OPTIMAL SAMPLING AND SWITCHING POLICIES FOR LINEAR STOCHASTIC SYSTEMS

BY

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THESIS

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ABSTRACT

For systems with limited capacity for storage, processing and transmission of data, the choice of sampling policy is critical. Although most systems determine their sampling instants in advance, for example periodically, this results in unnecessary use of samples if little changes occur between sampling times. Instead, to optimize the utilization of the samples, the decision to take a sample can be adaptively made based on the importance of the change in the state of the system. This calls for development of event-triggered sampling policies. In this thesis, we study the optimal event-triggered sampling policies under a constraint on the frequency of sampling. We first investigate the optimal sampling policies to minimize the estimation error over the infinite horizon. The optimal policies are provided for multidimensional Wiener processes and scalar linear diffusion processes. Then, we address an infinite horizon control problem with a stochastic process driven by a bang-bang controller. We obtain the optimal times to switch the control signal that determines the drift rate of the process. For the cases handled in this thesis, the results suggest the optimality of the simplest event-triggered sampling policy with constant thresholds over the infinite horizon.

To my parents, my sister Iraz, and my brother Ilgın

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LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^m$	m-dimensional Euclidian space
$\mathbb{E}$	Expectation
$\lvert \cdot \rvert$	Absolute value
$\lVert \cdot \rVert_2$	Euclidian norm
t	Time
T	Horizon length
δ_t, δ_k	Decision rule
$\tilde{\delta}_x$	Dirac delta function
U	Snell envelope
λ	Lagrange multiplier
V	Value function

CHAPTER 1

INTRODUCTION

Finite capacity for storage, processing and transmission of data renders sampling necessary for most systems. For dynamic systems, the sampling instants are determined by their sampling policies; and these policies can be considered in two categories, namely, time-triggered and event-triggered sampling policies.

In time-triggered sampling policies, sampling times are determined in advance as certain time instants. Since the policy is independent of how the process evolves over time, it can be calculated offline by a central unit as an open-loop sampling rule.

Periodic sampling, which separates the successive sampling instants by an equal amount of time, is the most commonly used time-triggered sampling method. Both the analysis and the design of periodically sampled systems are relatively simpler and well developed in control theory [1]. For example, for periodically sampled linear time invariant (LTI) systems, the characteristic equation of the systems can be written as a constant coefficient linear difference equation, and stability, controllability and observability of the systems can be checked easily.

Time-triggered sampling policies, however, lack the ability to take samples adaptively. For example, when a system is being sampled with a time-triggered sampling policy, little or no change might possibly occur between two consecutive samples and the new information obtained by the new sample might be inconsequential. Conversely, if the state of a control system deviates greatly from its estimate between two samples, the estimate and the control action need to be updated; however, this update cannot be performed in a timely fashion if the next sampling time is not close.

To improve the utilization of the samples, sampling policies can be augmented with the capacity to use the online information about the state of the system. Such sampling policies are referred as event-triggered throughout this thesis; however, they exist in the literature with various names, such as event-based sampling [2], state-dependent sampling [3], event-driven sampling [4], and Lebesgue sampling [5], [6].

Basically, event-triggered sampling policies define, for each time instant, a set consisting of the points in the state space for which the deviation from the estimate is regarded as unimportant. A new sample is taken at an instant only if the state leaves this set of unimportant events specified for that instant. The simplest event-triggered sampling policy is obtained if the sets of unimportant events are defined only based on the amount of deviation from the estimate, irrespective of time. For scalar valued processes, this corresponds to marking equally spaced level values for the state of the process and taking a sample whenever the state leaves one level and arrives at a neighboring one. Not using time information, such a sampling policy does not require synchronization. Consequently, this type of event-triggered policies is practical for many systems including multi-rate systems which encompass several subsystems operating at different frequencies [7].

Since event-triggered sampling policies call for an action only when it becomes necessary, they improve the usage of communication and computation resources [7] and play a critical role in optimization of sampling schemes.

1.1 Literature Review on Optimization of Sampling Schemes

Distributed systems may have their sensors, actuators, and controllers in different locations, and a communication network can be used for the transfer of information between these constituents of the systems [8]. However, usage of communication networks imposes practical limitations on the amount of information that can be transferred. For example, frequent attempts to use the communication links create congestion in the network, which leads to increased delays and increased rate of data dropout in the network. The network reliability degrades, and the effective data rate provided for the systems using the network diminishes.

Depending on the dynamics of the system, if the data dropout rate exceeds a critical value, the estimation error might grow unboundedly large for all possible estimators [9], [10]. Similarly, below a certain data rate, there may not exist any control scheme capable of stabilizing an unstable system [11]–[14]. Therefore, it is crucial to optimize the sampling scheme and the utilization of communication resources in networked control systems.

To improve the estimation and the control performance, the exact times for the information transmissions can be calculated by a central node. Since this calculation is made in advance, the scheduling scheme is prescribed as an open-loop time-triggered policy. The calculation of the scheduling policy is carried out based on the a priori knowledge of noise statistics and the dynamics of the processes. For example, for linear time-invariant stochastic diffusion processes with sampling constraints, periodic sampling is calculated to be the optimal open-loop policy to minimize the aggregated estimation error over a finite horizon [15], [16].

Optimization of open-loop sampling policies in control problems is also prevalent in the literature. A continuous time linear quadratic Gaussian (LQG) control problem with a limited number of samples is formulated in [17]. Only open-loop policies are considered and the possible sampling times are constrained to a finite set of fixed time instants. For this problem, the optimal control and the optimal timing of the measurements are suggested to be separable, which heavily depends on the fact that the policies are open-loop. Likewise, stated in [18] is a general finite horizon discrete time problem that can incorporate measurement costs or constraints. An optimal policy is suggested to be computable offline for the special LQG case. A similar control problem with an unknown initial state and a limited number of samples is handled in [19]. In this problem, even though the observations are noisy, the process is assumed to be deterministic.

If there are multiple alternatives for a measurement at each time, selection of measurements can also be optimized. For example, the optimal sequence of measurement vector selection is calculated offline in [20] to estimate a function of the state of a linear system at a desired time. Whereas only deterministic policies are considered in [20], a stochastic selection algorithm is sought in [21]. Upper and lower bounds on the expected covariance of the estimate are provided in [21], and an optimal probability distribution for the selection algorithm is obtained by minimizing the upper bound value. The solution for the network with the star structure in [21] is also extended in [22] to a multi-hop wireless sensor network, which has a tree structure. While the problems in [20]–[22] concern only estimation, an optimal control problem is formulated in [23] for an LQG process. An optimal measurement selection algorithm is obtained for both finite and infinite horizons.

To optimize the use of sampling or communication quota in a distributed system, event-triggered policies can also be used. This, however, requires online information about the evolution of the processes. Since this information cannot be transferred from every node to a central node continuously, the sampling instants need to be determined in a decentralized manner. In other words, every node in the network needs to decide when to take a sample on its own.

Event-triggered sampling appears, for example, in a discrete time problem with an estimator and a remote sensor in [24] and [25]. Jointly optimal estimation and scheduling policies are investigated for a sensor which can transmit only a limited number of its samples over a finite horizon. The optimal policies are obtained via dynamic programming for independent identically distributed and Gauss-Markov processes. The results are extended to vector processes in [26].

A similar problem is handled in [27] with an energy harvesting sensor. The number of samples that the sensor can transmit to the estimator is not fixed, but it depends on the amount of energy the sensor has collected, which is another random process. Majorization theory, in addition to dynamic programming, is employed to show the optimality of the policies suggested for a special class of processes.

While the problems in [24]–[27] are all in discrete time, an equivalent problem in continuous time is studied in [16]. To find the optimal sampling rule for a scalar Wiener process over finite horizon, the original problem is converted to an optimal multiple stopping problem, and the solution is obtained iteratively. A numerical solution procedure is also suggested for scalar linear stochastic processes with drift terms. An extension is presented in [28] for the vector case with noisy observations along with a suboptimal policy.

Instead of placing hard constraint on the transmission of samples, it is also possible to associate some cost for each transmission. For example, in [29], an aggregated cost of estimation error and number of transmissions is used over finite horizon. The jointly optimal estimation and sampling policies are given for scalar first-order LTI processes.

Even though scheduling the transmissions between a single sensor and an estimator is extensively studied, analysis of event-triggered scheduling for networks with several nodes appears in few works in the literature. For example, a special network architecture with event-triggered sampling is suggested in [30] to reduce the communication load. In this network, every node holds the estimate of all other nodes, and broadcasts its value only if its state deviates from its own estimate by a predetermined amount. In [31], each node of this architecture is defined as an LTI system and the communication network is assumed to have a constant delay. An optimal sampling policy is obtained which minimizes an infinite horizon average cost on estimation error and communication rate. Although probabilistic transmission strategies are also considered, the optimal policy is shown to be deterministic. A computationally feasible suboptimal sampling policy is also suggested in [32]. The stability of a similar design with data dropouts is analyzed in [33]. Stabilization of distributed networks over ad-hoc networks is shown to be possible even if the synchronization of the communication network is poor.

Event-triggered sampling can be integrated into several control schemes, such as proportional-integral-derivative (PID) control [34], impulse control [5], and on-off control [35]. However, when an event-triggered sampling policy is used in a closed-loop control system, the controller can intentionally drive the state out of its desired path to trigger a new sample and obtain a new estimate of the state. This phenomenon is called the dual effect [36]–[38], and renders the analysis of optimal solutions infeasible for most cases.

An event-triggered finite horizon control problem is formulated in [39] in continuous time. For a scalar system with simple dynamics, jointly optimal event-triggered sampling and control policies are investigated. Even though the control functions are constrained to be constant between sampling instants, obtaining an analytic solution for even only a few samples becomes infeasible.

An infinite horizon control problem with average cost is presented in [40]. Given a finite set of triggering levels for the state and a finite set of control values, existence of an optimal mapping from the triggering levels to the control values is shown. For this, the running cost is aggregated between sampling times and the problem is converted to a finite state Markov chain, similar to [41]. An iterative algorithm converging to the optimal policy is also provided in [40].

Despite the presence of similar problems with a finite horizon, optimal sampling policies have not been studied over the infinite horizon with hard constraints on the sampling. This problem constitutes the theme of this thesis.

1.2 Overview of Chapters

In Chapter 2, optimal sampling of multidimensional Wiener processes is studied. The aggregated estimation error is minimized subject to a hard constraint on the number of samples. Optimal event-triggered sampling policy is obtained for both finite and infinite horizons. Some simulation results are also provided to support the optimality of the policy suggested.

Optimal scheduling of samples over the infinite horizon is investigated for scalar linear diffusion processes in Chapter 3. By aggregating the cost on estimation error between sampling times, the original problem is converted to an optimal stopping problem. For scalar Wiener processes, the optimal solution is recalculated in this way and shown to be the same as the solution in Chapter 2. By using dynamic programming, the optimal policy for the processes with drift terms is also obtained.

In Chapter 4, an optimal switching problem is introduced. Optimal times to switch the control from one of two values to the other are examined for a bang-bang controller steering a scalar Wiener process. Average deviation from a reference level is minimized subject to a hard constraint on the average frequency of switching.

In Chapter 5, the results are discussed, possible directions for future research are provided, and the thesis is concluded.

CHAPTER 2

OPTIMAL SAMPLING OF MULTIDIMENSIONAL WIENER PROCESSES

The Wiener process is fundamental to the study of stochastic calculus and stochastic control [42]. In this chapter, we investigate the optimal event-triggered sampling policy for multidimensional Wiener processes.

Let $w_1(t), w_2(t), \cdots, w_m(t)$ denote independent standard Wiener processes. Introduce the multidimensional Wiener process

$$y_t = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

with

$$dy_t = \sum_{i=1}^{m} e_i dw_i,$$

where e_i is a column vector in $\mathbb{R}^m$ with 1 in the i^{th} entry and 0 in the others.

Let δ_t denote an event-triggered sampling rule:

$$\delta_t = \begin{cases} 1 & \text{if } y_t \text{ is sampled at time } t \\ 0 & \text{otherwise,} \end{cases}$$

and let $\hat{y}_t$ denote the estimate of y_t based on the samples obtained up to time t. In the following sections, we search for an optimal sampling policy, first in the finite horizon, and then over the infinite horizon.

