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EFFECTIVE MULTIPLE MIXING IN WEYL CHAMBER
ACTIONS

BY

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DISSERTATION

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Abstract

In this work, we prove effective decay of certain multiple correlation coefficients for Weyl chamber actions of semidirect products of semisimple groups with G -vector spaces. Using these estimates we get decay for corresponding actions in some semisimple groups of higher rank and homogeneous spaces of semidirect products of real algebraic groups.

To my family.

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Chapter 1

Introduction

The present work is concerned with the measurable dynamics of diagonal actions of connected algebraic groups over local fields of characteristic zero on probability spaces. By ‘diagonal’ action, we mean an action of a diagonal subgroup of a semisimple algebraic group. The groups we will consider will be either semisimple or semi-direct products of a semisimple group and a vector space over the local field on which the semisimple part acts by a suitable representation.

To a maximal split torus over a local field corresponds some split semisimple subgroup of the original group whose maximal torus coincides with the given one. The action will always be restricted to this split part of the original group. Rationality of finite dimensional representations are inherited by this restriction.

The actions we consider will always be measurable and probability measure-preserving. Further assumptions will imply ergodicity of the group action and strong mixing of the diagonal part. This will be our starting point: a key theorem of Mozes shows that the action is mixing of all orders; thus, multiple correlations of zero-mean test functions will decay.

Multiple mixing is a statement about divergent trajectories of tuples of group elements acting on arbitrary (bounded, zero-mean) tuples of functions. In this work, we will obtain quantitative decay for such correlations with some restrictions: the trajectories will be positive diagonal, so we only treat the positive Cartan part of the group action; the divergence we require of the tuple in higher rank is stronger than the usual mixing requirement: we require the divergence to be detectable from a fixed direction in the Weyl chamber, which we can take that of the highest root of the group; finally, the test functions will only comprise a dense subspace of all bounded functions in the Hilbert norm. Under additional hypotheses on the action, this subspace will contain all smooth, K -finite functions.

The central theme in our work is the idea that the geometric properties of the semisimple part are reflected in the spectral side of the abelian part. The abelian representation on square-integrable functions on the probability space provides a spectrum for each function; spectra are affected by the action of the semisimple part in various ways and they also reflect various operations among functions; in Figure 1 we see the spectrum of a sum of two functions with spectra distorted in two orthogonal directions by the action of the diagonal group of 2×2

real matrices of determinant 1.

The next chapter will provide the necessary background that underlies this work. The exposition is a collage taken from many sources, chief among which are [18], the crucial paper of Wang [24], the classical references [4] and [8] and the book [7].

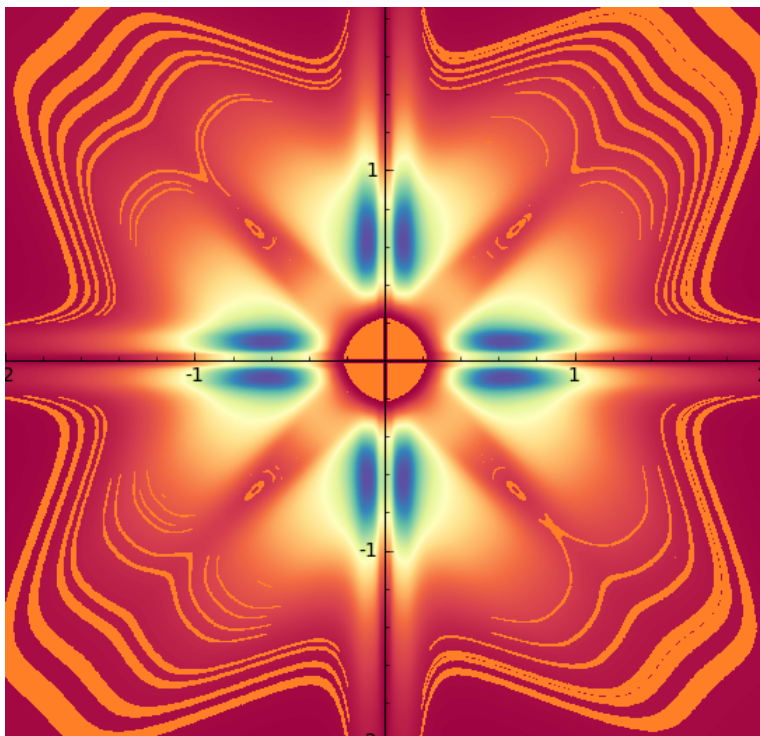


Figure 1.1: Part of the spectral fingerprint of a function in P_ϕ for ϕ with 4-fold symmetry.

Chapter 2

Preliminaries and Setup

2.1 Algebraic groups over locally compact fields

2.1.1 Local fields

Definition 2.1.1. A *local field* \mathcal{K} of characteristic zero is a field containing \mathbb{Q} which is complete with respect to a valuation v . \mathcal{K} is archimedean (resp. non-archimedean) if v has the corresponding property.

Remark 2.1.2. The real field \mathbb{R} , the complex field \mathbb{C} and all finite extensions of \mathbb{Q}_p for prime p exhaust the list of local fields of characteristic zero. Such fields carry a non-discrete, locally compact topology and can be characterized as such when $\text{char}(\mathcal{K}) = 0$. The topology comes from the metric $|\cdot|$ induced by the valuation on \mathcal{K} , which is the usual absolute value in the archimedean case or the unique valuation which extends the standard p -adic valuation when $\mathbb{Q}_p \subset \mathcal{K}$.

The following subsets of \mathcal{K} will be used later to define connected components and positive Weyl chambers in a \mathcal{K} -torus:

Definition 2.1.3. Let

$$\mathcal{K}^0 = \{x \in \mathbb{R} | x \geq 0\} \quad \text{and} \quad \overline{\mathcal{K}} = \{x \in \mathbb{R} | x \geq 1\}$$

when \mathcal{K} is archimedean. When \mathcal{K} is non-archimedean, we fix a uniformizer q (i.e. a generator of the maximal ideal of the ring of integers of \mathcal{K}) with $|q|^{-1}$ the cardinality of the residue field of \mathcal{K} . Then correspondingly

$$\mathcal{K}^0 = \{q^n | n \in \mathbb{Z}\} \quad \text{and} \quad \overline{\mathcal{K}} = \{q^{-n} | n \in \mathbb{N}\}.$$

2.1.2 Algebraic groups over local fields

Next we define affine algebraic groups over local fields. We will only consider groups defined over \mathbb{Q} in this work.

Definition 2.1.4. An *affine algebraic group* \mathbf{G} over a field k is an affine algebraic variety over k with a group structure whose multiplication and inverse laws

are morphisms ([4]); in this work k will always be \mathbb{Q} . Whenever we mention connectedness, we will always understand the notion as Zariski-connectedness of the variety unless explicitly mentioned otherwise. Finally, an algebraic group is called *simply connected* if for every algebraic group \mathbf{S} and every isogeny (surjective group morphism with finite kernel)

$$\mathbf{S} \rightarrow \mathbf{G},$$

the map is an algebraic group isomorphism; in other words, there are no non-trivial algebraic covers of \mathbf{G} . At the other extreme, \mathbf{G} is called *adjoint* if for every \mathbf{S} and every isogeny $\mathbf{G} \rightarrow \mathbf{S}$, the map is an isomorphism.

Remark 2.1.5. The definition of simple connectedness may seem odd, but it is the natural notion in the algebraic category: the kernel of a surjective algebraic group morphism will be an algebraic set; such a set has only finitely many connected components, so restrict to the connected component of the identity of \mathbf{S} . If this component had infinitely many elements, it would have positive dimension in \mathbf{S} . But then it would be a normal connected subgroup of positive dimension, and either the dimensions of \mathbf{S} and \mathbf{G} would be different or we would get a contradiction by surjectivity. In both cases, infinite kernel leads to higher dimension and thus does not correspond to a natural notion of cover.

Viewed as a functor

$$\mathcal{K} \xrightarrow{\mathbf{G}} \mathbf{G}(\mathcal{K}) =: G,$$

the image of \mathbf{G} consists of the group of \mathcal{K} -rational points of \mathbf{G} . The underlying set of G is the solution set of a finite number of polynomial equations with coefficients in k , so as a closed subset of some k^d it inherits the locally compact topology by which it becomes a topological group; for the topological structure of such groups refer to [18, Chapter 3].

Remark 2.1.6. Groups of the form $\mathbf{G}(\mathcal{K})$ have properties from two worlds: they inherit some structural properties from the overlying algebraic group \mathbf{G} (almost all relevant structural properties in the case of \mathcal{K} -split groups) and properties from their topological group structure. The algebraic group \mathbf{G} , a \mathbb{Q} -group-functor, and the group of \mathcal{K} -rational points, a topological group, are completely distinct entities and we will take care to distinguish them with the exception that we will be using the terminology ‘algebraic group’ both for \mathbf{G} and the group of \mathcal{K} -rational points G . Having fixed a local field \mathcal{K} once and for all, boldface letters will refer to algebraic groups and the corresponding italics to the group of its \mathcal{K} -rational points.

Definition 2.1.7. The Lie algebra of an algebraic group is defined as an algebraic variety to be the tangent space at the identity of the group. It affords a (well defined) Lie algebra structure by embedding the group as a closed subgroup of $\mathrm{GL}(n)$ for some n and defining the bracket $[X, Y] = XY - YX$ with the multiplication inherited from $\mathrm{GL}(n)$.

Definition 2.1.8. A finite dimensional *representation* of an algebraic group \mathbf{G} is a group morphism from \mathbf{G} to $\mathrm{GL}(n)$. Similarly, a representation of the Lie algebra is a Lie algebra homomorphism into some $\mathfrak{gl}(n)$. The finite dimensional representation theory of algebraic groups over characteristic zero is directly linked to the Lie algebra representation theory, see [8]. Representations defined over a local field \mathcal{K} are called \mathcal{K} -rational or simply \mathcal{K} -representations.

Definition 2.1.9. Let \mathbb{G}_m be the *multiplicative group*, i.e. the algebraic group whose \mathcal{K} -valued points is the group \mathcal{K}^* of invertible elements in the field (or ring, algebra etc.) \mathcal{K} .

An algebraic group G is called *diagonalizable* if there exists an embedding $G \hookrightarrow \mathrm{GL}(m)$ so that the image is conjugate to a subgroup of the group D of diagonal matrices. A connected diagonalizable group is called an *algebraic torus*. Any algebraic torus is isomorphic as an algebraic group to \mathbb{G}_m^d for some $d \geq 1$ called the dimension of the torus. If there exists an isomorphism to \mathbb{G}_m^d defined over a local field \mathcal{K} , the torus is said to be \mathcal{K} -split.

Example 2.1.10. Consider the group

$$T = \left\{ \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \mathrm{GL}(2) : u = x, v = -w, u^2 + v^2 = 1 \right\}.$$

The defining equations show that it is a Zariski-connected algebraic set and the map

$$\begin{pmatrix} u & v \\ w & x \end{pmatrix} \rightarrow \begin{pmatrix} u + iv & 0 \\ 0 & u - iv \end{pmatrix}$$

is an isomorphism to the multiplicative group \mathbb{G}_m . Note that if \mathcal{K} is a field that does not contain a square root of -1 , the isomorphism is not defined over \mathcal{K} . For instance, the \mathbb{R} -points of T are precisely the compact torus in $\mathrm{SL}(2, \mathbb{R})$.

Definition 2.1.11. A character of an algebraic group is an algebraic group morphism $G \rightarrow \mathbb{G}_m$. The commutative group of characters of G is denoted by $\mathbf{X}(G)$. If a character is defined over \mathcal{K} , we call it a \mathcal{K} -character and denote the corresponding group by $\mathbf{X}(G)_{\mathcal{K}}$. The dual group $\mathbf{X}^*(G)$ of cocharacters is the group of morphisms $\mathbb{G}_m \rightarrow G$. A cocharacter defined over a local field \mathcal{K} will also be called a *one-parameter subgroup* of G .

If $\mathbf{X}(G)_{\mathcal{K}} = 0$ then the group G is called \mathcal{K} -anisotropic. It turns out that this is equivalent to the statement that $G(\mathcal{K})$ is compact in the \mathcal{K} -topology.

Remark 2.1.12. A diagonalizable group is split over \mathcal{K} if and only if all its characters are defined over \mathcal{K} . As an example, take $\mathcal{K} = \mathbb{R}$ and try to find a character of T not defined over \mathbb{R} . It is easy to see that projections of either of the two entries of the matrix

$$\begin{pmatrix} u + iv & 0 \\ 0 & u - iv \end{pmatrix}$$

give characters of T which are, of course, not defined over \mathbb{R} .

Definition 2.1.13. If D is a torus define subsets D^0 and D^+ of D by

$$D^0 = \{d \in D \mid \chi(d) \in \mathcal{K}^0 \text{ for each } \chi \in \mathbf{X}(D)\},$$

$$D^+ = \{d \in D \mid \chi(d) \in \overline{\mathcal{K}} \text{ for each } \chi \in \mathbf{X}^+\}.$$

We call D^+ the positive Weyl chamber in D (relative to the prescribed data).

Next, we denote the centralizer of D in G by Z and transfer the ordering of $\mathbf{X}(D)$ to $\mathbf{X}(Z)$ by inclusion. Let

$$Z_+ = \{z \in Z \mid |\chi(z)| \geq 1 \text{ for each } \chi \in \mathbf{X}(Z)^+\}$$

and

$$Z_0 = \{z \in Z \mid |\chi(z)| = 1 \text{ for each } \chi \in \mathbf{X}(Z)^+\}.$$

Note that these are \mathcal{K} -subgroups of the \mathcal{K} -group Z .

Definition 2.1.14. An algebraic group \mathbf{G} is unipotent if there is an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}(n)$ so that the image consists entirely of unipotent matrices. Note that unipotent groups are nilpotent.

Definition 2.1.15. An algebraic group is called reductive if its unipotent radical is trivial, i.e. if the maximal connected unipotent normal subgroup is trivial. It is called semisimple if the radical is trivial, i.e. if the maximal connected solvable normal subgroup is trivial. It is called simple if it has no connected normal subgroups.

2.1.3 Examples

The first examples of algebraic groups are the additive group \mathbb{G}_a assigning the additive group of any field, and the multiplicative subgroup \mathbb{G}_m giving the group of invertible elements of a field. For example, $\mathbb{G}_a(\mathcal{K})$ (resp. \mathbb{G}_a^n) can be realized in $\mathrm{GL}(\mathcal{K}^n)$ as

$$k \rightarrow \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \quad (k_1, \dots, k_{n-1}) \rightarrow \begin{pmatrix} 1 & 0 & \cdots & k_1 \\ 0 & 1 & \cdots & k_2 \\ 0 & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \times n}.$$

The multiplicative group $\mathbb{G}_m(\mathcal{K})$ (resp. $\mathbb{G}_m(\mathcal{K}^n)$) can be realized in $\mathrm{GL}(2, \mathcal{K})$ (resp. $\mathrm{GL}(n, \mathcal{K})$) as

$$k \rightarrow \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad (k_1, \dots, k_{n-1}) \rightarrow \begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & (k_1 \cdots k_{n-1})^{-1} \end{pmatrix}_{n \times n}.$$

Examples of reductive groups include \mathbb{G}_m , $\mathbf{GL}(n) = \mathbb{G}_m(\text{End}(\mathbb{G}_a^n))$, $\mathbf{O}(2n)$ and norm tori $\mathbf{R}_{K/k}^{(1)}(\mathbf{G}_m)$. Simple groups include $\mathbf{SL}(n)$, $\mathbf{PGL}(n)$ and $\mathbf{SO}(2n)$. There is a finite list of connected, simply connected simple algebraic groups that includes four infinite families and five exceptional groups; the list is encoded in the combinatorial data given by the Dynkin diagram of the root system of the group. Algebraic groups covered by each simply connected group are obtained by isogenies ([18, Chapter 2.1]) in characteristic zero.

The isomorphism problem over a local field involves the notion of a \mathcal{K} -form of each group and the classification of such forms is a much more delicate issue than the classification over an algebraically closed field. A standard reference for classification of K -forms is [18, Chapter 2.2].

2.1.4 The Cartan decomposition of semisimple groups

Semisimple and reductive groups have a large number of very useful decompositions into simpler components, including the Bruhat decomposition, the Iwasawa decomposition and the Cartan decomposition over local fields. For the sequel we will need only the Cartan decomposition which we now define.

