QUASISYMMETRIC SPHERES CONSTRUCTED OVER QUASIDISKS

BY

VYRON SARANTIS VELLIS

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2014

Urbana, Illinois

Doctoral Committee:

Professor Jeremy Tyson, Chair
Professor Sergiy Merenkov, Co-Chair
Professor Jang-Mei Wu, Director of Research
Professor Aimo Hinkkanen

Abstract

In this thesis we construct concrete examples of quasispheres and quasisymmetric spheres. These examples are double-dome type surfaces in $\mathbb{R}^3$ over planar Jordan domains Ω. The thesis consists of three parts.

Let Ω be a Jordan domain and φ a self homeomorphism of $[0, +\infty)$. In the *Geometric construction*, the surface is the graphs of $\pm\varphi(\mathrm{dist}(x, \partial\Omega))$. We examine the properties of the Jordan domains Ω and of the height functions φ ensuring that these surfaces are either quasispheres or quasisymmetric equivalent to $\mathbb{S}^2$. As it turns out, the geometry of the sets of constant distance from $\partial\Omega$ plays a key role in the geometry of these surfaces.

The Geometric construction is the motivation of the second part, the study of *sets of constant distance from a planar Jordan curve* Γ. We ask what properties of Γ ensure that these sets are Jordan curves, or uniform quasicircles, or uniform chord-arc curves for all sufficiently small ϵ. Sufficient conditions are given in term of a scaled invariant parameter for measuring the local deviation of subarcs from their chords. The chordal conditions given are sharp.

In the third part, we discuss the *Analytic construction*. In this construction, the level sets of the height of the surface built over a Jordan domain Ω are the level sets of $|f|$ for some quasiconformal function f that maps Ω onto the unit disk. We investigate the properties of f which guarantee that these surfaces are either bi-Lipschitz or quasisymmetric equivalent to $\mathbb{S}^2$.

To my family.

Acknowledgements

I am most indebted to my adviser Jang-Mei Wu for suggesting this problem, guiding me through my research and patiently reading my manuscripts. Apart from the unique way of thinking in Mathematics that I learned from her, she also taught me how to be a careful writer. I am proud to be her student.

I would like to thank my family for their continuing support, love and encouragement and all my friends in Urbana and Greece, especially Devon, Dimitra, Dimitris, Katerina, Michalis and Minas who were always there for me.

I am grateful to my committee Aimo Hinkkanen, Sergiy Merenkov and Jeremy Tyson for their interest in my work and valuable comments on my thesis.

Part of the research that led to this dissertation was supported by the Bourgin Fellowship (Spring 2011) awarded by the department of Mathematics, a Research Assistantship (Fall 2011, Spring 2012) awarded by the Research Board of the University of Illinois and two NSF Grants (Summer 2010, Summer 2011).

The Department of Mathematics at the University of Illinois provided a remarkable academic and research environment. I benefited a lot from various courses offered and from stimulating discussions with several of the faculty members. Many thanks also to the Department of Mathematics at the Aristotle University and Professor Dimitrios Betsakos for an excellent undergraduate preparation.

Table of Contents

List of Abbreviations

LLC	Linear Local Connectivity
LJC	Level Jordan Curve property
LQC	Level Quasicircle property
LCA	Level Chord-Arc property

List of Symbols

$\lvert x\rvert, \lvert x-y\rvert$	the Euclidean norm and metric, respectively, in $\mathbb{R}^n$.
$\operatorname{dist}(a,B)$	$=\inf\{d(a,b)\colon B\in b\}]$ the distance of a point $a\in X$ from a set $B\subset X$ in a metric space (X,d).
$\operatorname{dist}(A,B)$	$=\inf\{d(a,b)\colon a\in A, b\in B\}$ the distance between two sets A,B in a metric space (X,d).
$\operatorname{diam} E$	$=\sup\{d(x,y)\colon x,y\in E\}$ the diameter of a set E in a metric space (X,d).
∂E, $\overline{E}$	the boundary and closure, respectively, of a set E in a metric space (X,d).
$B(x,r)$	$=\{y\in X\colon d(x,y)<r\}$, a ball of radius $r>0$ centered at a point x in a metric space (X,d).
$\overline{B}(x,r)$	$=\{y\in X\colon d(x,y)\le r\}$, a closed ball of radius $r>0$ centered at a point x in a metric space (X,d).
$B^n(x,r)$	$=\{y\in\mathbb{R}^n\colon \lvert x-y\rvert=r\}$
$S^{n-1}(x,r)$	$=\partial B^n(x,r)$, a ball and sphere respectively, of radius $r>0$ centered at a point $x\in\mathbb{R}^n$.
$\mathbb{B}^n$	$=\{x\in\mathbb{R}^n\colon \lvert x\rvert\le 1\}$, the unit ball in $\mathbb{R}^n$
$\mathbb{S}^{n-1}$	$=\partial\mathbb{B}^n$, the unit sphere in $\mathbb{R}^n$.
$\mathbb{R}^n_+, \mathbb{R}^n_-$	the upper and lower half-plane respectively.
$\pi(a)$	$=(a_1,\ldots,a_{n-1},0)$ the projection of a point $a=(a_1,\ldots,a_{n-1},a_n)$ on $\mathbb{R}^{n-1}\times\{0\}$.
$X\times\{a\}$	$=\{(x,a)\in\mathbb{R}^{n+1}\colon x\in X\}$ when $a\in\mathbb{R}$ and $X\subset\mathbb{R}^n$.
$[x,y]$	the line segment joining $x,y\in\mathbb{R}^n$ and $[x,y]=(x,y)\cup\{x,y\}$.
$l_{x,y}$	the infinite straight line passing through x,y.
$\gamma(x,y)$	the subarc of a Jordan curve γ connecting $x,y\in\gamma$ of smaller diameter, or, either subarc when both subarcs have the same diameter.
$\gamma'(x,y)$	the subarc of a Jordan curve γ connecting $x,y\in\gamma$ of shorter length, or, either subarc when both subarcs have the same length.
$\ell(\gamma)$	the length of a curve γ.
$u\lesssim v$	means that there is a constant $C>0$ such that $u\le Cv$. The constant, which we call comparison constant, may vary but is made clear.
$u\simeq v$	means that there is a constant $C>0$ such that $\frac{v}{C}\le u\le Cv$. The constant, which we call comparison constant, may vary but is made clear.

$\mathcal{F}$	the set of all self homeomorphisms φ of $[0,+\infty)$.
$\mathcal{F}_1$	the set of all self homeomorphisms φ of $[0,+\infty)$ that are Lipschitz in every interval $[\epsilon,+\infty)$ and $\liminf_{t\to 0}\varphi(t)/t>0$.
$\mathcal{F}_1^*$	the set of all self homeomorphisms φ of $[0,+\infty)$ that are Lipschitz in every interval $[\epsilon,+\infty)$ and are bi-Lipschitz in a neighbourhood of 0.
$\mathcal{F}_2$	the set of all self homeomorphisms φ of $[0,+\infty)$ such that $\limsup_{t\to 1}\frac{\varphi(1)-\varphi(t)}{1-t}<+\infty$ and $\liminf_{t\to 0}\varphi(t)/t>0$.
$\Sigma(\Gamma,\varphi)$	$=\{(x,z)\colon x\in\overline{\Omega}, z=\pm\varphi(\mathrm{dist}(x,\Gamma))\}$ where $\Gamma=\partial\Omega$.
$\Sigma^+(\Gamma,\varphi)$	the intersection of $\Sigma(\Gamma,\varphi)$ with $\mathbb{R}^n_+$.
$\Sigma^-(\Gamma,\varphi)$	the intersection of $\Sigma(\Gamma,\varphi)$ with $\mathbb{R}^n_-$.
$\tilde{\Sigma}(f,\varphi)$	$=\{(x,z)\colon x\in\overline{\Omega}, z=\pm\varphi(1-\lvert f(x)\rvert)\}$.
$\partial_\epsilon A$	$=\{x\in\mathbb{R}^2\colon \mathrm{dist}(x,A)=\epsilon\}$ the ϵ-boundary of A.
γ_ϵ	$=\{x\in\mathbb{R}^2\colon \mathrm{dist}(x,\Gamma)=\epsilon\}$ the ϵ-level set of Γ.
Δ_ϵ	$=\{x\in\mathbb{R}^2\colon \mathrm{dist}^*(x,\Gamma)>\epsilon\}$ if $\epsilon>0$ or $\{x\in\mathbb{R}^2\colon \mathrm{dist}^*(x,\Gamma)<\epsilon\}$ if $\epsilon<0$.
$\mathcal{S}$	a Rohde p-snowflake.
$\mathcal{W}_k$	the set of all words from $\{1,2,3,4\}$ with exactly k letters.
$\mathcal{W}$	the set of all words from $\{1,2,3,4\}$.
$\ell(w)$	the length of a word $w\in\mathcal{W}$.
$\langle w\rangle$	an edge of a polygon towards the construction of $\mathcal{S}$.
k_w	the number of of times that the first polygonal arc in Figure 2.1 has been used towards the construction of $\langle w\rangle$.
(p,k_w)	the coding of a snowflake $\mathcal{S}$.
$\zeta_\Gamma(x,y)$	$=\frac{1}{\lvert x-y\rvert}\sup_{z\in\Gamma(x,y)}\mathrm{dist}(z,l_{x,y})$.
ζ_Γ	$=\lim_{r_0\to 0}\sup_{x,y\in\Gamma,\lvert x-y\rvert\le r_0}\zeta_\Gamma(x,y)$, the chordal flatness of Γ.
γ_ϵ^Λ	$=\{x\in\gamma_\epsilon\colon \mathrm{dist}(x,\Lambda)=\lvert\epsilon\rvert\}$ where $\Lambda\subset\Gamma$.
Γ^λ	$=\{y\in\Gamma\colon \mathrm{dist}(y,\lambda)=\epsilon\}$ where $\lambda\subset\gamma_\epsilon$.
$\arg a$	the argument of $a\in\mathbb{C}$.
$\mathcal{C}_{a,h}(x,v)$	the truncated cone in $\mathbb{R}^n$ with vertex $x\in\mathbb{R}^n$, direction $v\in\mathbb{S}^{n-1}$, height $h>0$ and aperture $a>0$.
$S_\epsilon^+, S_\epsilon^-$	the pieces of $\Sigma^+(\Gamma,\varphi)$, $\Sigma^-(\Gamma,\varphi)$ respectively, such that $\pi(S_\epsilon^+)=\pi(S_\epsilon^-)=\Delta_\epsilon$.
$\mathcal{H}^Q$	the Q-dimensional Hausdorff measure.
$D(x_1,y_1,x_2,y_2)$	a rectangular piece on $\Sigma^+(\Gamma,\varphi)$.
Σ_n	topological n-sphere generated by iterating the Geometric construction on a Jordan curve $\Gamma\subset\mathbb{R}^2$.
$\mathcal{C}^n$	the double cone $\{(x,z)\in\mathbb{B}^{n-1}\times\mathbb{R}\colon \lvert z\rvert<1-\lvert x\rvert\}$.

$\mathcal{D}$	a Whitney decomposition of a domain Ω.
$\mathcal{D}_n,\ \mathcal{E}_n$	fixed Whitney decompositions of $\mathbb{B}^n, \mathcal{C}^n$ respectively.
Ω_n	topological ball in $\mathbb{R}^n$ generated by iterating the Geometric construction on a simply connected domain $\Omega \subset \mathbb{R}^2$.
(p, k_w, Γ_w, f)	a coding of a quasicircle Γ; for precise definition see Section 6.1.
$A(\Gamma_w, c)$	the set of minimal words u such that $\operatorname{diam}\Gamma_{wu} \geq c \operatorname{diam}\Gamma_w$ and $\operatorname{diam}\Gamma_{wui} < c \operatorname{diam}\Gamma_w$ for some $i \in \{1,2,3,4\}$.
$N(\Gamma_w, c)$	the number of elements that $A(\Gamma_w, c)$ has.
$M(\Gamma_w, c)$	$= \frac{1}{\operatorname{diam}\Gamma_w} \sum_{u \in A(\Gamma_w, c)} \operatorname{diam}\Gamma_{wu}$, a local chord-arc index.
$k(n)$	$= k_w$ for any $w \in \mathcal{W}_n$ (in the case that $\mathcal{S}$ is homogeneous).
$\tilde{\Sigma}(\varphi)$	$= \{(x,z) \in \overline{\mathbb{B}^2} \times \mathbb{R} \colon z = \pm\varphi(1-\lvert x\rvert)\}$ a surface obtained by revolving the graph of $\varphi(1-t)$, $t \in [0,1]$ around the vertical axis $\{0\} \times \mathbb{R}$ and reflecting with respect to the horizontal plane $\mathbb{R}^2 \times \{0\}$.
φ_{t_1,t_2}	the graph of the function $\varphi\colon [t_1,t_2] \to \mathbb{R}$.
$\tilde{\Sigma}_n$	topological n-sphere generated by iterating the Analytic construction on a Jordan curve $\Gamma \subset \mathbb{R}^2$.
$B_r,\ S_r$	the disc $\{x \in \mathbb{R}^2 \colon \lvert x\rvert < r\}$ and the circle $\{x \in \mathbb{R}^2 \colon \lvert x\rvert = r\}$ respectively.

Chapter 1

Introduction

A homeomorphism $f\colon D \to D'$ between two domains in $\mathbb{R}^2$ is called conformal if $f \in C^1$ and it satisfies the Cauchy-Riemann equations. An interesting property of conformal maps is that they "map infinitesimal disks to infinitesimal disks". Conformal maps can be defined in higher dimensions as follows. A sense preserving homeomorphism f from a a domain $D \subset \mathbb{R}^n$ into a domain $\mathbb{R}^n$ is conformal if, for each $x \in D$,

$$H_f(x) = \limsup_{r \to 0} \frac{\max\{|f(x) - f(y)|\colon |y - x| = r\}}{\min\{|f(x) - f(y)|\colon |y - x| = r\}} = 1.$$

While there are plenty of conformal maps between planar domains, by Liouville's rigidity theorem, only trivial examples exist in higher dimensions, namely the restrictions of Möbius transformations. This result was first proved under the assumption that $f \in C^4$ by Liouville in 1850, and in its present form by Gehring [12].

Quasiconformal maps are generalizations of conformal maps. A homeomorphism f of a domain D in $\mathbb{R}^n$ into $\mathbb{R}^n$ is said to be K-quasiconformal, $K \geq 1$, if for all $x \in D$,

$$H_f(x) \leq K.$$

Conformal mappings are 1-quasiconformal. Roughly speaking, quasiconformal mappings map "infinitesimal balls" to "infinitesimal ellipsoids of uniformly bounded eccentricity". The above definitions of quasiconformal mappings is known as the *metric definition*. Equivalent definitions of quasiconformality have been given in terms of the conformal modulus of curve families [39, Theorem 34.1], and of the distortion $|Df(x)|^n/|J_f(x)|$ [39, Theorem 34.6]. Here, Df denotes the formal differential matrix and J_f the Jacobian of f.

Planar quasiconformal mappings first appeared in 1932 in a paper of Grötzsch [19] and under this name in a paper of Ahlfors [1]. Since then, they have been one of the most important developments in Complex Analysis. A rigorous treatment of quasiconformal mappings in higher dimensions can be found in [39].

Quasisymmetric mappings are generalizations of quasiconformal mappings to abstract metric spaces. These mappings first appeared in $\mathbb{R}$ as the boundary values of quasiconformal self maps of the upper half-plane $\mathbb{R}^2_+$ in a paper of Beurling and Ahlfors [4], and were first studied in general metric spaces by

Tukia and Väisälä in [37]. An embedding f of a metric space (X, d) into a metric space (Y, d') is called η-quasisymmetric if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that, for all $x, a, b \in X$ and $t > 0$ with $d(x, a) \le td(x, b)$,

$$d'(f(x), f(a)) \le \eta(t) d'(f(x), f(b)).$$

Quasisymmetric mappings distort relative distances by a bounded amount.

It follows from its definition that an η-quasisymmetric embedding of a domain D in $\mathbb{R}^n$ into $\mathbb{R}^n$ is $\eta(1)$-quasiconformal. Quasisymmetry is a global notion while quasiconformality is an infinitesimal one. A K-quasiconformal mapping defined on a domain D in $\mathbb{R}^n$ is η-quasisymmetric on each compact set $G \subset D$ with η depending on K, n and $\operatorname{diam} G/\operatorname{dist}(G, \partial D)$ quantitatively [40, Theorem 2.4]. In $\mathbb{R}^n$ these two notions coincide: if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal then it is $\eta_{K,n}$-quasisymmetric. For the basic theory of quasisymmetric mappings, the reader is referred to [37] and [20].

1.1 Quasispheres and quasisymmetric spheres

An n-dimensional quasisphere is the image of the unit sphere $\mathbb{S}^n$ under a quasiconformal self map of $\mathbb{R}^{n+1}$. Unlike the case $n = 1$, where various characterizations of quasicircles have been found, little is known of quasispheres in higher dimensions. The only known characterization, until today, is due to Gehring [13] ,for $n = 2$, and Väisälä [41], for $n \ge 3$; a metric n-sphere Σ in $\mathbb{R}^{n+1}$ is a quasisphere if and only if the bounded component of $\mathbb{R}^{n+1} \setminus \Sigma$ is quasiconformally homeomorphic to $\mathbb{B}^{n+1}$ and the unbounded component of $\mathbb{R}^{n+1} \setminus \Sigma$ is quasiconformally homeomorphic to $\mathbb{R}^{n+1} \setminus \overline{\mathbb{B}^{n+1}}$.

A consequence of an extension theorem of Väisälä [42] is that a smooth metric n-sphere in $\mathbb{R}^{n+1}$ is a quasisphere. However, quasispheres can have interesting fractal type structure.

Several examples of fractal-like quasispheres have been found. Bishop [5] constructed quasispheres in $\mathbb{R}^3$ that contain no rectifiable curves. The same year, David and Toro [8] constructed quasispheres in $\mathbb{R}^{n+1}$ with some snowflake property: for each positive α sufficiently close to 0, they constructed a quasiconformal self map f of $\mathbb{R}^3$ satisfying $|f(x) - f(y)| \simeq |x - y|^{1-\alpha}$ for all $x, y \in \mathbb{R}^2 \times \{0\}$ with $|x - y| < 1$. Lewis and Vogel [26] built quasispheres in $\mathbb{R}^{n+1}$, for each $n \ge 2$, which are *pseudospheres*; that is, non-trivial metric n-spheres Σ, for which the n-dimensional Hausdorff measure equals the harmonic measure on Σ with respect to a fixed point. Meyer [30, 31] introduced *snowballs* in $\mathbb{R}^3$, domains whose boundaries are fractal type surfaces analogue to that of von Koch snowflakes in $\mathbb{R}^2$, and proved that their boundaries are quasispheres. Recently, Pankka and Wu [32] showed that the decomposition spaces of $\mathbb{S}^3$ by Antoine's necklaces constructed using long chains of tori, when equipped with a Semmes-type metric, are quasispheres in $\mathbb{R}^4$.

The search for intrinsic necessary and sufficient conditions for quasispheres remains a longstanding problem in the study of Geometric Analysis. At the moment, a geometric characterization of quasispheres seems to be out of reach.

Quasisymmetric spheres generalize quasispheres beyond the Euclidean setting. A metric n-sphere is defined to be a quasisymmetric n-sphere if it is quasisymmetric to $\mathbb{S}^n$. In $\mathbb{R}^2$ quasisymmetric circles are exactly the quasicircles; while in $\mathbb{R}^{n+1}$, for $n \geq 2$, the notion of quasisymmetric spheres is weaker than that of quasispheres. Complete characterizations of quasisymmetric spheres have been given in dimension 1 by Tukia and Väisälä [37] and in dimension 2 by Bonk and Kleiner [6]. There is no known characterization of quasisymmetric n-spheres for $n \geq 3$.

A necessary property for quasisymmetric spheres, and hence for quasispheres, is *linear local contractibility*. Moreover, a consequence of the main theorem in [6] states that a metric space which is a metric 2-sphere (topological condition), linearly locally contractible (geometric condition) and Ahlfors 2-regular (measure theoretic condition) is a quasisymmetric sphere. However, in $\mathbb{R}^4$, Semmes [34] constructed a metric 3-sphere which is linearly locally contractible and Ahlfors 3-regular but admits no quasisymmetric parametrization. Examples that satisfy the above conditions but fail to be quasisymmetric spheres have been constructed by Heinonen and Wu [21] in $\mathbb{R}^n$ for all dimensions $n \geq 4$.

The motivation of this thesis comes from the work of Gehring on slit domains [15] and the work of Väisälä on products of curves [43] and on cylindrical domains [44]. Gehring proved that if Ω is a planar domain then $\mathbb{R}^3 \setminus \overline{\Omega}$ is quasiconformally homeomorphic to the exterior of the unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ if and only if Ω is a quasidisk. Väisälä proved that the product of a simple arc Γ with an interval I is quasisymmetric embeddable into $\mathbb{R}^2$ if and only if Γ satisfies the chord-arc condition (2.2.2). Finally, Väisälä [44] showed among other results that if Ω is a Jordan domain in $\mathbb{R}^2$ then the cylindrical domain $\Omega \times \mathbb{R}$ is quasiconformally homeomorphic to the unit ball $\mathbb{B}^3$ if and only if $\partial\Omega$ satisfies the chord-arc condition (2.2.2).

In this thesis, we examine double-dome-like surfaces in $\mathbb{R}^3$ defined by the graphs of two functions on a Jordan domain $\Omega \subset \mathbb{R}^2$. If the height above each point of Ω is constant zero, then the exterior of the surface resembles a slit domain. On the other hand, a cylindrical domain occurs if the height above each point of Ω is infinite.

1.2 Geometric Construction

Define

$$\mathcal{F} = \{\varphi\colon [0, +\infty) \to [0, +\infty)\ ,\ \varphi \text{ is a homeomorphism}\}.$$

In the first construction, to which we refer as the *Geometric construction*, the height of the surface above and below a point $x \in \overline{\Omega}$ is a function of the distance of x from the boundary $\partial\Omega$. Precisely, if $\Gamma = \partial\Omega$ and $\varphi \in \mathcal{F}$, then define the 2-dimensional surface

$$\Sigma(\Gamma, \varphi) = \{(x, z) \colon x \in \overline{\Omega}, z = \pm\varphi(\operatorname{dist}(x, \Gamma))\}.$$

Our aim is to find the right conditions on the geometry of the base Ω and the growth of the gauge φ in order for these surfaces to be quasispheres, or quasisymmetric spheres, or bi-Lipschitz equivalent to $\mathbb{S}^2$.

In terms of this setting, a slit domain $\mathbb{R}^3 \setminus \Omega$ may be regarded as the complement of $\Sigma(\partial\Omega, \varphi)$ when $\varphi \equiv 0$, and a cylindrical domain $\Omega \times \mathbb{R}$ may be regarded as the domain enclosed by $\Sigma(\partial\Omega, \varphi)$ by choosing $\varphi \equiv \infty$.

For the gauge functions of the form $\varphi(t) = t^\alpha$ with $\alpha \in (0, 1)$, the surface $\Sigma(\Gamma, \varphi)$ near $\Gamma \times \{0\}$ resembles $\Gamma \times I$ for some small interval I. In view of Väisälä's result on products of curves [43], one expects that $\Sigma(\Gamma, t^\alpha)$ is quasisymmetric to $\mathbb{S}^2$ if and only if Γ is a chord-arc curve. As it turns out, the chord-arc property is necessary but not sufficient.

The geometry of the *level sets* γ_ϵ of Γ plays a key role in the properties of $\Sigma(\Gamma, \varphi)$. For $\epsilon > 0$, the ϵ-level set of Γ is defined to be

$$\gamma_\epsilon = \{x \in \Omega \colon \operatorname{dist}(x, \Gamma) = \epsilon\}.$$

A Jordan curve Γ is said to satisfy the *level chord-arc property* (or LCA *property*), if there exist $\epsilon_0 > 0$ and $C \geq 1$ such that γ_ϵ is a C-chord-arc curve for every $0 \leq \epsilon \leq \epsilon_0$.

The behaviour of φ near 0 is crucial in the behaviour of $\Sigma(\Gamma, \varphi)$. More specifically, if φ satisfies $\liminf_{t\to 0} \varphi(t)/t = 0$ then, for any Jordan curve Γ, the surface $\Sigma(\Gamma, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$; see Theorem 4.4.2. Therefore, we consider the following sub-collection of $\mathcal{F}$ in the study of these surfaces:

$$\mathcal{F}_1 = \{\varphi \in \mathcal{F} \colon \liminf_{t\to 0} \varphi(t)/t > 0 \text{ and } \varphi \text{ is Lipschitz in } [r, +\infty) \text{ for all } r > 0\}.$$

The main result from this construction is the following theorem.

Theorem 1.2.1. *Let Γ be a Jordan curve.*

1. *If Γ has the level chord-arc property then $\Sigma(\Gamma, \varphi)$ is a quasisymmetric sphere for all functions φ in $\mathcal{F}_1$.*
2. *If Γ does not have the level chord-arc property, then there exists a function φ in $\mathcal{F}_1$ which satisfies $\lim_{t\to 0} \varphi(t)/t = +\infty$ such that $\Sigma(\Gamma, \varphi)$ is not a quasisymmetric sphere.*

The assumption that φ is Lipschitz away from zero is necessary for the first claim; see Remark 4.4.1.

If $\varphi \in \mathcal{F}_1$ is bi-Lipschitz in a neighbourhood of 0 (e.g. $\varphi(t) = t$) and Γ is a quasicircle, the surface $\Sigma(\Gamma, \varphi)$ is a quasisphere. A partial converse is also true.

Theorem 1.2.2. *Suppose that Γ is a Jordan curve.*

1. *If Γ is a quasicircle then $\Sigma(\Gamma, \varphi)$ is a quasisphere for every $\varphi \in \mathcal{F}_1$ which is bi-Lipschitz in a neighbourhood of 0.*

2. *If $\Sigma(\Gamma, \varphi)$ is a quasisymmetric sphere for some $\varphi \in \mathcal{F}_1$ which is bi-Lipschitz in a neighbourhood of 0 then Γ is a quasicircle.*

The method of constructing quasispheres in Theorem 1.2.2 produces quasispheres in all dimensions; see Theorem 5.0.5.

If Γ is not a chord-arc curve then it does not satisfy the LCA property and, by Theorem 1.2.1, $\Sigma(\Gamma, \varphi)$ is not a quasisymmetric sphere for some $\varphi \in \mathcal{F}_1$ with $\lim_{t \to \infty} \varphi(t)/t = \infty$. If moreover we assume that Γ has *Assouad dimension* larger than 1 (see Chapter 6 for definition) then we have the following result.

Theorem 1.2.3. *Let Γ be a quasicircle with Assouad dimension larger than 1. Then, for any $\alpha \in (0, 1)$, the surface $\Sigma(\Gamma, t^\alpha)$ is not a quasisymmetric sphere.*

Since the Assouad dimension is larger than the Hausdorff, upper box counting and lower box-counting dimensions, the conclusion of Theorem 1.2.3 holds if any of these dimensions is bigger than 1.

1.3 Level sets of Jordan curves and chordal flatness

Sufficient conditions for a curve to satisfy the LCA property can be given in terms of the following flatness module. For $x, y \in \Gamma$ denote with $\Gamma(x, y)$ the subarc of Γ connecting x and y that has a smaller diameter and denote with $l_{x,y}$ the infinite line passing through x, y. Define

$$\zeta_\Gamma(x, y) = \frac{1}{|x - y|} \sup_{z \in \Gamma(x,y)} \operatorname{dist}(z, l_{x,y}).$$

Under this module of flatness, we prove the following result in Theorem 3.0.4, Theorem 3.0.5 and Theorem 3.0.6.

Theorem 1.3.1. *Suppose that Γ is a Jordan curve.*

1. *If there exists $r_0 > 0$ such that $\zeta_\Gamma(x, y) \le 1/2$ for all $x, y \in \Gamma$ with $|x - y| \le r_0$ then, there is $\epsilon_0 > 0$ depending on r_0 such that for each $\epsilon \in (0, \epsilon_0)$, the level set γ_ϵ is a Jordan curve.*

2. *If there exists $r_0 > 0$ and $\zeta_0 \in (0, 1/2)$ such that $\zeta_\Gamma(x, y) \le \zeta_0$ for all $x, y \in \Gamma$ with $|x - y| \le r_0$ then, there is $\epsilon_0 > 0$ depending on r_0 and K depending on r_0, ζ_0 such that for each $\epsilon \in (0, \epsilon_0)$, the level set γ_ϵ is a K-quasicircle.*

3. *The curve Γ has the level chord-arc property if and only if it is a chord-arc curve and for some $K > 1, \epsilon_0 > 0$ each level set γ_ϵ, $\epsilon \in (0, \epsilon_0)$ is a K-quasicircle.*

The number $1/2$ in the first two claims is sharp; see Remark 3.3.1 and Remark 3.3.2. For the LCA property we have the following sharp result.

Theorem 1.3.2. *If Γ satisfies a local C-chord-arc property with $1 \le C < \frac{\pi}{2}$, then Γ has the level chord-arc property. Moreover, there exists a Jordan curve Γ satisfying a local $\frac{\pi}{2}$-chord-arc property that does not have the level chord-arc property.*

1.4 Analytic Construction

In the second construction, to which we refer as the *Analytic construction*, the level sets of the surface are the level sets of $|f|$, where f is a quasiconformal mapping that maps Ω onto the unit disk. More precisely, suppose that f is a quasiconformal mapping that maps Ω onto $\mathbb{B}^2$. For a function $\varphi \in \mathcal{F}$, define the surface

$$\tilde{\Sigma}(f, \varphi) = \{(x, z) \colon x \in \overline{\Omega}, z = \pm\varphi(1 - |f(x)|)\}.$$

As with the Geometric construction, the behaviour of φ near zero is crucial to the behaviour of $\tilde{\Sigma}(f, \varphi)$. Define the following sub-collection of $\mathcal{F}$:

$$\mathcal{F}_2 = \{\varphi \in \mathcal{F} \colon \liminf_{t \to 0} \varphi(t)/t > 0 \text{ and } \limsup_{t \to 1} \frac{\varphi(1) - \varphi(t)}{1 - t} < \infty\}.$$

Note that $\mathcal{F}_1 \subset \mathcal{F}_2$. The second limit condition of $\mathcal{F}_2$ is satisfied by all functions φ which are locally Lipschitz at 1; hence all functions φ in $\mathcal{F}_1$.

The main result from this construction is the following theorem.

Theorem 1.4.1. *Suppose that Ω is a Jordan domain and $\Gamma = \partial\Omega$.*

1. *If Γ is a chord-arc curve then for any bi-Lipschitz mapping f from Ω onto $\mathbb{B}^2$ and for any $\varphi \in \mathcal{F}_2$, the surface $\tilde{\Sigma}(f, \varphi)$ is the image of $\mathbb{S}^2$ under a bi-Lipschitz self map of $\mathbb{R}^3$.*

2. *If Γ is not a chord-arc curve then for any quasiconformal mapping f from Ω onto $\mathbb{B}^2$ there exists a function $\varphi \in \mathcal{F}_2$ satisfying $\lim_{t\to 0} \varphi(t)/t = \infty$ such that $\tilde{\Sigma}(f,\varphi)$ is not a quasisymmetric sphere.*

A stronger version of the second claim of Theorem 1.4.1 is proved in Proposition 7.2.1. The necessity of the limit conditions in $\mathcal{F}_2$ for the first claim of Theorem 1.4.1 is illustrated in Remark 7.1.5.

1.5 Outline of the thesis

The thesis is organized as follows.

In Section 2.1 we establish our notation and in Section 2.2 we present some concepts of Geometric Analysis that appear throughout the thesis. The linear local connectivity (or LLC property) of metric spaces is discussed in Section 2.3 and the theory of Rohde's snowflakes is discussed in Section 2.4.

Chapter 3 deals with the theory of the level sets of Jordan curves and the proofs of Theorem 1.3.1 and Theorem 1.3.2. More precisely, in Section 3.1 we define the chordal flatness of Jordan curves and we compare it with other notions of flatness. The geometry of the level sets is investigated in Section 3.2. The main theorems of this chapter are proved in Section 3.3 and Section 3.4. In Section 3.5 we give some applications of Theorem 1.3.1 to Rohde's snowflakes.

Chapter 4 is devoted to the proof of Theorem 1.2.1. The proof requires the notions of linear local connectivity and Ahlfors 2-regularity. In Section 4.1 we establish necessary and sufficient conditions for $\Sigma(\Gamma,\varphi)$ to be linearly locally connected . These conditions are given in terms of the geometry of the level sets γ_ϵ. The Ahlfors 2-regularity is discussed in Section 4.2. A method of Väisälä is employed in Section 4.3 to complete the proof of Theorem 1.2.1, which is given in Section 4.4 along with additional remarks.

A multidimensional version of Theorem 1.2.2 is proved in Chapter 5. In Section 5.1, we compare the *Whitney decomposition* of the base Ω and that of the domain enclosed by $\Sigma(\Gamma,\varphi)$, and in Section 5.3 we discuss the *slit domains*. The proof of the main result Theorem 5.0.5 is given in Section 5.4.

The proof of Theorem 1.2.3 is given in Chapter 6. The structure of Rohde's snowflakes enables us to use combinatorial arguments.

Finally, the Analytic construction is presented in Chapter 7. We show the first claim of Theorem 1.4.1 in Section 7.1 and the second claim in Section 7.2.

Chapter 2

Preliminaries

2.1 Notation

The following notation is used throughout the thesis. If $x, y \in \mathbb{R}^n$, then $|x - y|$ denotes their Euclidean distance. For a set $E \subset \mathbb{R}^n$, we denote with ∂E its boundary and with $\overline{E}$ its closure in $\mathbb{R}^n$.

If $x \in \mathbb{R}^n$ and $r > 0$, let $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ and $S^{n-1}(x_0, r) = \partial B^n(x_0, r)$. In particular, we write $\mathbb{B}^n = B^n(0, 1)$ and $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$, the unit ball and sphere, respectively, in $\mathbb{R}^n$. Balls in abstract metric spaces (X, d) are denoted with $B(x, r)$. In addition, let $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ be the open upper, lower respectively, half-space of $\mathbb{R}^n$. For any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ let

$$\pi(a) = (a_1, \dots, a_{n-1}, 0)$$

be the projection of a on $\mathbb{R}^{n-1} \times \{0\}$.

A metric n-sphere is a metric space homeomorphic to the standard unit sphere $\mathbb{S}^n$. Metric 1-spheres are also called metric circles. A Jordan curve is a metric circle in $\mathbb{R}^2$. A Jordan domain is a bounded domain in $\mathbb{R}^2$ whose boundary is a Jordan curve.

For $x, y \in \mathbb{R}^n$, define $[x, y]$ to be the line segment that has x, y as its endpoints and $l_{x,y}$ to be the straight line passing from the points x, y. Given two points x, y on a Jordan curve γ, we denote with $\gamma(x, y)$ the subarc of γ connecting x and y that has a smaller diameter, or, to be either subarc when both subarcs have the same diameter. In same fashion, we denote with $\gamma'(x, y)$ the subarc of a rectifiable curve γ connecting x and y that has a shorter length, or, to be either subarc when both subarcs have the same length. The length of a curve γ is denoted by $\ell(\gamma)$.

If X is a subset of $\mathbb{R}^n$ and $a \in \mathbb{R}$ then, we denote with $X \times \{a\}$ the set $\{(x, a) \in \mathbb{R}^{n+1} : x \in X\}$. We often identify the plane $\mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$.

In the following, we write $u \lesssim v$ (resp. $u \simeq v$) when the ratio u/v is bounded above (resp. bounded above and below) by positive constants. The constants, which we refer to as the *comparison constants*, may vary but are made clear.

2.2 Background in Geometric Analysis

How can we decide whether two domains D, D' in $\mathbb{R}^n$ are quasiconformally equivalents? Furthermore, how can we decide if two metric spaces (X, d), (Y, d') are quasisymmetric equivalent? Both problems are still open and only partial answers exist, even when D, D' are topological balls or X, Y are metric spheres.

Definition 2.2.1. *A Jordan curve Γ in $\mathbb{R}^2$ is called a K-quasicircle if it is the image of $\mathbb{S}^1$ under a K-quasiconformal self map of $\mathbb{R}^2$. A metric n-sphere $\Sigma \subset \mathbb{R}^{n+1}$ is called a K-quasisphere if it is the image of $\mathbb{S}^n$ under a K-quasiconformal self map of $\mathbb{R}^{n+1}$. A quasidisk is a domain enclosed by a quasicircle.*

A metric space X is quasisymmetric to a metric space Y if there exists a quasisymmetric homeomorphism from X onto Y.

Definition 2.2.2. *A metric n-sphere is called a quasisymmetric sphere if it is quasisymmetric to $\mathbb{S}^n$.*

Beurling and Ahlfors [4] showed that a curve Γ is a quasicircle if and only if there is a quasisymmetric map $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ such that $f(\mathbb{S}^1) = \Gamma$. Therefore, quasicircles are planar quasisymmetric circles. Quasispheres in $\mathbb{R}^{n+1}$, $n \geq 2$, are quasisymmetric spheres [40, Theorem 2.4] but the converse is not always true [40, Theorem 6.3].

A geometric characterization of quasicircles was given by Ahlfors [2] who proved that a Jordan curve Γ is a quasicircle if and only if it satisfies the *2 points condition*:

$$\text{there exists } C > 1 \text{ such that for all } x, y \in \gamma,\ \operatorname{diam} \gamma(x, y) \leq C|x - y|, \tag{2.2.1}$$

where the distortion K and the constant C are quantitatively related. Rohde [33] constructed a catalogue of snowflake curves and proved that, up to bi-Lipschitz self maps of $\mathbb{R}^3$, all quasicircles are these snowflake curves; see Section 2.4. A long list of remarkably diverse characterizations of quasicircles has been found. See the monograph of Gehring [17] for informative discussion.

Quasisymmetric circles were first classified by Tukia and Väisälä [37] who showed that a metric circle is a quasisymmetric circle if and only if it is *doubling and of bounded turning*. Doubling spaces are those with finite Assouad dimension; see Chapter 6. The bounded turning property is essentially Ahlfors condition (2.2.1) for abstract metric circles. Herron and Meyer [22] produced a catalogue of snowflake type metric Jordan curves, similar to that of Rohde, that describes all quasisymmetric circles up to bi-Lipschitz equivalence.

On the other hand, there are few known characterizations for high dimensional quasispheres or quasisymmetric spheres. A necessary and sufficient condition for a metric n-sphere to be quasisphere was first given by Gehring [13] for $n = 2$ and later by Väisälä [41] for $n \geq 3$.

Theorem 2.2.3 ([41, Theorem 5.9], [13, Theorem]). *Let $n \geq 2$ and let Σ be a metric n-sphere in $\mathbb{R}^{n+1}$. Suppose that the bounded, unbounded component of $\mathbb{R}^{n+1} \setminus \Sigma$ is K-quasiconformally homeomorphic to $\mathbb{B}^{n+1}$, $\mathbb{R}^{n+1} \setminus \overline{\mathbb{B}^{n+1}}$ respectively. Then, there exists a K'-quasiconformal self map of $\mathbb{R}^{n+1}$ that maps Σ onto $\mathbb{S}^n$, with K' depending on K and n.*

Note that this theorem is false when $n = 1$.

Bonk and Kleiner [6] identified a necessary and sufficient condition for metric 2-spheres to be quasisymmetric spheres. A consequence of their main theorem is the following result.

Theorem 2.2.4 ([6, Theorem 1.1, Lemma 2.5]). *Let Z be metric 2-sphere which is Ahlfors 2-regular. Then Z is quasisymmetric to $\mathbb{S}^2$ if and only if Z is LLC.*

The LLC property is discussed in Section 2.3 and the definition of Ahlfors 2-regularity can be found in Section 4.2.

Definition 2.2.5. *A domain $D \subset \mathbb{R}^n$ is called a quasiball if it is quasiconformally homeomorphic to $\mathbb{B}^n$.*

By the Riemann mapping theorem, the quasiballs in $\mathbb{R}^2$ are exactly the bounded simply connected domains in $\mathbb{R}^2$. For $n > 2$, no geometric characterization of quasiballs exists. Strong linear local connectivity is a necessary condition for a domain $D \in \mathbb{R}^n$, $n \geq 3$ to be quasiball [18]; see Section 2.3. The following result of Gehring shows that one can characterize quasiballs D in terms of the part of D close to ∂D.

