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POINCARÉ INEQUALITIES IN NONCOMMUTATIVE  $L_P$  SPACES

BY

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DISSERTATION

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# Abstract

Let  $(\mathcal{N}, \tau)$  be a noncommutative  $W^*$  probability space, where  $\mathcal{N}$  is a finite von Neumann algebra and  $\tau$  is a normal faithful tracial state. Let  $(T_t)_{t \geq 0}$  be a normal, unital, completely positive, and symmetric semigroup acting on  $(\mathcal{N}, \tau)$ , which is also pointwise weak\* continuous. Denote by  $\Gamma$  the “carré du champ” associated to  $T_t$ . Let  $\text{Fix}$  be the fixed point algebra of  $T_t$  and  $E_{\text{Fix}} : \mathcal{N} \rightarrow \text{Fix}$  the corresponding conditional expectation. We are interested in the following  $L_p$  Poincaré inequalities

$$\|f - E_{\text{Fix}}f\|_p \leq C\sqrt{p} \max\{\|\Gamma(f, f)^{1/2}\|_p, \|\Gamma(f^*, f^*)^{1/2}\|_p\},$$

or a weaker version

$$\|f - E_{\text{Fix}}f\|_p \leq C\sqrt{p} \max\{\|\Gamma(f, f)^{1/2}\|_\infty, \|\Gamma(f^*, f^*)^{1/2}\|_\infty\}$$

for  $p \geq 2$  and  $f \in \mathcal{N}$ . We study when such inequalities hold as well as their consequences. A crucial condition is the  $\Gamma_2$ -criterion of Bakry and Emery. These inequalities lead to (noncommutative) transportation cost inequalities and concentration inequalities. Our approaches to prove such Poincaré inequalities are based on martingale inequalities and Pisier’s method on the boundedness of Riesz transforms.

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# Chapter 1

## Introduction

### 1.1 Background

In probability theory, functional inequalities for Markov diffusion semigroups have been studied extensively in different settings; see for example the recent monograph [BGL14] for an overview of this theory. Among various functional inequalities, Gross' log-Sobolev inequality (LSI) is of fundamental importance, because it implies many other inequalities, including spectral gap inequality, transportation cost inequalities, concentration inequalities, etc.. A sufficient condition for LSI is the  $\Gamma_2$ -criterion due to Bakry and Emery [BÉ85]. This is no longer the case in the non-diffusion setting. To give a simple example, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Then the operator  $Id - \mathbb{E}$  acting on  $L_2(\Omega, \mathcal{F}, \mathbb{P})$  generates a symmetric Markov semigroup which satisfies the  $\Gamma_2$ -criterion. However, LSI fails for  $Id - \mathbb{E}$ . This example will be revisited in Chapter 3. In the following, we will propose the  $L_p$  Poincaré inequalities with constant  $\sqrt{p}$  as a “weaker substitute” for LSI in more general setting, which still lead to spectral gap inequality, concentration inequalities, and some transportation cost inequalities.

Inspired by quantum physics, noncommutative (or quantum) probability provides a general framework to study classical (or commutative) probability, free probability, and random matrices; see for example [Mey93, Par92, VDN92]. On the other hand, the theory of noncommutative  $L_p$  spaces has become an important branch of functional analysis and operator algebras since the early development in 1950s; see for example [PX03, Ter81] for more details. In this framework, various noncommutative probability inequalities have been established. Among others, we mention the noncommutative Khintchine inequality established in 1980s due to Lust-Piquard [LP86], Lust-Piquard and Pisier [LPP91]. In the last two decades, a variety of noncommutative martingale inequalities were proved after the seminal work of Pisier and Xu [PX97]. Just as martingale theory plays an important role in analysis, noncommutative martingale inequalities have been applied to study noncommutative harmonic analysis in recent years; see for example [JM10, JMP10]. The matrix-valued special case of these inequalities is particularly interesting, because they can be used in some applied areas like compressed sensing; see for example [Tro, MJC<sup>+</sup>12]. The order of best constants in some

noncommutative martingale inequalities were determined in [JX05]. It is well known in probability theory that precise constants in inequalities could be the starting point of many applications.

Motivated by both probability theory and operator algebras as explained above, we apply noncommutative martingale inequalities to establish  $L_p$  Poincaré inequalities with satisfactory constants in the major part of this thesis. In some other cases, we also follow Pisier's idea on the boundedness of Riesz transforms [Pis88] to prove such inequalities.

## 1.2 Main results

Let us set up the framework. Unless specified otherwise, we consider a noncommutative  $W^*$  probability space  $(\mathcal{N}, \tau)$  where  $\mathcal{N}$  is a finite von Neumann algebra and  $\tau$  a normal faithful tracial state. The noncommutative  $L_p$  space  $L_p(\mathcal{N}, \tau)$  is the completion of  $\mathcal{N}$  with respect to  $\|f\|_p = \tau[(f^*f)^{p/2}]^{1/p}$  for  $0 < p < \infty$  and  $\|f\|_\infty = \|f\|$ . Here and in the following  $\|\cdot\|$  denotes the operator norm. It is well known that  $L_p(\mathcal{N}, \tau)$  is a Banach space for  $1 \leq p \leq \infty$ . For example, for a classical probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , we may take  $\mathcal{N} = L_\infty(\Omega, \mathbb{P})$  and  $\tau(f) = \mathbb{E}(f) = \int f d\mathbb{P}$  for  $f \in \mathcal{N}$ . Then  $L_p(\mathcal{N}, \tau) = L_p(\Omega, \mathbb{P})$ . Let  $(T_t)_{t \geq 0}$  be a *standard* semigroup acting on  $\mathcal{N}$  with generator  $A$ , i.e.,  $T_t = e^{-tA}$ . Here the standard semigroup is a noncommutative analogue of a symmetric Markov semigroup in classical probability theory. The precise definition is given in Section 2.4. The gradient form associated to  $A$  (Meyer's "carré du champs") is defined as

$$\Gamma^A(f_1, f_2) = \frac{1}{2}[A(f_1^*)f_2 + f_1^*A(f_2) - A(f_1^*f_2)]$$

for  $f_1, f_2$  in the domain of  $A$ . Let  $\text{Fix} = \{x \in \mathcal{N} : T_t x = x, \text{ for all } t > 0\}$  be the fixed point algebra of  $T_t$ . It was shown in [JX07] that  $\text{Fix}$  is a von Neumann subalgebra of  $\mathcal{N}$ , thus there exists a unique conditional expectation  $E_{\text{Fix}} : \mathcal{N} \rightarrow \text{Fix}$ . We are interested in the following Poincaré type inequalities: for  $2 \leq p < \infty$

$$\|f - E_{\text{Fix}}f\|_p \leq C\sqrt{p} \max\{\|\Gamma^A(f, f)^{1/2}\|_p, \|\Gamma^A(f^*, f^*)^{1/2}\|_p\}, \quad (1.1)$$

for  $f \in \mathcal{N}$ . We also consider the following weaker versions,

$$\|f - E_{\text{Fix}}f\|_p \leq C\sqrt{p} \max\{\|\Gamma^A(f, f)^{1/2}\|_\infty, \|\Gamma^A(f^*, f^*)^{1/2}\|_\infty\}, \quad (1.2)$$

and

$$\|f - E_{\text{Fix}}f\|_p \leq Cp \max\{\|\Gamma^A(f, f)^{1/2}\|_p, \|\Gamma^A(f^*, f^*)^{1/2}\|_p\}. \quad (1.3)$$



It is known in classical probability theory that these inequalities are closely related to concentration of measure phenomenon, which has a lot of applications in probability, analysis and geometry. In what follows, we will use  $C, C', C_1, c, c'$ , and so forth to denote absolute constants which may vary from line to line. We may simply write  $\Gamma$  for  $\Gamma^A$  to shorten the notation if the generator is clear from context.

It is standard in literature that the  $L_2$  Poincaré inequality (i.e.,  $p = 2$  in (1.1) or (1.3)) is also called the spectral gap inequality. To avoid describing the above Poincaré inequalities every time we mention them, we give the following definition.

**Definition 1.1.** Let  $T_t$  be a standard semigroup acting on  $(\mathcal{N}, \tau)$ .  $T_t$  is said to be subgaussian (resp. weak subgaussian, subexponential) if (1.1) (resp. (1.2), (1.3)) holds for all  $2 \leq p < \infty$ . We also call (1.1) (resp. (1.2), (1.3)) subgaussian (resp. weak subgaussian, subexponential) Poincaré inequality.

Recall that

$$\Gamma_2^A(f_1, f_2) = \frac{1}{2}[\Gamma^A(Af_1, f_2) + \Gamma^A(f_1, Af_2) - A\Gamma^A(f_1, f_2)]$$

whenever  $f_1$  and  $f_2$  are such that the right-hand side is well-defined. Bakry–Emery’s  $\Gamma_2$ -criterion [BÉ85] is the condition that there exists  $\alpha > 0$  such that  $\Gamma_2^A(f, f) \geq \alpha \Gamma^A(f, f)$  for all  $f \in \mathcal{N}$  for which both  $\Gamma_2^A(f, f)$  and  $\Gamma^A(f, f)$  are well-defined.

**Definition 1.2.** Let  $\rho$  be a positive  $\tau$ -measurable operators affiliated with  $(\mathcal{N}, \tau)$  (see Section 2.1 for definitions). The entropy of  $\rho \in L_1(\mathcal{N}, \tau)$  is defined as

$$\text{Ent}(\rho) = \tau(\rho \ln(\rho/\tau(\rho))).$$

Note that  $\text{Ent}(\rho) \geq 0$  by functional calculus and Jensen’s inequality (applied to the convex function  $f(x) = x \ln x$ ). This is different from the usual entropy used in physics and information theory where the trace (or measure) is not normalized. For example, the von Neumann entropy is defined to be  $S = -\text{Tr}(\rho \ln \rho)$  where  $\rho$  is the density matrix and  $\text{Tr}$  is the sum of diagonal entries. In the classical setting where  $(\mathcal{N}, \tau) = L_\infty(X, \mu)$  for some probability space  $(X, \mu)$ ,

$$\text{Ent}(g) = \int_X g \ln g d\mu - \int_X g d\mu \ln \int_X g d\mu$$

for any positive function  $g$ . We say that  $T_t$  satisfies the log-Sobolev inequality (LSI) if

$$\text{Ent}(f^2) \leq C \|\Gamma^A(f, f)\|_1$$

for all self-adjoint  $f \in \mathcal{N}$ . Here we only consider self-adjoint elements because our major concern of LSI is in the commutative setting.

### 1.2.1 Classical results

Let  $(X, d)$  be a metric space. Let  $\mu$  and  $\nu$  be probability measures on  $(X, d)$  with finite  $p$ -th moment. Recall that the  $p$ -Wasserstein distance between  $\mu$  and  $\nu$  is defined as

$$W_p(\mu, \nu) = \inf \left( \iint d(x, y)^p d\pi(x, y) \right)^{1/p}$$

where the infimum is taken over all probability measure  $\pi$  on the product space  $X \times X$  which is a coupling of  $\mu$  and  $\nu$ . Let  $\|f\|_{\text{Lip}}$  denote the Lipschitz constant of  $f$ . Suppose

$$\int e^{tf} d\mu \leq e^{ct^2/2} \tag{1.4}$$

for all  $t > 0$  and all  $f$  with  $\int f d\mu = 0$ ,  $\|f\|_{\text{Lip}} \leq 1$ . Then Bobkov and Götze showed in [BG99] that

$$W_1(\mu, \nu) \leq \sqrt{2cD(\nu||\mu)}, \tag{1.5}$$

for all  $\nu$  absolutely continuous with respect to  $\mu$ . Here

$$D(\nu||\mu) = \int \ln \frac{d\nu}{d\mu} d\nu = \int \frac{d\nu}{d\mu} \ln \frac{d\nu}{d\mu} d\mu = \text{Ent}\left(\frac{d\nu}{d\mu}\right)$$

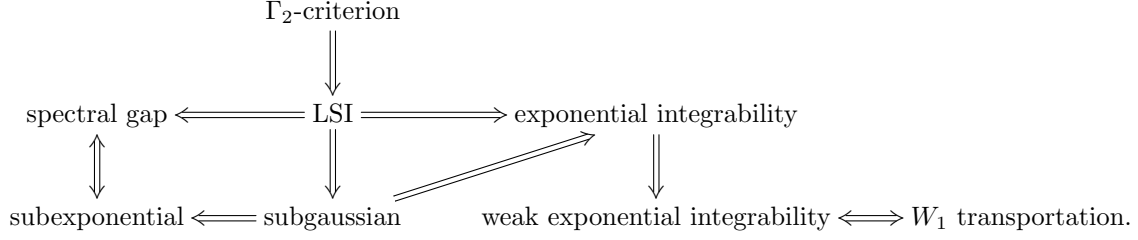
is the relative entropy. In fact, Bobkov and Götze showed that (1.4) and (1.5) are equivalent. (1.5) is called  $W_1$  transportation cost inequality, which can be used to derive a number of concentration and isoperimetric inequalities as explained in [BG99]. Moreover, they showed that a variant of LSI implies a stronger result than (1.4)

$$\int e^f d\mu \leq \int e^{c\Gamma(f, f)} d\mu. \tag{1.6}$$

In literature, results like (1.4) and (1.6) are called exponential integrability. In the following, we shall call (1.6) exponential integrability and (1.4) weak exponential integrability to distinguish them.

For a classical diffusion semigroup  $T_t$ , it is known that the functional inequalities we concern have the

following relationship:



Here “ $\Gamma_2$ -criterion $\Rightarrow$ LSI” was due to Bakry–Emery [BÉ85]; “LSI $\Rightarrow$ subgaussian semigroup” was due to Aida–Stroock [AS94]; the relationship among LSI,  $W_1$  transportation, and exponential integrability was studied by Bobkov–Götze [BG99] in the context of metric measure spaces (more general than classical diffusion). The rest of implications are either classical or folklore results; see [BGL14] for more historical accounts.

### 1.2.2 Main results

Given  $\tau$ -measurable (see Section 2.1 for definition) positive operators  $\rho$  and  $\sigma$  with  $\tau(\rho) = \tau(\sigma) = 1$ , we define the following analogues of classical Wasserstein distances

$$Q_1(\rho, \sigma) = \sup\{|\tau(x\rho) - \tau(x\sigma)| : x \text{ self-adjoint, } \|\Gamma(x, x)^{1/2}\|_\infty \leq 1\}$$

and

$$Q_\phi(\rho, \sigma) = \sup\{|\tau(x\rho) - \tau(x\sigma)| : x \text{ self-adjoint, } \|\Gamma(x, x)^{1/2}\|_\phi \leq 1\}$$

where  $\|y\|_\phi = \inf\{c > 0 : \tau[\phi(|y|/c)] \leq 1\}$  and  $\phi(t) = e^{t^2} - 1$ ; see Section 2.8 for more details. In fact, in the classical diffusion setting, if the generator  $A = -\Delta +$  first order differential operator,  $W_1(\mu, \nu) = Q_1(\frac{d\nu}{d\mu}, 1)$  by the Kantorovich–Rubinstein theorem, where the trace  $\tau$  in the definition of  $Q_1$  is given by  $\mu$ .

We introduce the following exponential integrability conditions analogous to (1.4) and (1.6): for all self-adjoint  $x \in \mathcal{N}$ ,

$$\tau(e^{x - E_{\text{Fix}} x}) \leq \tau(e^{c\Gamma^A(x, x)}), \quad (1.7)$$

and the weaker version

$$\tau(e^{x - E_{\text{Fix}} x}) \leq e^{c\|\Gamma^A(x, x)\|_\infty}. \quad (1.8)$$

Again, we call (1.7) (resp. (1.8)) the exponential integrability (resp. weak exponential integrability) condi-

tion. We also introduce the following analogues of the classical  $W_1$  transportation cost inequality:

$$Q_1(\rho, E_{\text{Fix}}\rho) \leq C\sqrt{\text{Ent}(\rho)}, \quad (1.9)$$

$$Q_\phi(\rho, E_{\text{Fix}}\rho) \leq C\max\{\sqrt{\text{Ent}(\rho)}, \text{Ent}(\rho)\} \quad (1.10)$$

for all  $\tau$ -measurable positive operators  $\rho$  with  $\tau(\rho) = 1$ . We call (1.9) (resp. (1.10)) the  $Q_1$  (resp.  $Q_\phi$ ) transportation cost inequality. To state our main results, we need some regularity condition on semigroups as in the classical setting. Following the terminology of Junge–Ricard–Shlyakhtenko [JRS14], a semigroup  $T_t$  is called noncommutative diffusion if  $\Gamma(f, f) \in L_1(\mathcal{N}, \tau)$  whenever  $f \in \text{Dom}(A^{1/2}) \cap \mathcal{N}$ . Here  $\text{Dom}(A^{1/2})$  is the domain of  $A^{1/2}$  in  $L_2(\mathcal{N}, \tau)$ . For a standard noncommutative diffusion semigroup  $T_t$ , we have the following results:

$$\text{spectral gap} \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \xrightarrow{\hspace{1cm}} \end{array} \text{subexponential} \xleftarrow{\hspace{1cm}} \Gamma_2\text{-criterion} \xrightarrow{\hspace{1cm}} \text{weak subgaussian}.$$

Here the dashed implication means that other conditions are in need. If  $\mathcal{N} = \mathcal{L}(G)$ , the group von Neumann algebra of a discrete group  $G$ , and  $T_t$  is given by a conditionally negative length function  $\psi$  such that  $T_t\lambda(g) = e^{-t\psi(g)}\lambda(g)$ , where  $\lambda$  is the left regular representation of  $G$ , then we have

$$\Gamma_2\text{-criterion} \Rightarrow \text{subgaussian}.$$

Since subgaussian Poincaré inequality is stronger than both weak subgaussian and subexponential Poincaré inequalities, we have better results in this case compared with the general case of noncommutative diffusion semigroups. Moreover, the  $L_p$  Poincaré inequalities have the following consequences. Note that here we do not need any regularity condition on the semigroup.

$$\begin{array}{ccccc} \text{subgaussian} & \xRightarrow{\hspace{1cm}} & \text{exponential integrability} & \xRightarrow{\hspace{1cm}} & Q_\phi \text{ transportation.} \\ \Downarrow & & \Downarrow & & \\ \text{weak subgaussian} & \xRightarrow{\hspace{1cm}} & \text{weak exponential integrability} & \iff & Q_1 \text{ transportation.} \end{array}$$

The above results were obtained in three papers. The relationship among  $\Gamma_2$ -criterion, weak subgaussian semigroups, weak exponential integrability, and  $Q_1$  transportation were discussed in joint work with Junge [JZ14]. The relationship among subgaussian semigroups, exponential integrability, and  $Q_\phi$  transportation were given in [Zen14b], where we also proved subgaussian Poincaré inequalities for certain group (Gaussian)

measure spaces using Pisier's method. The implication from  $\Gamma_2$ -criterion to subgaussian behavior for group von Neumann algebras was proved in joint work with Junge [JZ13a], where we also discussed the relationship between spectral gap and subexponential behavior for general standard noncommutative diffusion semigroups.

### 1.3 The structure of the thesis

The thesis is organized as follows. The preliminary material is provided in Chapter 2. In Chapter 3, we first develop some martingale inequalities and then use them to prove the  $L_p$  Poincaré inequality (1.1) (or its weaker versions (1.2) and (1.3)) under  $\Gamma_2$ -criterion for noncommutative diffusion semigroups. Pisier's method is presented in Chapter 4. As complimentary results, we discuss the relationship between the spectral gap and subexponential Poincaré inequalities in Chapter 5. The consequences of martingale inequalities and Poincaré inequalities are presented in Chapter 6, where we first derive Kolmogorov's law of the iterated logarithm for noncommutative martingales, and then prove concentration and transportation cost inequalities. We provide various examples which satisfy our Poincaré inequalities in Chapter 7. These examples include certain 1-cocycles on discrete groups and classical diffusion semigroups.

To conclude this chapter, we give some standard references for the unexplained facts used in this thesis. For probability background, we refer to [Dur10, RY99, KS91, DM82, Str11]. Our reference for basic functional analysis is [Con90]. We refer to [Tak02, KR83, KR86, BO08] for background in operator algebras and related group theory. The facts we use for noncommutative  $L_p$  spaces can be found in [PX03, Ter81]. Our development also relies on the theory of generalized singular numbers presented in [FK86].

# Chapter 2

## Preliminaries

### 2.1 Basic facts on von Neumann algebras

The facts on von Neumann algebras we mention here can be found in [Tak02, Chapter V]. Let  $H$  be a Hilbert space and  $B(H)$  the algebra of all bounded linear operators on  $H$ . A von Neumann algebra  $\mathcal{N}$  is a unital (i.e., the identity operator is in  $\mathcal{N}$ )  $*$ -subalgebra of  $B(H)$  which is closed in the strong (or equivalently weak) operator topology. Every von Neumann algebra  $\mathcal{N}$  has a unique predual  $\mathcal{N}_*$ , which is a Banach space itself and coincides with the space of linear normal functionals on  $\mathcal{N}$ . It follows that there are different topologies on  $\mathcal{N}$ .

A state  $\varphi$  on  $\mathcal{N}$  is a positive linear functional such that  $\|\varphi\| = \varphi(1) = 1$  (here the first 1 is the identity operator and the second is a scalar). A trace  $\tau$  on  $\mathcal{N}$  is a function defined on the positive cone  $\mathcal{N}_+$  with values in  $[0, +\infty]$  such that  $\tau(x + \lambda y) = \tau(x) + \lambda\tau(y)$  for  $x, y \in \mathcal{N}_+, \lambda \in \mathbb{R}_+$  and  $\tau(xx^*) = \tau(x^*x)$ .  $\tau$  is said to be faithful if  $\tau(x) > 0$  for any nonzero  $x \geq 0$ , finite if  $\tau(1) < +\infty$ , semifinite if every nonzero  $x \in \mathcal{N}_+$  majorizes some nonzero  $y \geq 0$  with  $\tau(y) < \infty$ , normal if  $\tau(\sup x_i) = \sup \tau(x_i)$  for every bounded increasing net  $(x_i) \subset \mathcal{N}_+$ .

Since every  $x \in \mathcal{N}$  can be written as a linear combination of four positive elements in  $\mathcal{N}_+$ , a finite trace on  $\mathcal{N}$  can be extended uniquely to a positive linear functional on  $\mathcal{N}$ . If  $\tau(1) = 1$ ,  $\tau$  is called a tracial state. A von Neumann algebra  $\mathcal{N}$  is finite if and only if it admits a normal faithful tracial state.  $\mathcal{N}$  is semifinite if and only if it admits a normal semifinite faithful trace.

Let  $\mathcal{M}$  be a finite von Neumann algebra with a normal faithful tracial state  $\tau$ , and  $\mathcal{N} \subset \mathcal{M}$  be a von Neumann subalgebra. Then there exists a unique normal faithful projection  $E : \mathcal{M} \rightarrow \mathcal{N}$  such that  $\tau = \tau \circ E$  and

$$E(axb) = aE(x)b, \quad a, b \in \mathcal{N}, x \in \mathcal{M}.$$

$E$  is called the conditional expectation of  $\mathcal{M}$  onto  $\mathcal{N}$ . In particular,  $E$  is  $*$ -preserving and positivity-preserving.  $E$  extends to contractions on  $L_p(\mathcal{M}, \tau)$  for all  $1 \leq p \leq \infty$ .

In this thesis, by a noncommutative  $W^*$  probability space (see [VDN92])  $(\mathcal{N}, \tau)$ , we mean that  $\mathcal{N}$  is a finite von Neumann algebra and  $\tau$  a normal faithful tracial state. This model includes the following examples, which are our most interesting examples in applications:

- Classical probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $\mathcal{N} = L_\infty(\mathbb{P})$  acts on  $L_2(\mathbb{P})$  by pointwise multiplication, and  $\tau = \mathbb{E}$ . Let  $\xi \in L_\infty(\mathbb{P})$  be a real valued random variable. Then the spectrum of  $\xi$ ,  $\sigma(\xi)$ , is its essential-range. We have  $\tau(1_{[t, \infty)}(\xi)) = \mathbb{P}(\xi \geq t)$  and  $\tau(f(\xi)) = \int f(\xi) d\mathbb{P}$  for a Borel function  $f$  defined on the spectrum of  $\xi$ .
- Random matrices.  $\mathcal{N} = L \otimes M_n(\mathbb{C})$  where, e.g., we put  $L = \bigcap_{1 \leq p < \infty} L_p(\mathbb{P})$  and  $\tau = \mathbb{E} \otimes tr$ , where  $tr$  is the normalized trace on  $M_n(\mathbb{C})$ .
- The group von Neumann algebra  $\mathcal{L}(G)$  of a countable discrete group  $G$  which is constructed as follows. Put  $\ell_2(G) = \text{span}\{\delta_g : g \in G\}$  where  $\delta_g$  is the unit vector with  $\delta_g(g) = 1$  and  $\delta_g(h) = 0$  for  $h \neq g$ . Let  $\lambda : G \rightarrow B(\ell_2(G))$  be the left regular representation given by  $\lambda(g)(\delta_h) = \delta_{gh}$ . The group von Neumann algebra  $\mathcal{L}(G)$  is the closure of linear span of  $\lambda(G) \cong \mathbb{C}G$  (the group algebra) in the weak operator topology.  $\mathcal{L}(G)$  admits a canonical normal faithful tracial state given by  $\tau(f) = \langle \delta_e, f \delta_e \rangle$  for  $f \in \mathcal{L}(G)$ , where  $e$  is the identity element of  $G$ . To be more specific, a generic element of  $\mathcal{L}(G)$  can be written as a Fourier series  $f = \sum_g \hat{f}(g) \lambda(g)$ . We have  $\tau(f) = \hat{f}(e)$ . As an example, by Fourier transform  $\mathcal{L}(\mathbb{Z}) = L_\infty(\mathbb{T})$ . In this sense, the analysis on group von Neumann algebras can be regarded as a natural generalization of Fourier analysis on  $\mathbb{Z}$ .

We also need the notion of measurable operators; see [FK86, Ter81]. In general, operators in a noncommutative  $L_p$  space for  $p < \infty$  can be unbounded operators. A closed operator  $x$  densely defined on  $H$  is said to be affiliated to  $\mathcal{N}$  if  $ux = xu$  for all unitary operators  $u \in \mathcal{N}'$ . Here  $\mathcal{N}'$  denotes the commutant of  $\mathcal{N}$  in  $B(H)$ , i.e.,  $\mathcal{N}' = \{y \in B(H) : ay = ya, \forall a \in \mathcal{N}\}$ . An operator  $x$  affiliated to  $\mathcal{N}$  is said to be  $\tau$ -measurable if for any  $\delta > 0$  there exists a projection  $e$  such that  $e(H) \subset \text{Dom}(x)$  and  $\tau(1 - e) \leq \delta$ . Here  $\text{Dom}(x)$  is the domain of  $x$  in  $H$ . An interesting fact is that all noncommutative  $L_p$  spaces are contained in the space of  $\tau$ -measurable operators, just as all  $L_p$  functions are measurable in the commutative setting.

## 2.2 Crossed products

We briefly recall the crossed product construction. Our reference is [Tak02, JMP10]. Let  $G$  be a discrete group with left regular representation  $\lambda : G \rightarrow B(\ell_2(G))$ . Given a noncommutative probability space  $(\mathcal{N}, \tau)$ , we may assume  $\mathcal{N} \subset B(H)$  for some Hilbert space  $H$ . Suppose a trace preserving action  $\alpha$  of  $G$  on  $\mathcal{N}$

is given, i.e., we have a group homomorphism  $\alpha : G \rightarrow \text{Aut}(\mathcal{N})$  (the  $*$ -automorphism groups of  $\mathcal{N}$ ) with  $\tau(x) = \tau(\alpha_g(x))$  for all  $x \in \mathcal{N}, g \in G$ . Identify  $\ell_2(G) \otimes H$  with  $\ell_2(G; H)$ . Consider the representation  $\pi$  of  $\mathcal{N}$  on  $\ell_2(G; H)$  given by

$$\pi(x) = \sum_{g \in G} \alpha_{g^{-1}}(x) \otimes e_{g,g},$$

where  $e_{g,h}$  is the matrix unit of  $B(\ell_2(G))$ . In other words,  $\pi(x)\xi(g) = \alpha_{g^{-1}}(x)\xi(g)$  for  $x \in \mathcal{N}, \xi \in \ell_2(G; H)$ . Then the crossed product of  $\mathcal{N}$  by  $G$ , denoted by  $\mathcal{N} \rtimes_\alpha G$ , is defined as the weak operator closure of  $1_{\mathcal{N}} \otimes \lambda(G)$  and  $\pi(\mathcal{N})$  in  $B(\ell_2(G; H))$ . We usually drop the subscript  $\alpha$  if there is no ambiguity. Clearly,  $\mathcal{N} \rtimes G$  is a von Neumann subalgebra of  $\mathcal{N} \overline{\otimes} B(\ell_2(G))$ . In the special case  $\mathcal{N} = \mathbb{C}$ , the complex number algebra,  $\mathbb{C} \rtimes G$  reduces to the group von Neumann algebra  $\mathcal{L}(G)$ . Therefore,  $\mathcal{L}(G)$  is a von Neumann subalgebra of  $\mathcal{N} \rtimes G$  and there exists a unique conditional expectation  $E_{\mathcal{L}(G)} : \mathcal{N} \rtimes G \rightarrow \mathcal{L}(G)$ . A generic element of  $\mathcal{N} \rtimes G$  can be written as

$$\begin{aligned} \sum_{g \in G} f_g \rtimes \lambda(g) &= \sum_{g \in G} \pi(f_g) \lambda(g) = \sum_{g,h,h'} (\alpha_{h^{-1}}(f_g) \otimes e_{h,h})(1_{\mathcal{N}} \otimes e_{gh',h'}) \\ &= \sum_{g,h} \alpha_{h^{-1}}(f_g) \otimes e_{h,gh^{-1}h}. \end{aligned}$$

There is a canonical trace on  $\mathcal{N} \rtimes G$  given by

$$\tau \rtimes \tau_G(f \rtimes \lambda(g)) = \tau \otimes \tau_G(f \otimes \lambda(g)) = \tau(f) \delta_{g=e},$$

where we denote by  $\tau_G$  the canonical trace on  $\mathcal{L}(G)$ . The arithmetic in  $\mathcal{N} \rtimes G$  is given by

$$(f \rtimes \lambda(g))^* = \alpha_{g^{-1}}(f^*) \rtimes \lambda(g^{-1})$$

and

$$(f_1 \rtimes \lambda(g_1))(f_2 \rtimes \lambda(g_2)) = (f_1 \alpha_{g_1}(f_2)) \rtimes \lambda(g_1 g_2).$$

In what follows, we may simply write  $f \lambda(g)$  instead of  $f \rtimes \lambda(g)$ . The group measure space is a special case of the crossed product, i.e.,  $\mathcal{N} = L_\infty(\Omega, \mu)$  for some (standard) probability space  $(\Omega, \mu)$ .

## 2.3 Hardy spaces associated to noncommutative martingales

We refer to [JM10, JP13, JKPX] for this section. Let  $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \dots \subset \mathcal{N}$  be a filtration of von Neumann algebras and  $E_k : \mathcal{N} \rightarrow \mathcal{N}_k$  be the conditional expectation. Let  $(x_k)$  be a martingale with martingale



differences  $dx_k = x_k - x_{k-1}$  where  $x_k \in \mathcal{N}_k$ . We need the conditional Hardy spaces associated to martingales given as follows. For  $1 \leq p < \infty$ , define

$$\|x\|_{h_p^d} = \left( \sum_k \|dx_k\|_p^p \right)^{1/p}, \quad \|x\|_{h_p^c} = \left\| \left( \sum_k E_{k-1}(dx_k^* dx_k) \right)^{1/2} \right\|_p,$$

and  $\|x\|_{h_p^r} = \|x^*\|_{h_p^c}$ .

We are going to use the continuous filtration  $(\mathcal{N}_t)_{t \geq 0} \subset \mathcal{N}$  in the following. Recall that a martingale  $x$  is said to have almost uniform (or a.u. for short) continuous path if for every  $T > 0$ , every  $\varepsilon > 0$  there exists a projection  $e$  with  $\tau(1 - e) < \varepsilon$  such that the function  $f_e : [0, T] \rightarrow \mathcal{N}$  given by  $f_e(t) = x_t e \in \mathcal{N}$  is norm continuous. Let  $\sigma = \{0 = s_0, \dots, s_n = T\}$  be a partition of the interval  $[0, T]$  and  $|\sigma|$  its cardinality. Put

$$\begin{aligned} \|x\|_{h_p^c([0, T]; \sigma)} &= \left\| \sum_{j=0}^{|\sigma|-1} E_{s_j} |E_{s_{j+1}} x - E_{s_j} x|^2 \right\|_{p/2}^{1/2}, \quad 2 \leq p \leq \infty, \\ \|x\|_{h_p^d([0, T]; \sigma)} &= \left( \sum_{j=0}^{|\sigma|-1} \|E_{s_{j+1}} x - E_{s_j} x\|_p^p \right)^{1/p}, \quad 2 \leq p < \infty, \end{aligned}$$

and  $\|x\|_{h_p^r([0, T]; \sigma)} = \|x^*\|_{h_p^c([0, T]; \sigma)}$ . Let  $\mathcal{U}$  be an ultrafilter refining the natural order given by inclusion on the set of all partitions of  $[0, T]$ . Let  $x \in L_p(\mathcal{N})$ . For  $2 \leq p < \infty$ , we define

$$\langle x, x \rangle_T = \lim_{\sigma, \mathcal{U}} \sum_{i=0}^{|\sigma|-1} E_{s_i} |E_{s_{i+1}} x - E_{s_i} x|^2.$$

Here the limit is taken in the weak\* topology and it is shown in [JKPX] that the convergence is also true in  $L_p$  norm  $\|\cdot\|_{p/2}$  for all  $2 < p < \infty$ . We define the continuous version of  $h_p$  norms for  $2 \leq p < \infty$ ,

$$\|x\|_{h_p^c([0, T])} = \lim_{\sigma, \mathcal{U}} \|x\|_{h_p^c([0, T]; \sigma)},$$

$$\|x\|_{h_p^d([0, T])} = \lim_{\sigma, \mathcal{U}} \|x\|_{h_p^d([0, T]; \sigma)}.$$

and  $\|x\|_{h_p^r([0, T])} = \|x^*\|_{h_p^c([0, T])}$  for  $2 \leq p < \infty$ . Then for all  $2 < p < \infty$

$$\|x\|_{h_p^c([0, T])} = \|\langle x, x \rangle_T\|_{p/2}^{1/2}. \quad (2.1)$$

A martingale  $x$  is said to be of vanishing variation if  $\|x\|_{h_p^d([0, T])} = 0$  for all  $T > 0$  and all  $2 < p < \infty$ . We

also write

$$\mathrm{var}_p(x) = \|x\|_{h_p^d([0,T])},$$

and let  $V_p(\mathcal{N})$  denote the  $L_2(\mathcal{N})$  closure of  $\{x \in L_p(\mathcal{N}) : \mathrm{var}_p(x) = 0\}$ .

The following results are proved in [JKPX]. For any  $y \in L_p(\mathcal{N})$ , we write  $d_j y = E_{s_j} y - E_{s_{j-1}} y$ . Put  $\|x\|_{L_p(\mathrm{var})} = \sup_{\sigma} \|(d_j x)\|_{L_p(\ell_1)}$ , where the supremum is taken over all finite partitions of  $[0, T]$ , and the norm  $\|\cdot\|_{L_p(\ell_1)}$  was defined in [Jun02], which we will not use after the next result.

**Theorem 2.1.** *Let  $2 < p < \infty$  and  $x \in L_p(\mathcal{N}_T)$ . Then for all  $\delta > 0$ , there exists a decomposition  $x = y^\delta + z^\delta$  satisfying the following*

1.  $\mathrm{var}_p(y^\delta) < \delta$ ,  $z^\delta \in L_p(\mathrm{var})$ .
2. Let  $P(x) = w^*\text{-}\lim_{\delta} y^\delta$ . Here  $w^*\text{-}\lim$  denotes the weak\* limit. Then  $P : L_p(\mathcal{N}) \rightarrow V_p(\mathcal{N})$  is an orthogonal projection.
3.  $P(x) = x$  for all  $x$  with vanishing variation.

One may take  $y^\delta = w^*\text{-}\lim_{\sigma} \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| \leq \delta]})$  where  $1_B$  is the spectral projection of  $d_j x$  restricted to the Borel set  $B$ .

**Lemma 2.2.** *If  $x$  has a.u. continuous path, then it is of vanishing variation.*

These results will be used to prove noncommutative Burkholder–Davis–Gundy type inequalities in Chapter 3. We remark that matrix-valued martingales driven by Brownian motions automatically have almost uniform continuous paths. We will need this fact for the subgaussian Poincaré inequalities of 1-cocycles on discrete groups.

## 2.4 Noncommutative diffusion semigroups

Let  $(T_t)_{t \geq 0}$  be a semigroup of operators acting on a noncommutative  $W^*$  probability space  $(\mathcal{N}, \tau)$ . Following [JM10, JM12] we say  $(T_t)$  is a *standard* semigroup if it satisfies the following assumptions:

1. Every  $T_t$  is a normal completely positive map on  $\mathcal{N}$  such that  $T_t(1) = 1$ ;
2. Every  $T_t$  is self-adjoint, i.e.  $\tau(T_t(x)y) = \tau(xT_t(y))$  for all  $x, y \in \mathcal{N}$ .
3. The family  $(T_t)$  is pointwise weak\* continuous. Equivalently,  $\lim_{t \rightarrow t_0} T_t x = T_{t_0} x$  with respect to the strong operator topology in  $\mathcal{N}$  for any  $x \in \mathcal{N}$ ; see [MS10].

It is well known that Assumption (3) is further equivalent to that  $(T_t)$  is a strongly continuous semigroup on  $L_2(\mathcal{N}, \tau)$ , where  $T_t$  extends to  $L_2(\mathcal{N}, \tau)$  by  $T_t \Lambda(x) = \Lambda(T_t x)$  for  $x \in \mathcal{N}$  and  $\Lambda : \mathcal{N} \rightarrow L_2(\mathcal{N}, \tau)$  is the natural embedding. By [DL92],  $(T_t)$  extends to a strongly continuous contraction semigroup on  $L_p(\mathcal{N})$  for every  $1 \leq p < \infty$  with generator  $A$ , i.e.  $T_t = e^{-tA}$ . Write for  $1 \leq p < \infty$

$$\text{Dom}_p(A) = \{f \in L_p(\mathcal{N}) : \lim_{t \rightarrow 0} (f - T_t f)/t \text{ converges in } L_p(\mathcal{N})\}.$$

Then the classical semigroup theory asserts that  $\text{Dom}_p(A)$  is dense in  $L_p(\mathcal{N})$  and that if  $x \in \text{Dom}_p(A)$  then  $T_t x \in \text{Dom}_p(A)$ . We also denote  $\text{Dom}(A) = \text{Dom}_2(A)$ . Note that  $A$  is a positive operator on  $L_2(\mathcal{N}, \tau)$ . The standard assumptions also imply that  $\tau(T_t x) = \tau(x)$  and thus  $T_t$ 's are faithful. In addition,  $T_t$  is a contraction on  $\mathcal{N}$ . Indeed, for  $x \in \mathcal{N}$ , we have

$$\|T_t x\|_\infty = \sup_{\|y\|_1 \leq 1} |\tau((T_t x)y)| = \sup_{\|y\|_1 \leq 1} |\tau(x(T_t y))| \leq \sup_{\|y\|_1 \leq 1} \|T_t y\|_1 \|x\|_\infty \leq \|x\|_\infty.$$

Recall that  $T_t$  is said to admit a reversed Markov dilation if

(H1) there exists a larger finite von Neumann algebra  $\mathcal{M}$  and a family  $\pi_t : \mathcal{N} \rightarrow \mathcal{M}$  of trace preserving  $*$ -homomorphism;

(H2) there is a decreasing filtration  $(\mathcal{M}_{[s]})_{0 \leq s < \infty}$  with  $\pi_r(x) \in \mathcal{M}_{[s]}$  for all  $r > s$  such that  $E_{[s]}(\pi_t(x)) = \pi_s(T_{s-t}x)$  for all  $t < s$  and  $x \in \mathcal{N}$ .

Here we have  $\mathcal{M}_{[t]} = E_{[t]}(\mathcal{M})$ . For elements  $x, y \in \text{Dom}(A)$  we may define the gradient form, which is called Meyer's "carré du champ" in the commutative theory,

$$2\Gamma(x, y) = A(x^*)y + x^*A(y) - A(x^*y)$$

and for  $x, y \in \text{Dom}(A^2)$  the second order gradient

$$2\Gamma_2(x, y) = \Gamma(Ax, y) + \Gamma(x, Ay) - A\Gamma(x, y).$$

Recall that  $(T_t)$  is called a noncommutative diffusion (or nc-diffusion for short) semigroup if  $\Gamma(x, x) \in L_1(\mathcal{N})$  for all  $x \in \text{Dom}(A^{1/2})$ . If  $(T_t)$  is nc-diffusion, then  $\Gamma(x, x) \in L_1(\mathcal{N})$  is well-defined for  $x \in \text{Dom}(A^{1/2})$  by extension. By duality,  $\Gamma(x, x) \in L_p(\mathcal{N})$  for  $1 \leq p < \infty$  if and only if there exists  $C > 0$  such that  $|\tau(\Gamma(x, x)y)| \leq C\|y\|_{p'}$  for all  $y$  and  $1/p + 1/p' = 1$ .

We will use the following crucial results proved by Junge, Ricard, and Shlyakhtenko in [JRS14], which is a noncommutative version of the Stroock–Varadhan martingale problem.