Theorem 2.1.16 (Cartan Decomposition). *Let G be the group of \mathcal{K} -points of a connected semisimple group. Recall definition 2.1.13. Consider a maximal \mathcal{K} -split torus D of G (defined over \mathcal{K}). Then there exists a maximal compact subgroup $K < G$ and a finite set $F \subset C_G(D)$ such that the following hold:*

1. $N_G(D) \subset KD$.
2. We have the decomposition $G = K(Z_+/Z_0)K$ such that for each $g \in G$, there exists a unique element z of Z_+ modulo Z_0 so that $g \in KzK$.
3. The decomposition above can be further elaborated as

$$G = K(D^+F)K$$

with the property that for each $g \in G$ there exist unique $d \in D^+$ and $f \in F$ so that $g \in KdfK$.

Remark 2.1.17. The presence of F in the Cartan decomposition above is a feature of the non-archimedean theory: when \mathcal{K} is archimedean, $F = 1$.

Example 2.1.18. Let $G = \mathrm{SL}(n, \mathbb{Q}_p)$. Then a maximal compact subgroup is $K = \mathrm{SL}(n, \mathbb{Z}_p)$ and the Cartan decomposition can be written as $G = KD^+K$ where

$$D^+ = \{\mathrm{diag}(q^{m_1}, \dots, q^{m_n}) : \sum m_i = 0, m_1 > \dots > m_n \geq 0\}.$$

Here as before q is a uniformizer and the m_i are non-negative integers.

2.1.5 Ergodic theory on semisimple groups

In order to do analysis on groups we will need to understand their measure theory. The basic measure on any locally compact group is the Haar measure.

Theorem 2.1.19 (Existence of Haar measure). *Let G be locally compact. There exists a unique (up to scaling) regular Borel measure dh on G that is invariant under left translations: $dh(g'g) = dh(g)$.*

Definition 2.1.20. Let G be a locally compact group defined over a complete field \mathcal{K} . The measure whose existence is asserted above is called the left Haar measure. Similarly we define the right Haar measure. If the left and right Haar measures of G coincide, the group is called *unimodular*. In general, if $dh(g)$ is a left invariant Haar measure, then $dh(gg') = \Delta_G(g')dh(g)$ for the so called modular function $\Delta_G(g)$. This function is a character of G which measures the deviation between left and right Haar measures.

Almost all the groups we will encounter will be unimodular: reductive groups, compact groups and of course abelian groups are all unimodular. On the other hand, non-unimodular groups make their appearance naturally as subgroups of unimodular groups and play a major role in the representation theory, structure theory and measurable action theory of semisimple groups.

Example 2.1.21. The most crucial non-trivial modular functions are those attached to certain solvable groups in archimedean fields. The simplest example involves the $ax + b$ group consisting of matrices in $\mathrm{SL}(2, \mathbb{R})$ of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0, b \in \mathbb{R}.$$

Let's look at a left-invariant Haar measure. The group is homeomorphic to $\mathbb{R}^* \times \mathbb{R}$ and since the Haar measure for the multiplicative group is $\frac{dx}{x}$ and that for the additive group is dt , we can try decomposing

$$dh(g) = \frac{da}{a} db.$$

If we apply $g' = \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}$ to the right of $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ we get

$$dh(gg') = \frac{d(aa')}{aa'} d(b'a + b) = \frac{da}{a} db$$

showing right invariance. But applying g' to the left we get

$$dh(g'g) = \frac{d(aa')}{aa'} d(ba' + b') = \frac{1}{a'} \frac{da}{a} db$$

so $\Delta_G(g) = a^{-1}$. It is trivial to check that this is a character of the $ax + b$ group.

Definition 2.1.22. 1. A *measurable action* σ of the locally compact group G of a Lebesgue probability space (X, μ) is a weakly measurable map $G \times X \rightarrow X$ with the property that $\sigma(gg', x) = \sigma(g, \sigma(g', x))$ (In fact, up to a set of measure zero that can be fixed once and for all, we can assume that X is a standard Borel space and the action is a Borel map; see [19].). We will usually denote an action by $\sigma(g) \cdot x$ or when the action is understood $g \cdot x$.

2. A measurable action σ is *measure preserving* if $\mu(\sigma(g) \cdot A) = \mu(A)$ for every measurable $A \subset X$.
3. A measure preserving action σ is *ergodic* if the only σ -invariant subsets of X in measure are either zero-measure or conull, i.e. if

$$\mu(A \Delta \sigma g^{-1} \cdot A) = \mu(A)$$

then $\mu(A) = 0$ or $\mu(A) = 1$.

4. A measure preserving action σ is *weakly mixing* if the *diagonal* action $g \mapsto \sigma(g) \times \sigma(g)$ on $X \times X$ is ergodic.
5. A sequence $g_n \in G$ *diverges to infinity* if for any compact set C we can find $n_0 \in \mathbb{N}$ such that $g_n \notin C$ for $n \geq n_0$. A measure preserving action σ is *mixing* if for any two measurable sets $A, B \subset X$ and any sequence $g_n \in G$ with $g_n \rightarrow \infty$, we have

$$\mu(A \cap g_n^{-1} \cdot B) \rightarrow \mu(A)\mu(B).$$

6. A sequence of k -tuples $(g_1(n), \dots, g_k(n))$ of elements of G diverges to infinity if for any compact set $C \subset G$ there exists n_0 such that if $n \geq n_0$ and $i \neq j$, $g_j(n)/g_i(n) \notin C$. A measure preserving action σ is *k-mixing* for any sequence of $k+1$ sets $A_0, \dots, A_k \subset X$ and any sequence of k -tuples $(g_i(n)) \in G$ which tends to infinity we have

$$\mu(A_0 \cap g_1(n)^{-1} \cdot A_1 \cap \dots \cap g_k(n)^{-1} \cdot A_k) \rightarrow \mu(A_0) \dots \mu(A_k).$$

An action is mixing of all orders if it is k -mixing for every $k \geq 2$.

We will need a characterization of the notions above in terms of function spaces.

Definition 2.1.23. Let σ be a measure preserving action of G on X . The *Koopman representation* associated to the action is the unitary representation $U_\sigma : G \rightarrow \text{Aut}(L^2(X))$ given by

$$(U_\sigma(g) \cdot f)(x) = f(\sigma^{-1}(g) \cdot (x)).$$

We will usually replace the notation above simply by $g \cdot f$; the caveat is that it is the *inverse* of g acting on the argument of f . We will also refer to $g \cdot f$ as the action of G on f .

Remark 2.1.24. We used L^2 to get a unitary representation on a Hilbert space, but the definition makes sense for any Banach or even topological vector space. Unitarity in L^2 is equivalent to the measure preserving property.

Proposition 2.1.25. Let $\sigma : G \curvearrowright X$ be measure preserving.

1. The action is ergodic if and only if 1 is a simple eigenvalue for U_σ on $L^2(X)$, i.e. if U_σ restricted to $L_0^2(X)$ has no fixed vectors.
2. The action is k -mixing if and only if for any $k+1$ functions f_i in $L^\infty(X) \subset L^2(X)$ and any sequence of tuples $(g_i(n))$ in G diverging to infinity we have

$$\int_X f_0 g_1(n) \cdot f_1 \cdots g_k(n) f_k d\mu \rightarrow \prod \int_X f_i d\mu$$

which is also equivalent to the statement that if the f_i are all in $L_0^\infty(X)$, then

$$\int_X f_0 g_1(n) \cdot f_1 \cdots g_k(n) f_k d\mu \rightarrow 0$$

as $n \rightarrow \infty$.

2.1.6 A brief excursion into the structure theory of semisimple groups

In this section we will review very briefly the structure theory of simple groups over algebraically closed fields in terms of their root systems; the main purpose is to set notation and fix conventions. The reader is referred to [4] and [8] for full expositions of the concepts we will introduced below.

Definition 2.1.26. The *adjoint representation* of an algebraic group \mathbf{G} is the map $\mathbf{G} \rightarrow \text{Aut}(\text{Lie}(\mathbf{G}))$ where $g \mapsto \text{Ad}(g)$ with formula $\text{Ad}(g)h = ghg^{-1}$.

By means of the adjoint representation we can define the root system of a semisimple group over an algebraically closed field. Note that since $D < G$ is

commutative and diagonalizable, matrices in D are simultaneously diagonalizable, so in particular the action of $Ad(d)$ on $Lie(G)$ splits into the kernel of Ad and one-dimensional eigenspaces whose eigenvalues depend multiplicatively on the acting element: $Lie(G) = u_0 \oplus \bigoplus_{a \neq 0} u_a$ where

$$u_a = \{g \in Lie(G) : Ad(d)(g) = a(d)g\} \neq 0.$$

Definition 2.1.27. The decomposition above gives distinguished characters $a \neq 1$ of D called the *roots* of the pair (G, D) ; the set of roots of (G, D) is called the root system and is denoted $R(D, G)$. The decomposition $Lie(G) = U_0 \oplus \bigoplus_{a \in R(D, G)} u_a$ is called the root space decomposition.

Definition 2.1.28. A semisimple group is \mathcal{K} -split if it has a maximal torus D which is defined over \mathcal{K} and \mathcal{K} -isomorphic to $\mathbb{G}_m^d(\mathcal{K})$. The \mathcal{K} -rank of a semisimple group is the dimension of any maximal \mathcal{K} -split torus; thus if a group is \mathcal{K} -split, its \mathcal{K} -rank equals its rank.

For \mathcal{K} -split semisimple groups, the structure theory over \mathcal{K} is identical to the structure theory over the algebraic closure; we will encounter higher rank split groups later in the text.

In the sequel we will usually restrict attention to simple groups. The following proposition describes how semisimple groups are built out of simple groups.

Theorem 2.1.29 ([18, Proposition 2.4]). *Let G be semisimple and G_i the (simple) minimal connected normal subgroups of G . Their number is finite and G is an almost direct product of the G_i , in the sense that the multiplication map*

$$\prod G_i \rightarrow G$$

is an isogeny. The root system of G is a disjoint union of pairwise orthogonal root systems $R(G_i)$.

Finally, we come to the main classification results for semisimple groups over algebraically closed fields.

Theorem 2.1.30 ([18, Theorem 2.6]). *Let G be semisimple.*

1. *There exists a simply connected group \tilde{G} , an adjoint group \bar{G} and isogenies $\tilde{G} \rightarrow G$ and $G \rightarrow \bar{G}$. The first isogeny is called a universal covering of G .*
2. *Any simply connected or adjoint group is a direct product of its minimal connected normal subgroups, and each such subgroup is simply connected or adjoint respectively.*
3. *If $R = R(G, D)$ is a root system for G and Π is the set of simple roots, then G is simply connected if $\mathbf{X}(D)$ has a base $\{\lambda_a : a \in \Pi\}$ such that $w_a \lambda_b = \lambda_b - \delta_{ab} \alpha_a$ where δ_{ab} is the Kronecker delta function and w_a the element of the Weyl group corresponding to the reflection about a . G is adjoint if Π spans the entire character group $\mathbf{X}(T)$.*

The kernel of $\tilde{G} \rightarrow G$ in a universal covering of G does not depend on the covering.

Definition 2.1.31. The (central) kernel of any universal covering of G is called the fundamental group of G .

The next theorem finishes the classification; its proof involves the greater part of the book [8].

Theorem 2.1.32. *Let \mathbf{G} be a connected semisimple algebraic group. Up to algebraic group isomorphism, \mathbf{G} is determined by its root system and its fundamental group.*

The theory outlined above works over algebraically closed fields and is called the absolute case. When we pass to a non-algebraically closed field such as a local field, most of the results above do not hold. However, given a maximal \mathcal{K} -split torus D in $G = \mathbf{G}(\mathcal{K})$ we can still define a root system as above; it still furnishes an abstract root system in the sense of [9], but the system may not be reduced. The differences are mainly reflected in the structure of U_a , leading to many more isomorphism classes and invariants than the absolute case; however, we will use little of the finer properties of the classification of semisimple groups over \mathcal{K} and refer to [18] for a good survey and the references to the original papers of Borel and Tits therein.

2.2 \mathcal{K} -rational representations of semisimple groups and semi-direct products

In this section we introduce certain finite dimensional representations of a semisimple group which will bind together with the group into a semidirect product; this composite group will be one of the main objects of our study.

When we speak of semisimple groups from now on we will understand to mean the groups of \mathcal{K} -rational points of the algebraic group for a fixed \mathcal{K} .

Definition 2.2.1. Fix \mathcal{K} . Let G be a semisimple group and V a finite dimensional \mathcal{K} -vector space. An algebraic group morphism

$$\rho : G \rightarrow \mathrm{GL}(V)$$

defined over \mathcal{K} is called a \mathcal{K} -rational representation of G .

Remark 2.2.2. In the case of $\mathcal{K} = \mathbb{R}$, all of our results hold in the much greater generality of continuous representations. We restrict notationally to \mathbb{R} -rational representations, but the reader will notice that rationality can be immediately replaced by continuity if we make the appropriate modifications (see [24, Section 1]).

Definition 2.2.3. Let \mathbf{G} be semisimple and \mathbf{G}_i its simple factors as in Theorem 2.1.29. For each i , $G_i = \mathbf{G}_i$ is either \mathcal{K} -isotropic or anisotropic. The product of all anisotropic factors of G will be denoted G_c and the isotropic factors G_s .

Definition 2.2.4 ([24]). A \mathcal{K} -rational representation ρ is called good if the \mathcal{K} -anisotropic part of the semisimple group G has no non-trivial fixed points in V . It is called excellent if for each \mathcal{K} -isotropic G_i , $\rho(G_i)$ has no non-trivial fixed points in V .

Example 2.2.5. 1. Let $\mathrm{SL}(2, \mathcal{K})$ act on \mathcal{K}^2 by the standard representation ρ_{std} . Since ρ is transitive, it is excellent.

2. Let $G = \mathrm{SL}(2, \mathcal{K}) \times \mathrm{SL}(2, \mathcal{K})$ act on $\mathcal{K}^2 \times \mathcal{K}^2$ by $\rho_{\mathrm{std}} \oplus \rho_{\mathrm{std}}$. Both factors of G are \mathcal{K} -anisotropic and there is no fixed vector in $\mathcal{K}^2 \times \mathcal{K}^2$ for the entire action, but each factor fixes an entire \mathcal{K}^2 summand, so the representation is good but not excellent.

3. If $H < G$ is closed in G and defined over \mathcal{K} , a well known theorem states that there exists an immersive representation $\rho : G \rightarrow \mathrm{GL}(E)$ and a line $L \subset E$ so that $H = \{g \in G : \rho(g)L = L\}$; see [4, Chapter II, Theorem 5.1]. In particular, if $L = \mathcal{K}\langle v \rangle$, then $\rho(g)v = w(g)v$ for some $w(g) \neq 0$. Being a representation, we have $w(g^{-1}) = w(g)^{-1}$. Now consider (ρ^*, E^*) , the contragredient representation and let $\rho_0 = \rho \otimes \rho^*|_H$. If $v^* \in E^*$ such that $v^*(v) = 1$, then note that

$$\rho_0(g)v \otimes v^* = \rho(g)v \otimes \rho^*(g)v^* = w(g)w(g)^{-1}v \otimes v^* = v \otimes v^*.$$

Thus ρ_0 is not good even if H is a big simple subgroup of G and ρ is an excellent representation of G (in fact, if H is simple, it has no non-trivial characters and thus $w(g) = 1$, so we do not need to consider the contragredient).

Next we formally define weights and weight spaces for ρ .

Definition 2.2.6. Let $D < G$ be a maximal \mathcal{K} -split torus. The *weights* of the representation $\rho : G \rightarrow \mathrm{GL}(V)$ are the non-zero \mathcal{K} -characters χ of D for which there exists $v \in V$ such that $\rho(d)v = \chi(d)v$ for $d \in D$. For a given $\chi \in \mathbf{X}(D)_{\mathcal{K}}$,

$$V_{\chi} = \{v \in V : \rho(d)v = \chi(d)v\}.$$

We will need two structural results concerning ρ and its weights.

Proposition 2.2.7 (Complete reducibility under ρ). *The space V splits as a finite direct sum $V = \bigoplus V_{\chi}$.*

Proposition 2.2.8 ([24, Lemma 2.5]). *Consider a \mathcal{K} -rational representation ρ of G on V . We have:*

1. If $\ker(\rho) \cap G_s \subset Z(G)$, then every root in G is a rational combination of weights of G .
2. The sum of weights of ρ is trivial.

Remark 2.2.9. The condition $\ker(\rho) \cap G_s \subset Z(G)$ is weaker than the notion of an excellent representation and coincides with it when ρ is irreducible.