Theorem 2.2.6 ([14, Theorem 2]). *A domain D in $\mathbb{R}^n$, $n \geq 3$, is quasiconformally equivalent to $\mathbb{B}^n$ if there exists a neighbourhood U of ∂D and a quasiconformal mapping g of $D \cap U$ into $\mathbb{B}^n$ such that $g(x) \to \mathbb{S}^{n-1}$ as $x \to \partial D$ in $D \cap U$.*

Definition 2.2.7. *An mapping f of a metric space (X, d) into a metric space (Y, d') is called L-Lipschitz, $L \geq 1$, if for all $x_1, x_2 \in X$*

$$d'(f(x_1), f(x_2)) \leq Ld(x_1, x_2).$$

An embedding f of a metric space (X, d) into a metric space (Y, d') is called L-bi-Lipschitz, $L \geq 1$, if both f, f^{-1} are $L - Lipschitz$.

An L-bi-Lipschitz mapping is η-quasisymmetric with $\eta(t) = L^2 t$. The difference between quasisymmetric maps and bi-Lipschitz maps is that the latter distort absolute distances by a bounded amount.

A metric space X is bi-Lipschitz to a metric space Y if there exists a bi-Lipschitz homeomorphism from X onto Y.

Definition 2.2.8. *A Jordan curve Γ in $\mathbb{R}^2$ is said to be a L-bi-Lipschitz curve if it is the image of $\mathbb{S}^1$ under an L-bi-Lipschitz self map of $\mathbb{R}^2$. A metric n-sphere $\Sigma \subset \mathbb{R}^{n+1}$ is called an L-bi-Lipschitz sphere if it is the image of $\mathbb{S}^n$ under an L-bi-Lipschitz self map of $\mathbb{R}^{n+1}$.*

Definition 2.2.9. *A rectifiable Jordan curve Γ in $\mathbb{R}^2$ is called a C-chord-arc curve, $C \geq 1$, if, for any two points $x, y \in \Gamma$, the length of the shorter component $\Gamma'(x, y)$ of $\Gamma \setminus \{x, y\}$ satisfies*

$$\ell(\Gamma'(x, y)) \leq C|x - y|. \tag{2.2.2}$$

It follows from its definition that an L-bi-Lipschitz curve is an L^2-chord-arc curve. Conversely, Jerison and Kenig [23] proved that if Γ is a C-chord-arc curve, then there exists $L \geq 1$ depending only on C such that Γ is an L-bi-Lipschitz curve. Moreover, a theorem by Tukia [36] implies that a curve Γ is a bi-Lipschitz curve if and only if there is a bilipschtz map $f\colon \mathbb{S}^1 \to \mathbb{R}^2$ with $f(\mathbb{S}^1) = \Gamma$.

Intrinsic characterizations of n-dimensional bi-Lipschitz spheres in $\mathbb{R}^{n+1}$, $n \geq 2$, are unknown. A necessary condition is Ahlfors n-regularity.

2.3 The LLC property

A natural extension of Ahlfors condition (2.2.1) to general metric spaces is the *linear local connectivity*, or *LLC* property.

Definition 2.3.1. *A metric space X is $\lambda - LLC$ for $\lambda \geq 1$ if the following two conditions are satisfied.*

1. *($\lambda - LLC_1$) If $x \in X$, $r > 0$ and $y_1, y_2 \in B(x, r) \cap X$, then there exists a continuum $E \subset B(x, \lambda r) \cap X$ containing y_1, y_2.*

2. *($\lambda - LLC_2$) If $x \in X$, $r > 0$ and $y_1, y_2 \in X \setminus B(x, r)$, then there exists a continuum $E \subset X \setminus B(x, r/\lambda)$ containing y_1, y_2.*

Recall that a continuum is a compact connected set consisting of more than one point.

It is an easy consequence of the definition that the LLC property is invariant under quasisymmetric mappings. As a result, *the LLC property is necessary for quasisymmetric spheres and quasispheres.*

Lemma 2.3.2. *Suppose that X is λ-LLC and f is an η-quasisymmetric mapping of X onto Y. Then Y is λ'-LLC for some λ' depending on λ, η.*

Proof. Let $\eta'(t) = \frac{1}{\eta^{-1}(1/t)}$. Recall that $f^{-1} \colon Y \to X$ is η'-quasisymmetric.

To verify the LLC_1 property take $y \in Y$, $r > 0$ and $y_1, y_2 \in B(y, r)$. Without loss of generality we may assume that $|y - y_2| \leq |y - y_1|$. Let x, x_1, x_2 be the preimages of y, y_1, y_2 respectively. Then, $|x_1 - x_2| \leq \eta'(2)|x - x_1|$. Since X is $\lambda-\mathrm{LLC}_1$, there exists a continuum $E \subset B(x, \lambda\eta'(2)|x - x_1|)$ containing x_1, x_2. Then, $f(E)$ is a continuum containing y_1, y_2. Furthermore, for each $w \in f(E)$, since $|f^{-1}(w) - x| \leq \lambda\eta'(2)|x - x_1|$, it follows that $|w - y| \leq \eta(\lambda\eta'(2))|y - y_1| \leq \eta(\lambda\eta'(2))r$.

To verify the LLC_2 property take $y \in Y$, $r > 0$ and $y_1, y_2 \in Y \setminus B(y, r)$. Let x, x_1, x_2 be the preimages of y, y_1, y_2 respectively. Without loss of generality, we may assume that $|y - y_2| \leq |y - y_1|$. Then, $|x - x_2| \leq \eta'(1)|x - x_1|$. Note that x_1, x_2 are outside of the ball $B(x, \frac{1}{2\eta'(1)}|x - x_2|)$. Since X is $\lambda - \mathrm{LLC}_2$, there exists a continuum $E \subset X \setminus B(x, \frac{1}{2\eta'(1)\lambda}|x - x_1|)$ containing x_1, x_2. Then, $f(E)$ is a continuum containing y_1, y_2. Furthermore, for each $w \in f(E)$, since $|f^{-1}(w) - x| \geq \frac{1}{2\eta'(1)\lambda}|x - x_2|$, it follows that $|w - y| \geq \frac{1}{\eta(2\eta'(1)\lambda)}|y - y_2| \geq \frac{1}{\eta(2\eta'(1)\lambda)} r$. □

The importance of linearly locally connected sets was first observed by Gehring and Väisälä [18] and the term first appeared as *strongly locally connected sets*, in a paper of Gehring [13]. In the latter, a set $X \subset \mathbb{R}^n$ is said to be λ-*strongly linearly locally connected* ifs conditions (1) and (2) in Definition 2.3.1 are satisfied for all $x \in \mathbb{R}^n$ instead of only $x \in X$.

Walker [46, Corollary 5.10] showed that strong linear local connectivity is invariant under quasiconformal mappings. The next remark follows from Walker's theorem and the fact that the unit circle $S^1(0,1) \times \{0\}$ is strongly linearly locally connected in $\mathbb{R}^3$.

Remark 2.3.3. *If $\Gamma \subset \mathbb{R}^2$ is a K-quasicircle then $\Gamma \times \{0\}$ is λ-strongly linearly locally connected in $\mathbb{R}^3$ for some $\lambda > 1$ depending on K.*

Bonk and Kleiner [6] introduced the stronger notion of *linearly locally contractible* spaces. A metric space is C-linearly locally contractible for some $C > 1$ if every small ball is contractible inside a ball whose radius is C times larger. The authors proved that linear local contractibility implies the LLC property; while for compact and connected topological 2-manifolds, linear local contractibility is equivalent to the LLC property [6, Lemma 2.5].

2.4 Rohde's snowflakes

In [33], Rohde gave an intrinsic characterization of planar quasicircles. He defined explicitly a family $\mathcal{F}$ of snowflake-type curves, then proceeded to prove that every quasicircle in the plane is the image of a member of this family under a bi-Lipschitz self map of $\mathbb{R}^2$.

Each of Rohde's snowflakes $\mathcal{S}$ is constructed as follows. Fix a number $p \in [\frac{1}{4}, \frac{1}{2})$, and let S_1 be the unit square. The polygon S_{n+1} is constructed by replacing each of the 4^n edges of S_n by a rescaled and rotated copy of one of the only two polygonal arcs allowed in Figure 2.1, in such a way that the polygonal regions are expanding. The curve $\mathcal{S}$ is obtained by taking the limit of S_n, just as in the construction of the usual von Koch snowflake.

It is easy to check that every p-snowflake satisfies (2.2.1) for some C depending on p. Therefore, every Rohde snowflake is a quasicircle. On the other hand, every quasicircle has the structure of a p-snowflake.

Theorem 2.4.1 ([33, Theorem 1.1])**.** *A Jordan curve Γ is a quasicircle if and only if there exist a snowflake curve $\mathcal{S}$ and a bi-Lipschitz self map f of $\mathbb{R}^2$ so that $\Gamma = f(\mathcal{S})$.*

We call the polygonal arc on the left, the *Type I arc* and the the figure on the right, the *Type II arc.*

Figure 2.1: The two polygonal arcs allowed in the construction of $\mathcal{S}$

Denote with $\mathcal{W}_k = \{1,2,3,4\}^k$ the set of all words with letters from the set $\{1,2,3,4\}$ of length equal to k and let $\mathcal{W} = \bigcup_{k\in\mathbb{N}} \mathcal{W}_k$. If $w \in \mathcal{W}_k$, set $\ell(w) = k$ to be the length the word.

The structure of $\mathcal{S}$ can be coded in the following way. In the first step of the construction we consider the unit square S_1 and denote with $\langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle$ its four edges in counter clockwise order. Inductively, in the n-th step we replace the edge $\langle w\rangle$ with a polygonal arc consisting of four edges, denoted with $\langle w1\rangle, \langle w2\rangle, \langle w3\rangle, \langle w4\rangle$ in counterclockwise order. Let $\mathcal{S}_w$ be the smallest subarc of $\mathcal{S}$ that has the same endpoints with $\langle w\rangle$.

For each $w \in \mathcal{W}$ let $k_w \in \mathbb{N}$ be the number such that diam $\langle w\rangle = 4^{-\ell(w)}(4p)^{k_w}$. In other words, k_w is the number of times that the Type I arc has been used towards the construction of $\langle w\rangle$. The pair (p, k_w) is *the coding of* $\mathcal{S}$.

A p-snowflake $\mathcal{S}$ is called *homogeneous* if, during the construction of $\mathcal{S}$, *all* of the 4^n line segments of the n-th generation are replaced by *the same* (rescaled and rotated) polygonal arc of Figure 2.1. In other words, k_w depends only on $\ell(w)$.

Chapter 3

Sets of constant distance from a Jordan curve

Let A be a compact subset of $\mathbb{R}^2$. For each $\epsilon > 0$, define the *ϵ-boundary* of A to be the set

$$\partial_\epsilon(A) = \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, A) = \epsilon\}.$$

Brown showed in [7] that for all but countably many ϵ, every component of $\partial_\epsilon(A)$ is a point, a simple arc, or a simple closed curve. In [9], Ferry showed, among other results, that $\partial_\epsilon(A)$ is a 1-manifold for almost all ϵ. Fu [10] generalized Ferry's results, and proved that for all ϵ outside a compact set of zero 1/2-dimensional Hausdorff measure, $\partial_\epsilon(A)$ is a Lipschitz 1-manifold. Papers [9] and [10] include theorems in higher dimensional Euclidean spaces; the work for dimensions $n \geq 3$ is more demanding.

Let Γ be a Jordan curve in $\mathbb{R}^2$ and Ω be the bounded component of $\mathbb{R}^2 \setminus \Gamma$. In this chapter we investigate the geometry of the part of $\partial_\epsilon\Gamma$ which is contained in Ω. The theory can easily be extended to the part of $\partial_\epsilon\Gamma$ contained in $\mathbb{R}^2 \setminus \overline{\Omega}$; see Corollary 3.0.7. As in Section 1.2, for $\epsilon > 0$ define the ϵ-level set

$$\gamma_\epsilon = \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \Gamma) = \epsilon\}.$$

What properties of Γ ensure that the ϵ-level sets are Jordan curves, or uniform quasicircles, or uniform chord-arc curves for *all* ϵ sufficiently close to 0?

Definition 3.0.2. *Let Γ be a Jordan curve in $\mathbb{R}^2$.*

We say that Γ has the level Jordan curve property (or LJC property), if there exists $\epsilon_0 > 0$ such that the level set γ_ϵ is a Jordan curve for every $0 < \epsilon \leq \epsilon_0$.

We say that Γ has the level quasicircle property (or LQC property), if there exist $\epsilon_0 > 0$ and $K \geq 1$ such that the level set γ_ϵ is a K-quasicircle for every $0 < \epsilon \leq \epsilon_0$.

We say that Γ has the level chord-arc property (or LCA property), if there exist $\epsilon_0 > 0$ and $C \geq 1$ such that γ_ϵ is a C-chord-arc curve for every $0 < \epsilon \leq \epsilon_0$.

It is not hard to see that if Γ has the LQC property then it is a quasicircle and if Γ satisfies the LCA

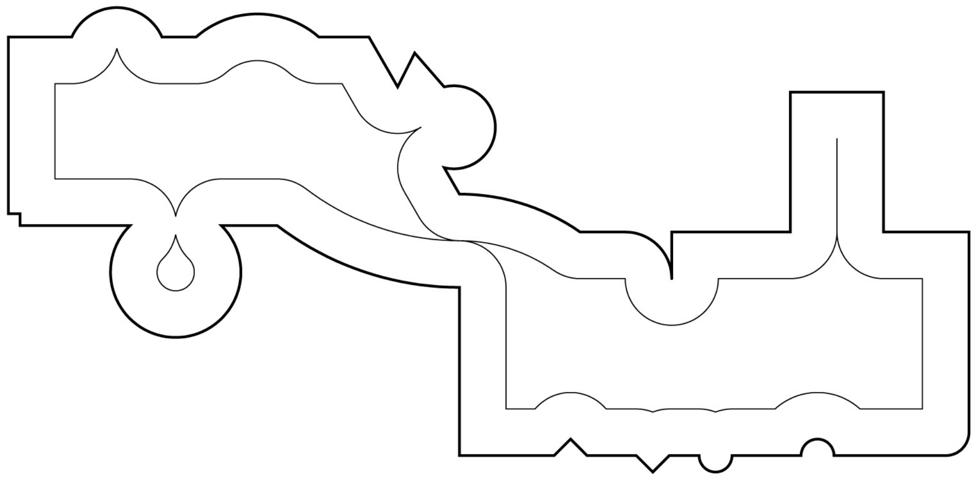

Figure 3.1: A level set of a Jordan curve.

property then it is a chord-arc curve; see Theorem 3.0.6.

Modeled on the *linear approximation property* of Mattila and Vuorinen [29], we define, for a Jordan curve Γ in the plane, a scaled invariant parameter to measure the local deviation of the subarcs from their chords. For points x, y on a Jordan curve Γ and the infinite line $l_{x,y}$ through x and y, we set

$$\zeta_\Gamma(x,y) = \frac{1}{|x-y|} \sup_{z \in \Gamma(x,y)} \operatorname{dist}(z, l_{x,y}).$$

Definition 3.0.3. *A Jordan curve Γ is said to have the (ζ, r_0)-chordal property for a certain $\zeta > 0$ and $r_0 > 0$, if*

$$\sup_{x,y \in \Gamma, |x-y| \le r_0} \zeta_\Gamma(x,y) \le \zeta.$$

We set

$$\zeta_\Gamma = \lim_{r_0 \to 0} \sup_{x,y \in \Gamma, |x-y| \le r_0} \zeta_\Gamma(x,y).$$

This notion of chord-likeness provides us a tool for studying the geometry of level sets.

Theorem 3.0.4. *Let Γ be a Jordan curve in $\mathbb{R}^2$. If Γ has the $(1/2, r_0)$-chordal property for some $r_0 > 0$, then Γ has the level Jordan curve property.*

Theorem 3.0.5. *Let Γ be a Jordan curve in $\mathbb{R}^2$. If $\zeta_\Gamma < 1/2$, then Γ has the level quasicircle property. In particular, if Γ has the (ζ, r_0)-chordal property for some $0 < \zeta < 1/2$ and $r_0 > 0$, then there exist $\epsilon_0 > 0$ and $K \ge 1$ depending on ζ, r_0 and the diameter of Γ so that the level sets γ_ϵ are K-quasicircles for all $0 < \epsilon < \epsilon_0$.*

Lemmas 3.2.13 and 3.2.15 lead naturally to the $(1/2, r_0)$-chordal condition for LJC in Theorem 3.0.4; they show that the behavior of the level set near branch points in Figure 3.1 is, in some sense, typical.

Condition $\zeta_\Gamma < 1/2$ in Theorem 3.0.5 is used to prove the Ahlfors 2-point condition (2.2.1) for level Jordan curves, thereby establishing the LQC property.

The chordal conditions in both theorems are sharp. The sharpness in Theorem 3.0.4 is given in Remark 3.3.1, and the sharpness in Theorem 3.0.5 is illustrated in Remark 3.3.2.

Moreover, using a lemma of Brown [7, Lemma 1], we are able to show the following.

Theorem 3.0.6. *A Jordan curve Γ in the plane satisfies the level chord-arc property if and only if it is a chord-arc curve and has the level quasicircle property.*

Theorem 3.0.4, Theorem 3.0.5 and Theorem 3.0.6 can easily be extended to the part of $\partial_\epsilon \Gamma$ outside of Ω.

Corollary 3.0.7. *Suppose that Ω is a Jordan domain and $\Gamma = \partial\Omega$.*

1. *If there exists $r_0 > 0$ such that $\zeta_\Gamma(x, y) \le 1/2$ for all $x, y \in \Gamma$ with $|x - y| \le r_0$ then, there is $\epsilon_0 > 0$ depending on r_0 such that for each $\epsilon \in (0, \epsilon_0)$, $\partial_\epsilon \overline{\Omega}$ is a Jordan curve.*

2. *If there exists $r_0 > 0$ and $\zeta_0 \in (0, 1/2)$ such that $\zeta_\Gamma(x, y) \le \zeta_0$ for all $x, y \in \Gamma$ with $|x - y| \le r_0$ then, there is $\epsilon_0 > 0$ depending on r_0 and K depending on r_0, ζ_0 such that for each $\epsilon \in (0, \epsilon_0)$, $\partial_\epsilon \overline{\Omega}$ is a K-quasicircle.*

3. *The sets $\partial_\epsilon \overline{\Omega}$ are chord-arc curves with uniform constant if and only if Γ is a chord-arc curve and $\partial_\epsilon \overline{\Omega}$ are quasicircles with uniform constant.*

This chapter is organized as follows. We discuss the chordal property in Section 3.1, and study geometric properties of level sets in Section 3.2. In Section 3.3, we prove Theorems 3.0.4 and 3.0.5 and give examples to show the sharpness of these theorems. We give the proof of Theorem 3.0.6 and Theorem 1.3.2 in Section 3.4. Finally in Section 3.5, we provide an additional example based on Rohde's p-snowflakes.

3.1 Chordal property of Jordan curves

For planar Jordan curves, the connection between the chordal property and the 2-point condition is easy to establish.

Proposition 3.1.1. *A Jordan curve Γ is a K-quasicircle if and only if Γ is (ζ, r_0)-chordal for some $\zeta > 0$ and $r_0 > 0$. Constants K and ζ_Γ are quantitatively related, with $\zeta_\Gamma \to 0$ as $K \to 1$.*

The converse of the second statement is not true since $\zeta_\Gamma = 0$ for every smooth Jordan curve Γ.

Proof. Suppose that Γ is a K-quasicircle and C is the constant in the Ahlfors 2-point condition (2.2.1) associated to K. Then Γ is $(C, \operatorname{diam}\Gamma)$-chordal.

Next suppose that Γ is (ζ, r_0)-chordal. We claim that Γ satisfies property (2.2.1). Let $x, y \in \Gamma$. If $|x-y| \geq r_0$, then

$$\operatorname{diam}\Gamma(x,y) \leq \frac{\operatorname{diam}\Gamma}{r_0}|x-y|.$$

So, we assume $|x-y| < r_0$, and let $[z,w]$ be the orthogonal projection of $\Gamma(x,y)$ on $l_{x,y}$, with points z, x, y and w listed in their natural order on the line. In the case that $z \neq x$, choose a point $z' \in \Gamma(x,y)$ whose projection on $l_{x,y}$ is z. Denote by l the line through x and orthogonal to $l_{x,y}$, and fix a subarc σ of $\Gamma(x,y)$ which contains z' and has endpoints, called z_1, z_2, on $\Gamma(x,y) \cap l$. Clearly $\sigma = \Gamma(z_1, z_2)$ and $l = l_{z_1,z_2}$; and by the (ζ, r_0)-chordal property, $\operatorname{dist}(z,l) = \operatorname{dist}(z',l) \leq \zeta|z_1 - z_2| \leq 2\zeta^2|x-y|$. It follows that, in all cases, $|z-w| \leq (4\zeta^2+1)|x-y|$. Therefore,

$$\operatorname{diam}\Gamma(x,y) \leq (4\zeta^2 + 2\zeta + 1)|x-y|.$$

So Γ satisfies property (2.2.1) with $C = \max\{4\zeta^2 + 2\zeta + 1, \frac{\operatorname{diam}\Gamma}{r_0}\}$ and is a K-quasicircle for some K depending on ζ, r_0 and $\operatorname{diam}\Gamma$.

The claim that $\zeta_\Gamma \to 0$ as $K \to 1$ follows from a lemma of Gehring [16, Lemma 7], which states that for each $\eta > 0$, there exists $K_0 = K_0(\eta) > 1$ such that if g is a K-quasiconformal mapping of $\mathbb{R}^2$ with $K \leq K_0$, and if g fixes two points z_1 and z_2, then

$$|g(z) - z| \leq \eta|z_1 - z_2|, \quad \text{when } |z - z_1| < |z_1 - z_2|.$$

Quasiconformality in [16] is defined using the conformal modulus of curve families, which is quantitatively equivalent to the notion of quasiconformality given in Chapter 1. This line of reasoning has been used by Mattila and Vuorinen in [29, Theorem 5.2]. □

By Proposition 3.1.1, the following is a corollary to Theorem 3.0.5.

Corollary 3.1.2. *There exists a constant $K_0 > 1$ such that all K_0-quasicircles have the* LQC *property.*

Mattila and Vuorinen [29] introduced the *linear approximation property* to study geometric properties of K-quasispheres with K close to 1. Let $k \in \{1, 2, \ldots, n-1\}$, $0 \leq \delta < 1$, and $r_0 > 0$. A set Z in $\mathbb{R}^n$ satisfies a

(k, δ, r_0)-linear approximation property if for each $x \in Z$ and each $0 < r < r_0$ there exists an affine k-plane P through x such that

$$\mathrm{dist}(z, P) \leq \delta r \quad \text{ for all } z \in Z \cap B^n(x, r).$$

In the same year, Jones [24] introduced a parameter, now known as the *Jones beta number*, to measure the oscillation of a set at all points and in all scales, for the investigation of the "traveling salesman problem". Later, beta number has been used by Bishop and Jones to study problems on harmonic measures and Kleinian groups. As it turns out, the Jones beta number and the δ-parameter of Mattila and Vuorinen are essentially equivalent.

For planar quasicircles, the chordal property and the linear approximation property are quantitatively related as follows.

Lemma 3.1.3. *Let Γ be a Jordan curve in $\mathbb{R}^2$. If Γ has the (ζ, r_0)-chordal property for some $0 < \zeta < 1/4$, then it is $(1, 4\zeta, r_1)$-linearly approximable, where $r_1 = \min\{\frac{r_0}{2}, \frac{\mathrm{diam}\,\Gamma}{C}\}$ and $C = C(\zeta, r_0, \mathrm{diam}\,\Gamma) > 1$ is a constant. On the other hand, if Γ is a K-quasicircle that has the $(1, \delta, r_0)$-linear approximation property, then it is $(C'^2\delta, r_0/C')$-chordal, for some constant $C' = C'(K) > 1$.*

Proof. Suppose that Γ is (ζ, r_0)-chordal. Then Γ is a K-quasicircle by Proposition 3.1.1, hence satisfies the 2-point condition (2.2.1) for some $C > 1$; here K and C depend on ζ, r_0 and $\mathrm{diam}\,\Gamma$. Let $0 < r < \min\{\frac{r_0}{2}, \frac{\mathrm{diam}\,\Gamma}{6C}\}$; take $x \in \Gamma$, and $x' \in \Gamma \setminus B^2(x, r)$ such that $|x - x'| \geq \frac{\mathrm{diam}\,\Gamma}{2}$. Let x_1, x_2 be the points in $\Gamma \cap S^1(x, r)$ with the property that one of the subarcs $\Gamma \setminus \{x_1, x_2\}$ contains x' and lies entirely outside $\overline{B}^2(x, r)$, and the other subarc, called τ, contains x. Since $\mathrm{diam}(\Gamma \setminus \tau) \geq |x - x'| - r \geq \frac{\mathrm{diam}\,\Gamma}{2} - r > 2Cr \geq C|x_1 - x_2|$, we have $\mathrm{diam}\,\tau \leq C|x_1 - x_2|$ and $\Gamma(x_1, x_2) = \tau$. Trivially, $\mathrm{dist}(x, l_{x_1,x_2}) \leq 2\zeta r$. Then, for $y \in \Gamma(x_1, x_2)$ and the line l through x and parallel to l_{x_1,x_2}, we have

$$\mathrm{dist}(y, l) \leq \mathrm{dist}(y, l_{x_1,x_2}) + \mathrm{dist}(l_{x_1,x_2}, l) \leq \zeta|x_1 - x_2| + 2\zeta r \leq 4\zeta r;$$

and the first claim follows.

For the second claim, suppose that Γ is a K-quasicircle that has the $(1, \delta, r_0)$-linear approximation property. So Γ satisfies the 2-point condition (2.2.1) for some $C = C(K) \geq 1$. Take $x, y \in \Gamma$ with $0 < |x - y| < \frac{r_0}{4C}$ and $r = (C+1)|x - y|$, then $\Gamma(x, y) \subset B^2(x, r)$. Since $r < r_0$, there exists a line l containing x such that

$$\Gamma(x, y) \subset \{z \in B^2(x, r) \colon \mathrm{dist}(z, l) \leq \delta r\}.$$

In particular, $\mathrm{dist}(y, l) \leq \delta r$. Given $z \in \Gamma(x, y)$, take a point $z' \in l \cap B^2(x, r)$ with $|z - z'| \leq \delta r$; then, from elementary geometry we get

$$\operatorname{dist}(z', l_{x,y}) \leq \frac{\operatorname{dist}(y,l)}{|x-y|} r \leq (C+1)\delta r.$$

It follows that $\operatorname{dist}(z, l_{x,y}) \leq |z-z'| + \operatorname{dist}(z', l_{x,y}) \leq (C+2)\delta r = (C+2)(C+1)\delta|x-y|$. Hence, $\zeta(x,y) < 6C^2\delta$, and the second claim is proved. □

3.2 Geometry of level sets

Let Γ be a Jordan curve in $\mathbb{R}^2$ and Ω be the bounded component of $\mathbb{R}^2 \setminus \Gamma$. For any $\epsilon > 0$, define the open set

$$\Delta_\epsilon = \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \Gamma) > \epsilon\}.$$

In general, Δ_ϵ need not be connected, and $\overline{\Delta_\epsilon}$ and $\Delta_\epsilon \cup \gamma_\epsilon$ may not be equal (see Figure 3.1).

However, *for any $\epsilon > 0$, the set $\overline{\Omega} \setminus \Delta_\epsilon$, is path-connected.* Indeed, given $x, y \in \overline{\Omega} \setminus \Delta_\epsilon$, take $x', y' \in \Gamma$ such that $|x-x'| = \operatorname{dist}(x,\Gamma)$ and $|y-y'| = \operatorname{dist}(y,\Gamma)$. Note that $[x,x']$ and $[y,y']$ are entirely in $\overline{\Omega} \setminus \Delta_\epsilon$. So x, y can be joined in $\overline{\Omega} \setminus \Delta_\epsilon$ by the arc $[x,x'] \cup \Gamma(x',y') \cup [y,y']$.

Remark 3.2.1. *Given $x \in \Omega$ let $x' \in \Gamma$ be such that $dist(x,\Gamma) = |x-x'|$. Then, for each $z \in [x,x']$, $dist(z,\Gamma) = |z-x'|$. Moreover, for each $\epsilon > 0$, the intersection $[x,x'] \cap \gamma_\epsilon$ contains either exactly one point (if $\epsilon \leq |x-x'|$) or no point (if $\epsilon > |x-x'|$).*

For the first claim observe that the ball $B^2(x, |x-x'|)$ does not intersect with Γ. Since $B^2(z, |z-x'|)$ is entirely inside $B^2(x, |x-x'|)$ and $x' \in S^1(z, |z-x'|)$, it follows that $\operatorname{dist}(z,\Gamma) = |z-x'|$. For the second claim it is clear that if $\epsilon > |x-x'|$ then $\gamma_\epsilon \cap [x,x'] = \emptyset$. If $0 < \epsilon \leq |x-x'|$, there is a unique point z in $[x,x']$ such that $|z-x'| = \epsilon$. By the first claim, $z \in \gamma_\epsilon \cap [x-x']$. If y was another point in $\gamma_\epsilon \cap [x-x']$ then, by the first claim, $|y-x'| = \epsilon$ which leads to $y = z$.

Remark 3.2.2. *Given $x, y \in \Omega$, let x', y' be points in Γ with the property that $|x-x'| = \operatorname{dist}(x,\Gamma)$ and $|y-y'| = \operatorname{dist}(y,\Gamma)$. If x, y, x', y' are not collinear then the segments $[x,x']$, $[y,y']$ do not intersect except perhaps at their endpoints.*

Indeed, if there is a point $z \in [x,x'] \cap [y,y']$ which is not x' or y', then, by Remark 3.2.1, $|z-x'| = |z-y'|$. By the triangle inequality,

$$|x-y'| < |x-z| + |z-y'| = |x-z| + |z-x'| = \operatorname{dist}(x,\Gamma),$$

which is a contradiction. The non-crossing property of $[x, x'], [y, y']$ is a special case of Monge's observation on optimal transportation; see [45, p. 163].

In the following, a point is considered as a degenerate arc.

Given a closed subset Λ of Γ and a number $\epsilon \neq 0$, we define

$$\gamma_\epsilon^\Lambda = \{x \in \gamma_\epsilon \colon \mathrm{dist}(x, \Lambda) = \epsilon\}.$$

In general, the set γ_ϵ^Λ may be empty even when Λ is a non-trivial arc (see Figure 3.1). However, γ_ϵ^Λ is an arc when γ_ϵ is a Jordan curve and Λ is connected, as we see from the following lemma.

Lemma 3.2.3. *Let Γ be a Jordan curve in $\mathbb{R}^2$, and assume that for some $\epsilon \neq 0$, the level set γ_ϵ is a Jordan curve. If Λ is a closed subarc of Γ and γ_ϵ^Λ is nonempty, then γ_ϵ^Λ is a subarc of γ_ϵ.*

Proof. It suffices to prove that if x and y are two distinct points in γ_ϵ^Λ then, one of the two subarcs λ_1, λ_2 of γ_ϵ connecting x and y is entirely in γ_ϵ^Λ.

We claim that if $\lambda_1 \setminus \gamma_\epsilon^\Lambda \neq \emptyset$ then $\lambda_2 \subset \gamma_\epsilon^\Lambda$. Take $z \in \lambda_1 \setminus \gamma_\epsilon^\Lambda$, $x', y' \in \Lambda$ and $z' \in \Gamma \setminus \Lambda$ such that

$$|x - x'| = |y - y'| = |z - z'| = \epsilon,$$

and let Λ_1 be the subarc of Λ that joins x', y' (Λ_1 could be just a point). We know that the open line segments $(x, x'), (y, y'), (z, z')$ and the Jordan curve γ_ϵ do not intersect one another. Let U_1 be the quadrilateral (possibly degenerated in the case $x' = y'$) enclosed by the Jordan curve $[x, x'] \cup \lambda_1 \cup [y, y'] \cup \Lambda_1$. Then the open arc $\lambda_2 \setminus \{x, y\}$ must be contained in U_1. For otherwise, $\lambda_2 \setminus \{x, y\}$ would intersect either the arc $[x, x'] \cup \lambda_1 \cup [y, y']$ or the segment $[z, z']$; this is impossible in view of properties of the distance function $\mathrm{dist}(\cdot, \Gamma)$. Therefore, the quadrilateral U_2 enclosed by the Jordan curve $[x, x'] \cup \lambda_2 \cup [y, y'] \cup \Lambda_1$ is contained in U_1. Suppose now that $\lambda_2 \subset \gamma_\epsilon^\Lambda$ is false. Then, by the argument above with the roles of λ_1 and λ_2 reversed, we get $U_1 \subset U_2$. Hence $U_1 = U_2$, which is impossible. This proves the claim. □

We show next that when two points x, y on a level Jordan curve γ_ϵ have a common closest point on Γ then $\gamma_\epsilon(x, y)$ is a circular arc.

Lemma 3.2.4. *Let Γ be a Jordan curve in $\mathbb{R}^2$, and assume that the level set γ_ϵ is a Jordan curve for some $\epsilon > 0$. Suppose that there exist $x, y \in \gamma_\epsilon$ and $z \in \Gamma$ such that $|x - z| = |y - z| = \epsilon$. Then $\gamma_\epsilon(x, y)$ is a circular arc on $S^1(z, \epsilon)$ of length at most $\pi\epsilon$.*

Proof. By Lemma 3.2.3, $\gamma_\epsilon^{\{z\}} = \{w \in \gamma_\epsilon \colon |w - z| = \epsilon\}$ is a subarc of $\gamma_\epsilon \cap S^1(z,\epsilon)$. Since $\{x, y\} \subset \gamma_\epsilon^{\{z\}}$, $\gamma_\epsilon(x, y)$ is one of the two subarcs of $S^1(z,\epsilon)$ that connects x and y.

Suppose that $\ell(\gamma_\epsilon(x, y)) > \pi\epsilon$. Then the domain U enclosed by the Jordan curve $\gamma_\epsilon(x, y) \cup [x, y]$ contains precisely one point from Γ, namely the point z; all other points on Γ are in the exterior of U. So, Γ intersects the segment $[x, y]$; consequently, $\mathrm{dist}(x,\Gamma) < \epsilon$ and $\mathrm{dist}(y,\Gamma) < \epsilon$. This is a contradiction. $\square$

Using the above result we can prove the following lemma.

Lemma 3.2.5. *Suppose that Γ is a K-quasicircle in $\mathbb{R}^2$. Then there exists $M \geq 1$ depending only on K such that for any $x \in \Gamma$ and for any $\epsilon > 0$, dist$(x, \gamma_\epsilon) \leq M\epsilon$.*

Proof. For any $\epsilon > 0$ let Γ^{γ_ϵ} be the points on Γ whose distance from γ_ϵ is ϵ.

Since Γ is a K-quasicircle, it satisfies Ahlfors condition (2.2.1) for some $C > 1$ depending on K. If $x \in \Gamma^{\gamma_\epsilon}$ then $\mathrm{dist}(x, \gamma_\epsilon) = \epsilon$ and the claim holds for $M = 1$. Suppose now that $x \in \Gamma \setminus \Gamma^{\gamma_\epsilon}$. Since Γ^{γ_ϵ} is closed in Γ, there exists a component Λ of $\Gamma \setminus \Gamma^{\gamma_\epsilon}$ that contains x. Denote with y', z' the endpoints of Λ. Then, $y', z' \in \Gamma^{\gamma_\epsilon}$ and therefore, there exist $y, z \in \gamma_\epsilon$ such that

$$|y - y'| = |z - z'| = \epsilon. \tag{3.2.1}$$

The points y, z can be chosen so that their distance is minimal over all pair of points in Λ satisfying (3.2.1). Then, it follows from Lemma 3.2.3 that $y = z$. Otherwise, γ_ϵ^Λ is either the subarc $\gamma_\epsilon(y, z)$ or $\Gamma \setminus \gamma_\epsilon(y, z)$ which are both nonempty. Therefore,

$$\mathrm{dist}(x, \gamma_\epsilon) \leq |x - y| + \epsilon \leq \mathrm{diam}\,\Lambda + \epsilon \leq C|y - z| + \epsilon \leq (2C + 1)\epsilon. \qquad \square$$

Suppose that $0 \leq \epsilon < \epsilon'$ and $\gamma_\epsilon, \gamma_{\epsilon'}$ are Jordan curves. The next corollary is an immediate consequence of the simple observation that $\gamma_{\epsilon'}$ is the $(\epsilon' - \epsilon)$-level set of γ_ϵ.

Corollary 3.2.6. *Suppose that Γ is a curve in $\mathbb{R}^2$ for which there exist $K > 1$ and $\epsilon_0 > 0$ such that γ_ϵ is a K-quasicircle whenever $\epsilon \in [0, \epsilon_0]$. There exists a constant M depending only on M such that if $0 \leq \epsilon < \epsilon' \leq \epsilon_0$ and $x \in \gamma_\epsilon$ then dist$(x, \gamma_{\epsilon'}) \leq M(\epsilon' - \epsilon)$.*

The following lemma shows that components of Δ_ϵ satisfy a weak form of the LLC_1 property introduced in Section 2.3. In particular, Δ_ϵ *has no inward cusps.*

Lemma 3.2.7. *Let Γ be a Jordan curve, $\epsilon > 0$ and D a connected component of Δ_ϵ. Then for any $x, y \in D$ with $|x - y| \leq 2\epsilon$, there exists a polygonal arc τ in D that joins x and y and has* $\mathrm{diam}\,\tau \leq 5|x - y|$.

Proof. Let x, y be two points in D with $|x-y| \leq 2\epsilon$. So, for each $z \in [x, y]$

$$\operatorname{dist}(z, \Gamma) > 2\epsilon - \min\{|z-y|, |z-x|\} > 0$$

and $[x, y] \cap \Gamma = \emptyset$. Consequently, since x, y are contained in the bounded component Ω of $\mathbb{R}^2 \setminus \Gamma$, the segment $[x, y]$ is contained in Ω as well. Let τ' be any curve in D that connects x to y. After approximating τ' by a polygonal curve, erasing the loops and making small adjustments near the segment $[x, y]$, we may assume that τ' is a simple polygonal curve which intersects $[x, y]$ in a finite set. In other words, τ' is the union of finitely many simple polygonal subarcs σ' in D, each of which meets $[x, y]$ precisely at its end points. The curve τ in the proposition is obtained by replacing each σ' with a polygonal arc σ in $D \cap B^2(x, \frac{5}{2}|x-y|)$ with the same end points.

Fix such a subarc σ' having end points $a, b \in [x, y]$. Assume that $\sigma' \setminus \overline{B}^2(x, 2|x-y|) \neq \emptyset$; otherwise, just let $\sigma = \sigma'$. Let U be the domain enclosed by the Jordan curve $\sigma' \cup [a, b]$. Since $\partial U \cap \Gamma = \emptyset$ and Ω is simply connected, $\overline{U} \subset \Omega$. We claim that

$$U \setminus B^2(x, 2|x-y|) \subset \Delta_\epsilon.$$

Otherwise, take a point $z \in U \setminus (B^2(x, 2|x-y|) \cup \Delta_\epsilon)$ and assume as we may, by the continuity of the distance function, that $z \in \gamma_\epsilon$. Let z' be a point on Γ for which $|z-z'| = \operatorname{dist}(z, \Gamma) = \epsilon$. Since $\overline{U} \subset \Omega$, $z' \notin \overline{U}$ and the open segment (z, z') intersects ∂U at some point z''. If z'' is in $[a, b] \subset [x, y]$ then

$$\operatorname{dist}(x, \Gamma) \leq |x-z'| \leq |x-z''| + |z''-z'| = |x-z''| + \epsilon - |z-z''| < \epsilon,$$

a contradiction. If z'' is in σ' then $\epsilon = |z-z'| > |z''-z'| \geq \operatorname{dist}(z'', \Gamma) > \epsilon$, again a contradiction. This proves the claim.

Let U' be the connected component of $U \cap B^2(x, 2|x-y|)$ that contains the segment $[a, b]$ in its boundary. Since U is a polygon, U' is simply connected and $\partial U'$ is a Jordan curve. In particular, $\partial U' \setminus (a, b)$ is composed of finitely many line segments in D and finitely many subarcs of $S^1(x, 2|x-y|)$. In view of the claim above, $\partial U' \setminus (a, b)$ is an arc contained in D. After replacing each maximal circular subarc of $\partial U' \setminus (a, b)$ by a polygonal arc nearby, we obtain a polygonal arc σ in $D \cap B^2(x, \frac{5}{2}|x-y|)$ connecting a to b. The arc τ in the proposition is given by the union of these new σ's. □

We next prove that the boundary of any connected component of Δ_ϵ is a Jordan curve. We need a theorem of Lennes in [25] which gives a sufficient condition for the frontier, of a bounded planar domain, to be a Jordan curve. Let D be a bounded domain and p a closed polygonal curve which encloses $\overline{D}$ in its interior. Let E' be the set of all points in the plane that can be joined to p by a continuous curve in the complement of D. The *frontier* F of D is the set of all common limit points of E' and D, that is, $F = \overline{E}' \cap \overline{D}$. Define moreover the *interior set of the frontier* F to be $I = \mathbb{R}^2 \setminus (E' \cup F)$ and the *exterior set of the frontier* F to be $E = E' \setminus F$. Observe that all the above definitions are independent of the choice of p.

