QUANTILE AUTOREGRESSION WITH CENSORED DATA

BY

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DISSERTATION

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Abstract

Quantile autoregression (QAR) provides an alternative way to study asymmetric dynamics and local persistence in time series. It is particularly attractive for censored data, where the classical autoregressive models are unidentifiable without further parametric assumptions on the distributions. There have been prominent works by Powell (1986), Portnoy (2003) and Peng and Huang (2008) on estimating the conditional quantile functions with censored data. However, unlike the standard regression models, the autoregressive models should take account of censoring on both response and regressors.

In this dissertation, we show that the existing censored quantile regression methods produce empirically consistent estimator on QAR models when using only observed part of regressors. A new algorithm is proposed to improve a censored quantile autoregression (CQAR) estimator by adopting an idea of imputation methods. The algorithm distributes probability mass of each censored point to any sufficiently large value appropriately, and iterates towards self-consistent solutions.

Monte Carlo simulations are conducted to examine the empirical consistency of the CQAR estimator. Also, empirical applications of the algorithm to the Samish river water quality study and dry decomposition of NH4 demonstrate the merits of the proposed method.

KEY WORDS: Censored time-series; Autoregression; Quantile; Self-consistent; Kaplan-Meier.

To my family.

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Table of Contents

List of Tables

List of Figures

List of Notations

CQAR	censored quantile autoregression.
CRQ	censored regression quantile.
MSE	mean squared error.
MAE	mean absolute error.
n	total number of observations.
Y_t	the response value at time t.
C_t	the censored value at time t.
$Q_Y(g_k\|x)$	the g_k-th conditional quantile of Y given x.
g_k	grid $\epsilon \leq g_1 < g_2 < ... < g_{M_n} \leq 1-\epsilon$ for a given $\epsilon > 0$, where M_n is the number of grid points.
δ_n	grid size $\delta_n = g_k - g_{k-1} = \frac{1-2\epsilon}{M_n}$.
p	the order of autoregressive model.
β_k	$\beta_k = \beta(g_k)$ true unknown quantile regression parameter at g_k.
Δ_t	indicator whether the observation at time t is censored or not, where $\Delta_t = I\{Y_t < C_t\}$.
$\tilde{Y}_t$	the observed value at time t, where $\tilde{Y}_t = \min\{Y_t, C_t\}$.
$w_t(g_k)$	weight for the right censored observation, where $\tilde{Y}_t$ when computing the g_k-th quantile.
$i.i.d.$	independent and identically distributed.

Chapter 1

Introduction

1.1 Motivation

Quantile Regression (QR) has received great attention due to its advantages over the classical Least Squares (LS) regression in terms of robustness and applicability to many complicated research questions which LS method may not answer. Whereas the LS regression relies only on the conditional mean function, QR focus on the conditional quantile functions which helps on modeling data with possible heterogeneity. Let Y and X be scalar random variables. Then the conditional quantile function of Y given X is the inverse of the corresponding conditional distribution function, i.e.

$$Q_Y(\tau|X) = F_Y^{-1}(\tau|X) = \inf\{y : F_Y(y|X) \geq \tau\},$$

where $F_Y(y|X) = P(Y \leq y|X)$. The conditional quantile function of Y given X fully describes the relationship between Y and X.

Based on the conditional quantile function, Koenker and Bassett (1978) established the QR model. Consider the following classical linear model,

$$y_i = x_i^\top \beta + u_i, \quad i = 1, \cdots, n,$$

with i.i.d errors $\{u_i\}$. Suppose that $\{u_i\}$ have a common distribution function F with associated density f, with $f(F^{-1})(\tau) > 0$. The QR estimator $\hat{\beta}$ is obtained based on the following optimization problem:

$$\hat{\beta}(\tau) = \min_{\beta} \sum_{i=1}^{n} \rho_{\tau}(y_i - x_i^{\top}\beta) \tag{1.1}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$, $\tau \in (0,1)$ and $I(\cdot)$ is the indicator function. Given $\hat{\beta}(\tau)$, the τ-th conditional quantile function of Y_t given X_t can be estimated by

$$\hat{Q}_{y_i}(\tau|x_i) = x_i^{\top}\hat{\beta}(\tau),$$

and the conditional density of y_i at $y = Q_{y_i}(\tau|x_i)$ can be estimated by the difference quotients,

$$\hat{f}_{y_i}(y|x_i) = \frac{2h}{\hat{Q}_{y_i}(\tau + h|x_i) - \hat{Q}_{y_i}(\tau - h|x_i)},$$

for some appropriately chosen sequence of $h = h(T) \to 0$.

Under appropriate regularity conditions, Koenker (2005) shows that solution of (1.1), $\hat{\beta}$, is a consistent estimate of β as follows.

$$\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \Longrightarrow N(0, \omega^2 D_0^{-1}), \tag{1.2}$$

where $\omega^2 = \tau(1-\tau)/f_i^2(F_{Y_i}^{-1}(\tau|x_i))$.
Further, in non-i.i.d error settings,

$$\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) \Longrightarrow N(0, \tau(1-\tau)D_1^{-1}D_0D_1^{-1}) \tag{1.3}$$

where $D_0 = \lim_{n\to\infty} n^{-1}\sum x_i x_i^{\top}$ and $D_1(\tau) = \lim_{n\to\infty} n^{-1}\sum f_i(F_{Y_i}^{-1}(\tau|x_i))x_i x_i^{\top}$.

Relaxation of the independence condition on the observations naturally extends applicability of QR method to time series data. Quantile regression not only provides a method of estimating the conditional quantiles of existing time series models, it also substantially expands the modeling options for time series analysis. There are substantial theoretical literature on quantile autoregression methods including Weiss (1991), Knight (1989, 1998), Koul and Saleh (1995) and Hallin and Jurečkova (1999). Recently, Koenker and Xiao (2006) generalized the Quantile autoregression model (QAR) as an competitive alternative of classical ARMA model. It allows to study asymmetric dynamics and local persistency in time series. More details about QAR model appear in Chapter 1.2.

However, time series measurements are often observed with a detection limit or other form of censoring. For instance, a water-quality monitoring device in the Samish River station, Washington, has a detection limit and it records the limit value of Ammonia-Nitrogen (NH3-N) measurements when the true value precedes the detection limit. This motivates us to consider censored time series models. In this paper, we first show the existing censored quantile regression methods still can play an important role in censored quantile autoregression models. And later, we propose a new algorithm to improve a censored quantile autoregression estimator.

For the standard regression model, Powell (1986) initially proposed a regression quantile estimator for responses with fixed censoring. Later, Portnoy (2003) and Peng and Huang (2008) introduced methods for random censoring. The Portnoy and Peng-Huang estimators can be viewed, respectively, as generalizations to regression of the Kaplan-Meier and Nelson-Aalen estimators of univariate quantiles for censored observations. In their paper, both methods achieves root-n consistency of censored quantile regression estimator. Further, on the Portnoy's estimator, a correction giving root-n consistency was presented in Neocleous et al. (2006), and under mild conditions asymptotic normality was shown in Portnoy and Lin (2010). Recently, Wang and Wang (2009) proposed a censored quantile regression estimator that employs a local reweighting scheme. Throughout the paper, we follow the grid method of Portnoy (2003), but other methods such as Powell (1986) and Peng and Huang (2008) can be used as alternatives.

Since censoring is a form of missing data, we improve our estimates by adopting an idea of imputation methods which is widely considered in many applications. Imputation methods for time series have been presented by several authors. Robinson (1980) suggested imputing the censored part with its conditional expectation given the completely observed part. Since the conditional expectation has the form of multiple incomplete integrals, he subgrouped the data vector so that each subgroup includes one censored observation, and thus requires a single integral. However, the method may not be feasible for many consecutive censored observations which can happen in many time series data. Zeger and Brookmeyer (1986) suggested a full likelihood estimation and approximate method for an autoregressive time series model. In addition, they suggested the use of pseudolikelihood estimation to overcome non-feasibility in case of high censoring rate in previous method. Later, Rubin (1987) introduced Multiple Imputation (MI) method to address the question of how to obtain valid inferences from imputed data. MI is a Monte Carlo technique in which the missing values are replaced by $m > 1$ simulated versions, where m is typically small (e.g. 3-10). In Rubin's method for 'repeated imputation' inference, each of the simulated complete datasets is analyzed by standard methods, and the results are combined to produce estimates and confidence intervals that incorporate missing-data uncertainty. Hopke et al. (2001) used multiple imputation based on a Bayesian approach. Recently, Park et al. (2007, 2009) presented parametric imputation method based on ARMA models and nonparametric estimation of autocovariance function. In quantile regression models, Wei et al. (2012) proposed a multiple imputation estimator for the quantile function when some covariates are missing at random.

In the rest of this chapter, we shall introduce quantile autoregression proposed by Koenker and Xiao (2006). In Chapter 2, we define censored quantile autoregression model and give a consistent estimate of autoregressive parameter. Then we introduce the Censored Quantile AutoRegression (CQAR) algorithm. In Chapter 3, a simulation study is conducted based on an QAR process to study the empirical consistency of the censored regression quantiles on time series data and show performance of the CQAR method compared with ordinary methods. Chapter 4 summarize our work and discuss some future directions in this area.

1.2 Introduction to Quantile Autoregression

The quantile autoregression (QAR) model was introduced in Koenker and Xiao (2006). Let $\{U_t\}$ be a sequence of *iid* standard uniform random variables, and consider the pth order autoregressive process,

$$y_t = \beta_0(U_t) + \beta_1(U_t)y_{t-1} + \cdots + \beta_p(U_t)y_{t-p}, \tag{1.4}$$

where the β_j's are unknown functions $[0,1] \to \mathbb{R}$ that we will want to estimate. Then, the τth conditional quantile function of y_t can be written as,

$$Q_{y_t}(\tau|y_{t-1}, ..., y_{t-p}) = \beta_0(\tau) + \beta_1(\tau)y_{t-1} + \cdots + \beta_p(\tau)y_{t-p}, \tag{1.5}$$

or more compactly as,

$$Q_{y_t}(\tau|\mathcal{F}_{t-1}) = x_t^\top \beta(\tau), \tag{1.6}$$

where $x_t = (1, y_{t-1}, ..., y_{t-p})^\top$, and $\mathcal{F}_t$ is the σ-field generated by $\{y_s, s \leq t\}$. This model is called the QAR(p) model in Koenker and Xiao (2006). The autoregression quantile estimator, $\hat{\beta}(\tau)$, is the minimizer over β of the objective function,

$$\sum_{t=1}^{n} \rho_\tau(y_t - x_t^\top \beta), \tag{1.7}$$

where $\rho_\tau(u) = u(\tau - I(u < 0))$. To facilitate asymptotic analysis, Koenker and Xiao (2006) reformulate the QAR(p) model in (1.4) in the more conventional random coefficient notation as,

$$y_t = \mu_0 + \alpha_{1,t} y_{t-1} + \cdots + \alpha_{p,t} y_{t-p} + u_t \tag{1.8}$$

where $\mu_0 = E\beta_0(U_t)$, $u_t = \beta_0(U_t) - \mu_0$, and $\alpha_{j,t} = \beta_j(U_t)$, for $j = 1, ..., p$. Thus, $\{u_t\}$ is an iid sequence of random variables with distribution function $F(\cdot) = \beta_0^{-1}(\cdot + \mu_0)$, and the $\alpha_{j,t}$ coefficients are functions of this u_t innovation random variable. The QAR(p) process in (1.8) can be expressed as an p-dimensional vector autoregression process of order 1:

$$Y_t = \Gamma + A_t Y_{t-1} + V_t$$

with

$$\Gamma = \begin{bmatrix} \mu_0 \\ 0_{p-1} \end{bmatrix}, A_t = \begin{bmatrix} A_{p-1,t} & \alpha_{p,t} \\ I_{p-1} & 0_{p-1} \end{bmatrix}, V_t = \begin{bmatrix} u_t \\ 0_{p-1} \end{bmatrix},$$

where $A_{p-1,t} = [\alpha_{1,t}, \cdots, \alpha_{p-1,t}]$, $Y_t = [y_t, \cdots, y_{t-p+1}]^\top$, and 0_{p-1} is the $(p-1)$ dimensional vector of zeros. Then, the QAR(p) process y_t given by (1.4) is covariance stationary and satisfies a central limit theorem,

$$\frac{1}{n} \sum_{t=1}^{n} (y_t - \mu_y) \Rightarrow N(0, \omega_y^2), \tag{1.9}$$

where $\mu_y = \mu_0/(1 - \sum_{j=1}^{p} \mu_j)$, $\omega_y^2 = \lim n^{-1} E[\sum_{t=1}^{n} (y_t - \mu_y)]^2$, and $\mu_j = E(\alpha_{j,t})$, $j = 1, ..., p$. The appropriate conditions are as follows:

1. $\{u_t\}$ are iid random variables with mean 0 and variance $\sigma^2 < 0$. The distribution function of u_t, F, has a continuous density f with $f(u) > 0$ on $\mathcal{U} = \{u : 0 < F(u) < 1\}$.

2. Let $E(A_t \bigotimes A_t) = \Omega_A$; the eigenvalues of Ω_A has a moduli less than unity.

3. Denote the conditional distribution function $P[y_t < \cdot | \mathcal{F}_{t-1}]$ as $F_{t-1}(\cdot)$ and its

derivative as $f_{t-1}(\cdot)$; f_{t-1} is uniformly integrable on $\mathcal{U}$.

Chapter 2

Quantile Autoregression with Censored data

In many practical situations we may not be able to observe $\{y_t\}$ directly. Here we focus on censored data. Specifically, suppose we observe $\{y_t\}$ only when $\{y_t\}$ is less than a constant value c_t. Let $\{\tilde{y}_t\}$ be the value we observe instead of $\{y_t\}$ due to censoring and c_t be the censored value at time t. Then, the censored response variable and censoring indicator can be denoted as

$$\tilde{Y}_t = \min\{Y_t, c_t\}, \qquad \Delta_t = I\{Y_t < c_t\}. \tag{2.1}$$

The main differences of this setting, compared with ordinary censored quantile regression, are that $\{y_t\}$ is dependent and x_t in (1.6) is also censored as well as y_t. This prevents the use of any previously introduced censored regression quantile methods. Here the regressor, X_t, and its censoring indicator can be denoted as

$$\tilde{X}_t = (1, \tilde{Y}_{t-1}, ..., \tilde{Y}_{t-p})^\top, \qquad \Gamma_t = I\{Y_{t-1} < c_{t-1}\} \cdots I\{Y_{t-p} < c_{t-p}\} = \prod_{k=1}^{p} \Delta_{t-k}. \tag{2.2}$$

On estimating a model with censored regressors, one can approach this as an estimation problem with missing data since censoring is a form of missing data. When censoring is conditionally independent of the response given the regressors, implying that censoring is not systematically related to the value of the response under study, estimation can proceed with complete cases only. That is, as long as conditional independence of the response and the censoring time given the regressors holds, one can still have consistent estimator using only observed part of regressors.

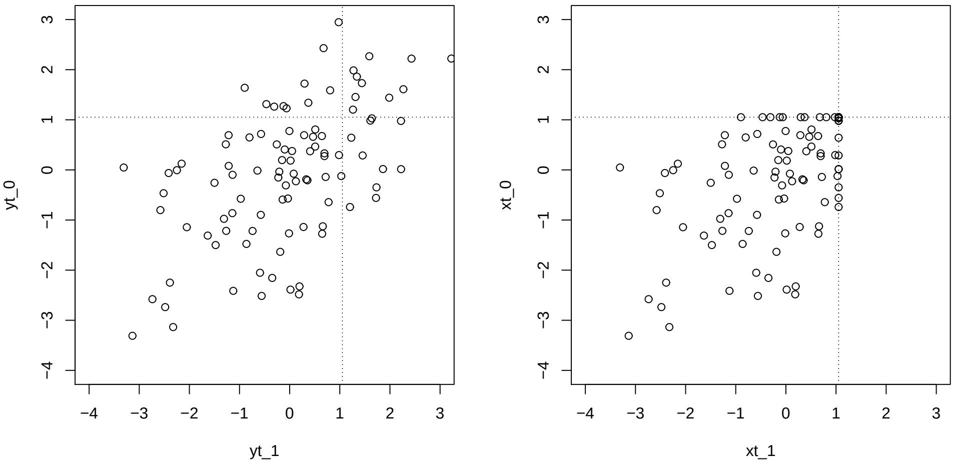

Figure 2.1: Scatter plot for simple QAR(1) model. Random time series with size 100 were generated. $\beta = .6$ and i.i.d Gaussian innovation were used. The left side is all observations before censoring and the right side is after censoring. Dashed line shows the cutoff value c which was generated to give a censoring rate of 20% approximately.

Working with complete part of observations, we have

$$\{\tilde{Y}_t, \tilde{X}_t, \Delta_t\} \quad \text{for } t \in \{t : \gamma_t = 1\}, \tag{2.3}$$

which leads to estimating problem for regression quantiles under dependency. Previous work by Cai (2001) showed that under some regularity conditions, the Kaplan-Meier estimator enjoys uniform consistency when estimating a distribution function for censored time series data. This motivates us to apply the grid method proposed by Portnoy (2003) to censored quantile autoregressive models since this method it-self is a natural generalization of the Kaplan-Meier estimator to the case of censored regression quantiles.

Following notations from Portnoy and Lin (2010), we define $\hat{\beta}(g_k)$ recursively along a grid $\{g_1, ..., g_M\}$. We assume that there is no censoring below some positive g_k, and so we can define $\hat{\beta}(g_1)$ (and $\hat{\beta}(g_2)$) by ordinary uncensored regression quantile methods (see Koenker (2005)). We then recursively define weights, $w_t(g)$, for each censored observation that is crossed between the regression quantiles at g_j and g_{j+1} and define $\hat{\beta}(g_{j+2})$ as a weighted regression quantile.

Before we define the weights, we need to define the true censoring probability, τ_t, so that $x_t'\beta(\tau_t) = c_t$. And the estimated censoring probability, $\hat{\tau}_t$, is defined as

$$\hat{\tau}_t = (1 - \hat{\alpha}_{tj})g_j + \hat{\alpha}_{tj} g_{j+1}; \quad \hat{\alpha}_{tj} = \frac{c_t - x_t'\hat{\beta}(g_{j+1})}{x_t'(\hat{\beta}(g_{j+1}) - \hat{\beta}(g_j))}. \tag{2.4}$$

Then, given $\hat{\tau}_t$ for censored observations crossed before g_j, we define the weights, $\hat{w}_t(g)$, as follows:

$$\hat{w}_t(g) = \begin{cases} \frac{g - \hat{\tau}_t}{1 - \hat{\tau}_t} & \text{if } \delta_t = 0 \text{ and } g \geq \hat{\tau}_t \\ 0 & \text{if } \delta_t = 0 \text{ and } g < \hat{\tau}_t \\ 1 & \text{if } \delta_t = 1 \end{cases} \tag{2.5}$$

Finally, given the weights we can define the regression quantile estimator, $\hat{\beta}(t_{j+2})$, recursively as the minimizer (over b) of the weighted objective function:

$$\begin{aligned} R_{j+2}(b) \equiv \; & \textstyle\sum_{\{t:\gamma_t=1\}} \{ I(\delta_t = 1)\rho(Y_t - x_t'b) + I(\delta_t = 0) \\ & \times [\hat{w}_t(g_{j+2})\rho(c_t - x_t'b) + (1 - \hat{w}_t(g_{j+2}))\rho(Y^* - x_t'b)] \} \end{aligned} \tag{2.6}$$

where Y^* is any sufficiently large value (specifically, larger than all the observations and fitted values), and ρ is the usual "check" function: $\rho_\tau(u) = u(\tau - I(u \leq 0))$.

