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SPECTRAL PROBLEMS ON TRIANGLES AND DISKS:
EXTREMIZERS AND GROUND STATES

BY

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DISSERTATION

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Abstract

In this dissertation we study how the energy levels of the Laplacian depend upon the shape of the domain, and identify the ground state energy level of the magnetic Laplacian on the disk. Although we consider these different questions we can find some kind of unity in the sense that we are looking at the eigenvalues (energy levels) of the operator. In the first case, with no magnetic field, we can say a lot about how the eigenvalues depend on the shape of the triangular domain. On the other hand, in the second case, with magnetic field, it is much harder to prove properties of the energy levels. Even identifying what the ground state is for the disk is a challenge.

In the first part of the thesis, we prove that among all triangles of given diameter, the equilateral triangle minimizes the sum of the first n eigenvalues of the Neumann Laplacian, when $n \geq 3$. The result fails for $n = 2$, because the second eigenvalue is known to be minimal for the degenerate acute isosceles triangle (rather than for the equilateral), while the first eigenvalue ($n=1$) is 0 for every triangle. We show the third eigenvalue is minimal for the equilateral triangle.

The second part of the thesis is concerned with the properties of Dirichlet eigenvalues for the magnetic Laplacian on the unit disk. We find an orthonormal basis of eigenfunctions for the magnetic Laplacian on a disk with Dirichlet boundary condition and explicitly identify the ground state of the magnetic Laplacian. Then we have the symmetry of eigenfunctions and eigenvalues with respect to angular momentum and magnetic field strength. Lastly we prove that positive angular momentum gives lower energy and establish the properties of magnetic eigenvalues; As angular momentum increases the energy level goes up, the ground state is radial and positive, and we find asymptotic behavior of the magnetic eigenvalues as the field strength tends to infinity.

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CHAPTER 1 : Introduction

Eigenvalues of the Laplacian represent interesting physical quantities. For example, eigenvalues of the Laplacian represent frequencies of wave motion, rates of decay in diffusion, and energy levels in quantum mechanics. These eigenvalues are closely connected with the geometry of the domain on which the operator is acting. For most domains, even simple polygons, there is no way to compute these eigenvalues precisely. And so, it is quite interesting to give intuition about how the eigenvalues depend on the shape of domain. We can get intuition by solving some extremal problems for these eigenvalues.

In the first part of my thesis, in Chapter 2, we will look at the special case of triangular domains and we will give sharp lower bounds on Neumann eigenvalues. Neumann eigenvalues correspond physically to insulated boundary conditions of heat flow. That part of the thesis is joint work with R. S. Laugesen and Z. C. Pan [28]. Dirichlet eigenvalue sums for triangles were studied by Laugesen and Siudeja [27]. We adapt their Method of the Unknown Trial Function to the Neumann case. The Neumann case throws up unexpected new phenomena. The numerical work I did leads to the discovery that the eigenvalues sums are not always minimal for the equilateral triangle, in the Neumann case. Following are my contributions to the project. I performed the numerical work for plotting the first five nonzero Neumann eigenvalues normalized by diameter ($\mu_j D^2$ for $j = 2, \dots, 6$) on a family of isosceles triangles. These eigenvalues were computed by the PDE Toolbox in Matlab and plotted as a function of the aperture angle α between the two equal sides (see Figure 2.1). Moreover, Pan and I developed explicit bounds of Weyl type on the eigenvalue counting function (see Lemma 2.4.1 and Lemma 2.4.3), and then applied the bounds to prove the “comparison of eigenvalue sums” lemma (see Lemma 2.2.3).

Let me explain why we study such domains and boundary conditions. First, why do we work on triangles? They are a test case for the general domain. The triangular domains are simple enough that we can transform them geometrically while still controlling (or estimating) the eigenvalues. This gives us much better information than we have known for general domains. It also gives us a intuition for general domain. If the minimizer among triangles is the equilateral triangle, which is the most symmetric triangle, then it suggests that among

convex domains of given diameter, the disk maybe minimizes the sum of eigenvalues of the Neumann Laplacian.

Second, why do we consider Neumann boundary conditions? The Dirichlet case has been done already, and provides some motivation. The Neumann case is more difficult, and the result is more interesting. The result of this chapter is that almost all (but not all) Neumann eigenvalue sums on triangles are minimal for equilaterals. The proof is fully rigorous except when $n = 4, 5, 7, 8, 9$. For those values of n , the proof relies on numerical estimation of the eigenvalues μ_2, \dots, μ_9 for one specific isosceles triangle. It is interesting that even though our proof is not fully rigorous for the small eigenvalues, after 10-th eigenvalues we will get this lovely result.

We just discussed the project of a lower bound. Chapter 3 is motivated by a classical upper bound. In 1950, G. Pólya and G. Szegő proved an upper bound on the first eigenvalue: If $f(z)$ is a conformal map of the unit disk D onto a bounded, simply connected plane domain Ω , normalized by requiring, $|f'(0)| = 1$, then $\lambda_1(\Omega) \leq \lambda_1(D)$. Note that the extremal domain is the one that is the most symmetric: the disk. (We consider the equilateral triangle in Chapter 2 and disk in Chapter 3 because of the symmetry idea.) As a first step towards finding a generalization of Pólya and Szegő's result for the Laplacian with the magnetic field, we need to understand the energy levels and mode shapes of the disk. Since no detailed analysis seem to be available in the literature, that is the contribution of the Chapter 3. We find the orthonormal basis (ONB) of eigenfunctions for the magnetic Laplacian on disk with Dirichlet boundary condition. We also develop the symmetry properties of eigenfunctions and eigenvalues with respect to angular momentum and magnetic field strength. The most significant contribution is to explicitly identify the ground state. We show that the ground state is radially symmetric, which is not the case for the Neumann (natural) boundary condition in general. The Neumann Laplacian ground state is constant and so is radially symmetric, but for large magnetic fields the Neumann ground state has angular dependence.

In Chapter 4 we develop formulas for parametric derivatives of the Kummer function. The formula implies a result needed by Laugesen and Siudeja [29] in their work on magnetic eigenvalues for perturbations of the disk..

Finally, in the last part of my thesis in Chapter 5 we will introduce open problems.

CHAPTER 2 : Neumann Eigenvalue Sums on Triangles are (mostly) Minimal for Equilaterals

2.1 Results

Eigenfunctions of the Neumann Laplacian satisfy $-\Delta u = \mu u$ with natural boundary condition $\frac{\partial u}{\partial n} = 0$, and the eigenvalues μ_j satisfy

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \rightarrow \infty.$$

We prove a geometrically sharp lower bound on sums of Neumann eigenvalues on triangular domains, under normalization of the diameter D .

Theorem 2.1.1. *Among all triangular domains of given diameter, the equilateral triangle minimizes the sum of the first n eigenvalues of the Neumann Laplacian, when $n \geq 3$.*

That is, if T is a triangular domain, E is equilateral, and $n \geq 3$, then

$$(\mu_2 + \cdots + \mu_n)D^2|_T \geq (\mu_2 + \cdots + \mu_n)D^2|_E$$

with equality if and only if T is equilateral.

Multiplying the eigenvalues by D^2 makes the quantity scale invariant. Note the eigenvalues of the equilateral triangle are known explicitly (see Section 2.6), so that the lower bound in the theorem is computable.

We prove the theorem in Sections 2.2–2.4. The proof is rigorous except when $n = 4, 5, 7, 8, 9$. For those values of n , the proof relies on numerical estimation of the eigenvalues μ_2, \dots, μ_9 for one specific isosceles triangle. See Proposition 2.2.4 and Table 2.1, below.

Theorem 2.1.1 is *geometrically* sharp, meaning there exists an extremal domain for each n . It is the first sharp lower bound on Neumann eigenvalue sums. (Upper bounds are due to Laugesen and Siudeja [26], under a moment of inertia normalization.) The theorem differs

from the Weyl-type bounds of Kröger [23], which are *asymptotically* sharp as $n \rightarrow \infty$, for each domain.

Two reasons for studying such sums are that the sum represents the energy needed to fill the lowest n quantum states under the Pauli exclusion principle, and that the eigenvalue sum provides a “summability” approach to studying the high eigenvalues (μ_n for large n), which are difficult to study directly.

We concentrate on triangular domains because they are the simplest domains whose eigenvalues cannot be computed explicitly. The “hot spots” conjecture of Jeffrey Rauch [21] about the maximum of the Neumann eigenfunction u_2 remains unsolved on acute triangles, in spite of Bañuelos and Burdzy’s proof for obtuse triangles by coupled Brownian motion [7], [38]. The triangular spectral gap conjecture of Antunes and Freitas [4], which claims that the difference of the first two Dirichlet eigenvalues is minimal for the equilateral, also remains unsolved in spite of considerable effort [30]. (The gap minimizer among convex domains is a degenerate rectangle [3], but that result sheds no light on the conjecture for triangles.) Clearly much remains to be discovered about triangles!

Theorem 2.1.1 fails for the second eigenvalue, $n = 2$, because $\mu_2 D^2$ is minimized not by the equilateral but by the degenerate acute isosceles triangle, as Laugesen and Siudeja showed when finding the optimal Poincaré inequality on triangles [25].

For the third eigenvalue we do prove minimality of the equilateral, in Section 2.5:

Corollary 2.1.2. *Among all triangles of given diameter, μ_3 is minimal for the equilateral triangle. That is, $\mu_3 D^2 \geq 16\pi^2/9$ for all triangular domains, with equality if and only if the triangle is equilateral.*

The fourth eigenvalue is *not* minimal for the equilateral, as shown by the numerical work in Figure 2.1. The minimum appears to occur at the intersection of two eigenvalue branches.

Now let us consider other shapes. Among rectangles of a given diameter, the square does not always minimize the sum of the first n Neumann eigenvalues. For example, by plotting the first 12 eigenvalues as a function of side-ratio, one finds that the square fails to minimize $(\mu_2 + \dots + \mu_n)D^2$ when $n = 5, 6, 7, 10, 11, 12$.

Ellipses behave more agreeably, for each individual eigenvalue, as we prove in Section 2.5:

Proposition 2.1.3. *Among ellipses of given diameter, the disk minimizes each eigenvalue of the Neumann Laplacian. That is, for each $j \geq 2$, the quantity $\mu_j D^2$ is strictly minimal when the ellipse is a disk .*

What about general convex domains? Our result for triangles in Theorem 2.1.1, together with Proposition 2.1.3 for ellipses, suggests that:

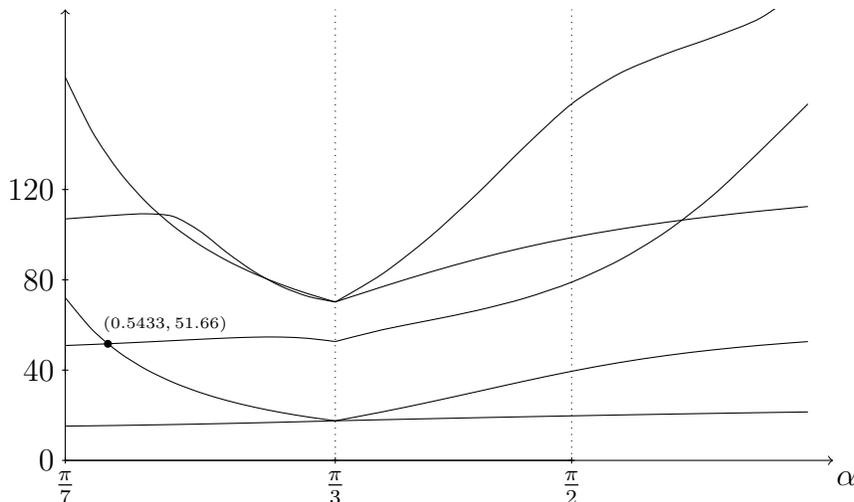


Figure 2.1: Numerical plot of the first five nonzero Neumann eigenvalues normalized by diameter ($\mu_j D^2$ for $j = 2, \dots, 6$) of an isosceles triangle, computed by the PDE Toolbox in Matlab and plotted as a function of the aperture angle α between the two equal sides. The minimum value of $\mu_4 D^2$ is approximately 51.66, occurring at $\alpha \simeq 0.5433$ (to 4 significant figures). The value at the equilateral triangle ($\alpha = \pi/3$) is larger: $\mu_4 D^2 = 3 \cdot 16\pi^2/9 \simeq 52.64$.

Conjecture 2.1.4. *Among convex domains of given diameter, the disk minimizes the sum of the first $n \geq 3$ eigenvalues of the Neumann Laplacian. That is, $(\mu_2 + \dots + \mu_n)D^2$ is minimal when the domain is a disk, for each $n \geq 3$.*

The conjecture fails for $n = 2$, because Payne and Weinberger proved $\mu_2 D^2$ is minimal for the degenerate rectangle (and not the disk) among all convex domains [34]. In other words, they proved that the optimal Poincaré inequality for convex domains is saturated by the degenerate rectangle.

2.1.1 Dirichlet and Robin boundary conditions

Minimality of *Dirichlet* eigenvalue sums for the equilateral, among all triangles of given diameter, was proved recently by Laugesen and Siudeja [27], for each $n \geq 1$. We will adapt their Method of the Unknown Trial Function to the Neumann case. The adaptation breaks down for triangles that are “close to equilateral” when $n = 4, 5, 7, 8, 9$, as we see in the next section. To overcome that obstacle we introduce a new triangle with which to compare, in Proposition 2.2.4. The eigenvalues of this triangle are not known explicitly, which necessitates a numerical evaluation for those exceptional n -values.

Similar results should presumably hold under Robin boundary conditions, although no such results have been proved. The Method of the Unknown Trial Function seems not to work

there, because the boundary integral in the Robin Rayleigh quotient transforms differently from the integrals over the domain, under linear maps.

For more information on isoperimetric-type eigenvalue inequalities in mathematical physics (the general topic of this part of the thesis), see the survey by Ashbaugh [5], and the monographs of Bandle [6], Henrot [18], Kawohl [21], Kesavan [22] and Pólya–Szegő [36].

We thank Bartłomiej Siudeja for suggesting that we investigate Neumann eigenvalues.

2.2 Method of the Unknown Trial Function: the proof of Theorem 2.1.1

Definition 2.2.1. The *aperture* of an isosceles triangle is the angle between its two equal sides. Call a triangle *subequilateral* if it is isosceles with aperture less than $\pi/3$, and *super-equilateral* if it is isosceles with aperture greater than $\pi/3$.

The theorem will be proved in three steps.

Step 1 — Reduction to subequilateral triangles. Suppose the given triangle is not equilateral. We may suppose it is subequilateral, as follows. Stretch the triangle in the direction perpendicular to its longest side, until one of the other two sides has the same length as the longest one. This subequilateral triangle has the same diameter as the original triangle, and has strictly smaller eigenvalue sums by Lemma 2.5.1 later in the paper. (When applying the equality statement of that lemma, notice that a second-or-higher Neumann eigenfunction of a triangle cannot depend only on x , because the boundary condition would force such a function to be constant.)

Thus it suffices to prove the theorem for subequilateral triangles.

Step 2 — Method of the Unknown Trial Function. Write

$$M_n = \mu_2 + \cdots + \mu_n$$

for the sum of the first n eigenvalues (where we may omit μ_1 from the sum because it equals 0). Define

$$T(a, b) = \text{triangle having vertices at } (-1, 0), (1, 0) \text{ and } (a, b),$$

where $a \in \mathbb{R}$ and $b > 0$. The triangle $T(a, b)$ is isosceles if $a = 0$, and subequilateral if in addition $b > \sqrt{3}$. We will prove the theorem for the subequilateral triangle $T(0, b)$ with $b > \sqrt{3}$.

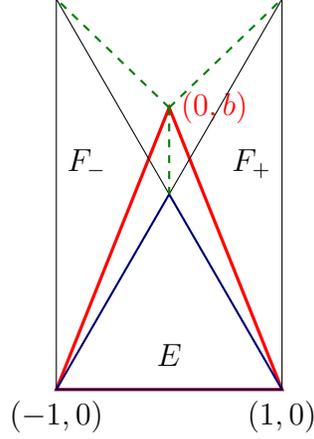


Figure 2.2: Linear maps to the subequilateral triangle $T(0, b)$, from the equilateral triangle E and right triangles F_+, F_- .

Further define three special triangles

$$\begin{aligned} E &= T(0, \sqrt{3}) = \text{equilateral triangle,} \\ F_+ &= T(+1, 2\sqrt{3}) = 30\text{-}60\text{-}90 \text{ right triangle,} \\ F_- &= T(-1, 2\sqrt{3}) = 30\text{-}60\text{-}90 \text{ right triangle.} \end{aligned}$$

The spectra of these triangles are explicitly computable, as we shall need in Step 3 below. Notice F_+ and F_- have the same spectra, by symmetry.

Our method involves transplanting the “unknown” eigenfunctions of the triangle $T(0, b)$ to obtain trial functions for the (known) eigenvalues of the special triangles E, F_+, F_- ; see Figure 2.2. By this technique we will prove:

Proposition 2.2.2. *For each $n \geq 2$:*

(a) *if $b > \sqrt{3}$ then*

$$M_n D^2|_{T(0,b)} > \min\{M_n D^2|_E, \frac{6}{11} M_n D^2|_{F_{\pm}}\};$$

(b) *if $b \geq 2.14$, then a better lower bound holds, namely*

$$M_n D^2|_{T(0,b)} > \min\{M_n D^2|_E, \frac{5}{8} M_n D^2|_{F_{\pm}}\}.$$

The proof is in Section 2.3.

Step 3 — Compare eigenvalues of right and equilateral triangles.

Lemma 2.2.3.

(a) $\frac{6}{11}M_n D^2|_{F_{\pm}} \geq M_n D^2|_E$ for $n = 3, 6$ and each $n \geq 10$.

(b) $\frac{5}{8}M_n D^2|_{F_{\pm}} \geq M_n D^2|_E$ for $n = 4, 5, 7, 8, 9$.

The lemma is proved in Section 2.4. The lemma is certainly plausible, because the Weyl asymptotic ($\mu_j \sim 4\pi j/A$ as $j \rightarrow \infty$) implies that $M_n D^2$ is about twice as large for the half-equilateral F_{\pm} as for the full equilateral E , when n is large.

Proposition 2.2.2 combined with Lemma 2.2.3 proves most of Theorem 2.1.1, by showing $M_n D^2|_{T(0,b)} > M_n D^2|_E$ for most of the needed n -values and subequilateral triangles. The remaining cases, where $n = 4, 5, 7, 8, 9$ and $\sqrt{3} < b < 2.14$, are treated in the next proposition.

Proposition 2.2.4. *The statement*

$$M_n D^2|_{T(0,b)} > M_n D^2|_E, \quad b \in (\sqrt{3}, 2.14),$$

is true when

$$n = 4 \text{ if } M_4 D^2|_G > 90.73,$$

$$n = 5 \text{ if } M_5 D^2|_G > 163.31,$$

$$n = 7 \text{ if } M_7 D^2|_G > 362.90,$$

$$n = 8 \text{ if } M_8 D^2|_G > 489.91,$$

$$n = 9 \text{ if } M_9 D^2|_G > 653.22.$$

Here G denotes the isosceles triangle $T(0, 2.14)$.

The proposition is proved in Section 2.3.

To verify the hypothesis of the proposition for each value of n , see Table 2.1.

n	$M_n D^2 _G$
4	94.59
5	176.73
6	259.48
7	379.58
8	530.54
9	712.65

Table 2.1: Numerical values of the diameter-normalized eigenvalue sum $M_n D^2 = (\mu_2 + \dots + \mu_n) D^2$ for the isosceles triangle $G = T(0, 2.14)$, computed using the PDE Toolbox in Matlab.

2.3 Linear transformation of unknown eigenfunctions: proof of Propositions 2.2.2 and 2.2.4

Write $\mu_j(a, b)$ for the Neumann eigenvalues of the triangle $T(a, b)$, and let the u_j be corresponding orthonormal eigenfunctions. Write

$$M_n(a, b) = \mu_2(a, b) + \cdots + \mu_n(a, b)$$

for the eigenvalue sum.

We need a lemma estimating the change in an eigenvalue sum when the triangle undergoes linear transformation.

Lemma 2.3.1 (Linear transformation and eigenvalue sums). *Let $a, c \in \mathbb{R}$ and $b, d > 0$. Take $C > 0$ and $n \geq 2$. Then the inequality*

$$M_n(a, b) > CM_n(c, d)$$

holds if

$$\frac{1}{d^2} \left[((a - c)^2 + d^2)(1 - \gamma_n) + 2b(a - c)\delta_n + b^2\gamma_n \right] < \frac{1}{C},$$

where

$$\gamma_n = \frac{\sum_{j=1}^n \int_{T(a,b)} u_{j,y}^2 dA}{\sum_{j=1}^n \int_{T(a,b)} |\nabla u_j|^2 dA} \quad \text{and} \quad \delta_n = \frac{\sum_{j=1}^n \int_{T(a,b)} u_{j,x} u_{j,y} dA}{\sum_{j=1}^n \int_{T(a,b)} |\nabla u_j|^2 dA}.$$

The lemma is due to Laugesen and Siudeja [27, Lemma 4.1] for Dirichlet boundary conditions and the Neumann proof is exactly the same, except that we omit the case $n = 1$, in order not to divide by zero in the denominators of γ_1 and δ_1 above.

