ADAPTIVE CONTROL OF A LINEAR SYSTEM WITH QUANTIZED STATE OBSERVATIONS

BY

ARAY KALIYEVA

THESIS

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Adviser:

Assistant Professor Maxim Raginsky

ABSTRACT

The thesis addresses a problem in networked control systems where quantization is a communication constraint. Control design together with parameter estimation algorithms lead to adaptive control techniques. A first order system with unknown parameters is estimated via projection algorithm recursively. Furthermore, a one-step-ahead control law is applied to regulate the output. In the second part of the thesis, the system with quantization is simplified to a system with white noise disturbance by applying a dithering method. It is shown that system with quantization error can be controlled with the same one-step-ahead control law and projection algorithm.

To my parents, for their love and support

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CHAPTER 1

COMMUNICATION CONSTRAINTS IN CONTROL SYSTEMS

1.1 Introduction

Networked control systems is an area posing new tasks time by time due to the conflicting preferences of communication and control aspects. Many works were done in the field of communication and control. The usual setting would be to set goals for control and restrictions for communication, and therefore to formulate new strategies which will meet both communication and control objectives. Control theory requires usual performance qualities such as perfect tracking, rise time, and stability of states. Meantime, communication restrictions prevent us from achieving control goals since there are bandwidth limits and quantization effects on the communication channel. With the availability of different types of quantizers and control system representations, various methods are given in [1], [2], and [3]. In [1] the authors thoroughly discuss the control system stability under the limited data rate, and derive universal lower bounds on the state norm. The lower bound describes the worst case scenario, where it states that the coder-controller should take actions faster than the system's output does. In other words, they formulate the rate of the communication channel that any coder-controller can be realized, and below this rate there cannot be a coder-controller which would stabilize the system. The basic approach they took to claim the lower bound on states was to consider the volumes of uncertainty, not the norm of vector states. Further, control system design has been proposed relying on joint coding and certainty equivalence principle. Depending on the types of quantizers they give, various coder-controllers were analyzed. When the certainty equivalent principle is used, the control law turns out to be a function of the state with some gain matrices as a solution of Riccati recursion. Using the memoryless logarithmic quantizer in [2], the authors show that the control law follows the logarithmic law, and it is achieved by minimization of quadratic Lyapunov function. This result is motivated by "coarsest" quantizer and coarser values the farther the state is from the origin. It also works for unstable systems, sampling time being dependent on unstable eigenvalues. The idea here is that states of the system and respectively the control actions can be inaccurate when the state is far from the origin. The resolution should be more accurate when the system starts hitting values close to the origin. In [3], asymptotic stability is achieved via finite quantizers with an adjustable width parameter. In other words, by changing the quantization width according to some law, a linear feedback control gives the asymptotic stable result. This is sometimes called zooming-in/zooming-out strategy. The system with the uniform finite quantizer with adjustable width forms a hybrid system, and the techniques developed work for both continuous- and discrete-time systems. When a controller takes measurements from an encoder through a digital channel, the problem is classified as a stabilization of the system with limited information. The latter problem is investigated in [4], where the relation between the state transition matrix and an encoder function is given in a way to achieve asymptotic stability in one sampling period time. In other words, by fixing the sampling period, positive integer number N for the encoder or controller, and dimension of the output n, the author establishes the conditions for asymptotic stability. In [5], problems on building a stable network control system over a digital communication channel are investigated. They construct an encoder, decoder and controller under some communication rate, and the rate which is given is a lower bound for stability of the linear discrete time system. This rate is lower bounded by the sum of logarithms of the moduli of the unstable eigenvalues of the system. Also, the authors give the upper bound on the communication rate with multiple sensors. In this case, the problem gets even more entangled because communication channel should be able to allocate all measurements from the multiple sensors.

1.2 Information-Constrained Control

In this thesis we pose a task of implementing an adaptive control in networked control systems where loss of information happens due to the quantization effects. This is again an information-constrained control problem, but with unknown parameters of the system model. Hence, tools from control theory, communication and signal processing will be utilized throughout the paper. A special branch of control theory, adaptive control, in fact, deals with the control of systems with unknown parameters. In addition to control theory mechanisms, many of the algorithms in the adaptive control are special formulations of mathematical optimization problems. In particular, adaptive control recursively solves the optimization problems. In [6] Tsypkin brings out excellent formulations of optimization problems as optimization algorithms. In essence, it helps to get a very intuitive understanding and basics of recursive adaptive control theory techniques. There are various adaptive control tools other than online recursive algorithms, but in many cases, realization of them in practice becomes challenging. In adaptive control theory, apart from control tasks, the system parameters have to be investigated as well. This, in turn, complicates the analysis of the given system. However, the techniques and analysis tools which will be taken in the paper are relatively easy to implement in practice. In particular, one-step-ahead control law for control design, and projection algorithm for parameter estimation will be used [7]. Detailed analysis of the above controller and parameter estimation algorithm for deterministic and stochastic cases are given in this book. Both of these algorithms are online recursive algorithms. The projection algorithm comes from the optimization problem with constraints where the square error between current and previous estimates is minimized. The constraint which is imposed is system's model written in a difference equation. Chapter 2 covers this parameter estimation algorithm and its convergence properties. The one-step-ahead control, in turn, is simple to implement yet gives appealing results. We should remind readers that our control law is designed for discrete-time systems. The notion behind using the one-step-ahead control is to adjust the controller at each time instant. Chapter 3 covers the adaptive control mechanisms for deterministic and stochastic systems separately. The analysis and proofs from [7] are simplified for a scalar first order system. Certainty equivalence principle is the core concept in the adaptive control. Based on the principle, the one-step-ahead control design is adapted for the deterministic system with some assumptions on the system. It turns out that this simple controller gives good performance results for our system under consideration. In a stochastic system, under the same principle the same control law with the projection algorithm will give a stable system. As output becomes unobservable due to noise, Kalman filter will give the best estimate of the output. The control law for the stochastic system is derived by minimizing the mean-square error between the output and desired value. It is different in structure from the one-step-ahead controller but it is simplified to the latter in the case of first-order single input single output system. Here we adapt the stochastic control analysis to a system with communication constraints. In particular, the constraint will be quantization noise. Hence, now the problem can be considered as an information-constrained control or control in communications. Since the quantization error (difference between the output and quantized output value) is dependent on input values, a dithering method is used [8] to decorrelate the quantization error from input value. Therefore, quantization noise will be approximated by an additive white noise disturbance. The dithering method is realized according to a method given in [9]. The one-step-ahead control law will be the one to stabilize a given system, and we get the boundedness of the state in a mean-square sense which depends on the white noise variance. Therefore, we will be able to achieve the asymptotic stability result for our system.

CHAPTER 2

PARAMETER ESTIMATION AND CONVERGENCE

2.1 Problem Formulation and Approach

The objective is to implement adaptive control goals such as output regulation $|y(t)| \longrightarrow 0$ and input boundedness $|u(t)| \leq L$ for $\forall t$ by means of projection algorithm for parameter estimation and certainty equivalence principle for adaptive control of linear deterministic and stochastic systems. A single-input single-output first-order system will be taken. Throughout the development we will see attractive features of adaptive control principles. Further, we use these combined algorithms in a system with a universal quantizer.

The process of quantization happens whenever conversion of analog signal to discrete signal takes place. The problem we encounter during quantization is that, when the analog signal is converted to a digital signal we end up with a distorted signal. The more the signal is distorted, the more we get false information. The quantization noise, the difference between the original signal and the quantized signal, is correlated with the original signal. In practice, so-called dithering methods are used to decorrelate the original signal from noise [8]. After applying this method, we can consider the system with quantization as a stochastic system with an additive white noise disturbance. This is simply a stochastic linear system, which we will be analyzing in the second part of the paper. Therefore, the system's behavior will depend on the range of the added white noise, and by applying the appropriate control law, we can control the effect of quantization noise on a given system. The goal is that we would like to achieve the asymptotic results for the stochastic case with the same control design structure that is built for a deterministic system. As we will see later, it is realizable when a certain dithering method is applied to the system. The advantage is that with the same control law we control the system's output within acceptable range of values for the stochastic case with quantization noise. The figures 2.1-2.3 given below show this approach systematically. As can be seen from figure 2.2, the dithering method plays the role of transforming the system with a uniform quantizer to a system in figure 2.3 with added noise disturbance. In the latter case, we already have tools to do stability analysis given in [7].

