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EXTREMAL PROBLEMS INVOLVING FORBIDDEN SUBGRAPHS

BY

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DISSERTATION

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Abstract

In this thesis, we study extremal problems involving forbidden subgraphs. We are interested in extremal problems over a family of graphs or over a family of hypergraphs. In Chapter 2, we consider improper coloring of graphs without short cycles. We find how sparse an improperly critical graph can be when it has no short cycle. In particular, we find the exact threshold of density of triangle-free $(0, k)$-colorable graphs and we find the asymptotic threshold of density of (j, k)-colorable graphs of large girth when $k \geq 2j + 2$. In Chapter 3, we consider other variations of graph coloring. We determine harmonious chromatic number of trees with large maximum degree and show upper bounds of r-dynamic chromatic number of graphs in terms of other parameters. In Chapter 4, we consider how dense a hypergraph can be when we forbid some subgraphs. In particular, we characterize hypergraphs with the maximum number of edges that contain no r-regular subgraphs. We also establish upper bounds for the number of edges in graphs and hypergraphs with no edge-disjoint equicovering subgraphs.

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Chapter 1

Overview

In this thesis, we are interested in extremal problems on graphs and hypergraphs. In other words, we study the relationship between several parameters over a family of graphs or hypergraphs with forbidden structures. Mainly, we want to determine the maximum or minimum value of a given parameter over a family of graphs or hypergraphs.

In Chapters 2 and 3, we explore variations of graph coloring problems. Part of why we study graph coloring problems is because they describe many real-life situations, and the solutions of the problems help us deal with the situations. For example, assigning frequencies to satellites can be modeled by an improper coloring problem on graphs, and distributing resources to people so that everyone has access to at least r different resources from their friends can be modeled by an r-dynamic coloring problem.

In Chapter 4, we discuss extremal problems for hypergraphs. These problems are important partially because hypergraphs are generalizations of many mathematical objects including graphs, matroids, and even topologies. Hence many problems in other areas can be stated in the language of hypergraphs. For example, many-body system problems in physics, and even chemical reaction networks can be modeled by hypergraphs.

1.1 Improper coloring of sparse graphs with given girth

Chapter 2 is based on [35] and [36] which are joint works with Alexandr V. Kostochka and Xuding Zhu. In Chapter 2, we study improper coloring of graphs with given girth.

A $(d_1, \ldots, d_k)$-*coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for every $1 \leq i \leq k$, the subgraph $G[V_i]$ of G induced by V_i has maximum degree at most d_i. If

$d_1 = \ldots = d_k = 0$, then a $(d_1, \ldots, d_k)$-coloring is simply a proper k-coloring. If at least one d_i is positive, then a $(d_1, \ldots, d_k)$-coloring is called *improper* or *defective*. Several papers on improper colorings of planar graphs with restrictions on girth and of sparse graphs have appeared.

In Chapter 2, we consider improper colorings with just two colors, the (j, k)-colorings. Even with two colors, testing (j, k)-colorability is a computationally difficult problem if $(j, k) \neq (0, 0)$. In particular, Esperet, Montassier, Ochem and Pinlou [24] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0, 1)$-coloring is NP-complete. Since the problem is hard, it is natural to consider related extremal problems.

The maximum average degree, $\mathrm{mad}(G)$, measures sparseness of G. Kurek and Ruciński [40] called graphs with low maximum average degree *globally sparse*. In particular,

$$\text{if } G \text{ is a planar graph of girth } g, \text{ then } \mathrm{mad}(G) < \tfrac{2g}{g-2}. \tag{1.1}$$

We will use the following slight refinement of the notion of $\mathrm{mad}(G)$. For $a, b \in \mathbf{R}$, a graph G is (a, b)-*sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. For example, every forest is $(1, 0)$-sparse, and every graph G with $\mathrm{mad}(G) < a$ is $(a/2, 0)$-sparse.

We also say that G is *almost* (a, b)-*sparse* if $|E(G)| = a|V(G)| + b$ and $|E(H)| < a|V(H)| + b$ for all $H \subsetneq G$. For example, every k-regular connected graph G is almost $(k/2, 0)$-sparse. Almost (a, b)-sparse graphs could be considered critical, that is, they become (a, b)-sparse after deleting any edge. In particular, every almost (a, b)-sparse graph is (a, b')-sparse for all $b' > b$.

For several pairs (j, k), recent research determines values a and g such that graphs with maximum average degree a are (j, k)-colorable and planar graphs with girth g are also (j, k)-colorable. However, it was not known whether planar graphs with girth 10 or 11 are $(0, 1)$-colorable or not. Theorem 2.1.1 is the first to determine the exact density threshold of triangle-free graphs for improper colorability. Theorem 2.1.1 together with (1.1) yields that every planar graph with girth at least 11 is $(0, 1)$-colorable.

In Chapter 2, instead of considering planar graphs with given girth, we consider graphs G with given girth that are (a, b)-sparse for small a. Let $F_{j,k}(g)$ denote the supremum of positive a such that there is some (possibly negative) b with the property that every (a, b)-sparse graph G with

girth g is (j,k)-colorable. The above-mentioned result in Section 2.1 implies $F_{0,1}(3) = \frac{12}{5} = 1.2$.

In Section 2.2, we consider graphs with any girth g. Theorem 2.2.3 gives values for this parameter by showing that $F_{j,k}(g) = 2 - \frac{k+2}{(j+2)(k+1)}$ for $k \geq 2j+2$.

1.2 Variations of coloring problems

In Chapter 3, we consider two variations of proper coloring, which are harmonious coloring and r-dynamic coloring.

Section 3.1 is based on [6] which is a joint work with Saieed Akbari and Alexandr V. Kostochka.

A *harmonious coloring* of G is a proper vertex coloring of G such that every pair of colors appears on at most one pair of adjacent vertices. The *harmonious chromatic number* of G, $h(G)$, is the minimum number of colors needed for a harmonious coloring of G. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard. Moreover, Edwards and McDiarmid [20] showed that the problem remains hard even when restricted to the class of trees. Since the problem is hard in the class of all trees, it makes sense to identify subclasses in which the problem is easier.

It is easy to show $h(G) \geq \Delta(G) + 1$ for every graph G. In [3] it was shown that if T is a tree of order n and $\Delta(T) \geq \frac{n}{2}$, then $h(T) = \Delta(T) + 1$. Moreover, the proof yields a polynomial-time algorithm for an optimal harmonious coloring of such a tree. We strengthen this result by showing that if an n-vertex forest T satisfies $\Delta(T) \geq \frac{n+2}{3}$, then

$$h(T) = \begin{cases} \Delta(T)+2, & \text{if } T \text{ has non-adjacent vertices of degree } \Delta(T); \\ \Delta(T)+1, & \text{otherwise.} \end{cases}$$

Moreover, there is a polynomial-time algorithm for an optimal harmonious coloring of such a forest.

Section 3.2 is based on [32] which is a joint work with Sogol Jahanbekam, Suil O, and Douglas B. West.

An *r-dynamic k-coloring* is a proper k-coloring c of G such that $|c(N(v))| \geq \min\{r, d(v)\}$ for each vertex v in $V(G)$, where $N(v)$ is the neighborhood of v and $c(U) = \{c(v)\colon\ v \in U\}$ for a vertex subset U. The *r-dynamic chromatic number*, introduced by Montgomery [45] and written as $\chi_r(G)$, is the minimum k such that G has an r-dynamic k-coloring.

In Section 3.2, we prove relations between $\chi_r(G)$ and other graph parameters like maximum degree, minimum degree, diameter, and chromatic number.

Regarding maximum degree of graphs, we prove $\chi_r(G) \leq r\Delta(G)+1$ and characterize the graphs achieving equality. When the minimum degree is not too small, we can greatly improve the bound in terms of the maximum degree. For an n-vertex graph G, we show that $\delta(G) \geq \frac{r+s}{s+1}(r+1)\ln n$ implies $\chi_r(G) \leq \Delta(G) + r + s$. In particular, $\delta(G) \geq 2r\ln n$ implies $\chi_r(G) \leq \Delta(G) + 2r$, and $\delta(G) \geq r(r+1)\ln n$ implies $\chi_r(G) \leq \Delta(G) + r$.

For regular graphs, we mention the inequality $\chi_r(G) \leq r\chi(G)$, which holds for k-regular graphs G with $k \geq (3 + o(1))r\ln r$. When k is not sufficiently large in terms of r, the ratio $\chi_r(G)/\chi(G)$ can grow superlinearly in r; we provide an example using Kneser graphs where $k = r$ and $\chi_r(G) > r^{1.37744}\chi(G)$.

For graphs with small diameter, we give a proof of $\chi_2(G) \leq \chi(G) + 2$ for every graph G with diameter 2. Furthermore, equality holds only for complete bipartite graphs and C_5. Moving on to diameter 3, we prove $\chi_2(G) \leq 3\chi(G)$ when $\mathrm{diam}(G) \leq 3$, and this is sharp. For graphs with diameter 4 there is no bound in terms of chromatic number. In contrast, χ_3 is unbounded already on bipartite graphs with diameter 2.

1.3 Hypergraphs with forbidden subgraphs

In Chapter 4, we consider two exteremal problems for hypergraphs. We characterize hypergraphs with no r-regular subgraphs, and we find upper bounds for the number of edges in hypergraphs without equicovering subgraphs.

Section 4.1 is based on [34] which is a joint work with Alexandr V. Kostochka.

In Section 4.1, we consider the question of how many edges a hypergraph H has when H

contains no r-regular subgraph. Some answers to this quesition are known for graphs, while only a little is known for hypergrphs. A hypergraph H is *r-free* if it has no r-regular sub(hyper)graphs. Mubayi and Verstraëte [46] proved that for every even integer $k \geq 4$, there exists n_k such that for each $n \geq n_k$, each n-vertex k-uniform 2-free hypergraph H has at most $\binom{n-1}{k-1}$ edges, and equality holds if and only if H is a *full k-star*, that is, H consists of all $\binom{n-1}{k-1}$ edges of size k containing a given vertex. In Theorem 4.1.1, they also proved that an n-vertex non-unifrom hypergraph H has at most 2^{n-1} edges, and equality holds if and only if H is a *full star*, that is, H consists of all 2^{n-1} edges containing a given vertex.

Our first result in Section 4.1 is Theorem 4.1.2 showing that the maximum number of edges in an n-vertex r-free hypergraph is $2^{n-1} + r - 2$. We also conjecture that r-free hypergraphs with the most edges must contain a full star. In Section 4.1, we prove this conjecture for graphs satisfying either $n \geq r + 2\lceil \log_2 r \rceil + 1$ or $n \geq 425$.

Section 4.2 is based on [17] which is a joint work with Ilkyoo Choi, Amelia Tebbe, and Douglas B. West.

In Section 4.2, we consider hypergraphs without t edge-disjoint subhypergraphs whose local behavior at vertices is somehow "the same". Lindström [42] introduced a precise notion of such a property. A hypergraph H satisfies the *t-Equal Union Property (t-EUP)* if H has t edge-disjoint distinct subhypergraphs $H_1, \ldots, H_t$ such that for all H_i the union of the edges is the same. If $H_1, \ldots, H_t$ are not edge-disjoint, then H satisfies the *weak t-EUP*. When the t-EUP holds, the union of the edges is the same for each subhypergraph. Lindström [42] proved an upper bound for the number of edges in a hypergraph not satisfying t-EUP.

We introduce the *t-Equal Valence Property* (t-EVP), a variation on the t-EUP. Trivially, the t-EVP is stronger than the t-EUP. A hypergraph H satisfies the *t-Equal Valence Property* (t-EVP) if H has t edge-disjoint distinct subhypergraphs $H_1, \ldots, H_t$ such that $d_{H_1}(v) = \cdots = d_{H_t}(v)$ for each $v \in V(H)$. If the valence condition holds but $H_1, \ldots, H_t$ are not edge-disjoint, then H satisfies the *weak t-EVP*.

In Section 4.2.1, we characterize the graphs satisfying the 2-EUP and those satisfying the 2-

EVP, yielding the exact maximum number of edges in graphs not satisfying 2-EUP and in those not satisfying the 2-EVP. In Section 4.2.2, we prove upper and lower bounds for the maximum number of edges in graphs not satisfying the t-EVP. In Section 4.2.3, we consider hypergraphs, and we prove upper bounds for the maximum number of edges in hypergraphs and k-uniform hypergraphs not satisfying 2-EVP and t-EVP.

1.4 Background material

For completeness, here we present background definitions used in this thesis.

A *hypergraph* H is a pair consisting of a *vertex set* $V(H)$ and an *edge set* $E(H)$, where each edge in $E(H)$ is a subset of $V(H)$. We abuse notation by denoting $|H| = |E(H)|$ be the number of edges in H. An edge e of H is a *k-edge* if $|e| = k$, and we do not allow the empty set to be an edge. For two hypergraphs H and H', we say a bijective function $f : V(H) \to V(H')$ is an *graph isomorphism* if f is also a bijection between $E(H)$ to $E(H')$ when $f(e) = \{f(v) : v \in e, e \in E(H)\}$. If there is a graph isomorphism between H and H', we say that they are *isomorphic*. When every edge of H is a k-edge, we call H a *k-uniform hypergraph*. In particular, we call G a *graph* if it is a 2-uniform hypergraph, in which case the two elements in an edge are called *endpoints*. We define the *rank* of H to be $\max_{e \in E(H)} |e|$ the *anti-rank* of H is to be $\min_{e \in E(H)} |e|$. For two hypergraphs H, H', we say H' is a *sub(hyper)graph* of H if $E(H') \subset E(H)$ and $V(H') \subset V(H)$. We then write $H' \subseteq H$ and say that "*H contains H'*". A hypergraph H' is an *induced* sub(hyper)graph of H if $V(H') = A \subset V(H)$ and $E'(H) = \{e : e \subset V(H'), e \in E(H)\}$"; we then write $H' = H[A]$ and say that A *induces* H' in H. We write $H - e$ or $H - M$ for the subgraph of H obtained by deleting an edge e or set of edges M from $E(H)$. We write $H - v$ or $H - S$ for the subgraph of H induced by $V(H) - v$ or $V(H) - S$. The number of edges containing v in H is the *degree* of v in H, written $d_H(v)$ or $d(v)$ when the hypergraph H is clear from the context, and we define the maximum degree of H to be $\Delta(H) = \max_{v \in V(H)} d(v)$, and the minimum degree of H be $\delta(H) = \min_{v \in V(H)} d(v)$. A hypergraph H is *regular* if all vertices in $V(H)$ have the same degree; it is r-regular if every vertex has degree r. The *span* of H, written $S(H)$, is the set of vertices with positive degree. The maximum average degree, $\text{mad}(H)$, of a hypergraph H is the maximum of $\frac{\sum_{e \in E(H')}}{|V(H)|}$, which is average vertex degree

of H, over all subgraphs H' of H. In particular, for a graph G, $\mathrm{mad}(G)$ is the maximum of $\frac{2|E(G')|}{|V(G')|}$ over all subgraphs G' of G.

Two vertices u, v are *adjacent* if there is an edge e containing both u and v. A vertex v and an edge e are *incident* if e contains v. Also two edges e and f are *incident* if e and f have a common element. A hypergraph is *finite* if its vertex set and edge set are finite. A *loop* is an edge of size 1. *Multiple edges* are edges having exactly the same elements. A *simple hypergraph* is a hypergraph having no loops or multiple edges.

We say that a set $A \subset V(H)$ is an *independent set* of H if A induces no edges; equivalently, $H[A]$ has no edges. A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G. A graph G is *k-partite* if $V(G)$ can be expressed as the union of k (possibly empty) independent sets. A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted K_n. A *complete bipartite graph* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have sizes r and s, the complete bipartite graph is denoted $K_{r,s}$. The *union* of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

Consider a graph whose vertex set can be indexed as $\{v_1, ..., v_n\}$ so that its edge set is $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$. We call the isomorphism class of this graph P_n. A copy of P_n is called a *path* of length n. Consider a graph with an n vertices and n edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. We call the isomorphism class of this graph C_n. A copy of C_n is called a *cycle* of length n.

A *walk* is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A *trail* is a walk with no repeated edge. A *u,v-walk* or *u,v-trail* has first vertex u and last vertex v; these are its *endpoints*. A *u,v-path* is a path whose vertices of degree 1 (its *endpoints*) are u and v; the others are *internal vertices*. The *length* of a walk, trail, path, or cycle is its number of edges. A walk or trail is *closed* if its endpoints are the same. A walk, trail or path is *odd* or *even* if its length is odd or even, respectively.

A graph G is *connected* if for all $u, v \in V(G)$, there is a path containing u and v. Otherwise, G

is *disconnected*. The *components* of a graph G are its maximal connected subgraphs. A component (or graph) is *trivial* if it has no edges; otherwise it is *nontrivial*. An *isolated vertex* is a vertex of degree 0. A graph is *Eulerian* if it has a closed trail containing all edges. We call a closed trail a *circuit* when we do not specify the first vertex but keep the list in cyclic order. An *Eulerian circuit* or *Eulerian trail* in a graph is a circuit or trail containing all the edges. An *even graph* is a graph with all vertex degrees even. A vertex is *odd* when its degree is odd, and it is *even* when its degree is even.

A graph G is *k-connected* if it has more than k vertices and there is no set of $k-1$ vertices whose deletion leaves a disconnects subgraph. We use $\kappa(G)$ to denote the *connectivity* of a graph, which is the largest k such that G is k-connected. Similarly, a graph G is *k-edge-connected* if there is no set of $k-1$ edges whose removal disconnects it. We use $\kappa'(G)$ to denote the *edge-connectivity* of a graph, which is the largest k such that G is k-edge-connected. A *cut-edge* or *cut-vertex* of a connected graph G is an edge or vertex whose deletion leaves a disconnected subgraph.

A *k-coloring* of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are *colors*; the vertices of one color form a *color class*. A k-coloring is *proper* if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.

Chapter 2

Improper coloring of sparse graphs with a given girth

2.1 On $(0,1)$-coloring of triangle-free graphs

A $(d_1,\dots,d_k)$-*coloring* of a graph G is a partition of $V(G)$ into sets $V_1,\dots,V_k$ such that for every $1 \le i \le k$, the subgraph $G[V_i]$ of G induced by V_i has maximum degree at most d_i.

A graph G is (a,b)-*sparse* if $|E(H)| < a|V(H)| + b$ for all $H \subseteq G$. We also say that G is *almost* (a,b)-*sparse* if $|E(G)| = a|V(G)| + b$ and $|E(H)| < a|V(H)| + b$ for all $H \subsetneq G$.

Glebov and Zambalaeva [26] proved that every planar graph G with girth at least 16 is $(0,1)$-colorable. Borodin and Ivanova [14] proved that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(0,1)$-colorable. By (1.1), this implies that every planar graph G with girth at least 14 is $(0,1)$-colorable. Borodin and Kostochka [15] proved that every graph G with $\mathrm{mad}(G) < \frac{12}{5}$ is $(0,1)$-colorable, and this is sharp. This implies that every planar graph G with girth at least 12 is $(0,1)$-colorable. As mentioned above, Esperet et al. [24] proved that the problem of verifying whether a given planar graph of girth 9 has a $(0,1)$-coloring is NP-complete. Dorbec, Kaiser, Montassier, and Raspaud [19] mention that because of these results, the remaining open question is whether all planar graphs with girth 10 or 11 are $(0,1)$-colorable. Corollary of the following main theorem of this section answers one of the questions.

Theorem 2.1.1. *If a graph G is triangle-free $(\frac{11}{9},\frac{4}{9})$-sparse, G is $(0,1)$-colorable. On the other hand, there are infinitely many non-$(0,1)$-colorable $(\frac{11}{9},\frac{5}{9})$-sparse graphs G with girth 5 whose subgraphs are all $(\frac{11}{9},\frac{4}{9})$-sparse.*

Theorem 2.1.1 together with (1.1) yields the following.

Corollary 2.1.2. *Every planar graph with girth at least 11 is $(0,1)$-colorable.*

This answers half of the question above by Dorbec et al. [19].

In Section 2.1.1, we introduce *potentials* of vertex subsets of graphs, restate the theorem in a more general way in the language of potentials, and construct infinitely many graphs for which the statement of Theorem 2.1.1 is sharp. In Section 2.1.2 we set up the proof, consider a smallest counter-example G and derive simple properties of G. In Sections 2.1.3 and 2.1.4 we describe the structure of subsets of $V(G)$ with small potential. In Sections 2.1.5 and 2.1.6 we analyze the structure of some special subgraphs of G, so-called *shovels* and $(1,1,1)$-*trees*. In Section 2.1.7 we describe the discharging method and in the last two sections we use it to obtain a contradiction.

2.1.1 Restatement and a construction

In $(0,1)$-coloring, we always will use colors 0 and 1, where the vertices of color 0 form an independent set and the set of vertices of color 1 induce a subgraph of maximum degree at most 1.

For technical reasons, we will consider pairs $R = (G, Z)$ where G is a graph and Z is a nonempty subset of $V(G)$ such that the vertices in Z have stronger restrictions on coloring. In $(0,1)$-coloring of such pairs, every $z \in Z$ must be colored with 1 and its neighbors must be colored with 0. Vertices in Z will be called *special* and vertices in $V(G) - Z$ *typical*. Since the vertices in Z could be isolated, this setting contains can deal with the original $(0,1)$-coloring problem on graphs. We also will use *potentials*: Given a pair $R = (G, Z)$, we let $\rho_R(A) = 11|A - Z| - 9|E(G[A])|$. When the pair R is clear from context, we will omit the subscript.

This definition is used because for all $A \subset V(G) - Z$,

$$\rho(A) \geq 0 \quad \text{if and only if} \quad \frac{2|E(G[A])|}{|A|} \leq \frac{22}{9}. \tag{2.1}$$

Also, by this definition,

$$\rho(A) + \rho(B) = \rho(A \cup B) + \rho(A \cap B) + 9|E_G(A - B, B - A)| \quad \text{for all } A, B \subseteq V(G). \tag{2.2}$$

So, instead of Theorem 2.1.1, we will prove the following slightly stronger theorem in the language of potentials.

Theorem 2.1.3. *If a pair $R = (G, Z)$ is such that G is triangle-free and $\rho(A) \geq -4$ for all $A \subseteq V(G)$, then G is $(0,1)$-colorable. On the other hand, there are infinitely many non-$(0,1)$-colorable pairs $R = (G, Z)$ such that G has girth 5, all vertices of Z are isolated, $\rho(V(G) - Z) = -5$ and $\rho(A) \geq -4$ for all $A \subsetneq V(G) - Z$.*

In Section 2.1.1, we prove the second part of the statement of Theorem 2.1.3, and the rest of the section 2.1 is devoted to the proof of the first part.

Let graph D be obtained from a 10-cycle $(x_1, \ldots, x_{10})$ by adding edge x_4x_8 (see Fig. 2.1). Let x_1 be the *root*, $r(D)$, of D and x_6 be the *top*, $t(D)$, of D. A useful property of $(0,1)$-colorings ϕ of D is the following.

(P1) *If $\phi(x_2) = \phi(x_{10}) = 0$, then $\phi(x_6) = 1$ and x_6 has a neighbor in D of color 1.*

Proof: Since $\phi(x_2) = \phi(x_{10}) = 0$, we have $\phi(x_3) = \phi(x_9) = 1$. Since $x_4x_8 \in E(G)$ and each of x_4 and x_8 has a neighbor of color 1, they cannot be colored with the same color. By symmetry, assume that $\phi(x_4) = 1$ and $\phi(x_8) = 0$. Then $\phi(x_5) = 0$ and $\phi(x_7) = 1$. This yields (P1). □

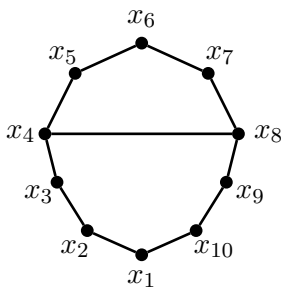


Figure 2.1 : $D = D_1$

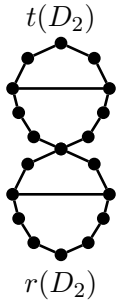


Figure 2.2 : D_2

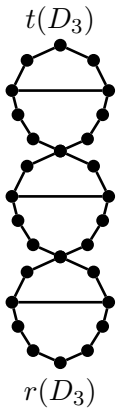


Figure 2.3 : D_3

Define the sequence $D_1, D_2, \ldots,$ of graphs as follows. Let $D_1 = D$. If D_i is constructed, then D_{i+1} is obtained from D_i by taking a copy D' of D disjoint from D_i, merging the root of D' with the top of D_i and choosing as the top, $t(D_{i+1})$, of D_{i+1} the top of D' and as the root, $r(D_{i+1})$, of D_{i+1} the root of D_i.

Similarly to (P1), for every i and every $(0,1)$-coloring ϕ of D_i the following holds.

(P2) *If the neighbors of $r(D_i)$ are colored with 0, then $\phi(t(D_i)) = 1$ and $t(D_i)$ has a neighbor in D_i of color 1.*

Proof: For $i = 1$ this is exactly (P1). Suppose (P2) holds for all $i \leq k$. Let ϕ be any $(0,1)$-coloring of D_{k+1} such that the neighbors of $r(D_{k+1})$ are colored with 0. Let D' be a copy of D. By definition, we obtain D_{k+1} by merging $t(D_k)$ in D_k with $r(D')$ of D' into a new vertex x. By the induction assumption, $\phi(t(D_k)) = 1$ and $t(D_k)$ has a neighbor of color 1 in D_k. Then both neighbors of x in D' must be colored with 0. So by (P1), $\phi(t(D')) = 1$ and $t(d') = t(D_{k+1})$ has a neighbor of color 1 in D'. □

(P3) *For every $i \geq 1$,*

(a) $\rho_{D_i}(V(D_i)) = 11$;

(b) $\rho_{D_i}(A) \geq 10$ for every nonempty $A \subseteq V(D_i)$,

(c) if $r(D_i), t(D_i) \in A$ and $A \neq V(D_i)$, then $\rho_{D_i}(A) \geq 12$.

Proof: Since D_i has $1 + 9i$ vertices and $11i$ edges, $\rho_{D_i}(V(D_i)) = 11(1 + 9i) - 9 \cdot 11i = 11$. This proves (a).

To prove (b) and (c), consider first D_1. Every proper subset A of $V(D_1)$ induces a subgraph with at most one cycle. If $D_1[A]$ is acyclic, then $|E(D_1[A])| \leq |A| - 1$ and so $\rho(A) \geq 11|A| - 9(|A| - 1) = 2|A| + 9$. If $|A| \geq 2$, this is at least 13. If $D_1[A]$ has exactly one cycle C, then $|E(D_1[A])| \leq |A|$ and so $\rho(A) \geq 11|A| - 9|A| = 2|A|$. Since $|C| \geq 5$, $2|A| \geq 10$. Moreover, if $|A| = 5$ and $D_1[A]$ has a cycle, then $A = \{x_4, x_5, x_6, x_7, x_8\}$ and so $x_1 = r(D_1) \notin A$. Otherwise, $|A| \geq 6$ and so $\rho(A) \geq 12$.

Suppose now that the claim holds for D_{i-1} and consider D_i for $i \geq 2$ as a copy of D_{i-1} to the top of which we merged the root of a copy D' of D. Let $x = t(D_{i-1}) = r(D')$. Let A be any nonempty proper subset of $V(D_i)$. Let $A' = A \cap V(D')$ and $A'' = A \cap V(D_{i-1})$. By induction, the claim holds for A' and A''. Thus if $A'' \subseteq \{x\}$ or $A' \subseteq \{x\}$, then (b) holds and (c) does not apply. So let $A'' - x \neq \emptyset$ and $A' - x \neq \emptyset$. If $x \notin A$, then $\rho_{D_i}(A) = \rho_{D'}(A') + \rho_{D_{i-1}}(A'') \geq 10 + 10 = 20$. Suppose $x \in A$. Then $\rho_{D_i}(A) = \rho_{D'}(A') + \rho_{D_{i-1}}(A'') - 11$ by 2.2. As $r(D') \in A'$ and $|A'| \geq 2$, by the argument in the previous paragraph for D_1, we know that either $\rho_{D'}(A') \geq 12$ or $A' = D'$ and $\rho_{D'}(A') = 11$. Therefore, $\rho_{D_i}(A) \geq \rho_{D_{i-1}}(A'')$ and hence (b) follows. Assume $r(D_i) = r(D_{i-1}) \in A$, $t(D_i) = t(D') \in A$ and $A \neq D_i$. If $A'' \neq D_{i-1}$, then $\rho_{D_i}(A) \geq \rho_{D_{i-1}}(A'') \geq 12$ (as $t(D') \in A''$). If

$A'' = D_{i-1}$, then $A' \neq D'$, and $\rho_{D'}(A') \geq 12$ and $\rho_{D_i}(A) \geq \rho_{D_{i-1}}(A'') + 1 \geq 12$. □

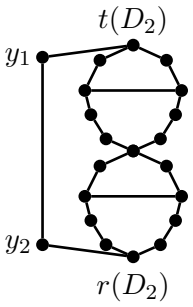


Figure 2.4 : $\widetilde{D_2}$

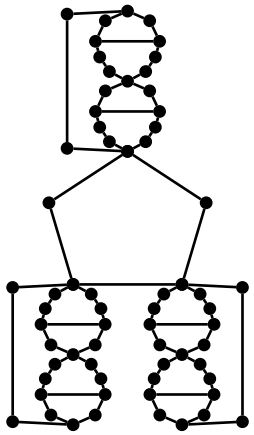


Figure 2.5 : G_2

Let $\widetilde{D}_i$ be obtained from D_i by adding path $(t(D_i), y_1, y_2, r(D_i))$, where y_1 and y_2 are new vertices (see Fig. 2.4).

(P4) *For every $i \geq 1$, and every nonempty $A \subseteq V(\widetilde{D}_i)$, $\rho_{\widetilde{D}_i}(A) \geq 6$. Moreover, if $1 \leq |A| < |V(\widetilde{D}_i)|$, then $\rho_{\widetilde{D}_i}(A) \geq 7$.*

Proof: Let $A \subseteq V(\widetilde{D}_i)$. If $\{y_1, y_2, r(D_i), t(D_i)\} \not\subseteq A$, then $\rho(A) \geq \rho(A - y_1 - y_2)$. Since $A - y_1 - y_2 \subseteq V(D_i)$, we get $\rho(A) \geq \rho(A - y_1 - y_2) \geq 10$. Suppose $\{y_1, y_2, r(D_i), t(D_i)\} \subseteq A$. Then by (P3), if $A = V(\widetilde{D}_i)$, then $\rho(A - y_1 - y_2) = 11$ and otherwise $\rho(A - y_1 - y_2) \geq 12$. Hence $\rho(A) = \rho(A - y_1 - y_2) + 11 \cdot 2 - 9 \cdot 3 \geq 11 + 22 - 27 = 6$, with equality only if $A = V(\widetilde{D}_i)$. □

We construct graph G_i from 3 copies, F_1, F_2 and F_3, of $\widetilde{D}_i$ and two vertices, u_1 and u_2, by adding 5 edges: $r(F_1)u_1, u_1r(F_2), r(F_2)u_2, u_2r(F_3)$ and $r(F_3)r(F_1)$ (see G_2 in Fig. 2.5).

(P5) *For every $i \geq 1$, $\rho_{G_i}(V(G_i)) = -5$, and for every $A \subsetneq V(G_i)$, $\rho_{G_i}(A) \geq -4$.*

Proof: Let $A \subsetneq V(G_i)$, $|A| \geq 2$. For $j \in \{1, 2, 3\}$, let $A_j = A \cap V(F_j)$. If $A_j = \emptyset$ for two indices j_1, j_2, then $\rho_{G_i}(A) \geq \rho_{F_{j_3}}(A_{j_3}) \geq 6$. If $A_j = \emptyset$ for exactly one index j_1, then $\rho_{G_i}(A) \geq \rho_{F_{j_3}}(A_{j_3}) + \rho_{F_{j_2}}(A_{j_2}) - 9 \geq 3$. If $A_j \neq \emptyset$ for all j, then

$$\rho_{G_i}(A) \geq \sum_{j=1}^{3} \rho_{F_j}(A_j) + 11 \cdot 2 - 9 \cdot 5 = \sum_{j=1}^{3} \rho_{F_j}(A_j) - 23,$$

with equality only in the case when $\{u_1, u_2, r(F_1), r(F_2), r(F_3)\} \subseteq A$. By (P4), for $j \in \{1, 2, 3\}$,

$\rho_{F_j}(A_j) \geq 6$ with equality only in the case when $A_j = V(F_j)$. Thus $\rho_{G_i}(A) \geq 6+6+6-23 = -5$ with equality only if $A = V(G_i)$. □

(P6) *For every $i \geq 1$, G_i is not $(0,1)$-colorable.*

Proof: Suppose f is a $(0,1)$-coloring of G_i. Then some two consecutive vertices of the cycle $C = (r(F_1), u_1, r(F_2), u_2, r(F_3))$ are colored with 1. Thus there exists $j \in \{1,2,3\}$ such that $f(r(F_j)) = 1$ and $r(F_j)$ has a neighbor in C of color 1. Then all neighbors of $r(F_j)$ in F_j are colored with 0. So by (P2), $f(t(F_j)) = 1$ and $t(F_j)$ has a neighbor of color 1 in the copy of D_i contained in F_j. But then both internal vertices y_1 and y_2 of the path $(r(F_j), y_1, y_2, t(F_j))$ in the corresponding copy of $\widetilde{D}_i$ must be colored 0, a contradiction. □

By (P5) and (P6), the sequence $\{G_i\}_{i=1}^{\infty}$ proves the second part of the statement of Theorem 2.1.3.

2.1.2 Simple properties of counterexamples

Assume that $R = (G, Z)$ is a counterexample to the first part of Theorem 2.1.3 with the fewest vertices, and among such pairs with the largest sum of degrees of the vertices in Z. Let $n = |V(G)|$. In the rest of the paper we will show that such R does not exist. In the current section we derive some basic properties of R.

Recall that $R = (G, Z)$ where G is a triangle-free graph with $\rho(A) \geq -4$ for all $A \subset V(G)$.

Claim 2.1.4. *$|Z| = 1$, i.e. R has exactly one special vertex.*

Proof. By assumption, R contains at least one special vertex. If there are more than one special vertices $z_1, \cdots, z_k$ in R, then we merge these vertices to get a new special vertex z' whose neighborhood is $\bigcup_{i=1}^{k} N_G(z_i)$. This gives us a smaller graph G' and the pair $R' = (G', \{z'\})$.

If G' has a triangle (u, v, w), then by construction, $z' \in \{u, v, w\}$, say $z' = u$, and there are $z_i, z_j \in Z$ such that the path (z_i, v, w, z_j) is in G. But then $\rho_R(\{z_i, v, w, z_j\}) \leq 3(-9)+2\cdot 0+2\cdot 11 = -5 < -4$, a contradiction. Thus G' is a triangle-free graph smaller than G. If $A \subset V(G')$ and $z' \notin A$, then $\rho_{R'}(A) = \rho_R(A) \geq -4$. If $z' \in A$, then $\rho_{R'}(A) = \rho_R(A \cup \{z_1, \ldots, z_k\} - z') \geq -4$. By the minimality of G, G' has a $(0,1)$-coloring ϕ'. Let $\phi(v) = \phi'(v)$ for each typical vertex $v \in V(G)$, and $\phi(u) = 1$ for each $u \in Z$. By construction, ϕ is a $(0,1)$-coloring of G, a contradiction. □

From now on, we assume that $Z = \{z\}$.

Claim 2.1.5. *If $v \in V(G) - z$, then $d(v) \geq 2$.*

Proof. Suppose, $N(v) \subseteq \{w\}$. Let $G' = G - v$ and $R' = (G', Z)$. By the minimality of G, G' has a $(0,1)$-coloring ϕ. Then letting $\phi(v) := 1 - \phi(w)$ we obtain a $(0,1)$-coloring of G, a contradiction. □

Lemma 2.1.6. *For every $\emptyset \neq A \subsetneq V(G)$, $\rho(A) \geq 0$. Moreover, $A = \{z\}$ is the only nonempty proper subset of $V(G)$ with $\rho(A) = 0$.*

Proof. Suppose that the lemma fails and choose the smallest $A \subsetneq V(G)$ such that $|A| \geq 2$ and $\rho(A) \leq 0$. If A is independent, then $\rho(A) \geq 11(|A| - 1) > 0$. So, by minimality we may assume that $G[A]$ is connected. If $|A| = 2$, then $\rho(A) \geq 11 - 9 = 2$, thus $|A| \geq 3$.

Since $A \subsetneq V(G)$, there is a $(0,1)$-coloring ϕ' of $G[A]$. Let $A_0 = \{v \in A : \phi'(v) = 0\}$, $A_1 = \{v \in A : \phi'(v) = 1\}$.

Let the new graph $G' = G'(\phi')$ have $V(G') = (V(G) - A) \cup \{z', w_0\}$ and

$$E(G') = E(G[V(G) - (A - z)]) \cup \{aw_0 : \exists b \in A_0, ab \in E(G)\} \cup \{az' : \exists b \in A_1, ab \in E(G)\} \cup \{w_0 z'\}.$$

Let $R' = (G', Z')$ where $Z' = (Z - A) + z'$. Since $|A| \geq 3$, G' is smaller than G.

If G' contains a triangle, then either there is a vertex $v \notin A$ with two neighbors in A or there are two adjacent vertices $v, u \notin A$ with neighbors in A. In the former case, $\rho(A+v) \leq 0 - 9 - 9 + 11 < -4$, a contradiction. In the latter case, $\rho(A + v + u) \leq -27 + 22 < -4$, a contradiction. Thus G' is triangle-free.

Suppose G' has a $(0,1)$-coloring ϕ''. Then by the definition of z', $\phi''(z') = 1$ and $\phi''(w_0) = 0$. Let $\phi(v) = \phi''(v)$ for $v \in V(G) - A$, and $\phi(v) = \phi'(v)$ if $v \in A$. Then ϕ is a $(0,1)$-coloring of G since every neighbor of $v \in A_1$ in $G - A$ is a neighbor of z' in G', and so is colored with 0, and every neighbor of $w \in A_0$ in $G - A$ is a neighbor of w_0 in G', and so is colored with 1. This contradicts the choice of G. So, G' has no $(0,1)$-coloring. By the minimality of G, there is $B \subset V(G')$ such that

$$\rho_{R'}(B) \leq -5. \tag{2.3}$$

Choose a largest $B \subset V(G')$ satisfying (2.3). Since adding z' to a set does not increase its potential, $z' \in B$. If $w_0 \in B$, then

$$\rho_R((B - w_0 - z') \cup A) = \rho_R(B - w_0 - z') + \rho_R(A) - 9|E_G[B - w_0 - z', A]|$$

$$\leq \rho_R(B - w_0 - z') + 0 - 9|E_{G'}(B - w_0 - z', \{w_0, z'\})|$$

$$< \rho_R(B - w_0 - z') + \rho_{R'}(\{w_0, z'\}) - 9|E_{G'}[B - w_0 - z', \{w_0, z'\}]| = \rho_{R'}(B).$$

By (2.3), this is at most -5, a contradiction. If $w_0 \notin B$, then

$$\rho_R((B - z') \cup A) = \rho_R(B - z') - 9|E[A, B - z']| + \rho(A) \leq \rho_{R'}(B - z') - 9|E_{G'}[z', B - z']| = \rho_{R'}(B),$$

again, a contradiction by (2.3). □

We say that a path P is a *k-path* if P has $k + 2$ vertices, and all internal vertices are typical and have degree 2 in G, and each of the endvertices of P either has degree at least 3 or is special. A vertex v is a *k-vertex* if $d(v) = k$. For $k \geq 2$, a k-vertex x is $(a_1, \ldots, a_k)$-*vertex* if x is typical and x is an end of distinct a_i-paths for $i = 1, 2, \ldots, k$.