**Theorem 2.3.** *Suppose that  $(T_t)_{t \geq 0}$  is a standard nc-diffusion semigroup. Then  $T_t$  admits a reversed Markov dilation  $(\pi_t)$  with a.u. continuous path, i.e. in addition to (H1) and (H2), for all  $x \in \text{Dom}(A)$  and all  $S > 0$ ,*

$$m_s(x) := \pi_s(T_s(x)), \quad 0 \leq s \leq S$$

*is a (reversed) martingale with a.u. continuous path.*

*Remark 2.4.* Let  $2 \leq p < \infty$ . For the purpose of our main result, we extend the theorem to  $x \in \text{Dom}(A^{1/2})$ . Indeed, since  $\text{Dom}(A^{1/2}) \cap \mathcal{N}$  is a  $*$ -subalgebra of  $\mathcal{N}$  by [DL92] and  $\text{Dom}(A)$  is dense in  $L_2(\mathcal{N})$ , there exists a sequence  $(x_n) \in \text{Dom}(A)$  such that  $\lim_{n \rightarrow \infty} \|x_n - x\|_2 = 0$ . But  $E_{[r]}(\pi_s T_s(x_n)) = \pi_r T_r(x_n)$  for  $s < r$ . Taking limits on both sides, we find  $E_{[r]}(\pi_s T_s(x)) = \pi_r T_r(x)$  in  $L_2(\mathcal{N})$ . According to [Luc08], the set of a.u. continuous path martingales is closed in  $L_2(\mathcal{N})$ . Hence  $\pi_s T_s(x)$  has a.u. continuous path. Similar argument applies to the forward martingales, but we only need the reversed martingales in this paper.

## 2.5 Conditionally negative length functions and 1-cocycles on discrete groups

Let  $G$  be a countable discrete group with the conditionally negative length (cn-length) function  $\psi : G \rightarrow \mathbb{R}_+$ . Recall that  $\psi$  is a cn-length function if it vanishes at the identity  $e$ ,  $\psi(g) = \psi(g^{-1})$  and is conditionally negative which means that  $\sum_g \xi_g = 0$  implies  $\sum_{g,h} \bar{\xi}_g \xi_h \psi(g^{-1}h) \leq 0$ . It is well known that  $\psi$  determines an affine representation which is given by an orthogonal representation  $\alpha : G \rightarrow O(H_\psi)$  over a real Hilbert space  $H_\psi$  together with a map  $b_\psi : G \rightarrow H_\psi$  satisfying the cocycle law, i.e.,  $b_\psi(gh) = b_\psi(g) + \alpha_g(b_\psi(h))$ ; see for example [BO08].  $b_\psi$  is called a 1-cocycle on  $G$ . For our later development, we need to construct  $b_\psi$  explicitly.

**Lemma 2.5.** *For any  $g, h \in G$ ,  $b_\psi(g) \in \ell_2(d)$  and  $\alpha_g(b_\psi(h))$  have at most finitely many nonzero coordinates.*

*Proof.* Let  $\mathbb{R}G$  be the algebraic group algebra of  $G$ , i.e.,

$$\mathbb{R}G = \{x : x = \sum_g c_g \delta_g, c_g \in \mathbb{R}\}$$

where the sum is over finitely many elements. Let  $K(g, h) = \frac{1}{2}(\psi(g) + \psi(h) - \psi(g^{-1}h))$  for  $g, h \in G$  and

define

$$[\sum_g a_g \delta_g, \sum_{g'} a_{g'} \delta_{g'}] = \sum_{g, g'} a_g a_{g'} K(g, g').$$

Since  $\psi$  is conditionally negative,  $K$  is a positive semidefinite matrix. Put  $N_\psi = \{x \in \mathbb{R}G : [x, x] = 0\}$ . Then we define an inner product on  $\mathbb{R}G/N_\psi$  as  $\langle x + N_\psi, y + N_\psi \rangle = [x, y]$  for  $x, y \in \mathbb{R}G$ . Clearly  $\langle \cdot, \cdot \rangle$  is a well-defined inner product. Let  $H_\psi$  be the norm closure of  $\mathbb{R}G/N_\psi$ . We define  $b_\psi : G \rightarrow H_\psi$  by  $b_\psi(g) = \delta_g + N_\psi$  and  $\alpha_g(b_\psi(h)) = b_\psi(gh) - b_\psi(g)$ . Then  $H_\psi, b_\psi, \alpha$  thus constructed satisfy the cocycle law. We may utilize the Gram–Schmidt procedure on  $(b_\psi(g))_{g \in G}$  and obtain an orthonormal basis  $(e_j)$  such that

$$b_\psi(g_k) = \sum_{j=1}^k b_{kj} e_j,$$

where  $(g_k)$  is an enumeration of  $G$ . Hence,  $b_\psi(g)$  only depends on finitely many  $e_j$ 's for all  $g \in G$  and  $H_\psi \cong \ell_2(d)$ .  $\square$

A direct consequence of this construction is  $\|b_\psi(g)\|^2 = \psi(g)$  for  $g \in G$ . We see from here that the conditionally negative length functions and 1-cocycles on discrete groups have one to one correspondence to each other.

Define a semigroup  $T_t$  acting on  $\mathcal{L}(G)$  by  $T_t \lambda(g) = e^{-t\psi(g)} \lambda(g)$ . The infinitesimal generator of  $T_t$  is given by  $A\lambda(g) = \psi(g)\lambda(g)$ . The associated gradient form is denoted by  $\Gamma^\psi$ . Let  $\mathcal{A} = \mathbb{C}G$  (the group algebra of  $G$ ) be the subalgebra of  $\mathcal{L}(G)$  which consists of elements that can be written as finite combinations of  $\lambda_g, g \in G$ . Then  $\mathcal{A}$  is weakly dense in  $\mathcal{L}(G)$  such that  $A\mathcal{A} \subset \mathcal{A}$  and  $T_t \mathcal{A} \subset \mathcal{A}$ .

**Lemma 2.6.**  *$\mathcal{A}$  is dense in  $\text{Dom}(A^{1/2})$  in the graph norm of  $A^{1/2}$  and  $T_t$  is a standard nc-diffusion semigroup acting on  $\mathcal{L}(G)$ .*

*Proof.* For  $f = \sum_{g_i \in G} \hat{f}(g_i) \lambda(g_i) \in \text{Dom}(A^{1/2})$ , put  $f_n = \sum_{i=1}^n \hat{f}(g_i) \lambda(g_i)$ . Note that  $f \in L_2(\mathcal{L}(G))$ . Then

$$\|f_n - f\|_{L_2(\mathcal{L}(G))} = \tau((f_n^* - f^*)(f_n - f)) = \sum_{i=n+1}^{\infty} |\hat{f}(g_i)|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since  $\langle Af, f \rangle = \sum_{i=1}^{\infty} \psi(g_i) |\hat{f}(g_i)|^2 < \infty$ , we have

$$\langle A(f_n - f), f_n - f \rangle_{L_2(\mathcal{L}(G), \tau)} = \sum_{i=n+1}^{\infty} \psi(g_i) |\hat{f}(g_i)|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore  $\mathcal{A}$  is dense in the graph norm. Since  $\psi$  is conditionally negative, Schoenberg's theorem implies that  $T_t$  is completely positive; see for example [BO08, Appendix D]. It can be directly checked that  $T_t$  is

normal and unital. Since  $\psi(g) = \psi(g^{-1})$ ,

$$\tau(T_t(x)y) = \langle \delta_e, \sum_g e^{-t\psi(g)} \hat{x}(g) \lambda_g \sum_h \hat{y}(h) \lambda_h \delta_e \rangle = \sum_g e^{-t\psi(g)} \hat{x}(g) \hat{y}(g^{-1}) = \tau(xT_t y).$$

Hence  $T_t$  is self-adjoint. To check that  $(T_t)$  is weak\* continuous on  $\mathcal{L}(G)$ , it suffices to verify that  $(T_t)$  is a strongly continuous semigroup on  $L_2(\mathcal{L}(G))$ . For  $f \in L_2(\mathcal{L}(G))$ , we have

$$\|T_t f - f\|_{L_2(\mathcal{L}(G))}^2 = \sum_{g \in G} (e^{-t\psi(g)} - 1)^2 |\hat{f}(g)|^2 \rightarrow 0 \text{ as } t \rightarrow 0.$$

We have proved that  $(T_t)$  is a standard semigroup. Let  $f \in \mathcal{A}$ . Then

$$\|\Gamma(f, f)\|_1 = \langle \delta_e, \sum_{g \in G} |\hat{f}(g)|^2 K_{g,g} \delta_e \rangle = \sum_{g \in G} \psi(g) |\hat{f}(g)|^2 = \|A^{1/2} f\|_2 < \infty.$$

Since  $\mathcal{A}$  is dense in  $\text{Dom}(A^{1/2})$  in the graph norm, by an approximation argument (see the proof of Lemma 3.12), the above equality holds for all  $f \in \text{Dom}(A^{1/2})$  and thus  $(T_t)$  is a nc-diffusion semigroup.  $\square$

## 2.6 Gaussian measure space construction

### 2.6.1 Basic construction

Our reference here is [Str11, Chapter 8] and [Nua06, Chapter 1]. Let  $H$  be a real Hilbert space of dimension  $d$ , where  $d \in \mathbb{N} \cup \{+\infty\}$ . Identify  $H$  as  $\ell_2(d)$ . Following the well known Gaussian measure space construction, we consider the linear map  $B : \ell_2(d) \rightarrow L_2(\mathbb{R}^d, \gamma_d)$  given by  $B(h)(y) = \sum_{i=1}^d \langle h, e_i \rangle y_i$ , where  $(e_i)$  is an orthonormal basis of  $\ell_2(d)$  and  $y_i$  is the  $i$ -th coordinate map. If  $d < \infty$ ,  $B(h)(y) = \langle h, y \rangle$ ; if  $d = \infty$ ,  $(\mathbb{R}^d, \gamma_d)$  is the measure space obtained from Kolmogorov's construction for which all the cylinder set measures are standard Gaussian measures. Note that  $\langle B(h), B(k) \rangle_{L_2(\mathbb{R}^d, \gamma_d)} = \langle h, k \rangle_{\ell_2(d)}$ . Let  $\alpha : G \rightarrow O(H)$  be an orthogonal representation of  $G$  on  $H$ . Then there exists a  $G$ -action  $\alpha^*$  on  $(\mathbb{R}^d, \gamma_d)$  preserving the Gaussian measure  $\gamma_d$ ; see [Str11, Theorem 8.3.14]. By abuse of notation, we simply write  $\alpha$  for  $\alpha^*$  because they are indeed the same if  $d < \infty$ . The action  $\alpha$  on  $(\mathbb{R}^d, \gamma_d)$  induces an action  $\hat{\alpha}$  on  $L_2(\mathbb{R}^d, \gamma_d)$ , such that  $\hat{\alpha}_g(B(h)) = B(\alpha_g(h))$  and

$$\hat{\alpha}_g(f)(x) = f(\alpha(g^{-1})x) = f(\alpha_{g^{-1}}(x)) \quad (2.2)$$

for  $f \in L_2(\mathbb{R}^d, \gamma_d)$ . Clearly,  $\hat{\alpha}$  extends naturally to isometric actions on  $L_p(\mathbb{R}^d, \gamma_d)$  for  $1 \leq p \leq \infty$ . In the following we will consider the von Neumann algebra  $\mathcal{M} = L_\infty(\mathbb{R}^d, \gamma_d) \rtimes_{\hat{\alpha}} G$  and simply forget the subscript

$\hat{\alpha}$  in the notation of  $\mathcal{M}$  if there is no ambiguity.

Let  $P_t$  be the Ornstein–Uhlenbeck semigroup acting on  $L_\infty(\mathbb{R}^d, \gamma_d)$ ; see [Fan05, Nua06] for the case  $d = \infty$ . Then  $P_t \otimes id_{\ell_2(G)}$  is a semigroup acting on  $L_\infty(\mathbb{R}^d, \gamma_d) \overline{\otimes} B(\ell_2(G))$ . Since the action  $\alpha : G \curvearrowright (\mathbb{R}^d, \gamma_d)$  is linear and measure preserving, by the Mehler formula or by the functoriality of the Gaussian functor  $\Gamma$  [BKS97],  $P_t$  is  $G$ -equivariant, i.e.,

$$P_t \circ \hat{\alpha}_g = \hat{\alpha}_g \circ P_t. \quad (2.3)$$

This will be the starting point of the  $L_p$  Poincaré inequalities (1.1) for group measure spaces in Chapter 4, because our extension of the Ornstein–Uhlenbeck semigroups to the group measure spaces relies on (2.3). Indeed, (2.3) implies

$$P_t \otimes id_{\ell_2(G)}(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G) \subset L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G.$$

Define  $P_t \rtimes id_G = P_t \otimes id_{\ell_2(G)}|_{L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G}$  and write  $T_t = P_t \rtimes id_G$ . Since the fixed point algebra of  $P_t$  is trivial, the fixed point algebra of  $T_t$  is  $\mathcal{L}(G)$ . It is well known that  $(T_t)$  extends to contractions on  $L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)$  and  $\lim_{t \rightarrow \infty} T_t f = E_{\mathcal{L}(G)} f$  for  $f \in L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)$ .

Define the Gaussian derivation

$$\begin{aligned} \delta_\psi : \mathcal{L}(G) &\rightarrow M_\infty := \bigcap_{0 < p < \infty} L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G), \\ \lambda(g) &\mapsto B(b_\psi(g)) \rtimes \lambda(g). \end{aligned}$$

Clearly,  $\delta_\psi$  is well-defined. Note that  $M_\infty$  is an  $\mathcal{L}(G)$ - $\mathcal{L}(G)$  bimodule with left and right actions given by  $\lambda(h)(f \rtimes \lambda(g)) = f \rtimes \lambda(hg)$  and  $(f \rtimes \lambda(g))\lambda(h) = f \rtimes \lambda(gh)$ . Then the derivation property  $\delta_\psi(f_1 f_2) = f_1 \delta_\psi(f_2) + \delta_\psi(f_1) f_2$  can be checked directly from the arithmetic in  $L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G$ .

## 2.6.2 Brownian motion construction

It turns out the previous construction is not good enough for our purpose. We need to add time parameter into the construction later. Note that the Hilbert space  $H \otimes L_2([0, \infty))$  is separable if  $H$  is separable. By the Gaussian space construction, there exists a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a linear map

$$\beta : L_2([0, \infty)) \otimes H \rightarrow L_2(\Omega, \mathbb{P})$$

such that  $\beta(1_{[0, t]} \otimes \xi)$  is Gaussian centered and

$$\mathbb{E}[\beta(1_{[0, t]} \otimes \xi) \beta(1_{[0, s]} \otimes \eta)] = 2 \langle 1_{[0, t]} \otimes \xi, 1_{[0, s]} \otimes \eta \rangle_{L_2([0, \infty)) \otimes H} = 2 \min\{t, s\} \langle \xi, \eta \rangle_H.$$

We simply write  $\beta_t(\xi) = \beta(1_{[0,t]} \otimes \xi)$  and denote by  $\mathcal{F}_t$  the  $\sigma$ -subalgebra of  $\mathcal{F}$  generated by  $\beta_s(\xi)$ , for all  $s \leq t$  and  $\xi \in H$ . By Kolmogorov's continuity criterion (see for example [RY99, Theorem I.2.1]),  $\beta_t(\xi)$  thus constructed is a  $\mathbb{R}^d$ -valued Brownian motion, where  $\mathbb{R}^d$  is viewed as an abstract Wiener space associated to  $H$  if  $d = \infty$ . Indeed, by construction the  $k$ -th component of  $\beta_t(\xi)$  is a 1-dimensional Brownian motion with mean 0 and variance  $2t|\xi_k|^2$ , where  $\xi_k$  is the  $k$ -th component of  $\xi$ , and all the components of  $\beta_t(\xi)$  are independent. More explicitly, we can simply take  $\Omega = (\mathbb{R}^d)^{[0,+\infty)}$ . Then  $\beta_t(\xi)(\omega) = \sqrt{2} \sum_{k=1}^d \xi_k \omega_t^k$ , where  $\omega_t^k$  is the  $k$ -th coordinate map at time  $t$ . It is readily seen that  $\beta_t(\xi)$  is a random variable in  $(\Omega, \mathbb{P})$  with variance  $2t\|\xi\|^2$ . Suppose  $\alpha$  is an orthogonal representation of  $G$  on  $H$ . By [Str11, Theorem 8.3.14],  $\alpha$  determines a Gaussian measure preserving action  $\alpha^*$  on  $(\Omega, \mathbb{P})$ . By abuse of notation, we still denote  $\alpha^*$  by  $\alpha$ . The  $G$ -action  $\alpha$  on  $\mathbb{R}^d$  induces an action  $\hat{\alpha}$  on  $L_2(\Omega, \mathbb{P})$ , such that  $\hat{\alpha}_g(\beta_t(h)) = \beta_t(\alpha_g(h))$ . It follows that

$$\hat{\alpha}_g(f)(\omega) = f(\alpha_{g^{-1}}(\omega))$$

for  $f \in L_2(\Omega, \mathbb{P})$ , where  $\alpha_g(\omega)_t = \alpha_g(\omega_t)$ . Clearly,  $\hat{\alpha}$  extends naturally to isometric actions on  $L_p(\Omega, \mathbb{P})$  for  $1 \leq p \leq \infty$ . In the following we will consider the von Neumann algebra  $L_\infty(\Omega, \mathbb{P}) \rtimes_{\hat{\alpha}} G$  and simply forget the subscript  $\hat{\alpha}$  in the notation. We will also consider  $H = H_\psi$ , where  $H_\psi$  is constructed in Section 2.5. We remark that in this case although  $H_\psi$  (and thus  $\beta_t(\xi)$ ) may be infinitely dimensional,  $\beta_t(b_\psi(g))$  is always a finite dimensional Brownian motion for all  $g \in G$  because  $b_\psi(g)$  only depends on finitely many nonzero coordinates.

## 2.7 Khintchine inequality

We briefly recall the modified Khintchine inequality derived in [JMP10, Section 4.1]. Let  $(\mathcal{M}, \tau)$  be a noncommutative probability space. Suppose a discrete group  $G$  acts on  $\mathcal{M}$  and preserves the trace  $\tau$ . By the Gaussian measure space construction explained previously, we may consider the linear map  $B : H \rightarrow L_2(\Omega, \mu)$  given by  $B(h) = \sum_k \langle h, e_k \rangle \zeta_k$ , where  $(\zeta_k)$  is a family of centered independent Gaussian random variables in a probability space  $(\Omega, \mu)$ , and  $(e_k)$  is an orthonormal basis of  $H$ . Put

$$G_p(\mathcal{M}) \rtimes G = \left\{ \sum_{h \in H} \sum_{g \in G} (B(h) \otimes f_{g,h}) \lambda(g) \right\} \subset L_p(L_\infty(\Omega, \mu; \mathcal{M}) \rtimes G),$$



where  $f_{g,h}$  is affiliated to  $\mathcal{M}$ . Recall that conditional expectations between von Neumann algebras extend to contractions between noncommutative  $L_p$  spaces. Consider the conditional expectation

$$E : L_p(L_\infty(\Omega, \mu; \mathcal{M}) \rtimes G) \rightarrow L_p(\mathcal{M} \rtimes G), \quad E\left(\sum_g f_g \lambda(g)\right) = \sum_g \left(\int_\Omega f_g d\mu\right) \lambda(g).$$

For  $F \in L_\infty(\Omega, \mu; \mathcal{M}) \rtimes G$ , define the conditional row (resp. column) space  $L_p^r(E)$  (resp.  $L_p^c(E)$ ) with norm  $\|F\|_{L_p^r(E)} = \|E(FF^*)^{1/2}\|_p$  (resp.  $\|F\|_{L_p^c(E)} = \|E(F^*F)^{1/2}\|_p$ ). Put  $L_p^{rc}(E) = L_p^c(E) \cap L_p^r(E)$ . Define  $RC_p(\mathcal{M}) \rtimes G$  as  $G_p(\mathcal{M}) \rtimes G$  with the norm inherited from  $L_p^{rc}(E)$ . The following Khintchine inequality was proved in [JMP10, Theorem 4.3] with the best order of constant obtained in [JZ13b].

**Theorem 2.7.** *Let  $2 \leq p < \infty$  and  $F \in L_\infty(\Omega, \mu; \mathcal{M}) \rtimes G$ . Then*

$$\|F\|_{G_p(\mathcal{M}) \rtimes G} \leq C\sqrt{p}\|F\|_{RC_p(\mathcal{M}) \rtimes G}.$$

We will use the case  $\mathcal{M} = L_\infty(\mathbb{R}^d, \gamma_d)$  and  $(\Omega, \mu) = (\mathbb{R}^d, \gamma_d)$ . Then we have  $L_\infty(\Omega, \mu; \mathcal{M}) \cong L_\infty(\mathbb{R}^{2d}, \gamma_{2d})$  and  $L_\infty(\mathbb{R}^{2d}, \gamma_{2d}) \rtimes G$  is simply extended from  $L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G$  by diagonal action  $\hat{\alpha}_g(\xi(\cdot)\eta(\cdot))(x, y) = (\hat{\alpha}_g\xi)(x)(\hat{\alpha}_g\eta)(y)$ . It follows from Theorem 2.7 that

$$\begin{aligned} & \left\| \sum_g [\xi_g(x)\eta_g(y)]\lambda(g) \right\|_{L_p(L_\infty(\mathbb{R}^{2d}, \gamma_{2d}) \rtimes G)} \\ & \leq C\sqrt{p} \max \left\{ \left\| \sum_{g,h} \left( \int \hat{\alpha}_{g^{-1}}(\bar{\xi}_g(x)\xi_h(x))\gamma_d(dx) \right) [\hat{\alpha}_{g^{-1}}(\bar{\eta}_g(y)\eta_h(y))]\lambda(g^{-1}h) \right\|_{L_{p/2}(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}^{1/2} \right. \\ & \quad \left. \left\| \sum_{g,h} \left( \int \hat{\alpha}_{g^{-1}}(\xi_g(x)\bar{\xi}_h(x))\gamma_d(dx) \right) [\hat{\alpha}_{g^{-1}}(\eta_g(y)\bar{\eta}_h(y))]\lambda(gh^{-1}) \right\|_{L_{p/2}(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}^{1/2} \right\}. \end{aligned}$$

Here we assumed that  $\xi$  is affiliated to  $L_\infty(\Omega, \mu)$  while  $\eta$  is affiliated to  $\mathcal{M}$ . Note that  $\hat{\alpha}_g$  preserves the Gaussian measure  $\gamma_d$ . In particular, if  $\xi_g = \langle b_\psi(g), \cdot \rangle$ , then

$$\int \bar{\xi}_g(x)\xi_h(x)\gamma_d(dx) = \langle b_\psi(g), b_\psi(h) \rangle_{\ell_2(d)}.$$

Similarly, for  $\mathcal{M} = \mathbb{C}$  and  $f \in \mathcal{L}(G)$ , we have

$$\|\delta_\psi f\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)} \leq C\sqrt{p} \max\{\|\Gamma^\psi(f, f)^{1/2}\|_{L_p(\mathcal{L}(G))}, \|\Gamma^\psi(f^*, f^*)^{1/2}\|_{L_p(\mathcal{L}(G))}\}. \quad (2.4)$$

The right-hand side of (2.4) follows from the arithmetic of crossed products as explained in Section 2.2. A detailed calculation was given in the proof of [JMP10, Theorem 4.6].

## 2.8 Noncommutative Orlicz spaces

Our reference of this section is [FK86]. Let  $\mathcal{M}$  be a semifinite von Neumann algebra with a normal semifinite faithful trace  $\tau$ . Given a  $\tau$ -measurable operator  $x$ , the distribution function of  $x$  is defined as

$$\lambda_s(x) = \tau(E_{(s,\infty)}(|x|)), \quad s > 0$$

where  $E_{(s,\infty)}(|x|)$  is the spectral projection of  $|x|$  corresponding to the interval  $(s, \infty)$ . The generalized singular number of  $x$  is given by

$$\mu_t(x) = \inf\{s \geq 0 : \lambda_s(x) \leq t\}.$$

Then [FK86, Corollary 2.8] asserts that, for any continuous increasing function  $f$  on  $[0, \infty)$  with  $f(0) = 0$  and any  $\tau$ -measurable operator  $x$ , one has

$$\tau(f(|x|)) = \int_0^\infty f(\mu_t(x)) dt. \quad (2.5)$$

Recall that a Young (or Orlicz in some literature) function  $\phi : \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}_+$  is convex, increasing with  $\phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \phi(t) = \infty$ . The noncommutative Orlicz space  $L_\phi(\mathcal{M}, \tau)$  is defined as the space of all  $\tau$ -measurable operators such that  $\tau(\phi(|x|/c)) < \infty$  for some  $c > 0$ .  $L_\phi(\mathcal{M}, \tau)$  is a Banach space with the norm

$$\|x\|_\phi = \inf\{c > 0 : \tau[\phi(|x|/c)] \leq 1\}.$$

$L_\phi(\mathcal{M}, \tau)$  can also be defined as a noncommutative symmetric function space; see for example [BC12, Xu91, DdPPS11, KS08] and the references therein for more information. When  $\tau$  is finite,  $L_\phi(\mathcal{M}, \tau)$  can be obtained from the completion of  $\mathcal{M}$  in this norm. In the following, we will mainly consider the Orlicz space with  $\phi(t) = e^{t^2} - 1$ . The following fact is standard. We include a quick proof for completeness.

**Proposition 2.8.** *Let  $a, b, x, y$  be  $\tau$ -measurable operators. Then,*

1.  $\|x\|_\phi = \|x^*\|_\phi = \||x|\|_\phi$ ;
2. if  $0 \leq x \leq y$ , then  $\|x\|_\phi \leq \|y\|_\phi$ .

*In particular,  $\|ax\|_\phi \leq \|a\|_\infty \|x\|_\phi$ ,  $\|xb\|_\phi \leq \|x\|_\phi \|b\|_\infty$ .*

*Proof.* Note that  $\psi(x) := \phi(x/c)$  is a Young function. Since  $\mu_t(x) = \mu_t(x^*) = \mu_t(|x|)$  (see [FK86, Lemma 2.5]), by (2.5), we have  $\tau(\psi(|x|)) = \tau(\psi(|x^*|))$ . Then the first assertion follows. If  $0 \leq x \leq y$ , then  $\mu_t(x) \leq \mu_t(y)$  for all  $t > 0$  ([FK86, Lemma 2.5]). Since  $\psi$  is increasing, using (2.5) again, we find  $\tau(\psi(x)) \leq \tau(\psi(y))$ .

This gives the second assertion. Notice that  $0 \leq x^*a^*ax \leq \|a\|_\infty^2 x^*x$  and that the square root is operator monotone. We have  $0 \leq |ax| \leq \|a\|_\infty |x|$ . It follows that  $\|ax\|_\phi \leq \|a\|_\infty \|x\|_\phi$ . The last inequality is immediate once we note that  $\|xb\|_\phi = \|b^*x^*\|_\phi$ .  $\square$

## Chapter 3

# The martingale approach

In the classical setting, one has the well-known Burkholder–Davis–Gundy inequality with best order of constant obtained in [BY82]

$$\|X_T\|_p \leq c\sqrt{p}\|[X, X]_T^{1/2}\|_p. \quad (3.1)$$

Here  $[X, X]$  is the quadratic variation of the continuous (local) martingale  $X$ ; see for example [RY99]. In this chapter, we first elaborate on analogous inequalities in the noncommutative setting, and then apply them to prove Poincaré inequalities. The starting point here is the Burkholder inequality first proved in [JX03]

$$\left\| \sum_k dx_k \right\|_p \leq c(p) \left( \left( \sum_k \|dx_k\|_p^p \right)^{\frac{1}{p}} + \left\| \left( \sum_k E_{k-1}(dx_k^* dx_k + dx_k dx_k^*) \right)^{1/2} \right\|_p \right), \quad (3.2)$$

where  $dx_k = x_k - x_{k-1}$  is the martingale difference associated to the martingale  $(x_k, \mathcal{N}_k)$  and  $E_k : \mathcal{N} \rightarrow \mathcal{N}_k$  is the conditional expectation. The optimal order of constant here is  $c(p) \sim cp$  in the noncommutative setting as observed in [JX05], and is due to Randrianantoanina [Ran07], while in the commutative theory [Hit90], the best order is  $c(p) \sim cp/\ln p$ . This suggests much richer objects in the category of noncommutative martingales so that (3.1) may not be true in the noncommutative generality. One way to obtain (3.1) is to separate the constant in (3.2) and consider

$$\left\| \sum_k dx_k \right\|_p \leq A(p) \left\| \left( \sum_k E_{k-1}(dx_k^* dx_k + dx_k dx_k^*) \right)^{1/2} \right\|_p + B(p) \left( \sum_k \|dx_k\|_p^p \right)^{\frac{1}{p}} \quad (3.3)$$

Although it is still unclear to us whether  $A(p)$  can be reduced to  $\sqrt{p}$  in the general noncommutative setting, we shall prove an analogue of (3.1) with  $L_\infty$  norm on the right-hand side. This is achieved by a noncommutative Bernstein type deviation inequality for noncommutative martingales. This will lead to the weak subgaussian Poincaré inequalities for noncommutative diffusion semigroups. Moreover, we will prove  $A(p) \leq C\sqrt{p}$  in (3.3) for a special martingale, which is good enough to establish the subgaussian Poincaré inequalities for 1-cocycles on discrete groups. Both proofs of Poincaré inequalities follow the same strategy.

However, in the latter case, the special structure of group von Neumann algebras allows us to construct the Markov dilation explicitly using Brownian motions. Then we apply a decoupling argument and reduce the martingale case to the independent case so that we can use the noncommutative Rosenthal inequality with optimal constant obtained in [JZ13b].

### 3.1 Noncommutative martingale deviation inequality

Our proof of the martingale deviation inequality relies on the well known Golden–Thompson inequality. The fully general case is due to Araki [Ara73]. The version for semifinite von Neumann algebras we used here was proved by Ruskai in [Rus72, Theorem 4].

**Lemma 3.1** (Golden–Thompson inequality). *Suppose that  $a, b$  are self-adjoint operators, bounded above and that  $a + b$  are essentially self-adjoint (i.e. the closure of  $a + b$  is self-adjoint). Then*

$$\tau(e^{a+b}) \leq \tau(e^{a/2} e^b e^{a/2}).$$

Furthermore, if  $\tau(e^a) < \infty$  or  $\tau(e^b) < \infty$  then

$$\tau(e^{a+b}) \leq \tau(e^a e^b). \tag{3.4}$$

**Lemma 3.2.** *Let  $(x_k)$  be a self-adjoint martingale with respect to the filtration  $(\mathcal{N}_k, E_k)$  and  $d_k := dx_k = x_k - x_{k-1}$  be the associated martingale differences such that*

$$i) \ \tau(x_k) = x_0 = 0; \text{ ii) } \|d_k\| \leq M; \text{ iii) } \sum_{k=1}^n E_{k-1}(d_k^2) \leq D^2 1.$$

Then

$$\tau(e^{\lambda x_n}) \leq \exp[(1 + \varepsilon)\lambda^2 D^2]$$

for all  $\varepsilon \in (0, 1]$  and all  $\lambda \in [0, \sqrt{\varepsilon}/(M + M\varepsilon)]$ .

*Proof.* We follow Oliveira’s original proof for matrix martingales [Imb09] and generalize it to the fully noncommutative setting. With the help of functional calculus, we actually have fewer technical issues. Let  $\varepsilon \in (0, 1]$ . Put  $y_n = \sum_{k=1}^n E_{k-1}(d_k^2)$ . Then  $y_n \leq D^2 1$ . We simply write  $D^2$  for the operator  $D^2 1 \in \mathcal{N}$  in the following. Let us first assume  $M = 1$ . Since  $e^{-((1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n)} \leq 1$ , it follows from (3.4) that

$$\begin{aligned} \tau(e^{\lambda x_n}) &\leq \tau(\exp[\lambda x_n + (1 + \varepsilon)\lambda^2 D^2 - (1 + \varepsilon)\lambda^2 y_n] \exp[-((1 + \varepsilon)\lambda^2 D^2 - (1 + \varepsilon)\lambda^2 y_n)]) \\ &\leq \tau(\exp[\lambda x_n + (1 + \varepsilon)\lambda^2 D^2 - (1 + \varepsilon)\lambda^2 y_n]). \end{aligned}$$

Put  $r_n = E_{n-1}d_n^2$ . Then  $y_n = y_{n-1} + r_n$ . Using (3.4) again we find

$$\begin{aligned} & \tau(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]) \\ &= \tau(\exp[\lambda x_{n-1} + \lambda d_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1} - (1+\varepsilon)\lambda^2 r_n]) \\ &\leq \tau(\exp[\lambda d_n - (1+\varepsilon)\lambda^2 r_n] \exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]). \end{aligned}$$

Since  $x_{n-1}, y_{n-1} \in \mathcal{N}_{n-1}$  and  $E_{n-1}$  is trace preserving, we obtain

$$\begin{aligned} & \tau(\exp[\lambda d_n - (1+\varepsilon)\lambda^2 r_n] \exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}]) \\ &= \tau(E_{n-1}[\exp(\lambda d_n - (1+\varepsilon)\lambda^2 r_n)] \exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}])). \end{aligned} \tag{3.5}$$

We claim that  $E_{k-1}[\exp(\lambda d_k - (1+\varepsilon)\lambda^2 r_k)] \leq 1$  for all  $k = 1, \dots, n$  and  $0 \leq \lambda \leq \sqrt{\varepsilon}/(1+\varepsilon)$ . Indeed,

$$\|\lambda d_k - (1+\varepsilon)\lambda^2 r_k\| \leq \frac{\sqrt{\varepsilon}}{1+\varepsilon} + (1+\varepsilon) \frac{\varepsilon}{(1+\varepsilon)^2} = \frac{\sqrt{\varepsilon} + \varepsilon}{1+\varepsilon} \leq 1.$$

Note that  $e^x \leq 1 + x + x^2$  for  $|x| \leq 1$ . It follows from functional calculus that  $e^A \leq 1 + A + A^2$  for any self-adjoint operator  $A$  with  $\|A\| \leq 1$ . Plugging in  $A = \lambda d_k - (1+\varepsilon)\lambda^2 r_k$  and using  $r_k \in \mathcal{N}_{k-1}$  and  $E_{k-1}d_k = 0$  we obtain

$$\begin{aligned} & E_{k-1}[\exp(\lambda d_k - (1+\varepsilon)\lambda^2 r_k)] \\ &\leq E_{k-1}[1 + \lambda d_k - (1+\varepsilon)\lambda^2 r_k + \lambda^2 d_k^2 - (1+\varepsilon)\lambda^3 d_k r_k - (1+\varepsilon)\lambda^3 r_k d_k + (1+\varepsilon)^2 \lambda^4 r_k^2] \\ &= 1 - \varepsilon \lambda^2 r_k + (1+\varepsilon)^2 \lambda^4 r_k^2. \end{aligned}$$

An elementary calculation shows that  $\varepsilon \lambda^2 x - (1+\varepsilon)^2 \lambda^4 x^2 \geq 0$  for all  $x \in [0, 1]$  and  $\lambda \in (0, \sqrt{\varepsilon}/(1+\varepsilon)]$ .

Using functional calculus of  $r_k$  again, we find

$$\varepsilon \lambda^2 r_k - (1+\varepsilon)^2 \lambda^4 r_k^2 \geq 0$$

which gives the claim. Combining with (3.5), we obtain

$$\begin{aligned} & \tau(\exp[\lambda x_n + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_n]) \\ &\leq \tau(\exp[\lambda x_{n-1} + (1+\varepsilon)\lambda^2 D^2 - (1+\varepsilon)\lambda^2 y_{n-1}])). \end{aligned}$$

Iteratively using (3.4) and the claim  $n - 1$  times yields

$$\tau(e^{\lambda x_n}) \leq \tau(\exp[(1 + \varepsilon)\lambda^2 D^2]) = \exp[(1 + \varepsilon)\lambda^2 D^2]$$

which completes the proof for  $M = 1$ . For arbitrary  $x_k$ , considering  $x'_k = x_k/M$  leads to the conclusion.  $\square$

We remark that the exponential inequality in this lemma is crucial for the proof of law of the iterated logarithms for noncommutative martingales in Section 6.1.

**Theorem 3.3.** *Let  $(x_k)$  be a self-adjoint martingale with respect to the filtration  $(\mathcal{N}_k, E_k)$  and  $d_k := dx_k = x_k - x_{k-1}$  be the associated martingale differences such that*

$$i) \tau(x_k) = x_0 = 0; \text{ ii) } \|d_k\|_\infty \leq M; \text{ iii) } \sum_{k=1}^n E_{k-1}(d_k^2) \leq D^2 1.$$

Then for  $t \geq 0$ ,

$$\text{Prob}(x_n \geq t) \leq \exp\left(-\frac{t^2}{4(1 + \varepsilon)D^2 + 2(1 + \varepsilon)tM/\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}Mt^3}{2(1 + \varepsilon)(2\sqrt{\varepsilon}D^2 + Mt)^2}\right)$$

for all  $0 < \varepsilon \leq 1$ .

Note that if  $\varepsilon = 1$  the first term in our upper bound reduces to the same estimate as Oliveira's. In fact, the first term is always dominating.

*Proof.* We assume  $M = 1$  first. Let  $\varepsilon \in (0, 1]$ . By exponential Chebyshev's inequality we have  $\tau(1_{[t, \infty)}(x_n)) \leq e^{-\lambda t} \tau(e^{\lambda x_n})$  for  $t > 0$ . It follows from Lemma 3.2 that

$$\tau(1_{[t, \infty)}(x_n)) \leq \exp(-\lambda t + (1 + \varepsilon)\lambda^2 D^2).$$

Now we set

$$\lambda = \frac{t}{2(1 + \varepsilon)D^2 + (1 + \varepsilon)t/\sqrt{\varepsilon}}$$

which is less than  $\sqrt{\varepsilon}/(1 + \varepsilon)$ . Then,

$$\begin{aligned} -\lambda t + (1 + \varepsilon)\lambda^2 D^2 &= -t^2 \cdot \frac{1 + t/(\sqrt{\varepsilon}D^2)}{4(1 + \varepsilon)D^2[1 + t/(2\sqrt{\varepsilon}D^2)]^2} \\ &= -\frac{t^2}{4(1 + \varepsilon)D^2[1 + t/(2\sqrt{\varepsilon}D^2)]} - \frac{\sqrt{\varepsilon}t^3}{2(1 + \varepsilon)(2\sqrt{\varepsilon}D^2 + t)^2}. \end{aligned}$$

Replacing  $t$  and  $D$  with  $t/M$  and  $D/M$  respectively yields the assertion.  $\square$

Similar to the classical probability theory, we have for positive  $a \in \mathcal{M}$  and for all  $0 < p < \infty$ ,

$$\|a\|_p^p = p \int_0^\infty t^{p-1} \text{Prob}(a > t) dt. \quad (3.6)$$

From here it is routine to estimate the  $p$ -th moment of  $x_n$  using Theorem 3.3.

**Proposition 3.4.** *Under the assumption of Theorem 3.3, for  $2 \leq p < \infty$  we have*

$$\|x_n\|_p \leq 2^{3/2}(1+\varepsilon)^{1/2} \sqrt{p} \left\| \sum_{i=1}^n E_{i-1}(dx_i^2) \right\|_\infty^{1/2} + 2^{5/2} \left( \frac{1+\varepsilon}{\sqrt{\varepsilon}} \right) p \sup_{i=1, \dots, n} \|dx_i\|_\infty \quad (3.7)$$

for all  $0 < \varepsilon \leq 1$ .

*Proof.* Our strategy is to integrate the first term in Theorem 3.3. The proof is similar to that of [JZ13b, Corollary 0.3]. Note that it follows from symmetry that

$$\text{Prob}(|x_n| \geq t) \leq 2 \exp \left( -\frac{t^2}{4(1+\varepsilon)D^2 + 2(1+\varepsilon)tM/\sqrt{\varepsilon}} \right).$$

Using (3.6), we obtain

$$\frac{\|x_n\|_p^p}{2p} \leq \int_0^{\frac{2\sqrt{\varepsilon}D^2}{M}} t^{p-1} \exp \left( -\frac{t^2}{8(1+\varepsilon)D^2} \right) dt + \int_{\frac{2\sqrt{\varepsilon}D^2}{M}}^\infty t^{p-1} \exp \left( -\frac{t\sqrt{\varepsilon}}{4(1+\varepsilon)M} \right) dt.$$

Let us estimate the first term on the right-hand side. Using the fact that  $\Gamma(x) \leq x^{x-1}$  for  $x \geq 1$ , we have

$$\begin{aligned} & \int_0^{\frac{2\sqrt{\varepsilon}D^2}{M}} t^{p-1} \exp \left( -\frac{t^2}{8(1+\varepsilon)D^2} \right) dt = 2^{3p/2-1} (1+\varepsilon)^{p/2} D^p \int_0^{\frac{\varepsilon D^2}{2M^2(1+\varepsilon)}} r^{p/2-1} e^{-r} dr \\ & \leq 2^{3p/2-1} (1+\varepsilon)^{p/2} D^p \int_0^\infty r^{p/2-1} e^{-r} dr \leq 2^{3p/2-1} (1+\varepsilon)^{p/2} D^p (p/2)^{p/2-1} \\ & \leq 2^p (1+\varepsilon)^{p/2} D^p p^{p/2-1}. \end{aligned}$$

For the second term on the right-hand side,

$$\begin{aligned} & \int_{\frac{2\sqrt{\varepsilon}D^2}{M}}^\infty t^{p-1} \exp \left( -\frac{t\sqrt{\varepsilon}}{4(1+\varepsilon)M} \right) dt \\ & \leq 4^p \left( \frac{1+\varepsilon}{\sqrt{\varepsilon}} \right)^p M^p \int_0^\infty r^{p-1} e^{-r} dr \leq 4^p \left( \frac{1+\varepsilon}{\sqrt{\varepsilon}} \right)^p M^p p^{p-1}. \end{aligned}$$

Hence, we find

$$\|x_n\|_p^p \leq 2^{p+1} (1+\varepsilon)^{p/2} D^p p^{p/2} + 2^{2p+1} \left( \frac{1+\varepsilon}{\sqrt{\varepsilon}} \right)^p M^p p^p.$$



This yields

$$\begin{aligned}\|x_n\|_p &\leq 2^{1+1/p}(1+\varepsilon)^{1/2}D\sqrt{p} + 2^{2+1/p}\left(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\right)Mp \\ &\leq 2^{3/2}(1+\varepsilon)^{1/2}D\sqrt{p} + 2^{5/2}\left(\frac{1+\varepsilon}{\sqrt{\varepsilon}}\right)Mp.\end{aligned}$$

Setting  $D^2 = \left\| \sum_{i=1}^n E_{i-1}(dx_i^2) \right\|$  and  $M = \sup_{i=1, \dots, n} \|dx_i\|$  gives the assertion.  $\square$

As indicated at the beginning of this chapter, another way to obtain (3.1) would be an improved Burkholder inequality for noncommutative martingales:

*Problem 3.5.* Is it true that for some function  $f(p)$  and constant  $C$ ,

$$\left\| \sum_k dx_k \right\|_p \leq C\sqrt{p} \left\| \left( \sum_k dx_k^* dx_k + dx_k dx_k^* \right)^{1/2} \right\|_p + f(p) \left( \sum_k \|dx_k\|_p^p \right)^{1/p} \quad (3.8)$$

holds for all noncommutative martingales.

