mathcal{F}$ is a rank 1 torsion-free sheaf on S with trivial first Chern class. The spectral sequence for $\mathcal{F}$ in Proposition 2.1.8 induces a K-complex with type $(n, 2n+1, n+1, n+1, \dots, n+1, n)$, and the morphism between such sheaves induces maps between homology groups and morphisms between their K-complexes.*

Proof. By Lemma 2.1.4, $E_1^{p,q}$ can be non-zero only at $E^{-k-2,k+1}$, $E^{-k-1,k+1}$ and $E^{i,i+1}$ for $k \geq i \geq 0$. Terms are $\mathrm{H}^1(\mathcal{F}(-2\ell_0)) \otimes \mathcal{O}(\ell_1 + \cdots + \ell_k - \ell_0)$, $\mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}(\ell_1 + \cdots + \ell_k)$, $\mathrm{H}^1(\mathcal{F}(-\ell_i - \ell_0)) \otimes \mathcal{O}_{\ell_i}(-1))$ for $k \geq i \geq 1$ and $\mathrm{H}^1(\mathcal{F}(-\ell_0)) \otimes \mathcal{O}(\ell_0)$, respectively. Since all of the $\mathcal{O}_{\ell_i}(-1)$'s are orthogonal in $\mathrm{D}^b(S)$, we may combine the first k pages of the spectral sequence in Proposition 2.1.8 together and simplify the whole picture as:

$$\begin{tikzcd} H_2 \otimes E_{k+2}^\vee[k] \arrow[r, "\mathrm{I}"] & H_1 \otimes E_{k+1}^\vee[k] \arrow[r, "\bigoplus \mathrm{L}_i"] \arrow[drr, "\mathrm{J}'' \text{: on the 2nd step}"'] & \bigoplus_i H_{T_i} \otimes E_i^\vee[i] & \\ & & & H_0 \otimes E_0^\vee. \end{tikzcd} \qquad (\triangle)$$

Here $(H_2, H_1, H_{T_1}, \dots, H_{T_k}, H_0) = (\mathrm{H}^1(\mathcal{F}(-2\ell_0)), \mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}), \mathrm{H}^1(\mathcal{F}(-\ell_1 - \ell_0)), \dots, \mathrm{H}^1(\mathcal{F}(-\ell_k - \ell_0)), \mathrm{H}^1(\mathcal{F}(-\ell_0))$. We denote I and L_i's as the map on the first page, and J'' as the map from $\ker \oplus L_i / \mathrm{im} I$ to $\mathrm{H}^1(\mathcal{F}(-\ell_0)) \otimes \mathcal{O}(\ell_0)$ on the next page. The subsheaf $\mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}$ in $\mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}(\ell_1 + \cdots + \ell_k)$ is always in the kernel of any morphism from $\mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}(\ell_1 + \cdots + \ell_k)$ to $\oplus_i(\mathrm{H}^1(\mathcal{F}(-\ell_i - \ell_0)) \otimes \mathcal{O}_{\ell_i}(-1))$. Denote the kernel of $\oplus L_i$ by $\mathcal{K}$. On the second page, J'' maps $\mathcal{K}/\mathrm{im} I$ to $\mathrm{H}^1(\mathcal{F}(-\ell_0)) \otimes \mathcal{O}(\ell_0)$. We define J to be the restriction of J'' on $\mathrm{Ext}^1(\pi^*\mathcal{T}, \mathcal{F}) \otimes \mathcal{O}$.

We then check that the data $(H_2, H_1, H_{T_1}, \dots, H_{T_k}, H_0)$; (I, L_i, J) gives a K-complex. The first page implies that $\oplus L_i \circ I$ is 0. Since the term $\mathrm{H}^1(\mathcal{F}(-\ell_i - \ell_0)) \otimes \mathcal{O}_{\ell_i}(-1)$ is not on the diagonal, L_i is always surjective. By the definition of J, $J|_{\ell_i}$ is 0 at the kernel part of L_i. J' is given by $\mathcal{K} \to \mathcal{K}/\mathrm{im} I \xrightarrow{J''} \mathcal{O}(\ell_0) \otimes H_0$. Condition 3 is clear. The dimension of each $\mathrm{Ext}^1(E_j, \mathcal{F})$ is computed in Lemma 2.1.4, which gives the type of $\mathbf{K}$. Since only $\mathcal{K}$ appears on the diagonal, the spectral sequence concentrates to $\mathrm{H}^0(\mathbf{K})$. By Proposition 2.1.8, this is the sheaf $\mathcal{F}$. $\square$

We denote the homology sheaf at the middle term of complex ($\triangle$) by $\mathrm{H}^0(\mathbf{K})$, the homology sheaf at $H_2 \otimes E_{k+2}^{\vee}[k]$ by $\mathrm{H}^{-1}(\mathbf{K})$. Invariants of a K-complex according to its type are listed in the table below.

Rank: $r(\mathbf{K}) := h_1 - h_2 - h_0$.

First Chern class: $c_1(\mathbf{K}) := (h_2 - h_0)\ell_0 + \sum_{i=1}^{k}(h_1 - h_2 - h_{T_i})\ell_i$.

Euler characteristic: $\chi(\mathbf{K}) := h_1 - 3h_0$.

Hilbert polynomial w.r.t divisor $H = s\ell_0 - \ell_1 - \cdots - \ell_k$:

$$p_{\mathbf{K}} := \frac{1}{2} r(\mathbf{K})t(t+3) + (c_1(\mathbf{K}) \cdot H)t + \chi(\mathbf{K}).$$

When the K-complex is induced from a sheaf $\mathcal{F}$ on S, these invariants coincide with those of the sheaf $\mathcal{F}$.

Definition 2.4.4. *A K-complex* $\boldsymbol{K}$ *is called* (semi)stable *(with respect to divisor* $s\ell_0 - \ell_1 - \cdots - \ell_k$*), if for every non-zero proper subcomplex* $\widetilde{\boldsymbol{K}}$ *of* $\boldsymbol{K}$*, one has* $r(\boldsymbol{K})p_{\widetilde{\mathbf{K}}} - r(\widetilde{\boldsymbol{K}})p_{\boldsymbol{K}} < 0$ *(resp.* ≤ 0*) under the lexicographic order on polynomials, i.e.*

1.$(r(\boldsymbol{K})c_1(\widetilde{\boldsymbol{K}}) - r(\widetilde{\boldsymbol{K}})c_1(\boldsymbol{K})) \cdot (r\ell_0 - \ell_1 - \cdots - \ell_k) \leq 0$*, and*

2. if '=' holds in 1, then $r(\boldsymbol{K})\chi(\widetilde{\boldsymbol{K}}) - r(\widetilde{\boldsymbol{K}})\chi(\boldsymbol{K}) < 0$ *(respectively* ≤ 0 *for semistable).*

Since Lemma 2.1.4 tells us the dimension of H_i's, we will focus on those semistable (actually stable) K-complexes of type $(n, 2n+1, n+1, \ldots, n+1, n)$. In our case, the semistable condition on a K-complex means that for any non-zero proper subcomplex $\widetilde{\mathbf{K}}$ of $\mathbf{K}$, $c_1(\widetilde{\mathbf{K}})(s\ell_0 - \ell_1 - \cdots - \ell_k) = (s-k)\widetilde{h}_2 + k\widetilde{h}_1 - s\widetilde{h}_0 - \sum \widetilde{h}_{T_i} \leq 0$, if '=' holds then $\widetilde{h}_1 - 3\widetilde{h}_0 - (\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0)(1-n) \leq 0$.

2.5 Homology sheaf of a K-complex

In this section, we show that the homology sheaves of any semistable K-complex of type $(n, 2n+1, n+1, \ldots, n+1, n)$ concentrate at the middle term and $\mathrm{H}^0(\mathbf{K})$ is a torsion-free sheaf of rank 1. That sets up a map from K-complexes to sheaves. The proof is purely linear algebra, which can be easily generalized to the non-commutative case.

For any point P on a del Pezzo surface S, we denote I_P as the morphism I at the fiber P. We may choose different bases $\{x_i, y_i, z_i\}$, $1 \leq i \leq k$ for $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(\ell_0))$ such that y_i and z_i span $\mathrm{Hom}(\mathcal{O}(\ell_i), \mathcal{O}(\ell_0))$. Under each base $\{x_i, y_i, z_i\}$, $I_\bullet$ ($J_\bullet$) can be written as $x_iI_{i1} + y_iI_{i2} + z_iI_{i3}$ (resp. $J_\bullet$), where $I_{it} \in \mathrm{Hom}\,(H_2, H_1)$, for $1 \leq t \leq 3$. Under such a decomposition, $L_i \circ I_{i1} = 0$ and J_{i1} factors through L_i.

Remark 2.5.1. *1. I_{i1} and J_{i1} do not depend on the choice of x_i (up to a scalar). Besides, for any point P that is not on the exceptional line, we may choose base $\{o, p, q\}$ for Hom$(\mathcal{O}, \mathcal{O}(\ell_0))$ such that o, q span the subspace of sections that vanish at P. If we write $I_\bullet = oI_1 + pI_2 + qI_3$, then up to a scalar I_2 is just the mapping matrix from H_2 to H_1 on the fiber at P.*
2. For any complex numbers a, b, c such that b and c are not both 0, we may choose different bases $\{x_i', y_i', z_i'\}$ such that if we decompose $I_\bullet$ under these new bases as $mx_i'I_{i1} + y_i'I_{i2}' + z_i'I_{i3}'$, then $I_{i2}' = aI_{i1} + bI_{i2} + cI_{i3}$.

Proposition 2.5.2. *For any semistable K-complex $\mathbf{K} = (I, J, L_1, \ldots, L_k; H_2 \ldots, H_0)$ of type $(n, 2n+1, n+1, \ldots, n+1, n)$, we have:*
1. each L_i is surjective;
2.$\mathcal{K} \xrightarrow{J'} H_0 \otimes \mathcal{O}(l_0)$ is surjective;
3. I is injective.
In particular the homology sheaves of this complex concentrate in degree 0. Furthermore, $H^0(\mathbf{K})$ is a rank 1 torsion free sheaf with trivial first Chern class.

Proof. 1. L_i is surjective: if L_i is not surjective for some i, we may consider the subcomplex $\mathbf{K}'$ that consists of spaces $H_2, H_1, H_{T_1}, \ldots, im(L_i), \ldots, H_0$. Now $(r(\mathbf{K})c_1(\mathbf{K}')-r(\mathbf{K}')c_1(\mathbf{K}))\cdot(r\ell_0-\ell_1-\cdots-\ell_k)$ $= n + 1$ - dim(im(L_1))> 0. This contradicts the semistable assumption on $\mathbf{K}$.

2. J' is surjective: surjectivity can be checked over fiber on each closed point. There are two different cases: the point is not on any exceptional line; the point is on an exceptional line.

Case 1: The point P is not on any exceptional line. Assume the map on that fiber is not surjective. By choosing a basis (o, p, q) for $\mathrm{Hom}(O, O(\ell_0))$ such that o, q span the subspace of sections that vanish at P, we can write $J_\bullet = oJ_1 + pJ_2 + qJ_3$, then J' is not surjective at P if and only if J_2 is not surjective. Let $\widetilde{H}_0$ =im(J_2), $\widetilde{H}_1 = J_1^{-1}(\widetilde{H}_0) \bigcap J_3^{-1}(\widetilde{H}_0)$, $\widetilde{H}_{T_i} = L_i(\widetilde{H}_1)$, $\widetilde{H}_2 = I_1^{-1}(\widetilde{H}_1)$. To show that $\widetilde{H}_i$'s form a subcomplex, we need to check $I(\widetilde{H}_2) \subset \widetilde{H}_1 \otimes$ Hom $(O, O(\ell_0))$, i.e. $I_2(\widetilde{H}_2)$, $I_3(\widetilde{H}_2) \subset \widetilde{H}_1$.

$J_3 \circ I_2(\widetilde{H}_2) = J_2 \circ I_3(\widetilde{H}_2) \subset \widetilde{H}_0$ which implies $I_2(\widetilde{H}_2) \subset J_3^{-1}(\widetilde{H}_0)$; similarly we know $I_2(\widetilde{H}_2) \subset J_1^{-1}(\widetilde{H}_0)$, hence $I_2(\widetilde{H}_2) \subset \widetilde{H}_1$. In almost the same way, $I_3(\widetilde{H}_2) \subset \widetilde{H}_1$.

Now we see that $\widetilde{H}_0 = \mathrm{im}(J_P)$, $\widetilde{H}_1 = J_\bullet^{-1}$ $(\widetilde{H}_0 \otimes \mathrm{Hom}(O, O(\ell_0)))$, $\widetilde{H}_2 = I_\bullet^{-1}(\widetilde{H}_1 \otimes \mathrm{Hom}(O, O(\ell_0)))$. Hence $\widetilde{H}_i$'s do not depend on the base choice of $\mathrm{Hom}(O, O(\ell_0))$, so that we may assume that given any exceptional line l_i, we may choose q such that p and q span the subspace Hom $(O(\ell_i), O(\ell_0))$. Let the codimension of $\widetilde{H}_0$ be c, then the codimension of $\widetilde{H}_1$ is less than or equal to $2c$. Since kerL_i $\subset$ ker$J_{x_i} \subset J_{x_i}^{-1}(\widetilde{H}_0)$, now choose base such that p and q span the subspace Hom $(O(\ell_i), O(\ell_0))$ (we can do this since P is not on any exceptional line) then $\widetilde{h}_{T_i} \leq \widetilde{h}_1 - \dim(\ker L_i \bigcap J_3^{-1}(\widetilde{H}_0)) \leq \widetilde{h}_1 - (n-c)$. Similarly, since im$I_1$ $\subset$ker$J_1 \subset J_1^{-1}(\widetilde{H}_0)$, then $\widetilde{h}_2 \geq n - c$. This contradicts the semistableness of the complex.

Case 2: The point P is on an exceptional line ℓ_1. Assume the map on that fiber is not surjective. We may choose a basis (x_1, y_1, z_1) for $\mathrm{Hom}(O, O(\ell_0))$ (recall that y_1 and z_1 span Hom $(O(\ell_1), O(\ell_0))$)

such that the zero locus of y is the union of ℓ_1 and the transverse image of a line across ℓ_1 at P. Write $J_\bullet = x_1 J_1 + y_1 J_2 + z_1 J_3$, then the morphism $\mathcal{K} \xrightarrow{J'} H_0 \otimes \mathcal{O}(\ell_0)$ restricts on ℓ_1 to the following map:

$$
\begin{tikzcd}
\mathrm{ker}L_1 \otimes \mathcal{O}_{\ell_1}(-1) \arrow[dr, bend left, "J_2 y_1' + J_3 z_1'"] & \\
\oplus & H_0 \otimes \mathcal{O}_{\ell_1} \\
(H_1/\mathrm{ker}L_1) \otimes \mathcal{O}_{\ell_1} \arrow[ur, bend right, "J_1"'] &
\end{tikzcd}
$$

y_1' is the morphism that vanishes on fiber P. From the picture, we know that the map is not surjective at P if and only if $\mathrm{im}J_1 + J_3(\mathrm{ker}L_1)$ is not the whole space of H_0. Let $\mathrm{im}J_1 + J_3(\mathrm{ker}L_1)$ be $\widetilde{H}_0$, then other $\widetilde{H}_i$'s can be defined similarly as that in Case 1. The only different thing here is that we use $\mathrm{ker}L_1 \subset J_3^{-1}(\ker L_1) \subset J_3^{-1}(\widetilde{H}_2)$ to estimate $\widetilde{h}_{T_1}$. Similarly, we get the contradiction.

3. Injectivity and torsion-freeness: we use the following lemma:

Lemma 2.5.3. *Let P be a closed point not on any exceptional line ℓ_i ($i \geq 1$), assume that $\mathrm{ker}I_p$ is not empty. Let H_2' be a 1-dim subspace in the kernel, then we can choose a subcomplex $\boldsymbol{K}'$, $(H_2', H_1', H_{T_1}', \ldots, H_{T_k}', H_0')$ with dimension $(1, 2, 1, \ldots, 1)$ such that the complex $0 \to H_2' \otimes \mathcal{O}(-\ell_0 + \ell_1 + \cdots + \ell_k) \to \mathcal{K}' \to H_0' \otimes \mathcal{O}(\ell_0)$ is exact. The same result holds for P on ℓ_i ($i \geq 1$), if $\mathrm{ker}I_{i1}$ contains a 1-dim subspace H_2' such that $L_i \circ I_{i2}(H_2')$ and $L_i \circ I_{i3}(H_2')$ are the same or either one of them is zero.*

Proof of the lemma. When P is not on any exceptional line, we may choose a base (o, p, q) as before and write $\boldsymbol{I}_\bullet = oI_1 + pI_2 + qI_3$ (resp. $\boldsymbol{J}_\bullet$). Let $H_1' = I_1(H_2') + I_3(H_2')$, $H_{T_i}' = L_i(H_1')$, $H_0' = J_1(H_1') + J_2(H_1') + J_3(H_3')$. It is easy to see that this is a subcomplex and does not depend on the choice of bases $\{o, p, q\}$. We only need show that it has the desired dimensions.
$h_1' = 2$: If h_1' is 0, this obviously contradicts the semistableness of the complex. If $h_1 = 1$, then H_0' is $J_3 \circ I_1(H_2') = -J_1 \circ I_3(H_2')$, but $(mI_1 + nI_3)(H_2') = 0$ for some $m, n \in \mathbb{C}$, hence $H_2' = 0$. This contradicts the semistableness.

$h'_{T_i} \leq 1$: since P is not on any ℓ_i for any i, we know $I_{i1}(H'_2) \neq 0$, else $h'_1 \leq 1$. Now $L_i \circ I_{T_i} = 0$ implies the inequality.

$h'_0 = 1$: $J_2(H'_1) = 0$, since $J_2 \circ I_j = J_j \circ I_2$ for all j. That means H'_0 is generated by $J_3 \circ I_1(e) = J_1 \circ I_3(e)$, so $h'_0 \leq 1$. Since the complex is semistable, the only possible case is that $h'_{T_i} = 1$ and $h'_0 = 1$.