2.3.1 Proof of Proposition 2.2.2 (a)

Part (a) of Proposition 2.2.2 was proved for Dirichlet boundary conditions by Laugesen and Siudeja [27, Proposition 3.1]. The Neumann argument is identical, but we include it here for the sake of completeness.

The equilateral triangle $E = T(0, \sqrt{3})$ has diameter 2, and the subequilateral triangle $T(0, b)$ has diameter $\sqrt{1 + b^2}$. Observe that the desired inequality

$$M_n D^2 \Big|_{T(0,b)} = M_n(0, b)(1 + b^2) > M_n(0, \sqrt{3})2^2 = M_n D^2 \Big|_E$$

holds by Lemma 2.3.1 with $a = c = 0, d = \sqrt{3}$ and $C = 4/(1 + b^2)$, if

$$(1 - \gamma_n) + \frac{1}{3}b^2\gamma_n < \frac{1 + b^2}{4}.$$

This last inequality is equivalent to $\gamma_n < 3/4$. Thus if $\gamma_n < 3/4$ then the Proposition is proved.

Suppose $\gamma_n \geq 3/4$, and remember $\gamma_n \leq 1$ by definition. Recall the right triangles $F_{\pm} = T(\pm 1, 2\sqrt{3})$, which have diameter 4. Observe that

$$M_n D^2|_{T(0,b)} = M_n(0, b)(1 + b^2) > \frac{6}{11} M_n(\pm 1, 2\sqrt{3})4^2 = \frac{6}{11} M_n D^2|_{F_{\pm}} \quad (2.3.1)$$

holds by Lemma 2.3.1 with $a = 0, c = \pm 1, d = 2\sqrt{3}$ and $C = \frac{6}{11} \frac{4^2}{1+b^2}$, if

$$\frac{1}{12} [13(1 - \gamma_n) \mp 2b\delta_n + b^2\gamma_n] < \frac{11}{6} \frac{1 + b^2}{4^2}.$$

The eigenvalues of F_+ and F_- are the same, and so we need only establish this last inequality for “+” or for “-”. It holds for at least one of these sign choices if

$$\frac{1}{12} [13(1 - \gamma_n) + b^2\gamma_n] < \frac{11}{6} \frac{1 + b^2}{4^2},$$

which is equivalent to

$$b^2 + \frac{50}{11 - 8\gamma_n} > 13.$$

This inequality is true because $b > \sqrt{3}$ and $\gamma_n \geq 3/4$. (Equality holds when $b = \sqrt{3}$ and $\gamma_n = 3/4$.) Hence (2.3.1) holds, which completes the proof.

2.3.2 Proof of Proposition 2.2.2 (b)

Now we prove part (b) of Proposition 2.2.2, by adapting Laugesen and Siudeja’s proof of the Dirichlet case. Assume $b > \sqrt{3}$. The equilateral triangle $E = T(0, \sqrt{3})$ has diameter 2, and the subequilateral triangle $T(0, b)$ has diameter $\sqrt{1 + b^2}$. The inequality

$$M_n D^2|_{T(0,b)} = M_n(0, b)(1 + b^2) > M_n(0, \sqrt{3})2^2 = M_n D^2|_E$$

will hold by Lemma 2.3.1 with $a = c = 0, d = \sqrt{3}$ and $C = 2^2/(1 + b^2)$ if

$$(1 - \gamma_n) + \frac{1}{3}b^2\gamma_n < \frac{1 + b^2}{2^2}.$$

This last inequality is equivalent to $\gamma_n < 3/4$. Thus if $\gamma_n < 3/4$ then part (b) of the Proposition is proved. Assume $\gamma_n \geq 3/4$ from now on.

The triangle $F_{\pm} = T(\pm 1, 2\sqrt{3})$ has diameter 4. The inequality

$$M_n D^2|_{T(0,b)} = M_n(0,b)(1+b^2) > \frac{5}{8} M_n(\pm 1, 2\sqrt{3}) 4^2 = \frac{5}{8} M_n D^2|_{F_{\pm}}$$

will hold by Lemma 2.3.1 with $a = 0, c = \pm 1, d = 2\sqrt{3}$ and $C = \frac{5}{8} \frac{4^2}{1+b^2}$ if

$$\frac{1}{12} [13(1 - \gamma_n) \mp 2b\delta_n + b^2\gamma_n] < \frac{8}{5} \frac{1+b^2}{4^2}.$$

We only need this inequality to hold for one of the choice of “+” or “-”, because F_+ and F_- have the same eigenvalues. Thus it suffices to show

$$\frac{1}{12} [13(1 - \gamma_n) + b^2\gamma_n] < \frac{8}{5} \frac{1+b^2}{4^2},$$

which is equivalent to

$$b^2 > 13 - \frac{19}{6 - 5\gamma_n}.$$

The maximum of the right hand side over all possible values of $\gamma_n \in [\frac{3}{4}, 1]$ is approximately $(2.134)^2$. Thus part (b) certainly holds under the assumption $b \geq 2.14$.

2.3.3 Proof of Proposition 2.2.4

In the previous proof we compared the eigenvalue sums of the subequilateral triangle $T(0, b)$ with those of the right triangles F_{\pm} , by means of the Method of the Unknown Trial Function. Those comparisons proved insufficient when $b < 2.14 = b_*$. So in this current proof we compare with the “endpoint” triangle $T(0, b_*)$. Unfortunately, the eigenvalues of this triangle are not explicitly computable, which explains why certain explicit estimates appear in the hypotheses of this Proposition.

We want to prove $M_n D^2|_{T(0,b)} > M_n D^2|_E$, for $\sqrt{3} < b < b_*$ and $n = 4, 5, 7, 8, 9$. The proof of Proposition 2.2.2 above proves this inequality when $\gamma_n < \frac{3}{4}$. So we assume $\gamma_n \geq \frac{3}{4}$.

Let $K = 0.967$. We will first prove

$$M_n D^2|_{T(0,b)} = M_n(0,b)(b^2 + 1) > K M_n(0, b_*)(b_*^2 + 1) = K M_n D^2|_{T(0,b_*)}. \quad (2.3.2)$$

This inequality holds by Lemma 2.3.1 with $a = c = 0, d = b_*$ and $C = K \frac{b_*^2+1}{b_*^2+1}$ if

$$1 - \gamma_n + \frac{b^2}{b_*^2} \gamma_n < \frac{1}{K} \frac{b^2 + 1}{b_*^2 + 1}. \quad (2.3.3)$$

We must show that this inequality holds for all $\gamma_n \in [\frac{3}{4}, 1]$ and all $b \in (\sqrt{3}, b_*)$. Fixing b temporarily, we see that the left side of inequality (2.3.3) is maximized when $\gamma_n = \frac{3}{4}$. Substituting $\gamma_n = \frac{3}{4}$ and then rearranging, we see it suffices to prove

$$K < \frac{4b_*^2}{3(b_*^2 + 1)} \frac{b^2 + 1}{b^2 + b_*^2/3}$$

for all $b \in (\sqrt{3}, b_*)$. The right side of this new inequality is an increasing function of b , since $b_*^2/3 > 1$. Thus it suffices to check the inequality at $b = \sqrt{3}$; one finds the right side equals approximately 0.9671, which exceeds our chosen value of $K = 0.967$ on the left side. Hence (2.3.2) is proved.

To complete the proof that $M_n D^2|_{T(0,b)} > M_n D^2|_E$, from (2.3.2), it would suffice to know

$$M_n D^2|_{T(0,b_*)} > \frac{1}{K} M_n D^2|_E.$$

The right hand side can be evaluated explicitly (using the eigenvalues of the equilateral triangle E as calculated in the Section 2.6). For $n = 4, 5, 7, 8, 9$ it equals 90.73, 163.31, 362.90, 489.91, 653.22, respectively. (We have rounded each number up in the second decimal place.) These calculations justify the appearance of the five numbers in the hypotheses of the proposition.

2.4 Comparison of eigenvalue sums: proof of

Lemma 2.2.3

Consider the eigenvalue counting function $N(\mu) = \#\{j \geq 0 : \mu_j(E_1) < \mu\}$, where E_1 is an equilateral triangle with sidelength 1. We develop explicit bounds of Weyl type on this counting function, and then apply the bounds to prove Lemma 2.2.3.

Lemma 2.4.1 (Neumann counting function $N(\mu)$). *The counting function satisfies*

$$\frac{\sqrt{3}}{16\pi} \mu + \frac{(6 - \sqrt{3})}{4\pi} \sqrt{\mu} + \frac{3}{2} > N(\mu) > \frac{\sqrt{3}}{16\pi} \mu + \frac{\sqrt{3}}{4\pi} \sqrt{\mu} - \frac{3}{2}, \quad \text{for all } \mu > 48\pi^2.$$

Hence for all $j \geq 26$,

$$\begin{aligned}
& \frac{16\pi}{\sqrt{3}}\left(j - \frac{3}{2}\right) - 8(2\sqrt{3} - 1)\sqrt{\frac{4\pi}{\sqrt{3}}\left(j - \frac{3}{2}\right) + 13 - 4\sqrt{3} + 8(13 - 4\sqrt{3})} \\
& \leq \mu_j(E_1) \\
& \leq \frac{16\pi}{\sqrt{3}}\left(j + \frac{1}{2}\right) - 8\sqrt{\frac{4\pi}{\sqrt{3}}\left(j + \frac{1}{2}\right) + 1 + 8}.
\end{aligned} \tag{2.4.1}$$

Proof of Lemma 2.4.1. The spectrum of the equilateral triangle E_1 under the Neumann Laplacian is well known (see Section 2.6):

$$\sigma_{m,n} = \frac{16\pi^2}{9}(m^2 + mn + n^2), \quad m, n \geq 0.$$

Hence the Neumann counting function equals

$$N(\mu) = \#\{(m, n) : m, n \geq 0, (m^2 + mn + n^2) < R^2\},$$

where $R = 3\sqrt{\mu}/4\pi$. The difference between this formula and the counting function $N_D(\cdot)$ for the Dirichlet eigenvalues is that in the Dirichlet case, m and n must be positive. Therefore by counting pairs (m, n) that have either $m = 0$ or $n = 0$, we can relate the two counting functions as follows:

$$N_D(\mu) + 2R + 1 > N(\mu) > N_D(\mu) + 2(R - 1) + 1, \tag{2.4.2}$$

where the “+1” counts the pair $(0, 0)$ (see Figures 2.3 and 2.4).

We will use some known estimates on the Dirichlet counting function N_D .

Lemma 2.4.2 (Dirichlet counting function N_D [27, Lemma 5.1]). *The counting function satisfies*

$$\frac{\sqrt{3}}{16\pi}\mu - \frac{\sqrt{3}}{4\pi}\sqrt{\mu} + \frac{1}{2} > N_D(\mu) > \frac{\sqrt{3}}{16\pi}\mu - \frac{(6 - \sqrt{3})}{4\pi}\sqrt{\mu} - \frac{1}{2}, \quad \forall \mu > 48\pi^2.$$

Hence for all $j \geq 17$,

$$\begin{aligned}
& \frac{16\pi}{\sqrt{3}} \left(j - \frac{1}{2} \right) + 8\sqrt{\frac{4\pi}{\sqrt{3}} \left(j - \frac{1}{2} \right) + 1} + 8 \\
& \leq \mu_j(E_1) \\
& < \frac{16\pi}{\sqrt{3}} \left(j + \frac{1}{2} \right) + \frac{4}{\sqrt{3}}(6 - \sqrt{3})\sqrt{\frac{16\pi}{\sqrt{3}} \left(j + \frac{1}{2} \right) + 4(13 - 4\sqrt{3}) + 8(13 - 4\sqrt{3})} \\
& < (29.03)j + 9.9\sqrt{29.03j + 39} + 64.
\end{aligned}$$

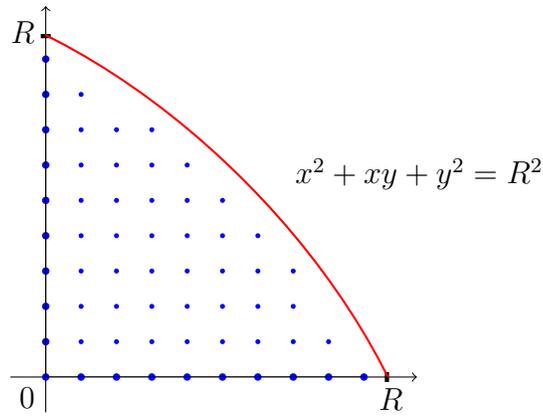


Figure 2.3: The upper bound (2.4.2) on the Neumann counting function $N(\mu)$ equals the number of lattice points inside the ellipse, $N_D(\mu)$, plus the number of lattice points along the axes, which is at most $2R + 1$.

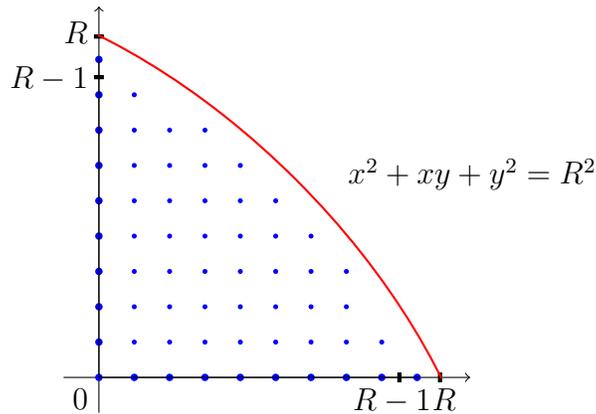


Figure 2.4: The lower bound (2.4.2) on the Neumann counting function $N(\mu)$ equals the number of lattice points inside the ellipse, $N_D(\mu)$, plus the number of lattice points along the axes, which is at least $2(R - 1) + 1$.

Now let us prove the estimates on the Neumann counting function in Lemma 2.4.1. By applying the upper estimate in Lemma 2.4.1 with $\mu = 48\pi^2 + 1$, we find $N(48\pi^2 + 1) < 26$. We conclude that $\mu_j \geq 48\pi^2 + 1$ whenever $j \geq 26$.

Now we deduce the eigenvalue bounds in (2.4.1). We will use a formula for inverting the counting function bounds; see Section 2.7. We know that the upper bound on the counting function is

$$N(\mu) < \frac{\sqrt{3}}{16\pi}\mu + \frac{(6 - \sqrt{3})}{4\pi}\sqrt{\mu} + \frac{3}{2}.$$

By the inversion formula in Section 2.7, we get

$$\mu_j \geq \frac{16\pi}{\sqrt{3}}\left(j - \frac{3}{2}\right) - 8(2\sqrt{3} - 1)\sqrt{\frac{4\pi}{\sqrt{3}}\left(j - \frac{3}{2}\right) + 13 - 4\sqrt{3} + 8(13 - 4\sqrt{3})}, \quad \text{for all } j \geq 26.$$

Similarly, the lower bound on the counting function is

$$N(\mu) > \frac{\sqrt{3}}{16\pi}\mu + \frac{\sqrt{3}}{4\pi}\sqrt{\mu} - \frac{3}{2}.$$

By the inversion formula in Section 2.7, we get

$$\mu_j \leq \frac{16\pi}{\sqrt{3}}\left(j + \frac{1}{2}\right) - 8\sqrt{\frac{4\pi}{\sqrt{3}}\left(j + \frac{1}{2}\right) + 1 + 8}, \quad \text{for all } j \geq 26.$$

□

Let $\mu_j^s(E_1)$ be the j th symmetric eigenvalue of the equilateral triangle E_1 (see Section 2.6), and write $N^s(\mu)$ for the symmetric counting function.

Lemma 2.4.3. *The symmetric counting function satisfies*

$$N^s(\mu) < \frac{\sqrt{3}}{32\pi}\mu + \frac{3}{4\pi}\sqrt{\mu} + \frac{5}{4}, \quad \text{for all } \mu > 48\pi^2.$$

Hence for all $j \geq 15$,

$$\mu_j^s(E_1) \geq \frac{32\pi}{\sqrt{3}}\left(j - \frac{5}{4}\right) - 32\sqrt{2\sqrt{3}\pi\left(j - \frac{5}{4}\right) + 9 + 96}.$$

Proof of Lemma 2.4.3. The symmetric eigenvalues of the equilateral triangle E_1 are

$$\sigma_{m,n} = \frac{16\pi^2}{9}(m^2 + mn + n^2), \quad m \geq n \geq 0,$$

so that

$$N^s(\mu) = \#\{(m, n) : m \geq n \geq 0, (m^2 + mn + n^2) < R^2\},$$

where $R = 3\sqrt{\mu}/4\pi$. Hence by symmetry,

$$2N^s(\mu) \leq N(\mu) + R/\sqrt{3} + 1, \quad (2.4.3)$$

where the term “ $+R/\sqrt{3} + 1$ ” estimates the number of pairs (m, n) with $m = n$. Now the upper bound on $N^s(\mu)$ in the lemma follows from the upper bound on $N(\mu)$ in Lemma 2.4.1.

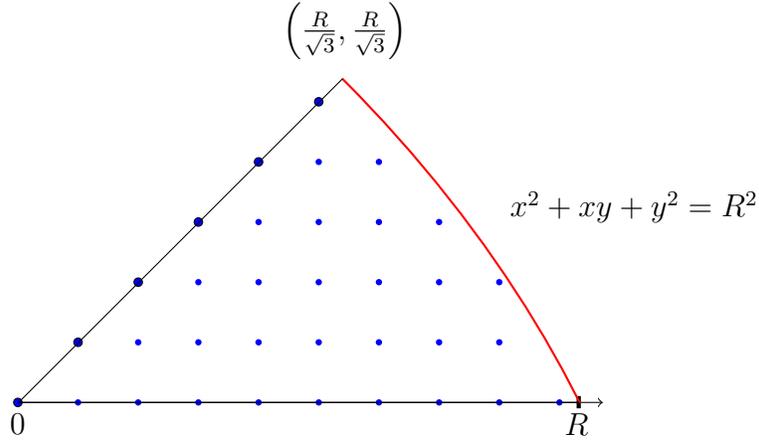


Figure 2.5: The upper bound in (2.4.3) on twice the Neumann symmetric counting function, $2N^s(\mu)$, equals the number of lattice points inside the ellipse, $N(\mu)$, plus the number of pairs (m, n) with $m = n$, which is at most $R/\sqrt{3} + 1$.

Next, by applying the upper estimate in this lemma with $\mu = 48\pi^2 + 1$, we find $N^s(48\pi^2 + 1) < 15$, so that $\mu_j^s \geq 48\pi^2 + 1$ whenever $j \geq 15$.

Start from the upper bound on $N^s(\mu)$.

$$N^s(\mu) < \frac{\sqrt{3}}{32\pi}\mu + \frac{3}{4\pi}\sqrt{\mu} + \frac{5}{4}, \quad \text{for all } \mu > 48\pi^2.$$

By the inverting formula in Section 2.7, we get

$$\mu_j^s \geq \frac{32\pi}{\sqrt{3}}\left(j - \frac{5}{4}\right) - 32\sqrt{2\sqrt{3}\left(j - \frac{5}{4}\right) + 9 + 96}, \quad \text{for all } j \geq 15.$$

Thus the counting function estimate in the Lemma 2.4.3 can be inverted to yield the stated bounds on μ_j^s , for each $j \geq 15$. \square

2.4.1 Proof of Lemma 2.2.3

The right triangle $F_+ = T(1, 2\sqrt{3})$ is half of an equilateral triangle. Thus the Neumann eigenvalues of F_+ are the symmetric eigenvalues of the equilateral triangle (the eigenvalues whose eigenfunctions are symmetric across the bisecting line). Therefore, after rescaling F_\pm to be half of E_1 , we see it suffices to show

$$\frac{M_j^s(E_1)}{M_j(E_1)} \geq \begin{cases} 11/6, & \text{for } j = 3, 6 \text{ and } j \geq 10, \\ 8/5, & \text{for } j = 4, 5, 7, 8, 9, \end{cases}$$

where E_1 is an equilateral triangle with diameter 1. For $j \leq 192$ this desired inequality follows by direct calculation of the eigenvalue sums (Lemma 2.6.1 in Section 2.6).

