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PROPAGATION OF ACCELERATION WAVES IN RANDOM MEDIA

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Theoretical and Applied Mechanics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2014

Urbana, Illinois

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ABSTRACT

The paper studies the effects of material spatial randomness on the evolution of acceleration waves. The deterministic analysis of acceleration waves assumes perfectly homogeneous materials and has led to the derivation of the following Bernoulli equation [13], which models the time evolution of the wave amplitude:

$$\frac{d\alpha}{dt} = \mu\alpha - \beta\alpha^2$$

The analysis also reveals that there exists a critical amplitude that determines whether the wave amplitude decays, or blows-up rapidly to infinity. In the latter case, as the wave amplitude increases, the wavefront becomes thinner tending to a shock, so we also calculate the distance to form a shock. The stochastic approach accounts for the microstructural inhomogeneity randomly distributed across the material. This suggests that the material coefficients μ and β be taken as random processes depending on the spatial distribution. Recalling the expressions for μ and β, the Bernoulli equation becomes the following position-dependent equation:

$$\frac{dz}{dx} = -\frac{G_0^{'}\rho^{1/2}}{2G_0^{3/2}}\alpha + \frac{\rho_R \tilde{E}_0}{2G_0^2}$$

Both the time-dependent and the position-dependent versions of the Bernoulli equation are investigated using a stochastic analysis. The material coefficients are taken to be one of the following random fields: (i) Gaussian White Noise, (ii) Gaussian with a Cauchy covariance function, (iii) Gaussian with a Dagum covariance function. Various coupling scenarios of the material coefficients are considered: (i) positive correlation, (ii) negative correlation, (iii) zero correlation. Similarly to the deterministic case, we are interested in how the material spatial randomness affects the average critical amplitude and the distance to form a shock. Both quantities become random variables and their behavior is compared to the deterministic results.

ACKNOWLEDGEMENTS

First of all, I would like to sincerely express my gratitude to my advisor Prof. Martin Ostoja-Starzewski. From the moment he approached me to join his research group, he believed in me. He guided me through my research with patience and kindness. He has also helped me out with life advice. I would also like to thank my research group, and Rishabh Narang from the Computational Science and Engineering department for helping me out with my codes. Finally, I would like to acknowledge the tremendous support from my family, boyfriend and friends from the Ivory Coast, the United States, Senegal, France and Spain. Their encouragements have carried me through difficult moments, and I wouldn't be where I am today without their support.

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LIST OF SYMBOLS

$\langle\rangle$	Ensemble average or mean
α	Jump in particle acceleration, wave amplitude
α_0	Initial condition to wave amplitude differential equation
α_{cr}	Critical wave amplitude
α_h	Homogeneous solution to wave amplitude differential equation
β	Elastic nonlinearity
c	Wave velocity
C	Integration constant
d	Characteristic grain size of microstructure
ε	Noise intensity
E_0	Instantaneous second-order tangent modulus
G_0	Instantaneous modulus
G_0 '	Dissipation coefficient
L	Wavefront thickness
μ	Dissipation
ω	Elementary outcome from Ω
Ω	Sample space
p	Propagation direction of wavefront
P	Probability
ρ_R	Mass density in the reference state
$\|R^D$	Set of real numbers of dimension D
R_Z	Autocorrelation function
t	Time
t_∞	Time to shock formation
$t_{\infty det}$	Deterministic time to shock formation
x_∞	Distance to form a shock
z	Inverse wave amplitude

z_0	Initial condition to inverse amplitude differential equation
z_{cr}	Inverse critical amplitude
z_{crdet}	Deterministic inverse critical amplitude
t_i	Initial time
t_f	Final time
Δt	Time step

CHAPTER 1: INTRODUCTION

1.1 Separation of Scales

The deterministic analysis of acceleration waves makes the assumption of perfect material homogeneity. However, realistically speaking, all materials present some amount of inhomogeneity. It is then important to define the context in which these fluctuations cannot be ignored.

Consider a wave front of finite thickness L propagating in the $\boldsymbol{p}$-direction in a random microstructure of characteristic grain size d. Three cases can then be distinguished: (Figure 1.1)

(a) $L >> d$

(b) L is finite relative to the grain size

(c) $L < d$

Case (a) pictures the classical (deterministic) continuum limit, where the wavefront is analogous to a Representative Volume Element in which fluctuations go to zero as $L/d \to \infty$. Here, the thickness L is large enough compared to the characteristic grain size, that the material disorder is negligible. The material appears homogeneous as the waves propagate. In case (b), L decreases so that it is finite relative to the grain size. The fluctuations then become significant. The wavefront becomes a statistical mesoscale window affected by random continuum type fluctuations of the microstructure [10].

In case (c), the wave's evolution is random and piecewise constant. The grains are uniform continua and the wavefront propagates as a random jump process across grain boundaries.

In this paper, our attention is focused on case (b) where μ and β are now random processes in x, so that α becomes a stochastic process driven by the $\{\mu,\beta\}x$ vector process according to the Bernoulli Equation. The competition between μ and β is therefore also stochastic and the critical amplitude α_c and the distance to form a shock x_∞ are random variables. It is important to note we are not dealing with specific materials here. We

assume various random processes to model the spatial randomness in the continuum and analyze the subsequent results. The goal is to eventually build a database of proven results that can be used to model a specific type of spatial randomness, given a material, and carry out further experiments based on those findings.

1.2 Review of Literature

There is extensive research on acceleration waves with the classical continuum limit assumption. Most of the papers however deal with a specific type of material. Hill [6] provides a study of the propagation of acceleration waves in solids, particularly elastic/plastic solids, where a homogeneous linear tensor relation between the stress-rate and the strain-rate characterizes the materials. Coleman, Gurtin and Herrera [3] focus on one-dimensional acceleration waves involving jump discontinuities in the gradient of the strain and the acceleration. They particularly deal with linear viscoelastic materials. Chen [2] presents results on the effect of hydrostatic stress on the growth of arbitrary acceleration waves propagating in an isotropic body. There is also a lot of literature about stochastic wave propagation, but very few focus on acceleration waves. Kliemann and Namachchivaya [7] for example discuss nonlinear dynamic systems subject to various types of random vibrations. Ostoja-Starzewski has carried out most of the studies on acceleration waves in random media (see references [9], [10], [11], [12]). The present paper revisits the case of White Noise already studied in the aforementioned papers, but also introduces two new models for the spatial randomness of the material: the random fields with Cauchy and Dagum covariance functions.

1.3 Outline of Paper

The paper is a continuation of Professor Martin Ostoja-Starzewski's theoretical investigation for modeling randomness in a material. The thesis is divided into a number of chapters that each addresses a particular aspect of the topic.
(a) Chapter 2 presents the model formulation for our investigation. The first section is dedicated to results from the classical analysis. The second section introduces the

stochastic assumptions. The third section gives some definitions of random processes and the last section deals with the simulation set-up.
(b) Chapter 3 presents the results and discussion for both versions of the Bernoulli equation with stochastic material coefficients.
(c) Chapter 4 summarizes the findings from the research and provides conclusions to draw from the analysis.

1.4 Figures

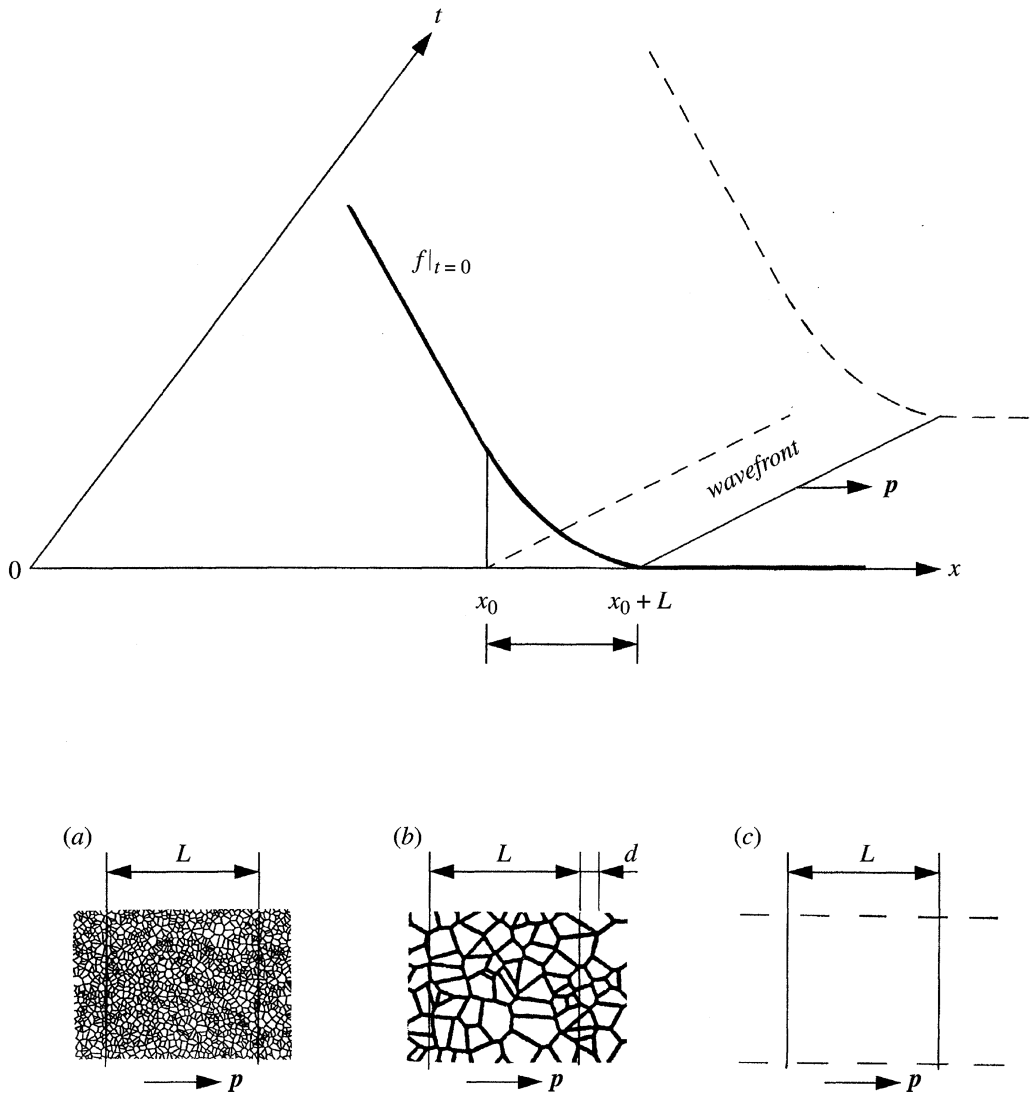


Figure 1.1*: Evolution of Wavefront in space-time domain seen as zone of finite thickness L (between x0 and x0 + L at time t=0), propagating in p-direction in microstructure with characteristic grain size d. Three cases are distinguished: (a) L>>d, which shows the trend of classical continuum limit, where fluctuations die out to zero; (b) L is finite relative to the grain size and fluctuations are significant; (c) L<<d which leads to random and piecewise-constant evolution.*

CHAPTER 2: MODEL FORMULATION

2.1 Time Evolution of Acceleration Waves: Deterministic Approach

Acceleration waves describe the propagation of a surface of discontinuity. The term 'wave' describes an isolated geometric surface, not necessary plane, which moves relative to the material and across which certain field variables are momentarily discontinuous. These variables generally represent the particle acceleration, and the spatial gradient of velocity [6]. Our study focuses on the jump in particle acceleration. Here, we derive the jump conditions that the acceleration must satisfy across the discontinuity surface as well as the time evolution of the acceleration amplitude.

Consider a discontinuity surface moving in the direction of positive X, material coordinate, and dividing the body into two not intersecting open pieces B_a^+ and B_a^-. The surface is identified by

$$f(x,t) = 0 \qquad (2.1)$$

where f is continuous with its first derivatives. By taking the material derivative of (2.1), we find that the propagation velocity is:

$$c = -\frac{\partial_t f}{|\nabla f|} \qquad (2.2)$$

Now, consider a field *a(x,t)* defined on the whole body B_a and continuous on both parts B_a^+ and B_a^-. The jump across the surface is defined by

$$[a] = a^+ - a^- \qquad (2.3)$$

If a(x,t) is continuous and presents a finite jump at the interface, the Hadamard's theorem [5] implies that

$$\left[\frac{\partial a}{\partial t}\right] = -c\left[\frac{\partial a}{\partial x}\right] \quad (2.4)$$

For the purpose of this paper, let a be the jump in particle acceleration α in a dissipative, non-linear elastic continuum. Accounting for microstructural stresses and evaluating the jumps in the time derivatives of equations of motion, we arrive at the following ordinary differential equation modeling the time evolution of the amplitude:

$$\frac{d\alpha}{dt} = \mu\alpha - \beta\alpha^2 \quad (2.5)$$

Equation (2.5) is referred to throughout the paper as the Bernoulli equation. (See Paoletti, 2012 for more details on the derivation of the amplitude time evolution). It is important to specify that the deterministic case occurs in the setting of case (a).
The question is whether the magnitude of the discontinuity decreases, remains constant, or increases as the surface propagates. When the discontinuity increases, we have formation of a shock. Finding an answer to this question leads to solving equation (2.5) subject to the initial condition:

$$\alpha(0) = \alpha_0 \quad (2.6)$$

Solving this equation leads to finding a homogeneous solution and a particular one. To find the particular solution, it is easier to introduce the change of variable $z = 1/\alpha$. Then:

$$\frac{dz}{dt} = \frac{dz}{d\alpha}.\frac{d\alpha}{dt}$$
$$\frac{dz}{dt} = -\frac{1}{\alpha^2}.(-\mu\alpha + \beta\alpha^2)$$
$$\frac{dz}{dt} = \frac{1}{\alpha}\mu - \beta$$

$$\frac{dz}{dt} = \mu z - \beta \quad (2.7)$$

with initial condition $z(0) = z_0 = 1/\alpha_0$ (2.8)

Equation (2.7) is a first-order linear differential equation, which also needs homogeneous and particular solutions. The homogeneous solution is

$$z_h(t) = C\exp(-\phi(t)) \quad (2.9)$$

where $\phi(t) = \int_0^t \mu(t)\,dt$

The particular solution to equation can be found using the method of integrating factors. We get:

$$z_p(t) = -\exp(\phi(t)).\int_0^t \beta(t).\exp(-\phi(t))\,dt \quad (2.10)$$

The solution to equation (2.7) is then:

$$z(t) = C\exp(-\phi(t)) - \exp(\phi(t)).\int_0^t \beta(t).\exp(-\phi(t))\,dt \quad (2.11)$$

If α_h is the homogeneous solution to equation (2.5), then the general solution is:

$$\alpha(t) = \alpha_h + \frac{1}{z(t)}$$

$$\alpha(t) = \alpha_h + \frac{1}{[Ce\exp(-\phi(t)) - \exp(\phi(t)).\int_0^t \beta(t).\exp(-\phi(t))\,dt} \quad (2.12)$$

For the sake of simplicity, we take μ and β to be constants. Equation (2.11) then becomes:

$$z(t) = \frac{\beta}{\mu} + (z_0 - \frac{\beta}{\mu})\exp(\mu t) \quad \text{and} \quad \alpha(t) = \frac{\mu}{[(\mu/\alpha_0) - \beta]e^{\mu t/\beta} + \beta} \quad (2.13)$$

We can observe that the sign of the coefficient $(z_0 - \frac{\beta}{\mu})$ determines whether we have an exponential growth or decay:

- If $z_0 = \frac{\beta}{\mu} \Leftrightarrow \alpha_0 = \frac{\mu}{\beta}$, then we have a constant curve $z(t) = \frac{\beta}{\mu} \Leftrightarrow \alpha(t) = \frac{\mu}{\beta}$
- If $z_0 \le \frac{\beta}{\mu} \Leftrightarrow \alpha_0 \ge \frac{\mu}{\beta}$ and $\mu > 0$, we have exponential growth of the amplitude, which leads to the formation of a shock. When $\mu < 0$, the amplitude decays exponentially.
- If $z_0 \ge \frac{\beta}{\mu} \Leftrightarrow \alpha_0 \le \frac{\mu}{\beta}$, and $\mu > 0$, the amplitude decays exponentially. When $\mu < 0$, the amplitude blows up rapidly to infinity.