Exactness: The complex $0 \to H'_2 \otimes \mathcal{O}(-\ell_0 + \ell_1 + \cdots + \ell_k) \to \mathcal{K}' \to H'_0 \otimes \mathcal{O}(\ell_0)$ is a resolution of the skyscraper sheaf $\mathcal{O}_P$. This can be checked fiber-wise: for a point that is not on any ℓ_i's, the $\mathcal{O}_{\ell_i}(-1)$'s can be ignored; for a point on an exceptional line, this can be done by restricting morphisms on that line.

If P is on the exceptional line. We may repeat the same procedure. The only different place is that h'_{T_i} might be 2 if $L_i \circ I_{i2}(H'_2)$ and $L_i \circ I_{i3}(H'_2)$ span a 2-dim space. That is why we make the requirement in the lemma. □

Back to the proof of the proposition: if I_P satisfies the requirements in the lemma for some P on S, then we get a subcomplex whose quotient complex (I'', J'', L''_i) has type $(n-1, 2n-1, n, \ldots, n, n-1)$. It is easy to see that the quotient complex is also semistable.

Since $H'_2 \otimes \mathcal{O}(-\ell_0 + \ell_1 + \cdots + \ell_k) \to \mathcal{K}'$ is injective, the injectivity of I'' implies the injectivity of I. Since the subcomplex is exact at 0-degree, the map of complexes $\mathbf{K} \to \mathbf{K}''$ induces an embedding of sheaves $\mathcal{F} \hookrightarrow \mathcal{F}''$. The torsion-freeness of $\mathcal{F}''$ implies the torsion-freeness of $\mathcal{F}$. According to the previous discussion, we may assume that I_P is injective if P is not on any exceptional line, and for any ℓ_i $(i \geq 1)$ and an element $e \in \ker I_{i1}$, we have $L_i \circ I_{i2}(e)$ and $L_i \circ I_{i3}(e)$ generate a 2-dim subspace in H_{T_i}.

Injectivity: If I is not injective, then the kernel sheaf of I is locally free, hence the dual morphism $I^T : \mathcal{O}(-\ell_1 - \cdots - \ell_k) \otimes H_1^* \to \mathcal{O}(\ell_0 - \ell_1 - \cdots - \ell_k) \otimes H_2^*$ is not surjective on any fiber, which implies $\mathrm{im} I_P^T \neq H_2^*$. Hence for any P, $\ker I_P$ is not empty, a contradiction.

Torsion-freeness: we only have to show that the cokernel of $\mathcal{O}(-\ell_0 + \ell_1 + \cdots + \ell_k) \otimes H_2 \xrightarrow{I} \mathcal{K}$ is torsion free. Since both sheaves are locally free, it is sufficient to show that the dual morphism $I^T : \mathcal{K}^* \to \mathcal{O}(\ell_0 - \ell_1 + \cdots + \ell_k) \otimes H_2^*$ is surjective. For any point P that is not on the exceptional line, the surjectivity of I^T on fiber P is the same as the injectivity of I_P which is known by the assumption. For a point that is on the exceptional line ℓ_i, restricting the morphism I^T on that line:

$$\begin{tikzcd}
(H_1^*/\mathrm{im}L_i^T) \otimes \mathcal{O}_{\ell_i}(1) \arrow[dr, bend left, "I_{i1}^T"] & \\
\oplus & H_2^* \otimes \mathcal{O}_{\ell_i}(1) \\
\mathrm{im}L_i^T \otimes \mathcal{O}_{\ell_i} \arrow[ur, bend right, "y_i I_{i2}^T + z_i I_{i3}^T"] &
\end{tikzcd}$$

The morphism at P (with coordinate (a, b)) is given by the matrix $aI_{i2}^T + bI_{i3}^T$ on $\mathrm{im}L_i^T$ part and I_{i1}^T on $(H_1^*/ \mathrm{im}L_i^T)$ part (since $I_{i1}^T \circ L_i^T = 0$, this is well-defined). If I^T is not surjective at P, then $\mathrm{im}I_{i1}^T + \mathrm{im}(aI_{i2}^T \circ L_i^T + bI_{i3}^T \circ L_i^T)$ is not the whole H_2^*, i.e., $\ker I_{i1} \cap \ker(aL_i \circ I_{i2} + bL_i \circ I_{i3}) \neq 0$. This contradicts the assumption that for any $e \in \ker I_{i1}$, $L_i \circ I_{i2}(e)$ and $L_i \circ I_{i3}(e)$ generates a 2-dimensional space.

4. Semistable: By the previous discussion, and easy computation of the invariants of sheaves, we know that $\mathcal{F} = \mathrm{H}^0(\mathbf{K})$ has rank 1 and trivial first Chern class. The torsion-freeness implies the semistability since $\mathcal{F}$ has rank 1. □

To finish the construction in proposition 2.5.5, we need the following lemma.

Lemma 2.5.4. Let $\mathcal{F}$ be a rank 1, torsion-free sheaf with trivial first Chern class on S. Then the K-complex $\mathbf{K}$ of $\mathcal{F}$ is semistable.

Proof. If $\mathbf{K}$ is not semistable, among all of the proper non-zero sub-K-complexes whose Hilbert polynomials are greater than the polynomial of $\mathbf{K}$, we may choose one $\mathbf{K}'$ with maximum $c_1(\mathbf{K}') \cdot H$ and $\chi(\mathbf{K}')$ in lexicographic order. Since $\mathrm{H}^{-1}(\mathbf{K}) = 0$, we have $\mathrm{H}^{-1}(\mathbf{K}') = 0$.

We claim the following: 1. each L'_i is surjective, $\mathcal{K}' \xrightarrow{J'} H'_0 \otimes O(\ell_0)$ is surjective; 2. $\mathrm{H}^{-1}(\mathbf{K}/\mathbf{K}') = 0$. Suppose both two claims are true, we get an injective map from $\mathrm{H}^0(\mathbf{K}')$ to $\mathrm{H}^0(\mathbf{K})$. Since the Hilbert polynomial of $\mathrm{H}^0(\mathbf{K}')$ is greater than $\mathrm{H}^0(\mathbf{K})$'s, we get the contradiction.

1. Surjectivity: If either surjectivity doesn't hold, then we may use the same argument as that in Proposition 2.5.2 and get a subcomplex $\tilde{\mathbf{K}}$ of $\mathbf{K}'$, which has greater $c_1 \cdot H$ or the same rank and greater χ. This subcomplex is non-zero since the quotient $\mathbf{K}/\tilde{\mathbf{K}}$ has Hilbert polynomial smaller than the Hilbert polynomial of $\mathbf{K}$.

2. $\mathrm{H}^{-1}(\mathbf{K}/\mathbf{K}') = 0$: write $\mathbf{K}''$ for the quotient K-complex. If $H''_2 \otimes O(\ell_1 + \cdots + \ell_k - \ell_0) \xrightarrow{I''} H''_1 \otimes O(\ell_1 + \cdots + \ell_k)$ is not injective, then for any point $P \in S$, I''_P is not injective. We may choose a point P which is not on any exceptional line ℓ_i Starting from a 1-dim subspace in $\ker I''_P$, we get a subcomplex in $\mathbf{K}''$ with positive $c_1 \cdot H$ or $c_1 \cdot H = r = 0$ and positive χ. Adding this part to $\mathbf{K}'$, we get a new subcomplex of $\mathbf{K}$ with greater $c_1 \cdot H, \chi$ in lexicographic order, this contradicts the properties of $\mathbf{K}'$. □

Let $\mathfrak{M}_S^{ss,H}(1, 0, n)$ be the moduli space of rank 1 torsion free sheaves with trivial first Chern class and Euler character χ equal to $1 - n$. By the discussion in [17] Example 4.3.6, $\mathrm{Hilb}^n S$ is canonically isomorphic to $\mathfrak{M}_S^{ss,H}(1, 0, n)$ obtained by sending subscheme $Z \subset S$ to the ideal sheaf $\mathcal{I}_Z$. On the other hand, we denote the moduli space of framed semistable K-complexes with type $(n, 2n + 1, n + 1, \ldots, n + 1, n)$ as $MK^{ss}(n)$. It is realized as a subvariety in $\mathrm{Hom}(H_2, H_1) \otimes \mathrm{Hom}(O(-\ell_0), O) \times \mathrm{Hom}(H_1, H_{T_1}) \times \ldots \times \mathrm{Hom}(H_1, H_{T_k}) \times \mathrm{Hom}(H_1, H_0) \otimes \mathrm{Hom}(O, O(\ell_0))$. We summarize this section in the following result.

Proposition 2.5.5. $\mathfrak{M}_S^{ss,H}(1, 0, n) \simeq MK^{ss}(n)/(G/\mathbb{C}^\times)$.

Proof. We will see in Proposition 3.4.1 that $\mathrm{MK}^{ss}(n)$ is smooth. Lemma 3.2.4 tells us the action of $G/\mathbb{C}^\times$ is free on $\mathrm{MK}^{ss}(n)$. Now by Luna's slice theorem [22], $MK^{ss}(n)$ is a principal $G/\mathbb{C}^\times$-bundle over $MK^{ss}(n)/(G/\mathbb{C}^\times)$. By Proposition 2.5.2 and Lemma 2.5.4, there a map between

$MK^{ss}(n)/(G/\mathbb{C}^{\times})$ and $\mathfrak{M}_S^{ss,H}(1,0,n)$ which is a set-theoretical bijection. The universal family of K-complex $\mathbf{K}_u$ on $\mathrm{MK}^{ss}(n)$ also has cohomology concentrated on the middle term. Since $\mathrm{MK}^{ss}(n)$ is smooth and the Hilbert polynomial is constant $\mathrm{H}^0(\mathbf{K}_u)$ is a flat family of sheaves on $\mathbf{P}^2$ over $\mathrm{MK}^{ss}(n)$. The map from $\mathrm{MK}^{ss}(n)$ to $\mathfrak{M}_S^{ss,H}(1,0,n)$ is a morphism. Since both $MK^{ss}(n)/(G/\mathbb{C}^{\times})$ and $\mathfrak{M}_S^{ss,H}(1,0,n)$ are smooth, they are isomorphic. $\square$

Chapter 3

Deformation of $\mathrm{Hilb}^n S$

3.1 Construction via Grassmannians

Let A be a noetherian ring, (μ_A, p_i) be a flat family of deformed del Pezzo surface as that in the appendix, H_1 be a projective module over A of rank $2n+1$. Consider the following product of Grassmanians:

$$\mathrm{Gr}_A = \mathrm{Gr}_n(H_1 \otimes_A V_A^*) \times \mathrm{Gr}^n(H_1 \otimes_A U_A) \times \mathrm{Gr}^{n+1}(H_1) \ldots (\text{k times}) \times \mathrm{Gr}^{n+1}(H_1),$$

where $\mathrm{Gr}_n(H_1 \otimes_A V_A^*)$ is the the Grassmannian of rank n subbundle of $H_1 \otimes_A V_A^*$, and $\mathrm{Gr}^n(H_1 \otimes_A U_A)$ is the rank n quotient bundles of $H_1 \otimes_A U_A$. As the case in [25], Gr_A corepresents the functor GR_A: $\mathrm{Ring}_A \to$Set: to an affine scheme $R = \mathrm{Spec} B \xrightarrow{f} \mathrm{Spec} A$, it associates the set of pairs:

$$\{(I, j, L_i) | I : H_2 \hookrightarrow H_{1,R} \otimes_{O_R} (f^* V_A^*), \quad j : H_{1,R} \otimes_{O_R} (f^* U_A) \twoheadrightarrow H_0, \quad L_i : H_{1,R} \twoheadrightarrow H_{T_i}\},$$

where $H_{1,R} := f^* H_1$, H_2 is a subbundle of rank n, H_0 is a quotient bundle of rank n, and H_{T_i}'s are of rank $n+1$. The map j induces a map

$$J : H_{1,R} \to H_{1,R} \otimes f^* U_A \otimes f^* U_A^* \xrightarrow{j \otimes \mathrm{id}} H_0 \otimes f^* U_A^*.$$

Compose J with I and we get a map from H_2 to $H_0 \otimes f^* U_A^* \otimes f^* V_A^*$.

We define N_A to be the subfunctor of GR_A which assigns pairs (I, J, L_i) for $R = \mathrm{Spec} B \xrightarrow{f} \mathrm{Spec} A$

satisfying the following conditions:

1. The image of $L_i \circ I : H_2 \to H_{T_i} \otimes_{O_R} (f^* V_A^*)$ is in $H_{T_i} \otimes_{O_R} (f^* Z_{\mu_A, p_i})$.

2. The image of $J \circ I : H_2 \to H_0 \otimes_{O_R} (f^* V_A^* \otimes f^* U_A^*$ is in $H_0 \otimes_{O_R} (f^* X_{\mu_A, p_i})$.

3. For any point Spec$F$ on R, and $1 \leq i \leq k$, J induces a map $c_{i,F}$ from $H_{1,F}$ to $H_{0,F}$ by $H_{1,F} \to H_{0,F} \otimes U_F^* \to H_{0,F} \otimes U_F^* / Y_{p_i,F} \simeq H_{0,F}$. We require that $c_{i,F}$ factors through $L_{i,F} : H_{1,F} \to H_{T_i,F}$.

(**)

Lemma 3.1.1. Let (μ_A, p_i) be a flat family of deformed noncommutative del Pezzo surface, and GR_A, N_A be as defined in Section 3.1, then there is a closed $\mathrm{GL}(H_1)$-invariant subscheme $\mathcal{N}_A \subset Gr_A$ that corepresents the subfunctor N_A.

Proof. Choose affine covers T_i =SpecA_i for Spec A such that the restricted bundles U_i, V_i, W_i, X_{ij}, Y_{ij}, Z_{ij}'s ($1 \leq j \leq k$) are free A_i modules and on each SpecA_i there are free bases of X_{ij}, Y_{ij}, Z_{ij} as A_i module, and they expand to free bases of $U_i^* \otimes V_i^*, U_i^*, V_i^*$. We can select such A_i's because of the flatness requirement (*) in the appendix on (μ_A, p_i). Consider the space of matrices $\mathrm{Hom}(A_i^n, A_i^{2n+1} \otimes V_i^*) \times \mathrm{Hom}(A_i^{2n+1} \otimes U_i, A_i^n) \times \prod_{1 \leq j \leq k} \mathrm{Hom}(A_i^{2n+1}, A_i^{n+1})$, $\mathrm{GL}(A_i, n) \times \mathrm{GL}(A_i, n) \times \prod_{1 \leq j \leq k} \mathrm{GL}(A_i, n+1)$ acts freely on an open set, and $\mathrm{Gr}_A(\mathrm{Spec}A_i)$ is just the quotient base space of this principle bundle. Lifting to the whole space, under the bases for X_{ij}, Y_{ij}, Z_{ij} the three requirements (**) for $N_A(\mathrm{Spec}A_i)$ are quadratic equations for coefficients, and the zero locus is $GL(A_i, n) \times GL(A_i, n) \times GL(A_i, n+1)$-invariant, hence the image in $Gr_A(\mathrm{Spec}A_i)$ is closed and it corepresents the functor $N_{\mathrm{Spec}A_i}$, we denote it by $\mathcal{N}_{\mathrm{Spec}A_i}$.

For different covers SpecA_s and SpecA_t, by flatness of (μ_A, p_i), each of their common fibers $\mathrm{Spec}A_s \leftarrow \mathrm{Spec}F \to \mathrm{Spec}A_t$, $\mathcal{N}_{\mathrm{Spec}A_s} \otimes$ Spec$F$ and $\mathcal{N}_{\mathrm{Spec}A_t} \otimes$Spec$F$ are naturally isomorphic to each other. That means we may glue those closed subschemes $\mathcal{N}_{\mathrm{Spec}A_i}$ in each part of $\mathrm{Gr}_A(\mathrm{Spec}A_i)$ together and get $\mathcal{N}_A$ in Gr_A which corepresents N_A. □

We may write N_A for $\mathcal{N}_A$ when there is no confusion. As an immediate result, for any ring morphism $A \to A'$, we have the induced $(\mu_{A'}, p_{A',i})$, then $N_{A'} = N_A \times_{\mathrm{Spec}A} \mathrm{Spec}A'$.

3.2 Stable locus

In this section we study the semistable locus of $\mathcal{N}$ under $SL(H_1)$ action with respect to a certain linearization sheaf. The variety Gr has natural ample line bundle $\mathcal{O}(s, l, m_1, \ldots, m_k)$ obtained by pulling back $\mathcal{O}(s)$, $\mathcal{O}(l)$ and $\mathcal{O}(m_i)$'s from projective spaces under the Plücker embeddings of the $(k+2)$ factors of Gr. We denote the open subset of $\mathcal{N}$ consisting of (semi)stable points under the $SL(H_1)$ action with respect to the line bundle $\mathcal{O}(s, l, m_1, \ldots, m_k)$ by $\mathcal{N}^s(s, l, m_1, \ldots m_k)$.

Lemma 3.2.1. Choose $t \gg 0$ according to n, then $\mathcal{N}^s((r-k)t+1-n, rt+2+n, t)(\mathcal{N}^{ss}((r-k)t+1-n, rt+2+n, t))$ corepresents the pairs (I, j, L) such that: for any $A \to F$, where F is an algeraically closed field, there is no nonzero proper subspaces $\widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_i}, \widetilde{H}_0$ of H_2, H_1, H_{T_i}, H_0 compatible with (I, j, L) such that $(r-k)\widetilde{h}_2 - r\widetilde{h}_0 + k\widetilde{h}_1 - \sum \widetilde{h}_{T_i} > 0$; or $(r-k)\widetilde{h}_2 - r\widetilde{h}_0 + k\widetilde{h}_1 - \sum \widetilde{h}_{T_i} = 0$ and $(n-1)(\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0) + \widetilde{h}_1 - 3\widetilde{h}_0 > 0 (\geq 0$ for semistable).