Claim 2.1.7. *G has no alternating cycle $C = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ such that all x_i are $(1, 1, 1)$-vertices (and thus all y_j are 2-vertices).*

Proof. Suppose G has such a cycle $C = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$. For $i = 1, \ldots, k$, let u_i be the neighbor of x_i outside of C and w_i be the neighbor of u_i distinct from x_i (it could be some x_j for $j \neq i$). Recall that by the definition of $(1, 1, 1)$-vertices, all x_i, y_i, u_i are typical. Consider $G' = G - \{x_1, y_1, u_1, \ldots, x_k, y_k, u_k\}$ and $R' = (G', \{z\})$. Then G' is a proper subgraph of G, so it has a $(0, 1)$-coloring ϕ'. Let $\phi(v) = \phi'(v)$ if $v \in V(G')$, and for $i = 1, \ldots, k$, let $\phi(x_i) = 1, \phi(y_i) = 0$ and $\phi(u_i) = 1 - \phi(w_i)$. Then ϕ is a $(0, 1)$-coloring for G, a contradiction. □

Let V_0 be the set of all $(1, 1, 1)$-vertices and their neighbors. By Claim 2.1.7, $G[V_0]$ is a forest. Let $T_1, \ldots, T_d$ be the components of $G[V_0]$. Each component T of $G[V_0]$ will be called a $(1, 1, 1)$-*tree*, and *the size*, $||T||$, of a $(1, 1, 1)$-tree T is the number of $(1, 1, 1)$-vertices in T.

Lemma 2.1.8. *Let T be a $(1,1,1)$-tree in G and ϕ be a $(0,1)$-coloring of $G-T$. Then exactly one vertex of T is adjacent to a vertex v of color 1 in ϕ, and either $v=z$ or v has a neighbor in $G-T$ of color 1.*

Proof. We use induction on the size of T. If the size of T is 1, suppose that x is the 3-vertex in T, $N(x)=\{y_1,y_2,y_3\}$, and $u_i \in N(y_i)-x$ for $i=1,2,3$. For $i=1,2,3$, let $\phi(y_i)=1-\phi(u_i)$. This coloring cannot be extended to x only if exactly one vertex in $N(x)$, say y_1 is colored with 0. Moreover, if in this case $u_1 \neq z$ and u_1 has no neighbor of color 1 in $V(G)-V(T)$, then we can recolor y_1 with 1 and color x with 0. But if $u_1=z$ or u_1 has a neighbor of color 1 in $V(G)-V(T)$, then ϕ does not extend to T.

Suppose now that the lemma holds for $(1,1,1)$-trees of size at most $i-1$ and let T be a $(1,1,1)$-tree of size i and ϕ be a $(0,1)$-coloring of $G-T$. A *quasi-leaf* of T is a 3-vertex adjacent to two leaves of T. Suppose T has a quasi-leaf x_1 such that the two leaves of T, y_1 and y_2, adjacent to x_1 are both adjacent to vertices in $G-T$ of color 0. Then in any extension of ϕ to $\{y_1,y_2,x\}$ we must have $\phi(y_1)=\phi(y_2)=1$ and so $\phi(x_1)=0$. By the choice of i, the new ϕ extends to the tree $T'=T-x_1-y_1-y_2$ if and only if exactly one vertex T' is adjacent to a vertex v of color 1 in ϕ, and either $v=z$ or v has a neighbor in $G-T'$ of color 1. But then by construction, this also holds for T.

Thus we may assume that for every quasi-leaf x of T, some leaf y of T adjacent to x has a neighbor in T of color 1. Since every $(1,1,1)$-tree of size at least two has at least two quasi-leaves, we may assume that x and x' are quasi-leaves and y_1 and y_2 are the leaves of T adjacent to x and u_1 and u_2 are their other neighbors. Let $\phi(x)=1$ and for $i=1,2$ let $\phi(y_i)=1-\phi(u_i)$. Since at least one of u_1 and u_2 has color 1, at least one of y_1 and y_2 gets color 0, and so the new partial coloring ϕ is a partial $(0,1)$-coloring. Then at least two leaves of the the tree $T'=T-x-y_1-y_2$ have neighbors of color 1, and so by the choice of i, the new ϕ extends to T'. Thus the original ϕ extends to T. □

In the next two sections we describe all sets of low potential.

2.1.3 Sets of potential at most 2

The goal of the section is to prove the following.

Lemma 2.1.9. *If $A \subset V(G)$, $2 \leq |A| \leq n-4$ and $\rho(A) \leq 2$, then A consists of one typical and one special vertex with an edge connecting them.*

We will do it using a sequence of smaller claims. It is straightforward to check the lemma in the case $|A| \in \{2,3\}$. Suppose now that $\rho(A) \leq 2$ and $4 \leq |A| \leq n-4$. Let A' be the set of vertices in A that have neighbors in $V(G) - A$.

Claim 2.1.10. *No vertex $v \in V(G) - A$ has more than one neighbor in A, and $G - A$ has no edge uw such that both u and w have neighbors in A. Furthermore, if $z \notin A$, then the distance from z to A is at least 3.*

Proof. If some $v \in V(G) - A$ has more than one neighbor in A, then $\rho(A+v) \leq \rho(A) + 11 - 9 \cdot 2 \leq 2 - 7 = -5$, a contradiction. If $uw \in E(G - A)$ and both u and w have neighbors in A, then $\rho(A+u+w) \leq \rho(A) + 11 \cdot 2 - 9 \cdot 3 \leq 2 - 5 = -3$, a contradiction to Lemma 2.1.6. Finally, if $z \notin A$ and there is a path $P_j = (z, v_1, \ldots, v_j)$ from z to A of length $j \leq 2$, then

$$\rho(A \cup V(P_j)) \leq \rho(A) + 11(j-1) - 9j \leq 2 - 9 + (11-9)(j-1) = -9 + 2j \leq -5,$$

a contradiction. □

Claim 2.1.11. *For each $(0,1)$-coloring ϕ of $G[A]$ and each $u \in A'$, $\phi(u) = 1$.*

Proof. Suppose that there is a $(0,1)$-coloring ϕ' of $G[A]$ and a vertex $u \in A'$ such that $\phi'(u) = 0$. Since $u \in A'$, there is $y \in V(G) - A$ such that $uy \in E(G)$. Let $A_0 = \{v \in A : \phi'(v) = 0\}$, $A_1 = \{v \in A : \phi'(v) = 1\}$. Consider the same $G' = G'(\phi')$ and R' as in the proof of Lemma 2.1.6. By Claim 2.1.10, G' is triangle-free. Since $\phi'(u) = 0$, w_0 has at least one neighbor in $V(G) - A$. If G' has a $(0,1)$-coloring ϕ'', then as in the proof of Lemma 2.1.6, letting $\phi(v) = \phi''(v)$ for $v \in V(G) - A$ and $\phi(v) = \phi'(v)$ if $v \in A$ yields a $(0,1)$-coloring ϕ of G, a contradiction. Thus G' has no such

coloring. Then by the minimality of G,

$$\text{there is } B \subset V(G') \text{ with } \rho_{R'}(B) \leq -5. \tag{2.4}$$

Choose a set B satisfying (2.4) with the smallest potential and among those—with the largest size. Then $z' \in B$. If $w_0 \in B$, then as in the proof of Lemma 2.1.6,

$$\rho_R((B - w_0 - z') \cup A) = \rho_R(B - w_0 - z') + \rho_R(A) - 9|E_G[B - w_0 - z', A]|$$

$$\leq \rho_R(B - w_0 - z') + 2 - 9|E_{G'}(B - w_0 - z', \{w_0, z'\})|$$

$$= \rho_R(B - w_0 - z') + \rho_{R'}(\{w_0, z'\}) - 9|E_{G'}[B - w_0 - z', \{w_0, z'\}]| = \rho_{R'}(B) \leq -5,$$

a contradiction. Suppose $w_0 \notin B$. Since $w_0 y \in E(G')$, if $y \in B$ then w_0 has two neighbors y and z' in B, hence moving w_0 into B would decrease its potential. So, $y \notin B$. Then as above by (2.4),

$$\rho_R((B - z') \cup A) = \rho_R(B - z') + \rho_R(A) - 9|E_G[B - z', A]|$$

$$\leq \rho_R(B - z') + 2 - 9|E_{G'}(B - z', z')| = \rho_{R'}(B) + 2 \leq -3.$$

But $A \neq \emptyset$ and $y \notin (B - z') \cup A$. So by Lemma 2.1.6, $\rho_R((B - z') \cup A) \geq 0$, a contradiction. □

Claim 2.1.12. *In every $(0,1)$-coloring ϕ' of $G[A]$, each typical vertex in A' has a neighbor of color 1 in A.*

Proof. Assume that for some $(0,1)$-coloring ϕ' of $G[A]$, a typical vertex $u \in A'$ has no neighbor of color 1 in A. Let y be a neighbor of u in $V(G) - A$. Consider the graph G'' with $V(G'') = (V(G) - A) \cup \{z', w_0, w_1\}$ and

$$E(G'') = E(G - A) \cup \{w_0 z', w_0 w_1\} \cup \{bz' : \exists x \in A' - u \text{ such that } bx \in E(G)\} \cup \{yw_1 : yu \in E(G)\}.$$

Let $R'' = (G'', Z'')$ where $Z'' = (Z - A) + z'$. By Claim 2.1.10, G'' is triangle-free.

Suppose G'' has a $(0,1)$-coloring ϕ''. Since $z'w_0, w_0 w_1 \in E(G'')$, $\phi''(w_0) = 0$ and $\phi''(w_1) = 1$.

Thus by Claim 2.1.11 and the choice of u, letting $\phi(v) = \phi''(v)$ for $v \in V(G) - A$, and $\phi(v) = \phi'(v)$ if $v \in A$ yields a $(0,1)$-coloring of G, a contradiction. Thus G'' has no such coloring. Then by the minimality of G,

$$\text{there is } B \subset V(G'') \text{ with } \rho_{R''}(B) \leq -5. \tag{2.5}$$

Choose a set B satisfying (2.5) with the smallest potential and among those—with the largest size. Then $z' \in B$. Since $N_{G''}(w_0) = \{z', w_1\}$, if $w_1 \in B$, then $w_0 \in B$, otherwise $w_0 \notin B$. Suppose $w_1 \in B$. Then $B \supseteq \{z', w_0, w_1\}$. So, since $|E_{G''}[\{z', w_0, w_1\}, B - \{z', w_0, w_1\}]| = |E_G[A, B - \{z', w_0, w_1\}]|$,

$$\rho_R(B \cup A - \{z', w_0, w_1\}) = \rho_{R''}(B) - \rho_{R''}(\{z', w_0, w_1\}) + \rho_R(A) \leq \rho_{R''}(B) - 4 + 2 \leq -7,$$

a contradiction.

Suppose $w_1 \notin B$. As in the proof of Claim 2.1.11, if $y \in B$, then moving w_0 and w_1 into B would decrease its potential. So, $y \notin B$. As at the end of the proof of Claim 2.1.11, by (2.5) we get

$$\rho_R((B - z') \cup A) \leq \rho_{R''}(B) + 2 \leq -3,$$

which contradicts Lemma 2.1.6, since $A \neq \emptyset$ and $y \notin (B - z') \cup A$. □

Claim 2.1.13. *If $u \in A'$, then $u = z$.*

Proof. Assume $u \in A'$ is typical. Consider the graph G' with $V(G') = A \cup \{w_1, w_2, w_3\}$ and $E(G') = E(G[A]) \cup \{uw_1, w_1w_2, w_2w_3\}$. Let $R' = (G', Z \cap A + w_3)$.

Since $|A| \leq n - 4$, $|V(G')| < n$. By Lemma 2.1.6, since $u \neq z$, $\rho(A) \geq 1$. Then $\rho_{R'}(B) \geq -4$ for every $B \subseteq V(G')$. By construction, G' is triangle-free. So by the minimality of G, G' has a $(0,1)$-coloring ϕ'. Since w_3 is special, $\phi'(w_3) = 1$ and $\phi'(w_2) = 0$. So $\phi'(w_1) = 1$. Now if $\phi'(u) = 0$, this contradicts to Claim 2.1.11 and if $\phi'(u) = 1$ then because of w_1, u has no neighbors of color 1 in A, a contradiction to Claim 2.1.12. □

Proof of Lemma 2.1.9. By Claim 2.1.13, $A' \subseteq \{z\}$. Since $4 \leq |A| \leq n - 4$, we can $(0,1)$-color $G[A]$ and $G[(V(G) - A) \cup A']$ by induction hypothesis, and combine the colorings to get a $(0,1)$-coloring for G since either $A' = \emptyset$ or $A' = \{z\}$, and z is always colored with 1. This is a contradiction. □

2.1.4 Sets of potential at most 4

The goal of this section is to show the following lemma.

Lemma 2.1.14. *If $A \subseteq V(G)$ and $4 \leq |A| \leq n-4$, then $\rho(A) \geq 5$. Moreover, if $|A| \leq 3$ and $\rho(A) \leq 4$, then $z \in A$ and $G[A]$ is connected.*

Suppose that the lemma fails and choose a smallest $A \subseteq V(G)$ with $4 \leq |A| \leq n-4$ for which it fails. Let $A' = \{v \in A : v \text{ has a neighbor in } V(G) - A\}$. If $B \subset V(G)$, $1 \leq |B| \leq 3$ and $z \notin B$, then $\rho(B) \geq 11$ since G has no triangle. Thus if $G[A]$ is not connected, then by this and the minimality of A, the vertex set C of some component of $G[A]$ has potential at least 5. Thus in this case, $\rho(A - C) \leq 4 - 5 = -1$, a contradiction to Lemma 2.1.6. So,

$$G[A] \text{ is connected.} \tag{2.6}$$

Practically repeating the proof of Claim 2.1.10, we obtain the similar claim:

Claim 2.1.15. *No vertex $v \in V(G) - A$ has more than one neighbor in A, and $G - A$ has no edge uw such that both u and w have neighbors in A.*

Claim 2.1.16. *Let ϕ be a $(0,1)$-coloring of $G[A]$. Then color class 0 of ϕ has at most one neighbor in $G - A$.*

Proof. Let $A_0 = \{v \in A : \phi(v) = 0\}$, $A_1 = \{v \in A : \phi(v) = 1\}$. Suppose there exist $x, y \in G - A$ with neighbors in A_0. Let G' be the graph with $V(G') = V(G) - A \cup \{w_0, w_0', z'\}$ and

$$E(G') = E(G[V(G) - A]) \cup \{uz' : \exists v \in A_1 \text{ with } uv \in E(G)\} \cup \{xw_0, z'w_0, z'w_0'\} \cup$$

$$\cup \{uw_0' : u \neq x, \exists v \in A_0 \text{ with } uv \in E(G)\}.$$

Let $R' = (G', (Z - A) + z')$.

By Claim 2.1.15, G' is triangle-free. Since $|A| \geq 4$, G' is smaller than G. If G' has a $(0,1)$-coloring ϕ', then by definition, $\phi'(z') = 1$ and $\phi'(w_0) = \phi'(w_0') = 0$. Thus ϕ and ϕ' together yield a $(0,1)$-coloring of G, a contradiction. So, G' has no such coloring. Then by the minimality of G, there

is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such set B with the smallest potential and among those—with the largest size. Then $z' \in B$. If $w_0, w_0' \in B$, then since $|E_{G'}[\{z', w_0, w_0'\}, B - \{z', w_0, w_0'\}]| = |E_G[A, B - \{z', w_0, w_0'\}]|$,

$$\rho_R(B \cup A - \{z', w_0, w_0'\}) = \rho_{R'}(B) - \rho_{R'}(\{z', w_0, w_0'\}) + \rho_R(A) \leq \rho_{R'}(B) - 4 + 4 \leq -5,$$

a contradiction.

Suppose now that $w_0' \in B$ and $w_0 \notin B$. Then by the minimality of $\rho_{R'}(B)$, $x \notin B$. Then

$$\rho_R(B \cup A - \{z', w_0'\}) = \rho_{R'}(B) - \rho_{R'}(\{z', w_0'\}) + \rho_R(A) - 9|E_G[B - w_0' - z', A]| + 9|E_{G'}[B - w_0' - z', \{z', w_0'\}]|$$

$$\leq \rho_{R'}(B) - 2 + 4 \leq -3,$$

contradicting Lemma 2.1.6, since $x \notin B$. In the case $w_0, w_0' \notin B$, the argument is the same, only the final inequality will be $\rho_R(B \cup A - z') \leq -1$. If $w_0 \in B$ and $w_0' \notin B$, then similarly $y \notin B$ and the the same calculations work. So in all cases we get a contradiction. □

The next claim is very similar to Claim 2.1.12 and has a similar proof with a twist.

Claim 2.1.17. *In any $(0,1)$-coloring ϕ of $G[A]$, each typical vertex of color 1 in A' has a neighbor of color 1.*

Proof. Assume that for some $(0,1)$-coloring ϕ' of $G[A]$, a typical vertex $u \in A'$ has no neighbor of color 1 in A. Let y be a neighbor of u in $V(G) - A$. By Claim 2.1.16, at most one vertex in $G - A$ has a neighbor of color 0 in A. If such a vertex exists, let it be y' and its neighbor in A be u' (by Claim 2.1.15, such u' should be unique). Consider the graph G'' with $V(G'') = (V(G) - A) \cup \{z', w_0, w_1\}$ and

$$E(G'') = E(G - A) \cup \{w_0 z', w_0 w_1, y' w_0\} \cup \{bz' : \exists x \in A' - u \text{ with } bx \in E(G)\} \cup \{yw_1 : yu \in E(G)\},$$

where possibly y' does not exist. Let $R'' = (G'', Z'')$ where $Z'' = (Z - A) + z'$. By Claim 2.1.15, G'' is triangle-free.

Suppose G'' has a $(0,1)$-coloring ϕ''. Since $z'w_0, w_0w_1 \in E(G'')$, $\phi''(w_0) = 0$ and $\phi''(w_1) = 1$. Thus by Claim 2.1.11 and the choice of u, letting $\phi(v) = \phi''(v)$ for $v \in V(G) - A$, and $\phi(v) = \phi'(v)$ if $v \in A$ yields a $(0,1)$-coloring of G, a contradiction. Thus G'' has no such coloring. Then by the minimality of G,

$$\text{there is } B \subset V(G'') \text{ with } \rho_{R''}(B) \leq -5. \tag{2.7}$$

Choose a set B satisfying (2.7) with the smallest potential and among those—with the largest size. Then $z' \in B$. Since $N_{G''}(w_0) \supseteq \{z', w_1\}$, if $w_1 \in B$, then $w_0 \in B$. Suppose $w_1 \in B$. Then $B \supseteq \{z', w_0, w_1\}$. So, since $|E_{G''}[\{z', w_0, w_1\}, B - \{z', w_0, w_1\}]| = |E_G[A, B - \{z', w_0, w_1\}]|$,

$$\rho_R(B \cup A - \{z', w_0, w_1\}) = \rho_{R''}(B) - \rho_{R''}(\{z', w_0, w_1\}) + \rho_R(A) \leq \rho_{R''}(B) - 4 + 4 \leq -5,$$

a contradiction.

Suppose $w_1 \notin B$. If $y \in B$, then moving w_1 (and w_0, if it is not there) into B would decrease its potential. So, $y \notin B$. If $w_0 \notin B$, then as at the end of the proof of Claim 2.1.11, by (2.5) we get $\rho_R((B - z') \cup A) \leq \rho_{R''}(B) + 4 \leq -1$, which contradicts Lemma 2.1.6, since $A \neq \emptyset$ and $y \notin (B - z') \cup A$. If $w_0 \in B$, then similarly,

$$\rho_R((B - z' - w_0) \cup A) = \rho_R(B - z' - w_0) + \rho_R(A) - 9|E_G[B - z' - w_0, A]|$$

$$\leq \rho_R(B - z' - w_0) + 4 - 9|E_{G''}(B - z' - w_0, \{z', w_0\})| = \rho_{R''}(B) + 4 - \rho_{R''}(\{z', w_0\}) \leq -3,$$

contradicting Lemma 2.1.6, since $y \notin B$. □

The next claim is an analogue of Claim 2.1.13.

Claim 2.1.18. *If $u \in A'$ is typical, then (a) there exists a coloring ϕ_u of $G[A]$ such that $\phi_u(u) = 0$ and $\phi_u(v) = 1$ for all $v \in A' - u$, and (b) u has exactly one neighbor in $G - A$.*

Proof. Assume $u \in A'$ is typical. Consider the graph G' with $V(G') = A \cup \{w_1, w_2, w_3\}$ and $E(G') = E(G[A]) \cup \{uw_1, w_1w_2, w_2w_3\}$. Let $R' = (G', Z \cap A + w_3)$.

Since $|A| \leq n - 4$, $|V(G')| < n$. By Lemma 2.1.6, since $u \neq z$, $\rho(A) \geq 1$. Then $\rho_{R'}(B) \geq -4$

for every $B \subseteq V(G')$. By construction, G' is triangle-free. So by the minimality of G, G' has a $(0,1)$-coloring ϕ'. Since w_3 is special, $\phi'(w_3) = 1$ and $\phi'(w_2) = 0$. So $\phi'(w_1) = 1$. If $\phi'(u) = 1$ then because of w_1, u has no neighbors of color 1 in A, a contradiction to Claim 2.1.17. Thus $\phi'(u) = 0$. Now Claim 2.1.16 yields (b), and together with Claim 2.1.15, also yields (a). □

Claim 2.1.19. *Set A' contains a typical vertex.*

Proof. If not, then $A' = \emptyset$ or $A' = \{z\}$. Since $4 \leq |A| \leq n-3$, both $G[V(G) - (A - A')]$ and $G[A]$ are smaller than G, and so by the minimality of G have $(0,1)$-colorings, say ϕ and ϕ'. Then $\phi \cup \phi'$ is a $(0,1)$-coloring of G, since if $A' \neq \emptyset$, then $A' = \{z\}$ and $\phi(z) = \phi'(z) = 1$. □

Claim 2.1.20. *If $z \in A'$, then z has no neighbor in A'.*

Proof. Suppose z has a neighbor v in A'. Then $vz \in E(G[A'])$. If $A' = \{v, z\}$, then we can color $G[A]$ and $G[V(G) - A \cup A']$ and combine the colorings, since v, z have the consistent colors in both colorings. So, there is $u \in A' - z - v$. Then by Claim 2.1.18(a), there is a $(0,1)$-coloring ϕ_u of $G[A]$ such that $\phi_u(v) = \phi_u(z) = 1$, a contradiction to the fact that $z \in Z$. □

Proof of Lemma 2.1.14. Assume $4 \leq |A| \leq n-4$ and $\rho(A) \leq 4$. Suppose $A' - \{z\} = \{u_1, \ldots, u_d\}$ and for $i = 1, \ldots, d$, x_i is the unique neighbor of u_i in $G - A$. Let $X = \{x_1, \ldots, x_d\}$. If $x_i = z$ for some i, then $\rho(A + x_i) = \rho(A) - 9 \leq -5$, a contradiction. Furthermore, if x_i is adjacent to z, then $\rho(A + x_i + z) \leq \rho(A) - 2 \cdot 9 + 11 \leq -3$ and $A + x_i + z \neq V(G)$, a contradiction to Lemma 2.1.6. Hence

$$z \notin X \cup N(X) - A'. \tag{2.8}$$

CASE 1: $A' = \{u_1\}$. Let G' be the graph with $V(G') = V(G) - A \cup \{w_0, z'\}$ and $E(G) = E(G[V(G) - A]) \cup \{u_1 w_0, w_0 z'\}$. Let $R' = (G', (Z - A) + z')$. If G' has a $(0,1)$-coloring ϕ, then $\phi(z') = 1$, $\phi(w_0) = 0$, and hence $\phi(u_1) = 1$. So, together with the coloring ϕ_{u_1} (see Claim 2.1.18(a)) of $G[A]$, ϕ will form a $(0,1)$-coloring of G, a contradiction. Thus G' has no $(0,1)$-coloring and is smaller than G, since $|A| \geq 4$. Hence there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z' \in B$. If $w_0 \notin B$, then z' is isolated in $G'[B]$ and $\rho_R(B - z') = \rho_{R'}(B - z') = \rho_{R'}(B) \leq -5$, a contradiction. So $w_0 \in B$ and hence

$u_1 \in B$. Since $B-z'-w_0 \subset V(G)$, we have $\rho_R(B-z'-w_0) = \rho_{R'}(B-z'-w_0) = \rho_{R'}(B)-11+2\cdot 9 \leq 2$. Since $|A| \geq 4$, $|B-z'-w_0| \leq n-4$. So by Lemmas 2.1.6 and 2.1.9, $|B-z'-w_0| \leq 2$ and $z \in B-z'-w_0$ is adjacent to $x_1 \in B - z' - w_0$, a contradiction to (2.8).

CASE 2: $A' = \{u_1, z\}$. Consider G' with $V(G') = A + w_0$, and $E(G') = E(G[A]) \cup \{zw_0, w_0u\}$ $E(G') = E(G[A]) \cup \{zw_0, w_0u_1\}$. Let $R' = (G', \{z\})$. If G' has a $(0,1)$-coloring ϕ, then $\phi(w_0) = 0$ and thus $\phi(u_1) = 1$. Recall that by Claim 2.1.18(a), there is a $(0,1)$-coloring ϕ_{u_1} of $G[A]$ such that $\phi_{u_1}(u_1) = 0$. By the minimality of G, $G - (A - z)$ has a $(0,1)$-coloring f. If $f(x_1) = 0$, then $f \cup \phi$ is a $(0,1)$-coloring of G; otherwise, $f \cup \phi_{u_1}$ is a $(0,1)$-coloring of G, a contradiction.

Thus G' has no $(0,1)$-coloring. Then there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Since $V(G')-w_0 \subset A \subset V(G)$, $w_0 \in B$. Hence $z, u_1 \in B$ and $\rho_R(B - w_0) \leq \rho_{R'}(B) - 11 + 2 \cdot 9 \leq 2$. Since $u_1z \notin E(G)$ by Claim 2.1.20, this contradicts Lemma 2.1.9.

CASE 3: $d \geq 2$. By Claim 2.1.15, all x_i are distinct and not adjacent to each other. Let G' be the graph with $V(G') = V(G) - (A - z) \cup \{w_0, w_1, z'\}$ and

$$E(G') = E(G[V(G) - (A - z)]) \cup \{zx_i : 3 \leq i \leq d\} \cup \{x_1x_2, z'w_0, w_0w_1, w_1x_1\}.$$

Observe that G' is triangle free. Otherwise x_1, x_2 have a common neighbor w. Then $\rho_R(A \cup \{w, x_1, x_2\}) \leq \rho_R(A) - 3 \leq 1$, contrary to Lemma 2.1.9. Let $R' = (G', \{z, z'\})$.

Suppose G' has a $(0,1)$-coloring ϕ. Then $\phi(z) = \phi(z') = 1$, $\phi(x_3) = \ldots = \phi(x_d) = \phi(w_0) = 0$ and hence $\phi(w_1) = 1$. If $\phi(x_2) = \phi(x_1)$, then since $x_1x_2 \in E(G')$, their common color is 1. But x_1 already has neighbor w_1 with $\phi(w_1) = 1$. Thus $\phi(x_2) \neq \phi(x_1)$. Then by Claim 2.1.18(a), either $\phi \cup \phi_{u_1}$ or $\phi \cup \phi_{u_2}$ is a $(0,1)$-coloring of G, a contradiction. So G' has no $(0,1)$-coloring and is smaller than G, since $|A| \geq 4$. Hence there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z \in B$. Since $G' - x_1 - z - w_0 - w_1 \subset G$ and $G'[B]$ has no isolated or pendant vertices, apart from z, z', we have $x_1 \in B$. Then $\{w_0, w_1\} \subset B$. Let $X' = \{x_1, \ldots, x_d\} \cap B$. Then $|E_G[A - z, B - w_0 - w_1 - z]| = |X'|$. So

$$\rho_R(A \cup (B - w_0 - w_1 - z)) \leq \rho_R(A) + \rho_R(B - w_0 - w_1 - z) - 9(|X'| + |E_G[\{z\} \cap A', B - w_0 - w_1 - z]|). \quad (2.9)$$

Because $x_1 \in X'$, if $x_2 \notin X'$, then $|E_{G'}[z, B-w_0-w_1-z]| = |E_G[\{z\} \cap A', B-w_0-w_1-z]|+|X'|-1$, and so the last expression in (2.9) is at most

$$4 + \rho_{R'}(B - z - w_0 - w_1) - 9(|E_{G'}[z, B - w_0 - w_1 - z]| + 1) = 4 + \rho_{R'}(B - w_0 - w_1) - 9$$

$$\leq 4 + (\rho_{R'}(B) - 2 \cdot 11 + 3 \cdot 9) - 9 = \rho_{R'}(B) \leq -5,$$

a contradiction.

If $x_2 \in B$, then on the one hand, G' has an extra edge x_1x_2, and on the other hand,

$$|E_{G'}[z, B - w_0 - w_1 - z]| = |E_G[\{z\} \cap A, B - w_0 - w_1 - z]| + |X'| - 2,$$

since $x_2 \in X'$. These edges cancel in the calculations, and we again get $\rho_R(A \cup (B - w_0 - w_1 - z)) \leq -5$, a contradiction.

To prove the "Moreover" part of the lemma, observe first that if $z \notin A$ and $1 \leq |A| \leq 3$, then $\rho(A) \geq 6$. Also, if $G[A]$ is disconnected and $1 \leq |A| \leq 3$, then $|E(G[A])| \leq |A| - 2$, and thus $\rho(A) \geq 11(|A| - 1) - 9(|A| - 2) = 2|A| + 7 \geq 9$. □

2.1.5 Shovels in G

Claim 2.1.21. *If G has a subgraph $P_4 = x_1x_2x_3x_4$ with $d(x_2) = d(x_3) = 2$, then $z \in \{x_1, x_2, x_3, x_4\}$.*

Proof. Assume $z \notin \{x_1, x_2, x_3, x_4\}$. By Claim 2.1.5, $d(x_4) \geq 2$. If $d(x_4) = 2$ and x_5 is the neighbor of x_4 distinct from x_3, let $G_0 = G - x_2 - x_3 - x_4$. By the choice of G, G_0 has a $(0, 1)$-coloring ϕ. We can extend it to G by letting $\phi(x_2) = 1 - \phi(x_1)$, $\phi(x_4) = 1 - \phi(x_5)$, and letting $\phi(x_3) = 0$ if $\phi(x_2) = \phi(x_4) = 1$ and $\phi(x_3) = 1$ otherwise. Since this contradicts the definition of G, $d(x_4) \geq 3$.

Let $G' = G - x_3x_4 + x_3z$ and $R' = (G', Z)$. Then by our definition, R' is smaller than R, since $|V(G)| = |V(G')|$ and $d_{G'}(z) > d_G(z)$. If G' has a triangle T, then $x_3z \in E(T)$. Since $N_{G'}(x_3) = \{x_2, z\}$, we need $V(T) = \{x_2, x_3, z\}$, but $x_1 \neq z$ by assumption. So G' is triangle-free.

Suppose G' has a $(0, 1)$-coloring ϕ'. If $\phi'(x_4) = 1$, then ϕ' is a $(0, 1)$-coloring of G, a contradiction. If $\phi'(x_4) = 0$, then recolor x_2 with $1 - \phi'(x_1)$ and x_3 with 1. Again, we get a $(0, 1)$-coloring of G.

So, G' has no $(0,1)$-coloring. Then by the minimality of G there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. By the choice, $z \in B$. If $x_3 \notin B$ or $x_4 \in B$, then $|E(G'[B])| \leq |E(G[B])|$ and $\rho_R(B) \leq -5$. Let $x_3 \in B$ and $x_4 \notin B$. Then by the minimality of $\rho_{R'}(B)$, $x_2 \in B$ and hence $x_1 \in B$. So by definition, $\rho_R(B) = \rho_{R'}(B) + 9 \leq 4$. Since $\{x_1, x_2, x_3, z\} \subseteq B$, by Lemma 2.1.14, $|B| \geq n - 3$. By the minimality of $\rho_{R'}(B)$, x_4 has at most one neighbor in $B - x_3$. So since $d(x_4) \geq 3$, x_4 has a neighbor y in $V(G) - B$ and $|B| \leq n - 2$. If $|B| = n - 2$ then by Claim 2.1.5, each of y and x_4 has a neighbor in B, and so $\rho_{R'}(B + y + x_4) \leq \rho_{R'}(B) + 2 \cdot 11 - 3 \cdot 9 < \rho_{R'}(B)$, a contradiction. So, we may assume that $V(G') - B = \{x_4, y, x\}$. Since G' has no triangles, by Claim 2.1.5, $\rho_{R'}(V(G')) \leq \rho_{R'}(B) + 3 \cdot 11 - 4 \cdot 9 < \rho_{R'}(B)$, a contradiction to the choice of B. □

Claim 2.1.22. *Suppose G contains a path $P = (x_1, x_2, x_3, z)$ such that $d(x_2) = d(x_3) = 2$. Then either $d(x_1) \geq 4$ or $d(x_1) = 3$ and x_1 belongs to a 5-cycle not containing x_2.*

Proof. By Claim 2.1.5, $d(x_1) \geq 2$. Suppose $d(x_1) = 2$ and $x_0 \neq x_2$ is a neighbor of x_1. If $x_0 = z$, then $\rho(\{x_1, x_2, x_3, z\}) = 3 \cdot 11 - 4 \cdot 9 = -3$, a contradiction to Lemma 2.1.6. So $x_0 \neq z$. This contradicts Claim 2.1.21 for the path (x_0, x_1, x_2, x_3). So $d(x_1) \geq 3$. If $d(x_1) = 3$, let x', x'' be the two neighbors of x_1 distinct from x_2.

Consider the graph G' with $V(G') = V(G) - x_1 - x_2 - x_3 - x' - x'' + x$ and

$$E(G') = E(G[V(G) - x_1 - x_2 - x_3 - x' - x'']) \cup \{xy : y \text{ is adjacent to } x' \text{ or } x'' \text{ in } G.\}$$

Let $R' = (G', Z)$. If x_1 does not belong to a 5-cycle not containing x_2, then G' is triangle-free. If G' has a $(0,1)$-coloring ϕ, then we extend it to G by letting $\phi(x') = \phi(x'') = \phi(x)$, $\phi(x_1) = 1 - \phi(x)$, $\phi(x_2) = 1$, and $\phi(x_3) = 0$. So, G' has no $(0,1)$-coloring. Then by the minimality of G there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size.

If $x \notin B$, then $\rho_R(B) = \rho_{R'}(B) \leq -5$. If $x \in B$, then $\rho_R(B - x + x' + x'' + x_1) = \rho_{R'}(B) + 2 \cdot 11 - 2 \cdot 9 \leq -5 + 4 = -1$, contradicting Lemma 2.1.6, since $x_2 \notin B - x + x' + x'' + x_1$. □

We need several new definitions. If G contains vertices x_1, x_2, x_3, z and a 5-cycle C_5 containing

x_1 with $d(x_1) = 3$ as in Claim 2.1.21, we call the subgraph induced by $V(C_5) \cup \{x_1, x_2, x_3, z\}$ an *intact shovel*. The path x_1, x_2, x_3, z is the *handle*, the cycle C_5 is the *head* and x_1 is the *joint* of this shovel. By definition, vertices x_1, x_2, x_3 in the shovel are not adjacent to any vertex outside the shovel.

A path x_1, x_2, x_3, z in G such that x_1 has degree $t \geq 4$ will be called a *broken shovel of degree* t. Below when we speak of shovels, we have in mind both, broken and intact shovels.

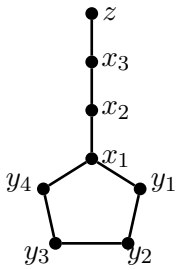


Figure 2.6 : An intact shovel

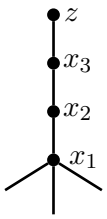


Figure 2.7 : A broken shovel of degree 4

Claim 2.1.23. *For every shovel S, $G[V(S)] = S$.*

Proof. Suppose G has an edge e not in S that connects two vertices in $V(S)$. If S is broken, then $\rho(V(S)) \leq 3 \cdot 11 - 4 \cdot 9 = -3$, and if S is intact, then $\rho(V(S)) \leq 7 \cdot 11 - 9 \cdot 9 = -4$. In both cases, we contradict Lemma 2.1.6, unless $V(S) = V(G)$ and $E(G) = E(S) + e$. Suppose the latter holds and the handle of S is x_1, x_2, x_3, z. If S is broken, then $d(v_1) = 2$ since $V(S) = V(G)$, a contradiction. If S is intact and its head is $C_5 = (x_1, y_1, y_2, y_3, y_4)$, then since G has no triangles, at least one end of e is outside of C_5 and so should be z. Let the other end be v. By degree restrictions on S, $v \notin \{x_2, x_1\}$. If $v \in \{y_1, y_4\}$, then it belongs to a 5-cycle C containing z and $\rho(V(C)) = 4 \cdot 11 - 5 \cdot 9 = -1$, a contradiction to Lemma 2.1.6. Otherwise, we may assume $v = y_2$. In this case, we color x_3, x_1 and y_2 with 0, and the rest with 1. □

Claim 2.1.24. *For any distinct shovels S and S', $V(S) \cap V(S') = \{z\}$.*

Proof. Let $S' \neq S$. Let the handle of S be x_1, x_2, x_3, z and the handle of S' be x'_1, x'_2, x'_3, z. Suppose $V(S) \cap V(S') = S_0 \neq \{z\}$.

CASE 1: $x_3 \in S_0$. Since $z \in S'$, by Claim 2.1.23, $x_3 = x'_3$. This yields that $x_2 = x'_2$ and $x_1 = x'_1$. Then since $S \neq S'$, we may assume that S is intact, and hence $d(x_1) = 3$. So S' also is

intact. Let the head of S be $(x_1, y_1, \ldots, y_4)$ and the head of S' be $(x_1, y'_1, \ldots, y'_4)$. Since $d(x_1) = 3$, we have $y'_1 = y_1$ and $y'_4 = y_4$. If $\{y_2, y_3\} \cap \{y'_2, y'_3\} = \emptyset$, then $\rho(V(S) \cup V(S')) \leq 9 \cdot 11 - 11 \cdot 9 = 0$. So by Lemma 2.1.6, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. This contradicts Claim 2.1.21 for the path y_1, y_2, y_3, y_4. Finally, if $y'_2 = y_2$, then $\rho(V(S) \cup V(S')) \leq 8 \cdot 11 - 10 \cdot 9 = -2$. Again, by Lemma 2.1.6, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. Then we color z, x_2, y_1, y_2, y_4 with 1 and the rest with 0.

CASE 2: $x_3 \notin S_0$. By symmetry, we may assume $x'_3 \notin S_0$. Then also $x_2, x'_2 \notin S_0$.