Therefore, the inverse critical amplitude is $z_{cr} = \frac{\beta}{\mu}$ and the critical amplitude is

$$\alpha_{cr} = \frac{\beta}{\mu} \quad (2.14)$$

Now to find the time to form a shock, we can simply solve *z(t)=0* for *t*, which gives us:

$$t_\infty = \frac{1}{\mu}\ln\left|\frac{\beta}{\beta - \mu z_0}\right| = -\frac{1}{\mu}\ln\left|1 - \frac{\mu}{\beta\alpha_0}\right| \quad (2.15)$$

2.2 Stochastic Analysis of Acceleration Waves

The deterministic study of acceleration waves assumes perfectly homogeneous materials. However, one can argue that every material presents some amount of inhomogeneity randomly distributed across the microstructure, which puts us in case (b). In this case, we can't talk about time dependence, but rather a spatial dependence. Material properties become physical properties depending on position x when spatial randomness is

introduced. μ and β are then taken to be random processes. Now, recall that the explicit formulas for the dissipation coefficient and the nonlinear elasticity are [11]:

$$\mu = -\frac{G_0^{'}}{2G_0} \quad \beta = -\frac{\tilde{E}_0}{2G_0 c} \quad (2.16)$$

with $c = \sqrt{\frac{G_0}{\rho_R}}$ (2.17)

To find the spatial evolution of the amplitude of the jump in particle acceleration, we apply the chain rule to equation (2.5), which becomes

$$\frac{d\alpha}{dx} = -\frac{1}{c}\mu(x)\alpha + \frac{1}{c}\beta(x)\alpha^2 \quad (2.18)$$

Replacing μ, β, and c by their formulas, we end up with the following new Bernoulli equation:

$$\frac{d\alpha}{dx} = \frac{G_0^{'}\rho_R^{1/2}}{2G_0^{3/2}}\alpha - \frac{\rho_R \tilde{E}_0}{2G_0^2}\alpha^2 \quad (2.19)$$

$\rho_R > 0$; $G_0 > 0$; $G'_0 \geq 0$; $\tilde{E}_0 \neq 0$

Here, equation (2.19) is the stochastic version of the Bernoulli equation, driven by a stochastic vector process $\{G_0, G_0^{'}, \rho_R, \tilde{E}_0\}_x$. A stochastic competition between the dissipation and the elastic nonlinearity occurs, and the critical amplitude and the distance to shock formation become random variables. The inverse amplitude differential equation converts to:

$$\frac{dz}{dx} = -\frac{G_0^{'}\rho^{1/2}}{2G_0^{3/2}}\alpha + \frac{\rho_R \tilde{E}_0}{2G_0^2} \quad (2.20)$$

2.3 Random Processes

First, let's recall some basic concepts about random variables. A scalar random variable is a function Z assigning to an elementary event $\omega \in \Omega$, a number z on a real line, that is [12]:

$$Z : \Omega \rightarrow R \ , \quad Z(\omega) = z \quad (2.21)$$

The probability distribution function (PDF) of Z is then:

$$F_z(z) = P\{Z \leq z\} \quad (2.22)$$

Now, we can introduce the concept of random process (or random field). Given a set X such that $X \subset R^D$, we say that Z is a scalar random process if it assigns to an elementary event a realization over X, that is:

$$Z : \Omega \times X \rightarrow R \ \ Z(\omega, x) = z \quad (2.23)$$

where $x \in X$. There are various ways to characterize random processes: the probability distribution, the mean, and the autocorrelation among others. In contrast to random variables, the probability distribution function has an order. Given n points x_1……, x_n, we consider n random variables $Z(x_1)$, ……., $Z(x_n)$. Their joint nth-order PDF is

$$F_n(z_1, ..., z_n; x_1, ..., x_n) = P\{Z(x_1) \leq z_1, ..., Z(x_n) \leq z_n) \quad (2.24)$$

The mean or ensemble average of the process is defined as:

$$\langle Z(x) \rangle = \int_R z \, dF_1(z; x) \quad (2.25)$$

The correlation describes how a change in on random variable affects the change in other random variable. The autocorrelation represents that relationship between values of the same process at different positions, and is defined as:

$$R_z(x_1,x_2)=\langle Z(x_1)Z(x_2)\rangle=\int_R z_1 z_2\, dF_2(z_1,z_2;x_1,x_2) \quad (2.26a)$$

The covariance is a measure of how two random variables change together. It measures the strength or weakness of correlation between random variables. The auto covariance is the covariance between two random variables of the same process and is defined as follows:

$$C_z(x_1,x_2)=\langle [Z(x_1)-\langle Z(x_1)\rangle][Z(x_2)-\langle Z(x_2)\rangle]\rangle=R_z(x_1,x_2)-\langle Z(x_1)\rangle\langle Z(x_2)\rangle \quad (2.26b)$$

Another important property of random processes is the homogeneity or stationarity. We distinguish two types of stationarities: the *strict-sense stationarity* (SSS) and the *wide-sense stationarity* (WSS). A process is SSS if all n-order distributions F_n are invariant with respect to arbitrary shifts *x'* that is

$$F_n(z_1,\ldots,z_n;x_1,\ldots,x_n)=F_n(z_1,\ldots,z_n;x_1+x',\ldots,x_n+x') \quad (2.27)$$

Next, a process is WSS if its mean is constant and its finite-valued autocorrelation depends only on $x = x_2 - x_1$:

$$\langle Z(x_1)Z(x_1+x)\rangle=R_Z(x)<\infty \quad (2.28)$$

For continuum mechanics problems, we take the random processes to be the superposition of a constant mean, and zero-mean fluctuations:

$$Z(\omega,x)=\langle Z\rangle+Z'(\omega,x) \qquad \langle Z'\rangle=0 \qquad \forall\omega,x \quad (2.29)$$

2.4 Examples of Random Processes

The following three WSS Gaussian random fields are to be used in our simulations: Gaussian white noise, Gaussian with a Cauchy covariance function, and Gaussian with a Dagum covariance function.

White noise is a stationary process used in random vibration to model approximately broadband inputs or inputs with very short memory [7]. In statistics, it refers to a discrete signal generating a sequence of uncorrelated random variables with zero mean and finite variance [4]. The noise is Gaussian if the random variables have a normal distribution.

A *Gaussian random field with a Cauchy covariance function*, is characterized by a two-parameter covariance function defines as follows [14]:

$$C_c(r;\theta,\eta) := (1+r^{\theta})^{-\eta/\theta} \quad (2.30)$$

where $\eta > 0$ and $0 < \theta \leq 2$ are necessary and sufficient conditions for positive definiteness.

A *Gaussian random field with a Dagum covariance function* has the added property of being isotropic. The covariance function is also biparametric [15]:

$$C_D(r;\delta,\varepsilon) := 1-(1+r^{-\delta})^{-\varepsilon/\delta} \quad (2.31)$$

where $0 < \varepsilon < \delta$ and $0 < \delta \leq 2$ are necessary and sufficient conditions for positive definiteness.

The interesting fact about Cauchy and Dagum covariance is that they can model fractals and the Hurst effect independently. A fractal is a roughness measure of a statistically self-similar profile or surface in $\mathbf{R}^n$, while the Hurst effect reflects a long distance dependence or long memory dependence in time series [14].

2.5 Simulation Set-up

A finite difference scheme is used to solve both the time evolution version of the Bernoulli equation, as well as the spatial evolution equation. For our simulations, we are using a Gaussian white noise with mean 0 and variance 1. We use a random number generator in MATLAB, to create Gaussian white noise. The Cauchy and Dagum random variables are generated through the R software.

2.5.1 Time Evolution With Random Coefficients

We mentioned earlier that a position-dependent Bernoulli equation would be more suited to study the effects of microstructure randomness on acceleration waves. However, since the equation was initially developed for a time-dependent evolution, it would be interesting to keep the deterministic format of the Bernoulli equation while making the dissipation and elastic nonlinearity coefficients random processes. For sake of simplicity, we work with the inverse amplitude. The corresponding finite difference equation is:

$$z_n = (1 + \mu\Delta t) z_{n-1} - \beta\Delta t \quad (2.32)$$

We apply equation (2.29) to the coefficients and get:

$$\mu = \langle\mu\rangle + \varepsilon_1\mu', \qquad \varepsilon_1 \geq 0 \quad (2.33)$$

$$\beta = \langle\beta\rangle + \varepsilon_2\beta', \qquad \varepsilon_2 \geq 0 \quad (2.34)$$

Here, μ' and β' are the fluctuations to be modeled using Gaussian white noise, and Gaussian random fields with Cauchy and Dagum covariance functions. Three possible coupling between the coefficients arise:

- Positive correlation: μ' = β'
- Negative correlation: μ' = -β'
- Zero correlation: μ' and β' are generated independently of each other

For sake of comparison, we start with an initial condition equal to the critical value in the deterministic case, that is:

$$z_0 = z_{cr_{\det}} = \frac{\langle\beta\rangle}{\langle\mu\rangle} \quad (2.35)$$

Because μ and β are random processes, we have multiple curves emerging from the same initial condition. We then use a Monte Carlo simulation to generate one hundred curves that will be compared to the deterministic result when the initial condition and the critical

amplitude are equal. For the Gaussian random fields with Cauchy and Dagum covariance functions, the code is harder to implement, so we only generate ten curves.
Next, we would like to find out what happens to the time to shock formation when the randomness is introduced. From section 2.1, we derived that with a positive elastic nonlinearity, the initial condition had to be smaller than the critical amplitude to have shock formation. Within this assumption, we again use a Monte Carlo simulation to generate multiple curves. Then we record the times at which they intersect the time axis, take the average of these values and compare them to the deterministic time to shock formation with the same initial condition.

2.5.2 Spatial Evolution

We follow the same procedure as for the time evolution, except that this time we use the stochastic version of the Bernoulli equation (2.19). We have for coefficients, each of them being a random process. Applying equation (2.29) to the coefficients, we get:

$$\rho_R = \langle \rho_R \rangle + \varepsilon_1 \rho_R^{'} \quad (2.36)$$

$$\tilde{E}_0 = \left\langle \tilde{E}_0 \right\rangle + \varepsilon_2 \tilde{E}_0' \quad (2.37)$$

$$G_0^{'} = \left\langle G_0^{'} \right\rangle + \varepsilon_3 (G_0^{'})' \quad (2.38)$$

$$G_0 = \langle G_0 \rangle + \varepsilon_4 G_0' \quad (2.39)$$

Again, we start with an initial condition equal to the deterministic critical value:

$$z_0 = z_{cr_{\det}} = \frac{(\langle \rho_R \rangle \langle G_0 \rangle)^{1/2} \langle \tilde{E}_0 \rangle}{\left\langle G_0^{'} \right\rangle} \quad (2.40)$$

Here, we have multiple coupling possibilities. However we focus on the following three cases:

- Positive correlation: all the coefficients are positive and equal.

- Negative correlation: three of the coefficients are equal and the last one is the opposite of the others.
- Zero correlation: all coefficients are independent of each other.

2.6 Tables

Variables	α_{crdet}	z_{crdet}	$Z_0 = z_{crdet}$	$Z_0 < z_{crdet}$	$\langle\mu\rangle$	$\langle\beta\rangle$	$\langle\rho_R\rangle$	$\langle E_0\rangle$	$\langle G_0\rangle$	$\langle G'_0\rangle$	ε	t_i	t_f	Δt
Values	**1**	**1**	**1**	**0.8**	**1**	**1**	**1**	**1**	**1**	**1**	**0.1**	**0**	**20**	**0.01**

Table 2.1*: Simulation Variables*

CHAPTER 3: RESULTS AND DISCUSSION

The results for both versions of the inverse amplitude Bernoulli equation with stochastic material coefficients are presented here. The results for the deterministic case are very straightforward and agree with the theoretical predictions. Figures 3.1 and 3.2 show that when the initial condition is equal to the critical amplitude, the solution to the Bernoulli equation is a constant equal to the value of the critical amplitude. When the initial condition is inferior to the critical amplitude, the solution blows up exponentially (Figure 3.2 shows exponential decay because we are using the inverse amplitude). As the wave amplitude increases, the wavefront becomes thinner tending to a shock. The time it takes for the shock to form is $t_{\infty_{\det}} = 16.1s$. Results for the stochastic differential equation however are very unpredictable. Each random field behaves very differently from the other. Specific findings for each of the random processes are provided below.

3.1 Gaussian White Noise

3.1.1 Time Evolution

As we mentioned earlier, when randomness is introduced, we have to consider the nature of the correlation between the material coefficients, in other words: negative correlation, positive correlation, and zero correlation. We chose average values of the dissipation and elastic nonlinearity such that the average critical amplitude was equal to the deterministic critical amplitude.

With an initial condition equal to that value, the plots showed a mixture of decaying and growing curves for each type of correlation, modeling an approximately equal competition between the dissipation and the elastic nonlinearity. For 100 curves plotted, the negative correlation case produced 44 exponentially growing curves. Fluctuations were most visible in this case as well. The positive correlation however displayed 97% of the curves blowing up. It seems as the dissipation and the elastic nonlinearity cancel each other out. The values of the inverse amplitude stay very close to 1 and the plot resembles the

deterministic one with initial condition equal to the critical amplitude. As for the zero correlation case, we also observe about 50% of the curves growing and the other half decaying.

Still with an initial condition of 1, the simulation revealed that the mixture of growing and decaying curves was observed over the small interval [0.993; 1.007] for all three types of correlation, but only the zero correlation is case is analyzed because results are more visible. Notice that the interval is centered at 1. This means instead of one fixed value for the critical amplitude, we have an interval of critical amplitude. Outside of this interval, all the curves either decay or grow. The probability of shock formation plot when the initial condition is picked for the critical interval shows the right half of a Gaussian distribution.

Now let's take a look at the results for an initial condition equal to 0.8. The solutions for all correlations blow up, which is similar to what happens in the deterministic case. In the positive and negative correlation cases, the curves almost collapse to the deterministic solution. We averaged the times to form shocks when the material coefficients have zero correlation and found $\langle t_\infty \rangle = 16.0984s$, which is equal to the deterministic time to form a shock.

3.1.2 Spatial Evolution

The spatial evolution solutions exhibit behaviors different from the time evolution. For an initial condition equal to 1, both the positive and zero correlation solutions all slowly grow exponentially. The solutions for the positive correlation almost collapse to the deterministic solution, while fluctuations are very pronounced with zero correlation. The negative correlation assumption on the other hand, displays an envelope of growing and decaying curves, that all seem to converge to 1. The bulk of the curves are concentrated in the middle. This behavior is observed on the small interval [0.969; 1.021].
When the initial condition is equal to 0.8, all the curves decay independently of the correlation. However, fluctuations are most visible for zero correlation.