Proof. Let F be an algebraically closed field, then Gr(SpecF) consists of pairs $(I : H_2 \to H_1 \otimes V^*, j : H_1 \otimes U \to H_0, L_i : H_1 \to H_{T_i})$. According to [21], such a point is in $\mathcal{N}^s(s, l, m_1, \ldots, m_k)$ if and only if for any proper nonzero F-linear subspace $\widetilde{H}'_1 \subset H_1$ with $\widetilde{H}'_2 := H_2 \cap (\widetilde{H}'_1 \otimes V^*)$, $\widetilde{H}'_0$:=Im$(j : \widetilde{H}'_1 \otimes U \to H_0)$, $\widetilde{H}'_{T_i}$:=Im$(L_i : \widetilde{H}'_1 \to H_{T_i})$, we have:

$$h_1(s\widetilde{h}_2 - l\widetilde{h}_0 - \sum_{i=1}^{k} m_i\widetilde{h}_{T_i}) - \widetilde{h}_1(sh_2 - lh_0 - \sum_{i=1}^{k} m_i h_{T_i}) < 0 (\leq 0 \text{ for semistable}). \tag{$\triangle$}$$

Here h_i and $\widetilde{h}_i$'s are the dimensions of H_i, $\widetilde{H}_i$'s. It is easy to see that for any proper non-zero subspaces $\widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_i}, \widetilde{H}_0$ which are compatible with (I, j, L_i),

the inequality ($\triangle$) still holds.

Let $s = (r-k)t + 1 - n$, $l = rt + 2 + n$ and $m_i = t$ for $t \gg 0$ determined by n. Now, the left hand side of $(\triangle)$ is $(2n+1)((r-k)\widetilde{h}_2 - r\widetilde{h}_0 + k\widetilde{h}_1 - \sum \widetilde{h}_{T_i})t + (2n+1)((n-1)(\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0) + \widetilde{h}_1 - 3\widetilde{h}_0)$. By choosing t large enough, $(\triangle)$ is equivalent to two inequalities in lexicographic order in the lemma. $\square$

We denote $\mathcal{N}^{s(ss)}(2t+1-n, 3t+2+n, t)$ by $\mathcal{N}^{s(ss)}(n)$ for short. The semistable and stable locus coincide in our case.

Lemma 3.2.2. *$\mathcal{N}^s(n)$ and $\mathcal{N}^{ss}(n)$ are the same.*

Proof. : Suppose $\mathcal{N}^{ss}(n)(\mathrm{Spec}F)$ has a non-stable pair (I, j, K), then there are non-zero proper subspaces $\widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_i}, \widetilde{H}_0$ compatible with (I, j, K) such that

$$(r-k)\widetilde{h}_2 - r\widetilde{h}_0 + k\widetilde{h}_1 - \sum_{i=1}^{k} \widetilde{h}_{T_i} = 0; \tag{3.1}$$

$$(n-1)(\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0) + \widetilde{h}_1 - 3\widetilde{h}_0 = 0. \tag{3.2}$$

However this cannot happen due to an elementary computation:

Case 1: $\widetilde{h}_1 - 3\widetilde{h}_0 \geq 0$.

Since each L_i is surjective, $\widetilde{h}_1 \geq \widetilde{h}_{T_i}$, hence by (3.1), $(r-k)\widetilde{h}_2 \leq r\widetilde{h}_0 \Longrightarrow \widetilde{h}_2 \leq \frac{r}{3(r-k)}\widetilde{h}_1 \Longrightarrow \widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0 > 0$. By (3.2), $n = 1$ and $\widetilde{h}_1 = 3\widetilde{h}_0$, hence $(\widetilde{h}_2, \widetilde{h}_1, \widetilde{h}_{T_i}, \widetilde{h}_0) = (0,0,0,0)$ or $(1,3,2,\ldots,2,1)$, that corresponds to the case that $\widetilde{H}_i$'s are zero or the whole spaces.

Case 2: $\widetilde{h}_1 - 3\widetilde{h}_0 < 0$.

We may assume $n > 1$. If $\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0 \geq 2$, then by (3.2), we have $2 - 2n \geq \widetilde{h}_1 - 3\widetilde{h}_0$, plug this into (3.1), we get $(2-2n)k + (3k-r)\widetilde{h}_0 + 2\widetilde{h}_2 - \sum \widetilde{h}_{T_i} \geq 0$, this implies $\widetilde{h}_2 \geq 1$. Since $\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0 \geq 2$, we know $\widetilde{h}_1 \geq 3 + \widetilde{h}_0$. This together with $\widetilde{h}_1 - 3\widetilde{h}_0 \leq 2 - 2n \Longrightarrow 3\widetilde{h}_0 \geq 2n + 1 + \widetilde{h}_0 \Longrightarrow \widetilde{h}_0 \geq \frac{2n+1}{2} > n$, contradiction.

Hence by (3.2), $\widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0 = 1$, we have the following equalities:

$$\begin{aligned} \widetilde{h}_1 - \widetilde{h}_2 - \widetilde{h}_0 &= 1; \\ 3\widetilde{h}_0 - \widetilde{h}_1 &= n - 1; \\ (r-k)\widetilde{h}_2 - r\widetilde{h}_0 + k\widetilde{h}_1 - \sum \widetilde{h}_{T_i} &= 0. \end{aligned}$$

Solve them in term of $\widetilde{h}_1$, we get $\sum \widetilde{h}_{T_i} = ((r+k)\widetilde{h}_1 - 2rn - r + 2k + kn)/3$, since each L_i is surjective, $\widetilde{h}_1 - \widetilde{h}_{T_i} \leq n$ which implies $\widetilde{h}_1 \geq 2n+1$. Hence $\widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_i}, \widetilde{H}_0$ must be the wholes spaces, contradiction. $\square$

Lemma 3.2.3. *Let (μ_F, p_i) be a deformed noncommutative del Pezzo surface, (I, j, L) be a stable pair in $\mathcal{N}^s(n)$. Then the stabilizers of (I, j, L_i) in $GL(H_1)$ are scalars.*

Proof. Let g_1 be a stablizor action, then $g_1/a \in \mathrm{SL}(H_1)$ for some $a \in F$. Since (I, j, L_i) is in the stable locus of $SL(H_1)$, g_1/a must have finite order and is semistable. H_1 decomposes as eigenspaces $H_{1,\lambda_1} \oplus H_{1,\lambda_2} \oplus \ldots H_{1,\lambda_s}$ of g_1 with different eigenvalues $\lambda_1, \ldots, \lambda_s$ respectively. Suppose g_1 is not a scalar, then $s \geq 2$.

First, we show that $H_{T_m} = \bigoplus_i L_m(H_{1,\lambda_i})$. Suppose $L_m(x_1 + x_2 + \ldots x_s) = 0$, where $x_i \in H_{1,\lambda_i}$. Since H_{1,λ_i}'s are eigenspaces of g_1, $L_j(\lambda_1^k x_1 + \lambda_2^k x_2 + \ldots \lambda_s^k x_s) = 0$ for $k = 1, \ldots, s$. The determinant of the matrix $[\lambda_i^s]$ is non zero, hence $L_m(x_i) = 0$ for all $i = 1, \ldots, s$. Since $H_{T_m} = \sum_i L_m(H_{1,\lambda_i})$, it is $\bigoplus_i L_m(H_{1,\lambda_i})$. By a similar argument, H_0 has such a decomposition.

Next, we show that $H_2 = \bigoplus I^{-1}(H_{1,\lambda_i} \otimes V^*)$. Here, we only need $H_2 = \sum I^{-1}(H_{1,\lambda_i} \otimes V^*)$. Let x be an element in H_2, $I(x)$ decomposes into $x_1 + x_2 + \cdots + x_s$, where $x_i \in H_{1,\lambda_i} \otimes V^*$. We make induction on the number of non-zero elements of x_i's to show that $x \in \sum I^{-1}(H_{1,\lambda_i} \otimes V^*)$. If $I(x)$ only has one factor, nothing to proof. We may assume x_1, x_2 are nonzero. As g_1 is a stabilizor of the pair (I, j, K), there exists $g_2 \in GL(H_2)$ such that $I \circ g_1 = g_2 \circ I$. Now x splits into two parts: $x = (\lambda_1 - \lambda_2)^{-1}(g_2 - \lambda_2)x + (\lambda_1 - \lambda_2)^{-1}(\lambda_1 - g_2)x$. Each part's image in H_1 has factors less than that of x. By induction, these two parts are in $\sum I^{-1}(H_{1,\lambda_i})$, so is x.

Now, H_2, H_{T_j}, H_0 decompose as direct sum of (I, j, L_i) invariant subspaces $H_{2,\lambda_1} \oplus H_{2,\lambda_2} \oplus \ldots H_{2,\lambda_s}$. That contradicts the stableness, since $\sum_i((r-k)h_{2,\lambda_i} - rh_{0,\lambda_i} + kh_{1,\lambda_i} - \sum_m h_{T_m,\lambda_i}) = 0$ and $\sum_i((n-1)(h_{1,\lambda_i} - h_{0,\lambda_i} - h_{2,\lambda_i}) + h_{1,\lambda_i} - 3h_{0,\lambda_i}) = 0$. $\square$

Corollary 3.2.4. Let (μ_A, p_i) be a flat family of deformed noncommutative del Pezzo surfaces, the projection $\mathcal{N}^s(n) \to \mathcal{N}^s(n) /\!/ PGL(n)$ is a principal bundle.

We denote the base space by $\mathcal{M}^s_{\mu_A,p_i}(n)$.

3.3 Noncommutative del Pezzo surface revisited

Let (μ, p_i) be the data of a non-commutative del Pezzo surface. When the degenerate locus of μ is a smooth cubic curve, the data $\mu : U \otimes V \to W$ can be rephrased in terms of an elliptic curve E and two degree 3 line bundles on E: $(E, \mathcal{L}_1, \mathcal{L}_2)$. Here, E is the degenerate locus in $P(U)$ and $P(V)$. $U^* = \mathrm{H}^0(E, \mathcal{L}_1)$, $V^* = \mathrm{H}^0(E, \mathcal{L}_2)$ and W^* is the kernel of the following map:

$$W^* \xrightarrow{\mu^*} U^* \otimes V^* = \mathrm{H}^0(E, \mathcal{L}_1) \otimes \mathrm{H}^0(E, \mathcal{L}_2) \to \mathrm{H}^0(E, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Each p_i is a point on E. Y_{p_i} is identified as $\mathrm{H}^0(E, \mathcal{L}_1(-p_i))$ or $\mathrm{Hom}(O_E(p_i), \mathcal{L}_1)$. Both of them are consisted by the map which is 0 on the fiber p_i, or equivalently, whose cokernel contains the subquotient O_{p_i}. Similarly, $Z_{p_i} = \mathrm{H}^0(E, \mathcal{L}_2(-p_i)) = \mathrm{Hom}(O_E(p_i), \mathcal{L}_2)$. X_μ is the image of W^* in $\mathrm{H}^0(E, \mathcal{L}_1) \otimes \mathrm{H}^0(E, \mathcal{L}_2)$. $\mathrm{H}^0(E, \mathcal{L}_1 \otimes \mathcal{L}_2(-p_i))$ is $Y_{p_i} \otimes V^* + U^* \otimes Z_{p_i}/X_\mu$.

The pair $K = (I, j, L_i) \in \mathcal{N}^s(n)$ associates to the following morphisms of sheaves on E.

$$I_{\mathcal{E}} : \mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2 \xrightarrow{I} \mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes V^* \otimes H_1 \to O_E(p_1 + \cdots + p_k) \otimes H_1.$$

$$L_{\mathcal{E}} : O_E(p_1 + \cdots + p_k) \otimes H_1 \xrightarrow{\oplus L_i} O_E(p_1 + \cdots + p_k) \otimes (\oplus H_{T_i}) \to \bigoplus (O_{p_i} \otimes H_{T_i}).$$

$$J_{\mathcal{E}} : O_E \otimes H_1 \xrightarrow{J} O_E \otimes U^* \otimes H_0 \to \mathcal{L}_1 \otimes H_0.$$

The restrictions (**) require $(I_{\mathcal{E}}, J_{\mathcal{E}}, L_{\mathcal{E}})$ satisfying the followings.

1. $L_{\mathcal{E}} \circ I_{\mathcal{E}} = 0$.

2. $J_{\mathcal{E}}$ factors through a morphism $\mathcal{O}_E \otimes H_1 \hookrightarrow \mathcal{K} \to \mathcal{L}_1 \otimes H_0$, where $\mathcal{K}$ is the kernel of the map $L_{\mathcal{E}}$. Since $\mathcal{O}_E \otimes H_1$ is always in the kernel of $L_{\mathcal{E}}$, it is identified as a subsheaf of $\mathcal{K}$.
3. $\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2 \xrightarrow{I_{\mathcal{E}}} \mathcal{K} \to \mathcal{L}_1 \otimes H_0$ is a complex.

The first two requirements are easy to check. The last requirement is due to $J \circ I \subset \mathrm{Hom}(H_2, H_0) \otimes X_\mu$ and the following diagram.

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathcal{L}_2^{-1}(\dots) \otimes H_2, \mathcal{K}) & \to & \mathrm{Hom}(\mathcal{L}_2^{-1} \otimes H_2, \mathcal{O} \otimes H_1) & \simeq & V^* \otimes \mathrm{Hom}(H_2, H_1) \\
\times & & \times & & \times \\
\mathrm{Hom}(\mathcal{K}, \mathcal{L}_1 \otimes H_0) & \to & \mathrm{Hom}(\mathcal{O} \otimes H_1, \mathcal{L}_1 \otimes H_0) & \simeq & U^* \otimes \mathrm{Hom}(H_1, H_0) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{L}_2^{-1}(\dots) \otimes H_2, \mathcal{L}_1 \otimes H_0) & \to & \mathrm{Hom}(\mathcal{L}_2^{-1} \otimes H_2, \mathcal{L}_1 \otimes H_0) & \simeq & U^* \otimes V^*/X_\mu \otimes \mathrm{Hom}(H_2, H_0).
\end{array}$$

The first horizontal arrow is the embedding of Hom $(\mathcal{L}_2^{-1}(p_1+\cdots+p_k)\otimes H_2, \mathcal{K})$ into Hom $(\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2, \mathcal{O}(p_1 + \cdots + p_k) \otimes H_1)$. The second horizontal arrow is applying $\mathrm{Hom}(-, \mathcal{L}_1 \otimes H_0)$ to $\mathcal{O} \otimes H_1 \to \mathcal{K}$. The third horizontal arrow is applying $\mathrm{Hom}(-, \mathcal{L}_1 \otimes H_0)$ to $\mathcal{L}_2^{-1} \otimes H_2 \to \mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2 \to \oplus \mathcal{O}_{p_i} \otimes H_2$. This diagram commutes, thus the composition of two elements in $\mathrm{Hom}(\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2, \mathcal{K})$ and $\mathrm{Hom}(\mathcal{K}, \mathcal{L}_1 \otimes H_0)$ is 0 when $J \circ I \subset \mathrm{Hom}(H_2, H_0) \otimes X_\mu$.

In the rest of the section, we assume $k = 8$. Apply $\mathrm{Hom}(-, \mathcal{L}_1 \otimes H_0)$ to $0 \to \mathcal{K} \to \mathcal{O}(p_1 + \cdots + p_8) \otimes H_1 \xrightarrow{L_{\mathcal{E}}} \oplus \mathcal{O}_{p_i} \otimes H_{T_i} \to 0$, we get

$$0 \to \mathrm{Hom}(\mathcal{K}, \mathcal{L}_1 \otimes H_0) \to \oplus \mathrm{Ext}^1(\mathcal{O}_{p_i} \otimes H_{T_i}, \mathcal{L}_1 \otimes H_0) \xrightarrow{L_{\mathcal{E}}} \mathrm{Ext}^1(\mathcal{L}^{-1}(p_1 + \cdots + p_8) \otimes H_2, \mathcal{L}_1 \otimes H_0) \to \cdots . \quad (\diamond)$$

Namely, $J_{\mathcal{E}} \in \mathrm{Hom}(\mathcal{K}, \mathcal{L}_1 \otimes H_0)$ is determined by its image in $\mathrm{Ext}^1(\mathcal{O}_{p_i} \otimes H_{T_i}, \mathcal{L}_1 \otimes H_0)$'s, in another word, by the data $M_i \in \mathrm{Hom}(H_{T_i}, H_0)$.

Remark 3.3.1. Let v_{i1}'s be the representatives for p_i's in V, J_{i1} is defined as $p_i \circ J$ and M_i is defined as the matrix such that $J_{i1} = M_i L_i$. In this way, M_i is well-defined not only up to a scalar. Besides, M_i is identified as an element in $\mathrm{Ext}^1(\mathcal{O}_{p_i} \otimes H_{T_i}, \mathcal{L}_1 \otimes H_0)$.

Moreover, we have the following complex $\mathcal{E}_K$ in $D^b(Sh(E))$:

$$\mathcal{E}_K : \mathcal{L}_2^{-1}(p_1 + \cdots + p_8) \otimes H_2 \xrightarrow{I_{\mathcal{E}}} \mathcal{O}(p_1 + \cdots + p_8) \otimes H_1 \xrightarrow{L_{\mathcal{E}}} \oplus \mathcal{O}_{p_i} \otimes H_{T_i} \xrightarrow{\oplus M_i} \mathcal{L}_1[1] \otimes H_0.$$

Here, 'complex' means: $L_{\mathcal{E}} \circ I_{\mathcal{E}} = 0$ and $\oplus M_i \circ L_{\mathcal{E}} = 0$. The second equation is due to ($\diamond$): $\oplus M_i$ is in the image of Hom $(\mathcal{K}, \mathcal{L}_1 \otimes H_1)$, hence the image of $\oplus M_i$ in $\mathrm{Ext}^1(\mathcal{O}(p_1 + \cdots + p_8) \otimes H_1, \mathcal{L}_1 \otimes H_0)$ is 0.