And it is also possible to define $\hat{\beta}(g_{j+2})$ using the subgradient, $\Psi_{j+2}(w(\underline{\hat{\beta}}_{j+1}, t_{j+2}), b)$

corresponding to (2.6), where

$$\begin{aligned}\Psi_k(\hat{w}, b) \equiv & \sum_{\{t:\gamma_t=1\}} x_t\{I(\delta_t = 1)\psi(Y_t - x_t'b, g_k) + I(\delta_t = 0) \\ & \times[\hat{w}_t(g_k)\psi(c_t - x_t'b, g_k) + (1 - \hat{w}_t(g_k))\psi(Y^* - x_t'b, g_k)]\}\end{aligned} \tag{2.7}$$

and where $\psi(u, \tau) = \tau - I(u \leq 0)$, and Y^* is as above. As described in Koenker (2005), the gradient conditions impose a bound of the form $\Psi_k(\hat{w}, b) = O(1)$ at $b = \hat{\beta}(g)$ (uniformly in k), as long as $\|x_t\|$ remain bounded.

The conditions needed for the asymptotic properties of the censored quantile regression estimator are listed here:

1. Y_t given by Equation (1.8) is a stationary time series.

2. All conditions restrict to $\epsilon \leq \tau \leq \min\{\bar{\tau}, 1-\epsilon\}$ where $\bar{\tau}$ is the largest identifiable τ-value. Furthermore, there is no censoring below $x_t'\beta(\epsilon)$. Hence, $\hat{\beta}(\epsilon')$ can be computed as an unweighted regression quantile for $\epsilon' < \epsilon$ with probability tending to one.

3. The conditional density $f_t(x_t'\beta(g))$ (conditional on $\{x_t\}$) has uniformly bounded derivative (with respect to g) on $\epsilon \leq \tau \leq \min\{\bar{\tau}, 1 - \epsilon\}$, and is strictly positive on this set.

4. $\|x_t\|$ has bounded support.

5. $\{g_1, ..., g_M\}$ is a grid with mesh $\delta_n = d_n n^{-a}$ for some a with $1/4 < a < 1/2$ and $d_n \to d$ (with $d > 0$).

6. The design matrix, X, satisfies $\frac{1}{n}X'X \to A$, where A is invertible.

Under the above conditions, a strong uniform root-n consistency result is proved in the next section. Note that all calculations below will be done conditionally on $\{x_t\}$ so that we may implicitly consider $\{x_t\}$ as fixed.

Remark 1. With heavy censoring in large τ, it is not unusual to limit range of $\tau \in [0,1]$ where a regression quantile is estimable. The value $\bar{\tau}$ in Condition (2) is the largest τ for which $\beta(\tau)$ is identifiable.

Remark 2. By letting u_t in (1.8) be serially uncorrelated, Condition (6) follows.

Remark 3. A further extension of the following theorem on random censoring would require some form of conditional independence of the response and the censoring time given the regressors.

2.1 Inductive Proof of Consistency under Dependence

Theorem 2.1 Let $\underline{\hat{\beta}}_M \equiv (\hat{\beta}(g_1)', ..., \hat{\beta}(g_M)')$ be the right censored quantile estimator along the grid $\epsilon = g_1 < g_2 < ... < g_M < \min\{\bar{\tau}, 1-\epsilon\}$ (where $\bar{\tau}$ is the largest identifiable τ-value). Under Conditions (1)-(6), we have

$$\| \hat{\beta}(g_k) - \beta(g_k) \| \leq 2r_1 n^{-1} d_{k,n}, \quad k = 1, ..., M.$$

where $M = o(n^{1/2})$, $d_{k,n} = R_n\sqrt{n}(1 + 2r_1r_2E_n^*\delta_n)^{k-1}$ with $E_n^* = O_p(1)$, and $R_n = O_p(1)$ and is defined by: $R_n = n^{-1/2}(\max_t\{\|x_t\|\})^{-1}\frac{(1-\bar{\tau})^2}{1-\epsilon}\max_k\{\|\Psi_{k+1}(\underline{w}_k, \beta(g_k))\| + E_{n,k}\}$. Here, Ψ_k is defined by Equation (2.7) and r_1 and r_2 are positive constants; and we show that $E_{n,k} = O_p(n^{1/4}\log n)$ uniformly in k, where $E_{n,k}$ is defined by Equation (2.11). Recall that $\beta(g_k)$ is the true regression quantile along the same grid, and $\delta_n = O(n^{-1/2})$ is defined in Condition (5).

Remark Note that since $M \leq 1/\delta_n$, the factor $(1 + 2r_1r_2E_n^*\delta_n)^{k-1} = O_p(1)$ uniformly in $k \leq M$. Thus, the uniform bound in Theorem is $O_p(n^{-1/2})$.

Proof. Let $CI_k = \{t : Y_t = c_t \ \text{ and } \ \max\{\hat{\tau}_t, \tau_t\} \leq g_k\}$ be the index set of the crossed censored observations. We shall use mathematical induction to show that for any $k = 1, 2, ..., M$,

$$\sum_{t \in CI_k} |\hat{\tau}_t - \tau_t| \leq d_{k,n}, \text{ and } \|\hat{\beta}(g_k) - \beta(g_k)\| \leq 2r_1 n^{-1} d_{k,n}. \tag{2.8}$$

First let $k = 1$, since $\hat{\beta}(g_1)$ is the quantile estimator at g_1 by applying the usual (uncensored) regression quantile, it is known that $\|\hat{\beta}(g_1) - \beta(g_1)\| = O_p(n^{-1/2})$ by Theorem 2 in Koenker and Xiao (2006). Thus,

$$\|\hat{\beta}(g_1) - \beta(g_1)\| = O_p(n^{-1/2}) \leq 2r_1 n^{-1} d_{1,n}$$

We can also see that all τ_t and $\hat{\tau}_t$ exceed g_1. So $\Sigma_{t \in CI_1} |\hat{\tau}_t - \tau_t| = 0$ since the sum is empty. Thus, Equation (2.8) is true for $k = 1$.

Assume that for $k = l$, Equation (2.8) is true. Now we can obtain a bound for the difference between the estimated weights and the true weights at the g_{k+1}th quantile:

$$\begin{aligned}\sum_{t=1}^{n} |w_t(\underline{\hat{\beta}}_l, g_{l+1}) - w_t(\underline{\beta}_l, g_{l+1})| &= \sum_{t \in CI_l} |w_t(\underline{\hat{\beta}}_l, g_{l+1}) - w_t(\underline{\beta}_l, g_{l+1})| \\ &+ \sum_{\{t\,:\, Y_t = c_t \text{ between } x_t'\hat{\beta}(g_l) \text{ and } x_t'\beta(g_l)\}} |w_t(\underline{\hat{\beta}}_l, g_{l+1}) - w_t(\underline{\beta}_l, g_{l+1})| \end{aligned} \tag{2.9}$$

For the second term above, each summand is bounded by the grid mesh, δ_n. From Equation (2.8) and the Condition (1), the number of summands in this second term is $O_p(n^{1/2})$. It follows that there are random bounds, $E_n = O_p(1)$ and $\tilde{E}_n = O_p(1)$, such that the difference of weights in Equation (2.9) is bounded by

$$
\begin{aligned}
\sum_{t\in CI_l}\Big|\frac{g_{l+1}-\hat{\tau}_t}{1-\hat{\tau}_t}-\frac{g_{l+1}-\tau_t}{1-\tau_t}\Big|+\sqrt{n}E_n\delta_n &= \sum_{t\in CI_l}\frac{(1-g_{l+1})|\hat{\tau}_t-\tau_t|}{(1-\hat{\tau}_t)(1-\tau_t)}+\sqrt{n}E_n\delta_n \\
&\le \sum_{t\in CI_l}\frac{1-\epsilon}{(1-\bar{\tau})^2}|\hat{\tau}_t-\tau_t|+\sqrt{n}E_n\delta_n \\
&\le \frac{1-\epsilon}{(1-\bar{\tau})^2}d_{l,n}+\sqrt{n}E_n\delta_n \\
&= \frac{1-\epsilon}{(1-\bar{\tau})^2}d_{l,n}\Big(1+\tilde{E}_n\delta_n\Big). \qquad (2.10)
\end{aligned}
$$

From Lemma 3.4 and 3.5 in Portnoy (1991), we can see that for any constant $C^*>0$, there is a constant $A_2>0$, such that for large enough values of $A_1>0$ and n,

$$
P\left(\sup_{\{\theta:\|\theta-\beta(g_{l+1})\|\le C^*n^{-1/2}\}}\|\eta_n(\theta,\beta(g_{l+1}))-E\eta_n(\theta,\beta(g_{l+1}))\|>A_1n^{1/4}\log n\right)\le A_2e^{-A_1n},
$$

where $\eta_n(\theta,\beta(g_{l+1}))=\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\theta)-\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\beta(g_{l+1}))$, with Ψ_{l+1} given by Equation (2.7). Thus, we have

$$
P\left(\max_{1\le l\le M}\sup_{\theta}\|\eta_n(\theta,\beta(g_{l+1}))-E\eta_n(\theta,\beta(g_{l+1}))\|>A_1n^{1/4}\log n\right)\le MA_2e^{-A_1n},
$$

That is, on $\{\theta:\|\theta-\beta(g_{l+1})\|\le C^*n^{-1/2}\}$,

$$
\begin{aligned}
E_{n,l}=&\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\theta)-\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\beta(g_{l+1})) \\
&-E\{\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\theta)-\Psi_{l+1}(w(\underline{\beta}_l,g_{l+1}),\beta(g_{l+1}))\}, \qquad (2.11)
\end{aligned}
$$

where $E_{n,l}=O_p(n^{1/4}\log n)$ uniformly in l.

The expectation in Equation (2.11) is as follows:

$$\sum_t x_t E\{[I(Y_t \le x_t'\theta) - I(Y_t \le x_t'\beta(g_{l+1}))]I(Y_t \le c_t)$$
$$+ w_t(g_{l+1})[I(c_t \le x_t'\theta) - I(c_t \le x_t'\beta(g_{l+1}))]I(Y_t \ge c_t)\}.$$

For c_t between $x_t'\theta$ and $x_t'\beta(g_{l+1})$, either $w_t(g_{l+1}) = 0$ (if c_t is uncrossed; that is, $c_t > x_t'\beta(g_{l+1})$), or $w_t(g_{l+1}) = O(n^{-1/2})$(from the definition of the weights (2.5), the fact that $\|\theta - \beta(t_{l+1})\| = O(n^{-1/2})$, and the definition of the interpolating weights (2.4)). Thus the second term above is $O(1)$(again using $\|\theta - \beta(t_{l+1})\| = O(n^{-1/2})$) and the expectation becomes:

$$\begin{aligned}
&\sum_t x_t E\{[I(Y_t \le x_t'\theta) - I(Y_t \le x_t'\beta(g_{l+1}))]I(Y_t \le c_t)\} + O(1) \\
&= \sum_t x_t\{[P(Y_t \le x_t'\theta) - P(Y_t \le x_t'\beta(g_{l+1}))]P(Y_t \le c_t)\} + O(1) \\
&= \sum_t x_t\{[F_t(x_t'\theta) - F_t(x_t'\beta(g_{l+1}))]F_t(c_t)\} + O(1) \\
&\le \sum_t x_t\{f_t(x_t'\beta(g_{l+1}))(x_t'\theta - x_t'\beta(g_{l+1}))F_t(c_t)\} + O(1) \\
&= \sum_t x_t\{f_t(x_t'\beta(g_{l+1}))F_t(c_t)\}x_t'(\theta - \beta(g_{l+1})) + O(1)
\end{aligned}$$

Then, the sum becomes $X'VX(\theta - \beta(g_{l+1})) + O(1)$ where X is the design matrix and V is a diagonal matrix with

$$V_{tt} \equiv f_t(x_t'\beta(g_{l+1}))F_t(c_t). \tag{2.12}$$

As a consequence, Equation (2.11) can be written as follows:

$$\Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \theta) - \Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \beta(g_{l+1})) + X'VX(\theta - \beta(g_{l+1})) = E_{n,l}, \tag{2.13}$$

where $E_{n,l}$ is modified by the $O(1)$ term in the expectation, but still $O_p(n^{1/4}\log n)$.

Now for $k = l+1$, and $\{\theta : \|\theta - \beta(g_{l+1})\| \le C^* n^{-1/2}\}$,

$$
\begin{aligned}
\Psi_{l+1}(w(\hat{\underline{\beta}}_l, g_{l+1}), \theta) &= \Psi_{l+1}(w(\hat{\underline{\beta}}_l, g_{l+1}), \theta) - \Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \theta) + \Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \theta) \\
&= \sum_{t \in CI_l} (\hat{w}_t - w_t) I(c_t < x_t'\theta) x_t + \Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \theta) \\
&= \sum_{t \in CI_l} (\hat{w}_t - w_t) I(c_t < x_t'\theta) x_t + \Psi_{l+1}(w(\underline{\beta}_l, g_{l+1}), \beta(g_{l+1})) \\
&\quad - X'VX(\theta - \beta(g_{l+1})) + E_{n,l}. \qquad (2.14)
\end{aligned}
$$

Note that, uniformly in l (and with $\underline{w}$ denoting $w(\underline{\beta}_l, g_{l+1})$),

$$
\begin{aligned}
\|\Psi_{l+1}(\underline{w}, \theta)\| &= O(n^{1/2}), \\
\left\| \sum_t (\hat{w}_t - w_t) I(c_t < x_t'\theta) x_t' \right\| &\le \sum_t \|x_t\| |\hat{w}_t - w_t| \\
&\le \max_t \{\|x_t\|\} \left\| \frac{1-\epsilon}{(1-\bar{\tau})^2} d_{l,n} \Big(1 + \tilde{E}_n \delta_n\Big) \right\| \\
&= \max_t \{\|x_t\|\} \left\| \frac{1-\epsilon}{(1-\bar{\tau})^2} d_{l,n} \right\| \\
&= O_p(n^{1/2}), \\
\|\Psi_{l+1}(\underline{w}, \beta(g_{l+1}))\| &= O_p(n^{1/2}), \quad \text{and} \\
E_{n,l} &= O_p(n^{1/4}\log n).
\end{aligned}
$$

An upper bound on the maximum eigen value, $\lambda_{\max}((X'VX)^{-1})$ is also needed. By Condition (3), $f(x_t'\beta(g))$ is bounded from below (uniformly in t) for $\epsilon \le g \le \min\{\bar{\tau}, 1-\epsilon\}$. Thus, $V_{tt} \ge a$ for some $a > 0$ (uniformly in t), and hence (using Condition (6)), for some $a_1 > 0$,

$$
\lambda_{\max}((X'VX)^{-1}) \le a^{-1} \lambda_{\max}((X'X)^{-1}) \le a_1 n^{-1}. \qquad (2.15)
$$

Now, since $\Psi_{l+1}(\underline{w}, \theta)$ is the gradient of a convex function ρ on θ, inserting these results into Equation (2.13) implies that the gradient condition cannot hold for $\{\theta : \|\theta - \beta(t_{l+1})\| > C^* n^{-1/2}\}$ if C^* is chosen large enough (see Theorem 2.1 in Koenker (2005)). Therefore, Equation (2.14) is true for $\theta = \hat{\beta}(g_{l+1})$. After inserting $\theta = \hat{\beta}(g_{l+1})$ in Equation (2.14), solving Equation (2.13), and using Equation (2.10) and (2.15), and the definition of R_n, we have

$$
\begin{aligned}
\|\hat{\beta}(g_{l+1}) - \beta(g_{l+1})\| &= \Big\|(X'VX)^{-1}\Big\{\sum_t (\hat{w}_t - w_t) I(c_t < x_t'\hat{\beta}(g_{l+1}))x_t \\
&\qquad - \Psi_{l+1}(\underline{\hat{w}}, \hat{\beta}(g_{l+1})) + \Psi_{l+1}(\underline{w}, \beta(g_{l+1})) + E_{n,l}\Big\}\Big\| \\
&\leq a_1 n^{-1}\Big\{\sum_t \|x_t\||(\hat{w}_t - w_t)| + \|\Psi_{l+1}(\underline{\hat{w}}, \hat{\beta}(g_{l+1}))\| \\
&\qquad + \|\Psi_{l+1}(\underline{w}, \beta(g_{l+1}))\| + \|E_{n,l}\|\Big\} \\
&\leq a_1 n^{-1}\Big\{\max_t\{\|x_t\|\}\frac{1-\epsilon}{(1-\bar{\tau})^2} d_{l,n}(1 + \tilde{E}_n\delta_n) \\
&\qquad + \|\Psi_{l+1}(\underline{w}, \beta(g_{l+1}))\| + \|E_{n,l}\|\Big\} \\
&\leq r_1(n^{-1}d_{l,n}(1 + \tilde{E}_n\delta_n) + n^{-1}R_n\sqrt{n}) \\
&\leq r_1 n^{-1}d_{l,n}(1 + \tilde{E}_n\delta_n) + r_1 n^{-1}d_{l,n} \\
&\leq 2r_1 n^{-1}d_{l,n}(1 + \tilde{E}_n\delta_n) \qquad (2.16) \\
&\leq 2r_1 n^{-1}d_{l+1,n}, \qquad (2.17)
\end{aligned}
$$

as long as $\tilde{E}_n \leq 2r_1r_2E_n^*$(see Equation (2.22)); thus providing the first part of the induction result.

Here we define

$$r_1 = a_1 \max_t\{\|x_t\|\}\frac{1-\epsilon}{(1-\bar{\tau})^2}. \qquad (2.18)$$

Note that, by definition of R_n in the hypothesis of the theorem,

$$R_n = n^{-1/2}(\max_t\{\|x_t\|\})^{-1}\frac{(1-\bar{\tau})^2}{1-\epsilon}\max_l\{\|\Psi_{l+1}(\underline{w}_l, \beta(g_{l+1}))\| + E_{n,l}\} = O_p(1).$$

Note also that the last steps in Equations (2.16) and (2.17) use the facts that $d_{l,n} \geq R_n\sqrt{n}$ and $d_{l,n}(1+\tilde{E}_n\delta_n) = d_{l+1,n}$ (by definitions in the hypotheses of the theorem). Also, applying the induction hypothesis, using the first inequality in Equation (2.16) in the next-to-last inequality below,

$$\begin{aligned}
&\sum_{t\in CI_{l+1}} |\hat{\tau}_t - \tau_t| \\
&\quad\leq \sum_{t\in CI_l} |\hat{\tau}_t - \tau_t| + \sum_t |\hat{\tau}_t - \tau_t| I(x_t'\hat{\beta}(g_l) \leq c_t \leq x_t'\hat{\beta}(g_{l+1})) && (2.19)\\
&\quad\leq \sum_{t\in CI_l} |\hat{\tau}_t - \tau_t| + \sum_t \Big(n^{1/2} f_t(x_t'\beta(g_l))x_t'B_l^* + O(\delta_n^2)\Big) I(x_t'\hat{\beta}(g_l) \leq c_t \leq x_t'\hat{\beta}(g_{l+1}))\\
&\quad\leq d_{l,n} + \sum_t \Big(n^{1/2} r_2\|B_l^*\| + O(\delta_n^2)\Big) I(x_t'\hat{\beta}(g_l) \leq c_t \leq x_t'\hat{\beta}(g_{l+1}))\\
&\quad\leq d_{l,n} + r_2\Big(2r_1 n^{-1} d_{l,n}(1+\tilde{E}_n\delta_n)\Big)\sum_t I(x_t'\hat{\beta}(g_l) \leq c_t \leq x_t'\hat{\beta}(g_{l+1}))\\
&\quad\leq d_{l,n} + 2r_1r_2 d_{l,n}(1+\tilde{E}_n\delta_n)\delta_n\\
&\quad\leq d_{l,n}(1+2r_1r_2E_n^*\delta_n) = d_{l+1,n} && (2.20)
\end{aligned}$$

where

$$B_l^* = \frac{1}{2}n^{-1/2}((\hat{\beta}(t_l) - \beta(t_l)) + (\hat{\beta}(t_{l+1}) - \beta(t_{l+1}))) \tag{2.21}$$

and E_n^* may be defined by

$$E_n^* \equiv \max\Big\{\frac{\tilde{E}_n}{(2r_1r_2)}, 1+\tilde{E}_n\delta_n\Big\} = O_p(1), \tag{2.22}$$

and where r_2 is a chosen constant satisfying

$$r_2 \geq \|x_t\| f_t(x_t'\beta(g_l)). \tag{2.23}$$

Note that (2.19) uses only the smoothness of the true $\beta(\tau)$ function and the definition of τ_t to find the bound for $|\hat{\tau}_t - \tau_t|$. And from (2.17), the indication function can be reformulated as

$$I(x_t'\hat{\beta}(g_l) \leq c_t \leq x_t'\hat{\beta}(g_{l+1})) \leq I(x_t'\beta(g_l) - M \leq c_t \leq x_t'\beta(g_{l+1}) + M) \tag{2.24}$$

where $M = 2r_1 n^{-1} d_{l,n}$. Since the interval is of length $O(\delta_n)$ and $x_t'\beta(t)$ is monotonically increasing at a rate that is at least linear, the indicator function can be non-zero only for $O(\delta_n)$ indices and thus the sum is $nO(\delta_n)$.