3.2 Gaussian with a Cauchy Covariance Function

3.2.1 Time evolution

For the negative correlation, all the curves decay, they all blow up in the case of positive correlation when the initial condition is equal to 1. The zero correlation results in half of the curves blowing up and the other half decaying. A similar behavior is observed with an initial condition of 0.8. For the positive correlation, the average time to shock formation is 0.07s, which is different from the deterministic time. In addition, there is no interval of critical amplitudes.

3.2.2 Spatial Evolution

When the initial condition is equal to 1, the solutions slowly exponentially grow, regardless of the correlation. Solutions for the negative correlation almost collapse to the deterministic one and fluctuations are most pronounced for zero correlation. There is no interval of critical amplitudes. When the initial condition is equal to 0.8, solutions for positive and zero correlation slowly decay while growing into shocks for the negative correlation. The average distance to shock formation is $\langle x_\infty \rangle = 16.792$ which differs from the deterministic value by 4.3%. The solutions start at the initial condition as expected.

3.3 Gaussian with a Dagum Covariance Function

3.3.1 Time Evolution

In the case of Gaussian random field with a Dagum covariance function, the fluctuations observed are more attenuated than those of the Gaussian white noise or Gaussian with a Cauchy covariance function. When the initial value is equal to the average critical amplitude, the positive correlation simulation yields one constant line equal to 1 just like the deterministic case. For the same initial condition, the negative correlation

solutions rapidly decay with curves being close to each other, while the zero correlation solutions blow up rapidly in an exponential manner. In the latter case, the curves appear to be grouped into two bunches emanating from the same initial condition. We do not observe any critical interval.

Looking at the solutions with initial condition equaling 0.8, we observe that in both the positive and zero correlation instances, all the curves blow up rapidly, in the same fashion. Both times, the average time to shock formation is 0.01s, which very small compared to the deterministic one. The negative correlation solutions all decay.

3.3.2 Spatial Evolution

The results for the spatial evolution are quite different from all our findings so far. Most of the cases behave very much like the deterministic case in the sense that for all ten simulations, we only get one curve instead of ten different curves. Again, we observe fluctuations smaller than those obtained with the other two random processes. The negative correlation simulations for both initial conditions yield to periodic oscillations. This can be easily explained by the fact that in equation (2.20), the coefficients in front of the variable become complex numbers. Matlab generates solutions that ignore the imaginary parts of those coefficients. The negative correlation solution is one decreasing straight line when the initial condition is equal to 1 or 0.8. For an initial condition of 1, the positive correlation solutions are exponentially increasing curves propagating in two separate groups. An initial condition of 0.8 produces one rapidly decaying curve when the correlation is positive.

3.4 Figures and Tables

3.4.1 Gaussian White Noise

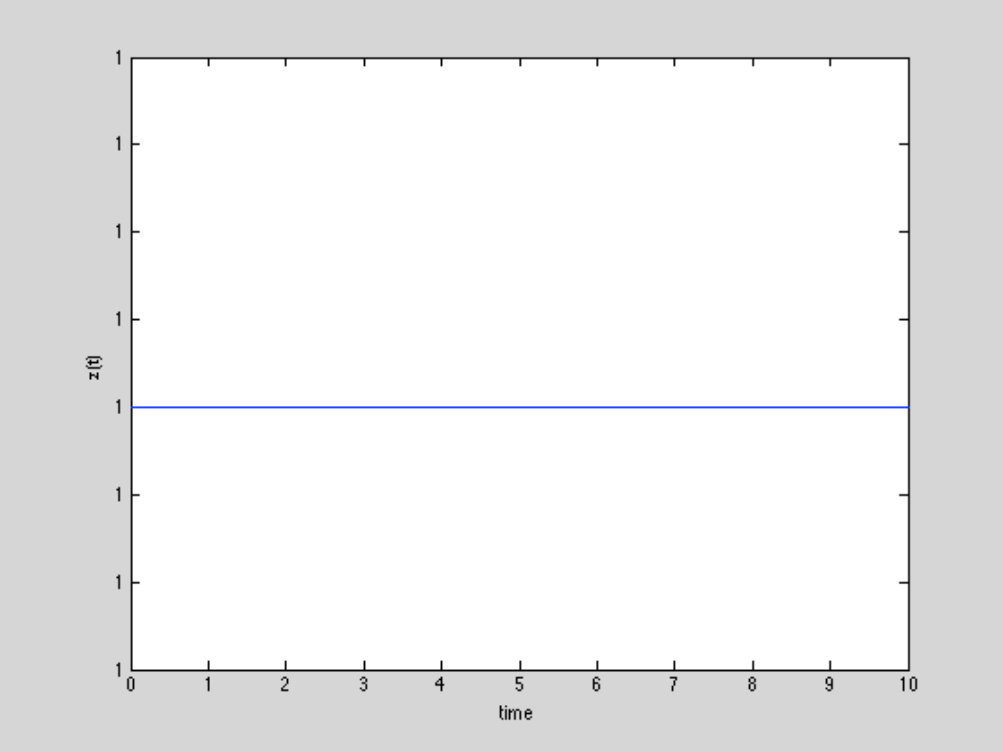


Figure 3.1: *Deterministic solution to Bernoulli equation with initial condition $z_0 = z_{cr} = 1$*

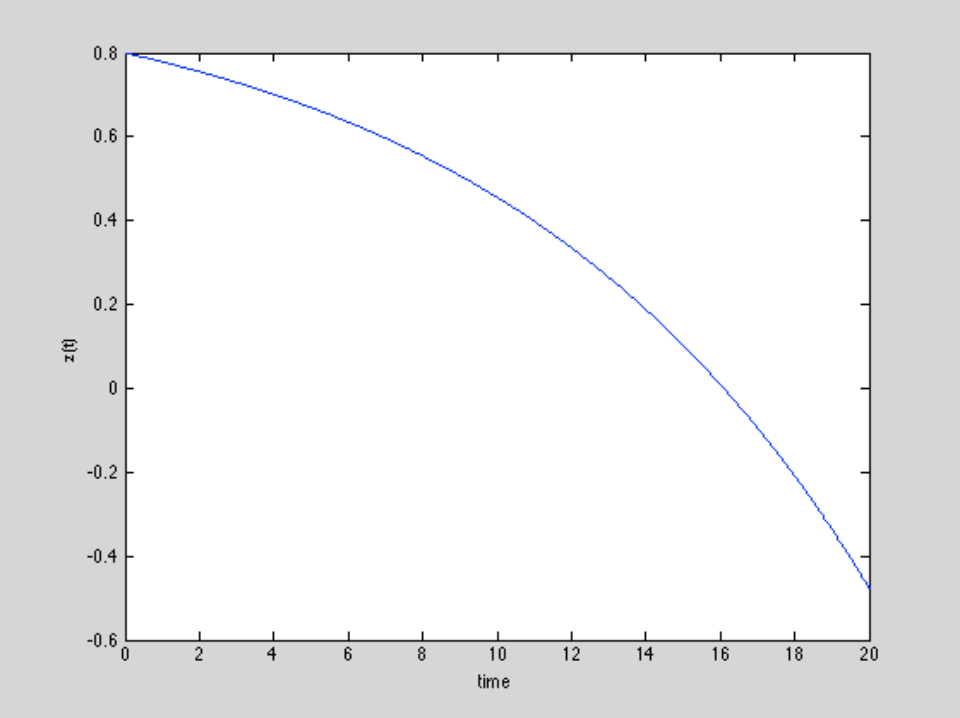


Figure 3.2*: Determinitic Solution to Bernoulli equation with initial condition $z_0 = 0.8$*

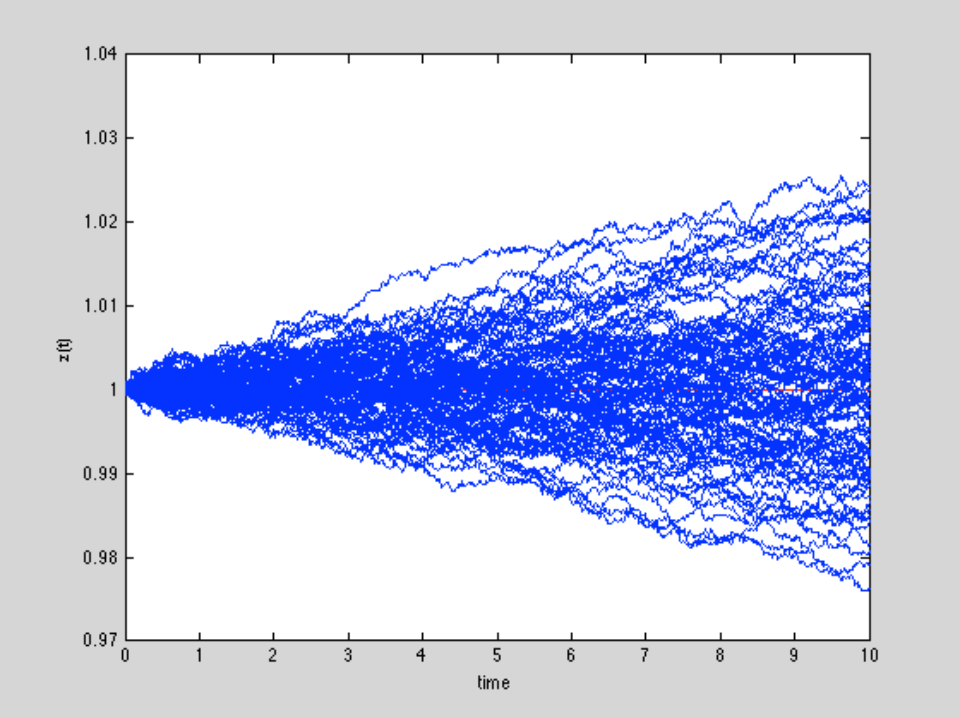


Figure 3.3: *Solution to time evolution Bernoulli equation with initial condition $z_0 = z_{cr} = 1$ and negative correlation*

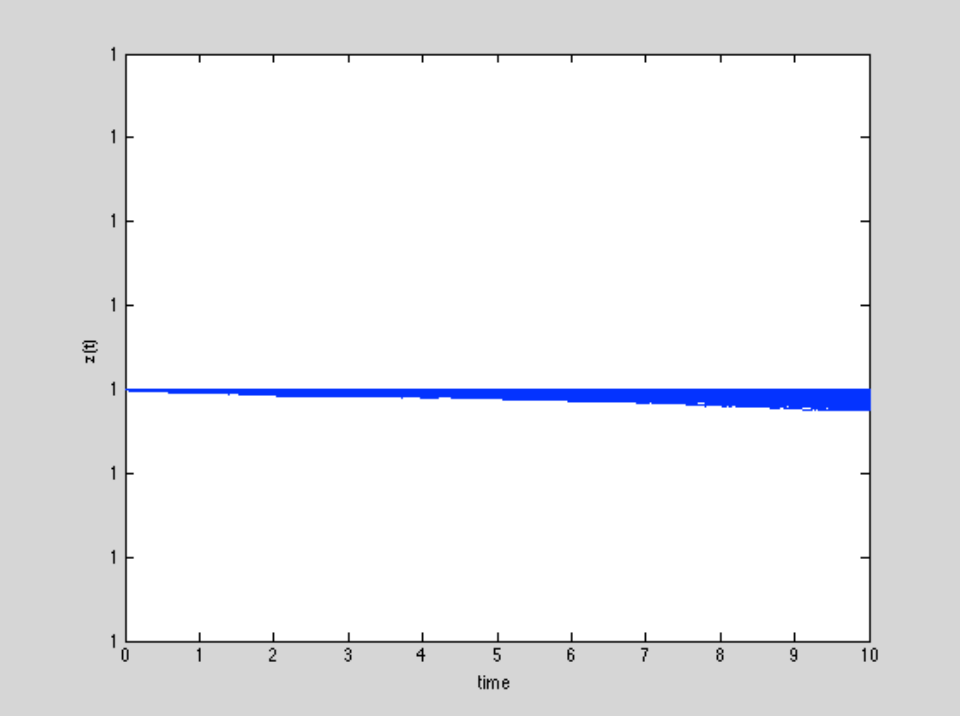


Figure 3.4: *Solution to time evolution Bernoulli equation with initial condition $z_0 = z_{cr} = 1$ and positive correlation*

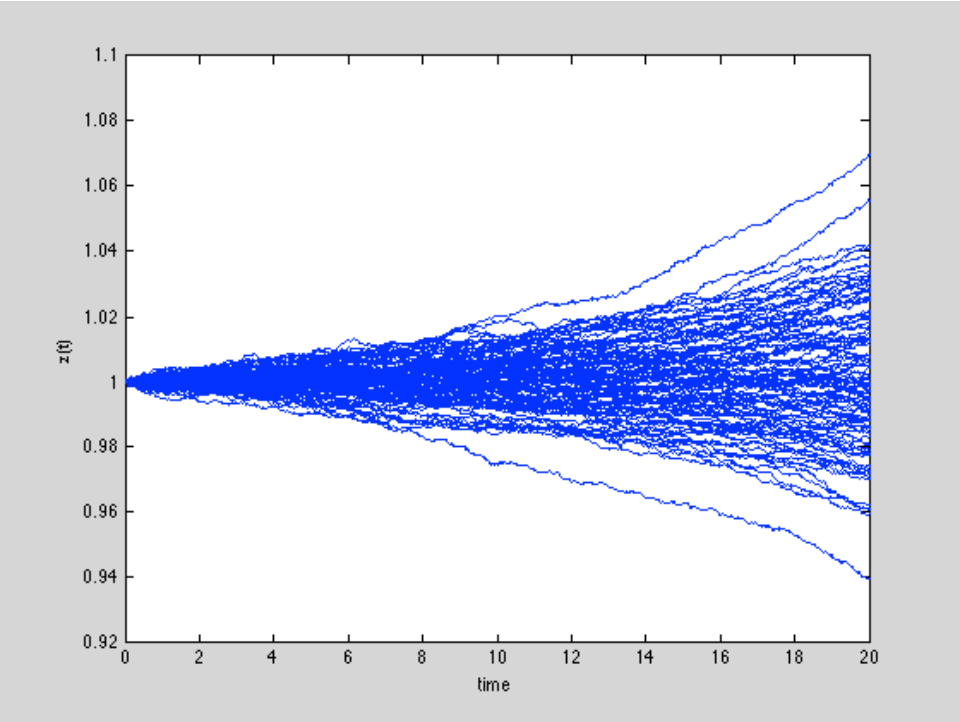


Figure 3.5: *Solution to time evolution Bernoulli equation with initial condition $z_0 = z_{cr} = 1$ and zero correlation*