3.3.1 Homological group of $\mathcal{E}_\mathbf{K}$

In this section we study the homological group of $\mathcal{E}_K$ and prove Lemma 3.3.2 that is an ingredient in the proof of Lemma 3.3.3. The method is almost the same as that in Proposition 2.5.2 for the commutative case. The different part is that J_k's and I_l's fail to satisfy some equations, for example the formula $J_k \circ I_l = J_l \circ I_k$ is not always true. That leads the construction of the subspaces $\widetilde{H}_i$'s failed. To solve this problem, we choose some suitable bases for U and V.

Suppose given $u_1 \neq u_2 \in E \subset \mathbf{P}(U)$. If im μ_{u_1} = im μ_{u_2}, then any point on the plane span$\{u_1, u_2\}$ is degenerate. This cannot happen since we assume the degenerate locus is a smooth elliptic curve. Any 3 points u_1, u_2, u_3 on E generate U (resp. V), if and only if $\mathcal{L}_1(-u_1 - u_2 - u_3) \neq \mathcal{O}$ (resp. $\mathcal{L}_2$). We may choose a $u_3 \in E$ such that u_3 together with u_1, u_2 form a base of U. Choose v_i to be a non-zero kernel of μ_{u_i} for $i = 1, 2, 3$. We may choose u_3 such that v_1, v_2, v_3 generate V and $\mu(u_2, v_3) \notin \mathrm{im}\mu_{u_1}$. Let $w_1 = \mu(u_2, v_3)$, $w_2 = \mu(u_1, v_2)$, $w_3 = \mu(u_1, v_3)$. By the previous discussion w_1, w_2, w_3 span W. Under these bases, the images of v_i's under μ_{u_j}'s are listed below.

$$\begin{array}{cc} im\mu_{u_1} & \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} im\mu_{u_2} & \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cc} im\mu_{u_3} & \begin{matrix} w_1 & w_2 & w_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Hence X_μ in $U^* \otimes V^*$ has bases:

$$w_1^* = a_{11}u_2^* \otimes v_1^* + a_{21}u_3^* \otimes v_1^* + a_{31}u_3^* \otimes v_2^* + u_2^* \otimes v_3^*,$$
$$w_2^* = a_{12}u_2^* \otimes v_1^* + a_{22}u_3^* \otimes v_1^* + a_{32}u_3^* \otimes v_2^* + u_1^* \otimes v_2^*,$$
$$w_3^* = a_{13}u_2^* \otimes v_1^* + a_{23}u_3^* \otimes v_1^* + a_{33}u_3^* \otimes v_2^* + u_1^* \otimes v_3^*.$$

Suppose $I = I_1v_1^* + I_2v_2^* + I_3v_3^*$, $J = J_1v_1^* + J_2u_2^* + J_3u_3^*$. Since $J \circ I \in \mathrm{Hom}\,(H_2, H_0) \otimes X$.

$$J_1I_1 = 0 \tag{3.3}$$
$$J_2I_2 = 0 \tag{3.4}$$
$$J_3I_3 = 0 \tag{3.5}$$
$$a_{11}J_2I_3 + a_{12}J_1I_2 + a_{13}J_1I_3 = J_2I_1 \tag{3.6}$$
$$a_{21}J_2I_3 + a_{22}J_1I_2 + a_{23}J_1I_3 = J_3I_1 \tag{3.7}$$
$$a_{31}J_2I_3 + a_{32}J_1I_2 + a_{33}J_1I_3 = J_3I_2 \tag{3.8}$$

We call such bases of U^*, V^* and W^* standard bases. Given $q \in U$ (V), we denote J_q by the maps $q \circ J$ from H_1 to H_0. When $q \in E$, J_q is just the morphism of $J_\mathcal{E}$ at fiber q up to a scalar. I_q is in a similar situation.

Lemma 3.3.2. *Let* $\mathbf{K}$= (I, J, L_i) *be a stable pair in* $\mathcal{N}(Spec\ \mathbb{C})$*, then:*

1. J_q *is surjective when* q *is not any* p_i*).* $ImJ_{p_i} + J_q(kerL_i) = H_0$*, if* $q \neq p_i$*.*

2. $kerI_{q_1} \cap kerI_{q_2} = 0$ *for any* $q_1 \neq q_2 \in V$*, i.e.,* $dim(I_{i1}(x) + I_{i2}(x) + I_{i3}(x)) \geq 2$ *for any non-zero* $x \in H_2$*.*

Proof. Statement 2: Since X_μ contains no nonzero element with form $v^* \otimes U^*$ for any element $v^* \in V^*$, if $\dim(I_{i1}(x) + I_{i2}(x) + I_{i3}(x)) = 1$, then $J \circ I|_x \in \mathrm{Hom}(x, H_0) \otimes (X_\mu \cap (v^* \otimes U^*)) = 0$. Consider the minimum subspaces of H_i's that are compatible with (I, J, L_i) and contain x, the subspace in H_0 is 0. This contradicts the stableness of $\mathbf{K}$.

Statement 1, when q is not any p_i: Choose standard bases for U, V and W with $u_2 = q$. In addition, we may choose the u_3 such that $\mathrm{im}\mu_{u_3} \neq \mathrm{im}\mu_{v_1}$. Suppose $J_q(= cJ_2)$ is not surjective. As in the commutative case, let $\widetilde{H}_0 = \mathrm{im}(J_q)$, $\widetilde{H}_1 = J_1^{-1}(\widetilde{H}_0) \bigcap J_3^{-1}(\widetilde{H}_0)$, $\widetilde{H}_{T_i} = L_i(\widetilde{H}_1)$ and

$\widetilde{H}_2 = I_3^{-1}(\widetilde{H}_1)$, if $a_{12} \neq 0$;

$\widetilde{H}_2 = I_1^{-1}(\widetilde{H}_1)$, if $a_{12} = 0$.

We first show these subspaces are compatible with (I, J, L_i). In the first case when $a_{12} \neq 0$, for any $x \in \widetilde{H}_2$, by equations (3.3), (3.5), $J_1 I_1 x$ and $J_3 I_3 x$ are in $\widetilde{H}_0$. By definition, $J_1 I_3 x$ and $J_2 I_l x$ are in $\widetilde{H}_0$ for all $l = 1, 2, 3$. Now by equations (3.6), (3.7), (3.8), the following elements are in $\widetilde{H}_0$: $a_{12} J_1 I_2 x$, $a_{22} J_1 I_2 x - J_3 I_1 x$, $a_{32} J_1 I_2 x - J_3 I_2 x$.

Since $a_{12} \neq 0$, $J_i I_l x$ is in $\widetilde{H}_0$ for any i, l.

When $a_{12} = 0$, we know that a_{13} and a_{22} are non-zero. By equations (3.3) and (3.5), $J_1 I_1 x$, $J_3 I_3 x$, $J_2 I_l x$ and $J_3 I_1 x$ are in $\widetilde{H_0}$. Then by the rest equations, the following elements are in $\widetilde{H}_0$: $a_{13} J_1 I_3 x$, $a_{22} J_1 I_2 x + a_{23} J_1 I_3 x$, $a_{32} J_1 I_2 x + a_{33} J_1 I_3 - J_3 I_2 x$.

Hence, $J_i I_l x$ is in $\widetilde{H}_0$ for any i, l. $(\widetilde{H}_2, \widetilde{H}_1, \widetilde{H}_{T_i}, \widetilde{H}_0)$ is compatible with (I, J, L_i) and does not depend on the choice of bases.

Now we may estimate the dimension of each linear subspace. Let the codimension of $\widetilde{H}_0$ be c, then the codimension of $\widetilde{H}_1$ is less than or equal to $2c$. To estimate $\tilde{h}_{T_i}$, we may choose $u_1 = p_i$. Since q is not any p_i, we can choose p_i, q, o_i as a standard base of U, then $\widetilde{H}_1 = J_{p_i}^{-1}(\widetilde{H}_0) \cap J_{o_i}^{-1}(\widetilde{H}_0)$. Since $\ker L_i \subset \ker J_{p_i} \subset J_{p_i}^{-1}(\widetilde{H}_0)$, we have $\widetilde{h}_{T_i} \leq \widetilde{h}_1 - \dim(\ker L_i \bigcap J_{o_i}^{-1}(\widetilde{H}_0)) \leq \widetilde{h}_1 - (n - c)$. Similarly, since $\mathrm{im} I_1 \subset \ker J_1 \subset J_1^{-1}(\widetilde{H}_0)$ (resp. I_3 version), $\widetilde{h}_2 \geq n - c$. This contradicts the stableness of the complex. Therefore, J_q is surjective.

The rest argument for $\mathrm{im} J_{p_i} + J_q(\ker L_i) = H_0$ is the same as that in the commutative case. $\square$

3.3.2 $\mathscr{H}om^\bullet$ complex of $\mathcal{E}_K$

We denote each term of $\mathcal{E}_K$ by $\mathcal{E}_3 = \mathcal{L}_2^{-1}(p_1 + \cdots + p_8) \otimes H_2$, $\mathcal{E}_2 = O(p_1 + \cdots + p_8) \otimes H_1$, $\mathcal{E}_1 = \oplus O_{p_i} \otimes H_i$, $\mathcal{E}_0 = \mathcal{L}_1[1] \otimes H_0$ and $\mathcal{E}_i = 0$ for $i \neq 0, 1, 2, 3$; each morphism by $e_3 = I_{\mathcal{E}}$, $e_2 = L_{\mathcal{E}}$, $e_1 = \oplus M_i$ and $e_i = 0$ for $i \neq 1, 2, 3$.

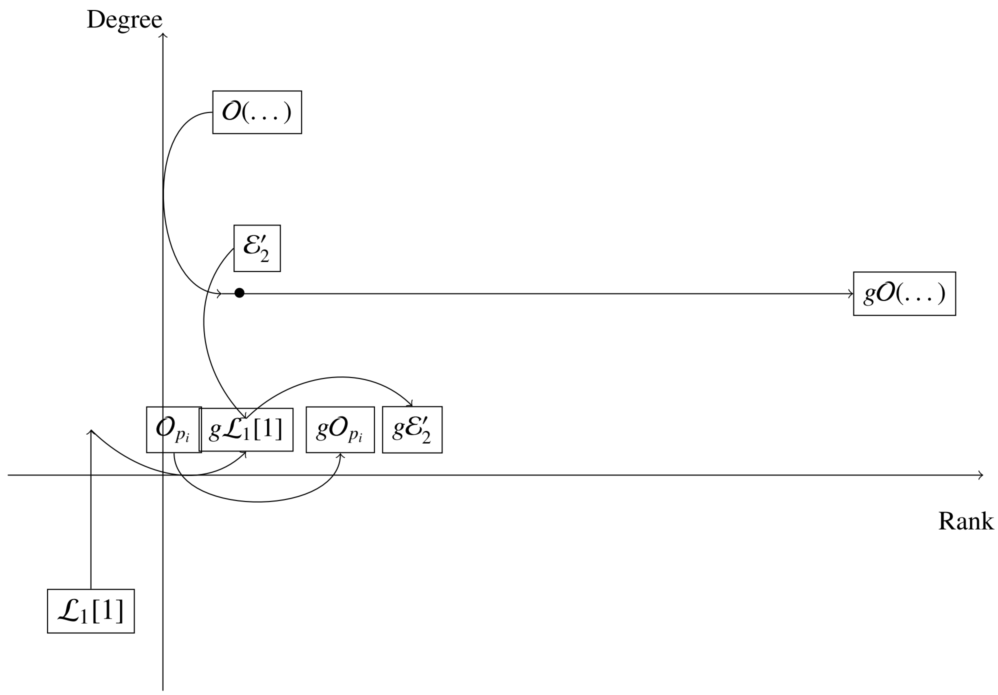
For $i = 1, 2, 3$, we may define the complex $\mathrm{Hom}^i(\mathcal{E}_K, \mathcal{E}_K)$ as

$$\mathrm{Hom}^i(\mathcal{E}_K, \mathcal{E}_K) := \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}(\mathcal{E}_{j+i}, \mathcal{E}_j).$$

For $i = 1, 2$, the derivative map $\overline{d^i}$: $\mathrm{Hom}^i(\mathcal{E}_K, \mathcal{E}_K) \to \mathrm{Hom}^{i+1}(\mathcal{E}_K, \mathcal{E}_K)$ is given as

$$(\phi_j)_{j \in \mathbb{Z}} \mapsto (e_{j+1}\phi_{j+1} + (-1)^{i+1}\phi_j e_{i+j+1})_{j \in \mathbb{Z}}.$$

We draw $\mathcal{L}_2^{-1}(p_1 + \cdots + p_8)$, $O(p_1 + \cdots + p_8)$, O_{p_i} and $\mathcal{L}_1[1]$ on the rank-degree coordinate plane of $D^b(Coh(E))$.



Since all $\mathcal{E}_i$'s are on a same half plane, there exists $g \in \mathrm{Aut}(D^b(Coh(E))) \supset \widetilde{SL_2}(\mathbb{Z}) \rtimes \mathrm{Pic}^0(E)$ that transits all $\mathcal{E}_i$'s to the right half plane. Since each $\mathcal{E}_i$ is semistable, $g(\mathcal{E}_i)$ is locally free. Let $\mathcal{F}_i := g(\mathcal{E}_i)$, $f_i := g(e_i)$ for all i. We get a new sequence $\mathcal{F}_K$ of locally free sheaves. $\mathcal{F}_K$ is a complex since $\mathcal{E}_K$ is so. Since slope($\mathcal{E}_3$) <slope($\mathcal{E}_2$) <slope($\mathcal{E}_1$) <slope($\mathcal{E}_0$), the $\mathrm{Hom}^\bullet(\mathcal{E}_K, \mathcal{E}_K)$ complex and $\mathrm{Hom}^\bullet(\mathcal{F}_K, \mathcal{F}_K)$ are naturally isomorphic.

The $\mathscr{H}om^\bullet$ complex of $\mathcal{F}_K$ is defined in the usual sense:

$$\mathscr{H}om^i(\mathcal{F}_K, \mathcal{F}_K) := \bigoplus_{j\in\mathbb{Z}} \mathscr{H}om_{O_E}(\mathcal{F}_{j+i}, \mathcal{F}_j), \text{ for all } i \in \mathbb{Z}.$$

The derivative map $d^i : \mathscr{H}om^i(\mathcal{F}_K, \mathcal{F}_K) \to \mathscr{H}om^{i+1}(\mathcal{F}_K, \mathcal{F}_K)$ is given by:

$$(\phi_j) \mapsto (f_j\phi_j + (-1)^{i+1}\phi_{j-1}f^{i+j}).$$

It is easy to check that $d^{i+1} \circ d^i = 0$ since $\mathcal{F}_K$ is a complex. d^i induces map from $\mathrm{H}^l(\mathscr{H}om^i(\mathcal{F}_K, \mathcal{F}_K))$ to $\mathrm{H}^l(\mathscr{H}om^{i+1}(\mathcal{F}_K, \mathcal{F}_K))$ for all $l \in \mathbb{Z}_{\geq 0}$, we denote them by $\overline{d^i}$. When $l = 0$, $\overline{d^i}$ is the same as the that for $\mathrm{Hom}^\bullet$ complex.

Lemma 3.3.3. Let $\mathcal{E}_K$ be the complex of sheaves induced by a stable pair $K = (I, J, M_i, L_i)$, g be an element in $\mathrm{Aut}(D^b(Coh(E)))$, such that $g(\mathcal{E}_i)$ is locally free sheaf for any i. Let $\mathcal{F}_K = g(\mathcal{E}_K)$, then $\mathrm{H}^0(\mathscr{H}om^1(\mathcal{F}_K, \mathcal{F}_K)) \xrightarrow{\overline{d^1}} \mathrm{H}^0(\mathscr{H}om^2(\mathcal{F}_K, \mathcal{F}_K)) \xrightarrow{\overline{d^2}} \mathrm{H}^0(\mathscr{H}om^3(\mathcal{F}_K, \mathcal{F}_K))$ is exact.

Proof. Since the complex $\mathscr{H}om^{-3}(\mathcal{F}_K, \mathcal{F}_K) \xrightarrow{d^{-3}} \mathscr{H}om^{-2}(\mathcal{F}_K, \mathcal{F}_K) \xrightarrow{d^{-2}} \mathscr{H}om^{-1}(\mathcal{F}_K, \mathcal{F}_K)$ is isomorphic to its dual complex $\mathscr{H}om^1(\mathcal{F}_K, \mathcal{F}_K) \xrightarrow{d^1} \mathscr{H}om^2(\mathcal{F}_K, \mathcal{F}_K) \xrightarrow{d^2} \mathscr{H}om^3(\mathcal{F}_K, \mathcal{F}_K)$. By Serre duality, the statement is equivalent to that $H^1(\mathscr{H}om^{-3}) \xrightarrow{\overline{d^{-3}}} H^1(\mathscr{H}om^{-2}) \xrightarrow{\overline{d^{-2}}} H^1(\mathscr{H}om^{-1})$ is exact.

By lemma 3.3.2, as $\mathbf{K}$ is stable, $\mathrm{H}^0(\mathcal{E}_K)$ is the only non-zero cohomological sheaf of $\mathcal{E}_K$. Write $\mathrm{H}^0(\mathcal{E}_K)$ as $\mathcal{L} \oplus Q$, where $\mathcal{L}$ is a line bundle with non-positive degree, and Q is the torsion part. It is not hard to see that $g(\mathcal{L}) = \mathcal{R}[-1]$ for some stable sheaf $\mathcal{R}$ with rank $2d - 1$ and degree d. gQ is the direct sum of some semi-stable sheaves with slope $1/2$.

Let the cokernel of $\mathcal{E}_3 \to \mathcal{E}_2$ and the extension sheaf of $\mathcal{E}_0$ by $\mathcal{E}_1$ be $Q_1 \oplus \mathcal{B}_1$ and $Q_0 \oplus \mathcal{B}_0$

respectively, where $\mathcal{Q}_i$'s are the torsion parts. Then the complex $\mathcal{G}_K := \{\ g\mathcal{Q}_1 \oplus g\mathcal{B}_1 \to g\mathcal{Q}_0 \oplus g\mathcal{B}_0\ \}$ has kernel $g\mathcal{Q}$ and cokernel $g\mathcal{L}[1]$. $\mathcal{G}_K$ and $\mathcal{F}_K$ are connected by two quasi-isomorphic maps of complexes, therefore $\mathscr{H}om^\bullet\ (\mathcal{G}_K, \mathcal{G}_K)$ and $\mathscr{H}om^\bullet\ (\mathcal{F}_K, \mathcal{F}_K)$ have the same hyper-cohomologies.