Combining Equations (2.16) and (2.20), the induction step is done. So Equation (2.8) is true for $k = 1, 2, ..., M$, and the induction proof is complete.

2.2 The Self-Consistent Algorithm for Censored QAR

In many applications, despite the fact that the estimation with complete cases only still produce a consistent estimator when censoring is conditionally independent of the response, there has been concern on precision of the estimates. The set of complete cases is often a very small fraction of the original data, so that estimation based only on complete cases involves substantially lower precision than with the full sample. This suggests that imputation of missing value may be useful, that is rather than removing censored observations, reasonable alternatives are to be filled in or "imputed". A variety of imputation approaches can be used that range from extremely simple to rather complex. These methods keep the full sample size, which can be advantageous for bias and precision. For a classical autoregressive time series model, Zeger and Brookmeyer (1986) suggested a full likelihood estimation and approximate method. Later Park et al. (2007) introduced an imputation algorithm using Gibbs sampling on conditional distribution of the censored part of an autoregressive model. However, there have not been any studies to adapt the imputation method to censored quantile autoregressive models. Here we propose a self-consistent algorithm to refine autoregressive coefficients for censored time series data. The algorithm estimates quantile autoregressive coefficients in a self-consistent manner by imputing regression predictions for censored observations.

Given the initial estimates, $\hat{\beta}_0(g_k)$, along a grid $\{g_1, ..., g_M\}$ from Equation (2.6), for censored observations in $\tilde{X}_t$ from (2.2), we can define $\hat{\tau}_t$ by

$$x'_{t-1}\hat{\beta}_0(\hat{\tau}_t) = c_t \tag{2.25}$$

Since the true value, x_t, is located somewhere above the censored observation, c_t, the corresponding τ_t^* which makes $x'_{t-1}\hat{\beta}_0(\tau_t^*) = x_t$ is somewhere in between $(\hat{\tau}_t, 1)$. By selecting random $\hat{\tau}_t^*$ from $(\hat{\tau}_t, 1)$, each censored observation is replaced by its quantile regression prediction, $x_t^{new} = x'_{t-1}\hat{\beta}_0(\hat{\tau}_t^*)$. After imputing all censored observations in $\tilde{X}_t$, we re-estimate $\beta(g_k)$ using full sample. Given the weights defined in (2.5), the quantile autoregressive estimator, $\hat{\beta}_1(g_{j+2})$, is defined recursively as minimizer (over b) of the weighted objective function:

$$\begin{aligned} R_{j+2}(b) \equiv \ & \textstyle\sum_t \{I(\delta_t = 1)[I(\gamma_t = 1)\rho(Y_t - x_t'b) + I(\gamma_t = 0)\rho(Y_t - x_t^{new'}b)] \\ & +I(\delta_t = 0)[I(\gamma_t = 1)[\hat{w}_t(g_{j+2})\rho(c_t - x_t'b) + (1 - \hat{w}_t(g_{j+2}))\rho(Y^* - x_t'b)] \\ & +I(\gamma_t = 0)[\hat{w}_t(g_{j+2})\rho(c_t - x_t^{new'}b) + (1 - \hat{w}_t(g_{j+2}))\rho(Y^* - x_t^{new'}b)]]\} \end{aligned} \tag{2.26}$$

and corresponding subgradient:

$$
\begin{aligned}
\Psi_k(\hat{w}, b) \equiv \; & \textstyle\sum_t x_t\{I(\delta_t = 1)[I(\gamma_t = 1)\psi(Y_t - x_t' b, g_k) + I(\gamma_t = 0)\psi(Y_t - x_t^{new'} b, g_k)] \\
& + I(\delta_t = 0)[I(\gamma_t = 1)[\hat{w}_t(g_k)\psi(c_t - x_t' b, g_k) + (1 - \hat{w}_t(g_k))\psi(Y^* - x_t' b, g_k)] \\
& + I(\gamma_t = 0)[\hat{w}_t(g_k)\psi(c_t - x_t^{new'} b, g_k) + (1 - \hat{w}_t(g_k))\psi(Y^* - x_t^{new'} b, g_k)]]\}
\end{aligned}
\tag{2.27}
$$

where Y^* is any sufficiently large value, and ρ and Ψ are as described previously.

Next we repeatedly carry out the imputing and estimating steps in an iterative fashion. Then, the censored autoregressive quantile estimator, $\hat{\beta}(\tau)$ is computed in a self-consistent manner. Formally, the censored quantile autoregression (CQAR) algorithm is described as follows:

- Step 1 (initialization). Obtain the initial estimates $\hat{\beta}(\tau)$. We can estimate $\hat{\beta}(g_k)$ recursively along a grid $\{g_1, \ldots, g_M\}$ by using only the uncensored part of $\tilde{x}_t$.

- Step 2 (randomly assign $\tau_t^* \geq \tau_t$). For $\tilde{Y}_t$, starting from the first observation, when the censoring is encountered, find the smallest $\tilde{g}_k$ which satisfies $\tilde{Y}_{t-1}\hat{\beta}(g_k) \geq c_t$. Then, randomly select $\tau_t^* \sim Unif(\tilde{g}_k, 1)$ for each censored observation.

- Step 3 (recomputing censored quantile regression). Impute a censored value c_t with $\tilde{x}_{t-1}\hat{\beta}(\hat{\tau}_t^*)$. Now re-estimate $\hat{\beta}(t_k)$ using newly imputed $\tilde{X}_t$.

- Step 4 Repeat step 3 until a stopping rule is satisfied. Here the stopping rule is that either there are little changes in $\tilde{X}_t$ from two consecutive steps or the maximum iteration number is reached.

Remark 1. A stopping rule used later in our simulation is that the Absolute Mean Difference of $\tilde{X}_t$ in two consecutive steps is less than δ, where $\delta = 10^{-3}$. And, the maximum number of steps is 20.

Remark 2. If "Self-consistent" iteration converges, then $Q_{y_t|\tilde{x}_t}(\tau) \to \tilde{x}_t'\beta(\tau)$

2.3 Inference on the Quantile Autoregression Estimator

Among several possible approaches to construct confidence intervals on the quantile regression estimator, the bootstrap could be the simplest to adapt and is indeed known to have asymptotically correct coverage probabilities. However, previous literature relies on the assumption that resampling triples (Y_i, C_i, x_i) are i.i.d. Since this assumption no longer holds in time series data, we look for more appropriate bootstrap methods.

The first type of bootstrap we considered is the so-called "xy-paired" bootstrap, which was originally proposed by Freedman (1981). The idea is to resample entire observations from the original data in the form of (Y_t, C_t, x_t) triples. Each bootstrap sample consists of some of original triples once, some of them more than once, and some of them not at all. Although it does not appear to be directly applicable to a autoregressive model, the xy-paired bootstrap can be used with autoregressive models that have serially independent error terms (See Gonçalves and Kilian (2004)). Especially, in Buhlmann (1994), if the data follow the AR(p) model, the optimal choice of block length (l) in the block bootstrap with respect to the mean square error of the bootstrap variance is $l = p$, which implies that the paired bootstrap is as good as the block bootstrap and works in autoregression models.

The second bootstrap method we used in our simulation is the block bootstrap, which was originally proposed by Kunsch (1989). By its nature, the block bootstrap captures the dependence structure of neighbored observations. Buhlmann (1994) showed that within the class of AR-models the block bootstrap is more robust than the paired bootstrap with respect to model miss-specification. The idea is to divide (Y_t, C_t, x_t) triples that are being resampled into blocks of b consecutive observations, and then resample blocks. The blocks may be either overlapping or non-overlapping.

In either case, it is required to specify a block length, b. If b is too small, the bootstrap sample fails to capture the patterns of dependence in the original data, because these patterns are lost whenever one block ends and the next begins. On the other hand, if b is too large, the bootstrap samples will tend to be excessively influenced by the random characteristics of the actual sample. Therefore, finding the optimal block length is evidently very important and it makes this procedure much complicated compared to the paired bootstrap.

It is generally believed that both methods work competitively, but the block bootstrap has advantage on its greater generality on dependent data and therefore it is more wildly used and might be a better choice in our study. We will investigate the performance of two methods in the next chapter.

Chapter 3

Simulation Studies

In this chapter, we shall investigate the performance of the proposed CQAR methods through Monte Carlo simulation studies. The performance shall be evaluated in terms of consistency and efficiency of the estimator. Throughout the simulations, we apply the CQAR methods under several circumstances. Since the CQAR method can be applied to both fixed and random censoring, we shall design the simulation under these two types of censoring to investigate performance of the CQAR estimator. Also, although we only consider right-censoring where we are only allowed to observe $\tilde{Y}_t = \min(Y_t, C_t)$, the CQAR method is applicable to left-censored data with simple adjustments. To avoid technical redundance, we restricts order of autoregressive models up to the first and second.

Three experiments are performed as follows. The first shows the consistency of the initial censored regression estimator that relies only on those observations where the regressor is not censored. The second experiment allows us to compare the performance of self-consistent method with ordinary censored regression method and other types of existing methods. The third experiment is to verify a proper bootstrap inference method under dependance. Although it is impossible to consider all the situations, we hope to investigate the performance of the proposed estimator in a variety of settings so that the comparison can be more informative and convincing.

3.1 Experiment 1 : Empirical Consistency of CRQ estimator under dependence

In this simulation we show that we can have a consistent estimator of β by using only part of observations where the regressor, X_t, is not censored. After generating a random time series with size n, following Portnoy (2003), we estimate coefficients twice based on different sets of data. The first set is using all observations on regressor by assuming that we know true x_t. And the second set is using only the uncensored part of observations by discarding all censored observation on regressor. Two estimates are denoted by $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. Since $\hat{\theta}_1$ is a consistent estimate of β at least in stationary time series as shown in previous chapter, $\hat{\theta}_2$ is also likely to be a consistent estimate if it converges to $\hat{\theta}_1$ as the sample size (n) increase. Among all the simulations, the censoring rate for each data set was approximately either 20% or 30% by calibrating censoring constants for fixed censoring or adjusting distribution of censoring times in case of random censoring. For each model, we repeated simulation 1000 times (N). Then, we observed mean absolute differences of two estimates, $1/N \sum |\hat{\theta}_1 - \hat{\theta}_2|$. To show compactly, we only reported the results at 3 different quantiles: .25, .5, .75. for 3 different sample sizes: 500, 1000, 2000. The true models used to generate random samples are as follows.

Model 1. (QAR(1) with Fixed Censoring) $Y_t = \beta_0(U_t) + \beta_1(U_t)Y_{t-1}$, where $\beta_1 = .55 + .25U_t$, $\beta_0 = \Phi^{-1}(U_t)$, and $U_t \sim \mathcal{U}(0,1)$. Hereafter, $\mathcal{U}(\cdot)$ stands for uniform distribution. For censoring constants, C_t, 1.1 and 0.7 were used to get approximately 20% and 30% censoring, respectively.

Model 2. (QAR(1) with Random Censoring) $Y_t = \beta_0(U_t) + \beta_1(U_t)Y_{t-1}$, where $\beta_1 = .55 + .25U_t$, $\beta_0 = \Phi^{-1}(U_t)$, and $U_t \sim \mathcal{U}(0,1)$. Censoring times $C_t \sim -1 + \exp(1/4.5)$ for 20% and $C_t \sim -1 + \exp(1/2.8)$ for 30% censoring. Here Y_t and C_t are mutually independent.

Model 3. (QAR(2) with Fixed Censoring) $Y_t = \beta_0(U_t) + \beta_1(U_t)Y_{t-1} + \beta_2(U_t)Y_{t-2}$, where $\beta_1 = .35 + .25U_t$, $\beta_2 = .25 + .20U_t$, $\beta_0 = \Phi^{-1}(U_t)$, and $U_t \sim \mathcal{U}(0,1)$. For censoring constants, C_t, 1.25 was used to get approximately 20% censoring.

Model 4. (QAR(2) with Random Censoring) $Y_t = \beta_0(U_t) + \beta_1(U_t)Y_{t-1} + \beta_2(U_t)Y_{t-2}$, where $\beta_1 = .35 + .25U_t$, $\beta_2 = .25 + .20U_t$, $\beta_0 = \Phi^{-1}(U_t)$, and $U_t \sim \mathcal{U}(0,1)$. Censoring times $C_t \sim -1 + \exp(1/4.5)$ for 20% censoring.

Model 1 and 2 are QAR(1) models with same coefficients, but their censoring distributions are different. Censoring times in Model 1 are fixed constants and in Model 2 are random. Model 3 and 4 are QAR(2) models sharing same coefficients and being censored at a fixed constant or randomly, respectively.

The results of this experiment are organized in Figures 3.1-3.4 and Tables 3.1-3.4. They show the mean absolute differences of two estimates get smaller as sample size n increase. Note that for Model 3 and 4, we only considered 20% censoring since 30% censoring in Y_t results in overall censoring rate of over 50% in X_t. Although, here we only considered the case where Y_t and C_t are mutually independent, we can extend to the case where Y_t and C_t are conditionally independent given X_t.

From this experiment, we can see that;

- The CRQ estimator converges to true parameter at least empirically, when it is estimated using only part of observations where the regressor is not censored.

- Although we only considered fixed censoring in previous chapter, our proposed algorithm can be applied in random censoring.

3.2 Experiment 2 : The Self-consistent Algorithm

As the second experiment, we compare the performance of six different methods including multiple imputation and the self-consistent algorithm. The data in experiments were generated from models in Experiment 1. Details for each method are as follows.

Method 1. The regular quantile regression method is used on complete data, y_t.

Method 2. Censoring is applied on complete data. Instead of having all y_t, we observe $\tilde{y}_t = \min\{y_t, c_t\}$. Then we use the regular quantile regression using all observations. Here we treat the censored values as observed.

Method 3. Portnoy's censored regression quantile method is used on all observations leaving censored x_t as observed.

Method 4. Portnoy's censored regression quantile method is used on part of observations where the regressor is not censored.

Method 5. The multiple imputation (MI) method is used. Since Step 1-3 in CQAR algorithm enables us to fill-in the missing data with plausible values, the MI method, proposed by Rubin (1987), can be adopted. Repeating Step 1-3 m times generates m simulated complete data. For a given τ, estimate $\hat{\beta}_i(\tau)$, $i = 1, ..., m$ based on simulated data. Then, the MI estimator of $\beta(\tau)$ is $\tilde{\beta}(\tau) = m^{-1}\sum_{i=1}^{m}\hat{\beta}_i(\tau)$.

Method 6. The self-consistent algorithm proposed in Chapter 2.2 is implemented.

Note that Method 1 is omniscient since we already know true y_t and estimate β using all y_t before censored. Method 2-3 are naturally biased since they do not take account of censoring. $\beta(\tau)$ estimated by Method 4 is same as the initial estimate of CQAR algorithm. In Method 5, the number of simulation, m, is decided as suggested in Rubin (1987) where it showed that the efficiency of an estimate based on m sets of imputations is approximately $(1 + \frac{\gamma}{m})^{-1}$ with γ being the censoring rate. With around 20% to 30% censoring rate, $m = 5$ already achieves about 95% efficiency.

In the Tables 3.5-3.16, we compared the performance of method 1-6. For each model, we made the total censoring rate to be about 20% or 30%. We drew 1000 simulations with sample size $n = 300$ and 1000. Three different quantiles, $\tau = .25, .5, .75$ were used. To evaluate each estimates, Mean Bias, Standard Error(SE), Mean Squared Error(MSE), and Mean Absolute Error(MAE) were calculated. Also, all MSE and MAE results are expressed as efficiencies with respect to Method 1.

The results from Tables 3.5-3.16 can be summarized as follows;

- Method 2 performs the worst in most cases, followed by Method 3 since both methods produce biased estimator.
- Method 4 gives fairly reasonable results. With increasing sample size, it performs similarly with Method 5-6.
- When comparing Method 5 and Method 6, performance-wise Method 5 did generally better than Method 6. However, with large sample size, both methods produced competitive results. Also, in Method 6, the CQAR algorithm converged very fast, resulting in shorter computation time than the MI method.

Note that computing the entire regression quantile process in Method 6 on Model 1 required 1.39 seconds on a Intel(R) Core(TM) i5 CPU with 2.50GHz and Method 5 took 1.68 seconds.

3.3 Experiment 3 : Comparison of Bootstrap methods

As the third experiment, we compare performance of two bootstrap methods, the paired bootstrap and the block bootstrap, on constructing confidence intervals on the

CQAR estimator. In our simulation, both methods applied to Model 1 and Model 3, used in Chapter 3.1. For the block bootstrap, we allowed blocks to be overlapping and set the block length to be $n^{1/3}$.

When implementing bootstrap approaches, the ordinary percentile confidence intervals might be unreliable for large τ, where the conditional quantile function is unestimable because of heavy censoring at the top of distribution. Following the hybrid approach introduced by Portnoy (2003), we take the bootstrap estimate of the interquartile range and use normality. Specifically, take the bootstrap sample interquartile values $\beta^*_{.75} - \beta^*_{.5}$ and $\beta^*_{.5} - \beta^*_{.25}$, multiply 2.906, and add the values to $\beta^*_{.5}$ to get upper and lower 95% confidence bounds.

In Table 3.17-18, we compared the coverage probability of both types of bootstrap for QAR(1) and QAR(2) models with fixed censoring. The results show both methods give reliable coverage probability around 90% to 95%. It is worth to note that although the paired bootstrap showed slightly better results in our simulation, the block bootstrap is considered as an appropriate choice because of its generality on dependent data and its robustness with respect to model misspecification.

3.4 Summary

Motivated by Portnoy (2003) and Rubin (1987), the censored quantile autoregression (CQAR) algorithm has been proposed. The CQAR algorithm follows Portnoy's grid method to produce the initial estimator and adopts an idea of imputation methods to further refine the estimator in self-consistent manner.

Throughout the simulations, we can see that:

- The CRQ estimator on triples $(\tilde{Y}_t, \tilde{X}_t, \Delta_t)$ where $t \in \{\gamma_t = 1\}$ in (2.3) converges to the CRQ estimator on triples $(\tilde{Y}_t, X_t, \Delta_t)$, which implies that it converges to true parameter at least empirically, when it is estimated using only part of observations where the regressor is not censored.

- Among the simulation examples in Experiment 2, the CQAR algorithm performs competitively well. It works better than naive methods that treats censored values as observed. And compared to the MI method, it has faster computation time.

- Within the class of AR-models, the paired bootstrap gives reliable coverage probability compared to the block bootstrap, which is more generally used in constructing inference on time series. However, it is believed that the block bootstrap is more robust than the paired bootstrap with respect to model miss-specification.

- The computation time using the CQAR algorithm is very fast. In most time it converges within 10 steps, and since the objective function in each step is convex with respect to the regression coefficients, it can be efficiently solved by the standard linear programming algorithm or interior point methods for regression quantiles described in Koenker (2005).