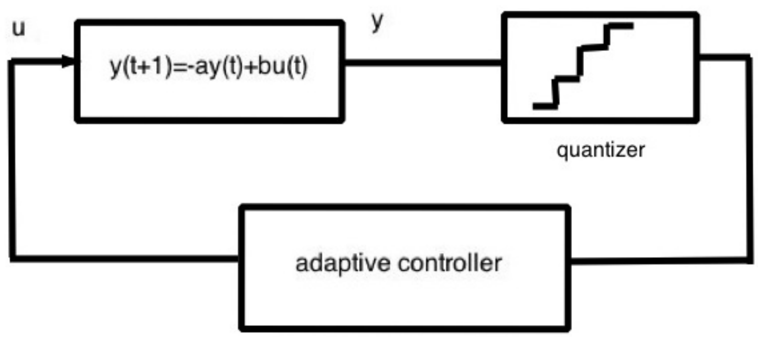

Figure 2.1: First-order feedback control system with uniform quantization of the output

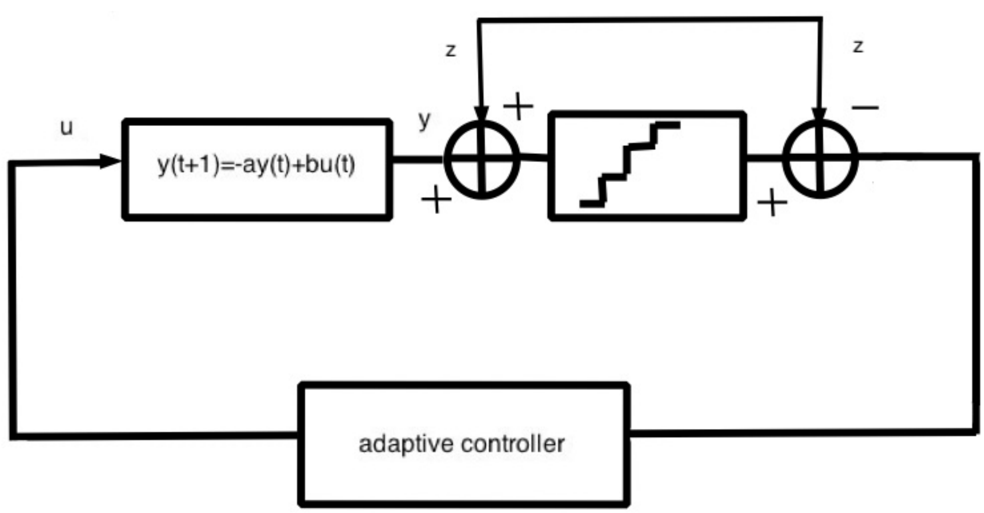

Figure 2.2: The system with dithering

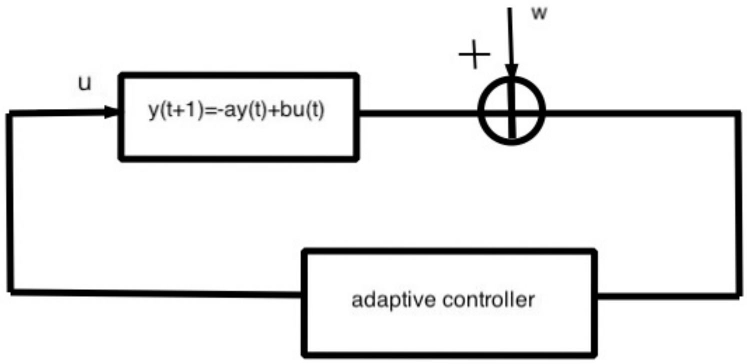

Figure 2.3: An equivalent representation of the system as a linear stochastic system

2.2 Projection Algorithm

For simplicity, let us take the first-order linear time-invariant deterministic system. There are many methods for realizing an adaptive control, and we will use the one-step-ahead control law. Together with the control law, parameter estimation is applied to a given system. A projection algorithm is used to estimate the system parameters. Here we give the detailed explanation of the algorithm with the control law. In essence, this method is called certainty equivalence principle, and it is given in [7]. We have a system with unknown parameters. We pretend to know the parameters of the closed-loop system, and we apply the one-step-ahead control law based on these parameters to drive the system's output to zero. In other words, the system's state in case of successfully reaching the right controller parameters will converge to zero. Also, additional objectives such as boundedness of input signal may be introduced. However, because we have replaced the parameters by some estimate, and derived the control law according to them, the output value will not reach zero. Here comes the need for the projection algorithm for parameter estimation. The parameter estimation algorithm is implemented using the current system measurements. There are some clarifications we should mention: the projection algorithm does not necessarily guarantee convergence of the estimated parameters to true parameters. It just drives prediction error and hence the output to zero. In fact, since our goal is output regulation, we do not worry about parameter estimate convergence.

The dynamic system is given by the equation:

$$y(t+1) = -a_0 y(t) + b_0 u(t) \tag{2.1}$$

or we can represent it as

$$y(t+1) = \phi(t)^T \theta_0$$

where $\phi(t)^T = [-y(t), u(t)]$ is our measurement vector and $\theta_0^T = [a_0\, b_0]$ are unknown plant parameters. From the viewpoint of the controller at time t, the system dynamics is of the form

$$y(t+1) = -ay(t) + bu(t) \tag{2.2}$$

where $\hat{\theta}(t) = [a(t), b(t)]^T$ are estimate parameters. We are given the general projection algorithm of the form:

$$\hat{\theta}(t+1) = \hat{\theta}(t) - M(t)\nabla_\theta \bar{e}(t+1, \hat{\theta})$$

where

$M(t) = \frac{1}{\phi(t)^T \phi(t)}$ - algorithm gain, $\phi(t)$- regressor vector (measurement function);

$$\bar{e}(t+1) = \frac{1}{2}\left[y(t+1) - \phi(t)^T \hat{\theta}(t)\right]^2$$

is the squared prediction error

In our case, the error is given by

$$\bar{e}(t+1, \theta) = \frac{1}{2}\left[y(t+1) - (-ay(t) + bu(t)\right]^2 = \frac{1}{2}\left[y(t+1) + ay(t) - bu(t)\right]^2,$$

so the gradient is given by

$$\frac{\partial}{\partial a}(\bar{e}(t+1, \theta)) = \left[y(t+1) + ay(t) - bu(t)\right] y(t)$$

$$\frac{\partial}{\partial b}(\bar{e}(t+1, \theta)) = -\left[y(t+1) + ay(t) - bu(t)\right] u(t)$$

$$\nabla_\theta \bar{e}(t+1, \theta) = -\left[y(t+1) - \phi(t)^T \hat{\theta}\right] \phi(t).$$

The resulting update is

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \frac{\phi(t)}{\|\phi(t)\|^2}[y(t+1) - \phi(t)^T\hat{\theta}(t)] \tag{2.3}$$

The above algorithm is motivated by an optimization problem, where the parameter error criterion J is minimized:

$$J = \frac{1}{2} \parallel \hat{\theta}(t+1) - \hat{\theta}(t) \parallel^2$$

Lemma 1 [7, p.51]. Given $\hat{\theta}(t)$ and $y(t+1)$, we need to determine $\hat{\theta}(t+1)$ so that

$$J = \frac{1}{2}\|\hat{\theta}(t+1) - \hat{\theta}(t)\|^2$$

is minimized subject to

$$y(t+1) = \phi(t)^T\hat{\theta}(t+1), \tag{2.4}$$

where $\hat{\theta}(t)$ is fixed.

Proof: Introducing a Lagrange multiplier for the constraint (2.4), we have

$$J_c = \frac{1}{2} \parallel \hat{\theta}(t+1) - \hat{\theta}(t) \parallel^2 + \lambda[y(t+1) - \phi(t)^T\hat{\theta}(t+1)]$$

Hence the necessary condition for a minimum are

$$\frac{\partial J_c}{\partial\hat{\theta}(t+1)} = 0$$

$$\frac{\partial J_c}{\partial\lambda} = 0$$

The equations become

$$\hat{\theta}(t+1) - \hat{\theta}(t) - \lambda\phi(t) = 0$$

$$y(t+1) - \phi(t)^T\hat{\theta}(t+1) = 0$$

Substituting $\theta(t+1)$ from the first equation into the second equation gives

$$y(t+1) - \phi(t)^T[\hat{\theta}(t) + \lambda\phi(t)] = 0$$

or

$$\lambda = \frac{y(t+1) - \phi(t)^T \hat{\theta}(t)}{\phi(t)^T \phi(t)}$$

Substituting this λ into $\hat{\theta}(t+1) - \hat{\theta}(t) - \lambda\phi(t) = 0$, we get the projection algorithm (2.3).