Let (g_1, g_2, g_3) be the morphism from $g\mathcal{Q}_1 \oplus g\mathcal{B}_1$ to $g\mathcal{Q}_0 \oplus g\mathcal{B}_0$ in $\mathcal{G}_K$, where $g_1 : g\mathcal{Q}_1 \to g\mathcal{Q}_0$, $g_2 : g\mathcal{Q}_1 \to g\mathcal{B}_0$, $g_3 : g\mathcal{B}_1 \to g\mathcal{B}_0$. Suppose an element $(\phi_1, \phi_2, \phi_3) \in \mathrm{Hom}\ (g\mathcal{Q}_0 \oplus g\mathcal{B}_0, g\mathcal{Q}_1 \oplus g\mathcal{B}_1)$ is in the kernel of $\overline{d^{-1}}\ \mathrm{Hom}^{-1}\ (\mathcal{G}_K, \mathcal{G}_K) \to \mathrm{Hom}^0\ (\mathcal{G}_K, \mathcal{G}_K)$. Here $\phi_1 : g\mathcal{Q}_0 \to \mathcal{Q}_1$, $\phi_2 : g\mathcal{B}_0 \to g\mathcal{Q}_1$ and $\phi_3 : g\mathcal{B}_0 \to g\mathcal{B}_1$. By the definition of $\overline{d^i}$, we have $\phi_1 g_1 = g_1 \phi_1 = 0$. Let $\mathcal{Q}'$ be the cokernel of $g\mathcal{Q}_1 \to g\mathcal{Q}_0$, then $0 \to g\mathcal{B}_1 \to \mathcal{Q}' \oplus g\mathcal{B}_0 \to g\mathcal{L}[1] \to 0$ is exact. ϕ_1 factors through $\mathcal{Q}'$. As $\phi_1 g_2 + \phi_2 g_3 = 0$, (ϕ_1, ϕ_2) factors through $g\mathcal{L}[1]$. Since the slope of $g\mathcal{L}[1]$ is greater than $1/2$, (ϕ_1, ϕ_2) is 0. Finally, since (g_2, g_3) on $g\mathcal{B}_1$ is injective, ϕ_3 is 0. Now $\mathrm{Hom}^{-1}\ (\mathcal{G}_K, \mathcal{G}_K) \to \mathrm{Hom}^0\ (\mathcal{G}_K, \mathcal{G}_K)$ is injective. Since $\mathcal{G}_K$ only has two non-zero terms, $\mathbf{H}^i\ (\mathscr{H}om^\bullet(\mathcal{G}_K,\ \mathcal{G}_K)) = 0$ unless $i = 0, 1$. In particular, the hypercohomology $\mathbf{H}^{-1}(\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K))$ is 0.

On the other hand, we may compute the hypercohomology of $\mathscr{H}om^\bullet\ (\mathcal{F}_K, \mathcal{F}_K))$ by the spectral sequence $E^1_{pq} = \mathrm{H}^q\ (\mathscr{H}om^p\ (\mathcal{F}_K,\ \mathcal{F}_K))$. The nonzero part of the page E^1_{pq} is shown below.

$$\begin{array}{cccccc}
H^1(\mathscr{H}om^{-3}) \xrightarrow{\overline{d^{-3}}} & H^1(\mathscr{H}om^{-2}) \xrightarrow{\overline{d^{-2}}} & H^1(\mathscr{H}om^{-1}) \xrightarrow{\overline{d^{-1}}} & H^1(\mathscr{H}om^{0}) & 0 & \dots \\
0 & 0 & 0 & H^0(\mathscr{H}om^{0}) \xrightarrow{\overline{d^{0}}} & H^0(\mathscr{H}om^{1}) \xrightarrow{\overline{d^{1}}} & \dots
\end{array}$$

Here we write $\mathscr{H}om^i$ instead of $\mathscr{H}om^i(\mathcal{F}_K, \mathcal{F}_K)$ for short. $\mathrm{H}^0(\mathscr{H}om^{-i})$ and $\mathrm{H}^1(\mathscr{H}om^i)$ are 0 for $i > 0$, since the slope of $\mathcal{F}_i$ is increasing. The next page reads:

$$\begin{tikzcd}
\bullet & \ker \overline{d^{-2}}/\mathrm{im}\overline{d^{-3}} \arrow[drr] & \bullet \arrow[drr] & \mathrm{coker}\ \overline{d^{-1}} \arrow[drr] & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{End}(\mathcal{F}_K) & \bullet & \bullet & \bullet
\end{tikzcd}$$

Since $\mathbf{H}^{-1}(\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K)) = 0$, $\ker\overline{d^{-2}}/\mathrm{im}\overline{d^{-3}}$ maps injectively into $\mathrm{End}(\mathcal{F}_K)$ on this page. By

Lemma 3.2.4, $\mathrm{End}(\mathcal{F}_K) = \mathbb{C}$. To get rid of the possibility that $\ker \overline{d^{-2}}/\mathrm{im}\overline{d^{-3}} = \mathbb{C}$, the rest part of the proof is to show that the map $\ker\overline{d^{-2}}/\mathrm{im}\overline{d^{-3}}$ is trivial.

Consider the natural embedding $O_E \to \mathscr{H}om^0(\mathcal{F}_i, \mathcal{F}_i)$ by mapping O_E to each factor $\mathscr{H}om(\mathcal{F}_i, \mathcal{F}_i)$ as the identity. Let $\widetilde{\mathscr{H}om}^0(\mathcal{F}_K, \mathcal{F}_K)$ be the quotient of the embedding. Since $d^0(O_E) = 0$, we get a quotient complex $\widetilde{\mathscr{H}om}^\bullet(\mathcal{F}_K, \mathcal{F}_K)$ by replacing $\mathscr{H}om^0$ by $\widetilde{\mathscr{H}om}^0$ while keeping the other terms. We show the following two things to finish the claim.

1. $\mathbf{H}^{-1}(\widetilde{\mathscr{H}om}^\bullet(\mathcal{F}_K, \mathcal{F}_K)) = 0$.
2. $\widetilde{\overline{d^0}} : \mathrm{H}^0(\widetilde{\mathscr{H}om}^0(\mathcal{F}_K, \mathcal{F}_K)) \to \mathrm{H}^0(\mathscr{H}om^1(\mathcal{F}_K, \mathcal{F}_K))$ is injective.

Fact 1: For any closed point $x \in E$, we consider the map d^{-1} of $\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K)$ on the fiber at x. The image of d_x^{-1} restricted on the factor $\mathscr{H}om(\mathcal{F}_{1,x}, \mathcal{F}_{1,x})$ is spanned by the maps $\mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \xrightarrow{f_{2,x}} \mathcal{F}_{1,x}$ and $\mathcal{F}_{1,x} \xrightarrow{f_{1,x}} \mathcal{F}_{0,x} \to \mathcal{F}_{1,x}$. Any element in $\ker f_{1,x} \setminus \mathrm{im} f_{2,x}$ is never mapped to itself by any morphism in the image of d_x^{-1} in $\mathscr{H}om(\mathcal{F}_{1,x}, \mathcal{F}_{1,x})$. $\mathrm{Ker} f_{1,x} \setminus \mathrm{im} f_{2,x}$ is not empty since the homological sheaf at $\mathcal{F}_1$ is non-zero. Thus the identity is not in the image of d^{-1}, i.e., $\mathrm{im} d^{-1}$ does not contain the image of O_E in $\mathscr{H}om^0(\mathcal{F}_K, \mathcal{F}_K)$. The homological sheaf of $\widetilde{\mathscr{H}om}^\bullet(\mathcal{F}_K, \mathcal{F}_K)$ at degree -1, which is isomorphic to $O_E \cap \mathrm{im} d^{-1}$, is a proper subsheaf of O_E. It doesn't have global section, therefore $\mathbf{H}^{-1}(\widetilde{\mathscr{H}om}^\bullet(\mathcal{F}_K, \mathcal{F}_K)) = 0$.

Fact 2: since $O_E \to \mathscr{H}om^0(\mathcal{F}_K, \mathcal{F}_K) \to \widetilde{\mathscr{H}om}^0$ splits by the trace map from $\mathscr{H}om^0(\mathcal{F}_K, \mathcal{F}_K)$ to O_E, $H^0(\widetilde{\mathscr{H}om}^0) = H^0(\mathscr{H}om^0)/H^0(O_E)$, that means $\widetilde{\overline{d^0}}$ is injective.
Now by the first claim $\ker\widetilde{\overline{d^{-2}}}/\mathrm{im}\widetilde{\overline{d^{-3}}}$ maps injectively into $\ker\widetilde{\overline{d^0}}$, which is 0 by the second claim. As $\widetilde{\overline{d^i}} = \overline{d^i}$ for $i = -2, -3$, the lemma holds. □

3.4 Smoothness of the family

Theorem 3.4.1. Let A be a finite type algebra over $\mathbb{C}$, and (μ_A, p_i) be a flat family of deformed non-commutative del Pezzo surfaces. Then $\mathcal{M}^s_{\mu_A, p_i}(n)$ is smooth over $\mathrm{Spec}\, A$.

Proof. As the same argument in the proof of [25] Theorem 8.1, the smoothness is equivelant to the smoothness of $\mathcal{N}^s(n) \to \text{Spec } A$. By the liftness criterion of smoothness, Proposition IV.17.7.7 in [13], we can prove this proposition by showing the following statement: given any local commutative $\mathbb{C}$-algebra R' with a factor ring $R = R'/\mathcal{I}$, where $\mathcal{I}^2 = 0$. Any stable pair (I, j, L_i) in $\mathcal{N}^s(n)(\text{Spec}R)$ can be lifted to a stable pair $(\widetilde{I}, \widetilde{j}, \widetilde{L_i})$ in $\mathcal{N}^s(n)(\text{Spec}R')$.

$$\begin{tikzcd}
\text{(tangent direction) Spec}R \arrow[r] \arrow[d] & \mathcal{N}^s(n) \arrow[d] \\
\text{(extending powers) Spec}R' \arrow[r] \arrow[ur, "?"] & \text{Spec}A
\end{tikzcd}$$

Suppose a stable pair (I, j, L_i) can be lifted to a pair $(\widetilde{I}, \widetilde{j}, \widetilde{L_i})$. Since $\mathcal{I}$ is nilpotent, it is contained in the kernel of any map $R' \to F$. As $R' \to F$ factors through a map $R \to F$, $(\widetilde{I}, \widetilde{j}, \widetilde{L_i}) \otimes F = (I, j, L_i) \otimes F$. Thus $(\widetilde{I}, \widetilde{j}, \widetilde{L_i})$ is a stable R'-pair.

Liftness. Since R, R' are local rings, projective R'-modules are free. Since (μ_A, p_i) satisfies the flatness condition (*), we can choose bases $\{u^*_{i2}, u^*_{i3}\}$ for Y_{p_i}(for the definition, see Appendix), $1 \leq i \leq k$. These bases can be extended to bases $\{u^*_{i2}, u^*_{i3}, u^*_{i1}\}$ for U^*. Similarly, we have bases $\{v^*_{i2}, v^*_{i3}, v^*_{i1}\}$ for V^*, where each $\{v^*_{i2}, v^*_{i3}\}$ is a base for Z_{μ,p_i}. In this way, $I = I_{i1} \otimes v^*_{i1} + I_{i2} \otimes v^*_{i2} + I_{i3} \otimes v^*_{i3}$, $J = J_{i1} \otimes u^*_{i1} + J_{i2} \otimes u^*_{i2} + J_{i3} \otimes u^*_{i3}$, where $I_{ij} \in \text{Hom }(H_{2,R'}, H_{1,R'})$ (resp. $J_{ij} \in \text{Hom }(H_{1,R'}, H_{0,R'})$). Up to a scalar I_{i1} and J_{i1} only depend on p_i. Now the restriction 1 in (**) is translated as

$$L_i \circ I_{i1} = 0.$$

The restriction 3 on the pairs in $\mathcal{N}(n)$ (Spec R') turns out to be

$$J_{i1} = M_i \circ L_i \text{ for an } M_i \in \text{ Hom}(H_{T_i,R'}, H_{0,R'}).$$

Since each L_i is surjective, M_i is determined by J_{i1} and L_i. Lifting (I, j, L_i) to $(\widetilde{I}, \widetilde{j}, \widetilde{L_i})$ is the same as lifting (I, J, M_i, L_i) to $(\widetilde{I}, \widetilde{J}, \widetilde{M}_i, \widetilde{L}_i)$.

To find a suitable lifted pair, first we lift (I, J, M_i, L_i) to a pair $(\widetilde{I}, \widetilde{J}, \widetilde{M}_i, \widetilde{L}_i)$ which may not satisfy the three restrictions in (**). Suppose $\widetilde{L}_i \circ \widetilde{I}_{i1} = C_i \in \mathrm{Hom}(H_2, H_{T_i})$; $\widetilde{J}_{p_i} - \widetilde{M}_i \circ \widetilde{L}_i = D_i \in \mathrm{Hom}(H_1, H_0)$; $\widetilde{J} \circ \widetilde{I} = B \in \mathrm{Hom}(H_2, H_0) \otimes (U^* \otimes V^*)$. Pay attention that under basis $\{u^*_{i1} \otimes v^*_{i1}, u^*_{i1} \otimes v^*_{i2} \dots, u^*_{i3} \otimes v^*_{i3}\}$ of $U^* \otimes V^*$, if we write B as $B_{i11} u^*_{i1} \otimes v^*_{i1} + B_{i12} u^*_{i1} \otimes v^*_{i2} + \cdots + B_{i33} u^*_{i3} \otimes v^*_{i3}$, then the factor $B_{i11} = M_i C_i + D_i I_{i1}$. Hence $B \in \bigcap_{1 \le i \le k} ((M_i C_i + D_i I_{i1}) u^*_{i1} \otimes v^*_{i1} + \mathrm{Hom}(H_2, H_0) \otimes (U^* \otimes Z_{\mu, p_i} + Y_{p_i} \otimes V^*))$.

Since the pair (I, J, M_i, L_i) satisfies the three restrictions in (**), we have $C_i \in \mathrm{Hom}(H_2, H_{T_i}) \otimes \mathcal{I}$, $D_i \in \mathrm{Hom}(H_1, H_0) \otimes \mathcal{I}$ and $B \in \bigcap_i ((M_i C_i + D_i I_{i1}) u^*_{i1} \otimes v^*_{i1} + \mathrm{Hom}(H_2, H_0) \otimes (U^* \otimes Z_{\mu, p_i} + Y_{p_i} \otimes V^*) \otimes \mathcal{I})$ modulo $\mathrm{Hom}(H_2, H_0) \otimes X_\mu$. Define $S_{(I_{11}, \dots, I_{k1}, M_1, \dots, M_k)}$ to be the submodule of $\bigoplus_i (\mathrm{Hom}(H_2, H_{T_i}) \oplus \mathrm{Hom}(H_1, H_0)) \oplus (\mathrm{Hom}(H_2, H_0) \otimes U^* \otimes V^* / X_\mu)$ as following.

$$S_{(I_{i1}, M_i)} := \{(C_1, \dots, C_k, D_1, \dots, D_k, B) |$$

$$B \in \bigcap_i (M_i C_i + D_i I_{i1}) \otimes u^*_{i1} \otimes v^*_{i1} + \mathrm{Hom}(H_2, H_0) \otimes ((U^* \otimes Z_{\mu, p_i} + Y_{p_i} \otimes V^*)/X_\mu)\}.$$

We need show that there is an adjustment $(I', J', M'_i, L'_i) \in \mathrm{Hom}(H_2, H_1 \otimes V^*) \otimes \mathcal{I} \times \mathrm{Hom}(H_1, H_0 \otimes U^*) \otimes \mathcal{I} \times \prod \mathrm{Hom}(H_{T_i}, H_0) \otimes \mathcal{I} \times \prod \mathrm{Hom}(H_1, H_{T_i}) \otimes \mathcal{I}$ such that:

1. $(\widetilde{L} + L') \circ (\widetilde{I}_1 + I'_1) = 0$ or equivalently, $LI'_1 + L'I_1 = -C_i$;
2. $(\widetilde{J}_{i1} + J'_{i1}) = (\widetilde{M}_i + M'_i) \circ (\widetilde{L}_i + L'_i)$ or equivalently, $J'_{i1} - M'_i L_i - M_i L'_i = -D_i$;
3. $(\widetilde{J} + J') \circ (\widetilde{I} + I') \in \mathrm{Hom}(H_2, H_0) \otimes X_\mu$ or equivalently, $JI' + J'I \in -B + \mathrm{Hom}(H_2, H_0) \otimes X_\mu$.

We summarize it as the following proposition.

Proposition 3.4.2. Let F be an algebraically closed field, and (μ_F, p_i) be a non-commutative del Pezzo surface. Let (I, j, L_i) be a stable pair in $\mathcal{N}^s_F(n)$ with $M_i \in \mathrm{Hom}(H_{T_i}, H_0)$ such that $J_{i1} = M_i \circ L_i$. Define the adjusting map as follow:

$\mathrm{Adj}_{I,J,M_i,L_i}$: $\mathrm{Hom}(H_2, H_1 \otimes V^*) \oplus \mathrm{Hom}(H_1, H_0 \otimes U^*) \oplus (\bigoplus_i \mathrm{Hom}(H_1, H_{T_i})) \oplus (\bigoplus_i \mathrm{Hom}\,(H_{T_i}, H_0))$

$\to S_{(I_{i1},M_i)}$.

$\mathrm{Adj}_{I,J,M_i,L_i}(I',J',M_i',L_i') := (L_iI_{i1}' + L_i'I_{i1}, J_{i1}' - M_i'L_i - M_iL_i', J'I + JI')$.