3.5 Figures and Tables

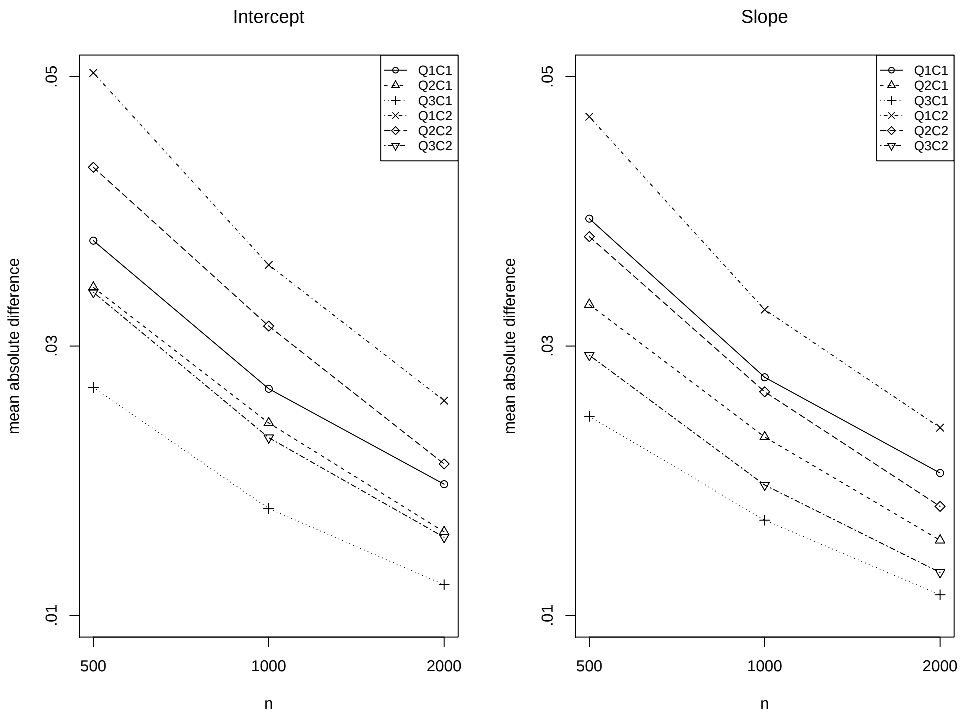

Figure 3.1: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 1 with three different sample sizes (500, 1000, 2000). Q1: the first quartile; Q2:the second quartile (median); Q3:the third quartile. C1: 20% censoring rate; C2: 30% censoring rate.

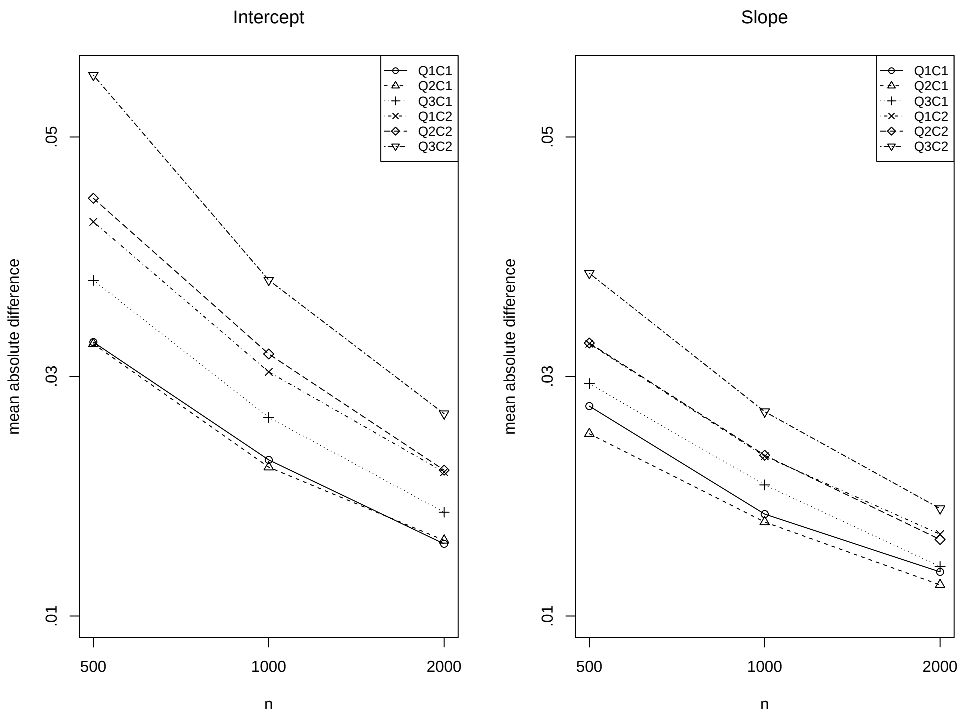

Figure 3.2: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 2 with three different sample sizes (500, 1000, 2000). Q1: the first quartile; Q2:the second quartile (median); Q3:the third quartile. C1: 20% censoring rate; C2: 30% censoring rate.

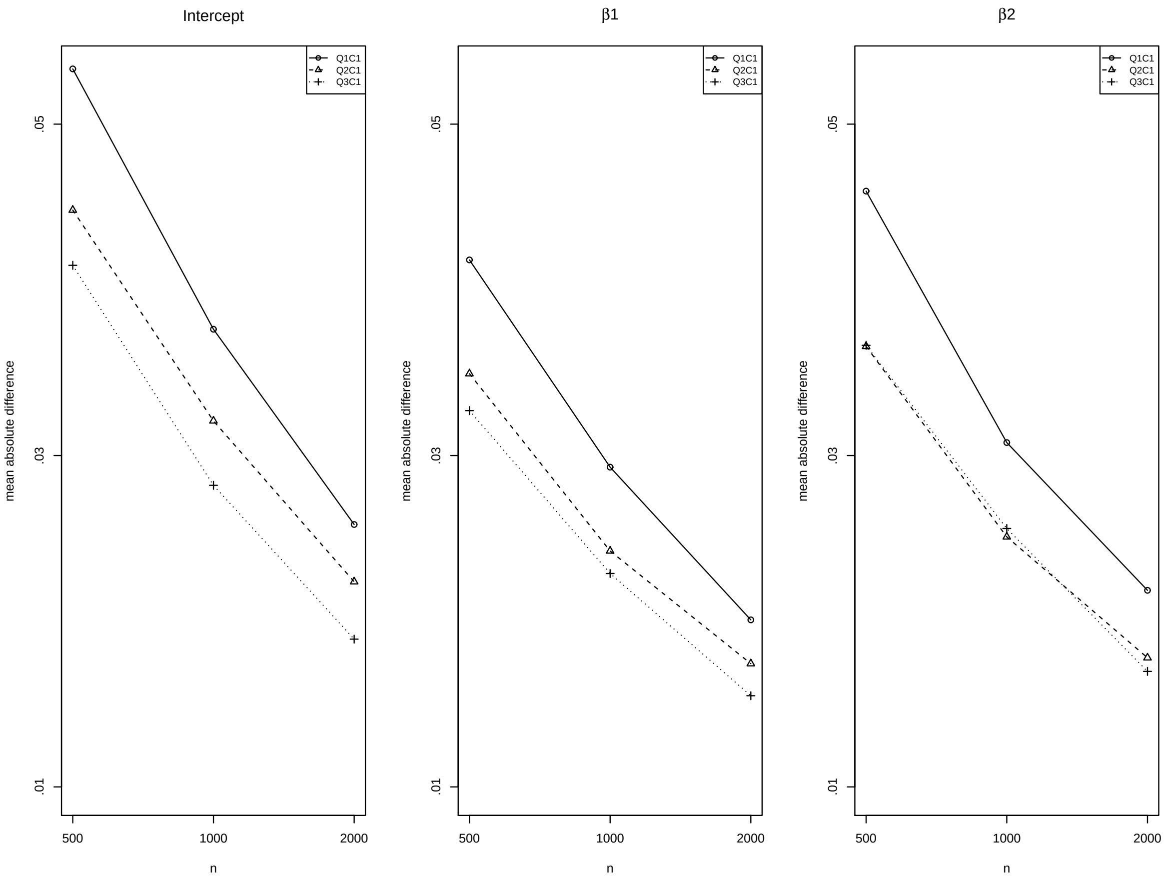

Figure 3.3: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 3 with three different sample sizes (500, 1000, 2000). Q1: the first quartile; Q2:the second quartile (median); Q3:the third quartile. C1: 20% censoring rate.

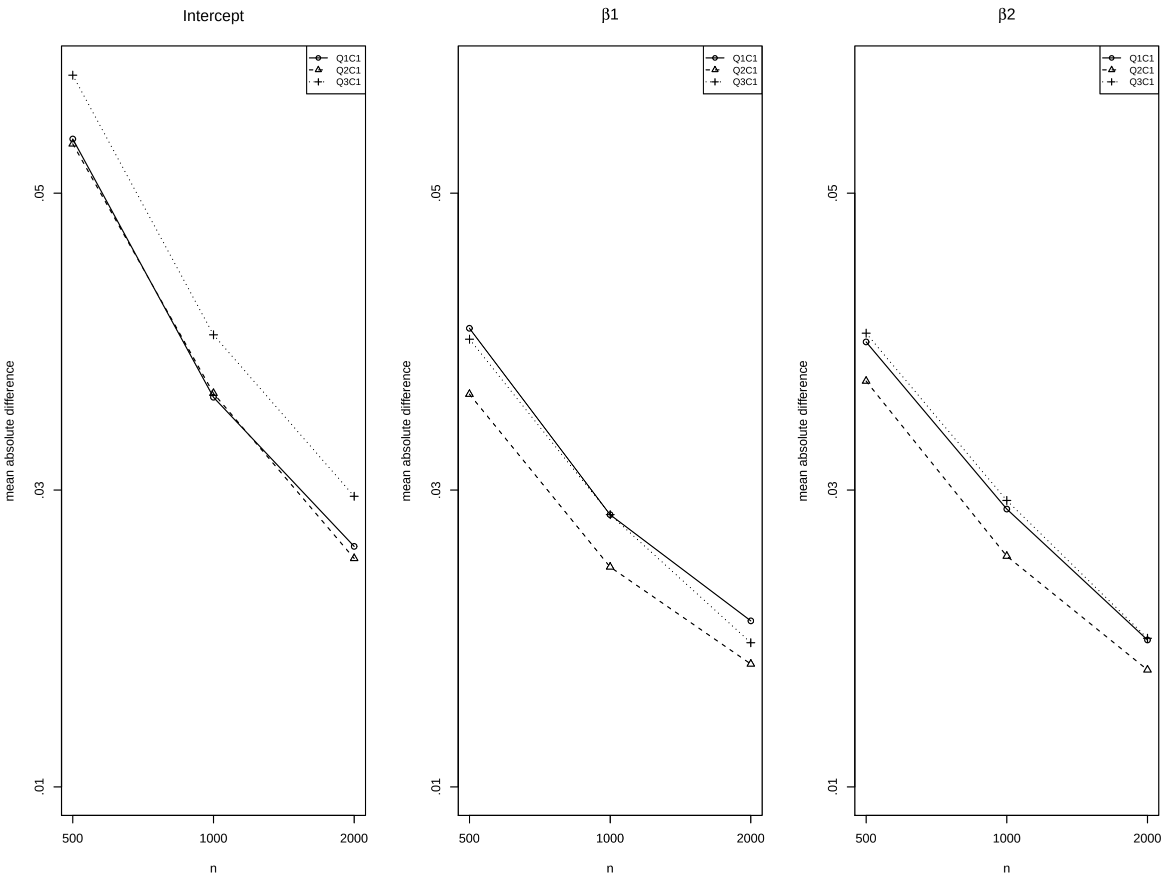

Figure 3.4: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 4 with three different sample sizes (500, 1000, 2000). Q1: the first quartile; Q2:the second quartile (median); Q3:the third quartile. C1: 20% censoring rate.

Table 3.1: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 1 with three different sample sizes (500, 1000, 2000).

	Intercept					
	20% Censoring			30% Censoring		
Quantiles	.25	.50	.75	.25	.50	.75
n=500	0.0378	0.0344	0.0269	0.0503	0.0433	0.0340
n=1000	0.0268	0.0243	0.0179	0.0360	0.0315	0.0232
n=2000	0.0197	0.0162	0.0123	0.0259	0.0213	0.0158
	Slope					
n=500	0.0395	0.0331	0.0248	0.0470	0.0381	0.0293
n=1000	0.0277	0.0233	0.0171	0.0327	0.0266	0.0197
n=2000	0.0206	0.0156	0.0115	0.0239	0.0181	0.0132

Table 3.2: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 2 with three different sample sizes (500, 1000, 2000).

	Intercept					
	20% Censoring			30% Censoring		
Quantiles	.25	.50	.75	.25	.50	.75
n=500	0.0329	0.0327	0.0380	0.0429	0.0449	0.0552
n=1000	0.0230	0.0224	0.0266	0.0304	0.0319	0.0380
n=2000	0.0160	0.0163	0.0187	0.0220	0.0222	0.0269
	Slope					
n=500	0.0275	0.0252	0.0294	0.0327	0.0328	0.0386
n=1000	0.0185	0.0178	0.0209	0.0233	0.0234	0.0271
n=2000	0.0137	0.0126	0.0141	0.0168	0.0164	0.0190

Table 3.3: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 3 with three different sample sizes (500, 1000, 2000). Approximately 20% censoring applied.

	Intercept			**b1**			**b2**		
Quantiles	.25	.50	.75	.25	.50	.75	.25	.50	.75
n=500	0.0533	0.0448	0.0415	0.0418	0.0349	0.0327	0.0460	0.0366	0.0367
n=1000	0.0376	0.0321	0.0282	0.0293	0.0242	0.0229	0.0308	0.0251	0.0256
n=2000	0.0258	0.0224	0.0189	0.0201	0.0174	0.0155	0.0219	0.0178	0.0170

Table 3.4: The Mean Absolute Difference ($|\hat{\theta}_1 - \hat{\theta}_2|$) for intercept and slope from Model 4 with three different sample sizes (500, 1000, 2000). Approximately 20% censoring applied.

	Intercept			**b1**			**b2**		
Quantiles	.25	.50	.75	.25	.50	.75	.25	.50	.75
n=500	0.0537	0.0533	0.0579	0.0409	0.0365	0.0402	0.0400	0.0373	0.0406
n=1000	0.0363	0.0365	0.0405	0.0283	0.0248	0.0283	0.0287	0.0255	0.0293
n=2000	0.0262	0.0254	0.0296	0.0212	0.0183	0.0197	0.0199	0.0179	0.0200

Table 3.5: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 1. Sample size: 300. Replication: 1000. Censoring rate: 20%.

	Intercept					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = \mathbf{.25}$						
RQ(complete obs.)	0.0047	0.0040	0.0048	0.0549	1.00	1.00
RQ(all obs.)	0.0998	0.0041	0.0150	0.1059	3.12	1.93
CRQ(all obs.)	0.0521	0.0041	0.0078	0.0716	1.62	1.31
CRQ(only uncensored)	-0.0539	0.0054	0.0115	0.0847	2.39	1.54
MI(CRQ)	-0.0406	0.0048	0.0086	0.0707	1.79	1.29
selfconsist(CRQ)	-0.0466	0.0049	0.0093	0.0734	1.92	1.34
$\tau = \mathbf{.50}$						
RQ(complete obs.)	0.0002	0.0035	0.0037	0.0489	1.00	1.00
RQ(all obs.)	0.1191	0.0036	0.0180	0.1209	4.88	2.47
CRQ(all obs.)	0.0854	0.0039	0.0118	0.0916	3.19	1.87
CRQ(only uncensored)	-0.0441	0.0048	0.0088	0.0749	2.38	1.53
MI(CRQ)	-0.0077	0.0040	0.0048	0.0552	1.31	1.13
selfconsist(CRQ)	-0.0153	0.0039	0.0049	0.0557	1.32	1.14
$\tau = \mathbf{.75}$						
RQ(complete obs.)	0.0020	0.0036	0.0039	0.0493	1.00	1.00
RQ(all obs.)	-0.1857	0.0030	0.0372	0.1858	9.62	3.77
CRQ(all obs.)	0.0485	0.0049	0.0097	0.0752	2.51	1.53
CRQ(only uncensored)	-0.0397	0.0051	0.0094	0.0777	2.43	1.57
MI(CRQ)	-0.0092	0.0045	0.0061	0.0618	1.57	1.25
selfconsist(CRQ)	-0.0155	0.0046	0.0066	0.0644	1.70	1.31
	Slope					
$\tau = \mathbf{.25}$						
RQ(complete obs.)	-0.0068	0.0034	0.0034	0.0467	1.00	1.00
RQ(all obs.)	0.0765	0.0041	0.0108	0.0863	3.15	1.85
CRQ(all obs.)	0.0724	0.0041	0.0102	0.0830	2.99	1.78
CRQ(only uncensored)	-0.0194	0.0047	0.0070	0.0663	2.05	1.42
MI(CRQ)	-0.0264	0.0045	0.0068	0.0629	1.99	1.35
selfconsist(CRQ)	-0.0330	0.0044	0.0070	0.0649	2.05	1.39
$\tau = \mathbf{.50}$						
RQ(complete obs.)	-0.0086	0.0030	0.0028	0.0422	1.00	1.00
RQ(all obs.)	0.0915	0.0035	0.0120	0.0964	4.25	2.28
CRQ(all obs.)	0.0906	0.0037	0.0123	0.0954	4.35	2.26
CRQ(only uncensored)	-0.0184	0.0042	0.0056	0.0591	1.98	1.40
MI(CRQ)	0.0017	0.0040	0.0048	0.0538	1.69	1.27
selfconsist(CRQ)	-0.0046	0.0037	0.0042	0.0506	1.48	1.20
$\tau = \mathbf{.75}$						
RQ(complete obs.)	-0.0084	0.0034	0.0036	0.0475	1.00	1.00
RQ(all obs.)	-0.1816	0.0027	0.0352	0.1816	9.78	3.83
CRQ(all obs.)	0.0636	0.0045	0.0100	0.0822	2.79	1.73
CRQ(only uncensored)	-0.0082	0.0047	0.0067	0.0650	1.87	1.37
MI(CRQ)	0.0093	0.0041	0.0052	0.0573	1.46	1.21
selfconsist(CRQ)	0.0046	0.0042	0.0052	0.0572	1.45	1.21

Table 3.6: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 1. Sample size: 1000. Replication: 1000. Censoring rate: 20%.

	Intercept					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = \mathbf{.25}$						
RQ(complete obs.)	-0.0003	0.0013	0.0017	0.0330	1.00	1.00
RQ(all obs.)	0.0972	0.0015	0.0116	0.0981	6.64	2.97
CRQ(all obs.)	0.0657	0.0015	0.0065	0.0697	3.71	2.11
CRQ(only uncensored)	-0.0331	0.0017	0.0039	0.0500	2.24	1.52
MI(CRQ)	-0.0306	0.0016	0.0034	0.0457	1.93	1.38
selfconsist(CRQ)	-0.0346	0.0015	0.0035	0.0472	2.01	1.43
$\tau = \mathbf{.50}$						
RQ(complete obs.)	-0.0011	0.0012	0.0015	0.0309	1.00	1.00
RQ(all obs.)	0.1196	0.0014	0.0164	0.1198	11.02	3.87
CRQ(all obs.)	0.0959	0.0015	0.0114	0.0965	7.63	3.12
CRQ(only uncensored)	-0.0265	0.0015	0.0030	0.0439	2.02	1.42
MI(CRQ)	-0.0003	0.0014	0.0019	0.0350	1.30	1.13
selfconsist(CRQ)	-0.0048	0.0014	0.0019	0.0342	1.26	1.11
$\tau = \mathbf{.75}$						
RQ(complete obs.)	-0.0022	0.0013	0.0017	0.0334	1.00	1.00
RQ(all obs.)	-0.1915	0.0009	0.0375	0.1915	21.60	5.74
CRQ(all obs.)	0.0588	0.0018	0.0065	0.0664	3.76	1.99
CRQ(only uncensored)	-0.0301	0.0017	0.0037	0.0492	2.11	1.47
MI(CRQ)	-0.0056	0.0015	0.0024	0.0394	1.37	1.18
selfconsist(CRQ)	-0.0093	0.0015	0.0025	0.0402	1.41	1.20
	Slope					
$\tau = \mathbf{.25}$						
RQ(complete obs.)	-0.0020	0.0010	0.0011	0.0268	1.00	1.00
RQ(all obs.)	0.0842	0.0013	0.0088	0.0847	8.11	3.16
CRQ(all obs.)	0.0813	0.0013	0.0083	0.0820	7.64	3.06
CRQ(only uncensored)	-0.0045	0.0015	0.0022	0.0373	1.98	1.39
MI(CRQ)	-0.0215	0.0015	0.0026	0.0391	2.38	1.46
selfconsist(CRQ)	-0.0252	0.0014	0.0025	0.0389	2.30	1.45
$\tau = \mathbf{.50}$						
RQ(complete obs.)	-0.0043	0.0009	0.0009	0.0237	1.00	1.00
RQ(all obs.)	0.0981	0.0012	0.0111	0.0983	12.44	4.16
CRQ(all obs.)	0.0958	0.0012	0.0107	0.0960	12.03	4.06
CRQ(only uncensored)	-0.0064	0.0012	0.0016	0.0317	1.78	1.34
MI(CRQ)	0.0054	0.0012	0.0015	0.0304	1.65	1.28
selfconsist(CRQ)	0.0019	0.0011	0.0013	0.0286	1.46	1.21
$\tau = \mathbf{.75}$						
RQ(complete obs.)	-0.0037	0.0011	0.0011	0.0271	1.00	1.00
RQ(all obs.)	-0.1765	0.0008	0.0319	0.1765	28.10	6.52
CRQ(all obs.)	0.0688	0.0015	0.0069	0.0722	6.08	2.67
CRQ(only uncensored)	-0.0033	0.0014	0.0020	0.0364	1.80	1.34
MI(CRQ)	0.0101	0.0014	0.0019	0.0353	1.70	1.30
selfconsist(CRQ)	0.0071	0.0013	0.0018	0.0345	1.63	1.28

Table 3.7: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 1. Sample size: 300. Replication: 1000. Censoring rate: 30%.