Remark 2.2.10. The significance of (1) in Proposition 2.2.8 is the following: roots of G reflect intrinsic properties of the group and will be used to understand actions of G on measure spaces. Weights for ρ on the other hand are specific to the representation. Item (1) implies that when the representation realizes a large enough portion of G as a set of automorphisms of V , the roots can be seen through the weights of V ; therefore, we can use properties of a simpler object, V to glean information about G . This idea in the context of unitary group representations was initiated by Mackey who introduced the system of imprimitivity in [14] and was exploited by many authors including R. Howe, C. Moore and others in work that we will mention as we use.

We are now in position to introduce the main object of our study.

Definition 2.2.11. Let G be semisimple, ρ a \mathcal{K} -rational representation on V . The *semidirect product*

$$\mathfrak{G} = G \ltimes_{\rho} V$$

of G and V with respect to ρ is the algebraic \mathcal{K} -group whose multiplication is given by

$$(g, v) * (g', v') = (gg', \rho(g)v' + v).$$

Whenever we talk about V as a group, we will mean the additive group of the vector space.

Note the claim that the abstract \mathcal{K} -group defined above is an affine algebraic group. The claim follows from [4, Corollary I.1.4] and the algebraic structure of the product of two algebraic varieties. Since V is a connected unipotent group, we see that \mathfrak{G} is not reductive; in fact V is the unipotent radical of \mathfrak{G} .

We close this section with a list of notations involving G, ρ and V . Recall that since G is semisimple and $\text{char}(\mathcal{K}) = 0$, the representation ρ is completely reducible and thus V breaks into irreducible components

$$V = \bigoplus_{i=1}^N V_i.$$

Definition 2.2.12. 1. $\|\cdot\|$ denotes a K -invariant norm on V .

2. The restriction of ρ on V_i is denoted by ρ_i .
3. Φ_i is the set of weights of ρ_i with respect to D on V_i .
4. λ_i (resp. ϱ_i) are the highest (resp. lowest) weights of ρ_i .

5. For each i and each weight w of ρ_i , V_w is the corresponding weight subspace of V_i (there will be no problem distinguishing irreducible components).
6. Φ is the set of roots of G with respect to D .
7. For each $\omega \in \Phi$, denote by \mathfrak{g}_ω the root space corresponding to the root.
8. δ_B is the modular function of the Borel subgroup B that determines the ordering on Φ .
9. $\{\omega_1, \dots, \omega_n\} \subset \Phi^+$ is the set of simple roots in Φ^+ .
10. $\mathfrak{q}_i := \left(\frac{1}{3}\right)^{\#\Phi_i - 1}$ if $\dim V_{\lambda_i} > 1$, otherwise $\mathfrak{q}_i := \left(\frac{1}{3}\right)^{\#\Phi_i - 2}$.

More details about the aspects of root systems and weights we will use can be found in [24, Section 3] and the references therein.

2.3 Unitary representations of algebraic groups

2.3.1 Ergodicity and fixed vectors

Let (X, μ) be a probability space, $\mathcal{H} = L_0^2(X)$ the Hilbert space of square integrable functions on X orthogonal to the constants, $\langle \cdot \rangle$ the inner product, $\mathcal{L} = L_0^\infty(X) \subset \mathcal{H}$ and σ a measure-preserving action of \mathfrak{G} on X ; we always use the notation $g \cdot x$ for $\sigma(g)(x)$. The Koopman representation on \mathcal{H} defined in 2.1.23 is unitary; we call $g \cdot f$ a translate of f by g , suppressing mention of the action. The Koopman representation is multiplicative, i.e. it distributes over pointwise (and a.e. pointwise) products of functions:

$$g \cdot (fh) = (g \cdot f)(g \cdot h).$$

Assume that for each irreducible component V_i of V , the representation $\sigma|_{V_i}$ has no fixed vectors in \mathcal{H} , so that the action of each V_i is ergodic. The basic *qualitative* theorem concerning semidirect products is the following, phrased in the language of unitary representations:

Theorem 2.3.1. *Let $\sigma : \mathfrak{G} \rightarrow \mathcal{H}$ be a unitary representation with the property that $\sigma|_V$ has no invariant vectors. Then all the matrix coefficients of $\sigma|_G$ decay to zero at infinity.*

The proof of Theorem 2.3.1 can be found in [26] in the case $\mathcal{K} = \mathbb{R}$. The only properties of \mathcal{K} used in the proof is self-duality under the Fourier transform, the structure of a locally compact field and the existence of an L^2 -dense space of functions whose Fourier transform has good properties. All these are satisfied for an arbitrary local field of characteristic zero and the proof carries over to that case. Later we will introduce all the necessary tools and deduce Theorem 2.3.1 from stronger quantitative versions.

We now give a flavor for the kind of arguments we will use later by proving the theorem in the special case when matrix coefficients of V decay at infinity.

Sketch of proof. Assume $\langle f, \sigma(v_n) \cdot h \rangle \rightarrow 0$ as $v_n \rightarrow \infty$. For a contradiction, consider an invariant function f for $\sigma|_G$. Form the matrix coefficient

$$\langle f, \sigma(v) \cdot f \rangle.$$

By invariance, we have

$$\langle f, \sigma(v) \cdot f \rangle = \langle \sigma(g) \cdot f, \sigma(v) \cdot f \rangle$$

and by unitarity

$$\langle f, \sigma(v) \cdot f \rangle = \langle f, \sigma(\rho(g^{-1}))\sigma(v) \cdot f \rangle$$

which becomes

$$\langle f, \sigma(g^{-1}, \rho(g^{-1})v) \cdot f \rangle.$$

Again by invariance we can replace the rightmost f by $\sigma(g, 0) \cdot f$ which cancels with the previous one to give

$$\langle f, \sigma(v) \cdot f \rangle = \langle f, \sigma(1, g^{-1}v) \cdot f \rangle = \langle f, \sigma|_V(g^{-1}v) \cdot f \rangle$$

By [2, Chapter III, Lemma 1.3], it is sufficient to show that matrix coefficients in a maximal \mathcal{K} -split torus D tend to infinity. Pick a $v \neq 0$ adequately close to 0 so that $\langle v \cdot f, f \rangle \sim \|f\|^2 \neq 0$. Write $v = \sum c_w v_w$ for some basis consisting of weight vectors of V for ρ . Applying d_n^{-1} to v for some sequence $d_n \rightarrow \infty$ in D , we see that the matrix coefficient above splits into a finite number of matrix coefficients of elements in V diverging to infinity, and thus tends to zero. This contradicts the fact that the left hand side is nonzero. \square

2.3.2 K -finite and smooth vectors

The Peter-Weyl theorem for compact groups (see [7, Chapter 1]) allows us to decompose an arbitrary unitary representation into a sum of irreducibles and provides a formula for the projection of vectors on each irreducible component. The following definition introduces a class of vectors for which the action of K is very tame.

Definition 2.3.2. Let K be a maximal compact subgroup of the semisimple group G for which the Cartan decomposition holds; such groups will be called *good* maximal compact subgroups. Consider a unitary representation σ of G on \mathcal{H} . An element $h \in \mathcal{H}$ is called K -finite if the vector space $\langle \sigma(K) \cdot h \rangle$ is finite dimensional. The space of K -finite vectors in a Hilbert space \mathcal{H} will be denoted by \mathcal{H}_K .

Remark 2.3.3. K -finite vectors are *dense* in the Hilbert space \mathcal{H} with respect to the topology given by the Hilbert norm. This follows from the Peter-Weyl theorem and will be used repeatedly without mention. It must be emphasized that other norms will be considered and density is not immediate with respect to broader topologies. In the sequel all of our functions will be inside some $L^2(X)$ and even some $L^\infty(X)$ realized as a subspace of L^2 ; the topology in both cases will be that of L^2 and all convergence statements will refer to L^2 convergence or Sobolev convergence for some $S^{2,k}(G)$; see 3.7.1 for the definition. In all cases, the topology will always come from some Hilbert space.

Example 2.3.4. Let $G = \mathrm{SL}(2, \mathbb{R})$, $K = \mathrm{SO}(2, \mathbb{R})$ and σ a unitary representation of G . Since K is *abelian*, $\sigma|_K$ decomposes into one dimensional representations for K indexed by integers n so that for any vector in the n -th eigenspace the eigenvalue is e^{int} for the matrix

$$\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Then K -finite vectors are analogs of *trigonometric polynomials* in the K direction of G .

Sometimes, K -finite vectors are too restrictive for applications. A different way of controlling the action of K follows by requiring the vector to be *smooth*.

Definition 2.3.5. Let $G, K, \mathcal{H}, \sigma$ as above. A vector $h \in \mathcal{H}$ is called *smooth* if:

1. \mathcal{K} is archimedean and the vector-valued function from G to \mathcal{H} that takes

$$g \mapsto \sigma(g) \cdot h$$

is smooth, i.e. for all $h' \in \mathcal{H}$, $\langle \sigma(g) \cdot h, h' \rangle$ is a smooth function.

2. \mathcal{K} is non-archimedean and the vector-valued function on G , $g \mapsto \sigma(g) \cdot h$ has an open centralizer in G . In other words, $\sigma(g) \cdot h$ is constant on the cosets of an open subgroup of G ; we also call such functions locally constant.

Similarly, a vector is in the *Schwartz space* if:

1. \mathcal{K} is archimedean and $\sigma(g) \cdot h$ is in the usual Schwartz space.
2. \mathcal{K} is non-archimedean $\sigma(g) \cdot h$ is locally constant and compactly supported.

So far we have dealt with Hilbert spaces since this is where unitary representations live. However, for the purposes of multiple mixing Hilbert spaces are not adequate (in fact, this deviation from the spectral picture that Hilbert spaces provide accounts for much of the trouble in dealing with multiple correlations). We amend this by the following definition:

Definition 2.3.6. Let G act on a probability space X by measure preserving transformations. The space of zero mean, bounded functions $L_0^\infty(X) \subset L_0^2(X)$ will be denoted by \mathcal{L} ; the subspace of K -finite vectors in \mathcal{L} is denoted by \mathcal{L}_K .

2.3.3 Pointwise decay of matrix coefficients in semisimple groups

In this section we describe the Howe-Moore theorem concerning decay of matrix coefficients in semisimple groups as well as a quantitative refinement.

Theorem 2.3.7 (Howe-Moore). *Let σ be a unitary representation of a semisimple group G on a Hilbert space \mathcal{H} such that no simple factor of G has invariant vectors in \mathcal{H} . Then all matrix coefficients of σ decay to zero.*

For the proof, see [2, Chapter III]. The remarkable consequence of this theorem is that an ergodic action of a simple group is automatically strongly mixing. It is known that strong mixing is a rare property for ergodic systems, so the theorem places semisimple groups in a very special category among all ergodic systems.

For applications, the qualitative statement of vanishing of matrix coefficients is usually not sufficient. The next theorem will give an optimal decay rate for matrix coefficients in certain cases. To state the theorem, we need the notion of Harish-Chandra function of G . Several measures of quantitative decay in semisimple groups involve this function.

Definition 2.3.8. Let G be semisimple group and B the minimal parabolic subgroup defined over \mathcal{K} given by the ordering of roots in the group. Let Δ_B denote the modular function of B . The Harish-Chandra function of G is defined by

$$\Xi_G(g) = \int_K \Delta_B(gk)^{-\frac{1}{2}} dk.$$

The Harish-Chandra function satisfies the following properties (see [25]):

Proposition 2.3.9. 1. *It is continuous, bi- K -invariant function $G \rightarrow (0, 1]$.*

2. *For any $\epsilon > 0$ there exist c_1, c_2 positive such that*

$$c_1 \Delta_B^{-\frac{1}{2}}(b) \leq \Xi_G(b) \leq c_2 \Delta_B^{-\frac{1}{2}+\epsilon}(b).$$

3. *Ξ_G is in $L^{2+\epsilon}(G)$ for any $\epsilon > 0$.*

Definition 2.3.10. A *strongly orthogonal system* S of roots of the semisimple group G is a subset of the roots such that for any $\phi, \psi \in S$ neither of $\phi + \psi$ and $\phi - \psi$ is a root.

Example 2.3.11. In $\mathrm{SL}(4, \mathbb{R})$ the roots $e_1 - e_2$ and $e_3 - e_4$ constitute a strongly orthogonal system consisting of simple roots, positive for the Borel subgroup of upper triangular matrices.

We can now state a quantitative version of the Howe-Moore theorem from [17]; the statement there is much more general. We omit the case $\mathcal{K} = \mathbb{C}$ which is a little more involved.

Theorem 2.3.12 ([17, Theorem 1.1.]). *Let \mathcal{K} be local of characteristic zero, $\mathcal{K} \neq \mathbb{C}$. Let G be a simple group over \mathcal{K} of rank greater than or equal to 2 and S a strongly orthogonal system of G . Then for any unitary representation σ without G -invariant vectors and for any K -finite vectors f, h we have the pointwise bound*

$$\langle \sigma(g)f, h \rangle \leq ([K : K \cap cKc^{-1}] \dim \langle K \cdot f \rangle \langle K \cdot h \rangle)^{\frac{1}{2}} \prod_{a \in S} \Xi(a(d))$$

where $\Xi = \Xi_{PGL(2, \mathcal{K})}$ and $g = k_1 c d k_2$, $d \in D^+$.

2.4 Fourier Transform and duality in additive groups of locally compact fields

Let $G = KD^+FK$ be a semisimple group with a Cartan decomposition with respect to a good maximal compact subgroup.

Definition 2.4.1. Given a finite dimensional normed vector space $(V, \|\cdot\|)$ over \mathcal{K} with a norm invariant under a fixed good maximal compact subgroup of G , we denote by \widehat{V} the unitary dual, i.e. the topological group of all additive unitary characters of V . For $\mathbf{x} = (x_i), \mathbf{y} = (y_i) \in V$ let

$$(\mathbf{x}, \mathbf{y}) = \sum x_i y_i$$

be the standard bilinear form on V . Choosing a fixed non trivial unitary character ζ of \mathcal{K} , define the map $V \rightarrow \widehat{V}$ by

$$v \rightarrow \zeta((v, \cdot)) =: \zeta_v.$$

This correspondence is a topological group isomorphism between V and \widehat{V} through which we will usually identify the two. In this situation, given $v, w \in V$, we denote $[v, w] = \zeta_v(w)$.

Under (\cdot, \cdot) we naturally define the transpose of a linear operator; define $\rho^* : G \rightarrow \text{GL}(V)$ to be the inverse transpose of ρ ,

$$\rho^*(v) := (\rho^{-1})^T(v).$$

This provides an identification of the dual action of G on V^* with the action ρ^* on V , given the topological isomorphism above. Furthermore, if ρ is irreducible and excellent on V , so is ρ^* ; finally, $\|\cdot\|$ is $\rho^*(K)$ -invariant as well. See Section 6.1 of [24] for these facts.

Definition 2.4.2. For $f \in L^1(V)$ and $\chi \in \widehat{V}$, define the Fourier transform

$$\widehat{f}(\chi) = \int_V \overline{\chi(v)} f(v) dm(v) \quad (2.4.1)$$

where $dm(v)$ is a Haar measure on V .

Using the topological identification of V and \widehat{V} , we can view the Fourier transform as a function on V by the formula

$$\widehat{f}(w) = \int_V \zeta_{-w}(v) f(v) dm(v) = \int_V [-w, v] f(v) dm(v) \quad (2.4.2)$$

in the bracket notation of the pairing.

We will use repeatedly the following theorems (Plancherel, inversion and duality):

Theorem 2.4.1. *There is a normalization of the dual Haar measure $dm(\chi)$ on \widehat{V} so that:*

1. *The Fourier transform extends to an isometry $L^2(V) \rightarrow L^2(\widehat{V})$.*
2. *If both f and \widehat{f} are integrable, then for almost every $v \in V$*

$$f(v) = \int_{\widehat{V}} \chi(v) \widehat{f}(\chi) dm(\chi). \quad (2.4.3)$$

3. *Every $v \in V$ defines a unitary character of \widehat{V} through the pairing $(v, \chi) \rightarrow \chi(v)$ which furnishes a canonical topological isomorphism between V and $\widehat{\widehat{V}}$.*

The Schwartz-Bruhat space $\mathfrak{S}(V)$ is just the usual Schwartz space when \mathcal{K} is archimedean; in the non-archimedean case, it consists of compactly supported, locally constant functions on V . The main properties of $\mathfrak{S}(V)$ are that its functions are dense in $L^2(V)$ and the Fourier transform furnishes a topological isomorphism $\mathfrak{S}(V) \simeq \mathfrak{S}(\widehat{V})$. For more details about the Fourier analysis facts we will use [23].