By the previous discussion, this map is well-defined. We claim that $\mathrm{Adj}_{I,J,M_i,L_i}$ is surjective.

Proof. Let $\mathcal{E}_K$ be the complex of sheaves induced by a stable $K = (I, J, L_i, M_i)$, then by Lemma 3.3.3, $\mathrm{Hom}^1(\mathcal{E}_K, \mathcal{E}_K) \xrightarrow{\overline{d^1}} \mathrm{Hom}^2(\mathcal{E}_K, \mathcal{E}_K) \xrightarrow{\overline{d^2}} \mathrm{Hom}^3(\mathcal{E}_K, \mathcal{E}_K)$ is exact. By Lemma 3.3.2, $I_{\mathcal{E},p}$ is injective for a general $p \in E$. We may add another blowing-up point by adding L_p and H_{T_p} into the pair, where I_p is injective and p is at a general position w.r.t p_i's and $\mathcal{L}_i$'s. L_p is given as an isomorphism from $H_1/\mathrm{im}I_p$ to H_{T_p}, since $J_p \circ I_p = 0$, J_p factor through L_p. Since I_p is injective, the new pair with the extra data L_p and H_{T_p} is still stable. In addition, if the new Adj is surjective, then the original one is surjective. We may assume $k = 8$. The rest part is due to a translation of the data.
By definition, Hom^1 $(\mathcal{E}_K, \mathcal{E}_K) = \mathrm{Hom}$ $(\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2, \mathcal{O}(p_1 + \cdots + p_k) \otimes H_1) \oplus \mathrm{Hom}$ $(\mathcal{O}(p_1 + \cdots + p_k) \otimes H_1, \bigoplus(\mathcal{O}_{p_i} \otimes H_{T_i})) \oplus \mathrm{Ext}^1$ $(\bigoplus$ $(\mathcal{O}_{p_i} \otimes H_{T_i}$ $, \mathcal{L}_1 \otimes H_0))$. Each direct sum factor corresponds to the data I', L_i', M_i' respectively.

Hom^2 $(\mathcal{E}_K, \mathcal{E}_K) = \mathrm{Hom}$ $(\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2, \bigoplus(\mathcal{O}_{p_i} \otimes H_{T_i})) \oplus$ Ext^1 $(\mathcal{O}(p_1 + \cdots + p_k) \otimes H_1, \mathcal{L}_1 \otimes H_0)$. The first factor corresponds to the data C_i's. Applying $\mathrm{Hom}(-, \mathcal{L}_1 \otimes H_0)$ to $0 \to \mathcal{O} \otimes H_1 \to \mathcal{O}(p_1 + \cdots + p_k) \otimes H_1 \to \oplus \mathcal{O}_{p_i} \otimes H_1 \to 0$, we get $0 \to \mathrm{Hom}(\mathcal{O} \otimes H_1, \mathcal{L}_1 \otimes H_0) \to \oplus_i \mathrm{Ext}^1(\mathcal{O}_{p_i} \otimes H_1, \mathcal{L}_1 \otimes H_0) \to \mathrm{Ext}^1(\mathcal{O}(p_1 + \cdots + p_k) \otimes H_1, \mathcal{L}_1 \otimes H_0) \to 0$. The second factor Ext^1 $(\mathcal{O}(p_1 + \cdots + p_k) \otimes H_1, \mathcal{L}_1 \otimes H_0)$ is Ext^1 $(\mathcal{O}_{p_i} \otimes H_1, \mathcal{L}_1 \otimes H_0)$ modulo $\mathrm{Hom}(\mathcal{O} \otimes H_1, \mathcal{L}_1 \otimes H_0)$.

Hom^3 $(\mathcal{E}_K, \mathcal{E}_K) = \mathrm{Ext}^1$ $(\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_1, \mathcal{L}_1 \otimes H_0)$. The datum $B \in \mathrm{Hom}(\mathcal{L}_2^{-1} \otimes H_2, \mathcal{L}_1 \otimes H_0)$ is determined by (C_i, D_i)'s, since $M_iC_i + D_iI_{i1}$ tells the morphism at point p_i and $k > 6$. $\overline{d^2}(C_1, \ldots, D_k) = 0$ only when $\bigcap_i((M_iC_i + D_iI_{i1})u_{i1}^* \otimes v_{i1}^* + \mathrm{Hom}(H_2, H_0) \otimes (U^* \otimes Z_{\mu,p_i} + Y_{p_i} \otimes V^*))/X_\mu$ is not empty. Now by the definitions of $\overline{d^1}$ and $\mathrm{Adj}_{I,J,M_i,L_i}$, the statement is clear. □

Back to the proof of the proposition, we show that Lemma 3.4.2 implies Proposition 3.4.1. In the proof of the proposition, we may consider the similar adjusting map of R' version. Consider the image of $\mathrm{Adj}_{I,J,M_i,L_i}$ as a R'-submodule of $S_{(I_{i1},M_i)}$. To show this is the whole space, by Nakayama Lemma, we only have to consider the tensor $R'/\mathfrak{m}$ version. And since $\mathcal{I}/\mathfrak{m}\mathcal{I}$ is a linear space over the residue field, we may assume it is one dimension. Since this is a linear map, by embedding the residue field k to its algebraic closure $\bar{k}$, the surjectiveness over $\bar{k}$ version implies the one over k. □

Recall that the quotient space $\mathcal{N}_A^s \,/\!/ \mathrm{PGL}(H_1)$ is denoted by $\mathcal{M}^s_{\mu_A,p_i}(n)$, our main technical result reads:

Theorem 3.4.3. Let A be a noetherian ring such that Spec$A$ is a smooth curve over $\mathbb{C}$ and (μ_A, p_i) be a flat family of deformed noncommutative del Pezzo surface with one fiber being the commutative S, i.e., there is a point Spec $\mathbb{C} \to$ Spec$A$ such that $(\mu_{\mathbb{C}}, p_{\mathbb{C},i})$ is the commutative datum. Then:
1. $\mathcal{M}^s_{\mu_A,p_i}(n)$ is smooth over Spec A.
2. For any closed point $b : A \to \mathbb{C}$, we have $\mathcal{M}^s_{\mu_A,p_i}(n) \otimes_b \mathbb{C} = \mathcal{M}^s_{\mu_b,p_{b,i}}(n)$ and it is a smooth, projective, irreducible scheme over $\mathbb{C}$ with dimension $2n$. Namely, $\mathcal{M}^s_{\mu_A,p_i}(n)$ is a family of deformations of the $\mathrm{Hilb}^n S$.

Proof. The smoothness of $f : \mathcal{M}^s_{\mu_A,p_i}(n) \to$ Spec$A$ is shown in Propostion 3.4.1.
The second part is proved in the same way as that in [25] Proposition 8.6. By Lemma 3.1.1, $\mathcal{M}^s_{\mu_A,p_i}(n) \otimes_b \mathbb{C} = \mathcal{M}^s_{\mu_b,p_{b,i}}(n)$. In particular, when $(\mu_b, p_{b,i})$ is the commutative data, $\mathcal{N}^s_{\mu_b,p_i}(n) = MK^s(n)$ which is defined in the previous section. By Theorem 3.4.1, $\mathcal{M}^s_{\mu_b,p_i}(n)$ is smooth and thus isomorphic to $\mathrm{Hilb}^n S$ by Proposition 2.5.5. By [27] Theorem 4, $\mathcal{M}^s(n) \to$Spec$A$ is proper, thus the image is close. By Theorem 3.4.1, this morphism is flat, thus the image is open. Now $\mathcal{M}^s_{\mu_b,p_{b,i}}(n)$ is non-empty, each fiber is non-empty. Now $f_*\mathcal{O}_{\mathcal{M}^s}$ is torsion a free sheaf over Spec$A$. As $\mathrm{H}^0(\mathcal{M}^s_{\mu_b,p_{b,i}}(n)) = \mathbb{C}$ and the function $\mathrm{h}^0(\mathcal{M}^s_{\mu_x,p_{x,i}}(n))$ is upper semi-continuous on Spec$A$ by Theorem III.12.8 in [15], $\mathrm{h}^0(\mathcal{M}^s_{\mu_x,p_{x,i}}(n))$ is actually constant 1. By Corollary III.12.9 in [15], $f_*\mathcal{O}_{\mathcal{M}^s(n)}$ is a rank 1 vector bundle. Thus each fiber $\mathcal{M}^s_{\mu_x,p_{x,i}}(n)$ is connected by Corollary III.11.3 in [15]. In addition, as each fiber is smooth, it is irreducible. □

Chapter 4

Deformation as holomorphic Poisson manifolds

4.1 Construction of the Poisson structure

Each space $\mathcal{M}^s_{\mu,p_i}(n)$ in the family carries a natural holomorphic Poisson structure that is generically symplectic. The construction of the Poisson structure is almost the same as that in [25] Section 9.1, we first recollect some of the notations from there.

Fix a positive integer n (n might be 3 or 4 in our case), let $\mathcal{M}''$ denote the moduli stack parameterizing $\mathcal{E} = (\mathcal{E}_i, e_i)_{0\le i\le n-1}$ of n-tuples of locally free sheaves $\mathcal{E}_i$ on an elliptic curve E and maps $e_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$. Let $\mathcal{M}'$ be the closed substack that parameterizes complexes of sheaves. Let $\mathcal{M} = \mathcal{M}(\mathcal{F}_n, \ldots, \mathcal{F}_1)$ be the locally closed substack of $\mathcal{M}'$ parameterizing $(\mathcal{E}_i, e_i)$ for which $\mathcal{E}_i \simeq \mathcal{F}_i$ for all i. Given a data $(\mathcal{E}_i, e_i)$, a complex $C = C(\mathcal{E}_i, e_i)$ is defined as: $C^i = \mathscr{H}om^i(\mathcal{E}, \mathcal{E})$ for $i = 0, 1$ and $C^i = 0$ for other i. By [26], the smooth locus $\mathcal{M}''_0$ in $\mathcal{M}''$ consists of points $(\mathcal{E}_i, e_i)$ whose hypercohomology of $C(\mathcal{E}_i, e_i)$ satisfies: $\mathbf{H}^2(C) = 0$ and $\mathbf{H}^0(C) = \mathbb{C}$. Let $\mathcal{M}'_0 := \mathcal{M}' \cap \mathcal{M}''_0$ and $\mathcal{M}_0 := \mathcal{M} \cap \mathcal{M}''_0$. The dual complex $C^\vee[-1]$ of C is isomorphic to the complex $\mathscr{H}om^{-1}(\mathcal{E}, \mathcal{E}) \xrightarrow{d^*} \mathscr{H}om^0(\mathcal{E}, \mathcal{E})$, concentrated in degree 0 and 1, with differential $d^* : (h_i) \mapsto (e_i h_i - h_{i-1} e_{i-1})$. By [26] Section 1, the tangent space and cotangent space to $\mathcal{M}''_0$ at $(\mathcal{E}_i, e_i)$ are identified as the hypercohomology spaces:

$$T_{(\mathcal{E}_i, e_i)}\mathcal{M}'' = \mathbf{H}^1(C); \quad T^*_{(\mathcal{E}_i, e_i)}\mathcal{M}'' = \mathbf{H}^1(C^\vee[-1]).$$

Define the maps $\psi : C^\vee[-1] \to C$ by: $\psi_0(h_i) = ((-1)^{i+1}(e_i h_i - h_{i-1} e_{i-1}))$ and $\psi_1 = 0$. By [26], Theorem 2.1, ψ globalizes to a map $\Psi'' : T^*\mathcal{M}''|_{\mathcal{M}'_0} \to T\mathcal{M}''|_{\mathcal{M}'_0}$ that at the fiber over $(\mathcal{E}_i, e_i)$ is exactly the map $\mathbf{H}^1(\psi)$.

Define a complex B by setting $B^0 = \mathscr{H}om^0(\mathcal{E}, \mathcal{E})$ and $B^1 = \mathscr{H}om^2(\mathcal{E}, \mathcal{E})$, with zero differential. Let $\Xi : C \to B$ be a map of complexes, where Ξ^0 is the identity and Ξ^1 maps $(h_i) \in C^1$ to $(e_i h_{i+1} + h_i e_{i+1})$. Now we can summarize the results in [25] Section 9.

Proposition 4.1.1 (Lemma 9.7 to Proposition 9.9 in [25])**.** *If $(\mathcal{E}_i, e_i)$ determines a point of $\mathcal{M}_0$, then under the identification that $T_{(\mathcal{E}_i,e_i)}\mathcal{M}'' = \boldsymbol{H}^1(C)$, we have $T_{(\mathcal{E}_i,e_i)}\mathcal{M}_0 = ker(\boldsymbol{H}^1(\Xi))$ and $T^*_{(\mathcal{E}_i,e_i)}\mathcal{M}_0$ is the cokernel of the dual map. $\Psi''|_{\mathcal{M}_0}$ factors through a map $\Psi : T^*\mathcal{M}_0 \to T\mathcal{M}_0$, and Ψ is a Poisson structure on $\mathcal{M}_0$.*

Now applying this theorem, we can construct the Poisson structure on the deformed Hilbert schemes $\mathcal{M}^s_{\mu,p_i}(n)$.

Proposition 4.1.2. *Suppose k =1, 7, or 8, let $(\mu_{\mathbb{C}}, p_i)$ be a data of noncommutative del Pezzo surface, then $\mathcal{M}^s_{\mu,p_i}(n)$ admits a Poisson structure which is generically symplectic.*

Proof. Let E be the smooth elliptic curve of the degenerate locus of μ; $\mathcal{L}_1$, $\mathcal{L}_2$ be the two degree 3 line bundles as before. When $k \geq 7$, there exists g in $\mathrm{Aut}(D^b(\mathcal{S}h(E)))$ such that $g\mathcal{L}_2^{-1}(p_1 + \cdots + p_k)$, $\ldots$, $g\mathcal{L}_1[1]$ are locally free sheaves concentrate on degree 0. Let $\mathcal{F}_i$ be the sheaves as that in the previous section: $\mathcal{F}_3 = g\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2$, $\mathcal{F}_2 = g\mathcal{O}(p_1 + \cdots + p_k) \otimes H_1$, $\mathcal{F}_1 = \oplus g\mathcal{O}_{p_i} \otimes H_{T_i}$, $\mathcal{F}_0 = g\mathcal{L}_1[1] \otimes H_0$.

$\mathcal{M}^s_{\mu,p_i}(n)$ is identified as a substack of $\mathcal{M}(\mathcal{F}_3, \ldots, \mathcal{F}_0)$. By Theorem 3.4.3, it is smooth with dimension $2n$, which is the dimension of $\mathcal{M}_0(\mathcal{F}_3, \ldots, \mathcal{F}_0)$, to show that $\mathcal{M}^s_{\mu,p_i}(n)$ admits a Poisson structure, we only need check that the stable pairs fall into the smooth locus of $\mathcal{M}''$, or equivalently, $\mathbf{H}^2(C(\mathcal{F}_3, \ldots, \mathcal{F}_0)) = 0$ and $\mathbf{H}^0(C) = \mathbb{C}$. The first equality is due to $\mathrm{H}^1(\mathscr{H}om^1(\mathcal{F}_K, \mathcal{F}_K)) = 0$. The second one: $\mathbf{H}^0(C) = \mathrm{End}(\mathcal{E}_K) = \mathbb{C}$ is due to Lemma 3.2.4.

When $k = 1$, let $\mathcal{F}_2 = \mathcal{L}_2^{-1}(p) \otimes H_2, \mathcal{F}_1 = \mathcal{K}, \mathcal{F}_0 = \mathcal{L}_1 \otimes H_0$, where $\mathcal{K}$ is $(O \otimes H_1/H_T) \oplus (O(p) \otimes H_T)$. Since the kernel of $L : O(p) \otimes H_1 \to O_p \otimes H_T$ is always isomorphic to $\mathcal{K}$, a stable pair associates to such a complex. The next lemma shows that $\mathcal{M}^s_{\mu,p}(n)$ is identified as a substack in $\mathcal{M}_0(\mathcal{F}_2, \mathcal{F}_1, \mathcal{F}_0)$.

Lemma 4.1.3. Let (I, J, L) be a pair in $\mathcal{N}^s(n)$, $\mathcal{F}_K$ be the complex of sheaves on E associate to it. Then $\mathcal{F}_K$ is exact except the middle term, and $\mathrm{End}_{O_E}(\mathcal{F}_K, \mathcal{F}_K) = \mathbb{C}$.

Proof. By Lemma 3.3.2, $J_{\mathcal{E}}$ is surjective and $I_{\mathcal{E}}$ is injective.

Suppose we have morphism (t_2, t_1, t_0) in $\mathrm{End}_{O_E}(\mathcal{E}_K, \mathcal{E}_K)$. Then $t_2 \in \mathrm{Hom}(\mathcal{L}_2^{-1}(p) \otimes H_2, \mathcal{L}_2^{-1}(p), \otimes H_2) \simeq \mathrm{Hom}(H_2, H_2)$, $t_0 \in \mathrm{Hom}(\mathcal{L}_1 \otimes H_0, \mathcal{L}_1, \otimes H_0) \simeq \mathrm{Hom}(H_0, H_0)$. Since $\mathcal{K}$ is always isomorphic to $O(p) \otimes \ker L \oplus H_1/\ker L \otimes O$, $t_1 \in \mathrm{Hom}(\mathcal{K}, \mathcal{K})$ is identified as an endomorphism $\widetilde{t_1}$ of H_1 which maps $\ker L$ to $\ker L$. That means we can write an endomorphism t_T of H_T such that $t_T L = L\widetilde{t_1}$. It is easy to check that $(t_2, \widetilde{t_1}, t_T, t_0)$ is an endomorphism of (I, J, M, L) by Lemma 3.2.4. Thus (t_2, t_1, t_0) is a scalar. □

Back to the proof of the proposition. The lemma implies $\mathbf{H}^0(C(\mathcal{F}_2, \mathcal{F}_1, \mathcal{F}_0)) = \mathbb{C}$, and since the slopes of $\mathcal{F}_i$'s are increasing, $\mathbf{H}^2(C) = 0$. This finishes the construction of Poisson structure for $k = 1$ case.