	Intercept					
300 obs, 30%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = \mathbf{.25}$						
RQ(complete obs.)	0.0140	0.0037	0.0044	0.0542	1.00	1.00
RQ(all obs.)	0.1784	0.0042	0.0372	0.1788	8.44	3.30
CRQ(all obs.)	0.1303	0.0043	0.0224	0.1330	5.08	2.46
CRQ(only uncensored)	-0.0384	0.0064	0.0136	0.0917	3.08	1.69
MI(CRQ)	-0.0280	0.0053	0.0091	0.0747	2.06	1.38
selfconsist(CRQ)	-0.0393	0.0052	0.0098	0.0774	2.22	1.43
$\tau = \mathbf{.50}$						
RQ(complete obs.)	0.0109	0.0032	0.0032	0.0449	1.00	1.00
RQ(all obs.)	0.1474	0.0018	0.0227	0.1474	7.14	3.28
CRQ(all obs.)	0.1777	0.0037	0.0356	0.1778	11.21	3.96
CRQ(only uncensored)	-0.0315	0.0056	0.0103	0.0805	3.24	1.79
MI(CRQ)	0.0255	0.0042	0.0060	0.0619	1.90	1.38
selfconsist(CRQ)	0.0107	0.0041	0.0051	0.0569	1.61	1.27
$\tau = \mathbf{.75}$						
RQ(complete obs.)	0.0147	0.0035	0.0039	0.0494	1.00	1.00
RQ(all obs.)	-0.3034	0.0024	0.0939	0.3034	23.80	6.14
CRQ(all obs.)	0.0988	0.0072	0.0251	0.1213	6.36	2.45
CRQ(only uncensored)	-0.0227	0.0067	0.0139	0.0934	3.53	1.89
MI(CRQ)	0.0206	0.0060	0.0112	0.0827	2.83	1.67
selfconsist(CRQ)	0.0092	0.0061	0.0111	0.0825	2.82	1.67
	Slope					
$\tau = \mathbf{.25}$						
RQ(complete obs.)	-0.0046	0.0032	0.0031	0.0442	1.00	1.00
RQ(all obs.)	0.1272	0.0044	0.0220	0.1302	7.16	2.94
CRQ(all obs.)	0.1233	0.0045	0.0212	0.1267	6.89	2.86
CRQ(only uncensored)	-0.0099	0.0056	0.0094	0.0776	3.07	1.75
MI(CRQ)	-0.0318	0.0053	0.0096	0.0747	3.11	1.69
selfconsist(CRQ)	-0.0423	0.0053	0.0101	0.0763	3.28	1.72
$\tau = \mathbf{.50}$						
RQ(complete obs.)	-0.0036	0.0030	0.0027	0.0413	1.00	1.00
RQ(all obs.)	0.1009	0.0025	0.0120	0.1018	4.45	2.47
CRQ(all obs.)	0.1477	0.0036	0.0258	0.1486	9.53	3.60
CRQ(only uncensored)	-0.0073	0.0050	0.0074	0.0679	2.74	1.64
MI(CRQ)	0.0171	0.0044	0.0060	0.0613	2.21	1.48
selfconsist(CRQ)	0.0062	0.0042	0.0052	0.0563	1.94	1.37
$\tau = \mathbf{.75}$						
RQ(complete obs.)	-0.0072	0.0034	0.0035	0.0470	1.00	1.00
RQ(all obs.)	-0.2675	0.0035	0.0752	0.2675	21.41	5.70
CRQ(all obs.)	0.0851	0.0057	0.0169	0.1054	4.82	2.24
CRQ(only uncensored)	-0.0018	0.0058	0.0101	0.0800	2.88	1.70
MI(CRQ)	0.0166	0.0052	0.0085	0.0732	2.42	1.56
selfconsist(CRQ)	0.0090	0.0053	0.0084	0.0720	2.39	1.53

Table 3.8: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 1. Sample size: 1000. Replication: 1000. Censoring rate: 30%.

	Intercept					
1000 obs, 30%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	0.0048	0.0013	0.0018	0.0342	1.00	1.00
RQ(all obs.)	0.1691	0.0016	0.0311	0.1691	17.07	4.95
CRQ(all obs.)	0.1371	0.0016	0.0213	0.1371	11.67	4.01
CRQ(only uncensored)	-0.0296	0.0020	0.0049	0.0563	2.70	1.65
MI(CRQ)	-0.0248	0.0018	0.0040	0.0497	2.20	1.45
selfconsist(CRQ)	-0.0325	0.0018	0.0041	0.0509	2.27	1.49
$\tau =$ **.50**						
RQ(complete obs.)	0.0048	0.0012	0.0014	0.0300	1.00	1.00
RQ(all obs.)	0.1489	0.0006	0.0225	0.1489	15.95	4.96
CRQ(all obs.)	0.1755	0.0012	0.0323	0.1755	22.92	5.85
CRQ(only uncensored)	-0.0228	0.0018	0.0037	0.0484	2.60	1.61
MI(CRQ)	0.0258	0.0015	0.0030	0.0437	2.14	1.46
selfconsist(CRQ)	0.0149	0.0015	0.0024	0.0386	1.69	1.29
$\tau =$ **.75**						
RQ(complete obs.)	0.0044	0.0013	0.0016	0.0320	1.00	1.00
RQ(all obs.)	-0.3089	0.0007	0.0960	0.3089	59.92	9.64
CRQ(all obs.)	0.0978	0.0023	0.0146	0.1010	9.13	3.15
CRQ(only uncensored)	-0.0239	0.0022	0.0053	0.0586	3.31	1.83
MI(CRQ)	0.0116	0.0020	0.0040	0.0491	2.47	1.53
selfconsist(CRQ)	0.0046	0.0020	0.0040	0.0501	2.53	1.56
	Slope					
$\tau =$ **.25**						
RQ(complete obs.)	0.0006	0.0010	0.0010	0.0249	1.00	1.00
RQ(all obs.)	0.1308	0.0014	0.0191	0.1308	19.85	5.24
CRQ(all obs.)	0.1277	0.0014	0.0183	0.1277	19.01	5.12
CRQ(only uncensored)	-0.0024	0.0017	0.0028	0.0428	2.94	1.72
MI(CRQ)	-0.0278	0.0018	0.0039	0.0472	4.03	1.89
selfconsist(CRQ)	-0.0345	0.0016	0.0038	0.0478	3.98	1.91
$\tau =$ **.50**						
RQ(complete obs.)	-0.0007	0.0009	0.0008	0.0230	1.00	1.00
RQ(all obs.)	0.1082	0.0007	0.0122	0.1082	14.48	4.70
CRQ(all obs.)	0.1441	0.0011	0.0219	0.1441	25.91	6.27
CRQ(only uncensored)	-0.0044	0.0015	0.0022	0.0375	2.57	1.63
MI(CRQ)	0.0166	0.0014	0.0022	0.0372	2.59	1.62
selfconsist(CRQ)	0.0085	0.0013	0.0016	0.0325	1.95	1.41
$\tau =$ **.75**						
RQ(complete obs.)	-0.0020	0.0010	0.0010	0.0258	1.00	1.00
RQ(all obs.)	-0.2597	0.0011	0.0686	0.2597	66.89	10.06
CRQ(all obs.)	0.0874	0.0017	0.0105	0.0896	10.20	3.47
CRQ(only uncensored)	-0.0014	0.0017	0.0030	0.0431	2.95	1.67
MI(CRQ)	0.0143	0.0016	0.0026	0.0407	2.57	1.58
selfconsist(CRQ)	0.0092	0.0016	0.0025	0.0393	2.45	1.52

Table 3.9: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 2. Sample size: 300. Replication: 1000. Censoring rate: 20%.

	Intercept					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = $ **.25**						
RQ(complete obs.)	-0.0033	0.0045	0.0060	0.0615	1.00	1.00
RQ(all obs.)	-0.1273	0.0043	0.0216	0.1296	3.62	2.11
CRQ(all obs.)	0.0435	0.0053	0.0102	0.0829	1.71	1.35
CRQ(only uncensored)	-0.0515	0.0055	0.0118	0.0853	1.98	1.39
MI(CRQ)	-0.0656	0.0048	0.0112	0.0854	1.87	1.39
selfconsist(CRQ)	-0.0695	0.0050	0.0122	0.0892	2.04	1.45
$\tau = $ **.50**						
RQ(complete obs.)	-0.0071	0.0041	0.0052	0.0579	1.00	1.00
RQ(all obs.)	-0.1825	0.0044	0.0390	0.1828	7.57	3.16
CRQ(all obs.)	0.0854	0.0053	0.0158	0.1034	3.06	1.79
CRQ(only uncensored)	-0.0460	0.0054	0.0110	0.0842	2.13	1.45
MI(CRQ)	-0.0403	0.0044	0.0075	0.0698	1.46	1.21
selfconsist(CRQ)	-0.0437	0.0047	0.0084	0.0728	1.63	1.26
$\tau = $ **.75**						
RQ(complete obs.)	-0.0081	0.0045	0.0061	0.0623	1.00	1.00
RQ(all obs.)	-0.1575	0.0053	0.0332	0.1607	5.45	2.58
CRQ(all obs.)	0.1356	0.0065	0.0312	0.1482	5.12	2.38
CRQ(only uncensored)	-0.0518	0.0060	0.0136	0.0927	2.24	1.49
MI(CRQ)	-0.0191	0.0050	0.0078	0.0715	1.28	1.15
selfconsist(CRQ)	-0.0223	0.0053	0.0090	0.0766	1.49	1.23
	Slope					
$\tau = $ **.25**						
RQ(complete obs.)	-0.0093	0.0035	0.0037	0.0483	1.00	1.00
RQ(all obs.)	-0.1648	0.0040	0.0319	0.1649	8.52	3.41
CRQ(all obs.)	0.0021	0.0043	0.0055	0.0591	1.48	1.22
CRQ(only uncensored)	-0.0113	0.0044	0.0059	0.0611	1.57	1.26
MI(CRQ)	-0.0483	0.0036	0.0061	0.0635	1.64	1.32
selfconsist(CRQ)	-0.0537	0.0039	0.0074	0.0695	1.96	1.44
$\tau = $ **.50**						
RQ(complete obs.)	-0.0112	0.0032	0.0032	0.0444	1.00	1.00
RQ(all obs.)	-0.1637	0.0038	0.0312	0.1640	9.72	3.69
CRQ(all obs.)	0.0077	0.0041	0.0052	0.0573	1.62	1.29
CRQ(only uncensored)	-0.0157	0.0042	0.0054	0.0579	1.69	1.30
MI(CRQ)	-0.0364	0.0034	0.0047	0.0546	1.48	1.23
selfconsist(CRQ)	-0.0400	0.0036	0.0055	0.0586	1.73	1.32
$\tau = $ **.75**						
RQ(complete obs.)	-0.0141	0.0035	0.0039	0.0480	1.00	1.00
RQ(all obs.)	-0.1365	0.0043	0.0242	0.1378	6.23	2.87
CRQ(all obs.)	0.0135	0.0051	0.0081	0.0717	2.09	1.49
CRQ(only uncensored)	-0.0161	0.0046	0.0065	0.0639	1.68	1.33
MI(CRQ)	-0.0215	0.0039	0.0051	0.0559	1.31	1.16
selfconsist(CRQ)	-0.0259	0.0041	0.0058	0.0600	1.48	1.25

Table 3.10: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 2. Sample size: 1000. Replication: 1000. Censoring rate: 20%.

	Intercept					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = $ **.25**						
RQ(complete obs.)	-0.0003	0.0013	0.0018	0.0336	1.00	1.00
RQ(all obs.)	-0.1277	0.0012	0.0178	0.1277	9.85	3.80
CRQ(all obs.)	0.0668	0.0017	0.0072	0.0726	4.00	2.16
CRQ(only uncensored)	-0.0340	0.0017	0.0039	0.0506	2.18	1.51
MI(CRQ)	-0.0493	0.0014	0.0044	0.0552	2.44	1.64
selfconsist(CRQ)	-0.0528	0.0015	0.0049	0.0583	2.73	1.74
$\tau = $ **.50**						
RQ(complete obs.)	-0.0005	0.0012	0.0015	0.0316	1.00	1.00
RQ(all obs.)	-0.1767	0.0013	0.0329	0.1767	21.52	5.59
CRQ(all obs.)	0.1090	0.0017	0.0147	0.1101	9.63	3.48
CRQ(only uncensored)	-0.0271	0.0016	0.0034	0.0465	2.22	1.47
MI(CRQ)	-0.0224	0.0014	0.0023	0.0388	1.53	1.23
selfconsist(CRQ)	-0.0255	0.0014	0.0026	0.0410	1.72	1.30
$\tau = $ **.75**						
RQ(complete obs.)	-0.0034	0.0014	0.0019	0.0345	1.00	1.00
RQ(all obs.)	-0.1531	0.0016	0.0260	0.1531	13.97	4.43
CRQ(all obs.)	0.1625	0.0022	0.0311	0.1628	16.74	4.72
CRQ(only uncensored)	-0.0337	0.0019	0.0046	0.0543	2.46	1.57
MI(CRQ)	-0.0002	0.0016	0.0024	0.0396	1.30	1.15
selfconsist(CRQ)	-0.0032	0.0016	0.0027	0.0411	1.43	1.19
	Slope					
$\tau = $ **.25**						
RQ(complete obs.)	-0.0026	0.0011	0.0011	0.0266	1.00	1.00
RQ(all obs.)	-0.1611	0.0012	0.0275	0.1611	24.49	6.07
CRQ(all obs.)	0.0110	0.0013	0.0018	0.0337	1.59	1.27
CRQ(only uncensored)	-0.0049	0.0013	0.0018	0.0332	1.59	1.25
MI(CRQ)	-0.0423	0.0011	0.0030	0.0455	2.68	1.71
selfconsist(CRQ)	-0.0472	0.0012	0.0036	0.0499	3.22	1.88
$\tau = $ **.50**						
RQ(complete obs.)	-0.0029	0.0010	0.0010	0.0248	1.00	1.00
RQ(all obs.)	-0.1572	0.0012	0.0261	0.1572	27.32	6.33
CRQ(all obs.)	0.0202	0.0013	0.0020	0.0357	2.07	1.44
CRQ(only uncensored)	-0.0053	0.0012	0.0016	0.0315	1.65	1.27
MI(CRQ)	-0.0267	0.0011	0.0018	0.0340	1.93	1.37
selfconsist(CRQ)	-0.0306	0.0011	0.0022	0.0370	2.27	1.49
$\tau = $ **.75**						
RQ(complete obs.)	-0.0036	0.0010	0.0011	0.0264	1.00	1.00
RQ(all obs.)	-0.1276	0.0013	0.0180	0.1276	16.34	4.83
CRQ(all obs.)	0.0292	0.0015	0.0032	0.0461	2.95	1.74
CRQ(only uncensored)	-0.0044	0.0014	0.0019	0.0343	1.72	1.30
MI(CRQ)	-0.0090	0.0012	0.0015	0.0307	1.39	1.16
selfconsist(CRQ)	-0.0129	0.0012	0.0017	0.0328	1.55	1.24

Table 3.11: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 2. Sample size: 300. Replication: 1000. Censoring rate: 30%.

	Intercept					
300 obs, 30%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	0.0070	0.0043	0.0056	0.0593	1.00	1.00
RQ(all obs.)	-0.1526	0.0038	0.0276	0.1527	4.90	2.58
CRQ(all obs.)	0.1145	0.0053	0.0216	0.1247	3.83	2.10
CRQ(only uncensored)	-0.0451	0.0061	0.0134	0.0923	2.37	1.56
MI(CRQ)	-0.0590	0.0048	0.0103	0.0806	1.83	1.36
selfconsist(CRQ)	-0.0682	0.0051	0.0124	0.0896	2.21	1.51
$\tau =$ **.50**						
RQ(complete obs.)	0.0048	0.0040	0.0047	0.0554	1.00	1.00
RQ(all obs.)	-0.2615	0.0042	0.0736	0.2615	15.50	4.72
CRQ(all obs.)	0.1806	0.0057	0.0424	0.1831	8.94	3.31
CRQ(only uncensored)	-0.0340	0.0061	0.0121	0.0884	2.56	1.60
MI(CRQ)	-0.0178	0.0044	0.0060	0.0619	1.27	1.12
selfconsist(CRQ)	-0.0266	0.0048	0.0075	0.0692	1.57	1.25
$\tau =$ **.75**						
RQ(complete obs.)	0.0042	0.0041	0.0051	0.0566	1.00	1.00
RQ(all obs.)	-0.2588	0.0053	0.0754	0.2589	14.87	4.57
CRQ(all obs.)	0.2684	0.0077	0.0900	0.2698	17.75	4.76
CRQ(only uncensored)	-0.0466	0.0069	0.0165	0.1049	3.26	1.85
MI(CRQ)	0.0166	0.0051	0.0082	0.0726	1.62	1.28
selfconsist(CRQ)	0.0050	0.0055	0.0090	0.0758	1.77	1.34
	Slope					
$\tau =$ **.25**						
RQ(complete obs.)	-0.0069	0.0035	0.0037	0.0482	1.00	1.00
RQ(all obs.)	-0.2134	0.0042	0.0507	0.2134	13.82	4.43
CRQ(all obs.)	0.0217	0.0046	0.0067	0.0658	1.82	1.37
CRQ(only uncensored)	-0.0100	0.0047	0.0068	0.0647	1.85	1.34
MI(CRQ)	-0.0587	0.0039	0.0080	0.0719	2.19	1.49
selfconsist(CRQ)	-0.0675	0.0040	0.0092	0.0779	2.52	1.62
$\tau =$ **.50**						
RQ(complete obs.)	-0.0081	0.0032	0.0032	0.0448	1.00	1.00
RQ(all obs.)	-0.2274	0.0041	0.0568	0.2274	17.85	5.07
CRQ(all obs.)	0.0390	0.0048	0.0083	0.0734	2.60	1.64
CRQ(only uncensored)	-0.0099	0.0046	0.0065	0.0649	2.05	1.45
MI(CRQ)	-0.0351	0.0038	0.0055	0.0587	1.73	1.31
selfconsist(CRQ)	-0.0424	0.0039	0.0064	0.0638	2.00	1.42
$\tau =$ **.75**						
RQ(complete obs.)	-0.0087	0.0035	0.0037	0.0482	1.00	1.00
RQ(all obs.)	-0.1974	0.0049	0.0460	0.1976	12.35	4.10
CRQ(all obs.)	0.0598	0.0064	0.0157	0.1026	4.20	2.13
CRQ(only uncensored)	-0.0122	0.0054	0.0088	0.0735	2.35	1.52
MI(CRQ)	-0.0121	0.0044	0.0058	0.0597	1.56	1.24
selfconsist(CRQ)	-0.0203	0.0047	0.0070	0.0659	1.88	1.37

Table 3.12: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 2. Sample size: 1000. Replication: 1000. Censoring rate: 30%.