2.1 Sampling over Finite Horizon

In this section, we want to find an optimal event-triggered sampling policy which minimizes

$$\mathbb{E}\left[\int_0^T \|y_t - \hat{y}_t\|_2^2 dt\right] \tag{2.1a}$$

over causal sampling policies with a constraint on the number of samples:

$$\sum_{t\in(0,T]} \delta_t = N. \tag{2.1b}$$

A discrete time version of this problem has already been studied in [27]. Being a Wiener process, the process y_t is neat [27], i.e., it has independent and zero-mean increments, and increments of small size are more likely than those with larger size. The cost function is quadratic in the estimation error as well; therefore, the assumptions in [27] hold. Although the results therein are derived for discrete time processes, they suggest that the jointly optimal estimator-scheduler pair could be given as

$$\hat{y}_t = y_{t_{\text{last}}}, \tag{2.2a}$$

$$\delta_t = \begin{cases} 1 & \text{if } \|y_t - \hat{y}_t\|_2^2 \geq \Delta(t, r_t) \\ 0 & \text{otherwise,} \end{cases} \tag{2.2b}$$

where r_t is the number of remaining samples to be used in $[t, T]$, Δ is a function which is referred as the (thresholding) envelope, and

$$t_{\text{last}} = \max\left\{s \leq t \mid \delta_s = 1\right\}.$$

As a typical event-triggered sampling policy, a new sample is taken whenever the estimation error reaches the thresholding envelope.

To find the optimal sampling policy for the problem (2.1), we extend the results obtained in [16] to multidimensional processes. With the estimator in (2.2a), if we define the estimation error as

$$x_t = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} := y_t - \hat{y}_t,$$

$x_i = 0$ at sampling instants and $dx_i = dw_i$ between sampling instants for all $i = 1, 2, \cdots, m$.

When the process y_t is not sampled,

$$\begin{aligned}\mathbb{E}\left[\int_0^T \|y_t-\hat{y}_t\|_2^2 dt\right] &= \mathbb{E}\left[\int_0^T \|x_t\|_2^2 dt\right] \\ &= \mathbb{E}\left[\int_0^T \sum_{i=1}^m x_i^2(t)dt\right] \\ &= m\frac{T^2}{2}.\end{aligned}$$

Using this as the base condition, we will show by iteration that the minimum value for (2.1a) subject to (2.1b) is proportional to T^2, and its factor depends on the number of samples N.

Assume that the minimum error that can be obtained taking only $(n-1)$ samples over an interval is proportional to the square of the length of the interval, i.e.,

$$\inf_{\delta^{n-1}} \mathbb{E}\left[\int_{t_1}^{t_2} \|x_t\|_2^2\ dt\right] = \frac{\theta_{n-1}}{2}(t_2-t_1)^2, \tag{2.3}$$

where t_1 is a moment at which $x_{t_1}=0$, δ^{n-1} is the set of causal sampling rules that allow only $(n-1)$ samples in the interval $(t_1,t_2]$, and θ_{n-1} is a positive constant. With this assumption, sampling policies taking n samples can be associated with those taking $(n-1)$ samples through a dynamic programming argument. If δ^{n*} is the optimal sampling policy with n samples and τ^* is the optimal time to take the first sample, δ^{n*} must coincide with $\delta^{(n-1)*}$ after τ^* and the expected cost incurred from τ^* to T must be $\frac{\theta_{n-1}}{2}(T-\tau^*)^2$. Hence, the optimal time τ^* to take the first of n samples can be obtained by minimizing

$$\begin{aligned}J(\tau) &= \mathbb{E}\left[\int_0^\tau \|x_t\|_2^2 dt + \frac{\theta_{n-1}}{2}(T-\tau)^2\right] \\ &= \mathbb{E}\left[\int_0^\tau \sum_{i=1}^m x_i^2(t)dt + \frac{\theta_{n-1}}{2}(T-\tau)^2\right].\end{aligned} \tag{2.4}$$

To eliminate the integral term in (2.4), using the Itô rule [43], we note that

$$d\left((T-t)\sum_i x_i^2\right) = -\sum_i x_i^2 dt + 2(T-t)\sum_i x_i dw_i + m(T-t)dt.$$

Since $\mathbb{E}\left[\int_0^\tau (T-t)\sum_i x_i dw_i\right] = 0$,

$$\begin{aligned}\mathbb{E}\left[\int_0^\tau \sum_i x_i^2 dt\right] &= \mathbb{E}\left[-\int_0^\tau d\left((T-t)\sum_i x_i^2\right) + \int_0^\tau m(T-t)dt\right] \\ &= \mathbb{E}\left[-(T-\tau)\sum_i x_i^2(\tau) + m\frac{T^2}{2} - m\frac{(T-\tau)^2}{2}\right].\end{aligned} \tag{2.5}$$

Inserting (2.5) into (2.4), we obtain

$$J(\tau) = m\frac{T^2}{2} - \mathbb{E}\left[(T-\tau)\sum_i x_i^2(\tau) + \frac{m-\theta_{n-1}}{2}(T-\tau)^2\right]. \tag{2.6}$$

Therefore, minimizing $J(\tau)$ is equivalent to maximizing

$$\mathbb{E}\left[2(T-\tau)\sum_i x_i^2(\tau) + (m-\theta_{n-1})(T-\tau)^2\right]. \tag{2.7}$$

To solve this optimal stopping problem [44], we can find its Snell envelope [45] $U(t,x)$ which satisfies

$$U_t + \frac{1}{2}\sum_i U_{x_i x_i} = 0, \tag{2.8a}$$

$$U(t,x) \geq 2(T-t)\sum_i x_i^2(t) + (m-\theta_{n-1})(T-t)^2, \tag{2.8b}$$

and attains equality in (2.8b) for some value of $\|x_t\|_2$ for each t. Then, for any $\tau \in (t,T]$,

$$\begin{aligned}\mathbb{E}\left[U(\tau,x_\tau)|x_t\right] &= \mathbb{E}\left[U(t,x_t) + \int_t^\tau dU(s,x_s)\middle| x_t\right] \\ &= \mathbb{E}\left[U(t,x_t) + \int_t^\tau \left[U_s(s,x_s)ds \right.\right. \\ &\quad \left.\left. + \sum_i U_{x_i}(s,x_s)dw_i + \frac{1}{2}\sum_i U_{x_i x_i}(s,x_s)\right]\middle| x_t\right] \\ &= U(t,x_t)\end{aligned}$$

and

$$\begin{aligned} U(t, x_t) &= \mathbb{E}\left[U(\tau, x_\tau) | x_t\right] \\ &\geq \mathbb{E}\left[2(T-\tau)\sum_i x_i^2(\tau) + (m-\theta_{n-1})(T-\tau)^2 \middle| x_t\right]. \end{aligned}$$

This shows that if

$$2(T-t)\|x_t\|_2^2 + (m-\theta_{n-1})(T-t)^2$$

reaches $U(t, x)$, no $\tau > t$ can yield a larger value in expectation, and thus, t is the optimal time to stop.

Conditions (2.8a) and (2.8b) are satisfied by

$$U(t, x) = \beta_n \left[m(T-t)^2 + 2(T-t)\|x\|_2^2 + \frac{\|x\|_2^4}{m+2}\right] \tag{2.9a}$$

with

$$\beta_n = \frac{4 + m + \theta_{n-1} - \sqrt{\theta_{n-1}^2 + (8+2m)\theta_{n-1} + m^2}}{4}. \tag{2.9b}$$

Therefore, (2.9) serves as the Snell envelope for maximizing (2.7). Utilizing this Snell envelope in (2.6), the optimal cost with n samples can be calculated as

$$\begin{aligned} \inf_\tau J(\tau) &= m\frac{T^2}{2} - \frac{1}{2}U(0, x_0) \\ &= m\frac{T^2}{2} - \frac{1}{2}m\beta_n T^2 \\ &= m(1-\beta_n)\frac{T^2}{2} \\ &= \theta_n \frac{T^2}{2}, \end{aligned} \tag{2.10}$$

where we defined

$$\theta_n := m(1-\beta_n). \tag{2.11}$$

The optimal cost in (2.10) verifies that the assumption (2.3) holds for n

samples as well. Then, the optimal stopping time can be found as

$$\inf\left\{t \geq 0 \;\middle|\; 2(T-t)\|x_t\|_2^2 + (m-\theta_{n-1})(T-t)^2 \geq U(t,x_t)\right\}$$
$$= \inf\left\{t \geq 0 \;\middle|\; \|x_t\|_2^2 \geq \sqrt{\frac{(m+2)(\theta_{n-1}-\theta_n)}{1-\frac{\theta_n}{m}}}(T-t)\right\}.$$

As a result, from (2.9b) and (2.11), for $n = 1, 2..., N$, θ_n is calculated iteratively by

$$\theta_n = \frac{m}{4}\left(-m - \theta_{n-1} + \sqrt{\theta_{n-1}^2 + (8+2m)\theta_{n-1} + m^2}\right) \tag{2.12a}$$

with $\theta_0 = m$, and the optimal time τ_n^* to take the n^{th} sample is obtained as

$$\inf\left\{t \geq \tau_{n-1}^* \;\middle|\; \|x_t\|_2^2 \geq \sqrt{\frac{(m+2)(\theta_{N-n}-\theta_{N-n+1})}{1-\frac{\theta_{N-n+1}}{m}}}(T-t)\right\} \tag{2.12b}$$

with $\tau_0^* = 0$.

Comparing (2.12b) with the scheduling rule in (2.2b), the optimal thresholding envelope for the finite horizon problem (2.1) is given as

$$\Delta(t, r_t) = \sqrt{\frac{(m+2)(\theta_{r_t-1}-\theta_{r_t})}{1-\frac{\theta_{r_t}}{m}}}(T-t).$$

2.2 Solution over the Infinite Horizon

In this section, our aim is to find an event-triggered sampling policy over the infinite horizon which minimizes the average estimation error

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \|y_t - \hat{y}_t\|_2^2 dt\right] \tag{2.13a}$$

subject to a constraint on the frequency of sampling:

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t\in(0,T]} \delta_t\right] = \frac{1}{c}, \tag{2.13b}$$

where c is a positive constant.

We can approach this infinite horizon optimal sampling problem by using the solution for the finite horizon problem (2.1). If the final time T is driven to infinity while preserving the ratio of number of samples to the length of horizon, $\frac{N}{T}$, the policy for taking the very first sample coincides with the optimal policy for (2.13).

Let $N = n$ and $T = cn$, where n is a positive integer. If the sampling policy in (2.12) is used to determine sampling times,

$$\frac{1}{T}\sum_{t\in(0,T]}\delta_t = \frac{1}{c}$$

is satisfied for all n values. Then, the first sampling time is given by

$$\tau_1^* = \inf\left\{ t \geq 0 \;\middle|\; \|x_t\|_2^2 \geq \sqrt{\frac{(m+2)(\theta_{n-1}-\theta_n)}{1-\frac{\theta_n}{m}}}(cn-t) \right\}. \tag{2.14}$$

To find the infinite horizon policy, we will drive n to infinity in (2.14). For this, we need to calculate the following limit

$$\lim_{n\to\infty} \frac{(m+2)(\theta_{n-1}-\theta_n)}{1-\frac{\theta_n}{m}}(cn-t)^2 \tag{2.15}$$

if it exists. The following three lemmas show the existence of this limit and give its value.

Lemma 2.1: $\{\theta_n\}$ is a strictly decreasing positive sequence converging to 0.