Furthermore, a point $x \in F$ is said to be *externally accessible* if there exists a finite or a continuous infinite polygonal path $\tau\colon [0,1] \to \mathbb{R}^2$ such that $\tau([0,1)) \subset E$ and $\tau(1) = x$. And a point $x \in F$ is said to be *internally accessible* if there exists a finite or a continuous infinite polygonal path $\tau\colon [0,1] \to \mathbb{R}^2$ such that $\tau([0,1)) \subset I$ and $\tau(1) = x$. Lennes proved the following.

Lemma 3.2.8 ([25, Theorem 5.3])**.** *If every point of a frontier F possesses both the internal and the external accessibility, then F is a Jordan curve.*

We now apply the theorem of Lennes to prove the following.

Lemma 3.2.9. *Let Γ be a Jordan curve and $\epsilon > 0$. Then, the boundary of every connected component of Δ_ϵ is a Jordan curve.*

Proof. Recall that Ω is the bounded component of $\mathbb{R}^2 \setminus \Gamma$. Let D be a connected component of Δ_ϵ, and p be a closed polygonal curve that encloses $\overline{\Omega}$ in its interior. Every point $x \in \Omega \setminus D$ can be joined to one of its closest points on Γ by a line segment entirely outside D, then to p by a curve in $\mathbb{R}^2 \setminus \Omega$. Therefore, $E' = \mathbb{R}^2 \setminus D$, $F = \overline{E}' \cap \overline{D} = \partial D$ and $I = D$, and any point in ∂D is externally accessible.

To check the internal accessibility, we take $x \in \partial D$, and a sequence $\{x_n\}$ in D with distance $|x_n - x| < 2^{-n}\epsilon$ for every $n \geq 1$. By Lemma 3.2.7, there exist a family of polygonal arcs $\{\tau_n\}_{n\in\mathbb{N}}$ in D such that τ_n joins x_n to x_{n+1} and has $\operatorname{diam} \tau_n \leq 5|x_n - x_{n+1}| \leq 2^{1-n}\epsilon$. Then, take τ to be the infinite polygonal path $\{x\} \cup \bigcup_{n\geq 1} \tau_n$. This proves that x is internally accessible, and by Lennes' theorem, we conclude that ∂D is a Jordan curve. ☐

Components of Δ_ϵ satisfy LLC_1, when Γ is a quasicircle.

Lemma 3.2.10. *Suppose that Γ is a K-quasicircle. Then, there exists a constant $M > 0$ depending only on K such that for any $\epsilon > 0$, for any connected component D of Δ_ϵ, and for any two points $x, y \in D$, there exists a curve τ in D joining x and y such that* $\operatorname{diam} \tau \leq M|x - y|$.

Proof. In view of Lemma 3.2.7, we consider points x and y in D with distance $|x - y| > 2\epsilon$ only. The proof follows closely that of Lemma 3.2.7; however, the segment $[x, y]$ here may intersect Γ.

Since Γ is a K-quasicircle, it satisfies condition (2.2.1) for some $C = C(K) > 1$. Fix a simple polygonal curve τ' in D joining x and y that intersects $[x, y]$ in a finite set. As in Lemma 3.2.7, we replace each subarc σ' of τ' that has end points in $[x, y]$ and does not intersect $[x, y]$ anywhere else, by a new arc σ in $D \cap B^2(x, (C+2)|x-y|)$ having the same end points.

Fix such a subarc σ' having end points $a, b \in [x, y]$. Assume that $\sigma' \setminus \overline{B}^2(x, (C+2)|x-y|) \neq \emptyset$; otherwise, just set $\sigma = \sigma'$. The domain U enclosed by the Jordan curve $\sigma' \cup [a, b]$ may now contain points outside Ω. We claim nevertheless that

$$U \setminus B^2(x, (C+2)|x-y|) \subset \Delta_\epsilon. \tag{3.2.2}$$

Suppose $U \setminus B^2(x, (C+2)|x-y|) \not\subseteq \Delta_\epsilon$. As before, we may pick a point $z \in (U \setminus B^2(x, (C+2)|x-y|)) \cap \gamma_\epsilon$ and a point $z' \in \Gamma$ such that $|z - z'| = \operatorname{dist}(z, \Gamma) = \epsilon$. Suppose $z' \notin U$; the segment $[z, z']$ must intersect ∂U at some point z''. Because $|x-y| > 2|\epsilon|$, the point z'' cannot be in $[x, y]$, therefore $z'' \in \sigma'$. Hence $\epsilon = |z-z'| > |z''-z'| \geq \operatorname{dist}(z'', \Gamma) > \epsilon$, a contradiction. So z' must be in U, therefore $z' \in (U \cap \Gamma) \setminus B^2(x, (C+1)|x-y|)$.

Since $\epsilon > 0$, Γ cannot be entirely in U, so $\Gamma \cap \partial U \neq \emptyset$. Since $\partial U = \sigma' \cup [a, b]$ and $\sigma' \subset D$, $\Gamma \cap [a, b] \neq \emptyset$. Let z_1, z_2 be the points in $[a, b] \cap \Gamma$ with the property that the open subarc Γ' of Γ connecting z_1 to z_2 and containing the point z', is entirely in U. So $|z_1 - z_2| < |a - b| \leq |x - y|$ and

$$\operatorname{diam} \Gamma' \geq \operatorname{dist}(z', [z_1, z_2]) \geq |z' - x| - |x - y| \geq C|x - y| > C|z_1 - z_2|.$$

From the 2-point condition (2.2.1) it follows that the diameter of the subarc $\Gamma'' = \Gamma \setminus \Gamma'$ is at most $C|z_1 - z_2|$. Therefore, $\Gamma'' \subset B^2(x, (C+1)|x-y|)$, and $\Gamma \subset U \cup B^2(x, (C+1)|x-y|)$.

Let w be one of the points on σ' that is furthest from x. Since $\Gamma' \setminus \{z_1, z_2\}$ is contained in the open set U, $|x - w| > \max_{u \in \Gamma'} |x - u|$; furthermore $|x - w| \geq (C+2)|x-y| > \max_{u \in \Gamma''} |x - u|$. As a consequence, w, also σ', is contained in the unbounded component of $\mathbb{R}^2 \setminus \Gamma$. This is impossible because $\sigma' \subset D \subset \Delta_\epsilon \subset \Omega$. Claim (3.2.2) is proved.

Let U' be the component of $U \cap B^2(x, (C+2)|x-y|)$ whose boundary contains $[a, b]$. As in Lemma 3.2.7, σ' is replaced by the subarc $\sigma = \partial U' \setminus (a, b) \subset D \subset \Delta_\epsilon$. The curve τ in the proposition is the union of these new σ's. $\square$

Remark 3.2.11. *Both Lemmas 3.2.7 and 3.2.10 can be strengthened to include the case when x and y are in $\overline{D}$. In such case, curves τ satisfying the diameter estimates in the lemmas are contained in D with the exception of their endpoints.*

We now state an elementary geometric fact needed in the following two lemmas. Given $0 < \delta < \epsilon$ and a point $a = \delta e^{i\alpha}$ in $B^2(0, \epsilon)$, then

$$S^1(a, \epsilon) \setminus B^2(0, \epsilon) = \{a + \epsilon e^{i\theta} \colon |\theta - \alpha| \leq \pi - \cos^{-1}(\frac{\delta}{2\epsilon})\},$$

and the circular arc is contained in the sectorial region $\{z \in \mathbb{R}^2 \colon |\arg z - \alpha| \le \cos^{-1}(\frac{\delta}{2\epsilon})\}$.

Given any $x_0 \in \gamma_\epsilon$, $\epsilon > 0$, set

$$\Gamma^{\{x_0\}} = \{y \in \Gamma \colon |x_0 - y| = \epsilon\}.$$

Lemma 3.2.12. *Suppose $\epsilon > 0$ and x_0 is a non-isolated point in γ_ϵ. Then the set $\Gamma^{\{x_0\}}$ lies entirely in a semi-circular subarc of $S^1(x_0, \epsilon)$.*

Proof. Choose a sequence of points a_n on γ_ϵ that converges to x_0; set $\delta_n = |a_n - x_0|$ and assume as we may that $0 < \delta_n < \epsilon$. Since $\operatorname{dist}(a_n, \Gamma) = \epsilon$, we have $\operatorname{dist}(a_n, \Gamma^{\{x_0\}}) \ge \epsilon$, for all $n \ge 1$. In particular $\Gamma^{\{x_0\}}$ is contained in the part of $S^1(x_0, \epsilon)$ that is outside $B^2(a_n, \epsilon)$, which is a circular arc of length $(2\pi - 2\cos^{-1}(\frac{\delta_n}{2\epsilon}))\,\epsilon$. The claim follows by letting $n \to \infty$. $\square$

Fix a non-isolated point x_0 on γ_ϵ, we examine the geometry of the level set γ_ϵ near x_0.

Let $X = \Gamma^{\{x_0\}}$. Since X is compact, there exist $x_1, x_2 \in X$ (possibly $x_1 = x_2$) such that $|x_1 - x_2| = \operatorname{diam} X \le 2\epsilon$; and by Lemma 3.2.12, X lies in a subarc Σ (possibly degenerated) of $S^1(x_0, \epsilon)$ having endpoints x_1 and x_2 and of length at most $\pi\epsilon$. Let $U = B^2(x_1, \epsilon) \cup B(x_2, \epsilon)$; clearly $\gamma_\epsilon \cap U = \emptyset$. Set $\epsilon_0 = (\epsilon^2 - |x_1 - x_2|^2/4)^{1/2}$. For $0 < \delta < \epsilon_0$, the set $S^1(0, \delta) \setminus U$ is a connected arc when $|x_1 - x_2| < 2\epsilon$, and it has two components when $|x_1 - x_2| = 2\epsilon$. Let S_δ be a component of $S^1(0, \delta) \setminus U$.

Lemma 3.2.13. *Suppose $\epsilon > 0$ and x_0 is a non-isolated point in γ_ϵ. There exists $\delta_0 \in (0, \epsilon_0)$ such that if $0 < \delta < \delta_0$ then the set $\gamma_\epsilon \cap S_\delta$ contains at most two points.*

Specifically, if $0 < \delta < \delta_0$, $a \in \gamma_\epsilon \cap S_\delta$, and a' is a point in Γ with $|a - a'| = \epsilon$, then at least one of the two components (maybe empty) $S^1_{\delta,a}, S^2_{\delta,a}$ of $S_\delta \setminus \{a\}$ is contained entirely in the disk $B^2(a', \epsilon)$; in other words, there exists $j \in \{1, 2\}$ such that every point in $S^j_{\delta,a}$ has distance strictly less than ϵ from Γ.

Remark 3.2.14. *For every non-isolated point x_0 on γ_ϵ and every $\delta \in (0, \epsilon_0)$, the set $\gamma_\epsilon \cap S^1(x_0, \delta)$ contains at most two points when $|x_1 - x_2| < 2\epsilon$, and at most four points when $|x_1 - x_2| = 2\epsilon$. See Figure 3.1 for some of the possibilities.*

Proof. Assume as we may that $x_0 = 0$, $x_1 = \epsilon e^{i\tau}, x_2 = \epsilon e^{i(2\pi - \tau)}$ with $\tau \in [\pi/2, \pi]$, and that $\Sigma \subset \{z \colon \operatorname{Re} z \le 0\}$. Consider from now on only those δ in $(0, \epsilon_0)$. Consequently, $S_\delta \subset \{z \colon \operatorname{Re} z > 0\}$ when $0 < |x_1 - x_2| < 2\epsilon$ and $S^1(0, \delta) \cap \{z \colon \operatorname{Re} z \ge 0\} \subset S_\delta$ when $x_1 = x_2 = -\epsilon$; assume therefore without loss of generality that $S_\delta \subset \{z \colon \operatorname{Re} z \ge 0\}$ when $|x_1 - x_2| = 2\epsilon$. It is straightforward to check that

$$-(\tau - \cos^{-1}(\frac{\delta}{2\epsilon})) \le \arg z \le \tau - \cos^{-1}(\frac{\delta}{2\epsilon}) \quad \text{for all } z \in S_\delta. \tag{3.2.3}$$

Fix a number $\xi \in (0, \frac{\pi}{24})$ depending on τ so that $\tau - \xi > \pi/2$ when $\tau > \pi/2$, and $\xi = \pi/48$ when $\tau = \pi/2$. Fix also a number $\delta_0 \in (0, \epsilon_0)$, satisfying $\cos^{-1}(\frac{\delta_0}{2\epsilon}) > \frac{5\pi}{12}$, and having the property that for any $a \in \gamma_\epsilon \cap B^2(0, \delta_0)$ and any point a' on Γ nearest to a, i.e., $|a - a'| = \epsilon$, we have

$$\tau - \xi \leq \arg a' \leq 2\pi - \tau + \xi. \tag{3.2.4}$$

If there were no such δ_0, X would contain a point outside Σ.

Suppose the assertion in the lemma is false. Then, there exist $\delta \in (0, \delta_0)$, $a \in \gamma_\epsilon \cap S_\delta, a$, a point $a' \in \Gamma$ with $|a - a'| = \epsilon$, $b_1 \in S^1_{\delta,a}$ and $b_2 \in S^2_{\delta,a}$ such that $b_1, b_2 \notin B^2(a', \epsilon)$. Assume as we may that

$$-(\tau - \cos^{-1}(\frac{\delta}{2\epsilon})) \leq \arg b_1 < \arg a < \arg b_2 \leq \tau - \cos^{-1}(\frac{\delta}{2\epsilon}).$$

Let l_1 (resp. l_2) be the line that bisects the segment $[a, b_1]$ (resp. $[a, b_2]$). Since $|b_j - a'| \geq \epsilon = |a - a'|$ for $j = 1$ and 2, the point a' lies in the closure of the component of $\mathbb{R}^2 \setminus \{l_1, l_2\}$ that contains a. In particular by (3.2.3),

$$-(\tau - \cos^{-1}(\frac{\delta}{2\epsilon})) \leq \frac{\arg b_1 + \arg a}{2} \leq \arg a' \leq \frac{\arg b_2 + \arg a}{2} \leq \tau - \cos^{-1}(\frac{\delta}{2\epsilon}),$$

which is impossible in view of (3.2.4) and the fact that $\xi < \pi/24 < \cos^{-1}(\frac{\delta_0}{2\epsilon})$. This proves the second assertion and the lemma. □

Lemma 3.2.15. *Suppose that for some $\epsilon > 0$, there exist a connected component D of Δ_ϵ and a connected component G of $\gamma_\epsilon \cup \Delta_\epsilon$ such that $\overline{D} \subsetneq G$. Then, there exists a point $x_0 \in \partial D$ and points $x_1, x_2 \in \Gamma$ such that x_0, x_1, x_2 are collinear and*

$$|x_0 - x_1| = |x_0 - x_2| = \epsilon.$$

Furthermore, $\Gamma^{\{x_0\}} = \{x_1, x_2\}$.

Proof. Let $E = G \setminus \overline{D}$. Since G is connected, we have that $\overline{E} \cap \overline{D} \neq \emptyset$ and $\overline{E} \cap \partial D \neq \emptyset$. Fix a point $x_0 \in \overline{E} \cap \partial D$; clearly x_0 is a non-isolated point in γ_ϵ. Define $X = \Gamma^{\{x_0\}}$, the shortest subarc Σ of $S^1(x_0, \epsilon)$ containing X, its end points x_1, x_2, the open set U, and the number $\delta_0 > 0$, relative to the point x_0 as in Lemma 3.2.13.

Suppose that $|x_1 - x_2| < 2\epsilon$. Then for $\delta \in (0, \epsilon_0)$, $S^1(x_0, \delta) \setminus U$ is the arc S_δ. Since $x_0 \in \partial D$, there exists a number $\delta_1 = \delta_1(x_0, D, \epsilon) > 0$ such that $D \cap S_\delta$ contains a non-trivial arc for every $0 < \delta < \delta_1$. Therefore $\partial D \cap S_\delta$ contains at least two points in γ_ϵ. Hence, by Lemma 3.2.13, $E \cap S_\delta = \emptyset$ when $0 < \delta < \min\{\delta_0, \delta_1\}$. This contradicts the assumption $x_0 \in \overline{E}$. Therefore $|x_1 - x_2| = 2\epsilon$ and x_0, x_1 and x_2 are collinear.

We now prove $\Gamma^{\{x_0\}} = \{x_1, x_2\}$. Assume, as in Lemma 3.2.13, that $x_0 = 0$, $\Sigma \subset \{\operatorname{Re} z \leq 0\}$, $x_1 = \epsilon e^{i\pi/2}$ and $x_2 = \epsilon e^{i3\pi/2}$. Suppose there exists another point $x_3 \in \Gamma^{\{x_0\}} \setminus \{x_1, x_2\}$; so $\operatorname{Re} x_3 < 0$. Observe, by elementary calculations, that there exists $\delta_2 = \delta_2(x_3, \epsilon) \in (0, \epsilon_0)$ so that for any y in the half disk $B^2(0, \delta_2) \cap \{\operatorname{Re} z < 0\}$, one of the numbers $|y - x_1|$, $|y - x_2|$, $|y - x_3|$ is strictly less than ϵ. Therefore, $(\Delta_\epsilon \cup \gamma_\epsilon) \cap B^2(0, \delta_2) \subset \{\operatorname{Re} z \geq 0\} \setminus U$. Since $x_0 \in \overline{D}$, $\partial D \cap S_\delta$ contains at least two points in γ_ϵ for all sufficiently small δ. As before, it follows from Lemma 3.2.13 that $E \cap S_\delta$ must be empty for all sufficiently small δ, a contradiction. This proves that $\Gamma^{\{x_0\}} = \{x_1, x_2\}$, and the lemma. □

The next two propositions lead naturally to the $(1/2, r_0)$-chordal condition for the LJC property in Theorem 3.0.4.

Proposition 3.2.16. *Suppose that for some $\epsilon > 0$, $\Delta_\epsilon \neq \emptyset$, $\gamma_\epsilon \cup \Delta_\epsilon$ is connected, and $\overline{\Delta}_\epsilon \subsetneq \gamma_\epsilon \cup \Delta_\epsilon$. Then, there exist points $x_0 \in \gamma_\epsilon$ and $x_1, x_2 \in \Gamma$ which are collinear such that*

$$|x_0 - x_1| = |x_0 - x_2| = \epsilon.$$

Moreover, $\Gamma^{\{x_0\}} = \{x_1, x_2\}$.

From the assumptions, there exist a connected component D of Δ_ϵ and a connected component G of $\gamma_\epsilon \cup \Delta_\epsilon$ such that $D \subset \overline{D} \subsetneq G$. The proposition follows from Lemma 3.2.15.

Remark 3.2.17. *The point x_0 in Proposition 3.2.16, which is chosen according to Lemma 3.2.15, lies, in fact, on the boundary of a component of Δ_ϵ.*

Proposition 3.2.18. *Suppose that $\Delta_\epsilon \neq \emptyset$ and $\gamma_\epsilon \cup \Delta_\epsilon$ is not connected for some $\epsilon > 0$. Then, there exist points $x_0 \in \Omega$ and $x_1, x_2 \in \Gamma$ which are collinear such that*

$$|x_0 - x_1| = |x_0 - x_2| = \mathrm{dist(x_0, \Gamma)} < \epsilon.$$

Moreover, $\Gamma^{\{x_0\}} = \{x_1, x_2\}$.

Proof. Choose a connected component D of Δ_ϵ, a point $x \in D$, and a point y in a connected component of $\Delta_\epsilon \cup \gamma_\epsilon$ that does not meet $\overline{D}$, and define

$$d_0 = \sup\{\delta > 0 \colon x, y \text{ are in a common component of } \Delta_\delta\}.$$

Since Ω is path connected, $d_0 > 0$; and since x and y are in two different components of the closed set $\gamma_\epsilon \cup \Delta_\epsilon$, $d_0 < \epsilon$.

For $\delta \in (0, d_0)$, let G_δ be the component of Δ_δ that contains x and y. Then, for $0 < \delta < \delta' < d_0$ we have $G_{\delta'} \subset \overline{G_{\delta'}} \subset G_\delta$. Since $\{\overline{G_\delta}\}_{\delta \in (0,d_0)}$ is a nested family of compact connected sets, the intersection $G = \bigcap_{0<\delta<d_0} \overline{G}_\delta$ is a connected subset of $\gamma_{d_0} \cup \Delta_{d_0}$ that contains $\overline{D} \cup \{y\}$.

We claim that G is the component of $\gamma_{d_0} \cup \Delta_{d_0}$ that contains x, y. Indeed, let $\tilde{G}$ be the component of $\gamma_{d_0} \cup \Delta_{d_0}$ that contains x, y. Clearly $G \subset \tilde{G}$. Since the set $\bigcup_{x \in \tilde{G}} B^2(x, \delta)$ is open and connected for every $\delta \in (0, d_0)$,

$$\tilde{G} \subset \bigcup_{x \in \tilde{G}} B^2(x, \delta) \subset G_{d_0 - \delta} \qquad \text{for each} \quad \delta \in (0, d_0).$$

So $\tilde{G} \subset G$ and G is the component of $\gamma_{d_0} \cup \Delta_{d_0}$ that contains x, y. Hence, $\overline{D} \subsetneq G$ and the proposition now follows from Lemma 3.2.15. ☐

Remark 3.2.19. *The point x_0 in Proposition 3.2.18, chosen according to Lemma 3.2.15, lies on the boundary of a component of Δ_{d_0} for some $0 < d_0 < \epsilon$.*

3.3 Level curves and level quasicircles

In this section, we give the proofs of Theorem 3.0.4 and Theorem 3.0.5 along with two examples that show the sharpness of the conditions.

Proof of Theorem 3.0.4. By the assumption of the theorem, there exists $r_0 > 0$ such that $\zeta_\Gamma(x, y) \le 1/2$, for all $x, y \in \Gamma$ with $|x - y| \le r_0$.

First we claim that $\Delta_\epsilon \cup \gamma_\epsilon$ is connected for all $\epsilon \in (0, r_0/2)$. Otherwise, by Proposition, 3.2.18, there exist $d_0 \in (0, r_0/2)$ and collinear points $x_0 \in \gamma_{d_0}$ and $x_1, x_2 \in \Gamma$ such that $\Gamma^{\{x_0\}} = \{y \in \Gamma \colon |x_0 - y| = d_0\} = \{x_1, x_2\}$. The line l that contains x_0 and is perpendicular to l_{x_1,x_2} intersects $\Gamma(x_1, x_2)$ at some point z. Note that $|x_1 - x_2| = 2d_0 < r_0$ and that

$$\operatorname{dist}(z, l_{x_1,x_2}) = |x_0 - z| > \operatorname{dist}(x_0, \Gamma) = d_0.$$

So $\zeta_\Gamma(x_1, x_2) > 1/2$, a contradiction.

Next we claim that Δ_ϵ must be connected for all $\epsilon \in (0, r_0/2)$. Otherwise, for some $\epsilon \in (0, r_0/2)$ the open set Δ_ϵ would have at least two components, called D_1, D_2. By the continuity of the distance function, each $D_j, j = 1, 2$, would contain a point z_j of distance ϵ' to Γ, for some $\epsilon' \in (\epsilon, r_0/2)$. This would imply that $\Delta_{\epsilon'} \cup \gamma_{\epsilon'}$ is not connected; this contradicts the previous claim.

Therefore, by Lemma 3.2.9, $\partial\Delta_\epsilon$ is a Jordan curve for every $\epsilon \in (0, r_0/2)$. It remains to check that

$\gamma_\epsilon = \partial\Delta_\epsilon$ for all $\epsilon \in (0, r_0/2)$. Suppose $\partial\Delta_\epsilon \subsetneq \gamma_\epsilon$ for some $\epsilon \in (0, r_0/2)$. Then $\overline{\Delta}_\epsilon \subsetneq \Delta_\epsilon \cup \gamma_\epsilon$. Therefore, by Proposition 3.2.16, we can find collinear points $x_0 \in \gamma_\epsilon$ and $x_1, x_2 \in \Gamma$ such that $\Gamma^{\{x_0\}} = \{x_1, x_2\}$. As before, this leads to the inequality $\zeta_\Gamma(x_1, x_2) > 1/2$, a contradiction. So $\gamma_\epsilon = \partial\Delta_\epsilon$.

This completes the proof of the theorem. $\square$

Remark 3.3.1. *The* $(1/2, r_0)$*-chordal condition is sharp for the conclusion of Theorem 3.0.4.*

We construct a chord-arc curve Γ with $\zeta_\Gamma = \frac{1}{2}$ which satisfies

(i) There exist two sequences of points $\{x_n\}, \{y_n\}$ on Γ such that $|x_n - y_n| \to 0$ and $\zeta_\Gamma(x_n, y_n) = \frac{1}{2} + 2^{-n}$.

(ii) There exists a decreasing sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that γ_{ϵ_n} is not a Jordan curve.

as follows. Let Γ be the boundary of the domain

$$D = [-1, 2] \times [-3, 0] \cup \bigcup_{n=0}^{\infty} [2^{-n} - 2^{-n-2}, 2^{-n}] \times [0, 2^{-n-2}(1/2 + 2^{-n})].$$

Observe that Γ is a Jordan curve and it is not difficult to show that Γ is also a chord-arc. Set, for any $n \in \mathbb{N}$,

$$x_n = (2^{-n} - 2^{-n-2}, 0) \text{ and } y_n = (2^{-n}, 0).$$

Note that $\zeta_\Gamma(x_n, y_n) = \frac{1}{2} + 2^{-n}$ and that it is not hard to check that $\zeta_\Gamma = \frac{1}{2}$. Let $\Lambda_n = \Gamma(x_n, y_n)$ and $\epsilon_n = 2^{-n-3}$. Then, the set $\gamma_{\epsilon_n}^{\Lambda_n} = \{x \in \gamma_{\epsilon_n} : \mathrm{dist}(x, \Lambda_n) = \epsilon_n\}$ is the union of the line segment $\{x_n + 2^{-n-3}\} \times [0, 2^{-2n-2}]$ and two quarter-circles $\{x_n + \epsilon_n e^{i\theta} : \frac{3\pi}{2} \leq \theta \leq 2\pi\} \bigcup \{y_n + \epsilon_n e^{i\theta} : \pi \leq \theta \leq \frac{3\pi}{2}\}$. It follows that γ_{ϵ_n} is not a Jordan curve.

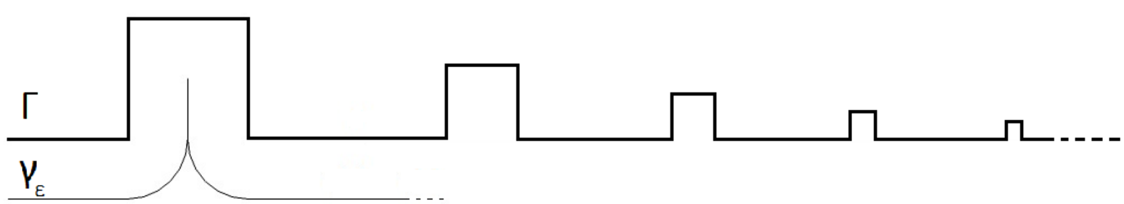

Figure 3.2: A Jordan curve with $\zeta_\Gamma(x_n, y_n) = \frac{1}{2} + 2^{-n}$ that does not satisfy the LJC property.

We now apply Lemma 3.2.7, Lemma 3.2.10, and Theorem 3.0.4 to prove Theorem 3.0.5. Recall from Lemma 3.2.7 that Δ_ϵ, if a Jordan domain, has no inward cusp. Condition $\zeta_\Gamma < 1/2$, together with the estimates (3.3.1) below, shows that Δ_ϵ *has no outward cusps.*

Proof of Theorem 3.0.5. By the assumption of the theorem, there exist $\zeta \in (0, 1/2)$ and $r_0 > 0$ such that $\zeta_\Gamma(x, y) \le \zeta$ for all $x, y \in \Gamma$ with $|x - y| \le r_0$. From Theorem 3.0.4 and its proof, γ_ϵ is a Jordan curve for every $\epsilon \in (0, r_0/2)$; by Lemma 3.1.1, Γ is a $K(\zeta)$-quasicircle, therefore satisfies the 2-point condition (2.2.1) for some constant $C(\zeta) > 1$. Constants below depend only on ζ.

We now prove that there exists $K' > 1$ depending only on ζ such that γ_ϵ is a K'-quasicircle for any

$$0 < \epsilon < \min\{\frac{r_0}{10}, \frac{\operatorname{diam}\Gamma}{20C(\zeta)}\}.$$

By the 2-point condition, it suffices to prove that there exists $M > 1$, depending only on ζ, such that

$$\operatorname{diam}\gamma_\epsilon(x, y) \le M|x - y| \quad \text{for all } x, y \in \gamma_\epsilon.$$

Given x and y in γ_ϵ, choose $x', y' \in \Gamma$ such that $|x - x'| = |y - y'| = \epsilon$; segments $[x, x']$ and $[y, y']$ do not meet except possibly at x' and y'. By Remark 3.2.11, there exists a curve $\tau_{x,y}$, with $\tau_{x,y} \setminus \{x, y\} \subset \Delta_\epsilon$, that connects x to y, and satisfies $|x - y| \le \operatorname{diam}\tau_{x,y} \le C_1(\zeta)|x - y|$ for some constant $C_1(\zeta) > 1$. Consider the domain D enclosed by the Jordan curve $[x, x'] \cup \Gamma(x', y') \cup [y, y'] \cup \tau_{x,y}$. Let $\gamma_\epsilon(x, y)^*$ be the component of $\gamma_\epsilon \setminus \{x, y\}$ that is contained in D; note that $\gamma_\epsilon(x, y)^*$ and $\gamma_\epsilon(x, y)$ are not necessarily the same arc. It suffices to show that

$$\operatorname{diam}\gamma_\epsilon(x, y)^* \simeq |x - y|.$$

We consider four cases according to the ratios $|x' - y'|/\epsilon$ and $|x - y|/\epsilon$.

Case 1. $|x' - y'| \ge 4(1 - \zeta)\epsilon$. In this case, $|x' - y'| - 2\epsilon \le |x - y| \le |x' - y'| + 2\epsilon$, which implies

$$\frac{1 - 2\zeta}{2 - 2\zeta}|x' - y'| \le |x - y| \le \frac{3 - 2\zeta}{2 - 2\zeta}|x' - y'|.$$

Since $0 < \zeta < 1/2$, $\operatorname{diam}\tau_{x,y} \simeq |x - y|$ and Γ is a $K(\zeta)$-quasicircle, we have $\operatorname{diam} D \simeq |x - y|$. Hence, $\operatorname{diam}\gamma_\epsilon(x, y)^* \simeq |x - y|$.

Case 2. $x' = y'$. In this case, $\gamma(x, y)^* = \gamma(x, y)$. By Lemma 3.2.4, $\gamma_\epsilon(x, y)$ is a subarc of $S^1(x', \epsilon)$ of length at most $\pi\epsilon$, hence $\operatorname{diam}\gamma_\epsilon(x, y) = |x - y|$.

Case 3. $0 < |x' - y'| < 4(1 - \zeta)\epsilon$ and $|x - y| \ge \epsilon(1 - 2\zeta)^2/10$. Since $\operatorname{diam} D \simeq \epsilon$ and $\gamma_\epsilon(x, y)^* \subset D$, we have $\operatorname{diam}\gamma_\epsilon(x, y)^* \simeq |x - y| \simeq \epsilon$.

Case 4. $0 < |x' - y'| < 4(1 - \zeta)\epsilon$ and $0 < |x - y| < \epsilon(1 - 2\zeta)^2/10$. In view of Lemma 3.2.10 and Remark 3.2.11, we may assume that $\operatorname{diam}\tau_{x,y} \le 5|x - y| < \epsilon/2$. It is easy to check that in this case $\gamma(x, y)^* = \gamma(x, y)$. However, there is no relation between $|x - y|$ and $|x' - y'|$, and $\operatorname{diam} D$ may be much bigger than $|x - y|$.

We construct a new domain D' whose closure contains $\gamma_\epsilon(x, y)$ and has $\operatorname{diam} D' \simeq |x - y|$.

First, let $R(x', y')$ be the rectangular domain whose boundary has two sides parallel to the line $l_{x',y'}$ of length $a = |x' - y'|$, and two other sides having mid-points x' and y' and of length $b = 2(\epsilon - \zeta|x' - y'|)$. Then define a domain

$$U(x', y') = B^2(x', \epsilon) \cup B^2(y', \epsilon) \cup R(x', y').$$

It is possible that $R(x', y')$ is contained in $B^2(x', \epsilon) \cup B^2(y', \epsilon)$ for some pairs x' and y'. Nevertheless, $\partial U(x', y')$ are K''-quasicircles for some constant $K'' > 1$ depending only on ζ, in particular not on x' and y'. This observation follows from the inequalities: $0 < \zeta < 1/2$,

$$0 < a = |x' - y'| < 4(1 - \zeta)\epsilon, \text{ and } 0 < \epsilon(1 - 2\zeta)^2 < \frac{b}{2} = \epsilon - \zeta|x' - y'| < \epsilon. \tag{3.3.1}$$

Next, we claim that $U(x', y') \cap \overline{\Delta}_\epsilon = \emptyset$. Indeed, for any $z \in R(x', y')$ the line containing z and perpendicular to $l_{x',y'}$ must intersect the arc $\Gamma(x', y')$ at some point z'. Note that $\operatorname{dist}(z, \Gamma) \leq \operatorname{dist}(z, \Gamma(x', y')) \leq |z - z'| \leq \operatorname{dist}(z, l_{x',y'}) + \operatorname{dist}(z', l_{x',y'}) < \frac{b}{2} + \zeta|x' - y'| = \epsilon$. Clearly, $\operatorname{dist}(z, \Gamma) < \epsilon$ for all $z \in B^2(x', \epsilon) \cup B^2(y', \epsilon)$.

Recall that $x \in \partial B^2(x', \epsilon) \cap \partial U(x', y')$ and $y \in \partial B^2(y', \epsilon) \cap \partial U(x', y')$. Let $T_{x,y}$ be the subarc of $\partial U(x', y')$ connecting x to y that has the smaller diameter. Then, $T_{x,y} \subset \mathbb{R}^2 \setminus \Delta_\epsilon$, and $\operatorname{diam} T_{x,y} \simeq |x - y|$ because $\partial U(x', y')$ is a K''-quasicircle.

To summarize, Δ_ϵ is a Jordan domain, x and y are two points on $\partial \Delta_\epsilon$, and $\tau_{x,y}$, $\gamma_\epsilon(x, y)$, and $T_{x,y}$ are arcs connecting x to y, with $\tau_{x,y} \setminus \{x, y\} \subset \Delta_\epsilon$, $\gamma_\epsilon(x, y) \subset \partial \Delta_\epsilon$, and $T_{x,y} \subset \mathbb{R}^2 \setminus \Delta_\epsilon$.

Let D' be the domain enclosed by the Jordan curve $\tau_{x,y} \cup T_{x,y}$. We claim that $\gamma_\epsilon(x, y)$ is contained in $\overline{D'}$. Otherwise, $\tau_{x,y}$ would be contained in the closure of the domain D'' enclosed by the Jordan curve $\gamma_\epsilon(x, y) \cup T_{x,y}$. By the connectedness of Δ_ϵ, the entire Δ_ϵ would be contained in D''. A preliminary estimate of $\operatorname{diam} \gamma_\epsilon(x, y)$ from the fact $\gamma_\epsilon(x, y) \subset D$ shows that

$$\operatorname{diam} \gamma_\epsilon(x, y) \leq 5|x - y| + 2\epsilon + C(\zeta)|x' - y'| \leq 7C(\zeta)\epsilon.$$

Therefore,

$$\operatorname{diam} \Delta_\epsilon \leq \operatorname{diam} D'' \leq \operatorname{diam} \gamma_\epsilon(x, y) + \operatorname{diam} U(x', y')$$

$$\leq 7C(\zeta)\epsilon + 4\epsilon + |x' - y'| \leq 15C(\zeta)\epsilon < \frac{3}{4} \operatorname{diam} \Gamma < \operatorname{diam} \Delta_\epsilon,$$

a contradiction. So $\gamma_\epsilon(x,y) \subset \overline{D'}$, and therefore

$$\operatorname{diam} \gamma_\epsilon(x,y) \leq \operatorname{diam} D' \leq \operatorname{diam} \tau_{x,y} + \operatorname{diam} T_{x,y} \simeq |x-y|.$$

This completes the proof of $\operatorname{diam} \gamma_\epsilon(x,y) \simeq |x-y|$ for Case 4, and the theorem. $\square$

Remark 3.3.2. *The condition $\zeta_\Gamma < 1/2$ is sharp for the conclusion of Theorem 3.0.5.*

We first make an observation. Given $\alpha \in [0, \pi/12]$, let σ be the circular arc $\{e^{i\theta} : \alpha \leq \theta \leq \pi - \alpha\}$, and Γ' be the infinite simple curve obtained by replacing the segment $[e^{i\alpha}, e^{i(\pi-\alpha)}]$ on $l_{e^{i\alpha}, e^{i(\pi-\alpha)}}$ by σ. The set of points below Γ' that have unit distance to Γ' is a simple arc γ' consisting of two horizontal semi-infinite lines and two circular arcs τ_1 and τ_2, where τ_1 is a subarc of the circle $S^1(e^{i\alpha}, 1)$ connecting 0 and $-i + e^{i\alpha}$, and τ_2 is a subarc of the circle $S^1(e^{i(\pi-\alpha)}, 1)$ connecting 0 and $-i + e^{i(\pi-\alpha)}$. Since τ_1 and τ_2 meet at an angle 2α, the arc γ' is a $K(\alpha)$-quasiline with $K(\alpha) \to \infty$ as $\alpha \to 0$.

Fix now a decreasing sequence α_n converging to 0 with $\alpha_1 = \pi/12$, and another sequence $\epsilon_n = 4^{-n-2}$. Let p_n be the point having coordinates $(2^{-n}, -\epsilon_n \sin \alpha_n)$ and σ_n be the subarc of $S^1(p_n, \epsilon_n)$ above the real axis; and let ω be the simple curve that has end points -1 and 1 and is the union of circular arcs $\bigcup_{n \geq 1} \sigma_n$ and a countable number of horizontal segments in $[0,1]$. Fix a large $N \in \mathbb{N}$, and let P be the boundary of a regular N-polygon in the lower half-plane which has $[-1,1]$ as one of its edges. Let Γ be the Jordan curve obtained from P by replacing the edge $[-1,1]$ by ω.

It is not hard to see that for sufficiently large N, Γ is a K-quasicircle for some $K > 1$ independent of N, that $\zeta_\Gamma(x,y) < 1/2$ for all $x, y \in \Gamma$ with $|x-y| \leq 1/2$, and that $\zeta_\Gamma = 1/2$.

On the other hand, every level curve γ_{ϵ_n} is a K_n-quasicircle which contains two circular arcs, with the same curvature, meeting at an angle $2\alpha_n$. Since $\alpha_n \to 0$, K_n's cannot have a uniform upper bound. So Γ does not satisfy the LQC property.

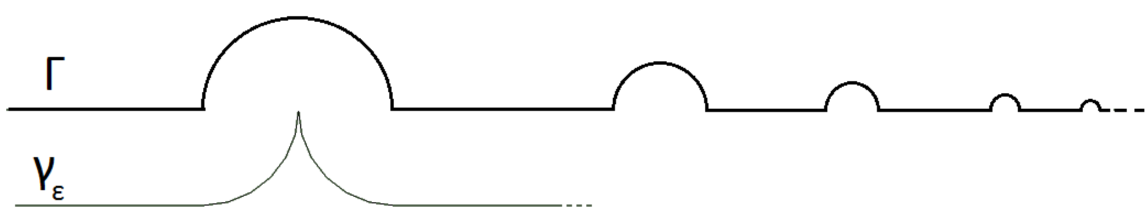

Figure 3.3: A Jordan curve with $\zeta_\Gamma(x_n, y_n) < \frac{1}{2}$ that does not satisfy the LQC property.