Next, from the upper bound in Lemma 2.4.1 and the lower bound in Lemma 2.4.3, and an elementary estimate, we find

$$\frac{\mu_j^s}{\mu_j} > \frac{\frac{32\pi}{\sqrt{3}}(j - \frac{5}{4}) - 32\sqrt{2\sqrt{3}\pi}(j - \frac{5}{4}) + 9 + 96}{\frac{16\pi}{\sqrt{3}}(j + \frac{1}{2}) - 8\sqrt{\frac{4\pi}{\sqrt{3}}}(j + \frac{1}{2}) + 1 + 8} > \frac{11}{6}$$

for all $j \geq 193$. Notice that the ratio of leading eigenvalues is $32/16 > 11/6$. Hence the inequality $M_j^s/M_j \geq 11/6$ extends from $j = 192$ to all $j \geq 193$.

2.5 Proof of Corollary 2.1.2 and Proposition 2.1.3

Our results rely on a special kind of domain monotonicity that holds for Neumann eigenvalues even though general domain monotonicity fails in the Neumann situation.

Lemma 2.5.1 (Stretching). *Let Ω be a Lipschitz domain in the plane. For $t > 1$, let $\Omega_t = \{(x, ty) : (x, y) \in \Omega\}$ be the domain obtained by stretching Ω by the factor t in the y direction. Then*

$$\mu_j(\Omega_t) \leq \mu_j(\Omega), \quad j \geq 2.$$

If equality holds for some $j \geq 2$, then there exists a corresponding eigenfunction on Ω that depends only on x .

Proof of Lemma 2.5.1. The eigenvalue problem $-(v_{xx} + v_{yy}) = \mu v$ on Ω_t has Rayleigh quotient

$$R[v] = \frac{\int_{\Omega_t} (v_x^2 + v_y^2) dx dy}{\int_{\Omega_t} v^2 dx dy}.$$

We pull back to Ω by writing $u(x, y) = v(x, ty)$, so that $R[v]$ equals

$$R_t[u] = \frac{\int_{\Omega} (u_x^2 + t^{-2}u_y^2) dx dy}{\int_{\Omega} u^2 dx dy}.$$

This quotient is smaller for $t > 1$ than for $t = 1$, and so $\mu_j(\Omega_t) \leq \mu_j(\Omega)$ by the variational characterization of eigenvalues [6, p. 97].

We prove the equality statement for $j = 2$, and leave the higher values of j to the reader. Suppose $\mu_2(\Omega_t) = \mu_2(\Omega)$. Let u be a second Neumann eigenfunction on Ω . Then u has mean value 0 on Ω , so that v has mean value 0 on Ω_t . Hence v is a valid trial function for $\mu_2(\Omega_2)$, and so

$$\mu_2(\Omega_t) \leq R[v] = R_t[u] \leq R[u] = \mu_2(\Omega).$$

Because equality holds in the second inequality, we conclude that $u_y \equiv 0$. That is, u depends only on x . □

2.5.1 Proof of Corollary 2.1.2

Consider a non-equilateral triangle T . We may assume T is subequilateral, for if not then it can be stretched in the direction perpendicular to its longest side, until one of the other two sides has the same length as the longest one; this subequilateral triangle has smaller μ_3 than the original one, by Lemma 2.5.1, and has the same diameter.

Among subequilateral triangles, $\mu_2 D^2$ is maximal for the equilateral by a result of Lauge-sen and Siudeja [25, Section 6]:

$$\mu_2 D^2|_T < \mu_2 D^2|_E.$$

Furthermore, $(\mu_2 + \mu_3) D^2$ is minimal for the equilateral by Theorem 2.1.1:

$$(\mu_2 + \mu_3) D^2|_T > (\mu_2 + \mu_3) D^2|_E.$$

Subtracting these two inequalities shows for subequilateral triangles that

$$\mu_3 D^2|_T > \mu_3 D^2|_E.$$

2.5.2 Proof of Proposition 2.1.3

Each ellipse can be stretched to a circle of the same diameter. The Neumann eigenvalues strictly decrease under such stretching (for $j \geq 2$), by Lemma 2.5.1.

2.6 Equilateral triangles, rectangles and their eigenvalues

The frequencies of the equilateral triangle were derived roughly 150 years ago by Lamé [24, pp. 131–135]. For our Neumann situation, one can adapt the treatment of the Dirichlet case given by Mathews and Walker’s text [31, pp. 237–239], or in the paper by Pinsky [35]; or else see the exposition of the Neumann case by McCartin [32].

The equilateral triangle E_1 with sidelength 1 has Neumann eigenvalues forming a doubly-indexed sequence:

$$\sigma_{m,n} = (m^2 + mn + n^2) \cdot \frac{16\pi^2}{9}, \quad m, n \geq 0.$$

For example,

$$\begin{aligned} \mu_1 = 0 = \sigma_{0,0}, & & \mu_2 = \mu_3 = 1 \cdot \frac{16\pi^2}{9} = \sigma_{1,0} = \sigma_{0,1}, \\ \mu_4 = 3 \cdot \frac{16\pi^2}{9} = \sigma_{1,1}, & & \mu_5 = \mu_6 = 4 \cdot \frac{16\pi^2}{9} = \sigma_{2,0} = \sigma_{0,2}. \end{aligned}$$

Now consider a line of symmetry of E_1 . Indices with $m > n$ correspond to eigenfunctions that are antisymmetric across that line (see McCartin [32]). Indices with $m \leq n$ correspond to symmetric eigenfunctions. Denote the corresponding “symmetric eigenvalues” by $0 = \mu_1^s < \mu_2^s \leq \mu_3^s \leq \dots$

Lemma 2.6.1. *For $j = 3, 6$, and for $10 \leq j \leq 200$, we have*

$$(\mu_2^s + \dots + \mu_j^s) > \frac{11}{6}(\mu_2 + \dots + \mu_j). \quad (2.6.1)$$

For $j = 4, 5, 7, 8, 9$, we have a weaker inequality,

$$(\mu_2^s + \dots + \mu_j^s) \geq \frac{8}{5}(\mu_2 + \dots + \mu_j), \quad (2.6.2)$$

with equality for $j = 4$ and strict inequality for $j = 5, 7, 8, 9$.

Proof of Lemma 2.6.1. Begin by computing the first 200 eigenvalues μ_j and symmetric eigenvalues μ_j^s , using the indices m and n listed in Table 2.2. Estimates (2.6.1) and (2.6.2) can then easily be checked. As a shortcut for (2.6.1), one can verify that $\mu_j^s > \frac{11}{6}\mu_j$ whenever $28 \leq j \leq 200$, so that (2.6.1) holds for $28 \leq j \leq 200$ as soon as the case $j = 27$ has been checked. \square

[0,0]	[0,1]	[1,1]	[0,2]	[1,2]	[0,3]	[2,2]	[1,3]	[0,4]	[2,3]
(0,0)	(0,1)	(1,0)	(1,1)	(0,2)	(2,0)	(1,2)	(2,1)	(0,3)	(3,0)
[1,4]	[0,5]	[3,3]	[2,4]	[1,5]	[0,6]	[3,4]	[2,5]	[1,6]	[4,4]
(2,2)	(1,3)	(3,1)	(0,4)	(4,0)	(2,3)	(3,2)	(1,4)	(4,1)	(0,5)
[0,7]	[3,5]	[2,6]	[1,7]	[4,5]	[3,6]	[0,8]	[2,7]	[1,8]	[5,5]
(5,0)	(3,3)	(2,4)	(4,2)	(1,5)	(5,1)	(0,6)	(6,0)	(3,4)	(4,3)
[4,6]	[3,7]	[0,9]	[2,8]	[1,9]	[5,6]	[4,7]	[3,8]	[0,10]	[2,9]
(2,5)	(5,2)	(1,6)	(6,1)	(4,4)	(0,7)	(3,5)	(5,3)	(7,0)	(2,6)
[6,6]	[5,7]	[1,10]	[4,8]	[3,9]	[0,11]	[2,10]	[6,7]	[5,8]	[1,11]
(6,2)	(1,7)	(7,1)	(4,5)	(5,4)	(3,6)	(6,3)	(0,8)	(8,0)	(2,7)
[4,9]	[3,10]	[0,12]	[2,11]	[7,7]	[6,8]	[5,9]	[4,10]	[1,12]	[3,11]
(7,2)	(1,8)	(8,1)	(5,5)	(4,6)	(6,4)	(3,7)	(7,3)	(0,9)	(9,0)
[0,13]	[7,8]	[6,9]	[2,12]	[5,10]	[4,11]	[1,13]	[3,12]	[8,8]	[7,9]
(2,8)	(8,2)	(1,9)	(5,6)	(6,5)	(9,1)	(4,7)	(7,4)	(3,8)	(8,3)
[0,14]	[6,10]	[2,13]	[5,11]	[4,12]	[1,14]	[3,13]	[8,9]	[7,10]	[6,11]
(0,10)	(10,0)	(2,9)	(9,2)	(6,6)	(5,7)	(7,5)	(1,10)	(10,1)	(4,8)
[0,15]	[2,14]	[5,12]	[4,13]	[1,15]	[9,9]	[8,10]	[3,14]	[7,11]	[6,12]
(8,4)	(3,9)	(9,3)	(0,11)	(11,0)	(2,10)	(10,2)	(6,7)	(7,6)	(5,8)
[0,16]	[2,15]	[5,13]	[4,14]	[9,10]	[1,16]	[8,11]	[7,12]	[3,15]	[6,13]
(8,5)	(1,11)	(4,9)	(9,4)	(11,1)	(3,10)	(10,3)	(0,12)	(12,0)	(2,11)
[0,17]	[5,14]	[2,16]	[10,10]	[4,15]	[9,11]	[8,12]	[1,17]	[7,13]	[3,16]
(7,7)	(11,2)	(6,8)	(8,6)	(5,9)	(9,5)	(4,10)	(10,4)	(1,12)	(12,1)
[6,14]	[0,18]	[5,15]	[2,17]	[10,11]	[9,12]	[4,16]	[8,13]	[1,18]	[7,14]
(3,11)	(11,3)	(0,13)	(7,8)	(8,7)	(13,0)	(6,9)	(9,6)	(2,12)	(12,2)
[3,17]	[6,15]	[0,19]	[5,16]	[11,11]	[2,18]	[10,12]	[9,13]	[8,14]	[4,17]
(5,10)	(10,5)	(4,11)	(11,4)	(1,13)	(13,1)	(3,12)	(12,3)	(8,8)	(7,9)
[7,15]	[1,19]	[3,18]	[6,16]	[11,12]	[5,17]	[10,13]	[0,20]	[2,19]	[9,14]
(9,7)	(0,14)	(6,10)	(10,6)	(14,0)	(2,13)	(13,2)	(5,11)	(11,5)	(4,12)
[8,15]	[4,18]	[7,16]	[1,20]	[3,19]	[6,17]	[12,12]	[11,13]	[10,14]	[5,18]
(12,4)	(1,14)	(14,1)	(3,13)	(8,9)	(9,8)	(13,3)	(7,10)	(10,7)	(6,11)
[0,21]	[9,15]	[2,20]	[8,16]	[4,19]	[7,17]	[1,21]	[6,18]	[3,20]	[12,13]
(11,6)	(0,15)	(15,0)	(2,14)	(14,2)	(5,12)	(12,5)	(4,13)	(13,4)	(1,15)
[11,14]	[10,15]	[5,19]	[9,16]	[0,22]	[2,21]	[8,17]	[4,20]	[7,18]	[1,22]
(15,1)	(9,9)	(8,10)	(10,8)	(3,14)	(7,11)	(11,7)	(14,3)	(6,12)	(12,6)
[13,13]	[12,14]	[6,19]	[11,15]	[3,21]	[10,16]	[9,17]	[5,20]	[0,23]	[2,22]
(0,16)	(16,0)	(2,15)	(5,13)	(13,5)	(15,2)	(4,14)	(14,4)	(9,10)	(10,9)
[8,18]	[4,21]	[7,19]	[13,14]	[12,15]	[1,23]	[11,16]	[6,20]	[3,22]	[10,17]
(1,16)	(8,11)	(11,8)	(16,1)	(7,12)	(12,7)	(3,15)	(15,3)	(6,13)	(13,16)
[9,18]	[5,21]	[0,24]	[8,19]	[2,23]	[4,22]	[14,14]	[7,20]	[13,15]	[12,16]
(0,17)	(17,0)	(5,14)	(14,5)	(2,16)	(16,2)	(10,10)	(4,15)	(9,11)	(11,9)

Table 2.2: Pairs of integers (m, n) giving the first 200 eigenvalues μ_j along with pairs $[m, n]$ giving the first 200 symmetric eigenvalues μ_j^s , for an equilateral triangle. The index j increases from 1 to 10 across the first row, and so on.

2.7 Inverting counting function bounds

Lemma 2.7.1 (Inverting counting function: upper bound). *Counting function satisfies*

$$N(\mu) < a\mu - b\sqrt{\mu} + c, \quad \text{where } a > 0, b > 0.$$

Hence for all $j \geq c$,

$$\mu_j \geq \frac{j-c}{a} + \frac{b}{a} \sqrt{\frac{(j-c)}{a} + \frac{b^2}{4a^2}} + \frac{b^2}{2a^2}.$$

Proof. Let $j \geq 1$, $\mu = \mu_j + \epsilon$, $\epsilon > 0$. Then

$$\begin{aligned} \mu_j &< \mu \\ j &\leq N(\mu) < a\mu - b\sqrt{\mu} + c \\ j &\leq a\mu_j - b\sqrt{\mu_j} + c && \text{(by letting } \epsilon \rightarrow 0) \\ 0 &\leq a\mu - b\sqrt{\mu} + (c-j) && (2.7.1) \\ \sqrt{\mu_j} &\geq \frac{b + \sqrt{b^2 + 4a(j-c)}}{2a} && \text{if } j \geq c \quad \text{(by quadratic formula)} \\ \mu_j &\geq \frac{b\sqrt{\mu_j} - (c-j)}{a} && \text{(by (2.7.1))} \\ \mu_j &\geq \frac{j-c}{a} + \frac{b}{a} \sqrt{\frac{(j-c)}{a} + \frac{b^2}{4a^2}} + \frac{b^2}{2a^2} && \text{if } j \geq c \end{aligned}$$

□

Lemma 2.7.2 (Inverting the counting function: lower bound). *Counting function satisfies*

$$N(\mu) \geq a\mu - b\sqrt{\mu} + c, \quad \text{where } a > 0, b > 0.$$

Hence for all $j \geq c+1$,

$$\mu_j \leq \frac{j-c}{a} + \frac{b}{a} \sqrt{\frac{(j-c)}{a} + \frac{b^2}{4a^2}} + \frac{b^2}{2a^2}.$$

Proof. Let $j \geq 1$, $\mu = \mu_j$. Then

$$\begin{aligned} j - 1 &\geq N(\mu_j) \geq a\mu - b\sqrt{\mu} + c \\ 0 &\geq a\mu - b\sqrt{\mu} + (c - j + 1) \end{aligned} \tag{2.7.2}$$

$$\sqrt{\mu_j} \leq \frac{b + \sqrt{b^2 + 4a(j - c - 1)}}{2a} \quad \text{if } j \geq c + 1 \quad (\text{by quadratic formula})$$

$$\mu_j \leq \frac{b\sqrt{\mu_j} + (j - c - 1)}{a} \quad (\text{by (2.7.2)})$$

$$\mu_j \leq \frac{j - c - 1}{a} + \frac{b}{a} \sqrt{\frac{(j - c - 1)}{a} + \frac{b^2}{4a^2}} + \frac{b^2}{2a^2} \quad \text{if } j \geq c + 1$$

□

CHAPTER 3 : Harmonic Oscillator and Magnetic Eigenvalues on the Disk, with Dirichlet Boundary Condition

3.1 Introduction

Eigenvalues of the Laplacian represent frequencies of wave motion, rates of decay in diffusion and energy levels in quantum mechanics. While the eigenvalues of the Laplacian have been studied intensively, much less is known about properties of eigenvalues when a transverse magnetic field is introduced. It is known that eigenvalues of the harmonic oscillator and eigenvalues of the magnetic Laplacian are related. By using properties of eigenvalues of the harmonic oscillator we will investigate the properties of eigenvalues for the magnetic Laplacian.

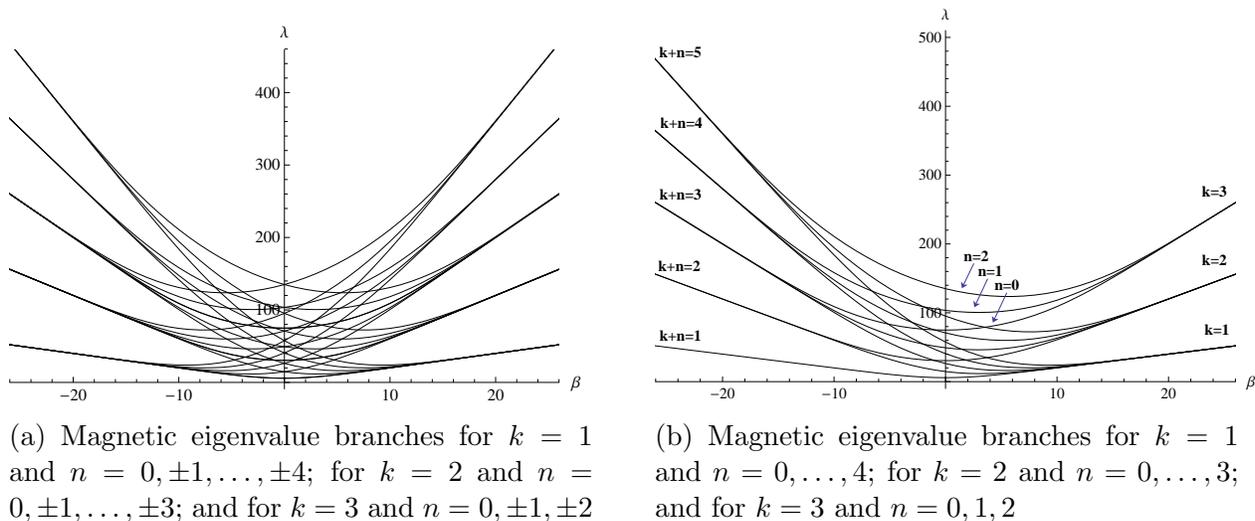


Figure 3.1: Structure of energy levels of the magnetic Laplacian. Here n is angular momentum and k counts oscillations in the radial direction. (See Figure 3.3 and 3.4 later.)

Figure 3.1 shows that there is fairly intricate structure of energy levels. Even in this simple case of the unit disk the way that the energy levels are related to each other is quite complicated. However we can see some patterns and structure. Here are our main results.

First, we find the ONB of eigenfunctions for the magnetic Dirichlet Laplacian, explicitly identify the ground state and the symmetry of eigenfunctions and eigenvalues with respect to angular momentum (denoted by the parameter n) and magnetic field strength (denoted by the parameter β). Secondly, we prove the positive angular momentum n -values give lower energy than $-n$. Lastly, we prove the properties of eigenvalues for the magnetic Laplacian. As angular momentum increases the energy level goes up, the ground state is radial and positive, and asymptotic behavior of the magnetic eigenvalues is such that for large magnetic field strength, each eigenvalue branch has the same slope, $2(2k - 1)$.

The ONB of eigenfunctions for the harmonic oscillator on the disk found by separation of variables are eigenfunctions for the magnetic Laplacian also. Hence we will begin to investigate the properties of eigenvalues for harmonic oscillator.

3.2 Harmonic oscillator eigenvalue problem on disk

First, we need to know what the eigenvalues of 2-dimensional harmonic oscillator are.

Definition 3.2.1 (Eigenvalues of 2-dimensional harmonic oscillator).

The eigenvalue problem for the harmonic oscillator is

$$\begin{cases} -\Delta u + Vu &= Eu & \text{in } \mathbb{D}, \\ u &= 0 & \text{on } \partial\mathbb{D}, \end{cases} \quad (3.2.1)$$

where the eigenfunction $u(x)$ is complex-valued, the quadratic potential is $V(x) = \beta^2(x_1^2 + x_2^2)$, and the eigenvalue E positive.

By using the definition of harmonic oscillator eigenvalue problem we find an ONB of eigenfunctions for harmonic oscillator and we have the following symmetry properties of eigenfunctions and eigenvalues with respect to n and β .

Theorem 3.2.2 (ONB of eigenfunctions for harmonic oscillator).

1. Let $\beta \in \mathbb{R}$. There exists an ONB of smooth eigenfunctions for $L^2(\mathbb{D}; \mathbb{C})$ of the form $R_{n,k,\beta}(r)e^{in\theta}$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$, with corresponding eigenvalues denoted $E(n, k, \beta)$.
2. Symmetry with respect to n, β :

$$\begin{aligned} R_{-n,k,\beta} &= R_{n,k,\beta} & \text{and} & & R_{n,k,-\beta} &= R_{n,k,\beta}, \\ E(-n, k, \beta) &= E(n, k, \beta) & \text{and} & & E(n, k, -\beta) &= E(n, k, \beta). \end{aligned}$$

3. Fix $\beta > 0$. The eigenvalue $E(n, k, \beta)$ is the k -th positive root of $E \mapsto M\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, \beta\right)$, for $E > 0$, where M is the Kummer function.