CASE 2.1: $x_1 \in S_0$. If S' is broken, this means that $x_1 = x'_1$ and $\rho(V(S') + x_2 + x_3) = 5 \cdot 11 - 6 \cdot 9 = 1$. Let $U = V(G) - S' - x_2 - x_3$. By Lemma 2.1.9, $|U| \leq 3$. If there is a vertex $u \in Y$ which has at least 2 neighbors in $S' + x_2 + x_3$, then $\rho(S' + x_2 + x_3 + u) \leq -6$, a contradiction. As $d(x_1) \geq 4$, x_1 has at least two neighbors in U. As G has no vertex of degree 1, we conclude that $U + x$ induces a 4-cycle (x, u_1, u_2, u_3). Also $d(u_2) = 2$, otherwise we get a contradiction by $\rho(V(G)) \leq 8 \cdot 11 - 11 \cdot 9 < -4$. Thus G is bipartite, and hence has a proper $(0, 1)$-coloring.

So, S' is intact. Let the head of S' be $(x'_1, y'_1, \ldots, y'_4)$. Then $\rho(V(S') + x_2 + x_3) \leq \rho(V(S')) + 2 \cdot 11 - 3 \cdot 9 = 0$, and by Lemma 2.1.6, $V(G) = V(S') + x_2 + x_3$ and $E(G) = E(S') \cup \{zx_3, x_3x_2, x_2x_1\}$. Since $d(x'_1) = 3$, by symmetry, we may assume that $x_1 \in \{y'_1, y'_2\}$. In both cases, we color x_3, x'_3, x'_1, y'_2 with 0 and the rest with 1.

CASE 2.2: $x_1 \notin S_0$. By symmetry, we may assume $x'_1 \notin S_0$. In particular, both S and S' are intact. Since by Claim 2.1.23, $G[S - x_1]$ is acyclic, $G[S_0]$ is acyclic. Also, z is isolated in $G[S_0]$ and thus $\rho(S_0) \geq 11|S_0 - z| - 9(|S_0 - z| - 1) = 9 + 2|S_0 - z|$. So,

$$\rho(V(S) \cup V(S')) \leq \rho(S) + \rho(S') - \rho(S_0) \leq 5 + 5 - 9 - 2|S_0 - z| = 1 - 2|S_0 - z|. \tag{2.10}$$

Hence $|S_0 - z| \leq 2$ and by Lemma 2.1.6, $V(G) = V(S) \cup V(S')$ and $E(G) = E(S) \cup E(S')$. By Claim 2.1.21, each of the sets $\{y'_1, y'_2\}$ and $\{y'_3, y'_4\}$ has a vertex common with S. Thus $|S_0 - z| = 2$. Also, if $G[S_0 - z]$ has no edge, then we would have an extra -9 in (2.10), a contradiction. So $S_0 - z = \{y'_2, y'_3\} = \{y_2, y_3\}$. Then we can color $x_3, x'_3, y_1, y_4, y'_1, y'_4$ with color 0 and the rest with 1. □

Claim 2.1.25. *Let S be an intact shovel with the handle (z, x_3, x_2, x_1) and the head $C = (x_1, y_1, y_2, y_3, y_4)$. Then at least 3 edges connect C with $V(G - S)$.*

Proof. Let E' be the set of edges connecting $V(C)$ with $V(G - S)$. Suppose to the contrary that $|E'| \leq 2$. Then since $z \notin V(C)$, by Claim 2.1.21, $|E'| = 2$ and the ends in C of the edges in E' are distinct. So we can suppose $E' = \{y_i u, y_j v\}$, where $u, v \in V(G - S)$.

By Claim 2.1.21 and symmetry, we may assume that $j = 3$ and either $i = 1$ or $i = 2$. Let G' be obtained from $G - (V(S) - z)$ by adding a vertex z' and edge uz'. Let $R' = (G', \{z, z'\})$. Since edge uz' is a cut edge, G' is a triangle-free graph smaller than G. If G' has no $(0, 1)$-coloring, then by the minimality of G, there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. By the choice of G, $u, z' \in B$. Then the set $B - z'$ contains u, is disjoint from $S - z$ and $\rho_R(B - z') \leq -5 + 9 = 4$. So denoting $B' = (B - z') \cup V(S)$ and taking into account edge $y_i u$, we have

$$\rho_R(B') \leq \rho_R(B - z') + \rho_R(S) - 9 = 0. \tag{2.11}$$

By Lemma 2.1.6 and (2.11), $B' = V(G)$. But then $v \in B - z'$, and edge $y_j v$ was not count in (2.11). So instead of (2.11), we have $\rho_R(B') \leq \rho_R(B - z') + \rho_R(S) - 18 = -9$, a contradiction.

Thus G' has a $(0, 1)$-coloring ϕ. In particular, $\phi(u) = 0$. We extend ϕ from $G' - z'$ to the whole G as follows. First, we let $\phi(x_3) = 0$, $\phi(x_2) = 1$ and $\phi(y_3) = 1 - \phi(v)$.

CASE 1: $u = v$. Since G has no triangles, $i = 1$. So we extend ϕ to G by letting $\phi(y_4) = \phi(y_1) = 1$ and $\phi(x_1) = \phi(y_2) = 0$.

CASE 2: $u \neq v$ and $i = 1$. Let $\phi(x_1) = 0$, $\phi(y_4) = \phi(y_1) = 1$ and $\phi(y_2) = \phi(v)$.

CASE 3: $u \neq v$ and $i = 2$. If $\phi(y_3) = 1$, then let $\phi(x_1) = \phi(y_2) = 1$, and $\phi(y_1) = \phi(y_4) = 0$, otherwise let $\phi(x_1) = 0$, and $\phi(y_1) = \phi(y_4) = \phi(y_2) = 1$. □

2.1.6 On $(1, 1, 1)$-trees in G

The goal of this section is to prove that all $(1, 1, 1)$-trees in G are small: each of them has at most two $(1, 1, 1)$-vertices.

Claim 2.1.26. *For each $(1,1,1)$-tree T, if a vertex $x \in V(G) - V(T)$ is adjacent to more than one vertex in T, then $zx \in E(G)$.*

Proof. Suppose some $x \in V(G) - V(T)$ is adjacent to distinct $y, y' \in V(T)$. By the definition of $(1,1,1)$-trees, $|V(T)| \geq 4$ and $z \notin V(T)$. If $x = z$, then let $G_0 = G - V(T)$ and $R_0 = (G_0, Z)$. Since $G_0 \subset G$, it has a $(0,1)$-coloring ϕ_0, and $\phi_0(z) = 1$. Then by Lemma 2.1.8, we can extend ϕ_0 to G, a contradiction.

Assume $x \neq z$. Let G' be obtained from $G - V(T)$ by adding vertices w and z' and edges xw and wz'. Let $R' = (G', \{z, z'\})$. Since new edges xw and wz' are cut edges, G' is a triangle-free graph. Since $|V(T)| \geq 4$, $|V(G')| < n$. If G' has a $(0,1)$-coloring ϕ, then $\phi(w) = 0$ and hence $\phi(x) = 1$. So by Lemma 2.1.8, we can extend ϕ to G, a contradiction. Then by the minimality of G, there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Choose such B with the smallest potential and among those—with the largest size. Then $z' \in B$. If $x \notin B$, then $w \notin B$ and $\rho(B - z') = \rho_{R'}(B) \leq -5$, a contradiction. So, $x \in B$ and thus $w \in B$. Then $\rho(B - z' - w) \leq \rho_{R'}(B) - 11 + 18 \leq 2$. Since $|B| \leq n - |V(T)| \leq n - 4$ and $x \neq z$, by Lemma 2.1.9, $B - z' - w = \{x, z\}$ and $xz \in E(G)$, as claimed. □

Lemma 2.1.27. *If a graph G' is obtained from G by replacing a $(1,1,1)-$tree T with another $(1,1,1)$-tree T' with the same set of vertices and the same set of leaves, then the pair $R' = (G', Z)$ is still a minimum counterexample to the theorem.*

Proof. Assume w is a common neighbor of two $(1,1,1)$-vertices v_1 and v_2 in a $(1,1,1)-$tree T. For $i = 1, 2$, let w_i and w_i' be the neighbors of v_i distinct from w and u_i be the neighbor of w_i distinct from v_i. By the definition of $(1,1,1)$-vertices, $d(w) = d(w_1) = d(w_2) = d(w_1') = d(w_2') = 2$. By Claim 2.1.7, $w_2 \neq w_1$. If $u_1 = u_2$, then $u_1 \notin T$ and by Claim 2.1.26, $zu_1 \in E(G)$. Then for $W = \{z, u_1, w_1, w_2, v_1, v_2, w\}$, we have $\rho(W) \leq 6 \cdot 11 - 7 \cdot 9 = 3$. So by Lemma 2.1.14, $n \leq 10$. If w_1' has a neighbor in $W - v_1$, then $\rho(W + w_1') \leq \rho(W) + 11 - 18 \leq -4$, contradicting Lemma 2.1.6, since $w_2' \notin W + w_1'$. Similarly, w_2' has no neighbor in $W - v_2$. Thus, since $n \leq 10$, there is a unique $y \in V(G) - W - w_1' - w_2'$, and $yw_1', yw_2' \in E(G)$. By Claim 2.1.21, y has a neighbor in W. So, $\rho(V(G)) \leq \rho(W) + 3 \cdot 11 - 5 \cdot 9 \leq -9$, a contradiction. Thus $u_1 \neq u_2$. Also, since v_1 and v_2 are adjacent only to 2-vertices in T, and T has no cycles, $u_1v_2, u_2v_1 \notin E(G)$.

Let G' be obtained from G by deleting edges w_1u_1, w_2u_2 and adding edges w_1u_2, w_2u_1. Let $R' = (G', Z)$. By construction, $|V(G')| = n$ and degrees of all vertices in G' are the same as in G. Since $u_1v_2, u_2v_1 \notin E(G)$, G' has no triangles. If $\rho_{R'}(B) \leq -4$ for all $B \subseteq V(G')$, then the lemma holds for the $(1,1,1)$-tree T' obtained from T by deleting edges w_1u_1, w_2u_2 and adding edges w_1u_2, w_2u_1. So consider $B \subset V(G')$ with the smallest potential in R' and assume $\rho_{R'}(B) \leq -5$. Let $U = \{u_1, u_2, w_1, w_2\}$. If $|B \cap U| \leq 1$ or $|B \cap U| \geq 3$, then $\rho_R(B) = \rho_{R'}(B) \leq -5$, a contradiction. Furthermore, if $\rho_{R'}(B) < \rho_R(B)$, then either $B \cap U = \{w_1, u_2\}$ or $B \cap U = \{w_2, u_1\}$. By symmetry, let $B \cap U = \{w_1, u_2\}$. Then $\rho_R(B) = \rho_{R'}(B) + 9 \leq 4$ and by Lemma 2.1.14, either $|B| \leq 3$ or $|B| \geq n-3$. Since $N_{G'}(w_1) = \{v_1, u_2\}$, by the minimality of $\rho_{R'}(B)$, $v_1 \in B$. For the same reason at least one of w, w_1' is in B. So $|B| \geq 4$ and thus $|B| \geq n-3$. If $v_2 \notin B$, then $w \notin B$, a contradiction to $|B| \geq n-3$, since the case is that $u_1, w_2 \notin B$. Thus $v_2 \in B$. Then $\rho(B+w_2) \leq \rho(B)+11-18 \leq -3$, a contradiction to Lemma 2.1.6, since $u_1 \notin B + w_2$. This proves the lemma for our special T'.

By a sequence of switching pairs of edges as above, we can change T into any $(1,1,1)$-tree with the same set of vertices and the same set of leaves. □

Claim 2.1.28. *For every $(1,1,1)$-tree T, each vertex $x \in V(G) - V(T)$ is adjacent to at most one vertex in T.*

Proof. Suppose that T is a $(1,1,1)$-tree in G and $x \in V(G)-V(T)$ is adjacent to y_1 and y_2 in T. By Claim 2.1.26, $zx \in E(G)$. By Lemma 2.1.27, if we replace T with a $(1,1,1)$-tree T' with the same set of vertices and the same set of leaves such that y_1 and y_2 have a common neighbor v in T', then the pair $R' = (G', Z)$ also is a minimum counterexample to the theorem. But $\rho_{R'}(\{z, x, y_1, y_2, v\}) = -1$, a contradiction to Lemma 2.1.6. □

Recall that for a $(1,1,1)$-tree T, $||V(T)||$ is its size, i.e., the number of $(1,1,1)$-vertices in T. Let $L(T)$ denote the set of leaves of T. Then

$$|V(T)| = 3||V(T)|| + 1 \text{ and } |L(T)| = |N(T) - V(T)| = ||V(T)|| + 2 \text{ for each } (1,1,1)\text{-tree } T. \quad (2.12)$$

Claim 2.1.29. *If x is adjacent to T and $xz \in E(G)$, then $||V(T)|| = 1$.*

Proof. Suppose $||V(T)|| \geq 2$ and $x \in V(G) - V(T)$ is adjacent to z and to $y \in V(T)$. Then $d(y) = 2$ and the neighbor of y in T is some $(1,1,1)$-vertex v. By Lemma 2.1.27, we can assume that $N(v) = \{y, y', y''\}$, the other neighbor x' of y' is in $V(G) - V(T)$ and the other neighbor x'' of y'' is in T. Let G' be obtained from $G - \{y, v, y'\}$ by adding edge $y''x'$ and T' be the tree induced in G' by $V(T) - \{y, v, y'\}$. Let $R' = (G', Z)$. If R' has a $(0,1)$-coloring ϕ, then $\phi(x) = 0$ and the number of vertices of color 1 in $N_G(T) - V(T) - x$ is not 1. But then by Lemma 2.1.8, we can extend the coloring of $G' - V(T')$ to a $(0,1)$-coloring of G.

So, by the minimality of R, there exists $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Since $\rho_R(B) \geq -4$, $y'', x' \in B$. Then $\rho_R(B + v + y') = \rho_{R'}(B) + 9 + 2(11) - 3(9) \leq -5 + 4 = -1$. Since $y \notin B + v + y'$, this contradicts Lemma 2.1.6. □

Claim 2.1.30. *Suppose T is a $(1,1,1)$-tree, $V' = \{x_1, \ldots, x_t\} = N(T) - V(T)$ and $t \geq 5$. Then*

(a) Each $x \in V'$ is at distance at least 3 from z;

(b) V' is an independent set;

(c) Each $w \in V(G)$ is adjacent to at most 2 vertices in V';

(d) Each $x \in V'$ is at distance 2 in $G - V(T)$ from at most one $x' \in V'$.

Proof. Since $t \geq 5$, by (2.12), $||V(T)|| \geq 3$ and so $|V(T)| \geq 10$.

(a) Since $||V(T)|| \geq 3$, by Claim 2.1.29, $z \notin N(V')$. So, if (a) does not hold, then there is $x \in V'$ such that either $x = z$ or $dist(x, z) = 2$. In both cases there is $x' \in V' - x$ at distance at least two from z. Let G' be obtained from $G - V(T)$ by adding a new vertex w adjacent to x' and z. Let $R' = (G', Z)$. If R' has a $(0,1)$-coloring ϕ, then $\phi(x) = \phi(x') = 1$. Thus by Lemma 2.1.8, we can extend ϕ to $V(T)$ to obtain a $(0,1)$-coloring of R. So R' has no $(0,1)$-coloring, and is smaller than R, since $|V(T)| \geq 10$. Then by the minimality of G there is a set $B \subset V(G')$ with $\rho_{G'}(B) \leq -5$. In order to have $\rho_{G'}(B) \leq -5$, we need $w \in B$ and both its neighbors, z and x', be in B. Then $\rho_R(B - w) \leq -5 + 2 \cdot 9 - 11 \leq 2$. Hence by Lemma 2.1.9, either $B - w = \{z, x'\}$ or $|B - w| \geq n - 3$. The former is impossible, since $zx' \notin E(G)$ and the latter does not hold, since $|V(T)| \geq 10$. This proves (a).

(b) Assume $x, x', x'' \in V'$ and $xx'' \in E(G)$. As in the proof of (a), let G' be obtained from $G - V(T)$ by adding a new vertex w adjacent to x' and z. Let $R' = (G', Z)$. If R' has a $(0,1)$-

coloring ϕ, then $\phi(x') = 1$ and either $\phi(x) = 1$ or $\phi(x'') = 1$. So by Lemma 2.1.8, we can extend ϕ to $V(T)$. Then we repeat the rest of the argument of the proof of (a) word by word and come to a contradiction.

(c) Assume that some $x_1, x_2, x_3 \in V'$ are adjacent to the same vertex w. By Lemma 2.1.27, we may assume that x_1, x_2, x_3 and some 6 vertices in T form a tree T' with 9 vertices and 8 edges. If there is a path (w, w', z) from w to z, then $\rho_R(\{z, w', w\} \cup V(T')) = 11 \cdot 11 - 9 \cdot 13 = 4$. However, this set has more than 3 and fewer than $|V(G)| - 3$ vertices. Thus we get a contradiction to Lemma 2.1.14. Together with (a) this yields that

$$\text{the distance from } w \text{ to } z \text{ is at least 3.} \tag{2.13}$$

Consider G' obtained from $G - V(T)$ by adding an edge wz. Let $R' = (G', Z)$. By (2.13), G' is triangle-free. If R' has a $(0,1)$-coloring ϕ, then $\phi(w) = 0$ and hence $\phi(x_1) = \phi(x_2) = 1$. Thus by Lemma 2.1.8, ϕ can be extended to a $(0,1)$-coloring for G, a contradiction. Since G' is smaller than G, there is a set $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Then $w, z \in B$, so $\rho_G(B) = -5 + 9 = 4$. By Lemma 2.1.14, either $|B| \leq 3$ or $|B| \geq n - 3$. However, B misses at least 7 vertices in T. So, $|B| \leq 3$ and $w, z \in B$. By (2.13), $G[B]$ is disconnected. This contradicts the last statement of Lemma 2.1.14.

(d) Assume that $x_1, x_2, x_3 \in V'$ are such that x_3 is at distance two from x_1 and from x_2. Then by (c) there are distinct u_1, u_2 such that for $i = 1, 2$, u_i is the common neighbor of x_3 and x_i. Consider the graph G' obtained from $G - V(T)$ by adding a new vertex w and edges zw and wx_3. Let $R' = (G', Z)$. Again, if R' has a $(0,1)$-coloring ϕ, then $\phi(x_3) = 1$ and hence one of u_1, u_2, say u_1 has color 0. But then $\phi(x_1) = 1$, and by Lemma 2.1.8, ϕ extends to a $(0,1)$-coloring for G. So again, there is $B \subset V(G')$ with $\rho_{R'}(B) \leq -5$. Then $w \in B$ and thus $z, x_3 \in B$. Therefore, $\rho_R(B - w) \leq -5 + 2(9) - 11 \leq 2$. By Lemma 2.1.9, either $B = \{z, x_3\}$ or $|B| \geq n - 3$. But by Claim 2.1.29, $x_3 z \notin E(G)$, and at least 7 vertices of T are not in B. This contradiction proves (d). □

We need a bit more notation. For a pair $R' = (G', Z')$ and a set $S \subseteq V(G')$, let $m_{R'}(S)$ be the minimal potential that may have subsets of $V(G')$ containing S. If we have two sets B_1 and B_2 containing S with $\rho_{R'}(B_1) = \rho_{R'}(B_2) = m_{R'}(S)$, then by the submodularity of $\rho_{R'}$, also

$\rho_{R'}(B_1 \cap B_2) = \rho_{R'}(B_1 \cup B_2) = m_{R'}(S)$. Thus the family

$$\mathcal{M}_{R'}(S) = \{B \subseteq V(G') \,:\, B \supseteq S,\, \rho_{R'}(B) = m_{R'}(S)\}$$

contains a unique inclusion minimal member $M_{R'}(S)$ contained in every other member of $\mathcal{M}_{R'}(S)$.

Claim 2.1.31. *Let T be a $(1,1,1)$-tree with $||V(T)|| \geq 3$. Let $V' = N(T) - V(T) = \{x_1, \ldots, x_t\}$. Let $G' = G - V(T)$ and $R' = (G', Z)$. Then for any distinct $x_1, x_2, x_3 \in V'$, (a) $m_{R'}(\{x_1\}) = 5$, $m_{R'}(\{x_1, x_2\}) = 8$, $m_{R'}(\{x_1, x_2, x_3\}) = 11$, and (b) $M_{R'}(\{x_1, x_2\}) \subset M_{R'}(\{x_1, x_2, x_3\})$.*

Proof. Order the vertices of V' as needed: $x_1, \ldots, x_t$. For $i = 1, \ldots, t$, let y_i be the neighbor of x_i in T and v_i be the neighbor of y_i in T. By Claim 2.1.30(b,d) and the symmetry between x_2, x_3, we may assume that the distance between x_1 and x_2 in G' is at least 3. By Lemma 2.1.27, we may assume that $v_2 = v_1$, $v_{t-1} = v_t$ and that for $j = 2, \ldots, t-2$, vertices v_j and v_{j+1} have a common neighbor u_j in T. First, we prove a half of (a):

$$\text{for any } 1 \leq j \leq 3, \quad m_{R'}(\{x_1, \ldots, x_j\}) \geq 3j + 2. \tag{2.14}$$

Indeed, suppose that $m_{R'}(\{x_1\}) \leq 4$ and $B = M_{R'}(\{x_1\})$. By Lemma 2.1.14, either $|B| \geq n - 3$ or $|B| \leq 3$. The former cannot hold, since $|V(T)| \geq 10$, so $|B| \leq 3$. Then again by Lemma 2.1.14, $z \in B$ and $G[B]$ is connected. This contradicts Claim 2.1.30(a). So Case $j = 1$ of (2.14) is proved. Suppose now that $m_{R'}(\{x_1, x_2, x_3\}) \leq 10$ and $B_1 = M_{R'}(\{x_1, x_2, x_3\})$. Let $B' = B_1 \cup \{y_1, y_2, y_3, v_2, u_2, v_3\}$. Then

$$\rho_R(B') \leq \rho_R(B_1) + 11(6) - 9(8) = 10 + 66 - 72 = 4. \tag{2.15}$$

As above, by Lemma 2.1.14, since $|B'| > 4$, we have $|B'| \geq n - 3$. Since $t \geq 5$, $u_4, u_3, v_4, y_4, y_5 \notin B'$, implying $|B'| < n - 3$, a contradiction. This proves Case $j = 3$ of (2.14). Case $j = 2$ is very similar but simpler, so we leave it for the reader.

Let G'' be obtained from G' by adding a vertex w and edges x_1x_2, x_3w, and wz. Let $R'' = (G'', Z)$. By Claim 2.1.30 and the choice of x_2 and x_1, G'' has no triangles. If R'' has a $(0,1)$-

coloring ϕ, then $\phi(x_3) = 1$ and at least one of x_1, x_2 also has color 1. So by Lemma 2.1.8, ϕ can be extended to a $(0,1)$-coloring of R. Thus by the minimality of G, there is a set $B \subset V(G'')$ with $\rho_{R''}(B) \leq -5$. By the choice of G, B contains $\{x_1, x_2\}$ or $\{w, x_3\}$, or both. If B does not contain w, then it contains $\{x_1, x_2\}$ and $\rho_R(B) \leq \rho_{R''}(B) + 9 \leq 4$. In this case, by Lemma 2.1.14 either $|B| \leq 3$ or $|B| \geq n - 3$. The latter cannot hold, since $|V(T)| \geq 10$, so $|B| \leq 3$. Then by the last part of Lemma 2.1.14, $z \in B$ and $G[\{x_1, x_2, z\}]$ is connected. This contradicts Claim 2.1.30(a). Thus $w \in B$ and hence $\{w, x_3, z\} \subseteq B$. Let $B_1 = B - w$.

If B does not contain $\{x_1, x_2\}$, then $\rho_R(B_1) \leq \rho_{R''}(B) + 7 \leq 2$. In this case, by Lemma 2.1.9 either $B_1 = \{x_3, z\}$ and $zx_3 \in E(G)$ or $|B_1| \geq n - 3$. The former contradicts Claim 2.1.30(a) and the latter cannot hold, since $|V(T)| \geq 10$. Thus

$$B \supseteq \{x_1, x_2, x_3, w, z\} \quad \text{and} \quad \rho_R(B_1) \leq -5 + 9 + 7 = 11. \tag{2.16}$$

Together with (2.14), this yields

$$\rho_R(B_1) = m_{R'}(\{x_1, x_2, x_3\}) = 11. \tag{2.17}$$

Let $M_1 = M_{R'}(\{x_1, x_2, x_3\})$. By the comment above the claim, $M_1 \subseteq B_1$. We now want to prove

$$M_1 \cap V' = \{x_1, x_2, x_3\}. \tag{2.18}$$

Indeed, suppose that $|V' \cap M_1| \geq 4$, say $\{x_1, \ldots, x_4\} \subseteq M_1$. Let $B' = M_1 \cup \{y_1, \ldots, y_4, v_2, u_2, v_3, u_3, v_4\}$. Then

$$\rho_R(B') \leq \rho_R(M_1) + 11(9) - 9(12) = 11 + 99 - 108 = 2. \tag{2.19}$$

By Lemma 2.1.9, since $|B'| > 4$, we have $|B'| \geq n - 3$. If $t \geq 6$, then $y_5, u_5, v_5, y_6 \notin B'$, a contradiction. So, $t = 5$ and $y_5 \notin B'$. If $x_5 \in B'$, then $\rho_R(B' + y_5) \leq \rho(B') + 11 - 9(2) \leq -5$, a contradiction. Now, let $t = 5$ and $x_5 \notin B'$. Recall that $d(x_5) \geq 3$. If $V(G) - B' = \{x_5, y_5\}$, then by (2.19),

$$\rho_R(V(G)) \leq \rho_R(B') + 11(2) - 9(4) \leq 2 + 22 - 36 = -12,$$

a contradiction to the choice of G. Similarly, if $V(G) - B' = \{x_5, y_5, u'\}$ for some u', then since G has no triangles, at least 5 edges are incident with x_5, y_5 or u'. In this case, $\rho_R(V(G)) \leq \rho_R(B') + 11(3) - 9(5) \leq 2 + 33 - 45 = -10$, a contradiction again. This proves (2.18).

Similarly to M_1, let $M_2 = M_{R'}(\{x_1, x_2, x_4\})$. Next, we prove

$$\rho_R(M_1 \cup M_2) \geq 14. \tag{2.20}$$

Indeed, if $\rho_R(M_1 \cup M_2) \leq 13$ then similarly to the preceding paragraph, we let $B'' = M_1 \cup M_2 \cup \{y_1, \ldots, y_4, v_2, u_2, v_3, u_3, v_4\}$ and observe that $\rho_R(B'') \leq \rho_R(M_1 \cup M_2) + 11(9) - 9(12) = 13 + 99 - 108 = 4$. By Lemma 2.1.14, since $|B''| > 4$, we have $|B''| \geq n - 3$. Now we proceed as in the end of the preceding paragraph, excluding the case $x_5 \in B'$, since $x_5 \notin M_1 \cup M_2$ by (2.18). Thus (2.20) holds.

By (2.2) and (2.20), $\rho_R(M_1 \cap M_2) \leq \rho_R(M_1) + \rho_R(M_2) - \rho_R(M_1 \cup M_2) \leq 11 + 11 - 14 = 8$. Together with (2.14), this proves $m_{R'}(\{x_1, x_2\}) = 8$ and part (b) of the claim. Moreover, since $M_1 \cap M_2 \cap V' = \{x_1, x_2\}$, we conclude

$$M_{R'}(\{x_1, x_2\}) \cap V' = \{x_1, x_2\}. \tag{2.21}$$

Now, let $D_1 = M_{R'}(\{x_1, x_2\})$ and $D_2 = M_{R'}(\{x_1, x_3\})$. Again by (2.2) and (2.20),

$$\rho_R(D_1 \cap D_2) \leq \rho_R(D_1) + \rho_R(D_2) - \rho_R(D_1 \cup D_2) \leq 8 + 8 - m_{R'}(\{x_1, x_2, x_3\}) = 16 - 11 = 5.$$

Together with (2.14), this completes the proof of (a) and thus of all claims. □

Now we are ready to prove the main result of the section.

Lemma 2.1.32. *Each $(1,1,1)$-tree T in G has $||V(T)|| \leq 2$.*

Proof. Assume $||V(T)|| \geq 3$. Let $V' = N(T) - V(T) = \{x_1, \ldots, x_t\}$. Since $||V(T)|| \geq 3$, $t \geq 5$. Let $G' = G - V(T)$ and $R' = (G', Z)$. For brevity, we will use M_i for $M_{R'}(\{x_i\})$, $M_{i,j}$ for $M_{R'}(\{x_i, x_j\})$,

and $M_{i,j,k}$ for $M_{R'}(\{x_i, x_j, x_k\})$. First, we prove

$$\text{for any distinct } i,j,k, \qquad M_{i,j} \cup M_{j,k} = M_{i,j,k}. \tag{2.22}$$

Indeed, by Claim 2.1.31(b), $M_{i,j} \cup M_{j,k} \subseteq M_{i,j,k}$. If the containment is strict, then by the minimality of $M_{i,j,k}$, $\rho_{R'}(M_{i,j} \cup M_{j,k}) \geq 12$. Then by Claim 2.1.31(a) and the submodularity of $\rho_{R'}$,

$$8 + 8 = \rho_{R'}(M_{i,j}) + \rho_{R'}(M_{j,k}) \geq \rho_{R'}(M_{i,j} \cup M(j,k)) + \rho_{R'}(M_{i,j} \cap M_{j,k}) \geq 12 + m_{R'}(\{x_j\}) = 17.$$

This contradiction proves (2.22).

For brevity, denote $F_i(j,k) = M_{i,j} \cap M_{i,k}$ and $A(i,j,k) = M_{i,j} \cap M_{i,k} \cap M_{j,k}$. By (2.22) and (2.2), $\rho_{R'}(F_i(j,k)) \leq \rho_{R'}(M_{i,j}) + \rho_{R'}(M_{i,k}) - \rho_{R'}(M_{i,j,k}) = 8+8-11 = 5$. Together with Claim 2.1.31(a), this gives $\rho_{R'}(F_i(j,k)) = 5$. Similarly, $\rho_{R'}(F_j(i,k)) = \rho_{R'}(F_k(j,i)) = 5$. By definition and (2.22),

$$F_i(j,k) \cup F_j(i,k) = M_{i,j} \cap (M_{i,k} \cup M_{j,k}) = M_{i,j} \cap M_{i,j,k} = M_{i,j}.$$

Since $A(i,j,k) = F_i(j,k) \cap F_j(i,k)$, by (2.2),

$$\rho_{R'}(A(i,j,k)) = \rho_{R'}(F_i(j,k)) + \rho_{R'}(F_j(i,k)) - \rho_{R'}(M_{i,j}) - 9|E_{G'}(F_i(j,k) - F_j(i,k), F_j(i,k) - F_i(j,k)|$$

$$= (5+5-8) - 9|E_{G'}(F_i(j,k) - F_j(i,k), F_j(i,k) - F_i(j,k)|.$$

Since $\rho_{R'}(A(i,j,k)) \geq -4$, we conclude that

$$|E_{G'}(F_i(j,k) - F_j(i,k), F_j(i,k) - F_i(j,k)| = 0 \tag{2.23}$$

and $\rho_{R'}(A(i,j,k)) = 2$. Then by Lemma 2.1.9, there is $w = w_{i,j,k} \in V(G)$ such that $A(i,j,k) = \{w, z\}$ and $wz \in E(G)$. We now claim that

$$\text{all } w_{i,j,k} \text{ coincide.} \tag{2.24}$$

Indeed, if (2.24) fails, then there are distinct i, j, k, m such that $w' = w_{i,j,m} \neq w = w_{i,j,k}$. By (2.23), $A(i,j,k)$ and $A(i,j,m)$ are separating sets in $G[M_{i,j}]$. It follows that w and w' are cut vertices in $G[M_{i,j}] - z$, and each of them separates x_j from x_i. By the symmetry between $F_i(j,k)$ and $F_j(i,k)$, we may assume that $w' \in F_j(i,k)$. Then $w \in F_i(j,m)$ and $F_j(i,k) \cup F_i(j,m) = M_{i,j}$. So, by (2.2),

$$\rho_{R'}(F_j(i,k) \cap F_i(j,m)) \leq \rho_{R'}(F_j(i,k)) + \rho_{R'}(F_i(j,m)) - \rho_{R'}(M_{i,j}) = 5 + 5 - 8 = 2.$$

Since $w, w' \in F_j(i,k) \cap F_i(j,m)$, this contradicts Lemma 2.1.9. Thus (2.24) holds. So we denote by w_0 the vertex that is $w_{i,j,k}$ for all i, j, k.

Let $F_1 = F_1(2,3)$, $F_2 = F_2(1,3)$, and for $j = 3, \ldots, t$, let $F_j = F_j(1,2)$. Let $F_0 = \bigcup_{j=1}^{t} F_j$. By (2.24), $F_i \cap F_j = \{w_0, z\}$ for all $1 \leq i < j \leq t$. By (2.23), $E_G(F_i - F_j, F_j - F_i) = \emptyset$ for all $1 \leq i < j \leq t$. Then by induction on j we have $\rho_R(\bigcup_{i=1}^{j} F_i) = 3j + 2$ and thus $\rho_R(F_0) = 3t + 2$. Now we claim

$$F_0 = V(G) - V(T). \tag{2.25}$$

Indeed, by (2.12)

$$\rho_R(F_0 \cup V(T)) = \rho_R(F_0) + 11(3(t-2)+1) - 9(3(t-2)+t) = (3t+2) + (33t-55) - (36t-54) = 1.$$

Let $W = V(G) - V(T) - F_0$. Since $\rho_R(F_0 \cup V(T)) = 1$, by Lemma 2.1.9, $|W| \leq 3$. Recall that $z \in F_0$. If $W = \{v\}$, then by Claim 2.1.5, $d_G(v) \geq 2$, and so $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11 - 9(2) = -6$, a contradiction. If $W = \{v_1, v_2\}$ and at least four edges are incident with v_1 or v_2, then $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11(2) - 9(4) = -13$. If $W = \{v_1, v_2\}$ and at most 3 edges are incident with v_1 or v_2, then $d_G(v_1) = d_G(v_2) = 2$ and $v_1v_2 \in E(G)$. Then by Claim 2.1.21, we may assume that $zv_1 \in E(G)$. Let y be the neighbor of v_2 distinct from v_1. Since $v_2 \notin V'$, $y \in F_0$, and thus there is $j \geq 1$ such that $y \in F_j$. But then $\rho_{R'}(F_j \cup W) = \rho_{R'}(F_j) + 11(2) - 9(3) = 0$, a contradiction. Suppose now that $W = \{v_1, v_2, v_3\}$. If at least 5 edges are incident with W, then $\rho_R(V(G)) \leq \rho_R(F_0 \cup V(T)) + 11(3) - 9(5) = -11$. Since G has no triangles, to have only 4 edges incident with 3 vertices of degree at least two, we need $|E(G[W])| = 2$ and each of them have degree exactly 2. This contradicts Claim 2.1.22. Thus, $W = \emptyset$, as claimed.

For $j = 1, \ldots, t$, let H_j be obtained from $G[F_j]$ by adding a new vertex u_j and edges $u_j z$ and $x_j u_j$. Let $R_j = (H_j, Z)$. Since $m_{R'}(\{x_j\}) = \rho_{R'}(F_j) = 5$, the potential of every subset of $V(H_j)$ containing both, x_j and u_j, is at least $5 + 11 - 9(2) = -2$. The potential of any set not containing u_i is at least -4 by the choice of G. If a set $U \subset V(H_j)$ contains u_j but does not contain x_j, then $\rho_{R_j}(U) > \rho_{R_j}(U - u_j) \geq -4$ again. So by the minimality of G. Each of R_j has a $(0,1)$-coloring ϕ_j. By construction, $\phi_j(x_j) = \phi_j(z) = 1$ and $\phi_j(u_j) = 0$. Since $F_j - \{w, z\}$ has no neighbors in $F_i - \{w, z\}$ for distinct i and j, $\phi = \bigcup_{j=1}^{t} \phi_j$ is a $(0,1)$-coloring of $G - V(T)$. Furthermore, $\phi(x_j) = 1$ for every j. Then by Lemma 2.1.8, ϕ can be extended to a $(0,1)$-coloring of G, a contradiction. □

2.1.7 Discharging

Consider the following vertex subsets of G:

$$\begin{aligned}
V_0 &= \bigcup_{T \text{ is a } (1,1,1)\text{-tree in } G} V(T), \\
V_1 &= \bigcup_{S \text{ is a shovel}} V(S) - z, \\
V_2 &= \{v \in V(G) - V_0 - V_1 : d_G(v) = 2\}, \\
V_3 &= V(G) - V_0 - V_1 - V_2.
\end{aligned}$$

By definition, $V(G) = V_0 \cup V_1 \cup V_2 \cup V_3$. For $i = 0, 1, 2, 3$, let $G_i = G[V_i]$.

We use discharging as follows. An *item* is a vertex in $V_2 \cup V_3$ or a shovel S or a component T of G_0. Initially, every vertex or edge x has the charge $ch(x)$ equal to its ρ-value. At the end of discharging, every item U will have a charge $ch^*(U)$ so that

$$\text{the sum of } ch^*(U) \text{ over all items is equal to } \textstyle\sum_{x \in V(G) \cup E(G)} \rho(x). \tag{2.26}$$

We use the following rules.

Rule 1: If an edge e not in a shovel is incident to exactly one 2-vertex distinct from z, then

e receives 5.75 from this 2-vertex and 3.25 from the other vertex. (Recall that by Claims 2.1.21 and 2.1.22, two adjacent 2-vertices can be only in a shovel.)

Rule 2: If an edge e not in a shovel is incident neither to z nor to any 2-vertex, then e receives 4.5 from each of the ends.

Rule 3: If an edge e not in a shovel is incident to z, then it gets 3.25 from z and 5.75 from the other end.

Rule 4: Every edge in a shovel gets 9 from the shovel.

Rule 5: For every $(1,1,1)$-tree T, let $ch^*(T)$ be the sum of the resulting charges of its vertices.

Rule 6: For every shovel S, let $ch^*(S)$ be the sum of the resulting charges of its vertices.

Claim 2.1.33. *After the discharging by Rules 1–6 above, we have*

(a) $ch^*(e) = 0$ *for every* $e \in E(G)$*;*

(b) $ch^*(v) \leq 0$ *for every* $v \in V_2 \cup V_3$*; moreover, if* $v \in V_3$ *and* $d_G(v) \geq 4$ *or* $d_{G_3}(v) \geq 2$ *, then* $ch^*(v) \leq -1.25$*;*

(c) $ch^*(v) \leq -0.5$ *for* $d_G(v) = 2$*,* v *not in a shovel.*

(d) $ch^*(T) \leq 0$ *for every* $(1,1,1)$*-tree* T*; moreover, if* T *contains only one* $(1,1,1)$*-vertex, then* $ch^*(T) \leq -1/4$*;*

(e) $ch^*(S) \leq -3.75$ *for every broken shovel* S*;*

(f) $ch^*(S) \leq -4.75$ *for every intact shovel* S

Proof. Part (a) immediately follows from Rules 1–4.

Let $v \in V_3$. If $v = z$ and is adjacent to exactly x vertices not in shovels, then by Rule 3 it gives out charge $3.25x$. Thus (b) holds for $v = z$. Suppose $v \in V_3 - z$. Then $d(v) \geq 3$. Moreover, if $d(v) = 3$ the v has a neighbor w that either is z or has degree at least 3. If $d(v) \geq 4$, then it gives away by Rule 1 at least $3.25d(v)$, so $ch^*(v) \leq 11 - 3.25d(v) \leq -1.25$. If $d(v) = 3$, then by Rules 1–3, $ch^*(v) \leq \max\{11 - 4.5 - 2(3.25), 11 - 2(4.5) - (3.25)\} = 0$. Moreover, if $d_{G_3}(v) \geq 2$, then $ch^*(v) \leq 11 - 2(4.5) - 3.25 = -1.25$. This proves (b).