3.4 Level chord-arc property

In this section we give the proof of Theorem 3.0.6. We start by recalling a known fact: *if a bounded starlike domain in* $\mathbb{R}^2$ *satisfies a strong interior cone property then its boundary is a chord-arc curve.*

For $a \in (0, \pi), h > 0$, $x \in \mathbb{R}^2$ and $v \in \mathbb{S}^1$, denote by

$$\mathcal{C}_{a,h}(x, v) = \{z \in \mathbb{R}^2 \colon \cos(a/2)\,|z - x| \leq v \cdot (z - x) \leq h\}$$

the truncated cone with vertex x, direction v, height h and aperture a.

Suppose that $U \subset \mathbb{R}^2$ is a bounded *starlike domain* with respect to a point $x_0 \in U$, i.e., for every $x \in \partial U$ the line segment $[x_0, x]$ intersects ∂U only at the point x. Suppose in addition (U, x_0) satisfies the *strong interior cone property*, i.e., there exist $a \in (0, \pi), h > 0$ so that the truncated cone $\mathcal{C}_{a,h}(x, v_x) \setminus \{x\}$, in the direction $v_x = (x_0 - x)/|x_0 - x|$, is contained in U for every $x \in \partial U$. Assume from now on $x_0 = 0$, and set

$$\rho = \max\{|x| \colon x \in \partial U\}.$$

We obtain, by elementary geometry, positive constants $c_1 = c_1(a, \frac{h}{\rho}), c_2 = c_2(a), c_3 = c_3(a)$ such that

$$c_2\,|x - y| \leq |x - |x|\frac{y}{|y|}| \leq c_3|x - y|, \text{ for all } x, y \in \partial U \text{ with } |\frac{x}{|x|} - \frac{y}{|y|}| \leq c_1.$$

Let $\psi\colon \partial U \to \mathbb{S}^1$ be the map $x \mapsto \frac{x}{|x|}$. Then $\rho\psi$ is L-bi-Lipschitz for some constant $L > 1$ depending only on a and h/ρ. Therefore ∂U is a C-chord-arc curve for some constant $C > 1$ depending only on a and h/ρ.

Essential to our proof of Theorem 3.0.6 is a lemma of Brown [7] on sets of constant distance from a compact subset A of $\mathbb{R}^2$. Recall from the Introduction that for a given $\epsilon > 0$, the *ϵ-boundary* of A is the set

$$\partial_\epsilon(A) = \{x \in \mathbb{R}^2 \colon \mathrm{dist}(x, A) = \epsilon\}.$$

In Lemma 1 of his paper, Brown proved that *if* $\epsilon > \operatorname{diam} A$*, then* $\partial_\epsilon(A)$ *is the boundary of a starlike domain* U_ϵ *with respect to any point* $x_0 \in A$. In fact, whenever $\epsilon > 3 \operatorname{diam} A$, (U_ϵ, x_0) also possesses the strong interior cone property, namely, the cone $\mathcal{C}_{\frac{\pi}{3}, \frac{\epsilon}{3}}(x, (x_0 - x)/|x_0 - x|) \setminus \{x\}$ with vertex $x \in \partial_\epsilon(A)$ is contained in U_ϵ. Since $2\epsilon < \mathrm{diam}(\partial_\epsilon(A)) < 3\epsilon$, we have the following.

Lemma 3.4.1. *There is a universal constant* $c_0 > 1$ *for the following. Suppose that* A *is a compact subset of* $\mathbb{R}^2$ *and that* $\epsilon > 3 \operatorname{diam} A$*. Then the* ϵ-boundary $\partial_\epsilon(A)$ *of* A *is a* c_0*-chord arc curve.*

We now apply Lemma 3.4.1 locally and repeatedly to prove Theorem 3.0.6.

Proof of Theorem 3.0.6. For the necessity, we only need to check that Γ is a chord-arc curve. By the LCA property, there exist $L > 1$, $n_0 \in \mathbb{N}$, and for each $n \geq n_0$, an L-bi-Lipschitz homeomorphism f_n of $\mathbb{R}^2$ such that $f_n(\overline{\mathbb{B}^2}) = \overline{\Delta_{\frac{1}{n}}}$. Since $f_n|\overline{\mathbb{B}^2}$ are equicontinuous, by Arzela-Ascoli, there is a subsequence $f_{k_n}|\overline{\mathbb{B}^2}$ which converges to a homeomorphism f. It is not hard to see that f is bi-Lipschitz and maps $\overline{\mathbb{B}^2}$ onto $\overline{\Omega}$. Therefore, $\Gamma = f(\partial\mathbb{B}^2)$ is a chord-arc curve.

To show the sufficiency, we assume that Γ is a C_1-chord-arc curve, and that there exist $\epsilon_0 > 0$ and $K > 1$ such that the Jordan curves γ_ϵ are K-quasicircles for all $\epsilon \in (0, \epsilon_0]$. In the rest of the proof, constants are understood to depend on C_1 and K only, in particular independent of ϵ.

For $\epsilon \in (0, \epsilon_0]$ and for a closed subset $\lambda \subset \gamma_\epsilon$, we set

$$\Gamma^\lambda = \{y \in \Gamma \colon |y - x| = \epsilon \text{ for some } x \in \lambda\} = \{y \in \Gamma \colon \operatorname{dist}(y, \lambda) = \epsilon\}.$$

In general, Γ^λ need not be connected, and there is no relation between the diameter of λ and the diameter of Γ^λ.

We prove now that γ_ϵ is a chord-arc curve. Since γ_ϵ is a K-quasicircle, it suffices to check

$$\ell(\lambda) \lesssim \operatorname{diam} \lambda \quad \text{ for all subarcs } \lambda \subset \gamma_\epsilon.$$

We consider three cases according to the diameter of Γ^λ.

Case 1. $\operatorname{diam} \Gamma^\lambda \leq \epsilon/10$. Set

$$\partial_\epsilon(\Gamma^\lambda) = \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x, \Gamma^\lambda) = \epsilon\}.$$

After a moment of reflection, we see that $\lambda \subset \partial_\epsilon(\Gamma^\lambda)$. By Lemma 3.4.1, there exists a universal constant $c_0 > 1$ such that, for any $x, y \in \partial_\epsilon(\Gamma^\lambda)$,

$$\ell(\partial_\epsilon(\Gamma^\lambda)(x, y)) \leq c_0|x - y|;$$

recall that $\partial_\epsilon(\Gamma^\lambda)(x, y)$ is the subarc of $\partial_\epsilon(\Gamma^\lambda)$ connecting x and y that has the smaller diameter. We deduce from this the following

$$\ell(\lambda) \leq c_0 \operatorname{diam} \lambda.$$

To prepare for the next two cases, we take Λ to be the subarc of Γ that contains Γ^λ having the smallest diameter. Subdivide Λ into subarcs $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$ which have mutually disjoint interiors and satisfy the condition

$$\epsilon/100 \leq \operatorname{diam} \Lambda_n < \epsilon/10 \quad \text{for all } n = 1, \ldots, N.$$

Since Γ is a quasicircle, $\operatorname{diam} \Lambda \simeq \operatorname{diam} \Gamma^\lambda$; since Γ is a C_1-chord-arc curve $N\epsilon \simeq \operatorname{diam} \Lambda$. So,

$$N \simeq \epsilon^{-1} \operatorname{diam} \Lambda \simeq \epsilon^{-1} \operatorname{diam} \Gamma^\lambda.$$

Set $\lambda_n = \gamma_\epsilon^{\Lambda_n} \cap \lambda$ for $n = 1, \ldots, N$. Again, after a moment of reflection, we see that $\lambda = \bigcup_{n=1}^N \lambda_n$. Recall, from Lemma 3.2.3, that $\gamma_\epsilon^{\Lambda_n} = \{x \in \gamma_\epsilon \colon \operatorname{dist}(x, \Lambda_n) = \epsilon\}$ are arcs whenever they are nonempty, so λ_n are subarcs of γ_ϵ. Note however that some of $\{\lambda_n\}$ may overlap. We now apply Lemma 3.4.1 to the ϵ-boundary $\partial_\epsilon(\Lambda_n)$ of Λ_n. Since λ_n is also a subarc of $\partial_\epsilon(\Lambda_n)$, it follows, as in Case 1, that

$$\ell(\lambda_n) \leq c_0 \operatorname{diam} \lambda_n \lesssim \epsilon.$$

Case 2. $\epsilon/10 < \operatorname{diam} \Gamma^\lambda \leq 10\epsilon$. From the estimates above, we obtain

$$\ell(\lambda) \leq \sum_{n=1}^N \ell(\lambda_n) \leq \sum_{n=1}^N c_0 \operatorname{diam} \lambda_n \leq N c_0 \operatorname{diam} \lambda \simeq \operatorname{diam} \lambda.$$

Note that in this case, diameter of λ might be much smaller than ϵ.

Case 3. $10\epsilon < \operatorname{diam} \Gamma^\lambda$. In this case, it is geometrically evident that

$$\operatorname{diam} \Gamma^\lambda - 2\epsilon \leq \operatorname{diam} \lambda \leq \operatorname{diam} \Gamma^\lambda + 2\epsilon,$$

hence $\operatorname{diam} \lambda \simeq \operatorname{diam} \Gamma^\lambda$. Therefore,

$$\ell(\lambda) \leq \sum_{n=1}^N \ell(\lambda_n) \leq \sum_{n=1}^N c_0 \operatorname{diam} \lambda_n \lesssim N\epsilon \simeq \operatorname{diam} \Gamma^\lambda \simeq \operatorname{diam} \lambda. \qquad \square$$

Remark 3.4.2. *Suppose that Γ is a Jordan curve. The proof of the previous theorem shows that each level set γ_ϵ, with $\epsilon \neq 0$, is contained in a finite union of c_0-chord-arc curves, and that if γ_ϵ is a quasicircle with $\epsilon \neq 0$, then it is a $C(\Gamma, \epsilon)$-chord-arc curve.*

We conclude this section with the proof of Theorem 1.3.2. The following simple lemma is needed in the proof.

Lemma 3.4.3. *The radial projection* $f\colon \{x \in \mathbb{R}^2 \colon |x| > 1\} \to \mathbb{S}^1$ *with* $f(re^{i\theta}) = e^{i\theta}$ *is* 1*-Lipschitz.*

Proof. It suffices to check the 1-Lipschitz condition for $x_1 = 1, x_2 = re^{i\theta}$ with $r \geq 1$. We have,

$$|x_1 - x_2| = \geq \sqrt{r^2 - 2r\cos\theta + 1} \geq \sqrt{1 - 2\cos\theta + 1} = |f(x_1) - f(x_2)|. \qquad \square$$

Proof of Theorem 1.3.2. A Jordan curve which satisfies a local $\pi/2$-chord-arc condition but fails the LCA property is given in Remark 3.3.2. It remains to prove the first claim. In view of Theorem 3.0.6, it suffices to show that if Γ satisfies a local C-chord-arc property with $C \in [1, \pi/2)$ then Γ satisfies the LQC property.

Fix $C \in [1, \pi/2)$ and suppose that there exists ϵ_0 such that, for each $x, y \in \Gamma$ with $|x - y| < \epsilon_0$, $\ell(\Gamma'(x, y)) < C|x - y|$. We may further assume that $\epsilon_0 < \frac{1}{4C}\operatorname{diam}\Gamma$. In this case, the arc $\Gamma'(x, y)$ of shorter length joining x, y coincides with the arc $\Gamma(x, y)$ of shorter diameter joining x, y.

We claim that for any $\epsilon \in (0, \epsilon_0]$ sufficiently small and any $x \in \gamma_\epsilon$, the set $\Gamma^{\{x\}}$ is contained in a subarc of $S^1(x, \epsilon)$ of length at most $2C\epsilon$. By Lemma 3.2.12 we already know that $\Gamma^{\{x\}}$ is contained in a semicircle of $S^1(x, \epsilon)$. Let S be the smallest subarc of $S^1(x, \epsilon)$ containing $\Gamma^{\{x\}}$ and let x_1, x_2 be the endpoints of S. It is easy to see that $x_1, x_2 \in \Gamma^{\{x\}}$. Since $\operatorname{dist}(x, \Gamma) = \epsilon$ the curve $\Gamma(x_1, x_2)$ can not intersect $B^2(x, \epsilon)$. Since the radial projection of $\Gamma(x_1, x_2)$ on $S^1(x, \epsilon)$ is 1-Lipschitz, and $\ell(\Gamma(x_1, x_2)) \leq C|x_1 - x_2| \leq 2C\epsilon$, we have that x_1, x_2 are in a subarc of $S^1(x, \epsilon)$ of length at most $2C\epsilon$ and the claim follows.

Next, we show that Γ has the LJC property. Suppose that for some $\epsilon > 0$ sufficiently small, the set Δ_ϵ is not connected. Then for a suitable $\epsilon' \in (\epsilon, 2\epsilon)$, the set $\gamma_{\epsilon'} \cup \Delta_{\epsilon'}$ is not connected. By Proposition 3.2.18, there exist $\delta > 0$ and points $x_0 \in \gamma_\delta$, $x_1, x_2 \in \Gamma$ such that $|x_0 - x_1| = |x_0 - x_2| = \delta$. The latter contradicts the above claim. Suppose now that for some $\epsilon > 0$ sufficiently small, $\gamma_\epsilon \cup \Delta_\epsilon$ is connected, and $\overline{\Delta}_\epsilon \subsetneq \gamma_\epsilon \cup \Delta_\epsilon$. Then by Proposition 3.2.16 there exist points $x_0 \in \gamma_\epsilon$, $x_1, x_2 \in \Gamma$ such that $|x_0 - x_1| = |x_0 - x_2| = \epsilon$. The latter contradicts the above claim again. Combining these observations with Lemma 3.2.9, we conclude that there exists $r_0 > 0$ such that γ_ϵ is a Jordan curve for each $\epsilon \in [0, r_0]$.

We now prove that there exists $K' > 1$ depending only on C such that γ_ϵ is a K'-quasicircle for any

$$0 < \epsilon < \min\{\frac{r_0}{10}, \frac{\operatorname{diam}\Gamma}{20C}\}.$$

By the 2-point condition, it suffices to prove that there exists $M > 1$, depending only on ζ, such that

$$\operatorname{diam}\gamma_\epsilon(x, y) \leq M|x - y| \quad \text{for all } x, y \in \gamma_\epsilon.$$

Given x and y in γ_ϵ, choose $x', y' \in \Gamma$ such that $|x - x'| = |y - y'| = \epsilon$; segments $[x, x']$ and $[y, y']$ do not meet except possibly at x' and y'. Let $\tau_{x,y}$, D and $\gamma_\epsilon(x,y)^*$ be as in the proof of Theorem 3.0.5. It suffices to show that $\operatorname{diam}\gamma_\epsilon(x,y)^* \simeq |x-y|$. We consider four cases according to the ratios $|x'-y'|/\epsilon$ and $|x-y|/\epsilon$.

Case 1. $|x'-y'| \geq \frac{2\pi}{\pi+C}\epsilon$. The proof of this case is similar to that of Case 1 in Theorem 3.0.5.

Case 2. $x' = y'$. The proof of this case is similar to that of Case 2 in Theorem 3.0.5.

Case 3. $0 < |x'-y'| < \frac{2\pi}{\pi+C}\epsilon$ and $|x-y| \geq \epsilon/10$. The proof of this case is similar to that of Case 3 in Theorem 3.0.5.

Case 4. $0 < |x'-y'| < \frac{2\pi}{\pi+C}\epsilon$ and $0 < |x-y| < \epsilon/10$. The difference between the proof of this case and that of Case 4 in Theorem 3.0.5 is that instead of a rectangle we use a disk.

In view of Lemma 3.2.10 and Remark 3.2.11, we may assume that $\operatorname{diam}\tau_{x,y} \leq 5|x-y| < \epsilon/2$. It is easy to check that in this case $\gamma(x,y)^* = \gamma(x,y)$. However, there is no relation between $|x-y|$ and $|x'-y'|$, and $\operatorname{diam} D$ may be much bigger than $|x-y|$. We construct a new domain D' whose closure contains $\gamma_\epsilon(x,y)$ and has $\operatorname{diam} D' \simeq |x-y|$.

Let z be the midpoint of $[x',y']$ and $r = \epsilon - \frac{C}{\pi}|x'-y'|$. The assumptions on x', y' yield

$$r = \epsilon - \frac{C}{\pi}|x'-y'| > \frac{\pi - 2C}{\pi + C}\epsilon.$$

Then define a domain

$$U(x',y') = B^2(x',\epsilon) \cup B^2(y',\epsilon) \cup B^2(z,r).$$

It is possible that $B(z,r)$ is contained in $B(x',\epsilon) \cup B(y',\epsilon)$ for some pairs x' and y'. Nevertheless,

$$|x'-y'| - 2\epsilon < \frac{\pi - 2C}{\pi + C}\epsilon$$

which implies that $B^2(z,r)$ intersects both $B(x',\epsilon)$ and $B^2(y',\epsilon)$ and each intersection is a set of diameter comparable to ϵ. It follows that $\partial U(x',y')$ are K''-quasicircles for some constant $K'' > 1$ depending only on C, in particular not on x' and y'.

Next, we claim that $U(x',y') \cap \overline{\Delta}_\epsilon = \emptyset$. Clearly, $\operatorname{dist}(z,\Gamma) < \epsilon$ for all $z \in B^2(x',\epsilon) \cup B^2(y',\epsilon)$. Let $r' = \frac{C}{\pi}|x'-y'|$. Note that $\Gamma(x',y')$ intersects $\overline{B}^2(z,r')$. Otherwise, the radial projection of $\Gamma(x',y')$ on $S^1(z,r')$ would contain at least a semicircle and by Lemma 3.4.3 $\ell(\Gamma(x',y')) > \pi r' = C|x'-y'|$ which is a contradiction. Since $\Gamma(x',y')$ intersects $\overline{B}^2(z,r')$, for each $w \in B^2(z,r)$, we have that $\operatorname{dist}(w,\Gamma) \leq \operatorname{dist}(z,\Gamma) - |w-z| < r' - r = \epsilon$. The latter implies that $\Delta(\epsilon)$ does not intersect $B^2(z,r')$ and, consequently, $U(x',y') \cap \overline{\Delta}_\epsilon = \emptyset$.

The rest of the proof is similar to that of Case 4 in Theorem 3.0.5 replacing $C(\zeta)$ with C. $\square$

3.5 Examples from Rohde's snowflakes

Fix a natural number $N \geq 4$. Suppose that a regular N-gon, of unit side length, is used in place of the unit square in the first step of Rohde's construction, while the remaining steps are unchanged. So each snowflake-type curve is the limit of a sequence of polygons, having $N4^{n-1}$ edges at the n-th stage. Let $\mathcal{F}_N$ be the family of these snowflakes. Then Rohde's argument shows that every quasicircle in $\mathbb{R}^2$ is the image of a curve in $\mathcal{F}_N$ under a bi-Lipschitz homeomorphism of $\mathbb{R}^2$.

Let $\mathcal{F}_{N,p}$ be the subfamily of curves in $\mathcal{F}_N$ constructed using only the Type I and Type II polygonal arcs of Figure 2.1. The following result is a corollary of Theorem 3.0.5.

Corollary 3.5.1. *There exist $N_0 > 4$ and $p_0 \in (\frac{1}{4}, \frac{1}{2})$ for the following. Given $N \geq N_0$ and $1/4 \leq p \leq p_0$, there exists $0 < \zeta_{N,p} < 1/2$ and $r_{N,p} > 0$ such that every curve $\mathcal{S} \in \mathcal{F}_{N,p}$ has the $(\zeta_{N,p}, r_{N,p})$-chordal property, and therefore satisfies the LQC property.*

Chapter 4

Quasisymmetric spheres over quasidisks – Geometric construction

Let Ω be a Jordan domain with boundary Γ , φ be a homeomorphism of $[0, \infty)$ onto itself and

$$\Sigma(\Gamma, \varphi) = \{(x, z)\colon x \in \overline{\Omega}, z = \pm\varphi(\mathrm{dist}(x, \Gamma))\}.$$

It is easy to verify that $\Sigma(\Gamma, \varphi)$ is a topological 2-sphere. Define $\Sigma^+(\Gamma, \varphi) = \Sigma(\Gamma, \varphi) \cap \mathbb{R}^3_+$ and $\Sigma^-(\Gamma, \varphi) = \Sigma(\Gamma, \varphi) \cap \mathbb{R}^3_-$.

The goal of this chapter is to prove Theorem 1.2.1.

In Section 4.1 we show that the LQC property of the curves Γ and the LLC property of the surfaces $\Sigma(\Gamma, \varphi)$ are intimately related. In particular, Proposition 4.1.1 states that the surface $\Sigma(\Gamma, \varphi)$ is LLC for each $\varphi \in \mathcal{F}_1$ if and only if Γ has the LQC property.

In Proposition 4.2.2, assuming that Γ satisfies the LCA property, we show that $\Sigma(\Gamma, h)$ is Ahlfors 2-regular for each $\phi \in \mathcal{F}_1$. In fact, as it turns out, the limit property of functions in $\mathcal{F}_1$ is not necessary. The first claim of Theorem 1.2.1 follows from Theorem 2.2.4 of Bonk and Kleiner, Proposition 4.1.1 and Proposition 4.2.2.

The proof of claim (2) of Theorem 1.2.1 consists of two steps. By Proposition 4.1.1, we have that if $\Sigma(\Gamma, \varphi)$ has the LLC property for all $\varphi \in \mathcal{F}_1$ satisfying $\lim_{t\to 0} \varphi(t)/t = \infty$ then Γ has the LQC property. The necessity of the chord-arc condition follows from Proposition 4.3.1 which states that if Γ satisfies the LQC property but is not a chord-arc then there exists some function $\varphi \in \mathcal{F}_1$ such that $\Sigma(\Gamma, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$. Then, since Γ is a chord-arc surve and has the LQC property, Proposition 3.0.6 implies that Γ has the LCA property.

4.1 The LLC and the LQC properties

In this section we prove the following proposition which connects the notion of the LQC property for Γ and the LLC property for $\Sigma(\Gamma, \varphi)$.

Proposition 4.1.1. *Suppose that Γ is a Jordan curve.*

1. *If Γ has the level quasicircle property then $\Sigma(\Gamma,\varphi)$ has the LLC property for all $\varphi \in \mathcal{F}_1$.*

2. *If $\Sigma(\Gamma,\varphi)$ has the LLC property for all $\varphi \in \mathcal{F}_1$ satisfying $\lim_{t\to 0}\varphi(t)/t = \infty$ then Γ has the level quasicircle property.*

The proof of the first claim follows from Lemma 4.1.2 and Lemma 4.1.3 while the proof of the second claim follows from Lemma 4.1.5 and Lemma 4.1.6.

Lemma 4.1.2. *Suppose that Γ is a K-quasicircle such that γ_ϵ is a K-quasicircle for any $\epsilon \in (0,\epsilon_0]$. Let φ be a self homeomorphism of $[0,+\infty)$ which is L-Lipschitz in $[\epsilon_0,\infty)$ and, for some $M, t_0 > 0$, satisfies $\varphi(t) > Mt$ for any $t \in [0,t_0]$. Then, $\Sigma(\Gamma,\varphi)$ is $\lambda - LLC_1$ with λ depending on K, L, M, t_0, $\operatorname{diam}\Gamma$.*

Proof. Let $S^+_{\epsilon_0}, S^-_{\epsilon_0}$ be surfaces contained in $\Sigma^+(\Gamma,\varphi), \Sigma^-(\Gamma,\varphi)$ respectively such that their projection on $\mathbb{R}^2\times\{0\}$ are Δ_{ϵ_0}. We start by proving that $\overline{S^+_{\epsilon_0}}, \overline{S^-_{\epsilon_0}}$ are quasisymmetric to the closed unit disc. Since γ_{ϵ_0} is a K-quasicircle, Δ_{ϵ_0} is η'-quasisymmetric to $\mathbb{B}^2$ for some η' depending on K. Consider now the map $F_1\colon \Delta_{\epsilon_0} \to S^+_{\epsilon_0}$ with $F(x) = (x, \varphi(\operatorname{dist}(x,\Gamma))$. Since, φ is L-Lipschitz and the distance function is 1-Lipschitz,

$$|x-y| \le |F(x)-F(y)| \le |x-y| + |\varphi(\operatorname{dist}(x,\Gamma) - \varphi(\operatorname{dist}(y,\Gamma)| \le (L+1)|x-y|.$$

Thus, F is $(L+1)$-bi-Lipschitz and Δ_{ϵ_0} is η-quasisymmetric to $\mathbb{B}^2$ with η depending only on K, L. We work similarly for $S^-_{\epsilon_0}$.

Now we prove the LLC_1 property for $\Sigma(\Gamma,\varphi)$. It suffices to show that there exists a $\lambda > 1$ such that, for any two points $y_1, y_2 \in \Sigma(\Gamma,\varphi)$, there exists a curve γ in $\Sigma(\Gamma,\varphi)$ joining y_1, y_2 such that $\operatorname{diam}\gamma \le \lambda|y_1-y_2|$. The latter implies that $\Sigma(\Gamma,\varphi)$ is $(1+2\lambda) - \mathrm{LLC}_1$.

Since all level curves $\{\gamma_\epsilon\}_{0\le\epsilon<\epsilon_0}$ are K-quasicircles, they satisfy the 2-points condition (2.2.1) for some $C > 1$. The proof is now divided into the following three cases.

Case 1. Suppose that $y_1, y_2 \in \overline{S^+_{\epsilon_0}}$ or $y_1, y_2 \in \overline{S^-_{\epsilon_0}}$. Since $\overline{\mathbb{B}^2}$ is $1 - \mathrm{LLC}_1$ and F is η-quasisymmetric then $\overline{S^+_{\epsilon_0}}$ and $\overline{S^-_{\epsilon_0}}$ are $\lambda' - \mathrm{LLC}_1$ for some λ' depending on η, thus on K, L.

Case 2. Suppose that $y_1 \notin \overline{S^+_{\epsilon_0}} \cup \overline{S^+_{\epsilon_0}}$ and y_1, y_2 are in the same half-space. Assume, for instance, that $y_1, y_2 \in \Sigma^+(\Gamma,\varphi)$ and $y_1 \notin S^+_{\epsilon_0}$. Let $\epsilon_1 \le \epsilon_2$ be such that $\pi(y_1) \in \gamma_{\epsilon_1}$ and $\pi(y_2) \in \gamma_{\epsilon_2}$. Take y_1' in $\Sigma(\Gamma,\varphi)\cap\mathbb{R}^3_+$ such that $\pi(y_1') \in \gamma_{\epsilon_1}$ and $|\pi(y_2) - \pi(y_1')| = \epsilon_2 - \epsilon_1$.

Since φ is increasing in $[\epsilon_1, \epsilon_2]$ we have that

$$\begin{aligned}|y_2 - y_1'| &\leq |\pi(y_2) - \pi(y_1')| + \varphi(\epsilon_2) - \varphi(\epsilon_1)\\ &\leq |\pi(y_2) - \pi(y_1)| + \varphi(\epsilon_2) - \varphi(\epsilon_1)\\ &\leq 2|y_1 - y_2|\end{aligned}$$

and

$$\begin{aligned}|y_1 - y_1'| &= |\pi(y_1) - \pi(y_1')|\\ &\leq |\pi(y_1) - \pi(y_2)| + |\pi(y_2) - \pi(y_1')|\\ &\leq 2|\pi(y_1) - \pi(y_2)|\\ &\leq 2|y_1 - y_2|.\end{aligned}$$

Since γ_{ϵ_1} satisfies (2.2.1) for some $C > 1$, there exists a path $\sigma_1' \subset \gamma_{\epsilon_1}$ joining $\pi(y_1'), \pi(y_1)$ such that

$$\operatorname{diam} \sigma_1' \leq C|\pi(y_1') - \pi(y_1)| \leq C|y_1' - y_1| \leq 2C|y_1 - y_2|.$$

Denote with $\sigma_1 \subset \Sigma^+(\Gamma, \varphi)$ the path joining y_1', y_1 such that $\pi(\sigma_1) = \sigma_1'$. Let σ_2 be a curve on $\Sigma^+(\Gamma, \varphi)$ such that $\pi(\sigma_2) = [\pi(y_2), \pi(y_1')]$. Since φ is increasing,

$$\operatorname{diam} \sigma_2 \leq \varphi(\epsilon_2) - \varphi(\epsilon_1) + \epsilon_2 - \epsilon_1 \leq 2|y_2 - y_1'| \leq 4|y_1 - y_2|.$$

Then, $\sigma = \sigma_1 \cup \sigma_2 \subset \Sigma(\Gamma, \varphi)$ joins y_1, y_2 and $\operatorname{diam} \sigma \leq (2C + 4)|y_1 - y_2|$.

Case 3. Suppose that y_1, y_2 are in different half-spaces. Assume, for instance, that $y_1 \in \Sigma^+(\Gamma, \varphi)$ and $y_2 \in \Sigma^-(\Gamma, \varphi)$. If $\pi(y_1) \in \gamma_{\epsilon_1}$ and $y_2 \in \pi(\gamma_{\epsilon_2})$ then $|y_1 - y_2| \geq \varphi(\epsilon_1) + \varphi(\epsilon_2)$. From the assumption for φ it follows that $\varphi(t) \geq M't$ for all $t \in [0, \operatorname{diam} \Gamma]$ with $M' = Mt_0/\operatorname{diam} \Gamma$.

Take $y_1', y_2' \in \Gamma$ such that $|\pi(y_1) - y_1'| = \epsilon_1$ and $|\pi(y_2) - y_2'| = \epsilon_2$. There are paths $\sigma_1 \subset \Sigma(\Gamma, \varphi) \cap \mathbb{R}^3_+$, $\sigma_2 \subset \Sigma(\Gamma, \varphi) \cap \mathbb{R}^3_-$ joining y_1, y_1' and y_2, y_2' respectively, such that $\pi(\sigma_1) = [\pi(y_1), y_1']$ and $\pi(\sigma_2) = [\pi(y_2), y_2']$. Since φ is increasing in $[0, \epsilon_1]$ and in $[0, \epsilon_2]$,

$$\operatorname{diam} \sigma_1 \leq |y_1 - y_1'| \leq \epsilon_1 + \varphi(\epsilon_1) \leq (1 + 1/M')\varphi(\epsilon_1) \leq (1 + 1/M')|y_1 - y_2|$$

Similarly, $\operatorname{diam} \sigma_2 \leq (1 + 1/M')|y_1 - y_2|$.

Since Γ is K-quasicircle, there exists path $\sigma_3 \subset \Gamma$ joining y_1', y_2' such that $\operatorname{diam} \sigma_3 \leq C|y_1' - y_2'|$ with $C > 1$ as above. It follows that

$$\operatorname{diam} \sigma_3 \leq C(|\pi(y_1) - \pi(y_2)| + \epsilon_1 + \epsilon_2) \leq C(2 + 1/M')|y_1 - y_2|.$$

Note that the path $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$ joins y_1, y_2 in $\Sigma(\Gamma, \varphi)$ and

$$\operatorname{diam} \sigma \leq C(4 + 3/M')|y_1 - y_2|. \qquad \square$$

Lemma 4.1.3. *Suppose that Γ is a K-quasicircle such that γ_ϵ is a K-quasicircle for any $\epsilon \in (0, \epsilon_0]$. Let φ be a self homeomorphism of $[0, +\infty)$ which is L-Lipschitz in $[\epsilon_0/3, \infty)$. Then, $\Sigma(\Gamma, \varphi)$ is $\lambda - LLC_2$ with λ depending on K, L, ϵ_0, φ and* $\operatorname{diam} \Gamma$.

Proof. Let $x \in \Sigma(\Gamma, \varphi)$, $r > 0$ and $y_1, y_2 \in \Sigma(\Gamma, \varphi) \setminus B(x, r)$. If the set $\Sigma(\Gamma, \varphi) \setminus B(x, r)$ is nonempty then

$$r < \operatorname{diam} \Sigma(\Gamma, \varphi) \leq \operatorname{diam} \Gamma + 2\varphi(\operatorname{diam} \Gamma).$$

We need to prove that there exists a $\lambda > 1$ such that for any $y_1, y_2 \in \Sigma(\Gamma, \varphi) \setminus B^3(x, r)$ there exists a continuum $E \subset \Sigma(\Gamma, \varphi) \setminus B^3(x, r/\lambda)$ that contains y_1, y_2. It suffices to show this claim for some $\lambda' > 1$ when

$$r \leq r^* = \min\{\epsilon_0/3, \varphi(\epsilon_0/3)\}.$$

Then, if $r > r^*$, note that y_1, y_2 can be connected in $\Sigma(\Gamma, \varphi) \setminus B^3(x, r/(H\lambda'))$ where

$$H = \frac{\operatorname{diam} \Gamma + 2\varphi(\operatorname{diam} \Gamma)}{r^*}.$$

As in Lemma 4.1.2, for each $\epsilon > 0$, we denote with S_ϵ^+ (resp. S_ϵ^-) the subset of $\Sigma^+(\Gamma, \varphi)$ (resp. $\Sigma^-(\Gamma, \varphi)$) whose projection $\pi(S_\epsilon^+)$ (resp. $\pi(S_\epsilon^-)$) is the domain Δ_ϵ. Since $r < \epsilon_0/3$, it is enough to divide the proof in the following two cases.

Case 1. Suppose that $B^3(x, r) \cap \Sigma(\Gamma, \varphi) \subset \overline{S_{\epsilon_0/3}^+} \cup \overline{S_{\epsilon_0/3}^-}$. Assume that $x \in S_{\epsilon_0/3}^+$. Note that since $\pi(x) \in \Delta_{\epsilon_0/3}$, $r < \varphi(\epsilon_0/3)$ and φ is increasing we have that $B^3(x, r) \cap \mathbb{R}^3_- = \emptyset$ and, thus, $B^3(x, r) \cap \Sigma(\Gamma, \varphi) \subset \overline{S_{\epsilon_0/3}^+}$. We know from the proof of Lemma 4.1.2 that $\overline{S_{\epsilon_0/3}^+}$ is η-quasisymmetric to $\overline{\mathbb{B}^2}$ where η depends only on K, L. Since the LLC property is invariant under quasisymmetric maps and $\mathbb{B}^2 \times \{0\}$ is $1 - \mathrm{LLC}$, we have that $\overline{S_{\epsilon_0/3}^+}$ is $\lambda_1 - \mathrm{LLC}_2$ for some λ_1 depending on K, L.

Case 1.1. If $y_1, y_2 \in \overline{S_{\epsilon_0/3}^+}$, since $\overline{S_{\epsilon_0/3}^+}$ is $\lambda_1 - \mathrm{LLC}_2$, we can find a curve $\gamma \subset \overline{S_{\epsilon_0/3}^+} \setminus B^3(x, r/\lambda_1)$ that connects y_1 with y_2.

Case 1.2. If $y_1, y_2 \in \Sigma(\Gamma,\varphi)\setminus S^+_{\epsilon_0/3}$, then note that $E = \Sigma(\Gamma,\varphi)\setminus S^+_{\epsilon_0/3}$ is a continuum in $\Sigma(\Gamma,\varphi)\setminus B(x,r)$ that contains y_1, y_2.

Case 1.3. If $y_1 \in S^+_{\epsilon_0/3}$ and $y_2 \in \Sigma(\Gamma,\varphi)\setminus S^+_{\epsilon_0/3}$, let y_1' be a point in $\gamma_{\epsilon_0/3}\times\{\varphi(\epsilon_0/3)\}$ that does not belong to $B^3(x,r)$. Then, apply Case 1.1 for y_1, y_1' and Case 1.2 for y_2, y_1'.

Case 2. Suppose that $B^3(x,r)\cap(S^+_{\epsilon_0}\cup S^-_{\epsilon_0}) = \emptyset$. Note that $B^2(\pi(x),r)\cap\Delta_{\epsilon_0} = \emptyset$.

Case 2.1. If $y_1, y_2 \in S^+_{\epsilon_0}$ or $y_1, y_2 \in S^-_{\epsilon_0}$, note that both $S^+_{\epsilon_0}\setminus B^3(x,r)$ and $S^-_{\epsilon_0}\setminus B^3(x,r)$ are continua.

Case 2.2. If $y_1, y_2 \in \Sigma(\Gamma,\varphi)\setminus(S^+_{\epsilon_0}\cup S^-_{\epsilon_0})$ let $\epsilon_1,\epsilon_2 \in [0,\epsilon_0]$ be such that $\pi(y_1)\in\gamma_{\epsilon_1}$ and $\pi(y_2)\in\gamma_{\epsilon_2}$. For simplicity, we may assume that $y_1 \in \mathbb{R}^3_+$ and $y_1 \in \mathbb{R}^3_-$. The other cases are treated similarly. Fix $x_0 \in \Delta_{\epsilon_0}$ and a half-line $l \subset \mathbb{R}^2$, with starting point x_0, such that $B^2(\pi(x),r)\cap l = \emptyset$. We use the line l and the continuity of the distance function to find points $y_1', y_2' \in \Sigma(\Gamma,\varphi)$ such that $y_1' = (\pi(y_1),\varphi(\epsilon_1))$, $y_2' = (\pi(y_2), -\varphi(\epsilon_2))$ and $\pi(y_1'), \pi(y_2') \in l$. Since $\gamma_{\epsilon_1}, \gamma_{\epsilon_2}$ are K-quasicircles, by Remark 2.3.3, there exists λ_2 depending on K, such that y_1, y_1' are contained in a curve $\gamma_1 \subset \Sigma(\Gamma,\varphi)\setminus B^3(x, r/\lambda_2)$ and y_2, y_2' are contained in a curve $\gamma_2 \subset \Sigma(\Gamma,\varphi)\setminus B^3(x,r/\lambda_2)$. Finally, consider a curve $\gamma_3 \subset \Sigma(\Gamma,\varphi)$ which joins y_1', y_2' and $\pi(\gamma_3)\subset l$. Then, y_1, y_2 can be joined in $\Sigma(\Gamma,\varphi)\setminus B^3(x,r/\lambda_2)$ via $\gamma = \gamma_1\cup\gamma_2\cup\gamma_3$.

Case 2.3. If $y_1 \in \Sigma(\Gamma,\varphi)\setminus(S^+_{\epsilon_0}\cup S^-_{\epsilon_0})$ and $y_2 \in S^+_{\epsilon_0}\cup S^-_{\epsilon_0}$, as with Case 2.2, fix a half-line $l' \subset \mathbb{R}^2$ with starting point $\pi(y_2)$ that does not intersect $B^2(\pi(x),r)$. Since the distance function is continuous, there exists a point $y_2' \in \Sigma(\Gamma,\varphi)$ on the same horizontal plane as y_2 such that $\pi(y_2') \in \gamma_{\epsilon_0}\cap l'$. Now apply Case 2.1 for y_2, y_2' and Case 2.2 for y_1, y_2'.

Case 2.4. If $y_1 \in S^+_{\epsilon_0}$ and $y_2 \in S^-_{\epsilon_0}$ then, as with Case 2.1, fix half-lines $l_1, l_2 \subset \mathbb{R}^2$ such that for each $i = 1,2$ the line l_i has starting point $\pi(y_i)$ and does not intersect $B^2(\pi(x),r)$. For each $i = 1,2$ find point $y_i' \in \Sigma(\Gamma,\varphi)$, on the same horizontal plane as y_i such that $\pi(y_i') \in \gamma_{\epsilon_0}\cap l_i$. Now apply Case 2.1 for the pairs y_1, y_1' and y_2, y_2' and Case 2.2 for the pair y_1', y_2'. □

We now turn to the proof of the second claim of Proposition 4.1.

Lemma 4.1.4. *Suppose that Γ is a Jordan curve and that φ is a function in $\mathcal{F}$ whose almost everywhere derivative satisfies $\lim_{t\to 0}\varphi'(t) = +\infty$. If $\Sigma(\Gamma,\varphi)$ is $\lambda - LLC_1$ then there exists $\epsilon_0 > 0$ depending on λ, φ such that the set γ_ϵ is Jordan curve for any $0 < \epsilon \leq \epsilon_0$.*

Proof. Suppose that $\Sigma(\Gamma,\varphi)$ is $\lambda - \mathrm{LLC}_1$ for some $\lambda > 1$. Since $\lim_{t\to 0}\varphi'(t) = +\infty$, there exists t_0 such that

$$\varphi(t_2) - \varphi(t_1) > 6\lambda(t_2 - t_1) \text{ for any } 0 < t_1 \leq t_2 \leq t_0. \tag{4.1.1}$$

The proof contains two steps. In the first step we prove that the set Δ_ϵ is connected when $\epsilon < t_0$.

By Lemma 3.2.9, the latter implies that Δ_ϵ is a Jordan domain. Then, in the second step, we prove that $\gamma_\epsilon = \partial\Delta_\epsilon$ when $\epsilon < t_0$ which, combined with Step 1, gives that γ_ϵ is a Jordan curve when $\epsilon < t_0$.