Chapter 3

Tools and main results

3.1 Spectral fingerprints

Given a Schwartz function ϕ on \widehat{V} and $f \in \mathcal{L}_K$ define

$$P_\phi(f) := \int_V \widehat{\phi}(x) (x \cdot f) dm(x). \quad (3.1.1)$$

Here we use the formulation of [12], Chapter 11 for Banach-space valued (Bochner) integrals. Because of the rapid decay of the Fourier transform $\widehat{\phi}$, $P_\phi(f)$ retains differentiability properties of f and the inequality

$$\|P_\phi(f)\|_p \leq \|\widehat{\phi}\|_{L^1(V)} \|f\|_p, \quad 1 \leq p \leq \infty \quad (3.1.2)$$

shows that it is bounded on all the L^p spaces.

Proposition 3.1.1. *The operator P_ϕ satisfies the following properties:*

- P_ϕ is self-adjoint (with respect to the inner product of \mathcal{H}) for real ϕ ; more generally,

$$P_\phi^* = P_{\bar{\phi}}.$$

- Operator multiplication transforms to pointwise multiplication of functions:

$$P_{\phi\psi} = P_\phi \circ P_\psi \quad (3.1.3)$$

This property plus linearity in the subscript shows that P is a homomorphism from the pointwise algebra of Schwartz functions to self adjoint operators on \mathcal{H} .

- For $g \in G$,

$$\sigma(g)P_\phi\sigma(g^{-1}) = P_{\phi(\rho(g^{-1})\cdot)}. \quad (3.1.4)$$

When ϕ is K -invariant, this equation implies that P_ϕ commutes with the K -action and thus K -finite vectors are L^2 -dense in the range of P_ϕ .

Proof. For the case $\mathcal{K} = \mathbb{R}$, all statements above are proven in [7, Chapter I and Chapter V]. No properties germane to \mathbb{R} are used and the proof carries over to all local fields of characteristic zero. \square

We will identify $\phi \in \mathfrak{S}(\widehat{V})$ with $\phi(\zeta) \in \mathfrak{S}(V)$. With that identification, the action of G in (3.1.4) corresponds to the contragredient representation ρ^* on $\mathfrak{S}(V)$, i.e. when we think of ϕ as a function on V , we have

$$\sigma(g)P_\phi\sigma(g^{-1}) = P_{\phi(\rho^*(g^{-1})\cdot)}. \quad (3.1.5)$$

Convergence and limits involving P are obtained using positivity: for $\phi \geq 0$, P_ϕ is a positive semidefinite operator. To see this, simply use (3.1.3) and self-adjointness:

$$\langle P_\phi(f), f \rangle = \langle P_{\sqrt{\phi}}(P_{\sqrt{\phi}}(f)), f \rangle = \langle P_{\sqrt{\phi}}(f), P_{\sqrt{\phi}}(f) \rangle \geq 0.$$

This way we see that $P_\phi \geq P_\psi$ and thus $\|P_\phi\|_2 \geq \|P_\psi\|_2$ when $\phi \geq \psi$. Thus, if ϕ_j increase or decrease monotonically to a bounded function on V , the P_{ϕ_j} converge strongly to a bounded, self-adjoint operator on \mathcal{H} .

Definition 3.1.2. Let S be an admissible set and ϕ_n a sequence of Schwartz functions that decrease monotonically to S . Then

$$P_S(f) = \lim_{n \rightarrow \infty} P_{\phi_n}(f).$$

The P_S as defined above is a self-adjoint projection operator on \mathcal{H} . Although we will mostly deal directly with the P_ϕ , since we cannot guarantee control on the L^∞ norm for the limits in general, we will use P_S as a tool to simplify calculations and their use is always permissible when we do not require $P_S(f)$ to have any boundedness conditions other than being in L^2 .

3.2 Spectrally restricted functions

Definition 3.2.1. Let S be a subset of V with the property that its characteristic function χ_S can be pointwise approximated by a sequence of decreasing compactly supported Schwartz functions; we call such sets *admissible*; all sets we will deal with will be admissible.

In particular, we will be interested in the annuli

$$\text{Ann}(s) := \{x \in V \mid s^{-1} < \|x\| < s\}.$$

Recall that the norm on V is assumed K -invariant. The characteristic function of each annulus $\chi_{\text{Ann}(s)}$ can be approximated pointwise from above by a sequence of smooth functions with the properties

$$\begin{aligned}
\phi_s^k &\equiv 1 \text{ on } \text{Ann}(s) \\
\text{supp}(\phi_s^k) &\subset \text{Ann}\left(s + \frac{1}{k}\right) \\
\phi_s^k &\leq \phi_s^l \text{ for } l \leq k
\end{aligned} \tag{3.2.1}$$

From this definition, the sequence $P_{s,k} := P_{\phi_s^k}$ consists of positive, decreasing, self-adjoint (see [7] for the easy computation) bounded operators on $L^2(X)$ and thus has a strong limit for fixed s as k tends to infinity which by (3.1.3) is idempotent, since $\phi_s^k \rightarrow \chi_{\text{Ann}(s)}$. Note that the image under $P_s = \lim P_{s,k}$ of $L_0^\infty(X)$ is L_0^2 -dense in L_0^∞ since the P_s form a system of projections that tends to the identity operator in $L_0^2(X)$ as s goes to infinity.

It will be convenient to fix a class of functions satisfying (3.2.1) once and for all. We do this as follows:

In the non-archimedean case, the characteristic function of the annulus is itself a smooth (even Schwartz) function. In this case we will simply take $\phi_s^k = \chi_{\text{Ann}(s)}$.

In the archimedean case, we will use functions ϕ_s^k constructed as follows: define

$$I(r) = e^{\frac{r^2-1}{r^2}} \tag{3.2.2}$$

for $0 \leq r \leq 1$ and extend it to \mathbb{R} by 0 for $r < 0$ and 1 for $r > 1$. This function is in $C^\infty(\mathbb{R})$. Construct ϕ_s^k using a homothetic copy of $I(r)$ rescaled to realize the transition from 0 to 1 (radially) at the annulus described above. What we gain is the guarantee that at the transitional annulus (the ‘corona’), the function ϕ_s^k satisfies

$$|(1 - \Delta)^a \phi_s^k(v)| \leq s^{la} \tag{3.2.3}$$

for any $a \in \mathbb{N}$ and some $l \geq 2$ depending on the real dimension of V . Here Δ is the usual Laplacian on V seen as a *real* vector space.

Recall the definition of P_S for an admissible set S . The properties of P_ϕ listed in 3.2.1 imply the following important facts:

- If $\text{supp}(\phi) \subset S$, then

$$P_S(P_\phi) = P_\phi \tag{3.2.4}$$

- If S is invariant under rotations, then for any $g \in K$,

$$\sigma(g)P_S\sigma(g^{-1}) = P_S \tag{3.2.5}$$

- By the previous property, when S or ϕ are K -invariant, P_S or P_ϕ commutes with the action of K and thus K -finite vectors are dense in the range of P_S or P_ϕ .

Definition 3.2.2. The following terminology will be convenient: if $P_S(f) = f$

for some set S , we say that the *spectral support* of f lies in S ; when S is replaced in the subscript by a Schwartz function, by spectral support we will refer to the support of the function.

Intuitively, P_S restricts the spectrum of f to lie in S , so a function which is unaffected by this application is justified in being called spectrally supported in S . Note that $P_S(L^2)$ is a closed vector subspace of L^2 for each S since P_S is a projection operator.

With this notion in hand, we can define explicitly the dense subspace of \mathcal{L}_K where we will bound the coefficients effectively.

Definition 3.2.3. Let

$$\mathcal{D}_s = \bigcup_{k > s^2} P_{\phi_s^k}(\mathcal{L}_K)$$

and

$$\mathcal{D} = \bigcup_{s > 0} \mathcal{D}_s$$

and call it the space of spectrally bounded functions in \mathcal{L}_K .

It is easy to see that this space is L^2 -dense in \mathcal{L}_K . The specific choice $k > s^2$ is not important: we just need some leeway for approximations and we do not want k to be too small as to cause problems with stretching annuli.

The next lemma describes how pointwise multiplication of functions behaves with respect to the operators P_ϕ . Recall the identification of \widehat{V} with V and the two dual Haar measures by the isomorphism in 2.4.1 (compatibility in the computations below is guaranteed by (2.4.3)). Recall the notation $[u, z] = \zeta_u(z)$ for $u, z \in V$ (keep in mind the standard case $[u, z] = e^{i\langle u, z \rangle}$).

Lemma 3.2.1. *Let $\phi, \psi \in \mathfrak{S}(V)$ and $f, g \in L^2(X)$ be such that the pointwise (a.e.) product $P_\phi(f)P_\psi(g)$ is in $L^2(X)$. Suppose $\omega \in \mathfrak{S}(V)$ is identically equal to one on $\text{supp}(\phi) + \text{supp}(\psi)$; then $P_\omega(P_\phi(f)P_\psi(g)) = P_\phi(f)P_\psi(g)$.*

Proof. Compute:

$$\begin{aligned} & P_\omega(P_\phi(f)P_\psi(g)) \\ &= \int \widehat{\omega}(z) \int \widehat{\phi}(x) \rho(z+x) f \, dm(x) \int \widehat{\psi}(y) \rho(z+y) g \, dm(y) \, dm(z) \\ &= \int \widehat{\omega}(z) \int \widehat{\phi}(x-z) \rho(x) f \, dm(x) \int \widehat{\psi}(y-z) \rho(y) g \, dm(y) \, dm(z) \\ &= \iint \rho(x) f \, \rho(y) g \int \widehat{\omega}(z) \widehat{\phi}(x-z) \widehat{\psi}(y-z) \, dm(z) \, dm(x) \, dm(y). \end{aligned}$$

Now expand the inner integral using the definition of the Fourier transform,

valid for L^1 functions:

$$\begin{aligned}
& \int \widehat{\omega}(z) \widehat{\phi}(x-z) \widehat{\psi}(y-z) dm(z) \\
&= \iiint \omega(u_3)[-z, u_3] \phi(u_1)[-(x-z), u_1] \\
&\quad \cdot \psi(u_2)[-(y-z), u_2] dm(u_2) dm(u_1) dm(u_3) dm(z) \\
&= \iint \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2] \\
&\quad \cdot \left(\iint \omega(u_3)[-z, u_3] [z, u_1] [z, u_2] dm(u_3) dm(z) \right) dm(u_1) dm(u_2) \\
&= \iint \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2] \\
&\quad \cdot \left(\int [z, u_1 + u_2] \int \omega(u_3)[-z, u_3] dm(u_3) dm(z) \right) dm(u_1) dm(u_2).
\end{aligned}$$

The integral in the parentheses is simply

$$\int [z, u_1 + u_2] \widehat{\omega}(z) dm(z) = \omega(u_1 + u_2) = 1$$

by Fourier inversion and the fact that $u_1 \in \text{supp}(\phi)$, $u_2 \in \text{supp}(\psi)$. Untangling the remaining integrals we get the required result. \square

Corollary 3.2.2. *Let $\text{supp}(\phi) \subset S$ and $\text{supp}(\psi) \subset T$ for admissible sets S and T . Then*

$$P_{S+T}(P_\phi P_\psi) = P_\phi P_\psi. \quad (3.2.6)$$

The relations (3.2.6) and (3.2.4) form the core of the main computation.

3.3 Bounds for the L^2 -norm in sectors

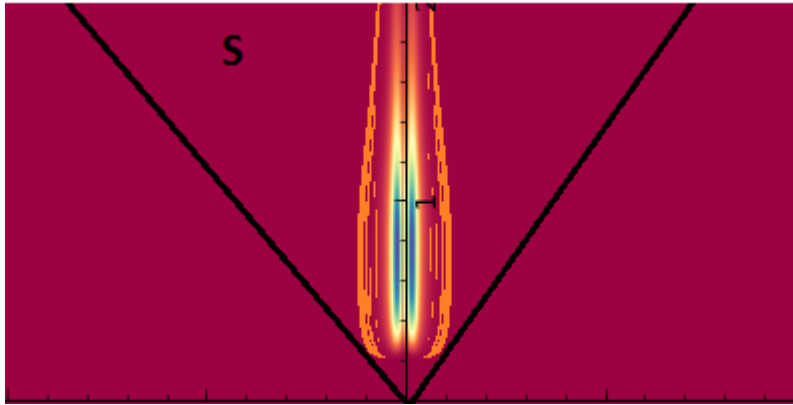


Figure 3.1: The spectral fingerprint of a function $P_S(f)$ for a sector in \mathbb{R}^2 .

In the sequel, we will examine how restricting a unit (in the L^2 -norm) K

-finite vector f to the image of an approximate projection P_ϕ for suitable ϕ affects its norm. This was accomplished in greater generality in [24] from which we will draw notation and results, noting the places in that paper where they are treated. The idea of estimating matrix coefficients (non-uniformly) by looking at the effect the representation has on their spectral support and then estimating norms of functions with restricted spectral support is a major theme in chapter 5 of [7] where the idea is applied to $G = \mathrm{SL}(2, \mathbb{R})$ and $V = \mathbb{R}^2$. The maximal compact subgroup in that setting is $K = \mathrm{SO}(2, \mathbb{R})$, a connected torus. In our setting, the non-commutativity of K increases the complexity of this method considerably. However, the detailed analysis in [24] allows one to carry it out effectively.

In order to state the second main lemma and principal ingredient for bounding norms of projected vectors, we need some additional concepts from [24]. Recall the list of notations in definition 2.2.12 and assume that $\rho = \rho_1$ is *irreducible* with highest and lowest weights λ and ϱ respectively. For $\psi \in \Phi_1$, let $\pi_\psi(v)$ be the projection of v on the weight space V_ψ .

Define the 'cones'

$$\begin{aligned} \mathrm{Cone}_1(c, s) &= \{v \in V : \|\pi_\lambda(v)\| \leq c \text{ and } \|v\| \geq s\}, \\ \mathrm{Cone}_2(c, s) &= \{v \in V : \|\pi_\varrho(v)\| \leq c \text{ and } \|v\| \geq s\}. \end{aligned}$$

See [24, Proposition 6.1] for the fundamental properties of these sets. We will not use Proposition 6.1 itself here, but we will follow verbatim the computations in Proposition 7.1 which uses Proposition 6.1 in a crucial way.

Observe that the norm $\|\cdot\|_\infty$ on V defined by

$$\|v\|_\infty = \max_{\phi \in \Phi_1} \|\pi_\phi(v)\|$$

is equivalent to the given norm since $\dim(V) < \infty$, so in particular $\|v\|_\infty \leq C\|v\|$; we will use this observation below.

Lemma 3.3.1. *Let f be K -finite with $\|f\|_2 = 1$, $\dim\langle K \cdot f \rangle = d_f$, $\mathbf{a}^i \in D^+$ for $i = 1, \dots, k$ ordered in increasing $\pi_\lambda(\mathbf{a}^i)$ with sufficiently large minimum depending only on the action, $\mathrm{Ann}(s)$ the annulus defined in Section 2.4 and $F_s \in \mathfrak{S}(V)$ with compact support inside the set*

$$X_1(\mathbf{a}, s) = \mathrm{Ann}(s) \cap \left(\sum_i \rho^*(\mathbf{a}^i) (\mathrm{Ann}(s^{-1}, s)) \right).$$

Then for some positive C independent of \mathbf{a} , s and f we have the bounds

$$\|P_{F_s}(f)\|_2 \leq C s^q d_f^{\frac{1}{2}} \left| \sum_i \varrho(\mathbf{a}^i)^{-1} \right|^{-\frac{q}{2}}. \quad (3.3.1)$$

Similarly, if the support of F_s is in the set

$$X_2(\mathbf{a}, s) = \text{Ann}(s) \cap \left(\sum_i \rho^* \left(\frac{\mathbf{a}^i}{\mathbf{a}^k} \right) (\text{Ann}(s^{-1}, s)) \right),$$

then as above

$$\|P_{F_s}(f)\|_2 \leq C s^q d_f^{\frac{1}{2}} \left| \sum_i \lambda \left(\frac{\mathbf{a}^k}{\mathbf{a}^i} \right) \right|^{-\frac{q}{2}}. \quad (3.3.2)$$

Proof. The proof is essentially contained in the proof of Proposition 7.1 of [24]. We indicate how to extract the relevant parts for our lemma and explain the correspondences. All references to numbered sections will belong to [24]. We will only examine the first situation, since the second one is identical.