Proof: We define the following continuous function

$$g(\xi) = \frac{m}{4}\left(-m-\xi+\sqrt{(\xi+m+4)^2-8m-16}\right)$$

so that $\theta_n = g(\theta_{n-1})$. Then,

$$\frac{dg(\xi)}{d\xi} = \frac{m}{4}\left(-1+\frac{\xi+m+4}{\sqrt{(\xi+m+4)^2-8m-16}}\right)$$

is positive for all $\xi \geq 0$, implying that g is strictly increasing on $[0,\infty)$. Since

$$0 < g(m) = \frac{m}{2}\left(\sqrt{m^2+2m}-m\right) < \frac{m}{2} < m,$$

we have $0 < \theta_1 = g(m) < \theta_0 = m$. Considering that g is an increasing function on $[0, \infty)$, given $0 < \theta_n < \theta_{n-1}$,

$$g(0) < g(\theta_n) < g(\theta_{n-1}) \implies 0 < \theta_{n+1} < \theta_n.$$

Therefore, $0 < \theta_n < \theta_{n-1}$ for all $n \geq 1$, i.e., $\{\theta_n\}$ is a decreasing sequence bounded below from 0. Since 0 is the only non-negative stationary point of g, the sequence $\{\theta_n\}$ converges to 0. ■

Lemma 2.2:

$$\lim_{n\to\infty} n\theta_n = \frac{m^2}{m+2}$$

Proof: After multiplying both sides of (2.12a) with n, we define $\alpha_n := n\theta_n$ and write the iteration for α_n as

$$\begin{aligned}\alpha_n &= \frac{m}{4}\left(-mn - \frac{n\alpha_{n-1}}{n-1} + \sqrt{\frac{n^2\alpha_{n-1}^2}{(n-1)^2} + \frac{(8+2m)n^2\alpha_{n-1}}{(n-1)} + m^2n^2}\right)\\ &=: f_n(\alpha_{n-1}) \end{aligned} \qquad (2.16)$$

for $n \geq 2$, with $\alpha_1 = \theta_1 = \frac{m}{2}\left(\sqrt{m^2+2m} - m\right)$.

From (2.16), if $\alpha_{n-1} > 0$, then $\alpha_n > 0$ as well. Since $\alpha_1 > 0$,

$$\alpha_n > 0 \text{ for all } n \geq 1. \qquad (2.17)$$

Subtracting α_{n-1} from both sides of (2.16),

$$\alpha_n - \alpha_{n-1} < 0 \text{ if and only if } h_n(\alpha_{n-1}) := \left(\frac{m+2}{m^2} - \frac{2}{m^2 n}\right)\alpha_{n-1} > 1. \qquad (2.18)$$

Then, either of the following cases is true regarding $h_n(\alpha_{n-1})$.

- Case 1: $h_n(\alpha_{n-1}) \leq 1$ for all $n \geq 2$

Due to reverse implication in (2.18), $\alpha_n \geq \alpha_{n-1}$ for all $n \geq 2$. In addition,

$$h_n(\alpha_{n-1}) = \left(\frac{m+2}{m^2} - \frac{2}{m^2 n}\right)\alpha_{n-1} \leq 1 \text{ for all } n \geq 2$$

implies

$$\alpha_n \leq \frac{m^2}{m+1} \text{ for all } n \geq 1.$$

Therefore, the sequence $\{\alpha_n\}$ is a nondecreasing sequence bounded from above.

- Case 2: $h_{n_0}(\alpha_{n_0-1}) > 1$ for some $n_0 \geq 2$

If $h_n(\alpha_{n-1}) > 1$, i.e.,

$$\alpha_{n-1} > \frac{m^2 n}{(m+2)n-2},$$

then $h_{n+1}(\alpha_n) > 1$ as well, since

$$\frac{m^2 n}{(m+2)n-2} > \frac{m^2(n^2-1)\left((m+2)n^2+(m+2)n+2\right)}{n\left((m+2)n+m\right)\left((m+2)n^2-m\right)} \quad \text{for all } n \geq 2$$

and

$$h_{n+1}(\alpha_n) > 1 \text{ if and only if } \alpha_{n-1} > \frac{m^2(n^2-1)\left((m+2)n^2+(m+2)n+2\right)}{n\left((m+2)n+m\right)\left((m+2)n^2-m\right)}.$$

Therefore, $h_n(\alpha_{n-1}) > 1$ for all $n \geq n_0$, which implies that the sequence $\{\alpha_n\}_{n_0-1}^{\infty}$ is a decreasing sequence. From (2.17), it is also bounded from below by 0.

As a result, in either case, $\overline{\alpha} := \lim_{n\to\infty} \alpha_n$ exists and it is finite. Then, $\overline{\alpha}$ must solve

$$\lim_{n\to\infty} [\overline{\alpha} - f_n(\overline{\alpha})] = 0$$

with f_n defined in (2.16). This yields

$$\lim_{n\to\infty} \left[\frac{\left(m^2\overline{\alpha} - (m+2)\overline{\alpha}^2\right)n}{m^2(n-1)} + \frac{2\overline{\alpha}^2}{m^2(n-1)}\right] = 0,$$

and the only non-zero solution for $\overline{\alpha}$ is

$$\overline{\alpha} = \lim_{n\to\infty} n\theta_n = \frac{m^2}{m+2}.$$

$\blacksquare$

Lemma 2.3:
$$\lim_{n\to\infty} n^2 \frac{(m+2)(\theta_{n-1}-\theta_n)}{1-\frac{\theta_n}{m}} = m^2$$

Proof: We rearrange the iterative definition of θ_n given in (2.12a) as

$$\frac{4}{m}\theta_n + \theta_{n-1} + m = \sqrt{\theta_{n-1}^2 + (8+2m)\theta_{n-1} + m^2},$$

and squaring both sides, after some cancellation, we obtain

$$\theta_{n-1} - \theta_n = \frac{2\theta_n^2}{m^2} + \frac{\theta_n \theta_{n-1}}{m}.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} n^2 \frac{(m+2)(\theta_{n-1} - \theta_n)}{1 - \frac{\theta_n}{m}} &= \lim_{n\to\infty} n^2 \frac{(m+2)\left(\frac{2\theta_n^2}{m^2} + \frac{\theta_n\theta_{n-1}}{m}\right)}{1 - \frac{\theta_n}{m}} \\
&= \lim_{n\to\infty} \frac{m+2}{m^2}\left(2n^2\theta_n^2 + \frac{mn}{n-1} n\theta_n (n-1)\theta_{n-1}\right) \\
&= \frac{(m+2)^2}{m^2}\left(\lim_{n\to\infty} n\theta_n\right)^2 \\
&= m^2
\end{aligned}$$

since $\lim_{n\to\infty} \theta_n = 0$ by Lemma 2.1 and $\lim_{n\to\infty} n\theta_n = \frac{m^2}{m+2}$ by Lemma 2.2.

■

Finiteness of the limit in Lemma 2.3 also implies

$$\lim_{n\to\infty} \frac{(m+2)(\theta_{n-1} - \theta_n)}{1 - \frac{\theta_n}{m}} = \lim_{n\to\infty} n \frac{(m+2)(\theta_{n-1} - \theta_n)}{1 - \frac{\theta_n}{m}} = 0.$$

Hence, the limit in (2.15) can be calculated as

$$\lim_{n\to\infty} \frac{(m+2)(\theta_{n-1} - \theta_n)}{1 - \frac{\theta_n}{m}} (cn - t)^2 = m^2 c^2$$

for all t. As a result, the rule to take the very first sample (2.14) converges to

$$\tau_1^* = \inf\left\{t \geq 0 \;\middle|\; \|x_t\|_2^2 \geq mc\right\}.$$

This yields to the following theorem.

Theorem 2.1: The optimal sampling policy that minimizes (2.13a) subject to (2.13b) is the following constant threshold rule:

$$\delta_t^* = \begin{cases} 1 & \text{if } \|y_t - \hat{y}_t\|_2^2 \geq mc \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

This shows that the simplest form of event-triggered sampling is the optimal policy to sample a multidimensional Wiener process under a hard constraint on the average frequency of sampling.

The minimum cost obtained with (2.19) can be computed using the limit of the time average of the optimal finite horizon cost (2.10):

$$\begin{aligned}\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \|y_t - \hat{y}_t\|_2^2 dt\right] &= \lim_{n\to\infty} \frac{1}{cn}\theta_n \frac{(cn)^2}{2} \\ &= \frac{\overline{\alpha} c}{2} \\ &= \frac{m^2}{2(m+2)} c. \end{aligned} \tag{2.20}$$

By contrast, conventional periodic sampling with the same average frequency leads to

$$\frac{1}{c}\,\mathbb{E}\left[\int_0^c \|x_t\|_2^2 dt\right] = \frac{1}{c} m \frac{c^2}{2} = \frac{m}{2} c.$$

Therefore, the ratio of the optimal cost to the periodic sampling cost is

$$\frac{m}{m+2}.$$

Let $\{\tau_n\}$ denote the sequence of time durations between consecutive sampling times. Even though the sampling constraint is expressed with the frequency of sampling in (2.13b), an equivalent constraint can be imposed on the expected time between consecutive sampling times. In fact, while increasing n to infinity to derive the optimal policy for the infinite horizon, we kept $\frac{T}{N}$ constant at c. Therefore, the average of the durations between consecutive samples is c. Assuming ergodicity, this implies that the sampling rule (2.19) yields an expected time of c between consecutive samples:

$$\mathbb{E}[\tau_n] = c. \tag{2.21}$$

Similar to [5], this can also be shown using the fact that

$$\|x_t\|_2^2 - mt = \sum_{i=1}^{m} \left(x_i^2(t) - t\right)$$

is a martingale in the interval $[0, \tau_1]$:

$$\mathbb{E}\left[\ \|x_t\|_2^2 - mt \ \middle|\ x_0 = 0\ \right] = 0.$$

Then, at the first sampling instant,

$$\mathbb{E}\left[\ \|x_{\tau_1}\|_2^2 - m\tau_1 \ \middle|\ x_0 = 0\ \right] = 0,$$

which implies

$$\mathbb{E}[\tau_1] = \frac{1}{m}\,\mathbb{E}\left[\|x_{\tau_1}\|_2^2\right] = c.$$

In [5], the steady state distribution of a one-dimensional Wiener process confined inside an absorbing constant envelope is also provided. The steady state Fokker-Plank equation [43] for the estimation error is given as

$$0 = \frac{1}{2}\frac{\partial^2\rho(x)}{\partial x^2} - \frac{1}{2}\frac{\partial\rho(\sqrt{c})}{\partial x}\tilde{\delta}_x + \frac{1}{2}\frac{\partial\rho(-\sqrt{c})}{\partial x}\tilde{\delta}_x$$

with the boundary conditions $\rho(\sqrt{c}) = \rho(-\sqrt{c}) = 0$, where $\tilde{\delta}_x$ denotes the Dirac delta function. This has the following solution:

$$\rho(x) = \begin{cases} \frac{(\sqrt{c}-|x|)}{c} & \text{if } |x| \leq \sqrt{c} \\ 0 & \text{otherwise.} \end{cases}$$

With this probability density function, $\mathbb{E}[x^2]$ is calculated to be $\frac{c}{6}$, which is the same as (2.20) for $m = 1$.

2.3 Numerical Comparison with a Special Class of Envelopes

The envelope of the solution for the problem of minimizing (2.13a) subject to (2.13b) is expected to be monotonic. In this section, we estimate the costs incurred by a special class of monotonically increasing and monotonically decreasing thresholding envelopes, and show that they yield larger costs than that of a constant threshold.

Let x_t be a one-dimensional Wiener process. If

$$\tau = \inf\left\{t \geq 0 \middle| |x_t| = a\sqrt{t+b}\right\},$$

where $0 \le a < 1$ and $0 < b < \infty$, then [44]

$$\mathbb{E}[\tau] = \frac{a^2 b}{1 - a^2}.$$

Therefore, if we choose

$$b = \frac{1}{a^2} - 1,$$

the stopping rule

$$\tau = \inf\left\{ t \ge 0 \,\middle|\, |x_t| = a\sqrt{t + \left(\frac{1}{a^2} - 1\right)} \right\} \tag{2.22}$$

leads to $\mathbb{E}[\tau] = 1$ for all $a \in (0, 1)$. Note that the envelope is monotonically increasing for each value of a, and as a approaches 0, it becomes a constant threshold.

Figure 2.1 shows the estimated value of the cost (2.13a) when (2.22) is used for different values of a. Simulations were carried out with a step size of 10^{-4} and 30000 samples.