Because of this symmetry of the eigenfunctions and eigenvalues for the harmonic oscillator, we will consider only $n \geq 0$ from now on, and $\beta \in \mathbb{R}$. Note that the Kummer function, $M(a, b, x)$, is a solution to Kummer's differential equation. We will define it in Section 3.5 (see Definition 3.5.1 below).

Then, we have the properties of eigenvalues for the harmonic oscillator. See Figure 3.2.

Theorem 3.2.3 (Properties of $E(n, k, \beta)$). Assume $n \geq 0$, $k \geq 1$, $\beta \geq 0$.

1. $E(n, k, \beta)$ is increasing with respect to $n \in \mathbb{N} \cup \{0\}$, for fixed k, β .
2. $E(n, k, \beta)$ is increasing with respect to $\beta \geq 0$, for fixed n, k .
3. The lowest eigenvalue comes from $n = 0$ and $k = 1$. That is, the ground state is radial and positive with form $u = R_{0,1,\beta}(r) > 0$ for $r \in [0, 1)$, where $R_{0,1,\beta}(1) = 0$.
4. Asymptotic behavior of $E(n, k, \beta)$:

$$E(n, k, \beta) = 2(2k + n - 1)\beta + o(\beta) \quad \text{as } \beta \rightarrow \infty. \quad (3.2.2)$$

5. The asymptotic is a lower bound for all $n \in \mathbb{Z}$, meaning

$$E(n, 1, \beta) > 2(n + 1)\beta, \quad \text{for all } \beta \geq 0.$$

The last theorem gives the leading order asymptotic behavior of the energy for large coupling parameter β . By graphical evidence the error term should be $o(1)$ since it is much bigger than $o(\beta)$, but the error term is not very precise, which leads to the following questions.

Questions. How fast is the error in (3.2.2) decaying as $\beta \rightarrow \infty$?

Now we connect the harmonic oscillator to the magnetic eigenvalue problem on the disk.

3.3 Magnetic eigenvalue problem on disk, with Dirichlet BC

Let us define the eigenfunctions and eigenvalues of the magnetic Laplacian on the disk.

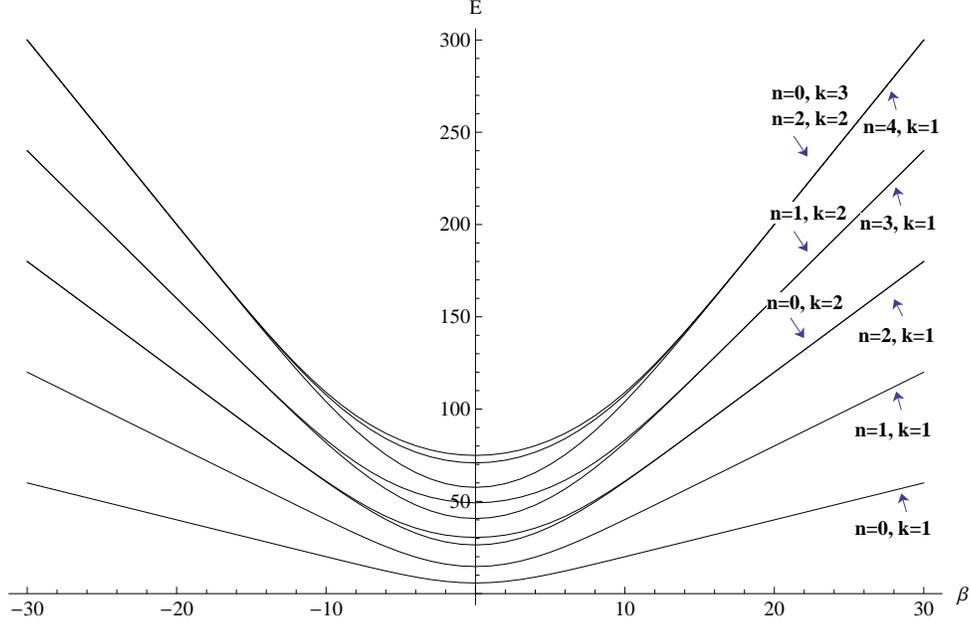


Figure 3.2: Harmonic oscillator eigenvalue branches for $k = 1, 2, 3$ and various small values of $n \in \mathbb{Z}$. The slope as $\beta \rightarrow \infty$ is $2(2k + n - 1)$, by Theorem 3.2.3.

Definition 3.3.1 (Eigenvalues of magnetic Laplacian).

We denote eigenfunctions and eigenvalues of the magnetic Dirichlet Laplacian on the unit disk \mathbb{D} by:

$$\begin{cases} (i\nabla + F)^2 u = \lambda u & \text{in } \mathbb{D}, \\ u = 0 & \text{on } \partial\mathbb{D}, \end{cases} \quad (3.3.1)$$

where $u(x)$ is complex-valued and the vector potential is $F(x) = \beta(-x_2, x_1)$. The magnetic field is $\nabla \times F = (0, 0, 2\beta)$, where $\beta \in \mathbb{R}$ is constant.

We will find the ONB of eigenfunctions for the magnetic Dirichlet Laplacian, and symmetry properties of eigenfunctions and eigenvalues of the magnetic Laplacian. The eigenfunctions still have separated form in polar coordinates, as one expects since the magnetic Laplacian (when written in polar coordinates) commutes with the angular derivative $i\partial/\partial\theta$.

Theorem 3.3.2 (ONB of eigenfunctions for the magnetic Dirichlet Laplacian).

1. Let $\beta \in \mathbb{R}$. There exists an ONB of eigenfunctions for $L^2(\mathbb{D}; \mathbb{C})$ of the form $R_{n,k,\beta}(r)e^{in\theta}$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$, with corresponding eigenvalues denoted $\lambda(n, k, \beta)$.
2. Symmetry with respect to n, β :

$$\lambda(-n, k, \beta) = \lambda(n, k, -\beta),$$

$$R_{-n,k,\beta} = R_{n,k,-\beta}.$$

In fact, $R_{-n,k,\beta} = R_{n,k,\beta}$ and $R_{n,k,-\beta} = R_{n,k,\beta}$.

3. Fix $\beta > 0$. The eigenvalue $\lambda(n, k, \beta)$ is the k -th root of $\lambda \mapsto M\left(\frac{1}{2}\left(1 - \frac{\lambda}{2\beta}\right), n + 1, \beta\right)$, where M is the Kummer function.

4. Changing the sign of n changes the energy by $4n\beta$:

$$\lambda(-n, k, \beta) - \lambda(n, k, \beta) = 4n\beta \quad \text{for all } \beta \in \mathbb{R}.$$

Due to this symmetry of the Dirichlet eigenfunctions and eigenvalues we will consider only $n \geq 0$ from now on, and $\beta \in \mathbb{R}$. Then we have the following corollary.

Corollary 3.3.3 (Positive n -values give lower energy). *Positive n -values give lower energy than negative n -values, when $\beta > 0$:*

$$\lambda(n, k, \beta) < \lambda(-n, k, \beta) \quad \text{for } n \geq 1 \text{ and } \beta > 0.$$

Proof of Corollary 3.3.3. To prove this Corollary we will use Theorem 3.3.2 part (4). If we change the sign of n -values from n to $-n$ the energy changes by $4n\beta$, so obviously, positive n -values give smaller energy than the negative n . Hence we can conclude that positive n -values give lower energy. \square

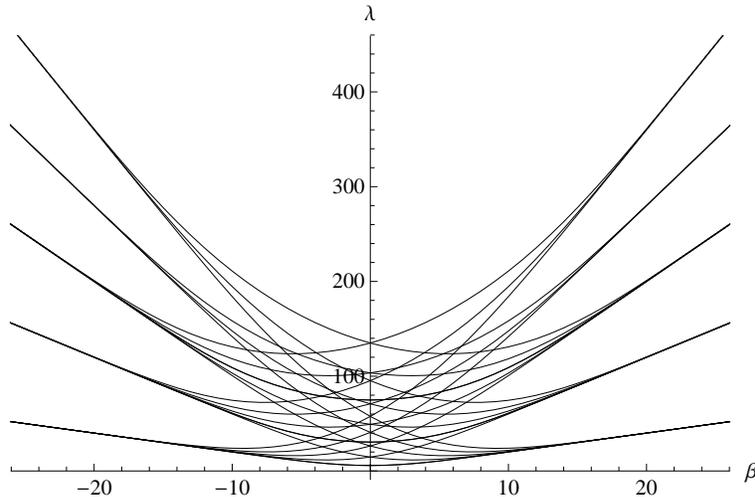


Figure 3.3: Magnetic eigenvalue branches for $k = 1, 2, 3$ and various small values of $n \in \mathbb{Z}$.

Figures 3.3 and Figure 3.4 show how eigenvalue branches depend upon β . Figure 3.3 is complicated. Fortunately by symmetry, we can get Figure 3.3 by reflecting all of the plots from Figure 3.4 across the λ -axis. Hence, we only need to look at $n \geq 0$ (see Figure 3.4).

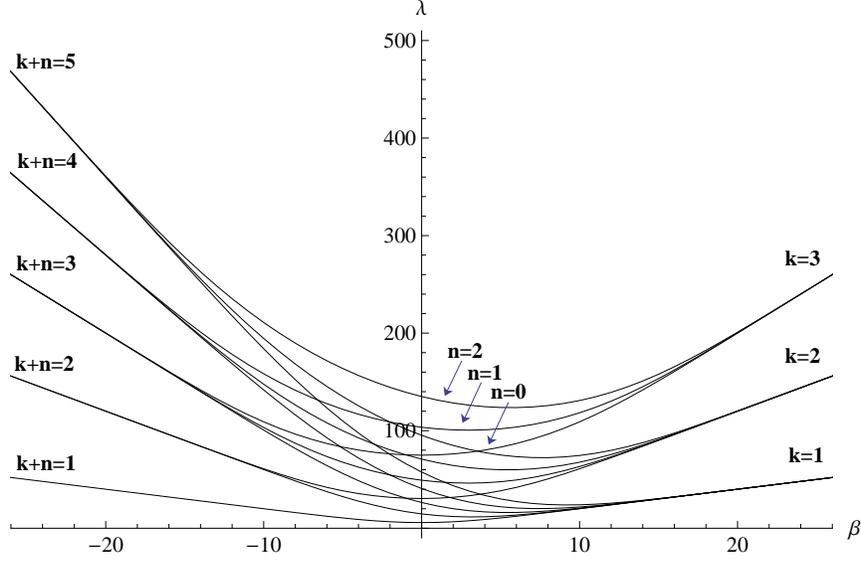


Figure 3.4: Magnetic eigenvalue branches for $k = 1$ and $n = 0, \dots, 4$; for $k = 2$ and $n = 0, \dots, 3$; and for $k = 3$ and $n = 0, 1, 2$.

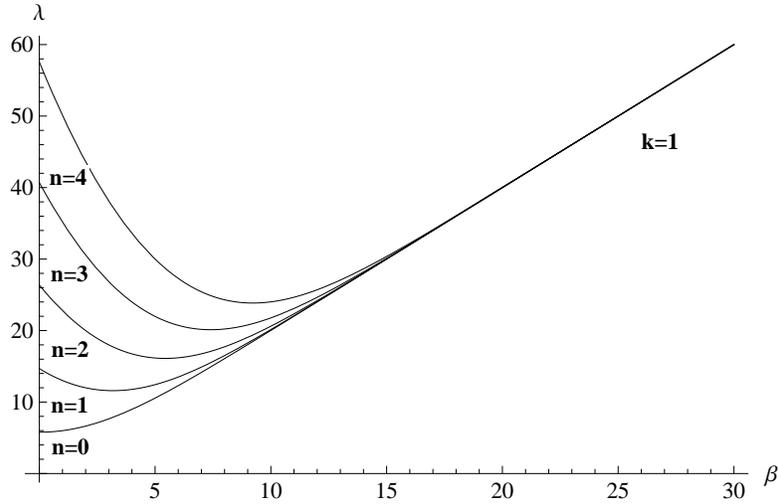


Figure 3.5: Magnetic eigenvalue branches for $k = 1$ and $n = 0, \dots, 4$.

Moreover, from Figure 3.5 we can observe that $n = 0$ gives the lowest eigenvalue, but every eigenvalue with $k = 1$ and $n \in \mathbb{Z}$ is collapsing onto the same curve as $\beta \rightarrow \infty$. So the fact that the lowest eigenvalue comes from $n = 0$ and $k = 1$ is subtle, when β is large.

We will explicitly identify the ground state in the following theorem.

Theorem 3.3.4 (Properties of $\lambda(n, k, \beta)$). *Assume $n \geq 0$.*

1. $\lambda(n, k, \beta)$ is increasing with respect to $n \in \mathbb{N} \cup \{0\}$, for fixed $k \in \mathbb{N}$ and $\beta \geq 0$.
2. The lowest eigenvalue comes from $n = 0$ and $k = 1$. That is the ground state is radial

and positive with form $u = R_{0,1,\beta}(r) > 0$ for $r \in [0, 1)$, where $R_{0,1,\beta}(1) = 0$.

3. *Monotonicity of the ground state energy with respect to field strength:*

$\lambda(0, 1, \beta)$ is an increasing function of $\beta \geq 0$.

4. *Asymptotic behavior of $\lambda(n, k, \beta)$:*

$$\lambda(n, k, \beta) = \begin{cases} 2(2k - 1)\beta + o(\beta) & \text{as } \beta \rightarrow \infty, \\ -2(2k + 2n - 1)\beta + o(\beta) & \text{as } \beta \rightarrow -\infty. \end{cases}$$

5. *The asymptotic is a lower bound for all $n \in \mathbb{Z}$:*

$$\lambda(n, 1, \beta) > 2\beta, \text{ for all } \beta \geq 0.$$

The question of monotonicity of the ground state energy arises in superconductivity, except using Neumann (natural) boundary conditions rather than Dirichlet boundary conditions. For example in the paper of Fournais and Helffer the authors prove that strong diamagnetism holds for sufficiently large magnetic field strength as we now describe.

Remark 3.3.5 (Strong diamagnetism for general domains Theorem 1.1 [15]). *Let β be the strength of the magnetic field, and let $\lambda_1(\beta)$ be the first eigenvalue of the magnetic Neumann Laplacian on a general domain. Then:*

$$\begin{cases} 0 \leq \beta & \Rightarrow & \lambda_1(0) \leq \lambda_1(\beta) & \text{(diamagnetic),} \\ 0 \ll \beta_1 < \beta_2 & \Rightarrow & \lambda_1(\beta_1) \leq \lambda_1(\beta_2) & \text{(strong diamagnetic).} \end{cases}$$

So monotonicity holds for large β , in the Neumann case, on general domains (see [11], [13]). Does it hold for all $\beta > 0$?

3.4 Rescaling and comparing to work of Erdős, and Landau levels

We will compare the asymptotic estimates in Theorem 3.3.4 with the eigenvalue bounds given by Erdős for the disk in his 1995 paper. Consider disk of radius R and the magnetic Laplacian with parameter β . Let $\lambda(n, k, \beta, R)$ be the eigenvalues. By using rescaling we can

find a formula for $\lambda(n, k, \beta, R)$ in terms of the eigenvalues of a disk of radius 1:

$$\lambda(n, k, \beta, R) = \frac{1}{R^2} \lambda(n, k, \beta R^2). \quad (3.4.1)$$

Then we can compare our result with the eigenvalue bounds given by Erdős for the disk. Here are the asymptotic estimates by Erdős.

Theorem 3.4.1 (Asymptotic estimates by Erdős [12, Proposition A.1]).

Consider $\lambda(0, 1, \beta)$, the lowest eigenvalue on unit disk with magnetic vector potential $\beta(-x_2, x_1)$ and field strength 2β . Then the following bounds hold for the rescaled eigenvalue:

$$2\beta + C_1 e^{-2\beta} \leq \lambda(0, 1, \beta) \leq 2\beta + C_2(1 + 4\beta^2)e^{-\beta/4}.$$

Hence in particular, $\lambda(0, 1, \beta) = 2\beta + o(1)$ as $\beta \rightarrow \infty$. This has the same form as the asymptotic estimate in Theorem 3.3.4 for $n = 0, k = 1$, except Erdős has a better error term than the term $o(\beta)$ in that theorem.. Then here comes the question about the error term.

Questions. *What is the “correct” error term for $\lambda(n, k, 2\beta)$? Also, can one find a formula for the next term in the asymptotic estimate?*

Now evaluate the limit of Equation (3.4.1) as the radius of the disk tends to infinity, with β fixed:

$$\begin{aligned} \lim_{R \rightarrow \infty} \lambda(n, k, \beta, R) &= \lim_{R \rightarrow \infty} \frac{1}{R^2} \lambda(n, k, \beta R^2) \\ &= \lim_{R \rightarrow \infty} \frac{1}{R^2} [2(2k - 1)\beta R^2 + o(\beta R^2)] \text{ by Theorem 3.3.4} \\ &= 2(2k - 1)\beta. \end{aligned}$$

The values that we found by letting $R \rightarrow \infty$ for the disk match the Landau levels for the magnetic Laplacian on the whole plane (see [19, Chapter 3]). We have infinite multiplicity because we get the same limit for each value of n . Therefore infinitely many energy levels are collapsing onto one in the limit as the disk expands to the whole plane.

In the next section we will prove the theorems for the harmonic oscillator case.

3.5 Harmonic oscillator in disk: proof of Theorem 3.2.2

The properties of eigenfunctions and eigenvalues for the harmonic oscillator are easier to compute and prove than for the magnetic Laplacian case. Thus, we will prove the theorems

of the harmonic oscillator first.

3.5.1 ONB of eigenfunctions for the harmonic oscillator equation

Start from the harmonic oscillator eigenvalue equation (3.2.1)

$$-\Delta u + \beta^2(x_1^2 + x_2^2)u = Eu.$$

This is the Schrödinger equation with quadratic potential $V(x) = \beta^2(x_1^2 + x_2^2)$ that is smooth and bounded on the disk. Standard spectral theory and elliptic regularity guarantees the existence of an L^2 -ONB of smooth eigenfunctions. We will find an explicit ONB by separation of variables. First, let us use polar coordinates (r, θ) :

$$-(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) + \beta^2 r^2 u = Eu. \quad (3.5.1)$$

For separation of variables, assume $u = R(r)T(\theta)$:

$$-(R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'') + \beta^2 r^2 RT = ERT.$$

Next, multiplying by $\frac{r^2}{RT}$ gives us:

$$-(r^2 \frac{R''}{R} + r \frac{R'}{R}) + \beta^2 r^4 - Er^2 = \frac{T''}{T}.$$

The left side depends only on r and the right side only on t . Hence both sides are constant, and by periodicity in θ we deduce $T(\theta) = e^{in\theta}$ with $n \in \mathbb{Z}$. Therefore (3.5.1) becomes

$$-(R'' + \frac{1}{r}R') + (\frac{n^2}{r^2} + \beta^2 r^2)R = ER. \quad (3.5.2)$$

We want to show that there exists ONB of eigenfunctions with respect to the radial weight, $r dr$, for each n . This follows from Weyl's criterion in Appendix B and Appendix C. The eigenfunctions will be classical solutions by the regularity theory of ODEs. That is the general theory and now we will look at the special function theory. In next section we will evaluate the solution of $R(r)$ in terms of Kummer function.

3.5.2 Finding solutions of harmonic oscillator on the disk

First we need to know the definition of Kummer function.

Definition 3.5.1 (Kummer function). *Kummer's equation is*

$$x \frac{d^2 w}{dx^2} + (b - x) \frac{dw}{dx} - aw = 0 \quad (3.5.3)$$

with a regular singular point at 0 and an irregular singular point at ∞ . It has two linearly independent solutions $M(a, b, x)$ called the Kummer function and $U(a, b, x)$ called the Tricomi function (see [33, Chapter 13]).

In Step 1 we find the eigenfunctions in terms of Kummer function. Fix $n \geq 0$.