In the course of proving that proposition, the author in [24] examines (pages 31-38) a Schwartz function¹ F_s and the projection of a unit vector with respect to that function, bounding the Hilbert space norm of

$$\widehat{\Pi}(F_s^{\frac{1}{2}})\eta$$

in the notation of that paper, where

$$F_s = (a\omega)^{-1}(h_s \cdot g_s)^{\frac{1}{2}};$$

in our notation η is f and $\widehat{\Pi}(F_s^{\frac{1}{2}})$ is $P_{F_s^{\frac{1}{2}}}$, $\omega = 1$ because we are only considering the positive Weyl chamber and the precise definition of F_s in [24] is irrelevant; in order to carry out the computations, we only need F_s to be Schwartz and its support contained in one of the two cones defined above, for specific $0 < c \ll 1$ and $s > 0$. There, it is claimed that how small c needs to be depends on s ; however, the only dependence of c on s that is necessary there is that $cs^{-1} < C$ with C depending only on the action. Since our c here is going to be of the form $c = sA$ where A does not involve s , we see that in order to ensure $cs^{-1} < C$ all we need is to bound A by a constant depending only on the action; this translates to the norm bounds on the \mathbf{a}^i in the statement. See p.27 of [24] for the specific requirements on c .

By the discussion above, the reductions from page 31 to page 34 in [24] carry over to our F_s , resulting in the situation where we want to bound

$$\|\tilde{P}_{F_s}\tilde{f}\|_2$$

where \tilde{P} is the approximate projection operator for the regular representation of $K \ltimes V$ on \mathcal{H} and \tilde{f} is a K -invariant unit norm vector. At that point we use the containment of the supports. In our case, observe that our $X_1(\mathbf{a}, s)$ is

¹The reason we use F_s for the function rather than the usual Greek letters is to facilitate the comparison with the computation in [24].

contained in the set

$$E_1 = \{v \in \text{Ann}(s) : s^{-1} \leq \left\| \sum_{i=1}^k \rho^*(\mathbf{a}^i)v \right\| \leq s\}$$

which becomes, after writing v in terms of the weights, applying ρ^* to each coordinate in V_ψ and switching summations

$$E_1 = \{v \in \text{Ann}(s) : s^{-1} \leq \left\| \sum_{\psi \in \Phi_1} \left(\sum_{i=1}^k \psi(\mathbf{a}^i)^{-1} \right) \pi_\psi(v) \right\| \leq s\}.$$

Note the inverses because we decomposed the vector v with respect to the weights Φ_1 of ρ ; then the weights of ρ^* act by inverses on each weight space π_ψ .

Using the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_\infty$ we see that this set is contained in

$$S_1 = \{v \in \text{Ann}(s) : \|\pi_\varrho(v)\| \leq Cs \left| \sum_{i=1}^k \varrho(\mathbf{a}^i)^{-1} \right|^{-1}\}.$$

Since $\mathbf{a}^i \in D^+$, for large $\min_{1 \leq i \leq k} |\mathbf{a}^i|$ (in any norm on G) the coefficient on the right hand side of the definition on S_1 is going to be small. In particular, S_1 will be contained in the cone

$$\text{Cone}_1(Cs \left| \sum_{i=1}^k \varrho(\mathbf{a}^i)^{-1} \right|^{-1}, s^{-1}).$$

Having this containment, the argument from pages 34-37 goes through without change leading to the desired conclusion analogous to (7.22), (7.23) there. \square

Since admissible sets can be approximated from above by Schwartz functions and the operators P_i are monotone, we get the corollary

Corollary 3.3.2. *With notation as in Lemma 3.3.1 and S an admissible set contained in one of the $X_i(\mathbf{a}, s)$, we have the corresponding bound from that lemma for $\|P_S(f)\|_2$.*

Remark: In the case of $G = \text{SL}(2, \mathbb{R})$, K is commutative and a much easier proof of the Lemma 3.3.1 follows from [7, Chapter V, Theorem 3.3.1].

3.4 Decay of multiple correlations

Consider $k \geq 2$ distinct elements $\mathbf{a}^i \in D^+$, $i = 1 \cdots k$ as above and $k+1$ functions $f_i \in \mathcal{D}$, $i = 0 \cdots k$. Order the \mathbf{a}^i in increasing highest weight valuations and define $\mathbf{a}^0 = \mathbf{I}$. We want to bound the correlation integral

$$\int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x). \quad (3.4.1)$$

Since $P_{s,k}(f) \rightarrow f$ for any $f \in \mathcal{L}$ and we only have finitely many f_i , we can assume that all f_i are in the image of $P_{s,l}$ for some s, l (and thus certainly in the image of $P_{s'}$ where $s' = s + \frac{2}{l}$).

Definition 3.4.1. Assume that the elements \mathbf{a}^i are such that $\mathbf{a}^0 = I$ and if $0 < i < j$ then

$$1 \leq |\lambda(\mathbf{a}^i)| \leq |\lambda(\mathbf{a}^j)|.$$

Define the sums

$$\mathcal{L}(\mathbf{a}) = \sum_{i=0}^{k-1} \lambda\left(\frac{\mathbf{a}^k}{\mathbf{a}^i}\right)$$

and

$$\mathcal{R}(\mathbf{a}) = \sum_{i=1}^k \varrho(\mathbf{a}^i)^{-1}. \quad (3.4.2)$$

At least one of these two sums will be large as the \mathbf{a}^i go out in the positive Weyl chamber; in order to treat the bound uniformly, define

$$\mathfrak{R}(\mathbf{a}) = \min(1, |\mathcal{L}(\mathbf{a})|) \cdot \min(1, |\mathcal{R}(\mathbf{a})|).$$

For notational convenience, abbreviate $\text{Ann}(s)$ by (s) and denote its image under $\rho^*(\mathbf{a}^i)$ simply by $\mathbf{a}^i(s)$. We will also denote the action of $\rho^*(\mathbf{a}^i)$ on the ϕ_s^k defined above by $\mathbf{a}^i(s, k)$.

Theorem 3.4.1. Let \mathbf{a}^i, f_i, s be as above. Let

$$d_i = \dim\langle K \cdot f_i \rangle.$$

There exists a positive constant C' independent of the f_i such that if

$$\max\left(\min_{i=0, \dots, k-1} |\lambda(\frac{\mathbf{a}^k}{\mathbf{a}^i})|, \min_{i=1, \dots, k} |\varrho(\mathbf{a}^i)^{-1}|\right) > C', \quad (3.4.3)$$

we have the bound

$$\begin{aligned} & \int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x) \\ & \leq s^{2q} d_0^{\frac{1}{2}} \|f_0\|_2 d_k^{\frac{1}{2}} \|f_k\|_2 \left(\prod_{i=1}^{k-1} \|f_i\|_\infty \right) \mathfrak{R}(\mathbf{a})^{-\frac{q}{2}}. \end{aligned} \quad (3.4.4)$$

The proof is based on an examination of the effect on the spectrum of $P_\phi(f)$ of Cartan elements, of taking pointwise products, and finally of correlating with other $P_{\phi'}(f)$. In Figure 3.4 we see a spectral picture of an approximate annulus ϕ applied to the spectrum of a sample f ; as Cartan elements act on this spectrum it is distorted as in Figure 3.4. Observe the gradual degeneration of the spectrum into a long, thin strip with a central ball removed.

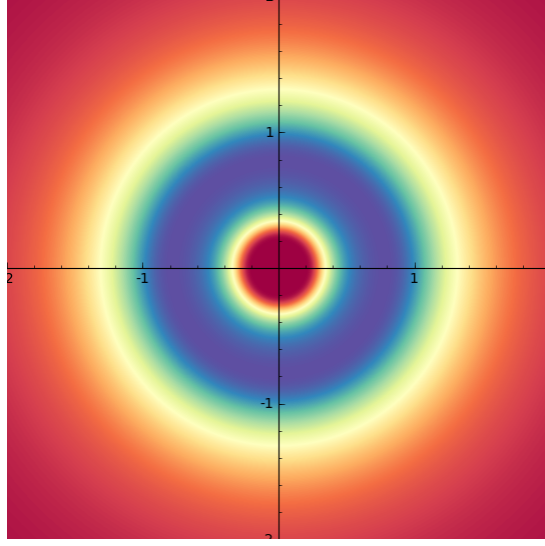


Figure 3.2: The spectral fingerprint of a function $P_\phi(f)$.

Proof. The correlation can be written as

$$\int_X P_{s,l}(f_0) \mathbf{a}^1 \cdot P_{s,l}(f_1) \cdots \mathbf{a}^k \cdot P_{s,l}(f_k) \quad (3.4.5)$$

which by (3.1.4) becomes

$$\int_X P_{s,l}(f_0) P_{\mathbf{a}^1(s,l)}(\mathbf{a}^1 \cdot f_1) \cdots P_{\mathbf{a}^k(s,l)}(\mathbf{a}^k \cdot f_k). \quad (3.4.6)$$

We now use Lemma 3.2.1 repeatedly to conclude that

$$P_{\mathbf{a}^1(s,l)}(\mathbf{a}^1 \cdot f_1) \cdots P_{\mathbf{a}^k(s,l)}(\mathbf{a}^k \cdot f_k) \in P_\Sigma(\mathcal{L})$$

where Σ is the iterated sum set $\sum_{i=1}^k \mathbf{a}^i(s')$; thus in particular if

$$z := P_{\mathbf{a}^1(s,l)}(\mathbf{a}^1 \cdot f_1) \cdots P_{\mathbf{a}^k(s,l)}(\mathbf{a}^k \cdot f_k)$$

then $P_\Sigma(z) = z$. Thus the integral in (3.4.6) becomes

$$\int_X P_{s,l}(f_0) P_\Sigma(z) \quad (3.4.7)$$

Now P_Σ is an orthogonal projection so we can transfer P_Σ from z to $P_s(f_0)$, getting

$$P_\Sigma(P_{s,l}(f_0)) = P_{\chi_\Sigma \phi_s^l}(f_0).$$

Here we are abusing notation a little bit, since the last expression need not be a bounded function; we will take this shortcut to mean that we have an arbitrary Schwartz function ϕ dominating the function χ_Σ and we are applying P_ϕ to

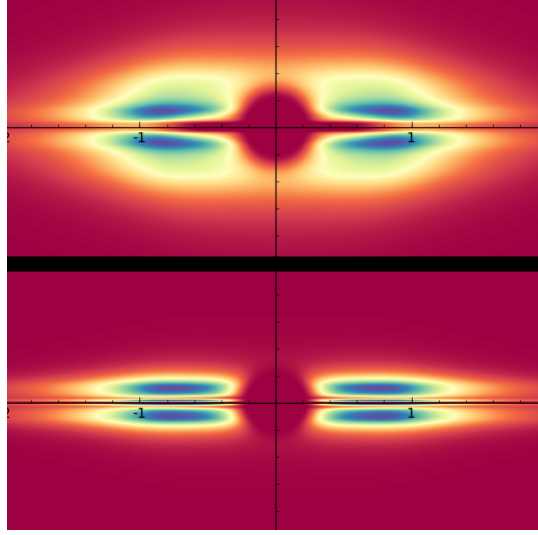


Figure 3.3: Evolution of the fingerprint of multiple autocorrelations $f(x)f(\mathbf{a}^1 \cdot x)f(\mathbf{a}^2 \cdot x)$ of the previous function as the Cartan elements diverge from each other.

both terms of the ‘inner product’; the rightmost term is unaffected, while the leftmost has spectral support approximately equal to that of $\chi_\Sigma \phi_s^l$ since ϕ is arbitrary and the support of $\chi_\Sigma \phi_s^l$ is easily seen to be an admissible set (also see Corollary 3.3.2). Thus the integral becomes

$$\begin{aligned} \int_X P_{\chi_\Sigma \phi_s^l}(f_0) z d\mu &= \int_X P_{\chi_\Sigma \phi_s^l}(f_0) P_{\mathbf{a}^1(s,l)}(\mathbf{a}^1 \cdot f_1) \cdots P_{\mathbf{a}^k(s,l)}(\mathbf{a}^k \cdot f_k) d\mu \\ &= \int_X P_{\chi_\Sigma \phi_s^l}(f_0) \mathbf{a}^1 \cdot P_{s,l}(f_1) \cdots \mathbf{a}^k \cdot P_{s,l}(f_k) d\mu \end{aligned}$$

Write $U_0 := \chi_\Sigma \phi_s^l$ and apply $(\mathbf{a}^k)^{-1}$ to all terms of the integral, giving

$$\int_X (\mathbf{a}^k)^{-1} \cdot P_{U_0}(f_0) P_{s,l}(f_k) \prod_{i=1}^{k-1} (\mathbf{a}^k)^{-1} \cdot \mathbf{a}^i \cdot P_{s,l}(f_i) d\mu \quad (3.4.8)$$

By unitarity, the value of the integral is not affected. So, now we can repeat the reasoning above, summing the indices for all factors except for $P_{s,l}(f_k)$, and conclude that this integral is equal to

$$\begin{aligned} &\int_X (\mathbf{a}^k)^{-1} \cdot P_{U_0}(f_0) P_{s,l}(f_k) \prod_{i=1}^{k-1} (\mathbf{a}^k)^{-1} \cdot \mathbf{a}^i \cdot P_{s,l}(f_i) d\mu \\ &= \int_X P_{s,l}(f_k) P_{\Sigma_k}(z_k) d\mu \end{aligned} \quad (3.4.9)$$

$$\begin{aligned} &= \int_X P_{\chi_{\Sigma_k} \phi_s^l}(f_k) z_k d\mu \\ &= \int_X P_{U_0}(f_0) \mathbf{a}^k \cdot P_{U_k}(f_k) \prod_{i=1}^{k-1} \mathbf{a}^i \cdot P_{s,l}(f_i) d\mu \end{aligned} \quad (3.4.10)$$

where

$$\Sigma_k = \sum_{j=0}^{k-1} (\mathbf{a}^k)^{-1} \cdot \mathbf{a}^j(s'),$$

$$z_k = \prod_{i=0}^{k-1} (\mathbf{a}^k)^{-1} \cdot \mathbf{a}^i \cdot P_{s,l}(f_i)$$

and $U_k = \chi_{\Sigma_k} \phi_s^l$.

Denote by U_0 and U_k respectively also the supports of the corresponding functions (which, note, are bounded above by 1 and thus by the characteristic functions of the supports). We can now immediately apply Lemma 3.3.1 with U_0 and U_k in the place of the two situations for F_s considered there, bounding

$$\|P_{U_0}(f_0)\|_2 \text{ and } \|\mathbf{a}^k \cdot P_{U_k}(f_k)\|_2 = \|P_{U_k}(f_k)\|_2.$$

In order to finish the proof, we simply bound (3.4.10) by the L^∞ norms of the functions f_i for $i \neq 0, k$ and then use Cauchy's inequality on the two remaining terms to finish the proof. \square

3.5 Examples

In this section we see what the bounds obtained above mean for two particular cases of semidirect products. The choice of these examples is not arbitrary: these groups will occur as subgroups (locally) of split simple groups of higher rank.

First consider the case of $\mathfrak{G} = \mathrm{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2$ where the action is the standard matrix action on 2-vectors. The action is irreducible and there are only two weights; the roots of $\mathrm{SL}(2, \mathcal{K})$ are

$$\mathrm{diag}(a, a^{-1}) \rightarrow a^{\pm 2}$$

and the weights of the standard representation on \mathcal{K}^2 are

$$\mathrm{diag}(a, a^{-1}) \rightarrow a^{\pm 1}$$

with the obvious weight spaces $V_1 = \{(v_1, 0) \in \mathcal{K}^2\}$ and $V_2 = \{(0, v_2) \in \mathcal{K}^2\}$. Take the weight

$$\mathrm{diag}(a, a^{-1}) \rightarrow a$$

to be positive, so this is the highest weight. The highest weight space is one dimensional, so the exponent \mathfrak{q} defined in the list 2.2.12 is in our case 1. Write

$$\mathbf{a}^i = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}$$

for $k+1$ elements of the positive Weyl chamber with $1 = a_0 < a_1 < \cdots < a_k$

and if $i > j$, $\frac{a_i}{a_j} > C_0$ for some C_0 depending on the action of \mathfrak{G} on X . Applying the preceding discussion to the hypotheses of Theorem 3.4.1, we get

Corollary 3.5.1. *In the setting of Theorem 3.4.1 and $\mathfrak{G} = SL(2, \mathcal{K}) \ltimes \mathcal{K}^2$ with the standard action and $\|f_i\|_\infty$ normalized to 1, we get the bound*

$$\begin{aligned} & \int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x) \\ & \leq C s^2 d_0^{\frac{1}{2}} d_k^{\frac{1}{2}} \left| \sum_{i=0}^{k-1} \frac{a_k}{a_i} \right|^{-\frac{1}{2}} \left| \sum_{i=1}^k a_i \right|^{-\frac{1}{2}}. \end{aligned}$$

Note how in the case $k = 1$ we recover the bound from Chapter 5 of [7].