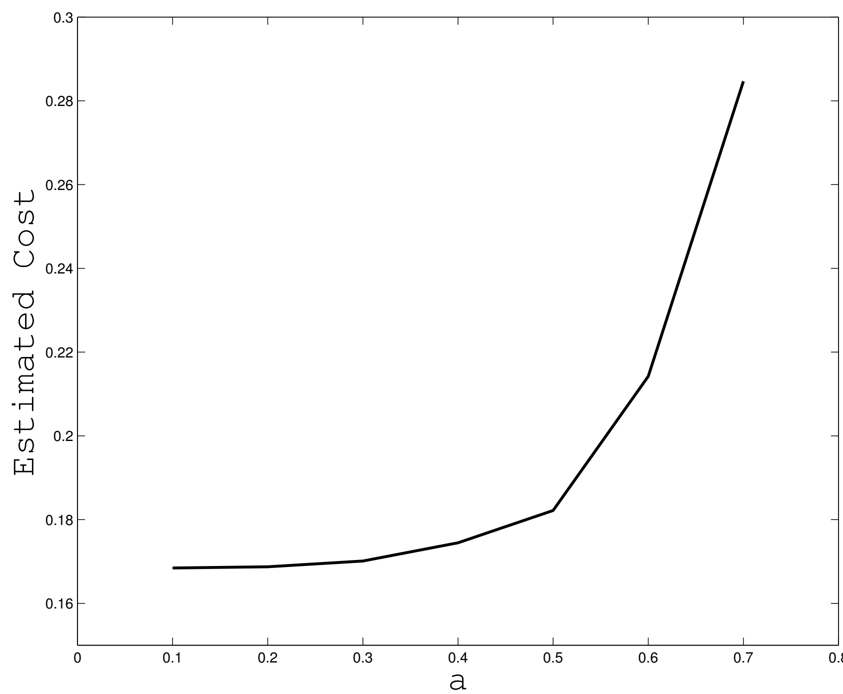

Figure 2.1: Estimated cost (2.13a) with the policy (2.22) for different values of a

Similarly, defining

$$\tau = \inf\left\{t \geq 0 \middle| |x_t| = a\sqrt{b-t}\right\},$$

where $0 < a < \infty$ and $0 < b < \infty$, we have

$$\mathbb{E}[\tau] = \frac{a^2 b}{1+a^2}.$$

Then, the following stopping rule

$$\tau = \inf\left\{t \geq 0 \middle| |x_t| = a\sqrt{\left(\frac{1}{a^2}+1\right) - t}\right\} \tag{2.23}$$

yields $\mathbb{E}[\tau] = 1$ for all $a \in (0, \infty)$. For each value of a, the stopping rule has a monotonically decreasing envelope, which becomes constant as a tends to 0. The estimated costs for different values of a are plotted in Figure 2.2.

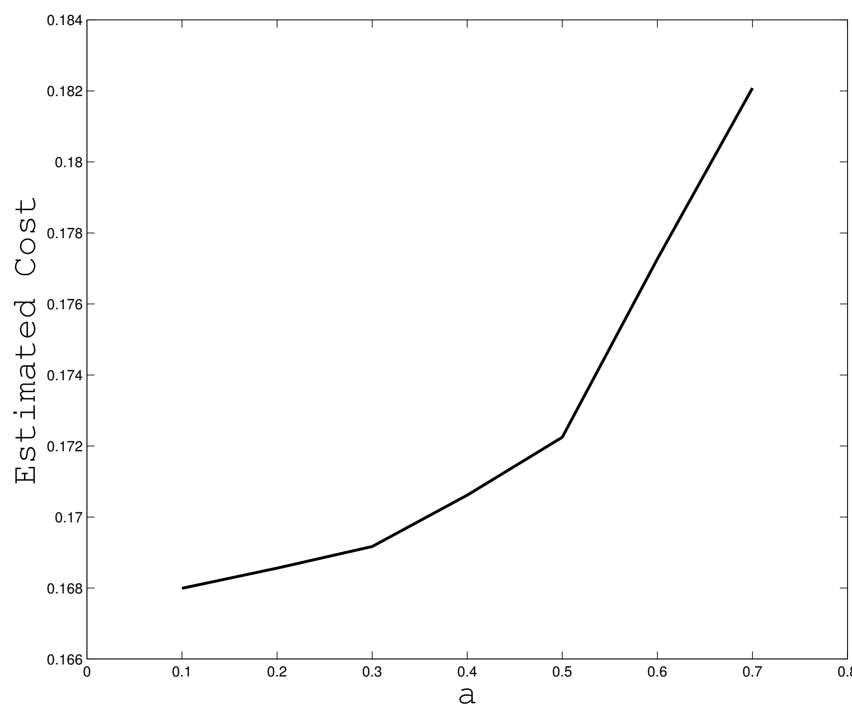

Figure 2.2: Estimated cost (2.13a) with the policy (2.23) for different values of a

We observe that deviating from a constant envelope to an increasing or a decreasing envelope in these classes leads to a larger cost even though the average time between consecutive sampling instants is the same. This result corroborates the optimality of the constant envelope sampling rule.

CHAPTER 3

OPTIMAL SAMPLING OF LINEAR DIFFUSION PROCESSES

In this chapter, we study the optimal infinite horizon sampling policy for scalar linear diffusion processes. Unlike the case for the Wiener processes in Chapter 2, analytic computation of an optimal finite horizon sampling policy turns out to be infeasible for linear diffusion processes. Therefore, we develop a new approach, which does not require the finite horizon solution, to solve the infinite horizon problem.

Let y_t be a one-dimensional stochastic linear diffusion process defined by a stochastic differential equation:

$$dy = aydt + dw$$

for some $a \in \mathbb{R}$. Let $\hat{y}_t$ denote its estimate based on the samples received by the estimator up to time t. Define the estimation error as

$$x_t := y_t - \hat{y}_t.$$

As the estimator, we set

$$\hat{y}_t = y_t$$

at sampling instants, and update it according to

$$d\hat{y} = a\hat{y}dt$$

between sampling instants. Then, the estimation error

$$x_t = 0 \tag{3.1a}$$

at sampling times, and

$$\begin{aligned} dx &= dy - d\hat{y} \\ &= aydt + dw - a\hat{y}dt \\ &= axdt + dw \end{aligned} \tag{3.1b}$$

between sampling times. This results in a causal unbiased estimator:

$$\mathbb{E}[y_t - \hat{y}_t] = \mathbb{E}[x_t] = 0.$$

Furthermore, at any time t, the probability distribution of the estimation error is symmetric around 0 and unimodal [27].

We refer to the sampling policies that depend on the time elapsed after the last sample, but not on the absolute time, as time-homogeneous policies. In other words, if t_{last} denotes the last sampling time,

$$\delta_t = \begin{cases} 1 & \text{if } |x| \geq \Delta(t - t_{\text{last}}) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

is a time-homogeneous policy, where $\Delta : [0, \infty) \to [0, \infty)$ denotes the thresholding envelope of the event-triggered policy. As we are working on an infinite horizon problem, restriction to time-homogeneous sampling policies is reasonable.

Let τ_k denote the length of time between k^{th} and $(k-1)^{\text{th}}$ samples. When a time-homogeneous sampling policy is used, τ_k's are independent and identically distributed.

Our goal is to find the optimal time-homogeneous sampling policy δ^* which minimizes

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T x^2 dt \tag{3.3a}$$

subject to

$$\mathbb{E}[\tau_k] = \mathbb{E}[\tau_1] = c. \tag{3.3b}$$

Note that taking expectation is not needed in (3.3a) since we use time-homogeneous policies, and hence, the estimation error is ergodic.

3.1 Reformulation of the Problem

To eliminate the limit in the problem statement (3.3), we use the following proposition.

Proposition 3.1: If a time-homogeneous sampling policy is used, then

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T x^2dt\right]=\frac{\mathbb{E}[\int_0^{\tau_1} x^2dt]}{\mathbb{E}[\tau_1]}.$$

Proof:

$$\begin{aligned}\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T x^2dt\right] &= \lim_{k\to\infty}\left[\frac{1}{k\mathbb{E}[\tau_1]}\int_0^{k\mathbb{E}[\tau_1]} x^2dt\right]\\ &= \frac{1}{\mathbb{E}[\tau_1]}\lim_{k\to\infty}\left[\frac{1}{k}\int_0^{k\mathbb{E}[\tau_1]} x^2dt\right]\end{aligned}$$

For any finite T, assume that a time-homogeneous sampling policy is used in $[0,T)$ and one last sample is taken at time $T=k\mathbb{E}[\tau_1]$. Define n_k as the number of samples taken in the interval $(0,k\mathbb{E}[\tau_1]]$. Then, sampling times are given as τ_1, $(\tau_1+\tau_2)$, $\cdots,(\tau_1+...+\tau_{n_k})$. Let $z_1,z_2,\cdots,z_{n_k}$ denote $\int_0^{\tau_1} x^2dt, \int_{\tau_1}^{\tau_1+\tau_2} x^2dt,\cdots,\int_{\tau_1+...+\tau_{n_k-1}}^{\tau_1+...+\tau_{n_k}} x^2dt$, respectively.

$$\begin{aligned}\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T x^2dt\right] &= \frac{1}{\mathbb{E}[\tau_1]}\lim_{k\to\infty}\left[\frac{1}{k}\int_0^{\tau_1} x^2dt+\cdots+\int_{\tau_1+...+\tau_{n_k-1}}^{\tau_1+...+\tau_{n_k}} x^2dt\right]\\ &= \frac{1}{\mathbb{E}[\tau_1]}\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{n_k} z_i\\ &= \frac{1}{\mathbb{E}[\tau_1]}\lim_{k\to\infty}\frac{n_k}{k}\frac{1}{n_k}\sum_{i=1}^{n_k} z_i \qquad (3.4)\end{aligned}$$

Since the first (n_k-1) samples are determined by a time-homogeneous sampling policy, $z_1,z_2,\cdots,z_{n_k-1}$ are independent and identically distributed random variables. The z_{n_k} is not identically distributed with the other terms; however, $\mathbb{E}[\tau_{n_k}]\le c<\infty$ and

$$\lim_{k\to\infty}\frac{1}{k}\int_{\tau_1+...+\tau_{n_k-1}}^{\tau_1+...+\tau_{n_k}} x^2dt=0$$

with probability 1. Hence, considering $z_1, z_2, \cdots, z_{n_k}$ as independent and identically distributed random variables in the following steps is of no consequence.

Since $\mathbb{E}[\tau_1] < \infty$, as k is driven to infinity, n_k goes to infinity as well. In fact,

$$\lim_{k\to\infty} \frac{n_k}{k} = 1. \tag{3.5}$$

This can also be verified from

$$\begin{aligned}
\mathbb{E}[\tau_1] &= \lim_{k\to\infty} \frac{\tau_1 + \tau_2 + \cdots + \tau_{n_k}}{n_k} \\
&= \lim_{k\to\infty} \frac{k\mathbb{E}[\tau_1]}{n_k} \\
&= \mathbb{E}[\tau_1] \lim_{k\to\infty} \frac{k}{n_k},
\end{aligned}$$

where the second equality follows from the definition of n_k.

Then, by the strong law of large numbers,

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} z_i = \lim_{n_k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} z_i = \mathbb{E}[z_1]. \tag{3.6}$$

Inserting (3.5) and (3.6) into (3.4), we obtain

$$\lim_{k\to\infty} \left[\frac{1}{T} \int_0^T x^2 dt \right] = \frac{\mathbb{E}[z_1]}{\mathbb{E}[\tau_1]} = \frac{\mathbb{E}[\int_0^{\tau_1} x^2 dt]}{\mathbb{E}[\tau_1]}.$$

■

As a result of Proposition 3.1, our goal is equivalent to minimizing

$$\mathbb{E}\left[\int_0^{\tau_1} x^2 dt \right] \tag{3.7a}$$

subject to

$$\mathbb{E}[\tau_1] = c. \tag{3.7b}$$

Note that it is also possible to show that (3.3) and (3.7) are equivalent by following a procedure used in [40], where an infinite horizon control problem with time-homogeneous control policies is transformed into a Markov chain.

3.2 Brownian Motion

Before handling stochastic processes including diffusion terms, we first address the problem (3.7) for Wiener processes. Let y_t be a Wiener process

$$dy = dw,$$

and $\hat{y}_t$ be its estimate given by the last sampled value. Then the estimation error x_t is 0 at sampling times, and it is governed by $dx = dw$ between sampling times.