Step 1 — Finding eigenfunctions in terms of Kummer function

In order to use this definition, we need to transform the Equation (3.5.2) into the form of Kummer's equation (see [39, Chapter 5]). Assume $n \geq 0$, $\beta > 0$ and define $R_{-n, k, \beta} = R_{n, k, \beta}$ when $n \geq 0$. Let us make the general transformation

$$R(r) = r^n e^{-\beta r^2/2} w(r). \quad (3.5.4)$$

We find that $w(r)$ now satisfies the equation

$$\frac{d^2 w}{dr^2} + \left(\frac{2n+1}{r} - 2\beta r \right) \frac{dw}{dr} - (2(n+1)\beta - E)w = 0. \quad (3.5.5)$$

Let us put $x = \beta r^2$ and use the chain rule:

$$\begin{aligned} \frac{dw}{dr} &= \frac{dw}{dx} \frac{dx}{dr} = 2\beta r \frac{dw}{dx}, \\ \frac{d^2 w}{dr^2} &= \frac{d}{dr} \left(\frac{dw}{dr} \right) = \frac{d}{dr} \left(2\beta r \frac{dw}{dx} \right) = 4\beta^2 r^2 \frac{d^2 w}{dx^2} + 2\beta \frac{dw}{dx}. \end{aligned}$$

So we have from (3.5.5) that

$$x \frac{d^2 w}{dx^2} + (n+1-x) \frac{dw}{dx} - \frac{1}{2} \left(n+1 - \frac{E}{2\beta} \right) w = 0. \quad (3.5.6)$$

A solution of Equation (3.5.6) is $w(x) = M\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, x\right)$.

In fact, there is another solution which is a form of Tricomi's function, $U\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, x\right)$. However this cannot give an eigenfunction because $R(r) = r^n e^{-\beta r^2/2} U\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, \beta r^2\right)$ is not in $L^2((0, 1); r dr)$ as we show now. To verify this we need to know how Tricomi's function behaves at the origin. From Claim 5.3.1 in Chapter 5, the leading order term as a power

of r , for small r , is:

$$U\left(\frac{1}{2}\left(n+1-\frac{E}{2\beta}\right), n+1, \beta r^2\right) = -\frac{\Gamma(n)}{\Gamma\left(\frac{1}{2}\left(n+1-\frac{E}{2\beta}\right)\right)}\beta^{-n}r^{-2n} + o(r^{-2n}), \text{ if } n > 0$$

$$\text{and } U\left(\frac{1}{2}\left(1-\frac{E}{2\beta}\right), 1, \beta r^2\right) = -\frac{2}{\Gamma(a)}\ln r + O(1), \text{ if } n = 0.$$

Then we can find the behavior of $R(r)$ at the origin.

$$R(r) = -\frac{\Gamma(n)}{\Gamma\left(\frac{1}{2}\left(n+1-\frac{E}{2\beta}\right)\right)}\beta^{-n}e^{-\beta r^2/2}r^{-n} + o(r^{-n}), \text{ if } n > 0$$

$$\text{and } R(r) = -\frac{2}{\Gamma(a)}e^{-\beta r^2/2}\ln r + O(1), \text{ if } n = 0.$$

Lastly, we need to check whether $R(r)$ is in $L^2((0, 1); r dr)$ or not. Start from finding the leading order term for the integrand of the integral, $\int_0^1 R(r)^2 r dr$. For $n \geq 1$, the last formula gives

$$R(r)^2 r = (\text{const.})r^{-2n+1} + o(r^{-2n+1}).$$

Notice that r^{-2n+1} is not integrable near $r = 0$ since $-2n+1 \leq -1$. So, $R(r)$ is not in $L^2((0, 1); r dr)$ and hence is not an eigenfunction. The remaining case is when $n = 0$. For $n = 0$,

$$R(r)^2 r = \left(\frac{2}{\Gamma(a)}\right)^2 r(\ln r)^2 + O(r).$$

$R(r)^2 r$ is integrable because $r(\ln r)^2$ is integrable near $r = 0$. However for an eigenfunction we also need to have the condition, $\int_{\mathbb{D}} |\nabla u|^2 dx < \infty$. We can check this condition from the leading order term for the integrand of the integral, $\int_0^1 R'(r)^2 r dr$, which for $n = 0$ we obtain from:

$$R'(r) = -\frac{2}{\Gamma(a)}e^{-\beta r^2/2}\left(-\beta r \ln r + \frac{1}{r}\right) + O(1) = (\text{const.})r^{-1} + O(1),$$

$$R'(r)^2 r = (\text{const.})r^{-1} + O(r),$$

which is not integrable near $r = 0$. So the function $R'(r)$ is not in $L^2((0, 1); r dr)$.

The same arguments will work if we take a linear combination of M and U . Hence we choose $M(a, b, x)$ for the solution of (3.5.6).

Then, we can find a solution, $R(r)$, of the differential equation (3.5.2) which is a form of

Kummer function:

$$R(r) = r^n e^{-\beta r^2/2} M\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, \beta r^2\right). \quad (3.5.7)$$

Since we consider Dirichlet boundary condition, set $R(1) = 0$, which determines the eigenvalue E :

$$R(1) = e^{-\beta/2} M\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, \beta\right) = 0. \quad (3.5.8)$$

The eigenfunctions to the harmonic oscillator are therefore

$$R_{n,k,\beta}(r)e^{in\theta} = r^n e^{-\beta r^2/2} M\left(\frac{1}{2}(n+1 - \frac{E}{2\beta}), n+1, \beta r^2\right) e^{in\theta}. \quad (3.5.9)$$

The corresponding eigenvalues are the numbers $E = E(n, k, \beta)$ satisfying (3.5.8). The infinite increasing sequence, $E(n, k, \beta)$, $k = 1, 2, 3, \dots$, is real and positive by the following lemma.

Lemma 3.5.2. *If $a \geq 0$, $b \geq 0$, $x \geq 0$, then $M(a, b, x) \neq 0$.*

Proof. The number of positive zeros of $M(a, b, x)$ is zero for $a \geq 0$ and $b \geq 0$ by NIST formula 13.9.2 (see [33, Chapter 13]). \square

The lemma implies that the energy levels E that satisfy (3.5.8) all have $a < 0$, that is

$$n+1 - \frac{E}{2\beta} < 0 \quad (\text{i.e., } E > 2(n+1)\beta.)$$

We let $a = \gamma$. So to find the eigenvalues, we need to find all γ -zeros of $M(-\gamma, n+1, \beta)$, with $\gamma > 0$, and then relate γ to E by

$$\gamma = -\frac{1}{2}(n+1 - \frac{E}{2\beta})$$

Still we need to explain why there exists an infinite increasing sequence of γ -zeros, for each fixed $\beta > 0$. We will explain this in our later analysis (see Section 4.3.1).

Then the eigenfunctions $u(x, y)$ are perfectly nice function at the origin by the proof of smoothness below.

Claim 3.5.3 (Smoothness of eigenfunctions). *The eigenfunctions $u = R(r)e^{in\theta}$ are smooth.*

Proof of Claim 3.5.3.

$$\begin{aligned}
u(r, \theta) &= R(r)e^{in\theta} \\
&= M\left(\frac{1}{2}\left(n+1-\frac{E}{2\beta}\right), n+1, \beta r^2\right) e^{-\beta r^2/2} (re^{i\theta})^n \\
u(x, y) &= M\left(\frac{1}{2}\left(n+1-\frac{E}{2\beta}\right), n+1, \beta(x^2+y^2)\right) e^{-\beta(x^2+y^2)/2} (x+iy)^n,
\end{aligned}$$

which is clearly a smooth function of $(x, y) \in \mathbb{R}^2$. □

In Step 2 let us show the orthogonality of eigenfunctions.

Step 2 — Eigenfunctions are orthogonal in $L^2(\mathbb{D}; \mathbb{C})$.

Consider the eigenfunctions for $L^2(\mathbb{D}; \mathbb{C})$ of the form $R_{n,k,\beta}(r)e^{in\theta}$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$. Next, we will justify that these eigenfunctions are orthogonal in $L^2(\mathbb{D}; \mathbb{C})$. This space is endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{D}} u \bar{v} \, dA(x)$$

Let $u = R_{n,k,\beta}(r)e^{in\theta}$ and $v = R_{m,l,\beta}(r)e^{im\theta}$ for $n, m \in \mathbb{Z}$, $k, l \in \mathbb{N}$.

$$\begin{aligned}
\langle u, v \rangle &= \int_0^{2\pi} \int_0^1 (R_{n,k,\beta}(r)e^{in\theta}) \overline{(R_{m,l,\beta}(r)e^{im\theta})} \, r \, dr \, d\theta \\
&= \int_0^1 R_{n,k,\beta}(r) R_{m,l,\beta}(r) \, r \, dr \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta
\end{aligned}$$

Hence if $n \neq m$ then $\langle u, v \rangle = 0$. Also, for $n = m$ and $k \neq l$

$$\langle u, v \rangle = 2\pi \int_0^1 R_{n,k,\beta}(r) R_{n,l,\beta}(r) \, r \, dr = 0, \tag{3.5.10}$$

since the eigenfunctions are orthogonal with respect to the weight function r by the self-adjointness of the Sturm-Liouville operator from Appendix B. Therefore we have proved that the functions $R_{n,k,\beta}(r)e^{in\theta}$ form an orthogonal collection in the space $L^2(\mathbb{D}; \mathbb{C})$.

In Step 3 we will prove the completeness of this family, so that it forms a basis. This is a standard part of the separation of variable argument, but we will give a short proof for the sake of completeness.

Step 3 — Justify the collection of eigenfunctions is complete in $L^2(\mathbb{D}; \mathbb{C})$.

Write $u_{n,k,\beta}(r, \theta) = R_{n,k,\beta}e^{in\theta}$ for the eigenfunctions that we have found for the harmonic oscillator on the disk. We want to show this orthogonal collection of functions is complete in $L^2(\mathbb{D}; \mathbb{C})$. Suppose some function $f \in L^2(\mathbb{D}; \mathbb{C})$ is orthogonal to each eigenfunction in our collection, meaning

$$\langle f, u_{n,k,\beta} \rangle = 0 \quad \text{for all } n \in \mathbb{Z}, k \geq 1.$$

We will show f must be the zero function ($f(x) = 0$ a.e. in \mathbb{D}), which shows that the eigenfunctions span $L^2(\mathbb{D}; \mathbb{C})$.

To show $f = 0$, define

$$f_n(r) = \int_0^{2\pi} f(r, \theta)e^{-in\theta} d\theta \quad \text{for } n \in \mathbb{Z}.$$

Then f_n is orthogonal to each of the $R_{n,k,\beta}$ functions with respect to the measure rdr , because

$$\int_0^1 f_n(r)R_{n,k,\beta}(r) r dr = \langle f, u_{n,k,\beta} \rangle = 0.$$

Since the $\{R_{n,k,\beta}\}$ form an ONB with respect to rdr , we conclude that $f_n(r) = 0$, say for $r \in (0, 1) \setminus E_n$ for some exceptional set E_n of zero measure. Then $E = \bigcup_n E_n$ also has zero measure. So far each $r \in (0, 1) \setminus E$ we have $f_n(r) = 0$ for all $n \in \mathbb{Z}$. That is all the Fourier coefficients of the function $\theta \mapsto f(r, \theta)$ are zero, and hence $f(r, \theta) = 0$ for almost every θ , and so $f = 0$ a.e. on \mathbb{D} . This completes the proof of completeness \square

Lastly, we will prove the symmetry condition and complete the proof of the Theorem 3.2.2.

3.5.3 Symmetry with respect to n, β

Recall the ODE Equation (4.2.5)

$$-(R'' + \frac{1}{r}R') + (\frac{n^2}{r^2} + \beta^2 r^2)R = ER.$$

Since the operator on the left side is the same whether the sign of n and β is positive or negative we define $R_{-n,k,\beta} = R_{n,k,\beta}$ when $n > 0$ and $R_{n,k,-\beta} = R_{n,k,\beta}$ when $\beta > 0$. So the eigenvalues are unchanged also. Hence $E(-n, k, \beta) = E(n, k, \beta)$ and $E(n, k, -\beta) = E(n, k, \beta)$ \square

CHAPTER 4 : Proofs of Spectral Properties of Harmonic Oscillator and Magnetic Laplacian

4.1 Properties of harmonic oscillator in disk: proof of Theorem 3.2.3

In this section we will show two monotonicity properties: the energy level goes up as angular momentum increases and as field strength increases.

4.1.1 Monotonicity of $E(n, k, \beta)$ with respect to $n \geq 0$

We know that the eigenfunction is the form of $R(r)e^{in\theta}$ by separation of variables. Then we will apply the method of Poincaré's minimax characterization of the eigenvalues. Define the Rayleigh quotient of u to be

$$\frac{\int_{\mathbb{D}} (|\nabla u(x)|^2 + \beta^2|x|^2|u(x)|^2) dA(x)}{\int_{\mathbb{D}} |u(x)|^2 dA(x)} \quad (4.1.1)$$

The following remark is the results of the Rayleigh quotient of u when $u(x) = R(r)e^{in\theta}$.

Remark 4.1.1. *Let $u(x) = R(r)e^{in\theta}$. Then we have*

1. $|u(x)|^2 = |R(r)|^2$
2. $|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2}|u_\theta|^2 = |R'(r)|^2 + \frac{n^2}{r^2}|R(r)|^2$
3. $\beta^2|x|^2|u|^2 = \beta^2r^2|R(r)|^2$

Hence for fixed n and β , we can find the k -th eigenvalue $E(n, k, \beta)$:

$$E(n, k, \beta) = \min_S \max_{R \in S} \frac{\int_0^1 \left[|R'(r)|^2 + \left(\frac{n^2}{r^2} + \beta^2r^2\right) |R(r)|^2 \right] r dr}{\int_0^1 |R(r)|^2 r dr}, \quad (4.1.2)$$

where a space of functions, S is an arbitrary k -dimensional subspace of K . Here we consider two infinite dimensional spaces \mathcal{H} and \mathcal{K} (see Appendix B).

$$\mathcal{H} = L^2((0, 1); r dr),$$

$$\mathcal{K} = \left\{ R \in W_{loc}^{1,2}(0, 1) : R' \in L^2((0, 1); r dr), R \in L^2((0, 1); \left(\frac{n^2}{r^2} + \beta^2 r^2 + 1\right)r dr), R(1) = 0 \right\},$$

Notice that the Rayleigh quotient satisfied the requirements for the Spectral theorem (see Appendix C). By Lemma B.2 we can justify the proof of the monotonicity of $E(n, k, \beta)$ with respect to n . The point is that the coefficient functions in formula (4.1.2) go up when n goes up, and so the hypotheses of Lemma B.2 are satisfied. Therefore we conclude that k -th eigenvalue $E(n, k, \beta)$ is increasing with respect to $n \geq 0$.

4.1.2 Monotonicity of $E(n, k, \beta)$ with respect to $\beta \geq 0$

By the similar argument like the previous Section 4.1.1, for fixed n , the k -th eigenvalue for the R -function is increasing with respect to β .

4.1.3 Lowest eigenvalue comes from $n = 0$ and $k = 1$

This is immediate result from monotonicity with respect to n .

To finish the proof of Theorem 3.2.3 we need to show the asymptotic behavior of harmonic oscillator eigenvalues, as $\beta \rightarrow \infty$. We will do that in Section 4.3.6.

We have just finished proving all theorems for harmonic oscillator. In next section we will prove theorems for magnetic Laplacian.

4.2 Magnetic Laplacian in disk: proof of Theorem

3.3.2

4.2.1 ONB of eigenfunctions for the magnetic Dirichlet Laplacian

Consider the Dirichlet conditions for the magnetic eigenfunction equation.

$$(i\nabla + F)^2 u = \lambda u, \tag{4.2.1}$$

where the vector potential $F(x) = \beta(-x_2, x_1)$ creates the magnetic field $\nabla \times F = (0, 0, 2\beta)$. Start from the Equation (4.2.1) and we get:

$$-\Delta u + 2i\nabla u \cdot F + |F|^2 u = \lambda u.$$

Notice that $\nabla u \cdot F = \beta u_\theta$ and $|F|^2 = \beta^2(x_1^2 + x_2^2) = \beta^2 r^2$, then

$$-\Delta u + 2i\beta u_\theta + \beta^2 r^2 u = \lambda u.$$

When we use polar coordinates (r, θ) we can get:

$$-(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) + 2i\beta u_\theta + \beta^2 r^2 u = \lambda u. \quad (4.2.2)$$

Here separation of variables does not work for the following reason. Assume $u = R(r)T(\theta)$:

$$-(R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'') + 2i\beta RT' + \beta^2 r^2 RT = \lambda RT.$$

Next, multiplying by $\frac{r^2}{RT}$ gives us:

$$\begin{aligned} -(r^2 \frac{R''}{R} + r \frac{R'}{R} + \frac{T''}{T}) + 2i\beta r^2 \frac{T'}{T} + \beta^2 r^4 &= \lambda r^2. \\ -(r^2 \frac{R''}{R} + r \frac{R'}{R}) + \beta^2 r^4 - \lambda r^2 &= \frac{T''}{T} - 2i\beta r^2 \frac{T'}{T}. \end{aligned} \quad (4.2.3)$$

Then we can see that the factor of r^2 in the Equation (4.2.3) prevents us from separating the r and θ variables. Instead, to solve Equation (4.2.2) we simply assume trigonometric dependence on θ , that is, we assume the form $u = R(r)e^{in\theta}$. So we can get the ODE:

$$-(R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R) + \beta^2 r^2 R = (\lambda + 2n\beta)R. \quad (4.2.4)$$

This equation relates energies of the magnetic Laplacian to harmonic oscillator energies. By letting $E = \lambda + 2n\beta$ where E is the energy level for the harmonic oscillator, we get:

$$-(R'' + \frac{1}{r}R') + (\frac{n^2}{r^2} + \beta^2 r^2)R = ER. \quad (4.2.5)$$

Notice that the LHS of the ODE in Equation (4.2.4) is the same as the ODE for eigenfunctions of 2-dimensional harmonic oscillator in polar coordinates. Since separation of variables for the harmonic oscillator on the disk does provide an L^2 -ONB of eigenfunctions of the form

$R(r)e^{in\theta}$, we conclude that these same eigenfunctions work for the magnetic Laplacian on the disk, except with the energy shifted by $2n\beta$, as in Equation (4.2.4). Note that n can be either positive or negative, so we can shift energy up or down. Moreover, since $\lambda = E - 2n\beta$, for $n > 0$ and $\beta > 0$ the energy shifts down and for $n < 0$ and $\beta > 0$ the energy shifts up.

4.2.2 Symmetry with respect to n, β

By using Theorem 3.2.2, for the formulas for R , we have

$$R_{-n,k,\beta} = R_{n,k,\beta} = R_{n,k,-\beta}.$$

So $R_{-n,k,\beta} = R_{n,k,-\beta}$.

For the formulas for λ , we have

$$\lambda(n, k, \beta) = E(n, k, \beta) - 2n\beta, \tag{4.2.6}$$

and E is even with respect to the n and β variables by Theorem 3.2.2. So $\lambda(-n, k, \beta) = \lambda(n, k, -\beta)$.

4.2.3 Changing the sign of n changes the energy by $4n\beta$

By (4.2.6) and evenness of E , we see $\lambda(-n, k, \beta) - \lambda(n, k, \beta) = 4n\beta$.

4.3 Properties of magnetic Laplacian in disk: proof of Theorem 3.3.4

Our goal is to show that the monotonicity of eigenvalues with respect to the n value (i.e., as angular momentum, n , increases the energy level goes up.) This is hard to prove directly, so we will start from the monotonicity of β -zeros of $M(-\gamma, n+1, \beta)$ with respect to $n \in \mathbb{Z}$ because this follows from a known interlacing property. Then we will deduce the monotonicity of γ -zeros of $M(-\gamma, n+1, \beta)$ with respect to n by using the properties of β -zeros and γ -zeros. Lastly, we will get the monotonicity of eigenvalues with respect to n from the monotonicity of γ -zeros.

4.3.1 Monotonicity of β -zeros of $M(-\gamma, n + 1, \beta)$ with respect to $n \in \mathbb{Z}$

In this section we are going to find that if we fix n and γ there are finitely many β -zeros and those zeros are interlaced; if we increase the n -value then the zeros interlace with the zeros of the previous n -value. Let us consider the Kummer function $M(-\gamma, n + 1, \beta)$ and denote $\beta(n, k, \gamma)$ for β -zeros of $M(-\gamma, n + 1, \beta)$, with $\gamma > 0$ and $n \geq 0$ fixed. Figure 4.1 shows a few of these functions, for $n = 0$. Notice that the numbers of β -zeros increases as γ -increases.