If $d_G(v) = 2$, and v is not in a shovel, then by Rule 1, it gives 5.75 to two incident edges. Thus $ch^*(v) = 11 - 2(5.75) = -0.5$. This proves (c).

Suppose a $(1,1,1)$-tree T contains k $(1,1,1)$-vertices for $k = 1$ or 2. Then T has exactly

$2k+1$ vertices of degree two. Each 2-vertex in T has charge -0.5 and each $(1,1,1)$-vertex has charge $11-3(3.25)=1.25$. So, $ch^*(T)=1.25k-0.5(2k+1)\leq 0$. Moreover, if $k=1$, then $ch^*(T)\leq 0.5-3(0.25)\leq -0.25$. This proves (d).

A broken shovel S contains three typical vertices and three edges. By Rules 1 and 2, the joint w sends out at least $(d(w)-1)(3.25)$. So, by Rule 6, $ch^*(S)\leq 3(11)-3(9)-(d(w)-1)3.25\leq 6-3.25(3)=-3.75$. This proves (e).

An intact shovel S contains 7 typical vertices and 8 edges. Also by Claim 2.1.25, S has at least three neighbors outside of it. So its total charge is at most $7(11)-8(9)-3(3.25)=-4.75$. □

Since the discharging preserves the total charge, it must be at least -4. Thus by Claim 2.1.33, if some set of items will have the total new charge less than -4, then we get a contradiction.

We will use the following immediate corollary of Lemma 2.1.8.

Claim 2.1.34. *If $G-V_0$ has a $(0,1)$-coloring ϕ such that at most one vertex in $V_3\cap N_G(V_0)$ is colored with 0, then ϕ can be extended to V_0. Moreover, if every component of G_0 has at least two $(1,1,1)$-vertex, and at most two vertices in $V_3\cap N_G(V_0)$ have color 0, then ϕ also can be extended to V_0.*

Claim 2.1.35. *G has no shovels.*

Proof. Suppose S is a shovel in G.

Case 1: S is a broken shovel with the handle (v_0,v_1,v_2,z) and $d(v_0)\geq 4$. If $d(v_0)\geq 5$, then $ch^*(S)\leq 3(11)-3(9)-4(3.25)<-4$, a contradiction. If $d(v_0)=4$ and v_0 has a neighbor of degree at last 3, then by Rules 6, 1 and 2, $ch^*(S)\leq 3(11)-3(9)-4.5-2(3.25)=-5<-4$, a contradiction. Assume $d(v_0)=4$ and all neighbors of v_0 are 2-vertices. Then $ch^*(S)\leq 6-3(3.25)=-3.75$.

If G_3 contains a P_3, then by Claim 2.1.33(b), the middle vertex w of this P_3 satisfies $ch^*(w)\leq -1.25$. So, $ch^*(S)+ch^*(w)<-4$, a contradiction. Thus G_3 does not contain P_3. Also by Claim 2.1.33(d), G has no shovels apart from S. We color all vertices in V_3 with color 1, and all vertices in V_2 with color 0, and color S so that v_0 is colored by 1. Then by Claim 2.1.34 we can extend the coloring to V_0, a contradiction.

Case 2: S is intact. Immediately follows from Claim 2.1.33(f). □

Claim 2.1.36. *The special vertex z is isolated in G.*

Proof. By Claim 2.1.35, $V_1 = \emptyset$. Thus by Rules 3 and 5, if $d(z) \geq 2$, then $ch^*(z) \leq -3.25 \cdot 2 = -6.5$, a contradiction.

Let $d(z) = 1$ and u be the unique neighbor of z. If $d(u) \geq 3$, then by Rules 3, $ch^*(z) = -3.25$ and $ch^*(u) \leq 11 - 5.75 - 2 \cdot 3.25 = -1.25$, a contradiction. So, let $d(u) = 2$. Then by Rule 3, $ch^*(z) \leq -3.25$. Hence $\Delta(G_3) \leq 1$, since each $w \in V_3$ of degree at least 2 in G_3 has $ch^*(w) \leq -1.25$. Also, z forms a component in G_3. Then we color V_3 with 1 and V_2 with 0. By Claim 2.1.34 the obtained coloring extends to V_0. □

Since z is isolated, from now on we consider coloring of $G - z$.

2.1.8 Structure of G_3

For $W \subseteq V_2 \cup V_3$, a coloring ϕ' is *standard on* W if $\phi'(v) = 1$ for $v \in V_3 \cap W$, and $\phi'(v) = 0$ for $v \in V_2 \cap W$.

Claim 2.1.37. *G_3 has no cycles, and no P_k with $k \geq 5$.*

Proof. Suppose G_3 contains C_k. Since G is triangle-free, $k \geq 4$. By (b) of Claim 2.1.33, $ch^*(C_k) \leq -1.25k < -4$, a contradiction. So G_3 is acyclic.

Suppose now that G_3 contains P_k with $k \geq 5$. Then by Claim 2.1.33 (b), $k \leq 5$.

In order to have $ch^*(P_5) \geq -4$, every vertex in the P_5 has degree three and is adjacent to a vertex of degree two.

Also, $\Delta(G_3 - P_5) \leq 1$. Otherwise, $ch^*(P_5) + ch^*(P_3) \leq -1.25 \cdot 3 - 1.25 < -4$, a contradiction. Let $P_5 = (v_1, \ldots, v_5)$ and x_3' be the neighbor of v_3 outside of P_3.

If $x_3' \in V_1$, then $ch^*(x_3') = -0.5$ and $ch^*(P_5) + ch^*(x_3') \leq -1.25 * 3 - 0.5 < -4$, a contradiction. So $x_3' \notin V_1$, then we color all vertices in $G_3 - P_5$ with color 1, all vertices in V_1 with 0 and the vertices in P_5 as follows: $1, 1, 0, 1, 1$. By Claim 2.1.34 this coloring extends to V_0. □

Claim 2.1.38. *G_3 does not contain $K_{1,3}$.*

Proof. Assume V_3 contains a subset X with $G[X] = K_{1,3}$ with center x.

If $d(x) \geq 4$, then $ch^*(x) \leq 11 - 3 \cdot 4.5 - 3.25 = -5.75$, a contradiction. So, $d(x) = 3$ and $ch^*(x) = -2.5$. If $\Delta(G_3 - X) \leq 1$, then we use the standard coloring on $V_2 \cup V_3 - x$ and color x with 0. By Claim 2.1.34 this coloring extends to V_0. So we may assume that for some $y \in V_3 - X$, $d_{G_3}(y) \geq 2$. Then $ch^*(y) \leq -1.25$. Thus in order the total charge to be at least -4, we need that every vertex in $V_3 - X - y$ has degree at most 1 in G_3, and $V_2 = \emptyset$. Then we use the standard coloring on $V_2 \cup V_3 - x - y$ and color x and y with 0. Since x has no neighbors in V_0, by Lemma 2.1.8, the coloring extends to V_0. □

So each component of G_3 is a K_1, K_2, P_3, or P_4. Note that if a vertex x is a component in G_3, then it is not a $(1,1,1)$-vertex. So in this case $d(x) \geq 4$, and $ch^*(x) \leq -2$.

Claim 2.1.39. *G_3 does not contain P_4.*

Proof. Suppose G_3 contains $P_4 = (v_1, v_2, v_3, v_4)$. Let $W = \{v_1, v_2, v_3, v_4\}$. Then $ch^*(v_2) + ch^*(v_3) \leq -1.25 + (-1.25) = -2.5$. Moreover, if $d(v_2) \geq 4$ or $d(v_3) \geq 4$, then $ch^*(v_2) + ch^*(v_3) \leq -1.25 + (-4.5) < -4$, a contradiction. So, $d(v_2) = d(v_3) = 3$.

If G_3 contains another 4-path $P_4' = (y_1, y_2, y_3, y_4)$ disjoint from W, we get the potential less than $2 * (-2.5) < -4$, a contradiction. Thus this P_4 is the only P_4 in G_3.

If $G_3 - W$ contains two disjoint 3-paths $P_3 = (y_1, y_2, y_3)$ and $P_3' = (u_1, u_2, u_3)$, then $ch^*(P_4) + ch^*(y_2) + ch^*(u_2) \leq -2.5 - 1.25 - 1.25 < -4$, a contradiction. Thus $G_3 - W$ contains at most one P_3.

CASE 1: $G_3 - W$ contains exactly one 3-path $P_3 = (y_1, y_2, y_3)$.
If V_2 is not empty, it contains v'. Then $ch^*(\{v_2, v_3, y_2, v'\}) \leq -1.25 - 1.25 - 1.25 - 0.5 = -4.25 < -4$, a contradiction. So V_2 is empty.

Then we use the standard coloring on $V_2 \cup V_3 - v_2 - y_2$ and color v_2 and y_2 with 0. By the choice of v_2 and the case for y_2, this coloring is a $(0,1)$-coloring for $G[V_2 \cup V_3]$.

If it extends to V_0, then we are done. Suppose not. Then some $(1,1,1)$-tree T of size 1 is adjacent to v_2 and y_2. Then $ch^*(\{v_2, v_3, y_2\}) + ch^*(T) \leq -3.75 - 0.25 = -4$. If there is another $(1,1,1)$-tree T' of size 1, then $ch^*(\{v_2, v_3, y_2\}) + ch^*(T) + ch^*(T') \leq -3.75 - 0.25 - 0.25 < -4$, a

contradiction. Thus T is the only $(1,1,1)$-tree of size 1. Since T is adjacent to exactly 3 vertices, we may assume that y_3 is not adjacent to T. We recolor y_2 with 1 and y_3 with 0, and can extend the new coloring to V_0.

CASE 2: $\Delta(G_3 - W) \leq 1$.

If some $v \in \{v_2, v_3\}$ has no neighbors in V_2, then we use the standard coloring on $V_2 \cup V_3 - v$ and color v with 0. By Claim 2.1.34 this coloring extends to V_0. So we may assume that for $j \in \{2,3\}$, v_j has a neighbor $v'_j \in V_2$. Since G is triangle-free, $v'_3 \neq v'_2$. Therefore, $ch^*(\{v_2, v_3, v'_2, v'_3\}) \leq -1.25 - 1.25 - 0.5 - 0.5 = -3.5$. So, all vertices in $V_3 - v_2 - v_3$ are $(1,1,0)$-vertices and $|V_2| \leq 3$.

CASE 2.1: $|V_2| = 3$, i.e. there is $w \in V_1 - v'_2 - v'_3$.

Then $ch^*(w) = -0.5$ and there are no $(1,1,1)$-trees of size 1. For $j = 2,3$, let $N(v'_j) = \{v_j, v''_j\}$.

Suppose first that $v''_3 = v''_2$. Then we color $V_3 + v'_2 + v'_3 - v''_2 - v_2 - v_4$ with 1 and v_2, v''_2, v_4 and w with 0. Since v_2 has no neighbors in V_0, if this coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$, then it extends to V_0. So, this is not the case. The only possibility for it is that $wv_4 \in E(G)$. By the symmetric argument, $wv_1 \in E(G)$. Then we use the standard coloring on $V_2 \cup V_3 - v_1 - v_4 - w$ and color v_1 and v_4 with 0 and w with 1. The obtained coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$ and extends to V_0.

Suppose now that $v''_3 \neq v''_2$. We use the standard coloring on $V_2 \cup V_3 - v_2 - v'_2 - v''_2$ and color v_2 and v''_2 with 0 and v'_2 with 1. The only possibility that this is not a $(0,1)$-coloring of $G[V_2 \cup V_3]$ is that $wv''_2 \in E(G)$. By the symmetric argument, $wv''_3 \in E(G)$. If $v''_2 \neq v_4$ and $v''_3 \neq v_1$, then we use the standard coloring on $V_2 \cup V_3 - v_1 - v_4$ and color v_1 and v_4 with 0. Let $v''_2 = v_4$. If $v''_3 \neq v_1$, then we use the standard coloring on $V_2 \cup V_3 - v_3 - v'_3 - v''_3 - w$ and color v_3 and v''_3 with 0 and v'_3 and w with 1. In both cases, the coloring extends to V_0. Finally, if also $v''_3 = v_1$, then $V(G) = W \cup \{v'_2, v'_3, w\}$. In this case, we color v_1, v_3 and v'_2 with 0 and the rest with 1.

CASE 2.2: $V_2 = \{v'_2, v'_3\}$.

If $v''_2 = v_4$, then we use the standard coloring on $V_2 \cup V_3 - v_2 - v'_2 - v''_2$ and color v_2 and v''_2 with 0 and v'_2 with 1. Since G is triangle-free, $v_4 \neq v''_3$ and so this is a $(0,1)$-coloring of $G[V_2 \cup V_3]$. Since v_2 has no neighbors in V_0, the coloring extends to V_0. Thus $v''_2 \neq v_4$.

If $v''_3 = v''_2$, then we use the standard coloring on $V_2 \cup V_3 - v_2 - v'_2 - v''_2$ and color v_2 and w''_2 with

0 and v_2' with 1. If $v_3'' \neq v_2''$, then let w be the neighbor of v_2'' in G_3, use the standard coloring on $V_2 \cup V_3 - v_2 - v_2' - w$ and color v_2 and w with 0 and v_2' with 1. In both cases, this is a $(0,1)$-coloring of $G[V_2 \cup V_3]$. Since v_2 has no neighbors in V_0, the coloring extends to V_0. □

2.1.9 Proof of Theorem 2.1.3

By the claims in section 2.1.8, each component of G_3 is isomorphic to K_1, K_2 or P_3. If $\Delta(G_3) \leq 1$, then the standard coloring of $V_2 \cup V_3$ is a $(0,1)$-coloring and extends to V_0. So, let (x_1, x_2, x_3) be a path in G_3 and $X = \{x_1, x_2, x_3\}$.

CASE 1: $\Delta(G_3 - X) \leq 1$.

We use the standard coloring on $V_2 \cup V_3 - x_2$ and color x_2 with 0. If this is a $(0,1)$-coloring of $G[V_2 \cup V_3]$, then by Claim 2.1.34 it extends to V_0. So, it is not. Then x_2 has a neighbor $w_1 \in V_2$. Let y_1 be the other neighbor of w_1. Since $ch^*(x_2) + ch^*(w_1) = -1.75$, every vertex $u \in V_3 - x_2$ is either a $(1,1,0)$-vertex as $d_{G_3}(u) \leq 1$ or a $(1,1,1,1)$-vertex (otherwise $ch^*(u) \leq -3.25$ and so $ch^*(\{x_2, w_1, u\}) \leq -1.25 - 0.5 - 3.25 < -4$).

We recolor w_1 with 1. If y_1 is a $(1,1,1,1)$-vertex, this yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$ which extends to V_0. So, let y_1 be a $(1,1,0)$-vertex with neighbors $w_1, w_2 \in V_2 \cup V_0$ and $y_1' \in V_3$. We now recolor y_1 with 0. If $w_2 \in V_0$, then this yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$. So suppose $w_2 \in V_2$ and the other neighbor of w_2 is y_2. Recolor w_2 with 1. If $y_2 \in \{x_1, x_3\}$ or is a $(1,1,1,1)$-vertex, then this again yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$. So let y_2 be a $(1,1,0)$-vertex with neighbors $w_2, w_3 \in V_2 \cup V_0$ and $y_2' \in V_3$. We now recolor y_2 with 0. Again, if $w_3 \notin V_2$, then we are done; otherwise we again consider the other neighbor y_3 of w_3 and so on. Since $ch^*(w_i) = -0.5$ for $i \geq 2$, in order to have $ch^*(V_2 \cup V_3) \geq 4$, the recoloring will stop after at most 4 steps.

CASE 2: There are exactly two 3-paths in G_3, (x_1, x_2, x_3) and (y_1, y_2, y_3).

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Similarly to Case 1, every vertex in $V_3 - x_2 - y_2$ is either a $(1,1,0)$-vertex or a $(1,1,1,1)$-vertex. Also $d(x_2) = d(y_2) = 3$, since otherwise $ch^*(x_2) + ch^*(y_2) \leq -1.25 - 4.5 < -4$. We use the standard coloring on $V_2 \cup V_3 - x_2 - y_2$ and color x_2 and y_2 with 0.

If this is a $(0,1)$-coloring of $G[V_2 \cup V_3]$ and extends to V_0, then we are done. Thus one of the two subcases below holds.

CASE 2.1: Our coloring is not a $(0,1)$-coloring of $G[V_2 \cup V_3]$.

Case 2.1.1: Both x_2 and y_2 have neighbors in V_2.

Let $x_2w_1, y_2u_1 \in E(G)$, $N(w_1) = \{x_2, q_1\}$ and $N(u_1) = \{y_2, v_1\}$. Recolor w_1 and u_1 with 1. If $q_1 \in Y$ and $v_1 \in X$ (in particular, if $w_1 = u_1$), then this gives a $(0,1)$-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0, a contradiction. So by symmetry we may assume that $q_1 \notin Y$. Then $ch^*(\{x_2, y_2, w_1, u_1\}) \leq -1.25 - 1.25 - 0.5 - 0.5 = -3.5$. It follows that G has no $(1,1,1,1)$-vertices and $|V_2| \leq 3$. So, q_1 is a $(1,1,0)$-vertex. Let $N(q_1) = \{w_1, q_1', w_2\}$ where $q_1' \in V_3$.

Suppose first that $u_1 \in \{x_1, x_3, q_1, q_1'\}$. Choose $q \in \{q_1, q_1'\}$ so that $q = u_1$ if $w_1 \in \{q_1, q_1'\}$, and otherwise q is not adjacent to $V_2 - w_1$, if possible. Recolor q with 0. If the new coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$, then it extends to V_0, since x_2 and y_2 have no neighbors in V_0. So it is not. If $q \neq w_1$, then by the choice of q, each of q_1, q_1' has a neighbor (say, r_1 and r_1') in $V_2 - w_1 - u_1$. Since G has no triangles, $r_1 \neq r_1'$, and so $ch^*(\{x_2, y_2, w_1, u_1, r_1, r_1'\}) \leq -1.25 - 1.25 - 0.5 - 0.5 - 0.5 - 0.5 < -4$. Thus $q = w_1$. Since the coloring is not a $(0,1)$-coloring of $G[V_2 \cup V_3]$, $q = q_1'$ and q_1' has another neighbor $w_2 \in V_2$. Again, $w_2, q_1 \notin E(G)$, because G is triangle-free. Since $|V_2| \leq 3$, we know that $V_2 = \{w_1, u_1, w_2\}$ and each $(1,1,1)$-tree has size at least 2. So, after recoloring q_1' back with 1 and coloring q_1 with 0 we obtain a $(0,1)$-coloring of $G[V_2 \cup V_3]$, which extends to V_0.

We may assume now that $u_1 \notin \{x_1, x_3, q_1, q_1'\}$. Let u_1' be the neighbor of u_1 in V_3. Since $|V_2| \leq 3$, we may assume that either $V_2 = \{w_1, u_1\}$ or $V_2 = \{w_1, u_1, r\}$. Then we can choose some $u \in \{u_1, u_1'\}$ and $q \in \{q_1, q_1'\}$ so that (a) either both or neither of u and q are adjacent to r (if exists), and (b) there is no $(1,1,1)$-tree of size 1 adjacent to both, q and u. Recoloring q and u with 0 and r, if it is adjacent to both q and u, with 1 yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0, a contradiction.

By the symmetry between x_2 and y_2, the remaining subcases are the following.

Case 2.1.2: Only x_2 has a neighbor $w_1 \in V_2$, and y_2 is adjacent to a $(1,1,1)$-tree T of size 1.

Let $N(w_1) = \{x_2, q_1\}$. Since $ch^*(\{x_2, y_2, w_1\}) + ch^*(T) \leq -1.25 - 1.25 - 0.5 - 0.25 = -3.25$, as in Case 1, every vertex $u \in V_3 - x_2 - y_2$ is a $(1,1,0)$-vertex. Moreover, $|V_2| \leq 2$, and we may assume that $V_2 = \{w_1\}$ or $V_2 = \{w_1, w_2\}$. Recolor w_1 with 1. If q_1 belongs to Y this gives a $(0,1)$-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0. So we may assume that $q_1 \notin Y$. Let

q_1' be the neighbor of q_1 in V_3. If there is $q \in \{q_1, q_1'\}$ such that q has neighbors neither in $V_2 - w_1$ nor in T, then recoloring q with 0 yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, each of q_1 and q_1' has a neighbor in $V_2 - w_1$ or in T. Similarly, if there is $y \in \{y_1, y_3\}$ not adjacent to V_2 and not adjacent to T, then we recolor y_2 with 1 and q_1 and y with 0. Since we know the neighbors of x_2 and q_1, if the obtained coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$, then it extends to V_0. So, each of y_1 and y_3 has a neighbor V_2 or in T. If some $x \in \{x_1, x_3\}$ is adjacent neither to T nor to V_2, then we can use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. This will be a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0. Thus every vertex in $Q = \{x_1, x_3, y_1, y_3, q_1, q_1'\}$ has a neighbor in T or in $V_2 - w_1$. It is impossible, since $|V_2| \leq 2$ and T has at most two neighbors in Q.

Case 2.1.3: Only x_2 has a neighbor $w_1 \in V_2$, and y_2 is not adjacent to any $(1,1,1)$-tree of size 1. Let $N(w_1) = \{x_2, q_1\}$. Since $ch^*(\{x_2, y_2, w_1\}) \leq -1.25 - 1.25 - 0.5 = -3$, $|V_2| \leq 3$. Recolor w_1 with 1. If q_1 belongs to Y or is a $(1,1,1,1)$-vertex, this gives a $(0,1)$-coloring of $G[V_2 \cup V_3]$. Since x_2 has no neighbors in V_0, it extends to V_0. So we may assume that $q_1 \notin Y$ and is a $(1,1,0)$-vertex. Let q_1' be the neighbor of q_1 in V_3. If there is $q \in \{q_1, q_1'\}$ such that q does not have neighbors in $V_2 - w_1$, then recoloring q with 0 yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, each of q_1 and q_1' has a neighbor in $V_2 - w_1$. If some $x \in \{x_1, x_3\}$ is not adjacent to V_2, then we can use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. This will be a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0. Thus every vertex in $Q = \{x_1, x_3, q_1, q_1'\}$ has a neighbor in $V_2 - w_1$. It follows that $|V_2| = 3$ and each $w \in V_2 - w_1$ is adjacent only to vertices in Q. Since no vertex is adjacent to both q_1 and q_1' (because G is triangle-free), there are $x \in \{x_1, x_3\}$ and $w \in V_2 - w_1$ such that $N(w) = \{x, q_1\}$. Then we use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - q - w_1$ and color x_2 and y_2 with 0 and w_1. Since we know the neighbors of all vertices in V_2, this will be a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

CASE 2.2: Our coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$ but does not extend to V_0.

Then by Lemma 2.1.8, there exists a $(1,1,1)$-tree T of size 1 adjacent to both, x_2 and y_2, and such that the third neighbor of T in V_3, say q, has a neighbor of color 1. So, q is a $(1,1,0)$-vertex not in $X \cup Y$. By Rules 5, $ch^*(T) \leq -0.25$. So, $|V_2| \leq 2$. If some $x \in \{x_1, x_3\}$ is not adjacent to V_2,

then we use the standard coloring on $V_2 \cup V_3 - x - y_2$ and color x and y_2 with 0. Since $q \notin X$, this will be a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0. So, both x_1 and x_3 are adjacent to V_2. Symmetrically, both y_1 and y_3 are adjacent to V_2. Then q is not adjacent to V_2. So we use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - q$ and color x_2, q and y_2 with 0. By the case, this is a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

CASE 3: There are exactly three 3-paths in G_3, (x_1, x_2, x_3), (y_1, y_2, y_3), and (u_1, u_2, u_3).

Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $U = \{u_1, u_2, u_3\}$. Similarly to Case 1, every vertex in $V_3 - x_2 - y_2 - u_2$ is either a $(1,1,0)$-vertex. Also $d(x_2) = d(y_2) = d(u_2) = 3$ and $|V_2| \leq 2$.

If V_2 is not empty, $w \in V_2$, so $ch^*(\{x_2, y_2, u_2, w\} \leq -1.25 - 1.25 - 1.25 - 0.5 < -4$, so $V_2 = \emptyset$.

Thus every vertex in $V_3 - x_2 - y_2 - u_2$ is a $(1,1,0)$-vertex.

We use the standard coloring on $V_2 \cup V_3 - x_2 - y_2 - u_2$ and color x_2, y_2 and u_2 with 0. Since this coloring is a $(0,1)$-coloring of $G[V_2 \cup V_3]$, so it does not extend to V_0.

Thus there is a $(1,1,1)$-tree T adjacent to j vertices of $\{x_2, y_2, u_2\}$ with $j = 1$ or 2. And no $(1,1,1)$-tree of size 1 distinct from T is adjacent to $\{x_2, y_2, u_2\}$. Let w be the neighbor of T outside of $\{x_2, y_2, u_2\}$. Then recoloring w with 0 yields a $(0,1)$-coloring of $G[V_2 \cup V_3]$ that extends to V_0.

Since there are at most three P_3s in G_3, this finishes the proof. □

2.2 On (j,k)-coloring of graphs with large girth

In Seciton 2.2, we consider graphs with girth g with arbitrary g and determine values of $F_{j,k}(g)$. For j, k with $k \geq 2j + 2$, the following theorem gives us the value of $F_{j,k}(3)$.

Theorem 2.2.1 ([16])**.** *Let $k \geq 2j + 2$ and G be a graph. If G is $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse, then it is (j,k)-colorable. Moreover, the result is sharp in the sense that there are infinitely many almost $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse graphs that are not (j,k)-colorable.*

Our first result in this section gives triangle-free sharpness examples for Theorem 2.2.1.

Theorem 2.2.2. *Let $j \geq 0$ and $k \geq j + 1$. Then there are infinitely many triangle-free almost $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse graphs that are not (j,k)-colorable. Furthermore, for every $k \geq 1$, there are infinitely many almost $\left(2 - \frac{k+2}{2(k+1)}, \frac{1}{k+1}\right)$-sparse graphs of girth 5 that are not $(0,k)$-colorable.*

When $k \geq 2j+2$, the graphs we construct in Theorem 2.2.2 are (j,k)-*critical* in the sense that each proper subgraph of every such graph is (j,k)-colorable by Theorem 2.2.1 but the graphs themselves are not.

Let $F_{j,k}(g)$ denote the supremum of positive a such that there is some (possibly negative) b with the property that every (a,b)-sparse graph G with girth g is (j,k)-colorable. The above mentioned result in [15] implies $F_{0,1}(3) = \frac{12}{5} = 1.2$. In Section 2.1 we prove the exact result: $F_{0,1}(4) = F_{0,1}(5) = \frac{11}{9}$ and also find the best possible value of b. In this section we extend this result in two directions: to large girth and to (j,k)-colorings instead of $(0,1)$-colorings.

Since $F_{0,0}(4)$ and $F_{0,1}(4)$ are already known, with Theorem 2.2.2 we have the values of $F_{0,k}(4)$ for all $k \geq 0$.

Our second result in this section concerns graphs with large girth.

Theorem 2.2.3. *For all $k \geq j \geq 0$ and $g \geq 3$, $F_{j,k}(g) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}$.*

So, we have $F_{0,1}(3) = 1.2$, $F_{0,1}(4) = F_{0,1}(5) = \frac{22}{9} = 1.222\ldots$, $F_{0,1}(g) \leq 1.25$ for all g, and if $k \geq 2j+2$ then $F_{j,k}(g) = 2 - \frac{(k+2)}{(j+2)(k+1)}$ for all g.

2.2.1 On (j,k)-coloring of triangle-free graphs

We consider a (j,k)-coloring of a graph G as a 2-coloring of $V(G)$ with color j and color k such that the vertices of color j (respectively, k) induce a subgraph with maximum degree at most j (respectively, k). For a graph G and $W \subseteq V(G)$, let *the (j,k)-potential of W in G* be defined as

$$\phi(W,G) = \phi_{j,k}(W,G) = (2 - \frac{k+2}{(j+2)(k+1)})|W| - |E(G[W])|.$$

(We will drop the subscripts j,k if they are clear from the context.)

Note that for a graph G, condition

$$\phi_{j,k}(W,G) > -\tfrac{1}{k+1} \text{ for all } W \subseteq V(G), \tag{2.27}$$

is equivalent to the statement that G is $\left(2 - \frac{k+2}{(j+2)(k+1)}, \frac{1}{k+1}\right)$-sparse.

In this section we prove Theorem 2.2.2, i.e. we show that for all $k \geq j+1$ there are infinitely many triangle-free graphs G with $\phi_{j,k}(W,G) \geq -\frac{1}{k+1}$ for all $W \subset V(G)$, but not (j,k)-colorable and for all $k \geq 2$ there are infinitely many graphs G of girth 5 with $\phi_{0,k}(W,G) \geq -\frac{1}{k+1}$ for all $W \subset V(G)$, and not $(0,k)$-colorable. Together with Theorem 2.2.1, this means that for all $k \geq 2j+2$, $F_{j,k}(4) = F_{j,k}(3)$. Recall that this is not the case for $(j,k) = (0,1)$ by our result in [35].

Let graph $L(j,k)$ be defined as follows. Let

$$V(L(j,k)) = \{x,w\} \cup \{u_1,\ldots,u_{j+1}\} \cup \bigcup_{i=1}^{k+1} \{y_{i,1},\ldots,y_{i,j+1},y_i\}.$$

Vertex x is adjacent to all vertices in $\{u_1,\ldots,u_{j+1}\} \cup \{y_1,\ldots,y_{k+1}\}$, vertex w is adjacent to all vertices in $\{u_1,\ldots,u_{j+1}\} \cup \bigcup_{i=1}^{k+1}\{y_{i,1},\ldots,y_{i,j+1}\}$, for every $i \in [1,k+1]$, vertex y_i is adjacent to all vertices in $\{y_{i,1},\ldots,y_{i,j+1}\}$, and there are no other edges (See Fig 2.8). We will call x *the base* and w *the top* of $L(j,k)$.

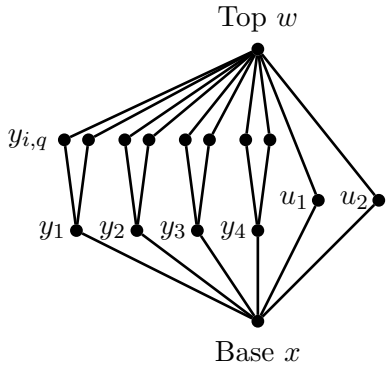


Figure 2.8 : Graph $L(1,3)$.

By construction, $L(j,k)$ is triangle-free and $L(0,k)$ has girth 5. We need the following simple property of $L(j,k)$.

Claim 2.2.4. *In every (j,k)-coloring f of $L(j,k)$, x has a neighbor of color k.*

Proof. Suppose $f(y_1) = \ldots = f(y_{k+1}) = f(u_1) = \ldots = f(u_{j+1}) = j$. Then for every $1 \leq i \leq k+1$ at least one of $y_{i,1},\ldots,y_{i,j+1}$ must be colored with k. So, w has at least $k+1$ neighbors of color k and $j+1$ neighbors $u_1,\ldots,u_{j+1}$ of color j, a contradiction to the definition of (j,k)-coloring. □

A (j,k)-*flag* in a graph G is a pendant block isomorphic to $L(j,k)$ whose unique cut vertex is the base vertex x in $L(j,k)$. Claim 2.2.4 immediately implies the following.

Claim 2.2.5. *In every (j,k)-coloring f of a graph G for any $x \in V(G)$,*
(a) if x is the base of $k+1$ distinct (j,k)-flags, then $f(x) = j$;
(b) if x is the base of k distinct (j,k)-flags and $f(x) = k$, then x has no neighbors of color k outside of these k blocks.

Another helpful property of (j,k)-flags is that they are sparse:

Claim 2.2.6. *Let graph G consist of q distinct (j,k)-flags, $W_1, \dots, W_q$ with the common base x, and for $i = 1, \dots, q$, let w_i be the top of W_i.*

(a) If $\emptyset \neq W \subseteq W_i$, then $\phi(W) \geq \phi(\{x\}) - \frac{1}{k+1}$, and equality holds only for $W = W_i$.

(b) If $\emptyset \neq W \subseteq V(G)$, then $\phi(W) \geq \phi(\{x\}) - \frac{q}{k+1}$ and equality holds only for $W = V(G)$.

Proof. To prove (a), choose among the nonempty subsets of W_i a set W of the smallest potential $\phi(W)$. Since deleting of an isolated or pendant vertex from a set increases the potential and the claim holds for a 1-element W, we may assume

$$\delta(G[W]) \geq 2. \tag{2.28}$$

If $\emptyset \neq W \subset W_i$ and $w_i \notin W$, then W induces a forest, a contradiction to (2.28). So $w_i \in W$.

Since adding to a set U of vertices a vertex with at least two neighbors in U decreases the potential, by (2.28),

$$\text{for all } 1 \leq h, h' \leq j+1,\ u_h \in W \text{ if and only if } x \in W \text{ and } y_{h,h'} \in W \text{ if and only if } y_h \in W. \tag{2.29}$$

Suppose $x \notin W$. Then by (2.29), $W \cap \{u_1, \dots, u_{j+1}\} = \emptyset$. Also, if in this case $y_h \in W$ then by (2.29), all $y_{h,1}, \dots, y_{h,j+1}$ are in W and

$$\phi(W) - \phi(W - \{y_{h,1}, \dots, y_{h,j+1}, y_h\}) \geq \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2) - (2j+2) = \frac{k}{k+1},$$

a contradiction to the choice of W. Thus $x \in W$. Then by (2.29), $\{u_1, \ldots, u_{j+1}\} \subset W$. Also adding each y_h together with $y_{h,1}, \ldots, y_{h,j+1}$ decreases potential by exactly $\frac{1}{k+1}$. So, the unique subset of W_i with the minimum possible potential is W_i itself and

$$\phi(W_i) - \phi(\{x\}) = \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(|W_i| - 1) - |E(G[W_i])|$$

$$= \left(2 - \frac{k+2}{(j+2)(k+1)}\right)(j+2)(k+2) - ((2j+3)(k+2) - 1) = -\frac{1}{k+1},$$

as claimed. This proves (a).

To prove (b), suppose that W intersects exactly $r > 0$ of $W_1, \ldots, W_q$. If $x \notin W$, then

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) > r(\phi(\{x\}) - \frac{1}{k+1}) \geq \phi(\{x\}) - \frac{r}{k+1}.$$

If $x \in W$, then $r = q$ and

$$\phi(W) = \sum_{i=1}^{q} \phi(W \cap W_i) - (q-1)\phi(\{x\}) \geq \phi(\{x\}) - \frac{q}{k+1}. \tag{2.30}$$

By (a), equality in (2.30) holds only when $W \cap W_i = W_i$ for all i, which means $W = V(G)$. □

Basic construction. We construct a graph $H_0 = H_0(j,k)$ from a star $K_{1,j+1}$ with the center x_0 and leaves $x_1, \ldots, x_{j+1}$ by adding $k+1$ (j,k)-flags to each of $x_0, x_1, \ldots, x_{j+1}$. (When we say "add (j,k)-flags to a vertex x", we mean that x will be the base of the added flags.)

By construction, $H_0(j,k)$ is triangle-free and $H_0(0,k)$ has girth 5. If H_0 has a (j,k)-coloring f, then by Claim 2.2.5(a), $f(x_0) = \ldots = f(x_{j+1}) = j$, and vertex x_0 of color j has $j+1$ neighbors $x_1, \ldots, x_{j+1}$ of color j, a contradiction. Thus

$$H_0 \text{ is not } (j,k)\text{-colorable.} \tag{2.31}$$

Now we want to prove that H_0 satisfies (2.27).

Claim 2.2.7. *If $W \subseteq V(H_0)$, then $\phi(W) \geq -\frac{1}{k+1}$, and equality holds only for $W = V(H_0)$.*

Proof. Choose a largest $W \subset V(H_0)$ among the sets with minimum $\phi(W)$. As in the proof of Claim 2.2.6, $\delta(H_0[W]) \geq 2$. By Claim 2.2.6(a), if L is any (j,k)-flag in H_0 and $W \cap L \neq \emptyset$, then $L \subseteq W$. It follows that if we know which vertices in $X = \{x_0, \ldots, x_{j+1}\}$ are in W, then we know W. Similarly, if $x_0 \in W$ and $x_i \notin W$ for some i, then by Claim 2.2.6(a), adding to W vertex x_i and all the $k+1$ (j,k)-flags containing x_i we get a set W' with

$$\phi(W') \leq \phi(W) + \phi(\{x_i\}) - \frac{k+1}{k+1} - 1 < \phi(W),$$

a contradiction to the minimality of $\phi(W)$. So, $W = V(H_0)$ is the unique set of minimum potential among the sets containing x_0.

If $x_0 \notin W$, then every component of $H_0[W]$ is a subgraph of a graph G described in Claim 2.2.6 and so has a nonnegative potential. So in this case $\phi(W) \geq 0$. □

Thus H_0 is the first in the series of examples proving Theorem 2.2.2.

In order to generalize H_0, we need one more notion. A vertex v in a graph G is a *remote* (j,k)*-base* if it is the base of $k+1$ (j,k)-flags $W_1, \ldots, W_{k+1}$ in G and has exactly one neighbor outside of $W_1 \cup \ldots \cup W_{k+1}$. This unique neighbor of v will be called *the main neighbor* of v.

Claim 2.2.8. *Suppose a graph H has no (j,k)-colorings, and $v \in V(H)$ is a remote (j,k)-base contained in (j,k)-flags $W_1, \ldots, W_{k+1}$ with the main neighbor x.*
(a) for any (j,k)-coloring f' of $H' = H - (W_1 - v)$ (if exists), $f'(v) = k$ and v has k neighbors of color k in H';
(b) for any (j,k)-coloring f'' of $H'' = H - \bigcup_{i=1}^{k+1} W_i$ (if exists), $f''(x) = j$ and x has j neighbors of color j in H''.

Proof. If H' has a (j,k)-coloring f' with $f'(v) = j$, then f' can be extended to W_1 by coloring all neighbors of v in W_1 and the top vertex of W_1 with k and the remaining vertices with j. But H has no (j,k)-colorings. Thus if a (j,k)-coloring f' of H' exists, then $f'(v) = k$, and by Claim 2.2.4 each of $W_2, \ldots, W_{k+1}$ contains a neighbor of v of color k. This proves (a).

Similarly, if H'' has a (j,k)-coloring f'' with either $f''(x) = k$ or with $f''(x) = j$ and at most $j-1$ neighbors of color j, then we can extend f'' to the whole H by letting $f''(v) = j$, coloring all

its neighbors in $W_1 \cup \ldots \cup W_{k+1}$ and the tops of $W_1, \ldots, W_{k+1}$ with k, and the remaining vertices in $W_1 \cup \ldots \cup W_{k+1}$ with j. $\square$

General construction. Recall that $H = H_0$ has the following properties:

(P1) H is not (j,k)-colorable;

(P2) H has no triangles and if $j = 0$, then H has girth 5;

(P3) $\phi(W) \geq -\frac{1}{k+1}$ for each $W \subseteq V(H)$, and equality holds only for $W = V(H)$

(P4) H has at least two remote bases (if $j = 0$, then x_0 also is a remote base in $H_0(0,k)$).