To eliminate the integration in $\mathbb{E}[\int_0^{\tau_1} x^2 dt]$, consider $f(t,x) = \frac{x^4}{6}$. Applying the Itô rule,

$$\begin{aligned} df &= f_t dt + f_x dx + \frac{1}{2} f_{xx} dt \\ &= x^2 dt + \frac{2}{3} x^3 dw. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left[\int_0^{\tau_1} x^2 dt\right] &= \mathbb{E}\left[\int_0^{\tau_1} df - \frac{2}{3}\int_0^{\tau_1} x^3 dw\right] \\ &= \mathbb{E}\Big[f(\tau_1, x_{\tau_1}) - f(0, x_0)\Big] \\ &= \frac{1}{6}\,\mathbb{E}[x_{\tau_1}^4]. \end{aligned}$$

As a result, for Brownian motion, the goal in (3.7) can be stated as:

$$\begin{aligned} &\text{minimize} && \mathbb{E}[x_{\tau_1}^4] \\ &\text{subject to} && \mathbb{E}[\tau_1] = c. \end{aligned}$$

This can also be written as an unconstrained optimization problem with a Lagrange multiplier λ:

$$\text{minimize} \quad \mathbb{E}[\ x_{\tau_1}^4 - \lambda \tau_1\ ]. \tag{3.8}$$

Note that $\lambda \geq 0$, because otherwise $(x_{\tau_1}^4 - \lambda\tau_1) > 0$ for all $\tau_1 > 0$ and $(x_0^4 - \lambda 0) = 0$ for $\tau_1 = 0$, and thus, the optimal stopping rule would be to sample at $t = 0$. However, this cannot satisfy $\mathbb{E}[\tau_1] = c$ for $c > 0$.

The Snell envelope $U(t,x)$ for the optimal stopping problem (3.8) satisfies

$$U_t + \frac{1}{2}U_{xx} = 0, \tag{3.9a}$$

$$U(t,x) \leq x^4 - \lambda t \quad \forall t, \forall x, \tag{3.9b}$$

and attains equality in (3.9b) for some x value for each t. The following function

$$\lambda x^2 - \lambda t - \frac{\lambda^2}{4}$$

solves (3.9a) and

$$(x^4 - \lambda t) - \left(\lambda x^2 - \lambda t - \frac{\lambda^2}{4}\right) = \left(x^2 - \frac{\lambda}{2}\right)^2. \tag{3.10}$$

Thus, (3.9b) is also satisfied. This suggests that the Snell envelope for (3.8) is given by

$$U(t,x) = \lambda x^2 - \lambda t - \frac{\lambda^2}{4}.$$

Then, the optimal time to take a new sample is the first time $(x^4 - \lambda t)$ reaches $U(t,x)$ after the last sample. From (3.10), this corresponds to the instant at which

$$\left(x^2 - \frac{\lambda}{2}\right)^2 = 0$$

is satisfied. Consequently, the optimal infinite horizon sampling rule is given as

$$\delta_t^* = \begin{cases} 1 & \text{if } x_t^2 = \frac{\lambda}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

The sampling rule in (3.11) is a constant threshold delta-sampling rule and the value of the threshold is determined based on $\mathbb{E}[\tau_1] = c$. This also verifies the result of Chapter 2 for scalar Wiener processes, suggesting that the value of the Lagrange multiplier λ is $2c$.

3.3 Linear Stochastic Processes

In this section, we address the optimal sampling of scalar linear diffusion processes with drift terms. Let us specify the one-dimensional random process

y_t as

$$dy = aydt + dw,$$

where $a \neq 0$ and use the following estimate:

$$\begin{aligned} \hat{y} &= y && \text{at sampling times,} \\ d\hat{y} &= a\hat{y}dt && \text{otherwise.} \end{aligned}$$

Then, as shown in (3.1), the estimation error is 0 at sampling times and obeys

$$dx = axdt + dw \tag{3.12}$$

between sampling times.

To remove the integration in (3.7a), consider $f(t,x) = \frac{1}{2a}(x^2 - t)$. Again, by the Itô rule,

$$\begin{aligned} df &= f_t dt + f_x dx + \frac{1}{2} f_{xx} dt \\ &= f_t dt + f_x (axdt + dw) + \frac{1}{2} f_{xx} dt \\ &= x^2 dt + \frac{x}{a} dw. \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}\left[\int_0^{\tau_1} x^2 dt\right] &= \mathbb{E}\left[\int_0^{\tau_1} df - \int_0^{\tau_1} \frac{x}{a} dw\right] \\ &= \mathbb{E}\Big[f(\tau_1, x_{\tau_1}) - f(0, x_0)\Big] \\ &= \frac{1}{2a}\, \mathbb{E}\left[x_{\tau_1}^2 - \tau_1\right]. \end{aligned}$$

Therefore, for the estimation error expressed with (3.12) where $a \neq 0$, the problem (3.7) can be stated as

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2a}\, \mathbb{E}\left[x_{\tau_1}^2 - \tau_1\right] \\ \text{subject to} \quad & \mathbb{E}[\tau_1] = c. \end{aligned}$$

By using a Lagrange multiplier λ and considering $a > 0$ and $a < 0$ cases separately, this problem becomes equivalent to

$$\text{minimize} \quad \mathbb{E}[x_{\tau_1}^2 - \lambda\tau_1] \quad \text{for } a > 0, \tag{3.13a}$$

and

$$\text{maximize} \quad \mathbb{E}[x_{\tau_1}^2 - \lambda\tau_1] \quad \text{for } a < 0. \tag{3.13b}$$

In the sequel, only the case $a > 0$ is handled, but the solution procedure for $a < 0$ is identical.

To minimize $\mathbb{E}[x_{\tau_1}^2 - \lambda\tau_1]$, we can look for the Snell envelope. The Snell envelope $U(t, x)$ for this stopping problem must satisfy

$$U_t + axU_x + \frac{1}{2}U_{xx} = 0,$$

$$U(t, x) \leq x^2 - \lambda t \quad \forall t, \forall x, \tag{3.14}$$

and must attain equality in (3.14) for some x value for each t. Finding $U(t, x)$, however, turns out to be infeasible.

Instead, we first discretize the problem as

$$x_{k+1} = \alpha x_k + w_k,$$

where w_k's are independent and identically distributed zero-mean Gaussian random variables with variance σ^2, and $\alpha > 1$ since $a > 0$. Then, we create a truncated problem with finite horizon N, and use dynamic programming to find the characteristics of the value function, which we denote by $V^N(x, k)$.

At the end of the horizon, the process has to be stopped:

$$V^N(x, N) = x^2 - \lambda N.$$

At step $(N - 1)$:

$$\begin{aligned} V^N(x, N-1) &= \min\{x^2 - \lambda(N-1)\ ,\ \mathbb{E}[V^N(\alpha x + w_{N-1}, N)]\} \\ &= \min\{x^2 - \lambda N + \lambda\ ,\ \mathbb{E}[(\alpha x + w_{N-1})^2 - \lambda N]\} \\ &= \min\{x^2 - \lambda N + \lambda\ ,\ \alpha^2 x^2 + \sigma^2 - \lambda N\}. \end{aligned}$$

If $V^N(x, N-1)$ is equal to $x^2 - \lambda N + \lambda$, the process is stopped at step $(N-1)$, otherwise it proceeds to step N.

Claim:

$$\lambda > \sigma^2$$

Proof: Assume $\lambda \le \sigma^2$. Then,

$$x^2 - \lambda N + \lambda \ \le \ \alpha^2 x^2 + \sigma^2 - \lambda N \quad \forall x,$$

and

$$V^N(x, N-1) = x^2 - \lambda(N-1).$$

By iteration, it can be shown that

$$V^N(x,k) = x^2 - \lambda k \quad \text{for } k = 1, 2, \cdots, N.$$

This suggests that stopping at the very first step is optimal. However, this contradicts with $\mathbb{E}[\tau_1] = c$, and therefore, the assumption was wrong. ■

Then, the value function at step $(N-1)$ is given as

$$V^N(x, N-1) = \begin{cases} \alpha^2 x^2 + \sigma^2 - \lambda N & \text{if } |x| \le \frac{\lambda - \sigma^2}{\alpha^2 - 1} =: \Delta_{N-1}^N \\ x^2 + \lambda - \lambda N & \text{otherwise.} \end{cases} \tag{3.15}$$

The process is stopped at a step when $|x|$ exceeds some threshold value determined for that step, as suggested by (3.2). Thus, $V^N(x,k) = x^2 - \lambda k$ for large values of $|x|$ for each k. The $V^N(x, N-1)$ given in (3.15) is also in this form.

For the discretized and truncated problem with horizon length N, define the threshold value of the envelope at step k as

$$\Delta_k^N = \min_x \{|x| \ : \ V^N(x,k) = x^2 - \lambda k\}. \tag{3.16}$$

The following two lemmas show that the thresholding envelope gets narrower toward the end of the horizon for fixed N, i.e., the sequence $\{\Delta_k^N\}_{k=1}^N$ is monotonically decreasing.

Lemma 3.1: Let w denote a zero-mean Gaussian random variable with variance σ^2. Then,

$$\mathbb{E}[V^N(x+w, k-1)] < \mathbb{E}[V^N(x+w, k)] + \lambda \quad \forall x, \ \text{for } k = 2, 3, \cdots, N.$$

Proof: For $k = N$:

$$\begin{aligned} V^N(x, N-1) &= \min\{x^2 - \lambda N + \lambda \ , \ \alpha^2 x^2 + \sigma^2 - \lambda N\} \\ &= \min\{V^N(x, N) + \lambda \ , \ \alpha^2 x^2 + \sigma^2 - \lambda N\}. \end{aligned}$$

In fact,

$$\begin{aligned} V^N(x, N-1) &= V^N(x, N) + \lambda \qquad \text{for } |x| \geq \Delta_{N-1}^N, \\ V^N(x, N-1) &< V^N(x, N) + \lambda \qquad \text{for } |x| < \Delta_{N-1}^N. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[V^N(x+w, N-1)] &= \int_{-\infty}^{\infty} V^N(x+w, N-1) p_W(w) dw \\ &< \int_{-\infty}^{\infty} \left(V^N(x+w, N) + \lambda\right) p_W(w) dw \\ &= \mathbb{E}[V^N(x+w, N)] + \lambda, \end{aligned}$$

where $p_W(w)$ denotes the probability density function of w.

Now, to prove by iteration, assume for an arbitrary step $k \leq N-1$,

$$\mathbb{E}[V^N(x+w, k)] < \mathbb{E}[V^N(x+w, k+1)] + \lambda \tag{3.17}$$

holds for all x. At step $(k-1)$,

$$V^N(x, k-1) = \min\{x^2 - \lambda k + \lambda \ , \ \mathbb{E}[V^N(\alpha x + w_{k-1}, k)]\}$$

and

$$V^N(x, k) + \lambda = \min\{x^2 - \lambda k + \lambda \ , \ \mathbb{E}[V^N(\alpha x + w_k, k+1)] + \lambda\}.$$

By assumption (3.17),

$$\mathbb{E}[V^N(\alpha x + w_{k-1}, k)] < \mathbb{E}[V^N(\alpha x + w_k, k+1)] + \lambda,$$

and therefore,

$$V^N(x, k-1) \leq V^N(x, k) + \lambda \quad \forall x.$$

Specifically, when $|x| \geq \max\{\Delta_{k-1}^N, \Delta_k^N\}$,

$$V^N(x, k-1) = V^N(x, k) + \lambda = x^2 - \lambda k + \lambda,$$

and otherwise

$$V^N(x, k-1) < V^N(x, k) + \lambda.$$

Then, for any x at step $(k-1)$,

$$\begin{aligned}\mathbb{E}[V^N(x+w, k-1)] &= \int_{-\infty}^{\infty} V^N(x+w, k-1) p_W(w) dw \\ &< \int_{-\infty}^{\infty} \left(V^N(x+w, k) + \lambda\right) p_W(w) dw \\ &= \mathbb{E}[V^N(x+w, k)] + \lambda.\end{aligned}$$

As a result, the assumption (3.17) holds for step $(k-1)$ as well. ■

Lemma 3.2: $\{\Delta_k^N\}_{k=1}^N$ is a monotonically decreasing sequence.
Proof: The value function at step $k \neq N$,

$$V^N(x, k) = \min\{x^2 - \lambda k\ ,\ \mathbb{E}[V^N(\alpha x + w_k, k+1)]\}$$

is even in x and consists of two segments on $[0, \infty)$: $\mathbb{E}[V^N(\alpha x + w_k, k+1)]$ and $x^2 - \lambda k$. By definition, Δ_k^N is the value of x at which both segments meet, i.e.,

$$x^2 - \lambda k = \mathbb{E}[V^N(\alpha x + w_k, k+1)] \tag{3.18}$$

is satisfied at $x = \Delta_k^N$. In addition, by Lemma 3.1, at $x = \Delta_k^N$,

$$\mathbb{E}[V^N(\alpha x + w_k, k+1)] + \lambda > \mathbb{E}[V^N(\alpha x + w_{k-1}, k)]. \tag{3.19}$$

On the other hand,

$$V^N(x, k-1) = \min\{x^2 - \lambda k + \lambda\ ,\ \mathbb{E}[V^N(\alpha x + w_{k-1}, k)]\},$$

and (3.18) and (3.19) imply that at $x = \Delta_k^N$,

$$x^2 - \lambda k + \lambda = \mathbb{E}[V^N(\alpha x + w_k, k+1)] + \lambda > \mathbb{E}[V^N(\alpha x + w_{k-1}, k)].$$

Therefore, $\mathbb{E}[V^N(\alpha x + w_{k-1}, k)]$ meets $x^2 - \lambda k + \lambda$ at a value of $|x|$ larger than Δ_k^N, i.e.,

$$\Delta_{k-1}^N > \Delta_k^N.$$

■

The following lemma shows that if the horizon length N is changed, the thresholding envelope only shifts along the time axis while maintaining its shape.