	Intercept					
1000 obs, 30%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = $ **.25**						
RQ(complete obs.)	0.0003	0.0014	0.0019	0.0347	1.00	1.00
RQ(all obs.)	-0.1562	0.0011	0.0257	0.1562	13.30	4.50
CRQ(all obs.)	0.1212	0.0018	0.0178	0.1218	9.21	3.51
CRQ(only uncensored)	-0.0357	0.0019	0.0050	0.0566	2.58	1.63
MI(CRQ)	-0.0520	0.0015	0.0050	0.0586	2.57	1.69
selfconsist(CRQ)	-0.0616	0.0016	0.0063	0.0665	3.24	1.92
$\tau = $ **.50**						
RQ(complete obs.)	-0.0014	0.0012	0.0015	0.0313	1.00	1.00
RQ(all obs.)	-0.2641	0.0012	0.0711	0.2641	46.07	8.43
CRQ(all obs.)	0.1819	0.0020	0.0371	0.1819	24.04	5.80
CRQ(only uncensored)	-0.0293	0.0019	0.0043	0.0523	2.82	1.67
MI(CRQ)	-0.0170	0.0014	0.0023	0.0383	1.52	1.22
selfconsist(CRQ)	-0.0250	0.0015	0.0029	0.0431	1.88	1.38
$\tau = $ **.75**						
RQ(complete obs.)	-0.0013	0.0014	0.0019	0.0351	1.00	1.00
RQ(all obs.)	-0.2605	0.0016	0.0705	0.2605	36.57	7.42
CRQ(all obs.)	0.2741	0.0026	0.0821	0.2741	42.57	7.81
CRQ(only uncensored)	-0.0349	0.0023	0.0064	0.0645	3.30	1.84
MI(CRQ)	0.0217	0.0017	0.0033	0.0462	1.74	1.32
selfconsist(CRQ)	0.0143	0.0018	0.0035	0.0463	1.79	1.32
	Slope					
$\tau = $ **.25**						
RQ(complete obs.)	-0.0036	0.0010	0.0011	0.0258	1.00	1.00
RQ(all obs.)	-0.2064	0.0013	0.0442	0.2064	41.88	8.00
CRQ(all obs.)	0.0258	0.0014	0.0025	0.0400	2.39	1.55
CRQ(only uncensored)	-0.0070	0.0015	0.0022	0.0373	2.05	1.45
MI(CRQ)	-0.0544	0.0012	0.0044	0.0571	4.18	2.22
selfconsist(CRQ)	-0.0667	0.0013	0.0060	0.0681	5.71	2.64
$\tau = $ **.50**						
RQ(complete obs.)	-0.0034	0.0009	0.0009	0.0236	1.00	1.00
RQ(all obs.)	-0.2206	0.0012	0.0501	0.2206	57.11	9.36
CRQ(all obs.)	0.0447	0.0014	0.0040	0.0519	4.60	2.20
CRQ(only uncensored)	-0.0068	0.0014	0.0020	0.0354	2.23	1.50
MI(CRQ)	-0.0311	0.0011	0.0022	0.0388	2.55	1.65
selfconsist(CRQ)	-0.0409	0.0012	0.0031	0.0459	3.48	1.95
$\tau = $ **.75**						
RQ(complete obs.)	-0.0039	0.0010	0.0011	0.0266	1.00	1.00
RQ(all obs.)	-0.1889	0.0014	0.0375	0.1889	33.91	7.10
CRQ(all obs.)	0.0699	0.0018	0.0081	0.0767	7.33	2.88
CRQ(only uncensored)	-0.0071	0.0016	0.0026	0.0402	2.33	1.51
MI(CRQ)	-0.0066	0.0013	0.0017	0.0330	1.55	1.24
selfconsist(CRQ)	-0.0163	0.0014	0.0021	0.0364	1.92	1.37

Table 3.13: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 3. Sample size: 300. Replication: 1000. Censoring rate: 20%.

	Intercept					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = \mathbf{.25}$						
RQ(complete obs.)	0.0014	0.0039	0.0046	0.0536	1.00	1.00
RQ(all obs.)	0.1445	0.0041	0.0258	0.1454	5.57	2.71
CRQ(all obs.)	0.0949	0.0042	0.0142	0.1011	3.06	1.89
CRQ(only uncensored)	-0.0483	0.0066	0.0154	0.0973	3.32	1.81
MI(CRQ)	-0.0386	0.0049	0.0087	0.0720	1.87	1.34
selfconsist(CRQ)	-0.0482	0.0049	0.0095	0.0766	2.05	1.43
$\tau = \mathbf{.50}$						
RQ(complete obs.)	0.0029	0.0032	0.0030	0.0439	1.00	1.00
RQ(all obs.)	0.1132	0.0020	0.0140	0.1134	4.65	2.59
CRQ(all obs.)	0.1355	0.0034	0.0219	0.1361	7.26	3.10
CRQ(only uncensored)	-0.0360	0.0057	0.0109	0.0818	3.62	1.86
MI(CRQ)	-0.0059	0.0039	0.0047	0.0539	1.55	1.23
selfconsist(CRQ)	-0.0176	0.0039	0.0049	0.0555	1.61	1.27
$\tau = \mathbf{.75}$						
RQ(complete obs.)	0.0023	0.0037	0.0040	0.0502	1.00	1.00
RQ(all obs.)	-0.2020	0.0036	0.0448	0.2021	11.06	4.03
CRQ(all obs.)	0.0890	0.0052	0.0161	0.1000	3.98	1.99
CRQ(only uncensored)	-0.0424	0.0061	0.0128	0.0899	3.17	1.79
MI(CRQ)	-0.0102	0.0046	0.0065	0.0639	1.60	1.27
selfconsist(CRQ)	-0.0197	0.0048	0.0072	0.0674	1.78	1.34
	b1					
$\tau = \mathbf{.25}$						
RQ(complete obs.)	-0.0052	0.0042	0.0054	0.0582	1.00	1.00
RQ(all obs.)	0.0631	0.0048	0.0110	0.0838	2.05	1.44
CRQ(all obs.)	0.0589	0.0049	0.0107	0.0824	2.01	1.42
CRQ(only uncensored)	-0.0044	0.0057	0.0097	0.0792	1.80	1.36
MI(CRQ)	-0.0037	0.0044	0.0059	0.0609	1.11	1.05
selfconsist(CRQ)	-0.0085	0.0046	0.0064	0.0648	1.21	1.11
$\tau = \mathbf{.50}$						
RQ(complete obs.)	-0.0041	0.0039	0.0047	0.0538	1.00	1.00
RQ(all obs.)	0.0427	0.0041	0.0068	0.0651	1.46	1.21
CRQ(all obs.)	0.0645	0.0042	0.0094	0.0786	2.00	1.46
CRQ(only uncensored)	-0.0053	0.0049	0.0072	0.0678	1.53	1.26
MI(CRQ)	0.0086	0.0042	0.0054	0.0572	1.15	1.06
selfconsist(CRQ)	0.0062	0.0042	0.0053	0.0574	1.12	1.07
$\tau = \mathbf{.75}$						
RQ(complete obs.)	-0.0017	0.0043	0.0054	0.0583	1.00	1.00
RQ(all obs.)	-0.1076	0.0045	0.0177	0.1128	3.26	1.93
CRQ(all obs.)	0.0457	0.0048	0.0089	0.0747	1.64	1.28
CRQ(only uncensored)	-0.0031	0.0053	0.0084	0.0717	1.55	1.23
MI(CRQ)	0.0115	0.0046	0.0066	0.0648	1.22	1.11
selfconsist(CRQ)	0.0094	0.0047	0.0066	0.0655	1.22	1.12

Table 3.13: (Continued)

	b2					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = $ **.25**						
RQ(complete obs.)	-0.0106	0.0042	0.0055	0.0594	1.00	1.00
RQ(all obs.)	0.0478	0.0050	0.0096	0.0797	1.76	1.34
CRQ(all obs.)	0.0444	0.0050	0.0095	0.0791	1.75	1.33
CRQ(only uncensored)	-0.0136	0.0055	0.0093	0.0772	1.70	1.30
MI(CRQ)	-0.0196	0.0042	0.0057	0.0601	1.05	1.01
selfconsist(CRQ)	-0.0245	0.0045	0.0066	0.0649	1.22	1.09
$\tau = $ **.50**						
RQ(complete obs.)	-0.0115	0.0038	0.0045	0.0536	1.00	1.00
RQ(all obs.)	0.0306	0.0040	0.0057	0.0607	1.28	1.13
CRQ(all obs.)	0.0523	0.0042	0.0080	0.0728	1.80	1.36
CRQ(only uncensored)	-0.0149	0.0048	0.0070	0.0669	1.58	1.25
MI(CRQ)	-0.0140	0.0039	0.0048	0.0556	1.09	1.04
selfconsist(CRQ)	-0.0211	0.0040	0.0052	0.0574	1.17	1.07
$\tau = $ **.75**						
RQ(complete obs.)	-0.0143	0.0044	0.0059	0.0608	1.00	1.00
RQ(all obs.)	-0.1112	0.0047	0.0191	0.1170	3.22	1.93
CRQ(all obs.)	0.0357	0.0050	0.0086	0.0742	1.46	1.22
CRQ(only uncensored)	-0.0160	0.0054	0.0091	0.0754	1.54	1.24
MI(CRQ)	-0.0141	0.0047	0.0067	0.0656	1.14	1.08
selfconsist(CRQ)	-0.0186	0.0047	0.0069	0.0666	1.16	1.10

Table 3.14: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 3. Sample size: 1000. Replication: 1000. Censoring rate: 20%.

	Intercept					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = .25$						
RQ(complete obs.)	-0.0025	0.0013	0.0016	0.0318	1.00	1.00
RQ(all obs.)	0.1404	0.0015	0.0219	0.1405	13.86	4.42
CRQ(all obs.)	0.1089	0.0015	0.0140	0.1092	8.87	3.44
CRQ(only uncensored)	-0.0324	0.0020	0.0052	0.0567	3.32	1.79
MI(CRQ)	-0.0260	0.0016	0.0032	0.0450	2.04	1.42
selfconsist(CRQ)	-0.0331	0.0016	0.0035	0.0477	2.24	1.50
$\tau = .50$						
RQ(complete obs.)	-0.0045	0.0011	0.0012	0.0282	1.00	1.00
RQ(all obs.)	0.1126	0.0006	0.0130	0.1126	10.42	4.00
CRQ(all obs.)	0.1429	0.0014	0.0223	0.1429	17.86	5.07
CRQ(only uncensored)	-0.0281	0.0018	0.0042	0.0520	3.34	1.85
MI(CRQ)	-0.0015	0.0014	0.0019	0.0347	1.51	1.23
selfconsist(CRQ)	-0.0108	0.0014	0.0020	0.0357	1.61	1.27
$\tau = .75$						
RQ(complete obs.)	-0.0046	0.0012	0.0015	0.0304	1.00	1.00
RQ(all obs.)	-0.2101	0.0011	0.0453	0.2101	31.19	6.92
CRQ(all obs.)	0.0967	0.0017	0.0122	0.0976	8.40	3.21
CRQ(only uncensored)	-0.0327	0.0019	0.0047	0.0553	3.25	1.82
MI(CRQ)	-0.0060	0.0015	0.0022	0.0378	1.53	1.25
selfconsist(CRQ)	-0.0121	0.0015	0.0025	0.0399	1.72	1.31
	b1					
$\tau = .25$						
RQ(complete obs.)	-0.0017	0.0013	0.0016	0.0320	1.00	1.00
RQ(all obs.)	0.0595	0.0015	0.0057	0.0630	3.54	1.97
CRQ(all obs.)	0.0567	0.0015	0.0054	0.0609	3.37	1.90
CRQ(only uncensored)	-0.0040	0.0016	0.0026	0.0414	1.63	1.29
MI(CRQ)	-0.0016	0.0014	0.0019	0.0345	1.18	1.08
selfconsist(CRQ)	-0.0040	0.0014	0.0019	0.0346	1.19	1.08
$\tau = .50$						
RQ(complete obs.)	-0.0019	0.0011	0.0013	0.0288	1.00	1.00
RQ(all obs.)	0.0439	0.0012	0.0033	0.0481	2.52	1.67
CRQ(all obs.)	0.0646	0.0013	0.0058	0.0667	4.42	2.32
CRQ(only uncensored)	-0.0051	0.0014	0.0020	0.0356	1.50	1.24
MI(CRQ)	0.0107	0.0013	0.0017	0.0322	1.28	1.12
selfconsist(CRQ)	0.0080	0.0013	0.0017	0.0319	1.25	1.11
$\tau = .75$						
RQ(complete obs.)	-0.0026	0.0012	0.0016	0.0319	1.00	1.00
RQ(all obs.)	-0.1072	0.0013	0.0132	0.1074	8.45	3.37
CRQ(all obs.)	0.0454	0.0014	0.0040	0.0520	2.57	1.63
CRQ(only uncensored)	-0.0048	0.0016	0.0025	0.0396	1.58	1.24
MI(CRQ)	0.0113	0.0014	0.0021	0.0367	1.34	1.15
selfconsist(CRQ)	0.0096	0.0014	0.0021	0.0370	1.36	1.16

Table 3.14: (Continued)

	b2					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	-0.0066	0.0013	0.0017	0.0321	1.00	1.00
RQ(all obs.)	0.0548	0.0015	0.0052	0.0613	3.14	1.91
CRQ(all obs.)	0.0526	0.0015	0.0050	0.0595	3.00	1.85
CRQ(only uncensored)	-0.0041	0.0016	0.0026	0.0411	1.57	1.28
MI(CRQ)	-0.0131	0.0013	0.0019	0.0337	1.11	1.05
selfconsist(CRQ)	-0.0172	0.0013	0.0020	0.0353	1.22	1.10
$\tau =$ **.50**						
RQ(complete obs.)	-0.0073	0.0012	0.0014	0.0297	1.00	1.00
RQ(all obs.)	0.0382	0.0012	0.0029	0.0450	2.12	1.52
CRQ(all obs.)	0.0584	0.0013	0.0051	0.0614	3.66	2.06
CRQ(only uncensored)	-0.0044	0.0014	0.0018	0.0350	1.33	1.18
MI(CRQ)	-0.0064	0.0012	0.0014	0.0298	1.02	1.00
selfconsist(CRQ)	-0.0112	0.0012	0.0015	0.0311	1.09	1.04
$\tau =$ **.75**						
RQ(complete obs.)	-0.0078	0.0013	0.0017	0.0325	1.00	1.00
RQ(all obs.)	-0.1034	0.0015	0.0128	0.1037	7.62	3.19
CRQ(all obs.)	0.0412	0.0014	0.0037	0.0499	2.19	1.53
CRQ(only uncensored)	-0.0047	0.0015	0.0024	0.0394	1.44	1.21
MI(CRQ)	-0.0064	0.0014	0.0019	0.0347	1.13	1.07
selfconsist(CRQ)	-0.0095	0.0014	0.0020	0.0351	1.17	1.08

Table 3.15: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 4. Sample size: 300. Replication: 1000. Censoring rate: 20%.

	Intercept					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	-0.0029	0.0042	0.0052	0.0582	1.00	1.00
RQ(all obs.)	-0.1259	0.0041	0.0210	0.1273	4.04	2.19
CRQ(all obs.)	0.1067	0.0053	0.0197	0.1179	3.80	2.02
CRQ(only uncensored)	-0.0599	0.0065	0.0162	0.1016	3.12	1.74
MI(CRQ)	-0.0632	0.0046	0.0102	0.0808	1.97	1.39
selfconsist(CRQ)	-0.0707	0.0048	0.0119	0.0873	2.30	1.50
$\tau =$ **.50**						
RQ(complete obs.)	-0.0078	0.0037	0.0041	0.0508	1.00	1.00
RQ(all obs.)	-0.1571	0.0040	0.0295	0.1575	7.25	3.10
CRQ(all obs.)	0.1642	0.0054	0.0359	0.1669	8.82	3.29
CRQ(only uncensored)	-0.0435	0.0064	0.0141	0.0950	3.47	1.87
MI(CRQ)	-0.0367	0.0039	0.0058	0.0606	1.44	1.19
selfconsist(CRQ)	-0.0435	0.0042	0.0072	0.0679	1.78	1.34
$\tau =$ **.75**						
RQ(complete obs.)	-0.0104	0.0042	0.0055	0.0597	1.00	1.00
RQ(all obs.)	-0.1115	0.0054	0.0213	0.1234	3.90	2.07
CRQ(all obs.)	0.2353	0.0072	0.0710	0.2377	13.00	3.98
CRQ(only uncensored)	-0.0610	0.0076	0.0209	0.1159	3.83	1.94
MI(CRQ)	-0.0138	0.0048	0.0072	0.0679	1.32	1.14
selfconsist(CRQ)	-0.0220	0.0053	0.0089	0.0761	1.63	1.28
	b1					
$\tau =$ **.25**						
RQ(complete obs.)	-0.0053	0.0043	0.0055	0.0595	1.00	1.00
RQ(all obs.)	-0.0954	0.0042	0.0143	0.1025	2.62	1.72
CRQ(all obs.)	0.0227	0.0045	0.0066	0.0646	1.22	1.08
CRQ(only uncensored)	-0.0122	0.0055	0.0091	0.0765	1.67	1.28
MI(CRQ)	-0.0138	0.0041	0.0053	0.0586	0.98	0.98
selfconsist(CRQ)	-0.0181	0.0044	0.0060	0.0621	1.11	1.04
$\tau =$ **.50**						
RQ(complete obs.)	-0.0068	0.0040	0.0048	0.0556	1.00	1.00
RQ(all obs.)	-0.0922	0.0040	0.0132	0.0978	2.74	1.76
CRQ(all obs.)	0.0318	0.0043	0.0065	0.0634	1.36	1.14
CRQ(only uncensored)	-0.0095	0.0051	0.0078	0.0713	1.62	1.28
MI(CRQ)	-0.0065	0.0040	0.0050	0.0567	1.03	1.02
selfconsist(CRQ)	-0.0110	0.0041	0.0053	0.0575	1.09	1.03
$\tau =$ **.75**						
RQ(complete obs.)	-0.0088	0.0043	0.0057	0.0610	1.00	1.00
RQ(all obs.)	-0.0753	0.0046	0.0120	0.0898	2.11	1.47
CRQ(all obs.)	0.0397	0.0052	0.0098	0.0783	1.71	1.28
CRQ(only uncensored)	-0.0140	0.0059	0.0105	0.0819	1.83	1.34
MI(CRQ)	0.0037	0.0048	0.0068	0.0660	1.20	1.08
selfconsist(CRQ)	-0.0010	0.0050	0.0076	0.0690	1.33	1.13

Table 3.15: (Continued)

	b2					
300 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau = $ **.25**						
RQ(complete obs.)	-0.0095	0.0042	0.0053	0.0576	1.00	1.00
RQ(all obs.)	-0.0534	0.0039	0.0074	0.0699	1.41	1.21
CRQ(all obs.)	0.0344	0.0047	0.0078	0.0713	1.48	1.24
CRQ(only uncensored)	-0.0105	0.0054	0.0090	0.0751	1.70	1.30
MI(CRQ)	-0.0281	0.0040	0.0055	0.0600	1.05	1.04
selfconsist(CRQ)	-0.0301	0.0043	0.0065	0.0650	1.23	1.13
$\tau = $ **.50**						
RQ(complete obs.)	-0.0113	0.0039	0.0047	0.0546	1.00	1.00
RQ(all obs.)	-0.0508	0.0039	0.0072	0.0686	1.55	1.26
CRQ(all obs.)	0.0485	0.0044	0.0083	0.0725	1.77	1.33
CRQ(only uncensored)	-0.0111	0.0048	0.0070	0.0662	1.50	1.21
MI(CRQ)	-0.0241	0.0039	0.0050	0.0565	1.08	1.03
selfconsist(CRQ)	-0.0249	0.0040	0.0055	0.0584	1.18	1.07
$\tau = $ **.75**						
RQ(complete obs.)	-0.0122	0.0043	0.0057	0.0613	1.00	1.00
RQ(all obs.)	-0.0268	0.0049	0.0080	0.0719	1.40	1.17
CRQ(all obs.)	0.0743	0.0056	0.0151	0.0990	2.63	1.62
CRQ(only uncensored)	-0.0120	0.0057	0.0100	0.0799	1.75	1.30
MI(CRQ)	-0.0154	0.0047	0.0069	0.0667	1.21	1.09
selfconsist(CRQ)	-0.0164	0.0051	0.0081	0.0715	1.41	1.17

Table 3.16: Bias, SE, MSE, MAE and Ratios for method 1 - 6 on Model 4. Sample size: 1000. Replication: 1000. Censoring rate: 20%.