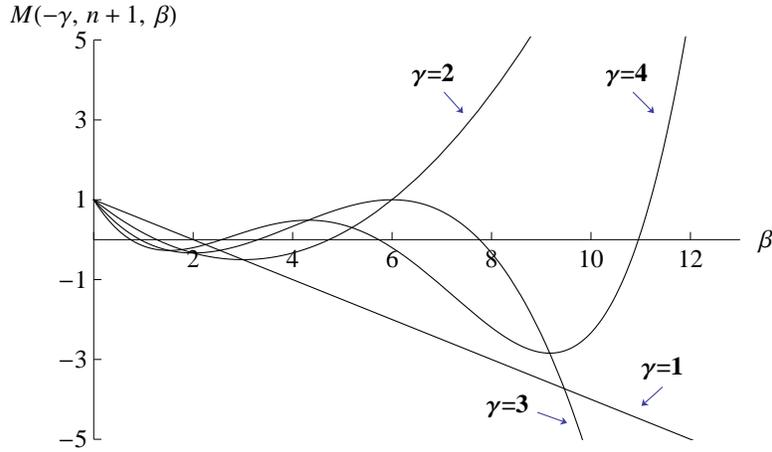


Figure 4.1: Kummer function $M(-\gamma, n + 1, \beta)$ for $n = 0$ and $\gamma = 1, 2, 3, 4$

Remark 4.3.1 (Number of β -zeros of $M(-\gamma, n + 1, \beta)$ [33, Chapter 13]).

When $\gamma, n \in \mathbb{R}$ the number of real zeros is finite. Let $p(-\gamma, n + 1)$ be the number of positive β -zeros of $M(-\gamma, n + 1, \beta)$. Then for $\gamma > 0, n + 1 \geq 0$,

$$p(-\gamma, n + 1) = \lceil \gamma \rceil,$$

$$p - 1 < \gamma \leq p.$$

First, here is the definition of interlacing we are going to use.

Definition 4.3.2 (Interlacing of the zeros of functions).

Consider a family of functions X_n all with same number p of positive zeros. Let $x_{n,k}$ where $k = 1, 2, \dots, p$ denote the positive zeros of X_n , arranged in increasing order. Then interlacing of the zeros of functions X_n and X_{n+1} means that:

$$0 < x_{n,1} < x_{n+1,1} < x_{n,2} < x_{n+1,2} < \dots < x_{n,p} < x_{n+1,p}.$$

By using Segura's on global interlacing properties for Laguerre functions (see [37, p. 397]), we can deduce the interlacing properties for Kummer functions.

Theorem 4.3.3 (Global interlacing properties for Kummer functions).

Let $\gamma, \gamma' > 0$ and $n, n' \geq 0$. The β -zeros of the Kummer functions $M(-\gamma, n+1, \beta)$ and $M(-\gamma', n'+1, \beta)$ interlace in $(0, \infty)$ when the differences $\delta\gamma \stackrel{\text{def.}}{=} \gamma' - \gamma \in \mathbb{Z}$ and $\delta n \stackrel{\text{def.}}{=} n' - n \in \mathbb{Z}$ satisfy:

1. $|\delta\gamma| \leq 1$
2. $|\delta\gamma + \delta n| \leq 1$

Proof of Theorem 4.3.3. To prove this theorem we relate to the known properties of the Laguerre functions.

Remark 4.3.4 (Relation between Kummer functions and Laguerre functions [37, p. 397]).

$$L_{\gamma}^n(\beta) = \frac{(\gamma+1)_n}{n!} M(-\gamma, n+1, \beta) = \frac{(\gamma+1) \cdots (\gamma+n)}{n!} M(-\gamma, n+1, \beta).$$

Now the theorem follows from the Global interlacing properties for Laguerre functions (see [37, p. 397]). \square

The Monotonicity result follows.

Corollary 4.3.5 (Monotonicity of $\beta(n, k, \gamma)$ with respect to n).

Fix $\gamma > 0$, $n \geq 0$. Let $p = \lceil \gamma \rceil$. Let $\{\beta(n, k, \gamma)\}_{k=1}^p$ and $\{\beta(n+1, k, \gamma)\}_{k=1}^p$ be the positive β -zeros of $M(-\gamma, n+1, \beta)$ and $M(-\gamma, n+2, \beta)$, respectively. Then $\beta(n, k, \gamma)$ and $\beta(n+1, k, \gamma)$ interlace in $(0, \infty)$, with

$$\beta(n, 1, \gamma) < \beta(n+1, 1, \gamma) < \beta(n, 2, \gamma) < \beta(n+1, 2, \gamma) < \cdots$$

Therefore $\beta(n, k, \gamma)$ increases as n increases, with k and γ fixed.

Proof of Corollary 4.3.5. Let $\gamma' = \gamma$ and $n' = n+1$. The differences $\delta\gamma = 0 \in \mathbb{Z}$ and $\delta n = n' - n = 1 \in \mathbb{Z}$ satisfy:

1. $|\delta\gamma| = 0 \leq 1$
2. $|\delta\gamma + \delta n| = 1 \leq 1$

So Theorem 4.3.3 applies, which gives interlacing of the β -zeros. We still have to show that the “ n ”-zeros come before the “ $n+1$ ”-zeros. That is, we have to show $\beta(n, 1, \gamma) <$

$\beta(n+1, 1, \gamma)$. Let $\beta_n = \beta(n, 1, \gamma), \beta_{n+1} = \beta(n+1, 1, \gamma) > 0$. We know that these β -zeros exist by Remark 4.3.1. First note that M has initial conditions $M(-\gamma, n+1, 0) = 1$ and $M'(-\gamma, n+1, 0) < 0$. Note $u(\beta) \stackrel{\text{def.}}{=} M(-\gamma, n+1, \beta)$ satisfies the Kummer equation, which can be rewritten as

$$0 = \frac{d}{d\beta}[\beta^{n+1}e^{-\beta}u'] + \gamma\beta^n e^{-\beta}u = \frac{d}{d\beta}[Pu'] + Qu,$$

say. Notice that $P, Q > 0$ for $\beta > 0$, with $P(0) = 0$. So,

$$\frac{d}{d\beta}[Pu'] = -Qu < 0 \quad \text{for } \beta \in (0, \beta_n).$$

That means that Pu' decreases from $\beta = 0$ until $\beta = \beta_n$. Also, $Pu' = 0$ at $\beta = 0$. So $Pu' < 0$ on $(0, \beta_n)$. Hence $u' < 0$ on $(0, \beta_n)$, meaning u decreases for as long as it stays positive. Therefore

$$M'(-\gamma, n+1, \beta) < 0 \quad \text{for } 0 < \beta < \beta_n. \quad (4.3.1)$$

Suppose $\beta_{n+1} < \beta_n$. By the recurrence relations and definition of β_{n+1} ,

$$\begin{aligned} (n+1+\gamma)M(-\gamma, n+2, \beta_{n+1}) &= (n+1)(M(-\gamma, n+1, \beta_{n+1}) - M'(-\gamma, n+1, \beta_{n+1})) \\ 0 &= M(-\gamma, n+1, \beta_{n+1}) - M'(-\gamma, n+1, \beta_{n+1}) \end{aligned}$$

Then we have

$$M(-\gamma, n+1, \beta_{n+1}) = M'(-\gamma, n+1, \beta_{n+1}), \quad (4.3.2)$$

which is a contradiction since for $0 < \beta < \beta_n$, $M(-\gamma, n+1, \beta) > 0$ and $M'(-\gamma, n+1, \beta) < 0$ by (4.3.1). Thus, we conclude $\beta_n \leq \beta_{n+1}$ so that the “ n ”-zeros come before the “ $n+1$ ”-zeros. Also β_n cannot equal β_{n+1} , since if $\beta_n = \beta_{n+1}$ then M and M' both equal zero at β_{n+1} , by (??), which is impossible by the uniqueness theorem for ODEs.

Hence $\beta(n, k, \gamma)$ increase as n increases, with k and γ fixed. □

4.3.2 Properties of β -zeros of $M(-\gamma, n+1, \beta)$

Notice that the graph of $\beta(n, k, \gamma)$ in Figure 4.2 is decreasing as a function of γ . Now we prove that fact.

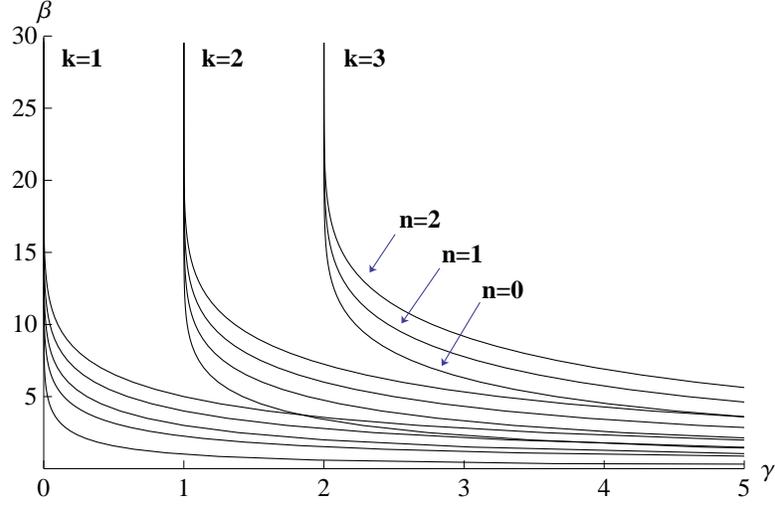


Figure 4.2: Behavior of $\beta(n, k, \gamma)$ as a function of γ , for fixed n .

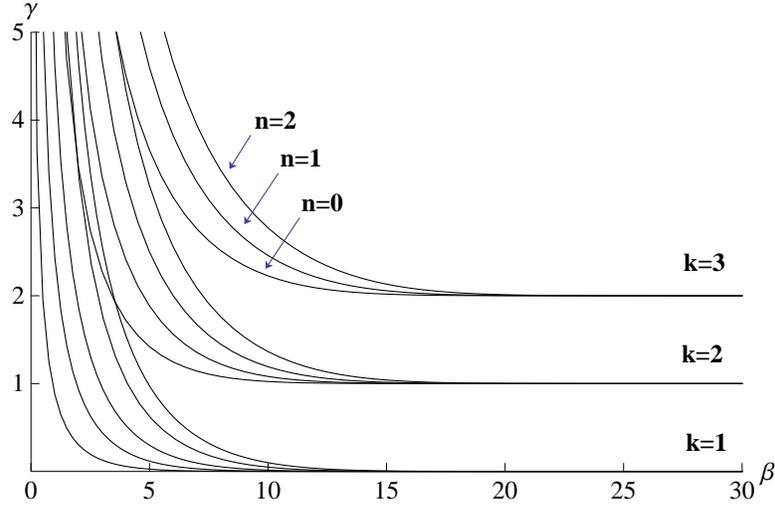


Figure 4.3: Behavior of $\gamma(n, k, \beta)$ as function of β , for fixed n .

Property I — $\beta(n, k, \gamma)$ is strictly decreasing as γ increases, for $\gamma > k - 1$ and fixed $n \geq 0$

This is a well known result of Tricomi (see [41, Chapter 3.5]). Start from the Kummer differential equation with parameter $a = -\gamma$, $b = n + 1$:

$$\beta y''(\beta) + (n + 1 - \beta)y'(\beta) + \gamma y(\beta) = 0, \quad y(0) = 1.$$

To use Sturm's comparison theorem method we need to change the ODE. By substitution

$$y(\beta) = \beta^{-(n+1)/2} e^{\beta/2} u(\beta),$$

the ODE can be transformed into the normal form

$$u''(\beta) + \left[\frac{\gamma}{\beta} + \frac{(n+1)(2\beta - n + 1) - \beta^2}{4\beta^2} \right] u(\beta) = 0, \quad u(0) = 0.$$

For fixed n , we have that the positive β -zeros of $M(\gamma, n + 1, \beta)$ is decreasing function of γ by Sturm's comparison Theorem 4.3.6: fix $\gamma_1 < \gamma_2$ and let's compare two zeros of solutions of

$$\begin{cases} u_1''(\beta) + q_1(\beta)u_1(\beta) = 0 \\ u_2''(\beta) + q_2(\beta)u_2(\beta) = 0 \end{cases}$$

where

$$\left[q_1 = \frac{\gamma_1}{\beta} + \frac{(n+1)(2\beta - n + 1) - \beta^2}{4\beta^2} \right] < \left[q_2 = \frac{\gamma_2}{\beta} + \frac{(n+1)(2\beta - n + 1) - \beta^2}{4\beta^2} \right]$$

because $\gamma_1 < \gamma_2$ and $\beta > 0$. Therefore Sturm's comparison theorem shows that the zeros $\beta(n, k, \gamma_2)$ occur before the zeros of $\beta(n, k, \gamma_1)$:

$$\beta(n, k, \gamma_1) > \beta(n, k, \gamma_2) \quad \text{for } \gamma_1 < \gamma_2 \text{ and fixed } n \geq 0, k = 1, \dots, \lceil \gamma_2 \rceil.$$

Hence $\beta(n, k, \gamma)$ is strictly decreasing function of $\gamma > k - 1$, for fixed n and k .

Theorem 4.3.6 (Sturm's comparison theorem [9, p. 33]).

Let $f(x)$ and $g(x)$ be nontrivial solutions of the differential equations

$$u'' + p(x)u = 0 \quad \text{and} \quad v'' + q(x)v = 0,$$

respectively, where $p(x) \geq q(x)$. Then $f(x)$ vanishes at least once between any two zeros of $g(x)$, unless $p(x) \equiv q(x)$ and f is a constant multiple of g .

Property II — $\beta(n, k, \gamma)$ is smooth function of $\gamma > 0$

Seek a proof using implicit function theorem applied to the equation $M(-\gamma, n + 1, \beta) = 0$.

Theorem 4.3.7 (Implicit function theorem [10, Chapter 9]).

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable, let $f(x_0, y_0) = 0$, and suppose that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Then there is an open interval U about x_0 with the following property: there exists a continuously differentiable function $g : U \rightarrow \mathbb{R}$ such that

$$g(x_0) = y_0 \quad \text{and} \quad f(x, g(x)) = 0.$$

These properties determine g uniquely.

In order to apply the Implicit Function Theorem we need the condition $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. In our case we should prove the condition $\frac{\partial M}{\partial \beta}(-\gamma_0, n_0 + 1, \beta_0) \neq 0$ where $M(-\gamma_0, n_0 + 1, \beta_0) = 0$. We can verify the condition by the existence and uniqueness theorem for second order ODE.

Theorem 4.3.8 (Existence and Uniqueness [40, Section 3.1]). *Consider the linear ordinary differential equation of order m*

$$a_0(x) \frac{d^m f}{dx^m} + a_1(x) \frac{d^{m-1} f}{dx^{m-1}} + \cdots + a_{m-1}(x) \frac{df}{dx} + a_m(x) f = F(x) \quad (4.3.3)$$

where a_0, a_1, \dots, a_m and F are continuous real functions on a real interval $a \leq x \leq b$ and $a_0(x) \neq 0$ for all x on $a \leq x \leq b$. If x_0 is any point of the interval $a \leq x \leq b$ and c_0, c_1, \dots, c_{m-1} are m arbitrary constants, then there exists a unique solution f of Equation (4.3.3) such that

$$f(x_0) = c_0, \quad f'(x_0) = c_1, \quad \dots, \quad f^{(m-1)}(x_0) = c_{m-1},$$

and this solution is defined over the entire interval $a \leq x \leq b$.

Corollary 4.3.9 (Uniqueness of solutions [40, Section 3.1]). *Let f be the solution of the m -th order homogeneous linear differential equation*

$$a_0(x) \frac{d^m f}{dx^m} + a_1(x) \frac{d^{m-1} f}{dx^{m-1}} + \cdots + a_{m-1}(x) \frac{df}{dx} + a_m(x) f = 0$$

such that

$$f(x_0) = 0, \quad f'(x_0) = 0, \quad \dots, \quad f^{(m-1)}(x_0) = 0$$

where x_0 is a point of the interval $a \leq x \leq b$ in which the coefficient a_0, a_1, \dots, a_m are all continuous and $a_0(x) \neq 0$. Then $f(x) = 0$ for all x on $a \leq x \leq b$.

Thus, we get the result $\frac{\partial M}{\partial \beta}(-\gamma_0, n_0 + 1, \beta_0) \neq 0$ where $M(-\gamma_0, n_0 + 1, \beta_0) = 0$.

Claim 4.3.10. *If $M(-\gamma_0, n_0 + 1, \beta_0) = 0$ then $\frac{\partial M}{\partial \beta}(-\gamma_0, n_0 + 1, \beta_0) \neq 0$.*

Proof of Claim 4.3.10. By the definition of Kummer function (see Definition 3.5.1), $M(-\gamma_0, n_0 + 1, \beta)$ satisfies: the homogeneous linear ODE

$$\beta \frac{d^2 M}{d\beta^2} + (n_0 + 1 - \beta) \frac{dM}{d\beta} + \gamma_0 M = 0.$$

Assume M and $\frac{\partial M}{\partial \beta}$ both equal 0 at $\beta = \beta_0$. Then the zero function also satisfies the ODE with the same initial conditions at β_0 . Then by the Uniqueness theorem, M should be the zero function. This contradicts the fact that the Kummer function is not a zero function since $M = 1$ when $\beta = 0$. Hence either $M(-\gamma_0, n_0 + 1, \beta_0) \neq 0$ or the first derivative, $\frac{\partial M}{\partial \beta}(-\gamma_0, n_0 + 1, \beta_0) \neq 0$. \square

Finally, $\beta(n, k, \gamma)$ is continuous function of $\gamma > 0$ by Implicit function Theorem 4.3.7. \square

4.3.3 Monotonicity of γ -zeros of $M(-\gamma, n + 1, \beta)$ with respect to $n \in \mathbb{Z}$

We denote the γ -zeros of $M(-\gamma, n + 1, \beta)$ by $\gamma(n, k, \beta)$, for $n \geq 0, k \geq 0$. Justify this we will use the relation between $\beta(n, k, \gamma)$ and $\gamma(n, k, \beta)$. For fixed n and k there will be an inverse function $\gamma(n, k, \beta)$ as a function of β that is strictly decreasing and continuously differentiable, since $\beta(n, k, \gamma)$ as a function of γ is strictly monotonic and continuous differentiable function (see Property I and Property II in Section 4.3.2). This inverse function $\gamma(n, k, \beta)$ is defined for all $\beta > 0$ because $\beta(n, k, \gamma)$ has range $(0, \infty)$ by Claim 4.3.13 below.

Getting monotonicity of γ -zeros with respect to n , by Inverse Function Theorem

Lemma 4.3.11 ([14, Section 14]). *Let f be a continuous, strictly increasing or decreasing function on $[a, b]$ with $x = f(a)$ and $y = f(b)$. Then f has a continuous, strictly increasing or decreasing inverse function defined on $[x, y]$.*

Now let us construct the proof of Section 4.3.3 in several steps. From the previous sections, we know the following.

Proposition 4.3.12 (Properties of $\beta(n, k, \gamma)$ and $\gamma(n, k, \beta)$).

1. $\beta(n, k, \gamma)$ is strictly increasing with respect to n , when k and γ are fixed.
2. $\beta(n, k, \gamma)$ is strictly decreasing with respect to γ , when n and k are fixed.
3. $\beta(n, k, \gamma)$ is a continuous function of γ , when n and k are fixed.
4. $\gamma(n, k, \beta)$ is the inverse function of $\beta(n, k, \gamma)$, for fixed n and k .
5. $\gamma(n, k, \beta)$ is strictly decreasing with respect to β , when n and k are fixed.

Proof. Part (1) follows from the Section 4.3.1. For part (2) and part (3), see Section 4.3.2. Part (4) and part (5) are the result from the previous Lemma 4.3.11. \square

Next we show that the range of $\gamma \mapsto \beta(n, k, \gamma)$ contains all positive numbers.