Step 1. We claim that for any $0 < \epsilon < t_0$, the open set Δ_ϵ is connected. On the contrary, assume that there exists $\epsilon < t_0$ such that $\gamma_\epsilon \cup \Delta_\epsilon$ is not connected. Then, the open set Δ_ϵ would have at least two components, called D_1, D_2. By the continuity of the distance function, each $D_j, j = 1, 2$, would contain a point z_j of distance ϵ' to Γ, for some $\epsilon' \in (\epsilon, t_0)$. This would imply that $\Delta_{\epsilon'} \cup \gamma_{\epsilon'}$ is not connected.

By Remark 3.2.19, there exist $d_0 < \epsilon'$, a component D of Δ_{d_0} and three distinct co-linear points $x_0 \in \partial D$ and $x_1, x_2 \in \Gamma$ such that

$$|x_0 - x_1| = |x_0 - x_2| = d_0.$$

By Lemma 3.2.9, we know that D is a Jordan domain. Moreover, we observe that D is exterior to balls $\overline{B}^2(x_1, d_0)$ and $\overline{B}^2(x_2, d_0)$, therefore D has a cusp at x_0 and that $\Gamma \cap B^2(x_0, d_0) = \emptyset$.

Fix a point $w_0 \in D$ and a simple arc σ in $D \cup \{x_0\}$ joining w_0 to x_0. We observe that the set

$$W = \{(x, z) \colon x \in \sigma, |z| \leq \varphi(\operatorname{dist}(x, \Gamma))\}$$

may be served as a tall, wide wall, that prevents two points on two sides of W, but near each other, to be joined by a path without travelling far. This contradicts the LLC_1.

To this end, we pick a point $y_0 \in \sigma$ such that

$$|y_0 - x_0| < \min\left\{\frac{|w_0 - x_0|}{2}, \frac{d_0}{16\lambda}\right\},$$

and let $r_0 = |y_0 - x_0|$. Simple geometric consideration shows that

$$\operatorname{dist}(y_0, \overline{B}^2(x_i, d_0)) \leq \frac{r_0^2}{d_0} < \frac{r_0}{16\lambda}.$$

Set $\delta_0 = d_0 - \frac{r_0}{8\lambda}$. Since $\operatorname{dist}(y_0, \Gamma) > d_0$, by the continuity of the distance function we can find points $z_1 \in [y_0, x_1] \cap \gamma_0$ and $z_2 \in [y_0, x_2] \cap \gamma_{\delta_0}$. Note that for $i = 1, 2$,

$$\delta_0 \leq |z_i - x_i| < d_0.$$

The lower estimate follows from that fact that z_1, z_2 are on γ_{δ_0}; the second inequality, if false, would imply $\operatorname{dist}(z_i, \Gamma) \geq \operatorname{dist}(y_0, \Gamma) - |z_i - y_0| \geq d_0 - \frac{r_0}{16\lambda} > \delta_0$, a contradiction. Therefore, z_1 and z_2 are in two different components of $B^2(y_0, r_0) \setminus \sigma$, and for $i = 1, 2$

$$|z_i - y_0| = |y_0 - x_i| - |z_i - x_i| < \frac{r_0}{8\lambda} + \frac{r_0}{16\lambda} = \frac{3r_0}{16\lambda}.$$

Let $\hat{z}_1 = (z_1, \varphi(\delta_0))$ and $\hat{z}_2 = (z_2, \varphi(\delta_0))$ be the lifts of z_1 and z_2 on $\Sigma(\Gamma, \varphi)$. Since

$$|\hat{z}_1 - \hat{z}_2| = |z_1 - z_2| \leq |y_0 - z_1| + |y_0 - z_2| < \frac{3r_0}{8\lambda},$$

$\hat{z}_1$ and $\hat{z}_2$ are contained in the ball

$$B = B^3\left(\hat{z}_1, \frac{3r_0}{8\lambda}\right).$$

Since $\Sigma(\Gamma, \varphi)$ is $\lambda - \mathrm{LLC}_1$, the points $\hat{z}_1, \hat{z}_2$ are contained in a continuum E in $\lambda B \cap \Sigma(\Gamma, \varphi)$, where $\lambda B = B^3\left(\hat{z}_1, \frac{3r_0}{8}\right)$. If $w \in \pi(\lambda B)$, then

$$|w - y_0| \leq |w - z_1| + |z_1 - y_0| \leq \frac{3r_0}{8} + \frac{3r_0}{16\lambda} < r_0,$$

which implies that $\pi(E) \subset \pi(\lambda B) \subset B^2(y_0, r_0)$.

Note for any $w \in \lambda B$ that $w = (\pi(w), \varphi(\mathrm{dist}(\pi(w), \Gamma)))$ and

$$|\varphi(\mathrm{dist}(\pi(w), \Gamma)) - \varphi(\delta_0))| \leq |\varphi(\mathrm{dist}(\pi(w), \Gamma)) - \varphi(\mathrm{dist}(z_1, \Gamma)| \leq |w - \hat{z}_1| \leq \frac{3r_0}{8\lambda}.$$

However, since $d_0 < t_0$, by (4.1.1),

$$\varphi(d_0) - \varphi(\delta_0) > \frac{3r_0}{8\lambda}.$$

It follows that $\mathrm{dist}(\pi(z), \Gamma) < d_0$ for any $z \in E$, and as a consequence $\pi(E)$ does not intersect σ. The latter contradicts the fact that $\pi(E)$ is a continuum joining two points z_1 and z_2 lying in two separate components of $B^2(y_0, r_0) \setminus \sigma$.

Step 2. We claim that $\partial\Delta_\epsilon = \gamma_\epsilon$ for each $\epsilon < t_0$. Suppose the contrary. Pick $\epsilon < t_0$ such that $\partial\Delta_\epsilon \subsetneqq \gamma_\epsilon$. Then, by Remark 3.2.17, there exists a component D of Δ_ϵ and collinear points $x_0 \in \partial D$ and $x_1, x_2 \in \Gamma$ such that

$$|x_0 - x_1| = |x_0 - x_2| = \epsilon.$$

The rest of the proof for this step is similar to the proof in *Step 1.* □

We have proved that, if $\Sigma(\Gamma, \varphi)$ is $\lambda - \mathrm{LLC}_1$ for some $\varphi \in \mathcal{F}_1$ then, for all ϵ small enough, γ_ϵ is a

Jordan curve. In the next lemma we show that if Γ is not a quasicircle, then there exists $f \in \mathcal{F}_1$ such that $\lim_{t\to 0} \varphi(t)/t = +\infty$ and $\Sigma(\Gamma, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$.

Lemma 4.1.5. *Suppose that Γ is a Jordan curve. If Γ is not a quasicircle then there exists a height function $\varphi \in \mathcal{F}_1$ such that $\lim_{t\to\infty} \varphi(t)/t = +\infty$ and $\Sigma(\Gamma, \varphi)$ is not LLC_1.*

Proof. Suppose that Γ is not a quasicircle. Then, for any $C > 1$, the curve Γ does not satisfy (2.2.1). In particular, for any $n \in \mathbb{N}$, we can find points $x_n, x'_n \in \Gamma$ such that if γ_n, γ'_n are the two components of $\Gamma \setminus \{x_n, x'_n\}$ then

$$\min\{\operatorname{diam}\gamma_n, \operatorname{diam}\gamma'_n\} \geq 2n|x_n - x'_n|.$$

Furthermore, we may choose x_n, x'_n so that the sequence $\{n|x_n - x'_n|\}$ is decreasing and convergent to 0. Find points $y_n \in \gamma_n$ and $y'_n \in \gamma'_n$ such that

$$\min\{|x_n - y_n|, |x_n - y'_n|\} \geq n|x_n - x'_n|.$$

After choosing the points x_n, x'_n, y_n, y'_n, we fix a path $\sigma_n \subset \Omega$ that joins y_n, y'_n.

Consider the ball $B = B^3(x_n, 2|x_n - x'_n|)$ and note that $x_n, x'_n \in B \cap \Sigma(\Gamma, \varphi)$ for any $\varphi \in \mathcal{F}$. Define

$$\epsilon_n = \operatorname{dist}(\sigma_n \cap \frac{n}{3}\pi(B), \Gamma).$$

Note that since y_n, y'_n are not in $\frac{n}{3}\pi(B)$, the number ϵ_n is nonzero. Choosing σ_n carefully we may assume that ϵ_n is decreasing and $\epsilon_n < (n|x_n - x'_n|)^2$.

Define a function $\varphi\colon \{\epsilon_n\} \to \mathbb{R}_+$ with $\varphi(\epsilon_n) = n|x_n - x'_n|$. By our assumptions for x_n, x'_n, ϵ_n it follows that φ is increasing and $\lim_{n\to\infty} \varphi(\epsilon_n) = 0$. Moreover, $\varphi(\epsilon_n) > \sqrt{\epsilon_n}$ for any $n \in \mathbb{N}$. Therefore, φ can be extended to a function $\varphi \in \mathcal{F}_1$ that satisfies $\varphi(t) \geq \sqrt{t}$.

It remains to prove that $\Sigma(\Gamma, \varphi)$ is not LLC_1. Fix $n \in \mathbb{N}$. We claim that x_n, x'_n are in different components of $\Sigma(\Gamma, \varphi) \cap \frac{n}{3}B$. Suppose, otherwise, that there exists a continuum $E \subset \Sigma(\Gamma, \varphi) \cap \frac{n}{3}B$ containing x_n, x'_n. Then, $\pi(E)$ is a continuum in $\pi(\frac{n}{3}B) \cap \overline{\Omega}$ that contains x_n, x'_n. However, $\pi(E)$ intersects with σ_n which implies that there exists a point $z \in E$ such that $\pi(z) \in \sigma$. Hence, $\operatorname{dist}(\pi(z), \Gamma) \geq \epsilon_n$ which, by the choice of $\varphi(\epsilon_n)$, implies that $z \notin \frac{n}{3}B$. The claim follows from this contradiction and we conclude that $\Sigma(\Gamma, \varphi)$ is not $\frac{n}{3} - \mathrm{LLC}_1$ for any $n \in \mathbb{N}$. ∎

We show next that if Γ is a quasicircle and $\Sigma(\Gamma, \varphi)$ is $\lambda - \mathrm{LLC}_1$ for some $\varphi \in \mathcal{F}_1$ satisfying $\lim_{t\to 0} \varphi(t)/t + \infty$, then Γ satisfies the LQC property.

Lemma 4.1.6. *Suppose that Γ is a K-quasicircle and φ is a function in $\mathcal{F}$ whose almost everywhere derivative satisfies $\lim \varphi'(t) \to \infty$ as $t \to 0$. If $\Sigma(\Gamma, \varphi)$ is $\lambda - LLC_1$ then Γ satisfies LQC. In particular, there exist $\epsilon_0 > 0$ depending on λ, φ and $K' > 1$ depending on K, λ such that γ_ϵ is a K'-quasicircle for any $\epsilon \leq \epsilon_0$.*

Proof. Since Γ is a K-quasicircle, there exists $C > 1$ depending on K so that

$$\operatorname{diam} \Gamma(x, y) \leq C|x - y|, \quad \text{for all } x, y \in \Gamma.$$

Fix $t_0 > 0$ so that

$$\varphi'(t) \geq 10\lambda \text{ a.e. } t \in (0, t_0). \tag{4.1.2}$$

By Lemma 4.1.4, there exists $\epsilon_0 \in (0, t_0)$ depending on λ, φ such that γ_ϵ is a Jordan curve for any $\epsilon \leq \epsilon_0$. It suffices to show that for every $\epsilon \in (0, \epsilon_0)$, the curve γ_ϵ satisfies the 2-points condition

$$\operatorname{diam} \gamma(x, y) \leq 50\lambda C|x - y|, \quad \text{for all } x, y \in \gamma_\epsilon. \tag{4.1.3}$$

The latter implies that there exists $K' = K'(\lambda, C)$ such that γ_ϵ is a K'-quasicircle. Assume the opposite. Then there exist $\epsilon \in (0, \epsilon_0)$ and points x_1, x_2, x_3, x_4 on γ_ϵ in cyclic order such that x_3 and x_4 are on two different components of $\gamma_\epsilon \setminus \{x_1, x_2\}$ and that

$$|x_3 - x_1|, |x_4 - x_1| > 25\lambda C|x_1 - x_2|.$$

We claim that

$$|x_1 - x_2| < \frac{\epsilon}{6\lambda}.$$

Set $d = |x_1 - x_2|$. Fix, for each $i = 1, \ldots, 4$, a point p_i on Γ that is nearest to x_i, so $|p_i - x_i| = \epsilon$. By Lemma 3.2.3, points p_1, p_2, p_3, p_4 follow the same order as that of x_1, x_2, x_3, x_4. Note, however, that some of the points p_i might coincide. Note that $|p_1 - p_2| \leq 2\epsilon + |x_1 - x_2|$. On the other hand, the 2-points condition (2.2.1) on Γ yields

$$\begin{aligned} C|p_1 - p_2| &\geq \min\{|p_1 - p_3|, |p_1 - p_4|\} \\ &\geq \min\{|x_1 - x_3|, |x_1 - x_4|\} - 2\epsilon \\ &\geq 25\lambda C|x_1 - x_2| - 2\epsilon. \end{aligned}$$

Hence, $|x_1 - x_2| < \frac{\epsilon}{6\lambda}$.

Since p_i is a nearest point on Γ to x_i, the intersection $[x_i, p_i] \cap \gamma_{\epsilon-d}$ contains a single point, let us call this point z_i. Let $\hat{z_i} = (z_i, \varphi(\epsilon - d))$ be the lift of z_i on the surface $\Sigma(\Gamma, \varphi)$.

Consider the ball $B = B^3(\hat{z}_1, 5d)$. Since $|z_1 - z_2| \leq |z_1 - x_1| + |x_1 - x_2| + |x_2 - z_2| \leq 3d$ we have that $\hat{z}_2 \in B$. By the $\lambda - \mathrm{LLC}_1$, there is a continuum E in $\lambda B \cap \Sigma(\Gamma, \varphi)$ that contains $\hat{z}_1, \hat{z}_2$.

Note for any $w \in \lambda B$ that $w = (\pi(w), \varphi(\mathrm{dist}(\pi(w), \Gamma)))$ and that

$$10\lambda|\pi(w) - z_1| \leq |\varphi(\mathrm{dist}(\pi(w), \Gamma)) - \varphi(\mathrm{dist}(z_1, \Gamma))| \leq |w - \hat{z}_1| < 5\lambda d.$$

Thus $\pi(E)$ is contained in the annular region $A = \overline{\Delta}_{\epsilon-d} \setminus \Delta_\epsilon$ bordered by two Jordan curves $\gamma_{\epsilon-d}$ and γ_ϵ. Since $\pi(E)$ is a continuum in $\pi(\lambda B) \cap \overline{\Omega}$ that contains z_1, z_2, it must intersect at least one of the two components in $A \setminus ([x_3, p_3] \cup [x_4, p_4])$. From this it follows that

$$\mathrm{diam}(\pi(E)) \geq \min\{|x_1 - x_3|, |x_1 - x_4|\} - |x_1 - z_1| - |x_2 - z_2| \geq 25\lambda C d - 12d > 10\lambda d,$$

which is a contradiction to $E \subset \lambda B$. Therefore (4.1.3) must hold. $\square$

4.2 Ahlfors 2-regularity

Definition 4.2.1. *A metric space X is called Ahlfors Q-regular if there is a constant $C > 0$ such that the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ of every open ball $B(a, r)$ in X satisfies*

$$C^{-1} r^Q \leq \mathcal{H}^Q(B(a, r)) \leq C r^Q, \tag{4.2.1}$$

whenever $0 < r \leq \mathrm{diam}\, X$.

Our goal in this section is to prove the following proposition which, combined with Theorem 2.2.4 and Proposition 4.1.1, completes the proof of the first claim of Theorem 1.2.1.

Proposition 4.2.2. *Suppose that Γ satisfies the level chord-arc property and $\varphi \in \mathcal{F}_1$. Then $\Sigma(\Gamma, \varphi)$ is Ahlfors 2-regular.*

The statement is quantitative in the following sense: Suppose that, for some $\epsilon_0 > 0$ and $c > 1$, the level set γ_ϵ is a c-chord-arc curve for every $\epsilon \in [0, \epsilon_0]$. Suppose also that φ is a self-homeomorphism of $[0, \infty)$ which is L-Lipschitz in $[\epsilon_0/3, \infty)$. Then $\Sigma(\Gamma, \varphi)$ satisfies (4.2.1) with C depending only on $\epsilon_0, c, L, \mathrm{diam}\, \Gamma$.

If $a \in \Sigma^+(\Gamma, \varphi)$ then by symmetry of $\Sigma(\Gamma, \varphi)$ with respect to $\mathbb{R}^2 \times \{0\}$ we have

$$\mathcal{H}^2(B^3(a,r) \cap \Sigma^+(\Gamma,\varphi)) \leq \mathcal{H}^2(B^3(a,r) \cap \Sigma(\Gamma,\varphi)) \leq 2\mathcal{H}^2(B^3(a,r) \cap \Sigma^+(\Gamma,\varphi)).$$

Therefore, it is sufficient to show that there exists a constant $C > 1$ depending on ϵ_0, c, L, $\operatorname{diam}\Gamma$ such that for all $r \leq \operatorname{diam}\Sigma(\Gamma,\varphi)$ and $a \in \Sigma^+(\Gamma,\varphi)$,

$$r^2/C \leq \mathcal{H}^2(B^3(a,r) \cap \Sigma^+(\Gamma,\varphi)) \leq Cr^2.$$

Recall that $S^+_{\epsilon_0/3}$ is the piece of $\Sigma^+(\Gamma,\varphi)$ such that the projection $\pi(S^+_{\epsilon_0/3})$ is the domain $\Delta_{\epsilon_0/3}$ of all points enclosed by Γ that their distance from Γ is greater than $\epsilon_0/3$. Recall also that $S^+_{\epsilon_0/3}$ is $(L+1)$-bi-Lipschitz to $\Delta_{\epsilon_0/3}$. Since $\gamma_{\epsilon_0/3}$ is a c-chord-arc curve, the domain $\Delta_{\epsilon_0/3}$ is Ahlfors 2-regular with the constant depending on L, c. Hence, $S^+_{\epsilon_0/3}$ is the graph of a Lipschitz function defined on a 2-regular domain and therefore, $S^+_{\epsilon_0/3}$ is Ahlfors 2-regular. Suppose that $\pi(a) \in \Delta_{2\epsilon_0/3}$. Then, $B^3(a,r) \cap \Sigma(\Gamma,\varphi) \subset S^+_{\epsilon_0/3}$ and $\mathcal{H}^2(B^3(a,r) \cap \Sigma(\Gamma,\varphi)) \simeq r^2$. Thus, for the rest of this section, we may assume that $\pi(a) \in \gamma_\epsilon$ with $\epsilon \leq 2\epsilon_0/3$.

Furthermore, we may assume that $r \leq \min\{\epsilon_0/3, \frac{1}{160c^2}\operatorname{diam}\gamma_{\epsilon_0/3}\}$.

The 2-regularity of $\Sigma^+(\Gamma,\varphi)$ is first checked for some *rectangular pieces* on $\Sigma^+(\Gamma,\varphi)$ which we define below.

Suppose that x_1, y_1, x_2, y_2 are points in $\Sigma^+(\Gamma,\varphi)$ such that

$$\pi(x_1), \pi(y_1) \in \gamma_{t_1} \text{ and } \pi(x_2), \pi(y_2) \in \gamma_{t_2} \text{ for some } 0 \leq t_2 < t_1 \leq \epsilon_0, \tag{4.2.2}$$

$$|\pi(x_1) - \pi(x_2)| = |\pi(y_1) - \pi(y_2)| = t_1 - t_2, \tag{4.2.3}$$

$$3c|x_1 - x_2| \leq |x_1 - y_1| \leq \frac{\operatorname{diam}\gamma_{\epsilon_0}}{10c}. \tag{4.2.4}$$

Property (4.2.2) implies that x_1 is on the same horizontal plane with y_1 and x_2 is on the same horizontal plane with y_2. Thus, $|\pi(x_1) - \pi(y_1)| = |x_1 - y_1|$ and $|\pi(x_2) - \pi(y_2)| = |x_2 - y_2|$. By property (4.2.3), $\pi(x_2)$ is a point in γ_{t_2} which is closest to $\pi(x_1)$ and $\pi(y_2)$ is a point in γ_{t_2} which is closest to $\pi(y_1)$. Hence $|x_1 - x_2| = |y_1 - y_2|$.

By Remark 3.2.2, the segments $[\pi(x_1), \pi(x_2)]$ and $[\pi(y_1), \pi(y_2)]$ do not cross each other; see the discussion in the beginning of Section 3.2. Moreover, by Remark 3.2.1, each of the curves $\gamma_{t_1}(\pi(x_1), \pi(y_1))$, $\gamma_{t_2}(\pi(x_2), \pi(y_2))$ intersect with each of the segments $[\pi(x_1), \pi(x_2)]$, $[\pi(y_1), \pi(y_2)]$ at exactly one point. Define $D = D(x_1, y_1, x_2, y_2)$ to be the subset of $\Sigma^+(\Gamma,\varphi)$ such that its projection is the Jordan domain bounded by the following four arcs: $[\pi(x_1), \pi(x_2)]$, $[\pi(y_1), \pi(y_2)]$, $\gamma_{t_1}(\pi(x_1), \pi(y_1))$, $\gamma_{t_2}(\pi(x_2), \pi(y_2))$.

Remark 4.2.3. *If $t \in [t_2, t_1]$, $x \in \gamma_t \cap [\pi(x_2), \pi(x_1)]$ and $y \in \gamma_t \cap [\pi(y_2), \pi(y_1)]$ then $\gamma_t(x, y) \subset \pi(D)$.*

To prove the remark, note first that both $\gamma_t \cap [\pi(x_2), \pi(x_1)]$ and $\gamma_t \cap [\pi(y_2), \pi(y_1)]$ contain only one point. Consider the components γ', γ'' of $\gamma_t \setminus \{x, y\}$ and observe that if $\gamma' \subset \pi(D)$ then $\operatorname{diam} \gamma' \leq 4c|x_1 - y_1| < \frac{1}{2} \operatorname{diam} \gamma_{\epsilon_0} < \frac{1}{2} \operatorname{diam} \gamma_t$. Hence, $\gamma' = \gamma_t(x, y)$ and the remark follows.

Remark 4.2.4. *The diameter of D satisfies $|x_1 - y_1| \leq \operatorname{diam} D \leq 6c|x_1 - y_1|$.*

To prove the remark, note that

$$\operatorname{diam} \gamma_{t_2}(\pi(x_2), \pi(y_2)) \leq c|x_2 - y_2| \leq c|x_1 - y_1| + 2c|x_1 - x_2| \leq 2c|x_1 - y_1|$$

which yields

$$\operatorname{diam} \pi(D) \leq 2|x_1 - x_2| + \operatorname{diam} \gamma_{t_1}(\pi(x_1), \pi(y_1)) + \operatorname{diam} \gamma_{t_2}(\pi(x_2), \pi(y_2)) \leq 4c|x_1 - y_1|.$$

Moreover, if $x \in D$ we have that

$$\begin{aligned}
|x - x_2| &\leq \operatorname{dist}(x, \gamma_{t_2} \times \{h(t_2)\}) + \operatorname{diam} \gamma_{t_2}(\pi(x_2), \pi(y_2)) \\
&\leq |x_1 - x_2| + 2c|x_1 - y_1| \\
&\leq 3c|x_1 - y_1|
\end{aligned}$$

Therefore, $|x_1 - y_1| \leq \operatorname{diam} D \leq 6c|x_1 - y_1|$.

Suppose that the points x_1, y_1, x_2, y_2 are in $\Sigma^+(\Gamma, \varphi)$ and satisfy (4.2.2) – (4.2.4). We say that the rectangular piece $D(x_1, y_1, x_2, y_2)$ is a *square piece* if $\frac{1}{6c}|x_1 - y_1| \leq 3|x_1 - x_2|$.

In Lemma 4.2.5,we show that $B^3(a, r) \cap \Sigma^+(\Gamma, \varphi)$ contains and is contained to two square pieces of diameters comparable to r. Then, it follows from Lemma 4.2.6 that each square piece D satisfies $\mathcal{H}^2(D) \simeq (\operatorname{diam} D)^2$.

Lemma 4.2.5. *Let Γ be a chord-arc curve such that, for some $r_0 > 0$, $c > 1$, each level set γ_ϵ, with $\epsilon \in [0, 3r_0]$, is a c-chord-arc curve. Then, for each $a \in \Sigma^+(\Gamma, \varphi) \setminus S^+_{2r_0}$ and each $r \leq \min\{r_0, \frac{1}{160c^2} \operatorname{diam} \gamma_{r_0}\}$, there exist two square pieces D_1, D_2 on $\Sigma^+(\Gamma, h)$ with $\operatorname{diam} D_1 \geq \frac{3}{22c^2} r$, $\operatorname{diam} D_2 \leq 96c^3 r$ and*

$$D_1 \subset B(a, r) \cap \Sigma^+(\Gamma, \varphi) \subset D_2.$$

Proof. We first construct D_1. Set $\rho = \frac{r}{22c^2}$. There exists unique $\epsilon' > \epsilon$ such that the graph of φ from

$(\epsilon, \varphi(\epsilon))$ to $(\epsilon', \varphi(\epsilon'))$ has length equal to ρ. Note that $\epsilon' - \epsilon \le \rho$. By Corollary 3.2.6, we can find a point $x_1 \in \gamma_{\epsilon'} \times \{\varphi(\epsilon')\}$ such that $|\pi(a) - \pi(x_1)| \le 3c(\epsilon' - \epsilon)$. Choose a point $y_1 \in \gamma_{\epsilon'} \times \{\varphi(\epsilon')\}$ such that

$$\ell(\gamma_{\epsilon'}(\pi(x_1), \pi(y_1))) = 3c\rho$$

and points $x_2, y_2 \in \gamma_\epsilon \times \{\varphi(\epsilon)\}$ such that

$$|\pi(x_1) - \pi(x_2)| = |\pi(y_1) - \pi(y_2)| = \epsilon' - \epsilon.$$

Observe that the points x_1, y_1, x_2, y_2 satisfy (4.2.2) – (4.2.4) and let $D_1 = D(x_1, y_1, x_2, y_2)$. If $z \in D$,

$$|z - a| \le |x_1 - a| + \operatorname{diam} D \le |x_1 - a| + 6c|x_1 - y_1| \le 21c^2\rho < r.$$

Thus, $D_1 \subset B^3(a, r)$. Moreover $\operatorname{diam} D_1 \ge |x_1 - y_1| \ge \frac{3}{22c^2} r$.

We now construct D_2. Suppose that $\pi(a) \in \gamma_\epsilon$. Define ϵ_1, ϵ_2 be the maximum, respectively minimum, of numbers $\delta \ge 0$ for which the closed ball $\overline{B}^3(a, r)$ intersects with $\gamma_\delta \times \{\varphi(\delta)\}$. Note that $0 \le \epsilon_2 \le \epsilon < \epsilon_1 \le 3r_0$ and $|\epsilon_i - \epsilon| \le r$ for each $i = 1, 2$. Let $z_1 \in \gamma_{\epsilon_1} \times \{\varphi(\epsilon_1)\}$ be such that $|a - z_1| \le r$.

Take distinct points $x_1, y_1 \in \gamma_{\epsilon_1} \times \{\varphi(\epsilon_1)\}$ such that $\pi(z_1) \in \gamma_{\epsilon_1}(\pi(x_1), \pi(y_1))$ and

$$\ell(\gamma_{\epsilon_1}(\pi(x_1), \pi(z_1))) = \ell(\gamma_{\epsilon_1}(\pi(z_1), \pi(y_1))) = 8cr.$$

Take also points $x_2, y_2 \in \gamma_{\epsilon_2} \times \{\varphi(\epsilon_2)\}$ such that

$$|\pi(x_1) - \pi(x_2)| = |\pi(y_1) - \pi(y_2)| = \epsilon_1 - \epsilon_2 \le 2r.$$

We claim that x_1, y_1, x_2, y_2 satisfy (4.2.2) – (4.2.4). The properties (4.2.2) – (4.2.3) are immediate; we verify (4.2.4). Since $|x_1 - y_1| = |\pi(x_1) - \pi(y_1)|$, we have $16r \le |x_1 - y_1| \le 16cr$. On the other hand, it is easy to see that the distance of $\gamma_{\epsilon_1} \times \{\varphi(\epsilon_1)\}$ and $\gamma_{\epsilon_2} \times \{\varphi(\epsilon_2)\}$ is at least r and at most $2r$. Hence, $|x_1 - y_1| \le |x_1 - x_2| \le 16c|x_1 - x_2|$. Finally, $|x_1 - y_1| \le 16cr < \frac{1}{10c} \operatorname{diam} \gamma_{r_0}$ and the claim follows.

Define the square piece $D_2 = D(x_1, y_1, x_2, y_2)$. Note that $\operatorname{diam} D_2 \le 6c|x_1 - y_1| \le 96c^3 r$. To show that $B^3(a, r) \cap \Sigma^+(\Gamma, \varphi)$ is contained in D_2 we first prove that $a \in D$. Let $z_2 \in \gamma_{\epsilon_2} \times \{\varphi(\epsilon_2)\}$ be such that

$|\pi(z_2) - \pi(a)| = \epsilon - \epsilon_2 \leq r$. Note that

$$\begin{aligned}
|\pi(z_2) - \pi(x_2)| &\geq |\pi(z_1) - \pi(x_1)| - |\pi(z_1) - \pi(z_2)| - |\pi(x_1) - \pi(x_2)| \\
&\geq 8cr - (|\pi(z_1) - \pi(a)| - |\pi(z_2) - \pi(a)|) - (\epsilon_1 - \epsilon_2) \\
&\geq (8c - 4)r.
\end{aligned}$$

The same lower bound can be obtained for $|\pi(z_2) - \pi(y_2)|$. Let $\sigma = \gamma_{\epsilon_2} \setminus \gamma_{\epsilon_2}(\pi(x_2), \pi(y_2))$. Then

$$\begin{aligned}
\operatorname{dist}(\pi(a), \sigma) &\geq \operatorname{dist}(\pi(z_2), \sigma) - |\pi(a) - \pi(z_2)| \\
&\geq \frac{1}{c} \min\{|\pi(z_2) - \pi(x_2)|, |\pi(z_2) - \pi(y_2)|\} - |\pi(a) - \pi(z_2)| \\
&\geq (8 - \frac{4}{c} - 1)r \\
&> 3r.
\end{aligned}$$

The latter implies that $z_2 \in \gamma_{\epsilon_2}(\pi(x_2), \pi(y_2))$ and, by Lemma 3.2.3, $\pi(a) \in \pi(D)$. Therefore, if $x \in B^3(a, r) \cap \Sigma^+(\Gamma, \varphi)$,

$$\operatorname{dist}(\pi(x), \gamma_2) \geq \operatorname{dist}(\pi(a), \gamma_2) - |\pi(x) - \pi(a)| \geq 2r$$

which implies that $\pi(x) \in \pi(D_2)$. It follows that $B^3(a, r) \cap \Sigma^+(\Gamma, \varphi) \subset D_2$. □

The proof of Proposition 4.2.2 follows now from the next lemma.

Lemma 4.2.6. *Suppose that $\varphi \in \mathcal{F}$ and Γ is a chord-arc curve such that, for some $c > 1$ and $\epsilon_0 > 0$, the level set γ_ϵ is a c-chord-arc curve for each $\epsilon \in [0, \epsilon_0]$. Let x_1, y_1, x_2, y_2 be points in $\Sigma^+(\Gamma, \varphi)$ satisfying properties (4.2.2) – (4.2.4) and $D(x_1, y_1, x_2, y_2)$ be the rectangular piece on $\Sigma^+(\Gamma, \varphi)$ defined as above. Then, there exists $C > 1$ depending only on c such that*

$$C^{-1} \leq \frac{\mathcal{H}^2(D(x_1, y_1, x_2, y_2))}{|x_1 - y_1||x_1 - x_2|} \leq C.$$

Proof. For simplicity we write $D = D(x_1, y_1, x_2, y_2)$. Since φ is increasing,

$$\frac{1}{2}(t_1 - t_2 + \varphi(t_1) - \varphi(t_2)) \leq |x_1 - x_2| = |y_1 - y_2| \leq t_1 - t_2 + \varphi(t_1) - \varphi(t_2). \tag{4.2.5}$$

For the lower bound we consider the following two cases.

Case 1. Suppose that $t_1 - t_2 < \varphi(t_1) - \varphi(t_2)$. It follows from (4.2.5) that $\varphi(t_1) - \varphi(t_2) \geq |x_1 - x_2|/2$. Let $l_{\pi(x_1), \pi(y_1)}$ be the infinite straight line passing through $\pi(x_1), \pi(y_1)$. We claim that there exists a line segment $[a,b] \subset l_{\pi(x_1),\pi(y_1)}$, of length $|x_1 - y_1|/3$, such that the projection of D on the plane $l_{\pi(x_1),\pi(y_1)} \times \mathbb{R}$ contains the rectangle $[a,b] \times [\varphi(t_2), \varphi(t_1)]$. Assuming the claim,

$$\mathcal{H}^2(D) \geq \mathcal{H}^2([a,b] \times [\varphi(t_2), \varphi(t_1)]) \geq \frac{|x_1 - x_2||x_1 - y_1|}{6}.$$

Let a, b be the points in $[\pi(x_1), \pi(y_1)]$ such that $a \in S^1(\pi(x_1), |x_1 - x_2|)$ and $b \in S^1(\pi(y_1), |x_1 - x_2|)$. Such points exist since $|x_1 - y_1| > 2|x_1 - x_2|$. Note that $|a - b| \geq |x_1 - y_1| - 2|x_1 - x_2| \geq |x_1 - y_1|/3$. Let $t \in [t_2, t_1]$ and w be a point on the segment $[a,b]$. We show that there exists a point $w' \in D$ whose projection on $[a,b] \times [h(t_2), h(t_1)]$ is the point $(w, h(t))$. Let $x \in \gamma_t \cap [x_1, x_2]$ and $y \in \gamma_t \cap [y_1, y_2]$. By Remark 4.2.3, $\gamma_t(x,y) \subset \pi(D)$. Observe that the line l perpendicular to $[a,b]$ and passing through w, separates x, y. Thus, there exists a point $w'' \in \gamma_t(x,y) \cap l$. Let $w' = (w'', h(t))$ and note that w' projected on $[a,b] \times [h(t_2), h(t_1)]$ is the point $(w, h(t))$.

Case 2. Suppose that $t_1 - t_2 \geq \varphi(t_1) - \varphi(t_2)$. By (4.2.5), $t_1 - t_2 \geq |x_1 - x_2|/2$. We claim that, $\pi(D)$ contains at least $|x_1 - y_1|/|x_1 - x_2|$ mutually disjoint discs of radius $|x_1 - x_2|/8$. Assuming the claim,

$$\mathcal{H}^2(D) \geq \mathcal{H}^2(\pi(D)) \geq \frac{|x_1 - y_1|}{|x_1 - x_2|} \frac{|x_1 - x_2|^2}{64} \geq \frac{|x_1 - y_1||x_1 - x_2|}{64}.$$

Set $t_3 = (t_1 + t_2)/2$ and let $x_3, y_3 \in D$ be such that $\pi(x_3) \in \gamma_{t_3} \cap [x_1, x_2]$ and $\pi(y_3) \in \gamma_{t_3} \cap [y_1, y_2]$. By Remark 4.2.3, $\pi(D)$ contains $\gamma_{t_3}(\pi(x_3), \pi(y_3))$. Note that

$$\operatorname{diam} \gamma_{t_3}(\pi(x_3), \pi(y_3)) \geq \operatorname{diam} \gamma_{t_1}(\pi(x_1), \pi(y_1)) - 2|x_1 - x_2| \geq c|x_1 - y_1|.$$

Since γ_{t_3} is rectifiable, we can find consecutive points $\pi(x_3) = z_0, z_1, \ldots, z_n = \pi(y_3)$ on $\gamma_{t_3}(\pi(x_3), \pi(y_3))$ such that

$$\frac{c}{2}(t_1 - t_2) \leq \ell(\gamma_{t_3}(z_i, z_{i+1})) \leq c(t_1 - t_2).$$

Since $\ell(\gamma_{t_3}(\pi(x_3), \pi(y_3))) \geq c|x_1 - y_1|$,

$$n \geq \frac{\ell(\gamma_{t_3}(\pi(x_3), \pi(y_3)))}{c(t_1 - t_2)} \geq \frac{|x_1 - y_1|}{t_1 - t_2} \geq \frac{|x_1 - y_1|}{|x_1 - x_2|}.$$

The c-chord-arc property of γ_{t_3} implies that, for any $i \neq j$,

$$|z_i - z_j| \geq \frac{\ell(\gamma_{t_3}(z_i, z_j))}{c} \geq \frac{t_1 - t_2}{2}.$$

Moreover, for any $i = 1, 2, \dots, n-1$, we have that

$$\operatorname{dist}(z_i, \gamma_{t_1}) = \operatorname{dist}(z_i, \gamma_{t_2}) = \frac{t_1 - t_2}{2}$$

and

$$\operatorname{dist}(z_i, [\pi(x_1), \pi(x_2)]) \geq |z_i - z_1| - \frac{t_1 - t_2}{4} \geq \frac{t_1 - t_2}{4}.$$

Similarly, $\operatorname{dist}(z_i, [\pi(x_1), \pi(x_2)]) \geq \frac{t_1-t_2}{4}$. Therefore, the balls $B^2(z_i, \frac{t_1-t_2}{4})$ are mutually disjoint and inside $\pi(D)$.

To establish the upper bound, we claim that, for every $\epsilon \in (0, |x_1 - x_2|/3)$, D can be covered by at most $\frac{64c}{\epsilon^2}|x_1 - y_1||x_1 - x_2|$ balls of radius ϵ. Assuming the claim,

$$\mathcal{H}^2(D) \leq \frac{64c}{\epsilon^2}|x_1 - y_1||x_1 - x_2|\epsilon^2 \leq 64c|x_1 - y_1||x_1 - x_2|.$$

Fix $\epsilon \in (0, |x_1 - x_2|/3)$. Since φ is a homeomorphism, by Lemma 7.1.2, its graph is a 2-chord-arc curve and we can find numbers

$$t_2 = \tau_n < \cdots < \tau_{i+1} < \tau_i < \cdots < \tau_0 = t_1$$

such that the length of the graph of φ from τ_{i+1} to τ_i satisfies

$$\epsilon/4 \leq \ell(\{(t, \varphi(t)) \colon \tau_{i+1} \leq t \leq \tau_i\}) \leq \epsilon/2.$$

Then,

$$n \leq \frac{\ell(\{(t, \varphi(t)) \colon t_2 \leq t \leq t_1\})}{\epsilon/4} \leq \frac{2|x_1 - x_2|}{\epsilon/4} = \frac{8|x_1 - x_2|}{\epsilon}.$$

Fix $i \in \{0, \dots, n\}$ and let $\sigma = \gamma_{\tau_i} \times \{\varphi(\tau_i)\}$ and w, w' be the unique points on σ such that $\pi(w) \in [\pi(x_1), \pi(x_2)] \cap \gamma_{\tau_i}$ and $\pi(w') \in [\pi(y_1), \pi(y_2)] \cap \gamma_{\tau_i}$. Since the curve $\sigma(w, w')$ is rectifiable, it can be divided into disjoint subarcs $\sigma_1, \dots, \sigma_N$ such that $\epsilon/4 \leq \ell(\sigma_j) \leq \epsilon/2$. The c-chord-arc property of γ_{τ_i} yields that

$$N \leq \frac{c|w - w'|}{\epsilon/4} \leq 4c\frac{|x_1 - y_1| + 2|x_1 - x_2|}{\epsilon} \leq \frac{8c|x_1 - y_1|}{\epsilon}.$$

For each $j = 1, \dots, N$ let $w_j \in \sigma_j$ and note that $\sigma_j \subset B^3(w_j, \epsilon/2)$. Consequently, the $\epsilon/2$-neighborhood of $\sigma(w, w')$ can be covered by at most $\frac{8c}{\epsilon}|x_1 - y_1|$ balls of radius ϵ. Therefore, the part of D which is between the planes $\mathbb{R}^2 \times \{\varphi(\tau_i)\}$ and $\mathbb{R}^2 \times \{\varphi(\tau_{i+1})\}$ can be covered by at most $\frac{8c}{\epsilon}|x_1 - y_1|$ balls of radius ϵ centered at $\gamma_{\tau_i} \times \{\varphi(\tau_i)\}$. It follows that D can be covered by at most $\frac{64c}{\epsilon^2}|x_1 - y_1||x_1 - x_2|$ balls of radius ϵ. $\square$

4.3 Chord-arc curves and Väisälä's method

In this section we prove that the chord-arc property of Γ is necessary for $\Sigma(\Gamma,\varphi)$ to be quasisymmetric to $\mathbb{S}^2$ for all $\varphi \in \mathcal{F}_1$ with $\lim_{t\to 0}\varphi(t)/t = \infty$. Combined with Proposition 4.1.6 and Proposition 3.0.6, the following proposition completes the proof of the second claim of Theorem 1.2.1.