We now show how to use a graph H satisfying (P1)–(P4) to construct a larger graph satisfying (P1)–(P4). Take two copies, H_1 and H_2 of H. For $h = 1, 2$, choose in H_h a remote base v_h contained in (j,k)-flags $W_{h,1}, \ldots, W_{h,k+1}$ with the main neighbor x_h. Let $H' = H_1 - (W_{1,1} - v_1)$ and $H'' = H_2 - \bigcup_{i=1}^{k+1} W_{2,i}$. We get the new graph $\widetilde{H}$ by adding to $H' \cup H''$ a new vertex z adjacent to v_1 in $V(H')$ and to x_2 in $V(H'')$.

Property (P2) for $\widetilde{H}$ directly follows from (P2) for H_1 and H_2. Since $H_1 \cup H_2$ had at least four remote bases and we destroyed only two of them when creating H' and H'', (P4) holds for $\widetilde{H}$.

Suppose $\widetilde{H}$ has a (j,k)-coloring f. Then by Claim 2.2.8(a), $f(v_1) = k$ and v_1 has k neighbors of color k in $V(H')$. Thus we need $f(z) = j$. But by Claim 2.2.8(b), $f(x_2) = j$ and x_2 has j neighbors of color j in $V(H'')$. This contradiction proves (P1) for $\widetilde{H}$.

To prove (P3), consider a set W of minimum potential in $\widetilde{H}$. If $z \notin W$, then by (P3) for H, $\phi(W) = \phi(W \cap V(H')) + \phi(W \cap V(H'')) \geq 0 + 0 = 0$. Suppose $z \in W$. Then, similarly to (2.28), $v_1, x_2 \in W$. Let $W' = W \cap V(H')$ and $W'' = W \cap V(H'')$. Since adding to W'' vertex v_2 together with all $k+1$ (j,k)-flags containing v_2 would decrease the potential of W'' by $\frac{k+2}{(j+2)(k+1)}$, we conclude that $\phi(W'') \geq \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1}$ with equality only when $W'' = V(H'')$. Similarly, $\phi(W') \geq 0$ with equality only when $W' = V(H')$. Thus

$$\phi(W) \geq \phi(W') + \phi(W'') + \phi(\{z\}) - 2 \geq 0 + \frac{k+2}{(j+2)(k+1)} - \frac{1}{k+1} + (2 - \frac{k+2}{(j+2)(k+1)}) - 2 \geq \frac{-1}{k+1},$$

with equality only when $W = V(\widetilde{H})$.

This construction yields Theorem 2.2.2:

2.2.2 On (j,k)-coloring of graphs with large girth

In this section, we prove Theorem 2.2.3.

First, we inductively define the tree $T'_d(j,k)$ which will be a gadget to construct graphs we want. For $i = 1, \ldots, k+1$, let S_i be a copy of the star $K_{1,j+1}$ with the center c_i. We subdivide each of the $j+1$ edges of each star S_i once and add edges c_1c_i for $i = 2, \ldots, k+1$. The resulting tree is $T_1(j,k)$ and c_1 is called *the center of* $T_1(j,k)$. Note that $T_1(j,k)$ has $(k+1)(j+1)$ leaves. Assume we already have defined the tree $T_{d-1}(j,k)$ and it has $(k+1)^{d-1}(j+1)^{d-1}$ leaves. Let T^0 be a copy $T_1(j,k)$ with the center c_0 and $T^1, \ldots, T^{(k+1)(j+1)}$ be disjoint copies of $T_{d-1}(j,k)$ with the centers $c_1, \ldots, c_{(k+1)(j+1)}$. Let $x_1, \ldots, x_{(k+1)(j+1)}$ be the leaves of T^0. The tree $T_d(j,k)$ with *the center* c_0 is obtained by merging c_i with x_i for all $i = 1, \ldots, (k+1)(j+1)$. Finally, the tree $T'_d(j,k)$ is obtained from two disjoint copies of $T_d(j,k)$ by adding an edge connecting their centers. The example of $T'_1(2,3)$ is in Fig. 2.9.

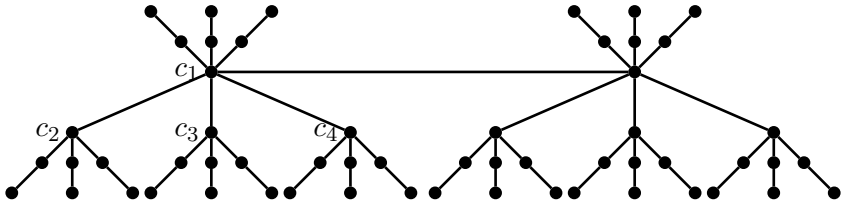


Figure 2.9 : $T'_1(2,3)$.

Claim 2.2.9. *Let f be a (j,k)-coloring of $T_d(j,k)$ with the center c_0 such that every neighbor of a leaf has color j. Then $f(c_0) = k$ and c_0 has k neighbors of color k.*

Proof. We use induction on d.

Let L be the set of all leaves of $T_1(j,k)$. If all the neighbors of L are all colored with the color j, then each of the remaining non-leaf vertices is adjacent to $j+1$ vertices of color j, and thus has color k. These vertices form a star $K_{1,k}$ with the center c_0, which yields the claim for $d = 1$.

Assume the statement holds for $d-1$. Let $T^0, T^1, \ldots, T^{(k+1)(j+1)}$ be the trees from the definition of $T_d(j,k)$ and $c_0, c_1, \ldots, c_{(k+1)(j+1)}$ be their centers. Let f be a (j,k)-coloring of $T_d(j,k)$ such that every neighbor of a leaf has color j. By the induction assumption, for each $i = 1, \ldots, (k+1)(j+1)$, $f(c_i) = k$ and c_i has k neighbors of color k in T^i. It follows that the neighbor of c_i in T^0 has color j. Again by the induction assumption, the conclusion holds for c_0. □

Lemma 2.2.10. *For $k \geq j$, in every (j,k)-coloring of $T'_d(j,k)$, some neighbor of a leaf has color k.*

Proof. Tree $T'_d(j,k)$ contains two disjoint copies T_1 and T_2 of $T_d(j,k)$ with centers c_1, c_2 connected by edge c_1c_2. If f is a (j,k)-coloring of $T'_d(j,k)$ such that every neighbor of a leaf has color j, then by the Claim 2.2.9, for $i = 1, 2$ the center c_i of T^i has color k and has k neighbors of color k in T^i. Since c_1 and c_2 are adjacent, each of them has $k+1$ neighbors of the color k, a contradiction. □

Claim 2.2.11. *Let $k \geq j$. Let L be the set of leaves in $T_d(j,k)$ and $B = V(T_d(j,k)) - L$. Then for every subgraph T of $T_d(j,k)$,*

$$|E(T)| \leq (2 - \frac{(k+2)}{(j+2)(k+1)})|B \cap V(T)|. \tag{2.32}$$

Moreover, if $T \neq T_d(j,k)$, then $|E(T)| \leq (2 - \frac{(k+2)}{(j+2)(k+1)})|B \cap V(T)| - 1$.

Proof. First, suppose that $d = 1$. Recall that in this case, $B = C \cup D$, where D is the set of vertices of degree 2 adjacent to L, $|D| = |L| = (j+1)(k+1)$, $C = \{c_1, \ldots, c_{k+1}\}$ is the set of centers of the original stars, each c_i is adjacent to $j+1$ vertices in D, and in addition c_1 is adjacent to each vertex in $C - c_1$. Thus there are three types of edges: Type 1 — the edges connecting D with L, Type 2 — the edges connecting D with C, and Type 3 — the edges connecting c_1 with $C - c_1$. We will prove (2.32) using discharging. Let every $e \in E(T)$ have charge $ch(e) = 1$ so that $\sum_{e \in E(T)} ch(e) = |E(T)|$. Now each $e \in E(T)$ distributes its charge between its neighbors according to the following rules.

Rule 1: Each edge $d\ell$ of Type 1 gives all its charge to the end $d \in D$.

Rule 2: Each edge c_id of Type 2 gives charge $1 - \frac{(k+2)}{(j+2)(k+1)}$ to the end $d \in D$ and charge $\frac{(k+2)}{(j+2)(k+1)}$ to the end $c_i \in C$.

Rule 3: Each edge c_1c_i of Type 3 gives charge $\frac{k}{k+1}$ to $c_i \in C - c_1$ and charge $\frac{1}{k+1}$ to c_1.

By the rules, only vertices of $V(T) \cap B$ may receive a positive charge and total charge on them will be exactly $|E(T)|$. Thus it is enough to prove that for every $v \in V(T) \cap B$,

$$ch(v) \leq 2 - \frac{(k+2)}{(j+2)(k+1)}. \tag{2.33}$$

If $v \in D$, then it gets at most 1 by Rule 1 and at most $1 - \frac{(k+2)}{(j+2)(k+1)}$ by Rule 2, so (2.33) holds for v. If $v = c_i$ for some $2 \leq i \leq k+1$, then it gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $\frac{k}{k+1}$ by Rule 3, so

$$ch(v) \leq (j+1)\frac{(k+2)}{(j+2)(k+1)} + \frac{k}{k+1} = 2 - \frac{(k+2)}{(j+2)(k+1)}.$$

Finally, if $v = c_1$, then it again gets at most $(j+1)\frac{(k+2)}{(j+2)(k+1)}$ by Rule 2 and at most $k\frac{1}{k+1}$ by Rule 3, so again (2.33) holds for v. This proves Case $d = 1$.

Suppose now that $d \geq 2$. Then $T_d(j,k)$ is obtained from several copies of $T_1(j,k)$ by merging leaves of some copies with the centers of some others. So if we do the discharging from $E(T)$ to $V(T) \cap B$ in each copy of $T_1(j,k)$ forming $T_d(j,k)$ by the Rules 1–3 above, then again only vertices of $V(T) \cap B$ may receive a positive charge and the total charge on them will be exactly $|E(T)|$. Moreover, since by Rule 1 the leaves of each copy of $T_1(j,k)$ will get zero charge from this copy, as we have checked above, (2.33) will hold for every $v \in V(T) \cap B$. This proves the claim. □

Proof of Theorem 2.2.3. Our goal is to show that for any $\epsilon > 0$, $g \geq 3$ and $k \geq j \geq 0$,

$$\text{there is an } (2 - \tfrac{(k+2)}{(j+2)(k+1)} + \epsilon, 0)\text{-sparse non-}(j,k)\text{-colorable graph } G \text{ of girth } g. \tag{2.34}$$

We use induction on $j + k$. If $j = k = 0$, then any odd cycle of length at least g is almost $(1,0)$-sparse and not $(0,0)$-colorable. Assume that $k \geq 1$ and (2.34) is proved for all pairs (j',k') with $j' + k' < j + k$ and $j' \leq k'$.

CASE 1: $j < k$. Then there is a graph G_0 with girth g which is not $(j, k-1)$-colorable and with

$$Mad(G_0) < 4 - \frac{2(k+1)}{(j+2)k} + 2\epsilon \leq 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{2.35}$$

Let $V(G_0) = \{v_1, \ldots, v_n\}$. Fix an integer $d > \frac{1}{\epsilon}$. Let M be the number of leaves in $T'_d(j,k)$. By an old result of Erdős and Hajnal, there exists a non-n-colorable nM-uniform hypergraph H with girth g. We construct our graph G using H and many copies of G_0 and $T'_d(j,k)$ as follows:

(i) Partition each $e \in E(H)$ into n subsets $e_1, \ldots, e_n$ of size M;

(ii) Replace each vertex x in H with a copy $G_0(x)$ of G_0;

(iii) For each $e \in H$ and $1 \leq i \leq n$, if $e_i = \{x_1, \ldots, x_M\}$, we take a copy $T(e,i)$ of $T'_d(j,k)$ with the set of leaves, say, $L(e,i) = \{\ell_1, \ldots, \ell_M\}$ and for $h = 1, \ldots, M$, merge ℓ_h with the vertex v_i in the copy $G_0(x_h)$ of G_0. We will say that $T(e,1), \ldots, T(e,n)$ *belong to* e and denote $B(e,i) = V(T(e,i)) - L(e,i)$.

Let us check that the obtained graph G has the properties we need: (a) the girth of G is at least g, (b) G is not (j,k)-colorable, and (c) $Mad(G) < 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$.

For an edge $e \in E(H)$, let $G(e)$ denote the subgraph of G formed by the copies $G_0(x)$ of G_0 for all nM vertices $x \in e$ plus all the copies $T(e,i)$ of $T'_d(j,k)$ for $i = 1, \ldots, n$. If G has a cycle C of length less than g, then C is not contained in a copy of G_0 since G_0 has girth g. Moreover, then C is not contained in any $G(e)$, since all edges of $G(e)$ in $\bigcup_{i+1}^{n} T(e,i)$ are cut-edges in $G(e)$. Since H is a linear hypergraph, C yields a (hypergraph) cycle in H, and any such cycle has at least g edges, a contradiction to the choice of C. This proves (a).

Suppose we have a (j,k)-coloring f of G. Since G_0 is not $(j,k-1)$-colorable, each graph $G_0(x)$ has a vertex v_i of color k with k neighbors in $G_0(x)$ of color k in f. Let $i(x)$ be the minimum i such that $G_0(x)$ has a vertex v_i of color k with k neighbors in $G_0(x)$ of color k in f. We define a coloring ϕ of H as follows: for each $x \in V(H)$, let $\phi(x) = i(x)$. Then ϕ is an n-coloring of H, and H has no proper n-colorings. Thus there is a monochromatic $e \in E(H)$. Suppose $f(x) = i$ for each $x \in e$. By construction, all the leaves of the copy $T(e,i)$ of $T'_d(j,k)$ are in e_i; each of these leaves is of color k and has k neighbors of color k in $\bigcup_{x \in e_i} G_0(x)$. Thus none of these leaves has a neighbor of color k in $T(e,i)$. This contradicts Lemma (2.2.10). Thus (b) holds.

In order to prove (c), consider some $W \subseteq V(G)$ with the largest $\frac{|E(G[W])|}{|W|}$. If this ratio is at most 1.5, then (c) holds; otherwise by the maximality of the average degree, $G[W]$ has no isolated vertices and no leaves. Let $W' = \bigcup_{x \in V(H)} (W \cap V(G_0(x)))$. Then $W - W' = \bigcup_{e \in E(H)} \bigcup_{i=1}^{n} (W \cap B(e,i))$. Since each component of $G[W']$ is contained in some $G_0(x)$, by (2.35), the average degree of $G[W']$ is less than $4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon$. We can obtain W from W' by a sequence of adding the sets

$W \cap B(e,i)$, one by one. We will show that after every such step,

$$\text{the average degree of the obtained subgraph remains less than } 4 - \tfrac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{2.36}$$

Indeed, suppose it is the turn to add to a current set W'' the set $W \cap B(e,i)$. Let c_1c_1' be the edge in $T(e,i)$ connecting the centers c_1 and c_1' of the two disjoint copies of $T_d(j,k)$. If $\{c_1, c_1'\} \not\subset W$, then by Claim 2.2.11, adding $W \cap B(e,i)$ to W'' adds at most $(2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges, as claimed. So let $\{c_1, c_1'\} \subset W$. Since $G[W]$ has no leaves, W contains the vertices of disjoint paths from c_1 and c_1' to $L(e,i)$ and thus $|W \cap B(e,i)| \geq 4d$. Again by Claim 2.2.11, adding $W \cap B(e,i)$ to W'' adds at most $1 + (2 - \frac{(k+2)}{(j+2)(k+1)})|W \cap B(e,i)|$ edges. Since $d > 1/\epsilon$ and $|W \cap B(e,i)| \geq 4d$, the last expression less than $(2 - \frac{(k+2)}{(j+2)(k+1)} + \epsilon)|W \cap B(e,i)|$, as claimed. This proves (c).

CASE 2: $0 < j = k$. Then there is a graph G_0 with girth g which is not $(k-1,k)$-colorable and with

$$Mad(G_0) < 4 - \frac{2(k+2)}{(k+1)^2} + 2\epsilon \leq 4 - \frac{2(k+2)}{(j+2)(k+1)} + 2\epsilon. \tag{2.37}$$

Now we simply repeat the proof of Case 1 with the only twist that using $j = k$, we consider G_0 as not $(k, k-1)$-colorable instead of not $(k-1,k)$-colorable. □

Concluding remark. Studying improper colorings with more colors, one can consider the function $F_{a_1,\ldots,a_t}(g)$ generalizing $F_{j,k}(g)$. Using similar techniques, we can prove the following extension of Theorem 2.2.3.

Theorem 2.2.12. *Let $a_1 \leq a_2 \leq \cdots \leq a_t$, $t \geq 2$ and $g \geq 3$. Then $F_{a_1,\ldots,a_t}(g) \leq t - \frac{(a_2+2)}{(a_1+2)(a_2+1)}$.*

Since we do not know how sharp is this bound, we do not supply a proof of Theorem 2.2.12.

Chapter 3

Variations of coloring problems

3.1 Harmonious coloring of trees with large maximum degree

We define *harmonious coloring* and *harmonious chromatic number*, $h(G)$ in 1.2. The first paper [25] on harmonious coloring appeared in 1982. However, the proper definition of this notion is due to Hopcroft and Krishnamoorthy [28]. Harmonious coloring of a graph is essentially an edge-injective homomorphism from a graph G to a complete graph and the harmonious chromatic number of G is the minimum order of a complete graph that admits such homomorphism from G. Paths and cycles are among the first graphs whose harmonious chromatic numbers have been established [25]. It was shown by Hopcroft and Krishnamoorthy that the problem of determining the harmonious chromatic number of a graph is NP-hard. Moreover, Edwards and McDiarmid [20] showed that the problem remains hard even restricted to the class of trees. Since the problem is hard in the class of all trees, it makes sense to identify subclasses in which the problem is easier.

Since vertices of distance at most two in a graph G must have distinct colors in any harmonious coloring of G, $h(G) \geq \Delta(G) + 1$ for every graph G. In [3] it was shown that if T is a tree of order n and $\Delta(T) \geq \frac{n}{2}$, then $h(T) = \Delta(T) + 1$. Moreover, the proof yields a polynomial-time algorithm for an optimal harmonious coloring of such a tree. We strengthen this result by finding a wider class of trees T for which $h(T) = \Delta(T) + 1$.

Theorem 3.1.1. *Let*

$$\Delta(T) \geq \frac{n+2}{3}. \tag{3.1}$$

If T is a forest of order n, then

$$h(T) = \begin{cases} \Delta(T)+2, & \text{if } T \text{ has non-adjacent vertices of degree } \Delta(T); \\ \Delta(T)+1, & \text{otherwise.} \end{cases}$$

Moreover, there is a polynomial-time algorithm for an optimal harmonious coloring of such a forest.

We prove the lower bounds in the theorem and show that the bound $\Delta(T) \geq \frac{n+2}{3}$ is sharp before we prove the upper bounds in the theorem.

Since in each harmonious coloring f of a graph G, the colors of all neighbors of a vertex v are different and distinct from $f(v)$,

$$h(G) \geq 1 + \Delta(G) \qquad \text{for every graph } G. \tag{3.2}$$

Claim 3.1.2. *Let $k \geq 1$. If a graph G contains two non-adjacent vertices, say u_1 and u_2, of degree k, then $h(G) \geq k+2$.*

Proof. Suppose that G has a harmonious $(k+1)$-coloring f with colors in $A + \{\alpha_1, \ldots, \alpha_{k+1}\}$. Then by (3.2), for each $j = 1, \ldots, k+1$, the set $f^{-1}(\alpha_j)$ has a vertex in $N_G[u_1]$ and a vertex in $N_G[u_2]$. If $f(u_1) \neq f(u_2)$, then the pair $\{f(u_1), f(u_2)\}$ appears on two pairs of adjacent vertices: one pair in $N_G[u_1]$ and one pair in $N_G[u_2]$. And if $f(u_1) = f(u_2)$, then for each $\alpha \in A - f(u_1)$, the pair $\{f(u_1), \alpha\}$ appears on two pairs of adjacent vertices. So, f is not harmonious. Thus $h(G) \geq k+2$.

Now for every $D \geq 3$ we present a tree T_D such that (i) $|V(T)| = 3D-1$, (ii) $\Delta(T) = D$, (iii) T has no non-adjacent vertices of degree D, and (iv) $h(T) \geq D+2 = \Delta(T)+2$. These examples show that the restriction (3.1) in Theorem 3.1 cannot be weakened.

Let T_D be obtained from a 4-vertex path (v_1, v_2, v_3, v_4) by adding $D-1$ leaves adjacent to v_1, $D-2$ leaves adjacent to v_2, and $D-2$ leaves adjacent to v_4. Tree T_4 is depicted in Fig. 3.1.

By construction, T_D is a tree with maximum degree D and $|V(T_D)| = 4 + (D-1) + (D-2) + (D-2) = 3D-1$. So, (i) and (ii) hold; and (iii) is also evident. We establish (iv) by proving the following.

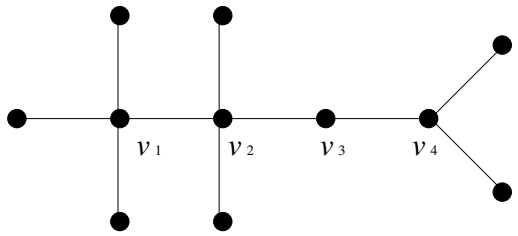


Figure 3.1: Tree T_4.

Claim 3.1.3. *$h(T_D) = D + 2$.*

Proof. Suppose f is a harmonious coloring of T_D with $D + 1$ colors. We may assume that $f(v_1) = 1, f(v_2) = 2$, and $f(v_3) = 3$. Also we may assume that $f(N(v_1) - v_2) = \{3, 4, \ldots, D + 1\}$. Then $f(N(v_2) - v_1 - v_3) = \{4, \ldots, D + 1\}$. Since $d_{T_D}(v_1) = d_{T_D}(v_2) = D$, no other vertices can be colored 1 or 2 in a harmonious $(D + 1)$-coloring. Thus, we may assume $f(v_4) = 4$. Then $f(N(v_4)-v_3) \subset \{5, 6, \ldots, D+1\}$ should hold. But $N(v_4)-v_3$ has $D-2$ vertices and only $D-3$ colors are available. Therefore we cannot complete the coloring with $D + 1$ colors. Thus $h(T_D) > D + 1$.

Now, we present polynomial-time coloring procedures yielding that

(*) *if (3.1) holds and T is an n-vertex forest with maximum degree $\Delta(T)$ such that T has no non-adjacent vertices of degree $\Delta(T)$, then $h(T) = \Delta(T) + 1$.*

First, observe that the statement holds for $\Delta(T) \leq 2$: By (3.1), $n \leq 3\Delta(T)-2$. So, if $\Delta(T) \leq 1$, then $n \leq 1$, and so $h(T) \leq 1 \leq 1 + \Delta(T)$. If $\Delta(T) = 2$, then $n \leq 4$ and hence T is a subgraph of the 4-vertex path P_4 whose harmonious chromatic number is $3 = \Delta(T) + 1$. So, everywhere below

$$\Delta(T) \geq 3. \tag{3.3}$$

Second, let us check that it is enough to prove (*) for trees. Indeed, if T is a disconnected n-vertex forest satisfying (3.1) and (3.3) with no non-adjacent vertices of degree $\Delta(T)$, then by adding an edge connecting two leaves or isolated vertices from different components of T, we again get a forest with these properties and fewer components. Thus in Section 3.1 we will assume that T is a tree.

Let $v \in V(T)$ be a vertex of degree $\Delta(T)$. We will construct a harmonious coloring $f : V(T) \to$

$V(K_{\Delta(T)+1})$ step by step. The vertices of $H := K_{\Delta(T)+1}$ will by denoted by Greek letters so that we do not mix them with the vertices of T. We start from mapping v and the $\Delta(T)$ neighbors of v in T into the all different $\Delta(T)+1$ vertices of H. Let $f(w)$ denote the color of w. If $f(w)$ is not defined yet, then w is an *uncolored vertex*, otherwise it is a *colored vertex*.

We consider several cases.

Case 0: T consists of two stars with a path joining them. This case is straightforward.

Case 1: v is the only vertex of degree $\Delta(T)$ in T, and T has no vertices of degree $\Delta(T)-1$. Suppose that we have already defined $f(w)$ for some vertices $w \in V(T)$ (in particular, $f(w)$ is defined for $w \in N_T[v]$). For $\alpha \in V(H)$, let

$$d(\alpha) := \sum_{x \in f^{-1}(\alpha)} d_T(x).$$

Also, we will say that vertices α and β of H are T-*adjacent*, if there are $x \in f^{-1}(\alpha)$ and $y \in f^{-1}(\beta)$ such that $xy \in E(T)$. Our procedure will color one vertex at each step. It works as follows:

(a) Choose a vertex $w \in V(T)$ such that $f(w)$ is defined and w has a neighbor u for which f is not defined and u is not a leaf. If there are no such vertex, then choose u which is a leaf.

(b) If there is $\gamma \in V(H) - f(w)$ such that (i) γ is not T-adjacent to $f(w)$ and (ii) $d(\gamma)+d_T(u) \leq \Delta(T)$, then we let $f(u)$ be any γ satisfying (i)–(ii) and go to (a) of the next step.

(c) If no $\gamma \in V(H) - f(w)$ satisfies (i)–(ii), then we stop.

We need to prove that we do not stop until we embed all T. Note that after the initial coloring of $N_T[v]$, for every $\alpha \in V(H)$ we have $|f^{-1}(\alpha)| = 1$, and hence $d(\alpha) \leq \Delta(T)$.

Suppose that we stopped in some step, before $f(x)$ was defined for every $x \in V(T)$. This means that at the moment of stopping, either every $\gamma \in V(H) - f(w)$ is T-adjacent to $f(w)$ or

$$d(\gamma) + d_T(u) \geq \Delta(T)+1 \quad \text{for every } \gamma \in V(H) - f(w) \text{ not } T\text{-adjacent to } f(w). \tag{3.4}$$

If the former holds, then since $f(u)$ is not defined yet, at the moment of defining $f(w)$ we had already had $d(f(w))+d_T(w) \geq \Delta(T)+1$ and should have stopped then. Thus some $\gamma \in V(H)-f(w)$ is not T-adjacent to $f(w)$, and (3.4) holds. We may assume that $f(w) = \gamma_0$. Let $\gamma_1, \ldots, \gamma_r$ be the vertices

of $H-\gamma_0$ not T-adjacent to γ_0, and $\gamma_{r+1},\ldots,\gamma_{\Delta(T)}$ be the vertices of H that are T-adjacent to γ_0. And let $\gamma_{\Delta(T)}=f(v)$. By the above, $r\geq 1$.

By the choice of u, $u\neq v$. So in our case $d(u)\leq \Delta(T)-2$. Thus by (3.4),

$$d(\gamma_i)\geq \Delta(T)+1-d_T(u)\geq 3 \quad \text{for every } 1\leq i\leq r. \tag{3.5}$$

Since $f(v)$ is T-adjacent to every other vertex in H, according to our rules, $f(x)\neq f(v)$ for every $x\neq v$.

For every $W\subseteq V(T)$,

$$n-1=|E(T)|\geq \sum_{w\in W} d_T(w)-|E(T[W])|. \tag{3.6}$$

We may assume that $d(\gamma_1)\leq d(\gamma_2)\leq\ldots\leq d(\gamma_r)$. Let

$$W:=\{v,u\}\cup f^{-1}(\{\gamma_0,\gamma_1,\ldots,\gamma_r\}).$$

Case 1.1: $r=1$. Then, since γ_0 is T-adjacent to $\Delta(T)-1$ vertices in H and $uw\in E(T)$, $d(\gamma_0)\geq\Delta(T)$. By (3.4), $d_T(u)+d(\gamma_1)\geq\Delta(T)+1$. So since γ_1 is not T-adjacent to γ_0, the graph $T[W]$ has exactly 3 edges, uw, vx_0 and vx_1, where x_i is a neighbor of v with $f(x_i)=\gamma_i$, for $i=0,1$. Thus using this W in (3.6), we have

$$|E(T)|\geq d_T(v)+d(\gamma_0)+d_T(u)+d(\gamma_1)-3\geq\Delta(T)+\Delta(T)+(\Delta(T)+1)-3=3\Delta(T)-2.$$

So, $3\Delta(T)-2\leq n-1$, i.e., $\Delta(T)\leq\frac{n+1}{3}$, a contradiction.

Case 1.2: $r=2$. Similarly to Case 1.1, $d(\gamma_0)\geq\Delta(T)-1$ and $d_T(u)+d(\gamma_1)\geq\Delta(T)+1$. Now γ_1 and γ_2 are not T-adjacent to γ_0. So, graph $T[W]$ has at most 5 edges. Thus, using this W in (3.6), we have (using also (3.5) to estimate $d(\gamma_2)$)

$$|E(T)|\geq d_T(v)+d(\gamma_0)+d_T(u)+d(\gamma_1)+d(\gamma_2)-5\geq\Delta(T)+(\Delta(T)-1)+(\Delta(T)+1)+3-5=3\Delta(T)-2,$$

a contradiction as in Case 1.1.

Case 1.3: $r \geq 3$. Now $d(\gamma_0) \geq \Delta(T) - r + 1$ and $\gamma_1, \ldots, \gamma_r$ are not T-adjacent to γ_0. So,

$$|E(T[W])| \leq (1 + r) + 1 + \sum_{i=1}^{r} \frac{d(\gamma_i) - 1}{2} \tag{3.7}$$

(here $r + 1$ counts the edges incident with v, 1 stands for the edge uw and $\sum_{i=1}^{r} \frac{d(\gamma_i)-1}{2}$ estimates from above the number of edges both ends of which are in $\{\gamma_1, \ldots, \gamma_r\}$). So by (3.6),

$$n - 1 = |E(T)| \geq d_T(v) + d_T(u) + \sum_{i=0}^{r} d(\gamma_i) - r - 2 - \sum_{i=1}^{r} \frac{d(\gamma_i) - 1}{2}$$

$$\geq \Delta(T) + (\Delta(T) - r + 1) + (d_T(u) + d(\gamma_1)) - d(\gamma_1) - r - 2 + \sum_{i=1}^{r} \frac{d(\gamma_i) + 1}{2}$$

$$\geq (2\Delta(T) - r + 1) + (\Delta(T) + 1) - d(\gamma_1) - r - 2 + r\frac{d(\gamma_1) + 1}{2} = 3\Delta(T) - 2r + (r - 2)\frac{d(\gamma_1)}{2} + \frac{r}{2}$$

$$= 3\Delta(T) - 3 + (d(\gamma_1) - 3)\frac{r - 2}{2} \geq 3\Delta(T) - 3 + \frac{d(\gamma_1) - 3}{2}.$$

Thus if $d(\gamma_1) \geq 4$ or if (3.7) is a strict inequality, or if $d(\gamma_0) > \Delta(T) - r + 1$, then we have $n - 1 > 3\Delta(T) - 3$, which yields $\Delta(T) \leq \frac{n+1}{3}$, a contradiction. So, by (3.5) we may suppose that $d(\gamma_1) = 3$, $d(\gamma_0) = \Delta(T) - r + 1$, and (3.7) holds with equality. In particular, by (3.5), $d_T(u) = \Delta(T) - 2$. Since $r \geq 3$, we have $\Delta(T) \geq 1 + r \geq 4$, and so u is not a leaf. Since we do not have Case 0, there is a leaf l not adjacent to u and not adjacent to v. Thus, according to our rule (a), l is not colored yet. Since $d(\gamma_0) = \Delta(T) - r + 1$, and (3.7) holds with equality, l is adjacent neither to any vertex in $f^{-1}(\gamma_0)$ nor to any vertex in $f^{-1}(\{\gamma_1, \ldots, \gamma_r\})$. Hence the right-hand side of (3.6) does not count the edge incident with l. So, we have $n - 2 \geq 3\Delta(T) - 3$, a contradiction to $\Delta(T) > \frac{n+1}{3}$.

Therefore we do not stop until we color all the vertices in T.

Let $(u_1, \ldots, u_n)$ be an ordering of the vertices of T such that $u_1 = v$ and $d_T(u_1) \geq d_T(u_2) \geq \ldots \geq d_T(u_n)$. In these terms, Case 1 was the case $d_T(u_2) \leq \Delta(T) - 2$. Let t be chosen so that $d_T(u_t) \geq 2$ and $d_T(u_{t+1}) = 1$.

Case 2: $d_T(u_3) \geq \Delta(T) - 1$. Let $W' := \{u_1, \ldots, u_t\}$. Since T is connected, $T[W']$ is also connected. Then

$$\sum_{w \in W'} d_T(w) \geq \Delta(T) + (\Delta(T) - 1) + (\Delta(T) - 1) + 2(t-3) \quad \text{and} \quad |E(T[W'])| = t - 1.$$

So by (3.6),

$$n - 1 = |E(T)| \geq 3\Delta(T) - 2 + 2(t-3) - (t-1) \geq (n+2) + t - 7 = n + t - 5.$$

It follows that $t \leq 4$, and that if $t = 4$, then $d_T(u_2) = \Delta(T) - 1$ and $d_T(u_4) = 2$.

Case 2.1: $t = 3$. The only 3-vertex tree is the 3-vertex path. So, $T[W']$ is the path (w_1, w_2, w_3). By (*), we may assume that $w_3 = u_3$. For $i = 1, 2, 3$, we let $f(w_i) = \gamma_{i-1}$. We place the leaves adjacent to w_1 into any $d_T(w_1) - 1$ vertices in $V(H) - \gamma_0 - \gamma_1$, the leaves adjacent to w_2 into any $d_T(w_2) - 2$ vertices in $V(H) - \gamma_0 - \gamma_1 - \gamma_2$, and the $\Delta(T) - 2$ leaves adjacent to w_3 into the vertices in $V(H) - \gamma_0 - \gamma_1 - \gamma_2$.

Case 2.2: $t = 4$. As it was mentioned, in this case $d_T(u_2) = d_T(u_3) = \Delta(T) - 1$ and $d_T(u_4) = 2$. Since $T[W']$ is connected and $d_T(u_4) = 2$, we may assume that u_3 is adjacent either to u_1 or to u_2. For $i = 1, 2, 3, 4$, we let $f(u_i) = \gamma_{i-1}$. Then for $j = 1, 2, 3, 4$, we place the leaves adjacent to u_j into the vertices of $H - \gamma_0 - \ldots - \gamma_{j-1}$ not occupied by the neighbors of u_j in W'. We can do it for $j = 1, 2$, since $d_T(u_j) = 1 + \Delta(T) - j$ for these j. And u_3 was chosen so that $\gamma_2 = f(u_3)$ is T-adjacent to $\{\gamma_0, \gamma_1\}$. Finally, since $T[W']$ is connected, u_4 has at most one adjacent leaf. So if $\Delta(T) \geq 4$ or u_4 has no adjacent leaves, then we are done. Thus we need only to handle the situation when $\Delta(T) = 3$ and each vertex of degree 2 in T has an adjacent leaf. Then $T[W'] = K_{1,3}$. For $i = 2, 3, 4$, let w_i be the leaf adjacent to u_i. In this case, we again let $f(u_i) = \gamma_{i-1}$ for $i = 1, 2, 3, 4$ and then let $f(w_2) = \gamma_2$, $f(w_3) = \gamma_3$, and $f(w_4) = \gamma_1$.

Case 3: $d_T(u_2) = \Delta(T)$. Under Condition (*), $vu_2 \in E(G)$, and by Case 2, $d_T(u_3) \leq \Delta(T) - 2$. Since $vu_2 \in E(G)$, $f(u_2)$ was defined at the first step. Then we can apply the procedure of Case 1, and the argument goes through since $d_T(u_3) \leq \Delta(T) - 2$.

The only case, we have not yet considered is:

Case 4: $d_T(u_2) = \Delta(T) - 1$ and $d_T(u_3) \leq \Delta(T) - 2$. Let $P = (v_1, \ldots, v_q)$ be the path in T connecting $v_1 = v$ with $v_q = u_2$. Suppose v has exactly p adjacent non-leaves in T. We claim that

$$q + p \leq \Delta(T) + 2, \tag{3.8}$$

since otherwise

$$n \geq q+(p-1)+(d_T(v)-1)+(d_T(u_2)-1) \geq (\Delta(T)+3-1)+(\Delta(T)-1)+(\Delta(T)-1-1) = 3\Delta(T)-1,$$

a contradiction to (3.1).

By (3.8), we can place all the vertices of P and all remaining non-leaf neighbors of v into distinct vertices of H. After that, we place the leaves adjacent to v into distinct vertices of H not containing v or its neighbors. Then we again apply the procedure of Case 1 and the argument goes through since $d_T(u_3) \leq \Delta(T) - 2$.

Thus this completes the algorithm for the case where T has no non-adjacent vertices of degree $\Delta(T)$. If T has non-adjacent vertices v and z of degree $\Delta(T)$. We add a vertex w to T and make w adjacent to v to get a tree T' with maximum degree $\Delta(T) + 1$. Then we may apply Case 1 to T' and color T' with $\Delta(T) + 2$ colors. This harmonious coloring of T' gives a harmonious coloring of T with $\Delta(T) + 2$ colors. This completes the proof of Theorem 1.

Remark. Since we colored vertices one by one with no recolorings, and the choice of every next vertex took polynomial time, the total time taken by our algorithm is polynomial.

3.2 On r-dynamic chromatic number of graphs

Imagine that students in a class are to be placed in groups. To encourage communication, friends must be in different groups, and each student should encounter many groups among his or her friends. Requiring all friends of each student to be in different groups splinters the students into too many groups. Instead, we specify a threshold r; the friends of a student with d friends must represent at least $\min\{r, d\}$ groups. We ask how many groups are needed.

We defined *r-dynamic k-coloring* and *r-dynamic chromatic number*, $\chi_r(G)$ in 1.2.

The 1-dynamic chromatic number of a graph G is its chromatic number $\chi(G)$. The 2-dynamic chromatic number was introduced as *dynamic chromatic number* by Montgomery [45]; he conjectured $\chi_2(G) \leq \chi(G) + 2$ when G is regular, which remains open. Alishahi [8] showed that for all k there is a k-chromatic regular graph G with $\chi_2(G) \geq \chi(G) + 1$. Akbari et al. [4] proved Montgomery's conjecture for bipartite regular graphs. Lai, Montgomery, and Poon [41] proved $\chi_2(G) \leq \Delta(G) + 1$ when $\Delta(G) \geq 3$ and no component is the 5-cycle C_5.

Akbari et al. [5] strengthened this to the list context: $\mathrm{ch}_d(G) \leq \Delta(G) + 1$ under the same conditions, where $\mathrm{ch}_d(G)$ is the least k such that a 2-dynamic coloring can be chosen from any lists of size k assigned to the vertices. Kim and Park [37] proved $\mathrm{ch}_d(G) \leq 4$ for planar G with girth at least 7, and $\mathrm{ch}_d(G) \leq k$ when $k \geq 4$ and G has maximum average degree at most $\frac{4k}{k+2}$ (both results are sharp). Kim, Lee, and Park [38] proved $\chi_2(G) \leq 4$ when G is planar and no component is C_5; also, $\mathrm{ch}_d(G) \leq 5$ whenever G is planar.