Claim 4.3.13. *Given $\tilde{\beta} > 0$, there exists $\tilde{\gamma} > 0$ such that $\tilde{\beta}(n, k, \tilde{\gamma}) = \tilde{\beta}$.*

Proof. By Proposition 4.3.12 we already know that $\beta(n, k, \gamma)$ is strictly decreasing and continuous function of γ . So it is enough to prove that for small γ , $\beta(n, k, \gamma)$ blows up and for large γ , $\beta(n, k, \gamma)$ approaches to zero. More precisely:

1. $\lim_{\gamma \rightarrow k-1} \beta(n, k, \gamma) = \infty$
2. $\lim_{\gamma \rightarrow \infty} \beta(n, k, \gamma) = 0$

First part follows from the proof of the asymptotic behavior in Section 4.3.8 below. For the second part we need the asymptotic formula of zeros of Kummer function $M(a, b, x)$ (see, [33, Section 13.9.8]):

$$\phi_r = \frac{(j_{b-1,r})^2}{2b-4a} \left[1 + \frac{2b(b-2) + (j_{b-1,r})^2}{3(2b-4a)^2} \right] + O\left(\frac{1}{a^5}\right), \quad \text{as } a \rightarrow -\infty,$$

when $a < 0$, $b > 0$ and ϕ_r , $r = 1, 2, 3, \dots$, are the positive zeros of $M(a, b, x)$ arranged in increasing order of magnitude, and $j_{b-1,r}$ is the r -th positive zero of the Bessel function $J_{b-1}(x)$. Now apply this formula to our case, then we get:

$$\beta(n, k, \gamma) = \frac{(j_{n,k})^2}{2(n+1) + 4\gamma} \left[1 + \frac{2(n+1)(n-1) + (j_{n,k})^2}{3(2(n+1) + 4\gamma)^2} \right] + O\left(\frac{1}{\gamma^5}\right), \quad \text{as } \gamma \rightarrow \infty.$$

Hence for fixed n, k :

$$\lim_{\gamma \rightarrow \infty} \beta(n, k, \gamma) = 0.$$

\square

In order to get the monotonicity of $\gamma(n, k, \beta)$ with respect to n , fix $\tilde{\beta} > 0$ and let $\gamma_1 = \gamma(n_1, k, \tilde{\beta})$, $\gamma_2 = \gamma(n_2, k, \tilde{\beta})$. We have the inequality (by Proposition 4.3.12) that

$$\beta(n_1, k, \gamma_2) < \beta(n_2, k, \gamma_2) = \tilde{\beta} = \beta(n_1, k, \gamma_1).$$

Since β is a strictly decreasing function of γ , we conclude $\gamma_2 > \gamma_1$. Hence for fixed $\tilde{\beta}$,

$$\gamma(n_2, k, \tilde{\beta}) > \gamma(n_1, k, \tilde{\beta})$$

We have the monotonicity of $\gamma(n, k, \beta)$ with respect to n . □

4.3.4 Monotonicity of $\lambda(n, k, \beta)$ with respect to $n \in \mathbb{Z}$

Recall that we let $\gamma = -\frac{1}{2}(n+1 - \frac{E}{2\beta}) = -\frac{1}{2}(1 - \frac{\lambda}{2\beta})$. We proved the monotonicity of $\gamma(n, k, \beta)$ in Section 4.3.3. Thus the monotonicity of $\lambda(n, k, \beta)$ with respect to $n \in \mathbb{Z}$ follows from the relation between $\gamma(n, k, \beta)$ and $\lambda(n, k, \beta)$. □

4.3.5 Monotonicity of $\lambda(0, 1, \beta)$ with respect to $\beta \geq 0$

Notice that when $n = 0$, $\lambda(n, k, \beta) = E(n, k, \beta)$. Hence we have the result (see section 4.1.2).

4.3.6 Asymptotic behavior of $E(n, k, \beta)$

Recall that we set

$$\gamma = -\frac{1}{2}(n+1 - \frac{E}{2\beta}). \quad (4.3.4)$$

Then let us consider the Kummer function $M(-\gamma, n+1, \beta)$ and denote $\beta(n, k, \gamma)$, $k = 1, 2, 3, \dots, \lceil \gamma \rceil$, for the sequence of positive β -zeros of $M(-\gamma, n+1, \beta)$ from now on. In Figure 4.2 we can see the asymptotic behavior of the numerical values of the zeros of $M(-\gamma, n+1, \beta)$. By understanding the numerical plot in order to prove the asymptotic behavior of $E(n, k, \beta)$ we need to show

$$\lim_{\gamma \rightarrow (k-1)^+} \beta(n, k, \gamma) = +\infty.$$

This can be proved directly from the result of Tricomi (see [41, Chapter 3.5]). When γ approaches from the right to a nonnegative integer there is always a β -zero of $M(-\gamma, n+1, \beta)$ that tends to infinity. Moreover Tricomi found that k -th positive zero of $M(-\gamma, n+1, \beta)$ is a decreasing function of $\varkappa = \frac{n+1}{2} + \gamma$ and appears at infinity for $\gamma = k - 1$ (see [42, p. 270]). By (4.3.4) we have that

$$\gamma = -\frac{1}{2}(n+1 - \frac{E}{2\beta}) \rightarrow k - 1$$

is equivalent to

$$\frac{E}{2\beta} \rightarrow 2k + n - 1 \quad \text{as } \beta \rightarrow \infty.$$

4.3.7 The asymptotic is a lower bound for all $n \in \mathbb{Z}$

Let us prove this inequality

$$E(n, 1, \beta) > 2(n+1)\beta, \quad \text{for all } \beta \geq 0. \quad (4.3.5)$$

Rewrite (4.3.5) like below:

$$\frac{E(n, 1, \beta)}{\beta} > 2(n + 1), \quad \text{for all } \beta \geq 0.$$

By (4.3.4) in the previous section,

$$\frac{E(n, 1, \beta)}{\beta} = 2(n + 1) + 4\gamma.$$

Then the right side is always greater than 2 since the property of the Kummer function, $\gamma > 0$ (see Remark 3.5.2).

4.3.8 Asymptotic behavior of $\lambda(n, k, \beta)$ with respect to $n \in \mathbb{Z}$

This follows from Theorem 3.2.3, as follows. We know that

$$E(n, k, \beta) = 2(2k + n - 1)\beta + o(\beta) \quad \text{as } \beta \rightarrow \infty. \quad (4.3.6)$$

Then we use the relationship $E(n, k, \beta) = \lambda(n, k, \beta) + 2n\beta$.

$$\begin{aligned} \lambda(n, k, \beta) &= E(n, k, \beta) - 2n\beta \\ &= 2(2k + n - 1)\beta + o(\beta) - 2n\beta \quad \text{as } \beta \rightarrow \infty. \end{aligned}$$

Hence

$$\begin{aligned} \lambda(n, k, \beta) &= 2(2k - 1)\beta + o(\beta) \quad \text{as } \beta \rightarrow \infty, \\ \lambda(n, k, \beta) &= -2(2k + 2n - 1)\beta + o(\beta) \quad \text{as } \beta \rightarrow -\infty. \end{aligned} \quad (4.3.7)$$

We get the Equation (4.3.7) by Equation (4.3.6).

$$\begin{aligned} \lambda(n, k, -\beta) &= E(n, k, -\beta) + 2n\beta \\ &= E(n, k, \beta) + 2n\beta \quad \text{by symmetry of the eigenvalues for the harmonic oscillator} \\ &= 2(2k + 2n - 1)\beta + o(\beta), \quad \text{for } \beta > 0. \end{aligned}$$

Therefore

$$\lambda(n, k, \beta) = -2(2k + 2n - 1)\beta + o(\beta) \quad \text{as } \beta \rightarrow -\infty,$$

since the asymptotic behavior of $E(n, k, \beta)$ works for both positive and negative β .

4.3.9 The asymptotic is a lower bound for all $n \in \mathbb{Z}$

Let us prove this inequality

$$\lambda(n, 1, \beta) > 2\beta, \quad \text{for all } \beta \geq 0.$$

We get the result by Section 4.3.7.

$$\lambda(n, 1, \beta) = E(n, 1, \beta) - 2n\beta > 2(n+1)\beta - 2n\beta > 2\beta.$$

All theorems for the magnetic Laplacian are completed.

CHAPTER 5 : Parametric derivatives of the Kummer function

5.1 Motivation

The derivatives of Kummer function, $M(a, b, z)$, with respect to the parameter z are well-known in the literature. However there is a situation where one needs to know information on the parametric derivatives of the Kummer function with respect to the first two parameters, a and b , and useful formulas are hard to find in literature. One example is in work of Laugesen and Siudeja (see [29]) where it is necessary to show that $\frac{\partial M}{\partial a}$ is non-zero at certain values of a, b , and z . They use our Corollary 5.2.2 below. Also another example is in paper of Ancarani and Gasaneo, where the authors investigate the derivatives of any order of the Kummer function with respect to a and b (see [2]). But the formulas they give do not seem to be useful for what we need to do. Therefore we will develop new formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$. The next section gives the integral formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ in terms of $M(a, b, z)$ and $U(a, b, z)$.

5.2 Parametric derivative formula

Theorem 5.2.1. *The integral formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ are following:*

$$\begin{aligned} \frac{\partial M}{\partial a} = & M(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) d\zeta \\ & - U(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta, \end{aligned} \quad (5.2.1)$$

$$\begin{aligned} \frac{\partial M}{\partial b} = & - M(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M'(a, b, \zeta) d\zeta \\ & + U(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M'(a, b, \zeta) d\zeta, \end{aligned} \quad (5.2.2)$$

for all $a \in \mathbb{R}, b \geq 1, z > 0$ (assuming a is not a nonpositive integer).

Recall that $M(a, b, z)$ does not exist when b is a nonpositive integer. We will prove this theorem in next section.

Corollary 5.2.2. *If $M(a, b, z) = 0$ then $\frac{\partial M}{\partial a}(a, b, z) \neq 0$, assuming that $a \in \mathbb{R}, b \geq 1, z > 0$ and a is not a nonpositive integer.*

Proof. The proof follows from Theorem 5.2.1 above. Consider the integral formula (5.2.1):

$$\begin{aligned} \frac{\partial M}{\partial a} = & M(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) d\zeta \\ & - U(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta. \end{aligned}$$

If $M(a, b, z) = 0$ then the first term vanishes. Next we want to show that the second term is not zero. Notice that the integral $\int_0^z e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta$ is clearly positive because that is integral of a real-valued square, $M(a, b, z)^2$. Hence we need to show that if $M(a, b, z) = 0$ then $U(a, b, z) \neq 0$. Recall that M and U are two standard solutions of Kummer's equation,

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0.$$

Since M and U solve the same second order differential equation, if they are both zero at some point then they are multiples of each other. This contradicts the fact that M and U are the independent solutions of Kummer's equation. \square

5.3 Proof of Theorem 5.2.1

5.3.1 Method of variation of parameters

First recall that a confluent hypergeometric function, $M(a, b, z)$, is a solution to confluent hypergeometric differential equation,

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0. \quad (5.3.1)$$

Now start from the Equation:

$$z \frac{\partial^2 M}{\partial z^2} + (b - z) \frac{\partial M}{\partial z} - aM = 0. \quad (5.3.2)$$

Differentiate with respect to a :

$$z \left(\frac{\partial M}{\partial a} \right)'' + (b - z) \left(\frac{\partial M}{\partial a} \right)' - a \left(\frac{\partial M}{\partial a} \right) = M,$$

where $' = \frac{\partial}{\partial z}$. Then divide by z to get the non homogeneous equation:

$$\left(\frac{\partial M}{\partial a}\right)'' + \frac{b-z}{z}\left(\frac{\partial M}{\partial a}\right)' - \frac{a}{z}\left(\frac{\partial M}{\partial a}\right) = \frac{M}{z} = f \quad (5.3.3)$$

The method of variation of parameters gives the general solution of the non homogeneous Equation (5.3.3):

$$\begin{aligned} \frac{\partial M}{\partial a} = & M(a, b, z) \int \frac{-U(a, b, z)}{W\{M, U\}} \frac{M(a, b, z)}{z} dz \\ & + U(a, b, z) \int \frac{M(a, b, z)}{W\{M, U\}} \frac{M(a, b, z)}{z} dz, \end{aligned}$$

where $M(a, b, z)$ and $U(a, b, z)$ are linearly independent solutions of the homogeneous Equation (5.3.1) and $W\{M, U\}$ denotes the Wronskian of M and U (see [33, Chapter 13]). The Wronskian can be evaluated as

$$W\{M, U\} = -\frac{\Gamma(b)e^z}{\Gamma(a)z^b}$$

by [13.2.34] NIST (just multiply that formula by $\Gamma(b)$). Hence we get the general solution of the non homogeneous Equation (5.3.3):

$$\begin{aligned} \frac{\partial M}{\partial a} = & M(a, b, z) \int \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} U(a, b, z) M(a, b, z) dz \\ & - U(a, b, z) \int \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} M(a, b, z) M(a, b, z) dz, \end{aligned}$$

Similarly, we can find $\frac{\partial M}{\partial b}$. Start from the Equation (5.3.2) then differentiate with respect to b :

$$z\left(\frac{\partial M}{\partial b}\right)'' + (b-z)\left(\frac{\partial M}{\partial b}\right)' - a\left(\frac{\partial M}{\partial b}\right) = -M'. \quad (5.3.4)$$

Hence we get:

$$\begin{aligned} \frac{\partial M}{\partial b} = & -M(a, b, z) \int \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} U(a, b, z) M'(a, b, z) dz \\ & + U(a, b, z) \int \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} M(a, b, z) M'(a, b, z) dz. \end{aligned}$$

Each of these indefinite integrals has the constants of integration implicitly in it. To fix the constants of integration we will fix the lower limit of integration to be $z = 0$. Then we have to justify that this is the correct choice to get $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$. Especially, we will show that the integrands as a function of z are integrable near $z = 0$ and that this gives the correct choice

for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ at $z = 0$ by checking the conditions for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$.

5.3.2 Finding a limit of integration

First we have to show that the integral makes sense. Then we need to figure out what would be the leading order of z . That follows from the Claim 5.3.1 and Claim 5.3.2.

Claim 5.3.1. *The leading order terms as a power of z , for small z are following:*

1. $M(a, b, z) = 1 + O(z)$
2. $M'(a, b, z) = \frac{a}{b} + O(z)$
3. $U(a, b, z) = -\frac{\Gamma(b)}{(1-b)\Gamma(a)}z^{1-b} + o(z^{1-b})$, if $b > 1$
and $U(a, 1, z) = -\frac{1}{\Gamma(a)}\ln z + O(1)$, if $b = 1$
4. $U'(a, b, z) = -\frac{\Gamma(b)}{\Gamma(a)}z^{-b} + o(z^{-b})$

Proof of Claim 5.3.1. By the series definition of M , we can easily check (1) and (2) (see [33, Chapter 13]). Also by NIST formulas 13.2.16 ~ 13.2.19 [33, Chapter 13], we get the result (3):

$$U(a, b, z) = -\frac{\Gamma(b)}{(1-b)\Gamma(a)}z^{1-b} + o(z^{1-b}), \quad \text{if } b > 1$$

$$\text{and } U(a, 1, z) = -\frac{1}{\Gamma(a)}\ln z + O(1), \quad \text{if } b = 1$$

Next, verify (4).

$$\begin{aligned} U'(a, b, z) &= -aU(a+1, b+1, z), \quad \text{by NIST formula 13.3.22} \\ &= -a \left(-\frac{\Gamma(b+1)}{(-b)\Gamma(a+1)}z^{-b} + o(z^{-b}) \right), \quad \text{by part (3)} \\ &= -\frac{\Gamma(b)}{\Gamma(a)}z^{-b} + o(z^{-b}). \end{aligned}$$

□

Then we have the next claim, where we show that the integral makes sense when the lower limit of integration is zero.

Claim 5.3.2. *The leading order terms as a power of z , for small $z > 0$ are following:*

1. $\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) d\zeta = \frac{1}{b-1} z + o(z)$, if $b > 1$
and $\int_0^z \Gamma(a) e^{-\zeta} U(a, 1, \zeta) M(a, 1, \zeta) d\zeta = -z \ln z + O(z)$, if $b = 1$
2. $\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta = \frac{\Gamma(a)}{b\Gamma(b)} z^b + O(z^{b+1})$
3. $\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M'(a, b, \zeta) d\zeta = \frac{a}{b(b-1)} z + o(z)$, if $b > 1$
and $\int_0^z \Gamma(a) e^{-\zeta} U(a, 1, \zeta) M'(a, 1, \zeta) d\zeta = -az \ln z + O(z)$, if $b = 1$
4. $\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M'(a, b, \zeta) d\zeta = \frac{a\Gamma(a)}{b^2\Gamma(b)} z^b + O(z^{b+1})$

Proof of Claim 5.3.2. To compute the integrals in Claim 5.3.2 we need to start from finding the leading order term for the integrand of integral. For $b > 1$,

$$\begin{aligned} \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) &= \left(\frac{\Gamma(a)}{\Gamma(b)} \right) \zeta^{b-1} \left(-\frac{\Gamma(b)}{(1-b)\Gamma(a)} \zeta^{1-b} \right) + O(\zeta) \\ &= \frac{1}{b-1} + o(1). \end{aligned}$$

Hence,

$$\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) d\zeta = \frac{1}{b-1} z + o(z).$$

For $b = 1$,

$$\Gamma(a) e^{-\zeta} U(a, 1, \zeta) M(a, 1, \zeta) = \Gamma(a) \left(-\frac{1}{\Gamma(a)} \ln \zeta \right) + O(1) = -\ln \zeta + O(1).$$

Then,

$$\int_0^z \Gamma(a) e^{-\zeta} U(a, 1, \zeta) M(a, 1, \zeta) d\zeta = -z \ln z + O(z).$$

Similarly, we can check (2).

$$\frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) = \frac{\Gamma(a)}{\Gamma(b)} \zeta^{b-1} + O(\zeta^b).$$

Therefore,

$$\int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta = \frac{\Gamma(a)}{b\Gamma(b)} z^b + O(z^{b+1}).$$

The proof of (3) and (4) are similar. □

We have showed that the integral makes sense, so lastly we will verify the conditions on $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$.

Remark 5.3.3. *Conditions on $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ are following:*

1. $\frac{\partial M}{\partial a}(a, b, 0) = 0$
2. $\left(\frac{\partial M}{\partial a}\right)' \Big|_{z \rightarrow 0} = \frac{1}{b}$
3. $\frac{\partial M}{\partial b}(a, b, 0) = 0$
4. $\left(\frac{\partial M}{\partial b}\right)' \Big|_{z \rightarrow 0} = -\frac{a}{b^2}$

The results in Remark 5.3.3 are derived directly from the series definition of the Kummer function, $M(a, b, z)$ (see, [1, Chapter 13]). The conditions will enable us to evaluate the constants of integration in $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$. Next, we need to show that our formulas in Theorem 5.2.1 satisfy those conditions. Since we have got two constants of integration in each of the derivative function we should prove that the formulas satisfy the correct initial conditions when we choose zero for the lower limit of integration. We will use the leading order terms in Claim 5.3.1 and Claim 5.3.2 to verify the conditions in Remark 5.3.3.

Claim 5.3.4. *The integral formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ in Theorem 5.2.1 satisfy the four conditions in Remark 5.3.3.*

Proof of Claim 5.3.4 . By Claims 5.3.1 and 5.3.2 we have:

$$\begin{aligned} \frac{\partial M}{\partial a} \Big|_{z \rightarrow 0} &= \lim_{z \rightarrow 0} 1 \left(\frac{1}{b-1} z \right) - \frac{\Gamma(b)}{(b-1)\Gamma(a)} z^{1-b} \frac{\Gamma(a)}{b\Gamma(b)} z^b = 0. \\ \left(\frac{\partial M}{\partial a}\right)' \Big|_{z \rightarrow 0} &= \lim_{z \rightarrow 0} M'(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M(a, b, \zeta) d\zeta \\ &\quad - U'(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M(a, b, \zeta) d\zeta. \end{aligned}$$

(Notice that two more terms of $\left(\frac{\partial M}{\partial a}\right)'$ by product rule are canceled:

$$M(a, b, z) \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} U(a, b, z) M(a, b, z) - U(a, b, z) \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} M(a, b, z) M(a, b, z) = 0.)$$

Thus,

$$\begin{aligned}
\left(\frac{\partial M}{\partial a}\right)' \Big|_{z \rightarrow 0} &= \lim_{z \rightarrow 0} \left(\frac{a}{b}\right) \frac{1}{b-1} z + \frac{\Gamma(b)}{\Gamma(a)} z^{-b} \frac{\Gamma(a)}{b\Gamma(b)} z^b = \frac{1}{b}. \\
\frac{\partial M}{\partial b} \Big|_{z \rightarrow 0} &= \lim_{z \rightarrow 0} (-1) \frac{a}{b(b-1)} z + \frac{\Gamma(b)}{(b-1)\Gamma(a)} z^{1-b} \frac{a\Gamma(a)}{b^2\Gamma(b)} z^b = 0. \\
\left(\frac{\partial M}{\partial b}\right)' \Big|_{z \rightarrow 0} &= \lim_{z \rightarrow 0} -M'(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} U(a, b, \zeta) M'(a, b, \zeta) d\zeta \\
&\quad + U'(a, b, z) \int_0^z \frac{\Gamma(a)}{\Gamma(b)} e^{-\zeta} \zeta^{b-1} M(a, b, \zeta) M'(a, b, \zeta) d\zeta.
\end{aligned}$$

(Notice that two more terms of $\left(\frac{\partial M}{\partial b}\right)'$ by product rule are canceled:

$$-M(a, b, z) \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} U(a, b, z) M'(a, b, z) + U(a, b, z) \frac{\Gamma(a)}{\Gamma(b)} e^{-z} z^{b-1} M(a, b, z) M'(a, b, z) = 0.)$$

Thus,

$$\left(\frac{\partial M}{\partial b}\right)' \Big|_{z \rightarrow 0} = \lim_{z \rightarrow 0} \left(-\frac{a}{b}\right) \frac{a}{b(b-1)} z - \frac{\Gamma(b)}{\Gamma(a)} z^{-b} \frac{a\Gamma(a)}{b^2\Gamma(b)} z^b = -\frac{a}{b^2}$$

The case $b=1$ is similar, using the formulas for the case $b=1$ from Claim 5.3.2. \square

Lastly, we can prove the main result Theorem 5.2.1.