For independent increments this has recently been proved in [JZ13b]. One would actually expect  $f(p) = p$ . As will become clear in the following, the validity of (3.8) would improve our main results and imply a number of results in different contexts. At the time of this writing we are unable to decide whether (3.8) holds. However, the commutative case was known to be true due to the work of Pinelis [Pin94], who attributed it to Hitczenko.

## 3.2 Noncommutative Burkholder–Davis–Gundy type inequalities

We follow the notation of Section 2.3. Let  $\sigma = \{0 = s_0, \dots, s_n = T\}$  be a partition of the interval  $[0, T]$  and  $|\sigma|$  its cardinality. Let  $\mathcal{U}$  be an ultrafilter refining the natural order given by inclusion on the set of all partitions of  $[0, T]$ .

**Theorem 3.6.** *Let  $x$  be a mean 0 martingale with a.u. continuous path. Then for every  $T > 0$ , we have*

1. *For  $2 \leq p < \infty$ , if  $x$  is self-adjoint, then*

$$\|E_T x\|_p \leq C\sqrt{p} \liminf_{\sigma, \mathcal{U}} \|x\|_{h_\infty^c([0, T]; \sigma)}.$$

*If  $x$  is not necessarily self-adjoint, then*

$$\|E_T x\|_p \leq C\sqrt{p} \liminf_{\sigma, \mathcal{U}} \left( \|x\|_{h_\infty^c([0, T]; \sigma)} + \|x\|_{h_\infty^r([0, T]; \sigma)} \right),$$

where we may take  $C = 2\sqrt{2}$ .

2. For all  $2 \leq p < \infty$ ,

$$\|E_T x\|_p \leq C' p \max \{ \|x\|_{h_p^c([0,T])}, \|x\|_{h_p^r([0,T])} \}.$$

*Proof.* (1) First assume that  $x$  is self-adjoint and that  $x \in \mathcal{N}_T$ . We follow the strategy used in the proof of Theorem 2.1. Fix a partition  $\sigma$  of  $[0, T]$ . We write  $h_p(\sigma)$  for  $h_p([0, T]; \sigma)$  in the following proof. Let  $\delta > 0$ . We have  $d_j x = d_j x 1_{[|d_j x| > \delta]} + d_j x 1_{[|d_j x| \leq \delta]}$ . Conditioning again, we obtain

$$d_j x = d_j(d_j x 1_{[|d_j x| > \delta]}) + d_j(d_j x 1_{[|d_j x| \leq \delta]}).$$

Put  $z_\sigma^\delta = \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| > \delta]})$  and  $y_\sigma^\delta = \sum_{j=1}^{|\sigma|} d_j(d_j x 1_{[|d_j x| \leq \delta]})$ . Then clearly

$$\sup_{j=1, \dots, |\sigma|} \|d_j(d_j x 1_{[|d_j x| \leq \delta]})\|_\infty \leq 2\delta.$$

Using Proposition 3.7 for some fixed  $0 < \varepsilon \leq 1$ , we find

$$\|y_\sigma^\delta - \tau(y_\sigma^\delta)\|_p \leq 2^{3/2}(1 + \varepsilon)^{1/2} \sqrt{p} \|y_\sigma^\delta\|_{h_\infty^c(\sigma)} + 2^{7/2} \left( \frac{1 + \varepsilon}{\sqrt{\varepsilon}} \right) p \delta. \quad (3.9)$$

Note that

$$\begin{aligned} 0 &\leq E_{s_{j-1}} |d_j(d_j x 1_{[|d_j x| < \delta]})|^2 = E_{s_{j-1}} [(d_j x 1_{[|d_j x| < \delta]})^2] - [E_{s_{j-1}}(d_j x 1_{[|d_j x| < \delta]})]^2 \\ &\leq E_{s_{j-1}} [(d_j x 1_{[|d_j x| < \delta]})^2] \leq E_{s_{j-1}} [(d_j x)^2]. \end{aligned}$$

Then we have

$$\begin{aligned} \|y_\sigma^\delta\|_{h_\infty^c(\sigma)} &= \left\| \sum_{i=0}^{|\sigma|-1} E_{s_{j-1}} |d_j(d_j x 1_{[|d_j x| < \delta]})|^2 \right\|_\infty^{1/2} \\ &\leq \left\| \sum_{i=0}^{|\sigma|-1} E_{s_{j-1}} |d_j x|^2 \right\|_\infty^{1/2} = \|x\|_{h_\infty^c(\sigma)}. \end{aligned}$$

According to Theorem 2.1 (in our context,  $y^\delta = w^* - \lim_\sigma y_\sigma^\delta$ ) and Lemma 2.2, we have  $x = w^* - \lim_{\delta \rightarrow 0} y^\delta$ .

Hence for any  $\lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1$ , we have

$$x = w^* - \lim_{\substack{\delta_i \rightarrow 0 \\ i=1, \dots, k}} \sum_{i=1}^k \lambda_i y^{\delta_i}.$$

Since in a Banach space the weak closure and the norm closure of a convex set are the same, by the reflexivity of  $L_p(\mathcal{N})$  we can find a net  $x_\alpha$  in the convex hull of  $\{y^\delta\}$  such that  $x_\alpha \rightarrow x$  in  $L_p(\mathcal{N})$ . Therefore by sending  $\delta \rightarrow 0$ , we deduce from (3.9) that

$$\|x\|_p \leq 2\sqrt{2}(1+\varepsilon)^{1/2}\sqrt{p} \liminf_{\sigma, \mathcal{U}} \|x\|_{h_\infty^c(\sigma)},$$

for all  $0 < \varepsilon \leq 1$ . Sending  $\varepsilon \rightarrow 0$  yields the first assertion. If  $x$  is not self-adjoint, we write  $x = \Re(x) + i\Im(x)$  where  $\Re(x) = \frac{x+x^*}{2}$  and  $\Im(x) = \frac{x-x^*}{2i}$ . Then the second assertion follows from the self-adjoint case by triangle inequality.

(2) Since  $x$  is of vanishing variation,  $\|x\|_{h_p^d([0,T])} = 0$  for all  $2 < p < \infty$ . Using (3.2), we have

$$\|x\|_p \leq C'p(\|x\|_{h_p^c([0,T];\sigma)} + \|x\|_{h_p^r([0,T];\sigma)}) + p\|x\|_{h_p^d([0,T];\sigma)}.$$

Taking limits on the right hand side yields the assertion for  $2 < p < \infty$ . The case  $p = 2$  is proved by sending  $p \downarrow 2$ .  $\square$

### 3.3 $L_p$ Poincaré inequalities for noncommutative diffusion semigroups

We follow the notation of Section 2.4. Let  $T_t$  be a standard nc-diffusion semigroup. Recall that  $m_t = \pi_t(T_t x)$  is a (reversed) martingale. Put  $L_p^0(\mathcal{N}) = \{x \in L_p(\mathcal{N}) : \lim_{t \rightarrow \infty} T_t x = 0\}$  for  $1 \leq p \leq \infty$ . Here the limit is taken with respect to  $\|\cdot\|_{L_p(\mathcal{N})}$  for  $1 \leq p < \infty$  and with respect to the weak\* topology for  $p = \infty$ . Let  $\text{Fix} = \{x \in \mathcal{N} : T_t x = x \text{ for all } t \geq 0\}$ . It was shown in [JX07] that  $\text{Fix}$  is a von Neumann subalgebra and  $\text{Fix}^\perp = L_\infty^0(\mathcal{N})$ . Denote by  $E_{\text{Fix}} : \mathcal{N} \rightarrow \text{Fix}$  the conditional expectation which extends to a contraction on  $L_p(\mathcal{N})$ . Then for all  $x \in L_p(\mathcal{N})$  we have  $x - E_{\text{Fix}} x \in L_p^0(\mathcal{N})$  and  $L_p^0(\mathcal{N})$  is a complemented subspace of  $L_p(\mathcal{N})$ .

**Lemma 3.7.** *Let  $2 \leq p < \infty$  and  $(T_t)_{t \geq 0}$  be a standard nc-diffusion semigroup. Then for all  $0 \leq s < t \leq \infty$ , and  $x \in \text{Dom}(A^{1/2}) \cap L_p^0(\mathcal{N})$  with  $\Gamma(T_r x, T_r x)$  uniformly bounded for  $r \geq 0$  in  $L_p(\mathcal{N})$  we have*

$$\|m(x)\|_{h_p^c([s,t])} = \left\| 2 \int_s^t \pi_r(\Gamma(T_r x, T_r x)) dr \right\|_{p/2}^{1/2}.$$

*Proof.* By Theorem 2.3, Remark 2.4 and Lemma 2.2,  $\text{var}_p(m) = 0$  for all  $2 < p < \infty$ . (2.1) implies for

$2 < p < \infty$ ,

$$\|m\|_{h_p^c([s,t])} = \|\langle m, m \rangle_t - \langle m, m \rangle_s\|_{p/2}^{1/2}.$$

It follows from [JM10, Lemma 2.4.1] and uniform boundedness that

$$\langle m, m \rangle_s - \langle m, m \rangle_t = 2 \int_s^t \pi_r(\Gamma(T_r x, T_r x)) dr.$$

Here the integral when  $t = \infty$  is well-defined for  $x \in L_p^0(\mathcal{N})$  according to [JM10, Proposition 2.4.3]. This gives the assertion for  $2 < p < \infty$ . The case  $p = 2$  follows by sending  $p \downarrow 2$ .  $\square$

We are now ready to state our main result of this section.

**Theorem 3.8.** *Suppose  $2 \leq p < \infty$ . Let  $T_t = e^{-tA}$  be a standard nc-diffusion semigroup and  $\Gamma$  the gradient form associated with  $A$ . Assume  $x \in L_p(\mathcal{N}) \cap \text{Dom}(A^{1/2})$  satisfies*

$$\tau(y\Gamma(T_t x, T_t x)) \leq e^{-2\alpha t} \tau(y\Gamma(x, x)), \quad y \in \mathcal{N}, y \geq 0, \quad (3.10)$$

for some  $\alpha > 0$ . Then we have the following Poincaré type inequalities

$$\|x - E_{\text{Fix}} x\|_p \leq C \sqrt{p/\alpha} \max\{\|\Gamma(x, x)^{1/2}\|_\infty, \|\Gamma(x^*, x^*)^{1/2}\|_\infty\}, \quad (3.11)$$

$$\|x - E_{\text{Fix}} x\|_p \leq C' \alpha^{-1/2} p \max\{\|\Gamma(x, x)^{1/2}\|_p, \|\Gamma(x^*, x^*)^{1/2}\|_p\}, \quad (3.12)$$

where we can take  $C = 4\sqrt{2}$  in general and  $C = 2\sqrt{2}$  if  $x$  is self-adjoint.

*Proof.* First assume  $2 < p < \infty$ . Notice that  $E_{\text{Fix}} x$  is in the multiplicative domain of  $T_t$ . Then  $\Gamma(x, x) = \Gamma(x - E_{\text{Fix}} x, x - E_{\text{Fix}} x)$ . Without loss of generality we may assume  $x \in L_p^0(\mathcal{N})$ , which implies  $\lim_{t \rightarrow \infty} T_t x = E_{\text{Fix}}(x) = 0$  in  $L_p$ . Fix a constant  $0 < M < \infty$  and consider the reversed martingale  $m_t(x)$  in Theorem 2.3 for  $t \in [0, M]$ . By Theorem 3.6 (applied to reversed martingales), noticing that  $m_0(x) = \pi_0(x)$ , we have

$$\begin{aligned} & \|m_0 - E_{[M]}(m_0)\|_p \\ & \leq C \sqrt{p} \liminf_{\sigma, \mathcal{U}} (\|m_0 - E_{[M]}(m_0)\|_{h_\infty^c([0, M]; \sigma)} + \|m_0 - E_{[M]}(m_0)\|_{h_\infty^r([0, M]; \sigma)}). \end{aligned}$$

Using the reversed Markov dilation and [JM10, Lemma 2.4.1 (iii)], we find (see [JM10, (2.12)])

$$\begin{aligned}
\|m_0 - E_{[M]}(m_0)\|_{h_\infty([0,M];\sigma)} &= \left\| \sum_{j=0}^{|\sigma|-1} E_{[s_{j+1}]} |m_{s_j}(x) - m_{s_{j+1}}(x)|^2 \right\|_\infty^{1/2} \\
&= \left\| \sum_{j=0}^{|\sigma|-1} E_{[s_{j+1}]} (\pi_{s_j}(|T_{s_j}x|^2)) - \pi_{s_{j+1}}(|T_{s_{j+1}}x|^2) \right\|_\infty^{1/2} \\
&= \left\| \sum_{j=0}^{|\sigma|-1} \pi_{s_{j+1}}(T_{s_{j+1}-s_j}|T_{s_j}x|^2 - |T_{s_{j+1}}x|^2) \right\|_\infty^{1/2} \\
&= \left\| 2 \sum_{j=0}^{|\sigma|-1} \pi_{s_{j+1}} \left( \int_0^{s_{j+1}-s_j} T_{s_{j+1}-s_j-r}(\Gamma(T_{r+s_j}x, T_{r+s_j}x)) dr \right) \right\|_\infty^{1/2} \\
&= \left\| 2 \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} E_{[s_{j+1}]} \pi_r(\Gamma(T_r x, T_r x)) dr \right\|_\infty^{1/2}.
\end{aligned}$$

Since  $E_{[s_{j+1}]}$  and  $\pi_r$  are contractions, we deduce from (3.10) that

$$\begin{aligned}
\|m_0 - E_{[M]}(m_0)\|_{h_\infty([0,M];\sigma)} &\leq \sqrt{2} \left( \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} \|\Gamma(T_r x, T_r x)\|_\infty dr \right)^{1/2} \\
&= \sqrt{2} \left( \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} \sup_{y \geq 0, y \in \mathcal{N}, \|y\|_1 \leq 1} \tau(y \Gamma(T_r x, T_r x)) dr \right)^{1/2} \\
&\leq \sqrt{2} \left( \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} e^{-2\alpha r} \sup_{y \geq 0, y \in \mathcal{N}, \|y\|_1 \leq 1} \tau(y T_r \Gamma(x, x)) dr \right)^{1/2} \\
&= \sqrt{2} \left( \sum_{j=0}^{|\sigma|-1} \int_{s_j}^{s_{j+1}} e^{-2\alpha r} \|T_r \Gamma(x, x)\|_\infty dr \right)^{1/2} \leq \sqrt{2} \left( \int_0^M e^{-2\alpha r} \|\Gamma(x, x)\|_\infty dr \right)^{1/2} \\
&\leq \alpha^{-1/2} \|\Gamma(x, x)\|_\infty^{1/2}.
\end{aligned}$$

Similarly,

$$\|m_0 - E_{[M]}(m_0)\|_{h_\infty([0,M];\sigma)} \leq \alpha^{-1/2} \|\Gamma(x^*, x^*)\|_\infty^{1/2}.$$

Hence we have

$$\|m_0 - E_{[M]}(m_0)\|_p \leq C \left( \frac{p}{\alpha} \right)^{1/2} (\|\Gamma(x, x)\|_\infty^{1/2} + \|\Gamma(x^*, x^*)\|_\infty^{1/2}).$$

By the reversed Markov dilation,

$$\|E_{[M]}(m_0)\|_p = \|E_{[M]}(\pi_0(x))\|_p = \|\pi_M T_M x\|_p \leq \|T_M x\|_p.$$

Note that  $\lim_{M \rightarrow \infty} \|T_M x\|_p = 0$  and that  $\|x\|_p = \|m_0\|_p \leq \|m_0 - E_{[M]}(m_0)\|_p + \|E_{[M]}(m_0)\|_p$ . Sending

$M \rightarrow \infty$  gives the first assertion for  $2 < p < \infty$ . Sending  $p \downarrow 2$  gives the case  $p = 2$ .

For (3.12), note that (3.10) implies  $\Gamma(T_t x, T_t x)$  is uniformly bounded in  $L_p(\mathcal{N})$ . Then Theorem 3.6 and Lemma 3.7 imply that for  $M > 0$ ,  $2 < p < \infty$ , and  $x \in \text{Dom}(A^{1/2})$ , we have

$$\begin{aligned} & \|m_0 - E_{[M]}(m_0)\|_p \\ & \leq \sqrt{2}C'p \max \left\{ \left\| \int_0^M \pi_r(\Gamma(T_r x, T_r x)) dr \right\|_{p/2}^{1/2}, \left\| \int_0^M \pi_r(\Gamma(T_r x^*, T_r x^*)) dr \right\|_{p/2}^{1/2} \right\}. \end{aligned}$$

Similar to the above argument, (3.10) yields

$$\left\| \int_0^M \pi_r(\Gamma(T_r x, T_r x)) dr \right\|_{p/2}^{1/2} \leq \frac{1}{\sqrt{2\alpha}} \|\Gamma(x, x)\|_{p/2}^{1/2}.$$

The rest of proof is the same as that of the first assertion.  $\square$

If  $A$  has a spectral gap in  $L_p$ , we can deduce the second inequality (3.12) from the main result of [JM10] on the noncommutative Riesz transform. However, we have explicit order  $p$  here. So far as we know, no previous method has achieved the order  $\sqrt{p}$  in the first inequality in the noncommutative setting.

*Remark 3.9.* In fact, if (3.10) holds for all  $x \in \text{Dom}(A^{1/2})$ , then  $T_t$  is a nc-diffusion semigroup. Indeed, it was proved in [JRS14] that  $T_t \Gamma(x, x) \in L_1(\mathcal{N})$  for  $t > 0$ . Then (3.10) implies that  $\Gamma(T_t x, T_t x) \in L_1(\mathcal{N})$ . Taking limit gives  $\Gamma(x, x) \in L_1(\mathcal{N})$ .

Condition (3.10) is not convenient to check. In practice, we may pose stronger assumptions which are easy to verify. The following lemma is of course well known in the commutative case.

**Lemma 3.10.** *Let  $T_t = e^{-tA}$  be a standard nc-diffusion semigroup. Let  $x \in \mathcal{N}$  be such that  $\Gamma_2(x, x)$  is well-defined. Then  $\Gamma_2(x, x) \geq \alpha \Gamma(x, x)$  implies (3.10).*

*Proof.* Since  $T_t$  is positive,  $\alpha T_t \Gamma(x, x) \leq T_t \Gamma_2(x, x)$ . Let  $\tilde{T}_t = e^{2\alpha t} T_t$ . Consider the function

$$f(s) = \tilde{T}_{t-s} \Gamma(\tilde{T}_s x, \tilde{T}_s x) = e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x).$$

Due to the assumption,  $f(s)$  is differentiable. Then

$$\begin{aligned} f'(s) &= 2\alpha e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x) + e^{2\alpha(t+s)} T_{t-s} A \Gamma(T_s x, T_s x) \\ &\quad - e^{2\alpha(t+s)} T_{t-s} [\Gamma(AT_s x, T_s x) + \Gamma(T_s x, AT_s x)] \\ &= 2\alpha e^{2\alpha(t+s)} T_{t-s} \Gamma(T_s x, T_s x) - 2e^{2\alpha(t+s)} T_{t-s} \Gamma_2(T_s x, T_s x) \leq 0 \end{aligned}$$

for all  $0 < s < t$ . We have by continuity  $\Gamma(\tilde{T}_t x, \tilde{T}_t x) = f(t) \leq f(0) = \tilde{T}_t \Gamma(x, x)$ , or  $\Gamma(T_t x, T_t x) \leq e^{-2\alpha t} \tilde{T}_t \Gamma(x, x)$  which implies (3.10).  $\square$

For the purpose of future development, let us recall the definition of positive forms. Suppose  $\Theta : \mathcal{N} \times \mathcal{N} \rightarrow L_1(\mathcal{N}, \tau)$  is a sesquilinear form whose domain is a weakly dense  $*$ -subalgebra  $\text{Dom}(\Theta)$  such that  $1 \in \text{Dom}(\Theta)$ . In this paper, we follow the convention that a sesquilinear form is conjugate linear in the first component.  $\Theta$  is said to be positive if for all  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in \text{Dom}(\Theta)$ ,  $(\Theta(x_i, x_j))_{i,j=1}^n$  is positive in  $M_n(L_1(\mathcal{N})) \cong L_1(M_n \bar{\otimes} \mathcal{N})$ . Given another sesquilinear form  $\Phi$ ,  $\Theta \geq \Phi$  if  $\Theta - \Phi \geq 0$ . We refer the readers to [Sau89, Pet09] for more details. For any  $n \in \mathbb{N}$ , and any  $a_1, \dots, a_n \in \text{Dom}(A)$  the  $n \times n$  matrix  $(\Gamma(a_i, a_j))_{i,j=1}^n$  with entries in  $\mathcal{N}$  is positive in  $M_n(\mathcal{N})$ . The following useful fact was due to Peterson [Pet09]; see also [Sau89] for the implication “ $\Rightarrow$ ”.

**Theorem 3.11.** *Let  $(T_t)$  be a strongly continuous semigroup on  $L_2(\mathcal{N})$ . Then  $(T_t)$  is a completely positive semigroup if and only if  $\Gamma$  is a positive form.*

As in the commutative case, the domain of  $\Gamma$  and  $\Gamma_2$  is a delicate issue. Theorem 3.8 avoided this difficulty by considering individual element. In many cases we are interested in the Poincaré type inequalities for the whole space. Our next result is meant for this purpose.

**Corollary 3.12.** *Let  $T_t = e^{-tA}$  be a standard nc-diffusion semigroup. Suppose that there exists a weakly dense self-adjoint subalgebra  $\mathcal{A} \subset \mathcal{N}$  such that*

*i)  $A(\mathcal{A}) \subset \mathcal{A}$ ; ii)  $T_t(\mathcal{A}) \subset \mathcal{A}$ ; iii)  $\mathcal{A}$  is dense in  $\text{Dom}(A^{1/2})$  in the graph norm of  $A^{1/2}$ .*

*Assume  $\Gamma_2(x, x) \geq \alpha \Gamma(x, x)$  for some  $\alpha > 0$  and for all  $x \in \mathcal{A}$ . Then (3.10) holds for all  $x \in \text{Dom}(A^{1/2})$ . Moreover, all  $x \in L_p(\mathcal{N})$  satisfies (3.11) and (3.12).*

*Proof.* For  $x \in \text{Dom}(A^{1/2})$  we deduce from assumption iii) that there exist  $(x_n) \subset \mathcal{A}$  with  $\Gamma_2(x_n, x_n) \geq \alpha \Gamma(x_n, x_n)$  such that  $\|x_n - x\|_2 \rightarrow 0$  and  $\|A^{1/2}x_n - A^{1/2}x\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . By [CS03, Section 9], we have  $\|\Gamma(x, x)\|_1 = \langle A^{1/2}x, A^{1/2}x \rangle_{L_2(\mathcal{N})}$ . Since  $\Gamma$  is a complete positive form, we have for  $x, y \in \text{Dom}(A^{1/2})$ ,

$$\begin{pmatrix} \Gamma(x, x) & \Gamma(x, y) \\ \Gamma(y, x) & \Gamma(y, y) \end{pmatrix} \geq 0.$$

Note that  $\Gamma(x, y)^* = \Gamma(y, x)$ . Then  $\|(\Gamma(x, x) + \varepsilon 1)^{-1/2} \Gamma(x, y) (\Gamma(y, y) + \varepsilon 1)^{-1/2}\|_\infty \leq 1$  for any  $\varepsilon > 0$ . Hence

$$\begin{aligned} & \|\Gamma(x, y)\|_1 \\ & \leq \|(\Gamma(x, x) + \varepsilon 1)^{1/2}\|_2 \|(\Gamma(x, x) + \varepsilon 1)^{-1/2} \Gamma(x, y) (\Gamma(y, y) + \varepsilon 1)^{-1/2}\|_\infty \|(\Gamma(y, y) + \varepsilon 1)^{1/2}\|_2 \\ & \leq \|\Gamma(x, x) + \varepsilon 1\|_1^{1/2} \|\Gamma(y, y) + \varepsilon 1\|_1^{1/2}. \end{aligned}$$

Sending  $\varepsilon \rightarrow 0$ , we have  $\|\Gamma(x, y)\|_1 \leq \|\Gamma(x, x)\|_1^{1/2} \|\Gamma(y, y)\|_1^{1/2}$ . It follows that

$$\begin{aligned} & \|\Gamma(x_n, x_n) - \Gamma(x, x)\|_1 \leq \|\Gamma(x_n - x, x_n - x)\|_1 + 2\|\Gamma(x_n - x, x)\|_1 \\ & \leq \langle A^{1/2}(x_n - x), A^{1/2}(x_n - x) \rangle_{L_2(\mathcal{N}, \tau)} + 2\langle A^{1/2}(x_n - x), A^{1/2}(x_n - x) \rangle_{L_2(\mathcal{N}, \tau)}^{1/2} \|\Gamma(x, x)\|_1^{1/2}. \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \Gamma(x_n, x_n) = \Gamma(x, x)$  in  $L_1(\mathcal{N})$ . Notice that  $T_t$  and  $A^{1/2}$  commute. Then for all  $t \geq 0$  and  $x \in \text{Dom}(A^{1/2})$ ,  $T_t x \in \text{Dom}(A^{1/2})$  and a similar argument as above gives that  $\lim_{n \rightarrow \infty} \Gamma(T_t x_n, T_t x_n) = \Gamma(T_t x, T_t x)$  in  $L_1(\mathcal{N})$ . Since Lemma 3.10 implies

$$\tau(y \Gamma(T_t x_n, T_t x_n)) \leq e^{-2\alpha t} \tau(y T_t \Gamma(x_n, x_n))$$

for all  $y \in \mathcal{N}$ ,  $y \geq 0$ , sending  $n \rightarrow \infty$  on both sides yields the first assertion. For the “moreover” part, note that we only need to prove (3.11) and (3.12) for

$$\max\{\|\Gamma(x, x)^{1/2}\|_p, \|\Gamma(x^*, x^*)^{1/2}\|_p\} < \infty.$$

Recall that  $\Gamma(x, x)$  is understood as the weak\* limit of  $\Gamma^{A_\varepsilon}(x, x)$  where  $A_\varepsilon = (I + \varepsilon A)^{-1} A$  (see [CS03, (3.2)]). If this limit exists in  $L_{p/2}$  for  $p > 2$ , then  $\tau(\Gamma(x, x))$  is finite and hence  $x \in \text{Dom}(A^{1/2})$ . The individual result Theorem 3.8 then comes into play and completes the proof.  $\square$

The condition  $\Gamma_2(x, x) \geq \alpha \Gamma(x, x)$  we posed here is usually called the curvature condition or  $\Gamma_2$ -criterion. The expression  $\max\{\|\Gamma(x, x)^{1/2}\|_\infty, \|\Gamma(x^*, x^*)^{1/2}\|_\infty\}$  is the so-called Lipschitz norm in the commutative theory. In the classical diffusion setting, Bakry and Emery [BÉ85] showed that the  $\Gamma_2$ -criterion implies log-Sobolev inequality, which in turn yields the  $L_p$  Poincaré inequalities with constant  $C\sqrt{p}$  due to Aida and Stroock [AS94]; see also [AW13] for another proof. In the general non-diffusion setting, we will show that the first implication is not true. At the time of this writing, we do not know whether the  $L_p$  Poincaré inequalities follow from LSI in full generality. However, adapting our theory to the classical diffusion setting will result in a shortcut. Namely, we can directly show that  $\Gamma_2$ -criterion implies the  $L_p$  Poincaré inequalities with constants



$C\sqrt{p}$ . Let  $T_t = e^{-tL}$  be a symmetric classical diffusion semigroup with infinitesimal generator  $L$  acting on a probability space  $(\mathbb{R}^d, \mu)$ . By the well known diffusion theory (see for example [RY99, Section VII.2]), under certain regularity conditions on  $L$ , one can always construct a diffusion process  $X_t$  corresponding to  $T_t$  by solving a martingale problem.

**Theorem 3.13.** *Assume that  $X_t$  is a classical diffusion process defined on  $(\Omega, \mathbb{P})$  for the semigroup  $T_t = e^{-tL}$ . Suppose  $\Gamma_2(f, f) \geq \alpha \Gamma(f, f)$  for the real-valued function  $f$ . Then for  $2 \leq p < \infty$ ,*

$$\|f - E_{\text{Fix}} f\|_p \leq C\sqrt{p} \|\Gamma(f, f)^{1/2}\|_p.$$

*Proof.* The proof follows the same strategy as for (3.12). Assume  $2 < p < \infty$  and  $\lim_{t \rightarrow \infty} T_t f = 0$  in  $L_p$ . By approximation, we may assume that  $f$  is compactly supported. Recall that for the diffusion process  $X_t$ , we have

$$\mathbb{E}_x[f(X_t)] = f(x) + \mathbb{E}_x \left[ \int_0^t -L f(X_s) ds \right],$$

where  $\mathbb{E}_x$  is the expectation conditioning on  $\{X_0 = x\}$ . We obtain a martingale

$$M_t^f = f(X_t) - f(X_0) + \int_0^t L f(X_s) ds$$

adapted to  $\mathcal{F}_t := \sigma(X_s : s \leq t)$ . Fix a large constant  $K > 0$ . For  $0 \leq t \leq K$ , define  $\tilde{X}_t = X_{K-t}$ ,  $\tilde{\mathcal{F}}_t = \mathcal{F}_{K-t}$ ,  $N_t^f = (T_t f)(X_{K-t})$ . Note that in this setting, the reversed Markov dilation is given by  $\pi_t f = f(\tilde{X}_t)$ . Then  $(N_t^f)_{0 \leq t \leq K}$  is a reversed martingale with respect to the filtration  $\tilde{\mathcal{F}}_t$ . Indeed, by the Markov property, for  $s < t$

$$\begin{aligned} E[N_s^f | \tilde{\mathcal{F}}_t] &= E[(T_s f)(X_{K-s}) | \mathcal{F}_{K-t}] \\ &= T_{(K-s)-(K-t)} T_s f(X_{K-t}) = T_t f(X_{K-t}) = N_t^f. \end{aligned}$$

By Lemma 3.7 with  $\pi_r f = f(X_{K-r})$ , we have

$$\langle N^f, N^f \rangle_K - \langle N^f, N^f \rangle_0 = 2 \int_0^K \Gamma(T_r f, T_r f)(X_{K-r}) dr.$$

Applying the BDG inequality (3.1) (see [BY82, Proposition 4.2]) to  $N_t^f$  on  $[0, K]$ , we have

$$\|T_K f(X_0) - f(X_K)\|_p \leq C\sqrt{p} \|\langle N^f, N^f \rangle_K - \langle N^f, N^f \rangle_0\|_{p/2}^{1/2}.$$

Here we used the continuity of the sample paths of the diffusion process  $X_t$  so that the conditional square function and the unconditional square functions coincide in continuous time (see for example [DM82, Remark VII.43(b)] or [RW87, VI.34]). Since  $\mu$  is the invariant measure, we have for any  $0 \leq t \leq K$ ,

$$\|f(X_t)\|_p^p = \int |f(X_t)|^p d\mathbb{P} = \int \mathbb{E}_x |f(X_t)|^p \mu(dx) = \int T_t |f|^p(x) \mu(dx) = \int |f|^p d\mu.$$

It follows that  $\|(T_K f)(X_0)\|_p = \|T_K f\|_p \rightarrow 0$  as  $K \rightarrow \infty$ . By the triangle inequality,

$$\|f\|_p = \|f(X_K)\|_p \leq \|f(X_K) - T_K f(X_0)\|_p + \|T_K f(X_0)\|_p.$$

The rest of the proof is the same as that of (3.12). □

Our first example is very simple, but it clarifies that  $\Gamma_2$ -criterion no longer implies LSI in the general non-diffusion setting. Let us recall the following generalized Schwartz inequality, which is called Choi's inequality; see [Cho74, Corollary 2.8].

**Lemma 3.14.** *Let  $\phi : \mathcal{M} \rightarrow \mathcal{N}$  be a contractive completely positive map between von Neumann algebras. Then  $[\phi(x_i^* x_j)] \geq [\phi(x_i^*) \phi(x_j)]$  for any  $n \in \mathbb{N}$  and any  $x_1, \dots, x_n \in \mathcal{M}$ .*

*Proof.* Since  $\phi$  is complete positive,  $\phi \otimes I_n$  is 2-positive, here  $I_n$  is the identity matrix. Let

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By [Pau02, Exercise 3.4],  $[\phi \otimes I_n(X)]^* [\phi \otimes I_n(X)] \leq \phi \otimes I_n(X^* X)$ . The proof is complete. □

**Example 3.15** (Conditional expectation). Let  $E : \mathcal{M} \rightarrow \mathcal{N}$  be the conditional expectation and  $A = I - E$ . For  $x, y \in \mathcal{M}$ , a calculation gives

$$2\Gamma(x, y) = x^* y - E(x^*) y - x^* E(y) + E(x^* y).$$

By Lemma 3.14,  $2[\Gamma(x_i, x_j)] \geq [(x_i - E x_i)^* (x_j - E x_j)] \geq 0$  for  $x_1, \dots, x_n \in \mathcal{M}$ . We deduce from Theorem 3.11 that  $A$  generates a completely positive semigroup  $T_t = e^{-tA}$  acting on  $\mathcal{M}$ . It is easy to check  $T_t$  is a standard nc-diffusion semigroup. Let  $\Gamma, \Gamma_2$  be the gradient forms associated to  $T_t$ .

**Proposition 3.16.**  $\Gamma_2 \geq \frac{1}{2}\Gamma$  in  $\mathcal{M}$ .

*Proof.* Note that  $AE = EA = 0$ . We find

$$4\Gamma_2(x, y) = x^*y - E(x^*)y - x^*E(y) - 2E(x^*)E(y) + 3E(x^*y).$$

Hence  $(\Gamma_2 - \frac{1}{2}\Gamma)(x, y) = \frac{1}{2}(E(x^*y) - E(x^*)E(y))$ . Since  $E$  is contractive completely positive, it follows from Lemma 3.14 that  $\Gamma_2 - \frac{1}{2}\Gamma$  is a positive form.  $\square$

The log-Sobolev inequality fails, however. Indeed, the LSI reads as follows in this case: for  $x \geq 0$ ,

$$\tau(x^2 \ln(|x|/\|x\|_2)) \leq C \left( \|x\|_2^2 - \frac{1}{2}[\tau(E(x^*)x) + \tau(x^*E(x))] \right).$$

It is easy to see this is not true. Indeed, let us consider the Lebesgue probability space  $([0, 1], dt)$  with  $E$  being the expectation, i.e.  $E(x) = \int_{[0, 1]} x(t)dt = \tau(x)$ . Set  $x_n(t) = \sqrt{n}1_{[0, 1/n]}(t)$ . Then  $\|x_n\|_2 = 1$  and the left-hand side is  $\tau(x_n^2 \ln x_n) = \frac{1}{2} \ln n$ . However, the right-hand side is less than a constant  $C > 0$ , which is impossible for large  $n$ .

### 3.4 $L_p$ Poincaré inequalities for 1-cocycles on discrete groups

Consider the semigroup  $T_t$  acting on  $\mathcal{L}(G)$  given by  $T_t\lambda(g) = e^{-t\psi(g)}\lambda(g)$ . As proved in Section 2.5,  $(T_t)$  is a noncommutative diffusion semigroup in the sense of Section 2.4, i.e.,  $\Gamma^\psi(x, x) \in L_1(\mathcal{L}(G))$  for all  $x \in \text{Dom}(A^{1/2})$  where  $A$  is the generator of  $T_t$ . According to Theorem 2.3,  $T_t$  admits a Markov dilation with almost uniformly continuous path. In fact, we can write down the dilation explicitly in this setting. Following the notation of Section 2.6.2, we define

$$\pi_t : \mathcal{L}(G) \rightarrow L_\infty(\Omega, \mathcal{F}_t) \rtimes G, \quad \pi_t(\lambda(g)) = e^{i\beta_t(b_\psi(g))(\omega)} \rtimes \lambda(g),$$

and the Markov property can be checked directly

$$E_s(\pi_t\lambda(g)) = \pi_s(T_{t-s}\lambda(g)),$$

for  $s < t$  and  $g \in G$ . General elements in the above formula can be obtained by linearity and density. It follows that

$$m_t(x) = \pi_t(x) - \pi_0(x) + \int_0^t \pi_s(Ax)ds \tag{3.13}$$

is a martingale with almost uniformly continuous path for  $x \in \mathcal{L}(G)$ . We will need the reversed martingale.

To this end, let us fix a large constant  $L > 0$ , and define

$$v_t(x) = \pi_t T_{L-t} x$$

for  $t < L$ . It is easy to check that  $(v_t)_{0 \leq t \leq L}$  is a martingale.

We follow the notation in Section 2.5. For  $\xi \in H_\psi$  with finitely many nonzero coordinates, we write  $\beta_t(\xi) = \sum_{k=1}^{\infty} \xi_k B_t^k$ , where  $B_t^k$ 's are independent Brownian motion with generator  $d^2/dx^2$ , and can be given by  $B_t^k(\omega) = \sqrt{2}\omega_t^k$ . By Ito's formula,

$$e^{i\beta_t(\xi)} = 1 + i \sum_k \xi_k \int_0^t e^{i\beta_s(\xi)} dB_s^k - \|\xi\|^2 \int_0^t e^{i\beta_s(\xi)} ds.$$

It follows that

$$\pi_t(\lambda(g)) = \lambda(g) + i \sum_k b_\psi(g)_k \left( \int_0^t e^{i\beta_s(b_\psi(g))} dB_s^k \right) \rtimes \lambda(g) - \int_0^t \pi_s(A(\lambda(g))) ds, \quad (3.14)$$

where  $b_\psi(g)_k$  is the  $k$ -th coordinate of  $b_\psi(g)$ . Combining (3.13) and (3.14), we have

$$m_t(\lambda(g)) = i \sum_k b_\psi(g)_k \left( \int_0^t e^{i\beta_s(b_\psi(g))} dB_s^k \right) \rtimes \lambda(g).$$

Note that  $v_t(\lambda(g)) = e^{-(L-t)\psi(g)} e^{i\beta_t(b_\psi(g))} \lambda(g)$ . By Ito's formula, we have

$$v_t(\lambda(g)) = v_0(\lambda(g)) + i \sum_k b_\psi(g)_k \left( \int_0^t e^{-(L-s)\psi(g)} e^{i\beta_s(b_\psi(g))} dB_s^k \right) \rtimes \lambda(g).$$

It follows that

$$\begin{aligned} \pi_L(\lambda(g)) - \pi_0(T_L \lambda(g)) &= \int_0^L dv_s(\lambda(g)) \\ &= i \sum_k b_\psi(g)_k \left( \int_0^L e^{-(L-s)\psi(g)} e^{i\beta_s(b_\psi(g))} dB_s^k \right) \rtimes \lambda(g). \end{aligned}$$

Let  $x = \sum_{g \in G} x_g \lambda(g) \in \mathbb{C}G$  be a finite sum. Then

$$\pi_L(x) - T_L(x) = i \sum_{g \in G, k \in \mathbb{N}} x_g b_\psi(g)_k \left( \int_0^L e^{-(L-s)\psi(g)} e^{i\beta_s(b_\psi(g))} dB_s^k \right) \rtimes \lambda(g). \quad (3.15)$$

We consider the discretized stochastic integral (assuming  $n = L$ ), or martingale transform

$$M_n(x) = i \sum_{g \in G, j \in \mathbb{N}} \sum_{k=0}^{n-1} x_g b_\psi(g)_j e^{-(L-t_k)\psi(g)} [e^{i\beta_{t_k}(b_\psi(g))} dB_{t_k}^j] \rtimes \lambda(g), \quad (3.16)$$

where  $dB_{t_k}^j = B_{t_{k+1}}^j - B_{t_k}^j$ . It is well known that this martingale converges to the stochastic integral in  $L_p$  for  $2 \leq p < \infty$ . We need a precise Burkholder inequality for this (noncommutative) martingale in order to derive the subgaussian property. As indicated at the beginning of this chapter, however, the upper bounds in known inequalities are not good enough for our purpose. Our approach here relies on the decoupling technique thanks to the special structure in the martingale transform.

Let us consider the discrete time martingale  $x_n \in L_\infty(\Omega, \mathcal{F}_n) \rtimes G$  given by

$$x_n = \sum_{k=0}^{n-1} \sum_{g \in G, j \in \mathbb{N}} [f_g^j(\beta_k(g)) dB_{k+1}^j] \rtimes \lambda(g) \quad (3.17)$$

where  $f_g^j$  is a continuous function, for any  $j \in \mathbb{N}$ ,  $(B_k^j)_k$  is a martingale with independent martingale differences  $dB_{k+1}^j = B_{k+1}^j - B_k^j$ . In what follows we will simply write  $\sum_{g,j}$  instead of  $\sum_{g \in G, j \in \mathbb{N}}$  and this always means a finite sum.