For the second example, consider the action of $SL(2, \mathcal{K})$ on its Lie algebra over \mathcal{K} , denoted simply by \mathfrak{g} and being equivalent to $S^2(\mathcal{K}^2)$, the second symmetric power of \mathcal{K}^2 (in the case $\text{char}(\mathcal{K}) = 0$ that we are considering). The weights and weight spaces in this case coincide with the roots and the highest weight space (pick $\text{diag}(a, a^{-1}) \rightarrow a^2$ as positive) is again one dimensional. Therefore, by the same procedure as above, we have:

Corollary 3.5.2. *In the setting of Theorem 3.4.1 and $\mathfrak{G} = SL(2, \mathcal{K}) \ltimes \mathfrak{g}$ with the adjoint action on the Lie algebra and $\|f_i\|_\infty$ normalized to 1, we get the bound*

$$\begin{aligned} & \int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x) \\ & \leq C s^2 d_0^{\frac{1}{2}} d_k^{\frac{1}{2}} \left| \sum_{i=0}^{k-1} \left(\frac{a_k}{a_i} \right)^2 \right|^{-\frac{1}{2}} \left| \sum_{i=1}^k (a_i)^2 \right|^{-\frac{1}{2}}. \end{aligned}$$

3.6 Spectral norms and extensions of decay

Observe that whenever we have an estimate of the form

$$\|P_{s,s^2}(f) - f\|_2 \leq C \|f\|' s^{-A} \quad (3.6.1)$$

for all $s > 0$, C and A independent of f and $\|\cdot\|'$ an appropriate norm, we can use a 2ϵ argument plus the uniform Hölder inequality to eliminate s :

$$\begin{aligned} \left| \int_X f_0 \cdots \mathbf{a}^k \cdot f_k d\mu \right| & \leq \sum_{i=0}^k \prod_{j \neq i} \|f_j\|_{\infty, s} \|P_{s,s^2}(f_i) - f_i\|_2 \\ & + \left| \int_X P_{s,s^2}(f_0) \cdots \mathbf{a}^k \cdot \mathbf{a}^i \cdot P_{s,s^2}(f_k) d\mu \right| \\ & \leq C \left(\sum_{i=0}^k \|f_i\|' \prod_{j \neq i} \|f_j\|_{\infty, s} \right) s^{-A} + C' s^{2q} d_0^{\frac{1}{2}} d_k^{\frac{1}{2}} \mathfrak{R}(\mathbf{a}) \end{aligned}$$

where $\|f\|_{\infty,s} = \max(\|f\|_{\infty}, \|P_{s,s^2}(f)\|_{\infty})$. Choosing $s = \Re(\mathbf{a})^{\epsilon}$ and optimizing for ϵ to get the best overall exponent, we can get a bound for all K -finite vectors with finite $\|\cdot\|'$ -norm which is uniform if it happens that $\|P_{s,s^2}(f)\|_{\infty} \leq s^M$ for some $M \geq 1$ and all $f \in \mathcal{L}_K$. Here we chose $l = s^2$ in the approximate projection $P_{s,l}$ for convenience; all we need is l to grow as a power of s , and since our main bound in Theorem 3.4.1 did not depend on l , we are free to make this choice.

In order to axiomatize this estimate, we introduce, for each $A > 0$, the norms

$$\begin{aligned} \|f\|_{\infty,s} &= \max(\|f\|_{\infty}, \|P_{s,s^2}(f)\|_{\infty}) \\ \|f\|_{-,A} &= \sup_{0 < s < \infty} s^{-A} \|P_{\psi_{0,s}}(f)\|_2, \\ \|f\|_{+,A} &= \sup_{0 < s < \infty} s^A \|P_{\psi_{\infty,s}}(f)\|_2, \\ \|f\|_{\pm,A} &= \|f\|_{-,A} + \|f\|_{+,A}, \end{aligned} \tag{3.6.2}$$

where

$$1 - \phi_t^{t^2} = \psi_{0,t^{-1}} + \psi_{\infty,t},$$

the first function supported in a small $B(0, t^{-1})$ and the other one in the complement of a large ball $B(0, t)$; the choice of exponents here indicate that for $\|\cdot\|_{-,A}$ it is small s that matter, while for $\|\cdot\|_{+,A}$ it is large s . We will study the two norms separately below, so there will be no danger of confusing the role of s .

Define \mathcal{L}_K^A to be the subspace of \mathcal{L}_K where both norms $\|f\|_{\pm,A}$ are finite. Note that for each $A > 0$,

$$P_{s,s^2}(\mathcal{L}_K) \subset \mathcal{L}_K^A$$

for all $s > 0$ and thus \mathcal{L}_K^A is L^2 -dense in \mathcal{L}_K (since $P_{s,s^2}(f) \rightarrow f$ in L^2).

The spaces \mathcal{L}_K^A form the broadest category of spaces where our method extends to give effective bounds. This should be understood in the sense that if we use as inputs only Theorem 3.4.1 and the basic structure of the projection operators P , we definitely need an estimate like (3.6.1) to remove the dependence on s . Let $f = (f_0, \dots, f_k)$ and denote the aggregate of the norms appearing above by

$$\mathcal{N}_{\pm,A,s}(f) = \max \left(\sum_{i=0}^k \|f_i\|_{\pm,A} \prod_{j \neq i} \|f_j\|_{\infty,s}, d_0^{\frac{1}{2}} d_k^{\frac{1}{2}} \|f_0\|_2 \|f_k\|_2 \prod_{i=1}^{k-1} \|f_i\|_{\infty,s} \right). \tag{3.6.3}$$

In this class of functions, our main result takes the form

Theorem 3.6.1. *Let $f_i \in \mathcal{L}_K^A$, \mathbf{a}^i and d_i as in Theorem 3.4.1. Under the*

assumption (3.4.3), we have the bound

$$\begin{aligned} & \int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x) \\ & \leq \mathcal{N}_{\pm, A, s}(f) \left(\left| \sum_{i=1}^{k-1} \lambda\left(\frac{\mathbf{a}^k}{\mathbf{a}^i}\right) \right| \left| \sum_{i=1}^k \varrho(\mathbf{a}^i)^{-1} \right| \right)^{-\frac{Aq}{2(A+2q)}}. \end{aligned} \quad (3.6.4)$$

Remark 3.6.1. At this point, the structure of the space \mathcal{L}_K^A is not apparent. However, it needs to be emphasized that the finiteness of $\|\cdot\|_{\pm, A}$ for all Sobolev functions in the archimedean case is strongly linked with mixing properties of the action of V on X , which is not typical of homogeneous actions of semidirect products. Mixing actions of V usually occur when \mathfrak{G} acts in a mixing manner as a subgroup of a higher rank group in one of the homogeneous spaces of the latter. In the case of homogeneous actions of semidirect products, the results of Chapter 4.1 complement the ones presented above.

3.7 Spectral norms in mixing actions of V

The purpose of this section is to show that for certain actions of \mathfrak{G} , \mathcal{L}_K^A contains all smooth, K -finite vectors and the spectral norms defined above can be replaced by norms involving derivatives of smooth vectors.

3.7.1 The archimedean case

Definition 3.7.1. The Sobolev norm of order k in an algebraic (or Lie) group G is

$$S_k(f) = \sup_{\substack{D \in \mathcal{U}(\mathfrak{g}) \\ \deg(D) \leq k}} \|D(f)\|_2. \quad (3.7.1)$$

The *Sobolev space* $\mathcal{S}^{2,k}(G)$ is the Hilbert space of C^k functions whose derivatives are all square integrable with norm given by (3.7.1).

We make the following assumptions on the action:

- \mathcal{K} is archimedean; in fact, take it to be \mathbb{R} without loss of generality; we will only rely on the additive and smooth structures of V below.
- The smooth matrix coefficients satisfy an estimate of the form

$$|\langle \mathbf{v} \cdot f, h \rangle| \leq C_k \mathcal{S}_k(f) \mathcal{S}_k(h) \frac{1}{(1 + \|\mathbf{v}\|)^\epsilon} \quad (3.7.2)$$

for some $\epsilon > 0$ and here we indicate the order of the Sobolev norm and the dependence of C_k on that order. The crucial part is the fact that these matrix coefficients, restricted to the action of V , are in $L^p(V)$ for some p . Note that ϵ , and therefore p do not depend on the functions.

Note that we did not specify the group G on which we take the Sobolev norms. This is because we usually apply these results when \mathfrak{G} sits inside a larger group G and we take Sobolev norms with respect to that group. All we really require of the expressions S_k apart from the norm axioms is that for any element D in the enveloping algebra of \mathfrak{G} of degree d and any smooth f to satisfy

$$S_k(Df) \leq S_{d+k}(f).$$

Below we assume that the K -invariant norm on V is the Euclidean norm for simplicity. The comparability of norms in V makes it easy to modify the statement for any other norm.

We have the following comparison result:

Proposition 3.7.2. *For any smooth, compactly supported f with zero mean there exists an $A = A(p)$ so that*

$$\|f\|_{\pm, A} \leq C(p, k) S_k(f) \quad (3.7.3)$$

for sufficiently large k .

Proof. First we treat $\|\cdot\|_{-, A}$. Assume $s < 1$. By monotonicity,

$$\|P_{\psi_{0,s}} f\| \leq \|P_{C(s)} f\|$$

where $C(s)$ is the cube of side $2s$ centered at the origin of V . Furthermore, the characteristic function of the cube satisfies

$$\|\widehat{\chi_{C(s)}}\|_p \leq C_p^d s^{\frac{d}{p'}} \quad (3.7.4)$$

for any $p > 1$ where $d = \dim_{\mathbb{R}}(V)$ and p' is the conjugate exponent; explicitly,

$$C_p = \left(\int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^p dt \right)^{\frac{1}{p}}.$$

Now, as $P_{C(s)}$ is an orthogonal projection, we have

$$\begin{aligned} \|P_{C(s)} f\|^2 &= \langle P_{C(s)} f, P_{C(s)} f \rangle \\ &= \langle P_{C(s)} f, f \rangle \\ &= \int_V \widehat{\chi_{C(s)}}(v) \langle v \cdot f, f \rangle d\mu(v), \end{aligned}$$

where the last expression makes sense because the integrand is in $L^1(V)$ and equals $\langle P_{C(s)} f, f \rangle$ by an approximation argument: the equality holds for the approximate projections decreasing to $P_{C(s)}$, so the dominated convergence theorem gives the equality at the limit in view of (3.7.4) and (3.7.2).

Define $\kappa(v) = \langle v \cdot f, f \rangle$. Using Hölder's inequality and (3.7.2), we get

$$\begin{aligned} \|P_{C(s)}f\|^2 &\leq \|\widehat{\chi_{C(s)}}\|_{p'} \|\kappa\|_p \\ &\leq C'_p C_k C_{p'}^d s^{\frac{d}{p}} S_k(f)^2 \end{aligned} \quad (3.7.5)$$

where the various constants depend only on p and d . We see, therefore, that if $A < \frac{d}{2p}$, the $\|\cdot\|_{-,A}$ norm is finite and satisfies the claimed bound (of course, we only treated $s < 1$ since the range $s > 1$ is immediate for $\|\cdot\|_{-,A}$, being bounded by $\|f\|_2$, which can be absorbed in the Sobolev norm by increasing C by an absolute factor).

Next we treat $\|\cdot\|_{+,A}$ and assume $s > 1$. Let $\psi_M = \psi_{M,s}$ be a smooth approximation from below to $\psi_{\infty,s}$: it equals $\psi_{\infty,s}$ in a very large ball $B(0, M')$ and then drops slowly (and smoothly) to zero outside $B(0, M)$ for $M \gg M' \gg s$. We will take care so that our bounds do not depend on M, M' in order to take the limit and recover $\psi_{\infty,s}$. From the general construction of ϕ_s^k we conclude that

$$|(1 - \Delta)^a \psi_{M,s}| \leq \chi_{B(0, M') \setminus B(0, s)} + s^{la} \chi_{R(s)} \quad (3.7.6)$$

for any $a \in \mathbb{N}$ and some $l \geq 2$ depending on d . The set $R(s)$ is the inner ‘corona’ of the approximation, i.e. the thin annulus $\text{Ann}(s - \frac{1}{s^2}, s)$.

We have

$$\begin{aligned} \|P_{\psi_{\infty,s}}f\|^2 &= \lim_{M \rightarrow \infty} \|P_{\psi_{M,s}}f\|^2 \\ &= \lim_{M \rightarrow \infty} \int_V \widehat{\psi_M}(v) \kappa(v) d\mu(v) \end{aligned}$$

with κ as before. Using the identity

$$\widehat{\psi_M}(v) = \frac{\mathfrak{F}((1 - \Delta)^a \psi_M)}{(1 + \|v\|^2)^a} \quad (3.7.7)$$

where \mathfrak{F} is the Fourier transform, we create a gain to transfer to the κ term and form a function in $L^1 \cap L^2$. Define

$$\lambda(v) := \frac{\kappa(v)}{(1 + \|v\|^2)^a}$$

for a fixed a large enough to put λ in $L^1 \cap L^2$ (recall that $\kappa(v)$ is smooth and

bounded). We get

$$\begin{aligned}
\int_V \widehat{\psi_M}(v) \kappa(v) d\mu(v) &= \int_V \mathfrak{F}((1 - \Delta)^a \psi_M) \lambda(v) d\mu(v) \\
&= \int_V ((1 - \Delta)^a \psi_M) \mathfrak{F}(\lambda)(v) d\mu(v) \\
&= \int_V ((1 - \Delta)^a \psi_M) \frac{\mathfrak{F}((- \Delta)^\beta(\lambda))(v)}{\|v\|^{2\beta}} d\mu(v)
\end{aligned}$$

Now choose $\beta = \gamma + \delta$ large enough so that

$$\frac{s^{2la}}{\|v\|^{2\delta}} = o_s(1)$$

on the corona (e.g. take $\delta > 2la$) and also renders

$$\frac{\chi_{B(0,M) \setminus B(0,s)}}{\|v\|^{2\delta}} \quad (3.7.8)$$

integrable as $M \rightarrow \infty$ (e.g. take $\delta > d$). The remaining part γ will be our gain. Note that

$$\Delta^\beta \langle v \cdot f, f \rangle = \langle v \cdot (d\sigma((- \Delta)^\beta) \cdot f), f \rangle \quad (3.7.9)$$

since the Laplace operator commutes with translations. But $d\sigma((- \Delta)^\beta) \cdot f$ is again a smooth compactly supported function, so the assumption on matrix coefficients persists (with k crudely replaced by $k + \beta$): the matrix coefficient above is in $L^p(V)$ so by the way a was chosen we deduce that $\mathfrak{F}((- \Delta)^\beta(\lambda))$ is in $L^1 \cap L^2$.

Using the uniform Hölder inequality twice, the inequality

$$\|\mathfrak{F}((- \Delta)^\beta(\lambda))\|_\infty \leq \|(- \Delta)^\beta(\lambda)\|_1$$

and taking (3.7.9) into consideration, we finally get

$$\|P_{\psi_{M,s}} f\|^2 \leq C_d C_{k+\beta} s^{-\gamma} S_{k+\beta}(f). \quad (3.7.10)$$

Since β only depended on d and p , the constants satisfy the requirement set in (3.7.3). Furthermore, there is no dependence on M , so taking limits we obtain the bound for $\|P_{\psi_{\infty,s}} f\|$ bounding the norm $\|\cdot\|_{+,A}$ for $A < \gamma$. \square

3.7.2 The non-archimedean case

The discussion here parallels the above but the details are easier to work out: smooth functions are locally constant and Fourier transforms of balls are easy to compute. We make some simplifying assumptions: $\mathcal{K} = \mathbb{Q}_p$ and G is such that the supremum norm on $V = \mathbb{Q}_p^d$ is K -invariant.

Definition 3.7.3. Let $\|\cdot\|$ be the K -invariant norm on V . We denote by

$$\Phi_k = \chi_{B(0, p^k)}$$

the ball of radius p^k and Haar volume p^{lk} , where the Haar measure on V is normalized to be self-dual with respect to the Fourier transform. We define the operator τ_x on functions on V to be translation by x : $\tau_x(f) = f(x + \cdot)$.