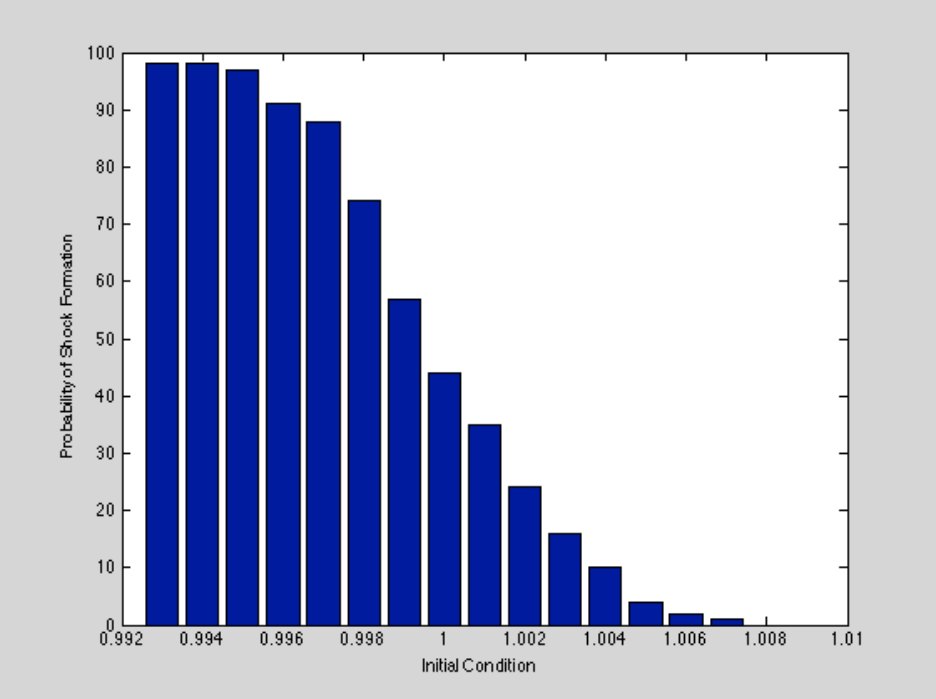


Figure 3.6: *Probability of Shock formation when initial condition belongs to critical value interval, for the case of time evolution Bernoulli equation with zero correlation.*

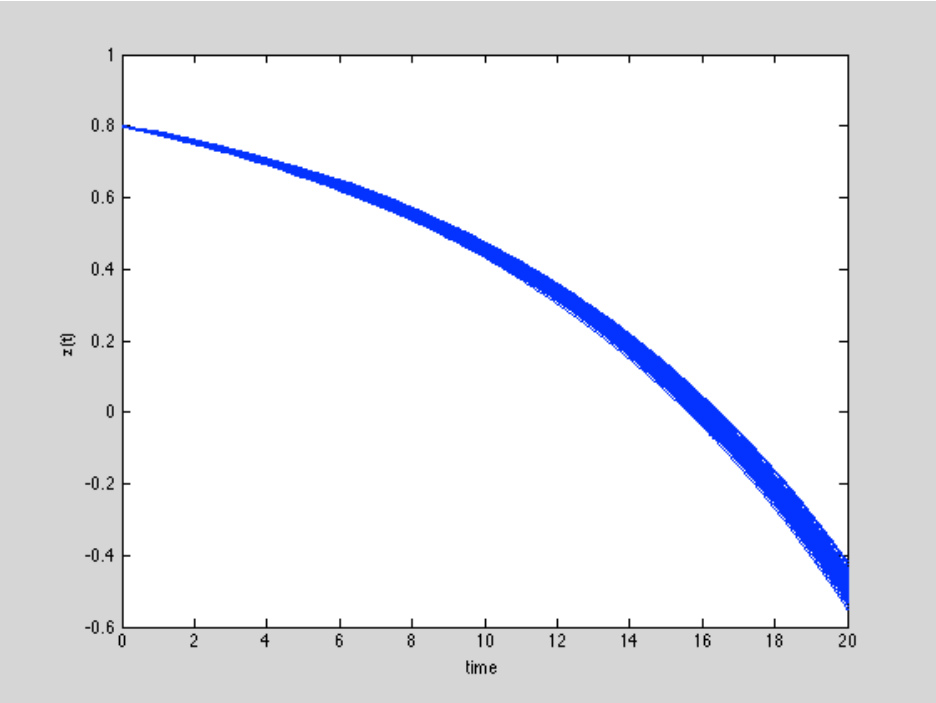


Figure 3.7: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 0.8$ and zero correlation*

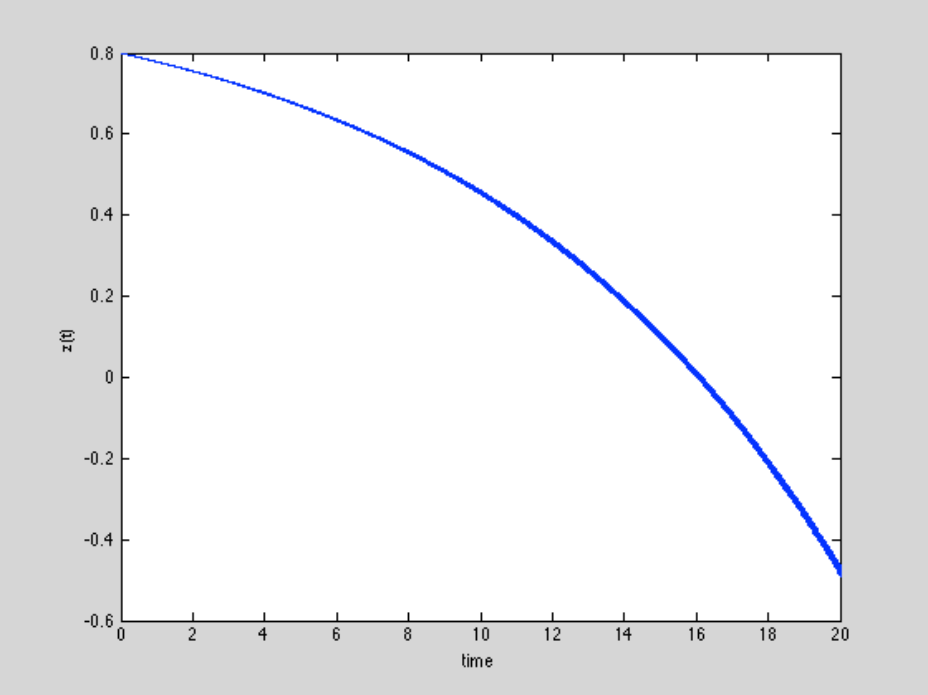


Figure 3.8: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 0.8$ and positive correlation*

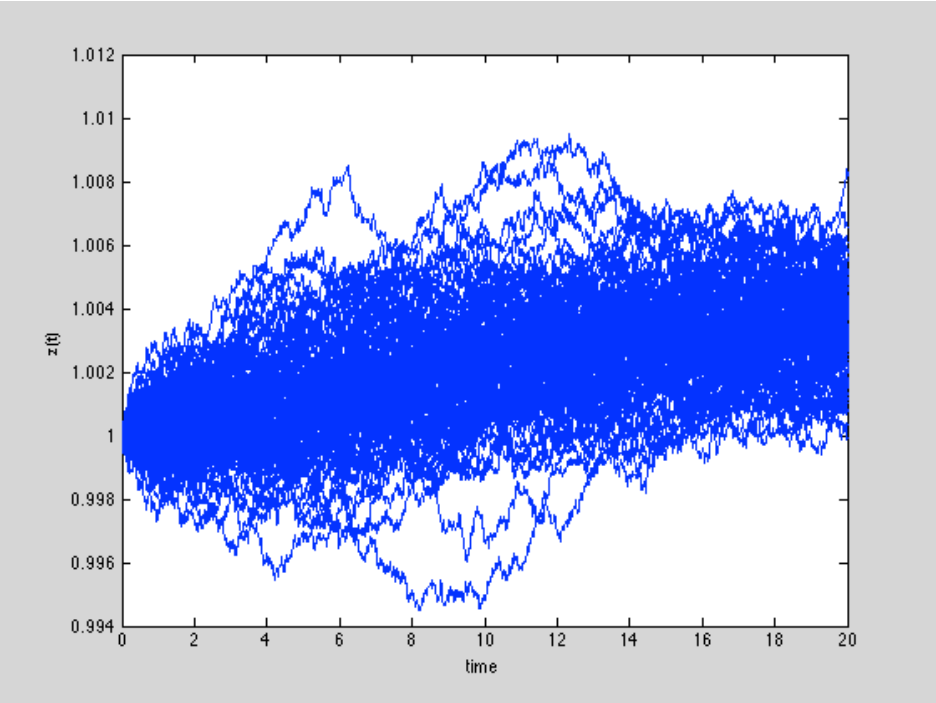


Figure 3.9: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and negative correlation*

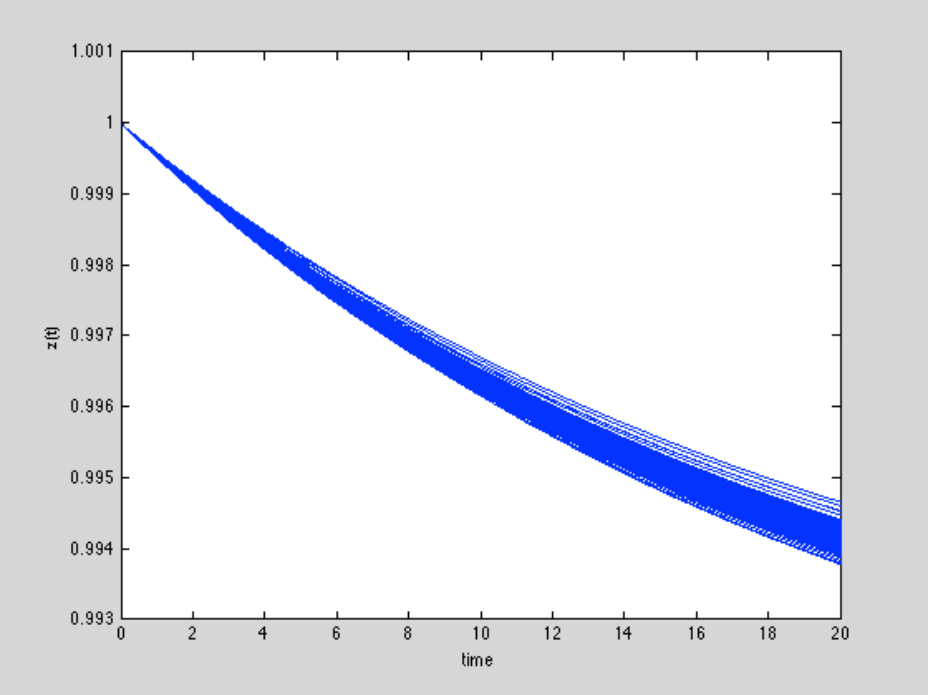


Figure 3.10: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and positive correlation*

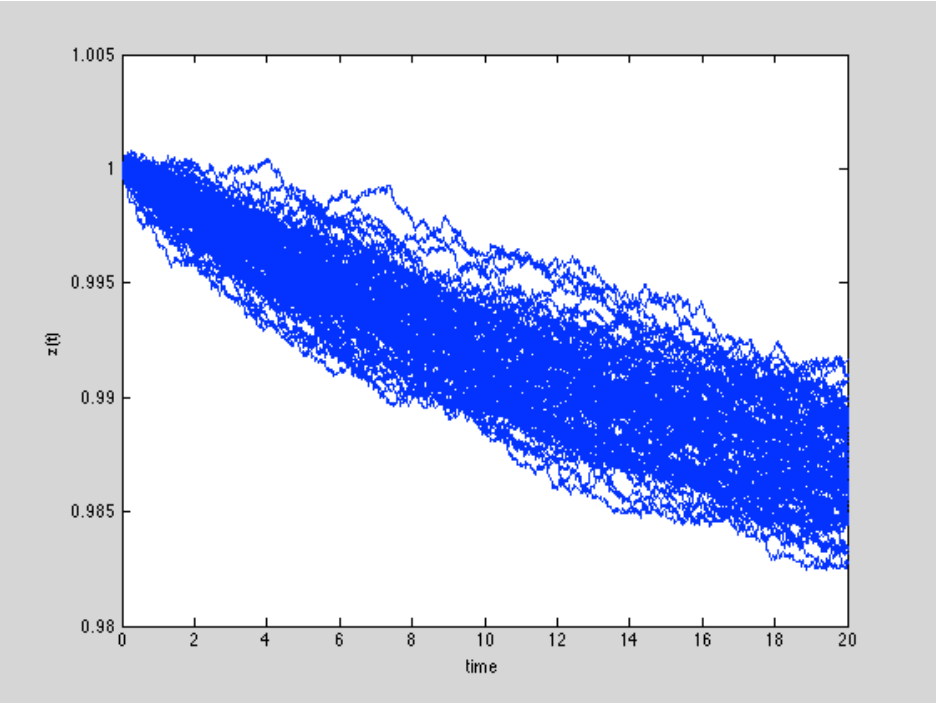


Figure 3.11: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and zero correlation*

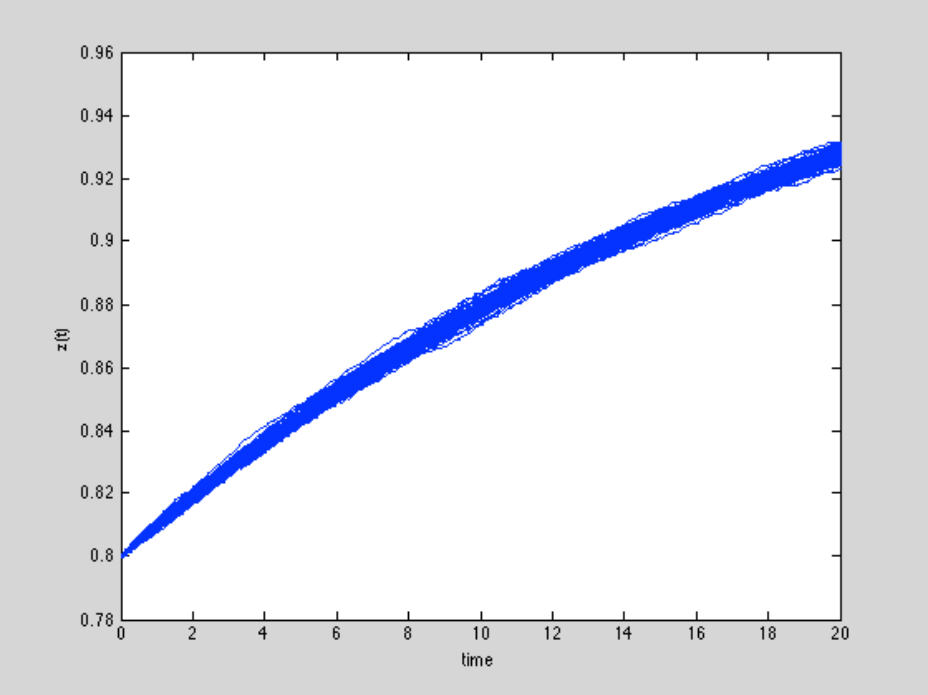


Figure 3.12: *Solution to spatial Bernoulli equation with initial condition $z_0 = 0.8$ and negative correlation*