■

At each iteration, we assume that, by applying the calculated control effort to the system with estimated parameters, the state of the system will become zero. The assumption comes from the certainty equivalence principle, and the state goes to zero if the estimate is correct. Thus, the gradient term in the update equation (2.2) will be simply the output measurement $y(t)$ of true system: $\phi(t)^T \hat{\theta}(t) = 0$ and

$$\nabla_\theta \bar{e} = \phi(t) y(t+1)$$

Hence, our final projection algorithm takes the form

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \frac{1}{\|\phi(t)\|^2} \phi(t) y(t+1)$$

To avoid division by zero, a modified projection algorithm is used:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + \frac{q\phi(t)}{c + \phi(t)^T \phi(t)} [y(t+1) - \phi(t)^T \hat{\theta}(t)] \tag{2.5}$$

with $\hat{\theta}(0)$ given and $c > 0$; $0 < q < 2$.

Some properties of the projection algorithm are summarized in the following:

Lemma 2 [7, p.52-53]. For the algorithm given above and subject to (2.1), it follows that

$$(i) \parallel \hat{\theta}(t) - \theta_0 \parallel \leq \parallel \hat{\theta}(t-1) - \theta_0 \parallel \leq \parallel \hat{\theta}(0) - \theta_0 \parallel ; t > 1 \tag{2.6}$$

$$(ii) \lim_{N\to\infty} \sum_{t=1}^{N} \frac{e(t)^2}{c + \phi(t-1)^T \phi(t-1)} < \infty \tag{2.7}$$

and this implies

$$a) \lim_{t\to\infty} \frac{e(t)}{[c + \phi(t-1)^T \phi(t-1)]^{1/2}} = 0 \tag{2.8}$$

$$b)\lim_{N\to\infty}\sum_{t=1}^{N}\frac{\phi(t-1)^T\phi(t-1)e(t)^2}{[c+\phi(t-1)^T\phi(t-1)]^2}<\infty \tag{2.9}$$

$$c)\lim_{N\to\infty}\sum_{t=1}^{N}\parallel \hat{\theta}(t)-\hat{\theta}(t-1)\parallel^2<\infty \tag{2.10}$$

$$d)\lim_{N\to\infty}\sum_{t=k}^{N}\parallel \hat{\theta}(t)-\hat{\theta}(t-k)\parallel^2<\infty \tag{2.11}$$

$$e)\lim_{t\to\infty}\parallel \theta(\hat{t})-\hat{\theta}(t-k)\parallel=0 \tag{2.12}$$

Intuitively, Lemma 2 shows the boundedness of the parameter error and the output error.

The proof is given in Appendix A.

CHAPTER 3

ADAPTIVE CONTROL TECHNIQUES

3.1 Adaptive Control of Linear Deterministic Systems

Here, we will develop one scheme of adaptive control: one-step-ahead adaptive control. Consequently, we will analyze stability of the system. First, we have to design the adaptive control system. As it is given in [7], the adaptive control design approach is called certainty equivalence adaptive control. The meaning of this approach is to combine the parameter estimation algorithm with a certain control law, matched to the estimated parameters. A block scheme of the adaptive control system consists of the system, parameter estimator, and a control law. There are two classes of algorithms: 1) indirect and 2) direct. These are the ways of implementing the adaptive control. In the direct algorithm the system parameters are given according to the control law parameters. In other words, the system parameters are estimated online, and they are directly used in the control law. That means the adaptive control law is very simple yet it will give desired results. In contrast, in the indirect approach the system parameters are estimated online, and the estimated parameters are used for design calculations. We will be using the direct approach, particularly the one-step-ahead controller design. This adaptive controller is classified as a self-tuning regulator, which has proven to be the optimal controller [7]. Initially, the one-step-ahead controller was introduced for non-adaptive control systems. The control action needs to be calculated at every time instant to bring the future output value to a desired one. From this context, it can be seen that the control law is online and it is very simple. If we have a system model, by writing it as a difference equation, the control law will become clear. In the adaptive case, the same one-step-ahead control law is rewritten in terms of the estimated parameters. The results are shown in the next section.

Before applying adaptive control techniques to the LTI system, we need to give the ***Key Technical Lemma*** which is the basis for all adaptive control algorithms:

Lemma 3 [7, p.181]. Suppose the following conditions are satisfied for some given sequences $\{s(t)\}$, $\{\sigma(t)\}$, $\{b_1(t)\}$, and $\{b_2(t)\}$:

(1)

$$\lim_{t\to\infty} \frac{s(t)^2}{b_1(t)+b_2(t)\sigma(t)^T\sigma(t)} = 0 \tag{3.1}$$

where $\{b_1(t)\}$, $\{b_2(t)\}$ and $\{s(t)\}$ are real scalar sequences and $\{\sigma(t)\}$ is a real $(p \times 1)$ vector sequence.

(2) Uniform boundedness condition

$$0 < b_1 < K < \infty, 0 < b_2 < K < \infty \tag{3.2}$$

for all $t \geq 1$.

(3) Linear boundedness condition

$$\| \sigma(t) \| < C_1 + C_2 \max \mid s(\tau) \mid \tag{3.3}$$

where $0 < C_1 < \infty$ and $0 < C_2 < \infty$. Then it follows that

(i) $\lim_{t\to\infty} s(t) = 0$

(ii) $\{\| \sigma(t) \|\}$ is bounded.

Proof. If $\{s(t)\}$ is a bounded sequence, then by (3.3) $\{\sigma(t)\}$ is a bounded sequence. Then by (3.2) and (3.1) it follows that $\lim_{t\to\infty} s(t) = 0$

Suppose $\{s(t)\}$ is unbounded. Then there exists a subsequence t_n such that

$$\lim_{t_n\to\infty} \mid s(t_n) \mid = \infty$$

and

$$\mid s(t) \mid \leq \mid s(t_n) \mid$$

for $t \leq t_n$. The inequality further can be extended to

$$\left| \frac{s(t_n)}{[b_1(t_n)+b_2(t_n)\sigma(t_n)^T\sigma(t_n)]^{1/2}} \right| \geq \frac{\mid s(t) \mid}{[K + K \| \sigma(t_n) \|^2]^{1/2}}$$

using (3.2), and the right-hand side is further lower-bounded by

$$\geq \frac{| s(t) |}{K^{1/2} + K^{1/2} \parallel \sigma(t_n \parallel}$$

$$\geq \frac{| s(t) |}{K^{1/2} + K^{1/2}[C_1 + C_2 \mid s(t_n) \mid]}$$

using (3.3).

Therefore,

$$\left| \frac{s(t_n)}{[b_1(t_n) + b_2(t_n)\sigma(t_n)^T\sigma(t_n)]^{1/2}} \right| \geq \frac{1}{K^{1/2}C_2} > 0$$

but this contradicts (3.1) and the assumption that $\{s(t)\}$ is unbounded is not true and the result follows.

The Key Technical Lemma shows that, under some assumptions, the error between the output and predicted output goes to zero. $\phi(t)$ (input and output values) stay bounded. For our case, $s(t)$ corresponds to $e(t)$, $\sigma(t)$ to $\phi(t)$, and $b_1 = 1$, $b_2 = 1$. The lemma will be used in the prove of Global Convergence Theorem which will be stated later. It is an important tool in the adaptive control theory, and helps to conclude asymptotic results for a control system.

3.1.1 One-Step-Ahead Control (The SISO Case)

In order to perform an adaptive control of a SISO system, we will combine a parameter estimation algorithm and direct adaptive control. In other words, the control law parameters are directly estimated from system observation, assuming a dynamical model of the form

$$y(t+1) = -ay(t) + bu(t)$$

The desired output sequence is denoted by $y^*(t)$. The objective is to design an adaptive control law to achieve closed-loop stability and asymptotically achieve zero tracking error:

$$\lim_{t\to\infty}[y(t) - y^*(t)] = 0$$

The true model of the system in regression form is $y(t+1) = \phi(t)^T\theta_0$, where $\phi(t)^T = [-y(t), u(t)]$ and $\theta_0^T = [a_0, b_0]$ is the vector of true system coefficients.

The tracking error is $e(t+1) := y(t+1) - y^*(t+1) = \phi(t)^T\theta_0 - y^*(t+1)$.