Recall the duality statements from Section 2.4. The fact that makes analysis of smooth vectors in non-archimedean fields easy is the following (see [23] for the proof):

Lemma 3.7.1. *We have the identity*

$$\widehat{\tau_x \Phi_k} = p^{lk} \chi_x \Phi_{-k}.$$

Combined with the fact that smooth vectors are linear combinations of characteristic functions of translated balls, this lemma allows the computation of all Fourier transforms of smooth vectors.

We need another simple but important observation: smooth vectors in the non-archimedean case are *automatically K -finite*. This is because smooth vectors have an open compact stabilizer. Since the stabilizer of a smooth vector f is compact and contains the identity, it intersects K ; since it is open, their intersection is again an open compact subgroup of G . But since it is open, it has finite index in K and therefore K transforms the smooth vector f only by the finite number of cosets of the stabilizer in K .

Now as in the archimedean case we assume that smooth matrix coefficients are in some $L^p(V)$ with p independent of the functions. We have the following estimate:

Proposition 3.7.4. *Let $f \in \mathcal{L}_K$ be smooth with respect to the V action, with matrix coefficients satisfying an estimate of the form*

$$\langle v \cdot f, g \rangle \leq (d_f d_g)^{\frac{1}{2}} \|f\|_2 \|g\|_2 A(v)$$

with $A(v)$ q -integrable for some $q \geq 2$. Write $s = p^k$. Then we have

$$\|P_{(s)} f - f\|_2 \leq (d_f)^{\frac{1}{2}} \|f\|_2 p^{-\frac{lk}{q}}$$

when s is large enough so that f is constant on cosets of Φ_{-k} .

Proof. Write

$$P_s(f) = \int_X \mathfrak{F}(\Phi_k - \Phi_{-k-1}) v \cdot f d\mu$$

which becomes after the lemma above

$$P_s(f) = \int_X p^{lk} \Phi_{-k} v \cdot f d\mu - \int_X p^{-lk} \Phi_{k+1} v \cdot f d\mu.$$

Owing to the local constancy of $v \cdot f$, after $s = p^k$, the first term above becomes equal to $0 \cdot f = f$ and subtracting it we get

$$P_s(f) - f = p^{-lk} \int_X \Phi_{k+1} v \cdot f \, d\mu. \quad (3.7.11)$$

Now we form the inner product of the two sides with an arbitrary smooth, zero mean g (recall that such g are dense in $L_0^2(X)$) and use Hölder's inequality as in the archimedean case to finish the proof. \square

Remark 3.7.5. Using the same procedure as above and the inequality of Bochner integrals

$$\left\| \int_X F(v) \, d\mu(v) \right\|_B \leq \int_X \|F(v)\|_B \, d\mu(v),$$

we can also get the bound

$$\|P_s(f)\|_\infty \leq 2\|f\|_\infty$$

thereby completing the description of the norms from (3.6.2) in the non-archimedean case.

Chapter 4

Homogeneous Spaces and higher rank actions

In this chapter we apply the non-uniform results obtained previously to homogeneous spaces of semi-direct products and of actions of higher rank groups.

4.1 Homogeneous spaces of semidirect products in \mathbb{R}

Here we obtain exponential decay for all sufficiently smooth vectors in the case of an action on a homogeneous space of a semidirect product over an archimedean field (which we may take to be \mathbb{R} , since we will only use the additive structure of V). Note that if an algebraic group G over a non-archimedean field \mathcal{K} contains a lattice (discrete, cofinite subgroup), then it is necessarily *reductive*. In particular, there are no finite volume homogeneous spaces of semidirect products over non-archimedean fields. Thus, we necessarily restrict to the archimedean case, where we first aim to understand how lattices are situated in semidirect products over \mathbb{R} and then proceed to use the structure (especially the Fourier analysis on the compact fiber of the homogeneous space) to improve on our main result.

4.1.1 Lattices in real semidirect products

Let G be the group of real points of a semisimple algebraic group \mathbf{G} , V a real vector space identified with \mathbb{R}^n by a choice of basis fixed once and for all, and a faithful representation $\rho : G \rightarrow \mathrm{SL}(V)$ by means of which we will identify the group in this note with a subgroup of the general linear group of V . The semidirect product

$$\mathfrak{G} = G \ltimes V$$

is a real Lie group and carries a natural Haar measure which is the product of the Haar measures of the factors with a fixed normalization. Fix a lattice Λ in \mathfrak{G} . We aim to understand its structure in terms of lattices in each of the factors and ρ . Let

$$G(\mathbb{Z}) = G \cap \mathrm{SL}(n, \mathbb{Z}),$$

a lattice in G by reduction theory. Denote $L = \Lambda \cap (\{1\} \ltimes V)$ and

$$\Gamma = p_1(\Lambda)$$

where p_1 is the projection onto the left (semisimple) factor, a surjective group homomorphism and continuous map.

Theorem 4.1.1. *With the notation above, we have the following: L is a complete lattice $g\mathbb{Z}^n$ in V for some $g \in GL(V)$, Γ is an arithmetic lattice in G which, after conjugating the group G by g and identifying $L = \mathbb{Z}^n$, becomes a finite index subgroup of $G(\mathbb{Z})$. We have an exact sequence*

$$0 \longrightarrow L \xrightarrow{\iota} \Lambda \xrightarrow{p_1} \Gamma \longrightarrow 0. \quad (4.1.1)$$

More specifically, there is a 1-cocycle $\lambda : \Gamma \rightarrow \mathbb{R}^n/\mathbb{Z}^n$ so that

$$\Lambda = \bigsqcup_{\gamma \in \Gamma} \gamma \ltimes (\lambda(\gamma) + \mathbb{Z}^n). \quad (4.1.2)$$

In general, there is a bijective correspondence between lattices with prescribed (L, Γ) data, where L is a complete lattice in \mathbb{R}^n and

$$\Gamma < G \cap \text{Stab}_{GL(V)}(L),$$

and the group of 1-cocycles $C^1(\Gamma, V/L)$. These lattices are all isomorphic to $\Gamma \ltimes L$ but no lattice corresponding to a non-trivial λ splits as

$$(\Lambda \cap (G \ltimes 0)) \ltimes (\Lambda \cap (1 \ltimes V)).$$

Proof. First of all, L is a discrete additive subgroup of V . If it were not a complete lattice, there would exist a free direction $v \in V$ so that $v\mathbb{Z} \hookrightarrow V/L$ is injective. Then for any sufficiently small neighborhood U of $(1, 0) \in \mathfrak{G}$ the translates $(1, v)\mathbb{Z} + U$ will be disjoint in \mathfrak{G}/Λ , and since they all have the same volume this contradicts the finiteness of the volume of the quotient.

From now on, conjugate G by an appropriate element to transform L into \mathbb{Z}^n . Take $(\gamma, \lambda) \in \Lambda$ and $(1, \lambda') \in 1 \ltimes L$. Conjugating the second element by the first we get

$$(\gamma, \lambda)(1, \lambda')(\gamma, \lambda)^{-1} = (1, \gamma\lambda').$$

Since $1 \ltimes L = \Lambda \cap (1 \ltimes V)$, $\gamma\lambda' \in L$ and since λ' was arbitrary, this implies that $\gamma \in G \cap \text{SL}(n, \mathbb{Z})$ so $\Gamma = p_1(\Lambda) < G(\mathbb{Z})$. Suppose Γ has infinite index in $G(\mathbb{Z})$. The projection p_1 makes \mathfrak{G}/Λ a fiber bundle over G/Γ with compact fiber isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$ and the Haar measure on the total space is the projection of the product measure on the quotient by the discrete subgroup Λ , showing that all fibers have the same measure (recall $G \subset \text{SL}(V)$). But if the index of Γ in $G(\mathbb{Z})$ is infinite, the base of the bundle has infinite measure and since all

fibers have the same measure, the total space has infinite measure, contradicting the cofiniteness of Λ .

The two paragraphs above imply the assertions about L and Γ and exhibit the exact sequence claimed in the theorem statement. Next take an arbitrary $(\gamma, \lambda) \in \Lambda$ and apply it to $1 \times \mathbb{Z}^n$, getting $(\gamma, \lambda + \mathbb{Z}^n) \subset \Lambda$. So for any $(\gamma, \lambda) \in \Lambda$, the entire coset $(\gamma, \lambda + \mathbb{Z}^n)$ is contained in Λ . Discreteness implies that for each level γ , there are only finitely many distinct cosets modulo \mathbb{Z}^n , but we can do better than that as follows: let $a = (\gamma, \lambda), b = (\gamma, \lambda')$ in Λ , and form the element $aba^{-2} \in \Lambda$. Explicitly,

$$aba^{-2} = (1, \gamma(\lambda' - \lambda)) \in \Lambda$$

so as before $\gamma(\lambda' - \lambda) \in \mathbb{Z}^n$ and since γ is invertible, $\lambda' - \lambda \in \mathbb{Z}^n$, showing that the two arbitrary elements at level γ lie in the same coset of \mathbb{Z}^n .

This allows us to define a mapping $\gamma \mapsto \lambda(\gamma)$ for a unique $\lambda(\gamma) \in \mathbb{R}^n/\mathbb{Z}^n$. Now note:

$$(\gamma, \lambda(\gamma) + \mathbb{Z}^n)(\gamma', \lambda(\gamma') + \mathbb{Z}^n) = (\gamma\gamma', \gamma\lambda(\gamma') + \lambda(\gamma) + \mathbb{Z}^n)$$

and by uniqueness of the assignment mod \mathbb{Z}^n , it follows that

$$\gamma\lambda(\gamma') + \lambda(\gamma) = \lambda(\gamma\gamma').$$

This shows that $\gamma \mapsto \lambda(\gamma)$ defines a cocycle with values in $\mathbb{R}^n/\mathbb{Z}^n$.

Different cocycles will give different values on at least some level, which distinguishes individual lattices. Furthermore, every cocycle furnishes a lattice by the assignment (4.1.2). Each such group is a lattice because it is discrete (reduce to an accumulation point on a level and obtain a contradiction) and cofinite (by the same fiber bundle argument as above). Now we show that $\Lambda \simeq \Gamma \ltimes L$. Define the natural map

$$I : \Gamma \ltimes L \rightarrow \Lambda \quad (\gamma, v) \mapsto (\gamma, \lambda(\gamma) + v);$$

it is invertible, one to one and onto and writing down the homomorphism property one sees it is equivalent to the cocycle property for λ . Therefore every Λ is isomorphic to $\Gamma \ltimes L$ by transitivity. \square

Remark 4.1.2. Note that conjugacy of two lattices with fixed (Γ, L) is equivalent to the respective cocycles being cohomologous by an easy computation. Therefore, $H^1(\Gamma; V/L)$ enumerates conjugacy classes of lattices with the prescribed factors.

4.1.2 Decay of multiple correlations for fiberwise zero mean functions

Now that we understand the structure of lattices in real (and complex) semidirect products, we are in position to bound multiple correlations in their homogeneous spaces.

If \mathfrak{G} has a homogeneous action on a finite measure homogeneous space $X = \mathfrak{G}/\Lambda$, then necessarily $\Lambda \cap V$ will be a complete lattice L in V by the previous result. This means that the Hilbert space valued functions $\sigma(\text{id} \times v)f : V \rightarrow L^2(X)$ are L -periodic (in particular, matrix coefficients will never decay, as this action is a translation on a torus and cannot be mixing). We also think of f as $f(g, v)$ by a homeomorphism locally to cosets of $G \times V$ as a topological space. This motivates the following considerations:

Let $\mathcal{H} = L_0^2(X)$ where the subscript now denotes *fiberwise* zero-mean functions, i.e. functions for which

$$\int_{V/L} f(g, v) dv = 0$$

for any g in the base. Let D be the differential operator

$$D = \frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_d}$$

in the universal enveloping algebra of V . Denote by $\sigma_V : V \rightarrow \mathcal{H}$ the restriction $\sigma_V := \sigma|_{\text{id} \times V}$; for $l = 1, 2, \dots$ consider the normed space Π_K^l of all bounded, smooth K -finite vectors f in \mathcal{H} such that $\|D^l(f)\|_{L^2(X)} < \infty$.

Proposition 4.1.1. *For $f \in \Pi_K^l$, $\dim_{\mathbb{R}}(V) = d$ and $l \geq 3$ we have the estimate*

$$\|P_{(s)}(f) - f\|_2 \leq C\sqrt{d} \frac{1}{s^{l-2}} \|D^l(f)\|_{L^2(X)}. \quad (4.1.3)$$

Proof. Without loss of generality assume $L = \mathbb{Z}^d$; a different lattice changes the estimate by a multiplicative constant at worst. We make repeated use of the fiber bundle structure of the space X with V/L as the fiber and for simplicity assume that the cocycle associated to Λ is trivial (or a coboundary, in which case Λ is conjugate to a product lattice).

Now consider the norm of $P_\phi(f)$ where ϕ a Schwartz function supported outside (s) ; recall that $f \in L_0^2(X)$. As the support of ϕ expands in a pointwise increasing manner at infinity, but still outside (s) , $P_{(s)}(f)$ tends in L^2 to the required difference.

Write $\tilde{f}(v) = \sigma_V(v)(f)$. By the Fourier expansion in the v -variable we get the expansion

$$\tilde{f} = \sum_{k \in L} \hat{\tilde{f}}(k) \exp(i\langle k, v \rangle)$$

and by orthogonality and the fiber bundle structure

$$\|f\|_{L^2(X)} = \sum_{k \in L} \|\widehat{f}(k)\|_{L^2(G/\Gamma)}.$$

Taking $L^2(G/\Gamma)$ norms in the identity

$$\widehat{f}(k) = \frac{1}{|k_1 \cdots k_d|^l} \int_{V/L} D^l(\tilde{f}(v)) \exp(i\langle k, v \rangle),$$

we get the estimate

$$\|\widehat{f}(k)\|_2 \leq \frac{1}{|k_1 \cdots k_d|^k} \|D^l(f)\|_{L^2(X)}. \quad (4.1.4)$$

Note that the \tilde{f} disappeared since we the iterated integral results in an integral over the entire bundle to recover f . Here the operator D^l is seen as an element of the universal enveloping algebra of the fiber embedded in the corresponding algebra of X .

Applying the Fourier expansion of \tilde{f} into the $P_\phi(f)$ and interchanging summation and integral, we get

$$P_\phi(f) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) \int_V \widehat{\phi}(v) \exp(i\langle v, k \rangle) d\mu(v)$$

which by Fourier inversion (with the toral Fourier transform properly normalized) becomes

$$P_\phi(f) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) \phi(k).$$

Recall that ϕ is supported outside the annulus (s) . Now we take the L^2 norm and use the triangle inequality; combined with the Fourier coefficient estimate above, we get

$$\|P_\phi(f)\|_2 \leq \sum_{|\min_i(k_i)| > \frac{s}{\sqrt{d}}} \|\widehat{f}(k)\|_2 \leq \frac{C\sqrt{d}}{s^{l-2}} \|D^l(f)\|_{L^2(X)}$$

where C comes from the summable remainder after we split off $l-2$ copies of the $\frac{1}{|k_1 \cdots k_d|^l}$ factor. The small disk around zero that remains does not contain any lattice points other than zero for large s (this restriction on s comes *only from the lattice*), so the only Fourier coefficient captured by ϕ is the zeroth which is zero since $f \in L_0^2(X)$. Note that this bound does not depend on any property of ϕ other than the support, so by taking limits the same bound holds for the required difference, which finishes the proof of the proposition. \square

Combining this proposition with the remarks at the beginning of the section, we get

Corollary 4.1.2. *For $f_i \in \Pi_K^l$, $\|\mathbf{a}^i\| > C'$ with C' independent of f_i , we have the bound*

$$\begin{aligned} & \int_X f_0(x) \mathbf{a}^1 \cdot f_1(x) \cdots \mathbf{a}^k \cdot f_k(x) d\mu(x) \\ & \leq C \prod_{i=0}^k \mathcal{S}(f) d_0^{\frac{1}{2}} d_k^{\frac{1}{2}} \left| \sum_{i=1}^{k-1} \lambda\left(\frac{\mathbf{a}^k}{\mathbf{a}^i}\right) \right|^{-\epsilon(l)} \left| \sum_{i=1}^k \varrho(\mathbf{a}^i)^{-1} \right|^{-\epsilon(l)} \end{aligned} \quad (4.1.5)$$

where the expressions $\mathcal{S}(f)$ are Sobolev norms of an appropriate order depending only on l .