Lemma 3.3:

$$\Delta_k^N = \Delta_{k+1}^{N+1}$$

Proof: For the problem with horizon length $(N+1)$, at the last step,

$$V^{N+1}(x, N+1) = x^2 - \lambda(N+1) = V^N(x, N) - \lambda.$$

Assume

$$V^{N+1}(x, k+1) = V^N(x, k) - \lambda$$

for an arbitrary step $(k+1)$. Then,

$$\begin{aligned}
V^{N+1}(x,k) &= \min\{x^2 - \lambda k, \mathbb{E}[V^{N+1}(\alpha x + w, k+1)]\} \\
&= \min\{x^2 - \lambda k + \lambda - \lambda, \mathbb{E}[V^N(\alpha x + w, k) - \lambda]\} \\
&= \min\{x^2 - \lambda k + \lambda, \mathbb{E}[V^N(\alpha x + w, k)]\} - \lambda \\
&= V^N(x, k-1) - \lambda,
\end{aligned}$$

which shows that the assumption also holds for step k. Therefore, by iteration,

$$V^{N+1}(x, k+1) = V^N(x, k) - \lambda \quad \forall x, \text{ for } k = 1, 2, \cdots, N.$$

As a result, by definition,

$$\begin{aligned}
\Delta_k^N &= \min\{|x| : V^N(x,k) = x^2 - \lambda k\} \\
&= \min\{|x| : V^N(x,k) - \lambda = x^2 - \lambda k - \lambda\} \\
&= \min\{|x| : V^{N+1}(x,k+1) = x^2 - \lambda(k+1)\} \\
&= \Delta_{k+1}^{N+1}.
\end{aligned}$$

■

Corollary 3.1: From Lemma 3.2 and Lemma 3.3, $\{\Delta_1^N\}_{N=1}^{\infty}$ is a monotonically increasing sequence.

Proposition 3.2: $\lim_{N\to\infty} \Delta_1^N$ is finite.

Proof: The sequence $\{\Delta_1^N\}_{N=1}^{\infty}$ gives the threshold values of the infinite horizon envelope, in the reverse order. If $\lim_{N\to\infty} \Delta_1^N = \infty$, the process x_k will not be stopped at infinitely many steps, and thus, $\mathbb{E}[\tau_1] = \infty$. Therefore, we must have

$$\lim_{N\to\infty} \Delta_1^N < \infty.$$

Theorem 3.1: Let $\Delta := \lim_{N\to\infty} \Delta_1^N < \infty$. Then, $\Delta_k^\infty = \Delta$ for all $k \geq 1$. In other words, the optimal sampling rule for the infinite horizon is event-triggered sampling with a constant threshold:

$$\delta_k^* = \begin{cases} 1 & \text{if } |x_k| \geq \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Proof: For any $k \geq 1$,

$$\Delta_k^\infty = \lim_{N\to\infty} \Delta_k^N = \lim_{N\to\infty} \Delta_k^{N+k-1}.$$

From Lemma 3.3, $\Delta_k^{N+k-1} = \Delta_1^N$. Therefore,

$$\Delta_k^\infty = \lim_{N\to\infty} \Delta_1^N = \Delta.$$

■

Lemmas 3.1–3.3 do not depend on the degree of fineness of the discretization. Consequently, the optimal sampling rule for the original infinite horizon problem in continuous time is also event-triggered sampling with a constant threshold. The threshold value is determined based on the frequency of sampling given by (3.3b).

In Chapter 2, the simplest form of event-triggered sampling, which uses a constant threshold, was shown to be the optimal policy to sample and estimate Wiener processes over the infinite horizon. In this chapter, we showed that this result is also valid for scalar linear diffusion processes.

CHAPTER 4

OPTIMAL SWITCHING FOR A BANG-BANG CONTROLLER

Chapter 2 and Chapter 3 studied the optimal infinite horizon sampling policies for estimation problems. In this chapter, we address a control problem with infinite horizon.

Most control systems with feedback controllers update their control signals upon receiving a new sample. For distributed networks with a large number of subsystems, these updates may require substantial computational capacity. Moreover, if the controllers and plants are located in different places of the system, a communication network is also needed for the transfer of control signals.

To handle the systems with limited computational or communicational capacity, we formulate a problem with a constraint on the sampling frequency.

4.1 Problem Formulation

Let x_t be a continuous time stochastic process with a drift term determined by a bang-bang controller:

$$dx = u(t)dt + dw, \quad u(t) \in \{u_1, -u_2\}, \tag{4.1}$$

where u is either u_1 or $-u_2$ at each t, and $u_1, u_2 \in (0, \infty)$. We want to develop an event-triggered time-homogeneous switching policy which has a hard constraint on the average time between consecutive switches and minimizes

$$\lim_{T \to \infty} \frac{1}{T} \left[\int_0^T x^2 dt \right]. \tag{4.2}$$

Note that taking expectation is not needed in (4.2) since we consider only time-homogeneous switching policies. However, because the values of the process at switching instants are not known a priori, we will need to define different time instants as reference points for the switching policies.

When $u_1 = u_2$, the process can be driven into both directions with the same drift rate. In this case, we can conjecture that the optimal switching policy does not switch from $-u_2$ to u_1 if $x_t > 0$, and u_1 to $-u_2$ if $x_t < 0$.

For our problem stated for the process (4.1) with the cost function (4.2) and the constraint on the frequency of switching, we will assume the following.

Assumption 4.1: There exists a critical level θ such that the optimal switching policy can switch from u_1 to $-u_2$ only if $x_t > \theta$. Likewise, switching from $-u_2$ to u_1 is possible only if $x_t < \theta$.

Let t_0' be an instant at which u switches from $-u_2$ to u_1. By assumption, $x_{t_0'} < \theta$. Since $u = u_1 > 0$, x_t crosses θ in finite time with probability 1. To show this, we can define t_0 as the first time x_t reaches θ after t_0'. Then,

$$\begin{aligned}
\mathbb{E}[t_0 - t_0'] &= \mathbb{E}\left[\int_{t_0'}^{t_0} dt\right] \\
&= \mathbb{E}\left[\int_{t_0'}^{t_0} \frac{1}{u_1}(dx - dw)\right] \\
&= \frac{1}{u_1}\mathbb{E}\left[x_{t_0} - x_{t_0'}\right] \\
&= \frac{\theta - x_{t_0'}}{u_1} < \infty.
\end{aligned}$$

Therefore, t_0 is well defined. Further define

- $(t_0 + \tau_1)$ as the first time u switches from u_1 to $-u_2$ after t_0,
- $(t_0 + \tau_1')$ as the first time x_t reaches θ after $(t_0 + \tau_1)$,
- $(t_0 + \tau_1' + \tau_2)$ as the first time u switches from $-u_2$ to u_1 after $(t_0 + \tau_1')$,
- $(t_0 + \tau_1' + \tau_2')$ as the first time x_t reaches θ after $(t_0 + \tau_1' + \tau_2)$.

These time instants are shown on a sample path in Figure 4.1.

The event-triggered switching policies we consider are time-homogeneous, in the sense that if $\bar{t}_1$ and $\bar{t}_2$ are two instants at which u switches from $-u_2$ to u_1 with $x_{\bar{t}_1} = x_{\bar{t}_2}$, then the thresholding envelopes following these samples are identical until the next sampling instants. However, by Assumtion 4.1, we already know that a new sample is not taken until x_t reaches θ; therefore,

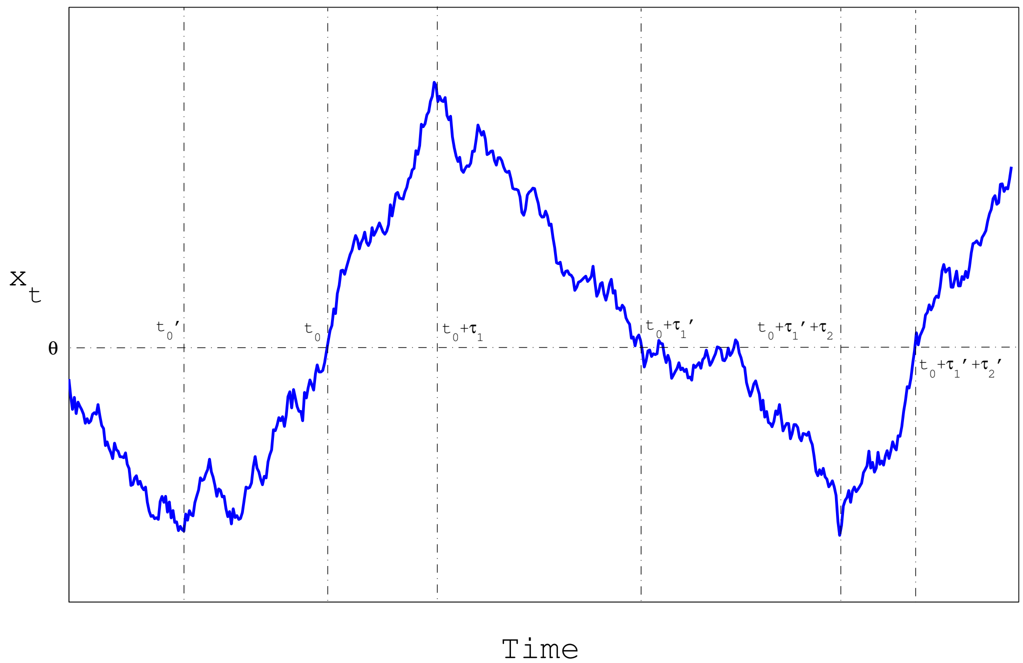

Figure 4.1: Defined time instants on a sample path

the time that elapses between a switching instant and the nearest time at which x_t reaches θ is unimportant. Hence, for the policies we consider, the time τ_1 is determined based on $(t - t_0)$, but not based on t_0. Similarly, the time τ_2 is determined based on $(t - t_0 - \tau_1')$, but not based on $t_0 + \tau_1'$. Then, the hard constraint on the average time between consecutive switches can be stated as

$$\mathbb{E}[\tau_1' + \tau_2'] = c,$$

which can also be separated as

$$\mathbb{E}[\tau_1'] = c_1$$

$$\mathbb{E}[\tau_2'] = c_2$$

$$c_1 + c_2 = c,$$

where c_1 and c_2 are two constants to be determined later.