Proposition 4.3.1. *Suppose that Γ is a quasicircle but not a chord-arc curve. Then, there exists $\varphi \in \mathcal{F}_1$ with $\lim_{t\to 0}\varphi(t)/t = \infty$ such that $\Sigma(\Gamma,\varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

If Γ does not satisfy the LQC property then the conclusion of the proposition is immediate by Proposition 4.1.6. Thus, we can assume that there exists $\epsilon_0 > 0$ and $C \geq 1$ such that for each $\epsilon \in [0,\epsilon_0]$, the set γ_ϵ is a quasicircle satisfying (2.2.1) with constant C.

We define inductively numbers $\epsilon_n > 0$ and arcs $\Gamma_n \subset \Gamma$ as follows. Let ϵ_0 be as above and set and $\Gamma_0 = \Gamma$. Suppose that ϵ_{n-1}, Γ_{n-1} have been defined. Since Γ is not a chord-arc curve, there exists a subarc $\Gamma_n \subset \Gamma$ such that $\operatorname{diam}\Gamma_n \leq \frac{1}{2}\operatorname{diam}\Gamma_{n-1}$ and $\ell(\Gamma_n) \geq 2n\operatorname{diam}\Gamma_n$. Find consecutive points $x_1^n, x_2^n, \dots, x_{N_n}^n$ on Γ_n such that for $i \in 1,\dots,N_n-1$, $|x_{i+1}^n - x_i^n| \leq \frac{\epsilon_{n-1}}{9C}$ and

$$\sum_{i=1}^{N_n} |x_i^n - x_{i-1}^n| \geq n\operatorname{diam}\Gamma_n. \tag{4.3.1}$$

Define

$$\epsilon_n = \frac{1}{36c}\min\{\min_{1\leq i\leq N}|x_i^n - x_{i-1}^n|, (\operatorname{diam}\Gamma_n)^2\}.$$

Each of the subarcs $\Gamma(x_{i-1}^n, x_i^n)$ is compact and can be further subdivided so that the new collection of endpoints, which by abuse of nations we still denote with $x_1^n,\dots,x_{N_n}^n$, satisfies

$$9C\epsilon \leq |x_i^n - x_{i-1}^n| \leq 36C\epsilon.$$

It follows from their constructions that the sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$, $\{\operatorname{diam}\Gamma_n\}_{n\in\mathbb{N}}$ are decreasing and converging to zero. Define

$$\varphi\colon \{\epsilon_n\}_{n\in\mathbb{N}} \to \mathbb{R} \quad \text{with} \quad \varphi(\epsilon_n) = \operatorname{diam}\Gamma_n.$$

Note that $\varphi(\epsilon_n) \geq \sqrt{\epsilon_n}$. We extend φ on $[0,+\infty)$ so that the extension is in $\mathcal{F}_1$ and satisfies $\varphi(t) \geq \sqrt{t}$ for all $t \geq 0$. Clearly, $\lim_{t\to 0}\varphi(t)/t = \infty$.

The proof of Proposition 4.3.1 follows now from the next lemma. The main idea is due to Väisälä [43].

Lemma 4.3.2. *Let Γ be a quasicircle such that for any $\epsilon \in [0,\epsilon_0]$, the level set γ_ϵ is a quasicircle satisfying*

(2.2.1) for some $C > 1$. Suppose that for any $n \in \mathbb{N}$ there exist consecutive points $x_1^n, \dots, x_{N_n}^n$ on Γ and a positive number $\epsilon_n < \min\{\frac{1}{9C} \operatorname{diam} \Gamma(x_1^n, x_{N_n}^n), \epsilon_0\}$ such that

1. $\sum_{i=1}^{N_n - 1} |x_i^n - x_{i-1}^n| \geq n \operatorname{diam} \Gamma(x_0^n, x_{N_n}^n)$
2. $9C\epsilon_n \leq |x_i^n - x_{i-1}^n|$ *for all* $i = 1, \dots, N_n - 1$.

Let $\varphi \in \mathcal{F}$ be such that, for some $L > 1$ and for each $n \in \mathbb{N}$

$$L^{-1} \operatorname{diam} \Gamma(x_1^n, x_{N_n}^n) \leq \varphi(\epsilon_n) \leq L \operatorname{diam} \Gamma(x_1^n, x_{N_n}^n)$$

Then $\Sigma(\Gamma, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$.

Proof. Suppose, on the contrary, that there exists an η-quasisymmetric mapping that maps $\Sigma(\Gamma, \varphi)$ onto $\mathbb{S}^2$. Post-composing this mapping with an inversion, we may assume that there exists an η-quasisymmetric map $F\colon \Sigma^+(\Gamma, \varphi) \to \mathbb{B}^2$.

Fix $n \in \mathbb{N}$. For simplicity we write $N = N_n$, $x_i^n = x_i$, $\epsilon_n = \epsilon$ and the dependence of quantities, points and sets on n is not recorded. However, the comparison constants in $\simeq$ and $\lesssim$ depend only on C, η.

As in the discussion before Lemma 4.3.2, by adding points in the collection $\{x_i\}$ we may further assume that, for each $i = 1, \dots, N-1$, we have $|x_{i+1} - x_i| \leq 36C\epsilon$.

Since Γ is a quasicircle, by Lemma 3.2.5, for each $i = 1, \dots, N$ there exists $w_i \in \gamma_\epsilon$ such that $|x_i - w_i| \leq 3C\epsilon$. Choose also points $w_i' \in \Gamma$ such that $|w_i - w_i'| = \epsilon$. Note that $w_i' \in B^2(x_i, 4C\epsilon) \cap \Gamma$ and thus the points $w_1', \dots, w_N'$ have the same orientation as the points $x_1, \dots, x_N$. By Lemma 3.2.3, $w_1, \dots, w_N$ have the same orientation as the points $x_1, \dots, x_N$. The points w_i' have been chosen so that the line segments $[w_i, w_i']$ intersect every level set at most once. This intersecting property follows from Remark 3.2.1 and is not necessarily true for the segments $[x_i, w_i]$.

Observe that for $i, j \in \{1, 2, \dots, N\}$ with $i \neq j$,

$$|w_i - w_j| \geq |x_i - x_j| - |x_i - w_i| - |x_j - w_j| \geq 9C\epsilon - 6C\epsilon \gtrsim \epsilon$$

and

$$|w_i - w_{i+1}| \leq |x_i - x_{i+1}| + |x_i - w_i| + |x_{i+1} - w_{i+1}| \lesssim \epsilon.$$

Similarly we deduce $|w_i' - w_{i+1}'| \simeq \epsilon$. Note that

$$\operatorname{diam} \Gamma(w_1', w_N') \simeq \operatorname{diam} \gamma_\epsilon(w_1, w_N) \simeq |w_1 - w_N| \simeq |w_1' - w_N'| \simeq \varphi(\epsilon). \tag{4.3.2}$$

Moreover, since γ_ϵ satisfies (2.2.1), for all $i = 1, \ldots, N-1$

$$\operatorname{diam}\Gamma(w_i', w_{i+1}') \simeq \operatorname{diam}\gamma_\epsilon(w_i, w_{i+1}) \simeq |w_i - w_{i+1}| \simeq |w_i' - w_{i+1}'| \simeq \epsilon.$$

Denote by Λ the subset of $\Sigma^+(\Gamma, \varphi)$ whose projection on $\mathbb{R}^2$ is the Jordan domain bounded by the curves $\Gamma(w_1', w_N')$, $[w_1, w_1']$, $[w_N, w_N']$ and $\gamma_\epsilon(w_1, w_N)$. As with the square pieces defined in Section 4.2, the piece Λ is well defined. In fact, the points $X_1 = (w_1', h(\epsilon))$, $Y_1 = (w_N', h(\epsilon))$, $X_2 = (w_1, 0)$, $Y_2 = (w_N, 0)$ satisfy properties (4.2.2)–(4.2.3). They also satisfy property (4.2.4) but with different constants. From (4.3.2) it follows that $\operatorname{diam}\Lambda \simeq \varphi(\epsilon)$.

Define

$$\beta = \min\{|F(x) - F(y)| \colon x \in \Gamma(w_1', w_N')\ ,\ y \in \gamma_\epsilon(w_1, w_N) \times \{\varphi(\epsilon)\}\}.$$

We claim that

$$\beta^2 n \lesssim \mathcal{H}^2(F(\Lambda)). \tag{4.3.3}$$

Assume for the moment the claim; the proof then completes as follows. Let $x^* \in \Gamma(w_1', w_N')$ and $y^* \in \gamma_\epsilon(w_1, w_N) \times \{\varphi(\epsilon)\}$ be the points for which β is realized. Then, for any $x \in \Lambda$, $|x - x^*| \lesssim |x^* - y^*|$ which implies that $|F(x) - F(x^*)| \lesssim \beta$ and $\mathcal{H}^2(F(\Lambda)) \lesssim \beta^2$. Since $\beta \neq 0$, the last inequality and (4.3.3) imply that $n \leq C$ for some C depending on η, which is a contradiction.

We now show (4.3.3). For $i = 1, \ldots, N-1$ define Λ_i be the subset of Λ whose projection on $\mathbb{R}^2$ is the Jordan domain bounded by the curves $\Gamma(w_i', w_{i+1}')$, $[w_i, w_i']$, $[w_{i+1}, w_{i+1}']$ and $\gamma_\epsilon(w_i, w_{i+1})$.

Let k be the integer part of $\varphi(\epsilon)/\epsilon - 1$. The curves

$$\sigma_j = \begin{cases} \Gamma & \text{for } j = 0, \\ \gamma_{\varphi^{-1}(j\epsilon)} \times \{j\epsilon\} & \text{for } 1 \leq j \leq k \\ \gamma_\epsilon \times \{\varphi(\epsilon)\} & \text{for } j = k+1. \end{cases}$$

divide Λ into horizontal strips. Therefore, Λ is divided into smaller square-like pieces Λ_{ij} with $i = 1, \ldots, N$ and $j = 1, \ldots k+1$. More precisely, Λ_{ij} is the subset of Λ whose projection on $\mathbb{R}^2$ is the Jordan domain bounded by $[w_i, w_i']$, $[w_{i+1}, w_{i+1}']$, $\gamma_{\varphi^{-1}(j\epsilon)}$, $\gamma_{\varphi^{-1}((j+1)\epsilon)}$.

For each $i = 1, \ldots, N-1$ let $\tau_i = \{(x, \varphi(\operatorname{dist}(x, \Gamma))) \colon x \in [w_i, w_i']\}$. Note that τ_i is isometric to the graph of φ from 0 to ϵ.

Fix a piece Λ_{ij} and define its four vertices

$$A_{ij} = \tau_i \cap \sigma_j \ , \ B_{ij} = \tau_{i+1} \cap \sigma_j \ , \ C_{ij} = \tau_{i+1} \cap \sigma_{j+1} \ , \ D_{ij} = \tau_i \cap \sigma_{j+1}.$$

Since

$$\operatorname{diam} \sigma_j(A_{ij}, B_{ij}) \simeq \operatorname{diam} \sigma_j(C_{ij}, D_{ij}) \simeq \epsilon$$

there exist points $y'' \in \sigma_j(A_{ij}, B_{ij})$, $y' \in \sigma_j(C_{ij}, D_{ij})$ and

$$y \in \Lambda_{ij} \cap (\gamma_{\varphi^{-1}((j+1/2)\epsilon))} \times \{(j+1/2)\epsilon\})$$

such that

$$\operatorname{dist}(y, \tau_{i+1}(B_{ij}, C_{ij})) \simeq \operatorname{dist}(y', \tau_{i+1}(B_{ij}, C_{ij})) \simeq \operatorname{dist}(y'', \tau_{i+1}(B_{ij}, C_{ij})) \simeq \epsilon$$
$$\operatorname{dist}(y, \tau_i(A_{ij}, D_{ij})) \simeq \operatorname{dist}(y', \tau_i(A_{ij}, D_{ij})) \simeq \operatorname{dist}(y'', \tau_i(A_{ij}, D_{ij})) \simeq \epsilon$$

and

$$\operatorname{dist}(y, \tau_i(A_{ij}, D_{ij})) \simeq \operatorname{dist}(y, \tau_{i+1}(B_{ij}, C_{ij})) \simeq \epsilon.$$

In a sense, the points y, y', y'' are the "centers" of $\Lambda_{ij}, \sigma_j(C_{ij}, D_{ij}), \sigma_j(A_{ij}, B_{ij})$ respectively.

Write $\beta_{ij} = |F(y'') - F(y')|$. Let $u \in \partial\Lambda_{ij}$ be the point at which

$$|F(u) - F(y)| = \operatorname{dist}(F(y), \partial F(\Lambda_{ij})) = r.$$

Then, $|u-y| \gtrsim |w'_i - w'_{i+1}|$ and since $|y-y'| \lesssim |w'_i - w'_{i+1}|$ the quasisymmetry of F implies that $|F(y)-F(y')| \lesssim r$. The same inequality is true with y' replaced by y''. Hence, $\beta_{ij} \lesssim r$ which implies $\beta_{ij}^2 \lesssim \mathcal{H}^2(F(\Lambda_{ij}))$. By Schwarz inequality this yields

$$\beta^2 \leq \left(\sum_{j=1}^{k+1} \beta_{ij} \right)^2 \lesssim (k+1)\mathcal{H}^2(F(\Lambda_i)).$$

Note that $(k+1)\epsilon \simeq (k+1)|w'_i - w'_{i+1}| \simeq \operatorname{diam} \Gamma(x_0, x_N)$. Thus,

$$\beta^2 |w'_i - w'_{i+1}| \lesssim \operatorname{diam} \Gamma(x_0, x_N) \mathcal{H}^2(F(\Lambda_i)).$$

Since $|w'_i - w'_{i-1}| \simeq |x_i - x_{i-1}|$, summing over i we obtain (4.3.3) from (4.3.1). $\square$

4.4 Proof of Theorem 1.2.1

In this section we give the proof of Theorem 1.2.1 and some remarks on the assumptions of the theorem.

Proof of Theorem 1.2.1. Suppose that Γ satisfies the LCA property and $\varphi \in \mathcal{F}_1$. By Proposition 4.1, the surface $\Sigma(\Gamma, \varphi)$ is LLC and by Proposition 4.2.2, the surface $\Sigma(\Gamma, \varphi)$ is 2-regular. The first claim follows now from Theorem 2.2.4.

Conversely, suppose that $\Sigma(\Gamma, \varphi)$ is a quasisymmetric sphere for all functions $\varphi \in \mathcal{F}_1$ which satisfy $\lim_{t\to 0} \varphi(t)/t = \infty$. By Proposition 4.1, we know that Γ is a quasicircle that satisfies the LQC property. Then, Proposition 4.3.1 gives that Γ is a chord-arc curve and the second claim follows from Proposition 3.0.6. $\square$

Remark 4.4.1. *The assumption that* $\liminf_{t\to 0} \varphi(t)/t > 0$ *is necessary for the first claim of Theorem 1.2.1.*

For example, if $\varphi(t) = t^2$ and $\Gamma = \mathbb{S}^1$ then it is easy to see that $\Sigma(\Gamma, \varphi)$ is not LLC_1 and therefore not quasisymmetric to $\mathbb{S}^2$. With some effort this observation can be generalized into the following result.

Theorem 4.4.2. *Let* φ *be a self homeomorphism of* $[0, +\infty)$ *that satisfies* $\liminf_{t\to 0} \varphi(t)/t = 0$. *Then, for any Jordan curve* Γ*, the surface* $\Sigma(\Gamma, \varphi)$ *is not* LLC_1.

Proof. Contrary to the claim, assume that $\Sigma(\Gamma, \varphi)$ is λ-LLC_1.

Fix $x_0 \in \Omega$ and find $t_0 \in (0, \text{dist}(x_0, \Gamma)/2)$ such that $h(t_0)/t_0 < (4\lambda)^{-1}$. Suppose that $y_0 \in \Gamma$ is such that $\text{dist}(x_0, \Gamma) = |x_0 - y_0|$. It is easy to see that for any $x \in [x_0, y_0]$, $\text{dist}(x, \Gamma) = |x - y_0|$ which implies that $B(x, |x - y_0|) \subset \Omega$. Let $z_0 \in [x_0, y_0] \cap \gamma(t_0)$, $z_1 = (z_0, \varphi(t_0))$ and $z_2 = (z_0, -\varphi(t_0))$.

Consider the ball $B = B^3(z_1, 3\varphi(t_0))$ and note that $z_1, z_2 \in B$. Suppose that there is a continuum $E \subset \lambda B \cap \Sigma(\Gamma, \varphi)$ containing z_1, z_2. Then, E intersects $\Gamma \times \{0\}$. However the choice of t_0 implies that $\pi(E) \subset \pi(\lambda B) \subset \Omega$ and thus $E \cap \Gamma \times \{0\} = \emptyset$. $\square$

Remark 4.4.3. *The assumption that* φ *is Lipschitz in* $[\epsilon, +\infty)$ *for any* $\epsilon > 0$ *is necessary for Theorem 1.2.1.*

In particular, let $\mathcal{F}_1'$ be the set of all $\varphi \in \mathcal{F}$ such that $\liminf_{t\to 0} \varphi(t)/t > 0$ and φ is Lipschitz in $[\delta, \infty)$ for some $\delta > 0$. Clearly $\mathcal{F}_1 \subset \mathcal{F}_1'$ and the inclusion is strict. For $\varphi \in \mathcal{F}_1'$ it turns out that the property "$\Sigma(\Gamma, \varphi)$ is quasisymmetric to $\mathbb{S}^2$" is not in general invariant under dilations, that is, there exists $h \in \mathcal{F}_1'$, quasicircle Γ and a dilation function T such that $\Sigma(\Gamma, \varphi)$ is quasisymmetric to $\mathbb{S}^2$ but $\Sigma(T(\Gamma), \varphi)$ is not. For example, if $\Gamma = \mathbb{S}^1$, $T(x) = x/2$ and

$$\varphi(t) = \begin{cases} 1 - \sqrt{1-t} & t \in [0, 1] \\ t & t \in [1, +\infty) \end{cases}$$

then, by Theorem 1.2.1, $\Sigma(T(\mathbb{S}^1), \varphi)$ is quasisymmetric to $\mathbb{S}^2$ but the surface $\Sigma(\mathbb{S}^1, \varphi)$ is not LLC_2 and therefore not quasisymmetric to $\mathbb{S}^2$. With a slight modification in the proofs of Lemma 4.1.2, Lemma 4.1.3 and Proposition 4.2.2, the following result can be deduced.

Corollary 4.4.4. *Let Γ be a quasicircle. If Γ satisfies the level chord-arc property and $\varphi \in \mathcal{F}_1'$ then there exists a dilation T such that the surface $\Sigma(T(\Gamma), \varphi)$ is quasisymmetric to $\mathbb{S}^2$.*

Chapter 5

Quasispheres constructed over quasidisks

In this chapter we show that an iteration of the Geometric construction, using functions $\varphi_1(t), \varphi_2(t), \dots \in \mathcal{F}_1$ which are bi-Lipschitz when t is close to 0, yields quasispheres of any dimension. Theorem 1.2.2 follows as a corollary.

Let Γ be a Jordan curve and

$$\mathcal{F}_1^* = \{\varphi \in \mathcal{F}_1 \colon \varphi \text{ is bi-Lipschitz in a neighborhood of } 0\}.$$

Suppose that φ is L-bi-Lipschitz in $[0, \epsilon]$ and L-Lipschitz in $[\epsilon, +\infty)$. In the following, the numbers ϵ, L are called *the data of* φ.

Let $\varphi_1, \varphi_2, \dots \in \mathcal{F}_1^*$ and Γ be a planar Jordan curve. Define $\Sigma_1 = \Gamma$ and Ω_2 to be the bounded component of $\mathbb{R}^2 \setminus \Gamma$. For $n \in \mathbb{N}$, define inductively

$$\Sigma_n = \Sigma(\Sigma_{n-1}, \varphi_{n-1}) = \{(x, z) \in \overline{\Omega_n} \times \mathbb{R} \colon z = \pm \mathrm{dist}(x, \Sigma_{n-1})\}$$

and Ω_{n+1} to be the bounded component of $\mathbb{R}^{n+1} \setminus \Sigma_n$. The main result of this chapter is the following theorem.

Theorem 5.0.5. *Let $\varphi_1, \varphi_2, \dots \in \mathcal{F}_1^*$ be such that, for any $n \in \mathbb{N}$, the function φ_n is L_n-bi-Lipschitz in $[0, \epsilon_n]$ and L_n-Lipschitz in $[\epsilon_n, \infty)$.*

1. *If Γ is a K-quasicircle then, for each $n \in \mathbb{N}$, the surface Σ_n is an n-dimensional K'-quasisphere in $\mathbb{R}^{n+1}$ with K' depending on $K, n, \mathrm{diam}\,\Gamma$ and the data of $\varphi_1, \dots, \varphi_{n-1}$.*

2. *If Γ is a C-chord-arc curve then, for each $n \in \mathbb{N}$, the surface Σ_n is an n-dimensional L'-bi-Lipschitz sphere in $\mathbb{R}^{n+1}$ with L' depending on $C, n, \mathrm{diam}\,\Gamma$ and the data of $\varphi_1, \dots, \varphi_{n-1}$.*

For the first part of Theorem 5.0.5 we use Theorem 2.2.3. In Section 5.2 we prove that the bounded component of $\mathbb{R}^{n+1} \setminus \Sigma_n$ is quasiconformally homeomorphic to $\mathbb{B}^{n+1}$ and in Section 5.4 we show that the unbounded component of $\mathbb{R}^{n+1} \setminus \Sigma_n$ is quasiconformally homeomorphic to $\mathbb{R}^{n+1} \setminus \mathbb{B}^{n+1}$.

The second part is proved in Section 5.4 by constructing a piecewise bi-Lipschitz self map of $\mathbb{R}^{n+1}$ that maps Σ_n onto $\mathbb{S}^n$ and showing that this mapping is bi-Lipschitz in $\mathbb{R}^{n+1}$.

5.1 Whitney decomposition

For the following, we define the *dyadic Whitney decomposition* of an open set. Recall that a dyadic cube $W \subset \mathbb{R}^n$ is a set of the form

$$W = 2^m(k + [0,1]^n) = \{2^m(k + x) \colon x \in [0,1]^n\}$$

for some $m \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Two such dyadic cubes W_1, W_2 are called disjoint if they have disjoint interiors. Note that if W_1, W_2 are two dyadic cubes of $\mathbb{R}^n$ with a common interior point then, either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Setting $c = 3\sqrt{n}$ in the proof in [35, pp. 167–168], the following lemma can be established.

Lemma 5.1.1 ([35, Theorem IV.1.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set which is not all of $\mathbb{R}^n$. Then, there exists a collection $\mathcal{D} = \{W_1, W_2, \dots\}$ of dyadic cubes so that*

1. *the cubes W_k are mutually disjoint,*
2. *$\Omega = \bigcup_{W \in \mathcal{D}} W$,*
3. *$2 \operatorname{diam} W \leq \mathit{dist}(W, \partial\Omega) \leq 6 \operatorname{diam} W$, for each $W \in \mathcal{D}$.*

As a result of the proof, we can further require that if a cube $W \in \mathcal{D}$, of edge length 2^{-k}, intersects the set

$$\Omega(k) = \{x \colon 3\sqrt{n}2^{-k} < \operatorname{dist}(x, \partial\Omega) \leq 3\sqrt{n}2^{-k+1})\}$$

then, either $W \in \mathcal{D}$ or W is a subset of a cube $W' \in \mathcal{D}$. Therefore, if a cube $W \in \mathcal{D}$ intersects $\Omega(k)$ then

$$\sqrt{n}2^{-k-1} \leq \operatorname{diam} W \leq 3\sqrt{n}2^{-k}.$$

For a domain $\Omega \subsetneq \mathbb{R}^n$, a collection $\mathcal{D}$ satisfying the above properties is called *a dyadic Whitney decomposition* of Ω and its elements are called Whitney cubes.

For each $n \in \mathbb{N}$, we fix $\mathcal{D}_n$ a dyadic Whitney decomposition of $\mathbb{B}^n$ and $\mathcal{E}_n$ a dyadic Whitney decomposition of the double cone

$$\mathcal{C}^n = \{(x, z) \in \mathbb{B}^{n-1} \times \mathbb{R} \colon |z| < 1 - |x|\}.$$

Remark 5.1.2. *For any $W \in \mathcal{E}_n$, there exists a unique $W' \in \mathcal{D}_{n-1}$ such that the projection $\pi(W)$ is contained in W'.*

Assume that $W \in \mathcal{E}_n$ and $\operatorname{diam} W = 2^{-k}\sqrt{n}$. Then $\operatorname{diam} \pi(W) = 2^{-k}\sqrt{n-1}$ and by Lemma 5.1.1,

$$\operatorname{dist}(\pi(W), \partial\mathbb{B}^{n-1}) \geq \operatorname{dist}(W, \partial\mathcal{C}_n) > 2^{1-k}\sqrt{n-1}.$$

Suppose that $\pi(W)$ intersects $\mathbb{B}^{n-1}(k)$. Then either $\pi(W) \in \mathcal{D}_{n-1}$ or $\pi(W) \subset W'$ for some $W' \in \mathcal{D}_{n-1}$. Suppose that $\pi(W)$ does not intersect the annulus $\mathbb{B}^{n-1}(k)$. Then, we have that

$$\operatorname{dist}(\pi(W), \partial\mathbb{B}^{n-1}) > 2^{-k+1}3\sqrt{n-1}.$$

If $W' \subset \pi(W)$ for some $W' \in \mathcal{D}_{n-1}$ then

$$\frac{\operatorname{dist}(W', \partial\mathbb{B}^{n-1})}{\operatorname{diam} W'} \geq \frac{\operatorname{dist}(\pi(W), \partial\mathbb{B}^{n-1})}{\operatorname{diam} \pi(W)} > 6$$

which is false by Lemma 5.1.1. Hence $\pi(W) \subseteq W'$ for some $W' \in \mathcal{D}_{n-1}$.

For the uniqueness, note that if there are $W', W'' \in \mathcal{D}_{n-1}$ such that $\pi(W) \subset W', W''$ then, since $W' \cap W'' \neq \emptyset$, we would have that either $W' \subset W''$ or $W'' \subset W'$. Since both are Whitney cubes, it follows that $W' = W''$ and the the proof of Remark 5.1.2 is complete.

We conclude this section with the following result by Gehring [12].

Lemma 5.1.3 ([12, Theorem 11]). *Let F be a K-quasiconformal mapping of a domain $D \subset \mathbb{R}^n$ onto a domain $D' \subset \mathbb{R}^n$. If ∂D is not empty, then $\partial D'$ is not empty. Moreover, there exists a strictly increasing and continuous function $\Theta_K : [0,1) \to [0,+\infty)$ such that*

$$\frac{|F(x) - F(y)|}{dist(F(x), \partial D')} \leq \Theta_K\left(\frac{|x-y|}{dist(x, \partial D)}\right)$$

for all x, y in D with $|x-y| < dist(x, \partial D)$. The function Θ_K depends only on K and satisfies $\lim_{t\to 1}\Theta_K(t) = +\infty$.

As a corollary note that if $f\colon \Omega \to \Omega'$ is K-quasiconformal, $\mathcal{D}$ is a dyadic Whitney decomposition of Ω and $W \in \mathcal{D}$ then for all $x, y \in W$,

$$\frac{|F(x) - F(y)|}{\operatorname{dist}(F(x), \partial\Omega')} \leq \Theta_K\left(\frac{|x-y|}{\operatorname{dist}(x, \partial\Omega)}\right) \leq \Theta_K\left(\frac{\operatorname{diam} W}{\operatorname{dist}(W, \partial\Omega)}\right) \leq \Theta_K(1/2).$$

Therefore, there exists a number $C_1(K)$ such that

$$\frac{\operatorname{diam} F(W)}{\operatorname{dist}(F(W), \partial\Omega')} \leq C_1(K).$$

In similar fashion, using the function F^{-1}, we have that

$$\frac{\operatorname{diam} F(W)}{\operatorname{dist}(F(W), \partial\Omega')} \geq C_2(K)$$

for some positive number $C_2(K)$.

It is clear now that, if $\mathcal{D}'$ is a dyadic Whitney decomposition of D', the image of a Whitney cube $W \in \mathcal{D}$ intersects a finite number of cubes in $\mathcal{D}'$. In particular, the number of cubes in $\mathcal{D}'$ that W intersects, is bounded by a constant depending on K. Indeed, suppose that $k \in \mathbb{N}$ is the smallest natural number such that $\operatorname{dist}(F(W), \Omega') \geq 3\sqrt{n}2^{-k}$. Then,

$$3C_2(K)\sqrt{n}2^{-k} \leq \operatorname{diam} F(W) \leq 6C_1(K)\sqrt{n}2^{-k}.$$

Following the discussion after Lemma 5.1.1, if $W' \in \mathcal{D}'$ intersects $F(W)$ then $\operatorname{diam} W' \geq 2\sqrt{n}2^{-k}$. Thus, $F(W)$ can be covered with at most $(6\sqrt{n}C_1(K))^n$ Whitney cubes of $\mathcal{D}'$. Therefore, it is natural to say that *quasiconformal functions map Whitney cubes to "Whitney type objects"*.

5.2 A class of quasiballs

We show that for any bounded simply connected domain Ω and for any $\varphi_1, \varphi_2, \ldots, \varphi_{n-2} \in \mathcal{F}_1^*$, the domain enclosed by the double-dome-like surface Σ_{n-1} is quasiconformally equivalent to the unit ball $\mathbb{B}^n$.

Proposition 5.2.1. *Let $\Omega_2 \subset \mathbb{R}^2$ be a simply connected domain and $\varphi_1, \varphi_2, \cdots \in \mathcal{F}_1^*$. Suppose that, for each $n \in \mathbb{N}$, the function φ_n is L_n-bi-Lipschitz in $[0, \epsilon_n]$ and L_n-Lipschitz in $[\epsilon_n, +\infty)$. For $n > 2$, define inductively*

$$\Omega_n = \{(x, z) \in \Omega_{n-1} \times \mathbb{R} \colon |z| < \varphi_{n-2}(dist(x, \partial\Omega_{n-1}))\}.$$

Then, there exists a quasiconformal mapping F that maps $\mathbb{B}^n$ onto Ω_n. Moreover, if $\mathcal{D}_n = \{W_k\}_{k\in\mathbb{N}}$, then F is a (λ_k, L)-quasisimilarity in every Whitney cube W_k with $L > 1$ depending on n, the data of $\varphi_1, \ldots, \varphi_{n-2}$ and $\operatorname{diam} \Omega_2$.

Recall that a a mapping f between two domains $D, D' \subset \mathbb{R}^n$ is a (λ, L)-*quasisimilarity* if

$$\frac{\lambda}{L}|x-y| \leq |f(x-f(y))| \leq \lambda L|x-y|$$

for all $x, y \in D$ and some constants $L \geq 1$ and $\lambda > 0$.

Suppose that $D, D' \subset \mathbb{R}^n$ and $\mathcal{D}$ is a Whitney decomposition of D. It is easy to see that if $F\colon D \to D'$ is a (λ_k, L)-quasisimilarity in each $W_k \in \mathcal{D}$, then F is K-quasicondormal for some $K > 1$ depending only on L.

The proof of Proposition 5.2.1 is done by induction. For the first step of the induction we use some well-known inequalities from the classical function theory.

Lemma 5.2.2 (Koebe 1/4-Theorem)**.** *Let f be a conformal map from the unit disk $\mathbb{B}^2$ onto a simply-connected domain Ω. Then for all $z \in \mathbb{B}^2$*

$$\frac{1}{4}|f'(z)| \leq \frac{dist(f(z), \partial\Omega)}{1-|z|^2} \leq |f'(z)|.$$

For a proof of Koebe 1/4-Theorem see J. Garnett and D. Marshall [11, Theorem.4.3]. The next corollary is an easy consequence of Koebe's theorem.

Corollary 5.2.3. *There is an absolute constant $A > 0$ such that if f is a conformal map from the unit disk $\mathbb{B}^2$ onto a simply-connected domain Ω. Then f is (λ_k, L)-quasisimilarity in each Whitney cube $W_k \in \mathcal{D}_2$.*

The induction step is proved in the following lemma.

Lemma 5.2.4. *Suppose that Ω is a domain in $\mathbb{R}^n$ and f is a K-quasiconformal map that maps $\mathbb{B}^n$ onto Ω and is a (λ_k, L)-quasisimilarity in each dyadic cube $w_k \in \mathcal{D}_n$. Given a function φ which is L_0-bi-Lipschitz in $[0, \epsilon]$ and L_0-Lipschitz in $[\epsilon, +\infty)$ consider the domain*

$$\mathcal{K}(\Omega, \varphi) = \{(x, z) \in \Omega \times \mathbb{R}\colon |z| < \varphi(dist(x, \partial\Omega))\}.$$

Then, there exists a K'-quasiconformal function F that maps $\mathbb{B}^{n+1}$ onto $\mathcal{K}(\Omega, \varphi)$ and is (λ'_m, L')-quasisimilarity in each Whitney cube of $W_m \in \mathcal{D}_{n+1}$. Here L' depend only on L, n, $\operatorname{diam}\Omega$ *and the data of φ.*

Proof. Recall that for $n \in \mathbb{N}$,

$$\mathcal{C}^{n+1} = \{(x, z) \in \mathbb{B}^n \times \mathbb{R}\colon |z| < 1 - |x|\}$$

denotes the double cone constructed over $\mathbb{B}^n$. It is easy to see that $\mathcal{C}^{n+1}$ is C_{n+1}-bi-Lipschitz to $\mathbb{B}^{n+1}$ with

C_{n+1} depending only on $n+1$. Therefore, it is sufficient to construct a function that maps $\mathcal{C}^{n+1}$ onto $\mathcal{K}(\Omega, \varphi)$ and satisfies the properties of the lemma.

By Lemma 5.1.3, there exist constants $0 < c < C$ depending on K, hence on L, such that

$$c \leq \frac{\operatorname{diam} f(w_k)}{\operatorname{dist}(f(w_k), \partial\Omega)} \leq C. \tag{5.2.1}$$

Consider the fixed dyadic Whitney decomposition $\mathcal{E}_{n+1} = \{W_k\}_{k \in \mathbb{N}}$ of $\mathcal{C}^{n+1}$, satisfying the properties of Lemma 5.1.1. By Remark 5.1.2, the projection of each $W_k \in \mathcal{E}_{n+1}$ on $\mathbb{R}^n \times \{0\}$ is contained in a unique Whitney cube $w_j \in \mathcal{D}_n$ with $j = j(k)$.

Define the homeomorphism $G\colon \mathcal{C}^{n+1} \to \mathcal{K}(\Omega, \varphi)$ such that for $(x, z) \in \mathcal{C}^{n+1}$

$$G(x, z) = (f(x), M_x z) \text{ where } M_x = \frac{\varphi(\operatorname{dist}(f(x), \partial\Omega))}{1 - |x|}.$$

For the rest of the proof let $\delta(y) = \operatorname{dist}(y, \partial\Omega)$ when $y \in \Omega$.

Take $(x_1, z_1), (x_2, z_2) \in W_k$. By Lemma 5.1.2, there exists unique $j = j(k) \in \mathbb{N}$ such that $x_1, x_2 \in w_j$. Note that,

$$\begin{aligned} |M_{x_1} - M_{x_2}| &= \frac{|(1 - |x_2|)\varphi(\delta(f(x_1))) - (1 - |x_1|)\varphi(\delta(f(x_2)))|}{(1 - |x_1|)(1 - |x_2|)} \\ &\leq \frac{(1 - |x_2|)L_0|\delta(f(x_1)) - \delta(f(x_2))| + \varphi(\delta(f(x_2)))|x_1 - x_2|}{(1 - |x_1|)(1 - |x_2|)} \\ &\leq \frac{L_0 \lambda_j L |x_1 - x_2|}{1 - |x_1|} + M_{x_2} \frac{|x_1 - x_2|}{1 - |x_1|}. \end{aligned}$$

These inequalities follow from the fact that the distance function $\delta(x)$ is 1-Lipschitz, φ is L_0-Lipschitz and f is (λ_j, L)-quasisimilarity in w_j.

We claim that if $x \in w_j$ then $M_x \simeq \lambda_j$. Applying (5.2.1), we have that

$$M_x \leq \frac{L_0 \delta(f(x))}{1 - |x|} \leq L_0 \frac{\operatorname{diam} f(w_j) + \operatorname{dist}(f(w_j), \partial\Omega)}{\operatorname{dist}(w_j, \mathbb{S}^{n-1})} \leq \frac{\lambda_j L_0 L(c+1)}{2c}.$$

For the lower bound of M_x, take $x \in w_j$ and consider two cases. If $\delta(f(x)) \leq \epsilon$ then

$$M_x \geq \frac{1}{L_0} \frac{\delta(f(x))}{1 - |x|} \geq \frac{1}{L_0} \frac{\operatorname{dist}(f(w_j), \partial\Omega)}{\operatorname{diam} w_j + \operatorname{dist}(w_j, \mathbb{S}^{n-1})} \geq \frac{\lambda_j}{7 L_0 C L}.$$

If $\delta(f(x)) \geq \epsilon$ then

$$M_x \geq \frac{\varphi(\epsilon)}{\operatorname{diam} w_j + \operatorname{dist}(w_j, \mathbb{S}^{n-1})} \geq \frac{\epsilon}{7L_0 \operatorname{diam} w_j} \geq \frac{\lambda_j \epsilon}{7L_0 L \operatorname{diam} \Omega}.$$

Combining the two estimates, it follows that, for $x \in w_j$,

$$\frac{1}{R}\lambda_j \leq M_x \leq R\lambda_j$$

where $R > 1$ depends only on $L, L_0, \epsilon, \operatorname{diam} \Omega$.

Therefore, for $(x_1, z_1), (x_2, z_2) \in W_k$,

$$\begin{aligned} |M_{x_1} - M_{x_2}| &\leq \frac{L_0 \lambda_j L |x_1 - x_2|}{1 - |x_1|} + M_{x_2} \frac{|x_1 - x_2|}{1 - |x_1|} \\ &\leq (L_0 L + R)\lambda_j \frac{|x_1 - x_2|}{1 - |x_1|}. \end{aligned}$$

Set $A = L_0 L + R$ and take (x_1, z_1), (x_2, z_2) in W_k. The inequalities above yield

$$\begin{aligned} |G(x_1, z_1) - G(x_2, z_2)| &\leq |f(x_1) - f(x_2)| + |M_{x_1} z_1 - M_{x_2} z_2| \\ &\leq L\lambda_j |x_1 - x_2| + M_{x_1}|z_1 - z_2| + |z_1||M_{x_1} - M_{x_2}| \\ &\leq (L + A)\lambda_j |x_1 - x_2| + A\lambda_j |z_1 - z_2| \\ &\leq 4A\lambda_j |(x_1, z_1) - (x_2, z_2)|. \end{aligned}$$

Set $B = 1 + 2A^2$. If $|z_1 - z_2| \leq B|x_1 - x_2|$ then,

$$|G(x_1, z_1) - G(x_2, z_2)| \geq |f(x_1) - f(x_2)| \geq \frac{\lambda_j}{L}|x_1 - x_2| \geq \frac{\lambda_j}{2L}|(x_1, z_1) - (x_2, z_2)|.$$

On the other hand, if $|z_1 - z_2| \geq B|x_1 - x_2|$ then,

$$\begin{aligned} |G(x_1, z_1) - G(x_2, z_2)| &\geq |M_{x_1} z_1 - M_{x_2} z_2| \\ &\geq M_{x_2}|z_1 - z_2| - |(M_{x_1} - M_{x_2}) z_1| \\ &\geq \frac{\lambda_j}{A}|z_1 - z_2| - A\lambda_j |x_1 - x_2| \\ &\geq \frac{\lambda_j}{2A}|(x_1, z_1) - (x_2, z_2)|. \end{aligned}$$

Set $\Lambda_k = \lambda_{j(k)}$ and $\Lambda = \max\{4A, 2L\}$. Then, for $(x_1, z_1), (x_2, z_2) \in W_k$,

$$\frac{\Lambda_k}{\Lambda} \leq \frac{|G(x_1, z_1) - G(x_2, z_2)|}{|(x_1, z_1) - (x_2, z_2)|} \leq \Lambda\Lambda_k.$$

Thus, G is a (Λ_k, Λ)-quasisimilarity in every Whitney cube W_k.