If we could choose $u(t)$ to satisfy

$$\phi(t)^T\theta_0 = y^*(t+1), \tag{3.4}$$

that would mean the tracking error is zero, but as true parameters θ_0 are unknown we replace (3.4) by the parameters estimated using the projection algorithm on the basis of online system measurements. The certainty equivalence principle discussed above enters the picture here. We will use the estimated parameters as if they were the true parameters: accordingly, the control $u(t)$ at time t will be chosen so that $\phi(t)^T\theta(\hat{t}) = y^*(t+1)$, where $\theta(\hat{t})$ is the estimate of θ_0 at time instant t. Hence, the control law $u(t)$ that would bring the systems' output to the desired value is given by

$$\phi(t)^T\theta(\hat{t}) = y^*(t+1) \tag{3.5}$$

$$-y(t) \times \hat{a} + u(t) \times \hat{b} = y^*(t+1) \Rightarrow u(t) = \frac{y^*(t+1) + y(t) \times \hat{a}}{\hat{b}}, \tag{3.6}$$

assuming $\hat{b} \neq 0$.

Next, we will show that this algorithm has nice convergence properties, and since we are carrying out the procedure for an LTI system, this algorithm is globally convergent. Before moving any further, we need to make the following assumptions:

(i) All modes of inverse of the model (z-transform) lie inside the closed unit disk.

(ii) All controllable modes of the inverse of the model lie strictly inside the unit disk.

(iii) Any modes of the inverse of the model on the unit circle have a Jordan block size of 1.

$(i)-(iii)$ are necessary to achieve perfect tracking and closed-loop stability for the one-step-ahead controller.

Global convergence result. Theorem 1 [7]:

Subject to assumptions given above, the one-step-ahead adaptive control algorithm (3.6) applied to the system $y(t+1) = -ay(t) + bu(t)$ has the following properties:

$$1.\ y(t), u(t) \text{ are bounded sequences} \tag{3.7}$$

$$2.\ \lim_{t\to\infty}[y(t) - y^*(t)] = 0 \tag{3.8}$$

$$3.\ \lim_{t\to\infty}\sum[y(t) - y^*(t)]^2 < \infty \tag{3.9}$$

Proof: Property 1. From Lemma 2 (2.8), (2.12):

$$\lim_{t\to\infty}\frac{e(t+1)}{[c+\phi(t)^T\phi(t)]^{1/2}} = 0 \tag{3.10}$$

where

$$e(t+1) = y(t+1) - \phi(t)^T\theta\hat{(t)} \tag{3.11}$$

$$\lim_{t\to\infty} \| \hat{\theta}(t+1) - \hat{\theta}(t) \| = 0 \tag{3.12}$$

Now, if we define the tracking error $\epsilon(t)$ as

$$\epsilon(t) = y(t+1) - y^*(t+1) \tag{3.13}$$

then from (2.2) and (3.4)

$$\epsilon(t+1) = y(t+1) = \phi(t)^T\theta_0 - \phi(t)^T\theta\hat{(t)} = -\phi(t)\tilde{\theta}(t) \tag{3.14}$$

where

$$\tilde{\theta}(t+1) = \hat{\theta}(t+1) - \theta_0 \tag{3.15}$$

If we take the limit as $t \to \infty$, the right-hand side becomes zero from (3.10) and (3.12). Therefore,

$$\lim_{t\to\infty}\frac{\epsilon(t)^2}{c+\phi(t)^T\phi(t)} = 0$$

Property 2: Here we present the use of the Key Technical Lemma. Let us define $s(t) = \epsilon(t)$, $\sigma(t) = \phi(t-d)$, $b_1(t) = c$, $b_2(t) = 1$, where d is a delay parameter.

$u(k-d) \leq m_3 + m_4 \max \mid y(\tau) \mid$

$\| \phi(t-d) \| \leq pm_3 + [\max(1, m_4)] \max \mid y(\tau) \mid$, where p is the dimension of ϕ. It can be seen that $\phi(t-d)$ corresponds to the condition (3.3) in the Key Technical Lemma. But $\mid \epsilon(t) \mid \geq \mid y(t) \mid - \mid y^*(t) \mid \geq \mid y(t) \mid - m_1$; $m_1 < \infty$

Thus,

$\| \phi(t-d) \| \leq pm_3 + [\max(1, m_4)] \max(\mid \epsilon(\tau) + m_1) = C_1 + C_2 \max \mid \epsilon(\tau) \mid$, where $0 \leq C_1 < \infty$, $0 < C_2 < \infty$ and linear boundedness condition holds. Now equations (3.7) and (3.8) follow immediately from Lemma 3. Last, from Lemma 2, part (ii),

$$\lim_{t \to \infty} \sum \frac{e(t)^2}{c + \phi(t-d)^T \phi(t-d)} < \infty$$

$$\lim_{t \to \infty} \sum \parallel \hat{\theta}(t) - \hat{\theta}(t-k) \parallel^2 < \infty$$

Thus, using the first equation above, the Cauchy-Schwarz inequality, and the boundedness of $\phi(t)$, we conclude (3.9).

■

Finally, the results from the Theorem 1 are given below:

1) Closed-loop stability is achieved.

2) The output tracking error asymptotically goes to zero (tracking goal is achieved).

3) The convergence rate is better than $1/t$.

We have implemented the algorithm described earlier in this section on a linear system with parameters $[a(t), b(t)]$. The evolution of the state, output prediction error, and the control signal is shown in Figure 3.1.

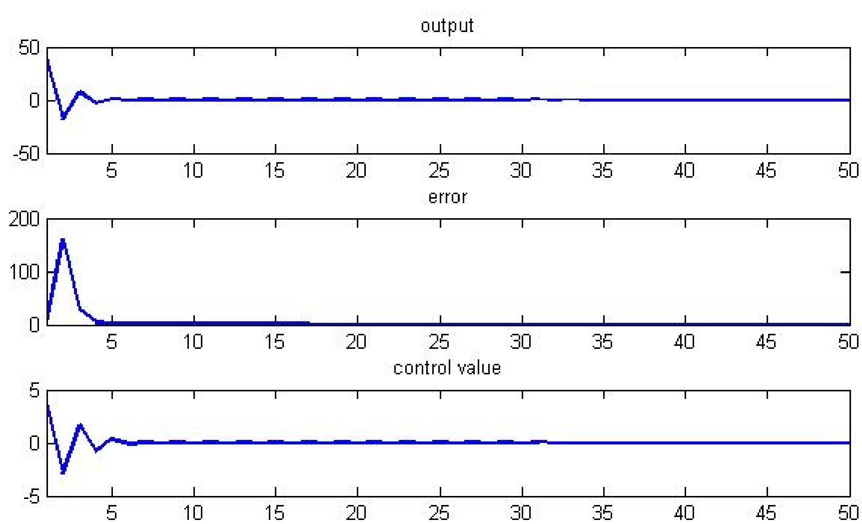

Figure 3.1: Evolution of output state, error, and input values

As we can see, the algorithm does indeed achieve the desired goal of output regulation, while keeping the control signal uniformly bounded. The Matlab code is given in Appendix B.

3.2 Adaptive Control of Stochastic Systems

3.2.1 Stochastic Model and Kalman Filter

This section is devoted to a stochastic version of the model (2.1). Stochastic nature is modeled by noise, specifically with white noise sequences. A stochastic linear system in state-space form is given by [7]:

$$x(t+1) = Ax(t) + Bu(t) + \vartheta_1(t)$$

$$y(t) = Cx(t) + \vartheta_2(t),$$

where $x(t_0)$ has mean $\bar{x_0}$ and covariance Σ_0, and the additive stochastic disturbances $\vartheta_1(t)$ and $\vartheta_2(t)$ have zero mean and

$$E\begin{bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{bmatrix}\begin{bmatrix} \vartheta_1(t) \\ \vartheta_2(t) \end{bmatrix}^T = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta(t-\tau) \tag{3.16}$$

When $\vartheta_2(t) = 0$, then $R = 0$ and $S = 0$.

Since random noise is added to the system, the true state values are unobservable. Then the Kalman filter gives the best linear estimate of the state. The Kalman filter which recursively calculates the state estimates is given below [7]:

$$\hat{x}(t+1) = A\hat{x}(t) + K(t)[y(t) - C\hat{x}(t)] + Bu(t)$$

$$\hat{x}(t_0) = \bar{x}_0$$

$$K(t) = [A\Sigma(t)C^T + S][C\Sigma(t)C^T + R]^{-1}$$

where $K(t)$ is the filter gain.