As in Section 3.1, we can define

$$z_1 = \int_{t_0}^{t_0+\tau_1'+\tau_2'} x^2 dt,$$

and if $\left(t_0 + \sum_{i=1}^4 \tau_i'\right)$ is the first time x_t crosses θ after the next switch of u from $-u_2$ to u_1,

$$z_2 = \int_{t_0+\sum_{i=1}^2 \tau_i'}^{t_0+\sum_{i=1}^4 \tau_i'} x^2 dt,$$

and so on. Then, we can show that

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\left[\int_{t_0}^{t_0+T} x^2 dt\right] &= \lim_{k\to\infty} \frac{1}{k\mathbb{E}[\tau_1'+\tau_2']}\sum_{i=1}^{n_k} z_i \\
&= \frac{\mathbb{E}[z_1]}{\mathbb{E}[\tau_1'+\tau_2']} \\
&= \frac{1}{\mathbb{E}[\tau_1'+\tau_2']}\mathbb{E}\left[\int_{t_0}^{t_0+\tau_1'+\tau_2'} x^2 dt\right],
\end{aligned}$$

where n_k is defined similar to that in Section 3.1.

As a result, our goal becomes equivalent to

$$\text{minimize } \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1'+\tau_2'} x^2 dt\right] \quad \text{subject to } \mathbb{E}[\tau_1'] = c_1,\ \mathbb{E}[\tau_2'] = c_2,$$

which, again due to the time-homogeneity of the switching policy, can be separated into two problems as:

$$\text{minimize } \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1'} x^2 dt\right] \quad \text{subject to } \mathbb{E}[\tau_1'] = c_1, \tag{4.3a}$$

$$\text{minimize } \mathbb{E}\left[\int_{t_0+\tau_1'}^{t_0+\tau_1'+\tau_2'} x^2 dt\right] \quad \text{subject to } \mathbb{E}[\tau_2'] = c_2. \tag{4.3b}$$

Note that these two problems are almost identical and independent except for the coupling constraint $c_1 + c_2 = c$.

4.2 Conversion to an Optimal Stopping Problem

To eliminate the integration inside the expectation in (4.3a), we consider the function $f^u(x) = \frac{x^3}{3u} - \frac{x^2}{2u^2} + \frac{x}{2u^3}$. Applying Itô's differentiation rule,

$$
\begin{aligned}
df^u &= f_t^u dt + f_x^u dx + \frac{1}{2} f_{xx}^u dt \\
&= f_t^u dt + f_x^u (udt + dw) + \frac{1}{2} f_{xx}^u dt \\
&= x^2 dt + \left(\frac{x^2}{u} - \frac{x}{u^2} + \frac{1}{2u^3} \right) dw.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathbb{E}\left[\int_{t_0}^{t_0+\tau_1'} x^2 dt\right] &= \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1} df^{u_1}\right] + \mathbb{E}\left[\int_{t_0+\tau_1}^{t_0+\tau_1'} df^{-u_2}\right] \\
&= \mathbb{E}\Big[f^{u_1}(x_{t_0+\tau_1}) - f^{u_1}(x_{t_0})\Big] \\
&\quad + \mathbb{E}\Big[f^{-u_2}(x_{t_0+\tau_1'}) - f^{-u_2}(x_{t_0+\tau_1})\Big] \\
&= \mathbb{E}\left[\left(\frac{1}{u_1} + \frac{1}{u_2}\right)\frac{x_{t_0+\tau_1}^3}{3}\right] + \mathbb{E}\left[\left(\frac{1}{u_2^2} - \frac{1}{u_1^2}\right)\frac{x_{t_0+\tau_1}^2}{2}\right] \\
&\quad + \mathbb{E}\left[\left(\frac{1}{u_1^3} + \frac{1}{u_2^3}\right)\frac{x_{t_0+\tau_1}}{2}\right] - f^{u_1}(\theta) + f^{-u_2}(\theta). \qquad (4.4)
\end{aligned}
$$

Furthermore, we can express the condition $\mathbb{E}[\tau_1'] = c_1$ in terms of $\mathbb{E}[x_{t_0+\tau_1}]$:

$$
\begin{aligned}
\mathbb{E}[\tau_1'] &= \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1} dt + \int_{t_0+\tau_1}^{t_0+\tau_1'} dt\right] \\
&= \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1} \frac{dx - dw}{u_1} + \int_{t_0+\tau_1}^{t_0+\tau_1'} \frac{dx - dw}{-u_2}\right] \\
&= \mathbb{E}\left[\frac{1}{u_1}\int_{t_0}^{t_0+\tau_1} dx - \frac{1}{u_2}\int_{t_0+\tau_1}^{t_0+\tau_1'} dx\right] \\
&= \left(\frac{1}{u_1} + \frac{1}{u_2}\right)\left(\mathbb{E}\left[x_{t_0+\tau_1}\right] - \theta\right).
\end{aligned}
$$

As a result, ignoring the deterministic terms in (4.4), the problem in (4.3a) can be equivalently stated as

$$\text{minimize} \;\; \mathbb{E}\left[\left(\frac{1}{u_1}+\frac{1}{u_2}\right)\frac{x_{t_0+\tau_1}^3}{3}+\left(\frac{1}{u_2^2}-\frac{1}{u_1^2}\right)\frac{x_{t_0+\tau_1}^2}{2}+\left(\frac{1}{u_1^3}+\frac{1}{u_2^3}\right)\frac{x_{t_0+\tau_1}}{2}\right]$$
$$\text{subject to} \;\; \mathbb{E}[x_{t_0+\tau_1}]=\theta+\left(\frac{1}{u_1}+\frac{1}{u_2}\right)^{-1}c_1. \tag{4.5}$$

The solution for this problem can be obtained by minimizing

$$\mathbb{E}\left[\left(\frac{1}{u_1}+\frac{1}{u_2}\right)\frac{x_{t_0+\tau_1}^3}{3}+\left(\frac{1}{u_2^2}-\frac{1}{u_1^2}\right)\frac{x_{t_0+\tau_1}^2}{2}+\left(\frac{1}{u_1^3}+\frac{1}{u_2^3}+\lambda\right)\frac{x_{t_0+\tau_1}}{2}\right] \tag{4.6}$$

for a Lagrange multiplier $\lambda \in \mathbb{R}$. Even though we have an expectation in (4.6), the stochasticity of the problem is lost since we have been able to express both the cost and the constraint only in terms of $x_{t_0+\tau_1}$. As $\left(\frac{1}{u_1}+\frac{1}{u_2}\right) > 0$, the expression inside the expectation in (4.6) attains its unique minimum in the region $[0,\infty)$ at some constant value $\overline{x}$:

$$\overline{x}=\arg\min_{x\geq 0}\left[\left(\frac{1}{u_1}+\frac{1}{u_2}\right)\frac{x^3}{3}+\left(\frac{1}{u_2^2}-\frac{1}{u_1^2}\right)\frac{x^2}{2}+\left(\frac{1}{u_1^3}+\frac{1}{u_2^3}+\lambda\right)\frac{x}{2}\right].$$

Then the optimal time to switch from u_1 to $-u_2$ is the first time x_t reaches $\overline{x}$ after t_0. This gives an event-triggered switching rule defined by a constant thresholding envelope. The value λ, and thus, $\overline{x}$ are determined from (4.5) based on the condition $\mathbb{E}[\tau_1']=c_1$:

$$\overline{x}=\mathbb{E}[x_{t_0+\tau_1}]=\theta+\left(\frac{1}{u_1}+\frac{1}{u_2}\right)^{-1}c_1.$$

For the problem (4.3b), we can follow a similar procedure to obtain the optimal time to switch from $-u_2$ to u_1 as the first time x_t reaches

$$\underline{x}:=\theta-\left(\frac{1}{u_1}+\frac{1}{u_2}\right)^{-1}c_2.$$

Then, the optimal switching policy is given by

$$\delta^*=\begin{cases}\text{switch to } -u_2 & \text{if } x_t=\overline{x} \text{ and } u=u_1,\\ \text{switch to } u_1 & \text{if } x_t=\underline{x} \text{ and } u=-u_2,\\ \text{do not switch} & \text{otherwise.}\end{cases} \tag{4.7}$$

Now, knowing the value of the process at switching times in terms of c_1 and c_2, we can optimally adjust c_1 and c_2 values. Note that the following conditions are equivalent:

$$c_1 + c_2 = c \quad \Longleftrightarrow \quad \overline{x} - \underline{x} = \left(\frac{1}{u_1} + \frac{1}{u_2}\right)^{-1} c. \tag{4.8}$$

With deterministic values of the process at switching instants, we can write

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^{t_0+\tau_1'+\tau_2'} x^2 dt\right] &= \mathbb{E}\left[\int_{t_0}^{t_0+\tau_1} df^{u_1}\right] + \mathbb{E}\left[\int_{t_0+\tau_1}^{t_0+\tau_1'+\tau_2} df^{-u_2}\right] \\
&\quad + \mathbb{E}\left[\int_{t_0+\tau_1'+\tau_2}^{t_0+\tau_1'+\tau_2'} df^{u_1}\right] \\
&= \left(\frac{1}{u_1} + \frac{1}{u_2}\right)\frac{\overline{x}^3}{3} + \left(\frac{1}{u_2^2} - \frac{1}{u_1^2}\right)\frac{\overline{x}^2}{2} + \left(\frac{1}{u_1^3} + \frac{1}{u_2^3}\right)\frac{\overline{x}}{2} \\
&\quad - \left(\frac{1}{u_1} + \frac{1}{u_2}\right)\frac{\underline{x}^3}{3} - \left(\frac{1}{u_2^2} - \frac{1}{u_1^2}\right)\frac{\underline{x}^2}{2} - \left(\frac{1}{u_1^3} + \frac{1}{u_2^3}\right)\frac{\underline{x}}{2} \\
&= \left(\frac{1}{u_1} + \frac{1}{u_2}\right)\frac{\overline{x}^3 - \underline{x}^3}{3} + \left(\frac{1}{u_2^2} - \frac{1}{u_1^2}\right)\frac{\overline{x}^2 - \underline{x}^2}{2} \\
&\quad + \left(\frac{1}{u_1^3} + \frac{1}{u_2^3}\right)\frac{\overline{x} - \underline{x}}{2}.
\end{aligned} \tag{4.9}$$

The unique $\overline{x}^*$ and $\underline{x}^*$ minimizing (4.9) subject to (4.8) are calculated as

$$\overline{x}^* = \frac{1}{2}\left(\frac{1}{u_1} - \frac{1}{u_2}\right) + \frac{c}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right)^{-1}, \tag{4.10a}$$

$$\underline{x}^* = \frac{1}{2}\left(\frac{1}{u_1} - \frac{1}{u_2}\right) - \frac{c}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right)^{-1}. \tag{4.10b}$$

Therefore, (4.10) provides the optimal threshold values for the optimal policy (4.7). As the expected time between consecutive switches decreases, the optimal upper and lower threshold levels get closer. If we let $c \to 0$, which allows sampling infinitely often,

$$\lim_{c\to 0} \overline{x}^* = \lim_{c\to 0} \underline{x}^* = \frac{1}{2}\left(\frac{1}{u_1} - \frac{1}{u_2}\right).$$

Remark: Note that the value of θ has not been needed in any of the steps. Only the assumption that there exists such a constant level has been important to find out that the optimal switching rule is given by a constant envelope rule. Then, to find the optimal levels for the envelope, $\overline{x}^*$ and $\underline{x}^*$, we minimized the expression (4.9) subject to (4.8), neither of which depend on the value of θ.

4.3 Comparison with Discounted Cost Minimization

A similar infinite horizon optimal control problem has also been formulated in [46] although it uses a discounted cost and has no constraint on the frequency of switching.

Let $\tilde{x}_t$ be a process driven by a bounded control input:

$$d\tilde{x} = u(t)dt + dw,$$

$$-u_2 \leq u(t) \leq u_1, \qquad \forall t \geq 0.$$

Note that the set of possible actions is not binary, but a compact subset of the set of real numbers containing 0. The optimal control input adapted to the process $\tilde{x}_t$ which minimizes

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\tilde{x}_t^2 dt\right]$$

has been shown in [46] to be a bang-bang controller:

$$u(t) = \begin{cases} -u_2 & \text{if } \tilde{x}_t \geq \theta^\beta \\ u_1 & \text{if } \tilde{x}_t < \theta^\beta, \end{cases} \tag{4.11a}$$

where

$$\theta^\beta = \frac{1}{\sqrt{u_1^2 + 2\beta} + u_1} - \frac{1}{\sqrt{u_2^2 + 2\beta} + u_2}. \tag{4.11b}$$

The discount rate β affects only the switching level θ^β in (4.11), and the bang-bang structure of the control law is independent of β otherwise. As the discount rate β approaches 0, the switching level tends to

$$\lim_{\beta \to 0} \theta^\beta = \frac{1}{2}\left(\frac{1}{u_1} - \frac{1}{u_2}\right). \tag{4.12}$$

We cannot directly claim that the control law obtained by taking the limit of (4.11) will be the optimal rule for the infinite horizon problem with an undiscounted cost function. However, we observe that the limit value in (4.12) coincides with the average of the optimal upper and lower threshold values given in (4.10). In fact, (4.12) is the only value that lies between $\underline{x}^*$ and $\overline{x}^*$ for all values of c.