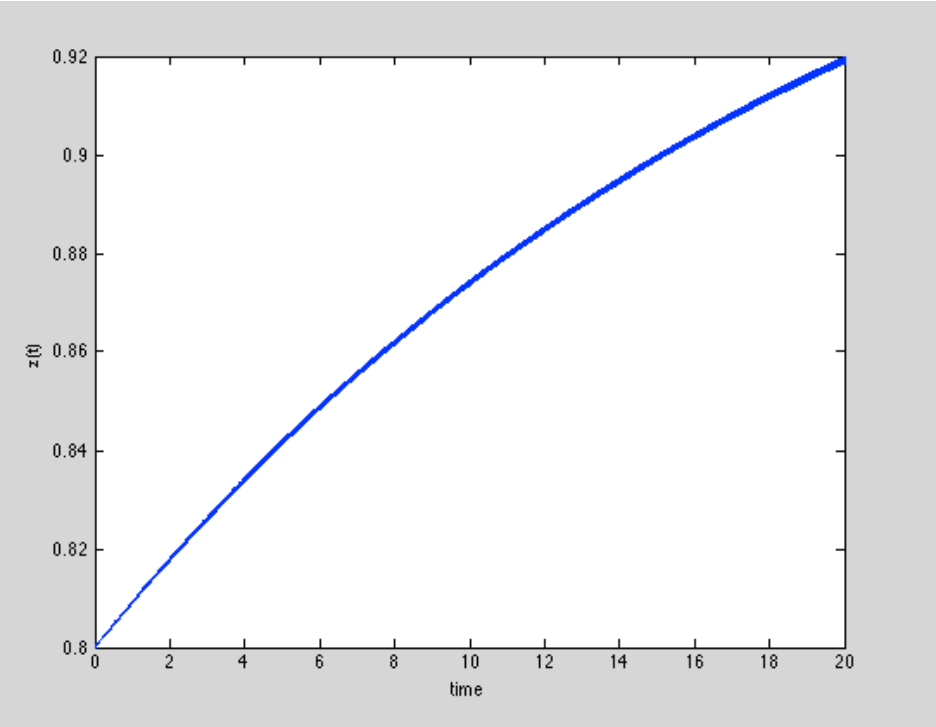


Figure 3.13: *Solution to spatial Bernoulli equation with initial condition z_0 =0.8 and positive correlation*

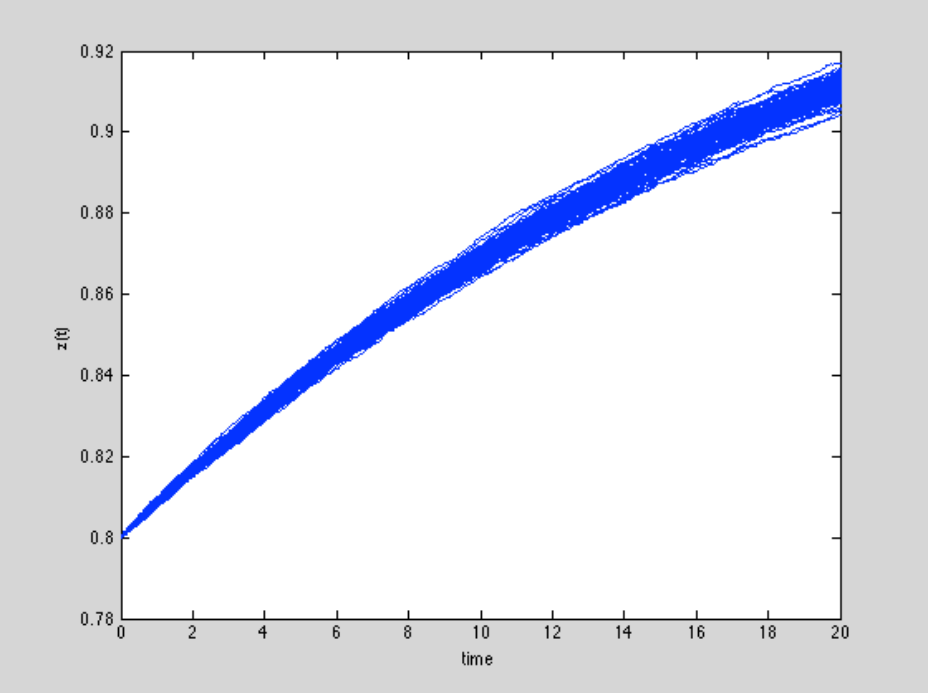


Figure 3.14: *Solution to spatial Bernoulli equation with initial condition z_0 =0.8 and zero correlation*

Initial Condition z_0	Probability of Shock Formation (%)
0.993	**98**
0.994	**98**
0.995	**97**
0.996	**91**
0.997	**88**
0.998	**74**
0.999	**57**
1	**44**
1.001	**35**
1.002	**24**
1.003	**16**
1.004	**10**
1.005	**4**
1.006	**2**
1.007	**4**

Table 3.1: *Probability of Shock formation when initial condition belongs to critical value interval, for the case of time evolution Bernoulli equation with zero correlation.*

3.4.2 Gaussian with a Cauchy Covariance Function

Zoomed in graphs are provided to better show the fluctuations.

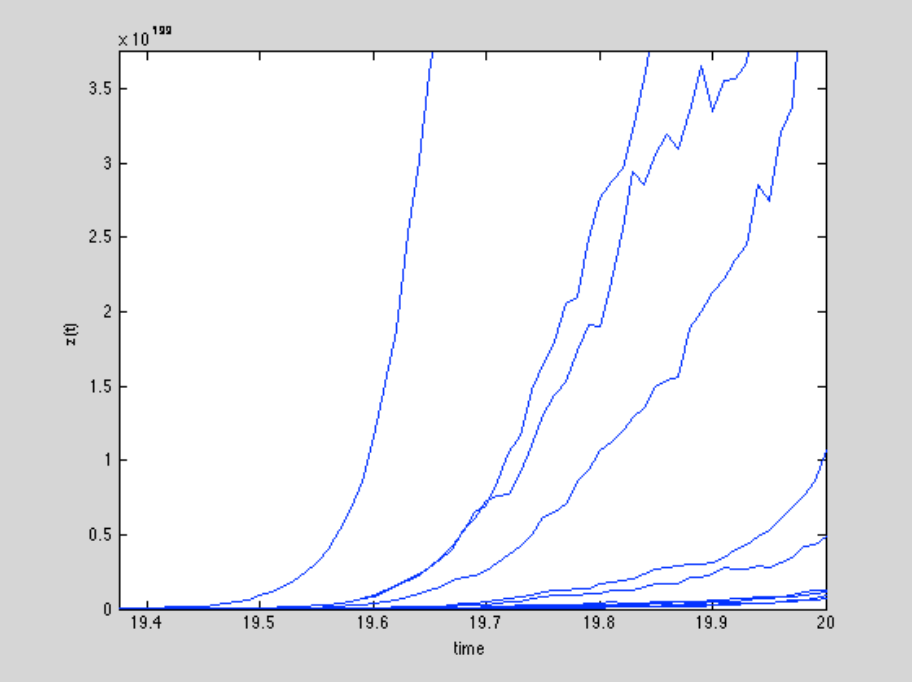


Figure 3.15: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and negative correlation*

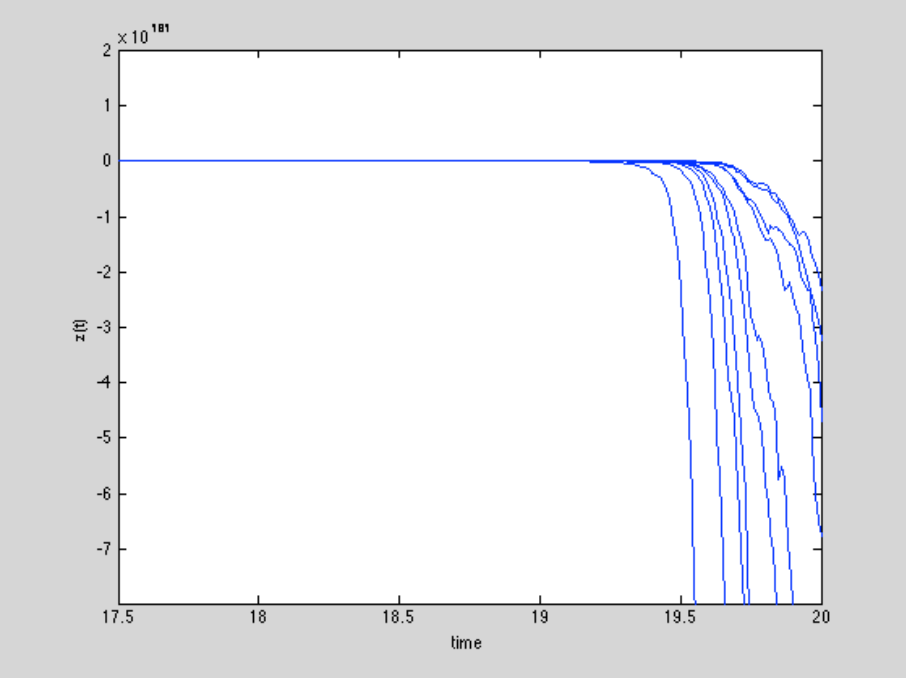


Figure 3.16: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and positive correlation*

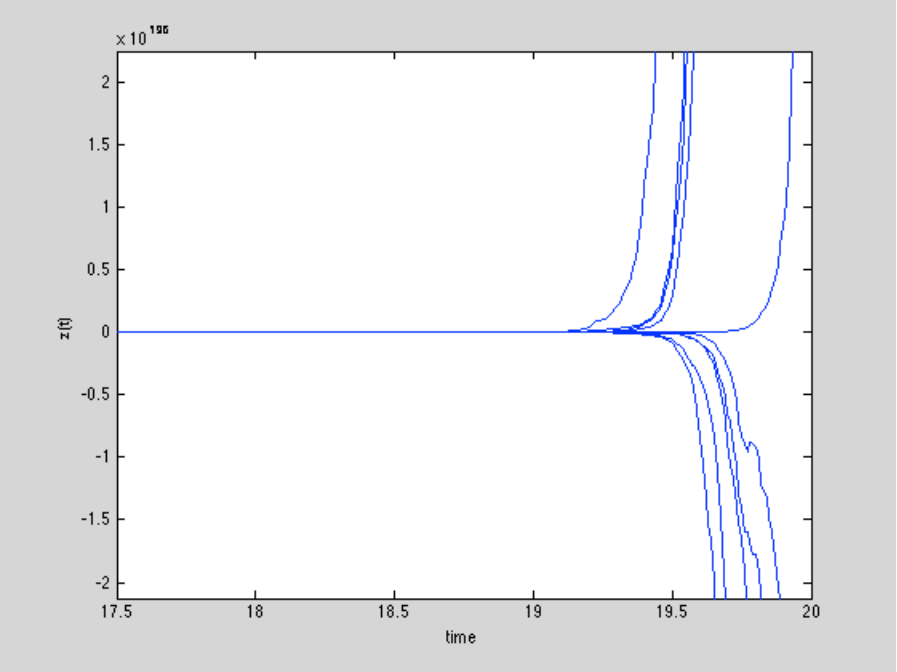


Figure 3.17: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and zero correlation*

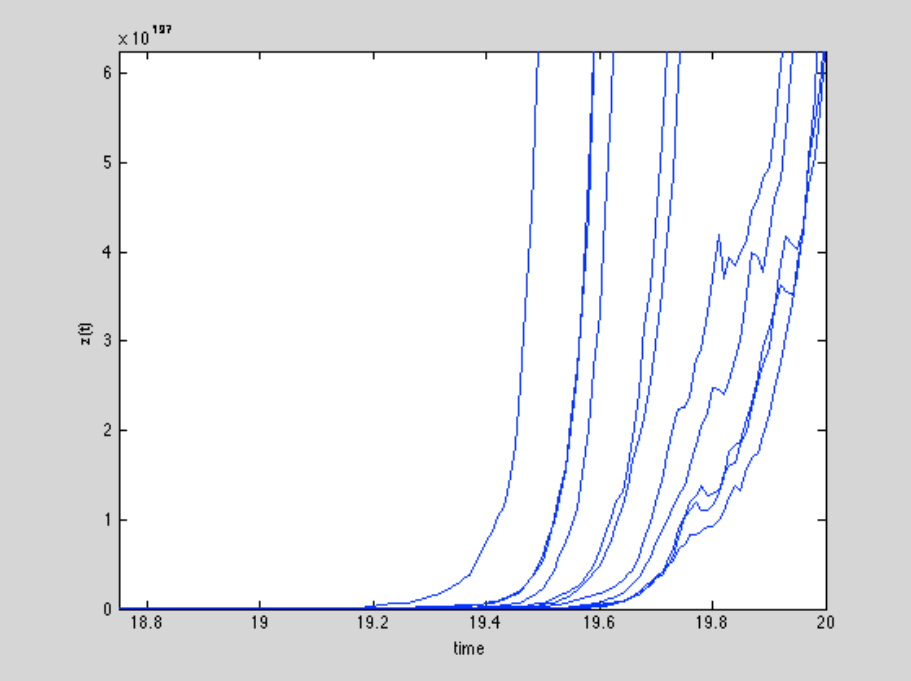


Figure 3.18: *Solution to time evolution Bernoulli equation with initial condition $z_0 = 0.8$ and negative correlation*

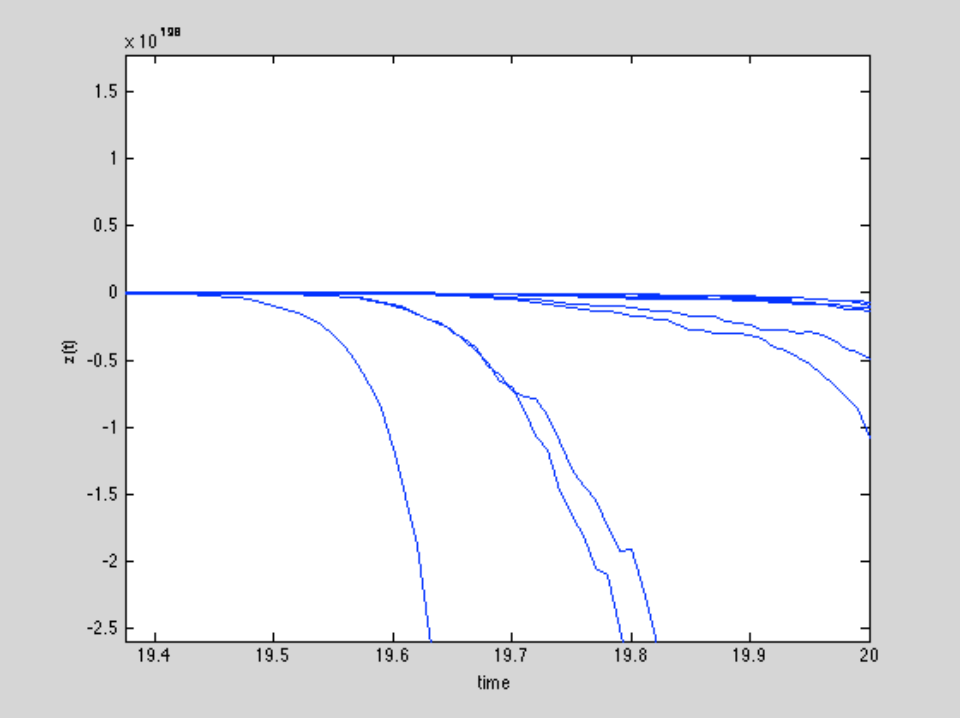


Figure 3.19: *Solution to time evolution Bernoulli equation with initial condition z_0 =0.8 and positive correlation*