$\Sigma(t)$ is the state error covariance, and it satisfies the following Riccati difference equation:

$$\Sigma(t+1) = A\Sigma(t)A^T + Q - K(t)[C\Sigma(t)C^T + R]K(t)^T$$

with the initial condition $\Sigma(t_0) = \Sigma_0$

If to define $\omega(t) = y(t) - C\hat{x}(t)$, the Kalman filter can be modified to this form:

$$\hat{x}(t+1) = A\hat{x}(t) + K(t)\omega(t) + Bu(t) \tag{3.17}$$

$$y(t) = \omega(t) + C\hat{x}(t)$$

Equation (3.17) is the model of the system interconnected with Kalman filter in state-space form. One comment on further derivation of algorithms for stochastic case is that, when we describe our first order system using an ARMA model, the model is already in the form of Kalman filter (3.17).

Some comments should be made about the stability of Kalman filter. Before doing this, assume that $S = 0$. Then the Kalman filter can be written as

$$\hat{x}(t+1) = A\hat{x}(t) + K(t)[y(t) - C\hat{x}(t)] + Bu(t)$$

$$\hat{x}(t_0) = \bar{x}_0$$

or, equivalently,

$$\hat{x}(t+1) = \bar{A}(t)\hat{x}(t) + K(t)y(t) + Bu(t)$$

where $\bar{A}(t)$ and $K(t)$ are the filter transition and filter gain matrices given by

$\bar{A}(t) = A - K(t)C$

$K(t) = A\Sigma(t)C^T(C\Sigma(t)C^T + R)^{-1}$

$$\Sigma(t+1) = A\Sigma(t)A^T - A\Sigma(t)C^T(C\Sigma(t)C^T + R)^{-1}C\Sigma(t)A^T + Q \tag{3.18}$$

As $t \to \infty$, the error covariance $\Sigma(t)$ and the Kalman gain $K(t)$ converge to steady-state values. If $\Sigma(t)$ converges, the limiting solution Σ will satisfy the algebraic Riccati equation (ARE), obtained from (3.18) by putting $\Sigma(t+1) = \Sigma(t) = \Sigma$:

$$\Sigma - A\Sigma A^T + A\Sigma C^T(C\Sigma C^T + R)^{-1}C\Sigma A^T - Q = 0 \tag{3.19}$$

For simplicity, we take error variance $R = 0$ and $Q = 1$. In particular, for our first-order scalar case,

$$x(t+1) = -ax(t) + bu(t) + \vartheta_1(t)$$

$$y(t) = x(t)$$

where $A = a$, $B = b$, and $C = 1$; $a, b \in R$

After solving the equation (3.19) for the scalar case, we obtain the value for Kalman filter gain:

$$\Sigma - a^2\Sigma + \frac{\Sigma^2 a^2}{\Sigma} - 1 = 0$$

$$\Sigma = 1$$

and

$$K = A\Sigma C^T (C\Sigma C^T + R)^{-1} = -a$$

Now let us relate the state-space and an ARMA model. The general expression for an ARMA model is

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(t, q^{-1})\omega(t); \tag{3.20}$$

where $\omega(t)$ is a white noise sequence and $C(q^{-1})$ is a filter of the form

$C(q^{-1}) = 1 + c_1 q^{-1} + ... + c_n q^{-n}$

$A(q^{-1}) = 1 + aq^{-1}$ (for our particular first order system)

$B(q^{-1}) = bq^{-1}$

We should keep in mind that it is assumed all roots of the polynomial $C(q^{-1})$ are inside of the unit circle. For the further analysis the equation is simplified to

$$y(t+1) = -ay(t) + bu(t) + \omega(t+1) + (k+a)\omega(t) \tag{3.21}$$

Also, from the derived result above: $k = -a$; c_n's in (3.20) correspond to $k+a$. Hence, the polynomial $C(q^{-1}) = 1$. The other terms in the polynomial $C(q^{-1})$ vanish as $c_1 = k_1 + a_1 = 0$, and the system order is 1.

$$y(t+1) = -ay(t) + bu(t) + \omega(t+1)$$

The equation is the final result for the stochastic model with Kalman filter.

3.2.2 Parameter Estimation

Parameter estimation for stochastic dynamic system has slight modifications. In particular, the parameter vector $\hat{\theta}(t)$ will include parameters of $C(q^{-1})$, which is $\hat{\theta}(t) = [\hat{a}(t)\ \hat{b}(t)\ \hat{c}(t)]^T$. We will use the algorithm based on pseudo-linear regressions:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t-1)\phi(t-1)[y(t) - \hat{y}(t)]$$

$$P(t-1) = P(t-2) - \frac{P(t-2)\phi(t-1)\psi(t-1)^T P(t-2)}{1 + \phi(t-1)^T P(t-2)\phi(t-1)}$$

$$\hat{y}(t) = \phi(t-1)^T \hat{\theta}(t-1)$$

$$\phi(t-1)^T = [y(t-1), u(t-1), -\hat{y}(t-1)]$$

where $P(0) = I$, and for the first-order system SISO $P(0) = 1$.

In general, the algorithm given above is used for systems with noise. $P(t)$ is a positive-definite matrix. The algorithm is derived based on the best fit criteria or "quality" of the approximation of system output to the true system output. The iteration process follows the gradient descent method. The $\phi(t)$ vector contains the observed output measurements, input values, and predicted output values. $P(t)$ can be viewed as some gain matrix that will change the steepness of the parameter estimate. The derivation of the algorithm is given in [7]. However, for the first-order stable system with noise for regulation problem (in particular, for desired zero output case), this algorithm reduces to the simple projection algorithm as we will see later.

3.2.3 Control of Stochastic Systems

Now we consider a first-order stochastic linear system of the form

$$y(t+1) = -ay(t) + bu(t) + \omega(t+1). \tag{3.22}$$

The noise sequence, $\omega(t)$, will be taken to satisfy our usual assumptions:

$$E[\omega(t)] = 0$$

$$E[\omega^2(t)] = \sigma^2 = 1$$

The controller that we will develop is called the minimum variance controller. It is the counterpart of the one-step-ahead controller discussed in previous sections. The idea in the deterministic case was to determine the input $u(t)$ to bring the future output $y(t+1)$ value to a desired value $y^*(t+1)$. In the stochastic case, the output cannot be predicted exactly. However, we can make it close to the desired output value in mean-square sense by minimizing cost function

$$J(t+1) = E[y(t+1) - y^*(t+1)]^2 \tag{3.23}$$

At each time t the minimization above is subject to the constraint that (3.22) be satisfied:

$$\hat{y}(t+1) = -ay(t) + bu(t) + \omega(t+1) \tag{3.24}$$

Theorem 2 [7]. Consider the control law

$$\beta_0 u(t) = q[\beta_0 - \beta(q^{-1})]u(t-1) + y^*(t+1) + [C(q^{-1}) - 1]\hat{y}(t+1) - \alpha(q^{-1})y(t) \tag{3.25}$$

(a) The effect of the control law (3.25) is to give

$$\hat{y}(t+1) = y^*(t+1) \tag{3.26}$$

(b) If (3.25) holds for all t, the control law can be written

$$\beta(q^{-1})u(t) + \alpha(q^{-1})y(t) = C(q^{-1})y^*(t+1) \tag{3.27}$$

Hence, to convert (3.25) for our case,

$$bu(t) = -(c-a)y(t) - cz^{-1}\hat{y}(t+1) + y^*(t+1) \tag{3.28}$$

The proof will be given for a general case. Hence, equations given so far will be rewritten for the general case.

Single-input single-output case will be considered. The ARMA model for our given system is

$$A(q^{-1})y(t) = B(q^{-1})u(t) + C(q^{-1})\omega(t) \tag{3.29}$$

where $u(t)$ and $y(t)$ denote the input and output, respectively, and $\omega(t)$ is a "white noise" sequence satisfying

$$E[\omega(t)] = 0 \tag{3.30}$$

$$E[\omega(t)^2] = \sigma^2 \tag{3.31}$$

We have a cost function:

$$J(t+d) = E[[y(t+d) - y^*(t+d)]^2]$$

We want to get $u(t)$ as a function of past inputs and outputs to minimize the above function. Conditional mean gives the minimum variance estimator. Therefore, we can write the above expression using the smoothing property. The optimal cost is given by

$$J^*(t+d) = E\left[\min_{u(t)} E[[y(t+d) - y^*(t+d)]^2 | \mathcal{F}_t]\right] \tag{3.32}$$

where $u(t)$ is constrained to be $\mathcal{F}_t$ measurable. $\mathcal{F}_t$ the σ-algebra generated by past input and output values up to time t.