Given a graph G, let G^2 denote the graph obtained from G by adding edges joining nonadjacent vertices that have a common neighbor. Properly coloring G^2 is a restricted version of properly coloring G, require vertices with the same color to be distance at least 3 apart. One motivation for the study of r-dynamic chromatic number is that it provides a spectrum of parameters between $\chi(G)$ and $\chi(G^2)$.

Observation 3.2.1. *Always* $\chi(G) = \chi_1(G) \leq \cdots \leq \chi_{\Delta(G)}(G) = \chi(G^2)$. *If* $r \geq \Delta(G)$, *then* $\chi_r(G) = \chi_{\Delta(G)}(G)$.

Observation 3.2.2. $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$, *with equality when* G *is a tree.*

Proof. The closed neighborhood of a vertex of maximum degree needs $\min\{\Delta(G), r\} + 1$ colors. When G is a tree, take one vertex as a root and iteratively color children of each vertex greedily with colors different from the parent; $\min\{\Delta(G), r\} + 1$ colors suffice. □

We begin in Section 3.2.1 with an analogue of Brooks' Theorem [13] for r-dynamic coloring. We prove $\chi_r(G) \leq r\Delta(G) + 1$ and characterize the graphs achieving equality. Our result is not a generalization of Brooks' Theorem, because although $\chi_r(G) = \chi(G)$ when $r = 1$, the characteriza-

tion of equality for $r \geq 2$ does not reduce to Brooks' Theorem when $r = 1$. In particular, for $r \geq 2$ equality holds if and only if G is an r-regular graph with diameter 2 and girth 5. Such graphs are known as *Moore graphs* and are quite rare; the only examples are C_5, the 3-regular Petersen graph, the 7-regular Hoffman–Singleton graph, and possibly a 57-regular graph not known to exist.

When the minimum degree is not too small, we can greatly improve the bound in terms of the maximum degree. For an n-vertex graph G, we show that $\delta(G) \geq \frac{r+s}{s+1}(r+1)\ln n$ implies $\chi_r(G) \leq \Delta(G) + r + s$. In particular, $\delta(G) \geq 2r \ln n$ implies $\chi_r(G) \leq \Delta(G) + 2r$, and $\delta(G) \geq r(r+1)\ln n$ implies $\chi_r(G) \leq \Delta(G) + r$.

In Section 3.2.2, we study bounds for regular graphs in terms of the chromatic number, motivated by Montgomery's conjecture. Akbari et al. [4] proved $\chi_2(G) \leq 2\chi(G)$ for every k-regular graph G. Alishahi [7] proved $\chi_2(G) \leq \chi(G)+14.06 \ln k+1$ (and later $\chi_2(G) \leq \chi(G)+2\lceil 4 \ln k + 1 \rceil$ [8]), which Taherkhani [52] improved to $\chi_2(G) \leq \chi(G) + \lceil 5.437 \ln k + 2.721 \rceil$. For general r, we mention our result in [32] showing $\chi_r(G) \leq r\chi(G)$ for k-regular graphs G with $k \geq (3 + o(1))r \ln r$. This result appeared in [48] and the same conclusion when $k \geq 7r \ln r$, and later Taherkhani [52] obtained a similar result by essentially the same method as ours. When k is not sufficiently large in terms of r, the ratio $\chi_r(G)/\chi(G)$ can grow superlinearly in r; we provide an example using Kneser graphs where $k = r$ and $\chi_r(G) > r^{1.37744}\chi(G)$.

In Section 3.2.3 we consider k-chromatic graphs with small diameter. We first give a short proof of $\chi_2(G) \leq \chi(G) + 2$ for every graph G with diameter 2. Furthermore, equality holds only for complete bipartite graphs and C_5. For regular graphs with $\chi(G) \geq 4$, Alishahi [8] proved $\chi_2(G) \leq \chi(G) + \alpha(G^2)$ (here $\alpha(G)$ is the maximum size of an independent set of vertices and G^2 is obtained from G by adding edges joining vertices with common neighbors; note that $\alpha(G^2) = 1$ when $\mathrm{diam}(G) = 2$).

Moving on to diameter 3, we prove $\chi_2(G) \leq 3\chi(G)$ when $\mathrm{diam}(G) \leq 3$, and this is sharp. For graphs with diameter 4 there is no bound in terms of chromatic number: subdividing every edge of the complete graph K_n yields a bipartite graph G with $\mathrm{diam}(G) = 4$ and $\chi_2(G) = n$. In contrast, χ_3 is unbounded already on bipartite graphs with diameter 2.

To study χ_2 on graphs with diameter 3 we use a related notion from hypergraph coloring. A

vertex coloring of a hypergraph H is *c-strong* if every edge e has at least $\min\{c, |e|\}$ distinct colors; this concept was introduced in [12]. An r-dynamic of a graph G is a proper coloring of G that also is an r-strong coloring of the hypergraph H with vertex set $V(G)$ whose edge set is the set of neighborhoods of vertices in G. Thus $\chi_r(G) \geq \chi(H; r)$, where $\chi(H; c)$ is the minimum number of colors in a c-strong coloring of H. More importantly, $\chi_r(G) \leq \chi(G)\chi(H; r)$, by combining a proper coloring of G with an r-strong coloring of H.

3.2.1 Bounds in Terms of Maximum Degree

We prove an upper bound on $\chi_r(G)$ in terms of $\Delta(G)$ and characterize (for $r \geq 2$) when equality holds. Not only is the result analogous to Brooks' Theorem, also the proof idea is like that of a well-known proof of Brooks' Theorem by Lovász [43].

Theorem 3.2.3. *$\chi_r(G) \leq r\Delta(G) + 1$, with equality for $r \geq 2$ if and only if G is r-regular with diameter 2 and girth 5.*

Proof. If G is r-regular, then vertices having a common neighbor must receive distinct colors in an r-dynamic coloring. If $\text{diam}(G) = 2$, then all pairs of vertices are adjacent or have a common neighbor. Hence $\chi_r(G) = |V(G)|$. The maximum value of $|V(G)|$ is $r^2 + 1$, which occurs if and only if G has girth 5. Hence equality holds for Moore graphs.

For the upper bound, we may assume that G is connected, by considering each component. Choose a vertex v_n, and use a spanning tree to order the vertices as $v_1, \ldots, v_n$ so that each vertex before v_n has a higher-indexed neighbor; this is an *ascending ordering* to v_n. Color the vertices in the order $v_1, \ldots, v_n$. A vertex is *dangerous* when its neighborhood does not yet have r colors. When coloring v_i, avoid each color used on a neighbor of v_i or on a neighbor of a dangerous neighbor of v_i. This is the *greedy coloring procedure* with respect to the ordering; the first $\min\{d(v), r\}$ colors assigned to the neighborhood of any vertex v are distinct. At most r colors must be avoided for each neighbor of v_i.

For $i < n$, the uncolored higher-indexed neighbor of v_i means that at most $r\Delta(G) - 1$ colors need to be avoided when coloring v_i. Hence $r\Delta(G)$ colors suffice, except possibly for the last vertex. When G is not regular, choosing v_n to be a vertex of minimum degree completes the proof, since

v_n needs to avoid at most $r(\Delta(G)-1)$ colors.

Hence we may assume that G is regular. If $\Delta(G) > r$, then when v_n is colored in an ascending ordering it is the last uncolored vertex. Under the greedy procedure, its neighbors already have at least r distinct colors in their neighborhoods, so none are dangerous. Therefore only $\Delta(G)$ colors need to be avoided in coloring v_n. If $\Delta(G) < r$, then Observation 3.2.1 yields $\chi_r(G) = \chi(G^2) \leq \Delta(G^2)+1 \leq \Delta(G)^2+1 \leq r\Delta(G)$.

Hence we may assume that G is r-regular. If G has a cycle of length at most 4, then let v_n be a vertex on a shortest cycle C and color greedily in an ascending ordering to v_n. At most r^2 colors are used in coloring $v_1, \ldots, v_{n-1}$. When coloring v_n, the two neighbors of v_n on C generate at least one common color to be avoided (on a vertex of C), leaving at most r^2-1 colors to be avoided. Thus r^2 colors suffice.

Hence we may assume that G has girth at least 5, which yields r^2 other vertices within distance 2 of each vertex. If $\mathrm{diam}(G) = 2$, then we are finished, so assume $\mathrm{diam}(G) \geq 3$. Let u and w be vertices at distance 3, with $\langle u, x, y, w\rangle$ an induced path. Let $T_1, \ldots, T_k$ be the components of $G - \{u, w\}$, with $x, y \in T_k$.

Color u and w first (each get color 1), and then use an increasing ordering in each T_i to a neighbor of u or w, leaving T_k last with an increasing ordering to x. As usual, at most r^2-1 colors must be avoided on any vertex of T_i before the last. For $i < k$, the last vertex v in T_i has an uncolored vertex at distance 2 (it is x or y), so it needs to avoid at most r^2-1 colors. Finally, when coloring x, the two vertices u and w have the same color, so again at most r^2-1 colors need to be avoided on x. □

We believe that also there is no graph G with $\chi_r(G) = r\Delta(G)$ other than cycles whose length is not divisible by 3 (when $r = 2$); that is, when $\Delta(G) > 2$ and Moore graphs are excluded, the upper bound should improve further. It is known for example that $\chi_2(G) \leq \Delta(G)+1$ when $\Delta(G) \geq 3$ and no component of G is C_5 [41]. We present a restricted construction where the bound cannot be improved by much.

Example 3.2.4. *Graphs with $\chi_r(G) = r\Delta(G)-1$ when $r = \Delta(G)$. When $r = \Delta(G)$, deleting an edge uv from a Moore graph with $\Delta(G) > 2$ yields a graph G with $\chi_r(G) = r\Delta(G)-1 = r^2-1$.*

For the lower bound, any two vertices in $V(G) - \{u, v\}$ are adjacent or have a common neighbor, so they must have distinct colors. For the upper bound, give distinct colors to $V(G) - \{u, v\}$, give u a color in $N(v)$, and give v a color in $N(u)$; now no color is on two vertices with a common neighbor in G.

Let G_i for $i \in \mathbb{Z}_k$ be a copy of this graph G. Add the edges joining the copy of v in G_i to the copy of u in G_{i+1}, for all $i \in \mathbb{Z}_k$. Since $\Delta(G) > 2$, the duplicated colors in successive copies of G can be chosen to be distinct. Thus infinitely many 2-connected graphs are constructed with r-dynamic chromatic number $r\Delta(G) - 1$, but only for $\Delta(G) = r$ and $r \in \{2, 3, 7\}$ (and possibly 57).

Qestion 3.2.5. *For fixed r and k, what is the best bound on $\chi_r(G)$ that holds for all but finitely many graphs G with maximum degree k?*

We show next that if the minimum degree is not too small relative to the number of vertices (for fixed r), then the bound in Theorem 3.2.3 can be improved by replacing the product with a sum involving $\Delta(G)$ and r. The idea is to modify the greedy coloring algorithm used in Theorem 3.2.3. There we ensured that the first r neighbors of a vertex would have distinct colors. Now we will allow r distinct colors to be obtained at any time.

Theorem 3.2.6. *If G is an n-vertex graph, and $\delta(G) \geq \frac{r+s}{s+1}(r+1)\ln n$, then $\chi_r(G) \leq \Delta(G)+r+s$. In particular, $\delta(G) \geq 2r \ln n$ implies $\chi_r(G) \leq \Delta(G) + 2r$, and $\delta(G) \geq r(r+1)\ln n$ implies $\chi_r(G) \leq \Delta(G) + r$.*

Proof. The special cases arise by setting $s = r$ and $s = 0$ in the general statement.

If $n \leq \Delta(G) + r + s$, then we may use distinct colors on the vertices, so we may assume $n > \Delta(G) + r + s$. Let $v_1, \ldots, v_n$ be any vertex ordering of G. Generate integers $a_1, \ldots, a_n$ by choosing each a_i randomly and uniformly from $\{1, \ldots, r + s\}$. Color the vertices in the order $v_1, \ldots, v_n$. When coloring v_i, choose the a_ith smallest color among the colors not yet used on neighbors of v_i. This produces a proper coloring with at most $\Delta(G) + r + s$ colors.

We claim that with positive probability the coloring is also r-dynamic. This fails at a vertex v only if the colors in $N(v)$ are confined to a particular set of $r - 1$ colors. The probability that this happens with a particular set of $r - 1$ colors is bounded by $(\frac{r-1}{r+s})^{\delta(G)}$, which in turn is bounded

by $e^{-\delta(G)\frac{s+1}{r+s}}$. There are $\binom{\Delta(G)+s}{r-1}$ choices of a set of $r-1$ colors, which is bounded by n^{r-1} since $\Delta(G)+r+s<n$.

Since G has n vertices, the probability of having a bad vertex is bounded by $n^r e^{-\delta(G)\frac{s+1}{r+s}}$. The constraint on $\delta(G)$ bounds this by $n^r n^{-(r+1)}$, which is less than 1. □

3.2.2 Regular Graphs and Chromatic Number

To strengthen $\chi_r(G) \le r\Delta(G)$ for non-Moore graphs, we want to replace $\Delta(G)$ with a value no larger; for example, $\chi(G) \le \Delta(G)$ by Brooks' Theorem when $\Delta(G) \ge 3$ and G is not complete. We prove $\chi_r(G) \le r\chi(G)$ for regular graphs with sufficiently large degree in terms of r. The Petersen graph shows that the inequality does not hold for all G when $r = 3$.

For k-regular graphs with k sufficiently large in terms of r, we use a random r-coloring of the vertices to show that some coloring puts r distinct colors into each vertex neighborhood. We then assign each vertex a pair consisting of its color under this r-coloring and its color under a proper $\chi(G)$-coloring. This ensures that adjacent vertices have distinct colors; in total, $r\chi(G)$ color pairs are used.

Theorem 3.2.7. *[32, 48] If G is a k-regular graph and $ep(k(k-1)+1) \le 1$, where $p = r\mathrm{e}^{-k/r}$, then $\chi_r(G) \le r\chi(G)$. In particular, if $k \ge (3+x)r\ln r$, where $x - \frac{2\ln\ln r}{\ln r}$ is a small positive constant, then $\chi_r(G) \le r\chi(G)$.*

We do not know the least k to guarantee $\chi_r(G) \le r\chi(G)$ when G is k-regular, but we can show that when $k = r$ the ratio $\chi_r(G)/\chi(G)$ can grow superlinearly in r. Let $[n]$ denote $\{1,\dots,n\}$.

Theorem 3.2.8. *For infinitely many r, there is an r-regular graph G such that $\chi_r(G) > r^{1.37744}\chi(G)$.*

Proof. The *Kneser graph* $K(n,t)$ is the graph whose vertices are the t-element subsets of $[n]$, with two sets adjacent when they are disjoint. Each vertex is adjacent to $\binom{n-t}{t}$ other vertices. Given $t \in \mathbb{N}$, let $G = K(3t-1,t)$ and $r = \binom{n-t}{t}$. Any two nonadjacent vertices in G have a common neighbor, since two intersecting t-sets in $[3t-1]$ omit at least t elements, so $\mathrm{diam}(G) = 2$. Since G is r-regular, we thus have $\chi_r(G) = |V(G)| = \binom{3t-1}{t}$. On the other hand, Lovász [44] and Bárany [11] proved $\chi(K(n-t)) = n-2t+2$, so $\chi(G) = t+1$.

It remains to express $\chi_r(G)$ in terms of r and $\chi(G)$. In terms of t, we have $r = \binom{2t-1}{t} = \frac{1}{2}\binom{2t}{t}$ and $\chi_r(G) = \binom{3t-1}{t} = \frac{2}{3}\binom{3t}{t}$. Note that $\chi_r(G)$ and r are both very much larger than $\chi(G)$, so what is important is the ratio between $\chi_r(G)$ and r as a function of r. For $c \in \{\frac{1}{2}, \frac{1}{3}\}$, we use the approximation

$$\binom{m}{cm} \approx \frac{(c^c(1-c)^{1-c})^{-m}}{\sqrt{c(1-c)2\pi m}}$$

that arises from Stirling's Formula $m! = m^m e^{-m}\sqrt{2\pi m}$. We compute

$$\frac{\chi_r(G)}{r\chi(G)} \approx \frac{\frac{2}{3}\binom{3t}{t}}{(t+1)\frac{1}{2}\binom{2t}{t}} \approx \frac{4}{3t}\frac{(\frac{27}{4})^t\sqrt{\pi t}}{4^t\sqrt{(4/3)\pi t}} \approx \frac{1}{t}\sqrt{\frac{4}{3}}\left(\frac{27}{16}\right)^t.$$

Setting this ratio to be r^x, where $r \approx \frac{1}{2}4^t/\sqrt{\pi t}$, we take logarithms to obtain $t\log_2(27/16) = (1+o(1))tx\log_2 4$, which simplifies to $x = \frac{1}{2}(\log_2 27 - 4) > .37744$. Thus $\chi_r(G) > r^{1.37744}\chi(G)$. □

Theorems 3.2.7 and 3.2.8 suggest the following question.

Qestion 3.2.9. *As a function of r, what is the least k such that $\frac{\chi_r(G)}{\chi(G)}$ is at most linear in r when G is k-regular? What is the least k such that $\frac{\chi_r(G)}{\chi(G)}$ is bounded by $r(\ln r)^c$ for some c?*

3.2.3 Diameter and Chromatic Number

In this section we study the relationship between $\chi_r(G)$ and $\chi(G)$ when G has small diameter. We first prove $\chi_2(G) \le \chi(G) + 2$ when $\mathrm{diam}(G) = 2$, regardless of whether G is regular, and we characterize when equality holds. The proof of the bound is easy, but the characterization of equality takes more work. Given a coloring f of a graph G, say that a vertex is *f-monochromatic* if its neighbors have the same color under f.

Theorem 3.2.10. *If $\mathrm{diam}(G) = 2$, then $\chi_2(G) \le \chi(G)+2$, with equality only when G is a complete bipartite graph or C_5.*

Proof. The claim holds trivially for stars. For non-stars with minimum degree 1, the graph obtained by deleting all vertices of degree 1 has the same chromatic number, and a 2-dynamic coloring of it extends to a 2-dynamic coloring of the original graph. Hence we may assume $\delta(G) \ge 2$. We claim

($*$) If f is a proper coloring of a graph with diameter 2, and v is f-monochromatic with neighborhood of color a, then $N(v) = \{u\colon f(u) = a\}$. In particular, nonadjacent f-monochromatic vertices have the same neighborhood.

To prove ($*$), note that the set of vertices with color a is independent, so a vertex outside $N(v)$ with color a cannot have a common neighbor with v. For the second statement, nonadjacent vertex must have a common neighbor, and then being f-monochromatic makes both adjacent to all vertices of that color.

We next prove the upper bound. Let f be a proper $\chi(G)$-coloring of a graph G with diameter 2; note that G is connected. If f is not 2-dynamic, some vertex v is f-monochromatic, with color a on all of $N(v)$. Modify f by giving a new color α to v and another new color β to one vertex x in $N(v)$.

The resulting coloring f' is a proper $(\chi(G) + 2)$-coloring. If f' is not 2-dynamic, then some vertex z is f'-monochromatic. By construction, z cannot be v or a neighbor of v (each neighbor of v has color α on exactly one neighbor). If z is not a neighbor of v, then by ($*$) we have $N(z) = N(v)$, and colors a and β both appear in $N(z)$.

For the characterization of equality, note first that if no vertex of $N(v)$ is f-monochromatic, then we do not need to introduce a new color on v, and we obtain $\chi_2(G) \leq \chi(G) + 1$. Hence we may assume that two adjacent vertices v and u are f-monochromatic. If no vertex lies outside $N(v) \cup N(u)$, then G is the complete bipartite graph with parts $N(v)$ and $N(u)$, since those sets are independent and $\mathrm{diam}(G) = 2$.

Hence if G is not a complete bipartite graph, then there is a vertex w outside $N(v) \cup N(u)$. Let a and b be the colors on $N(v)$ and $N(u)$, respectively. Since $\mathrm{diam}(G) = 2$, vertex w has neighbors in both $N(v)$ and $N(u)$ and hence has a third color, c. Let W be the set of all vertices with color c under f. Let U and V be the set of all f-monochromatic vertices having the same color as u and v, respectively; note that $U \subseteq N(v)$ and $V \subseteq N(u)$. By ($*$), there are no other f-monochromatic vertices.

All vertices in U have the same neighborhood, as do all vertices in V. If $|U| > 1$, then we change the color of u to c and use a new color d on v. This produces a 2-dynamic coloring with

$\chi(G)+1$ colors, since the neighbors of u now have colors a and c in their neighborhood. Similarly, if $|V|>1$ we can change the color of v to c and use d on u.

Hence we may assume $|U|=|V|=1$. The change still works unless $N(u)$ contains a vertex x whose neighbors other than u all have color c, and similarly $N(v)$ contains a vertex y whose neighbors other than v all have color c. In particular, x and y are not adjacent. Now changing x and y to a new color d produces the desired 2-dynamic coloring of G with $\chi(G)+1$ colors, unless there is a vertex $w \in W$ whose only neighbors are x and y.

In this case, reaching $N(u)-\{v,x\}$ in two steps from w requires $N(u)-\{v,x\} \subseteq N(y)$, and similarly $N(v)-\{u,y\} \subseteq N(x)$. Since vertices of $N(x)-\{u\}$ and $N(y)-\{v\}$ all have color c, we conclude that $N(u)=\{v,x\}$ and $N(v)=\{u,y\}$. We are now left with $G=C_5$ unless G has a vertex outside the 5-cycle induced by w,x,u,v,y in order. Since u and v have no other neighbors, and all other neighbors of x and y have color c, reaching u and v in two steps requires that all the remaining vertices have neighborhood $\{x,y\}$. Now we use colors $1,2,3,4$ in order on the path $\langle x,u,v,y\rangle$ and use colors 2 and 3 on the remaining independent set, each at least once. □

The graph obtained by subdividing every edge of an n-vertex complete graph has diameter 4. Its chromatic number is 2, but its 2-dynamic chromatic number is n. Hence for diameter 4 there is no bound in terms of the chromatic number, while for diameter 2 the bound is very tight. For diameter 3 we determine the best bound.

Theorem 3.2.11. *If G is a k-chromatic graph with diameter at most 3, then $\chi_2(G) \le 3k$, and this bound is sharp when $k \ge 2$.*

Proof. We begin with the sharpness construction. Let $H=K_{3k}$, and let F be a subgraph of H consisting of disjoint triangles $T_1,\dots,T_k$. Form G from H by subdividing each edge of F. In G the vertices belonging to T_i can receive color i, and their neighbors of degree 2 can receive another color. In a 2-dynamic coloring, all the original vertices of H must have distinct colors, so $3k$ colors are needed. The diameter is 3, because any two vertices of degree 2 have neighbors that are adjacent.

Now let G be any k-chromatic graph with diameter at most 3. As observed in the introduction, $\chi_r(G) \le kj$, where j is the minimum number of colors in an r-strong coloring of the hypergraph

H of vertex neighborhoods in G. More precisely, we give each vertex v a color pair $(f(v), h(v))$, where f is a proper k-coloring of G and h is an r-strong coloring of the subhypergraph H' of H whose edges are the edges of H that do not already have r colors under f. The first coordinate ensures that the resulting coloring g of G is proper, and the second coordinate ensures that it is r-dynamic.

Therefore, it suffices to show that when $\text{diam}(G) \leq 3$, the resulting hypergraph H' always has a 2-strong coloring with three colors. Call a hypergraph with such a coloring *good*. Note first that every intersecting hypergraph is good, where a hypergraph is *intersecting* if any two of its edges intersect. Given an intersecting hypergraph, just use colors 1 and 2 on a minimal edge and use color 3 on all vertices not in that edge (in fact, we only need one minimal edge that intersects all other edges).

Given the proper k-coloring f of G, we define k subhypergraphs of H'. Let $V_i = \{v \in V(G)\colon\ f(v) = i\}$. The subhypergraph H_i has vertex set V_i, and its edges are the vertex neighborhoods in G in which every vertex has color i. That is, $N(u) \in E(H_i)$ if $f(v) = i$ for all $v \in N(u)$. Since each vertex in H_i has color i under f, the vertex sets of these subhypergraphs are disjoint. Hence if each is good, then their union is good. We can define $h(v)$ arbitrarily for $v \notin V(H')$, because we do not need multiple colors in the second coordinate of g on a neighborhood where the first coordinate already provides multiple colors.

It remains only to show that H_i is an intersecting hypergraph when $\text{diam}(G) \leq 3$. Consider $x, y \in V(G)$ such that $N(x), N(y) \in E(H_i)$. If $xy \in E(G)$, then $y \in N(x)$ and $x \in N(y)$, and hence $f(x) = f(y) = i$, which contradicts that f is a proper coloring of G. Hence x and y are not adjacent in G. Also, no edge of G can join $N(x)$ and $N(y)$, since all vertices in $N(x) \cup N(y)$ have color i under f. If $N(x) \cap N(y) = \varnothing$, then a shortest path in G from x to y must pass through $N(x)$ and $N(y)$ and some other vertex between them. Hence the path has length at least 4, which contradicts $\text{diam}(G) \leq 3$. We conclude that H_i is an intersecting hypergraph and hence is good, as desired. □

These results are sharp in various ways. For larger r, there is no bound, not even on bipartite graphs with diameter 3 or on 3-chromatic graphs with diameter 2. Note that the only bipartite

graphs with diameter 2 are complete bipartite graphs, where r-dynamic chromatic number does not exceed $2r$.

Theorem 3.2.12. *For $3 \leq k < r$, the r-dynamic chromatic number is unbounded on the graphs with minimum degree $k+1$ that are bipartite and have diameter 3, and also on those that are 3-colorable and have diameter 2.*

Proof. Let H be the incidence graph of the k-subsets of $[n]$, where $n > k+1$. That is, H is bipartite, with one part being $[n]$ and the other being the family of k-subsets, and element j is adjacent to set A if $j \in A$. Form G by adding a single vertex v adjacent to all the k-sets, giving them degree $k+1$. The graph G is bipartite, with the added vertex in the same part as $[n]$. Any two elements of $[n]$ lie in a common k-set, so the distance between them is 2, and the distance between any k-set and an element not in it is 3. The added vertex has distance 2 from all of $[n]$ and ensures distance 2 between any two k-sets. Hence $\mathrm{diam}(G) = 3$.

In an r-dynamic coloring, the neighbors of any vertex with degree at most r receive distinct colors. In G, any two vertices of $[n] \cup \{v\}$ have a common neighbor with degree at most r, so $\chi_r(G) \geq n+1$ (equality holds).

For a construction with diameter 2, let the added vertex v be adjacent also to all of $[n]$. Now v is a dominating vertex, so the diameter is 2, but the chromatic number increases to 3. However, each k-set vertex still has $k+1$ neighbors, so the argument for $\chi_r(G) \geq n+1$ remains the same. □

Chapter 4

Hypergraphs with forbidden subgraphs

4.1 Maximum hypergraphs with no regular subgraphs

A natural question in graph theory is which graphs do not contain r-regular subgraphs. For $r \in \{1, 2\}$, the answer is easy, but for $r \geq 3$ it is not. It was a breakthrough when Tashkinov [53] proved the conjecture by Berge that every 4-regular graph contains a 3-regular subgraph. The questions on existence of r-regular subgraphs in regular or near-regular graphs were also considered in [9, 54]. Let $F(r, n)$ denote the maximum number of edges an n-vertex graph with no r-regular subgraphs have. For $r \geq 3$, it is not fully resolved how big $F(r, n)$ is. Pyber [49] showed that for every fixed r, $F(r, n) = O(n \ln n)$. On the other hand, Pyber, Rödl and Szemerédi [50] proved that $F(3, n) \geq cn \ln \ln n$.

Similar questions are also natural for hypergraphs. In 1.3, we define *r-free* hypergraphs, *full star* and *full k-star*. Mubayi and Verstraëte [46] proved that for every even integer $k \geq 4$, there exists n_k such that for each $n \geq n_k$, each n-vertex k-uniform 2-free hypergraph H has at most $\binom{n-1}{k-1}$ edges, and equality holds if and only if H is a *full k-star*. They also proved the following simpler result for non-uniform hypergraphs.

Theorem 4.1.1 ([46])**.** *For $n \geq 3$, every n-vertex 2-free hypergraph H satisfies $|H| \leq 2^{n-1}$, and equality holds if and only if H is a full star, that is, H consists of 2^{n-1} distinct edges containing a given vertex.*

Our first result in Section 4.1 is the following (simple) generalization of Theorem 4.1.1.

Theorem 4.1.2. *If $2 \leq r \leq 2^{n-1}$, then the maximum number of edges in an n-vertex r-free hypergraph is $2^{n-1} + r - 2$.*

Many examples of n-vertex r-free hypergraphs with $2^{n-1}+r-2$ edges are formed by a full star with $r-2$ other edges. If $r \geq n$, then some extremal examples do not contain full stars. For $r=2$, Theorem 4.1.1 says that if $n \geq 3$, then the only n-vertex 2-free hypergraph with 2^{n-1} edges is a full star. We conjecture the following.

Conjecture 4.1.3. *Let H be an n-vertex r-free hypergraph with $|H| = 2^{n-1}+r-2$. If $n > r$ and $r \geq 2$, then H contains a full star.*

The main results of Section 4.1 are the following.

Theorem 4.1.4. *Suppose H is an n-vertex r-free hypergraph with $|H| = 2^{n-1}+r-2$. If $n \geq r + 2\lceil \log_2 r \rceil + 1$, then H contains a full star.*

Theorem 4.1.5. *Suppose H is an n-vertex r-free hypergraph with $|H| = 2^{n-1}+r-2$. If $n > r$ and $n \geq 425$, then H contains a full star.*

In the next section we prove Theorem 4.1.2 and derive simple properties of dense r-free hypergraphs. In Section 4.1.2 we show that dense r-free hypergraphs have no small transversals. In Section 4.1.3 we prove Theorem 4.1.4. In Section 4.1.4 and Section 4.1.5 we prove Theorem 4.1.5.

4.1.1 Preliminaries

Proof of Theorem 4.1.2: Let H be an n-vertex r-free hypergraph with ground set N. Consider all 2^{n-1} pairs $\{A, N-A\}$ of subsets of N. In at most $r-1$ pairs of sets both sets are edges in H, otherwise we get an r-regular subgraph of H with vertex set N. If there are exactly $r-1$ such pairs, N cannot be an edge in H, since N together with those $r-1$ pairs would form an r-regular subgraph of H. Thus $|H| \leq 2^{n-1}+r-2$.

If $2 \leq r \leq 2^{n-1}$, then equality can be achieved. Let $N = [n]$ and

$$H = \{e : 1 \in e\} \cup \{r-2 \text{ smallest nonempty distinct subsets of } [n]-\{1\}\ \}.$$

Suppose that H has an r-regular subgraph $\mathcal{G}$. Let $C_1, \ldots, C_r$ be the edges of $\mathcal{G}$ that contain 1, and $D_1, \ldots, D_s$ be the remaining edges of $\mathcal{G}$. Let $C = \bigcup_{i=1}^{r} C_i$ and for $i \in [r]$ let $C_i' = C - C_i$.

Since all edges are distinct, $\sum_{i=1}^{r} |C'_i| \leq \sum_{j=1}^{s} |D_j|$ should hold. The left-hand side is the sum of cardinailities of at least $r-1$ nonempty distinct sets (possibly one C'_i is empty) not containing 1, and the right-hand side is the sum of cardinailities of at most $r-2$ smallest distinct sets not containing 1, and so, the right-hand side is less than the left-hand side. This contradiction shows that H has no r-regular subgraphs. □

Let N be a finite set, and $n = |N|$. Let $3 \leq r < n$. A hypergraph H is (N, r)-*strange* if H is an r-free hypergraph with $V(H) = N$ and $|H| = 2^{n-1} + r - 2$ such that H does not contain a *full star*, i.e., 2^{n-1} sets containing a given element.

For a set $A \subseteq N$, $\overline{A}$ is the *complement* of A to N, i.e., $\overline{A} = N - A$. A *full pair* in H is a pair $\{A, \overline{A}\}$ such that both A and $\overline{A}$ are in H. We let the set N by itself form a full pair.

In order to prove Theorems 4.1.4 and 4.1.5, we derive some properties of (N, r)-strange hypergraphs. If H is (N, r)-strange, then it contains at most $r-1$ full pairs, and so, since $|H| = 2^{n-1}+r-2$, it contains exactly $r-1$ full pairs. Moreover,

$$\text{for each } A \subset N \text{ with } N \neq A \neq \emptyset, \text{ either } A \in H \text{ or } \overline{A} \in H. \tag{4.1}$$

Furthermore, the following statements hold for each (N, r)-strange hypergraph H.

Lemma 4.1.6. *If $A, B \in H$, $A \cap B = \emptyset$ and both A and B are not in full pairs, then $A \cup B \in H$.*

Proof. If $A \cup B \notin H$, then $\overline{A \cup B} \in H$ by (4.1). Thus $A, B, \overline{A \cup B}$ with $r-1$ full pairs form an r-regular subfamily of H, a contradiction. □

Lemma 4.1.7. *If $A \in H$ and B and C are disjoint nonempty subsets of A such that $A = B \cup C$, then at least one of B and C is in H.*

Proof. Suppose that $A = B \cup C$ is a partition of A into nonempty sets and $B, C \notin H$. Then by (4.1), $\overline{B}$ and $\overline{C}$ are in H but not in full pairs. Thus the sets A, $\overline{B}$ and $\overline{C}$ together with $r-2$ full pairs not containing A form an r-regular subgraph of H, a contradiction. □

Corollary 4.1.8. *Every edge A of H contains an element x_A such that $\{x_A\} \in H$. In particular, the union S of 1-edges of H intersects each edge of H.*

Lemma 4.1.9. *Let A and B be edges of H such that $A \cap B \neq \emptyset$. If at least one of A and B is not in a full pair, then either $A \cap B$ or $A \cup B$ is in H.*

Proof. Suppose that $A \cap B, A \cup B \notin H$. Then $\overline{A \cap B}$ is in H, and $\overline{A \cup B}$ is either empty or in H. In both cases, the sets $A, B, \overline{A \cap B}$, and $\overline{A \cup B}$ cover every element of N exactly twice. Adding $r-2$ full pairs containing neither A nor B will give an r-regular subgraph of H. □

4.1.2 Sizes of transversals of (N, r)-strange hypergraphs

A set $A \subset V(H)$ is a *transversal* of a hypergraph H if every edge of H intersects A.

Let S be a smallest transversal of a hypergraph $\mathcal{F}$. Then S contains all 1-edges of $\mathcal{F}$. If H is (N, r)-strange, then by Corollary 4.1.8, S contains no other vertices. Thus S is exactly the union of 1-edges of H. It has several useful properties.

The goal of this section is to prove the following fact. Throughout the section, k denote $\lceil \log_2 r \rceil$.

Theorem 4.1.10. *Let $3 \leq r < n$ and N be a finite set with $|N| = n$. If S is the smallest transversal of an (N, r)-strange hypergraph $\mathcal{F}$, then $|S| \geq n - 3k - 2$.*

Let S be the smallest transversal of an (N, r)-strange hypergraph $\mathcal{F}$.

Lemma 4.1.11. *If a nonempty $S' \subset S$ is not in H, then every $S' \subseteq B \subseteq N - (S - S')$ is not in H, and hence every $S - S' \subseteq A \subseteq N - S'$ is in H.*

Proof. Suppose that such B is in H. By Lemma 4.1.7, either S' or $B - S'$ is in H. But $(B - S') \cap S = \emptyset$, and we know that S' is not in H, a contradiction. □

From now on, in this section, we will assume that

$$|S| \leq n - 2k - 2. \tag{4.2}$$

Note that to prove Theorem 4.1.10, we could make the stronger assumption that $|S| \leq n - 3k - 3$, but we plan to use these lemmas also in the next section.

For $S' \subseteq S$ and $M \subseteq N - S$, we say that M *belongs to* S' if $S' \cup M \in H$. A nonempty proper subset S' of S is *firm* if some $M \subset N - S$ with $|M| \geq 1 + k$ belongs to S'. In particular, S is firm

by the following reason. For a set $A \subset N - S$ with $|A| = k + 1$, one of $A \cup S$ and $N - S - A$ is in H by (4.1). Since S is a transversal, $N - S - A$ is not in H. Thus $S \cup A \in H$, so A belongs to S and S is firm.

Lemma 4.1.12. *Let $S' \subseteq S$ and $M \subseteq N - S$. If M belongs to S', then every $M' \subset M$ belongs to S'*

Proof. Since $M \cup S' \in H$, by Lemma 4.1.7, either $S' \cup M' \in H$ or $M - M' \in H$. But the latter does not hold, since $M \cap S = \emptyset$. This proves the lemma. □

Lemma 4.1.13. *For every partition $S = S' \cup S''$ of S into nonempty subsets, exactly one of S' and S'' is firm.*

Proof. Assume first that neither of S' and S'' is firm. Let M be a subset of $N - S$ with $|M| = 1 + k$. Since S' is not firm, $S' \cup M \notin H$. Then $N - (S' \cup M) \in H$, and $N - (S' \cup M) = S'' \cup (N - S - M)$. So by (4.2), $|N - S - M| \geq 2k + 2 - (1 + k)$, and thus S'' is firm. Assume now that both S' and S'' are firm. If a set $M \subset N - S$ with $|M| \geq k + 1$ belongs to both S' and S'', then we will find an r-regular subgraph $\mathcal{H}$ of H.

Since $2^{|M|} \geq r$, there are at least r subsets of M. Call them $A_1, \ldots, A_r$. Let $\mathcal{H} = \{A_i \cup S' : 1 \leq i \leq r\} \cup \{(M - A_i) \cup S'' : 1 \leq i \leq r\}$. By construction, $\mathcal{H}$ is r-regular, a contradiction.

If a set $M \subset N - S$ with $k \leq |M| \leq k + 2$ belongs to neither S' nor S'', then $N - S - M$ belongs to both, and again H has an r-regular subgraph. Thus each $M \subset N - S$ with $|M| = k + 1$ belongs to exactly one of S' and S''. Let $\mathcal{R}_{S'}$ (respectively, $\mathcal{R}_{S''}$) denote the family of $M \subset N - S$ with $|M| = k + 1$ that belongs to S' (respectively, to S''). By our assumption, both $\mathcal{R}_{S'}$ and $\mathcal{R}_{S''}$ are nonempty. Then there exist $M' \in \mathcal{R}_{S'}$ and $M'' \in \mathcal{R}_{S''}$ with $|M' \cap M''| = k$. Thus $M' \cap M''$ belongs to both S' and S'', and so H has an r-regular subgraph, a contradiction. □

Corollary 4.1.14. *If S' is a firm subset of S, then every $M \subset N - S$ with $1 \leq |M| \leq n - s - (k + 1)$ belongs to S'.*

Corollary 4.1.15. *Every two firm subsets of S intersect each other.*

Proof. Suppose that S_1 and S_2 are two disjoint firm subsets of S. Let $M \subset N - S$ with $|M| = k + 1$. By Corollary 4.1.14, M belongs to both S_1 and S_2. Then as in the proof of Lemma 4.1.13, H has an r-regular subgraph with vertex set $S_1 \cup S_2 \cup M$, a contradiction. □

Lemma 4.1.16. $s \leq r - 1$

Proof. Suppose $s \geq r$.