At the last, we show that the Poisson structure is generically symplectic. By Proposition 4.1.1, the tangent space at $(\mathcal{F}_i, f_i)$ of $\mathcal{M}_0(\mathcal{F}_i)$ is identified as the homological vector space at the middle term of the complex $\mathrm{H}^0(\mathscr{H}om^0(\mathcal{F}_K, \mathcal{F}_K)) \to \mathrm{H}^0(\mathscr{H}om^1(\mathcal{F}_K, \mathcal{F}_K)) \to \mathrm{H}^0(\mathscr{H}om^2(\mathcal{F}_K, \mathcal{F}_K))$. The cotangent space is given by the homological vector space at the middle term of the complex $\mathrm{H}^1(\mathscr{H}om^{-2}(\mathcal{F}_K, \mathcal{F}_K)) \to \mathrm{H}^1(\mathscr{H}om^{-1}(\mathcal{F}_K, \mathcal{F}_K)) \to \mathrm{H}^1(\mathscr{H}om^0(\mathcal{F}_K, \mathcal{F}_K))$. The Poisson map is given by the map between these two spaces on the third page of the spectral sequence that computes the hypercohomology of $\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K)$. By the previous discussion in Lemma 3.3.3, the two scalars $\mathbb{C}$ on the last picture would stay. The Poisson map is surjective if and only if the hypercohomology $\mathbf{H}^1(\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K))$ is $\mathbb{C}$. An equivalent description is that the homological sheaf of

$\mathcal{F}_K$ is concentrate on one degree and is stable. No matter what k is, that means the homological sheaf of $\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2 \to \mathcal{K} \to \mathcal{L}_1 \otimes H_0$ concentrates at the middle and has no torsion part, i.e., it is isomorphic to $(\mathcal{L}_2^{-1} \otimes \mathcal{L}_1)^{\otimes n}$. That corresponds to the deformation of $\mathrm{Hilb}^n(S \setminus E)$ for commutative del Pezzo surface cutting an anti-canonical elliptic curve. □

Remark 4.1.4. One can also calculate the rank of the Poisson map on the degenerating locus by the torsion part of the homological sheaf of $\mathcal{E}_K$. In the commutative case, this coincides with the result in [4].

4.2 Generic dimension of deformation space

In this part, we apply the result in [16] to show that the generic deformation of $\mathrm{Hilb}^n S$ has a $(k+2)$-dimensional space of moduli. Some notations and results in [16] are collected as the followings. Let $\sigma \in H^0(S, K^*)$ be a non-zero holomorphic Poisson structure on S, then by [4], it induces a Poisson structure τ on $\mathrm{Hilb}^n S$. When $n \geq 2$, let F be the exceptional divisor of the Hilbert-Chow map $\mathrm{Hilb}^n S \to \mathrm{Sym}^n S$, then $[F]$ stands in $\mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{T}^*)$. Theorem 1 in [16] tells us that any class $\tau([F]) \in \mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{T})$ is tangent to a deformation of complex structure. Moreover, there is a split exact sequence:

$$0 \to \mathrm{H}^1(S, \mathcal{T}) \to \mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{T}) \xrightarrow{\rho} \mathrm{H}^0(S, K^*) \to 0$$

Theorem 4.2.1 (Theorem 9 in [16])**.** $\rho(\tau([F])) = -2\sigma$. □

The theorem tells us that each deformation of the complex structure of $\mathrm{Hilb}^n S$ is induced by $\tau([F]) + \phi$ that relates to some Poisson structure σ on S and a class in $\mathrm{H}^1(S, \mathcal{T})$, which induces deformation of S. The following proposition is just an easy exercise after reading [16]'s Proposition 11.

Proposition 4.2.2. Let σ be a Poisson stucture on S whose zero set is a smooth elliptic curve, let M be the deformation of $\mathrm{Hilb}^n S$ induced by $\tau([F])$ for sufficiently small $t \neq 0$. Then $\dim H^1(M, \mathcal{T}) \leq$

$k+2$.

Proof. Repeat the argument in [16] Proposition 11, when $k \leq 4$, we replace all $\mathbf{P}^2$ there by S. Since S is rigid, the differences are the dimensions of some cohomology groups. The dimension of $\mathrm{H}^1(S^{[n]}, \mathcal{T})(\simeq \mathrm{H}^0(S, K^*))$ is $10-k$. $\mathrm{h}^0(S^{[n]}, \mathcal{T}) = \mathrm{h}^0(S, \mathcal{T}) = 8-2k$. Since there are only finite -1-curves on S, the holomorphic vector fields on S must fix those curves. Since the zero set of the Poisson structure σ is a smooth elliptic curve that intersects all exceptional curves, it is not fixed by any non-zero vector field as the case in $\mathbf{P}^2$. The dimension of $\mathrm{H}^1(M, T)$ is at most $\mathrm{H}^1(M, \mathcal{T}) - \mathrm{H}^0(M, \mathcal{T}) = 2+k$.

When $k > 4$, by the upper semi-continuity property, dim $\mathrm{H}^1(M, \mathcal{T}) \leq$ dim $\mathrm{H}^1(S^{[n]}, \mathcal{T}) =$ dim $\mathrm{H}^1(S, \mathcal{T}) +$ dim $\mathrm{H}^0(S, K^*) = (2k-8)+(10-k) = 2+k$. □

On the other hand, we have constructed a $(k+2)$-parameters family deformations of $\mathrm{Hilb}^n S$ with natural Poisson structures, hence H^1 (M, T) is a $(k+2)$-dimensional space. The Kodaira-Spencer class of each tangent direction at $\mathrm{Hilb}^n S$ can be explained explicitly. The variation of the positions of p_i's on E contributes to the factor $\mathrm{H}^1(S, \mathcal{T})$ (when it is non-zero) in $\mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{T})$. The variation of $\mathcal{L}_2^{-1} \otimes \mathcal{L}_1$ contributes to the factor $\tau([F])$, whose degenerate locus is E, in $\mathrm{H}^0(S, K^*)$. The last assertion is due to the following result in [16] and a computation of line bundle with first Chern class $[F]$ restricted on $\mathrm{Sym}^n E$.

Proposition 4.2.3 (Proposition 10 in [16])**.** Under the deformation of $\mathrm{Hilb}^n S$ induced by $\tau([F])$, the line bundle with Chern class $[F]$ restricted to the zero set $\mathrm{Sym}^n E$ of the Poisson structure varies linear in t in $\mathrm{H}^1(\mathrm{Sym}^n E, O^*)$. □

The $\mathrm{Sym}^n E$ in $\mathcal{M}_0''$ consists of pairs such that the Poisson map Ψ is 0, i.e., $\mathbf{H}^1(\mathscr{H}om^\bullet(\mathcal{F}_K, \mathcal{F}_K))$ is $\mathbb{C}^{2n+1}$, i.e., the middle term cohomological sheaf of $\mathcal{L}_2^{-1}(p_1 + \cdots + p_k) \otimes H_2 \to \mathcal{K} \to \mathcal{L}_1 \otimes H_0$ is $\mathcal{L} \oplus Q$ for some line bundle with degree $-n$. Q is a torsion sheaf of length n and is a quotient sheaf of O. Such torsion sheaves are naturally identified as $\mathrm{Sym}^n E$. The exceptional divisor F is more subtle. A K-complex $\mathbf{K}$ is in F if and only if it is S-equivalent (w.r.t to $G/\mathbb{C}^\times$ action and character (det,0,. . . ,0,det^{-1})) to another K-complex, i.e. it has a filtration $0 = \mathbf{K}_0 \subset \mathbf{K}_1 \ldots \subset \mathbf{K}_m = \mathbf{K}$ such that there is another $\mathbf{K}'$ non isomorphic to $\mathbf{K}$ who has an isomorphic filtration $0 = \mathbf{K}'_0 \subset \mathbf{K}'_1 \ldots \subset \mathbf{K}'_m = \mathbf{K}'$ in the sense that $\mathbf{K}_{j+1}/\mathbf{K}_j \simeq \mathbf{K}'_{j+1}/\ \mathbf{K}'_j$ has type $(l, *, \ldots, *, l)$.

We call a K-complex with type $(1, 2, 1, \ldots, 1)$ a resolution of point p if its associate complex $\mathcal{L}_2^{-1}(p_1 + \ldots p_k) \to \mathcal{K} \to \mathcal{L}_1$ on E has homological sheaves $\mathcal{O}_p$ and $\mathcal{O}_{\iota(p)}$ at the last two terms. ι is an automorphism of E and depends linearly on $\mathcal{L}_2^{-1} \otimes \mathcal{L}_1$. In the commutative case, when $\mathcal{L}_1 = \mathcal{L}_2$, $F \cap \mathrm{Sym}^n E$ is the big diagonal of $\mathrm{Sym}^n E$ where at least two points coincide. It consists of the K-complexes with a filtration $0 = \mathbf{K}_0 \subset \mathbf{K}_1 \ldots \subset \mathbf{K}_n = \mathbf{K}$ where both $\mathbf{K}_n/\ \mathbf{K}_{n-1} \simeq \mathbf{K}_{n-1}/\ \mathbf{K}_{n-2}$ have type $(1, 2, 1, \ldots, 1)$ and are resolutions of a same $p \in E$; each $\mathbf{K}_{j+1}/\ \mathbf{K}_j$ is a resolution of point on E or with type $(1, 3, 1, \ldots)$. In the non-commutative case, there is only one non-trivial extension of a resolution of p by itself, and the torsion sheaf $\mathcal{Q}$ is a quotient sheaf of $\mathcal{O}_E$. Defferent from the commutative case, containing two same factor is not the feature of $F \cap \mathrm{Sym}^n E$. On the other hand, there is a non-trivial extension of p-resolution complex by $\iota^{-1}(p)$-resolution complex. $F \cap \mathrm{Sym}^n E$ contains $\mathbf{K}$ which has two such factors. As a result, the line bundle with Chern class $[F]$ restricted to $\mathrm{Sym}^n E$ varies linearly in $\mathrm{H}^1(\ \mathrm{Sym}^n E, \mathcal{O}^*)$ along the deformation induced by the variation of $\mathcal{L}_2^{-1} \otimes \mathcal{L}_1$. By Theorem 4.2.3, $\tau([F])$ is tangent to this deformation direction.

Appendix

Extremal rays of the effective cone of D_n

We complete the proof of Lemma 1.2.4. All the arguments are elementary and well-known to the experts on unimodular lattice.

Claim: any tuple A of non-negative real numbers $(a_0, a_1, \dots, a_{9-n})$, $1 \leq n \leq 7$, satisfying the following constrains

$$
\begin{aligned}
a_i &\geq 0;\\
a_0^2 - (a_1^2 + \cdots + a_{9-n}^2) &\leq 0;\\
a_0 &\geq a_i + a_j;\\
2a_0 &\geq a_i + a_j + a_k + a_l + a_m;\\
3a_0 &\geq 2a_i + a_j + a_k + a_l + a_m + a_o + a_p;\\
4a_0 &\geq 2a_i + 2a_j + 2a_k + a_l + a_m + a_o + a_p + a_r;\\
5a_0 &\geq 2a_i + 2a_j + 2a_k + 2a_l + 2a_m + 2a_o + a_p + a_r;\\
6a_0 &\geq 3a_i + 2a_j + 2a_k + 2a_l + 2a_m + 2a_o + 2a_p + 2a_r
\end{aligned}
$$

is in the cone spanned by tuples listed in Lemma 1.2.4.

Proof. We may assume $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{9-n}$.

$n = 7$: $(a_0, a_1, a_2) = (1, 1, 1)$ or $(a, a, 0)$.

$n = 6$: We may assume $a_1 \geq a_0/2$, then $a_0^2 \geq a_1^2 + 2(a_0 - a_1)^2 \geq a_1^2 + a_2^2 + a_3^2$. The equality holds only when $a_1 = a_0$ and $a_2 = a_3 = 0$.

$n = 5$: Again, $a_0^2 \geq a_1^2 + 3(a_0 - a_1)^2$. When $a_0 \geq 1$, the equality holds only when $a_0 = a_1$ or $a_i = a_0/2$ for all $1 \leq i \leq 4$. In the last case, the tuple $(2n, n, n, n, n)$ is spanned by $(2, 1, 1, 0, 0)$ and $(2, 0, 0, 1, 1)$.

$n = 4$: $2a_1a_2 \leq a_0^2$ - a_1^2 - $a_2^2 \leq a_3^2$ $+a_4^2$ $+a_5^2 \leq 3a_2^2$.
$\Rightarrow a_2 \geq \frac{2}{3}a_1$ or $a_2 = 0$. If $a_2 = 0$, then $a_1 = a_0$ and other a_i's are zero. We may assume $a_2 \geq \frac{2}{3}a_1$. We may assume $a_5 > 0$, since else the problem reduces to $n = 5$ case. We may therefore assume $a_3 = a_2$, since else one may always use $(a_0, a_1, a_2, a_3 + a, a_4, a_5 - a)$ as a new tuple which also satisfies the constraints. Now we have the following two inequalities:

$$a_1 \geq a_4 + a_5;$$
$$2a_1a_2 \leq a_2^2 + a_4^2 + a_5^2.$$

These imply a_5 must be 0 and the tuple is in the form $(2n, n, n, n, n, 0)$.

$n = 3$: As we may replace (a_1, a_2) by $(a_1 + a, a_2 - a)$, for the similar reason as that in the $n = 4$ case, we may assume that $a_2 = a_3$. As we may also adjust (a_1, a_5), we may assume that $a_5 = a_6 > 0$. Case 1, $a_4 \neq a_5$. If $a_1 + a_2 \neq a_0$, then we may replace (a_1, a_4) by $(a_1 + a, a_4 - a)$. Therefore we may assume that $a_0 = a_1 + a_2$. Now we have the following inequalities:

$$a_1 \geq a_4 + a_5;$$
$$2a_1a_2 \leq a_2^2 + a_4^2 + 2a_5^2.$$

By the first inequality, $a_1^2 \geq a_4^2 + 3a_5^2$. Hence by the second inequality, we have $(a_1 - a_2)^2 \geq a_5^2$, in another word, $a_1 - a_2 \geq a_5$. The tuple is in the form $(2n, n, n, n, n, 0, 0)$.

Case 2, $a_4 = a_5$. If $a_1 + a_2 = a_0$, then by the same argument in Case 1, we get contradiction. We may assume $2a_0 = a_1 + a_2 + a_3 + a_4 + a_5 = a_1 + 2a_2 + 2a_5$, since else we may replace a_0 by some $a_0 - a$. Now we have the following inequalities:

$$a_5 \geq \frac{a_1}{2};$$
$$(\frac{a_1}{2} + a_2 + a_5)^2 \leq a_1^2 + 2a_2^2 + 3a_5^2.$$

As a quadric function of a_1, the second inequality holds either when $a_1 = 2a_5$ or when $a_1 = a_2$. In the case that $a_1 = 2a_5$, we have $a_2^2 - 4a_2a_5 + 3a_5^2 \geq 0$. Hence the tuple is in the form $(3n, 2n, n, n, n, n, n)$ which is spanned by $(1, 1, 1, 0, 0, 0, 0)$ and $(2, 1, 0, 1, 1, 1, 1)$. In the case that $a_1 = a_2$, we have $3a_2^2 - 12a_2a_5 + 8a_5 \geq 0$. Since $a_2 \geq a_5$, we have $a_2 > 2a_5 \geq a_1$, contradiction.

$n = 2$: Recall that the new constrain in this case is:

$$3a_0 \geq 2a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7.$$

By a similar discussion as that in the $n = 3$ case, we may assume $a_7 > 0$ and $a_5 = a_6$. In addition, $a_3 = a_2$ or $a_4 = a_5$.

Case 1, $a_1 \geq 2a_7$. Since we may replace the pair (a_1, a_7) by $(a_1 + a, a_7 - 2a)$, we may assume that either $a_0 = a_1 + a_2$ or $2a_0 = a_1 + a_2 + a_3 + a_4 + a_5$.

Case 1, $2a_0 > a_1 + a_2 + a_3 + a_4 + a_5$ and $a_0 = a_1 + a_2$. Since we may adjust the pair (a_3, a_7), we can assume that $a_3 = a_2$. For the same reason $a_4 = a_5 = a_2$. Now we have the following inequalities:

$$a_1 \geq 2a_2;$$

$$a_1 + 3a_2 \geq 5a_2 + a_7;$$

$$2a_1a_2 \leq 4a_2^2 + a_7^2.$$

The last two inequalities imply that $a_7 = 0$, contradiction.

Case 2, $a_0 > a_1 + a_2$ and $2a_0 = a_1 + a_2 + a_3 + a_4 + a_5$. We may adjust pair (a_2, a_3), (a_2, a_4) or (a_3, a_4), hence we may assume either $a_3 = a_4 = a_5$; $a_3 = a_2 = a_1$; or $a_1 = a_2$ and $a_5 = a_4$. Now we have the inequalities:

$$a_3 + a_4 + a_5 > a_1 + a_2;$$

$$a_2 + a_3 + a_4 \geq a_1 + a_5 + 2a_7;$$

$$(a_1 + a_2 + a_3 + a_4 + a_5)^2 \leq 4a_1^2 + 4a_2^2 + 4a_3^2 + 4a_4^2 + 8a_5^2 + 4a_7^2.$$

In the case that $a_3 = a_4 = a_5$, the inequalities becomes:

$$3a_3 > a_1 + a_2;$$

$$a_2 + a_3 \geq a_1 + 2a_7;$$

$$2a_1a_2 + 6a_1a_3 + 6a_2a_3 \leq 3a_1^2 + 3a_2^2 + 7a_3^2 + 4a_7^2.$$

Combining the last two inequalities, we have $a_1a_2 + 2a_1a_3 + a_2a_3 \leq a_1^2 + a_2^2 + 2a_3^2$. Hence $(a_1 - a_3)^2 + (a_2 - a_3)^2 \geq a_2(a_1 - a_3)$. We have $a_2 \geq a_1 - a_3 + a_2 - a_3$, contradiction.