	Intercept					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	-0.0033	0.0014	0.0018	0.0341	1.00	1.00
RQ(all obs.)	-0.1277	0.0012	0.0179	0.1277	9.67	3.75
CRQ(all obs.)	0.1298	0.0018	0.0201	0.1304	10.91	3.83
CRQ(only uncensored)	-0.0365	0.0020	0.0055	0.0586	2.98	1.72
MI(CRQ)	-0.0475	0.0014	0.0043	0.0544	2.35	1.60
selfconsist(CRQ)	-0.0581	0.0015	0.0056	0.0633	3.06	1.86
$\tau =$ **.50**						
RQ(complete obs.)	-0.0031	0.0012	0.0015	0.0314	1.00	1.00
RQ(all obs.)	-0.1541	0.0012	0.0252	0.1541	16.52	4.92
CRQ(all obs.)	0.1857	0.0021	0.0388	0.1857	25.38	5.92
CRQ(only uncensored)	-0.0304	0.0020	0.0048	0.0555	3.11	1.77
MI(CRQ)	-0.0229	0.0014	0.0024	0.0391	1.57	1.25
selfconsist(CRQ)	-0.0324	0.0014	0.0031	0.0447	2.00	1.42
$\tau =$ **.75**						
RQ(complete obs.)	-0.0048	0.0014	0.0019	0.0347	1.00	1.00
RQ(all obs.)	-0.1009	0.0017	0.0131	0.1023	6.78	2.95
CRQ(all obs.)	0.2656	0.0027	0.0780	0.2656	40.48	7.65
CRQ(only uncensored)	-0.0401	0.0022	0.0064	0.0655	3.34	1.89
MI(CRQ)	0.0023	0.0016	0.0026	0.0407	1.36	1.17
selfconsist(CRQ)	-0.0055	0.0017	0.0029	0.0435	1.51	1.25
	b1					
$\tau =$ **.25**						
RQ(complete obs.)	-0.0009	0.0013	0.0017	0.0330	1.00	1.00
RQ(all obs.)	-0.0966	0.0012	0.0108	0.0968	6.36	2.93
CRQ(all obs.)	0.0259	0.0013	0.0024	0.0393	1.41	1.19
CRQ(only uncensored)	-0.0055	0.0016	0.0027	0.0421	1.60	1.28
MI(CRQ)	-0.0104	0.0012	0.0016	0.0322	0.95	0.98
selfconsist(CRQ)	-0.0169	0.0013	0.0019	0.0356	1.14	1.08
$\tau =$ **.50**						
RQ(complete obs.)	-0.0014	0.0012	0.0014	0.0297	1.00	1.00
RQ(all obs.)	-0.0924	0.0012	0.0099	0.0925	7.19	3.12
CRQ(all obs.)	0.0352	0.0013	0.0029	0.0437	2.12	1.47
CRQ(only uncensored)	-0.0043	0.0015	0.0022	0.0378	1.62	1.28
MI(CRQ)	-0.0026	0.0012	0.0014	0.0306	1.03	1.03
selfconsist(CRQ)	-0.0084	0.0012	0.0016	0.0320	1.13	1.08
$\tau =$ **.75**						
RQ(complete obs.)	-0.0013	0.0013	0.0017	0.0326	1.00	1.00
RQ(all obs.)	-0.0697	0.0014	0.0069	0.0714	4.13	2.19
CRQ(all obs.)	0.0496	0.0016	0.0050	0.0582	3.02	1.79
CRQ(only uncensored)	-0.0038	0.0017	0.0030	0.0440	1.82	1.35
MI(CRQ)	0.0096	0.0014	0.0021	0.0373	1.28	1.15
selfconsist(CRQ)	0.0039	0.0015	0.0022	0.0373	1.33	1.14

Table 3.16: (Continued)

	b2					
1000 obs, 20%	Mean Bias	SE	MSE	MAE	R_{MSE}	R_{MAE}
$\tau =$ **.25**						
RQ(complete obs.)	-0.0044	0.0012	0.0015	0.0312	1.00	1.00
RQ(all obs.)	-0.0450	0.0012	0.0035	0.0495	2.32	1.59
CRQ(all obs.)	0.0475	0.0013	0.0040	0.0533	2.69	1.71
CRQ(only uncensored)	-0.0028	0.0016	0.0025	0.0394	1.65	1.26
MI(CRQ)	-0.0204	0.0012	0.0018	0.0338	1.18	1.09
selfconsist(CRQ)	-0.0247	0.0013	0.0022	0.0378	1.47	1.21
$\tau =$ **.50**						
RQ(complete obs.)	-0.0050	0.0011	0.0013	0.0291	1.00	1.00
RQ(all obs.)	-0.0411	0.0012	0.0031	0.0459	2.33	1.58
CRQ(all obs.)	0.0607	0.0013	0.0055	0.0635	4.08	2.19
CRQ(only uncensored)	-0.0061	0.0014	0.0021	0.0363	1.56	1.25
MI(CRQ)	-0.0157	0.0012	0.0016	0.0317	1.18	1.09
selfconsist(CRQ)	-0.0186	0.0012	0.0018	0.0341	1.37	1.18
$\tau =$ **.75**						
RQ(complete obs.)	-0.0055	0.0013	0.0017	0.0321	1.00	1.00
RQ(all obs.)	-0.0152	0.0014	0.0023	0.0385	1.41	1.20
CRQ(all obs.)	0.0846	0.0016	0.0098	0.0866	5.89	2.69
CRQ(only uncensored)	-0.0084	0.0017	0.0029	0.0429	1.73	1.34
MI(CRQ)	-0.0088	0.0014	0.0019	0.0353	1.16	1.10
selfconsist(CRQ)	-0.0103	0.0015	0.0023	0.0378	1.36	1.18

Table 3.17: The coverage probabilities of bootstrap methods: the xy-paired and the block bootstrap. (QAR(1) model, 95% Confidence level, n=100)

	xy-paired bootstrap			Block bootstrap		
Quantiles	.25	.50	.75	.25	.50	.75
Intercept	0.920	0.943	0.945	0.911	0.941	0.929
Slope	0.918	0.926	0.951	0.905	0.923	0.947

Table 3.18: The coverage probabilities of bootstrap methods: the xy-paired and the block bootstrap. (QAR(2) model, 95% Confidence level, n=100)

	xy-paired bootstrap			Block bootstrap		
Quantiles	.25	.50	.75	.25	.50	.75
Intercept	0.920	0.947	0.945	0.925	0.936	0.940
β_1	0.912	0.943	0.949	0.910	0.934	0.942
β_2	0.926	0.935	0.946	0.912	0.934	0.926

Chapter 4

Empirical data analysis

4.1 Samish River Example

As the first example, we shall apply our method to Ammonia-Nitrogen (NH3-N) measurements taken in the Samish River in Washington State from 1977-2009 (see Hallock (2009)). The Washington State Department of Ecology have monitored monthly NH3-N as one of conventional parameters indicating water quality level. Approximately, 360 measurements were taken over the period and we are interested in modeling how the past concentration measurements affect the present. Before going deep in the analysis there are a few features we need to consider.

- The data is left censored.
- The data is not evenly spaced.

Measurements of less than .01 percent were considered as undetectable and such measurements were listed as censored. This is called "left" censoring, which is different from our settings in previous chapters. But, this can be handled in software by multiplying the data by -1 and replacing τ by $(1-\tau)$ in quantile function. Another feature to be considered in this data is that it is not evenly spaced. In several periods of times, it is measured bi-monthly or once in three months. When this happens, we add a blank observation and treat it as a missing. Since we could think of missing values as censored values with no detection limits, those can be easily managed throughout the self-consistent algorithm. There are 361 observations, 104 censored observations, and

11 missing observations, which results in 68.1% uncensoring, 28.8% left censoring, and 3% missing. Note that with few exceptions, all of .01 measurements were listed as censored, and so overall censoring rate is about 30%, which can be considered as highly censored in autoregressive models since it results in only 51% of measurements to be uncensored in both regressor and response.

Figure 4.1 shows the data with fitted lines. The censored values in the plot are marked as "C". The lines in the plot corresponds to the .10,.25,.5,.75, and .9 (conditional) quantiles for two methods of estimation. The dashed lines use the usual quantile autoregression method by treating censored values as observed. The solid lines use the self-consistent algorithm.

Notice that as illustrated previously, the classical constant-coefficient linear time series models have limitation to have the reasonable fit to the heterogeneity in this data since it requires that the conditional quantiles have similar shapes. Also, due to heavy censoring at low quantiles, the censored regression quantile algorithm gives a defective distribution, which cannot estimate quantiles below .15. Also it results in limiting modeling options for the data.

The table 4.1-4.2 shows the estimates of intercept and slope parameters at quantiles .1, .25, .5, .75, and .9. Here we only considered autoregressive models of order 1 and 2. The ordinary QAR models which treat censored observations as uncensored do not seem unreasonable, but clearly do not take censoring into account and fails to provide consistent estimates.

From this example, we can see that;

- The CQAR method captures heterogeneity in data by its nature and provides empirically consistent estimates.
- The CQAR algorithm converges very fast, taking less than 5 steps until it satisfies the stopping rule.

- Due to heavy left censoring, conditional quantiles below .15 are not estimable from the CQAR method.

From the results in Figure 4.1-3 and Table 4.1-4, we can see that both intercept and slope coefficient estimates with CQAR algorithm are monotonically increasing as τ increase, which shows the heterogeneity that cannot be explained in classical constant-coefficient time series models. Also, coefficient estimates with CQAR algorithm are generally smaller than those with ordinary QAR method, which might give us reasons to taking the censored values into account. Unrealized data structure under the censored values may affect estimation of regression coefficients in upper quantiles. Notice that CQAR algorithm on QAR(2) model was able to estimate coefficients down to $\tau = .10$, which was unable on QAR(1) model. This might suggest that conditional quantiles below .10 can be models by allowing higher order in autoregression.

The bootstrap inferences are summarized in Table 4.3-4.4. Note that we used the block-bootstrap to capture the dependance in time series and make it robust with respect to model misspecification. However, as we discussed in Chapter 2.3, usual xy-paired bootstrap also gives reliable results. Figure 4.2-4.3 shows intercept and slope estimates of QAR(1) and QAR(2) models, evaluated at 17 equally spaced quantiles. Note that the shaded region is a .95 confidence band.

4.2 Dry Decomposition of NH4 Example

As the second example, we shall apply our method to time series data on the chemical composition of atmospheric deposition. The data was collected monthly by the Environmental Measurements Laboratory between 1977 and early 1980 at a number of sites in the United States on the composition of dry deposition (see Zeger and Brookmeyer (1986) for more details). The main objective of collecting the data was to study geographical differences and time trends in precipitation chemistry and concentration of pollutants in deposition.

We are interested in how the past monthly concentration of ammonia affects the current at the Lawrence Livermore, California, site. There are lower detection limits of the assays. In fact, as stated in Zeger and Brookmeyer (1986), the detection limit depends on the total quantity of deposition collected each month; smaller volumes collected give higher detection limits. There are 43 measurements; 34 uncensored measurements, 6 left censored measurements, and 3 missing measurements, which resulting in 79% uncensoring, 14% left censoring, and 7% missing. We have fit a first and second order autoregressive model to log-transformed concentration of ammonia (NH4).

Figure 4.4 shows the scatterplot with fitted lines. The censored observations in the plot are marked as "C". The dashed lines are estimated by using the usual QAR method of order 1 at .10,.25,.5,.75, and .90 quantiles. The solid lines represents corresponding quantiles by the censored QAR with self-consistent algorithm. Note that a quantile at .10 was unestimable. And also notice that there are possible crossing at .25 and .50 quantiles.

Table 4.5-4.6 give the QAR(1) and QAR(2) estimates of intercept and slope parameters at quantiles .1, .25, .5, .75, and .9. The ordinary QAR models which treat censored observations as uncensored fails to capture dynamics of the data especially at low quantiles. And Table 4.7-4.8 summarize the bootstrap inferences of the censored QAR(1) and QAR(2) models at 9 equally spaced quantiles from .10 to .90. The inferences are based on 1000 bootstrap samples and the block bootstrap methods was implemented. Figure 4.5-4.6 shows changes of intercept and slope estimates with 95% confidence band over 18 equally spaced quantiles.

4.3 Figures and Tables

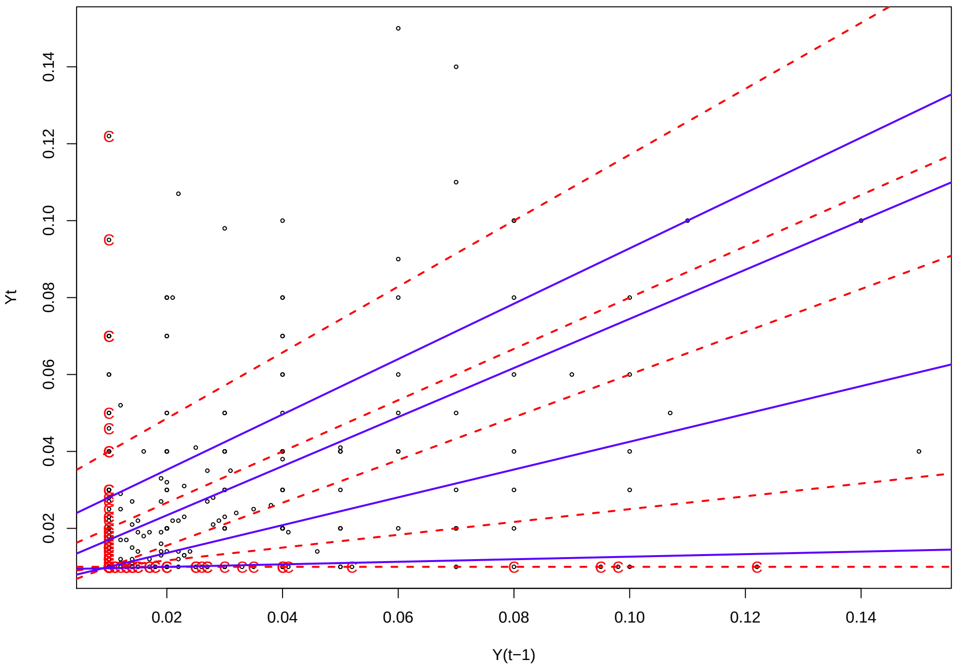

Figure 4.1: Scatter plot for NH3-N rate. Superimposed on the plots are the .10, .25, .50, .75, .90 quantile autoregression lines from bottom to top. The dotted lines are estimated from naive QAR(1) (treating the censored observations as uncensored); The solid lines are estimated from the self-consistent algorithm. Censored values are marked as C.

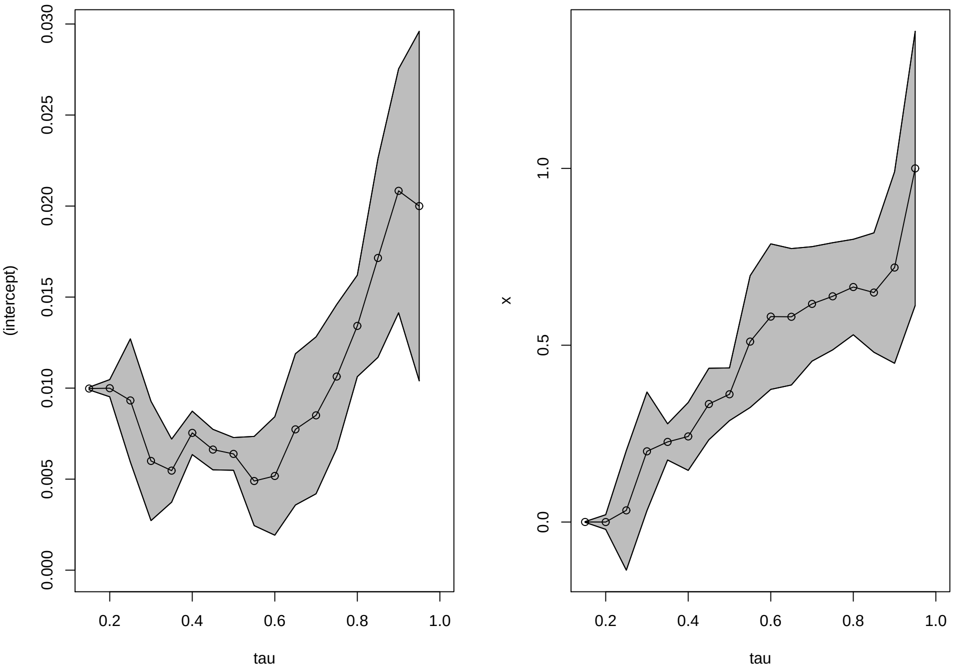

Figure 4.2: Estimating the QAR(1) Model using the self-consistent algorithm on the NH3-N rate data. The figure illustrates the QAR(1) intercept and slope estimates: the estimate at 17 equally spaced quantiles. The shaded region is a .95 confidence band.

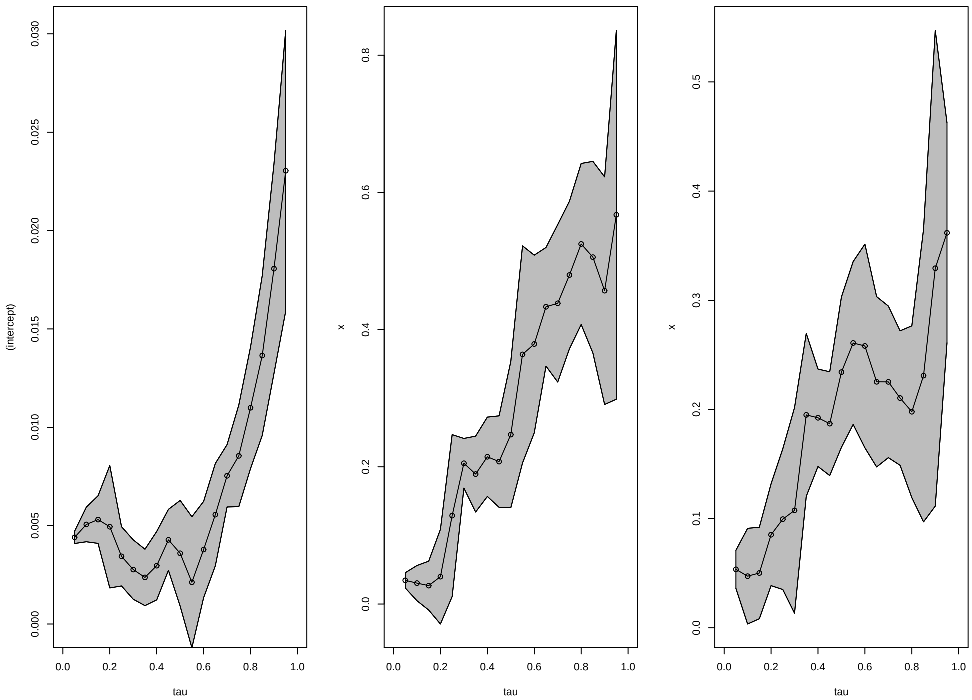

Figure 4.3: Estimating the QAR(2) Model using the self-consistent algorithm on the NH3-N rate data. The figure illustrates the QAR(2) intercept and slope estimates: the estimate at 19 equally spaced quantiles. The shaded region is a .95 confidence band.