With this relation between the problem with undiscounted cost and constraint on the frequency of switching and the unconstrained problem with discounted cost, we conclude this chapter.

CHAPTER 5

CONCLUSION

In this thesis, we studied the optimal event-triggered sampling and switching policies over the infinite horizon. To address the computational and communicational limitations in systems such as networked control systems, we integrated a sampling constraint into our analyses. Instead of associating a cost with each sample, we imposed a hard constraint on the frequency of sampling.

In Chapter 2, we examined the optimal event-triggered sampling policy to estimate multidimensional Wiener processes. We first investigated the solution for a finite horizon problem with a limited number of samples. After converting the finite horizon problem to a sequence of optimal stopping problems, we obtained its solution by providing its Snell envelope. Then, to solve the infinite horizon problem, we increased the length of the horizon to infinity while maintaining the ratio of the number of samples to the length of the horizon at a constant value. We showed that the policy to take the very first sample converges to the event-triggered sampling policy with a constant envelope and it gives the solution for the infinite horizon problem. To demonstrate the optimality of the policy suggested, we also compared it with a class of event-triggered policies via simulation.

With Wiener processes, it was possible to calculate the Snell envelope for the finite horizon optimal stopping problem. However, this was infeasible for stochastic linear processes with drift terms. Therefore, to find the optimal event-triggered sampling policies for these processes over the infinite horizon, we developed a new approach in Chapter 3.

To deal with linear diffusion processes with time-invariant coefficients, in Chapter 3, we used the fact that the aggregated estimation errors between consecutive sampling instants are independent and identically distributed random variables. This allowed us to formulate a simpler optimal stopping problem which could be expressed with a single Lagrangian cost function.

We were able to provide the Snell envelope of this optimal stopping problem for scalar Wiener processes, which yielded the same solution as the one in Chapter 2. For the processes with drift terms, however, we could not obtain an explicit expression for the Snell envelope. Instead, we created discretized and truncated versions of the problem. Resorting to dynamic programming, we obtained the characteristics of the thresholding envelope of the event-triggered sampling policy. Consequently, we proved the optimality of the sampling policy with a constant threshold for linear diffusion processes as well.

Whereas the problems in Chapter 2 and Chapter 3 concerned only the estimation of stochastic linear processes, the problem studied in Chapter 4 involved the control of a process. To focus on the optimality of the sampling policies, we selected a process whose drift rate was controlled with a bang-bang controller. Following an approach similar to that in Chapter 3, we obtained a simpler optimal stopping problem. Then, we removed the stochasticity of this problem by expressing both the cost and the constraint only in terms of the value of the process at the switching instants. As a result, we showed that the optimal switching rule for this process is also given by an event-triggered policy with a constant envelope.

Chapter 4 provided the optimal switching policy for a bang-bang controller. It is also possible to look for jointly optimal control and sampling policies, where the sampling instants are determined by the sampling policy to update the control. In addition, some cost could be associated with the control applied. Nevertheless, the dual effect arising with the use of an event-triggered sampling policy in closed-loop control systems renders the analysis of jointly optimal policies infeasible for most cases.

Our analysis in Chapter 3 was limited to scalar processes. The optimal event-triggered sampling policy for vector processes are also likely to have a constant thresholding envelope. However, the weights the policy will place on different states would possibly be different, and their computation clearly requires further investigation.

The hard constraints we imposed on the frequency of sampling and switching are important for networked systems. However, the problems we formulated address the issues that appear in networked systems from the perspective of a single process, ignoring the other processes in the network. Similar problems could also be studied with larger networks containing several subsystems. With such networks, it is possible to handle the problems that appear when different subsystems simultaneously request to transmit a sample or to update the control.

In our problems, we also considered observations to be noiseless and transmissions to be perfect. The problems could also be modified to include noisy observations, and the estimator might be changed to a Kalman filter with intermittent observations. Possible problems that arise in communication networks, such as delays, transmission errors, and data dropouts, could also be integrated into the formulation of the problems.

Nevertheless, despite their relative simplicity, the problems addressed in this thesis capture an important portion of the issues concerning networked control systems. The results obtained are critical as they show that the event-triggered sampling with constant thresholds, the simplest and the most commonly used type of event-triggered policies, optimally schedules the sampling instants over the infinite horizon.

REFERENCES

[1] K. J. Aström and B. Wittenmark, *Computer-Controlled Systems*, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, Inc., 1997.

[2] M. G. Cea and G. C. Goodwin, "Event based sampling in non-linear filtering," *Control Engineering Practice*, vol. 20, no. 10, pp. 963–971, Oct 2012.

[3] R. Schlueter and A. H. Levis, "The optimal linear regulator with state-dependent sampling," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 512–515, Oct 1973.

[4] A. Konak, A. Smith, and S. Kulturel-Konak, "New event-driven sampling techniques for network reliability estimation," in *Proceedings of the 2004 Winter Simulation Conference*, vol. 1, Dec 2004.

[5] K. J. Astrom and B. M. Bernhardsson, "Comparison of Riemann and Lebesgue sampling for first order stochastic systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, Dec 2002, pp. 2011–2016.

[6] R. McCann, A. K. Gunda, and S. R. Damugatla, "Improved operation of networked control systems using Lebesgue sampling," in *Conference Record of the 2004 IEEE Industry Applications Conference and the 39th IAS Annual Meeting*, vol. 2, Oct 2004, pp. 1211–1216.

[7] M. Miskowicz, "The event-triggered sampling optimization criterion for distributed networked monitoring and control systems," in *Proceedings of the IEEE International Conference on Industrial Technology*, vol. 2, Dec 2003, pp. 1083–1088.

[8] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, Jan 2007.

[9] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, Sept 2004.

[10] Y. Mo and B. Sinopoli, "A characterization of the critical value for Kalman filtering with intermittent observations," in *Proceedings of the 47th IEEE Conference on Decision and Control*, Dec 2008, pp. 2692–2697.

[11] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, Jan 2007.

[12] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, July 2004.

[13] J. Baillieul, "Feedback designs in information-based control," in *Stochastic Theory and Control*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, 2002, vol. 280, pp. 35–57.

[14] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback," *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049–1053, May 1999.

[15] M. Rabi, G. V. Moustakides, and J. S. Baras, "Multiple sampling for estimation on a finite horizon," in *Proceedings of the 45th IEEE Conference on Decision and Control*, Dec 2006, pp. 1351–1357.

[16] M. Rabi, G. V. Moustakides, and J. S. Baras, "Adaptive sampling for linear state estimation," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 672–702, Mar 2012.

[17] E. Skafidas and A. Nerode, "Optimal measurement scheduling in linear quadratic Gaussian control problems," in *Proceedings of the IEEE International Conference on Control Applications*, vol. 2, Sept 1998, pp. 1225–1229.

[18] L. Meier, J. Peschon, and R. M. Dressler, "Optimal control of measurement subsystems," *IEEE Transactions on Automatic Control*, vol. 12, no. 5, pp. 528–536, Oct 1967.

[19] H. Kushner, "On the optimum timing of observations for linear control systems with unknown initial state," *IEEE Transactions on Automatic Control*, vol. 9, no. 2, pp. 144–150, Apr 1964.

[20] M. Athans, "On the determination of optimal costly measurement strategies for linear stochastic systems," *Automatica*, vol. 8, no. 4, pp. 397–412, 1972.

[21] V. Gupta, T. H. Chung, B. Hassibi, and R. M. Murray, "On a stochastic sensor selection algorithm with applications in sensor scheduling and sensor coverage," *Automatica*, vol. 42, no. 2, pp. 251–260, Feb 2006.

[22] Y. Mo, E. Garone, A. Casavola, and B. Sinopoli, "Stochastic sensor scheduling for energy constrained estimation in multi-hop wireless sensor networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2489–2495, Oct 2011.

[23] W. Wu and A. Arapostathis, "Optimal sensor querying: General Markovian and LQG models with controlled observations," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1392–1405, July 2008.

[24] O. C. Imer and T. Başar, "Optimal estimation with limited measurements," in *Proceedings of the 44th IEEE Conference on Decision and Control and 2005 European Control Conference*, Dec 2005, pp. 1029–1034.

[25] O. C. Imer and T. Başar, "Optimal estimation with limited measurements," *International Journal of Systems, Control and Communications*, vol. 2, no. 1/2/3, pp. 5–29, Jan 2010.

[26] P. Bommannavar and T. Başar, "Optimal estimation over channels with limits on usage," in *Proceedings of the 17th IFAC World Congress*, 2008, pp. 6632–6637.

[27] A. Nayyar, T. Başar, D. Teneketzis, and V. V. Veeravalli, "Optimal strategies for communication and remote estimation with an energy harvesting sensor," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2246–2260, Sept 2013.

[28] L. Li, M. Lemmon, and X. Wang, "Event-triggered state estimation in vector linear processes," in *Proceedings of American Control Conference*, June 2010, pp. 2138–2143.

[29] G. M. Lipsa and N. C. Martins, "Remote state estimation with communication costs for first-order LTI systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 9, pp. 2013–2025, Sept 2011.

[30] J. K. Yook, D. M. Tilbury, and N. R. Soparkar, "Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 4, pp. 503–518, July 2002.

[31] Y. Xu and J. Hespanha, "Optimal communication logics in networked control systems," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, vol. 4, Dec 2004, pp. 3527–3532.

[32] R. Cogill, S. Lall, and J. P. Hespanha, "A constant factor approximation algorithm for event-based sampling," in *Proceedings of American Control Conference*, July 2007, pp. 305–311.

[33] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586–601, Mar 2011.

[34] K.-E. Arzen, "A simple event-based PID controller," in *Proceedings of the 14th World Congress of IFAC*, Mar 1999, pp. 423–428.

[35] H. Sira-Ramirez, "A geometric approach to pulse-width modulated control in nonlinear dynamical systems," *IEEE Transactions on Automatic Control*, vol. 34, no. 2, pp. 184–187, Feb 1989.

[36] A. A. Feldbaum, "Dual control theory I-II," *Automation and Remote Control*, vol. 21, pp. 184–187, 1033–1039, 1960.

[37] A. A. Feldbaum, "Dual control theory III-IV," *Automation and Remote Control*, vol. 22, pp. 1–12, 109–121, 1961.

[38] C. Ramesh, H. Sandberg, and K. H. Johansson, "Design of state-based schedulers for a network of control loops," *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 1962–1975, Aug 2013.

[39] M. Rabi, K. H. Johansson, and M. Johansson, "Optimal stopping for event-triggered sensing and actuation," in *Proceedings of the 47th IEEE Conference on Decision and Control*, Dec 2008, pp. 3607–3612.

[40] Y.-K. Xu and X.-R. Cao, "Lebesgue-sampling based optimal control problems with time aggregation," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1097–1109, May 2011.

[41] X.-R. Cao, Z. Ren, S. Bhatnagar, M. Fu, and S. Marcus, "A time aggregation approach to Markov decision processes," *Automatica*, vol. 38, no. 6, pp. 929–943, 2002.

[42] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed. New York: Springer, 1991.

[43] R. Brockett, "Stochastic control," 2009. [Online]. Available: http://www.eeci-institute.eu/pdf/M015/RogersStochastic.pdf

[44] A. N. Shiryaev, *Optimal Stopping Rules*, 3rd ed., ser. Stochastic Modelling and Applied Probability. Springer, 2008, vol. 8.

[45] D. Lamberton, "Optimal stopping and American options," 2009. [Online]. Available: http://www.fmf.uni-lj.si/finmath09/ShortCourseAmericanOptions.pdf

[46] V. E. Benes, L. A. Shepp, and H. S. Witsenhausen, "Some solvable stochastic control problems," *Stochastics*, vol. 4, no. 1, pp. 39–83, 1980.