In the case that $a_3 = a_2 = a_1$, the inequalities becomes:

$$a_4 + a_5 > a_1;$$

$$a_1 + a_4 \geq a_5 + 2a_7;$$

$$(3a_1 + a_4 + a_5)^2 \leq 12a_1^2 + 4a_4^2 + 8a_5^2 + 4a_7^2.$$

Combining the last two inequalities, we have $a_1a_4 + 2a_1a_5 + a_4a_5 \leq a_1^2 + a_4^2 + 2a_5^2$. For the same reason, we have $2a_5 \leq a_1$. We may rewrite the last inequality as:

$$6a_1a_4 + 6a_1a_5 + 2a_4a_5 \leq 3a_1^2 + 3a_4^2 + 11a_5^2.$$

Hence $a_5(6a_1 + 2a_4 - 11a_5) \leq 3(a_1 - a_4)^2 \leq 3a_5^2$. We have $2a_5 = a_1$ and $a_4 + a_5 = a_1$, contradiction.
In the case that $a_1 = a_2$ and $a_5 = a_4$, the inequalities becomes:

$$2a_3 + a_5 > 2a_1;$$

$$a_3 \geq 2a_7;$$

$$(2a_1 + a_3 + 2a_4)^2 \leq 8a_1^2 + 4a_3^2 + 8a_4^2 + 4a_7^2.$$

Combining the last two inequalities, we have $a_1a_3 + 2a_1a_4 + a_3a_4 \leq a_1^2 + a_3^2 + a_4^2$. In a similar way, we get contradiction.
Case 3, $a_0 = a_1 + a_2$ and $2a_0 = a_1 + a_2 + a_3 + a_4 + a_5$. We may assume $a_3 = a_2$ or $a_4 = a_5$. In any case, we have the following constrains:

$$a_3 + a_4 + a_5 = a_1 + a_2;$$

$$a_2 + a_3 + a_4 \geq a_1 + a_5 + 2a_7;$$

$$2a_1a_2 \leq a_3^2 + a_4^2 + 2a_5^2 + a_7^2.$$

In the case that $a_3 = a_2$, we have $a_1 = a_4 + a_5$ and inequalities:

$$a_2 \geq a_5 + a_7;$$

$$2a_2a_1 \leq a_2^2 + a_4^2 + 2a_5^2 + a_7^2.$$

We have $(a_1 - a_2)^2 \geq 2a_4a_5 - a_5^2 - a_7^2$. Since $a_2 \geq a_4$, we have $2a_4a_5 \leq 2a_5^2 + a_7^2$, which implies $a_4 \leq a_5 + \frac{a_7}{2} \leq a_2 - \frac{a_7}{2}$. Suppose $a_7 = la_5$, for some $0 < l \leq 1$, then $2a_4a_5 - a_5^2 - l^2a_5^2 \leq (1 - \frac{l}{2})^2a_5^2$. We have $l \geq \frac{4}{5}$ and $a_4 \leq \frac{9}{8}a_5$. On the other hand, $a_1 - a_2 \leq a_4 - a_7$. Hence $2a_4a_7 + 2a_4a_5 \leq a_4^2 + 2a_7^2 + a_5^2$. This implies $a_4 = a_5 = a_7$, and the tuple is in the form $(4n, 2n, 2n, 2n, n, n, n, n)$.

In the case that $a_4 = a_5$, we have the following constrains:

$$a_3 + 2a_4 = a_1 + a_2;$$

$$a_2 \geq a_4 + a_7;$$

$$2a_1a_2 \leq a_3^2 + 3a_4^2 + a_7^2.$$

Plug the the first inequalities into the third inequality, we have

$$2(a_2 - a_4)a_4 \leq (a_2 - a_3)^2 + 2(a_2 - a_4)^2.$$

Hence $3a_2 \geq a_3 + 4a_4$. Combining with the first inequality, we have $a_1 + 2a_4 \leq 2a_2$. Therefore, $a_2 \geq 2a_4$. By the third inequality, $2a_1a_2 \leq a_3^2 + a_2^2$. Hence the tuple is in the form $(4n, 2n, 2n, 2n, n, n, n, n)$.

Case 4, $a_0 > a_1 + a_2$ and $2a_0 > a_1 + a_2 + a_3 + a_4 + a_5$. We may assume $3a_0 = 2a_1 + a_2 + a_3 + a_4 + 2a_5 + a_7$. Since we may adjust the pair (a_2, a_7); (a_3, a_7); (a_4, a_7) and (a_5, a_7). We may assume that $a_2 = a_3 =$

$a_4 = a_5 = a_1$. We have the following inequalities:

$$a_1 \leq 2a_7;$$

$$3a_0 = 7a_1 + a_7;$$

$$a_0^2 \leq 6a_1^2 + a_7^2.$$

Plug the second equality into the last inequality, we have $(5a_1 - 4a_7)(a_1 - 2a_7) \geq 0$. Hence the tuple is in the form $(5n, 2n, 2n, 2n, 2n, 2n, 2n, n)$.

$\square$

Cones of $\text{Hilb}^n\mathbf{P}^2$

In this subsection, we discuss the birational geometry and deformation of $\text{Hilb}^n\mathbf{P}^2$ for some small numbers of n.

$n = 2$: Except for the Hilbert-Chow morphism, $\text{Hilb}^2\mathbf{P}^2$ has a morphism to $(\mathbf{P}^2)^*$, which is space of all lines on $\mathbf{P}^2$. We denote MF as the morphism. For $[Z] \in \text{Hilb}^2\mathbf{P}^2$ that supports at two points, $MF([Z])$ is the unique line passing through Z. For $[Z] \in \text{Hilb}^2\mathbf{P}^2$ that supports at one point, recall from the previous example that $[Z]$ determines a tangent direction at the point, $MF([Z])$ is the line with this tangent direction. The fiber of MF morphism at a point ℓ on $(\mathbf{P}^2)^*$ is the Hilbert scheme of 0-dimensional subschemes with length 2 on ℓ. Since $\text{Hilb}^2\ell \simeq \text{Sym}^2\ell \simeq \mathbf{P}^2$, MF realizes Hilb^2 as $\mathbf{P}^2$ fiber bundle over $(\mathbf{P}^2)^*$. Let T be the tangent sheaf of $(\mathbf{P}^2)^*$, a section σ of T at ℓ determines a line on $(\mathbf{P}^2)^*$. In another word, $\sigma|_\ell$ determines a point on ℓ. A section $\tau|_\ell$ in Sym^2T determines a pair of points in $\text{Sym}^2\ell$. This identification works globally, and the fiber bundle is given by the projectivization of Sym^2T. The deformations of Sym^2T induces deformations of $\text{Hilb}^2\mathbf{P}^2$. We have

$$Ext^1(Sym^2T, Sym^2T) = \mathbb{C}^{10}; Ext^2(Sym^2T, Sym^2T) = 0.$$

All the deformations of $\text{Hilb}^2\mathbf{P}^2$ are induced by the deformations of the sheaf Sym^2T.

$n = 3$: Except for the Hilbert-Chow morphism, $\text{Hilb}^3\mathbf{P}^2$ has a divisorial contraction by contracting $[Z]$ to $[\ell]$ if Z is a subscheme of a line ℓ. 5-dimensional locus on $\text{Hilb}^3\mathbf{P}^2$ are contracted to $(\mathbf{P}^2)^*$. The total base scheme of this divisorial contraction is isomorphic to the Kronecker model: $\{\mathcal{O}(-3)^{\oplus 2} \to \mathcal{O}(-2)^{\oplus 3}\}$. In another word, the moduli space of all $(3, -2)$-stable representations of quiver $\mathbb{C}^{\oplus 2} \Rrightarrow \mathbb{C}^{\oplus 3}$.

Another slightly different way to describe this contraction is as following. The canonical model of divisor $|4H - B|$ maps $\text{Hilb}^3\mathbf{P}^2$ into the Grassmanian $G(3, 6)$. Here $\mathbb{C}^6$ is the space $\text{Hom}(\mathcal{O}(-2), \mathcal{O})$. When $[Z]$ is not a subscheme of any line ℓ, $\text{Hom}(\mathcal{O}(-2), \mathcal{I}_Z)$ has dimension 3. When $[Z]$ is a subscheme of a line ℓ, $\mathrm{Hom}(\mathcal{O}(-2), \mathcal{I}_\ell)$ has dimension 3. In either case, $\mathrm{Hom}(\mathcal{O}(-2), *)$ is a 3-dimensional subspace in $\mathrm{Hom}(\mathcal{O}(-2), \mathcal{O})$, and realized as a point in $G(3, 6)$.

When $n \geq 4$, except for the Hilbert-Chow morphism, $\mathrm{Hilb}^n\mathbf{P}^2$ has no other non-trivial divisorial contraction or fiberation morphism. Since $\mathrm{Hilb}^n\mathbf{P}^2$ is a Mori dream space with Picard number 2, after doing finite steps of flip and flop, the new model has a unique divisorial contraction or fiberation morphism.

$n = 4$: The birational model of $\mathrm{Hilb}^4\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-4) \to \mathcal{O}(-2)^{\oplus 2}\}$ which is isomorphic to the Grassmanian $G(2, 6)$. The image of the exceptional locus has dimension 4.

$n = 5$: The birational model of $\mathrm{Hilb}^5\mathbf{P}^2$ has a fiberation contraction. Strictly speaking, in this case, not all fibers are isomorphic to each other. The base space of this fiberation has dimension 5, and is isomorphic to the Kronecker model $\{\mathcal{O}(-4)^{\oplus 2} \to \mathcal{O}(-3)^{\oplus 2}\}$. This model is not smooth. The smooth locus is isomorphic to $\mathbf{P}^5 \setminus \mathbf{P}^2 \times \mathbf{P}^2$, the fiber over the smooth locus is isomorphic to $\mathbf{P}^5$. On the singular locus, a generic fiber has dimension 4. The total fiber space over the singular locus on the Kronecker model $\{\mathcal{O}(-4)^{\oplus 2} \to \mathcal{O}(-3)^{\oplus 2}\}$ has codimension 2.

$n = 6$: The birational model of $\mathrm{Hilb}^6\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-4)^{\oplus 3} \to \mathcal{O}(-3)^{\oplus 4}\}$. The image of the exceptional locus has dimension 4.

$n = 7$: The birational model of $\mathrm{Hilb}^7\mathbf{P}^2$ has a divisorial contraction. The base space is the projective space $\mathbf{P}^{14}$. The image of the exceptional locus has dimension 7.

$n = 8$: The birational model of $\mathrm{Hilb}^8\mathbf{P}^2$ has a fiberation contraction. The base space is isomorphic to the projective space $\mathbf{P}^2$. Each fiber is isomorphic to $G(2, 9)$.

$n = 9$: The birational model of $\mathrm{Hilb}^9\mathbf{P}^2$ has a fiberation contraction. The base space contains a Kronecker model $\{\mathcal{O}(-5)^{\oplus 3} \to \mathcal{O}(-4)^{\oplus 3}\}$, which is not smooth. The base space has dimension 10.

$n = 10$: The birational model of $\mathrm{Hilb}^{10}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-5)^{\oplus 4} \to \mathcal{O}(-4)^{\oplus 5}\}$. The image of the exceptional locus has dimension 10.

$n = 11$: The birational model of $\mathrm{Hilb}^{11}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-5)^{\oplus 2} \to \mathcal{O}(-4)^{\oplus 4}\}$ with dimension 5.

$n = 12$: The birational model of $\mathrm{Hilb}^{12}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-6)^{\oplus 2} \to \mathcal{O}(-4)^{\oplus 3}\}$. The image of the exceptional locus has dimension 2.

$n = 13$: The birational model of $\mathrm{Hilb}^{13}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-6)^{\oplus 3} \to \mathcal{O}(-5)^{\oplus 2}\}$ with dimension 6.

$n = 14$: The birational model of $\mathrm{Hilb}^{14}\mathbf{P}^2$ has a divisorial contraction. The base space contains a Kronecker model $\{\mathcal{O}(-5)^{\oplus 4} \to \mathcal{O}(-4)^{\oplus 4}\}$, which is not smooth and has dimension 17.

$n = 15$: The birational model of $\mathrm{Hilb}^{15}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-6)^{\oplus 5} \to \mathcal{O}(-5)^{\oplus 6}\}$. The image of the exceptional locus has dimension 15.

$n = 16$: The birational model of $\mathrm{Hilb}^{16}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-6)^{\oplus 3} \to \mathcal{O}(-5)^{\oplus 5}\}$ with dimension 12.

$n = 17$: The birational model of $\mathrm{Hilb}^{17}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-7)^{\oplus 2} \to \mathcal{O}(-5)\}$, which is is isomorphic to $G(2, 6)$ with dimension 8.

$n = 18$: The birational model of $\mathrm{Hilb}^{18}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{T}(-8) \to \mathcal{O}(-5)^{\oplus 3}\}$, which is isomorphic to $G(3, 15)$. The image of the exceptional locus has dimension 11.

$n = 19$: The birational model of $\mathrm{Hilb}^{19}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-7)^{\oplus 4} \to \mathcal{O}(-6)^{\oplus 3}\}$ with dimension 12.

$n = 20$: The birational model of $\mathrm{Hilb}^{20}\mathbf{P}^2$ has a fibration contraction. The base space contains a Kronecker model $\{\mathcal{O}(-7)^{\oplus 5} \to \mathcal{O}(-6)^{\oplus 5}\}$ with dimension 26.

$n = 21$: The birational model of $\mathrm{Hilb}^{21}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-7)^{\oplus 6} \to \mathcal{O}(-6)^{\oplus 7}\}$ with dimension 21.

$n = 22$: The birational model of $\mathrm{Hilb}^{22}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-7)^{\oplus 4} \to \mathcal{O}(-6)^{\oplus 6}\}$ with dimension 21.

$n = 23$: The birational model of $\mathrm{Hilb}^{23}\mathbf{P}^2$ has a fiberation contraction. The base space is a Kronecker model $\{\mathcal{O}(-7)^{\oplus 2} \to \mathcal{O}(-6)^{\oplus 5}\}$ which is isomorphic to $\mathbf{P}^2$.

$n = 24$: The birational model of $\mathrm{Hilb}^{24}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-8)^{\oplus 3} \to \mathcal{O}(-6)^{\oplus 4}\}$.

$n = 25$: The birational model of $\mathrm{Hilb}^{25}\mathbf{P}^2$ has a fibration contraction. The base space contains a Kronecker model $\{\mathcal{O}(-8)^{\oplus 4} \to \mathcal{O}(-7)^{\oplus 2}\}$ with dimension 5.

$n = 26$: The birational model of $\mathrm{Hilb}^{26}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-8)^{\oplus 5} \to \mathcal{O}(-7)^{\oplus 4}\}$ with dimension 20.

$n = 27$: The birational model of $\mathrm{Hilb}^{27}\mathbf{P}^2$ has a fibration contraction. The base space contains a Kronecker model $\{\mathcal{O}(-8)^{\oplus 6} \to \mathcal{O}(-6)^{\oplus 6}\}$ with dimension 37.

$n = 28$: The birational model of $\mathrm{Hilb}^{28}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-8)^{\oplus 7} \to \mathcal{O}(-7)^{\oplus 8}\}$ with dimension 28.

$n = 29$: The birational model of $\mathrm{Hilb}^{29}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-8)^{\oplus 5} \to \mathcal{O}(-7)^{\oplus 7}\}$ with dimension 32.

$n = 30$: The birational model of $\mathrm{Hilb}^{30}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-8)^{\oplus 3} \to \mathcal{O}(-7)^{\oplus 6}\}$ with dimension 10.

$n = 31$: The birational model of $\mathrm{Hilb}^{31}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 3} \to \mathcal{O}(-7)^{\oplus 2}\}$ with dimension 24.

$n = 32$: The birational model of $\mathrm{Hilb}^{32}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9) \to \mathcal{O}(-7)^{\oplus 4}\}$ which is isomorphic to $G(2, 6)$ with dimension 8.

$n = 33$: The birational model of $\mathrm{Hilb}^{33}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 5} \to \mathcal{O}(-8)^{\oplus 3}\}$ with dimension 12.

$n = 34$: The birational model of $\mathrm{Hilb}^{34}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 6} \to \mathcal{O}(-8)^{\oplus 5}\}$ with dimension 30.

$n = 35$: The birational model of $\mathrm{Hilb}^{35}\mathbf{P}^2$ has a fibration contraction. The base space contains a

Kronecker model $\{\mathcal{O}(-9)^{\oplus 7} \to \mathcal{O}(-8)^{\oplus 7}\}$ with dimension 50.

$n = 36$: The birational model of $\mathrm{Hilb}^{36}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 8} \to \mathcal{O}(-8)^{\oplus 9}\}$ with dimension 36.

$n = 37$: The birational model of $\mathrm{Hilb}^{37}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 6} \to \mathcal{O}(-8)^{\oplus 8}\}$ with dimension 45.

$n = 38$: The birational model of $\mathrm{Hilb}^{38}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-9)^{\oplus 4} \to \mathcal{O}(-8)^{\oplus 7}\}$ with dimension 20.

$n = 39$: The birational model of $\mathrm{Hilb}^{39}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-10)^{\oplus 3} \to \mathcal{T}(-9)^{\oplus 2}\}$, the image of the exceptional locus has dimension 11.

$n = 40$: The birational model of $\mathrm{Hilb}^{40}\mathbf{P}^2$ has a divisorial contraction. The base space is a Kronecker model $\{\mathcal{O}(-10)^{\oplus 4} \to \mathcal{O}(-8)^{\oplus 5}\}$, the image of the exceptional locus has dimension 34.

$n = 41$: The birational model of $\mathrm{Hilb}^{41}\mathbf{P}^2$ has a fibration contraction. The base space is a Kronecker model $\{\mathcal{O}(-10)^{\oplus 5} \to \mathcal{O}(-9)^{\oplus 3}\}$ with dimension 12.

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