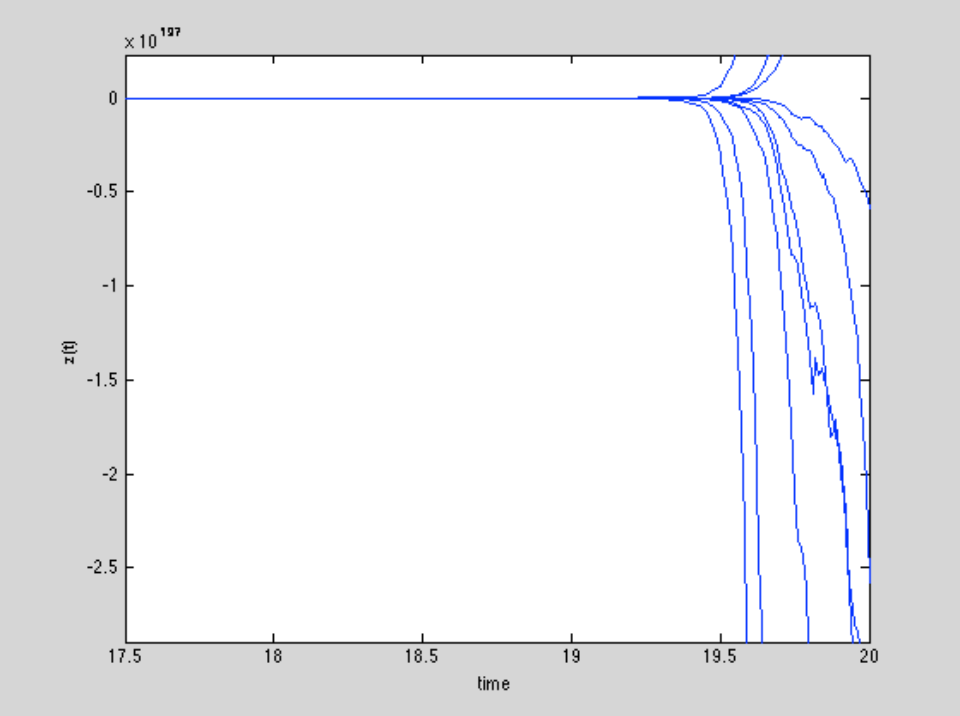


Figure 3.20: *Solution to time evolution Bernoulli equation with initial condition z_0 =0.8 and zero correlation*

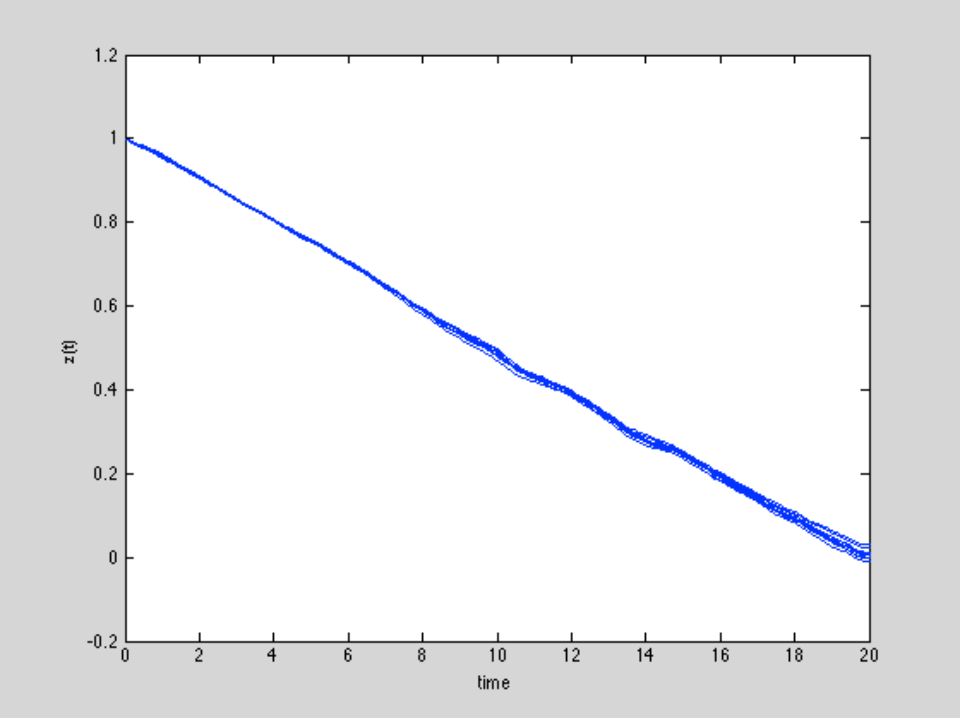


Figure 3.21: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and negative correlation*

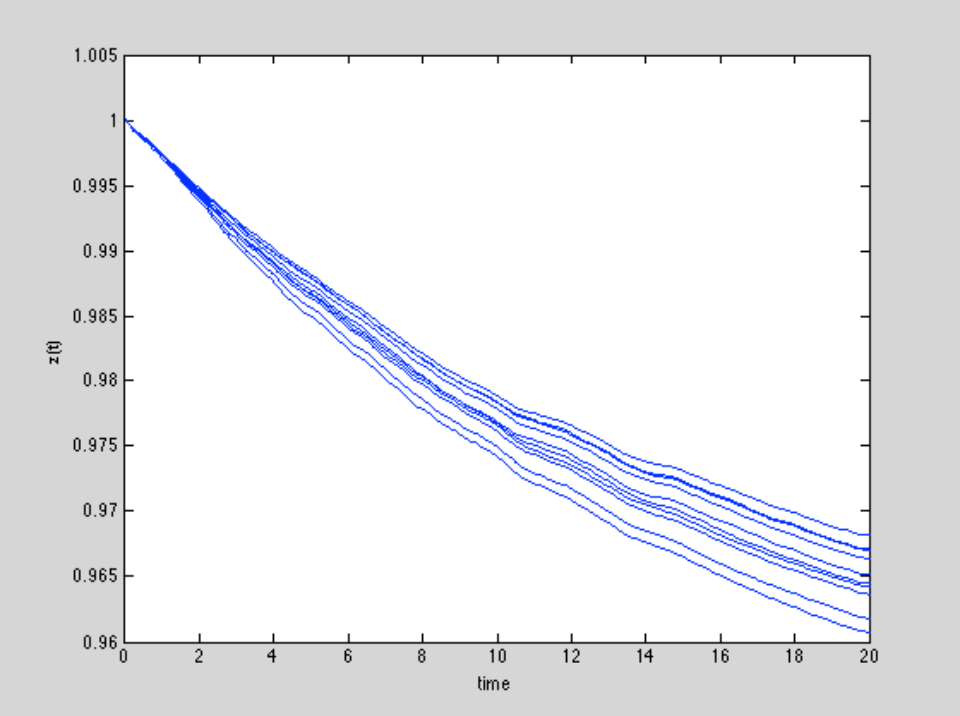


Figure 3.22: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and positive correlation*

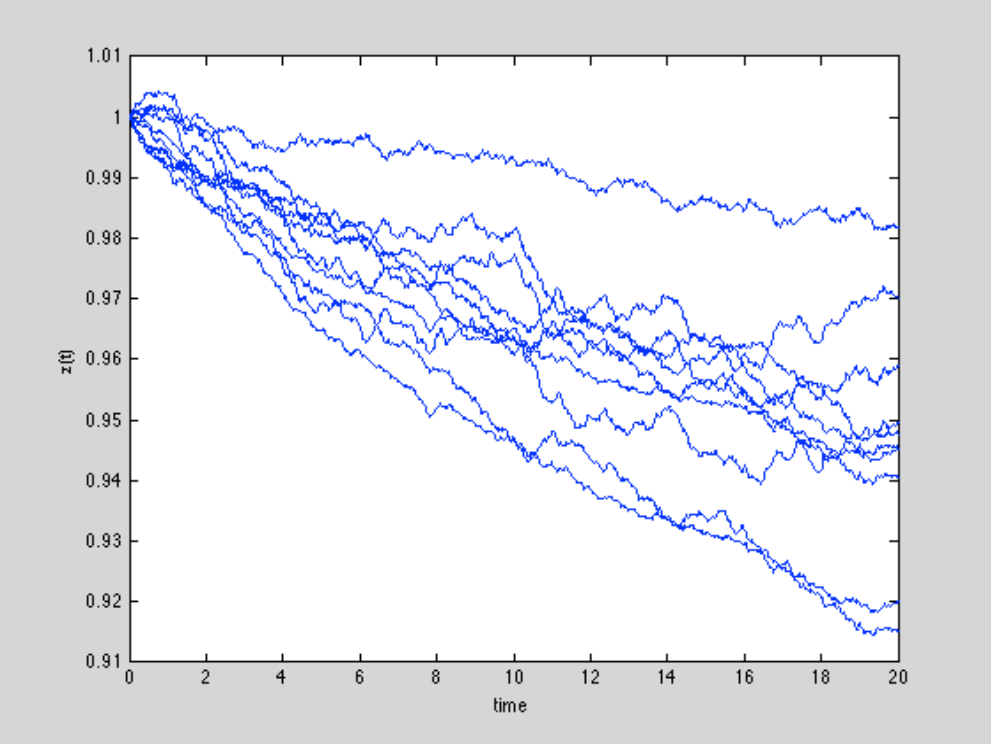


Figure 3.23: *Solution to spatial Bernoulli equation with initial condition $z_0 = 1 = z_{cr}$ and zero correlation*

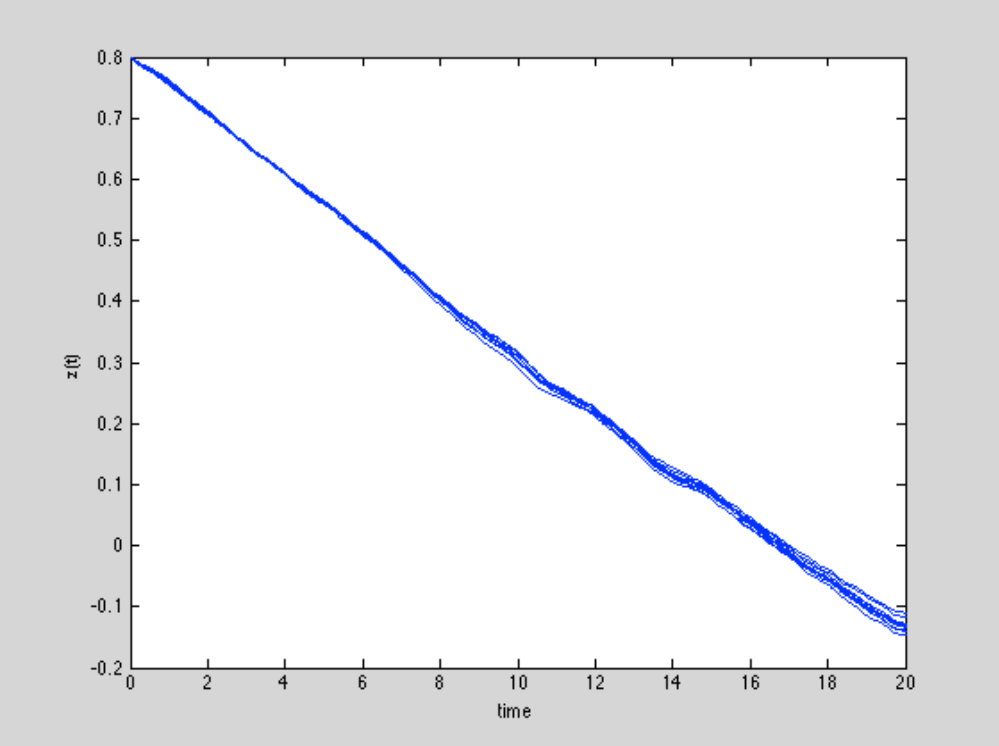


Figure 3.24: *Solution to spatial Bernoulli equation with initial condition $z_0 = 0.8$ and negative correlation*

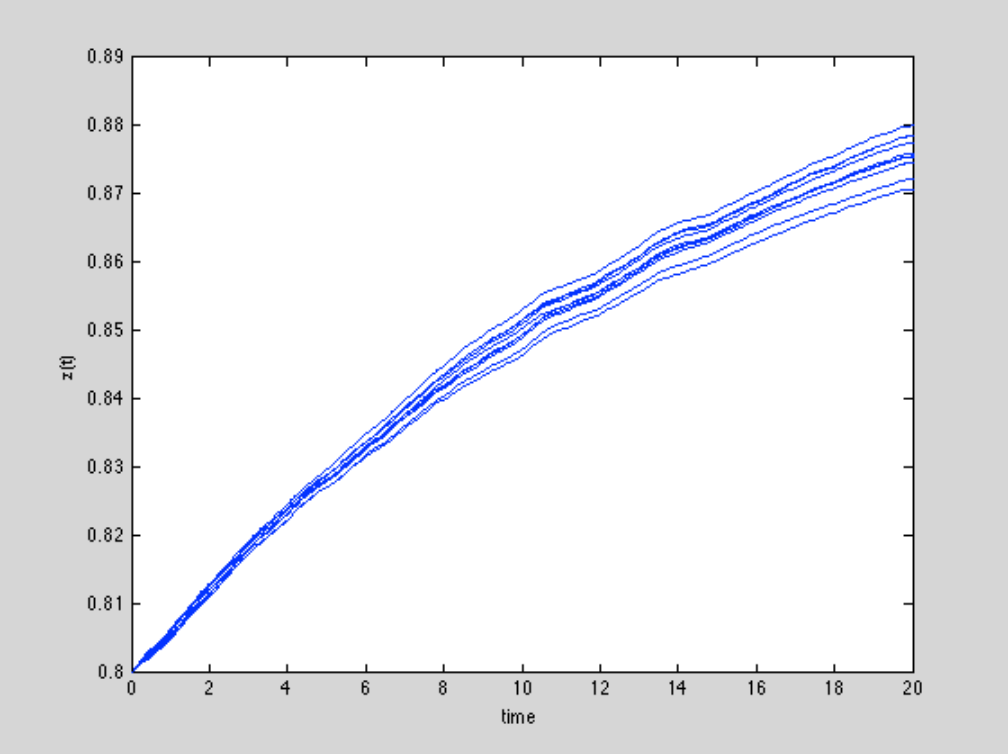


Figure 3.25: *Solution to spatial Bernoulli equation with initial condition z_0 =0.8 and positive correlation*