**Lemma 3.17** (Decoupling). *Suppose  $\beta_k(g)$  is measurable with respect to  $\sigma(B_m^j, m \leq k, j \in \mathbb{N})$  for  $g \in G$  and  $k = 0, \dots, n-1$  and  $(\tilde{B}_k^j)$  an independent copy of  $(B_k^j)$ . Then for  $2 \leq p < \infty$ ,*

$$\left\| \sum_{k=0}^{n-1} \sum_{g, \ell} [f_g^\ell(\beta_k(g)) dB_{k+1}^\ell] \rtimes \lambda(g) \right\|_p^p \leq 4^p \left\| \sum_{k=0}^{n-1} \sum_{g, \ell} [f_g^\ell(\beta_k(g)) d\tilde{B}_{k+1}^\ell] \rtimes \lambda(g) \right\|_p^p.$$

*Proof.* To shorten the notation, we simply write  $\beta_k$  for  $\beta_k(g)$ . Consider independent random selectors  $\delta_k, k = 0, \dots, n$  with  $\mathbb{E}(\delta_k) = 1/2$ . Define  $\Delta = \{j \in \{0, \dots, n\} : \delta_j = 1\}$ . Then  $\mathbb{E}(\delta_j(1 - \delta_k)) = 1/4$  for  $j \neq k$ . The left-hand side is

$$\begin{aligned} A &:= \left\| \sum_{g, \ell} \sum_{j, k=0}^n 1_{\{j=k+1\}} [f_g^\ell(\beta_k) dB_j^\ell] \rtimes \lambda(g) \right\|_p^p \\ &= 4^p \left\| \sum_{g, \ell} \sum_{j, k=0}^n 1_{\{j=k+1\}} \mathbb{E}_\delta \delta_k (1 - \delta_j) [f_g^\ell(\beta_k) dB_j^\ell] \rtimes \lambda(g) \right\|_p^p. \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} A &\leq 4^p \mathbb{E}_\delta \left\| \sum_{g,\ell} \sum_{j,k=0}^n 1_{\{j=k+1\}} \delta_k (1 - \delta_j) [f_g^\ell(\beta_k) dB_j^\ell] \rtimes \lambda(g) \right\|_p^p \\ &\leq 4^p \mathbb{E}_\delta \left\| \sum_{g,\ell} \sum_{k \in \Delta, j \notin \Delta} 1_{\{j=k+1\}} [f_g^\ell(\beta_k) dB_j^\ell] \rtimes \lambda(g) \right\|_p^p. \end{aligned}$$

Since  $\beta_k$  and  $dB_{k+1}^\ell$  are independent for all  $\ell$ , and taking expectation of  $\beta_k$ 's commutes with the group action, we have

$$A \leq 4^p \mathbb{E}_\delta \left\| \sum_{g,\ell} \sum_{k \in \Delta, j \notin \Delta} 1_{\{j=k+1\}} [f_g^\ell(\beta_k) d\tilde{B}_j^\ell] \rtimes \lambda(g) \right\|_p^p$$

for any independent copy  $(\tilde{B}_k^j)$  of  $(B_k^j)$ . We may and do fix a realization of  $\delta$  and thus fix a partition  $\Delta_0 \sqcup \Delta_0^c = \{0, \dots, n\}$  so that

$$A \leq 4^p \left\| \sum_{g,\ell} \sum_{k \in \Delta_0, k+1 \notin \Delta_0} [f_g^\ell(\beta_k) d\tilde{B}_{k+1}^\ell] \rtimes \lambda(g) \right\|_p^p. \quad (3.18)$$

Now the von Neumann algebra has been enlarged to  $(L_\infty(\Omega, \mathcal{F}) \overline{\otimes} L_\infty(\tilde{\Omega}, \tilde{\mathcal{F}})) \rtimes G$  by diagonal extension of the group action. Here  $(\tilde{\Omega}, \tilde{\mathcal{F}})$  is generated by  $(\tilde{B}_k^\ell)$ . Let

$$id \otimes E_{\widetilde{\Delta_0^c}} : L_\infty(\Omega, \mathcal{F}) \overline{\otimes} L_\infty(\tilde{\Omega}, \tilde{\mathcal{F}}) \rightarrow L_\infty(\Omega, \mathcal{F}) \overline{\otimes} L_\infty(\sigma(\tilde{B}_{k+1}^\ell : \ell \in \mathbb{N}, k+1 \in \Delta_0^c))$$

denote the conditional expectation. Note that  $d\tilde{B}_{k+1}^\ell$ 's are mean zero. Then we may rewrite (3.18) as

$$A \leq 4^p \left\| \left( id \otimes E_{\widetilde{\Delta_0^c}} \left( \sum_{k \in \Delta_0} \sum_{k+1=1}^n \sum_{g,\ell} [f_g^\ell(\beta_k) d\tilde{B}_{k+1}^\ell] \right) \right) \rtimes \lambda(g) \right\|_p^p.$$

Observing that  $k \in \Delta_0$  if and only if  $k+1 \in \Delta_0 + 1$ , we have

$$A \leq 4^p \left\| id \otimes E_{\widetilde{\Delta_0+1}} \left( id \otimes E_{\widetilde{\Delta_0^c}} \left( \sum_{k=0}^{n-1} \sum_{g,\ell} [f_g^\ell(\beta_k) d\tilde{B}_{k+1}^\ell] \right) \right) \rtimes \lambda(g) \right\|_p^p,$$

where  $id \otimes E_{\widetilde{\Delta_0+1}}$  is defined similarly to  $id \otimes E_{\widetilde{\Delta_0^c}}$  as above. Since conditional expectations extend to contractions on  $L_p$ , the proof is complete.  $\square$

*Remark 3.18.* The general decoupling argument is a very powerful tool in various applications. In fact, a more general version of Lemma 3.17 holds. Namely, we can remove the condition that  $dB_{k+1}^j$ 's are martingale differences, only require them to be independent from  $\beta_k(g)$ . The proof follows the general decoupling

technique developed for  $U$ -statistics due to de la Peña [dlP92]; see also the proof of [dlPG99, Theorem 3.1.1]. We refer the interested reader to the monograph [dlPG99] for an extensive discussion of decoupling methods. We keep the current version for simplicity.

Let us denote by  $\tilde{x}_n$  the decoupled version of  $x_n$ , i.e.,

$$\tilde{x}_n = \sum_{k=0}^{n-1} \sum_{g,j} [f_g^j(\beta_k(g)) d\tilde{B}_{k+1}^j] \rtimes \lambda(g).$$

Consider the von Neumann subalgebra  $\mathcal{N} = L_\infty(\Omega, \mathcal{F}) \rtimes G$ . By the noncommutative Rosenthal inequality proved in [JZ13b], we have for  $2 \leq p < \infty$ ,

$$\|\tilde{x}_n\|_p \leq C \max \left\{ \sqrt{p} \left\| \left( \sum_{j=1}^n E_{\mathcal{N}}(d\tilde{x}_j^* d\tilde{x}_j + d\tilde{x}_j d\tilde{x}_j^*) \right)^{1/2} \right\|_p, p \left( \sum_{j=1}^n \|d\tilde{x}_j\|_p^p \right)^{1/p} \right\}$$

where  $d\tilde{x}_j = \sum_{g,\ell} [f_g^\ell(\beta_{j-1}(g)) d\tilde{B}_j^\ell] \rtimes \lambda(g)$  for  $j = 1, \dots, n$ . Now it is crucial to observe that

$$E_{\mathcal{N}}(d\tilde{x}_j^* d\tilde{x}_j + d\tilde{x}_j d\tilde{x}_j^*) = E_{j-1}(dx_j^* dx_j + dx_j dx_j^*),$$

and  $\|d\tilde{x}_j\|_p = \|dx_j\|_p$ , where  $dx_j = x_j - x_{j-1}$  is the martingale difference. We have then for  $2 \leq p < \infty$ ,

$$\|x_n\|_p \leq C \max \left\{ \sqrt{p} \left\| \left( \sum_{k=1}^n E_{k-1}(dx_k^* dx_k + dx_k dx_k^*) \right)^{1/2} \right\|_p, p \left( \sum_{k=1}^{n-1} \|dx_k\|_p^p \right)^{1/p} \right\}.$$

In other words,

$$\|x_n\|_p \leq C \max \{ \sqrt{p} \|x_n\|_{h_p^c}, \sqrt{p} \|x_n\|_{h_p^r}, p \|x_n\|_{h_p^d} \}. \quad (3.19)$$

Now apply (3.19) to the discretized martingale (3.16). By the facts on Hardy spaces presented in Section 2.3,

$$\|v(x)\|_{h_p^c([0,L])} = \|\langle v(x), v(x) \rangle_L^{1/2}\|_p = \lim_{n \rightarrow \infty} \|M_n(x)\|_{h_p^c}.$$

Since  $v(x)$  is driven by Brownian motion, it has continuous path and is of vanishing variation, i.e.,

$$\|v(x)\|_{h_p^d([0,L])} = \lim_{n \rightarrow \infty} \|M_n(x)\|_{h_p^d} = 0.$$

Combining things together, we have shown the following result.

**Lemma 3.19** (Burkholder–Davis–Gundy inequality). *Let  $v_t(x) = \pi_t T_{L-t}(x)$  be the martingale associated*

to  $x \in \mathbb{C}G \subset \mathcal{L}(G)$  as before, then for  $2 \leq p < \infty$ ,

$$\|v_L(x) - v_0(x)\|_p \leq C\sqrt{p} \max\{\|v(x)\|_{h_p^c([0,L])}, \|v(x)\|_{h_p^r([0,L])}\}.$$

Let  $\tilde{\pi}_t = \pi_{L-t}$ ,  $\tilde{\mathcal{F}}_t = \mathcal{F}_{L-t}$  and  $E_t = E_{L-t}$  for  $t < L$ . Define  $n_t(x) = \pi_{L-t}(T_t x)$ . It is easy to check that  $(\tilde{\pi}_t, \tilde{\mathcal{F}}_t)$  is a reversed Markov dilation, i.e., for  $s < t$ ,

$$E_{[t]}\tilde{\pi}_s x = E_{L-t}\pi_{L-s}(x) = \pi_{L-t}T_{t-s}x = \tilde{\pi}_t T_{t-s}x.$$

It follows that  $(n_t(x), E_t)$  is a reversed martingale and  $v_L(x) - v_0(x) = n_0(x) - n_L(x)$ .

**Theorem 3.20.** *Let  $G$  be a countable discrete group with cn-length function  $\psi$  and  $\mathcal{L}(G)$  its group von Neumann algebra. Suppose  $f \in \mathcal{L}(G)$  satisfies  $\Gamma_2^\psi(f, f) \geq \alpha \Gamma^\psi(f, f)$  for some  $\alpha > 0$ . Then for  $2 \leq p < \infty$ ,*

$$\|f - E_{\text{Fix}}f\|_p \leq C\sqrt{p/\alpha} \max\{\|\Gamma^\psi(f, f)^{1/2}\|_p, \|\Gamma^\psi(f^*, f^*)^{1/2}\|_p\}. \quad (3.20)$$

*Proof.* The proof follows the same idea as for Theorem 3.8. We give a sketch for completeness. By approximation, we may assume  $x = \sum_{g \in G} x_g \lambda(g)$  is a finite linear combination. By replacing  $x$  by  $x - E_{\text{Fix}}x$ , we may assume  $E_{\text{Fix}}x = 0$ . It follows that  $\lim_{L \rightarrow \infty} \|T_L x\|_p = 0$ . The Bakry–Emery condition implies uniform boundedness for  $\Gamma(T_r x, T_r x)$  for  $r \geq 0$  in  $L_p(\mathcal{L}(G))$ . By Lemma 3.7 and Lemma 3.19, we have

$$\begin{aligned} \|\pi_L x\|_p - \|\pi_0 T_L x\|_p &\leq \|\pi_L x - \pi_0 T_L x\| = \|n_0(x) - n_L(x)\|_p \\ &\leq C\sqrt{p} \max\left\{\left\|\int_0^L \tilde{\pi}_r(\Gamma(T_r x, T_r x))dr\right\|_{p/2}^{1/2}, \left\|\int_0^L \tilde{\pi}_r(\Gamma(T_r x^*, T_r x^*))dr\right\|_{p/2}^{1/2}\right\}. \end{aligned}$$

Then the Bakry–Emery condition gives  $\Gamma(T_t x, T_t x) \leq e^{-2\alpha t} T_t \Gamma(x, x)$ ; see Lemma 3.10. Since  $\tilde{\pi}_t$  and  $T_t$  are contractions, we have

$$\left\|\int_0^L \tilde{\pi}_r(\Gamma(T_r x, T_r x))dr\right\|_{p/2}^{1/2} \leq \left(\frac{1}{2\alpha} - \frac{1}{2\alpha e^{2\alpha L}}\right)^{1/2} \|\Gamma(x, x)\|_{p/2}^{1/2}.$$

Similar inequality holds for  $x^*$ . Since  $\pi_t$  is trace preserving  $*$ -homomorphism,  $\|\pi_t x\|_p = \|x\|_p$  for all  $t \geq 0$  and  $x \in \mathcal{L}(G)$ . We also have  $\lim_{L \rightarrow \infty} T_L x = E_{\text{Fix}}x = 0$  in  $L_p(\mathcal{L}(G))$ . Now sending  $L \rightarrow \infty$  completes the proof.  $\square$

Notice that the group algebra  $\mathbb{C}G$  is weakly dense in  $\mathcal{L}(G)$ , and  $\mathbb{C}G$  is also dense in  $\text{Dom}(A^{1/2})$  in the graph norm of  $A^{1/2}$  by Lemma 2.6. Repeating the proof of Corollary 3.12 gives the following result.



**Corollary 3.21.** *Suppose the  $\Gamma_2$ -criterion holds for the  $cn$ -length function  $\psi$  on a group  $G$ . Then we have the  $L_p$  Poincaré inequalities (3.20) for all  $f \in \mathcal{L}(G)$  and  $2 \leq p < \infty$  whenever the right-hand side of (3.20) is finite. Therefore, the 1-cocycle  $b_\psi$  is subgaussian.*

The strategy we used here can be applied to other settings as long as the martingale obtained from the Markov dilation can be approximated by a martingale transform like (3.16). Let us consider an application to the Lindblad operator in quantum dynamical system; see [Lin76, Par92]. Let  $(a_j)_{j=1}^m \subset M_n$  be a family of mutually commuting Hermitian matrices, where  $M_n$  is the matrix algebra of dimension  $n^2$ . Define  $A$  acting on  $M_n$  by

$$A(x) = \sum_{j=1}^m x a_j^2 + a_j^2 x - 2a_j x a_j.$$

Consider the semigroup  $T_t = e^{-tA}$  acting on  $M_n$  generated by  $A$ . Then we have the following result.

**Theorem 3.22.** *Suppose there exists  $\alpha > 0$  such that  $\Gamma_2^A(x, x) \geq \alpha \Gamma^A(x, x)$  for  $x \in M_n$ . Then for  $2 \leq p < \infty$ ,*

$$\|x - E_{\text{Fix}} x\|_p \leq C \sqrt{p/\alpha} \max\{\|\Gamma^A(x, x)^{1/2}\|_p, \|\Gamma^A(x^*, x^*)^{1/2}\|_p\}. \quad (3.21)$$

*Sketch of proof.* The proof follows that of Theorem 3.20 with appropriate modification. We sketch the main steps here. Let  $u_t = \exp(i \sum_{j=1}^m B_t^j a_j)$ , where  $B_t^j$ 's are independent Brownian motions with generator  $d^2/dx^2$ . Since we assumed  $a_j$ 's are mutually commuting, by Ito's formula,  $u_t$  satisfies the following stochastic differential equation

$$du_t = - \sum_{k=1}^m a_k^2 u_t dt + i \sum_{k=1}^m u_t a_k dB_t^k.$$

Let  $\pi_t x = u_t^* x u_t$  for  $x \in M_n$ . Then it was shown in [JRS14] that  $\pi_t$  is a Markov dilation for  $T_t$ , i.e.,  $E_s \pi_t x = \pi_s T_{t-s} x$  for  $s < t$ .

Fix  $L > 0$ . Let  $v_t(x) = \pi_t T_{L-t} x$ . It is a martingale for  $0 < t < L$ . By Ito's formula,

$$\pi_L x - \pi_0 T_L x = i \sum_k \int_0^L u_s^* (-a_k T_{L-t} x + T_{L-t} x a_k) u_s dB_s^k.$$

Then we can discretize the stochastic integral and apply a decoupling argument to find the BDG inequality for  $v_t(x)$ , as what we did in the proof of Theorem 3.20. Note that  $n_t(x) := \pi_{L-t}(T_t x)$  is a reversed martingale and  $n_0(x) - n_L(x) = \pi_L x - T_L x$ . Combining Lemma 3.7 with the  $\Gamma_2$ -criterion, we arrive at the assertion.  $\square$

# Chapter 4

## Pisier's method

Our goal in this chapter is to prove the subgaussian Poincaré inequalities for certain group measure spaces following Pisier's method.

### 4.1 $L_p$ Poincaré inequalities for Gaussian measures

Let  $-L = \Delta - x \cdot \nabla$  be the generator of Ornstein–Uhlenbeck semigroup  $P_t$  in  $\mathbb{R}^d$  for  $d < \infty$ . Let  $\gamma_d$  denote the standard Gaussian measure on  $\mathbb{R}^d$ . Then by e.g. [Pis88] the Mehler formula for  $\cos^L \theta := P_{-\ln \cos \theta}$  holds: for all  $f \in L_2(\gamma_d)$  and  $0 \leq \theta \leq \pi/2$

$$(\cos^L \theta f)(x) = \int_{\mathbb{R}^d} f(x \cos \theta + y \sin \theta) \gamma_d(dy). \quad (4.1)$$

Following [Pis86, ELP08], we have the Poincaré type inequality for Gaussian measures, which is a classical result due to Pisier [Pis86, Corollary 2.4]; see [AW13] for another proof based on LSI. We present the proof here because it is our guideline for the group measure space setting. We write  $|\nabla f| = (\sum_{i=1}^d (\frac{\partial f}{\partial x_i})^2)^{1/2}$ .

**Proposition 4.1.** *Let  $p \geq 2$  and  $\phi \in L_1([0, \pi/2])$ . Then for all  $f \in L_\infty(\mathbb{R}^d, \gamma_d)$ ,*

$$\left\| \int_0^{\pi/2} \phi(\theta) \frac{\partial}{\partial \theta} \cos^L \theta(f) d\theta \right\|_{L_p(\gamma_d)} \leq C \sqrt{p} \|\phi\|_{L_1([0, \pi/2])} \|\nabla f\|_{L_p(\gamma_d)}.$$

*In particular,*

$$\|f - \int f d\gamma_d\|_{L_p(\gamma_d)} \leq C \sqrt{p} \|\nabla f\|_{L_p(\gamma_d)}.$$

*Proof.* By approximation, we may assume that  $f$  is a bounded  $C^1$  function with  $|\nabla f|$  bounded by some polynomial so that we can differentiate under integral in the following. Since  $\lim_{t \rightarrow \infty} P_t f = \int f d\gamma_d$ , by

(4.1), we have

$$\begin{aligned} \int_0^{\pi/2} \phi(\theta) \frac{\partial \cos^L \theta(f)}{\partial \theta}(x) d\theta &= \int_0^{\pi/2} \phi(\theta) \frac{\partial}{\partial \theta} \int f(x \cos \theta + y \sin \theta) \gamma_d(dy) d\theta \\ &= \int_0^{\pi/2} \int_{\mathbb{R}^d} \phi(\theta) \mathcal{R}_\theta(\nabla f(x) \cdot y) \gamma_d(dy) d\theta, \end{aligned}$$

where  $\mathcal{R}_\theta$  is a measure preserving automorphism of  $(\mathbb{R}^d \times \mathbb{R}^d, \gamma_d \times \gamma_d)$  given by  $(\mathcal{R}_\theta F)(x, y) = F(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$ . By Minkowski's integral inequality and Hölder's inequality,

$$\begin{aligned} &\left\| \int_0^{\pi/2} \phi(\theta) \frac{\partial}{\partial \theta} \cos^L \theta(f) d\theta \right\|_{L_p(\gamma_d)} \\ &\leq \int_0^{\pi/2} |\phi(\theta)| \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\mathcal{R}_\theta(\nabla f(x) \cdot y)|^p \gamma_d(dx) \right)^{1/p} \gamma_d(dy) d\theta \\ &\leq \int_0^{\pi/2} |\phi(\theta)| \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\mathcal{R}_\theta(\nabla f(x) \cdot y)|^p \gamma_d(dx) \gamma_d(dy) \right)^{1/p} d\theta \\ &\leq \|\phi\|_{L_1([0, \pi/2])} \left\| \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x) y_i \right\|_{L_p(\gamma_d \times \gamma_d)}. \end{aligned}$$

The first assertion follows from Khintchine's inequality. Taking  $\phi(\theta) = 1_{[0, \pi/2]}(\theta)$  gives the second one as in [ELP08].  $\square$

We remark that by approximation the above result also holds for the standard Gaussian measure on  $\mathbb{R}^\infty$ , because we still have the Mehler formula in this setting; see for example [Nua06].

## 4.2 $L_p$ Poincaré inequalities for group (Gaussian) measure spaces

The idea of our proof for the Poincaré type inequalities goes back to Pisier [Pis88] where he deduced a magic formula to connect the Riesz transform and the Gaussian measure space. This strategy was further developed by Lust-Piquard in various situations. In particular, in [ELP08] Efraim and Lust-Piquard proved the Poincaré type inequalities for Walsh systems and CAR algebras following Lust-Piquard's earlier works, which motivates our proof.

**Lemma 4.2.** *Let  $\hat{\alpha} : G \rightarrow \text{Aut}(L_\infty(\mathbb{R}^d, \gamma_d))$  be the measure preserving action given by (2.2). Suppose  $f \in L_\infty(\mathbb{R}^d, \gamma_d)$  is differentiable and depends on finitely many coordinates if  $d = \infty$ . Then for  $g \in G$ ,*

$$\frac{\partial \hat{\alpha}_g(f)}{\partial x_i}(x) = \langle (\nabla f)(\alpha_{g^{-1}}(x)), \alpha_{g^{-1}}(e_i) \rangle,$$

where  $(e_i)$  is the standard basis of  $\mathbb{R}^d$ . Therefore,  $(\nabla \hat{\alpha}_g(f))(x) = \alpha_g[(\nabla f)(\alpha_{g^{-1}}(x))]$  and

$$\langle (\nabla \hat{\alpha}_g(f))(x), y \rangle = \langle (\nabla f)(\alpha_{g^{-1}}(x)), \alpha_{g^{-1}}(y) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\ell_2(d)$ .

*Proof.* This is just the chain rule. Here is the direct calculation.

$$\begin{aligned} \frac{\partial}{\partial x_i} \hat{\alpha}_g(f)(x) &= \lim_{t \rightarrow 0} \frac{\hat{\alpha}_g(f)(x + te_i) - \hat{\alpha}_g(f)(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f(\alpha(g^{-1})(x) + t\alpha(g^{-1})(e_i)) - f(\alpha(g^{-1})(x))}{t} \\ &= \langle (\nabla f)(\alpha(g^{-1})(x)), \alpha(g^{-1})(e_i) \rangle = \langle \alpha(g)[(\nabla f)(\alpha(g^{-1})(x))], e_i \rangle. \end{aligned}$$

This gives the gradient of  $\hat{\alpha}_g(f)$  at  $x$ . □

We follow the notation in Section 2.6.1. Let  $f(x, y)$  be a measurable function on  $(\mathbb{R}^d \times \mathbb{R}^d, \gamma_d \times \gamma_d)$ . Recall that  $\mathcal{R}_\theta$  is the measure preserving automorphism on  $L_\infty(\mathbb{R}^d \times \mathbb{R}^d, \gamma_d \times \gamma_d)$  given by  $(\mathcal{R}_\theta f)(x, y) = f(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$ , and that  $T_t = P_t \rtimes id_G$  is the semigroup acting on  $L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G$ , which is a natural extension of the Ornstein–Uhlenbeck semigroup  $P_t$  on  $L_\infty(\mathbb{R}^d, \gamma_d)$ .

**Lemma 4.3.** *Suppose that  $\xi$  is a bounded  $C^1$  function on  $\mathbb{R}^d$  with  $|\nabla \xi|$  bounded by some polynomial and  $\xi$  depends on only finitely many coordinates. Then for  $g \in G$  and  $0 < \theta < \pi/2$ ,*

$$\frac{\partial}{\partial \theta} T_{-\ln \cos \theta}(\xi \rtimes \lambda(g))(x) = \sum_h \int \mathcal{R}_\theta[\langle (\nabla \xi)(\alpha_h(x)), \alpha_h(y) \rangle] \gamma_d(dy) \otimes e_{h, g^{-1}h}.$$

*Proof.* Since the Mehler formula (4.1) can be extended naturally to  $T_t$ , we have

$$\begin{aligned} T_{-\ln \cos \theta}(\xi \rtimes \lambda(g))(x) &= \sum_h P_{-\ln \cos \theta}((\hat{\alpha}_{h^{-1}}\xi))(x) \otimes e_{h, g^{-1}h} \\ &= \sum_h \int (\hat{\alpha}_{h^{-1}}\xi)(x \cos \theta + y \sin \theta) \gamma_d(dy) \otimes e_{h, g^{-1}h}. \end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned} \frac{\partial}{\partial \theta} (\hat{\alpha}_{h^{-1}}\xi)(x \cos \theta + y \sin \theta) &= \langle (\nabla \hat{\alpha}_{h^{-1}}(\xi))(x \cos \theta + y \sin \theta), -x \sin \theta + y \cos \theta \rangle \\ &= \langle (\nabla \xi)(\alpha_h(x \cos \theta + y \sin \theta)), \alpha_h(-x \sin \theta + y \cos \theta) \rangle \\ &= \mathcal{R}_\theta(\langle (\nabla \xi)(\alpha_h(x)), \alpha_h(y) \rangle). \end{aligned}$$

We have assumed  $\xi$  to be a nice function so that we can differentiate entrywise under integral and find

$$\frac{\partial}{\partial \theta} T_{-\ln \cos \theta}(\xi \rtimes \lambda(g))(x) = \sum_h \int \mathcal{R}_\theta[\langle (\nabla \xi)(\alpha_h(x)), \alpha_h(y) \rangle] \gamma_d(dy) \otimes e_{h, g^{-1}h}. \quad \square$$

Recall that the fixed point algebra of  $T_t$  is  $\mathcal{L}(G)$ . Let  $L$  denote the generator of  $P_t$  and  $L \rtimes I$  the generator of  $T_t$ .

**Theorem 4.4.** *Let  $2 \leq p < \infty$  and  $f \in L_\infty(\mathbb{R}^d, \gamma_d) \rtimes_{\hat{\alpha}} G$  where the action  $\hat{\alpha}$  is the measure preserving action determined by the orthogonal representation  $\alpha$  given by (2.2). Then*

$$\begin{aligned} & \|f - E_{\mathcal{L}(G)} f\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)} \\ & \leq C\sqrt{p} \max\{\|\Gamma_{L \rtimes I}(f, f)^{1/2}\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}, \|\Gamma_{L \rtimes I}(f^*, f^*)^{1/2}\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}\}. \end{aligned} \quad (4.2)$$

*Proof.* We follow the strategy of Proposition 4.1 and take advantage of the techniques developed in [JMP10]. By approximation, we may assume that  $f = \sum_{g \in G} f_g \lambda(g) \in L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G$  for finitely many  $g \in G$  and that  $f_g$ 's satisfy the assumption of Lemma 4.3. Note that  $L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G \subset L_\infty(\mathbb{R}^d, \gamma_d) \overline{\otimes} B(\ell_2(G))$ . Then we have

$$T_{-\ln \cos(\pi/2-\varepsilon)} f - f = \int_0^{\pi/2-\varepsilon} \frac{\partial}{\partial \theta} T_{-\ln \cos \theta} f d\theta.$$

Sending  $\varepsilon \rightarrow 0$ , we have

$$E_{\mathcal{L}(G)} f - f = \int_0^{\pi/2} \frac{\partial}{\partial \theta} T_{-\ln \cos \theta} f d\theta.$$

We can extend the action  $\hat{\alpha} : G \rightarrow \text{Aut}(L_\infty(\mathbb{R}^d, \gamma_d))$  to

$$G \curvearrowright L_\infty(\mathbb{R}^{2d}, \gamma_{2d}) = L_\infty(\mathbb{R}^d, \gamma_d) \overline{\otimes} L_\infty(\mathbb{R}^d, \gamma_d)$$

by diagonal action

$$\hat{\alpha}_g(\xi(\cdot)\eta(\cdot))(x, y) = (\hat{\alpha}_g \xi)(x)(\hat{\alpha}_g \eta)(y).$$

Noticing that  $\hat{\alpha}_g \circ \mathcal{R}_\theta = \mathcal{R}_\theta \circ \hat{\alpha}_g$  for  $g \in G$  and  $\theta \in [0, \pi/2]$ , by Lemma 4.3, we have

$$\begin{aligned}
E_{\mathcal{L}(G)}f - f &= \int_0^{\pi/2} \sum_{g,h} \int \mathcal{R}_\theta[\langle \nabla f_g(\alpha_h(x)), \alpha_h(y) \rangle] \gamma_d(dy) \otimes e_{h,g^{-1}h} d\theta \\
&= \int_0^{\pi/2} \sum_{g,h} \int \hat{\alpha}_{h^{-1}}[\mathcal{R}_\theta(\langle \nabla f_g(x), y \rangle)] \gamma_d(dy) \otimes e_{h,g^{-1}h} d\theta \\
&= \int_0^{\pi/2} E_{L_\infty^x(\mathbb{R}^d, \gamma_d) \rtimes G} \left[ \sum_g \mathcal{R}_\theta[\langle \nabla f_g(x), y \rangle] \rtimes \lambda(g) \right] d\theta \\
&= \int_0^{\pi/2} E_{L_\infty^x(\mathbb{R}^d, \gamma_d) \rtimes G} \left[ \sum_g (\mathcal{R}_\theta \otimes id_{\ell_2(G)})[(\langle \nabla f_g(x), y \rangle) \rtimes \lambda(g)] \right] d\theta. \tag{4.3}
\end{aligned}$$

Here we used the facts that  $\sum_{g,h} \hat{\alpha}_{h^{-1}}[\mathcal{R}_\theta(\langle \nabla f_g(x), y \rangle)] \otimes e_{h,g^{-1}h}$  is in  $L_p(L_\infty(\mathbb{R}^{2d}, \gamma_d) \rtimes G)$  for  $2 \leq p < \infty$  and that the conditional expectation

$$E_{L_\infty^x(\mathbb{R}^d, \gamma_d) \rtimes G} : L_\infty(\mathbb{R}^{2d}, \gamma_{2d}) \rtimes G \rightarrow L_\infty^x(\mathbb{R}^d, \gamma_d) \rtimes G$$

extends to a contraction on  $L_p(L_\infty(\mathbb{R}^{2d}, \gamma_{2d}) \rtimes G)$ . It follows from (2.3) or the chain rule that  $\hat{\alpha}_g \circ L = L \circ \hat{\alpha}_g$ .

By the arithmetic of crossed products as explained in Section 2.2, we have

$$\begin{aligned}
&\Gamma_{L \rtimes I}(f, f) \\
&= \frac{1}{2} \sum_{g,h} [L(\hat{\alpha}_{g^{-1}}(\bar{f}_g))\hat{\alpha}_{g^{-1}}(f_h) + \hat{\alpha}_{g^{-1}}(\bar{f}_g)\hat{\alpha}_{g^{-1}}(Lf_h) - L(\hat{\alpha}_{g^{-1}}(\bar{f}_g)\hat{\alpha}_{g^{-1}}(f_h))]\lambda(g^{-1}h) \\
&= \sum_{g,h} \hat{\alpha}_{g^{-1}}(\langle \nabla f_h, \nabla f_g \rangle) \lambda(g^{-1}h) \\
&= \sum_{g,h} \langle \nabla f_h(\alpha_g(\cdot)), \nabla f_g(\alpha_g(\cdot)) \rangle \rtimes \lambda(g^{-1}h).
\end{aligned}$$

We write  $E_x$  for  $E_{L_\infty^x(\mathbb{R}^d, \gamma_d) \rtimes G}$ . By the definition of  $L_p^c(E)$ , we have

$$\begin{aligned}
&\| \sum_g \langle \nabla f_g(x), y \rangle \rtimes \lambda(g) \|_{L_p^c(E_x)} \\
&= \| \sum_{g,h} \int \hat{\alpha}_{g^{-1}}(\overline{\langle \nabla f_g(x), y \rangle} \langle \nabla f_h(x), y \rangle) \gamma_d(dy) \rtimes \lambda(g^{-1}h) \|_{L_{p/2}(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}^{1/2} \\
&= \| \sum_{g,h} \langle \nabla f_h(\alpha_g(x)), \nabla f_g(\alpha_g(x)) \rangle \rtimes \lambda(g^{-1}h) \|_{L_{p/2}(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}^{1/2} \\
&= \| \Gamma_{L \rtimes I}(f, f) \|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}^{1/2}. \tag{4.4}
\end{aligned}$$

Similarly,

$$\left\| \sum_g \langle \nabla f_g(x), y \rangle \rtimes \lambda(g) \right\|_{L_p^r(E_x)} = \|\Gamma_{L \rtimes I}(f^*, f^*)^{1/2}\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}.$$

It follows from Theorem 2.7, (4.3) and (4.4) that

$$\begin{aligned} & \|f - E_{\mathcal{L}(G)}f\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)} \\ & \leq \frac{\pi}{2} \left\| \sum_g \langle \nabla f_g(x), y \rangle \rtimes \lambda(g) \right\|_{L_p(L_\infty(\mathbb{R}^d \times \mathbb{R}^d, \gamma_d \times \gamma_d) \rtimes G)} \\ & \leq \frac{C\pi}{2} \sqrt{p} \max\{\|\Gamma_{L \rtimes I}(f, f)^{1/2}\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}, \|\Gamma_{L \rtimes I}(f^*, f^*)^{1/2}\|_{L_p(L_\infty(\mathbb{R}^d, \gamma_d) \rtimes G)}\}. \end{aligned}$$

This completes the proof. □

## Chapter 5

# The spectral gap and $L_p$ Poincaré inequalities

In the classical diffusion setting, if the probability measure is non-atomic, it is known that the  $L_2$  Poincaré inequality (a.k.a. the spectral gap inequality) implies the  $L_p$  Poincaré inequalities with constant  $Cp$  for  $2 \leq p < \infty$ ; see for example [Mil09, Proposition 2.5]. In this chapter, we show that such implication still holds in the noncommutative setting under some conditions. A crucial ingredient in the commutative theory is the chain rule for the gradient form, which is not available in general non-diffusion setting. Our approach relies on the derivation property of  $\Gamma$ . Let us recall a result of Junge–Ricard–Shlyakhtenko [JRS14] proved using free dilation theory.

**Theorem 5.1.** *Let  $\Gamma$  be the gradient form associated to a noncommutative diffusion semigroup  $T_t = e^{-tA}$  acting on  $(\mathcal{N}, \tau)$ . Then there exists a trace preserving  $*$ -homomorphism  $\pi$  from  $\mathcal{N}$  into a finite von Neumann algebra  $\mathcal{M}$  and a  $(\pi)$ -derivation  $\delta : \text{Dom}(\delta) \rightarrow L_2(\mathcal{M})$  such that for all  $x, y \in \text{Dom}(A) \cap \mathcal{N}$ ,  $\tau[\Gamma(x, y)] = \tau[\delta(x)^* \delta(y)]$  and for all  $1 < p < \infty$*

$$C^{-1} \|\delta(x)\|_p \leq \max\{\|\Gamma(x, x)^{1/2}\|_p, \|\Gamma(x^*, x^*)^{1/2}\|_p\} \leq C \|\delta(x)\|_p.$$

for some absolute constant  $C$ .

The following technical result is standard.

**Lemma 5.2.** *Let  $1 \leq p < \infty$ . Then*

$$\|\Gamma(x, y)\|_p \leq \|\Gamma(x, x)\|_p^{1/2} \|\Gamma(y, y)\|_p^{1/2}.$$

*Proof.* The case  $p = 1$  was proved in Corollary 3.12. The general case follows from the same argument with the help of Hölder's inequality.  $\square$



**Lemma 5.3.** *Let  $f_1, \dots, f_m, g_1, \dots, g_n \in \text{Dom}(A)$  be self-adjoint elements. Then*

$$\tau[\Gamma(f_1 \cdots f_m, g_1 \cdots g_n)] \leq C \sum_{i=1}^m \sum_{j=1}^n \|\Gamma(f_i, f_i)^{1/2}\|_{m+n} \|\Gamma(g_j, g_j)^{1/2}\|_{m+n} \prod_{k \neq i, \ell \neq j} \|f_k\|_{m+n} \|g_\ell\|_{m+n}.$$

*Proof.* By Theorem 5.1 and the derivation property, we have

$$\tau[\Gamma(f_1 \cdots f_m, g_1 \cdots g_n)] = \sum_{i,j} \tau[(f_1 \cdots f_{i-1} \delta(f_i) f_{i+1} \cdots f_m)^* (g_1 \cdots g_{j-1} \delta(g_j) g_{j+1} \cdots g_n)].$$

Using Hölder's inequality, we obtain

$$\begin{aligned} & \tau[\Gamma(f_1 \cdots f_m, g_1 \cdots g_n)] \\ & \leq \sum_{j,k} \|f_1\|_{m+n} \cdots \|f_{j+1}\|_{m+n} \|\delta(f_j)\|_{m+n} \|f_{j-1}\|_{m+n} \cdots \|f_m\|_{m+n} \\ & \quad \|g_1\|_{m+n} \cdots \|g_{k-1}\|_{m+n} \|\delta(g_k)\|_{m+n} \|g_{k+1}\|_{m+n} \cdots \|g_n\|_{m+n}. \end{aligned}$$

Applying Theorem 5.1 again, we complete the proof.  $\square$

**Theorem 5.4.** *Let  $T_t$  be an ergodic noncommutative diffusion semigroup acting on a diffuse probability space  $(\mathcal{N}, \tau)$ . Suppose the spectral gap inequality holds: for  $f \in \mathcal{N}$ ,*

$$\|f - \tau(f)\|_2 \leq C \max\{\|\Gamma(f, f)^{1/2}\|_2, \|\Gamma(f^*, f^*)^{1/2}\|_2\}.$$

*Then for all even integer  $p \geq 2$  and all  $f \in \mathcal{N}$ ,*

$$\|f - \tau(f)\|_p \leq C' p \max\{\|\Gamma(f, f)^{1/2}\|_p, \|\Gamma(f^*, f^*)^{1/2}\|_p\}.$$

*Proof.* Let  $g \in \mathcal{N}$  be a self-adjoint element and  $p = 2q$  be an even integer. Without loss of generality we may assume  $\tau(g) = 0$ . Since  $\mathcal{N}$  is diffuse, the scalar spectral measure of  $g$  is non-atomic. We can find a function  $\text{sgn}(\cdot) : \text{spec}(g) \rightarrow \{\pm 1\}$  such that  $\tau[\text{sgn}(g)g^{p/2}] = 0$ , where  $\text{spec}(g)$  denotes the spectrum of  $g$ . Let  $f = \text{sgn}(g)g^{p/2}$ . Applying the spectral gap inequality on  $f$ , we have

$$\tau(g^{2q}) \leq C^2 \tau[\Gamma(g^q, g^q)]. \quad (5.1)$$

By Lemma 5.3,

$$\tau[\Gamma(g^q, g^q)] \leq C' q^2 \|g\|_{2q}^{2q-2} \|\Gamma(g, g)^{1/2}\|_{2q}^2$$

Hence,

$$\|g\|_{2q} \leq C' q \|\Gamma(g, g)^{1/2}\|_{2q}. \quad (5.2)$$

For general mean zero element  $f \in \mathcal{N}$ , write  $f = \Re(f) + i\Im(f)$ , where  $\Re(f) = \frac{f+f^*}{2}$  and  $\Im(f) = \frac{f-f^*}{2i}$ . Using the triangle inequality and (5.2), we obtain

$$\|f\|_p \leq \|\Re(f)\|_p + \|\Im(f)\|_p \leq C' q (\|\Gamma(\Re(f), \Re(f))^{1/2}\|_p + \|\Gamma(\Im(f), \Im(f))^{1/2}\|_p).$$

By Lemma 5.2, we find

$$\begin{aligned} \|\Gamma(\Re(f), \Re(f))^{1/2}\|_p &= \|\Gamma(\Re(f), \Re(f))\|_q^{1/2} \\ &= \frac{1}{2} [\|\Gamma(f, f)\|_q + 2\|\Gamma(f, f^*)\|_q + \|\Gamma(f^*, f^*)\|_q]^{1/2} \\ &\leq \frac{1}{2} [\|\Gamma(f, f)\|_q + 2\|\Gamma(f, f)\|_q^{1/2} \|\Gamma(f^*, f^*)\|_q^{1/2} + \|\Gamma(f^*, f^*)\|_q]^{1/2} \\ &\leq \frac{1}{2} [\|\Gamma(f, f)\|_q^{1/2} + \|\Gamma(f^*, f^*)\|_q^{1/2}]. \end{aligned}$$

Similar argument applies to  $\|\Gamma(\Im(f), \Im(f))^{1/2}\|_p$  and the proof is complete.  $\square$

Note that the diffuse and ergodic assumptions are indispensable in the above argument. We provide some results without these assumptions in the following.

**Theorem 5.5.** *Let  $T_t$  be a noncommutative diffusion semigroup acting on a probability space  $(\mathcal{N}, \tau)$  such that*

$$\|E_{\text{Fix}} g\|_2 \leq C_1 \tau(g) \quad (5.3)$$

for all  $g \geq 0$ . Suppose the spectral gap inequality holds: for  $f \in \mathcal{N}$ ,

$$\|f - E_{\text{Fix}}(f)\|_2 \leq C_2 \max\{\|\Gamma(f, f)^{1/2}\|_2, \|\Gamma(f^*, f^*)^{1/2}\|_2\}.$$

Then we have for all  $f \in \mathcal{N}$  and  $k \in \mathbb{N}$ ,

$$\|f - E_{\text{Fix}}(f)\|_{2^k} \leq C_3 2^k \max\{\|\Gamma(f, f)^{1/2}\|_{2^k}, \|\Gamma(f^*, f^*)^{1/2}\|_{2^k}\}.$$

*Proof.* By the same argument as for Theorem 5.4, it suffices to consider the self-adjoint element  $f$ . Since  $\Gamma(f - E_{\text{Fix}}f, f - E_{\text{Fix}}f) = \Gamma(f, f)$ , we may assume  $E_{\text{Fix}}f = 0$ . Note that  $k = 1$  is the spectral gap inequality.