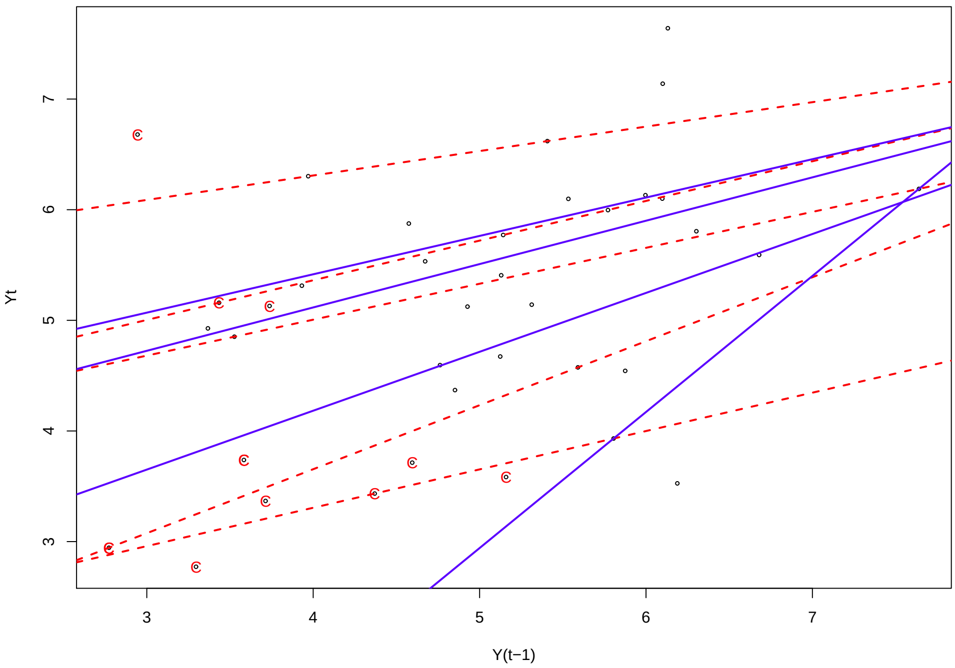

Figure 4.4: Scatter plot of dry decomposition for NH4. Superimposed on the plots are the .10,.25,.50,.75,.90 quantile autoregression lines from bottom to top. The dotted lines are estimated from naive QAR(1) (treating the censored observations as uncensored); The solid lines are estimated from Censored QAR(1) with the self-consistent algorithm. Censored values are marked as C.

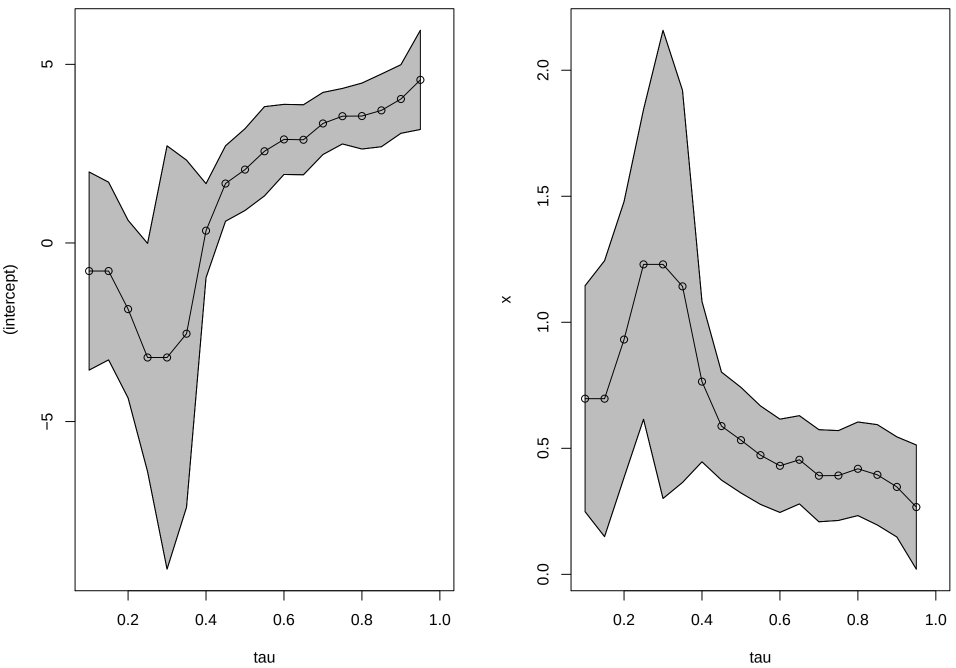

Figure 4.5: Estimating the censored QAR(1) Model using the self-consistent algorithm on dry decompostion for NH4. The figure illustrates the QAR(1) intercept and slope estimates: the estimate at 18 equally spaced quantiles. The shaded region is a .95 confidence band.

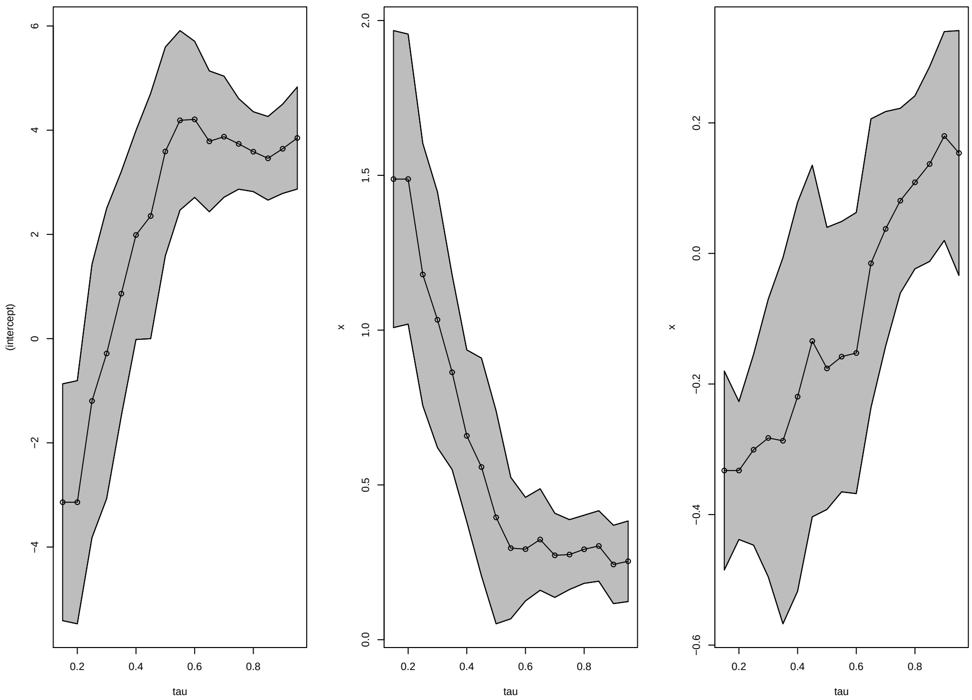

Figure 4.6: Estimating the censored QAR(2) Model using the self-consistent algorithm on dry decompostion for NH4. The figure illustrates the QAR(2) intercept and slope estimates: the estimate at 17 equally spaced quantiles. The shaded region is a .95 confidence band.

Table 4.1: Estimates of a QAR(1) model for the NH3-N measurements from different methods (The ordinary QAR method and the CRQ with the self-consistent algorithm). NA means the "quantreg" package fails.

	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
QAR method (treating the censored observations as uncensored)					
Intercept	0.01000	0.00833	0.00444	0.01333	0.03143
slope	0.00000	0.16667	0.55556	0.66667	0.85714
CRQ method with self-consistent algorithm					
Intercept	NA	0.00932	0.00639	0.01064	0.02084
Slope	NA	0.03310	0.36135	0.63832	0.71966

Table 4.2: Estimates of a QAR(2) model for the NH3-N measurements from different methods (The ordinary QAR method and the CRQ with the self-consistent algorithm).

	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
QAR method (treating the censored observations as uncensored)					
Intercept	0.01000	0.00728	0.00389	0.00932	0.02679
β_1	0.00000	0.13289	0.36111	0.52273	0.62500
β_2	0.00000	0.09967	0.25000	0.25000	0.32143
CRQ method with self-consistent algorithm					
Intercept	0.00506	0.00344	0.00360	0.00855	0.01806
β_1	0.03048	0.12890	0.24694	0.47956	0.45675
β_2	0.04729	0.09952	0.23418	0.21042	0.32933

Table 4.3: Results from a censored QAR(1) model for the NH3-N measurements with 1000 Bootstrap samples.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.20	Intercept	0.0100	0.0096	0.0104	0.0002	50.6563	0.0000
	Slope	0.0000	-0.0188	0.0189	0.0096	0.0041	0.9967
.30	Intercept	0.0060	0.0026	0.0094	0.0017	3.4455	0.0006
	Slope	0.2000	0.0236	0.3764	0.0900	2.2219	0.0263
.40	Intercept	0.0075	0.0062	0.0089	0.0007	10.6247	0.0000
	Slope	0.2421	0.1516	0.3326	0.0462	5.2422	0.0000
.50	Intercept	0.0064	0.0052	0.0076	0.0006	10.3339	0.0000
	Slope	0.3613	0.2656	0.4571	0.0488	7.3971	0.0000
.60	Intercept	0.0052	0.0014	0.0090	0.0019	2.6587	0.0078
	Slope	0.5809	0.3664	0.7953	0.1094	5.3081	0.0000
.70	Intercept	0.0085	0.0049	0.0121	0.0018	4.6290	0.0000
	Slope	0.6168	0.4592	0.7745	0.0804	7.6701	0.0000
.80	Intercept	0.0134	0.0091	0.0177	0.0022	6.1534	0.0000
	Slope	0.6646	0.5093	0.8199	0.0792	8.3876	0.0000
.90	Intercept	0.0208	0.0144	0.0273	0.0033	6.3052	0.0000
	Slope	0.7197	0.4439	0.9955	0.1407	5.1143	0.0000

Table 4.4: Results from a censored QAR(2) model for the NH3-N measurements with 1000 Bootstrap samples.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.10	Intercept	0.0051	0.0043	0.0059	0.0004	12.4559	0.0000
	β_1	0.0305	0.0198	0.0412	0.0055	5.5782	0.0000
	β_2	0.0473	0.0114	0.0831	0.0183	2.5860	0.0097
.20	Intercept	0.0049	0.0016	0.0083	0.0017	2.8888	0.0039
	β_1	0.0399	-0.0251	0.1049	0.0332	1.2021	0.2293
	β_2	0.0852	0.0342	0.1363	0.0260	3.2747	0.0011
.30	Intercept	0.0028	0.0012	0.0044	0.0008	3.4044	0.0007
	β_1	0.2052	0.1472	0.2632	0.0296	6.9389	0.0000
	β_2	0.1075	0.0036	0.2114	0.0530	2.0284	0.0425
.40	Intercept	0.0030	0.0012	0.0047	0.0009	3.3863	0.0007
	β_1	0.2146	0.1638	0.2655	0.0259	8.2708	0.0000
	β_2	0.1924	0.1316	0.2531	0.0310	6.2091	0.0000

Table 4.4: Cont.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.50	Intercept	0.0036	0.0010	0.0061	0.0013	2.7673	0.0057
	β_1	0.2469	0.1754	0.3185	0.0365	6.7665	0.0000
	β_2	0.2342	0.1360	0.3324	0.0501	4.6742	0.0000
.60	Intercept	0.0038	0.0009	0.0067	0.0015	2.5534	0.0107
	β_1	0.3790	0.2536	0.5044	0.0640	5.9241	0.0000
	β_2	0.2581	0.1561	0.3602	0.0521	4.9575	0.0000
.70	Intercept	0.0075	0.0051	0.0100	0.0012	6.0579	0.0000
	β_1	0.4383	0.3701	0.5066	0.0348	12.5863	0.0000
	β_2	0.2253	0.1474	0.3031	0.0397	5.6706	0.0000
.80	Intercept	0.0110	0.0080	0.0140	0.0015	7.2010	0.0000
	β_1	0.5248	0.4119	0.6377	0.0576	9.1088	0.0000
	β_2	0.1979	0.0930	0.3028	0.0535	3.6989	0.0002
.90	Intercept	0.0181	0.0134	0.0227	0.0024	7.6151	0.0000
	β_1	0.4567	0.3133	0.6002	0.0732	6.2404	0.0000
	β_2	0.3293	0.0854	0.5732	0.1244	2.6463	0.0081

Table 4.5: Estimates of a QAR(1) model for the dry decomposition from different methods (The ordinary QAR method and the CRQ with the self-consistent algorithm).

	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
QAR method (treating the censored observations as uncensored)					
Intercept	1.9188	1.3407	3.7066	3.9276	5.4255
slope	0.3468	0.5784	0.3248	0.3586	0.2209
CRQ method with self-consistent algorithm					
Intercept	-0.7861	-3.2068	2.0538	3.5486	4.0292
Slope	0.6969	1.2297	0.5323	0.3920	0.3468

Table 4.6: Estimates of a QAR(2) model for the dry decomposition from different methods (The ordinary QAR method and the CRQ with the self-consistent algorithm). NA means the “quantreg” package fails.

	$\tau = 0.10$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.90$
QAR method (treating the censored observations as uncensored)					
Intercept	2.21957	3.08977	3.79464	3.47857	4.00989
β_1	0.64813	0.35349	0.32462	0.29642	0.02319
β_2	-0.32529	-0.12349	-0.01412	0.15164	0.48438
CRQ method with self-consistent algorithm					
Intercept	NA	-1.19688	3.59142	3.73737	3.64332
β_1	NA	1.17959	0.39505	0.27497	0.24303
β_2	NA	-0.30065	-0.17611	0.08080	0.17989

Table 4.7: Results from a censored QAR(1) model for the dry decomposition with 1000 Bootstrap samples.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.10	Intercept	-0.7861	-3.2732	1.7011	1.2690	-0.6195	0.5356
	Slope	0.6969	0.1494	1.2443	0.2793	2.4950	0.0126
.20	Intercept	-1.8522	-4.3394	0.6350	1.2690	-1.4596	0.1444
	Slope	0.9315	0.3841	1.4790	0.2793	3.3351	0.0009
.30	Intercept	-3.2066	-9.1454	2.7322	3.0300	-1.0583	0.2899
	Slope	1.2297	0.2981	2.1613	0.4753	2.5871	0.0097
.40	Intercept	0.3437	-0.8336	1.5211	0.6007	0.5722	0.5672
	Slope	0.7645	0.4466	1.0824	0.1622	4.7133	0.0000
.50	Intercept	2.0538	0.8562	3.2515	0.6111	3.3611	0.0008
	Slope	0.5323	0.3099	0.7547	0.1135	4.6910	0.0000
.60	Intercept	2.8984	1.8624	3.9345	0.5286	5.4833	0.0000
	Slope	0.4306	0.2482	0.6130	0.0930	4.6276	0.0000
.70	Intercept	3.3446	2.4862	4.2031	0.4380	7.6366	0.0000
	Slope	0.3912	0.2160	0.5664	0.0894	4.3758	0.0000
.80	Intercept	3.5526	2.5980	4.5071	0.4870	7.2946	0.0000
	Slope	0.4187	0.2318	0.6057	0.0954	4.3901	0.0000
.90	Intercept	4.0292	3.0962	4.9622	0.4760	8.4643	0.0000
	Slope	0.3468	0.1612	0.5323	0.0947	3.6630	0.0002

Table 4.8: Results from a censored QAR(2) model for the dry decomposition with 1000 Bootstrap samples.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.20	Intercept	-3.1401	-5.7490	-0.5312	1.3311	-2.3590	0.0183
	β_1	1.4878	1.0431	1.9326	0.2269	6.5568	0.0000
	β_2	-0.3325	-0.4449	-0.2202	0.0573	-5.8028	0.0000
.30	Intercept	-0.2853	-2.8711	2.3004	1.3193	-0.2163	0.8288
	β_1	1.0334	0.6655	1.4012	0.1877	5.5056	0.0000
	β_2	-0.2827	-0.4660	-0.0994	0.0935	-3.0227	0.0025
.40	Intercept	1.9888	-0.0310	4.0085	1.0305	1.9299	0.0536
	β_1	0.6585	0.3711	0.9460	0.1467	4.4898	0.0000
	β_2	-0.2194	-0.5031	0.0642	0.1447	-1.5162	0.1295
.50	Intercept	3.5914	1.4881	5.6948	1.0732	3.3466	0.0008
	β_1	0.3951	0.0332	0.7569	0.1846	2.1396	0.0324
	β_2	-0.1761	-0.3777	0.0255	0.1028	-1.7124	0.0868

Table 4.8: Cont.

τ		Coeff	Lower Bd	Upper Bd	Std Error	T Value	P-value
.60	Intercept	4.2080	2.7001	5.7160	0.7694	5.4695	0.0000
	β_1	0.2925	0.1031	0.4820	0.0967	3.0267	0.0025
	β_2	-0.1525	-0.3651	0.0601	0.1085	-1.4063	0.1596
.70	Intercept	3.8751	2.7412	5.0089	0.5785	6.6986	0.0000
	β_1	0.2726	0.1285	0.4167	0.0735	3.7074	0.0002
	β_2	0.0376	-0.1351	0.2103	0.0881	0.4267	0.6696
.80	Intercept	3.5876	2.7500	4.4253	0.4274	8.3944	0.0000
	β_1	0.2921	0.1808	0.4035	0.0568	5.1417	0.0000
	β_2	0.1090	-0.0186	0.2365	0.0651	1.6748	0.0940
.90	Intercept	3.6433	2.7812	4.5055	0.4399	8.2824	0.0000
	β_1	0.2430	0.1227	0.3633	0.0614	3.9596	0.0001
	β_2	0.1799	0.0366	0.3231	0.0731	2.4614	0.0138

Chapter 5

Conclusions and Future Work

The quantile autoregression (QAR) models has advantages over classical constant coefficient linear time series models. It can capture systematic influences of conditioning variables on the location, scale and shape of the conditional distribution of the response whereas classical models are restricted only on observing a location shift. Having coefficients functionally dependent, the QAR model differs from the random-coefficient autoregressive (RCAR) model and substantially extends modeling options for time series data.

Quantile autoregression models with censored data have been rarely discussed in the literature. In this dissertation, we have proposed a censored quantile autoregression model which generally extends a censored regression method on standard regression model by adopting an idea of imputation methods. The censored quantile estimators can be easily implemented by using an existing R package “quantreg”, and we have implemented our proposed algorithm in R-language. The full R-code is available upon request to the authors. Throughout this paper, we can see that:

- The CQAR algorithm generates the empirically-consistent estimator in self-consistent manner.

- In the simulation experiments, the CQAR algorithm works much better than any currently available naive methods that treats censored values as observed.

- The computation time using the CQAR algorithm is very fast. In most time it converges within 10 steps, and since the objective function in each step is convex with respect to the regression coefficients, it can be efficiently solved by the standard linear programming algorithm or interior point methods for regression quantiles described in Koenker (2005).

- The CQAR algorithm can be generalized in various types of censoring: left- or right-fixed censoring and random censoring.

The results of the proposed algorithm from the simulation are very promising. However, The theoretical backgrounds of the proposed model are not well established yet and still need intensive investigation. Also, there are many interesting questions in this area. and we are particularly interested in:

- extending the algorithm in more complicated situation, where the order of autoregressive model is higher(i.e., ≥ 3).
- inference methods for censored quantile autoregression models.
- extensive comparison between quantile regression methods and other existing methods for censored time series.
- providing a R package that can handle different types of censoring on time series and allows users to obtain estimates and make inference based on desired bootstrap methods.

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