The minimization above is subject to constraint that (3.29) be satisfied. The optimal d-step-ahead prediction of $y(t)$ for the system (3.29) satisfies

$$C(q^{-1})y^0(t+d|t) = \alpha(q^{-1})y(t) + \beta(q^{-1})u(t) \tag{3.33}$$

where

$$y^0(t+d|t) = E[y(t+d)] = y(t+d) - F(q^{-1})\omega(t+d) \tag{3.34}$$

$$\alpha(q^{-1}) = G(q^{-1}) \tag{3.35}$$

$$\beta(q^{-1}) = F(q^{-1})B^{'}(q^{-1}) \tag{3.36}$$

and $G(q^{-1})$ and $F(q^{-1})$ are the unique polynomials satisfying

$$C(q^{-1}) = F(q^{-1})A(q^{-1}) + q^{-d}G(q^{-1}) \tag{3.37}$$

$$F(q^{-1}) = f_0 + f_1 q^{-1} + ... + f_{d-1}q^{-(d-1)}; f_0 = 1 \tag{3.38}$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + ... + g_{n-1} q^{-(n-1)} \tag{3.39}$$

Proof [7, p.412]. Substituting (3.34) into (3.32) gives

$$J^*(t+d) = E\left[\min_{u(t)} E[[y(t+d) - y^0(t+d|t) + y^0(t+d|t) - y^*(t+d)]^2|\mathcal{F}_t]\right] \tag{3.40}$$

$$\begin{aligned} J^*(t+d) = E[\min_{u(t)} E[[y(t+d) - y^0(t+d|t)]^2 + 2[y(t+d) - y^0(t+d|t)] \\ [y^0(t+d|t) - y^*(t+d)] + [y^0(t+d|t) - y^*(t+d)]^2|\mathcal{F}_t]] \\ = E\left[\min_{u(t)} \sum_{j=0}^{d-1} f^2\sigma^2 + [y^0(t+d|t) - y^*(t+d)]^2\right] \end{aligned}$$

using (3.30), (3.31), and (3.32). Substituting for $y^0(t+d)$ from (3.33) gives

$$\begin{aligned} J^*(t+d) = E\Bigg[\sum_{j=0}^{d-1} f^2\sigma^2 + \min_{u(t)}[(1 - C(q^{-1}))y^0(t+d|t) \\ + \beta(q^{-1})u(t) + \alpha(q^{-1})y(t) - y^*(t+d)]^2\Bigg] \end{aligned} \tag{3.41}$$

The input $u(t)$ minimizing (3.41) can be seen to be given by

$$\beta(q^{-1})u(t) = y^*(t+d) + [C(q^{-1}) - 1]y^0(t+d|t) - \alpha(q^{-1})y(t)$$

This establishes Theorem 2.

Remarks. The above proof was derived for general case. We need to simplify it for the first-order system. Again, we have a model given by (3.24) and noise assumptions.

$$J^*(t+1) = E\left[\min_{u(t)} E[[y(t+1) - \hat{y}(t+1) + \hat{y}(t+1) - y^*(t+1)]^2|\mathcal{F}_t]\right]$$

$$\begin{aligned} J^*(t+1) = E[\min_{u(t)} E[[y(t+1) - \hat{y}(t+1)]^2 + 2[y(t+1) - \hat{y}(t+1)] \\ [\hat{y}(t+1) - y^*(t+1)] + [\hat{y}(t+1) - y^*(t+1)]^2|\mathcal{F}_t]] \\ = E\left[\min_{u(t)} \sum_{j=0}^{d-1} \sigma^2 + [\hat{y}(t+1) - y^*(t+1)]^2\right] \end{aligned}$$

$$J^*(t+1) = E\left[\sigma^2 + \min[cz^{-1}\hat{y}(t+1) + bu(t) + (c-a)y(t) - y^*(t+1)]\right]$$

Now it is clear to see the control law minimizing the given cost function

$$bu(t) = -cz^{-1}\hat{y}(t+1) - (c-a)y(t) + y^*(t+1)$$

3.2.4 Adaptive Control of Stochastic Systems. Concept of Certainty Equivalence Control. Adaptive Minimum Variance Control

When it comes to stochastic systems, the control has a dual role. It takes into account both parameter uncertainty and noise effects. The concept of dual control consists of learning and regulation of the input. In other words, the input is given in a way in order to learn the unknown system dynamics and to follow the output, which is regulation. As theory shows, the concept of dual control is difficult in realization. Therefore, by ignoring the uncertainty in parameter perturbations we will use certainty equivalence control law that was applied in deterministic case. For this type of control law, we will use the pseudo-linear regression algorithm and one-step ahead control law. Together they have a name of self-tuning regulator or adaptive minimum variance controller.

We have a system model of the form

$$A(q^{-1})y(t) = q^{-1}B^{'}(q^{-1})u(t) + C(q^{-1})\omega(t)$$

or, in our case,

$$y(t+1) = -ay(t) + bu(t) + \omega(t+1)$$

An adaptive minimum variance control law is obtained by setting the predicted future output, $\hat{y}(t+1)$, of the system to a desired value, $y^*(t+1)$, to generate the control $u(t)$ at time t:

Adaptive Minimum Variance Controller with Least Square Parameter Estimation:

$$\hat{\theta}(t+1) = \hat{\theta}(t) + qP(t)\phi(t)e(t+1)$$

$$e(t+1) = y(t+1) - \hat{y}(t+1)$$

$$P(t) = P(t-1) - \frac{P(t-1)\phi(t-1)\phi(t-1)^T P(t-1)}{1+\phi(t)^T P(t-1)\phi(t)}$$

$$\hat{y}(t+1) = \phi(t)^T \hat{\theta}(t)$$

$$\phi(t)^T = [y(t), u(t), -\hat{y}(t)]$$

$$\hat{\theta}(t) = [\hat{c}(t) - \hat{a}(t), \hat{b}(t), \hat{c}(t)]$$

Again, the idea of one-step-ahead control (or minimum variance control) is applied where predicted output is equal to the desired output value. Finally the input is generated from the following feedback control law.

$$\phi(t)\hat{\theta}(t) = y^*(t+1) \tag{3.42}$$

From $\hat{y}(t+1) = y^*(t+1)$ we get

$$u(t) = \frac{-(\hat{c}-\hat{a})y(t) - \hat{c}z^{-1}y^*(t+1) + y^*(t+1)}{\hat{b}} = \frac{(1-\hat{c}z^{-1})y^*(t+1) - (\hat{c}-\hat{a})y(t)}{\hat{b}} \tag{3.43}$$

The assumptions for convergence of the above algorithm:

1. An upper bound is known for the orders of the polynomials in the system description.

2. $[1/C(z^{-1}) - 1/2]$ is positive real.

3. $B(z^{-1})$ is asymptotically stable.

So far, the algorithms given above for stochastic case were generalized versions. In our case, we used the zero output as the desired output value in deterministic case, and we again want the desired output to be zero in the stochastic case. The interesting point here is that the parameters in $C(q^{-1})$ may be removed from the problem and ordinary projection algorithm can be used [7]. That means, that if $y^*(t+1) = 0$, then (3.43) gives

$$\phi(t)^T \hat{\theta}(t+1) = 0 \tag{3.44}$$

Then, the desired zero output value, $y^*(t+1) = 0$ and $C(q^{-1}) = 1$ $(c = 0)$:

$$u(t) = -\frac{\hat{a}y(t)}{\hat{b}} \tag{3.45}$$

Equation (3.46) gives the control law to apply to the first order system with white noise sequences. Also, we can eliminate $\hat{c}$ value from $\hat{\theta}(t)$ and $\hat{y}(t)$ from $\phi(t)$.

3.3 Analysis of Quantization Effects on the First-Order System

It is shown [8] that the quantization error on a system can be replaced by an additive noise disturbance $\omega(t)$. One way to realize this assumption is dithering [9]. The dithering method is realized in the following steps for our first-order SISO system. Initially, we have a first-order system with intentionally added white noise sequences, then the corrupted with noise signal is passed through an quantizer. After that, the same white noise sequences are subtracted from the quantized signal, and further followed by Kalman filter and a controller. In other words, the process of adding white noise sequences before quantization and subtracting them after quantized signal is known as dithering. When the process of dithering is implemented in the system, one can separate quantization noise from system's output and write the resulting model just as the model is given below, where $\omega(t)$ is the dithered quantization noise.

$$y(t+1) = -ay(t) + bu(t) + \omega(t+1)$$

A particular way of implementing dither is given in [9], where properties of the dither signal $z(t)$ are given as following:

1) For any t, the dither signal $z(t)$ is supported on the range $(-\sqrt{3\sigma^2},\sqrt{3\sigma^2})$. This means, $E[z(t)] = 0$ and $Var[z(t)] = \sigma^2$.