We proceed by induction. Assume

$$\|f\|_{2^k} \leq A_k 2^k \|\Gamma(f, f)^{1/2}\|_{2^k},$$

where  $A_k$  is the best constant. Applying the spectral gap inequality to  $f^{2^k}$  and using the assumption (5.3), we have

$$\|f\|_{2^{k+1}}^{2^{k+1}} \leq C_2^2 \tau[\Gamma(f^{2^k}, f^{2^k})] + C_1^2 \tau(f^{2^k})^2.$$

By Lemma 5.3 and the induction hypothesis,

$$\|f\|_{2^{k+1}}^{2^{k+1}} \leq CC_2^2 2^{2k} \|\Gamma(f, f)^{1/2}\|_{2^{k+1}}^2 \|f\|_{2^{k+1}}^{2^{k+1}-2} + C_1^2 A_k^{2^{k+1}} 2^{k2^{k+1}} \|\Gamma(f, f)^{1/2}\|_{2^k}^{2^{k+1}} =: I + II.$$

Suppose  $I \leq II$ . Since  $\tau(1) = 1$ , we have

$$\|f\|_{2^{k+1}}^{2^{k+1}} \leq 2C_1^2 A_k^{2^{k+1}} 2^{k2^{k+1}} \|\Gamma(f, f)^{1/2}\|_{2^k}^{2^{k+1}}.$$

It follows that  $A_{k+1} \leq (\sqrt{2}C_1)^{1/2^k} 2^{-1} A_k$ . Suppose  $II \leq I$ . Then

$$\|f\|_{2^{k+1}}^{2^{k+1}} \leq 2CC_2^2 2^{2k} \|\Gamma(f, f)^{1/2}\|_{2^{k+1}}^2 \|f\|_{2^{k+1}}^{2^{k+1}-2}.$$

We have

$$\|f\|_{2^{k+1}} \leq \frac{1}{\sqrt{2}} \sqrt{C} C_2 2^{k+1} \|\Gamma(f, f)^{1/2}\|_{2^{k+1}}.$$

Hence  $A_{k+1} \leq \sqrt{C} C_2 / \sqrt{2}$ . It follows that

$$A_{k+1} \leq \max\{(\sqrt{2}C_1)^{1/2^k} 2^{-1} A_k, \sqrt{C} C_2 / \sqrt{2}\} \leq \max\{(\sqrt{2}C_1)^{1/2^k} A_k, \sqrt{C} C_2 / \sqrt{2}\}.$$

Note that we may assume without loss of generality  $\sqrt{2}C_1 \geq 1$  and  $C_1 \geq \sqrt{C}/2$ . Since we may take  $A_1 = C_2$ , inductively we have  $(\sqrt{2}C_1)^{1/2^k} A_k \geq \sqrt{C} C_2 / \sqrt{2}$  for all  $k \in \mathbb{N}$ . Let

$$B_k = C_2 \prod_{j=1}^{k-1} (\sqrt{2}C_1)^{1/2^j}.$$

Since  $\log B_k \leq \log(\sqrt{2}C_1 C_2)$ ,  $A_k \leq B_k$  is uniformly bounded. The proof is complete.  $\square$

To state a result for arbitrary  $p$ , let us recall the  $L_p$  regularity of Dirichlet forms due to Olkiewicz and

Zegarliniski [OZ99, Theorem 5.5]. In our context, their result reads as

$$\tau(f^{p/2} A f^{p/2}) \leq \frac{p^2}{4(p-1)} \tau(f^{p-1} A f)$$

for positive  $f$  and  $1 < p < \infty$ . By [CS03], we know that  $\tau(f A g) = \tau[\Gamma(f, g)]$ . It follows that

$$\tau[\Gamma(f^{p/2}, f^{p/2})] \leq \frac{p^2}{4(p-1)} \tau[\Gamma(f, f^{p-1})] \quad (5.4)$$

for  $f \geq 0$ .

**Theorem 5.6.** *Under the assumptions of Theorem 5.5, for  $p \geq 2$ , there exists a finite set  $F_p \subset [1, 2)$  determined by  $p$ , such that for all self-adjoint element  $f \in \mathcal{N}$  with  $E_{\text{Fix}}(f) = 0$ , we have*

$$\|f\|_p \leq C' \max \left\{ \max_{\alpha \in F_p} p^{1/\alpha} \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/\alpha}^{1/\alpha}, p \|\Gamma(f, f)^{1/2}\|_p \right\}. \quad (5.5)$$

*Proof.* We argue by induction on  $n$  for  $2^n \leq p \leq 2^{n+1}$ . Let  $2 \leq p < \infty$ . By the spectral gap inequality, we have

$$\|f\|_p^p = \| |f|^{p/2} \|_2^2 \leq C_2^2 \tau[\Gamma(|f|^{p/2}, |f|^{p/2})] + \|E_{\text{Fix}}(|f|^{p/2})\|_2^2.$$

Using Theorem 5.1 and (5.4), we have

$$\tau[\Gamma(|f|^{p/2}, |f|^{p/2})] \leq \frac{p^2}{4(p-1)} \tau[\delta(|f|)^* \delta(|f|^k |f|^\alpha)], \quad (5.6)$$

where  $[p-2] = k$ ,  $p = k+1+\alpha$  and  $1 \leq \alpha < 2$ . By the derivation property, Hölder's inequality and Theorem 5.1,

$$\begin{aligned} \tau[\delta(|f|)^* \delta(|f|^k |f|^\alpha)] &= \sum_{j=1}^k \tau[\delta(|f|)^* |f|^{k-j} \delta(|f|) |f|^{j-1} |f|^\alpha] + \tau[\delta(|f|)^* |f|^k \delta(|f|^\alpha)] \\ &\leq \sum_{j=1}^k \|\delta(|f|)^*\|_p \|f\|_p^{k-j} \|\delta(|f|)\|_p \|f\|_p^{j-1+\alpha} + \|\delta(|f|)^*\|_p \|f\|_p^k \|\delta(|f|^\alpha)\|_{p/\alpha} \\ &\leq C^2 [k \|\Gamma(|f|, |f|)^{1/2}\|_p^2 \|f\|_p^{k-1+\alpha} + \|\Gamma(|f|, |f|)^{1/2}\|_p \|f\|_p^k \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/\alpha}]. \end{aligned}$$

Let  $\mathcal{L}_{p,\alpha} = \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/\alpha} \|f\|_p^{p/2-\alpha}$ . Noticing the relationship among  $p, k, \alpha$ , we find

$$\tau[\delta(|f|)^* \delta(|f|^k |f|^\alpha)] \leq C^2 [(p-1-\alpha) \mathcal{L}_{p,1}^2 + \mathcal{L}_{p,1} \mathcal{L}_{p,\alpha}] \leq C^2 (p-\alpha) \max\{\mathcal{L}_{p,\alpha}^2, \mathcal{L}_{p,1}^2\}. \quad (5.7)$$

On the other hand, by assumption (5.3), we have for  $2 \leq p \leq 4$ ,

$$\begin{aligned} \|E_{\text{Fix}}(|f|^{p/2})\|_2^2 &\leq C_1^2 \tau(|f|^{p/2})^2 \leq C_1^2 \|f\|_2^p \\ &\leq C_1^2 C_2^p \|\Gamma(f, f)^{1/2}\|_2^p \leq C_1^2 C_2^p \|\Gamma(f, f)^{1/2}\|_p^p. \end{aligned}$$

Plugging into (5.6), we have

$$\|f\|_p^p \leq \frac{C^2 C_2^2 p^2}{4} \max\{\mathcal{L}_{p,\alpha}^2, \mathcal{L}_{p,1}^2\} + C_1^2 C_2^p \|\Gamma(f, f)^{1/2}\|_p^p =: I + II.$$

If  $I \leq II$ , we have  $\|f\|_p \leq 2^{1/p} C_1^{2/p} C_2 \|\Gamma(f, f)^{1/2}\|_p$ . Suppose  $II \leq I$ . Let  $F_p = \{1, \alpha\}$ . We may find  $\alpha_0 \in F_p$  such that  $\mathcal{L}_{p,\alpha_0}^2 = \max\{\mathcal{L}_{p,\alpha}^2, \mathcal{L}_{p,1}^2\}$ . It follows that

$$\|f\|_p^{2\alpha_0} \leq \frac{C^2 C_2^2 p^2}{2} \|\Gamma(|f|^{\alpha_0}, |f|^{\alpha_0})^{1/2}\|_{p/\alpha_0}^2.$$

Hence, we find

$$\|f\|_p \leq \frac{(CC_2)^{1/\alpha_0} p^{1/\alpha_0}}{2^{1/(2\alpha_0)}} \|\Gamma(|f|^{\alpha_0}, |f|^{\alpha_0})^{1/2}\|_{p/\alpha_0}^{1/\alpha_0} \leq C' \max_{\alpha \in F_p} p^{1/\alpha} \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/\alpha}^{1/\alpha}. \quad (5.8)$$

We have proven (5.5) for  $2 \leq p \leq 4$ . Assume (5.5) holds for  $2^{n-1} \leq p \leq 2^n$  and let  $A_n$  denote the best constant. By assumption (5.3) and the induction hypothesis,

$$\begin{aligned} \|E_{\text{Fix}}(|f|^{p/2})\|_2^2 &\leq C_1^2 \|f\|_{p/2}^p \\ &\leq C_1^2 A_n^p \max \left\{ \max_{\alpha \in F_{p/2}} \left(\frac{p}{2}\right)^{p/\alpha} \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/(2\alpha)}^{p/\alpha}, \left(\frac{p}{2}\right)^p \|\Gamma(f, f)^{1/2}\|_{p/2}^p \right\} =: III. \end{aligned}$$

Combining with (5.6) and (5.7), we get

$$\|f\|_p^p \leq \frac{C^2 C_2^2 p^2}{4} \max\{\mathcal{L}_{p,\alpha}^2, \mathcal{L}_{p,1}^2\} + III = I + III.$$

If  $I \geq III$ , we get (5.8) as above. In this case, we may take  $F_p = \{1, \alpha\}$  and

$$A_{n+1} \leq \sup_{1 \leq \alpha \leq 2} 2^{-1/(2\alpha)} (CC_2)^{1/\alpha}.$$

Now suppose  $I \leq III$ . Then

$$\begin{aligned} \|f\|_p &\leq 2^{1/p} C_1^{2/p} A_n \max \left\{ \max_{\alpha \in F_{p/2}} \left( \frac{p}{2} \right)^{1/\alpha} \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/(2\alpha)}^{1/\alpha}, \frac{p}{2} \|\Gamma(f, f)^{1/2}\|_{p/2} \right\} \\ &\leq 2^{1/p} C_1^{2/p} A_n \max \left\{ \max_{\alpha \in F_{p/2}} p^{1/\alpha} \|\Gamma(|f|^\alpha, |f|^\alpha)^{1/2}\|_{p/\alpha}^{1/\alpha}, p \|\Gamma(f, f)^{1/2}\|_p \right\}. \end{aligned}$$

In this case, we may take  $F_p = F_{p/2}$  and  $A_{n+1} \leq 2^{1/p} C_1^{2/p} A_n$ . Combining together, we may set  $F_p = F_{p/2} \cup \{\alpha\}$  and  $A_{n+1} \leq \max\{\sup_{1 \leq \alpha \leq 2} (CC_2/\sqrt{2})^\alpha, (2C_1^2)^{1/2^n} A_n\}$ . We may assume without loss of generality  $\sup_{1 \leq \alpha \leq 2} (CC_2/\sqrt{2})^\alpha \leq (2C_1^2)^{1/2} A_1$  and  $2C_1^2 \geq 1$ . Thus inductively

$$\sup_{1 \leq \alpha \leq 2} (C_2/\sqrt{2})^\alpha \leq (2C_1^2)^{1/2^n} A_n.$$

By the same argument as for Theorem 5.5,  $A_n$  is uniformly bounded and the proof is complete.  $\square$

*Remark 5.7.* It is not difficult to check that the assumption (5.3) is satisfied if the fixed point algebra of  $T_t$  is finite dimensional. The constant  $C_1$  depends on the dimension of the fixed point algebra and the trace on this algebra. In fact, finite dimensional von Neumann algebras are of the form  $\oplus_{i=1}^r M_{n_i}$ , where  $M_{n_i}$  is the matrix algebra of dimension  $n_i^2$ . For simplicity, let us illustrate the case  $\text{Fix} = M_n$ . For  $x \in M_n$ ,

$$\|x\|_2 = \left[ \frac{1}{n} \text{tr}(x^* x) \right]^{1/2} = \frac{1}{\sqrt{n}} \sum_{i=1}^n s_i^2,$$

where  $\text{tr}$  is the usual trace on  $M_n$ , and  $s_i$ 's are the singular values of  $x$ . Similarly,  $\|x\|_1 = \frac{1}{n} \sum_{i=1}^n s_i$ . Then  $\|x\|_2 \leq \sqrt{n} \|x\|_1$ . Hence, for  $g \in \mathcal{N}$ ,

$$\|E_{\text{Fix}} g\|_2 \leq \sqrt{n} \|E_{\text{Fix}} g\|_1 \leq \sqrt{n} \|g\|_1.$$

The general form  $\text{Fix} = \oplus_{i=1}^r M_{n_i}$  is slightly more complicated and we leave it to the interested reader.

*Remark 5.8.* Although it looks complicated, the inequality (5.5) is actually consistent with that in the classical diffusion theory. To simplify our calculation, let us consider the one-variable functions and assume  $\Gamma(f, f)^{1/2} = |f'|$ . Assume further that  $|f|$  is differentiable and  $\int f d\mu = 0$ . Then  $f'(x) = 0$  for  $f(x) = 0$ . For example,  $f(x) = x^2 \text{sgn}(x)$  defined on the Gaussian probability space  $(\mathbb{R}, \gamma)$  satisfies these conditions. Since  $\Gamma(f, f)^{1/2} = \Gamma(|f|, |f|)^{1/2}$  in this setting, we only need to consider the first term in (5.5). By Hölder's inequality, we get

$$\|(|f|^\alpha)'\|_{p/\alpha}^{1/\alpha} \leq \alpha^{1/\alpha} \| |f|^{\alpha-1} |f'| \|_{p/\alpha}^{1/\alpha} \leq \alpha^{1/\alpha} \|f\|_p^{1-1/\alpha} \|f'\|_p^{1/\alpha}.$$

After choosing the optimal  $\alpha$ , we have

$$\|f\|_p \leq C\alpha p \|f'\|_p = C\alpha p \|f'\|_p,$$

which is exactly the classical result deduced from the spectral gap inequality as in [Mil09]. In general,  $|f|$  may not be differentiable at the zeros of  $f$  even if  $f$  is smooth. In this case, one may use a smoothening procedure by convolution to deduce similar results.

# Chapter 6

## Applications

We give two types of applications in this chapter. The first is Kolmogorov’s law of the iterated logarithm for noncommutative martingales. A crucial ingredient in the proof is the exponential inequality in Lemma 3.2 for noncommutative martingales. The paper [Zen14a] is based on this result. The second type is motivated by the work of Bobkov–Götze [BG99] on the transportation inequality and exponential integrability. We deduce several results in this direction as consequences of the (weak) subgaussian Poincaré inequalities.

### 6.1 Kolmogorov’s law of the iterated logarithm

#### 6.1.1 Introduction

In probability theory, law of the iterated logarithm (LIL) is among the most important limit theorems and has been studied extensively in different contexts. The early contributions in this direction for independent increments were made by Khintchine, Kolmogorov, Hartman–Wintner, etc; see [Bau96] for more history of this subject. Stout generalized Kolmogorov and Hartman–Wintner’s results to the martingale setting in [Sto70a, Sto70b]. The extension of LIL for independent sums in Banach spaces were due to Kuelbs, Ledoux, Talagrand, Pisier, etc; see [LT91] and the references therein for more details in this direction. In the last decade, there has been new development for LIL results of dependent random variables; see [Wu07, ZW08] and the references therein for more details. However, it seems that the LIL in noncommutative (= quantum) probability theory has only been proved recently by Konwerska [Kon08, Kon12] for Hartman–Wintner’s version. Even the Kolmogorov’s LIL for independent sums in the noncommutative setting is not known. The goal of this paper is to prove Kolmogorov’s version of LIL for noncommutative martingales.

Let us first recall Kolmogorov’s LIL. Let  $(Y_n)_{n \in \mathbb{N}}$  be an independent sequence of square-integrable, centered, real random variables. Put  $S_n = \sum_{i=1}^n Y_i$  and  $s_n^2 = \text{Var}(S_n) = \sum_{i=1}^n \mathbb{E}(Y_i^2)$ . Here and in the following  $\mathbb{E}$  denotes the expectation and  $\text{Var}$  denotes the variance. For any  $x > 0$ , we define the notation



$L(x) = \max\{1, \ln \ln x\}$ . In 1929, Kolmogorov proved that if  $s_n^2 \rightarrow \infty$  and

$$|Y_n| \leq \alpha_n \frac{s_n}{\sqrt{L(s_n^2)}} \text{ a.s.} \quad (6.1)$$

for some positive sequence  $(\alpha_n)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , then

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{s_n^2 L(s_n^2)}} = \sqrt{2} \text{ a.s..} \quad (6.2)$$

Later on, Hartman–Wintner [HW41] proved that if  $(X_n)$  is an i.i.d. sequence of real, centered square-integrable random variables with variance  $\text{Var}(X_i) = \sigma^2$ , then

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{nL(n)}} = \sqrt{2}\sigma \text{ a.s..}$$

de Acosta [dA83] simplified the proof of Hartman–Wintner. To compare the two results, if the sequence  $(Y_n)$  are i.i.d. and uniformly bounded, then the two results coincide. Apparently, Hartman–Wintner’s LIL does not contain Kolmogorov’s version as a special case. However, Kolmogorov’s LIL can be used in a truncation procedure to prove other LIL results; see for example [Sto70b].

Kolmogorov’s LIL was generalized to martingales by Stout [Sto70a]. Let  $(X_n, \mathcal{F}_n)_{n \geq 1}$  be a martingale with  $\mathbb{E}(X_n) = 0$ . Let  $Y_n = X_n - X_{n-1}$  for  $n \geq 1$ ,  $X_0 = 0$  be the associated martingale differences. Put  $s_n^2 = \sum_{i=1}^n \mathbb{E}[Y_i^2 | \mathcal{F}_{i-1}]$ . Then Stout proved that if  $s_n^2 \rightarrow \infty$  and (6.1) holds, then  $\limsup_{n \rightarrow \infty} X_n / \sqrt{s_n^2 L(s_n^2)} = \sqrt{2} \text{ a.s..}$

In the noncommutative setting, following [Kon08], a sequence  $(x_n)$  of  $\tau$ -measurable operators is said to be almost uniformly bounded by a constant  $K \geq 0$ , denoted by  $\limsup_{n \rightarrow \infty} x_n \underset{a.u.}{\leq} K$ , if for any  $\varepsilon > 0$  and any  $\delta > 0$ , there exists a projection  $e$  with  $\tau(1 - e) < \varepsilon$  such that

$$\limsup_{n \rightarrow \infty} \|x_n e\| \leq K + \delta; \quad (6.3)$$

and  $(x_n)$  is said to be bilaterally almost uniformly bounded by a constant  $K \geq 0$ , denoted by  $\limsup_{n \rightarrow \infty} x_n \underset{b.a.u.}{\leq} K$ , if (6.3) is replaced by

$$\limsup_{n \rightarrow \infty} \|e x_n e\| \leq K + \delta.$$

Clearly,  $\limsup_{n \rightarrow \infty} x_n \underset{a.u.}{\leq} K$  implies  $\limsup_{n \rightarrow \infty} x_n \underset{b.a.u.}{\leq} K$ .

For a  $\tau$ -measurable operator  $x$  and  $t > 0$ , the generalized singular numbers [FK86] are defined by

$$\mu_t(x) = \inf\{s > 0 : \tau(1_{(s,\infty)}(|x|)) \leq t\}.$$

In this paper, we use  $1_A(a)$  to denote the spectral projection of an operator  $a$  on the Borel set  $A$ . According to [Kon08], a sequence of operators  $(x_i)$  is said to be uniformly bounded in distribution by an operator  $y$  if there exists  $K > 0$  such that  $\sup_i \mu_t(x_i) \leq K \mu_{t/K}(y)$  for all  $t > 0$ . Let  $(x_n)$  be a sequence of mean zero self-adjoint independent random variables. Konwerska [Kon12] proved that if  $(x_n)$  is uniformly bounded in distribution by a random variable  $y$  such that  $\tau(|y|^2) = \sigma^2 < \infty$ , then

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{nL(n)}} \sum_{i=1}^n x_i \underset{\text{b.a.u.}}{\leq} C\sigma.$$

Note that if the sequence  $(x_n)$  is i.i.d., which is the case in the original version of Hartman–Wintner’s LIL, then  $(x_n)$  is uniformly bounded in distribution by  $x_1$ . Essentially, the condition of uniform boundedness in distribution requires the sequence to be almost identically distributed.

Our main result is an extension of Stout’s result to the noncommutative setting. Let  $(x_n)_{n \geq 0}$  be a noncommutative self-adjoint martingale with  $x_0 = 0$  and  $d_i = x_i - x_{i-1}$  the associated martingale differences. Define  $s_n^2 = \|\sum_{i=1}^n E_{i-1}(d_i^2)\|_\infty$  and  $u_n = [L(s_n^2)]^{1/2}$ .

**Theorem 6.1.** *Let  $0 = x_0, x_1, x_2, \dots$  be a self-adjoint martingale in  $(\mathcal{N}, \tau)$ . Suppose  $s_n^2 \rightarrow \infty$  and  $\|d_n\|_\infty \leq \alpha_n s_n / u_n$  for some sequence  $(\alpha_i)$  of positive numbers such that  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then*

$$\limsup_{n \rightarrow \infty} \frac{x_n}{s_n u_n} \underset{\text{a.u.}}{\leq} 2.$$

A natural question is to ask for the lower bound of LIL. As observed in [Kon08], however, one can only expect an upper bound for LIL in the general noncommutative setting. Indeed, consider a free sequence of semicircular random variables  $(x_n)$  (the so-called free Gaussian random variables [VDN92]) such that the law of  $x_n$  is  $\gamma_{0,2}$  (in notation,  $x_n \sim \gamma_{0,2}$ ) for all  $n$ . Here  $\gamma_{0,2}$  has density function  $p(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$  for  $-2 \leq x \leq 2$ . Then it is well known in free probability theory that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \sim \gamma_{0,2}.$$

It follows that  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i / \sqrt{nL(n)} = 0$  in the norm topology since a random variable with law  $\gamma_{0,2}$  is bounded. Therefore there is no reasonable notion of the positive LIL lower bound for the free semicircular

sequence. Comparing our LIL results with classical ones, we lose a constant of  $\sqrt{2}$ . However, since there is no hope to obtain an LIL lower bound in the general noncommutative theory, we are more interested in the order of the fluctuation for general noncommutative martingales. It is also commonly acknowledged that going from the commutative theory to the noncommutative setting usually requires considerably more technologies [PX03]. Due to these reasons, it seems fair to have the constant 2 in the noncommutative martingale setting.

### 6.1.2 Some preliminary results

Let us recall the vector valued noncommutative  $L_p$  spaces for  $1 \leq p \leq \infty$  introduced by Pisier [Pis98] and Junge [Jun02]. Let  $(x_n)$  be a sequence in  $L_p(\mathcal{N})$  and define

$$\|(x_n)\|_{L_p(\ell_\infty)} = \inf\{\|a\|_{2p}\|b\|_{2p} : x_n = ay_nb, \|y_n\|_\infty \leq 1\}.$$

Then  $L_p(\ell_\infty)$  is defined to be the closure of all sequences with  $\|(x_n)\|_{L_p(\ell_\infty)} < \infty$ . It was shown in [JX07] that if every  $x_n$  is self-adjoint, then

$$\|(x_n)\|_{L_p(\ell_\infty)} = \inf\{\|a\|_p : a \in L_p(\mathcal{N}), a \geq 0, -a \leq x_n \leq a \text{ for all } n \in \mathbb{N}\}.$$

Similarly, Junge and Xu introduced in [JX07] the space  $L_p(\ell_\infty^c)$  with norm

$$\begin{aligned} & \|(x_i)_{i \in I}\|_{L_p(\ell_\infty^c)} \\ &= \inf\{\|a\|_p : a \in L_p(\mathcal{N}), a \geq 0, -a \leq x_i^* x_i \leq a \text{ for all } i \in I\} \\ &= \inf\{\|b\|_p : x_i = y_i b, \|y_i\|_\infty \leq 1 \text{ for all } i \in I\}. \end{aligned}$$

The following result is the noncommutative asymmetric version of Doob's maximal inequality proved by Junge [Jun02]. We add a short proof to elaborate on the constant which is implicit in the original paper.

**Theorem 6.2.** *Let  $4 \leq p \leq \infty$ . Then, for any  $x \in L_p(\mathcal{N})$ , there exists  $b \in L_p(\mathcal{N})$  and a sequence of contractions  $(y_n) \subset \mathcal{N}$  such that*

$$\|b\|_p \leq 2^{2/p} \|x\|_p \quad \text{and} \quad E_n x = y_n b, \text{ for all } n \geq 0.$$

*Proof.* This follows from [Jun02, Corollary 4.6]. Indeed, setting  $r = p \geq 4$  and  $q = \infty$ , we find  $E_n x = az_n b$  for  $a, z_n \in \mathcal{N}$  and  $b \in L_p(\mathcal{N})$ . Let  $y_n = az_n / \|az_n\| \in \mathcal{N}$  and  $b' = \|az_n\| b \in L_p(\mathcal{N})$ . Then  $(y_n)$  is a sequence

of contractions,  $E_n x = y_n b'$ , and

$$\|b'\|_p \leq \|a\|_\infty \|b\|_p \sup_n \|z_n\|_\infty \leq c(p, q, r) \|x\|_p,$$

where  $c(p, q, r) \leq c_{q/(q-2)}^{1/2} c_{r/(r-2)}^{1/2} = c_1^{1/2} c_{p/(p-2)}^{1/2}$  and  $c_p$  is the constant in the dual Doob's inequality. Note that  $1 \leq p/(p-2) \leq 2$ . By Lemma 3.1 and Lemma 3.2 of [Jun02], we find that  $c_p \leq 2^{2(p-1)/p}$  for  $1 \leq p \leq 2$ . It follows that  $c(p, q, r) \leq 2^{2/p}$ .  $\square$

Suppose  $(x_i)_{m \leq i \leq n}$  is a martingale in  $L_p(\mathcal{N})$ . According to Theorem 6.2, there exist  $b \in L_p(\mathcal{N})$  and contractions  $(y_i)_{m \leq i \leq n} \subset \mathcal{N}$  such that  $x_i = y_i b$  for  $m \leq i \leq n$  and  $\|b\|_p \leq 2^{2/p} \|x_n\|_p$  for  $p \geq 4$ . It follows that

$$\|(x_i)_{m \leq i \leq n}\|_{L_p(\ell_\infty^c)} \leq 2^{2/p} \|x_n\|_p.$$

Doob's inequality will be used in this form in the proof of our main result.

Another important tool in our proof is a noncommutative version of Borel–Cantelli lemma. To state this result, we recall from [Kon08] that for a self-adjoint sequence  $(x_i)_{i \in I}$  of random variables, the column version of tail probability is by definition

$$\begin{aligned} & \text{Prob}_c \left( \sup_{i \in I} \|x_i\| > t \right) \\ &= \inf \{ s > 0 : \exists \text{ a projection } e \text{ with } \tau(1 - e) < s \\ & \text{and } \|x_i e\|_\infty \leq t \text{ for all } i \in I \} \end{aligned}$$

for  $t > 0$ . It is immediate that

$$\text{Prob}_c(\sup_{i \in I} \|x_i\| > t) \leq \text{Prob}_c(\sup_{i \in I} \|x_i\| > r) \quad (6.4)$$

for  $t \geq r$  and that if  $a_i \geq 1$  for  $i \in I$ , then

$$\text{Prob}_c(\sup_{i \in I} \|x_i\| > t) \leq \text{Prob}_c(\sup_{i \in I} \|a_i x_i\| > t). \quad (6.5)$$

Using the notation  $\text{Prob}_c$ , we state two lemmas which are taken from [Kon08].

**Lemma 6.3** (Noncommutative Borel–Cantelli lemma). *Let  $\cup_n I_n = \{n \in \mathbb{N} : n \geq n_0\}$  for some  $n_0 \in \mathbb{N}$  and*

$(z_n)$  be a sequence of self-adjoint random variables. If for any  $\delta > 0$ ,

$$\sum_{n \geq n_0} \text{Prob}_c \left( \sup_{m \in I_n} \|z_m\| > \gamma + \delta \right) < \infty,$$

then

$$\limsup_{n \rightarrow \infty} z_n \leq \gamma \text{ a.u.}$$

**Lemma 6.4** (Noncommutative Chebyshev inequality). *Let  $(x_i)_{i \in I}$  be a self-adjoint sequence of random variables. For  $t > 0$  and  $1 \leq p < \infty$ ,*

$$\text{Prob}_c(\sup_n \|x_n\| > t) \leq t^{-p} \|x\|_{L_p(\ell_\infty^c)}^p.$$

### 6.1.3 Proof of LIL

According to [Bau96], the original proof of Kolmogorov's LIL is comparably expensive as that of Hartman–Wintner. However, our proof of Kolmogorov's LIL here seems to be relatively easier than (the upper bound of) Hartman–Wintner's version for the commutative case due to the exponential inequality (Lemma 3.2).

*Proof of Theorem 6.1.* Let  $\eta \in (1, 2)$  be a constant which we will determine later. To avoid annoying subscripts, we write  $s(k_i) = s_{k_i}$  in the following. Using the stopping rule in [Sto70a], we define  $k_0 = 0$  and for  $n \geq 1$ ,

$$k_n = \inf\{j \in \mathbb{N} : s_{j+1}^2 \geq \eta^{2n}\}.$$

Then  $s_{k_n+1}^2 \geq \eta^{2n}$  and  $s_{k_n}^2 < \eta^{2n}$ . Note that given  $\varepsilon' > 0$  there exists  $N_1(\varepsilon') > 0$  such that for  $n > N_1(\varepsilon')$ ,

$$\begin{aligned} & s_{k_n+1}^2 u_{k_n+1}^2 / (s(k_{n+1})^2 u(k_{n+1})^2) \\ & \geq \eta^{-2} \ln \ln \eta^{2n} / \ln \ln \eta^{2(n+1)} \geq (1 - \varepsilon')^2 \eta^{-2}. \end{aligned}$$

Then  $s_m u_m \geq (1 - \varepsilon') \eta^{-1} s(k_{n+1}) u(k_{n+1})$  for  $k_n < m \leq k_{n+1}$ . For any  $\delta' > 0$ , we can find  $\delta, \varepsilon' > 0$  and  $\eta \in (1, 2)$  such that  $1 + \delta' > \eta(1 + \delta)(1 - \varepsilon')^{-1}$ . Fix  $\beta > 0$  which will be determined later. Using the notation  $\text{Prob}_c$  with order relations (6.4) and (6.5), we have for  $n > N_1(\varepsilon')$

$$\begin{aligned} & \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{x_m}{s_m u_m} \right\| > \beta(1 + \delta') \right) \\ & \leq \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1}) u(k_{n+1})} \right\| > \lambda \beta(1 + \delta) \right). \end{aligned} \tag{6.6}$$

By Lemma 6.4 and Theorem 6.2, we have for  $p \geq 4$ ,

$$\begin{aligned}
& \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta) \right) \\
& \leq (\lambda\beta(1+\delta))^{-p} \left\| \left( \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right)_{k_n < m \leq k_{n+1}} \right\|_{L_p(\ell_\infty^c)}^p \\
& \leq (\lambda\beta(1+\delta))^{-p} (2^{2/p})^p \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right\|_p^p.
\end{aligned}$$

Using the elementary inequality  $|u|^p \leq p^p e^{-p}(e^u + e^{-u})$ , functional calculus and Lemma 3.2 with  $M = \alpha(k_{n+1})s(k_{n+1})/u(k_{n+1})$ ,  $D^2 = s(k_{n+1})^2$ , we find

$$\begin{aligned}
& \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right\|_p^p \\
& \leq p^p e^{-p} \tau \left( \exp \left( \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right) + \exp \left( - \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right) \right) \\
& \leq 2 \left( \frac{p}{e} \right)^p \exp \left( \frac{(1+\varepsilon)\lambda^2}{u(k_{n+1})^2} \right)
\end{aligned}$$

provided  $0 \leq \lambda \leq \frac{\sqrt{\varepsilon}u(k_{n+1})^2}{(1+\varepsilon)\alpha(k_{n+1})}$  and  $0 < \varepsilon \leq 1$ . Hence we obtain

$$\begin{aligned}
& \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta) \right) \\
& \leq 8 \left( \frac{p}{\lambda\beta(1+\delta)e} \right)^p \exp \left( \frac{(1+\varepsilon)\lambda^2}{u(k_{n+1})^2} \right).
\end{aligned}$$

Now optimizing in  $p$  gives  $p = \lambda\beta(1+\delta)$  and thus,

$$\begin{aligned}
& \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta) \right) \\
& \leq 8 \exp \left( \frac{(1+\varepsilon)\lambda^2}{u(k_{n+1})^2} - \beta(1+\delta)\lambda \right).
\end{aligned}$$

Put  $\lambda = \beta(1+\delta)u(k_{n+1})^2/(2(1+\varepsilon))$ . Since  $\alpha_n \rightarrow 0$ , for any  $\varepsilon > 0$ , there exists  $N_2 > 0$  such that for  $n > N_2$ ,  $0 < \alpha(k_{n+1}) \leq \frac{2\sqrt{\varepsilon}}{\beta(1+\delta)}$ , which ensures that we can apply Lemma 3.2. This also implies  $p \geq 4$  for large  $n$ . It follows that

$$\text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta) \right) \leq (\ln s(k_{n+1}))^2 e^{-\frac{\beta^2(1+\delta)^2}{4(1+\varepsilon)}}.$$

Notice that  $s(k_{n+1})^2 \geq s(k_n + 1)^2 \geq \eta^{2n}$ . Setting  $\beta = 2$  in the beginning of the proof, we have

$$\text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1 + \delta) \right) \leq [(2 \ln \eta)n]^{-\frac{(1+\delta)^2}{1+\varepsilon}}.$$

By choosing  $\varepsilon$  small enough so that  $(1 + \delta)^2/(1 + \varepsilon) > 1$ , we find that for  $n_0 = \max\{N_1, N_2\}$ ,

$$\sum_{n \geq n_0} \text{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1 + \delta) \right) < \infty.$$

Then (6.6) and Lemma 6.3 give the desired result.  $\square$

## 6.2 Transportation cost inequalities

### 6.2.1 Introduction

The readers are referred to [Vil09] for general questions on transportation problems. Let  $(\Omega, d)$  be a metric space. Let  $\mu$  and  $\nu$  be probability measures on  $(\Omega, d)$  with finite  $p$ -th moment. Recall that the  $p$ -Wasserstein distance between  $\mu$  and  $\nu$  is defined as

$$W_p(\mu, \nu) = \inf \left( \iint d(x, y)^p d\pi(x, y) \right)^{1/p}$$

where the infimum is taken over all probability measure  $\pi$  on the product space  $\Omega \times \Omega$  which is a coupling of  $\mu$  and  $\nu$ . Assume  $\nu$  is a probability measure absolutely continuous with respect to  $\mu$ . Suppose that there exists  $x_0 \in \Omega$  such that  $\int d(x, x_0) d\mathbb{P} < \infty$  for  $\mathbb{P} = \mu$  and  $\nu$ . Let  $g = \frac{d\nu}{d\mu}$ . By the Kantorovich–Rubinstein formula (see for example [Vil09]), the 1-Wasserstein distance can be written as

$$W_1(\mu, \nu) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int f g d\mu - \int f d\mu \right|. \quad (6.7)$$

There is also a functional representation of  $W_2$  (see for example [Rac91])

$$W_2^2(\mu, \nu) = \sup \int g d\nu - \int f d\mu \quad (6.8)$$

where the sup is taken over all bounded continuous function pairs  $(g, f)$  with  $g(y) - f(x) \leq d(x, y)^2$  for all  $x, y \in \Omega$ . As explained in Section 1.2, these functional representations motivate our definition of  $Q_1$  and  $Q_\phi$ , which can be regarded as the noncommutative analogues of Wasserstein distances. The precise definitions

will be given below.

### 6.2.2 Consequences of weak subgaussian Poincaré inequalities

As is well-known, the Poincaré inequality with constant  $\sqrt{p}$  implies the subgaussian concentration phenomenon. We are going to prove a noncommutative version of exponential integrability due to Bobkov and Götze [BG99] in the commutative case. The following variant was due to Efraim and Lust-Piquard in the case of Walsh system.

**Corollary 6.5.** *Under the assumptions of Theorem 3.8, we have*

$$\tau(e^{|x - E_{\text{Fix}}x|}) \leq 2 \exp \left( \frac{C}{\alpha} \max\{\|\Gamma(x, x)\|_\infty, \|\Gamma(x^*, x^*)\|_\infty\} \right), \quad (6.9)$$

and for  $t > 0$

$$\text{Prob}(|x - E_{\text{Fix}}x| \geq t) \leq 2 \exp \left( - \frac{\alpha t^2}{4C \max\{\|\Gamma(x, x)\|_\infty, \|\Gamma(x^*, x^*)\|_\infty\}} \right). \quad (6.10)$$

We may take  $C = 32e$  in general and  $C = 8e$  for  $x$  self-adjoint.

*Proof.* We follow the proof in the commutative case; see [ELP08, Corollary 4.1 and 4.2]. Since  $\Gamma(x, x) = \Gamma(x - E_{\text{Fix}}x, x - E_{\text{Fix}}x)$ , we may assume  $E_{\text{Fix}}(x) = 0$ . Put  $M = \max\{\|\Gamma(x, x)^{1/2}\|_\infty, \|\Gamma(x^*, x^*)^{1/2}\|_\infty\}$ . Note that  $\frac{k^k}{(2k-1)!!} \leq \left(\frac{e}{2}\right)^k$  for all  $k \in \mathbb{N}$ . By functional calculus and (3.11),

$$\begin{aligned} \frac{1}{2} \tau(e^{|x|}) &\leq \tau(\cosh x) = 1 + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \|x\|_{2k}^{2k} \\ &\leq 1 + \sum_{k=1}^{\infty} \frac{C^{2k} (2k)^k}{\alpha^k (2k)!} M^{2k} \leq 1 + \sum_{k=1}^{\infty} \frac{k^k (CM)^{2k}}{\alpha^k k! (2k-1)!!} \\ &\leq 1 + \sum_{k=1}^{\infty} \frac{(e/2)^k (CM)^{2k}}{\alpha^k k!} = \exp \left( \frac{eC^2 M^2}{2\alpha} \right). \end{aligned}$$

We have proved the first assertion for  $C = 32e$  and we can take  $C = 8e$  if  $x$  is self-adjoint. For the second inequality, we deduce from Chebyshev inequality that

$$\tau(1_{[t, \infty)}(|x|)) \leq e^{-\lambda t} \tau(e^{\lambda |x|}) \leq 2e^{-\lambda t + C\lambda^2 M^2 / \alpha}.$$

Then the assertion follows from minimizing the right hand side with respect to  $\lambda$ . □

The improvement in the situation of commutative diffusion in Theorem 3.13 also gives an intermediate



term in (6.9) for self-adjoint element  $x$ , i.e.

$$\tau(e^{|x-E_{\text{Fix}}x|}) \leq 2\tau \exp\left(\frac{C'}{\alpha}\Gamma(x,x)\right) \leq 2\exp\left(\frac{C'}{\alpha}\|\Gamma(x,x)\|_\infty\right).$$

We do not have such an intermediate term in the fully noncommutative generality without the help of (3.8). However, it seems the Lipschitz norm is the right choice in application to the concentration inequality. In this sense, we did not lose much even if we use a larger norm. The following result is simply a one side version of Corollary 6.5. We record it here for future references.

**Proposition 6.6.** *Under the hypotheses of Theorem 3.8, assume further that  $x$  is self-adjoint. Then for  $t \in \mathbb{R}$*

$$\tau(e^{t(x-E_{\text{Fix}}x)}) \leq e^{c\|\Gamma(x,x)\|_\infty t^2}, \quad (6.11)$$

and for  $t > 0$ ,

$$\text{Prob}(x - E_{\text{Fix}}x \geq t) \leq \exp\left(-\frac{t^2}{4c\|\Gamma(x,x)\|_\infty}\right) \quad (6.12)$$

where the constant  $c$  only depends on  $\alpha$ .

*Proof.* Again it suffices to consider (6.11) for  $x$  with  $E_{\text{Fix}}(x) = 0$  since  $\Gamma(x,x) = \Gamma(x - E_{\text{Fix}}x, x - E_{\text{Fix}}x)$ . From the proof of (6.9), we know there exists  $C > 0$  such that for  $t \in \mathbb{R}$

$$\tau(e^{tx}) \leq \tau(e^{tx}) + \tau(e^{-tx}) \leq 2e^{C\|\Gamma(x,x)\|_\infty t^2/\alpha}.$$

Then for  $t^2\|\Gamma(x,x)\|_\infty \geq 1$ , we have  $\tau(e^{tx}) \leq e^{(\ln 2 + C/\alpha)\|\Gamma(x,x)\|_\infty t^2}$ . For  $t^2\|\Gamma(x,x)\|_\infty < 1$ ,

$$\begin{aligned} \tau(e^{tx}) &= 1 + \sum_{k=2}^{\infty} \frac{t^k \tau(x^k)}{k!} \leq 1 + \sum_{k=2}^{\infty} \frac{t^k C^k k^{k/2} \|\Gamma(x,x)\|_\infty^{k/2}}{\alpha^{k/2} k!} \\ &\leq 1 + c\|\Gamma(x,x)\|_\infty t^2 \leq e^{c\|\Gamma(x,x)\|_\infty t^2} \end{aligned}$$

for some constant  $c = c(\alpha)$  since  $\tau(x) = 0$  and the series  $\sum_{k=2}^{\infty} \frac{k^{k/2}}{k!}$  converges. The second assertion follows in the same way as (6.10).  $\square$

The exponential integrability result (6.11) was proved by Bobkov and Götze [BG99] in the commutative case by using a variant of LSI. They also deduced a transportation inequality from (6.11). We will follow their approach to obtain a noncommutative version of transportation inequality. Since LSI is not available in our noncommutative theory, our Poincaré inequalities might be a more universal approach to the transportation inequality. Let us first define Wasserstein distance and entropy in the noncommutative setting.