4.2 Higher rank split simple groups

4.2.1 Semidirect products inside simple groups and one parameter groups

In this section we describe how to use the results obtained so far to get effective multiple mixing of correlations of one parameter subgroups (corresponding to coroots of the group) in simple split groups of higher rank.

We do this by locating semidirect products enveloping the one-parameter subgroup in question to which we can apply the main results. We keep notation from previous sections when referring to the functions f_i in the definition of the multiple correlation, the Cartan elements \mathbf{a}^i etc. In this section we will make heavy use of results from [17].

Our setting involves a simple algebraic group split over \mathcal{K} of rank greater than or equal to 2, a maximal \mathcal{K} -split torus D , root system $\Phi = \Phi(G, D)$ and ordering Φ^+ . Consider a mixing action σ of G on a standard probability space (X, μ) . We want to apply the results above to bound multiple correlation coefficients for a given one-parameter group $\sigma(\mathbf{a}(t))$ on X . In order to achieve this, following the proof of Proposition (1.6.2) in [15] we do the following: given the root $\omega \in \Phi^+$ corresponding to \mathbf{a} , we choose another positive root ω' that is not orthogonal to ω . Then from the Dynkin diagram this pair of roots corresponds to either an A_2 system, G_2 system or C_2 system, so we get a surjective morphism

$$\mathrm{SL}(3) \rightarrow \langle U_{\pm\omega}, U_{\pm\omega'} \rangle =: G_{\omega, \omega'},$$

$$\mathrm{Sp}(4) \rightarrow \langle U_{\pm\omega}, U_{\pm\omega'} \rangle =: G_{\omega, \omega'}$$

or

$$G_2 \rightarrow \langle U_{\pm\omega}, U_{\pm\omega'} \rangle =: G_{\omega, \omega'}$$

with finite central kernel. Furthermore, if ω corresponds to the root $\bar{\omega}$ in $G_{\omega, \omega'}$ then its kernel in $D \cap G_{\omega, \omega'}$ corresponds to the kernel in the diagonal $A \subset \mathrm{SL}(3)$ (resp. $A \subset \mathrm{Sp}(4), G_2$) of $\bar{\omega}$. The unipotent groups $U_{\pm\omega}$ along with $D \cap G_{\omega, \omega'}$ generate a copy of $\mathrm{SL}(2)$ which is situated inside the rank 2 group $G_{\omega, \omega'}$ in

one of four ways described by [17, Lemma 3.6]. From that lemma, we see that whenever ω is not conjugate to a long simple root in $\mathrm{Sp}(2n)$ or a short simple root in G_2 , the group $\langle U_{\pm\omega} \rangle$ comes with a linear action on a unipotent abelian group forming a semidirect product of one of the types treated at the end of Section 3.5, plus the symplectic action in the case of G_2 , which we omit for brevity; note that the G_2 system does not appear in any higher rank system so the parameters for that semidirect product are only relevant if our group G is locally isomorphic to G_2 .

Thus we get an isogeny from $\mathrm{SL}(2, \mathcal{K}) \ltimes V$ to its image in $G_{\omega, \omega'}$ with the positive diagonal in $\mathrm{SL}(2)$ going to the one parameter semigroup \mathbf{a} in the positive Weyl chamber of G . We will denote the copy of $\mathrm{SL}(2, \mathcal{K}) \ltimes V$ corresponding to the root ω by $\mathrm{SL}(2, \mathcal{K})_{\omega} \ltimes V_{\omega}$, and from now on any quantity defined in previous sections for semi-direct products subscripted with ω will refer to its definition over $\mathrm{SL}(2, \mathcal{K})_{\omega} \ltimes V_{\omega}$.

Now suppose an $\mathbf{a}(t)$ acts on a K -finite function f . Through the isogeny we get a corresponding action on f of $\mathrm{SL}(2)_{\omega} \ltimes V_{\omega}$ which we denote again simply by $\mathbf{a}(t) \cdot f$, and f retains K -finiteness for the action of the maximal compact subgroup of the $\mathrm{SL}(2)_{\omega}$ part. Thus f affords an action of $\mathrm{SL}(2)_{\omega} \ltimes V_{\omega}$ with no invariant vectors for V_{ω} on $L_0^2(X)$ (mixing descends to subgroups and an isogeny has finite kernel, so we get no invariant vectors for the V_{ω} factor).

Using the reduction above a correlation

$$\int_X f_0((\mathbf{a}(t_0))^{-1} \cdot x) f_1((\mathbf{a}(t_1))^{-1} \cdot x) \cdots f_k((\mathbf{a}(t_k))^{-1} \cdot x) d\mu(x) \quad (4.2.1)$$

of K -finite vectors f_i can be viewed as a correlation for the action of $\mathrm{SL}(2)_{\omega} \ltimes V_{\omega}$. Let \mathcal{D}_{ω} be the space of functions in the image of the projections P_{ϕ} corresponding to V_{ω} . One sees that this space is L^2 -dense by commutativity with the maximal compact subgroup. Applying Theorem 3.4.1, we get

Theorem 4.2.1. *Let G be the group of \mathcal{K} -rational points of a \mathcal{K} -split simple group of \mathcal{K} -rank at least 2. Consider a measure-preserving, mixing action*

$$G \curvearrowright X$$

on a probability space X ; let $\mathbf{a} : \mathcal{K}^0 \rightarrow D^0$ be a one-parameter subgroup of a maximal split torus D corresponding to a non-multipliable root ϕ of G that is not conjugate to a short simple root in the group G_2 or a long simple root in $\mathrm{Sp}(2n)$.

Given a $(k+1)$ -tuple $(f) = (f_0, \dots, f_k)$ in \mathcal{D}_s and $t_0, \dots, t_k \in \overline{\mathcal{K}}$ ordered

in increasing valuations we have

$$\left| \int_X f_0((\mathbf{a}(t_0))^{-1} \cdot x) f_1((\mathbf{a}(t_1))^{-1} \cdot x) \cdots f_k((\mathbf{a}(t_k))^{-1} \cdot x) d\mu(x) \right| \\ \leq C s^{2q} d_0^{\frac{1}{2}} \|f_0\|_2 d_k^{\frac{1}{2}} \|f_k\|_2 \left(\prod_{i=1}^{k-1} \|f_i\|_\infty \right) \left[\left(\sum_{i=1}^k \frac{t_i}{t_0} \right) \left(\sum_{i=0}^{k-1} \frac{t_k}{t_i} \right) \right]^{-\epsilon}$$

Remark 4.2.1. We can obtain similar results for any semisimple G without compact factors and one parameter subgroups corresponding to non-multipliable roots; note that one-parameter subgroups not corresponding to coroots can also be treated by this method provided they are sufficiently close to some root. .

A version of the theorem above had been obtained previously by T-H. Hui for real semisimple groups G and homogeneous $X = G/\Gamma$. In that context there was no restriction on the nature of the one parameter subgroup and the space of functions contained all smooth functions on X and many more; see [22, Chapter 4].

4.2.2 Beyond one parameter subgroups

The previous result requires the acting elements to be confined on a coroot and in general we cannot do better. In this section we indicate an extension and illustrate some of the difficulties in applying it. We finish with an example showing how to handle the extension in special cases.

From the proof of [17, Lemma 5.2], we can decompose D^0 in the archimedean case as

$$D^0 = \ker(\omega) D_\omega^0 \quad (4.2.2)$$

and in the non-archimedean case

$$2D^0 \subset \ker(\omega) D_\omega^0 \quad (4.2.3)$$

where $2D^0 = \{d^2 | d \in D^+\}$ and in both cases D_ω^+ corresponds to the positive diagonal of $\mathrm{SL}(2)$. We will see how to go from $2D^0$ to the full D^0 below.

The image of the $\mathrm{SL}(2)$ in $G_{\omega, \omega'}$ commutes with $\ker(\omega)$ (this follows from the observation that their Lie algebras commute). Therefore, any maximal compact subgroup K_ω of that image commutes with $\ker(\omega)$. This fact plus the K -finiteness of the f_i imply the K_ω -finiteness of the translates of the f_i by elements in $\ker(\omega)$; note that these translates are no longer necessarily K -finite when \mathcal{K} is archimedean. We need the K_ω -finiteness in order for Theorem 3.4.1 to be applicable to the action of $\mathrm{SL}(2, \mathcal{K})_\omega \ltimes V_\omega$.

Now suppose $\mathbf{A} \in D^+$ act on a K -finite function f ; we can write $\mathbf{A} = \mathbf{S}\mathbf{a}\mathbf{C}$ where $\mathbf{a}\mathbf{C} \in 2D^+$ ($\mathbf{S} = 1$ if \mathcal{K} is archimedean), \mathbf{a} being (the image in D^+ of) an element of the diagonal group of $\mathrm{SL}(2)$, \mathbf{C} centralized by the maximal compact of that $\mathrm{SL}(2)$. Then $\mathbf{A} \cdot f = \mathbf{a} \cdot \tilde{f}$ where \tilde{f} is K_ω -finite by the remarks above;

in the non-archimedean case, \mathbf{S} is not necessarily in $\ker(\omega)$ but in this case translates of K -finite vectors are still K -finite: see [17, Lemma 5.6]. Thus the translate \tilde{f} affords an action of $\mathrm{SL}(2)_\omega \ltimes V_\omega$ with no invariant vectors for V_ω on $L_0^2(X)$.

Doing this for all terms in a correlation $\mathbf{A}^i \cdot f_i$ we get K -finite vectors f for an action of $\mathrm{SL}(2, \mathcal{K})_\omega \ltimes V_\omega$. Then, given a correlation

$$\int_X f_0((\mathbf{A}^0)^{-1} \cdot x) f_1((\mathbf{A}^1)^{-1} \cdot x) \cdots f_k((\mathbf{A}^k)^{-1} \cdot x) d\mu(x) \quad (4.2.4)$$

of K -finite vectors f_i , we obtain a correlation

$$\int_X \tilde{f}_0((\mathbf{a}^0)^{-1} \cdot x) \tilde{f}_1((\mathbf{a}^1)^{-1} \cdot x) \cdots \tilde{f}_k((\mathbf{a}^k)^{-1} \cdot x) d\mu(x) \quad (4.2.5)$$

for the Cartan action of $\mathrm{SL}(2, \mathcal{K})_\omega \ltimes V_\omega$. In order to apply the results obtained so far to this context we assume the hypotheses of Theorem 3.4.1. From the reductions above, we need to understand how the hypotheses on the functions f_i are affected when we pass to the \tilde{f}_i . The following lemma is immediate from the discussion above and [17, Lemma 5.6].

Lemma 4.2.2. *The \tilde{f}_i are K_ω -finite functions with*

$$\dim(K_\omega \cdot \tilde{f}_i) \leq \left(\max_{t \in D^0/2D^0} [K_\omega : tK_\omega t^{-1} \cap K_\omega] \right) \dim(K_\omega \cdot f_i)$$

Theorem 4.2.3. *Let G , X as in Theorem 4.2.1. Let \mathbf{A} be a $(k+1)$ -tuple of Cartan elements in a maximal split torus D and Φ its root system; let Φ_+^+ be the subset of positive roots a not locally conjugate to a long simple root in $\mathrm{Sp}(2m)$ or a short root in G_2 .*

There exists an L^2 -dense normed space of functions $\mathcal{D}(G) = \cup \mathcal{D}_s(G)$ so that for every $(k+1)$ -tuple (f_i) in $\mathcal{D}_s(G)$ we have

$$\begin{aligned} & \left| \int_X f_0((\mathbf{A}^0)^{-1} \cdot x) f_1((\mathbf{A}^1)^{-1} \cdot x) \cdots f_k((\mathbf{A}^k)^{-1} \cdot x) d\mu(x) \right| \\ & \leq C s^{2\epsilon} \min_{\phi \in \Phi^{-1}} \left(d_{i_m}^{\frac{1}{2}} \|\tilde{f}_{i_m}\|_2 d_{i_M}^{\frac{1}{2}} \|\tilde{f}_{i_M}\|_2 \prod_{i \neq i_m, i_M} \|\tilde{f}_i\|_\infty \right) \\ & \cdot \left[\left(\sum_{i \neq i_m} \phi \left(\frac{\mathbf{A}^i}{\mathbf{A}^{i_m}} \right) \right) \left(\sum_{i \neq i_M} \phi \left(\frac{\mathbf{A}^{i_M}}{\mathbf{A}^i} \right) \right) \right]^{-\epsilon} \end{aligned}$$

where \mathbf{A}^{i_m} and \mathbf{A}^{i_M} are the elements where each root ϕ takes its largest and respectively smallest value. Here ϵ only depends on the action.

Remark 4.2.2. Unlike the previous decay estimates we obtained, this theorem does not guarantee a non-trivial bound. If the \mathbf{C} are large, the norms of the \tilde{f}_i cannot be controlled and may *beat* the bound. In general we can only guarantee

non-triviality when the \mathbf{C}^i are bounded, i.e. if essentially the \mathbf{A} diverges in the direction of \mathbf{a} . However, in special cases (including, for instance, homogeneous actions of $\mathrm{SL}(n), \mathrm{SO}(n)$ etc.) the contribution from \mathbf{C} is of smaller order of magnitude than the main term and we still get an exponential decay with weaker exponent.

Other results of this nature were recently obtained by Bjorklund, Einsiedler and Gorodnik [3] independently of this work. Their work treats semisimple groups over local fields as well as the adèles and obtains decay estimates for the entire G action under the assumption that a spectral gap exists for the action. Furthermore, the space of functions for which they obtain their results includes all Sobolev vectors and is uniform depending only on the acting elements and Sobolev norms of the test functions.

4.2.3 From $\mathrm{SL}(n, \mathcal{K})$ to $\mathrm{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2$

We illustrate the procedure above in the context of $\mathrm{SL}(n)$. The first step in the derivation of our bound is to drop from $\mathrm{SL}(n, \mathcal{K})$ to the semi-direct product $\mathrm{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2$. In this section, denote by $K(n)$ the maximal compact of $\mathrm{SL}(n)$.

First of all, $\mathrm{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2$ embeds in $\mathrm{SL}(n, \mathcal{K})$ in the following ways (all elements not depicted are zero, 1 on the diagonal).

$$\mathrm{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2 \ni \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$$

In our bound, we have diagonal elements acting. We can extract a single element in the diagonal group of $\text{SL}(2)$ as follows:

$$\mathbf{A} = \begin{pmatrix} \ddots & & & & \\ & a_j & & & \\ & & \ddots & & \\ & & & a_l & \\ & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b^{-1} & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & & & \\ & c & & & \\ & & \ddots & & \\ & & & c & \\ & & & & a_n \end{pmatrix}$$

where b and c are defined in the archimedean case by $c = \sqrt{a_j a_l}$ and $b = \frac{a_j}{c}$; in the non-archimedean case, we have the same definition if the difference of exponents of q is even, otherwise we will compensate by adding and subtracting a

$$\text{diag}(\dots, q^{\frac{1}{2}}, \dots, q^{-\frac{1}{2}}, \dots)$$

to bring the matrix entries back in \mathcal{K} ; since this defect is on a finite set $D^0/2D^0$, the decay is not affected by this tweak.

Note that \mathbf{C} commutes with the specific copy of $K(2)$ inside SL_{jl} so if f is a $K(n)$ -finite function, the new function $\tilde{f} = \mathbf{C} \cdot f$ is $K(2)$ -finite for the action of that SL_{jl} copy.

Therefore, in order to derive estimates on the initial correlation integral, we are led to consider (X, μ) with a mixing action of $G := \text{SL}(2, \mathcal{K}) \ltimes \mathcal{K}^2$ and bound integrals of the form

$$\int_X f_0(x) \mathbf{a}^1 \cdot \tilde{f}_1(x) \cdots \mathbf{a}^k \cdot \tilde{f}_k(x) d\mu(x) \quad (4.2.6)$$

where

$$\mathbf{a}^i = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}$$

and the \tilde{f}_i are bounded, zero mean functions satisfying K -finiteness properties inherited from the original f_i .

Note that in this case, when the \mathbf{A} are in the positive Weyl chamber of $\text{SL}(n, \mathcal{K})$, the elements c defined above are of smaller magnitude than the elements a (uniformly, as long as we stay a uniform distance away from the walls of the Weyl chamber). Now we can apply Theorem 4.2.3 to the correlation and get the corresponding bound, which in the case of 2-correlations and $\mathcal{K} = \mathbb{R}$ corresponds up to a constant depending on (f) to the bound for matrix coefficients in [7, Chapter 5].

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