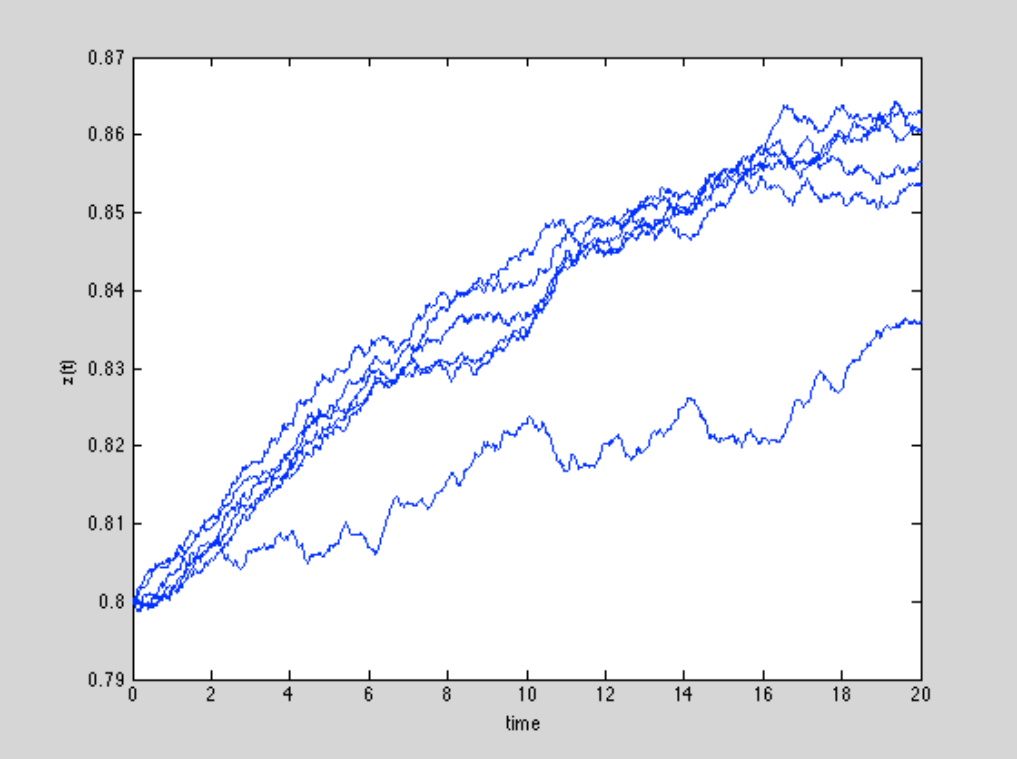


Figure 3.26: *Solution to spatial Bernoulli equation with initial condition z_0 =0.8 and zero correlation*

3.4.3 Gaussian with a Dagum Covariance Function

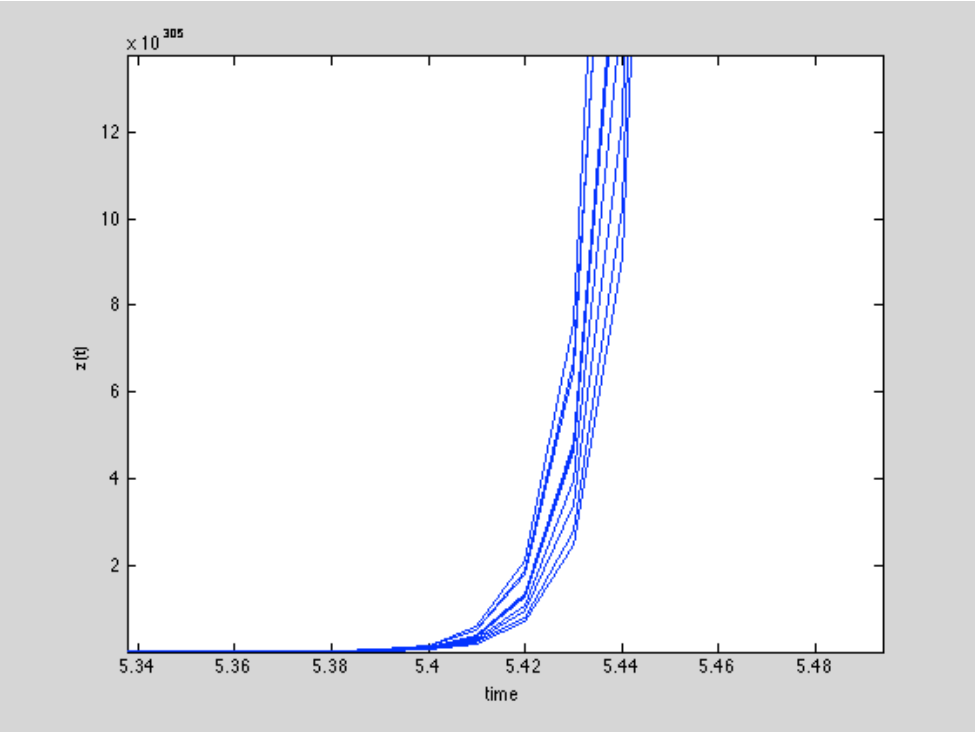


Figure 3.27: *Solution to time evolution Bernoulli equation with initial condition z_0 =1 and negative correlation*

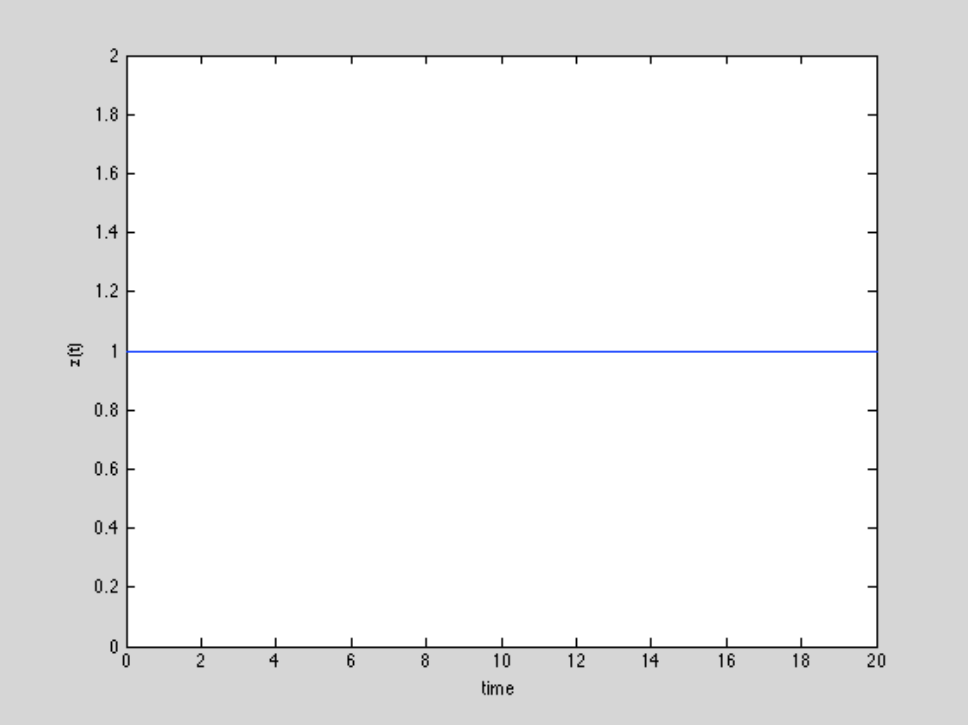


Figure 3.28: *Solution to time evolution Bernoulli equation with initial condition z_0 =1 and positive correlation*

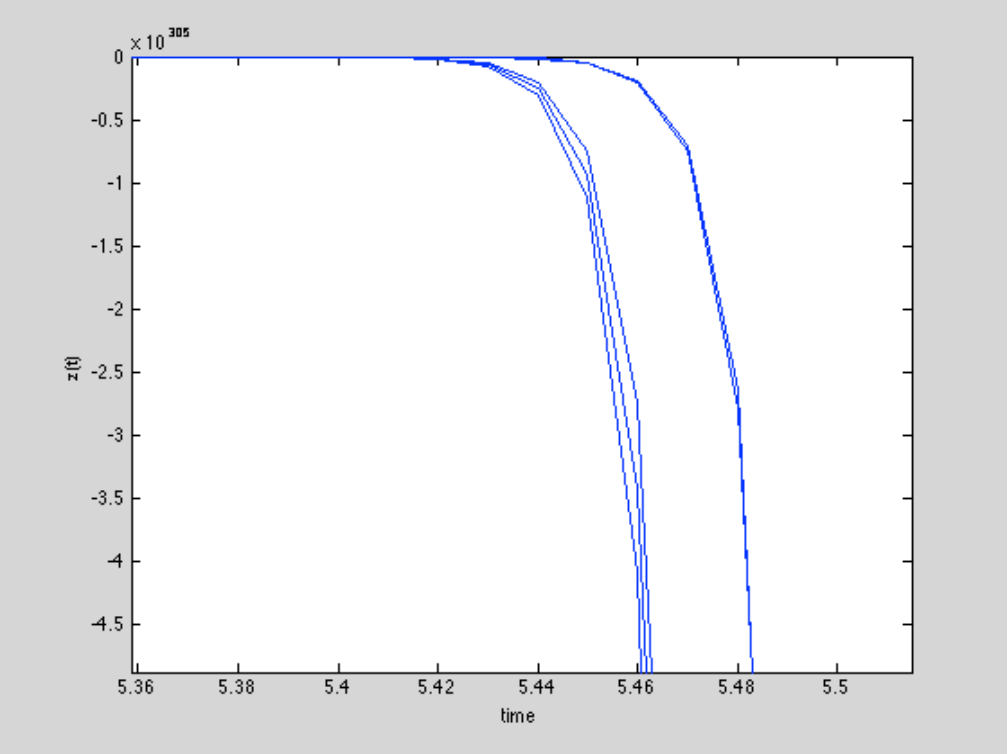


Figure 3.29: *Solution to time evolution Bernoulli equation with initial condition z_0 =1 and zero correlation*

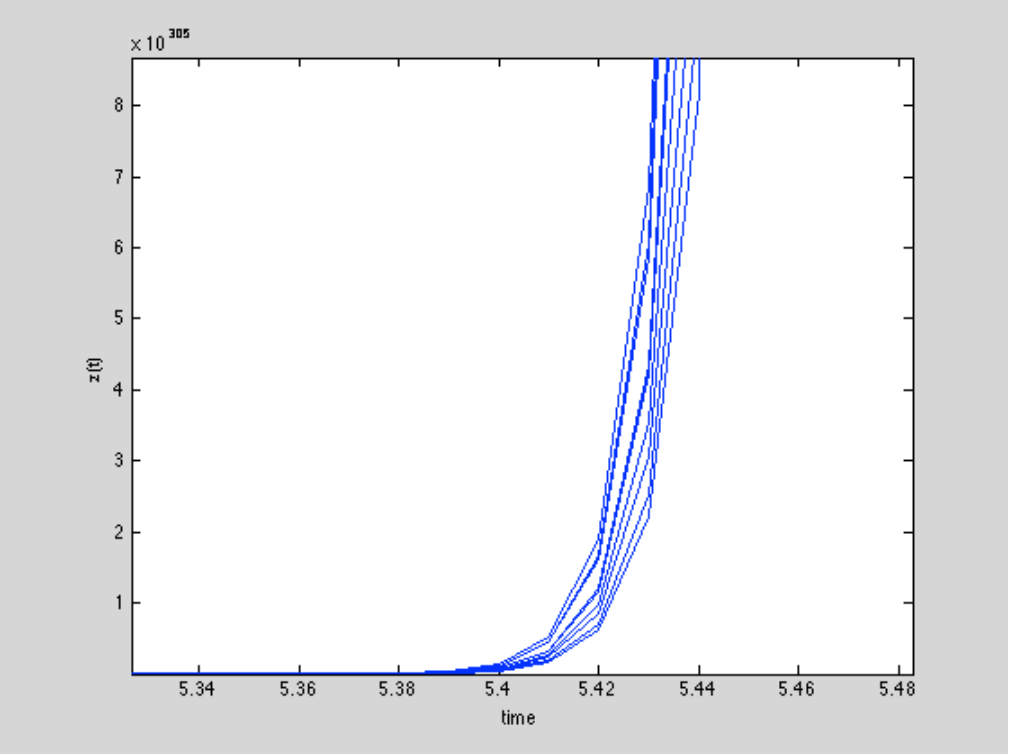


Figure 3.30: *Solution to time evolution Bernoulli equation with initial condition z_0 =0.8 and negative correlation*

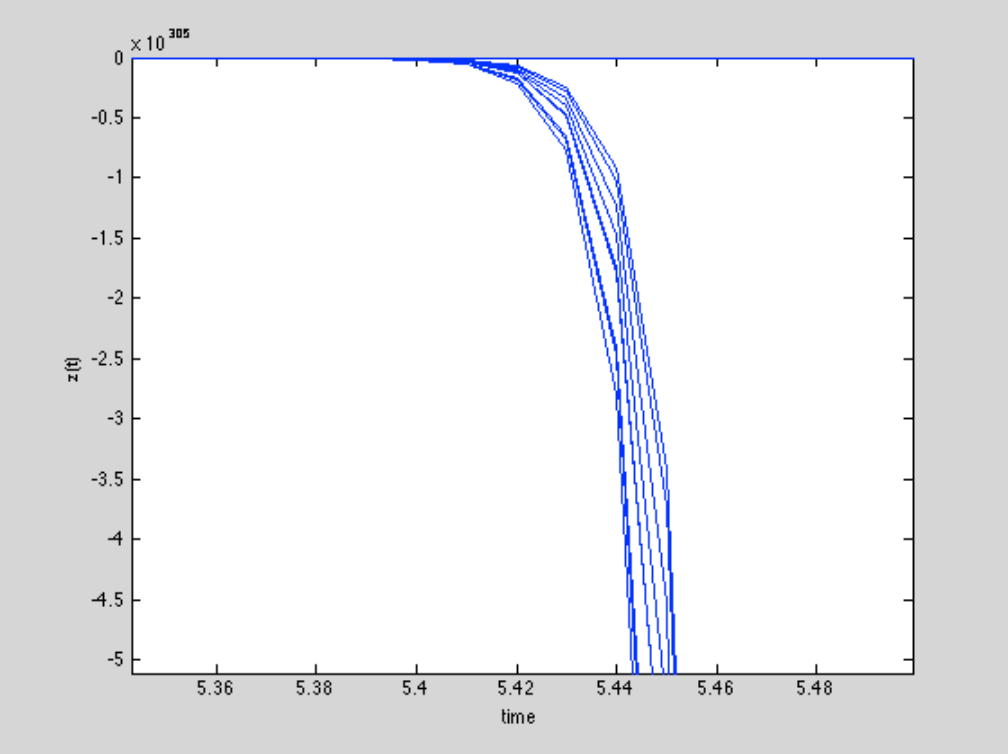


Figure 3.31: *Solution to time evolution Bernoulli equation with initial condition z_0 =0.8 and positive correlation*

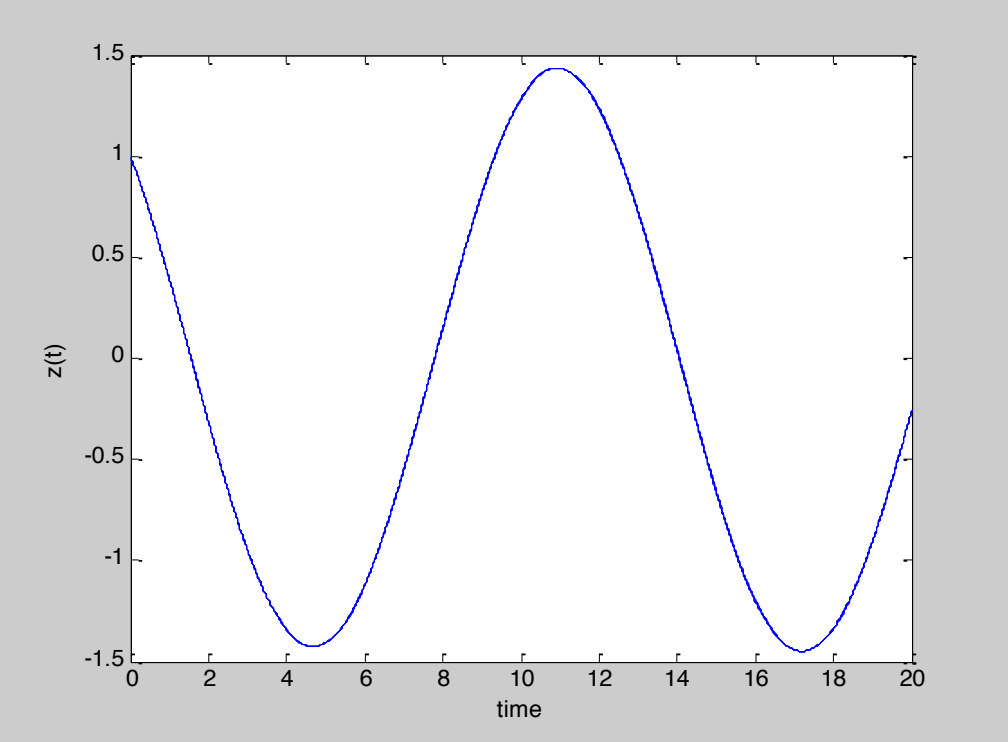


Figure 3.32: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =1 and negative correlation*