**Definition 6.7.** Let  $\rho$  and  $\sigma$  be positive  $\tau$ -measurable operators (e.g. density matrices) affiliated with  $(\mathcal{M}, \tau)$ . The noncommutative entropy of  $\rho \in L_1(\mathcal{M}, \tau)$  is given by

$$\text{Ent}(\rho) = \tau(\rho \ln(\rho/\tau(\rho))).$$

Let  $\phi$  and  $\psi$  be states on  $\mathcal{M}$ . We define an analogue of  $L_1$ -Wasserstein distance between  $\phi$  and  $\psi$  by

$$Q_1^A(\phi, \psi) = \sup\{|\phi(x) - \psi(x)| : x \text{ self-adjoint}, \|\Gamma(x, x)\|_\infty \leq 1\}.$$

We also put  $Q_1^A(\rho, \sigma) = Q_1^A(\phi_\rho, \phi_\sigma)$  for  $\phi_\rho(\cdot) = \tau(\cdot\rho)/\tau(\rho)$  and  $\phi_\sigma(\cdot) = \tau(\cdot\sigma)/\tau(\sigma)$ .

Here the superscript  $A$  in  $Q_1^A$  is to emphasize the dependence on the generator of the semigroup  $T_t$ . We may ignore the superscript  $A$  for simplicity in the following. It is easy to check that  $Q_1^A$  is a pseudometric but may not be a metric in general. Our definition of Wasserstein distance coincides with the classical definition in the commutative case due to the Kantorovich–Rubinstein theorem; see for example [Vil09, Theorem 5.10]. It is also closely related to the quantum metric in the sense of Rieffel [Rie04]. Now we state a general fact on the relationship between conditional expectation and entropy.

**Lemma 6.8.** *Let  $\rho \in L_1(\mathcal{M}, \tau)$  with  $\rho \geq 0$  and  $\tau(\rho) = 1$  and  $E : \mathcal{M} \rightarrow \mathcal{N}$  the conditional expectation onto subalgebra  $\mathcal{N}$ . Then*

$$\tau(E\rho \ln E\rho) \leq \tau(\rho \ln \rho).$$

*Proof.* Let  $\rho_n = \rho 1_{[0, n]}$ . Then  $\rho_n \in L_p(\mathcal{M}, \tau)$  for all  $p \geq 1$ . It is easy to see that  $\rho_n \rightarrow \rho$  in the measure topology. Notice that  $\tau[(\rho_n/\tau(\rho_n)) \ln(\rho_n/\tau(\rho_n))] = \lim_{p \downarrow 1} \frac{\|\rho_n/\tau(\rho_n)\|_p^p - 1}{p - 1}$ . This yields

$$\begin{aligned} \tau\left(\frac{E\rho_n}{\tau(\rho_n)} \ln \frac{E\rho_n}{\tau(\rho_n)}\right) &= \lim_{p \downarrow 1} \frac{\|E\rho_n/\tau(\rho_n)\|_p^p - 1}{p - 1} \\ &\leq \lim_{p \downarrow 1} \frac{\|\rho_n/\tau(\rho_n)\|_p^p - 1}{p - 1} = \tau\left(\frac{\rho_n}{\tau(\rho_n)} \ln \frac{\rho_n}{\tau(\rho_n)}\right). \end{aligned}$$

Let  $\mu$  be the distribution of  $\rho$ . Then

$$\tau\left(\frac{\rho_n}{\tau(\rho_n)} \ln \frac{\rho_n}{\tau(\rho_n)}\right) = \frac{1}{\tau(\rho_n)} \int_0^n x \ln x \mu(dx) - \ln \tau(\rho_n) \rightarrow \tau(\rho \ln \rho).$$

Following [FK86], we denote the generalized singular number of  $\rho$  by  $\mu_t(\rho)$ . Note that  $\|E\rho_n - E\rho\|_1 \leq$

$\|\rho_n - \rho\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ . We have for every  $t > 0$ ,

$$\mu_t(E\rho_n - E\rho) \leq t^{-1} \int_0^t \mu_s(E\rho_n - E\rho) ds \leq t^{-1} \|E\rho_n - E\rho\|_1.$$

Then  $\lim_{n \rightarrow \infty} \mu_t(E\rho_n - E\rho) = 0$ . By [FK86, Lemma 3.1],  $(E\rho_n)$  converges to  $E\rho$  in the measure topology. Since  $0 \leq E\rho_n \leq E\rho$ , by [FK86, Lemma 2.5],  $\mu_t(E\rho_n) \leq \mu_t(E\rho)$ . We deduce from [FK86, Lemma 3.4] that  $\lim_{n \rightarrow \infty} \mu_t(E\rho_n) = \mu_t(E\rho)$  for  $t > 0$ . Now consider  $g(x) = x \ln x = x \ln x 1_{[1, \infty)(x)} - (-x \ln x 1_{(0, 1)})$ . Both functions in the decomposition are nonnegative Borel functions vanishing at the origin. It follows from [FK86, (3)] that  $\tau(E\rho \ln E\rho) = \int_0^1 \mu_t(E\rho) \ln \mu_t(E\rho) dt$ . Since for fixed  $\varepsilon > 0$ ,  $[\mu_t(E\rho_n) \ln \mu_t(E\rho_n)]_n$  is uniformly bounded on  $t \in [\varepsilon, 1]$ , we have

$$\begin{aligned} \tau(E\rho \ln E\rho) &= \sup_{\varepsilon > 0} \int_{\varepsilon}^1 \mu_t(E\rho) \ln \mu_t(E\rho) dt \\ &= \sup_{\varepsilon > 0} \lim_{n \rightarrow \infty} \int_{\varepsilon}^1 \mu_t(E\rho_n) \ln \mu_t(E\rho_n) dt \leq \limsup_{n \rightarrow \infty} \tau(E\rho_n \ln E\rho_n) \\ &\leq \limsup_{n \rightarrow \infty} \tau(\rho_n \ln \rho_n). \end{aligned}$$

This completes the proof. □

The next result in the commutative setting is well known; see for example [DZ98, Section 6.2].

**Lemma 6.9.** *Let  $\sigma$  be a self-adjoint  $\tau$ -measurable operator. Then,*

$$\ln \tau(e^\sigma) = \sup\{\tau(\rho\sigma) - \tau(\rho \ln \rho) : \rho \geq 0, \tau(\rho) = 1\}. \quad (6.13)$$

Therefore, for all positive  $\rho \in L_1(\mathcal{M}, \tau)$

$$\text{Ent}(\rho) = \sup\{\tau(\sigma\rho) : \sigma \text{ self-adjoint}, \tau(e^\sigma) \leq 1\}. \quad (6.14)$$

*Proof.* Let  $\sigma$  be a self-adjoint operator  $\tau$ -measurable operator. Consider the von Neumann subalgebra  $\mathcal{N}$  generated by  $\{f(\sigma) : f : \mathbb{C} \rightarrow \mathbb{C} \text{ bounded measurable}\}$ . Then there exists a conditional expectation  $E : \mathcal{M} \rightarrow \mathcal{N}$  which can extend to a contraction  $L_p(\mathcal{M}, \tau) \rightarrow L_p(\mathcal{N}, \tau)$  for all  $1 \leq p \leq \infty$ . Assume  $\tau(\rho) = 1$ . Then  $E(\rho) \in L_1(\mathcal{N}, \tau)$ . But  $\mathcal{N}$  is commutative and  $\tau(E(\rho)) = \tau(\rho) = 1$ . After identifying  $\tau$  with a probability measure denoted still by  $\tau$ , we use Jensen's inequality for the measure  $E(\rho)d\tau$  to deduce that

$$\tau(\sigma E(\rho)) - \tau(E(\rho) \ln E(\rho)) = \tau(\ln(e^\sigma E(\rho)^{-1})E(\rho)) \leq \ln \tau(e^\sigma).$$

Using Lemma 6.8 and noticing that  $\tau(\sigma\rho) = \tau(\sigma E(\rho))$ , we find

$$\tau(\sigma\rho) - \tau(\rho \ln \rho) \leq \ln \tau(e^\sigma).$$

For the reverse inequality, put  $\sigma_n = \sigma 1_{(-\infty, n]}(\sigma)$  for  $n \in \mathbb{N}$  where  $1_{(-\infty, n]}(\sigma)$  is the spectral projection of  $\sigma$ .

Plugging  $\rho_n = e^{\sigma_n} / \tau(e^{\sigma_n})$  into the right hand side of (6.13), we have

$$\tau(\sigma\rho_n) - \tau(\rho_n \ln \rho_n) = \frac{\tau((\sigma - \sigma_n)e^{\sigma_n})}{\tau(e^{\sigma_n})} + \ln \tau(e^{\sigma_n}).$$

By the spectral decomposition theorem of  $\sigma$ ,  $\tau((\sigma - \sigma_n)e^{\sigma_n}) \geq 0$ . Then for all  $n$  we have

$$\sup\{\tau(\rho\sigma) - \tau(\rho \ln \rho) : \rho \geq 0, \tau(\rho) = 1\} \geq \ln \tau(e^{\sigma_n}).$$

By Fatou's lemma [FK86, Theorem 3.5]  $\liminf_{n \rightarrow \infty} \ln \tau(e^{\sigma_n}) \geq \ln \tau(e^\sigma)$ . This proves (6.13). For (6.14), note that, by (6.13),  $\tau(e^\sigma) \leq 1$  implies  $\tau(\sigma\rho) \leq \tau(\rho \ln \rho)$  for all positive  $\rho \in L_1(\mathcal{M}, \tau)$  with  $\tau(\rho) = 1$ . If  $\tau(\rho) \neq 1$ , we consider  $\rho' = \rho / \tau(\rho)$  and find  $\tau(\sigma\rho) \leq \tau(\rho \ln(\rho / \tau(\rho)))$ . The equality is achieved by  $\sigma = \ln \rho - \ln \tau(\rho)$ . This proves the second assertion.  $\square$

**Theorem 6.10.** *Let  $(\mathcal{M}, \tau)$  be a noncommutative probability space. Then*

$$Q_1(\rho, 1) \leq \sqrt{2c \operatorname{Ent}(\rho)}, \quad (6.15)$$

for all  $\rho \geq 0$  with  $\tau(\rho) = 1$  if and only if for every self-adjoint  $\tau$ -measurable operator  $x$  affiliated with  $\mathcal{M}$  such that  $\|\Gamma(x, x)\|_\infty \leq 1$  and  $\tau(x) = 0$ ,

$$\tau(e^{tx}) \leq e^{ct^2/2}, \text{ for all } t \in \mathbb{R}_+. \quad (6.16)$$

*Proof.* Thanks to the preceding two lemmas, the proof is the same as that in the commutative case in [BG99].

We provide it here for completeness. Setting  $\sigma = tx - ct^2/2$  in (6.14) and assuming (6.16) we find

$$\tau((tx - ct^2/2)\rho) \leq \operatorname{Ent}(\rho). \quad (6.17)$$

Since  $\tau(x) = 0$  and  $\tau(\rho) = 1$ , it follows that  $\tau(x\rho - x) \leq \frac{ct}{2} + \frac{1}{t} \operatorname{Ent}(\rho)$ . Minimizing right hand side gives

$$\tau(x\rho - x) \leq \sqrt{2c \operatorname{Ent}(\rho)}. \quad (6.18)$$

Note that  $\tau(x\rho - x) = \tau(\hat{x}\rho - \hat{x})$  for all  $x$  where  $x = \hat{x} + \tau(x)$ . Taking sup over all self-adjoint  $x$  with  $\|\Gamma(x, x)\|_\infty \leq 1$  on the left hand side of (6.18) gives (6.15). For the other direction, note that (6.17) is equivalent to (6.15) by reversing the above argument. Then (6.16) follows from (6.13) by setting  $\sigma = tx - ct^2/2$ .  $\square$

If  $\text{Fix} = \mathbb{C}1$  (i.e. the system  $(\mathcal{M}, T_t)$  is ergodic), then combining the above theorem with (6.11), we find the transportation inequality (6.15) under the assumptions of Theorem 3.8. In fact, we even have a non-ergodic version of transportation inequality.

**Corollary 6.11.** *Suppose*

$$\tau(e^{x - E_{\text{Fix}}x}) \leq e^{c\|\Gamma^A(x, x)\|_\infty} \quad (1.8)$$

*for all  $\tau$ -measurable self-adjoint operator  $x$  affiliated to  $\mathcal{M}$ . Then*

$$Q_1(\rho, E_{\text{Fix}}\rho) \leq \sqrt{2c \text{Ent}(\rho)}. \quad (6.19)$$

*for all  $\rho \geq 0$  with  $\tau(\rho) = 1$ . In particular, (6.15) holds under the additional assumption  $E_{\text{Fix}}\rho = 1$ .*

*Proof.* The proof modifies a little that of Theorem 6.10. Since  $\tau(\rho) = 1$ , we have  $\tau([t(x - E_{\text{Fix}}x) - ct^2/2]\rho) \leq \text{Ent}(\rho)$ . Then we deduce that  $\tau(\rho x - \rho E_{\text{Fix}}(x)) \leq \sqrt{2c \text{Ent}(\rho)}$ . Since  $\tau(\rho E_{\text{Fix}}(x)) = \tau(E_{\text{Fix}}(\rho)x)$ , we have

$$\tau(\rho x - E_{\text{Fix}}(\rho)x) \leq \sqrt{2c \text{Ent}(\rho)}.$$

Taking sup over all self-adjoint  $x$  with  $\|\Gamma(x, x)\|_\infty \leq 1$  gives the assertion.  $\square$

The same argument as for Theorem 6.10 shows that the converse implication in Corollary 6.11 also holds. By Proposition 6.6, the assumption (1.8) is fulfilled by the hypotheses of Theorem 3.8. The point here is that even though the fixed point algebra  $\text{Fix}$  is not trivial we still have a transportation inequality although in certain situation the inequality does fail.

*Remark 6.12.* Let  $\rho$  be a positive operator with  $\tau(\rho) = 1$ . For  $\rho \in \text{Fix}$ , define  $B(\rho) = \{f \in L_1(\mathcal{N}) : E_{\text{Fix}}(f) = \rho\}$ . Then for  $f_1, f_2 \in B(\rho)$ , we have  $Q_1(f_1, f_2) \leq Q_1(f_1, \rho) + Q_1(f_2, \rho) < \infty$ . However, if  $f_1 \in B(\rho_1), f_2 \in B(\rho_2)$  and  $\rho_1 \neq \rho_2$ , then

$$Q_1(\rho_1, \rho_2) \geq \sup\{|\tau(\rho_1 x - \rho_2 x)| : x \in \text{Fix}, \|\Gamma(x, x)\|_\infty \leq 1\} = \infty.$$

It follows that  $Q_1(f_1, f_2) \geq |Q_1(\rho_1, \rho_2) - Q_1(f_1, \rho_1) - Q_1(f_2, \rho_2)| = \infty$ . This yields an interesting geomet-

ric picture: operators in the same “fiber”  $B(\rho)$  have finite distance between one another while operators belonging to different “fibers” have infinite distance.

The following simple result provides another way (under the assumption of finite diameter) to obtain the transportation inequality.

**Corollary 6.13.** *Suppose for self-adjoint  $x \in \mathcal{N}$ ,  $E_{\text{Fix}}(x) = 0$  and  $\|\Gamma(x, x)\|_\infty \leq 1$  imply  $\|x\|_\infty \leq K$ . Then, (6.19) holds with  $c = K^2$  for all  $\rho \geq 0$  such that  $\tau(\rho) = 1$ .*

*Proof.* A calculation gives  $e^x - x \leq e^{x^2}$ . Assume  $E_{\text{Fix}}(x) = 0$  and  $\|\Gamma(x, x)\|_\infty \leq 1$ . Then for  $t > 0$ ,  $\tau(e^{tx}) = \tau(e^{tx} - tx) \leq \tau(e^{tK} - tK) \leq e^{K^2 t^2}$ . The claim now follows from Corollary 6.11.  $\square$

Suppose in Theorem 3.8 we only have  $\Gamma_2 \geq 0$  but not the  $\Gamma_2$ -condition. Junge and Mei proved in [JM10] as the main result

$$\|A^{1/2}x\|_p \leq c(p) \max\{\|\Gamma(x, x)^{1/2}\|_p, \|\Gamma(x^*, x^*)^{1/2}\|_p\}$$

in this setting. Using the proof of Theorem 1.1.7 in the same paper [JM10], it can be shown that if

$$\|T_t : L_1^0(\mathcal{N}) \rightarrow L_\infty(\mathcal{N})\| \leq Ct^{-n/2}, \quad (6.20)$$

then  $\|A^{-1/2} : L_p^0(\mathcal{N}) \rightarrow L_\infty(\mathcal{N})\| \leq C(n)$  for  $p > n$ . Indeed, we consider the composition of operators

$$L_p^0(\mathcal{N}) \xrightarrow{A^{-\alpha}} L_q^0(\mathcal{N}) \hookrightarrow L_{s,1}^0(\mathcal{N}) \xrightarrow{A^{-\beta}} L_\infty(\mathcal{N}),$$

where  $s, q, p, \alpha, \beta$  are chosen so that

$$1 < s < q, n < p < q < \infty, \alpha + \beta = \frac{1}{2}, \alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}), \beta = \frac{n}{2s}.$$

For example,  $p = 2n, q = 4n, s = \frac{4n}{3}$  satisfy these conditions. By [JM10, Corollary 1.1.4],  $\|A^{-\alpha} : L_p^0(\mathcal{N}) \rightarrow L_q^0(\mathcal{N})\| \leq C(n)$ . Using [JM10, Lemma 1.1.3], we have  $\|A^{-\beta} : L_{s,1}^0(\mathcal{N}) \rightarrow L_\infty(\mathcal{N})\| \leq C(n)$ . The embedding  $L_q^0(\mathcal{N}) \hookrightarrow L_{s,1}^0(\mathcal{N})$  follows from interpolation theory. Hence,  $\|A^{-1/2} : L_p^0(\mathcal{N}) \rightarrow L_\infty(\mathcal{N})\| \leq C(n)$ . This gives  $\|x\|_\infty \leq C(n)\|A^{1/2}x\|_p$  for large  $p$  and  $E_{\text{Fix}}(x) = 0$ . Assuming  $\|\Gamma(x, x)\|_\infty \leq 1$  for self-adjoint  $x$ , it follows that  $\|x\|_\infty \leq C(n, p)$ . By choosing, e.g.,  $p = 2n$ , the constant  $C(n, p)$  actually only depends on  $n$ . In light of Corollary 6.13, we obtain the following result.

**Corollary 6.14.** *Let  $T_t$  be a standard  $nc$ -diffusion semigroup acting on  $\mathcal{N}$  with  $\Gamma_2 \geq 0$ . Then (6.20) with finite dimension  $n$  implies the transportation inequality (6.19) for all  $\rho \geq 0$  such that  $\tau(\rho) = 1$ .*

In the commutative theory, the transportation inequality (6.15) implies isoperimetric type inequality by Marton's argument in [BG99]. So far it is not clear what isoperimetric inequality means in noncommutative probability. We hope to give a noncommutative analogue of isoperimetric inequality.

**Definition 6.15.** Let  $e, f \in (\mathcal{M}, \tau)$  be projections. The distance between  $e$  and  $f$  is

$$d(e, f) = \inf\{Q_1(\phi, \psi) : \phi \text{ and } \psi \text{ are states, } s(\phi) = e, s(\psi) = f\},$$

where  $s(\phi)$  is the support of  $\phi$ .

Here our definition generalizes directly the distance of sets in the commutative theory. Thus in general  $d$  is not a metric, as in the commutative setting. Then the following result follows from the same proof as in the commutative setting given in [BG99].

**Proposition 6.16.** Let  $e, f \in (\mathcal{M}, \tau)$  be projections. Then under the assumptions of Theorem 3.8 and assuming  $\text{Fix} = \mathbb{C}1$ ,

$$d(e, f) \leq \sqrt{-2c \ln \tau(e)} + \sqrt{-2c \ln \tau(f)}.$$

Equivalently, for every  $h > \sqrt{-2c \ln \tau(e)}$  and every projection  $p$  such that  $d(p, e) > h$ ,

$$\tau(p) \leq \exp\left(-\frac{1}{2c}\left(h - \sqrt{-2c \ln \tau(e)}\right)^2\right).$$

*Proof.* Put  $\phi_e(\cdot) = \tau(\cdot)/\tau(e)$  and  $\phi_f = \tau(\cdot)/\tau(f)$ . It is easy to see that  $d(e, f) \leq Q_1(\phi_e, \phi_f)$ . Then triangle inequality and (6.15) yield  $d(e, f) \leq \sqrt{2c \text{Ent}(e)} + \sqrt{2c \text{Ent}(f)}$ . By spectral decomposition theorem of the identity,  $\text{Ent}(e) = \int_0^1 \frac{1_A}{\tau(e)} \ln \frac{1_A}{\tau(e)} d\mu$  where  $A$  is a Borel set such that  $1_A(Id) = e$ . Hence we find  $\text{Ent}(e) = -\ln \tau(e)$ , which gives the first assertion. The equivalent formulation is a simple calculation.  $\square$

To conclude this section, we remark that the best possible  $\alpha$  in  $\Gamma_2 \geq \alpha\Gamma$  sometimes characterizes the dynamical system  $(\mathcal{M}, T_t)$ ; see the example of hyperfinite  $II_1$  factor below.

### 6.2.3 Consequences of subgaussian Poincaré inequalities

The starting point of this section is the subgaussian  $L_p$  Poincaré type inequalities (1.1), i.e., for  $2 \leq p < \infty$ ,

$$\|x - E_{\text{Fix}}x\|_p \leq C\sqrt{p} \max\{\|\Gamma^A(x, x)^{1/2}\|_p, \|\Gamma^A(x^*, x^*)^{1/2}\|_p\}. \quad (1.1)$$

We remark that this is stronger than the weak subgaussian Poincaré inequalities (1.2). Hence it implies all concentration and transportation results obtained using (1.2) in Section 6.2.2.

Assuming log-Sobolev inequality, Bobkov and Götze proved an exponential integrability result, which reads in our context as

$$\tau(e^{x-E_{\text{Fix}}x}) \leq \tau(e^{c\Gamma^A(x,x)}). \quad (1.7)$$

Our next result says that (1.7) can be derived from (1.1).

**Theorem 6.17.** *Suppose the  $L_p$  Poincaré inequalities (1.1) hold for all  $p \geq 2$  and all self-adjoint  $x \in \mathcal{N}$ . Then there exists  $c > 0$  such that (1.7) holds for all self-adjoint  $x \in \mathcal{N}$ .*

*Proof.* Since  $E_{\text{Fix}}x$  is in the multiplicative domain of  $T_t$ , we have  $\Gamma^A(x, x) = \Gamma^A(x - E_{\text{Fix}}x, x - E_{\text{Fix}}x)$ .

Without loss of generality, we may assume  $E_{\text{Fix}}x = 0$  and thus  $\tau(x) = 0$ . By the Taylor series, we have

$$\begin{aligned} \tau(e^x) &= 1 + \sum_{k=2}^{\infty} \frac{\tau(x^k)}{k!} \leq 1 + \sum_{k=2}^{\infty} \frac{C^k k^{k/2} \tau(\Gamma^A(x, x)^{k/2})}{k!} \\ &= 1 + \sum_{j=1}^{\infty} \frac{C^{2j} (2j)^j \tau(\Gamma^A(x, x)^j)}{(2j)!} + \sum_{j=1}^{\infty} \frac{C^{2j+1} (2j+1)^{j+1/2} \tau(\Gamma^A(x, x)^{j+1/2})}{(2j+1)!}. \end{aligned}$$

Choose  $\theta \in (0, 1)$  so that  $\frac{1}{j+1/2} = \frac{1-\theta}{j} + \frac{\theta}{j+1}$ . By the noncommutative Hölder inequality,

$$\|\Gamma^A(x, x)\|_{j+1/2} \leq \|\Gamma^A(x, x)\|_j^{1-\theta} \|\Gamma^A(x, x)\|_{j+1}^{\theta}.$$

By the definition of  $\theta$ , we have

$$\tau(\Gamma^A(x, x)^{j+1/2}) \leq \max\{\tau(\Gamma^A(x, x)^j), \tau(\Gamma^A(x, x)^{j+1})\}.$$

Note that for  $j \geq 1$ ,

$$\frac{(2j+1)^{j+1/2}}{(2j+1)!} \leq \min\left\{\frac{(2j)^j}{(2j)!}, \sqrt{2j+2} \frac{(2j+2)^{j+1}}{(2j+2)!}\right\}.$$

Since  $\sqrt{2j+2}$  is bounded by  $C^{2j+2}$  for some  $C > 1$ , we have

$$\begin{aligned} \tau(e^x) &\leq 1 + 2 \sum_{j=1}^{\infty} \frac{C'^{2j} (2j)^j \tau(\Gamma^A(x, x)^j)}{(2j)!} + \sum_{j=2}^{\infty} \frac{C'^{2j} (2j)^j \tau(\Gamma^A(x, x)^j)}{(2j)!} \\ &\leq 1 + 3 \sum_{j=1}^{\infty} \frac{C'^{2j} (2j)^j \tau(\Gamma^A(x, x)^j)}{(2j)!}. \end{aligned}$$

Notice the elementary inequality  $\frac{(2j)^j}{(2j)!} = \frac{j^j}{j!(2j-1)!!} \leq \left(\frac{e}{2}\right)^j \frac{1}{j!}$  for  $j \in \mathbb{N}$ . We have

$$\tau(e^x) \leq 1 + \sum_{j=1}^{\infty} \frac{C'^{2j} (e/2)^j \tau(\Gamma^A(x, x)^j)}{j!} = \tau(e^{c\Gamma^A(x,x)}).$$

□



Now that the better bounds in (1.1) (compared with the weak subgaussian condition) result in the stronger exponential inequality (1.7) (compared with the weak exponential integrability), we expect to have stronger transportation type inequality as well. On the other hand, Bobkov and Götze actually showed a sharper inequality based on log-Sobolev inequality in the classical setting in [BG99]

$$Q(\mu, \nu) := \sup_{\|f\|_{\text{Lip}} \leq 1} \|f\|_{L^2(d\nu)} - \|f\|_{L^2(d\mu)} \leq \sqrt{2cD(\nu|\mu)}. \quad (6.21)$$

It was observed in the same paper that  $W_1(\mu, \nu) \leq Q(\mu, \nu) \leq W_2(\mu, \nu)$ . Our goal here is to give a stronger version (compared with (6.19)) of the transportation type inequality in the spirit of (6.21). Since we have all the  $L_p$  norms of  $\Gamma^A(x, x)^{1/2}$  in the Poincaré type inequalities, it is natural to consider the situation where we do not have  $\|\Gamma^A(x, x)^{1/2}\|_\infty \leq 1$  but some mild control on  $\|\Gamma^A(x, x)^{1/2}\|_p$ . Our approach is related to Milman's generalization [Mil12] of transportation cost inequalities in the commutative setting.

Given a Young function  $\phi$ , the complementary function  $\phi^*$  is given by the Legendre transform  $\phi^*(s) = \sup_{t \geq 0} \{ts - \phi(t)\}$  for  $s \geq 0$ . The following lemma follows easily from Young's inequality. We include a proof for completeness following [BG99, Mil12]. Recall that  $\text{Ent}(\rho) = \tau(\rho \ln(\rho / \ln(\rho)))$  for a positive  $\tau$ -measurable operator  $\rho$ . We will need the exponential integrability

$$\tau(e^{t(x - E_{\text{Fix}}x)}) \leq e^{\varphi(t)}. \quad (6.22)$$

**Lemma 6.18.** *Let  $\varphi : [0, \infty) \rightarrow [0, \infty]$  be a strictly increasing Young function. Assume (6.22) holds for all  $t \geq 0$  and all self-adjoint  $\tau$ -measurable operator  $x$ . Then for all positive operator  $\rho \in \mathcal{N}$  with  $\tau(\rho) = 1$ , we have for any self-adjoint  $x$ ,*

$$\tau(x\rho - xE_{\text{Fix}}\rho) \leq (\varphi^*)^{-1}(\text{Ent}(\rho)).$$

Here  $\varphi^*$  is the Legendre transform of  $\varphi$ , and  $h^{-1} : [0, \infty) \rightarrow [0, \infty]$  is the inverse of the function  $h$  given by

$$h^{-1}(t) = \sup\{s : h(s) \leq t\}.$$

*Proof.* It follows from (6.22) that  $\tau(e^{t(x - E_{\text{Fix}}x) - \varphi(t)}) \leq 1$ . By Lemma 6.9, we find

$$\tau((x - E_{\text{Fix}}x)\rho) \leq \frac{\varphi(t) + \text{Ent}(\rho)}{t}.$$

Note that  $\tau((E_{\text{Fix}}x)\rho) = \tau(xE_{\text{Fix}}\rho)$ . Since  $\varphi^*$  is continuous, taking inf over  $t$  on the right-hand side gives the assertion.  $\square$

From now on, let us fix the Young function  $\phi(t) = e^{t^2} - 1$ . This function is usually called  $\psi_2$  in literature. It is well known that  $\|f\|_\phi \leq C_1$  is equivalent to (see for example [Ver12])

$$\|f\|_p \leq C_2 \sqrt{p} \text{ for } p \geq 1. \quad (6.23)$$

Given two positive  $\tau$ -measurable operators  $\rho$  and  $\sigma$  with  $\tau(\rho) = \tau(\sigma) = 1$ , recall from Section 1 that

$$Q_\phi(\rho, \sigma) = \sup\{|\tau(x\rho - x\sigma)| : x \text{ self-adjoint}, \|\Gamma^A(x, x)^{1/2}\|_\phi \leq 1\}.$$

The definition of  $Q_\phi$  is motivated by the functional representations of Wasserstein distance (6.7) and (6.8). Notice that  $Q_\phi$  is much bigger than  $Q_1$  in general. One may compare the following result with various transportation cost inequalities obtained in [BGL01, Section 5].

**Theorem 6.19.** *Suppose the exponential inequality (1.7) holds for all self-adjoint  $x$  with  $E_{\text{Fix}}x = 0$  and  $\|\Gamma^A(x, x)^{1/2}\|_\phi \leq 1$  in a noncommutative probability space  $(\mathcal{N}, \tau)$ . Then there exist absolute positive constants  $H_0, C_1, C_2, C_3$  such that for any positive  $\tau$ -measurable operator  $\rho$  affiliated to  $\mathcal{N}$  with  $\tau(\rho) = 1$ , we have*

$$Q_\phi(\rho, E_{\text{Fix}}\rho) \leq \begin{cases} C_1 \sqrt{\text{Ent}(\rho)}, & \text{if } \text{Ent}(\rho) < H_0, \\ C_2 \text{Ent}(\rho) + C_3, & \text{if } \text{Ent}(\rho) \geq H_0. \end{cases} \quad (6.24)$$

In particular, there exists  $C' > 0$  such that

$$Q_\phi(\rho, E_{\text{Fix}}\rho) \leq C' \max\{\sqrt{\text{Ent}(\rho)}, \text{Ent}(\rho)\}.$$

*Proof.* Assume  $x$  is self-adjoint and  $E_{\text{Fix}}x = 0$ . We can rewrite (1.7) as  $\tau(e^{tx}) \leq \tau(e^{ct^2\Gamma^A(x, x)})$  for all  $t \in \mathbb{R}$ .

Using (6.23) and under the assumption  $\|\Gamma^A(x, x)^{1/2}\|_\phi \leq 1$ , we have

$$\tau(e^{ct^2\Gamma^A(x, x)}) = \sum_{k=0}^{\infty} \frac{(ct^2)^k \tau(\Gamma^A(x, x)^k)}{k!} \leq 1 + \sum_{k=1}^{\infty} \frac{(ct^2)^k (C_2 \sqrt{2k})^{2k}}{k!} =: 1 + g(t).$$

Let  $C = 2cC_2^2$ . Then the series on the right-hand converges if  $t < \frac{1}{\sqrt{Ce}}$ . To get an explicit bound, note that for fixed  $0 < \varepsilon < 1$ , there exists  $C_\varepsilon > 0$  such that the series  $g(t)$  converges uniformly and is bounded by  $C_\varepsilon t^2$  for  $t \in [0, \frac{1-\varepsilon}{\sqrt{Ce}}]$ . We define

$$\varphi(t) = \begin{cases} C_\varepsilon t^2, & \text{if } 0 \leq t \leq \frac{1-\varepsilon}{\sqrt{Ce}}, \\ \infty, & \text{if } t > \frac{1-\varepsilon}{\sqrt{Ce}}. \end{cases}$$

Clearly  $\varphi(t)$  is a Young function and  $g(t) \leq \varphi(t)$ . The Legendre transform of  $\varphi$  is given by

$$\varphi^*(s) = \begin{cases} \frac{s^2}{4C_\varepsilon}, & 0 \leq s \leq \frac{2C_\varepsilon(1-\varepsilon)}{\sqrt{C_\varepsilon}}, \\ \frac{1-\varepsilon}{\sqrt{C_\varepsilon}}s - \frac{C_\varepsilon(1-\varepsilon)^2}{C_\varepsilon}, & s > \frac{2C_\varepsilon(1-\varepsilon)}{\sqrt{C_\varepsilon}}. \end{cases}$$

We find the inverse function

$$(\varphi^*)^{-1}(z) = \begin{cases} 2\sqrt{C_\varepsilon}z, & 0 \leq z < \frac{C_\varepsilon(1-\varepsilon)}{C_\varepsilon}, \\ \frac{\sqrt{C_\varepsilon}}{1-\varepsilon}z + \frac{C_\varepsilon(1-\varepsilon)}{\sqrt{C_\varepsilon}}, & z \geq \frac{C_\varepsilon(1-\varepsilon)}{\sqrt{C_\varepsilon}}. \end{cases}$$

Since  $\Gamma^A(x, x) = \Gamma^A(x - E_{\text{Fix}}x, x - E_{\text{Fix}}x)$ , it suffices to take sup over all self-adjoint  $x$  with  $E_{\text{Fix}}x = 0$  in the definition of  $Q_\phi$ . The proof is complete by Lemma 6.18.  $\square$

**Example 6.20.** Consider the Gaussian space  $(\mathbb{R}, \gamma)$  where  $\gamma$  is the standard Gaussian measure. Let  $\mu$  be a probability measure absolutely continuous with respect to  $\gamma$ . In order to compute  $Q_1(\gamma, \mu)$ , one takes supremum over essentially linear functions. On the other hand, one needs to take supremum over quadratic functions to compute  $Q_\phi(\gamma, \mu)$ .

Given  $a \in \mathbb{R}$ , let us consider  $\mu(B) = \gamma(a + B)$  for any Borel set  $B \subset \mathbb{R}$  as suggested in [Tal96]. Let  $f(x) = d\mu/d\gamma = e^{ax - a^2/2}$ . Then  $\text{Ent}(f) = \int f \ln f d\gamma = a^2/2$ . Let  $g(x) = \frac{x^2-1}{2}$ . Then  $\int g d\gamma = 0$ ,  $\|g'\|_\phi = 2\sqrt{2}/\sqrt{3}$  and

$$Q_\phi(\gamma, \mu) = \sup_{\|g'\|_\phi \leq 1} \int g f d\gamma - \int g d\gamma \geq \frac{\sqrt{3}a^2}{4\sqrt{2}}.$$

Let  $h(x) = x$ . Then  $\int h d\gamma = 0$ ,  $\|h'\|_\infty = 1$  and

$$Q_1(\gamma, \mu) = \sum_{\|h'\|_\infty \leq 1} \int h f d\gamma - \int h d\gamma \geq a.$$

Since  $Q_\phi \geq Q_1$ , we see that both the  $Q_1$  transportation inequality (6.19) and the  $Q_\phi$  transportation inequality (6.24) are sharp up to a constant.

In fact, we have the following immediate application in the Gaussian setting. Let  $H$  be a real separable Hilbert space. There exists a centered Gaussian family  $W = \{W(h) : h \in H\}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with  $\mathbb{E}(W(h)W(k)) = \langle h, k \rangle$ . Let  $\mathcal{H}_n$  denote the Wiener chaos of order  $n$ , spanned by  $\{H_n(W(h)) : h \in H, \|h\| = 1\}$  in  $L_2(\Omega, \mathcal{F}, \mathbb{P})$ , where  $H_n(x) = \frac{(-1)^n e^{x^2/2}}{\sqrt{n!}} \frac{d^n}{dx^n} e^{-x^2/2}$  is the Hermite polynomial of order  $n$ . In particular,  $H_1(x) = x$ ,  $H_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$ . Let  $\mathcal{G}$  be the  $\sigma$ -algebra generated by  $\{W(h) : h \in H\}$ . It is well known that  $L_2(\Omega, \mathcal{G}, \mathbb{P}) = \oplus_{n=0}^\infty \mathcal{H}_n$ . See [Nua06] for more details.

**Corollary 6.21.** *Let  $P_{\mathcal{H}_n} : L_2(\Omega, \mathcal{G}, \mathbb{P}) \rightarrow \mathcal{H}_n$  be the orthogonal projection. Then for any positive function  $f \in L_2(\Omega, \mathcal{G}, \mathbb{P})$  with  $\|f\|_1 = 1$ , we have*

$$\|P_{\mathcal{H}_1} f\|_2 \leq C \sqrt{\text{Ent}(f)}, \quad \|P_{\mathcal{H}_2}(f)\|_2 \leq C' \max\{\sqrt{\text{Ent}(f)}, \text{Ent}(f)\}.$$

*Proof.* We first consider the second inequality. Let  $(e_i)$  be an orthonormal basis of  $H$ . Then

$$\{H_2(W(e_i))\}_i \cup \{H_1(W(e_i))H_1(W(e_j))\}_{i \neq j}$$

gives an orthonormal basis of  $\mathcal{H}_2$ ; see [Nua06, Proposition 1.1.1]. Write  $g_i = W(e_i)$ . It suffices to show that

$$\langle f, h \rangle \leq C' \max\{\sqrt{\text{Ent}(f)}, \text{Ent}(f)\}$$

for all  $h = \sum_{i=1}^{\infty} a_{ii} H_2(g_i) + \sum_{1 \leq i < j} a_{ij} H_1(g_i) H_1(g_j)$  where  $\sum_{j \geq i \geq 1} a_{ij}^2 = 1$ . Note that  $(g_i)_i$  are independent standard Gaussian random variables. We have  $\int h d\mathbb{P} = 0$ . Consider  $h$  as a function in the Gaussian space  $(\mathbb{R}^{\mathbb{N}}, \gamma)$ . A computation yields

$$\|\nabla h\|_{L_p(\gamma)} = \|(\sqrt{2}a_{ii}g_i + \sum_{j>i} a_{ij}g_j)_{i=1}^{\infty}\|_{L_p(\ell_2)}.$$

Then by the Minkowski inequality and the Khintchine inequality (or directly, the Khintchine–Kahane inequality), we have

$$\|\nabla h\|_{L_p(\gamma)} \leq \|(\sum_{j=i+1}^{\infty} a_{ij}g_j + \sqrt{2}a_{ii}g_i)_{i=1}^{\infty}\|_{\ell_2(L_p)} \leq c_1 \sqrt{p} \left( \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} |a_{ij}|^2 \right)^{1/2} = c_1 \sqrt{p}.$$

Hence,  $\|\nabla h\|_{\phi} \leq c$  for some numerical constant  $c$ . It follows that

$$\|P_{\mathcal{H}_2}(f)\|_2 \leq \sup_{\|\nabla h\|_{\phi} \leq c, \int h d\gamma = 0} \langle f, h \rangle_{L_2(\Omega, \mathcal{G}, \mathbb{P})} \leq \sup_{\|\nabla h\|_{\phi} \leq c} \int h f d\gamma - \int h d\gamma.$$

Since the fixed point algebra for the Ornstein–Uhlenbeck semigroup is trivial, Proposition 4.1 and Theorem 6.19 yield the second inequality. The first one follows from the same argument with the help of (6.19).  $\square$

It is well known that the Walsh system (or Rademacher sequence) has similar properties to the Gaussian system. Let  $(\varepsilon_i)_{i \geq 1}$  be a Rademacher sequence. It can be realized as the coordinate functions of the discrete cube  $\Omega = \{-1, 1\}^{\mathbb{N}}$  with product uniform probability  $\mathbb{P}$ . The Walsh system is given by  $\{\varepsilon_B = \prod_{j \in B} \varepsilon_j | B \subset \mathbb{N}, |B| < \infty\}$ . It was shown in [ELP08] that the  $L_p$  Poincaré inequalities hold for the Walsh system with the

number operator. The same proof as for Corollary 6.21 gives an estimate of Rademacher chaos of order 1 and 2. Let  $\mathcal{K}_2 = \text{span}\{\varepsilon_i \varepsilon_j : i \neq j\}$  and  $\mathcal{K}_1 = \text{span}\{\varepsilon_i : i \in \mathbb{N}\}$ .

**Corollary 6.22.** *Let  $P_{\mathcal{K}_2} : L_2(\Omega, \mathbb{P}) \rightarrow \mathcal{K}_2$  be the orthogonal projection. Then for any positive function  $f \in L_2(\Omega, \mathbb{P})$  with  $\|f\|_1 = 1$ ,*

$$\|P_{\mathcal{K}_1}(f)\|_2 \leq C\sqrt{\text{Ent}(f)}. \quad \|P_{\mathcal{K}_2}(f)\|_2 \leq C' \max\{\sqrt{\text{Ent}(f)}, \text{Ent}(f)\}.$$

*Remark 6.23.* Talagrand showed in [Tal96] that for Gaussian measure  $\gamma$  on  $\mathbb{R}^{\mathbb{N}}$ ,

$$W_2(\gamma, \mu) \leq \sqrt{2cD(\mu|\gamma)} \tag{6.25}$$

for any probability measure  $\mu$  absolutely continuous with respect to  $\gamma$  with  $c = 1$ . An alternative proof was given in [BG99]. Otto–Villani [OV00] proved (6.25) in general Riemannian setting under the assumption of log-Sobolev inequality. A simplified proof was given by Bobkov–Gentil–Ledoux in [BGL01], still using LSI. It would be interesting to compare  $Q_1$  and  $Q_\phi$  with  $W_1$  and  $W_2$  in this commutative setting. Obviously,  $Q_\phi \geq Q_1$ , and in the Gaussian setting  $Q_\phi$  is no less than  $W_2$ , because Talagrand’s transportation cost inequality is sharp in this case and  $Q_\phi$  is of the same order as  $\text{Ent}$  if the entropy is large. But the relationship between  $Q_\phi$  and  $W_2$  is not clear to us in general.

The interesting fact in (6.24) is that a phase transition may happen, which is different from Bobkov–Götze and Talagrand’s inequality. As observed in Example 3.15, Poincaré inequalities may still hold when LSI fails for non-diffusion semigroups. Especially for the more general noncommutative setting, an estimate like (6.24) seems desirable without knowing further information such as LSI. It is also reasonably effective in deducing transportation type inequalities compared with LSI.