Randomized quantization (Dithering).

An input vector $Y = y_1, y_2, ..., y_i, ..., y_n$ and a one-dimensional uniform quantizer with code points $C_1 = [0, \pm 2\sqrt{3\sigma^2}, \pm i2\sqrt{3\sigma^2}, ...]$, $i = 0, 1, 2, 3, ...$ Let z be a random variable that is distributed uniformly in the interval $[-\sqrt{3\sigma^2}, \sqrt{3\sigma^2}]$ and is statistically independent of Y. Let $Z = z_1, z_2, z_3, ..., z_n$, where for every $i = 1, 2, ..., n, z_i = z$. Consider now the one-dimensional randomized quantization process where the vector $Y + Z$ is quantized and z is subtracted from quantized value at each step. Here we can use $n = 1$. It is shown that for any Y

$$\frac{1}{n} E \parallel Q_1(Y + Z) - Z - Y \parallel^2 = \sigma^2$$

$$\frac{1}{n} E \parallel Q_1(Y + Z) - Z - Y \parallel = 0$$

or, for y at time t we have

$$E \parallel Q_1(Y + Z) - Z - Y \parallel^2 = \sigma^2$$

$$E\left[Q_1(Y + Z) - Z - Y\right] = 0$$

where E denotes expectation over Z. Thus, the quantization mean-square distortion of a randomized input vector Y is equal to σ^2 for any value of Y. The proof is given in the Appendix C.

2) The dither signal $z(t)$ is a uniform random variable.

3) The dither signal is not correlated with the signal sequence $u(t)$.

The relation between quantization width Δ and variance of $z(t)$ is $\Delta = 2\sqrt{3\sigma^2}$.

Now we want to apply some control law to lead the system's output to zero. If we apply the control law which is $u(t) = -\frac{a}{b}y(t)$, the state will not converge to zero. It will take some value in the range $(-\sqrt{3\sigma^2}, \sqrt{3\sigma^2})$. When we apply the control law, the minimum mean-square error between the output and desired value will be equal to the variance term. Then we minimize the mean-square error of both sides with respect to time from $t = 1$ to $t = n$:

$$y(t + 1) = -ay(t) + bu(t) + \omega(t + 1)$$

After the control applied,

$$y(t + 1) = \omega(t + 1)$$

Next convergence analysis is done for the control algorithm in stochastic setting. This convergence analysis is summarized in *Stochastic Key Technical*

Lemma.

Conditions:

$$\lim_{N\to\infty} \sum_{t=1}^{N} \frac{[e(t)-\omega(t)]^2}{r(t-1)} < \infty \tag{3.46}$$

$$1) E[\omega(t)|\mathcal{F}_{t-1}] = 0 \text{ a.s.} \tag{3.47}$$

$$2) E[\omega(t)^2|\mathcal{F}_{t-1}] = 0 \text{ a.s.} \tag{3.48}$$

$$3) \lim_{N\to\infty} \sup \frac{1}{N} \sum_{t=1}^{N} \omega(t)^2 < \infty \text{ a.s.} \tag{3.49}$$

$$4) e(t) = y(t) - \hat{y}(t) \tag{3.50}$$

5) $r(t-1)$ is a nondecreasing nonnegative sequence such that $r(t-1)$ is $\mathcal{F}_{t-1}$ measurable.

Lemma 4 (The Stochastic Key Technical Lemma) [7]:

If condition (3.47) together with properties $1-5$ hold, and if there exist constants K_1, K_2, and N such that

$$\frac{1}{N} r(N-1) \leq K_1 + \frac{K_2}{N} \sum_{t=1}^{N} [e(t) - \omega(t)]^2 \tag{3.51}$$

then

1)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} [e(t) - \omega(t)]^2 = 0 \quad \text{a.s.} \tag{3.52}$$

2)

$$\lim_{N\to\infty} \sup \frac{1}{N} r(N-1) < \infty \text{ a.s.} \tag{3.53}$$

3)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[[y(t) - \hat{y}(t)]^2|\mathcal{F}_{t-1}] = \sigma^2 \text{ a.s.} \tag{3.54}$$

So we see that output mean-square error will be in the range of quantization width, and the output is mean-square bounded:

$\frac{1}{N}\sum_{t=1}^{N} y(t)^2 < \infty$

Part 3 of the Stochastic Key Technical Lemma is very crucial point, where it establishes that, subject to the noise, system, and signal assumptions, adaptive one-step-ahead control prediction, $\hat{y}(t)$ converges to the true output in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[(y(t) - \hat{y}(t))^2 | \mathcal{F}_{t-1}] = \sigma^2 \tag{3.55}$$

where $\hat{y}(t) = 0$ and it simplifies to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E[(y(t))^2 | \mathcal{F}_{t-1}] = \sigma^2 \tag{3.56}$$

Equation (3.56) defines the best fit of the parameter approximation measure. In other words, this is the best we can do with the given control law to track the system's true output.

The proof is given in Appendix D.

From Figure 3.2 we see that output does not asymptotically go to zero, and it is a consequence of quantization errors. However, they stay bounded by some value. Finally, we see the state error (output) is decreasing from each iteration. The state converges to zero.

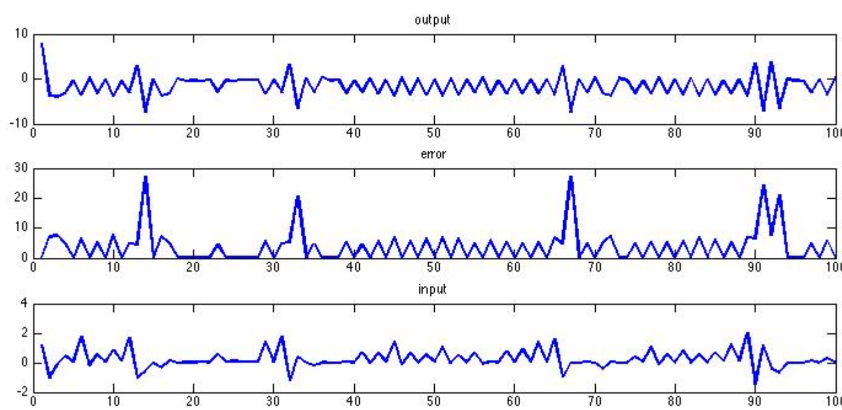

Figure 3.2: Evolution of output state, error, and input values

3.4 Conclusion

We studied the problem of stabilizing the first-order single-input single-output system with unknown parameters and the communication constraint. By applying a certain dithering method on a system with quantization, we substituted the quantization error with white noise disturbances. Then, applying the one-step-ahead control law in a recursive manner, we were able to achieve the asymptotic results. The method that was used to decorrelate the quantization error from the input value to the quantizer was to add uniform random variables before quantization, and subtract the same random variables after quantization.

APPENDIX A

PROJECTION ALGORITHM

A.1 Lemma 2

The following notations are introduced:

$$\tilde{\theta}(t+1) = \hat{\theta}(t) - \theta_0 \tag{A.1}$$

$$e(t+1) = y(t+1) - \phi(t)^T\hat{\theta}(t) = -\phi(t)^T\tilde{\theta}(t) \tag{A.2}$$

Proof [7]. (i) Subtracting θ_0 from both sides of (2.5) and using (2.1) and $(A.2)$, we obtain

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \frac{q\phi(t)}{c+\phi(t)^T\phi(t)}\tilde{\theta}(t)$$

Hence, using $(A.2)$,

$$\| \tilde{\theta}(t+1) \|^2 - \| \tilde{\theta}(t) \|^2 = q[-2 + \frac{q\phi(t)^T\phi(t)}{c+\phi(t)^T\phi(t)}]\frac{e(t+1)^2}{c+\phi(t)^T\phi(t)} \tag{A.3}$$

Now since $0 < q < 2$,$c > 0$, we have

$$q[-2 + \frac{q\phi(t)^T\phi(t)}{c+\phi(t)^T\phi(t)}] < 0 \tag{A.4}$$

and then(2.6) follows from $(A.3)$.