Observe that the distortion of G in the interior of each W_k depends only on Λ. Hence G is K_1-quasiconformal in each cube W_k with K_1 depending only on Λ. By a theorem of Väisälä for removable singularities [39, Theorem 35.1], G is K_2-quasiconformal on $\mathcal{C}^{n+1}$ for some K_2 depending only on Λ.

Suppose that H is a C_{n+1}-bi-Lipschitz self map of $\mathbb{R}^{n+1}$ that maps $\mathbb{B}^{n+1}$ onto $\mathcal{C}^{n+1}$. Then, $F = G \circ H$ is K_2C-quasiconformal and maps $\mathbb{B}^{n+1}$ onto $\mathcal{K}(\Omega, h)$.

If $\mathcal{D}_{n+1} = \{Q_m\}_{m\in\mathbb{N}}$ is the fixed Whitney decomposition of $\mathbb{B}^{n+1}$ then the image of each Whitney cube Q_m intersects with at most $N(C_n)$ Whitney cubes of $\mathcal{E}_{n+1}$. Therefore, F is a (λ'_m, Λ')-quasisimilarity with L' depending only on L, L_0, n, ϵ, $\operatorname{diam}\Omega$. □

We now prove Proposition 5.2.1

Proof of Proposition 5.2.1. We first prove the statement for $n = 2$ and then apply induction on n.

By Riemann Mapping Theorem, there exists a conformal mapping F that maps the unit disc $\mathbb{B}^2$ onto Ω. Let $\mathcal{D}_2 = \{W_k\}_{k\in\mathbb{N}}$ be the fixed Whitney decomposition of $\mathbb{B}^2$. By Lemma 5.1.1 and Lemma 5.2.2, for any $k \in \mathbb{N}$ and $x \in W_k$

$$\frac{2}{7}\frac{\operatorname{diam} F(W_k)}{\operatorname{diam} W_k} \leq |F'(x)| \leq 14\frac{\operatorname{diam} F(W_k)}{\operatorname{diam} W_k}.$$

Set $\lambda_k = \frac{\operatorname{diam} F(W_k)}{\operatorname{diam} W_k}$. It follows that F is 1-quasiconformal and a $(\lambda_k, 14)$-quasisimilarity in every Whitney square W_k.

Suppose now that the claim holds true for some n. By Lemma 5.2.4 it follows that the claim is also true for $n + 1$ and the induction is complete. □

5.3 Slit domains

A domain $D \subset \mathbb{R}^n$ is called a *slit domain* if $\mathbb{R}^n \setminus \overline{D} \subset \mathbb{R}^{n-1} \times \{0\}$. When $n = 3$, Gehring discovered the following elegant characterization of quasidisks.

Theorem 5.3.1 ([15, Theorem 5]). *Suppose that Ω is a planar Jordan domain. Then the slit domain $\mathbb{R}^3 \setminus \overline{\Omega}$ is quasiconformally homeomorphic to $\mathbb{R}^3 \setminus \overline{\mathbb{B}^3}$ if and only if Ω is a quasidisk.*

The sufficiency of this theorem can easily be generalized in higher dimensions by the Tukia and Väisälä extension theorem [38, Theorem 3.11]. The necessity is much more demanding.

It follows from Proposition 5.2.1 that the bounded component of $\mathbb{R}^{n+1}\setminus\Sigma_n$ is quasiconformally equivalent to $\mathbb{B}^{n+1}$. For the unbounded component of $\mathbb{R}^{n+1}\setminus\Sigma_n$ we use the following lemma.

Lemma 5.3.2. *Suppose that $h\colon \mathbb{R}^n \to \mathbb{R}$ is an L-Lipschitz function for some $L > 0$. Then the function $G(x,z) = (x, z + h(x))\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is $2(L+2)$-bi-Lipschitz.*

Proof. Note first that

$$|G(x_1,z_1) - G(x_2,z_2)| \le 2(L+2)|(x_1,z_1) - (x_2,z_2)|.$$

For the lower bound we consider two cases. If $|z_1 - z_2| \le (L+1)|x_1 - x_2|$ then

$$|G(x_1,z_1) - G(x_2 - z_2)| \ge |x_1 - x_2| \ge \frac{|(x_1,z_1) - (x_2,z_2)|}{L+2}.$$

If $|z_1 - z_2| > (L+1)|x_1 - x_2|$ then

$$|G(x_1,z_1) - G(x_2 - z_2)| \ge |z_1 - z_2| - |h(x_1) - h(x_2)| \ge \frac{|(x_1,z_1) - (x_2,z_2)|}{L+2}. \qquad \square$$

As a corollary we deduce that $\mathbb{R}^n \setminus \Omega_n$ is quasiconformally equivalent to a slit domain.

Corollary 5.3.3. *Let D be a bounded domain in $\mathbb{R}^n$ and $h_1, h_2\colon \mathbb{R}^n \to \mathbb{R}$ be two L-Lipschitz functions satisfying $h_1 = h_2 = 0$ in $\mathbb{R}^n \setminus D$ and $h_2 \le h_1$ in D. Then the domain*

$$\mathbb{R}^{n+1} \setminus \{(x,z)\colon x \in \overline{D},\ h_2(x) \le z \le h_1(x)\}$$

is K-quasiconformal to the slit domain $\mathbb{R}^{n+1} \setminus \overline{D}$ for some constant $K \ge 1$ depending only on L.

Proof. Define $H\colon \mathbb{R}^{n+1} \setminus \overline{D} \to \mathbb{R}^{n+1} \setminus \{(x,z)\colon x \in \overline{D}, h_2(x) \le z \le h_1(x)\}$ with

$$H(x,z) = \begin{cases} (x, z + h_1(x)) & \text{if } x \in \overline{D}\ ,\ z > 0 \\ (x, z + h_2(x)) & \text{if } x \in \overline{D}\ ,\ z < 0 \\ (x, 0) & \text{if } x \notin \overline{D}. \end{cases}$$

Since H is $(L+2)$-bi-Lipschitz in each of the half-spaces $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$, and is homeomorphism on $\mathbb{R}^{n+1}\setminus\overline{D}$, it is locally $(L+2)$-bi-Lipschitz, therefore K-quasiconformal for some K depending only L. $\square$

5.4 Proof of Theorem 5.0.5

We are now ready to prove the first part of Theorem 5.0.5. It is enough to show that $\mathbb{R}^n \setminus \overline{\Omega_n}$ is quasiconformally homeomorphic to $\mathbb{R}^n \setminus \overline{\mathbb{B}^n}$.

Proof of the first claim of Theorem 5.0.5. We apply induction on n. For $n = 1$, $\Sigma_1 = \Gamma$ which is a quasicircle and the claim holds true.

Suppose that Σ_n is a K_n-quasisphere in $\mathbb{R}^{n+1}$ with K_n depending on n, K, $\operatorname{diam}\Gamma$ and the data of $\varphi_1, \dots, \varphi_{n-1}$. Define Σ_{n+1} and let Ω_{n+2} be the bounded component of $\mathbb{R}^{n+2} \setminus \Sigma_{n+1}$. In view of Theorem 2.2.3 it is sufficient to show that Ω_{n+2} is quasiconformally homeomorphic to $\mathbb{B}^{n+2}$ and $\mathbb{R}^{n+2} \setminus \overline{\Omega_{n+2}}$ is quasiconformally homeomorphic to $\mathbb{R}^{n+2} \setminus \overline{\mathbb{B}^{n+2}}$. The first equivalence follows from Proposition 5.2.1. The rest of the proof is essentially a generalization of the proof of Theorem 5.3.1. We repeat it for the shake of completeness.

Since the distance function $\delta(\cdot) = \operatorname{dist}(\cdot, \Sigma_n)$ is 1-Lipschitz, by Corollary 5.3.3, we can find a K'-quasiconformal map G_1 that maps $\mathbb{R}^{n+2} \setminus \overline{\Omega_{n+2}}$ onto $\mathbb{R}^{n+2} \setminus \overline{\Omega_{n+1}}$ and a K''-quasiconformal map G_2 that maps $\mathbb{R}^{n+2} \setminus \overline{\mathcal{C}^{n+2}}$ onto $\mathbb{R}^{n+2} \setminus \overline{\mathbb{B}^{n+1}}$ with K', K'' being absolute constants.

By the induction step, there exists a K_n-quasiconformal mapping $f_n \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $f_n(\mathbb{B}^{n+1}) = \Omega_{n+1}$. By an extension theorem of Tukia and Väisälä [38, Theorem 3.12], f_n can be extended to a K_n'-quasiconformal function $F \colon \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$. Here, K_n' depends only on K_n, n, hence only on K, n and the data of $\varphi_1, \dots, \varphi_{n-1}$.

Finally, there exists a C_{n+2}-bi-Lipschitz map $H \colon \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ that maps $\mathbb{R}^{n+2} \setminus \overline{\mathbb{B}^{n+2}}$ onto $\mathbb{R}^{n+2} \setminus \overline{\mathcal{C}^{n+2}}$, where C_{n+2} depends only on $n+2$. Define $f_{n+1} \colon \mathbb{R}^{n+2} \setminus \overline{\mathbb{B}^{n+2}} \to \mathbb{R}^{n+2} \setminus \overline{\Omega_{n+2}}$ with

$$f_{n+1} = (G_1)^{-1} \circ F \circ G_2 \circ H.$$

Note that f_{n+1} is a $C_{n+2}K'K''K_n'$-quasiconformal and maps $\mathbb{R}^{n+2} \setminus \overline{\mathbb{B}^{n+2}}$ onto $\mathbb{R}^{n+2} \setminus \overline{\Omega_{n+2}}$. □

We now give the proof of Theorem 1.2.2.

Proof of Theorem 1.2.2. The first claim of Theorem 1.2.2 is a special case of Theorem 5.0.5. It remains to show the second claim.

Suppose that h is L-Lipschitz and $\Sigma(\Gamma, \varphi)$ is a quasisymmetric sphere. So $\Sigma(\Gamma, \varphi)$ is $\lambda - \mathrm{LLC}$ for some $\lambda > 1$ depending on the distortion K and the Lipschitz constant L. We claim that Γ satisfies the 2-point condition with $C = 16L\lambda^2$.

Suppose the claim is false. Then there exist consecutive points $x_1, x_2, x_3, x_4 \in \Gamma$ such that,

$$|x_2 - x_1|, |x_4 - x_1| > 16L\lambda^2|x_3 - x_1|. \tag{5.4.1}$$

Define

$$d = \inf\{\operatorname{diam}\sigma \colon \sigma \subset \overline{\Omega} \text{ is a continuum that contains } x_1 \text{ and } x_3\}.$$

We consider the following two cases.

Case 1. Suppose that $d > 2\lambda|x_1 - x_3|$. Note that x_1 and x_3 are included in $\Sigma(\Gamma, \varphi) \cap B^3(x_1, 2|x_1 - x_3|)$. The $\lambda - \text{LLC}_1$ condition implies the existence of a continuum $E \subset \Sigma(\Gamma, \varphi) \cap B^3(x_1, 2\lambda|x_1 - x_3|)$ containing x_1, x_3. Its projection $\pi(E)$ on $\mathbb{R}^2$ is a continuum in $\overline{\Omega}$ that contains x_1, x_3 and has $\operatorname{diam}\pi(E) \leq \operatorname{diam} E \leq 2\lambda|x_1 - x_3| < d$, which contradicts the definition of d.

Case 2. Suppose that $d \leq 2\lambda|x_1 - x_3|$. Let $\sigma \subset \Omega$ be a path joining x_1 with x_3 satisfying $\operatorname{diam}\sigma \leq 3\lambda|x_1 - x_3|$. Observe that each path $\gamma \subset \Omega$, which joins x_2 with x_4, intersects σ and satisfies $\operatorname{diam}\gamma \geq (16L\lambda^2 - 3\lambda)|x_1 - x_3|$. The first observation is trivial. For the second, note that if $\gamma \subset \Omega$ is a path joining x_2 with x_4 of diameter less than $(16L\lambda^2 - 3\lambda)|x_1 - x_3|$ then

$$|x_1 - x_2| \leq \operatorname{diam}\gamma + \operatorname{diam}\sigma \leq 16L\lambda^2|x_1 - x_3|$$

which contradicts (5.4.1).

Consider the ball $B = B^3(x_1, 8L\lambda^2|x_1 - x_3|)$. Using (5.4.1), note that $x_2, x_4 \in \Sigma(\Gamma, \varphi) \setminus B$ and $x_2, x_4 \in \Sigma(\Gamma, \varphi) \setminus \frac{1}{\lambda}B$. Suppose that $E \subset \Sigma(\Gamma, \varphi)$ is a continuum that contains x_2, x_4. Then, $\pi(E) \subset \overline{\Omega}$ is a continuum that contains x_2, x_4 and, therefore, intersects σ. Suppose that $x \in E$ is such that $\pi(x) \in \sigma$. Since φ is L-Lipschitz,

$$|x - x_1| \leq |\pi(x) - x_1| + L|\pi(x) - x_1| \leq 2L\operatorname{diam}\sigma \leq 6L\lambda|x_1 - x_3|.$$

Thus, $x \in B$ and $E \cap B \neq \emptyset$. The latter implies that $\Sigma(\Gamma, \varphi)$ is not $\lambda - \text{LLC}_2$, which is a contradiction. Moreover, x_2, x_4 are in different components of $\Sigma(\Gamma, \varphi) \setminus \frac{1}{\lambda}B$. $\square$

For the proof of the second claim of Theorem 5.0.5, we construct a self homeomorphism of $\mathbb{R}^n$ that maps $\mathbb{S}^n$ onto Σ_n which is of bounded length distortion.

Recall that a continuous map f between two path connected metric spaces is of *bounded length distortion* (abbv. λ-*BLD map*) if for every rectifiable path $\gamma \subset X$,

$$\frac{\ell(\gamma)}{\lambda} \leq \ell(f(\gamma)) \leq \lambda \ell(\gamma).$$

Trivially, every L-bi-Lipschitz map is L-BLD but the converse is not always true, even locally. However, if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a λ-BLD homeomorphism, then it is λ-bi-Lipschitz [28, Corollary 2.13].

Proof of the second claim of Theorem 5.0.5. We apply induction on n.

For $n = 1$, $\Sigma_1 = \Gamma$ is a C-chord-arc curve and by a theorem of Jerison and Kenig [23, Proposition 1.13], there exists a Λ_1-bi-Lipschitz transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $f(\Gamma) = \mathbb{S}^1$ and Λ_1 depending only on C and $\operatorname{diam}\Gamma$. Hence, the claim holds for $n = 1$.

Suppose that there exists a Λ_n-bi-Lipschitz map $f_n : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that maps Σ_n onto $\mathbb{S}^n$ with Λ_n depending only on $C, n, \operatorname{diam}\Gamma$ and the data of $\varphi_1, \dots, \varphi_{n-1}$. For $(x,z) \in \mathbb{R}^{n+1} \times \mathbb{R}$ define the function

$$F(x,z) = \begin{cases} (f_n(x), M_x z) & \text{if } (x,z) \in \mathcal{C}^{n+2} \\ (f_n(x), z - 1 + |x| - \varphi(\delta(f_n(x))) & \text{if } x \in \mathbb{B}^{n+1}, z > 1 - |x| \\ (f_n(x), z + 1 - |x| + \varphi(\delta(f_n(x))) & \text{if } x \in \mathbb{B}^{n+1}, z < |x| - 1 \\ (f_n(x), z) & \text{otherwise} \end{cases}$$

where M_x is as in the proof of Lemma 5.2.4 and $\delta(\cdot) = \operatorname{dist}(\cdot, \Sigma_n)$.

First, observe that F is a homeomorphism of $\mathbb{R}^{n+2}$ onto itself that maps the double cone $\mathcal{C}^{n+2}$ onto Ω_{n+2}. Since the boundary of the double cone is a C_{n+2}-bi-Lipschitz sphere, for some C_{n+2} depending on $n+2$, it suffices to show that F is bi-Lipschitz.

Clearly, F is Λ_n-bi-Lipschitz in $(\mathbb{R}^{n+1} \setminus \Omega_{n+1}) \times \mathbb{R}$. Also, following the proof of Lemma 5.2.4, it is easy to show that F is $L^{(1)}$-bi-Lipschitz in $\mathcal{C}^{n+2}$ for some $L^{(1)} > 1$ depending on $C, n, \operatorname{diam}\Gamma$ and the data of $\varphi_1, \dots, \varphi_n$. Furthermore, applying the proof of Corollary 5.3.3 twice, we conclude that F is $L^{(2)}$-bi-Lipschitz in $(\mathbb{B}^{n+1} \times \mathbb{R}) \setminus \mathcal{C}^{n+2}$ for some $L^{(2)} > 1$ depending only on Λ_n and the data of φ_n.

Note that F is piecewise $L^{(3)}$-bi-Lipschitz with $L^{(3)} = \max\{L^{(1)}, L^{(2)}\}$. Consequently, F is $L^{(3)}$-BLD homeomorphism, hence $L^{(3)}$-bi-Lipschitz with $L^{(3)}$ depending only on $C, n, \operatorname{diam}\Gamma$ and the data of $\varphi_1, \dots, \varphi_n$. $\square$

Chapter 6

Snowflake curves and Assouad dimension

In this chapter we use the theory of Rohde's snowflakes for the proof of Theorem 1.2.3. The basic notation for Rohde's snowflakes is presented in Section 2.4. In Section 6.1, we create an index that measures, on small scale, the deviation of a quasicircle from being a chord-arc curve. The proof of Theorem 1.2.3 is given in Section 6.2. Finally, in Section 6.3 we give examples of quasicircles Γ that have Assouad dimensions equal to 1, satisfy the level quasicircle property and $\Sigma(\Gamma, t^{\alpha})$ is not quasisymmetric to $\mathbb{S}^2$ for any $\alpha \in (0,1)$.

Definition 6.0.1. *The Assouad dimension of a metric space (X,d) is the infimum of all $s > 0$ that satisfy the following property: There exists $C > 1$ such that for any $\epsilon > 0$, any $Y \subset X$ can be covered by at most $C\epsilon^{-s}$ subsets of diameter at most $\epsilon \operatorname{diam} Y$.*

This dimension was first introduced by Assouad [3] under the name *metric dimension* as a tool to study the metric spaces which are bi-Lipschitz embeddable into some Euclidean space $\mathbb{R}^n$. If (X,d) is a metric space, we always have

$$\dim_H(X) \leq \underline{\dim}_B(X) \leq \overline{\dim}_B(X) \leq \dim_A(X)$$

where $\dim_H, \underline{\dim}_B, \overline{\dim}_B, \dim_A$ are the Hausdorff, lower box-counting, upper box-counting and Assouad dimension respectively. The main difference between Hausdorff and Assouad dimension is that that the former is related to the average small scale structure of sets, while the latter measures the size of sets in all scales. A detailed survey of the concept can be found in [27].

6.1 Coding of quasicircles and a chord-arc index

Let Γ be a quasicircle. Suppose that $\mathcal{S}$ is a p-snowflake and f a bi-Lipschitz self map of $\mathbb{R}^2$ that maps $\mathcal{S}$ onto Γ. The coding (p, k_w) of the construction of $\mathcal{S}$, as in Section 2.4, induces a coding of Γ. In particular, the subdivision $\{\mathcal{S}_w\}_{w\in\mathcal{W}}$, induces a subdivision of Γ by setting $\Gamma_w = f(\mathcal{S}_w)$. The quadruple (p, k_w, Γ_w, f) is called a *coding* of Γ and $\mathcal{S}$ is called the snowflake associated to the given coding.

Note that if f is L-bi-Lipschitz then for any $w, u \in \mathcal{W}$

$$\frac{\operatorname{diam}\Gamma_w}{4^{\ell(u)}L^2} \le \operatorname{diam}\Gamma_{wu} \le \min\{p^{\ell(u)}L^2, 1\}\operatorname{diam}\Gamma_w. \tag{6.1.1}$$

Fix a quasicircle Γ and a coding (p, k_w, Γ_w, f). We define an index that measures, on small scale, how much a subarc Γ_w deviates from being a chord-arc curve. For $w \in \mathcal{W}$ and $c \in (0,1)$ denote with $A(\Gamma_w, c)$ the set of all words $u \in \mathcal{W}$ that satisfy

1. $\operatorname{diam}\Gamma_{wu} \ge c\operatorname{diam}\Gamma_w$,
2. $\operatorname{diam}\Gamma_{wui} < c\operatorname{diam}\Gamma_w$ for some $i \in \{1,2,3,4\}$,
3. the words in $A(\Gamma_w, c)$ are *minimal* in the sense that if u is a word satisfying the above two conditions, then $uv \notin A(\Gamma_w, c)$ for each $v \in \mathcal{W}$.

We think of $A(\Gamma_w, c)$ as a decomposition of Γ_w into disjoint subarcs $\Gamma_{wu} \subset \Gamma_w$ that have diameter comparable to $c\operatorname{diam}\Gamma_w$. Denote with $N(\Gamma_w, c)$ the number of elements in $A(\Gamma_w, c)$. The dependence on the chosen coding is suppressed.

For $w \in \mathcal{W}$ and $c \in (0,1)$ define

$$M(\Gamma_w, c) = \frac{1}{\operatorname{diam}\Gamma_w}\sum_{u\in A(\Gamma_w,c)}\operatorname{diam}\Gamma_{wu}. \tag{6.1.2}$$

The number $M(\Gamma_w, c)$ is an approximation of the number $\ell(\Gamma_w)/\operatorname{diam}\Gamma_w$ with c being the scale of approximation. Since all the subarcs Γ_{wu} have diameter comparable to $c\operatorname{diam}\Gamma_w$, it easily follows that $M(\Gamma_w, c) = cN(\Gamma_w, c)$. The following lemma summarizes the properties of this index.

Lemma 6.1.1. *Let Γ be a quasicircle, (p, k_w, Γ_w, f) be a coding of Γ and f be L-bi-Lipschitz.*

1. *If $c \in (0,1)$ and $w \in \mathcal{W}$ then $M(\Gamma_w, c) \simeq cN(\Gamma_w, c)$ with the comparison constants depending on L.*
2. *If $0 < c_1 \le c_2 < 1$ then $M(\Gamma_w, c_2) \le M(\Gamma_w, c_1) \le mM(\Gamma_w, c_2)$ with $m > 1$ depending on p, L and the ratio c_2/c_1.*
3. *Suppose that $\mathcal{S}$ is the p-snowflake associated to the given coding. If $c \in (0,1)$ and $w \in \mathcal{W}$ then $M(\Gamma_w, c) \simeq M(\mathcal{S}_w, c)$ with the comparison constants depending on p, L.*
4. *For any $w \in \mathcal{W}$,*
$$\lim_{c\to 0} M(\Gamma_w, c) = \frac{\ell(\Gamma_w)}{\operatorname{diam}\Gamma_w}.$$

Proof. Property (1) is an immediate consequence of the definition.

For property (2) consider the sets $G_1 = \{\Gamma_w : w \in A(\Gamma_w, c_1)\}$ and $G_2 = \{\Gamma_w : w \in A(\Gamma_w, c_2)\}$. To obtain G_1 we subdivide, if necessary, each of the subarcs of G_2 until they satisfy the properties in the definition of $A(\Gamma_w, c_1)$. Thus, $M(\Gamma_w, c_1) \geq M(\Gamma_w, c_2)$. To show the other inequality, by property (1) of Lemma 6.1.1, it suffices to check the ratio $N(\Gamma_w, c_1)/N(\Gamma_w, c_2)$. Fix a subarc $\Gamma_{wv} \in G_2$ and assume that $\Gamma_{wvu} \in G_1$. We claim that $\ell(u) \leq N = N(p, L, c_2/c_1)$. Assuming the claim we have that

$$N(\Gamma_w, c_1) \leq 4^N N(\Gamma_w, c_2).$$

To show the claim apply (6.1.1) twice and note that for some $i \in \{1, 2, 3, 4\}$

$$c_1 \operatorname{diam} \Gamma_w \leq \operatorname{diam} \Gamma_{wvu} \leq L^2 p^{\ell(u)} \operatorname{diam} \Gamma_{wv} \leq 4L^4 p^{\ell(u)} \operatorname{diam} \Gamma_{wvui} \leq 4L^4 p^{\ell(u)} c_2 \operatorname{diam} \Gamma_w.$$

To prove (3), it suffices to show that $N(\Gamma_w, c) \simeq N(\mathcal{S}_w, c)$ with the comparison constants depending on p, L. Let

$$c' = \min_{v \in A(\mathcal{S}_w, c)} \left\{ \frac{\operatorname{diam} \Gamma_{wv}}{\operatorname{diam} \Gamma_w} \right\}.$$

Since f is L-bi-Lipschitz, the ratio c/c' is bounded above and below by constants depending on L. By (2) of Lemma 6.1.1, $N(\mathcal{S}_w, c) \leq N(\Gamma_w, c') \lesssim N(\Gamma_w, c')$. Similarly we obtain the inverse inequality.

For the proof of (4), fix $w \in \mathcal{W}$ and assume that $\ell(\Gamma_w) < \infty$. Similar arguments apply in the case $\ell(\Gamma_w) = +\infty$. Let $\epsilon > 0$ and $x_0, \ldots, x_N \in \Gamma_w$ be consecutive points such that x_0, x_N are the endpoints of Γ_w and $\sum_{i=1}^N |x_i - x_{i-1}| \geq \ell(\Gamma_w) - \epsilon \operatorname{diam} \Gamma_w$. For each $i \in \{0, \ldots, N\}$ find $v_i \in \mathcal{W}$ such that one of the two endpoints x_i' of Γ_{wv_i} satisfies $|x_i - x_i'| \leq \epsilon 2^{-i-1} \operatorname{diam} \Gamma_w$. If $c = (4L^2 \operatorname{diam} \Gamma_w)^{-1} \min_{1 \leq i \leq N} \operatorname{diam} \Gamma_{wv_i}$ then,

$$M(\Gamma_w, c) \geq \frac{1}{\operatorname{diam} \Gamma_w} \sum_{i=1}^N |x_i' - x_{i-1}'| \geq \frac{1}{\operatorname{diam} \Gamma_w} \sum_{i=1}^N |x_i - x_{i-1}| - \epsilon \geq \frac{\ell(\Gamma_w)}{\operatorname{diam} \Gamma_w} - 2\epsilon. \qquad \square$$

6.2 Proof of Theorem 1.2.3

The following lemma is a variation of Lemma 4.3.2 and gives a necessary condition for Γ and φ so that $\Sigma(\Gamma, \varphi)$ is quasisymmetric to $\mathbb{S}^2$.

Lemma 6.2.1. *Let Γ be a quasicircle which has the level quasicircle property, (p, k_w, Γ_w, f) a coding of Γ*

and $\varphi \in \mathcal{F}$. *Suppose that there exists a sequence* $\{\Gamma_{w_n}\}_{n\in\mathbb{N}}$ *such that for each* $n \in \mathbb{N}$,

$$c_n = \frac{\varphi^{-1}(\operatorname{diam}\Gamma_{w_n})}{\operatorname{diam}\Gamma_{w_n}} < 1$$

and $\lim_{n\to\infty} M(\Gamma_{w_n}, c_n) = +\infty$. *Then* $\Sigma(\Gamma,\varphi)$ *is not quasisymmetric to* $\mathbb{S}^2$.

Proof. Suppose that there exist $\epsilon > 0$ and $K > 1$ such that γ_ϵ is a K-quasicircle for all $\epsilon \in [0,\epsilon_0]$. Then, for all $\epsilon \in [0,\epsilon_0]$, the curves γ_ϵ satisfy Ahlfors 2-point condition (2.2.1) for the same $C > 1$. Suppose also that f is L-bi-Lipschitz.

For each $n \in \mathbb{N}$ set $\epsilon_n = c_n \operatorname{diam}\Gamma_{w_n}$ and $c_n' = 9C^2 c_n$. By Lemma 6.1.1 we have that $M(\Gamma_{w_n}, c_n') \geq \frac{1}{m}M(\Gamma_{w_n}, c_n)$ for some $m = m(L) > 1$. Passing to a subsequence, we may assume that $M(\Gamma_{w_n}, c_n') > Cn$. Let $x_1^n, \dots, x_{N_n}^n$ be the endpoints of the subarcs $\Gamma_{w_n v}$, $v \in A(\Gamma_{w_n}, c_n')$. We may assume that $x_1^n, \dots, x_{N_n}^n$ are consecutive. Then,

$$\sum_{i=1}^{N_n-1} |x_i^n - x_{i-1}^n| \geq \frac{1}{C} \sum_{v\in A(\Gamma_{w_n}, c_n')} \operatorname{diam}\Gamma_{w_n v} = \frac{M(\Gamma_{w_n}, c_n')\operatorname{diam}\Gamma_{w_n}}{C} > n \operatorname{diam}\Gamma(x_0^n, x_{N_n}^n).$$

Moreover, for all $i = 1, \dots, N_n - 1$, $|x_i^n - x_{i-1}^n| \geq \frac{1}{C} c_n' \operatorname{diam}\Gamma_{w_n} = 9C\epsilon_n$. Finally, $\varphi(\epsilon_n) = \operatorname{diam}(\Gamma(x_1^n, x_{N_n}^n))$. The assumptions of Lemma 4.3.2 are satisfied and $\Sigma(\Gamma,\varphi)$ is not quasisymmetric to $\mathbb{S}^2$. ☐

To prove Theorem 1.2.3 we need to show that if the Assouad dimension of a quasicircle Γ is bigger than 1 and $\varphi(t) = t^\alpha$ with $\alpha \in (0,1)$ then Γ, φ satisfy the assumptions of Lemma 6.2.1. The following two lemmas are required for the proof of Theorem 1.2.3.

Lemma 6.2.2. *Suppose that* Γ *has a coding* (p, k_w, Γ_w, f) *and* f *is* L*-bi-Lipschitz. There exist* $\alpha = \alpha(p) > 0$ *and* $A = A(L) > 0$ *with the following property: If* $\Gamma_{w_0} \subset \Gamma$, $\delta < \delta' < 1$ *and* $M(\Gamma_{w_0}, \delta) > M$ *for some* $M > 1$ *then there exists* $\Gamma_{w_0 w} \subset \Gamma_{w_0}$ *such that* $M(\Gamma_{w_0 w}, \delta') \geq AM^{\alpha\frac{\log\delta'}{\log\delta}}$.

Proof. Suppose that $\mathcal{S}$ is the p-snowflake associated to the given coding.

For $v \in \mathcal{W}$ and $n \in \mathbb{N}$, define $\mu(\Gamma_v, n) = 4^{-n}\sum_{u\in\mathcal{W}_n}(4p)^{k_{wu}-k_w}$. It is easy to see that if $v \in \mathcal{W}$ and $n \in \mathbb{N}$ then

$$\mu(\Gamma_v, n) = \frac{1}{\operatorname{diam}\mathcal{S}_v}\sum_{u\in\mathcal{W}_n}\operatorname{diam}\mathcal{S}_{vu} \simeq \frac{1}{\operatorname{diam}\Gamma_v}\sum_{u\in\mathcal{W}_n}\operatorname{diam}\Gamma_{vu}$$

with the comparison constants depending on L. Like $M(\Gamma_v, \cdot)$, the index $\mu(\Gamma_v, \cdot)$ is an approximation of the number $\ell(\Gamma_v)/\operatorname{diam}\Gamma_v$ but in a different sense. While in $M(\Gamma_v, \cdot)$ we divide Γ_v into subarcs of roughly the same diameter, in $\mu(\Gamma_v, \cdot)$ we divide Γ_v into subarcs of same generation.

If $v \in \mathcal{W}$, $n_1 \leq n_2$ then $\mu(\Gamma_v, n_1) \geq \mu(\Gamma_v, n_2)$. We claim that, for any $n \in \mathbb{N}$ and $v \in \mathcal{W}$

$$M(\Gamma_v, p^n) \lesssim \mu(\Gamma_v, n) \lesssim M(\Gamma_v, 4^{-n})$$

with the comparison constants depending on L. Indeed, since $M(\Gamma_v, c) \simeq M(\mathcal{S}_v, c)$ for any $c < 1$ we only need to prove the replacing Γ with $\mathcal{S}$. If $vu \in A(\mathcal{S}_v, 4^{-n})$ then $\ell(u) \geq n$. Thus,

$$M(\mathcal{S}_v, 4^{-n}) = \frac{1}{\operatorname{diam} \mathcal{S}_v} \sum_{u \in \mathcal{W}_n} \sum_{uu' \in A(\mathcal{S}_v, 4^{-n})} \operatorname{diam} \mathcal{S}_{vuu'} \geq \frac{1}{\operatorname{diam} \mathcal{S}_v} \sum_{u \in \mathcal{W}_n} \operatorname{diam} \mathcal{S}_{vu}.$$

We work similarly for the lower bound.

Let m be the integer part of $-\log \delta' / \log 4$ and N be the integer part of $\log \delta / \log p$. Since $\delta < \delta'$, it is easy to see that $m \lesssim N$ with the comparison constants depending on p. Take N' to be the smallest multiple of m that is bigger than N. If $m \geq N$ then $N' = m$ while if $m < N$ then $N' \leq N + m$. In each case, $N \simeq N'$. Hence,

$$\mu(\Gamma_{w_0}, N') \geq \mu(\Gamma_{w_0}, N) \gtrsim M(\Gamma_{w_0}, \delta) > M.$$

Set $C = \mu(\Gamma_{w_0}, N')$. We claim that there exists $w \in \mathcal{W}$ with $\ell(w) \leq N' - m$ such that $\mu(\Gamma_{w_0 w}, m) \geq C^{m/N'}$. Assuming the claim,

$$M(\Gamma_{w_0 w}, \delta') \gtrsim \mu(\Gamma_{w_0 w}, m) \geq C^{m/N'} \gtrsim M^{\alpha \frac{\log \delta'}{\log \delta}}$$

with $\alpha > 0$ depending on p and the comparison constants depending on L.

To prove the claim, assume the opposite. That is, for all $w \in \mathcal{W}$ with $\ell(w) \leq N' - m$ we have $\mu(\Gamma_{w_0 w}, m) < C^{m/N'}$. We apply induction to show that $\mu(\Gamma_{w_0}, im) < C^{im/N'}$ for all $i \in \{1, \ldots, N'/m\}$. For $i = 1$, it is true by our assumption. Suppose it is true for i. Then,

$$\begin{aligned}
\mu(\Gamma_{w_0},(i+1)m) &= 4^{-(i+1)m}\sum_{w\in\mathcal{W}_{(i+1)m}}(4p)^{k_{w_0w}-k_{w_0}}\\
&= 4^{-(i+1)m}\sum_{u\in\mathcal{W}_{im}}\sum_{v\in\mathcal{W}_m}(4p)^{k_{w_0uv}-k_{w_0}}\\
&= 4^{-(i+1)m}\sum_{u\in\mathcal{W}_{im}}(4p)^{k_{w_0u}-k_{w_0}}\sum_{v\in\mathcal{W}_m}(4p)^{k_{w_0uv}-k_{w_0u}}\\
&= 4^{-(i+1)m}\sum_{u\in\mathcal{W}_{im}}(4p)^{k_{w_0u}-k_{w_0}}4^m\mu(\Gamma_{w_0u},m)\\
&< 4^{-im}C^{m/N'}\mu(\Gamma_{w_0},im)4^{im}\\
&< C^{m(i+1)/N'}.
\end{aligned}$$

By induction the claim holds for $i=N'/m$ contradicting the fact that $\mu(\Gamma_{w_0},n)=C$. □

Lemma 6.2.3. *Suppose that $\epsilon>0$, Γ is a quasicircle and (p,k_w,Γ_w,f) is a coding of Γ. If the Assouad dimension of Γ is greater than $1+\epsilon$, then, for any $n\in\mathbb{N}$, there exist $w_n\in\mathcal{W}$ and $\delta_n\in(0,1)$ such that $N(\Gamma_{w_n},\delta_n)>n\delta_n^{-(1+\epsilon)}$.*

Proof. Suppose that f is L-bi-Lipschitz.

Assume the opposite, that is, $N(\Gamma_w,\delta)\lesssim\delta^{-(1+\epsilon)}$ for all $\delta>0$ and Γ_w. Fix a $\delta\in(0,1)$ and a set $X\subset\Gamma$ such that $\operatorname{diam}X\le\operatorname{diam}\Gamma/10$. Let Γ' be the smallest subarc of Γ that contains X. Since Γ satisfies (2.2.1), $\operatorname{diam}\Gamma'\le C\operatorname{diam}X$ for some $C>1$ depending only on Γ. Choose a maximal word w such that $\Gamma'\subset\Gamma_w$.

Suppose that Γ' contains a subarc Γ_{wi} for some $i=1,2,3,4$. If $v\in A(\Gamma_w,\delta C^{-1}(2L)^{-4})$ and $j\in\{1,2,3,4\}$, using (6.1.1) twice, we see that

$$\operatorname{diam}\Gamma_{wv}\le 4L^2\operatorname{diam}\Gamma_{wvj}\le\frac{\delta\operatorname{diam}\Gamma_w}{4CL^2}\le\frac{\delta\operatorname{diam}\Gamma_{wi}}{C}\le\delta\operatorname{diam}X.$$

Therefore,

$$X\subset\bigcup_{u\in A(\Gamma_w,\delta C^{-1}(2L)^{-4})}\Gamma_{wu}$$

where all such subarcs Γ_{wu} have diameter less or equal to $\delta\operatorname{diam}X$ and

$$N(\Gamma_w,\delta C^{-1}(2L)^{-4})\lesssim\delta^{-(1+\epsilon)}.$$

Suppose that Γ' does not contain one of the subarcs $\Gamma_{w1},\Gamma_{w2},\Gamma_{w3},\Gamma_{w4}$. Then, there exist $i\in\{1,2,3\}$

and maximal natural numbers r, q such that $\Gamma' \subseteq \Gamma_{wi4^r} \cup \Gamma_{w(i+1)1^q}$. In that case repeat the above argument for the subarcs $\Gamma' \cap \Gamma_{wi4^r}$ and $\Gamma' \cap \Gamma_{w(i+1)1^q}$.

In both cases, X can be covered by at most $M\delta^{-(1+\epsilon)}$ subsets of diameter at most δ and Γ has Assouad dimension less or equal to $1+\epsilon$. $\square$

We are now ready to prove Theorem 1.2.3. The basic idea of the proof is to show that if the Assouad dimension of Γ is bigger than 1 and $\alpha \in (0,1)$ then Γ and $\varphi(t) = t^\alpha$ satisfy the assumptions of Lemma 6.2.1.

Proof of Theorem 1.2.3. Suppose that Γ is a quasicircle with a coding (p, k_w, Γ_w, f) and Assouad dimension bigger than $1+\epsilon$ for some $\epsilon > 0$. Suppose also that f is L-bi-Lipschitz. In view of Proposition 4.1.6, we may assume that Γ satisfies the LQC property.

By Lemma 6.2.3, for each $n > 0$ there exist $w_n \in \mathcal{W}$ and $\delta_n \in (0,1)$ such that $N(\Gamma_{w_n}, \delta_n) > n\delta_n^{-(1+\epsilon)}$. We may assume that the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is decreasing and converges to 0. By Lemma 6.1.1, it follows that $M(\Gamma_{w_n}, \delta_n) \gtrsim n\delta_n^{-\epsilon}$. Set $c_n = \frac{\varphi^{-1}(\operatorname{diam}\Gamma_{w_n})}{\operatorname{diam}\Gamma_{w_n}}$ and consider the following two cases.

Case 1. Suppose that $c_n \leq \delta_n$. Then, by Lemma 6.1.1,

$$M(\Gamma_{w_n}, c_n) \geq M(\Gamma_{w_n}, \delta_n) \gtrsim n\delta_n^{-\epsilon} \gtrsim n$$

with the similarity constants depending on L.

Case 2. Suppose that $c_n > \delta_n$. By Lemma 6.2.2, there exists $\Gamma_{w_n v} \subset \Gamma_{w_n}$ such that

$$M(\Gamma_{w_n v}, c_n) \gtrsim \delta_n^{-\epsilon\alpha\frac{\log c_n}{\log \delta_n}} = c_n^{-\alpha\epsilon}$$

with the similarity constants depending on L. Set $c_n' = \frac{\varphi^{-1}(\operatorname{diam}\Gamma_{w_n v})}{\operatorname{diam}\Gamma_{w_n v}}$ and note that $c_n' < c_n$. It follows that

$$M(\Gamma_{w_n v}, c_n') \geq M(\Gamma_{w_n v}, c_n) \gtrsim c_n^{-\alpha\epsilon}.$$

By Case 1 and Case 2, there exists a sequence Γ_{v_n} of subarcs of Γ such that if $c_n = \frac{\varphi^{-1}(\operatorname{diam}\Gamma_{v_n})}{\operatorname{diam}\Gamma_{v_n}}$ then $\lim_{n\to\infty} M(\Gamma_{v_n}, c_n) = \infty$. The theorem follows now from Lemma 6.2.1. $\square$

6.3 Further examples from Rohde's snowflakes

For the rest of this section we give an example of quasicircles with Assouad dimension equal to 1, satisfying the LQC property such that the surface $\Sigma(\Gamma, t^\alpha)$ is not quasisymmetric to $\mathbb{S}^2$ for any $\alpha \in (0,1)$.