# Chapter 7

## Examples and illustrations

In this chapter, we investigate a variety of examples which satisfy the assumptions of Theorem 3.8. All the consequences of this theorem we derived previously will hold for these examples. The key point is to check  $\Gamma_2 \geq \alpha\Gamma$ . It may be of independent interest because it means a strictly positive Ricci curvature from the geometric point of view. We consider the  $\Gamma_2$ -criterion for group von Neumann algebras in the first section. As explained in Section 3.4, these examples in fact satisfy the subgaussian Poincaré inequalities which are stronger than the weak subgaussian and subexponential inequalities obtained in Theorem 3.8. In the second section, we prove that the  $\Gamma_2$ -criterion is stable under tensor products and free products. Thus we can construct new examples from old ones using these algebraic operations. In the last section, we revisit the well known classical diffusion semigroups, mainly as illustrations of our theory.

### 7.1 $\Gamma_2$ -criterion for group von Neumann algebras

Let  $G$  be a countable discrete group with the conditionally negative length (cn-length) function  $\psi : G \rightarrow \mathbb{R}_+$ . Recall that any  $f \in \mathcal{L}(G)$  can be written as

$$f = \sum_{g \in G} \hat{f}(g) \lambda(g).$$

We consider the semigroup  $T_t$  acting on  $\mathcal{L}(G)$  given by  $T_t \lambda(g) = e^{-t\psi(g)} \lambda(g)$ . As shown in Lemma 2.6,  $T_t$  thus defined is a standard nc-diffusion semigroup. Recall that the Gromov form is defined as

$$K(g, h) = K_{g, h} = \frac{1}{2}(\psi(g) + \psi(h) - \psi(g^{-1}h)), \quad g, h \in G.$$

Given  $f = \sum_{x \in G} \hat{f}(x) \lambda(x)$  and  $g = \sum_{y \in G} \hat{g}(y) \lambda(y)$ , a straightforward calculation gives

$$\Gamma(f, g) = \sum_{x, y \in G} \bar{\hat{f}}(x) \hat{g}(y) K(x, y) \lambda(x^{-1}y),$$

$$\Gamma_2(f, g) = \sum_{x, y \in G} \bar{\hat{f}}(x) \hat{g}(y) K(x, y)^2 \lambda(x^{-1}y).$$

By virtue of Lemma 2.6 and Corollary 3.12, our Poincaré inequalities will follow if the  $\Gamma_2$ -criterion holds. With the help of Lemma 2.6 and Corollary 3.12, we only need to check the  $\Gamma_2$ -criterion on the finitely supported elements in order to fulfill the hypotheses of our theorems. We call

$$[\Gamma_2(f^i, f^j)] \geq \alpha[\Gamma(f^i, f^j)] \text{ for any } n \in \mathbb{N} \text{ and } f^1, \dots, f^n \in \mathcal{A}$$

the algebraic  $\Gamma_2$ -condition (or  $\Gamma_2$ -criterion) and abbreviate it to “ $\Gamma_2 \geq \alpha\Gamma$  in  $\mathcal{L}(G)$ ”. This is the theme of two sections from now on. This condition is seemingly stronger than needed. However,  $\Gamma_2(f, f) \geq \alpha\Gamma(f, f)$  for all  $f = \sum_{g \text{ finite}} \hat{f}_g \lambda(g) \in \mathcal{L}(G)$  amounts to check  $[\Gamma_2(\lambda(g_i), \lambda(g_j))] \geq \alpha[\Gamma(\lambda(g_i), \lambda(g_j))]$  for  $g_i \in G$ . This algebraic condition is also easier to check because it can be reduced to check the positivity of certain matrices as will be shown below. The following technical lemmas will be used repeatedly.

**Lemma 7.1.** *Suppose  $K = (K_{g,h})_{g,h \in G}$  is a matrix indexed by  $G$  with entries in  $\mathbb{C}$  and define a sesquilinear form  $\Theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ ,  $\Theta(f^i, f^j) = \sum_{g,h \in G} \bar{\hat{f}}^i(g) \hat{f}^j(h) K_{g,h} \lambda(g^{-1}h)$ . Then  $K$  is positive semidefinite if and only if  $\Theta$  is positive.*

*Proof.* Our proof is based on Lance [Lan73, Proposition 2.1]. Assume  $K$  is positive semidefinite. Write  $K = X^*X$  for  $X = (x_{g,h})$ ,  $x_{g,h} \in \mathbb{C}$ . Then  $K = \sum_l (\sum_{g,h} x_{lg}^* x_{lh} \otimes e_{g,h})$ . Given  $f^1, \dots, f^n \in \mathcal{A}$ , we have

$$\begin{aligned} \Theta(f^i, f^j) &= \sum_{l \in G} \sum_{g,h \in G} \bar{\hat{f}}^i(g) \hat{f}^j(h) x_{lg}^* x_{lh} \lambda(g^{-1}h) \\ &= \sum_{l \in G} \left( \sum_g \hat{f}^i(g) x_{lg} \lambda_g \right)^* \left( \sum_h \hat{f}^j(h) x_{lh} \lambda_h \right). \end{aligned}$$

Here we understand all indices are finite. Put  $\theta_l^i = \sum_g \hat{f}^i(g) x_{lg} \lambda_g$ . We then have

$$\Theta = \sum_{l \in G} \sum_{i,j=1}^n \theta_l^{i*} \theta_l^j \otimes e_{i,j} = \sum_{l \in G} \left( \sum_{i=1}^n \theta_l^i \otimes e_{1,i} \right)^* \left( \sum_{j=1}^n \theta_l^j \otimes e_{1,j} \right),$$

which is positive in  $M_n(\mathcal{L}(G))$ . Conversely, let  $(x_i) = \sum_i x_i \delta_{k_i} \in \ell_2(G)$  and write  $g_i = k_i^{-1} \in G$ . Then  $(x_i) = \sum_{i \in \mathbb{N}} x_i \delta_{g_i^{-1}}$ . Let  $f^i = \lambda(k_i)$  so that  $(\Theta(f^i, f^j)) = (K_{i,j} \lambda(k_i^{-1} k_j))$  is positive in  $M_n(\mathcal{L}(G))$  for all

$n \in \mathbb{N}$ . Then for  $h_i = x_i \delta_{g_i} \in \ell_2(G)$ , we have for all  $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leq \langle [\Theta(f^i, f^j)](h_1, \dots, h_n), (h_1, \dots, h_n) \rangle_{\ell_2^n(\ell_2(G))} \\ &= \sum_{i,j=1}^n \langle K_{i,j} \lambda(k_i^{-1} k_j) x_j \delta_{g_i}, x_i \delta_{g_i} \rangle_{\ell_2(G)} = \sum_{i,j=1}^n K_{i,j} x_j \bar{x}_i, \end{aligned}$$

which implies that  $K$  is positive semidefinite.  $\square$

The next lemma is useful when we deal with the product of groups. Note that

$$\mathcal{L}\left(\prod_{i=1}^m G_i\right) \cong \bar{\otimes}_{i=1}^m \mathcal{L}(G_i).$$

The identification is given by  $\lambda(g_1, \dots, g_m) \mapsto \lambda(g_1) \otimes \dots \otimes \lambda(g_m)$  for  $g_i \in G_i$ . We associated the form  $\Gamma_2$  to a matrix  $K$  as follows:  $\Gamma_2^K(f, g) = \sum_{x,y \in G} \tilde{f}(x) \hat{g}(y) K_{x,y}^2 \lambda(x^{-1}y)$ . In what follows the matrix  $K$  will be the Gromov form. If  $K = K_1 \otimes K_2$ , then it is easy to check  $\Gamma^K(f^i \otimes g^i, f^j \otimes g^j) = \Gamma^{K_1}(f^i, f^j) \otimes \Gamma^{K_2}(g^i, g^j)$  for  $f_i \in \mathcal{L}(G_1), g_i \in \mathcal{L}(G_2)$ .

**Lemma 7.2.** *Let  $(K_i)_{i=1}^m$  be positive semidefinite matrices and  $\Gamma^{K_i}$  the associated gradient forms in the sense of Lemma 7.1. Suppose  $\Gamma_2^{K_i} \geq \alpha \Gamma^{K_i}$ . Then*

$$\Gamma_2^K \geq \alpha \Gamma^K,$$

where  $K = \sum_{i=1}^m \mathbb{1} \otimes \dots \otimes K_i \otimes \dots \otimes \mathbb{1}$  with  $K_i$  in the  $i$ -th position and in what follows  $\mathbb{1}$  always denotes the matrix with every entry equal to 1.

*Proof.* In light of Lemma 7.1 it suffices to verify  $K \bullet K \geq \alpha K$ . Here and in the following  $A \bullet B$  denotes the Schur product of matrix. Note that trivially  $\mathbb{1} \geq 0$ . Since  $K_i \geq 0$ , all the “cross terms” of the form

$$\mathbb{1} \otimes \dots \otimes K_{i_1} \otimes \dots \otimes K_{i_2} \otimes \dots \otimes \mathbb{1}$$

are nonnegative matrices for all  $1 \leq i_1 < i_2 \leq m$ . It follows that

$$K \bullet K \geq \sum_{i=1}^m \mathbb{1} \otimes \dots \otimes (K_i \bullet K_i) \otimes \dots \otimes \mathbb{1} \geq \alpha K. \quad \square$$



### 7.1.1 The free groups

Let  $\mathbb{F}_n$  denote the free group on  $n$  generators with length function  $\psi = |\cdot|$ , where for  $g \in \mathbb{F}_n$ ,  $|g|$  is the length of (the freely reduced form of)  $g$ . Note that the Gromov form  $K(g, h) = |\min(g, h)| := \max\{|w| : g = wg', h = wh'\}$  where  $\min(g, h)$  is the longest common prefix subword of  $g$  and  $h$ . It is well known that  $\psi$  is conditionally negative due to Haagerup [Haa78].

**Proposition 7.3.**  $\Gamma_2 \geq \Gamma$  holds in  $\mathcal{L}(\mathbb{F}_n)$  for the semigroup  $(e^{-t\psi})$  where  $\psi$  is defined as above.

*Proof.* For a freely reduced word  $x \in \mathbb{F}_n$ , write  $g_i \prec g$  for the prefix subword of  $g$  with length  $i$ . Following Haagerup's construction, we define a map

$$V : \mathbb{F}_n \rightarrow \ell_2(\mathbb{F}_n), \quad g \mapsto V(g) = \sum_{g_i \prec g} \sqrt{2(i-1)} \delta_{g_i}.$$

Then we have

$$\tilde{K}_{g,h} := K_{g,h}^2 - K_{g,h} = \langle V(g), V(h) \rangle_{\ell_2(\mathbb{F}_n)} = V(g)^* V(h),$$

where  $V(g)^*$  is a row vector and  $V(h)$  a column vector. It follows that  $\tilde{K} = (\tilde{K}_{g,h})_{g,h}$  is a positive semidefinite matrix. We deduce from Lemma 7.1 that  $\Gamma_2 \geq \Gamma$ .  $\square$

The particular case  $n = 1$  gives some interesting results in classical Fourier analysis. Indeed,  $\mathcal{L}(\mathbb{F}_1) = \mathcal{L}(\mathbb{Z}) = L_\infty(\mathbb{T})$  and  $L_p(\mathcal{L}(\mathbb{F}_1)) = L_p(\mathbb{T})$  after identifying  $\lambda(k)(x) = e^{2\pi i k x}$ . In this case

$$K(j, k) = \begin{cases} \min(|j|, |k|), & jk > 0, \\ 0, & \text{otherwise.} \end{cases}$$

**Corollary 7.4.** Let  $2 \leq p < \infty$ . Then there exists constants  $C$  such that for all  $f \in L_p(\mathbb{T})$ , we have

$$\|f - \hat{f}(0)\|_p \leq C\sqrt{p} \left\| \sum_{j,k \in \mathbb{Z}, jk > 0} \bar{\hat{f}}(j) \hat{f}(k) \min(|j|, |k|) e^{2\pi i(k-j)} \right\|_p^{1/2}.$$

*Remark 7.5.* Observe that this example is purely commutative. However, commutative probability theory seems insufficient to establish these inequalities. Intuitively, the multiplier  $|j|$  corresponds to  $\Delta^{1/2}$ . The Markov process generated by  $\Delta^{1/2}$  is the Cauchy process with discontinuous path. The classical diffusion theory does not apply here. But it is still nc-diffusion so that our noncommutative theory is essential in this regard. In general, whenever the process has discontinuous path but its semigroup still satisfies our assumptions, the noncommutative theory seems to be a natural choice due to the existence of Markov

dilation with a.u. continuous path as stated in Theorem 2.2. We will have more examples of this kind in the following.

*Remark 7.6.* It was shown in [JM10, Remark 1.3.2] that

$$\|T_t : L_1^0(\mathcal{L}(\mathbb{F}_n)) \rightarrow L_\infty(\mathcal{L}(\mathbb{F}_n))\| \leq Ct^{-3}.$$

Therefore, Corollary 6.11 and Corollary 6.14 give two different ways to prove the transportation inequality (6.19) for  $\mathcal{L}(\mathbb{F}_n)$ .

### 7.1.2 Application to the noncommutative tori $\mathcal{R}_\Theta$

We recall the definition following [JMP10]. Let  $\Theta$  be a  $d \times d$  antisymmetric matrix with entries  $0 \leq \theta_{ij} < 1$ . The noncommutative torus (or the rotation algebra) with  $d$  generators associated to  $\Theta$  is the von Neumann algebra  $\mathcal{R}_\Theta$  generated by  $d$  unitaries  $u_1, \dots, u_d$  satisfying  $u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$ . Every element of  $\mathcal{R}_\Theta$  is in the closure of the span of words of the form  $w_k = u_1^{k_1} \dots u_d^{k_d}$  for  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ .  $\mathcal{R}_\Theta$  admits a unique normal faithful trace  $\tau$  given by  $\tau(x) = \hat{x}(0)$  where  $x = \sum_{k \in \mathbb{Z}^d} \hat{x}(k) u_1^{k_1} \dots u_d^{k_d} \in \mathcal{R}_\Theta$ . Our goal is to show that  $\mathcal{R}_\Theta$  admits a standard nc-diffusion semigroup with the  $\Gamma_2$ -criterion. We start with the von Neumann algebra of  $\mathbb{Z}^d$ . It is well known that  $\mathcal{L}(\mathbb{Z}^d) \cong L_\infty(\mathbb{T}^d)$ . We define  $\psi(k) = \|k\|_1 = \sum_{i=1}^d |k_i|$  for  $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ . Clearly,  $\psi$  is a cn-length function and thus generate a standard nc-diffusion semigroup  $P_t$  by Lemma 2.6. In fact  $P_t = \tilde{P}_t^{\otimes d}$  where  $\tilde{P}_t$  is the Poisson semigroup on  $L_\infty(\mathbb{T})$ .

**Proposition 7.7.** *Let  $\Gamma$  be the gradient form associated to  $P_t$ . Then  $\Gamma_2 \geq \Gamma$  in  $\mathcal{L}(\mathbb{Z}^d)$ .*

*Proof.* Let  $K^d$  be the Gromov form associated with  $\psi$ . A calculation shows that  $K^d(j, k) = K(j_1, k_1) + \dots + K(j_d, k_d)$  for  $j = (j_1, \dots, j_d), k = (k_1, \dots, k_d) \in \mathbb{Z}^d$ , where  $K$  is the Gromov form of  $\mathbb{Z} = \mathbb{F}_1$  considered in the proceeding section. Alternatively, we may write  $K^d = \sum_{i=1}^d \mathbf{1} \otimes \dots \otimes K \otimes \dots \otimes \mathbf{1}$  where  $K$  is in the  $i$ th position. But we know from Proposition 7.3 that  $\Gamma_2 \geq \Gamma$  in  $\mathcal{L}(\mathbb{Z})$ . The assertion follows from Lemma 7.1.  $\square$

**Proposition 7.8.**  *$\mathcal{R}_\Theta$  admits a standard nc-diffusion semigroup with  $\Gamma_2 \geq \Gamma$ .*

*Proof.* Let  $k \in \mathbb{Z}^d$ . Consider an action  $\alpha : \mathbb{T}^d \rightarrow \text{Aut}(\mathcal{R}_\Theta)$  given by: for  $s \in \mathbb{T}^d$ ,

$$\alpha_s(u_1^{k_1} \dots u_d^{k_d}) = e^{2\pi i \sum_{j=1}^d k_j s_j} u_1^{k_1} \dots u_d^{k_d}.$$

It is easy to check that  $\alpha_s$  is a trace preserving automorphism. Define a map

$$\pi : \mathcal{R}_\Theta \rightarrow L_\infty(\mathbb{T}^d) \bar{\otimes} \mathcal{R}_\Theta, \quad w_k = u_1^{k_1} \cdots u_d^{k_d} \mapsto \pi(w_k)(s) = \alpha_s(w_k) = e^{2\pi i \langle k, s \rangle} u_1^{k_1} \cdots u_d^{k_d}.$$

Then  $\pi$  is an injective  $*$ -homomorphism. Define  $T_t : \mathcal{R}_\Theta \rightarrow \mathcal{R}_\Theta, T_t(w_k) = e^{-t\|k\|_1} w_k$ . We claim that  $(T_t)_{t \geq 0}$  is the desired semigroup. Indeed, by Lemma 7.22 and Proposition 7.7,  $P_t \otimes Id$  acting on  $L_\infty(\mathbb{T}^d) \bar{\otimes} \mathcal{R}_\Theta$  is a standard nc-diffusion semigroup and satisfies  $\Gamma_2 \geq \Gamma$ . Then since  $\pi$  is injective and

$$P_t \otimes Id(\pi(w_k)) = e^{-t\|k\|_1} e^{2\pi i \langle k, \cdot \rangle} \otimes u_1^{k_1} \cdots u_d^{k_d} = \pi(T_t(w_k))$$

leaves  $\pi(\mathcal{R}_\Theta)$  invariant, we deduce that  $T_t$  is a standard nc-diffusion semigroup acting on  $\mathcal{R}_\Theta$  with  $\Gamma_2 \geq \Gamma$ .  $\square$

### 7.1.3 The finite cyclic group $\mathbb{Z}_n$

We consider the group von Neumann algebra  $\mathcal{L}(\mathbb{Z}_n)$  in this section. Let  $(e_j)_{j=1}^n$  be the standard basis of  $\mathbb{C}^n$ . Each  $e_j$  can be regarded as a vector in  $\mathbb{R}^{2n}$  by canonical identification. Given  $k \in \mathbb{Z}_n$ , define the  $2n \times 2n$  diagonal matrix  $\alpha_k = (e^{2\pi i k j / n})_{j=0}^{n-1}$  where each  $e^{2\pi i k j / n}$  is on diagonal and is identified with the  $2 \times 2$  rotation matrix

$$\begin{pmatrix} \cos(2\pi k j / n) & -\sin(2\pi k j / n) \\ \sin(2\pi k j / n) & \cos(2\pi k j / n) \end{pmatrix}.$$

Consider the finite cyclic group  $\mathbb{Z}_n$  with 1-cocycle structure  $(b, \alpha, \mathbb{R}^{2n})$ , where

$$b(k) = \frac{1}{\sqrt{n}} \left( \sum_{j=1}^n \alpha_k(e_j) - e_j \right) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \begin{pmatrix} \cos(2\pi k(j-1)/n) - 1 \\ \sin(2\pi k(j-1)/n) \end{pmatrix} \otimes e_j.$$

Then the length function  $\psi$  given by  $\psi(g) = \|b(g)\|_2^2$  is conditionally negative; see for example [BO08, Appendix D].

**Lemma 7.9.** *Let  $K(k, h)$  be the Gromov form. Then  $K(k, h) = \langle b(k), b(h) \rangle$ .*

*Proof.* Since the length function  $\psi(k) = \|b(k)\|^2$ , by the cocycle property,

$$\begin{aligned} K(k, h) &= \frac{1}{2} (\|b(k)\|^2 + \|b(h)\|^2 - \|b(h-k)\|^2) \\ &= \frac{1}{2} (\|b(-k)\|^2 + \|\alpha_{-k}(b(h))\|^2 - \|\alpha_{-k}(b(h)) + b(-k)\|^2) \\ &= -\langle b(-k), \alpha_{-k}(b(h)) \rangle = -\langle \alpha_k(b(-k)), b(h) \rangle \\ &= \langle b(k), b(h) \rangle. \end{aligned}$$

$\square$

Clearly,  $K(k, h) = 0$  if  $k = 0$  or  $h = 0$ . For  $k, h \neq 0$ , a computation gives

$$\begin{aligned} K(k, h) &= \frac{1}{n} \sum_{j=0}^{n-1} [(1 - \cos(2\pi k j/n))(1 - \cos(2\pi h j/n)) + \sin(2\pi k j/n) \sin(2\pi h j/n)] \\ &= 1 + \frac{1}{n} \sum_{j=0}^{n-1} \cos\left(\frac{2\pi(k-h)j}{n}\right) = 1 + \delta_{k,h}, \end{aligned}$$

where  $\delta_{k,h}$  is the Kronecker delta function. It follows that  $\psi(k) = 2(1 - \delta_{k,0})$ . For reasons that will become clear later, we normalize  $\psi$  and still denote it by  $\psi$  so that  $\psi(k) = 1 - \delta_{k,0}$  for  $k \in \mathbb{Z}_n$ . Then the associated Gromov form satisfies  $K_{k,h} = \frac{1}{2}(1 + \delta_{k,h})$  for  $k, h \neq 0$  and  $(K_{k,h}^2 - \frac{1}{2}K_{k,h}) \geq 0$ . It is an immediate consequence of Lemma 7.1 that  $\Gamma_2 \geq \frac{1}{2}\Gamma$  in  $\mathcal{L}(\mathbb{Z}_n)$ . In fact, we can do better.

**Proposition 7.10.** *For all  $0 < \alpha \leq \frac{n+2}{2n}$ , we have  $\Gamma_2 \geq \alpha\Gamma$  in  $\mathcal{L}(\mathbb{Z}_n)$ . Moreover,  $\alpha_n = \frac{n+2}{2n}$  is the largest possible  $\alpha$  with the  $\Gamma_2$ -criterion.*

*Proof.* Note that the  $n \times n$  matrix  $K$  can be written as a block matrix

$$K = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}(I_{n-1} + \mathbb{1}_{n-1}) \end{pmatrix}$$

where  $I_{n-1}$  is the  $n-1$  dimensional identity matrix and every entry of  $\mathbb{1}_{n-1}$  is 1. Write  $\widehat{K} = \frac{1}{2}(I_{n-1} + \mathbb{1}_{n-1})$ . Since  $\mathbb{1}_{n-1} \leq (n-1)I_{n-1}$ , for  $0 < \alpha \leq \frac{n+2}{2n}$  we have

$$\begin{aligned} 4\widehat{K} \bullet \widehat{K} - 4\alpha\widehat{K} &= (3 - 2\alpha)I_{n-1} - (2\alpha - 1)\mathbb{1}_{n-1} \\ &\geq (2 + n - 2\alpha n)I_{n-1} \geq 0. \end{aligned}$$

Plugging  $\mathbf{x} = (\frac{1}{\sqrt{n-1}}, \dots, \frac{1}{\sqrt{n-1}})$  into  $\mathbf{x}'(\widehat{K} \bullet \widehat{K} - \alpha\widehat{K})\mathbf{x} \geq 0$  reveals that  $\alpha_n = \frac{n+2}{2n}$  is sharp. Then Lemma 7.1 leads to the  $\Gamma_2$ -criterion.  $\square$

#### 7.1.4 Word length on $\mathbb{Z}_n$

Since one may embed  $\mathbb{Z}_n$  to  $\mathbb{Z}_{2n}$ , we always assume that  $n$  is an even integer in this example. Consider the word length of  $k \in \mathbb{Z}_n$  in the Cayley graph of  $\mathbb{Z}_n$  given by  $\psi(k) = \min\{k, n-k\}$ . It is known that  $\psi$  is conditionally negative; see [JPPP13]. One can also show this fact from the following explicit construction

of 1-cocycles. Let  $(e_i)_{i=1}^{n/2}$  be an orthonormal basis of  $\mathbb{R}^{n/2}$ . Define  $b : \mathbb{Z}_n \rightarrow \mathbb{R}^{n/2}$  to be

$$b(k) = \begin{cases} 0, & k = 0, \\ \sum_{i=1}^k e_i, & k = 1, \dots, n/2, \\ \sum_{i=k-n/2+1}^{n/2} e_i, & k = n/2 + 1, \dots, n-1, \end{cases}$$

and  $\alpha : \mathbb{Z}_n \rightarrow O(\mathbb{R}^{n/2})$  given by  $\alpha_1(e_j) = e_{j+1}$  for  $j = 1, \dots, n/2 - 1$  and  $\alpha_1(e_{n/2}) = -e_1$ . It can be checked that  $b$  is a 1-cocycle into the representation  $(\alpha, \mathbb{R}^{n/2})$  and  $\psi(k) = \|b(k)\|^2$ . It follows that the Gromov form  $K$  is positive semidefinite. We will show that  $[K(i, j)^2 - K(i, j)]_{i, j=1}^{n-1}$  is a positive semidefinite matrix. By Lemma 7.1, we have the following result (compare it with  $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$  for the other choice of  $\psi$ ).

**Proposition 7.11.**  $\Gamma_2 \geq \Gamma$  in  $\mathcal{L}(\mathbb{Z}_n)$ .

We write  $K_n$  for the Gromov form of  $\mathbb{Z}_n$ . Let us take away the trivial  $K_n(0, i)$ 's and view  $K_n$  as an  $(n-1) \times (n-1)$  matrix. We need to show  $K_n \bullet K_n - K_n$  is positive definite. Here  $K_n \bullet K_n$  denotes the Schur product. For all even integers  $2 \leq m \leq n-2$ , we write  $\tilde{K}_m$  for the  $(n-1) \times (n-1)$  matrix obtained from enlarging the size of  $K_m$  by adding surrounding 0's so that  $K_m(m/2, m/2) = \tilde{K}_m(n/2, n/2)$ . In other words,

$$\tilde{K}_m(i, j) = K_m(i - \frac{n-m}{2}, j - \frac{n-m}{2}) \quad (7.1)$$

whenever the right-hand side is well-defined. We claim that

$$K_n \bullet K_n - K_n = 2 \sum_{\ell=1}^{n/2-1} \tilde{K}_{2\ell}. \quad (7.2)$$

Since each  $\tilde{K}_m$  is positive semidefinite, (7.2) will complete the proof.

In fact, note that  $K_n$  satisfies the symmetric property

$$K_n(j, i) = K_n(i, j) = K(n-j, n-i).$$

This is equivalent to saying that  $K_n$  is symmetric along the two diagonals. Therefore we only need to verify

(7.2) entrywise in the block  $B_n := \{(i, j) : 1 \leq i \leq j \leq n - i\}$ . In  $B_m$  for general even  $m$ , we have

$$K_m(i, j) = \begin{cases} i, & (i, j) \in B_m^1 := \{(i, j) : 1 \leq i \leq j \leq \frac{m}{2}\}, \\ \frac{m}{2} - j + i, & (i, j) \in B_m^2 := \{(i, j) : 1 \leq i \leq \frac{m}{2} < j \leq i + \frac{m}{2} - 1\}, \\ 0, & (i, j) \in B_m \setminus (B_m^1 \cup B_m^2). \end{cases}$$

By our construction, (7.2) is trivial if  $K_n(i, j) = 0$ . For  $(i, j) \in B_n^1$ ,  $\tilde{K}_{2\ell}$  is nonzero only if  $i \geq n/2 - \ell + 1, j \geq n/2 - \ell + 1$ , and for these  $(i, j)$ 's,  $(i - \frac{n}{2} + \ell, j - \frac{n}{2} + \ell)$ 's are in the block  $B_{2\ell}^1$  of  $K_{2\ell}$ . Hence, the right-hand side of (7.2) is

$$2 \sum_{\ell=n/2-i+1}^{n/2-1} (i - \frac{n}{2} + \ell) = i(i-1) = K_n(i, j)^2 - K_n(i, j).$$

For  $(i, j) \in B_n^2$ ,  $\tilde{K}_{2\ell}$  is nonzero only if  $j - i \leq \ell - 1$ , and for these  $(i, j)$ 's,  $(i - \frac{n}{2} + \ell, j - \frac{n}{2} + \ell)$ 's are in the block  $B_{2\ell}^2$  of  $K_{2\ell}$ . Then the right-hand side of (7.2) is

$$2 \sum_{\ell=j-i+1}^{n/2-1} (i - j + \ell) = \left(\frac{n}{2} + i - j\right) \left(\frac{n}{2} + i - j - 1\right) = K_n(i, j)^2 - K_n(i, j).$$

### 7.1.5 The discrete Heisenberg group $H_3(\mathbb{Z}_n)$

We consider the Heisenberg group  $H_3(\mathbb{Z}_n) = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$  over  $\mathbb{Z}_n$  as a subalgebra of  $M_3(\mathbb{Z}_n)$  as follows:

$$\begin{pmatrix} 1 & b & a \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b' & a' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+b' & a+a'+bc' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix};$$

see for example [Dav96, Section VII.5] for more details. We will write  $H$  for  $H_3(\mathbb{Z}_n)$  as long as there is no confusion. The multiplication here is given by

$$(a, b, c)(a', b', c') = (a + a' + bc', b + b', c + c'), \quad (a, b, c), (a', b', c') \in H.$$

Other multiplications have been considered in the literature.

**Proposition 7.12.** *Let  $\psi(a, b, c) = 2 - \delta_{b,0} - \delta_{c,0}$ . Then*

1.  $\psi$  is conditionally negative and thus the semigroup  $(T_t)$  determined by  $\psi$  is a standard nc-diffusion semigroup.

2. Let  $\Gamma$  be the gradient form associated to  $\psi$ . Then  $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$  in  $\mathcal{L}(H)$ .

*Proof.* (1) The length function of  $\mathbb{Z}_n$  considered in Section 7.1.3 is given by  $\tilde{\psi}(k) = (1 - \delta_{k,0})$ , which extends to  $\mathbb{Z}_n \times \mathbb{Z}_n$  as  $\tilde{\psi}(k, l) = (1 - \delta_{k,0}) + (1 - \delta_{l,0})$ . Define a group homomorphism

$$\beta : H \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n, \quad (a, b, c) \mapsto (b, c).$$

Since  $\tilde{\psi}$  is conditionally negative, it follows from the definition that  $\psi = \tilde{\psi} \circ \beta$  is also conditionally negative. Lemma 2.6 yields that  $(T_t)$  is a standard nc-diffusion semigroup.

(2) Let  $K$  and  $\tilde{K}$  be the Gromov form of  $(H, \psi)$  and  $(\mathbb{Z}_n, \tilde{\psi})$  respectively. A calculation shows that for indices  $(a, b, c), (a', b', c') \in H$ ,

$$\begin{aligned} K((a, b, c), (a', b', c')) &= \tilde{K}(b, b') + \tilde{K}(c, c') \\ &= (\tilde{K} \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{K})((b, b') \otimes (c, c')). \end{aligned}$$

By Proposition 7.10 and Lemma 7.2 with  $m = 2$ , we have  $\Gamma_2^K \geq \frac{n+2}{2n}\Gamma^K$  in  $\mathcal{L}(H)$ , as desired.  $\square$

Let  $e_{i,j}$  be the standard basis of the matrix algebra  $M_n(\mathbb{C})$  and  $\delta_j$  the standard basis of  $\ell_2(\mathbb{Z}_n)$ . Define the diagonal matrix  $u_k = \sum_{j=1}^n e^{2\pi i k(j-1)/n} \otimes e_{j,j}$  and the shift operator  $v_l(\delta_j) = \delta_{j+l}$  which is nothing but the left regular representation of  $\mathbb{Z}_n$  on  $\ell_2(\mathbb{Z}_n)$ . It is easy to see that  $u_k, v_l \in M_n = B(\ell_2(\mathbb{Z}_n))$  and they satisfy  $u_k v_l = e^{2\pi i k l/n} v_l u_k$ .

**Proposition 7.13.** *Let  $\mathcal{L}(H)$  be the group von Neumann algebra of  $H$ . Then*

$$\mathcal{L}(H) \cong L_\infty(\mathbb{Z}_n^2) \oplus M_n \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_{n-1},$$

where  $\mathcal{M}_x, x = 2, \dots, n-1$  are von Neumann algebras acting on  $\ell_2(\mathbb{Z}_n^2)$ . Moreover, if  $T_t$  is the semigroup associated to  $\psi(a, b, c) = 2 - \delta_{b,0} - \delta_{c,0}$ , then  $T_t$  leaves each component invariant and  $T_t|_{\mathcal{M}_x}$  is a standard nc-diffusion semigroup.

*Proof.* Let us first determine the center of  $\mathcal{L}(H)$  denoted by  $\mathcal{Z}$ . The identity

$$\lambda(a, b, c)\lambda(a', b', c') = \lambda(a', b', c')\lambda(a, b, c)$$

for all  $(a', b', c') \in H$  holds if and only if  $b = c = 0$ . Thus  $\mathcal{L}(\mathbb{Z}_n, 0, 0) \subset \mathcal{Z}$ . Let  $\mathcal{F}$  denote the discrete Fourier

transform of the first component on  $\ell_2(H)$ . For  $\delta_{(x,0,0)} \in \ell_2(\mathbb{Z}_n, 0, 0)$ , we have

$$\mathcal{F}(\delta_{(x,0,0)}) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-\frac{2\pi i k x}{n}} \delta_{(k,0,0)}.$$

A calculation gives

$$\mathcal{F}\lambda(a, 0, 0)\mathcal{F}^{-1}\delta_{(x,0,0)} = e^{-\frac{2\pi i a x}{n}} \delta_{(x,0,0)}.$$

This shows that  $\mathcal{F}\mathcal{L}(\mathbb{Z}_n, 0, 0)\mathcal{F}^{-1} = \{e^{-\frac{2\pi i a \cdot}{n}} : a \in \mathbb{Z}_n\}'' = L_\infty(\mathbb{Z}_n)$ . Since for fixed  $x \in \mathbb{Z}_n$ ,  $\delta_{(x, \cdot, \cdot)} \in \ell_2(\mathbb{Z}_n^2)$ , we have the Hilbert space decomposition  $\ell_2(H) = \bigoplus_{x \in \mathbb{Z}_n} \ell_2^x(\mathbb{Z}_n^2)$ , where the superscript  $x$  is used to distinguish different copies. We may drop  $x$  if there is no ambiguity. Then by the decomposition theorem of von Neumann algebras for subalgebras of the center (see for example [Tak02, Theorem IV.8.21]),

$$\mathcal{L}(H) \cong \bigoplus_{x \in \mathbb{Z}_n} \mathcal{M}_x,$$

where  $\mathcal{M}_x$  is determined by the unitary  $\mathcal{F}$ , the discrete Fourier transform on the first component. Let  $q_x \in \ell_\infty(\mathbb{Z}_n, 0, 0)$  be the central projection given by  $q_x : \ell_2(H) \rightarrow \text{span}\{\delta_{(x,y,z)} : y, z \in \mathbb{Z}_n\}$ . Put  $p_x = \mathcal{F}^{-1}q_x\mathcal{F} \in \mathcal{L}(\mathbb{Z}_n, 0, 0)$ . Then  $p_x$  is a central projection,  $\sum_{x=0}^{n-1} p_x = 1$  and  $p_x\mathcal{L}(H) = \mathcal{F}^{-1}\mathcal{M}_x\mathcal{F}$ . We observe that  $T_t\lambda(a, 0, 0) = \lambda(a, 0, 0)$ . Hence  $\mathcal{L}(\mathbb{Z}_n, 0, 0)$  is contained in the multiplicative domain of  $T_t$ . Then by the property of multiplicative domain (see for example [Pau02, Theorem 3.18]) for all  $x \in \mathbb{Z}_n, \xi \in \mathcal{L}(H)$ ,  $T_t(p_x\xi) = p_xT_t(\xi) \in p_x\mathcal{L}(H)$ . Let  $\eta \in \mathcal{M}_x$  with  $p_x\xi = \mathcal{F}^{-1}\eta\mathcal{F}$ . We define  $\tilde{T}_t\eta = \mathcal{F}T_t(p_x\xi)\mathcal{F}^{-1}$ . Since  $\mathcal{F}$  is a unitary,  $\tilde{T}_t$  restricted to  $\mathcal{M}_x$  is a standard nc-diffusion semigroup. By abuse of notation, we will write  $T_t$  for  $\tilde{T}_t$  on  $\mathcal{M}_x$  in the future.

To get more precise description of  $\mathcal{M}_x$ , we define a family of maps for  $x \in \mathbb{Z}_n$

$$\pi_x : \mathcal{L}(H) \rightarrow B(\ell_2^x(\mathbb{Z}_n^2)), \quad \lambda(a, b, c) \mapsto \pi_x(\lambda(a, b, c)),$$

where  $\pi_x(\lambda(a, b, c))$  acts on  $\delta_{(k,l)}$  by

$$\mathcal{F}\lambda(a, b, c)\mathcal{F}^{-1}\delta_{(x,k,l)} = e^{-\frac{2\pi i x(a+bl)}{n}} \lambda_{(b,c)}\delta_{(k,l)}.$$

Here  $\lambda(b, c)$  is the shift operator on  $\ell_2(\mathbb{Z}_n^2)$  given by  $\lambda(b, c)\delta_{(k,l)} = \delta_{(k+b, l+c)}$ . Then

$$\mathcal{M}_x = \{\pi_x(\lambda(a, b, c)) : (a, b, c) \in H\}'' = \{e^{-\frac{2\pi i a x}{n}} v_b \otimes (v_c u_{-xb}) : (a, b, c) \in H\}''.$$



Here we have used the convention  $\lambda(b, c) = v_b \otimes v_c$ . If  $x = 0$ , we have

$$\mathcal{M}_0 = \{\lambda(b, c) : (b, c) \in \mathbb{Z}_n^2\}'' = \mathcal{L}(\mathbb{Z}_n^2) = L_\infty(\mathbb{Z}_n^2).$$

If  $x = 1$ , it can be checked that  $\{v_c u_{-b} : (b, c) \in \mathbb{Z}_n^2\}'' = M_n$ ; see for example [Dav96, Theorem VII.5.1].

Define for  $(b, c) \in \mathbb{Z}_n^2$

$$\rho(v_b \otimes (v_c u_{-b})) = v_c u_{-b}.$$

Then  $\rho$  is a  $*$ -isomorphism and thus  $\mathcal{M}_1 = M_n$ . □

Consider the semigroup  $T_t$  acting on  $M_n(\mathbb{C})$  defined by  $T_t|_{M_n(\mathbb{C})}$  in the preceding proposition. Explicitly,  $T_t$  is determined by  $T_t(v_c u_b) = e^{-t\psi(b, c)}(v_c u_b)$  where  $\psi(b, c) = 2 - \delta_{b,0} - \delta_{c,0}$ . Then the  $\Gamma_2$ -criterion for  $M_n$  follows from Proposition 7.12. We record this fact below.

**Proposition 7.14.**  *$M_n$  admits a standard nc-diffusion semigroup  $(T_t)_{t \geq 0}$ . Let  $\Gamma^{M_n}$  be the gradient form associated to  $T_t$ . Then  $\Gamma_2^{M_n} \geq \frac{n+2}{2n} \Gamma^{M_n}$  in  $M_n$ .*

By the same argument as above, using the word length function on  $\mathbb{Z}_n$  as discussed in the preceding section, we find a new 1-cocycle on  $H_3(\mathbb{Z}_n)$  with cn-length function  $\psi(a, b, c) = |b| + |c|$  for  $(a, b, c) \in H_3(\mathbb{Z}_n)$ , where  $|b| = \min\{b, n - b\}$  is the word length. The semigroup generated by this 1-cocycle satisfies the  $\Gamma_2$ -criterion. In particular, let  $T_t(v_c u_b) = e^{-t(|c| + |b|)} v_c u_b$  act on  $M_n$ . By the same reasoning as above,  $T_t$  is a new semigroup acting on the matrix algebras which is subgaussian.

### 7.1.6 Application to the generalized Walsh system

Let us recall some basic facts about the Walsh system following [ELP08]. Let

$$\Omega_n^m = \{1, e^{2\pi i/n}, e^{2\pi i 2/n}, \dots, e^{2\pi i(n-1)/n}\}^m$$

be the  $m$ -dim discrete cube equipped with uniform probability measure  $P$ . Let  $\omega_j, j = 1, \dots, m$  denote the  $j$ th coordinate function on  $\Omega_n^m$ . For a nonempty subset  $B \subset \{1, \dots, m\}$  and  $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_n^m$ , define

$$\omega_B(\mathbf{x}) = \prod_{j \in B} \omega_j^{x_j},$$

and  $\omega_\emptyset = 1$ . Put  $G = \{\omega_B(\mathbf{x}) : B \subset \{1, \dots, m\}, \mathbf{x} \in \mathbb{Z}_n^m\}$ . Then  $G$  is clearly a group and  $L_\infty(\Omega_n^m)$  is spanned by the elements of  $G$ .