Proof of Theorem 5.2.1. From Claim 5.3.1, Claim 5.3.2, and Claim 5.3.4, we verify that the integral formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ make sense where the lower limit of integration is zero and they give the correct initial conditions at $z = 0$. That proves the integral formulas for $\frac{\partial M}{\partial a}$ and $\frac{\partial M}{\partial b}$ are correct, by the Uniqueness Theorem for linear ODEs. \square

CHAPTER 6 : Open problems

1 Generalize the Pólya-Szegő problem to the case of magnetic field

Extending the work of Pólya-Szegő mentioned in the Introduction, one can look for a sharp upper bound for the first eigenvalue for the magnetic Laplacian. Is the first eigenvalue for the magnetic Laplacian under conformal mapping normalization maximal for the disk? Then one might be interested in proving the following inequality,

$$\lambda_1(\Omega; \tilde{\beta}) \leq \lambda_1(\mathbb{D}; \beta),$$

where $f : \mathbb{D} \mapsto \Omega$ is a conformal map with $f'(0) = 1$. We need to try to prove this inequality for some choice of $\tilde{\beta}$ depending on Ω (probably, $\tilde{\beta} < \beta$, since otherwise the flux gets bigger on Ω than on \mathbb{D} , because Ω has bigger area than \mathbb{D}). Part of the problem is to figure out what normalization should be imposed on the magnetic field on Ω .

2 Magnetic ground state of annulus under the Dirichlet boundary condition

We have determined the ground state of disk on the Dirichlet boundary condition. What is the ground state for the annulus? Is the ground state ever radial? Or is the ground state always non radial?

3 Generalize the Pólya-Szegö problem to the domain of the annuli

If the disk case is successful, then we can try to tackle the first eigenvalue for the magnetic Laplacian under conformal mapping normalization maximal for the annulus.

Then we can look at questions for the Neumann boundary condition.

4 Understand the ground state energy for the Neumann boundary condition

We denote eigenfunctions and eigenvalues of the magnetic Neumann Laplacian on the unit disk \mathbb{D} by:

$$\begin{cases} (i\nabla + F)^2 u &= \mu u & \text{in } \mathbb{D}, \\ \vec{n} \cdot (\nabla - iF)u &= 0 & \text{on } \partial\mathbb{D}, \end{cases}$$

where $u(x)$ is complex-valued and the vector potential is $F(x) = \beta(-x_2, x_1)$. The magnetic field is $\nabla \times F = (0, 0, 2\beta)$, where $\beta \in \mathbb{R}$ is constant.

For the Neumann case it is not known rigorously what the ground state is. We know that it depends on magnetic field, β and angular momentum, n . We try to at least get some estimate. For example, if we given the β -value can we roughly estimate the n -value that gives the ground state?

We have a picture from D. Saint-James [20] below. What can we really prove about this

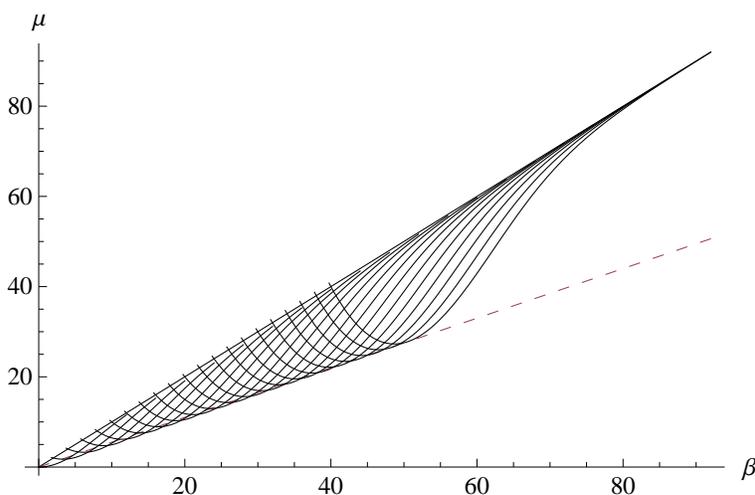


Figure 6.1: Neumann eigenvalue branches on the disk, for various n -values, as a function of magnetic field strength β .

picture? Especially, can we prove what n -value should be for the ground state? Can we prove the rate of asymptotic growth that is minimum? From the Figure 6.1, the minimum value is lower than where any particular curve ends up, as $\beta \rightarrow \infty$. At least we can investigate that asymptotic growth rate numerically.

Appendix A

A Discrete Spectral theorem

Hypotheses. Consider two infinite dimensional Hilbert spaces \mathcal{H} and \mathcal{K} over \mathbb{R} .

1. \mathcal{H} is separable.
2. \mathcal{K} is continuously and densely imbedded in \mathcal{H} , meaning there exists a continuous linear injection $\iota : \mathcal{K} \rightarrow \mathcal{H}$ with $\iota(\mathcal{K})$ dense in \mathcal{H} .
3. The imbedding $\mathcal{K} \hookrightarrow \mathcal{H}$ is compact, meaning if B is a bounded subset of \mathcal{K} then B is precompact when considered as a subset of \mathcal{H} .
4. We have a quadratic form $a : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}$ that is linear in each variable, continuous, and symmetric.
5. a is elliptic on \mathcal{K} , meaning

$$a(f, f) \geq c \|f\|_{\mathcal{K}}^2 \quad \forall f \in \mathcal{K},$$

for some $c > 0$.

Note that because of the ellipticity condition a , defines an inner product on \mathcal{K} that is equivalent to the original inner product in Hilbert space \mathcal{K} . Ellipticity gives a lower bound and the continuity gives an upper bound.

Theorem A.1 ([8, Chapter 6]). *Under the hypotheses above, there exist vectors $u_1, u_2, u_3, \dots \in \mathcal{K}$ and numbers*

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \dots \rightarrow \infty$$

such that:

1. u_j is an eigenvector of $a(\cdot, \cdot)$ with eigenvalue γ_j , meaning

$$a(u_j, v) = \gamma_j \langle u_j, v \rangle_{\mathcal{H}} \quad \text{for all } v \in \mathcal{K}, \quad (\text{A.1})$$

2. $\{u_j\}$ is an ONB for \mathcal{H} ,
3. $\{u_j/\sqrt{\gamma_j}\}$ is an ONB for \mathcal{K} with respect to the a -inner product.

The decomposition

$$f = \sum_j \langle f, u_j \rangle_{\mathcal{H}} u_j \quad (\text{A.2})$$

converges in \mathcal{H} for each $f \in \mathcal{H}$, and converges in \mathcal{K} for each $f \in \mathcal{K}$.

Appendix B

B Proof of Weyl's criterion

In this Appendix, we will prove some Weyl type criteria for discreteness of the spectrum, by using Appendix A. Alternatively, one could use the general criteria of Friedrichs [16]. We consider the Sturm-Liouville eigenvalue problem

$$-(p(x)u_x)_x + q(x)u = \lambda r(x)u, \quad 0 < x < L, \quad (\text{B.1})$$

under the assumptions: $L > 0$ and

1. $p, r > 0$ on $(0, L]$,
2. p, p_x, q , and r are real-valued and continuous on $(0, L]$,
3. Either

(i) $\frac{q(x)}{r(x)} \rightarrow \infty$ as $x \rightarrow 0$

or

- (ii) $q(x)/r(x)$ is bounded below on $(0, L)$, and $r, Pr \in L^1((0, L); dx)$ where $P(x)$ is an antiderivative of $1/p(x)$.

Also, consider two infinite dimensional Hilbert spaces \mathcal{H} and \mathcal{K} over \mathbb{R} . Let $m = \inf_{0 < x \leq L} \frac{q(x)}{r(x)} > -\infty$.

$$\mathcal{H} = L^2((0, L); r dx),$$

$$\mathcal{K} = \{f \in W_{loc}^{1,2}(0, L) : f' \in L^2((0, L); p dx), f \in L^2((0, L); (|q| + r) dx), f(L) = 0\},$$

with inner products

$$\begin{aligned}\langle f, g \rangle_{\mathcal{H}} &= \int_0^L f g r \, dx, \\ \langle f, g \rangle_{\mathcal{K}} &= \int_0^L (f' g' p + f g (|q| + r)) \, dx.\end{aligned}$$

Theorem B.1. *There exists an ONB $\{u_j\}$ for $L^2(\mathbb{D}, \mathbb{C})$ and corresponding eigenvalues which we denote $E_j > 0$ satisfying Equation (B.1) weakly in the sense that*

$$\int_0^L (u_j' v' p + u_j v q) \, dx = E_j \int_0^L u_j v r \, dx, \quad \text{for all } v \in \mathcal{K}.$$

The eigenfunction u_j satisfies the eigenfunction equation weakly, so that u_j is a weak eigenfunction of the Laplacian with eigenvalue E_j . Elliptic regularity theory gives that u_j is smooth (assuming p, q, r are smooth) and hence satisfies the eigenfunction equation classically. The boundary condition $u_j(L) = 0$ is satisfied classically also.

Let us check the following assumptions of the Discrete Spectral Theorem (Appendix A) in order to prove Theorem B.1.

1. \mathcal{H} is separable.
2. \mathcal{K} is continuously and densely imbedded in \mathcal{H} , meaning there exists a continuous linear injection $\iota : \mathcal{K} \rightarrow \mathcal{H}$ with $\iota(\mathcal{K})$ dense in \mathcal{H} .

Proof.

$$\begin{aligned}\|f\|_{\mathcal{H}}^2 &= \int_0^L f^2 r \, dx \\ &= \int_0^L f^2 (q - (m-1)r) \, dx, \quad \text{since } q - mr \geq 0 \\ &\leq (\text{const.}) \|f\|_{\mathcal{K}}^2, \quad \text{since } q - (m-1)r \leq |q| + (|m| + 1).\end{aligned}$$

So if $f \in \mathcal{K}$ then $f \in \mathcal{H}$. And \mathcal{K} is dense in \mathcal{H} since $\mathcal{C}_c^\infty(0, L) \subseteq \mathcal{K}$ and \mathcal{C}_c^∞ is dense in $L^2((0, L), r dx)$. \square

3. The imbedding $\mathcal{K} \hookrightarrow \mathcal{H}$ is compact, meaning if B is a bounded subset of \mathcal{K} then B is precompact when considered as a subset of \mathcal{H} . (Equivalently, every bounded sequence in \mathcal{K} has a subsequence that converges in \mathcal{H} .)

Proof. (i) Assume $q(x)/r(x) \rightarrow \infty$ as $x \rightarrow 0$. Suppose $\{f_k\}$ is a bounded sequence in \mathcal{K} , say with $\|f_k\|_{\mathcal{K}} \leq M$ for all k . We must prove the existence of a subsequence converging in \mathcal{H} . For each $\delta > 0$, note that p and $|q| + r$ are bounded below away from 0 on $[\delta, L]$. Hence the sequence is bounded in $W^{1,2}(\delta, L)$, for each $\delta \in (0, L)$. Take $\delta_2 = \frac{L}{2}$. The Rellich–Kondrachev theorem provides a subsequence that converges in $L^2(\delta_2, L)$ and hence in $L^2((\delta_2, L), r dx)$ (using here that r is bounded below away from 0 on $[\delta, L]$). Repeating with $\delta_3 = \frac{L}{3}$ provides a sub-subsequence converging in $L^2(\delta_3, L)$. Continue in this fashion and then consider the diagonal subsequence that converges in $L^2((\delta, L), r dx)$ for each $\delta \in (0, L)$. We will show this subsequence converges in $L^2((0, L), r dx)$. Denote it by $\{f_{k_\ell}\}$. Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{2(m+1)}$. Since $\frac{q(x)}{r(x)}$ grows to infinity as $x \rightarrow 0$, we may choose δ small enough that

$$\frac{q(x)}{r(x)} > \frac{1}{\varepsilon}, \quad \text{when } x \in (0, \delta).$$

Then

$$\begin{aligned} \int_0^\delta f_{k_\ell}^2 r \, dx &\leq \varepsilon \int_0^\delta f_{k_\ell}^2 q(x) \, dx \\ &\leq \varepsilon \int_0^\delta f_{k_\ell}^2 (q + r) \, dx \\ &\leq \varepsilon \|f_{k_\ell}\|_{\mathcal{K}}^2 \\ &\leq \varepsilon M^2 \end{aligned}$$

for all ℓ . Since also $\{f_{k_\ell}\}$ converges on $L(\delta, L)$, we have

$$\limsup_{\ell, n \rightarrow \infty} \|f_{k_\ell} - f_{k_n}\|_{L^2((0, L), r dx)} = \limsup_{\ell, n \rightarrow \infty} \|f_{k_\ell} - f_{k_n}\|_{L^2((0, \delta), r dx)} \leq 2\sqrt{\varepsilon}M.$$

Since ε was arbitrary, we conclude that $\limsup_{\ell, n \rightarrow \infty} \|f_{k_\ell} - f_{k_n}\|_{L^2((0, L), r dx)} = 0$. Therefore $\{f_{k_\ell}\}$ is Cauchy in $L^2((0, \delta), r dx)$, and hence converges.

- (ii) Assume $q(x)/r(x)$ is bounded below on $(0, L)$ (so that $m > -\infty$) and that $r(x)$ and $P(x)r(x)$ are integrable, where $P' = 1/p$. The proof goes like in part (i), except we need a different argument to show

$$\int_0^\delta f_{k_\ell}^2 r \, dx \leq \varepsilon M^2.$$

Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\int_0^\delta |P(x)|r(x) dx \leq \epsilon,$$

which is possible since Pr is integrable. Write $f = f_{kl}$ and $P(x) = -\int_x^L p(y)^{-1} dy$, so that P is antiderivative of $1/p$. Then

$$\begin{aligned} f(x) &= -\int_x^L f'(y) dy \quad \text{since } f(L) = 0 \\ &\leq \left(\int_x^L f'(y)^2 p(y) dy \right)^{1/2} \left(\int_x^L p(y)^{-1} dy \right)^{1/2} \quad \text{by Cauchy-Schwarz} \\ &\leq \|f\|_{\mathcal{K}} |P(x)|^{1/2}. \end{aligned}$$

So

$$\begin{aligned} \int_0^\delta f^2 r dx &\leq \|f\|_{\mathcal{K}}^2 \int_0^\delta |P(x)|r(x) dx \\ &\leq M^2 \epsilon \end{aligned}$$

by our choice of δ . That gives the inequality we needed. □

4. We have a quadratic form

$$\begin{aligned} a : \mathcal{K} \times \mathcal{K} &\rightarrow \mathbb{R} \\ a(f, g) &= \int_0^L f'g'p dx + \int_0^L fg[q - (m-1)r] dx \end{aligned}$$

that is linear, continuous, and symmetric, meaning

$$\begin{aligned} f &\mapsto a(f, g) \text{ is linear, for each fixed } g, \\ g &\mapsto a(f, g) \text{ is linear, for each fixed } f, \\ |a(f, g)| &\leq (\text{const.}) \|f\|_{\mathcal{K}} \|g\|_{\mathcal{K}} \\ a(f, g) &= a(g, f) \end{aligned}$$

Proof. $a(f, g)$ is clearly linear in f and g .

We have $|a(f, g)| \leq C \|f\|_{\mathcal{K}} \|g\|_{\mathcal{K}}$ by Cauchy-Schwarz inequality:

the key point is that

$$0 < q - (m - 1)r \leq (\text{const.})(|q| + r).$$

$a(f, g) = a(g, f)$ is obvious.

□

5. a is elliptic on \mathcal{K} , meaning

$$a(f, f) \geq c\|f\|_{\mathcal{K}}^2 \quad \forall f \in \mathcal{K},$$

for some $c > 0$.

Proof. We will show

$$\frac{q - (m - 1)r}{|q| + r} \geq (\text{const.}) > 0. \quad (\text{B.2})$$

Then

$$\begin{aligned} a(f, f) &= \int_0^L f'^2 p \, dx + \int_0^L f^2 [q - (m - 1)r] \, dx \\ &\geq \int_0^L f'^2 p \, dx + (\text{const.}) \int_0^L f^2 (|q| + r) \, dx \\ &\geq c\|f\|_{\mathcal{K}}^2, \end{aligned}$$

which is ellipticity.

We prove (B.2) with constant $1/(|m| + 1)$, by dividing into cases.

Case 1. Suppose $m \geq 0$. Then $q \geq mr \geq 0$ and so

$$\begin{aligned} \frac{q - (m - 1)r}{|q| + r} &= 1 - m \frac{r}{q + r} \\ &\geq 1 - m \frac{1}{m + 1} \quad \text{since } q \geq mr \\ &= \frac{1}{m + 1}. \end{aligned}$$

Case 2. Suppose $m < 0$ and $q(x) \geq 0$. Then

$$\frac{q - (m - 1)r}{|q| + r} = 1 + |m| \frac{r}{q + r} \geq 1.$$

Case 3. Suppose $m < 0$ and $q(x) < 0$. Then

$$\begin{aligned} \frac{q - (m - 1)r}{|q| + r} &= \frac{(|m| + 2)r}{|q| + r} - 1 \\ &\geq \frac{|m| + 2}{|m| + 1} - 1 \quad \text{since } |q| \leq |m|r \\ &= \frac{1}{|m| + 1}. \end{aligned}$$

□

Finishing the proof of Theorem B.1. We checked the assumptions of the Discrete Spectral Theorem, and so by Appendix A we get an ONB of weak eigenfunctions satisfying

$$a(u_j, v) = \gamma_j \langle u_j, v \rangle_{\mathcal{H}}$$

for all $v \in \mathcal{K}$. To finish proving Theorem B.1, just add $(m - 1)\langle u_j, v \rangle_{\mathcal{H}}$ to both sides and let

$$E_j = \gamma_j + m - 1.$$

Lemma B.2 (Monotonicity with respect to the weight). *Assume (p, q, r) and $(\tilde{p}, \tilde{q}, r)$ are two collections of weight functions satisfying the assumptions at the beginning of Appendix B.*

If $p \leq \tilde{p}$ and $q \leq \tilde{q}$, then the corresponding eigenvalues of the two problems are related by

$$E_k \leq \tilde{E}_k, \quad \forall k \geq 1.$$

Proof. Write \mathcal{K} and $\tilde{\mathcal{K}}$ for the spaces corresponding to the two problems. The assumption that $p \leq \tilde{p}$ and $q \leq \tilde{q}$ implies that

$$\mathcal{K} \supset \tilde{\mathcal{K}}.$$

By Poincaré's minimax characterization of the higher eigenvalues,

$$\tilde{E}_k = \min_{\tilde{S}} \max_{f \in \tilde{S}} \frac{\int_0^L (|f'|^2 \tilde{p} + f^2 \tilde{q}) dx}{\int_0^L f^2 r dx},$$

where \tilde{S} ranges over all k -dimensional subspaces of $\tilde{\mathcal{K}}$. Hence

$$\tilde{E}_k \geq \min_{\tilde{S}} \max_{f \in \tilde{S}} \frac{\int_0^L (|f'|^2 p + f^2 q) dx}{\int_0^L f^2 r dx}.$$

Every subspace of \tilde{K} is also a subspace of \mathcal{K} , and so

$$\tilde{E}_k \geq \min_S \max_{f \in S} \frac{\int_0^L (|f'|^2 p + f^2 q) dx}{\int_0^L f^2 r dx} = E_k$$

where S ranges over all k -dimensional subspaces of \mathcal{K} . □

Appendix C

C Application for $f(x)$ in harmonic oscillator equation

Now let us check the Weyl's criterion for our eigenfunction $f(x)$:

$$-(f'' + \frac{1}{x}f' - \frac{n^2}{x^2}f) + \beta^2 x^2 f = Ef.$$

Then rewrite this equation in form of Equation (B.1) in Appendix B.

$$-(xf')' + (\frac{n^2}{x^2} + \beta^2 x^2)xf = Exf$$

In other words,

$$-(xf_x)_x + (\frac{n^2}{x} + \beta^2 x^3)f = Exf, \quad 0 < x < 1.$$

Now check the assumptions in Appendix B:

1. $x > 0$ on $(0, 1]$.
2. x, x' and $\frac{n^2}{x} + \beta^2 x^3$ are real-valued and continuous on $(0, 1]$.
3. (i) When $n \neq 0$, we see $\frac{n^2}{x^2} + \beta^2 x^2 \rightarrow \infty$ as $x \rightarrow 0$.
(ii) When $n = 0$, we check that $\beta^2 x^2$ is bounded from below, and $x, xP(x)$ are integrable on $(0, 1)$, where $P(x) = \int 1/p(x) dx = \int 1/x dx = \ln x$.
Note: part (ii) actually works for all n , not just for $n = 0$.

One can find a lot more information about Sturm-Liouville problem in Zettl [43].

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