Case 1: $s \geq r + k$. Since the number of 1-edges in full pairs is at most $r - 1$, we can choose $k + 1 (\leq s - (r - 1))$ 1-edges of H that are not in full pairs. Let S' be the union of these edges. If some $A \subseteq S'$ is not in H, then $\overline{A}$ and the 1-edges contained in A cover N once, and together with the $r - 1$ full pairs we obtain an r-regular subgraph of H covering N, a contradiction. Thus all nonempty subsets of S' are in H, and the number of nonempty proper subsets of S' is at least $2^{k+1} - 2 \geq 2r - 2$. We can pair them up so that they are partitions of S'. At least $r - 1$ of such pairs exist, so together with S' they form an r-regular subgraph of H, a contradiction.

Case 2: $r \leq s \leq r+k-1$. By (4.2), $n-s \geq k+1$. If there are $v_1, v_2 \in S$ such that $S-v_1, S-v_2 \notin H$, then by Lemma 4.1.11, every $B \subseteq N - S$ satisfies $B + v_1 \in H$ and $B + v_2 \in H$. Since there are at least $2^{n-s} \geq r$ possible sets for B, we can find r pairs of sets $v_1 + B, v_2 + (N - S - B)$, and they will form an r-regular subgraph of H on $(N - S) + v_1 + v_2$.

Thus for some $r - 1$ vertices $v_1, ..., v_{r-1} \in S$, the sets $S - v_i$ are in H. Then the family $\{v_1, ..., v_{r-1}, S - v_1, ..., S - v_{r-1}, S\}$ covers every $v \in S$ exactly r times, a contradiction. □

Lemma 4.1.17. *No* 1*-edge of* H *is firm.*

Proof. Let $S = \{v_1, \ldots, v_s\}$. Suppose that $S_1 := \{v_1\}$ is firm. Then by Corollary 4.1.15, no subset of $S - v_1$ is firm. Hence by Lemma 4.1.13, the firm subsets of S are exactly the sets containing v_1.

Since not every subset of N containing v_1 is in H and $s \leq r-1$, there are at least $r-1-(s-1) = r - s$ edges $W_1, \ldots, W_{r-s}$ that are in H, not 1-edges and do not contain v_1. For $j = 1, \ldots, r - s$, let $M_j = W_j - S$. Let $M = \bigcup_{j=1}^{r-s} M_j$. Choose $W_1, \ldots . W_{r-s}$ so that to minimize $|M|$.

Case 1: $|M| \leq n - s - k - 1$. Denote by H' the family

$$\{S \cup M, \{v_2\}, \ldots, \{v_s\}, S \cup M - v_2, \ldots, S \cup M - v_s, W_1, \ldots, W_{r-s}, S \cup M - W_1, \ldots, S \cup M - W_{r-s}\}.$$

Since H' forms an r-regular hypergraph, H' is not a subgraph of H. But since $\{v_1\}$ is firm, by the choice of W_j and Corollary 4.1.14, every member of H' is in H, a contradiction. This proves Case 1.

Let $t = \max\{|A - S| : A \in H \text{ and } v_1 \notin A\}$ and let $A_0 \in H$ be such that $v_1 \notin A_0$ and $|A_0 - S| = t$.

Case 2: $t \geq k$. Let M_0 be any k-element subset of $A_0 - S$ and $W_0 = M_0 \cup (A_0 \cap S)$. Since $A_0 \in H$ and $S \cap (A_0 - W_0) = \emptyset$, $W_0 \in H$. Since $2^k \geq r$, M_0 contains some r distinct subsets $M'_1, \ldots, M'_r$. Let $W'_i = M'_i \cup (A_0 \cap S)$ for $i = 1, \ldots, r$. Since $(M_0 - M'_i) \cap S = \emptyset$, each of W'_i is in H. Moreover, since $|S| \leq n - 2k - 2$, $|M_0| = k$, and $\{v_1\}$ is firm, for every $1 \leq i \leq r$, the set $(S \cup M_0) - W'_i$ contains v_1 and has at most $n - s - (k + 1)$ vertices in $N - S$. This means that $(S \cup M_0) - W'_i$ is also in H. So, the family

$$\{W'_1, \ldots, W'_r, (S \cup M_0) - W'_1, \ldots, (S \cup M_0) - W'_r\}$$

forms an r-regular hypergraph, a contradiction.

Case 3: $\log_2(r - s) \leq t \leq k - 1$. Let $M_0 = A_0 - S$. In our case, $2^{|M_0|} = 2^t \geq r - s$. Let $M'_1, \ldots, M'_{r-s}$ be any distinct subsets of M_0, and for $i = 1, \ldots, r - s$, let $W'_i = M'_i \cup (A_0 \cap S)$. Similarly to Case 2, since $(M_0 - M'_i) \cap S = \emptyset$, each of W'_i is in H. Moreover, since $|S| \leq n - 2k - 2$, $|M_0| \leq k - 1$, and $\{v_1\}$ is firm, for every $1 \leq i \leq r$, the set $(S \cup M_0) - W'_i$ is also in H. By the same reason, for every $2 \leq j \leq s$, the set $(S \cup M_0) - v_j$ is in H. So, the family

$$\{W'_1, \ldots, W'_{r-s}, (S \cup M_0) - W'_1, \ldots, (S \cup M_0) - W'_{r-s}, \{v_2\}, \ldots, \{v_s\}, S \cup M_0 - v_2, \ldots, S \cup M_0 - v_s, S - \cup M_0\}$$

forms an r-regular hypergraph, a contradiction.

Case 4: $|M| \geq n - k - s$ and $t < \min\{k - 1, \log_2(r - s)\}$. Let $M_0 = A_0 - S$. We claim that $|M| \leq r - s - 2^t + t + 1$. To prove the claim, we show a way to choose $W_1, \ldots, W_{r-s}$ so that

$$|\bigcup_{j=1}^{i} W_j - S| \leq i - 2^t + t + 1. \tag{4.3}$$

for every $2^t - 1 \leq i \leq r - s$. The sets $W_1, \ldots, W_{2^t-1}$ are all the sets of the form $A_0 - X$ where $X \subseteq M_0, X \neq M_0$. So, for $i = 2^t - 1$, (4.3) holds. Suppose that for some $2^t - 1 \leq i_0 \leq r - s - 1$, we have found $W_1, \ldots, W_{i_0}$ satisfying (4.3) for $i = i_0$. Let $\mathcal{C}$ be the family of members of H not containing v_1 that are distinct from $W_1, \ldots, W_{i_0}$. Since $i_j \leq r - s - 1$, there is $C \in \mathcal{C} \neq \emptyset$. Let $C' = C - S - \bigcup_{j=1}^{i_0} W_j$. If $C' = \emptyset$, then we let $C := W_{i_0+1}$ and (4.3) holds for $i = i_0 + 1$. Suppose $x \in C'$. Since $(C' - x) \cap S = \emptyset$, the set $C - C' + x$ is in H, does not contain v_1, and is distinct from $W_1, \ldots, W_{i_0}$. So, letting $W_{i_0+1} = C - C' + x$ we again have that (4.3) holds for $i = i_0 + 1$. This proves the claim.

Let H' be the family defined in Case 1. Since it is r-regular, some $W' \in H'$ is not in H. Then $\overline{S \cup M - W'} \in H$ by (4.1). By the definition of t, $|N - (S \cup M)| \leq |\overline{S \cup M - W'} - S| \leq t$. Thus by (4.3), $n = |N| = |M| + |S| + |N - (M \cup S)| \leq (r - s - 2^t + t + 1) + s + t = r - 2^t + 2t + 1$.

If $t \geq 3$, we get $n \leq r - 1$, a contradiction.

If $t = 2$, we get $n \leq r + 1$, $|M| = r - s - 1$ and $|A_0 - S| = 2$. Then $(A_0 - S) \subset M$ with $|M| = r - s - 1$ and $|\overline{M \cup S - W'} - S| \geq 2$. Thus there are distinct A_0, A_1 with $|A_i - S| = 2$, $(A_0 - S) \cap (A_1 - S) = \emptyset$ and $v_1 \notin A_i$. Let $\{v_1, \ldots, v_{r-s-6}\} \subset M - A_0 - A_1$. For $x = 1, \ldots, r - s - 6$, let $v_x \in M_{j_x}$. By Lemma 4.1.12, $W_{j_x} - M_{j_x} + v_x \in H$ for every $x_1, \ldots, r - s - 6$. These edges with $A \cup (A_i \cap S)$ for nonempty $A \subseteq A_i - S$ yield that $|M| \leq r - s - 2$, a contradiction.

Thus $t = 1$. Then $n = r + 1$, $|M| = r - s$ and every edge not containing v_1 is either a 2-edge or a 1-edge. And $\overline{S \cup M - W'}$ is also a 2-edge, so W' is a 1-edge. Let $W' = \{v_2\}$.

Since $|M| = r - s$ and $\overline{S \cup M - v_2} = \{v_2\} \cup (N - M - S)$, for each $w \in N - S$ there is exactly one 2-edge not containing v_1 containing w. Since $|N - S| = r - s + 1$ and the number of 1-edges in $S - v_1$ is $s - 1$, H has exactly $r - s + 1 + s - 1 = r$ edges not containing v_1.

So we have exactly two sets containing v_1 that are not in H. Call them D_1, D_2. We have $D_1 = S \cup M - v_2$. Since $|N - S| = n - s \geq 2k + 2 \geq 4$ and every vertex in $N - S$ is contained in exactly one 2-edge not containing v_1, there are at least 4 different ways to choose W_is to get minimum $|M|$. Each way gives different M, let two of them be M and M'. Then, by the above logic, $D_2 = S \cup M' - v_3$ for some v_3, and $|M \cap M'| \geq 2$. Thus $D_1 \cap D_2 - S$ is not empty. Let $w \in D_1 \cap D_2 - S$. Then there are $2^{n-1} - 2$ edges in H containing both w and v_1, and exactly one

edge containing w but not v_1. Thus $H-w = \{E \in H : w \notin E\}$ is an $(n-1)$-vertex hypergraph with $2^{n-1}+r-1$ edges. By Theorem 4.1.2, $H-w$ contains an r-regular subgraph, a contradiction. □

Lemma 4.1.18. *$s \leq r-2$.*

Proof. Suppose that $s \geq r-1$. Then by Lemma 4.1.16, $s = r-1$. By Lemma 4.1.17, $S - v_i$ is firm (and so is in H) for every $i = 1, \ldots, s$. Then the $2r-1$ sets

$$\{v_1\}, \ldots, \{v_{r-1}\}, S - v_1, \ldots, S - v_{r-1}, S$$

form an r-regular subgraph of H, a contradiction. □

Lemma 4.1.19. *Let $k \geq 2$ and let B be a set with $|B| \geq 2k+2$. Then there are at least $2^{k-2}+1$ partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ of B such that $|B_{i,1}| = \lceil (k+1)/2 \rceil, |B_{i,2}| = \lceil (k+1)/2 \rceil, |B_{i,3}| \geq k+1$ and all $3\lceil 2^{k-2} \rceil + 3$ parts of these partitions are distinct.*

Proof. We will choose B_1 of size $\lceil (k+1)/2 \rceil$ and B_2 of size $\lceil (k+1)/2 \rceil$, so that $|B_3| = |B| - 2\lceil k/2 \rceil \geq k+1$.

For $k \geq 4$, we have

$$\begin{aligned}\binom{2k+2-\lceil \frac{k+1}{2} \rceil}{\lceil \frac{k+1}{2} \rceil} &= \frac{(2k+2-\lceil \frac{k+1}{2} \rceil)(2k+1-\lceil \frac{k+1}{2} \rceil)(2k-\lceil \frac{k+1}{2} \rceil)}{\lceil \frac{k+1}{2} \rceil (2k+2-2\lceil \frac{k+1}{2} \rceil)(2k+1-2\lceil \frac{k+1}{2} \rceil)} \binom{2k-2-\lceil \frac{k-2+1}{2} \rceil}{\lceil \frac{k-2+1}{2} \rceil} \\ &\geq 4\binom{2(k-2)+2-\lceil \frac{k-1}{2} \rceil}{\lceil \frac{k-1}{2} \rceil}. \end{aligned} \tag{4.4}$$

So, for even k,

$$\binom{2k+2-\lceil (k+1)/2 \rceil}{\lceil (k+1)/2 \rceil} \geq 4\binom{2(k-2)+2-\lceil (k-1)/2 \rceil}{\lceil (k-1)/2 \rceil} \geq \cdots \geq 4^{\frac{k-2}{2}}\binom{4}{2} = 6 \cdot 2^{k-2}.$$

For odd k,

$$\binom{2k+2-\lceil (k+1)/2 \rceil}{\lceil (k+1)/2 \rceil} \geq 4\binom{2(k-2)+2-\lceil (k-1)/2 \rceil}{\lceil (k-1)/2 \rceil} \geq \cdots \geq 4^{\frac{k-3}{2}}\binom{6}{2} = 15 \cdot 2^{k-3} \geq 6 \cdot 2^{k-2}.$$

First for each $i = 1, \ldots, 2^{k-2}+1$, choose a set $B_{i,1}$ of size $\lceil k/2 \rceil$ so that all chosen sets are distinct. Then one by one for each $i = 1, \ldots, 2^{k-2}+1$, choose a set $B_{i,2}$ of size $\lceil k/2 \rceil$ so that

(a) $B_{i,2}$ is distinct from all $2^{k-2}+1$ sets $B_{i',1}$ and previously chosen $B_{i',2}$, and

(b) $B_{i,1} \cup B_{i,2}$ is distinct from all already chosen $B_{i',1} \cup B_{i',2}$.

Even at the last step (Step $2^{k-2}+1$), the number of forbidden sets is at most $3 \cdot 2^{k-2}+1 < 6 \cdot 2^{k-2}$. So, by (4.4), we finish the construction. □

Corollary 4.1.20. *Let B be a set with $|B| \geq 3k+3$. Then there are at least $2^{k-2}+1$ partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ of B such that all $3\lceil 2^{k-2} \rceil + 3$ parts of these partitions are distinct, and each $B_{i,j}$ has size at least $k+1$.*

Proof. Let $A \subseteq B$ and $|A| = k+1$. Let $B' = B - A$. Partition A into $A_1 \cup A_2 \cup A_3$ with $|A_1| = |A_2| = k+1-\lceil \frac{k+1}{2} \rceil$. By Lemma 4.1.19, there are at least $2^{k-2}+1$ partitions $(B'_{i,1}, B'_{i,2}, B'_{i,3})$ of B' such that all parts of these paritions are distinct, and $|B'_{i,1}| = |B'_{i,2}| = \lceil \frac{k+1}{2} \rceil, |B'_{i,3}| \geq k+1$. Take $B_{i,j} = B'_{i,j} \cup A_i$. Then partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ satisfy all the conditions. □

Proof of Theorem 4.1.10. Suppose $s \leq n-3k-3$. Then all lemmas in this section hold, since $s \leq n-3k-3 \leq n-2k-2$.

Let S' be a smallest firm subset of S. Note that S' is not a 1-edge. Partition S' into nonempty subsets S_1 and S_2. By the minimality of S', sets S_1 and S_2 are not firm, and so $S - S_1$ and $S - S_2$ are firm.

Let $B := N - S$. Then $|B| = n - s \geq 3k+3$. So, by Corollary 4.1.20, there are $K := \lceil 2^k/6 \rceil + 1$ partitions

$$(B_{1,1}, B_{1,2}, B_{1,3}), (B_{2,1}, B_{2,2}, B_{2,3}), \ldots, (B_{K,1}, B_{K,2}, B_{K,3})$$

of B such that all $B_{i,j}$ are distinct and $|B_{i,j}| \geq k+1$. For every $i \in \{1, \ldots, K\}$ and every $j \in \{1,2,3\}$, the three sets $S' \cup (B - B_{i,j})$, $S_1 \cup (B - B_{i,j+1})$, and $S_2 \cup (B - B_{i,j+2})$ (where j counts modulo 3) are in H (by Corollary 4.1.14 and the fact that $|B - B_{i,j}| \leq n - s - (k+1)$) and cover every vertex in N exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1,2,3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples.

If r is odd, then we pick a full pair $(A, N-A)$. There are at most two triples $(S' \cup (B - B_{i,j}), S_1 \cup$

$(B-B_{i,j+1}), S_2\cup(B-B_{i,j+2}))$ containing A or $N-A$. Then we cover the set N once by the set A and $N-A$ and $r-1$ times with $\frac{r-1}{2} \leq 3K-2$ triples $(S'\cup(B-B_{i,j}), S_1\cup(B-B_{i,j+1}), S_2\cup(B-B_{i,j+2}))$ containing neither A nor $N-A$. This contradicts the choice of H. Therefore $|S| \geq n-3k-2$.

4.1.3 Proof of Theorem 4.1.4

If the theorem does not hold, then for some $3 \leq r < n$, $k = \lceil \log_2 r \rceil$ with $n \geq r+2k+1$, and for some n-vertex set N, there exists an (N,r)-strange hypergraph $\mathcal{F}$. Let S be the union of 1-edges in $\mathcal{F}$. By Lemma 4.1.18, $|S| \leq r-2 \leq n-2k-3$.

Let $\mathcal{S}_{nf}$ denote the family of non-firm subsets of S. For every $S' \in \mathcal{S}_{nf}$, let

$$H_{S'} := \{W \in H \,:\, W \cap S = S'\}.$$

Furthermore, let

$$H_{nf} := \bigcup_{S'\in\mathcal{S}_{nf}} H_{S'}.$$

Lemma 4.1.21. *Let $M := \bigcup_{W\in H_{nf}} W - S$. Then*
(a) $|M| \leq r-s-2$;
(b) $|H_{nf}| \leq r-2$.

Proof. Assume that (a) does not hold and that $w_1, \ldots, w_{r-s-1}$ are in M. Let $M' := \{w_1, \ldots, w_{r-s-1}\}$. For $j = 1, \ldots, r-s-1$, let W_j be a member of H_{nf} such that $w_j \in W_j$, and let $S_j = W_j \cap S$. By Lemma 4.1.12, $W'_j := S_j + w_j$ is in H for every $j = 1, \ldots, r-s-1$. Since each S_j is non-firm and $|N-S-M'| = n-s-(r-s-1) = n-r+1 \geq 2k+2$, by Corollary 4.1.14, every set of the form $S\cup M' - S_j - w_j$ or of the form $S\cup M' - v_i$ is in H. So, every member of the family

$$\{S\cup M', \{v_1\}, \ldots, \{v_s\}, S\cup M'-\{v_1\}, \ldots, S\cup M-\{v_s\}, W'_1, \ldots, W'_{r-s-1}, S\cup M'-W'_1, \ldots, S\cup M'-W'_{r-s-1}\}$$

is in H. Moreover, together they cover every vertex in $S\cup M'$ exactly r times. This proves (a).

Suppose now that $W_1, \ldots, W_{r-1}$ are in H_{nf}. Since $|M| \leq r-s-2$, every member of the family

$$\{S \cup M, W_1, \ldots, W_{r-1}, S \cup M - W_1, \ldots, S \cup M - W_{r-1}\}$$

is in H. Moreover, together they cover every vertex in $S \cup M$ exactly r times. This proves (b). $\square$

Remark 1. Since no member of H_{nf} contains any element in $N - S - M$, for every $w \in N - M - S$, every subset of $N - S - w$ belongs to every firm $S' \subset S$.

Let S' be a smallest firm subset of S. By Lemma 4.1.17 S' is not an 1-edge. Choose a partition $S' = S_1 \cup S_2$ of S' into nonempty subsets. By the minimality of S', sets S_1 and S_2 are not firm, and so $S - S_1$ and $S - S_2$ are firm.

Fix any element $z \in N-S-M$ and let $B := N-S-z$. Since $s \leq r-2$, $|B| \geq n-(r-2)-1-r+1 \geq 2k+2$. So, by Lemma 4.1.19, there are $K := \lceil 2^k/6 \rceil + 1$ partitions

$$(B_{1,1}, B_{1,2}, B_{1,3}), (B_{2,1}, B_{2,2}, B_{2,3}), \ldots, (B_{K,1}, B_{K,2}, B_{K,3})$$

of B such that all $B_{i,j}$ are distinct. For every $i \in \{1, \ldots, K\}$ and every $j \in \{1,2,3\}$, the three sets $S' \cup (B - B_{i,j})$, $S_1 \cup (B - B_{i,j+1})$, and $S_2 \cup (B - B_{i,j+2})$ (where j counts modulo 3) are in H (by Remark 1) and cover every vertex in $N - z$ exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1,2,3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples. If r is odd, then we pick a full pair $(A, N-A)$. Then we cover the set N once by the set A and $N-A$ and $r-1$ times with the triples $(S' \cup (B - B_{i,j}), S_1 \cup (B - B_{i,j+1}), S_2 \cup (B - B_{i,j+2}))$ for $\frac{r-1}{2} (\leq 3K-2)$ triples containing neither A nor $N - A$.

4.1.4 Size of almost H-free subsets

A set A is *almost H-free* if every $B \in H$ such that $B \subseteq A$ has size 1.

The aim of this section is to prove the following theorem.

Theorem 4.1.22. *If $n \geq 425$, then $|T| \leq n - 15k - 6$ for each almost H-free $T \subseteq N$.*

Observe that for $n \geq 425$,

$$n - 15k - 6 \geq \frac{n}{2} > 0 \text{ and } n > (4k+4)(\lceil \log_2(k) \rceil + 6) + 2k + 6. \tag{4.5}$$

We need some notation and lemmas. Let $Q = N - T$. Assume that $Q < 15k + 6$, i.e., $T > n - 15k - 6$. For $Q' \subseteq Q$ and $T' \subseteq T$, we say that T' *belongs to* Q' if $Q' \cup T' \in H$. A nonempty subset Q' of Q is *solid* if some $T' \subset T'$ with $|T'| \geq 3 + k$ belongs to Q'.

To show that Q is solid, let $B \subset T$ with $|B| = 2$. Since T is almosts H-free, $B \notin H$. Then $N - B = (T - B) \cup Q \in H$. By (4.5), $|T - B| \geq n/2 - |B| = n/2 - 2 \geq k + 3$, and so Q is solid.

Lemma 4.1.23. *Let $Q' \subseteq Q$ and $T' \subseteq T$. If T' belongs to Q', then every $T'' \subset T'$ with $|T''| \leq |T'| - 2$ belongs to Q'.*

Proof. Since $T' \cup Q' \in H$, by Lemma 4.1.7, either $Q' \cup T'' \in H$ or $T - T'' \in H$. But the latter does not hold, since T is almost H-free. This proves the lemma. □

Lemma 4.1.24. *For every partition $Q = Q' \cup Q''$ of Q into nonempty subsets, exactly one of Q' and Q'' is solid.*

Proof. Assume first that Q' is not solid. By (4.5), there exists a set $M \subset T$ with $|M| = 3 + k$. Since Q' is not solid, $Q' \cup M \notin H$. Then $N - (Q' \cup M) \in H$, and $N - (Q' \cup M) = Q'' \cup (T - M)$. So, since $|T - M| \geq n - 15k - 6 - (3 + k) \geq k + 3$, Q'' is solid.

Assume now that both Q' and Q'' are solid. We will show that if a set $M \subset T$ with $|M| \geq k + 3$ belongs to both Q' and Q'', then H has an r-regular subgraph with vertex set $Q \cup M$.

If $a \in M$, then the number of distinct subsets $A_1, \ldots, A_r$ of M containing a with $2 \leq |A_i| \leq |M| - 2$ is at least

$$2^{|M|-1} - (|M| + 1) = 2^{k+2} - k - 4 = 4r - k - 4 \geq r.$$

Note that $r \geq 2$, and $M - A_i \neq A_j$, since $a \in A_j$ and $a \notin M - A_j$. Let $\mathcal{H} = \{A_i \cup Q' : 1 \leq i \leq r\} \cup \{(M - A_i) \cup Q'' : 1 \leq i \leq r\}$. By construction, $\mathcal{H}$ is r-regular, a contradiction.

If a set $M \subset T$ with $|M| = k + 4$ belongs to neither of Q' and Q'', then $T - M$ belongs to both, and again H has an r-regular subgraph. Thus each $M \subset T$ with $|M| = k + 4$ belongs to exactly

one of Q' and Q''. Let $\mathcal{R}_{Q'}$ (respectively, $\mathcal{R}_{Q''}$) denote the family of $M \subset T$ with $|M| = k+4$ that belong to Q' (respectively, to Q''). By our assumption, both $\mathcal{R}_{Q'}$ and $\mathcal{R}_{Q''}$ are nonempty. Then there exist $M' \in \mathcal{R}_{Q'}$ and $M'' \in \mathcal{R}_{Q''}$ with $|M' \cap M''| = k+3$. By Lemma 4.1.23, $M' \cap M''$ belongs to both Q' and Q'', and so H has an r-regular subgraph, a contradiction. □

Corollary 4.1.25. *If Q' is a solid subset of Q, then every $M \subset T$ with $k+3 \leq |M| \leq |T|-(k+3)$ belongs to S'.*

Lemma 4.1.26. *The number of 1-edges not in full pairs of H is at most k.*

Proof. Assume that there are $k+1$ distinct 1-edges $\{a_1\}, \ldots, \{a_{k+1}\}$ not in full pairs. If some nonempty $B \subset A = \{a_1, \ldots, a_{k+1}\}$ is not in H, then $\overline{B} \in H$ by (4.1). Then $\overline{B}$ together with 1-edges contained in B cover N once and none of these is in a full pair. These sets together with $r-1$ full pairs cover N exactly r times, a contradiction. Thus every nonempty subset of A is in H.

There are 2^k distinct nonempty subsets of A containing a_1, call them $B_1, \ldots, B_{2^k}$. Then

$$B_1, \ldots, B_r, A - B_1, \ldots, A - B_r$$

are all in H, and they form an r-regular subgraph of H, a contradiction. Therefore the number of 1-edges not in full pairs of H is at most k. □

Lemma 4.1.27. *The number of 1-edges in full pairs in H is at least $n-4k-2$. Thus at most $8k-2$ elements in full pairs are neither 1-edges nor $(n-1)$-edges.*

Proof. By Theorem 4.1.10, $|S| \geq n-3k-2$, so the number of 1-edges is at least $n-3k-2$. If fewer than $n-4k-2$ of them are in full pairs, then we get $k+1$ distinct 1-edges $a_1, \ldots, a_{k+1}$ not in full pairs, a contradiction to Lemma 4.1.26. Thus every $B \subset \{a_1, \ldots, a_{k+1}\} = S'$ is in H. Then we can take $r \leq 2^{k+1-1}$ different pairs $\{B, S'-B\}$. They cover S' exactly r times. So, we get an r-regular subgraph, a contradiction. □

Lemma 4.1.28. *For each $a \in Q$, there is $A \in H$ with $2 \leq |A| \leq 3$ such that $\{a\} = A \cap Q$.*

Proof. Since $T \cup \{a\}$ is not almost H-free, there is $B \subset T \cup \{a\}$ such that $B \in H$ and $|B| \geq 2$. Take a smallest such B.

If $|B| = b \geq 4$, then there is $B' \subset B$ with $|B'| = b - 2 > 1$ and $B' \subset T$. Then $B' \notin H$, and by Lemma 4.1.7, $B - B' \in H$, so $A = B - B'$ is what we need. □

Lemma 4.1.29. *The set Q contains at least one solid 1-edge.*

Proof. Let B be a smallest solid set in Q. Suppose $|B| \geq 2$. Then there are disjoint nonempty $B'_1, B'_2 \subset B$ with $B'_1 \cup B'_2 = B$. By Lemma 4.1.24, $B_1 = Q - B'_1$ and $B_2 = Q - B'_2$ are solid.

By (4.5), $T \geq n - 15k - 6 \geq 3k + 9$. Let $K := \lceil 2^k/6 \rceil + 1$. Similarly to the proofs of Lemma 4.1.19 and Corollary 4.1.20, for each $i = 1, 2, \ldots, K$ there are partitions $(T_{i,1}, T_{i,2}, T_{i,3})$ of T such that all $T_{i,j}$ are distinct and $|T_{i,j}| \geq k + 3$ for all $i = 1, \ldots, K$ and $j = 1, 2, 3$.

For every $i \in \{1, \ldots, K\}$ and every $j \in \{1, 2, 3\}$, the three sets $B \cup (T - T_{i,j})$, $B_1 \cup (T - T_{i,j+1})$, and $B_2 \cup (T - T_{i,j+2})$ (where j counts modulo 3) are in H (by Corollary 4.1.25, and the fact that $|T - T_{i,j}| \leq |T| - (k + 3)$) and cover every vertex in N exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1, 2, 3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples.

If r is odd, then we pick a full pair $(A, N - A)$. There are at most two triples $(B \cup (T - T_{i,j}), B_1 \cup (T - T_{i,j+1}), B_2 \cup (T - T_{i,j+2}))$ containing A or $N - A$. Then we cover the set N once by the sets A and $N - A$ and $r - 1$ times by $\frac{r-1}{2} \leq 3K - 2$ triples $(B \cup (T - T_{i,j}), B_1 \cup (T - T_{i,j+1}), B_2 \cup (T - T_{i,j+2}))$ containing neither A nor $N - A$. This contradicts the choice of H. □

Lemma 4.1.30. $|Q| < 4k + 4$.

Proof. Suppose $|Q| \geq 4k + 4$. By Lemma 4.1.29, Q contains a solid 1-edge $\{a\}$. Let $Q - a = \{b_1, \ldots, b_{4k+3}, \ldots, b_{|Q|-1}\}$. By Lemma 4.1.28, for each $i = 1, 2, \ldots, 4k + 3$, we can find B_i with $2 \leq |B_i| \leq 3$ such that $B_i \cap Q = \{b_i\}$. Let $L := N - a - \bigcup_{i=1}^{4k+3} B_i$. By definition, $|\bigcup_{i=1}^{4k+3} B_i| \leq 12k + 9$. Since $n \geq 13k + 13$, $|L| \geq 13k + 3 - 1 - (12k + 9) = k + 3$. Let $L' \subseteq L$ with $|L| = k + 3$. Let $M = N - L'$. Then H contains at least $n - 4k - 5$ edges $\{a_1\}, \ldots, \{a_{n-4k-5}\}$ such that all $M - a_i$ are also in H, since $a \in M - a_i$ and $k + 3 \leq |M - a_i| \leq |T| - k - 3$. Recall that for each $i = 1, \ldots, 4k + 3$, $B_i \in H$ and $M - B_i \in H$. Since $r \leq n - 1$, the edges

$$\{a_1\}, \ldots, \{a_{r+1-4k-5}\}, M - a_1, \ldots, M - a_{r+1-4k-5}, B_1, \ldots, B_{4k+3}, M - B_1, \ldots, M - B_{4k+3}, M$$

form an r-regular subgraph of H, a contradiction. □

Lemma 4.1.31. *If $\{a\}$ is a solid 1-edge and $B \in H$ with $a \notin B$, then $|B \cap T| < \lceil \log_2 k \rceil + 5$.*

Proof. If there is a set B with $a \notin B$ and $|B \cap T| \geq \lceil \log_2 k \rceil + 5$, then by Lemma 4.1.23, we can find $B_1, B_2, \ldots, B_{8k} \in H$ such that $B_i \subset B$, $B \cap Q = B_i \cap Q$, since $2^{\lceil \log_2 k \rceil + 4} - (\lceil \log_2 k \rceil + 5) \geq 8k$. Let $X \subset N - (B \cup Q)$ with $|X| = k+3$ and let $M = N - X$. Since at least $n - 3k - 2 - (k+3) = n - 4k - 5$ of 1-edges $\{a_i\}$ are in M, the sets

$$B_1, \ldots, B_{4k+4}, M - B_1, \ldots, B - B_{4k+4}, \{a_1\}, \ldots, \{a_{r-4k-5}\}, M - a_1, \ldots, M - a_{r-4k-5}, M$$

form an r-regular subgraph of H, a contradiction. □

Lemma 4.1.32. *There are at most $4k + 3$ sets $A_i \in H$ such that no A_i is a 1-edge and no solid 1-edge a is contained in A_i.*

Proof. Suppose that there are $4k + 4$ such sets $A_1, \ldots, A_{4k+4}$. Then by Lemma 4.1.31,

$$|T \cap \bigcup_{i=1}^{4k+4} A_i| \leq (4k + 4)(\lceil \log_2 k \rceil + 5) \leq |T| - k - 3.$$

Thus, as in the proof of Lemma 4.1.31, we can find an r-regular subgraph of H by using A_i instead of B_i. □

Lemma 4.1.33. *If $\{a\}$ is a solid 1-edge, then there is at most one $D \notin H$ with $a \in D$.*

Proof. Suppose $D_1, D_2 \notin H$ with $a \in D_1 \cap D_2$. By Lemma 4.1.31, $|D_i \cap T| \geq |T| - k + 3$ for $i = 1, 2$. So, $|D_1 \cap D_2 \cap T| \geq |T| - 2k - 6$.

By Lemmas 4.1.31 and 4.1.32, at least $|T| - (4k + 4)(\log_2 k + 6)$ elements in T are covered only by 1-edges and sets containing a.

By (4.5), $|T| - (4k + 4)(\log_2 k + 6) - 2k - 6 > 0$. So there is $c \in D_1 \cap D_2$ such that c is not covered by any edge of size at least 2 not containing a. Since H is (n, r)-strange, Then at most $2^{n-1} + 1 - 2 = 2^{n-1} - 1$ edges of H contain c. Thus the family $H_c = \{A \in H : c \in A\}$ has at least $2^{n-2} + r - 1$ edges on $n - 1$ vertices, and by Theorem 4.1.2 we get an r-regular subgraph of H' which is also a subgraph of H, a contradiction. □

Proof of Theorem 4.1.22. By Lemma 4.1.29, H has a solid 1-edge $\{a\}$. By Lemma 4.1.33, there is at most one set $D \notin H$ with $a \in D$. Since H is (n,r)-strange, such D exists and exactly $r-1$ edges of H do not contain a, call them $B_1, \ldots, B_{r-1}$.

Case 1: $\bigcup_{i=1}^{r-1} B_i = N-a$. Let l be the minimum integer such that we can renumber $B_1, \ldots, B_{r-1}$ so that $\bigcup_{i=1}^{l} B_i = N-a$. Let $\mathcal{B} = \{B_{l+1}, \ldots, B_{r-1}\}$. Let $C_1 = B_1, C_2 = B_2 - B_1, C_3 = B_3 - B_2 - B_1, \ldots, C_l = B_l - B_1 - B_2 - \cdots - B_{l-1}$. By the minimality of l, $C_i \neq \emptyset$ for every $i = 1, \ldots, l$. By construction, $\{C_1, \ldots, C_l\}$ is a partition of $N-a$.

For every $i = 1, \ldots, l$, there are $2^{|C_i|} - 2$ ways to choose a nonempty proper subset A of C_i. By Lemma 4.1.7, for each proper subset A of C_i, one of A and $B_i - A$ is in H, and hence it is in $\mathcal{B}$. It follows that $\mathcal{B}$ contains at least $\frac{1}{2}(2^{|C_i|} - 2) = 2^{|C_i|-1} - 1 \geq |C_i| - 1$ sets B such that (i) $0 < |B \cap C_i| < |C_i|$ and (ii) $B \cap C_j = \emptyset$ for all $i+1 \leq j \leq l$. Since all C_is are disjoint, we conclude that $|\mathcal{B}| \geq \sum_{i=1}^{l}(|C_i| - 1) = n - 1 - l$. Together with $B_1, \ldots, B_l$, we have at least $n-1$ members of H not containing a. This contradicts the fact that H has only $r - 1 \leq n - 2$ sets not containing a.

Case 2: There is $y \in N - a - \bigcup_{i=1}^{r-1} B_i$. Since $N - D \in H$ and $a \notin N - D$, $y \notin N - D$. So, $y \in D$. Thus y belongs to at most $2^{n-2} - 1$ members of H containing a and to none not containing a. So, the family $H' = H - y$ has at least $2^{n-1} + r - 2 - (2^{n-2} - 1) = 2^{n-2} + r - 1$ members. By Theorem 4.1.2, H' has an r-regular subgraph, which is also a subgraph of H, a contradiction. □

4.1.5 Proof of Theorem 4.1.5

Suppose H is (n,r)-strange hypergraph on N. By Theorem 4.1.22,

$$\text{every } S \subseteq N \text{ with } |S| \geq n - 15k - 5 \text{ contains some } A \in H \text{ with } |A| \geq 2. \tag{4.6}$$

Let $B_1, \ldots, B_l$ be the 1-edges not in full pairs. Let $N_1 = N - B_1 - B_2 - \cdots - B_l$. By Lemma 4.1.26, $|N_1| \geq n - k$. So, by (4.6), N_1 contains some $B_{l+1} \in H$ with $|B_{l+1}| \geq 2$. Then by Lemma 4.1.7, we can choose such B_{l+1} with $2 \leq |B_{l+1}| \leq 3$. Let $N_2 = N_1 - B_{l+1}$. Since $|N_2| \geq (n-k) - 3$, again by (4.6) and Lemma 4.1.7, N_2 contains some $B_{l+2} \in H$ with $2 \leq |B_{l+2}| \leq 3$. Similarly, we find $B_{l+3}, \ldots, B_{5k+2}$. Since at least $n - 4k - 2$ of 1-edges are in full pairs, by Lemma 4.1.27, at most $4k+1$ full pairs have no 1-edges. Among the at most $8k+2$ sets in these full pairs, at most

$4k+1$ of the sets are in $\{B_1, \ldots, B_{5k+2}\}$, since $|B_i| \leq 3$ and $n \geq 425$. Thus some $k+1$ sets among $B_1, \ldots, B_{5k+2}$ are not in full pairs. Call them $A_1, \ldots, A_{k+1}$. Then for any $I \subset [k+1]$, $A_I = \bigcup_{i \in I} A_i$ is in H, otherwise $\overline{A_I}$ and $\{A_j : j \in I\}$ together with $r-1$ full pairs yield an r-regular subgraph of H. Therefore H contains $2^{k+1-1} \geq r$ different pairs of edges of the kind $A_I, A_{[k+1]-I}$. They form an r-regular subgraph of H covering $A_{[k+1]}$, a contradiction. □

4.2 Hypergraphs with no equicovering subgraphs

In Section 4.2, we consider hypergraphs without t edge-disjoint subhypergraphs whose local behavior at vertices is somehow "the same". Such structures may be useful for fault-tolerance of communication protocols.

In 1.3, we introduce the notion of t-EUP and t-EVP. Lindström [42] proved the following theorem finding a sufficient condition for t-EUP.