Let $N \geq 4$ be a natural number. As in Section 3.5, an (N,p)-snowflake is a snowflake constructed with Rohde's method except for the first step where a regular N-gon of side length 1 is used in place of the unit square.

Also, as in Section 2.4, $\mathcal{S}$ is called *homogeneous* if, for any n, at the n-th step of the construction of $\mathcal{S}$, all edges of the n-th generation are replaced by the same polygonal arc. In other words, k_w depends only on $\ell(w)$. If $\mathcal{S}$ is homogeneous then, for $n \in \mathbb{N}$, define $k(n) = k_w$ for some $w \in \mathcal{W}_n$. Intuitively, $k(n)$ represents the number of times that the Type I arc of Figure 2.1 was used in the first n steps.

By Corollary 3.5.1, there exist $N_0 \in \mathbb{N}$ and $p_0 \in (1/4, 1/2)$ such that, every (N,p)-snowflake, with $N \geq N_0$ and $p \leq p_0$, satisfies the LQC property.

Proposition 6.3.1. *Let $\alpha, \beta \in (0,1)$, $\varphi(t) = t^\alpha$ and $\mathcal{S}$ be a homogenous (N,p)-snowflake with $N \leq N_0$, $p \leq p_0$ and $k(n)$ equal to the integer part of n^β. Then, $\mathcal{S}$ has the level quasicircle property, has Assouad dimension equal to 1 and $\Sigma(\mathcal{S},\varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

The proof requires the following simple lemma.

Lemma 6.3.2. *Let $\beta \in (0,1)$. For any $\epsilon > 0$ there exists $M > 0$ such that, for any $x > M$ and $y > 0$, $|x^\beta - y^\beta| \leq \epsilon |x - y|$.*

Proof. Let $M = (2/\epsilon)^{\frac{1}{1-\beta}}$. If $x > M$ and $y \geq M/2$, by Mean Value Theorem, it is easy to see that $|x^\beta - y^\beta| \leq \epsilon|x-y|$. If $x > M$ and $y < M/2$ then

$$x - y > \frac{x}{2} > \frac{1}{\epsilon}x^\beta > \frac{1}{\epsilon}(x^\beta - y^\beta). \qquad \square$$

For the proof of Proposition 6.3.1, we need to show that $\mathcal{S}$ and φ satisfy the assumptions of Lemma 6.2.1.

Proof of Proposition 6.3.1. Since $N \geq N_0$ and $p \leq p_0$, by Corollary 3.5.1, the (N,p)-snowflake $\mathcal{S}$ satisfies the LQC property.

Fix $\epsilon > 0$; we claim that $\mathcal{S}$ has Assouad dimension less than $1+\epsilon$. Take $\delta > 0$ and $w \in \mathcal{W}_n$ for some $n \in \mathbb{N}$. Since $\mathcal{S}$ is homogenous, there exists $m \geq n$ such that $A(\mathcal{S}_w, \delta) = \mathcal{W}_{m-n}$. Then, $4^{n-m}(4p)^{k(m)-k(n)} \geq \delta$ which implies that

$$(m-n)\log 4 - (m^\beta - n^\beta)\log 4p \leq \log\left(\frac{1}{\delta}\right) + C \tag{6.3.1}$$

for some $C > 0$ depending only on p, β. By Lemma 6.3.2, there exists $M > 0$ such that for $x > M$ and $x > y$

$$x^\beta - y^\beta \leq \frac{\epsilon}{1+\epsilon}\frac{\log 4}{\log(4p)}(x-y). \tag{6.3.2}$$

If $m < M$ then clearly $N(\mathcal{S}_w, \delta) = 4^{m-n} \leq 4^M \left(\frac{1}{\delta}\right)^{1+\epsilon}$. If $m \geq M$ then by (6.3.2)

$$(m-n)\log 4 - (m^\beta - n^\beta)\log 4p \geq \frac{\log 4}{1+\epsilon}(m-n)$$

and (6.3.1) yields

$$(m-n)\log 4 \leq \log\left(\frac{1}{\delta}\right)^{1+\epsilon} + 2C_2.$$

Therefore,

$$N(\mathcal{S}_w, \delta) = 4^{m-n} \leq 4^M e^{2C_2}\left(\frac{1}{\delta}\right)^{1+\epsilon}$$

and the claim is proved. Since ϵ was chosen arbitrarily, it follows that $\mathcal{S}$ has Assouad dimension equal to 1.

We prove now that $\Sigma(\mathcal{S}, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$. Take $n \in \mathbb{N}$, $w \in \mathcal{W}_n$ and assume that $A(\mathcal{S}_w, (\operatorname{diam}\mathcal{S}_w)^{\frac{1}{\alpha}-1}) = \mathcal{W}_{m-n}$. If $v \in A(\mathcal{S}_w, (\operatorname{diam}\mathcal{S}_w)^{\frac{1}{\alpha}-1})$ then

$$p^\alpha(\operatorname{diam}\mathcal{S}_{wv})^\alpha \leq \operatorname{diam}\mathcal{S}_w \leq (\operatorname{diam}\mathcal{S}_{wv})^\alpha$$

and for some $0 < A_1 < A_2$ depending on p, α,

$$A_1\left(4^{-m}(4p)^{m^\beta}\right)^\alpha < 4^{-n}(4p)^{n^\beta} < A_2\left(4^{-m}(4p)^{m^\beta}\right)^\alpha.$$

Therefore,

$$\frac{\log A_1}{m} \leq \left(\alpha - \frac{n}{m}\right)\log 4 - \frac{1}{m^{1-\beta}}\left(\alpha - \left(\frac{n}{m}\right)^\beta\right)\log(4p) \leq \frac{\log A_2}{m}.$$

Note that as n goes to infinity, m goes to infinity and n/m goes arbitrarily close to α. Hence, for any $\epsilon > 0$ we can find sufficiently large n so that $|\frac{1}{\alpha} - \frac{m}{n}| < \epsilon$. Therefore, if $c = \operatorname{diam}{\mathcal{S}_w}^{1/\alpha-1}$

$$M(\mathcal{S}_w, c) \simeq N(\mathcal{S}_w, c)c = (4p)^{m^\beta - n^\beta} > (4p)^{\left((1/\alpha-\epsilon)^\beta-1\right)n^\beta}$$

which goes to infinity as n goes to infinity. The proposition follows from Lemma 6.2.1. □

Chapter 7

Quasisymmetric spheres over quasidisks – Analytic construction

Suppose that f is a quasiconformal mapping that maps a Jordan domain Ω onto the unit disk $\mathbb{B}^2$. For a function $\varphi \in \mathcal{F}$ we write

$$\tilde{\Sigma}(f,\varphi) = \{(x,z)\colon x \in \overline{\Omega}, z = \pm\varphi(1-|f(x)|)\}.$$

This chapter is devoted to the proof of Theorem 1.4.1.

7.1 Surfaces obtained by rotations of decreasing functions

Suppose that $\varphi \in \mathcal{F}_2$. Define

$$\tilde{\Sigma}(\varphi) = \tilde{\Sigma}(\mathrm{id},\varphi) = \{(x,z) \in \overline{\mathbb{B}^2} \times \mathbb{R}\colon z = \pm\varphi(1-|x|)\}.$$

Write $\mathbb{R}^3 = \{(t,s,z)\colon t,s,z \in \mathbb{R}\}$. Note that $\tilde{\Sigma}(\varphi)$ is the surface generated by first revolving the graph of $\varphi(1-t)$, $t \in [0,1]$ around the vertical axis $\{t = s = 0\}$ and then reflecting the resulting surface with respect to the horizontal plane $\{z = 0\}$.

To prove the first claim of Theorem 1.4.1, we first show the following result.

Proposition 7.1.1. *Suppose that $\varphi \in \mathcal{F}$ is such that for some $\delta \in (0,1)$ and $M > 0$, $\varphi(t) > Mt$ for each $t \in (0,\delta)$ and $\varphi(1) - \varphi(t) < M(1-t)$ for each $t \in (1-\delta,1)$. Then, the surface $\tilde{\Sigma}(\varphi)$ is a 2-dimensional bi-Lipschitz sphere.*

The following two simple observations are needed for the proof of Proposition 7.1.1.

Lemma 7.1.2. *If $\varphi : (a,b) \to \mathbb{R}$ is a monotone continuous function, then the graph of φ is a 2-chord-arc curve. More precisely, if φ_{t_1,t_2} is the graph of φ from t_1 to t_2 with $t_1 < t_2$, then,*

$$\ell(\varphi_{t_1,t_2}) \le 2|(t_1,\varphi(t_1)) - (t_2,\varphi(t_2))|.$$

Proof. First observe that

$$\operatorname{diam} \varphi_{t_1,t_2} = |(t_1, \varphi(t_1)) - (t_2, \varphi(t_2))| \geq \frac{1}{2}(|t_1 - t_2| + |\varphi(t_1) - \varphi(t_2)|).$$

Let $t_1 = x_0 < x_1 < \cdots < x_n = t_2$ and $z_i = (x_i, \varphi(x_i))$. Then

$$\begin{aligned}\sum_{i=1}^{n} |z_i - z_{i-1}| &\leq \sum_{i=1}^{n} (|x_1 - x_{i-1}| + |\varphi(x_i) - \varphi(x_{i-1})|) \\ &= |t_1 - t_2| + |\varphi(t_1) - \varphi(t_2)|.\end{aligned}$$

Therefore, $\ell(\varphi_{t_1,t_2}) \leq |t_1 - t_2| + |\varphi(t_1) - \varphi(t_2)|$ and the lemma follows. $\square$

Lemma 7.1.3. *Suppose that $f = (f_1, f_2)\colon \mathbb{R} \times [0, +\infty) \to \mathbb{R} \times [0, +\infty)$ is an L-bi-Lipschitz homeomorphism. For $r \geq 0, \nu \in \mathbb{S}^1$ and $z \in \mathbb{R}$ define $F(r\nu, z) = (f_1(r, z)\nu, f_2(r, z))$. Then F is an L'-bi-Lipschitz self map of $\mathbb{R}^3$ with L' depending only on L.*

Proof. Let $r, r' > 0$, $\nu, \nu' \in \mathbb{S}^1$ and $z \in \mathbb{R}$. We have that

$$\begin{aligned}|F(r\nu, z) - F(r'\nu', z')| &\simeq |f_1(r', z')\nu' - f_1(r, z)\nu| + |f_2(r', z') - f_2(r, z)| \\ &\simeq |f_1(r', z') - f_1(r, z)| + |f_1(r, z)||\nu - \nu'| + |f_2(r', z') - f_2(r, z)| \\ &\simeq |f_1(r', z') - f_1(r, z)| + |(r, z)||\nu - \nu'| + |f_2(r', z') - f_2(r, z)| \\ &\simeq |(r, z)||\nu - \nu'| + |(r, z) - (r', z')| \\ &\simeq |(r\nu, z) - (r'\nu', z')|\end{aligned}$$

with the comparison constants depending at most on L. $\square$

Proof of Proposition 7.1.1. Let $\psi(t) = \varphi(1 - t)$, $t \in [0, 1]$. By Lemma 7.1.2, the graph $\psi_{0,1}$ satisfies the chord-arc condition. Consider the points $a_1 = (0, 0)$, $a_2 = (1, 0)$, $a_3 = (0, 1)$ and $a_4 = (0, \varphi(1))$. The two limit conditions of $\mathcal{F}_2$ imply that the union of $\psi_{0,1}$ with the line segments $[a_1, a_2]$ and $[a_1, a_4]$ is a C-chord-arc curve for some C depending on M, δ. Using arc-length parametrization, it is easy to find a bi-Lipschitz mapping g that maps $\psi_{0,1}$ onto the line segment $[a_2, a_3]$, the points a_1, a_2, a_4 to a_1, a_2, a_3 respectively, and the half-lines $\{(0, t)\colon t \geq 0\}$, $\{(t, 0)\colon t \geq 0\}$ onto themselves. By Tukia's extension theorem [36, Theorem A], g can be extended to an L_1-bi-Lipschitz map $G = (g_1, g_2)$ of the first quadrant $[0, +\infty) \times [0, +\infty)$ onto itself. Here $L_1 > 1$ depends only on C.

Consider now the mapping $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that for each $\nu \in \mathbb{S}^1$, $r > 0$ and $z \geq 0$

$$F(r\nu, z) = (g_1(r,z)\nu, g_2(r,z)) \text{ and } F(r\nu, z) = (g_1(r,-z)\nu, g_2(r,-z)).$$

Clearly, F maps $\tilde{\Sigma}(\varphi)$ onto the boundary of the double cone $\mathcal{C}^3 = \{(x,z)\colon x \in \mathbb{B}^2, |z| < 1 - |x|\}$. By Lemma 7.1.3 we have that F is bi-Lipschitz and the claim follows from the fact that $\partial\mathcal{C}^3$ is the image of $\mathbb{S}^2$ under a bi-Lipschitz self map of $\mathbb{R}^3$. □

Proof of the first claim of Theorem 1.4.1. Suppose that $f\colon \Omega \to \mathbb{B}^2$ is L-bi-Lipschitz. It is easy to show that f can be extended to an L-bi-Lipschitz map of $\overline{\Omega}$. Moreover, by Tukia's extension theorem [36, Theorem A], f can be extended to an L'-bi-Lipschitz self map of $\mathbb{R}^2$ for some $L' > 1$ depending on L. Consider the function $G : \mathbb{R}^3 \to \mathbb{R}^3$ with $G(x,z) = (f(x), z)$, where $x \in \mathbb{R}^2$ and $z \in \mathbb{R}$. Note that, for each $(x_1,z_1),(x_2,z_2) \in \mathbb{R}^3$,

$$|G(x_1,z_1) - G(x_2,z_2)| \leq |f(x_1) - f(x_2)| + |z_1 - z_2| \leq (L+1)|(x_1,z_1) - (x_2,z_2)|$$
$$|G(x_1,z_1) - G(x_2,z_2)| \geq \frac{1}{2}|f(x_1) - f(x_2)| + \frac{1}{2}|z_1 - z_2| \geq |(x_1,z_1) - (x_2,z_2)|.$$

Hence, G is a bi-Lipschitz self map of $\mathbb{R}^3$ that maps $\tilde{\Sigma}(\varphi)$ onto $\tilde{\Sigma}(f,\varphi)$. By Proposition 7.1.1, $\tilde{\Sigma}(f,\varphi)$ is a bi-Lipschitz sphere. □

Remark 7.1.4. *Suppose that $f\colon \mathbb{B}^2 \to \Omega$ is bi-Lipschitz and $\varphi_1, \varphi_2 \in \mathcal{F}_2$. Define $\tilde{\Sigma}_1 = \Gamma$ and $F_1 = f$. By Theorem 1.4.1, there exists a bi-Lipschitz mapping $F_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ that maps the surface $\tilde{\Sigma}_2 = \tilde{\Sigma}(F_1,\varphi_1)$ onto $\mathbb{S}^2$. Let Ω' be the domain enclosed by Σ_2. Consider the 3-dimensional surface in $\mathbb{R}^4$*

$$\tilde{\Sigma}_3 = \tilde{\Sigma}(F_2,\varphi_2) = \{(x,z) \in \mathbb{R}^4\colon x \in \overline{\Omega'}, z = \pm\varphi_2(1 - |F_2(x)|)\}.$$

The proof of the first part of Theorem 1.4.1 shows that $\tilde{\Sigma}_3$ is the image of $\mathbb{S}^3$ under a bi-Lipschitz self map of $\mathbb{R}^4$.

Remark 7.1.5. *The limit assumptions of the functions $\varphi \in \mathcal{F}_2$ are necessary for the first claim of Theorem 1.4.1.*

For the necessity of the first limit assumption, take $\Gamma = \mathbb{S}^1$, $f =$ Id and $\varphi(t) = t^2$. It is not hard to see that $\tilde{\Sigma}(f,\varphi)$ is not LLC_1 and thus not quasisymmetric to $\mathbb{S}^2$. For the necessity of the second limit assumption, take $\Gamma = \mathbb{S}^1$, $f =$ Id and

$$\varphi(t) = \begin{cases} 1 - \sqrt{1-t} & t \in [0,1] \\ t & t \in [1, +\infty). \end{cases}$$

It is easy to see that $\tilde{\Sigma}(f, \varphi)$ is not LLC_2 and therefore not quasisymmetric to $\mathbb{S}^2$.

7.2 Proof of the second claim of Theorem 1.4.1

In this section we prove a statement stronger than the second part of Theorem 1.4.1. For each $r \in [0,1]$ let $B_r = B^2(O, r)$ and $S_r = \partial B_r$.

Proposition 7.2.1. *Suppose that f is a quasiconformal mapping of a Jordan domain Ω onto $\mathbb{B}^2$ such that $\tilde{\Sigma}(f, \varphi)$ is quasisymmetric to $\mathbb{S}^2$ for all $\varphi \in \mathcal{F}_2$ satisfying $\lim_{t \to 0} \varphi(t)/t = +\infty$. Then, for some $r_0 \in (0,1)$, the pre-images $f^{-1}(S_r)$ are chord-arc curves for each $r \in [r_0, 1]$.*

Suppose that Ω, f satisfy the assumptions of Proposition 7.2.1. In Lemma 7.2.2 we show that $\partial\Omega$ is a quasidisk. Then, in Proposition 7.2.3 we show that for some $r_0 \in (0,1)$, the pre-images $f^{-1}(S_r)$ are chord-arc curves for each $r \in [r_0, 1)$. The proof in the case $r = 1$ is identical to that of Proposition 7.2.3; see Corollary 7.2.6.

Lemma 7.2.2. *Suppose that f is quasiconformal mapping of a Jordan domain Ω onto $\mathbb{B}^2$. If $\partial\Omega$ is not a quasicircle then, there exists a function $\varphi \in \mathcal{F}_1$ satisfying $\lim_{t \to 0} \varphi(t)/t = +\infty$, such that the surface $\tilde{\Sigma}(f, \varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

Proof. The proof is quite similar to that of Lemma 4.1.5.

Suppose that x_n, x'_n, y_n, y'_n and ϵ_n, σ_n are as in the proof of Lemma 4.1.5. We construct φ so that $\tilde{\Sigma}(f, \varphi)$ fails the LLC_1 property.

Since Ω is a Jordan domain, f can be extended to a homeomorphism of $\overline{\Omega}$ onto $\overline{\mathbb{B}^2}$. For each $n \in \mathbb{N}$, find $r_n \in (0, \epsilon_n)$ such that $\mathrm{dist}(f^{-1}(S_{r_n}), \Gamma) < \epsilon_n$. The sequence $\{r_n\}_{n \in \mathbb{N}}$ can be chosen to be decreasing. Define a function

$$\varphi \colon \{r_n\} \to \mathbb{R}_+ \text{ with } \varphi(r_n) = n|x_n - x'_n|.$$

The arguments in the proof of Lemma 4.1.5 show that $\tilde{\Sigma}(f, \varphi)$ is not LLC_1. □

Proposition 7.2.3. *Let f be a quasiconformal mapping of a quasidisk Ω onto $\mathbb{B}^2$. Suppose that for a sequence $\{r_n\} \subset (0,1)$ with $r_n \uparrow 1$, the pre-images $f^{-1}(S_{r_n})$ are not chord-arc curves. Then, there exists a function $\varphi \in \mathcal{F}_2$, satisfying $\lim_{t\to 0} \varphi(t)/t = +\infty$, such that $\tilde{\Sigma}(f,\varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

It is well known that any quasiconformal map of a quasidisk Ω onto $\mathbb{B}^2$ can be extended to a quasiconformal self map of $\mathbb{R}^2$. Therefore, $f\colon \overline{\Omega} \to \overline{\mathbb{B}^2}$ is η-quasisymmetric for some homeomorphism η depending on the quasiconformal constant K of f.

For the rest we write $\Gamma_r = f^{-1}(S_r)$ and $O = (0,0)$.

Fix an increasing sequence $\{r_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\lim r_n = 1$ and Γ_{r_n} are not chord-arc curves. For each $n \in \mathbb{N}$, we define inductively a positive number r_n' and an arc $J_n \subset \Gamma_{r_n}$. Set $r_0' = r_0$ and $J_0 = \Gamma_{r_0}$. Suppose that r_{n-1}', J_{n-1} have been defined. Since Γ_{r_n} is not a chord-arc curve, there exists $J_n \subset \Gamma_{r_n}$ such that $\ell(J_n) > n \operatorname{diam} J_n$. Moreover, J_n can be chosen so that

$$\operatorname{diam} J_n \leq \frac{1}{2}(\sqrt{1-r_{n-1}} - \sqrt{1-r_n}). \tag{7.2.1}$$

Indeed, since Γ_{r_n} is not a chord-arc curve, it contains arbitrarily small subarcs which are not chord-arc curves.

Find consecutive points $P_1^n, \dots, P_{N_n}^n \in J_n$ such that $\Gamma_{r_n}(P_1^n, P_{N_n}^n) = J_n$ and

$$\sum_{i=1}^{N_n} |P_{i+1}^n - P_i^n| > n \operatorname{diam} J_n. \tag{7.2.2}$$

Define also $p_i^n = f(P_i^n)$. By adding more points in the collection $\{p_i^n\}_{i=1}^{N_n}$, we may assume that, for each $i,j \in \{1,\dots,N_n\}$,

$$\frac{1}{2}|p_{i+1}^n - p_i^n| \leq |p_{j+1}^n - p_j^n| \leq 2|p_{i+1}^n - p_i^n|.$$

Since f is η-quasisymmetric, for each $z \in \Gamma_{r_n}(P_i^n, P_{i+1}^n)$,

$$\frac{|P_i^n - P_{i+1}^n|}{|z - P_{i+1}^n|} \geq \eta^{-1}\left(\frac{|p_i^n - p_{i+1}^n|}{|f(z) - p_{i+1}^n|}\right) \geq \eta^{-1}(1).$$

Therefore, for each $i \in \{1,\dots,N_n - 1\}$, $\operatorname{diam} \Gamma_{r_n}(P_i^n, P_{i+1}^n) \leq \frac{2}{\eta^{-1}(1)}|P_i^n - P_{i+1}^n|$.

By uniform continuity of f, there exists a number $r_n' \in (r_{n-1}, r_n)$ such that

$$\operatorname{dist}(\Gamma_{r_n'}, \Gamma_{r_n}) \leq \min_{i=1,\dots,N_n} \{|P_{i+1}^n - P_i^n|\}.$$

We may also assume that r_n' is close enough to r_n so that

$$r_n - r_n' \leq \frac{1}{\eta(3)} \min_{i=1,\dots,N_n} \{|P_{i+1}^n - P_i^n|\} \tag{7.2.3}$$

and

$$\sqrt{1-r_n'} - \sqrt{1-r_n} \leq \operatorname{diam} J_n. \tag{7.2.4}$$

Define $\varphi\colon \{1-r_n\}_{n\in\mathbb{N}} \cup \{1-r_n'\}_{n\in\mathbb{N}} \to (0,+\infty)$ with

$$\varphi(1-r_n) = \sqrt{1-r_n} \quad \text{ and } \quad \varphi(1-r_n') = \sqrt{1-r_n} + \operatorname{diam} J_n.$$

It follows from (7.2.1) and (7.2.4) that φ is increasing. Moreover, by (7.2.4), $\varphi(1-r_n') \geq \sqrt{1-r_n'}$. Extend φ in $[0,1]$ so that $\varphi(t) \geq \sqrt{t}$ for each $t \in [0,1]$. The extension, which we still denote by φ, is in $\mathcal{F}_2$ and satisfies $\lim_{t\to 0} \varphi(t)/t = +\infty$.

The following lemma concludes the proof of Proposition 7.2.3.

Lemma 7.2.4. *Let f and φ be as above. Then, $\tilde{\Sigma}(f,\varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

Proof. Suppose, on the contrary, that there exists a quasisymmetric mapping that maps $\tilde{\Sigma}(f,\varphi)$ onto $\mathbb{S}^2$. Post-composing this mapping with an inversion, we may assume that there exists a θ-quasisymmetric map $F\colon \tilde{\Sigma}(f,\varphi) \cap \mathbb{R}^3_+ \to \mathbb{B}^2$.

Fix $n \in \mathbb{N}$. For simplicity we write $J_n = J$, $N_n = N$, $r_n = r$, $r_n' = r'$, $P_i^n = P_i$, $p_i^n = p_i$ and the dependence of quantities, points and sets on n will not be recorded. However, the constants in the comparisons $\simeq$ and $\lesssim$ are depending only on η, θ and not on n.

For each $i = 1,\dots,N$ let $q_i \in S_{r'} \cap [O,p_i]$ and $Q_i = f^{-1}(q_i)$. Then, the points $q_1,\dots,q_N$ are consecutive with q_1, q_N being the first and last points respectively. Moreover, by (7.2.3),

$$\frac{|P_i - P_{i+1}|}{|Q_i - P_i|} \geq \eta^{-1}\left(\frac{|p_i - p_{i+1}|}{|q_i - p_i|}\right) \geq 3.$$

Therefore, for any $i = 1,\dots,N-1$,

$$|Q_i - Q_{i+1}| \leq |P_i - P_{i+1}| + |P_i - Q_i| + |P_{i+1} - Q_{i+1}| \leq 2|P_i - P_{i+1}|$$

and

$$|Q_i - Q_{i+1}| \geq |P_i - P_{i+1}| - |P_i - Q_i| - |P_{i+1} - Q_{i+1}| \geq \frac{1}{3}|P_i - P_{i+1}|.$$

Thus, $|P_i - P_{i+1}| \simeq |Q_i - Q_{i+1}|$ which yields $\sum_{i=1}^{N-1} |Q_{i+1} - Q_i| \gtrsim n \operatorname{diam} J$.

Let Λ be the piece of $\tilde{\Sigma}(f,h)$ whose projection on $\mathbb{R}^2$ is the Jordan domain bounded by the arcs $\Gamma_r(P_1, P_N))$, $\Gamma_{r'}(Q_1, Q_N)$, $f([p_1, q_1])$, $f([p_N, q_N])$. Define also

$$\beta = \min\{|F(x) - F(y)| \colon x \in \Gamma_r(P_1, P_N)) \times \{\varphi(1-r)\},\, y \in \Gamma_{r'}(Q_1, Q_N) \times \{\varphi(1-r')\}\}.$$

As in the proof of Lemma 4.3.2, to finish the proof it suffices to show that

$$\beta^2 n \lesssim \mathcal{H}^2(F(\Lambda)). \tag{7.2.5}$$

For each $i = 1, \dots, N-1$ let Λ_i be the piece of Λ whose projection is the Jordan domain bounded by the four arcs $\Gamma_r(P_i, P_{i+1}))$, $\Gamma_{r'}(Q_i, Q_{i+1}))$, $f([p_i, q_i])$, $f([p_{i+1}, q_{i+1}])$. Denote with τ_i the arc on Λ such that $\pi(\tau_i) = f([q_i, p_i])$. Clearly, τ_i, τ_{i+1} are boundary arcs of Λ_i.

For each $i = 1, \dots, N-1$ let $\delta_i = |P_i - P_{i+1}|$, $d_i = \frac{\delta_i}{\operatorname{diam} J}$ and k_i be the integer part of $\frac{1}{d_i} - 1$. Note that $k_i |P_i - P_{i+1}| \simeq \operatorname{diam} \Gamma_r(P_1, P_N) \simeq \varphi(1-r') - \varphi(1-r)$.

Fix $i \in \{1, \dots, N-1\}$. For $j = 1, \dots, k_i + 1$, let $\rho_{ij} \in (0,1)$ be such that

$$\varphi(1 - \rho_{ij}) = \varphi(1-r) + j d_i \operatorname{diam} J.$$

The curves

$$\sigma_{ij} = \begin{cases} \Gamma_r \times \varphi(1-r) & \text{for } j = 0, \\ \Gamma_{\rho_{ij}} \times \{\varphi(1-\rho_{ij})\} & \text{for } 1 \le j \le k_i \\ \Gamma_{r'} \times \{\varphi(1-r')\} & \text{for } j = k_i + 1 \end{cases}$$

subdivide Λ_i into pieces Λ_{ij} with $1 \le j \le k_i + 1$. More precisely, Λ_{ij} is the piece of Λ whose projection on $\mathbb{R}^2$ is the Jordan domain bounded by $f([p_i, q_i])$, $f([p_{i+1}, q_{i+1}])$, $\Gamma_{\rho_{ij}}$, $\Gamma_{\rho_{i(j+1)}}$. The choice of ρ_{ij} implies that the height of each piece Λ_{ij} is in the range $[\delta_i, 2\delta_i]$.

Fix a piece Λ_{ij} and define

$$A_{ij} = \tau_i \cap \sigma_{ij}\ ,\ B_{ij} = \tau_{i+1} \cap \sigma_{ij}\ ,\ C_{ij} = \tau_{i+1} \cap \sigma_{i(j+1)}\ ,\ D_{ij} = \tau_i \cap \sigma_{i(j+1)}.$$

The points $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ can be thought as the four vertices of the piece Λ_{ij}.

Since A_{ij}, B_{ij} are on the same horizontal plane, $|A_{ij} - B_{ij}| = |\pi(A_{ij}) - \pi(B_{ij})|$. Since $\pi(A_{ij})$, $\pi(B_{ij})$ are in $\Gamma_{\rho_{ij}}$ and since f is η-quasisymmetric,

$$\frac{|P_i - P_{i+1}|}{|\pi(A_{ij}) - P_i|} \geq \eta^{-1}\left(\frac{|p_i - p_{i+1}|}{|f(\pi(A_{ij})) - p_i|}\right) \geq \eta^{-1}\left(\frac{|p_i - p_{i+1}|}{|q_i - p_i|}\right) \geq 3.$$

Therefore, $\frac{1}{3}\delta_i \leq |A_{ij} - B_{ij}| \leq 2\delta_i$. Similarly, $|C_{ij} - D_{ij}| \simeq \delta_i = |P_i - P_{i+1}|$.

Since f is quasisymmetric, it is easy to show that the curves $\{\Gamma_t\}_t$ satisfy (2.2.1) with $C = \frac{2}{\eta^{-1}(1)}$. Thus, there exist points $z'' \in \sigma_{ij}(A_{ij}, B_{ij})$, $z' \in \sigma_{i(j+1)}(C_{ij}, D_{ij})$ such that

$$\mathrm{dist}(z, \tau_{i+1}(B_{ij}, C_{ij})) \simeq \mathrm{dist}(z', \tau_{i+1}(B_{ij}, C_{ij})) \simeq \mathrm{dist}(z'', \tau_{i+1}(B_{ij}, C_{ij})) \simeq \delta_i$$
$$\mathrm{dist}(z, \tau_i(A_{ij}, D_{ij})) \simeq \mathrm{dist}(z', \tau_i(A_{ij}, D_{ij})) \simeq \mathrm{dist}(z'', \tau_i(A_{ij}, D_{ij})) \simeq \delta_i$$

Moreover, if $\rho'_{ij} \in (\rho_{i(j+1)}, \rho_{ij})$ satisfies $\varphi(1 - \rho'_{ij}) = \varphi(1 - \rho_{ij}) + \frac{1}{2}\delta_i$, then there exists a point $z \in \Lambda_{ij} \cap (\Gamma_{\rho'_{ij}} \times \{\varphi(\rho'_{ij})\})$ such that

$$\mathrm{dist}(y, \tau_i(A_{ij}, D_{ij})) \simeq \mathrm{dist}(y, \tau_{i+1}(B_{ij}, C_{ij})) \simeq \delta_i.$$

We think of the points z', z'' as the "centers" of $\sigma_{i(j+1)}(C_{ij}, D_{ij}), \sigma_{ij}(A_{ij}, B_{ij})$ respectively, and z as the "center" of Λ_{ij}.

Set $\beta_{ij} = |F(z'') - F(z')|$ and let $u \in \partial\Lambda_{ij}$ be the point at which

$$|F(u) - F(z)| = \mathrm{dist}(F(z), \partial F(\Lambda_{ij})) = R.$$

Then, $|u - z| \gtrsim |P_i - P_{i+1}|$ and by the quasisymmetry of F, since $|z - z'| \lesssim |P_i - P_{i+1}|$, it follows that $|F(z) - F(z')| \lesssim R$. The same inequality is true with z' replaced by z''. Hence, $\beta_{ij} \lesssim R$ which implies $\beta_{ij}^2 \lesssim \mathcal{H}^2(F(\Lambda_{ij}))$. By Schwarz inequality, this yields

$$\beta^2 \leq \left(\sum_{j=1}^{k_i+1} \beta_{ij}\right)^2 \lesssim (k_i + 1)\mathcal{H}^2(F(\Lambda_i)).$$

Since $(k_i + 1)\delta_i = (k_i + 1)|P_{i+1} - P_i| \simeq \operatorname{diam}\Gamma_r(P_1, P_N)$ we have

$$\beta^2 |P_i - P_{i+1}| \lesssim \operatorname{diam}\Gamma_r(P_1, P_N)\mathcal{H}^2(F(\Lambda_i)).$$

Summing over i, by (7.2.2), we obtain (7.2.5). $\square$

Remark 7.2.5. *The function φ constructed for the proof of Proposition 7.2.3 is in $\mathcal{F}_1$.*

Corollary 7.2.6. *Let f be a quasiconformal mapping of $\mathbb{B}^2$ onto a quasidisk Ω. Suppose that $\partial\Omega$ is not a chord-arc curve. Then, there exists $\varphi \in \mathcal{F}_2$ with $\lim_{t\to 0}\varphi(t)/t = +\infty$ such that $\tilde{\Sigma}(f,\varphi)$ is not quasisymmetric to $\mathbb{S}^2$.*

Proof. For the proof of the corollary, simply set $r_n = 1$ in the proof of Proposition 7.2.3 and follow the same arguments. The only thing that changes is the definition of the arcs J_n.

Define inductively numbers $r'_n > 0$ and arcs $J_n \subset \Gamma$ as follows. Set $r'_0 = 0$ and $J_0 = \Gamma$. Given r'_{n-1} and $J_{n-1} \subset \Gamma$, let $J_n \subset \Gamma$ be such that $\operatorname{diam} J_n < \frac{1}{2}\operatorname{diam} J_{n-1}$ and $\ell(J_n) > n \operatorname{diam} J_n$. Let $P_1^n, \dots, P_{N_n}^n \in J_n$ satisfying (7.2.2) and $\Gamma(P_1^n, P_{N_n}^n) = J_n$. For the rest of the proof, repeat the arguments in the proof of Proposition 7.2.3 setting $r_n = 1$. $\square$

Chapter 8

References

[1] L. Ahlfors. Zur Theorie der Überlagerungsflächen. *Acta Math.*, 65(1):157–194, 1935.

[2] L. V. Ahlfors. Quasiconformal reflections. *Acta Math.*, 109:291–301, 1963.

[3] P. Assouad. Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbf{R}^n$. *C. R. Acad. Sci. Paris Sér. A-B*, 288(15):A731–A734, 1979.

[4] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[5] C. J. Bishop. A quasisymmetric surface with no rectifiable curves. *Proc. Amer. Math. Soc.*, 127(7):2035–2040, 1999.

[6] M. Bonk and B. Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002.

[7] M. Brown. Sets of constant distance from a planar set. *Michigan Math. J.*, 19:321–323, 1972.

[8] G. David and T. Toro. Reifenberg flat metric spaces, snowballs, and embeddings. *Math. Ann.*, 315(4):641–710, 1999.

[9] S. Ferry. When ϵ-boundaries are manifolds. *Fund. Math.*, 90(3):199–210, 1975/76.

[10] J. H. G. Fu. Tubular neighborhoods in Euclidean spaces. *Duke Math. J.*, 52(4):1025–1046, 1985.

[11] J. B. Garnett and D. E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008. Reprint of the 2005 original.

[12] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.*, 103:353–393, 1962.

[13] F. W. Gehring. Extension of quasiconformal mappings in three space. *J. Analyse Math.*, 14:171–182, 1965.

[14] F. W. Gehring. Extension theorems for quasiconformal mappings in n-space. *J. Analyse Math.*, 19:149–169, 1967.

[15] F. W. Gehring. Quasiconformal mappings of slit domains in three space. *J. Math. Mech.*, 18:689–703, 1969.

[16] F. W. Gehring. Spirals and the universal Teichmüller space. *Acta Math.*, 141(1-2):99–113, 1978.

[17] F. W. Gehring. Characterizations of quasidisks. In *Quasiconformal geometry and dynamics (Lublin, 1996)*, volume 48 of *Banach Center Publ.*, pages 11–41. Polish Acad. Sci., Warsaw, 1999.

[18] F. W. Gehring and J. Väisälä. The coefficients of quasiconformality of domains in space. *Acta Math.*, 114:1–70, 1965.

[19] H. Grötzsch. Über möglichst konforme Abbildungen von schlichten Bereichen. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 84:114–120, 1932.

[20] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[21] J. Heinonen and J.-M. Wu. Quasisymmetric nonparametrization and spaces associated with the Whitehead continuum. *Geom. Topol.*, 14(2):773–798, 2010.

[22] D. Herron and D. Meyer. Quasicircles and bounded turning circles modulo bi-Lipschitz maps. *Rev. Mat. Iberoam.*, 28(3):603–630, 2012.

[23] D. S. Jerison and C. E. Kenig. Hardy spaces, A_∞, and singular integrals on chord-arc domains. *Math. Scand.*, 50(2):221–247, 1982.

[24] P. W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.

[25] N. J. Lennes. Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations. *Amer. J. Math.*, 33(1-4):287–326, 1911.

[26] J. L. Lewis and A. Vogel. On pseudospheres that are quasispheres. *Rev. Mat. Iberoamericana*, 17(2):221–255, 2001.

[27] J. Luukkainen. Assouad dimension: antifractal metrization, porous sets, and homogeneous measures. *J. Korean Math. Soc.*, 35(1):23–76, 1998.

[28] O. Martio and J. Väisälä. Elliptic equations and maps of bounded length distortion. *Math. Ann.*, 282(3):423–443, 1988.

[29] P. Mattila and M. Vuorinen. Linear approximation property, Minkowski dimension, and quasiconformal spheres. *J. London Math. Soc. (2)*, 42(2):249–266, 1990.

[30] D. Meyer. Quasisymmetric embedding of self similar surfaces and origami with rational maps. *Ann. Acad. Sci. Fenn. Math.*, 27(2):461–484, 2002.

[31] D. Meyer. Snowballs are quasiballs. *Trans. Amer. Math. Soc.*, 362(3):1247–1300, 2010.

[32] P. Pankka and J.-M. Wu. Geometry and quasisymmetric parametrization of semmes spaces. *Rev. Mat. Iberoamericana*, to appear.

[33] S. Rohde. Quasicircles modulo bilipschitz maps. *Rev. Mat. Iberoamericana*, 17(3):643–659, 2001.

[34] S. Semmes. Good metric spaces without good parameterizations. *Rev. Mat. Iberoamericana*, 12(1):187–275, 1996.

[35] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[36] P. Tukia. The planar Schönflies theorem for Lipschitz maps. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):49–72, 1980.

[37] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):97–114, 1980.

[38] P. Tukia and J. Väisälä. Quasiconformal extension from dimension n to $n+1$. *Ann. of Math. (2)*, 115(2):331–348, 1982.

[39] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin, 1971.

[40] J. Väisälä. Quasisymmetric embeddings in Euclidean spaces. *Trans. Amer. Math. Soc.*, 264(1):191–204, 1981.

[41] J. Väisälä. Quasi-Möbius maps. *J. Analyse Math.*, 44:218–234, 1984/85.

[42] J. Väisälä. Bi-Lipschitz and quasisymmetric extension properties. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 11(2):239–274, 1986.

[43] J. Väisälä. Quasisymmetric maps of products of curves into the plane. *Rev. Roumaine Math. Pures Appl.*, 33(1-2):147–156, 1988.

[44] J. Väisälä. Quasiconformal maps of cylindrical domains. *Acta Math.*, 162(3-4):201–225, 1989.

[45] C. Villani. *Optimal Transport, Old and New*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009.

[46] M. F. Walker. Linearly locally connected sets and quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 11(1):77–86, 1986.