(ii) We observe that $\| \tilde{\theta}(t+1) \|^2$ is a bounded nonincreasing function, and by summing $(A.3)$, we have

$$\| \tilde{\theta}(t+1) \|^2 = \| \tilde{\theta}(0) \|^2 + \sum_{j-1}^{t} q[-2 + \frac{q\phi(j)^T\phi(j)}{c+\phi(j)^T\phi(j)}]\frac{e(j+1)^2}{c+\phi(j)^T\phi(j)} \tag{A.5}$$

Since $\| \tilde{\theta}(t+1) \|^2$ is nonnegative, and since $(A.4)$ holds, we can conclude (2.7).

(a) Equation (2.8) follows immediately from (2.7).

(b) Noting that

$$\frac{e(t+1)^2}{c+\phi(t)^T\phi(t)} = \frac{[c+\phi(t)^T\phi(t)]e(t+1)^2}{[c+\phi(t)^T\phi(t)]^2}$$

establishes (2.9) using (2.7).

(c) Equation (2.9) immediately implies (2.10) by noting the form of the algorithm (2.5).

(d) It is clear that

$$\| \hat{\theta}(t+1)-\hat{\theta}(t-k-1) \|^2 = \| \hat{\theta}(t+1)-\hat{\theta}(t)+\hat{\theta}(t)-\hat{\theta}(t-1)+...\hat{\theta}(t-k-1) \|^2$$

Then using the Cauchy-Schwarz inequality

$$\| \hat{\theta}(t+1)-\hat{\theta}(t-k-1) \|^2 \leq k(\| \hat{\theta}(t+1)-\hat{\theta}(t) \|^2 + ... + \| \hat{\theta}(t-k)-\hat{\theta}(t-k-1) \|^2$$

Then the result follows immediately from (2.10) since k is finite.

(e) Equation (2.12) here follows from (2.11).

APPENDIX B

IMPLEMENTATION PROCEDURES

B.1 Deterministic System

The procedure of recursive parameter estimation flows this way: every time we have the values of $\hat{\theta}(t)$ and $y(t)$ ($\hat{\theta}(0)$ and $y(0)$ are given in the beginning), and let us say $\hat{\theta}(0)^T = (\hat{a}(0), \hat{b}(0)) = (0.7, 7)$ and $y(0) = 40$. Also, we need to specify true parameters $\theta_0^T = (a_0, b_0) = (0.5, 0.5)$ so as to be able to calculate the output measurement (in reality we don't calculate this value, we take $y(t)$ it from sensor's display). Parameter estimation begins at $t + 1$.

Algorithm:

```
y(1)=40; % y(1)  can be any arbitrary number;
a_hat(1)=0.7; %can be any arbitrary number less than 1
b_hat(1)=7; %can be any arbitrary number
theta=[a_hat b_hat]';
a=0.5; b=0.5;  % true parametres of the system


 for t=1:50
  u(t)=(theta(1)/theta(2))*y(t) % calculate control effort
  y(t+1)=y(t)*a+u(t)*b  % output measurement
  e(t+1)=(1/2)*y(t+1)^2;
  y_sqr(t)=y(t+1)^2;  % square of output
  theta=theta+(y(t+1)/(y(t)^2+u(t)^2))*[ y(t) u(t)]';
  w1(t,:)=theta(1,:)
  w2(t,:)=theta(2,:)
 end

 subplot(3,1,1);
```

```
plot(y,'LineWidth',2)
title('output')
subplot(3,1,2)
plot(e,'LineWidth',2)
title('error')
subplot(3,1,3);
plot(u,'LineWidth',2)
title('control value)
```

B.2 Stochastic System

Since we put our desired output value to zero, last terms from regression vector and parameter vector will be removed. Final control looks like

$$u(t) = -\frac{\hat{a}y(t)}{\hat{b}} \tag{B.1}$$

Algorithm:

```
y(1)=8; %can be any arbitrary number
y_hat=8;
a_hat(1)=0.6; %can be any arbitrary number less than 1
b_hat(1)=4; %can be any arbitrary number
theta=[a_hat b_hat]';
a=0.5; b=2;



for t=1:100
 u(t)=(theta(1)/theta(2))*y_hat(t);
 %noise with variance 1 and mean 0
 z=rand(1,length(t)) 0.5;
 y(t+1)=y(t)*a+u(t)*b;
 %noise added to the output
 w(t)=y(t+1)+z;
 %dithering
 y_hat(t+1)=(floor(w(t)/(2*sqrt(3))))*((2*sqrt(3))) z;
 e(t+1)=(1/2)*y_hat(t+1)^2;
```

```
 y_sqr(t)=y(t+1)^2;
 theta=theta+(y_hat(t+1)/(1+(y_hat(t)^2
+u(t)^2)))*[ y_hat(t) u(t)]';

 w1(t,:)=theta(1,:);
 w2(t,:)=theta(2,:);

end

subplot(3,1,1);
plot(y_hat,'LineWidth',2)
title('output')
subplot(3,1,2);
plot(e,'LineWidth',2)
title('error')
subplot(3,1,3);
plot(u,'LineWidth',2)
title('input')
```

APPENDIX C

DITHERING

Proof [9]:

Let $Q(Y = Z) = q_1, q_2, ..., q_n$, where $q_i = q_i(y_i + z_i) \in C_1$. Then

$$E(q_i - y_i - z_i) = \frac{1}{2\sqrt{3\epsilon}} \int_{x_{min}}^{x_{max}} (q(x) - x)dx,$$

$$x = y_i + z_i, x_{min} = y_i - \sqrt{3\epsilon}, x_{max} = y_i + \sqrt{3\epsilon},$$

$$= \frac{1}{2\sqrt{3\epsilon}} \int_{x_{min}}^{q(x_{min})+\sqrt{3\epsilon}} (q(x_{min}) - x)dx + \int_{x_{min}}^{q(x_{min})+\sqrt{3\epsilon}} (q(x_{min}) + 2\sqrt{3\epsilon} - x)dx$$

$$= \frac{1}{2\sqrt{3\epsilon}} \int_{q(x_{min})-\sqrt{3\epsilon}}^{q(x_{min})+\sqrt{3\epsilon}} (q(x_{min}) - x)dx = 0$$

In a similar way

$$E(q_i - y_i - z_i)^2 = \frac{1}{2\sqrt{3\epsilon}} \int_{q(x_{min})-\sqrt{3\epsilon}}^{q(x_{min})+\sqrt{3\epsilon}} (q(x_{min}) - x)^2 dx = \epsilon$$

APPENDIX D

KEY STOCHASTIC LEMMA

(1) If $r(t-1) < K_3 < \infty$, then (3.47) implies

$$\lim_{N\to\infty} \frac{1}{K_3} \sum_{t-1}^{N} [e(t) - v(t)]^2 < \infty \text{ a.s.} \tag{D.1}$$

and (3.53) follows trivially.

Alternatively, if $r(t-1)$ is unbounded, then since the sum in (3.47) is nondecreasing, we can apply Kronecker's lemma to conclude that

$$\lim_{N\to\infty} \frac{N}{r(N-1)} \frac{1}{N} \sum_{t=1}^{N} [e(t) - \omega(t)]^2 = 0 \text{ a.s.} \tag{D.2}$$

Substituting (3.52) into $(D.2)$ gives

$$\lim_{N\to\infty} \frac{\frac{1}{N}[e(t) - \omega(t)]^2}{K_1 + \frac{K_2}{N}\sum_{t=1}^{N}[e(t) - \omega(t)]^2} = 0 \text{ a.s.} \tag{D.3}$$

Equation (3.53) follows immediately.

(2) Equation (3.54) follows from (3.52) and (3.53).

(3) Note that

$$E[[y(t) - \hat{y}(t)]^2 | \mathcal{F}_{t-1}] = E[[y(t) - \hat{y}(t) - \omega(t) + \omega(t)]^2 | \mathcal{F}_{t-1}] = \tag{D.4}$$

$$E[[y(t) - \hat{y}(t) - \omega(t)]^2 + 2[y(t) - \hat{y}(t) - \omega(t)]\omega(t) + \omega(t)^2 | \mathcal{F}_{t-1}]$$

Since $y(t) - \omega(t)$ and $\hat{y}(t)$ are $\mathcal{F}_{t-1}$ measurable and using (3.58), we have

$$E[[y(t) - \hat{y}(t)]^2 | \mathcal{F}_{t-1}] = [e(t) - \omega(t)]^2 + E[\omega(t)^2 | \mathcal{F}_{t-1}] \tag{D.5}$$

Equation (3.55) now follows from (3.49) ,(3.53) and $(D.5)$.

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