Theorem 4.2.1. [42] *Every n-vertex hypergraph with more than $n(t-1)$ edges satisfies the t-EUP.*

We generalize the extremal problem studied by Lindström. Let $\mathrm{U}(n,t)$ denote the maximum number of edges in an n-vertex hypergraph not satisfying the t-EUP. Let $\mathrm{U}_k(n,t)$ denote the maximum number of edges in a k-uniform n-vertex hypergraph not satisfying the t-EUP. For the weak t-EUP, let $\mathrm{U}'(n,t)$ and $\mathrm{U}'_k(n,t)$ denote the corresponding extremal values for general hypergraphs and k-uniform hypergraphs, respectively. For the t-EVP, use V instead of U to define the corresponding extremal problems. By definition, $A(n,t) \geq A_k(n,t)$ for $A \in \{\mathrm{U}, \mathrm{U}', \mathrm{V}, \mathrm{V}'\}$.

n-vertex	t-EUP	weak t-EUP	t-EVP	weak t-EVP
graphs	$\mathrm{U}_2(n,t)$	$\mathrm{U}'_2(n,t)$	$\mathrm{V}_2(n,t)$	$\mathrm{V}'_2(n,t)$
k-uniform hypergraphs	$\mathrm{U}_k(n,t)$	$\mathrm{U}'_k(n,t)$	$\mathrm{V}_k(n,t)$	$\mathrm{V}'_k(n,t)$
general hypergraphs	$\mathrm{U}(n,t)$	$\mathrm{U}'(n,t)$	$\mathrm{V}(n,t)$	$\mathrm{V}'(n,t)$

In Section 4.2.1, we characterize the graphs satisfying the 2-EUP and those satisfying the 2-EVP, yielding $\mathrm{U}_2(n,2) = \mathrm{U}'_2(n,2) = n$ and $\mathrm{V}_2(n,2) = \mathrm{V}'_2(n,2) = \lfloor \frac{4}{3}n \rfloor - 1$. In Section 4.2.2, we prove $\mathrm{V}'_2(n,3) = \lfloor \frac{4}{3}n \rfloor$ and $\left(t - 1 + \frac{1}{2(t-1)}\right)(n-1) - (t-2)^2 \leq \mathrm{V}_2(n,t) \leq (2+o(1))(t-1)n$. In Section

4.2.3, we consider k-uniform hypergraphs. We prove $\mathrm{V}_k(n,2) \in O(n)$ and $\mathrm{V}(n,2) \in O(n \log n)$. We also prove $\mathrm{V}(n,t), \mathrm{V}_k(n,t) \in O\left(n^2 \left(\frac{\log n}{\log\log\log n}\right)^2\right)$ and obtain a lower bound for $\mathrm{V}_k(n,2)$.

We close this introduction with several simple observations used in later proofs.

Observation 4.2.2. *The weak 2-EVP and the 2-EVP are equivalent. Thus* $\mathrm{V}(n,2) = \mathrm{V}'(n,2)$, *and* $\mathrm{V}_k(n,2) = \mathrm{V}'_k(n,2)$ *for* $k \geq 2$.

Proof. If subhypergraphs H_1 and H_2 witness the weak 2-EVP, then $H_1 - E(H_1) \cap E(H_2)$ and $H_2 - E(H_1) \cap E(H_2)$ witness the 2-EVP, since each vertex loses the same number of incident edges from H_1 and H_2. □

Observation 4.2.3. $\mathrm{U}'(n,t) < n + \log_2 t$ *and* $\mathrm{U}'_k(n,t) < \log_2\left(\sum_{i=k}^{n} \binom{n}{i}\right) + \log_2 t$.

Proof. A hypergraph with m edges has 2^m distinct subhypergraphs. An n-vertex hypergraph H with $\lceil n + \log_2 t \rceil$ edges has at least $t2^n$ distinct subhypergraphs. Since there are only 2^n choices for the span of a subhypergraph of H, by the pigeonhole principle there are t distinct subhypergraphs with the same span. Thus H satisfies the weak t-EUP.

The second statement uses the fact that the span of nonempty k-uniform subhypergraphs has size at least k. □

Observation 4.2.4. $\mathrm{V}_k(n+1,t) \geq \mathrm{V}_{k-1}(n,t)$ *for* $k \geq 3$.

Proof. Let H be a $(k-1)$-uniform hypergraph not satisfying the t-EVP. Form H' from H by adding one vertex x belonging to every edge. If $H'_1, \ldots, H'_t$ witness the t-EVP in H', then deleting x from each edge in each H'_i yields $H_1, \ldots, H_t$ witnessing the t-EVP in H. □

Observation 4.2.5. *If H_1 and H_2 have equal degree at every vertex of G, then their symmetric difference contains no cut-edge of G.*

Proof. Let e be a cut-edge of G contained in H_1 but not H_2. Let G' be a component of $G - e$. The sum of the degrees of the subgraphs of H_1 and H_2 in G' differ by 1. Hence one of them is a graph with odd degree sum, which does not exist. □

Observation 4.2.6. *If $H_1, \ldots, H_t$ witness the weak t-EVP in G, with degree 0 at v or degree $d_G(v)$ at v, then deleting v yields graphs $H'_1, \ldots, H'_t$ witnessing the weak t-EVP in $G - v$.*

Proof. Since the usage of edges incident to v is the same in each H_i, ignoring those edges also contributes the same degree from all t subgraphs at each of the remaining vertices. □

4.2.1 The 2-EUP and the 2-EVP for Graphs

Our graphs have no repeated edges or loops. A *walk* in a graph is a list of vertices in which every two consecutive vertices are adjacent (edges may repeat). A *trail* is a walk that traverses each edge at most once (vertices may repeat). A walk or trail is *closed* if its first and last vertices are the same. A *circuit* is a closed trail. A *cycle* in a graph is a connected subgraph in which every vertex has degree 2. The *length* of a walk, trail, circuit, or cycle is the number of edges it traverses, and the object is *even* or *odd* as its length is even or odd.

We characterize the graphs satisfying the 2-EUP (Theorem 4.2.7) and the graphs satisfying the 2-EVP (Theorems 4.2.9–4.2.10). From the characterizations, we determine $\mathrm{U}_2(n,2)$, $\mathrm{U}'_2(n,2)$, $\mathrm{V}_2(n,2)$, and $\mathrm{V}'_2(n,2)$.

Theorem 4.2.7. *A graph G satisfies the 2-EUP if and only if G has an even cycle or has a component containing two odd cycles.*

Proof. (Sufficiency) Alternating edges along an even cycle gives two subgraphs witnessing the 2-EUP. Now, let G have a component containing odd cycles C_1 and C_2. If C_1 and C_2 share an edge, then G has an even cycle. Otherwise, let P be a shortest path joining $V(C_1)$ and $V(C_2)$, with endpoints $u \in V(C_1)$ and $v \in V(C_2)$. Let T be a trail that starts at u, follows C_1, moves from u to v through P, and follows C_2. Alternating edges along W gives two subgraphs with vertex set $V(C_1) \cup V(C_2) \cup V(P)$.

(Necessity) Let G_1 and G_2 be two subgraphs of G witnessing the 2-EUP, and take $u_0u_1 \in E(G_1) - E(G_2)$. Since u_1 must be covered by both $E(G_1)$ and $E(G_2)$, there exists $u_1u_2 \in E(G_2)$. Iteratively, we find $u_iu_{i+1} \in E(G_1)$ and $u_{i+1}u_{i+2} \in E(G_2)$ for even i. Since G is finite, $u_k = u_{k+r}$ for some r and k. Taking the first such repetition, $u_k, \ldots, u_{k+r-1}$ form a cycle. If r is even, then G has an even cycle.

If r is odd, then by symmetry we may assume that u_ku_{k+1} and u_ku_{k+r-1} are both in $E(G_1)$. Since G_1 and G_2 have the same span, we find v_1 such that $u_kv_1 \in E(G_2)$. We now generate

$v_1, v_2, \ldots$ as above, with $v_j v_{j+1} \in E(G_1)$ and $v_{j+1} v_{j+2} \in E(G_2)$ for odd j, until we obtain $v_l = v_{l+s}$ or $v_l = u_s$ for some l and s with $s > k$; consider the first such occurrence. (In particular, we ignore $u_0, \ldots, u_{k-1}$, as if they had never been defined.)

If $v_l = v_{l+s}$, then $v_l, \ldots, v_{l+s-1}$ form an even cycle when s is even. If s is odd, then $[v_l, \ldots, v_{l+s-1}]$ and $[u_k, \ldots, u_{k+r-1}]$ form two odd cycles in the same component of G.

If $v_l = u_s$, then the edges we have discovered form three internally disjoint paths joining u_k and u_s. The union of some two of them is an even cycle. □

Corollary 4.2.8. *If a graph G does not satisfy the 2-EUP, then $|E(G)| \leq n$. Equality holds if and only if every component of G has at most one cycle and all cycles in G have odd length. Thus $\mathrm{U}_2(n, 2) = n$.*

A *nontrivial graph* is a graph having an edge.

Theorem 4.2.9. *A graph G satisfies the 2-EVP if and only if it has an even circuit.*

Proof. (Sufficiency) If G has an even circuit, then alternating edges along it provides two subgraphs witnessing the 2-EVP.

(Necessity) Let H_1 and H_2 be two subgraphs witnessing the 2-EVP in G. Since H_1 and H_2 have the same number of edges and have equal degree at each vertex, their union H is a nontrivial graph whose components have an even number of edges and have even degree at each vertex. Hence H decomposes into cycles. If any such cycle has even length, then G contains an even circuit.

Otherwise, all cycles in the decomposition of a nontrivial component of H have odd length, and there are an even number of them. Since a connected union of cycles has no cut-edge, two such odd cycles C and C' are joined by two edge-disjoint paths P and P' in H. The endpoints of P and P' on C are connected by two paths of opposite parity along C, and the same is true along C'. Hence the union of P, P', and appropriate paths connecting them in C and C' is a subgraph traversed by an even circuit. □

A *cactus* is a connected graph in which every edge is in at most one cycle. A *strict odd cactus* is a cactus in which every cycle has odd length and no two cycles share a vertex.

Theorem 4.2.10. *A graph G fails to satisfy the 2-EVP if and only if every component is a strict odd cactus.*

Proof. (Sufficiency) Theorem 4.2.9 applies, since a strict odd cactus contains no even circuit.

(Necessity) We use induction on n, where $n = |V(G)|$. Let G be an n-vertex graph not satisfying the 2-EVP. For $n \leq 3$, the statement is true. For larger n, since every tree is a strict odd cactus, we may assume that G has a cycle C; avoiding the 2-EVP requires C to be odd. If $G = C$, then G is a strict odd cactus. Otherwise, both $G - V(C)$ and C have fewer than n vertices and do not satisfy the 2-EVP. By the induction hypothesis, each component of $G - V(C)$ is a strict odd cactus. If at most one edge joins $V(C)$ to any such component H, then again every component of G is a strict odd cactus.

Finally, consider edges xy and $x'y'$ with $x, x' \in V(C)$ and $y, y' \in V(H)$. Combining these two edges with a y, y'-path in H yields an x, x'-path that is disjoint from C except for its endpoints. Now adding one of the paths from x to x' along C completes an even circuit, witnessing the 2-EVP for G. Hence this possibility does not occur, and every component of G is a strict odd cactus. □

Theorem 4.2.11. *If an n-vertex graph G does not satisfy the 2-EVP, then $|E(G)| \leq \lfloor \frac{4}{3}n \rfloor - 1$. Also, the bound is sharp, so $\mathrm{V}_2(n, 2) = \lfloor \frac{4}{3}n \rfloor - 1$.*

Proof. By Theorem 4.2.10, every component of G is a strict odd cactus. A largest such graph on n vertices is connected. If G has k cycles, then $k \leq \lfloor n/3 \rfloor$, since each cycle has length at least 3 and the cycles share no vertices. Also, deleting one edge from each cycle yields a tree. Hence $|E(G)| = n - 1 + k \leq \lfloor \frac{4}{3}n \rfloor - 1$. Equality holds for any strict odd cactus having $\lfloor n/3 \rfloor$ triangles and j vertices not in triangles, where $n \equiv j \mod 3$ with $j \in \{0, 1, 2\}$. □

4.2.2 The t-EVP for Graphs

In this section we study $\mathrm{V}_2(n, t)$. We give an upper bound in Theorem 4.2.16 using a theorem of Alon, Friedland, and Kalai [9]. In Theorem 4.2.17, we construct a sequence of graphs to provide a lower bound. We close the section by determining $\mathrm{V}'_2(n, 3)$ exactly in Theorem 4.2.23. We begin with two well-known facts.

Fact 4.2.12. *Every graph G has a bipartite subgraph H with $|E(H)| \geq \frac{1}{2}|E(G)|$.*

Fact 4.2.13. [21] *For $k \geq 2$, there exists a prime number p such that $k \leq p < 2k$.*

For $q \in \mathbb{N}$, a *q-divisible* graph is a graph where every vertex degree is a multiple of q.

Theorem 4.2.14. [9] *If q is a prime power, then every n-vertex graph G with $|E(G)| \geq (q-1)n+1$ contains a q-divisible subgraph.*

Lemma 4.2.15. *Every t-divisible bipartite graph G satisfies the t-EVP.*

Proof. For $v \in V(G)$, let $c(v) = d_G(v)/t$. Form a graph G' by expanding each vertex $v \in V(G)$ into an independent set of size $c(v)$ in which each vertex inherits t of the edges incident to v. The graph G' is t-regular and bipartite, so G' decomposes into t edge-disjoint perfect matchings. These perfect matchings correspond to t edge-disjoint subgraphs $H_1, \ldots, H_t$ of G, each having degree $c(v)$ at v. Thus $H_1, \ldots, H_t$ witness the t-EVP for G. □

Theorem 4.2.16. *If t is a prime power, then $\mathrm{V}_2(n,t) \leq 2(t-1)n$. In general, $\mathrm{V}_2(n,t) \leq 2.4(t-1)n$ and $\mathrm{V}_2(n,t) \leq (2+o(1))(t-1)n$.*

Proof. Let G be an n-vertex graph. If $|E(G)| > m$ implies that G satisfies the t-EVP, then $\mathrm{V}_2(n,t) \leq m$.

Suppose first that t is a prime power and $|E(G)| > 2(t-1)n$. By Fact 4.2.12, G has a spanning bipartite subgraph H such that $|E(H)| > (t-1)n$. By Theorem 4.2.14, H contains a t-divisible subgraph Q. By Lemma 4.2.15, Q satisfies the t-EVP, and hence G does also.

Consider general t and $|E(G)| > 2.4(t-1)n$. By the preceding paragraph, it suffices to find a prime power q between t and $1.2t$. By Fact 4.2.13 (Bertrand's Postulate), there is a prime number between t and $2t-1$. For $t \geq 25$, Nagura [47] proved that there is a prime between t and $1.2t$. For $2 \leq t \leq 24$, there fails to be a prime in the desired range for $t \in \{4, 8, 14, 24\}$. However, for these values there is a prime power in the desired range.

Meanwhile, Iwaniec and Pintz [29] proved that there exists $t_0 \in \mathbb{N}$ such that for $x > t_0$, there is a prime between $x - x^{23/42}$ and x. This yields $\mathrm{V}_2(n,t) \leq (2+o(1))(t-1)n$ in the same way as above. The value $23/42$ was further reduced to .525 in [10]. □

Alon, Friedland, and Kalai [9] conjectured that the conclusion of Theorem 4.2.14 holds even when q is not a power of a prime. This would improve the upper bound on $\mathrm{V}_2(n,t)$ from $2.4(t-1)n$ to $2(t-1)n$.

A graph is *k-degenerate* if every subgraph has a vertex of degree at most k. In a $(t-1)$-degenerate graph, every nontrivial subgraph has a vertex that cannot occur in the span of t edge-disjoint subgraphs, so $(t-1)$-degenerate graphs cannot satisfy the t-EVP. There is an n-vertex $(t-1)$-degenerate graph with as many as $\binom{t}{2}+(t-1)(n-t)$ edges, so $\mathrm{V}_2(n,t) \geq (t-1)(n-t/2)$. We improve the leading coefficient.

Theorem 4.2.17. $\mathrm{V}_2(n,t) \geq \left(t-1+\frac{1}{2t-2}\right)(n-1)-(t-2)^2$.

Proof. The *k-fold wheel* $W_{k,r}$ is obtained from the complete bipartite graph $K_{k,r}$ by adding the edges of a cycle through the r vertices of degree k, giving them degree $k+2$. Call the vertices of degree r *centers*, the vertices of degree $k+2$ *rim vertices*, and the edges joining central and rim vertices *spokes*. In any copy of $W_{k,r}$, specify two distinct rim vertices as the *head* and *tail*. Note that $W_{k,r}$ has $k+r$ vertices and $r(k+1)$ edges.

For $t \geq 2$, let $B_1,\ldots,B_\ell$ be copies of $W_{t-2,t+1}$. Let z_i denote the tail of B_i (and z_0 denote the head of B_1). For $1 \leq i < \ell$, merge z_i with the head of B_{i+1}. Add edges joining all centers of B_i to all centers of B_{i+1}, for $1 \leq i < \ell$. Let G_t^ℓ be the resulting graph.

Note that G_t^ℓ has $\ell(2t-2)+1$ vertices and $\ell(t+1)(t-1)+(\ell-1)(t-2)^2$ edges. Writing $n=\ell(2t-2)+1$, the number of edges becomes $(t-1+\frac{1}{2t-2})(n-1)-(t-2)^2$. Hence it suffices to show that G_t^ℓ does not satisfy the t-EVP.

Suppose that $H_1,\ldots,H_t$ are subgraphs of G_t^ℓ witnessing the t-EVP. Let $H = H_1 \cup \cdots \cup H_t$; each vertex degree in H is a multiple of t. Say that the edges of H are "used". The edges of G_t^ℓ that are not spokes form a $(t-2)$-degenerate graph, so some spoke must be used.

Each rim vertex other than $z_0,\ldots,z_\ell$ has degree t. Let i be the least index such that some spoke in B_i is used. If only spokes incident to z_i are used, then some center of B_i has degree between 1 and $t-1$. Hence some spoke not incident to z_i is used. Since rim vertices have degree t in B_i and form a connected subgraph, it follows that all edges of B_i are used except possibly some spokes incident to z_i.

If any spoke incident to z_i is used, then its other endpoint has degree between $t+1$ and $2t-1$ in H ($t+1$ rim vertices and up to $t-2$ centers in B_{i+1}). Hence no spoke of B_i at z_i is used, but all other edges of B_i are used.

Now z_i has two neighbors in H that are rim vertices of B_i, plus up to t neighbors in B_{i+1}. Hence $d_H(z_i) = t$. This requires the rim edges of B_i at z_i to belong to H_a and H_b with $a \neq b$. The remaining vertices of B_i have neighbors only in $V(B_i)$ in H, and hence H_a must form a matching there. However, $|V(B_i)| = 2t-1$, so no such matching is possible.

When $n \not\equiv 1 \mod 2(t-1)$, we may add one small complete graph and possibly one copy of K_t to G_t^ℓ (where $\ell = \lfloor n/(2t-2) \rfloor$) to reach n vertices. □

The remainder of this section determines $\mathrm{V}_2'(n,3)$. We start with the lower bound.

Lemma 4.2.18. *A graph containing at most one even circuit does not satisfy the weak 3-EVP.*

Proof. Suppose that G satisfies the weak 3-EVP, witnessed by three distinct nontrivial subgraphs H_1, H_2, H_3 having equal degrees at each vertex. Since H_1 and H_2 are distinct and have the same vertex degrees, the symmetric difference of H_1 and H_2 is nontrivial, has even degree at each vertex, and has an even number of edges in each component (the same number from both H_1 and H_2). Hence each nontrivial component can be expressed as an even circuit. The same is true for H_1 and H_3, which cannot have the same components since H_2 and H_3 are distinct. Hence G contains at least two even circuits. □

Lemma 4.2.19. $\mathrm{V}_2'(n,3) \geq \lfloor \frac{4}{3}n \rfloor$ *for* $n > 3$.

Proof. Let j be the congruence class of n modulo 3, with $j \in \{0,1,2\}$. Let G be a maximal strict odd cactus with n vertices. As computed in Theorem 4.2.10, G has $\lfloor \frac{4}{3}n \rfloor - 1$ edges and no even circuit; also G has $\lfloor n/3 \rfloor$ triangles and j vertices not in triangles. Further restrict G when $n \equiv 2$ so that the vertices not in triangles are leaves with a common neighbor.

In each case, we add one edge to G. When $j = 1$, add one edge joining the vertex v that lies in no triangle to a vertex of a nearest triangle; this creates one even circuit (of length 4). When $j = 2$, add one edge joining the two leaves not in triangles, creating one even circuit (of length 6). By Lemma 4.2.18, the resulting graph does not satisfy the weak 3-EVP.

We construct an explicit example G' for $n = 6$, consisting of disjoint triangles on abc and xyz plus edges ax and by. Every proper induced subgraph of G' has at most one even circuit and does not satisfy the weak 3-EVP, by Lemma 4.2.18. Hence by Lemma 4.2.6 subgraphs witnessing it have degree 1 at c and z and have degree 1 or 2 at the other vertices. If all have degree 1, then the subgraphs are perfect matchings, but G' has only two perfect matchings. If all have degree 2, then they omit a perfect matching, and again there are only two.

Hence we may assume that exactly two vertices have degree 2. By symmetry, consider cases for the degrees of x and y, with $d(x) \geq d(y)$. If $d(x) > d(y)$, then xz must appear in each subgraph. By reasoning as in Lemma 4.2.6, $G' - z$ then satisfies the weak 3-EVP, which it does not. If $d(x) = d(y)$, then in each case a subgraph with the specified degrees is determined by choosing xz or yz, so again there are only two.

For larger multiples of 3, we use this 6-vertex subgraph G' plus a maximal strict odd cactus $\hat{G}$ on $n - 6$ vertices joined to G' by a single cut-edge e. If there exist H_1, H_2, H_3 witnessing the weak 3-EVP in G, then any two of them have equal degrees at all vertices. By Observation 4.2.5, the symmetric difference of any two cannot contain the cut-edge e. Hence all three agree on e. Hence $G - e$ satisfies the weak 3-EVP. This requires the weak 3-EVP to hold in $\hat{G}$ or G', which it does not. □

The upper bound needs several lemmas.

Lemma 4.2.20. *Let G be a graph.*

(a) If G has $\lceil \log_2 t \rceil$ edge-disjoint even circuits, then G satisfies the weak t-EVP.

(b) If G has three edge-disjoint trails with the same endpoints whose lengths have the same parity, then G satisfies the weak 3-EVP.

Proof. (a) Let $s = \lceil \log_2 t \rceil$, so $2^s \geq t$, and let $C_1, \ldots, C_s$ be the given edge-disjoint even circuits. There are two ways to take alternating edges on each circuit. Hence at least t distinct subgraphs of $C_1 \cup \cdots \cup C_s$ have the same degrees at all vertices.

(b) Consider three edge-disjoint u, v-trails. If all have even length, then take alternating edges in each trail, starting at u in two of the trails. Each way of doing this yields the same degree at all vertices. The three ways to do it yield the weak 3-EVP.

If all have odd length, then the union of two form an even circuit. Take alternating edges from the other so that u and v are not covered, plus alternating edges along the circuit. The three ways to choose a pair of trails and two ways to choose edges along the circuit yield six subgraphs having the same degree at all vertices. Thus G satisfies the weak 6-EVP. □

Lemma 4.2.21. *If G has a path through w, x, y, z such that $d_G(x) = d_G(y) = 2$, then G satisfies the weak t-EVP if and only if G' satisfies the weak t-EVP, where $G' = G - \{x, y\} \cup wz$.*

Proof. If $H'_1, \ldots, H'_t$ witness the t-EVP in G', then modify each H'_i as follows. If $wz \in E(H'_i)$, then put $wx, yz \in E(H_i)$ and $xy \notin E(H_i)$, leaving the usage of all other edges the same. If $wz \notin E(H'_i)$, then put $xy \in E(H_i)$ and $wx, yz \notin E(H_i)$, again leaving other edges as in H'_i. In each H_i, the degree is 1 at x and at y, and at all other vertices it is the same as in H'_i, so $H_1, \ldots, H_t$ witness the t-EVP in G'.

For the converse, let $H_1, \ldots, H_t$ witness the t-EVP in G. If each subgraph has degree 1 at both x and y, then the transformation above can be reversed. If the subgraphs all have degree 0 or degree 2 at x or y, then they all have the same usage of each edge in $\{wx, xy, yz\}$. Now letting $H'_i = H_i - \{x, y\}$ for all i yields subgraphs witnessing the t-EVP in G' (they all omit the edge wz). □

We apply Lemma 4.2.21 to subdivisions of K_4 that arise in the proof of the upper bound.

Lemma 4.2.22. *Let G be a subdivision of K_4, and call the paths joining branch vertices "threads". If G has an odd cycle C through three branch vertices, and the cycles through exactly two threads of C all have even length, then G satisfies the weak 3-EVP.*

Proof. By Lemma 4.2.21, we may assume that every thread has length 1 or 2.

When all threads in C have length 1, the three threads incident to the remaining vertex z all have the same parity. If they are single edges, then use the decomposition of K_4 into three matchings M_1, M_2, M_3. Otherwise, the threads at z each have length 2. Each subgraph H_i uses one edge incident to z and a path of length 4 joining the other two neighbors of z.

If only one thread in C has odd length, then the lengths of the two threads from its endpoints to z have the same parity. If they have length 1, then we can treat that cycle as C above. If they

have length 2, then the other thread at z has length 1; now exactly two threads (with no shared endpoints) have odd length, and our graph G is obtained from K_4 by subdividing the edges of a 4-cycle.

The 8-cycle has two crossing chords; call them uv and $u'v'$. Pairing uv with each of the two perfect matchings in the 8-cycle generates two subgraphs with the same vertex degrees. A third such subgraph uses $u'v'$ and the edges at u and v other than uv. Hence G satisfies the weak 3-EVP. □

Theorem 4.2.23. $\mathrm{V}'_2(n,3) = \lfloor \frac{4}{3}n \rfloor$. *That is, every n-vertex graph with more than $\lfloor \frac{4}{3}n \rfloor$ edges satisfies the weak 3-EVP, and this is sharp.*

Proof. We need only prove the upper bound. Let G be an n-vertex graph with $\lfloor \frac{4}{3}n \rfloor + 1$ edges. By using the component with largest average degree, we may take G to be connected.

Let G' be a subgraph of G with the most edges that contains no even circuit. Adding a cut-edge joining components does not create a circuit. Hence G' is a strict odd cactus spanning all of $V(G)$, by Theorem 4.2.9. By Theorem 4.2.10 and Theorem 4.2.11, $|E(G')| \le \lfloor \frac{4}{3}n \rfloor - 1$. Thus G' omits at least two edges of G; choose $e_1, e_2 \in E(G) - E(G')$. By the definition of G', adding e_i to G' creates an even circuit C_i. If C_1 and C_2 share no edge, then G satisfies the weak 4-EVP, by Lemma 4.2.20(a). Thus, we may assume that they share an edge.

Shrinking the odd cycles of G' into vertices yields a tree T. Call the vertices of T *nodes* to distinguish them from vertices of G'; they correspond to vertices or odd cycles in G'. Note that $C_1 - e_1$ and $C_2 - e_2$ shrink to paths P_1 and P_2 in T. Since C_1 and C_2 are not edge-disjoint, P_1 and P_2 intersect in T. If the intersection is one node of T, then C_1 and C_2 intersect in a single nontrivial path in G. Since both circuits have even length, the endpoints of this path are joined by three edge-disjoint trails of the same parity. Hence G satisfies the weak 3-EVP, by Lemma 4.2.20(b).

Hence we may assume that P_1 and P_2 intersect in a nontrivial path P' in T; let u and w be its endnodes. Let U and W be the subgraphs of G' corresponding to u and w, they may be single vertices or odd cycles in G'. Let Q_0 be the subgraph of G' corresponding to P'. Let x_0 and y_0 be the vertices of U and W with largest degree in Q_0, and let Q'_0 be an x_0, y_0-path through Q_0.

For $i \in \{1, 2\}$, let Q_i be the trail obtained from C_i by deleting $E(Q_0)$; note that $e_i \in E(Q_i)$. Let x_i and y_i be the endpoints of Q_i in U and W, respectively.

If Q_2 visits an odd cycle $\hat{C}$ in G', then let x and y be the vertices in U and W that are endpoints of $C_2 - E(C_1)$. By traversing $\hat{C}$ appropriately, we obtain a trail whose length has the same parity as the two x, y-trails along C_1, and Lemma 4.2.20(b) applies. Hence we may assume that Q_2 (and similarly Q_1) visits no such cycle.

Now combining Q_1 and Q_2 with appropriate traversals through U and W may yield an even circuit containing Q_1 and Q_2. If this happens, and Q_0' visits an odd cycle in G', then again Lemma 4.2.20(b) applies.

Hence two cases remain. In one case, $x_0 = x_1 = x_2$ and $y_0 = y_1 = y_2$, these are the only vertices of U and W, and Q_1 and Q_2 are paths with opposite parity. Now $Q_1 \cup Q_2$ is an odd cycle that shares vertices with no odd cycles in G'. Letting e' be the edge of Q_0' incident to U, we now have $G' - e' \cup \{e_1, e_2\}$ as a subgraph of G having no even circuit, contradicting the maximality of G'.

In the remaining case, the interior nodes of P' are single vertices in G', not odd cycles. The x_1, x_0-path through U in C_1 and the x_2, x_0-path through U in C_2 may or may not overlap. If they overlap, then taking the complement within U for each of these paths makes them disjoint but changes the parity of the circuits. However, if the same thing happens within W, then performing the complementations at both ends of P' leaves us with two even cycles C_1' and C_2' in G whose intersection is the single path P', and Lemma 4.2.20(b) applies. The same holds if complementation was not needed on either end.

We are left with the case where C_1 and C_2 both contain the x_1, x_2-path through U that avoids x_0, while in W they contain the edge-disjoint y_0, y_1-path and y_0, y_2-path. Deleting the edges and interior vertices of the y_1, y_2-path (if any exist), we obtain a subdivision of K_4. Three of the thread form the odd cycle U. Each of C_1 and C_2 is a union of four threads in this subdivision with total length even, omitting one of the threads in U. It follows that the circuit in the subgraph obtained by omitting the remaining thread in U also has even length. Hence Lemma 4.2.22 applies, and G satisfies the weak 3-EVP. □

4.2.3 The Weak t-EVP and t-EVP for Hypergraphs

In this section, we obtain further upper bounds. For the case $t = 2$, recall that the 2-EVP and weak 2-EVP are equivalent.

Theorem 4.2.24. *If $k \geq 3$, then $\mathrm{V}'_k(n,t) < (\log_2 k + 3\log_2\log_2 k)n + \log_2(t-1)$. In fact, more than this many edges forces the weak t-EVP for any hypergraph whose average edge-size is at most k.*

Proof. It suffices to prove that an n-vertex hypergraph H with m edges of average size at most k satisfies the weak t-EVP, where $m = \lceil (\log_2 k + 3\log_2\log_2 k)n + \log_2(t-1)\rceil$. Let $V(H) = \{v_1, \ldots, v_n\}$, and let $d_i = d_H(v_i)$. For every subhypergraph H', we have $d_{H'}(v_i) \leq d_i$. Hence there are at most $\prod_{i=1}^n (d_i+1)$ degree lists of subhypergraphs. Since $\sum_{i=1}^n d_i = km$, it follows that $\prod_{i=1}^n (d_i+1) \leq \left(\frac{km}{n}+1\right)^n$.

With m edges in H, there are 2^m subhypergraphs of H. If $2^m > (t-1)\left(\frac{km}{n}+1\right)^n$, then some t subhypergraphs have the same degree list. With $u = m/n$, we need $2^u > (t-1)^{1/n}ku + 1$. It suffices to have $u \geq \log_2 k + 3\log_2\log_2 k + \log_2(t-1)$. □

The constant 3 can be decreased to a constant $1 + c_k$ with $c_k > 0$ by carefully considering lower-order terms. As $k \to \infty$, actually $c_k \to 0$, but quite slowly. For $t = 2$, still $c_k > 0.01$ when $k = 10^{42}$. Below we show c_k and m for small values of k when $t = 2$.

k	3	4	5	6	7	8
c_k	1.9389	1.1314	0.8569	0.7162	0.6296	0.57041
m	$< 3.5377n$	$< 4.1314n$	$< 4.5787n$	$< 4.9364n$	$< 5.23417n$	$< 5.48904n$

When we say nothing about the average size of edges, Theorem 4.2.24 still gives an upper bound by setting $k = n$.

Corollary 4.2.25. $\mathrm{V}'(n,t) < n(\log_2 n + 3\log_2\log_2 n) + \log_2(t-1)$.

The upper bound in Theorem 4.2.24 is not valid for $\mathrm{V}_k(n,t)$ when $t \geq 3$ because the t subhypergraphs found with the same degree lists need not have the same pairwise intersections. This means there is no set of edges to discard from them all that preserves equality of degrees. To avoid this

problem, we use the fact that large families of sets contain large subfamilies with common pairwise intersections.

Definition 4.2.26. *A family of r sets is an r-Δ-system if the intersections of any two sets in the family are the same. Let $f(k,r)$ be the least integer q such that every k-uniform family of q sets contains an r-Δ-system. Let $g(m,r)$ be the least integer q such that every family of q subsets of an m-element set contains an r-Δ-system.*

Theorem 4.2.27 (Kostochka [39])**.** *For $r \geq 3$ and $\alpha > 1$, there exists $D(r,\alpha)$ such that $f(k,r) \leq D(r,\alpha)k!(\frac{(\log\log\log k)^2}{\alpha \log\log k})^k$.*

Theorem 4.2.28 (Erdős and Szemerédi [23])**.** $g(m,3) \leq 2^{(m-.1\sqrt{m})}$.

In proving this upper bound, Erdős and Szemerédi used $f(k,3) \leq (1+o(1))k!$, which was the best known upper bound at the time. Kostochka's later improvement in Theorem 4.2.27 yields a better upper bound for $g(m,r)$ when r is fixed and m is large. The proof is the same as that in [23], just invoking the improved bound on $f(k,r)$ at the appropriate point.

Lemma 4.2.29. *For $r \geq 3$ and $\epsilon > 0$, there exists $N(r,\epsilon)$ such that $g(m,r) < 2^{m-(1-\epsilon)\sqrt{m}\log\log\log m}$ for $m \geq N(r,\epsilon)$.*

Theorem 4.2.30. *If $t \geq 3$ and $\epsilon > 0$, then $\mathrm{V}(n,t) < (4+\epsilon)n^2(\frac{\log n}{\log\log\log n})^2$ for $n \geq N(t,\epsilon)$, where $N(t,\epsilon)$ is defined as in Lemma 4.2.29.*

Proof. Let H be an n-vertex hypergraph with $n \geq N(t,\epsilon)$ and $|E(H)| \geq (4+\epsilon)n^2(\frac{\log n}{\log\log\log n})^2$. We show that H satisfies the t-EVP.

As in Theorem 4.2.24, let $V(H) = \{v_1,\ldots,v_n\}$ and $d_i = d_H(v_i)$. For every subhypergraph H', we have $d_{H'}(v_i) \leq d_i$. Again there are at most $\prod_{i=1}^n (d_i+1)$ degree lists of subhypergraphs. Since $0 \leq d_i \leq m$ for all i, it follows that $\prod_{i=1}^n (d_i+1) \leq (m+1)^n$.

With m edges in H, there are 2^m subhypergraphs of H. Let $q = \left\lceil \frac{2^m}{(m+1)^n} \right\rceil$; there are distinct subhypergraphs $H_1,\ldots,H_q$ having the same degree list.

Since $m = (4+\epsilon)n^2\left(\frac{\log n}{\log\log\log n}\right)^2$, Theorem 4.2.29 yields

$$\begin{aligned}
\log q &> m - n\log(m+1) > m - n\log\left(n^2\left(\frac{\log n}{\log\log\log n}\right)^2\right) \\
&= m - 2n\log n - n\log\left(\left(\frac{\log n}{\log\log\log n}\right)^2\right) \\
&> m - (1-\epsilon)\sqrt{m}\log\log\log m - o(\sqrt{m}) \\
&> m - (1-\epsilon)\sqrt{m}\log\log\log m > \log g(m,t).
\end{aligned}$$

Thus, $q \geq g(m,t)$, which implies that among $H_1, \ldots, H_q$, each viewed as a subset of the set of m edges in H, there exists a Δ-system of size t. That is, we obtain t subhypergraphs that are pairwise disjoint except for the same set of edges shared by all of them. Deleting that common subset yields t subhypergraphs witnessing the t-EVP. □

The same argument yields a bound when the average size of edges is at most k. As in Theorem 4.2.24, use $\prod_{i=1}^{n}(d_i+1) \leq \left(\frac{km}{n}+1\right)^n$.

Theorem 4.2.31. *If $t \geq 3$, $k \geq 2$, and $\epsilon > 0$, then $\mathrm{V}_k(n,t) < (1+\epsilon)n^2\left(\frac{\log n}{\log\log\log n}\right)^2$ for $n \geq \max\{k, N(t,\epsilon)\}$.*

If the upper bound in Lemma 4.2.29 can be improved to $(2-\epsilon_r)^n$, then we obtain an upper bound of $C_t n\log k$ for Theorem 4.2.31 and an upper bound of $C_t n\log n$ for Theorem 4.2.30, where C_t is a constant depending on t.

Erdös and Rado conjectured that $f(n,t) \leq (c_1(t))^n$. If this conjecture is true, then in Lemma 4.2.29 it yields $g(n,t) \leq 2^{n-c\sqrt{n}\log n}$, and in Theorem 4.2.30 it yields $\mathrm{V}(n,t) \leq c_2(t)n^2$, where $c_1(t)$ and $c_2(t)$ are functions of t.

We close with a simple lower bound.

Lemma 4.2.32. *Let H' and H'' be disjoint k-uniform hypergraphs. Fix s with $s \leq \log_2 k$, and form a hypergraph H by adding s edges to $H' \cup H''$ so that the ith added edge has exactly 2^{i-1} vertices in $V(H')$. If both H' and H'' fail the 2-EVP, then also H fails the 2-EVP.*

Proof. Let F be the set of s added edges. Since H' and H'' fail the 2-EVP, a pair $\{H_1, H_2\}$ witnessing the 2-EVP in H must use some edge in F. Since H_1 and H_2 share no edges, and the contributions by edges of F to the total degree in $V(H')$ are distinct powers of 2, the contributions by $E(H_1) \cap F$ and $E(H_2) \cap F$ to the total degree in $V(H')$ are distinct. Since $\sum_{i=1}^{\lfloor \log_2 k \rfloor} 2^{i-1} < k$, the difference between the two amounts is less than k. However, the total degree contributed by edges in H' is a multiple of k (for both H_1 and H_2). Hence the total degrees in $V(H')$ for H_1 and H_2 are distinct, so they cannot witness the 2-EVP. □

Corollary 4.2.33. *If $k \geq 3$, then $\mathrm{V}_k(n+k-3, 2) \geq \frac{11n-31}{7}$.*

Proof. By Observation 4.2.4, it suffices to prove $\mathrm{V}_3(n, 2) \geq \frac{11n-31}{7}$.

By exhaustive computer search, we know that the 7-vertex 3-uniform hypergraph with 10 edges whose incidence matrix appears below does not satisfy the 2-EVP. Starting with r copies of this hypergraph, $r-1$ applications of Lemma 4.2.32 adds $r-1$ edges (since $\lfloor \log_2 3 \rfloor = 1$) and produces a 3-uniform hypergraph with $7r$ vertices and $11r-1$ edges. Similarly, we can add one edge for each vertex beyond a multiple of 7. Letting $n = 7r + j$ with $0 \leq j \leq 6$, we obtain $\mathrm{V}_3(n, 2) \geq 11\frac{n-j}{7} - 1 + j \geq \frac{11n-31}{7}$. □

	1	2	3	4	5	6	7
$\{1,2,3\}$	1	1	1	0	0	0	0
$\{1,2,4\}$	1	1	0	1	0	0	0
$\{1,2,5\}$	1	1	0	0	1	0	0
$\{1,3,4\}$	1	0	1	1	0	0	0
$\{1,5,6\}$	1	0	0	0	1	1	0
$\{1,6,7\}$	1	0	0	0	0	1	1
$\{2,3,6\}$	0	1	1	0	0	1	0
$\{2,4,7\}$	0	1	0	1	0	0	1
$\{3,5,7\}$	0	0	1	0	1	0	1
$\{4,5,6\}$	0	0	0	1	1	1	0

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