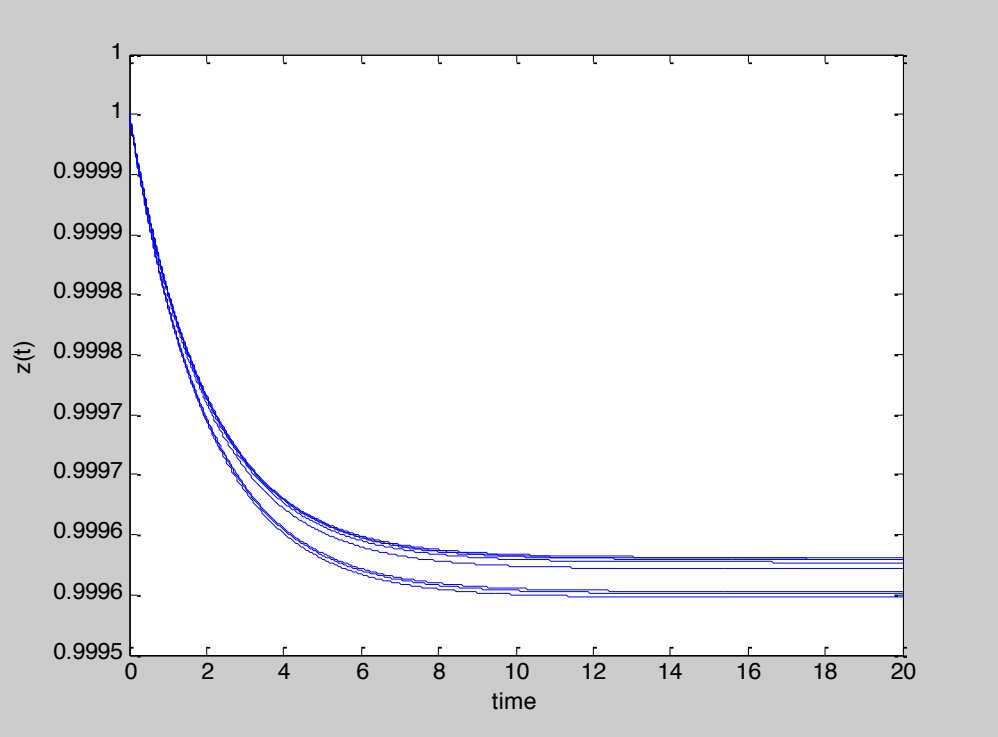


Figure 3.33: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =1 and positive correlation*

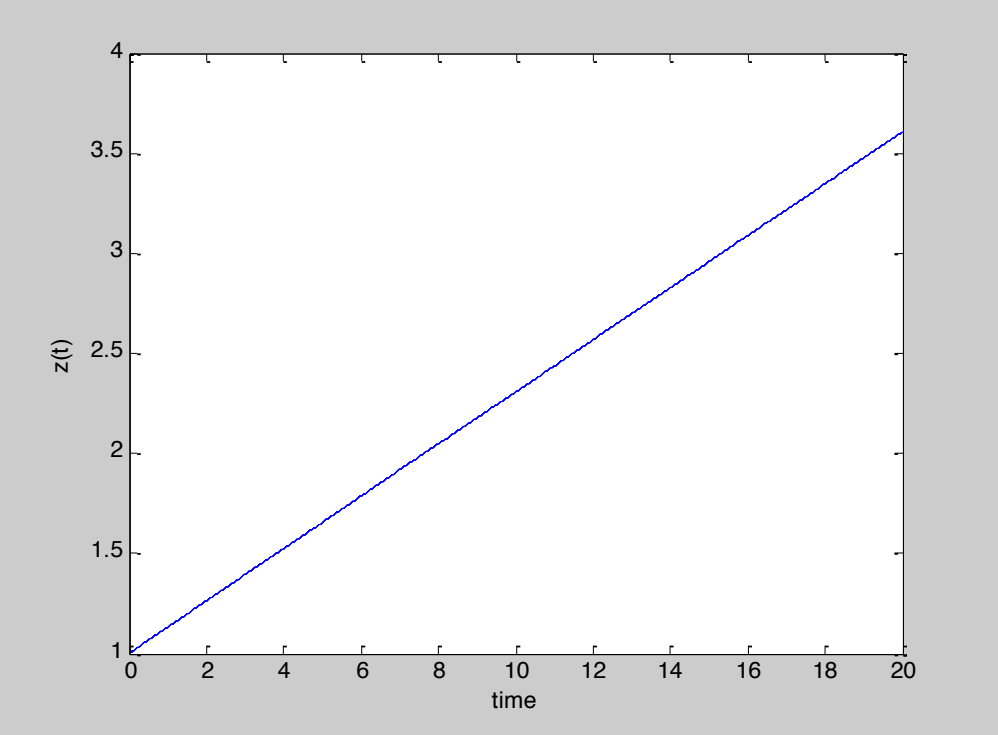


Figure 3.34: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =1 and zero correlation*

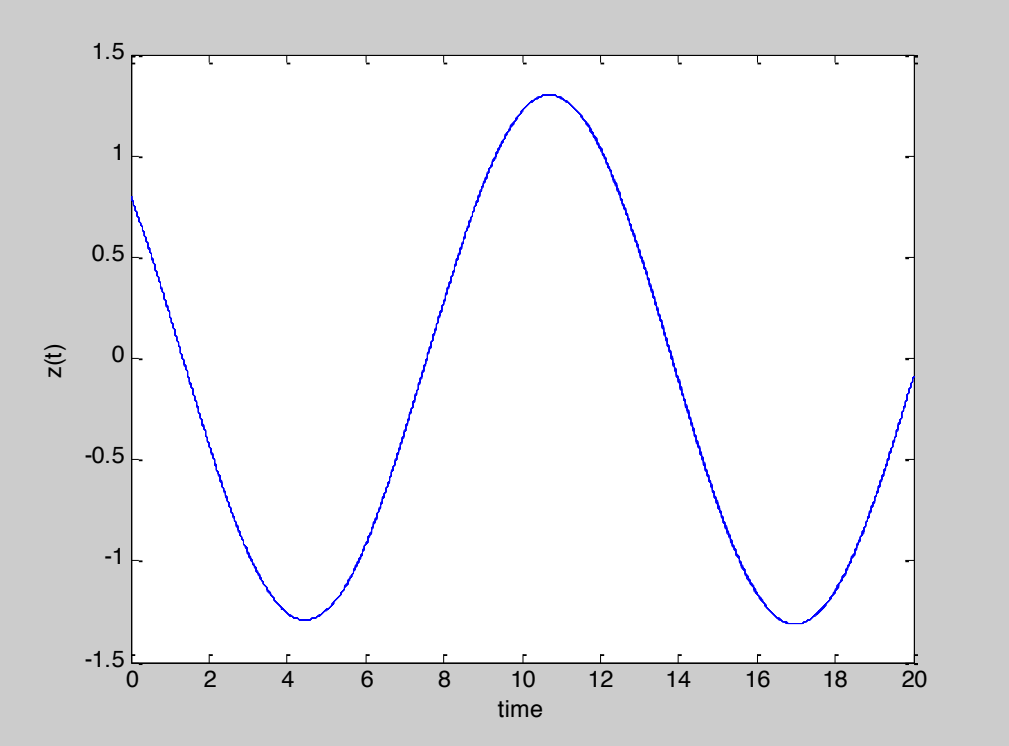


Figure 3.35: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =0.8 and negative correlation*

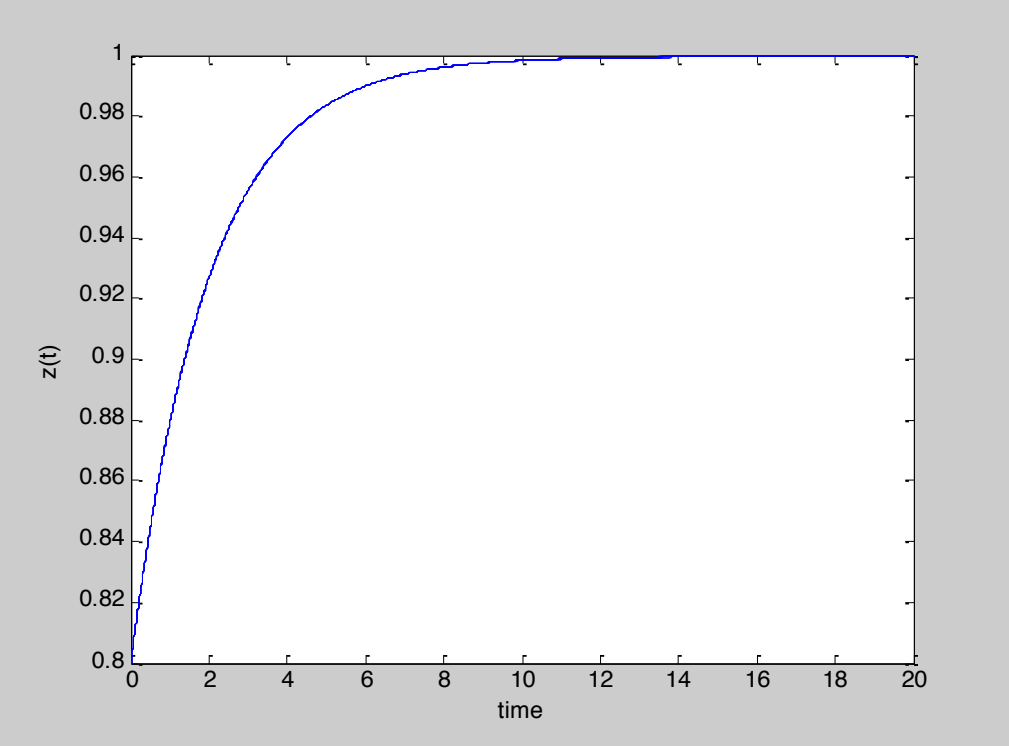


Figure 3.36: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =0.8 and positive correlation*

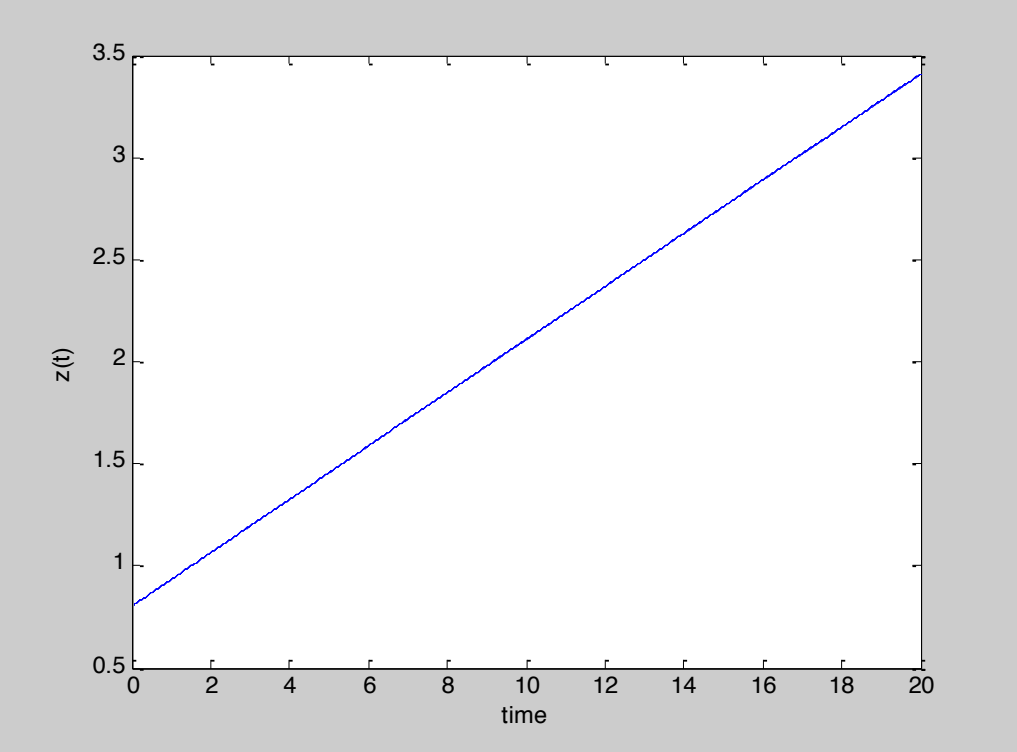


Figure 3.37: *Solution to spatial evolution Bernoulli equation with initial condition z_0 =0.8 and zero correlation*

CHAPTER 4: CONCLUSION

This thesis reports a study of material spatial randomness on the propagation of acceleration waves. In the reference deterministic medium case there is a competition between the dissipation and the elastic nonlinearity. The numerical simulation results agree with classical derivations in that there is a critical amplitude separating two regimes of wave evolutions (decay and blow-up) and the time (or distance) to blow-up.
In particular, it is found that:

- If the initial condition is smaller than the critical amplitude, the wave decays exponentially, indicating that the dissipation effects were predominant.
- If the initial condition is equal to the critical amplitude, both effects balance each other out and we have a constant value of acceleration at each position.
- If on the other hand, the initial value is greater than the critical amplitude, we observe rapid exponential growth of the wave and formation of a shock. Here it is convenient to introduce the inverse amplitude z in order to calculate the distance to form a shock.

The numerical simulations results agree with the theoretical ones. The stochastic analysis of acceleration waves suggested that a position-dependent Bernoulli equation would be more accurate to model the evolution of the wave amplitude. We then introduced perturbations of type Gaussian White Noise, Gaussian with Cauchy covariance function, and Gaussian with Dagum covariance function in the material coefficients for both the time and spatial evolutions. The positive, negative and zero correlations are considered. Although the results are unpredictable, they show that the correlation plays an important role in the final solution. Unlike the deterministic case, we observe an interval for critical amplitudes for the time evolution equation with Gaussian White Noise, and for the position-dependent equation with Cauchy coefficients. In those cases also, the time or distance to form a shock are nearly equal to the deterministic time to shock formation. For the Dagum simulations, we do not observe a critical interval, but we obtain periodic oscillations and straight lines unlike the other random fields. When all three random fields are considered, it turns out that an initial condition inferior to the average critical value does not determine a fixed behavior. Overall, the time evolution equation with Gaussian White Noise and Dagum noise,

and the spatial evolution equation with Cauchy noise exhibit behaviors most resembling the classical solutions.

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