We consider the abelian group  $\mathbb{Z}_n^m$ . Define

$$\psi(x_1, \dots, x_m) = (m - \delta_{x_1} - \dots - \delta_{x_m}), \quad (x_1, \dots, x_m) \in \mathbb{Z}_n^m,$$

where  $\delta_x = \delta_{x,0}$ . Given  $\mathbf{x} = (x_1, \dots, x_m)$ , put  $B_{\mathbf{x}} = \{i : x_i \neq 0, i = 1, \dots, m\}$ . Clearly  $\psi(x_1, \dots, x_m) = |B_{\mathbf{x}}|$ , where  $|B|$  is the cardinality of  $B$ . An argument similar to the proof of Proposition 7.12 shows that  $\psi$  is a cn-length function and the associated  $\Gamma$  form satisfies

$$\Gamma_2 \geq \frac{n+2}{2n} \Gamma \text{ in } \mathcal{L}(\mathbb{Z}_n^m). \quad (7.3)$$

Here the constant  $\frac{n+2}{2n}$  is given by Proposition 7.10. Define a map

$$\beta : \mathbb{Z}_n^m \rightarrow G, \quad \mathbf{x} = (x_1, \dots, x_m) \mapsto \prod_{j \in B_{\mathbf{x}}} \omega_j^{x_j}.$$

It is easy to check that  $\beta$  is a group isomorphism from  $\mathbb{Z}_n^m$  to  $G$ . The idea here is to convert addition to multiplication. Under the identification  $\beta$ ,  $\mathcal{L}(\mathbb{Z}_n^m) = L_{\infty}(\Omega_n^m)$  and thus every  $f \in \mathcal{L}(\mathbb{Z}_n^m)$  can be written as

$$f = \mathbb{E}(f) + \sum_{\mathbf{x} \in \mathbb{Z}_n^m, \mathbf{x} \neq \mathbf{0}} \hat{f}_{\mathbf{x}} \prod_{j \in B_{\mathbf{x}}} \omega_j^{x_j},$$

where  $\mathbb{E}(f) = \tau(f)$  is the expectation associated to the uniform probability. By abuse of notation, we still denote by  $\psi$  the cn-length function induced by  $\beta$  on  $\{\omega_B\}$ , i.e.

$$\psi(\omega_{B_{\mathbf{x}}}) = \psi(\beta(\mathbf{x})) := \psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}_n^m.$$

Then we have

$$\psi(\omega_{B_{\mathbf{x}}}) = \psi(x_1, \dots, x_m) = |B_{\mathbf{x}}|.$$

Therefore the infinitesimal generator  $A$  of the heat semigroup  $T_t$  in this case is the number operator for the generalized Walsh system which counts non-zero elements

$$A\omega_B = |B|\omega_B.$$

Moreover, it follows from (7.3) that  $\Gamma_2 \geq \frac{n+2}{2n} \Gamma$  in  $L_{\infty}(\Omega_n^m)$ . The case  $n = 2$  is of particular interest. Indeed,

we may write  $f \in \mathcal{L}(\mathbb{Z}_2^m)$  as

$$f = \mathbb{E}(f) + \sum_{B \subset \{1, \dots, m\}, B \neq \emptyset} \hat{f}_B \omega_B.$$

Note that in this case  $\omega_B^{-1} \omega_C = \omega_{B \Delta C}$ . Then the Gromov form of  $\{\omega_B\}$  is given by

$$K(\omega_B, \omega_C) = \frac{1}{2}(|B| + |C| - |B \Delta C|) = |B \cap C|.$$

Hence, we find the gradient form

$$\Gamma(f, f) = \sum_{B, C \subset \{1, \dots, m\}} \bar{\hat{f}}_B \hat{f}_C |B \cap C| \omega_{B \Delta C}.$$

Let  $e_j = (1, \dots, -1, \dots, 1)$  where  $-1$  is only at the  $j$ th position. For  $x \in \Omega_n^m$ , put  $(\partial_j f)(x) = \frac{1}{2}(f(x) - f(xe_j))$  and define the discrete gradient  $\nabla f = (\partial_j f)_{j=1}^m$ . Then a calculation gives  $\Gamma(f, f) = |\nabla f|^2$ , where  $|\cdot|$  is the Euclidean norm of a vector in  $\mathbb{C}^n$ . If we simply write  $|\nabla f| = \Gamma(f, f)^{1/2}$  for any  $n = 2, 3, \dots$ , our Poincaré inequalities for the generalized Walsh system is a dimension  $m$  free estimate.

**Corollary 7.15.** *Let  $2 \leq p < \infty$ . Then for all  $f \in L_p(\Omega_n^m, P)$ ,*

$$\|f - \mathbb{E}(f)\|_p \leq C \sqrt{\frac{2n}{n+2}} \sqrt{p} \|\nabla f\|_p.$$

One of Efraim and Lust-Piquard's main results in [ELP08] asserts that for  $2 < p < \infty$  and  $f \in L_p(\Omega_2^m, P)$ ,

$$\|f - \mathbb{E}(f)\|_p \leq C \sqrt{p} \|\nabla f\|_p.$$

Our result for the special case  $n = 2$  recovers this inequality.

### 7.1.7 The $q$ -Gaussian algebras

We first recall some definitions and basic facts following [BKS97]. Throughout this section  $-1 \leq q \leq 1$ . Let  $\mathcal{H}$  be a separable real Hilbert space with complexification  $\mathcal{H}_{\mathbb{C}}$ . Let  $(F_q(\mathcal{H}), \langle \cdot, \cdot \rangle_q)$  be the  $q$ -Fock space with vacuum vector  $\Omega$  and  $\Gamma_q(\mathcal{H})$  the  $q$ -Gaussian algebra which is the von Neumann algebra generated by  $s(f) = l(f) + l^*(f)$  for  $f \in \mathcal{H}$  where

$$l^*(f) f_1 \otimes \dots \otimes f_n = f \otimes f_1 \otimes \dots \otimes f_n$$

and

$$l(f)f_1 \otimes \cdots \otimes f_n = \sum_{j=1}^n q^{j-1} \langle f_j, f \rangle f_1 \otimes \cdots \otimes f_{j-1} \otimes f_{j+1} \otimes \cdots \otimes f_n$$

are the creation and annihilation operators respectively. The vacuum vector gives rise to a canonical tracial state  $\tau_q(X) = \langle X\Omega, \Omega \rangle_q$  for  $X \in \Gamma_q(\mathcal{H})$ . The  $q$ -Ornstein–Uhlenbeck semigroup  $T_t^q = \Gamma_q(e^{-t}I_{\mathcal{H}})$  is a standard semigroup and extends to a semigroup of contractions on  $L_p$  spaces. The generator on  $L_2$  is the number operator  $N^q$  which acts on the Wick product by

$$N^q W(f_1 \otimes \cdots \otimes f_n) = n W(f_1 \otimes \cdots \otimes f_n), \quad f_1, \dots, f_n \in \mathcal{H}_{\mathbb{C}},$$

where  $W$  is the Wick operator. It is easy to check that  $(T_t)$  is a nc-diffusion semigroup.

Let  $\ell_2^n$  be the real Hilbert space with dimension  $n$  and  $\{e_1, \dots, e_n\}$  an orthonormal basis. For  $j = 1, \dots, n$ , consider the embedding

$$\iota_j : \mathcal{H} \rightarrow \mathcal{H} \otimes \ell_2^n, \quad h \mapsto h \otimes e_j.$$

According to [BKS97, Theorem 2.11], there exists a unique map  $\Gamma_q(\iota) : \Gamma_q(\mathcal{H}) \rightarrow \Gamma_q(\mathcal{H} \otimes \ell_2^n)$  such that  $\Gamma_q(\iota_j)(s(h)) = s(h \otimes e_j)$ . The map  $\Gamma_q(\iota)$  is linear, bounded, unital completely positive and preserves the canonical trace. Define  $s_j^q(h) = s(h \otimes e_j)$ . If  $q = 1$ , we write  $g_j(h) = s(h \otimes e_j)$  and it is well-known  $g_j(h)$  is a standard Gaussian random variable if  $\|h\| = 1$ . For  $h \in \mathcal{H}$ , put

$$u_n(h) = \frac{1}{\sqrt{n}} \sum_{j=1}^n s_j^q(h) \otimes g_j(h).$$

Write  $\mathcal{M}_n^q = \Gamma_q(\mathcal{H} \otimes \ell_2^n)$ . We consider the von Neumann algebra ultraproduct  $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_n^q \bar{\otimes} \mathcal{M}_n^1$ . Any element of  $\mathcal{M}$  can be written as  $u_{\omega}(h) = (u_n(h))^{\bullet}$ . We need the following fact proved in [AJ12].

**Lemma 7.16.** *The map  $s(h) \mapsto u_{\omega}(h)$  extends to an injective trace preserving  $*$ -homomorphism  $\pi : \Gamma_q(\mathcal{H}) \rightarrow \mathcal{M}$ . Moreover, for  $x \in \Gamma_q(\mathcal{H})$ ,*

$$\pi(T_t^q(x)) = (Id \otimes T_t^1)^{\bullet} \pi(x).$$

**Proposition 7.17.** *For  $-1 \leq q \leq 1$ ,  $\Gamma_2 \geq \Gamma$  in  $\Gamma_q(\mathcal{H})$ .*

*Proof.* Since  $\pi$  is injective, it suffices to prove  $\pi\Gamma_2^{N^q} \geq \pi\Gamma^{N^q}$  in  $\mathcal{M}$ . By Lemma 7.16, we have  $\pi N^q(x) = (Id \otimes N^1)^{\bullet} \pi(x)$ . It follows that

$$\pi(\Gamma^{N^q}(x, y)) = \Gamma^{(Id \otimes N^1)^{\bullet}}(\pi(x), \pi(y)),$$

and similar identity is true for  $\Gamma_2^{N^q}$ . It is proved in [AJ12] by using the central limit theorem of Speicher [Spe92] that  $\Gamma_2 \geq \Gamma$  in  $\Gamma_1(\mathcal{H})$ . Here  $\Gamma_1(\mathcal{H})$  is the von Neumann algebra acting on the symmetric Fock space. It follows that for all  $n \in \mathbb{N}$  and in  $\mathcal{M}_n^q \otimes \mathcal{M}_n^1$ ,  $\Gamma_2^{Id \otimes N^1} \geq \Gamma^{Id \otimes N^1}$ . Hence, we find

$$\Gamma_2^{(Id \otimes N^1)^\bullet}(\pi(x_i), \pi(x_j)) \geq \Gamma^{(Id \otimes N^1)^\bullet}(\pi(x_i), \pi(x_j))$$

where for any  $m \in \mathbb{N}$  and  $x_i \in \Gamma_q(\mathcal{H}), i = 1, \dots, m$ , as desired.  $\square$

### 7.1.8 The hyperfinite $II_1$ factor

Our goal in this section is to show that the hyperfinite  $II_1$  factor  $R$  admits different standard nc-diffusion semigroups with  $\Gamma_2$ -criterion and that the best possible  $\alpha$  characterizes the corresponding dynamical system. It is well known that  $R$  can be approximated by matrix algebras  $\{M_{n^k} : k \in \mathbb{N}\}$ . We will embed  $M_{n^{m/2}}$  into the group von Neumann algebra of the generalized discrete Heisenberg group  $H_n^{m+1} = \mathbb{Z}_n/2 \times \mathbb{Z}_n^m$ .

Let  $\Theta = (\theta_{jk})$  be an antisymmetric  $m \times m$  matrix with  $\theta_{jk} = \frac{1}{2}$  if  $j < k$ . The multiplication in  $H_n^{m+1} = \mathbb{Z}_n/2 \times \mathbb{Z}_n^m$  is given by

$$(x, \xi)(y, \eta) = (x + y + B(\xi, \eta), \xi + \eta),$$

where  $B : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n/2$  is a bilinear form given by  $B(\xi, \eta) = \sum_{j,k=1}^m \theta_{jk} \xi_j \eta_k = \langle \xi, \Theta \eta \rangle$ . For  $(r, \xi) \in H_n^{m+1}$ , put  $\psi(r, \xi) = \sum_{j=1}^m 1 - \delta_{\xi_j, 0} = \#\{\xi_j \neq 0\}$ . Define a semigroup acting on  $\mathcal{L}(H_n^{m+1})$  by  $T_t \lambda(r, \xi) = e^{-t\psi} \lambda(r, \xi)$  for  $\lambda(r, \xi) \in \mathcal{L}(H_n^{m+1})$  and  $t \geq 0$ . Using Lemma 7.2, an argument similar to Proposition 7.12 shows that  $(T_t)_{t \geq 0}$  is a standard nc-diffusion semigroup and that the associated gradient form satisfies  $\Gamma_2 \geq \frac{n+2}{2n} \Gamma$  in  $\mathcal{L}(H_n^{m+1})$ .

**Lemma 7.18.** *Let  $m \geq 2$  be an even integer and  $n \geq 3$ . We have*

$$\mathcal{L}(H_n^{m+1}) \cong \bigoplus_{x \in \mathbb{Z}_n/2} \mathcal{M}_x,$$

where  $\mathcal{M}_2 = M_{n^{m/2}}$ . Furthermore,  $T_t$  leaves each  $\mathcal{M}_x$  invariant.

*Proof.* Most of the argument utilizes the proof of Proposition 7.13 and [JMP10, Lemma 5.3]. Note that  $\lambda(\mathbb{Z}_n/2, 0)$  lives in the center of  $\mathcal{L}(H_n^{m+1})$ . By the decomposition of von Neumann algebras for the subalgebras of the center we obtain the first assertion. Write  $e_j, j = 1, \dots, m$  for the canonical basis of  $\mathbb{Z}_n^m$  and put  $u_r^j = \lambda(0, re_j)$  for  $r \in \mathbb{Z}_n$ . Then these  $u_r^j$ 's generate  $\lambda(0, \mathbb{Z}_n^m)$  and  $u_r^j u_s^k (\delta_{(x, \cdot)}) = e^{2\pi i r s \theta_{jk} x / n} u_s^k u_r^j (\delta_{(x, \cdot)})$ .

Acting on  $H_2 := \text{span}\{\delta_{(2,\cdot)}\}$ ,  $u_r^j$ 's satisfy  $u_r^j u_s^k = e^{2\pi i r s/n} u_s^k u_r^j$  for  $j < k$  and  $u_r^j u_s^k = e^{-2\pi i r s/n} u_s^k u_r^j$  for  $j > k$ . It is clear that  $y(r_1, \dots, r_m) = u_{r_1}^1 \cdots u_{r_m}^m$  is a basis for  $\mathcal{M}_2$  which satisfies the equation

$$u_r^j y(r_1, \dots, r_m) u_r^{j*} = C(r, j, r_1, \dots, r_m) y(r_1, \dots, r_m),$$

where  $C(r, j, r_1, \dots, r_m) = \exp(2\pi i r(r_1 + \dots + r_{j-1} - r_{j+1} - \dots - r_m)/n)$ . In order to determine the center of  $\mathcal{M}_2$ , we consider the equation  $C(r, j, r_1, \dots, r_m) = 1$  for all  $r \in \mathbb{Z}_n, j = 1, \dots, m$ . This leads to a linear system over  $\mathbb{Z}_n$

$$\begin{aligned} -r_2 - \dots - r_m &= 0, \\ r_1 - r_3 - \dots - r_m &= 0, \\ &\vdots \\ r_1 + \dots + r_{m-1} &= 0. \end{aligned}$$

Solving this system, we find  $r_1 = \dots = r_m = 0$ . Here we used the crucial assumption that  $m$  is even. Hence  $\mathcal{M}_2$  has trivial center. Since it has dimension  $n^m$ , it follows that  $\mathcal{M}_2 = M_{n^{m/2}}$ , as desired. By restricting  $T_t$  to  $M_{n^{m/2}}$  and repeating the argument of Proposition 7.13, we can prove the last assertion.  $\square$

It follows from the lemma that  $M_{n^{m/2}}$  admits a standard nc-diffusion semigroup  $T_t$  with  $\Gamma_2 \geq \frac{n+2}{2n}\Gamma$  in  $M_{n^{m/2}}$  for all  $m \in 2\mathbb{N}$ . Since the hyperfinite  $II_1$  factor  $R$  is the weak closure of  $\cup_{k=1}^\infty M_{n^k}$ , we have proved the following result.

**Proposition 7.19.** *For any integer  $n \geq 2$ , there exists a standard nc-diffusion semigroup  $T_t^n$  acting on  $R$  such that the associated gradient form  $\Gamma^n$  satisfies  $\Gamma_2^n \geq \frac{n+2}{2n}\Gamma^n$  in  $R$ . The constant  $\alpha_n = \frac{n+2}{2n}$  is best possible.*

The last conclusion follows from Proposition 7.10. We want to show that the semigroups  $T_t^n$  for different  $n$  are different. Let us now recall a definition from dynamical systems; see for example [Wal82, Definition 2.4]. Let  $(X, \mathcal{B}_X, \mu, T)$  be a measure-preserving dynamic system (MPDS) where  $(X, \mathcal{B}_X, \mu)$  is a probability space and  $T$  is a measure-preserving transformation. A MPDS  $(Y, \mathcal{B}_Y, \nu, S)$  is said to be isomorphic to  $(X, \mathcal{B}_X, \mu, T)$  if there exist (i) full measure sets  $X_1 \subset X$  and  $Y_1 \subset Y$  such that  $T(X_1) \subset X_1$  and  $S(Y_1) \subset Y_1$ ; and (ii) an invertible measure-preserving measurable map  $\phi : X \rightarrow Y$  such that  $\phi(Tx) = S(\phi x)$  for all  $x \in X_1$ . This motivates our following definition.

**Definition 7.20.** Let  $S_t$  and  $T_t$  be standard semigroups acting on noncommutative probability spaces  $(\mathcal{N}, \tau)$  and  $(\mathcal{M}, \tau')$  respectively. We say  $(\mathcal{M}, T_t)$  and  $(\mathcal{N}, S_t)$  are isomorphic if there exist  $\lambda > 0$  and a trace-preserving  $*$ -isomorphism  $\phi : \mathcal{M} \rightarrow \mathcal{N}$  such that  $S_{\lambda t}(\phi x) = \phi(T_t x)$  for all  $x \in \mathcal{M}$ .

The following result shows that  $R$  admits infinitely many non-isomorphic standard nc-diffusion semigroups.

**Proposition 7.21.** *Let  $T_t^n$  be the semigroup considered in Proposition 7.19. If  $(R, T_t^n)$  and  $(R, T_t^{n'})$  are isomorphic, then  $\alpha_n = \alpha_{n'}$ .*

*Proof.* There exists a trace-preserving  $*$ -isomorphism  $\phi : R \rightarrow R$  such that

$$\phi(T_t^n x) = T_{\lambda t}^{n'}(\phi x) \quad (7.4)$$

for  $x \in \mathcal{M}$ . Let  $A^n$  be the generator of  $T_t^n$ .  $\Gamma_2^{A^{n'}} \geq \frac{n'+2}{2n'} \Gamma^{A^{n'}}$  implies  $\Gamma_2^{\lambda A^{n'}} \geq \lambda \frac{n'+2}{2n'} \Gamma^{\lambda A^{n'}}$ . This together with (7.4) gives  $\Gamma_2^{A^n} \geq \lambda \frac{n'+2}{2n'} \Gamma^{A^n}$ . But the best  $\alpha$  is  $\alpha_n = \frac{n+2}{2n}$ . Hence we have  $\frac{n+2}{2n} \geq \lambda \frac{n'+2}{2n'}$ . It is clear that  $\text{sp}(A^n) = \mathbb{N}$  and  $\text{sp}(\lambda A^n) = \lambda \mathbb{N}$ . Here  $\text{sp}(A^n)$  denotes the spectrum of  $A^n$ . (7.4) implies  $\text{sp}(\lambda A^n) = \text{sp}(A^n)$  and thus  $\lambda = 1$ . Hence  $n' \geq n$ . Repeating the argument by starting from  $\Gamma_2^n \geq \frac{n+2}{2n} \Gamma^n$  gives  $n \geq n'$ .  $\square$

## 7.2 Tensor products and free products

In this section we will construct further examples with the  $\Gamma_2$ -criterion based on the examples considered in the previous section. This is done via the powerful algebraic tools – tensor products and free products. It is not difficult to see that the property “standard nc-diffusion” is stable under tensor products and free products. Due to the reason explained in the previous section, it suffices to consider the algebraic  $\Gamma_2$ -condition. That is, we always work with a dense subalgebra contained in the domain of the form under consideration.

### 7.2.1 Tensor products

The following result is our starting point to understand tensor products.

**Lemma 7.22.** *Let  $\Theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{M}$  and  $\Phi : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{N}$  be positive sesquilinear forms, where  $\mathcal{A} \subset \mathcal{M}$  and  $\mathcal{B} \subset \mathcal{N}$  are dense subalgebras so that  $\Theta$  and  $\Phi$  are well-defined. Then  $\Theta \otimes \Phi : \mathcal{A} \otimes \mathcal{B} \times \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{M} \otimes \mathcal{N}$  is positive where for  $\xi^i = \sum_{k=1}^{n_i} x_k^i \otimes y_k^i \in \mathcal{A} \otimes \mathcal{B}$ ,*

$$\Theta \otimes \Phi(\xi^i, \xi^j) := \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j).$$

*Proof.* For  $r \in \mathbb{N}$ , let  $(x_k^i) \subset \mathcal{A}, (y_k^i) \subset \mathcal{B}$  where  $k = 1, \dots, n_i, i = 1, \dots, r$ . Put  $m = \sum_{i=1}^r n_i$ . Without loss of generality we may assume  $n_i = n$  for  $i = 1, \dots, r$ . Suppose  $\mathcal{M}$  and  $\mathcal{N}$  act on Hilbert spaces  $H$  and  $K$  respectively. Then  $(\Theta(x_k^i, x_l^j))_{k,l,i,j} = \sum_{i,j,k,l} \Theta(x_k^i, x_l^j) \otimes e_{(k,i),(l,j)} \geq 0$  as an operator on  $\ell_2^m(H)$  where  $1 \leq k, l \leq n$ . Similarly,  $(\Phi(y_k^i, y_l^j))_{k,l,i,j} \geq 0$  on  $\ell_2^m(K)$ . It follows that

$$\begin{aligned} & (\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'})) \\ &= \sum_{i,j,k,l,i',j',k',l'} \Theta(x_k^i, x_l^j) \otimes \Phi(y_{k'}^{i'}, y_{l'}^{j'}) \otimes e_{(k,i),(l,j)} \otimes e_{(k',i'),(l',j')} \geq 0 \end{aligned}$$

on  $\ell_2^m(H) \otimes \ell_2^m(K)$ . Define

$$v : \ell_2^m(H \otimes K) \rightarrow \ell_2^m(H) \otimes \ell_2^m(K), \quad \sum_s (\xi_t^s \otimes \eta_t^s) \otimes e_t \mapsto \sum_s (\xi_t^s \otimes e_t) \otimes (\eta_t^s \otimes e_t).$$

Here  $\xi_t^s \in H, \eta_t^s \in K$ , and  $(e_t)$  is the canonical basis of  $\ell_2^m$  for  $t = 1, \dots, m$ . Then

$$v^* \left[ \sum_s (\xi_t^s \otimes e_t) \otimes (\eta_t^s \otimes e_t) \right] = \sum_s (\xi_t^s \otimes \eta_t^s) \otimes e_t.$$

It is clear that  $v^*[(\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'}))]v \geq 0$ . But

$$\begin{aligned} v^*[(\Theta(x_k^i, x_l^j)) \otimes (\Phi(y_{k'}^{i'}, y_{l'}^{j'}))]v &= \sum_{i,j,k,l} \Theta(x_k^i, x_l^j) \otimes \Phi(y_{k'}^{i'}, y_{l'}^{j'}) \otimes e_{(k,i),(l,j)} \\ &= [\Theta(x_k^i, x_l^j) \otimes \Phi(y_{k'}^{i'}, y_{l'}^{j'})]_{(k,i),(l,j)}. \end{aligned}$$

Let  $\mathbf{1}_n$  be a  $n \times 1$  column vector with each entry equal to 1 and  $I_r$  the  $r \times r$  identity matrix. Define an operator  $w : \ell_2^r(H \otimes K) \rightarrow \ell_2^m(H \otimes K)$  by

$$w = \mathbf{1}_n \otimes I_r = \begin{pmatrix} \mathbf{1}_n & & & \\ & \mathbf{1}_n & & \\ & & \ddots & \\ & & & \mathbf{1}_n \end{pmatrix}.$$



$w$  is an  $m \times r$  matrix. Note that  $[\Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j)]$  is an  $m \times m$  matrix. Then

$$\begin{aligned} 0 &\leq w^* \left[ \sum_{i,j,k,l} \Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j) \otimes e_{(k,i),(l,j)} \right] w \\ &= \sum_{i,j=1}^r \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \Theta(x_k^i, x_l^j) \otimes \Phi(y_k^i, y_l^j) \otimes e_{i,j} \\ &= \sum_{i,j=1}^r \Theta \otimes \Phi(\xi^i, \xi^j) \otimes e_{i,j}, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 7.23.** *Let  $(T_t)_{t \geq 0}$  and  $(S_t)_{t \geq 0}$  be standard semigroups with generator  $A$  and  $B$  acting on finite von Neumann algebras  $\mathcal{M}$  and  $\mathcal{N}$  respectively such that  $\Gamma_2^A \geq \alpha \Gamma^A$  in  $\mathcal{M}$  and  $\Gamma_2^B \geq \alpha \Gamma^B$  in  $\mathcal{N}$ . Then  $\Gamma_2^{A \otimes I} \geq \alpha \Gamma^{A \otimes I}$ ,  $\Gamma_2^{I \otimes B} \geq \alpha \Gamma^{I \otimes B}$  and  $\Gamma_2^{A \otimes I + I \otimes B} \geq \alpha \Gamma^{A \otimes I + I \otimes B}$  all in  $\mathcal{M} \otimes \mathcal{N}$ .*

*Proof.* The first inequality follows from Lemma 7.22 with  $\Theta = \Gamma^A$  and  $\Phi(y_1, y_2) = y_1^* y_2$ . The second inequality can be shown similarly. For the last one, note that

$$\begin{aligned} &\Gamma_2^{A \otimes I + I \otimes B} - \alpha \Gamma^{A \otimes I + I \otimes B} \\ &= (\Gamma_2^{A \otimes I} - \alpha \Gamma^{A \otimes I}) + (\Gamma_2^{I \otimes B} - \alpha \Gamma^{I \otimes B}) + 2\Gamma^A \otimes \Gamma^B. \end{aligned}$$

Then the first two inequalities and Lemma 7.22 with  $\Theta = \Gamma^A$  and  $\Phi = \Gamma^B$  yield the assertion.  $\square$

**Proposition 7.24.** *Let  $A_j$  be self-adjoint generators of standard nc-diffusion semigroups  $(T_t^{A_j})$  acting on  $\mathcal{N}_j$  and  $\Gamma_2^{A_j} \geq \alpha \Gamma^{A_j}$  respectively for  $j = 1, \dots, n$  with the same constant  $\alpha > 0$ . Then the tensor product generator  $\otimes A_j(x_1 \otimes \dots \otimes x_n) = \sum_j x_1 \otimes \dots \otimes x_{j-1} \otimes A_j(x_j) \otimes x_{j+1} \otimes \dots \otimes x_n$  generates a standard nc-diffusion semigroup  $(T_t^{\otimes A_j})$  with*

$$\Gamma_2^{\otimes A_j} \geq \alpha \Gamma^{\otimes A_j}.$$

*Proof.* Note that  $T_t^{\otimes A_j} = T_t^{A_1} \otimes \dots \otimes T_t^{A_n}$ . Since  $(T_t^{A_j})$  is a standard nc-diffusion semigroup for  $j = 1, \dots, n$ , so is  $T_t^{\otimes A_j}$ . We prove the  $\Gamma_2$ -condition by induction. The case  $n = 2$  follows from Lemma 7.23. The general case follows by induction and repeatedly invoking Lemma 7.22 to deal with “cross terms” like  $\Gamma^{I \otimes \dots \otimes A_i \otimes \dots \otimes I} \otimes \Gamma^{I \otimes \dots \otimes A_j \otimes \dots \otimes I}$ .  $\square$

**Example 7.25** (Tensor product of matrix algebras). Let  $A$  be the generator of the semigroup  $T_t$  acting on  $M_n$  considered in Proposition 7.14. Let  $\Gamma$  be the gradient form associated to  $\sum_{i=1}^m I \otimes \dots \otimes A \otimes \dots \otimes I$  where  $A$  is in the  $i$ th position. Then it follows from Proposition 7.24 that  $\Gamma_2 \geq \frac{2+n}{2n} \Gamma$  in  $\otimes_{i=1}^m M_n$ .

**Example 7.26** (Random matrices). Let  $(\Omega, \mathbb{P})$  be a probability space. Consider  $I \otimes T_t$  acting on  $L_\infty(\Omega, \mathbb{P}) \otimes M_n$  where  $T_t$  is the semigroup considered in Proposition 7.14. By Lemma 7.23,  $I \otimes T_t$  is a standard nc-diffusion semigroup and satisfies  $\Gamma_2 \geq \frac{2+n}{2n} \Gamma$  in  $L_\infty(\Omega, \mathbb{P}) \otimes M_n$ . Hence our results apply for random matrices.

**Example 7.27** (Product measure). Here we consider  $A_j = I - E_j$  for  $E_j$  a conditional expectation on  $\mathcal{N}_j$  for  $j = 1, \dots, n$ . By example 3.15,  $A_j$  generates a standard nc-diffusion semigroup and  $\Gamma_2^{A_j} \geq \frac{1}{2} \Gamma^{A_j}$ . Then we deduce from Proposition 7.24 that  $\Gamma_2^A \geq \frac{1}{2} \Gamma^A$  for the tensor product generator  $A = \otimes A_j$ . For  $x = x_1 \otimes \dots \otimes x_n$ , put  $\Gamma_j(x, x) = x_1^* x_1 \otimes \dots \otimes \Gamma^{A_j}(x_j, x_j) \otimes \dots \otimes x_n^* x_n$ . Then we have

$$\Gamma(x, x) = \sum_{j=1}^n \Gamma_j(x, x).$$

We want to investigate an easy consequence of our general theory for the product measure space. Let  $(\Omega_i, \mathbb{P}_i)$ ,  $i = 1, \dots, n$  be a family of probability spaces and denote by  $(\Omega, \mathbb{P})$  the product probability space. Then  $L_\infty(\Omega, \mathbb{P}) = \otimes_{i=1}^n L_\infty(\Omega_i, \mathbb{P}_i)$ . Define  $E_i(f) = \int f d\mathbb{P}_i$  for  $f \in L_\infty(\Omega, \mathbb{P})$  and put  $A_i = I - E_i$ . Then

$$\begin{aligned} \Gamma_i(f, f) &= \frac{1}{2}(|f|^2 - f \int \bar{f} d\mathbb{P}_i - \bar{f} \int f d\mathbb{P}_i + \int |f|^2 d\mathbb{P}_i) \\ &= \frac{1}{2} \left( |f - \int f d\mathbb{P}_i|^2 + \int (|f|^2 - |\int f d\mathbb{P}_i|^2) d\mathbb{P}_i \right). \end{aligned}$$

It is straightforward to check that the fixed point subalgebra of the semigroup  $e^{-t(\otimes A_i)}$  is  $\mathbb{C}1$ . Hence  $E_{\text{Fix}} f = \mathbb{E} f$  for  $f \in L_\infty(\Omega, \mathbb{P})$  where  $\mathbb{E}$  is the expectation operator of  $\mathbb{P}$ . Then (6.12) yields

$$\begin{aligned} \mathbb{P}(f - \mathbb{E}(f) \geq t) &\leq \exp \left( - \frac{ct^2}{\| \sum_{i=1}^n \Gamma_i(f, f) \|_\infty} \right) \\ &\leq \exp \left( - \frac{2ct^2}{\sum_{i=1}^n \|f - \int f d\mathbb{P}_i\|_\infty^2 + \| \int (|f|^2 - |\int f d\mathbb{P}_i|^2) d\mathbb{P}_i \|_\infty} \right). \end{aligned} \tag{7.5}$$

Note that we do not impose any concrete condition on the probability spaces. This shows that the sub-Gaussian tail behavior is always true for product measures. We do not know whether such results were known before.

## 7.2.2 Free products with amalgamation

Here we want to prove that the condition  $\Gamma_2 \geq \alpha \Gamma$  is stable under free products. Our general reference is [VDN92]. We need some preliminary facts about free product of semigroups  $T_t = *_k T_t^{A_k}$  acting on  $\mathcal{N} := *_k \mathcal{N}_k$  with generators  $A_k$  acting on von Neumann algebra  $\mathcal{N}_k \supset \mathcal{D}$ . Here  $\mathcal{D}$  is a von Neumann subalgebra of all  $\mathcal{N}_k$ . Similar to the tensor products considered before, if  $(T_t^{A_k})$  is a standard nc-diffusion

semigroup for  $k = 1, \dots, n$ , so is  $*_k T_t^{A_k}$ . We assume that  $A_k$  commutes with the conditional expectation  $E : \mathcal{N}_k \rightarrow \mathcal{D}$  for which we amalgamate and even

$$A_k E = E A_k = 0.$$

Our first task is to calculate the gradient  $\Gamma$ . For simplicity of notation, we always assume the elements we consider are chosen so that  $\Gamma$  and  $\Gamma_2$  are well-defined. Let us now consider elementary words  $x = a_1 \cdots a_m$  and  $y = b_1 \cdots b_n$  of mean 0 elements  $a_k \in \mathcal{N}_{i_k}$ ,  $b_k \in \mathcal{N}_{j_k}$ . Recall that the free product generator is given by

$$A(b_1 \cdots b_n) = \sum_{l=1}^n b_1 \cdots b_{l-1} A_{j_l}(b_l) b_{l+1} \cdots b_n.$$

In the future we will ignore the index for  $A$ . If we want to apply the free product generator  $A$  on the product  $x^* y$ , we have to know the mean 0 decomposition

$$\begin{aligned} x^* y &= a_m^* \cdots a_1^* b_1 \cdots b_n \\ &= \sum_{k=1}^{\min(n,m)} a_m^* \cdots a_{k+1}^* \overbrace{a_k^* E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k}^{\circ} b_{k+1} \cdots b_n \\ &\quad + \begin{cases} E(a_m^* \cdots a_1^* b_1 \cdots b_m) b_{m+1} \cdots b_n, & \text{if } m \leq n, \\ a_m^* \cdots a_{n+1}^* E(a_n^* \cdots a_1^* b_1 \cdots b_m), & \text{if } m > n. \end{cases} \end{aligned}$$

Here  $\hat{x} = x - E(x)$ . Let  $k_0 = \inf\{i \in \mathbb{N} : k_i \neq j_i\}$ . The equality  $T_t(E(x)) = E(x)$  implies that

$$A(E(x)y) = \lim_{t \rightarrow 0} \frac{T_t(E(x)y) - E(x)y}{t} = E(x)A(y). \quad (7.6)$$

It is easy to see that all terms containing  $A(a_i^*)$ ,  $A(b_i)$  for  $i \geq k_0$  will cancel out in  $\Gamma(x, y)$  and thus

$$\begin{aligned} 2\Gamma(x, y) &= \sum_{i=1}^{k_0-1} a_m^* \cdots a_{i+1}^* A(a_i^*) a_{i-1} \cdots a_1^* b_1 \cdots b_n + \sum_{i=1}^{k_0-1} a_m^* \cdots a_1^* b_1 \cdots b_{i-1} A(b_i) b_{i+1} \cdots b_n \\ &\quad - a_m^* \cdots a_{k_0}^* A(a_{k_0-1}^* \cdots a_1^* b_1 \cdots b_{k_0-1}) b_{k_0} \cdots b_n \\ &= 2a_m^* \cdots a_{k_0}^* \Gamma(a_1 \cdots a_{k_0-1}, b_1 \cdots b_{k_0-1}) b_{k_0} \cdots b_n. \end{aligned}$$

**Lemma 7.28.** *Let  $a_i, b_i \in \mathcal{N}_{k_i}$  be mean 0 elements for  $i = 1, \dots, r$ . Then*

$$\Gamma(a_1 \cdots a_r, b_1 \cdots b_r) = a_r^* \Gamma(a_1 \cdots a_{r-1}, b_1 \cdots b_{r-1}) b_r + \Gamma(a_r, E(a_{r-1}^* \cdots a_1^* b_1 \cdots b_{r-1}) b_r).$$

*Proof.* Using the mean 0 decomposition, we have

$$\begin{aligned}
2\Gamma(a_1 \cdots a_r, b_1 \cdots b_r) &= A(a_r^*)a_{r-1}^* \cdots b_{r-1}b_r + a_r^*A(a_{r-1}^* \cdots a_1^*)b_1 \cdots b_r \\
&+ a_r^* \cdots a_1^*A(b_1 \cdots b_{r-1})b_r + a_r^* \cdots b_{r-1}A(b_r) \\
&- A(a_r^*)\left(\sum_{i=1}^{r-1} a_{r-1}^* \cdots a_{i+1}^* \overbrace{a_i^*E(a_{i-1}^* \cdots a_1^*b_1 \cdots b_{i-1})}^{\circ} b_i b_{i+1} \cdots b_{r-1}\right)b_r \\
&- a_r^*\left(\sum_{i=1}^{r-1} a_{r-1}^* \cdots a_{i+1}^* \overbrace{a_i^*E(a_{i-1}^* \cdots a_1^*b_1 \cdots b_{i-1})}^{\circ} b_i b_{i+1} \cdots b_{r-1}\right)A(b_r) \\
&- a_r^*A(a_{r-1}^* \cdots a_1^*b_1 \cdots b_{r-1})b_r - A(a_r^*E(a_{r-1}^* \cdots b_{r-1})b_r) \\
&= 2a_r^*\Gamma(a_1 \cdots a_{r-1}, b_1 \cdots b_{r-1})b_r + A(a_r^*)E(a_{r-1}^* \cdots b_{r-1})b_r \\
&+ a_r^*E(a_{r-1}^* \cdots b_{r-1})A(b_r) - A(a_r^*E(a_{r-1}^* \cdots b_{r-1})b_r)
\end{aligned}$$

which completes the proof with the help of (7.6). □

The recursion formula immediately yields that

**Lemma 7.29.** *Let  $k_0$  be as above. Then*

$$\Gamma(x, y) = \sum_{k=1}^{k_0-1} \Gamma^{(k)}[x, y].$$

where

$$\Gamma^{(k)}[x, y] = a_m^* \cdots a_{k+1}^* \Gamma[a_k, E(a_{k-1}^* \cdots a_1^*b_1 \cdots b_{k-1})b_k]b_{k+1} \cdots b_n.$$

In order to calculate  $\Gamma_2$ , we have to analyze

$$\Gamma^{(k)}[A(x), y] + \Gamma^{(k)}[x, A(y)] - A\Gamma^{(k)}[x, y].$$

Observe that for  $j < k$  all terms containing  $A(a_j^*)$  or  $A(b_j)$  appear inside the conditional expectation  $E$  in  $\Gamma^{(k)}[A(x), y] + \Gamma^{(k)}[x, A(y)]$  and there is no counterpart in  $A\Gamma^{(k)}$ . Hence we find

$$\begin{aligned}
I^{(k)}(x, y) &= a_m^* \cdots a_{k+1}^* \Gamma[a_k, E(A(a_{k-1}^* \cdots a_1^*)b_1 \cdots b_{k-1})b_k]b_{k+1} \cdots b_n \\
&+ a_m^* \cdots a_{k+1}^* \Gamma[a_k, E(a_{k-1}^* \cdots a_1^*A(b_1 \cdots b_{k-1}))b_k]b_{k+1} \cdots b_n.
\end{aligned}$$

For  $k \leq j < k_0$  we are left with the following terms

$$\begin{aligned}
II^{(k)}(x, y) &= \sum_{j=k}^{k_0-1} a_m^* \cdots A(a_j^*) \cdots a_{k+1}^* \Gamma[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] b_{k+1} \cdots b_n \\
&+ \sum_{j=k}^{k_0-1} a_m^* \cdots a_{k+1}^* \Gamma[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] b_{k+1} \cdots A(b_j) \cdots b_n \\
&- a_m^* \cdots A(a_{k_0-1}^* \cdots a_{k+1}^* \Gamma[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] b_{k+1} \cdots b_{k_0-1}) \cdots b_n
\end{aligned}$$

where we understand, when  $j = k$ , that  $A(a_k)$  and  $A(b_k)$  are inside the  $\Gamma$ -form. Since

$$\Gamma[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] \in \mathcal{N}_{i_k},$$

we are in the situation of Lemma 7.28. The recursion formula gives that

$$II^{(k)}(x, y) = \sum_{j=k+1}^{k_0-1} F_{jk}(x, y) + 2a_m^* \cdots a_{k+1}^* \Gamma_2[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] b_{k+1} \cdots b_n$$

where

$$\begin{aligned}
F_{jk}(x, y) &= a_m^* \cdots a_{j+1}^* \Gamma[a_j, E(a_{j-1}^* \cdots \Gamma[a_k, \\
&E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1}) b_k] \cdots b_{j-1}) b_j] b_{j+1} \cdots b_n.
\end{aligned}$$

Therefore we find  $\Gamma_2$ .

**Lemma 7.30.** *Using the above notation, we have*

$$\begin{aligned}
2\Gamma_2(x, y) &= \sum_{k=1}^{k_0-1} \Gamma^{(k)}[A(x), y] + \Gamma^{(k)}[x, A(y)] - A\Gamma^{(k)}[x, y] \\
&= \sum_{k=1}^{k_0-1} I^{(k)}(x, y) + II^{(k)}(x, y)
\end{aligned}$$

In order to show  $\Gamma_2 \geq \alpha\Gamma$ , we need a technical lemma which is an application of the Hilbert  $W^*$ -module theory; see [Pas73, Lan95].

**Lemma 7.31.** *Let  $\Phi : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{N}$  be a sesquilinear form where  $\mathcal{A}$  is a separable  $*$ -algebra contained in the domain of  $\Phi$  and  $\mathcal{N}$  is a von Neumann algebra. Then  $\Phi$  is a positive form if and only if there exists a map  $v : \mathcal{A} \rightarrow C(\mathcal{N})$  such that  $\Phi(x, y) = v(x)^* v(y)$  for  $x, y \in \mathcal{A}$  where  $C(\mathcal{N}) = \ell_2 \otimes \mathcal{N}$  denotes the Hilbert  $\mathcal{N}$ -module, or the column space of  $\mathcal{N}$ .*

*Proof.* The sufficiency is obvious. Conversely, following the KSGNS construction [Pas73, Lan95], we consider the algebraic tensor product  $\mathcal{A} \otimes \mathcal{N}$  and define  $\langle \sum_i x_i \otimes a_i, \sum_j y_j \otimes b_j \rangle = \sum_{i,j} a_i^* \Phi(x_i, y_j) b_j$  for  $x_i, y_j \in \mathcal{A}$  and  $a_i, b_j \in \mathcal{N}$ . Set  $\mathcal{K} = \{x \in \mathcal{A} \otimes \mathcal{N} : \langle x, x \rangle = 0\}$ . Then  $\mathcal{A} \otimes \mathcal{N} / \mathcal{K}$  is a pre-Hilbert  $\mathcal{N}$ -module with  $\mathcal{N}$ -valued inner product  $\langle x + \mathcal{K}, y + \mathcal{K} \rangle = \langle x, y \rangle$  for  $x, y \in \mathcal{A} \otimes \mathcal{N}$ . Let  $\mathcal{A} \otimes_{\Phi} \mathcal{N}$  be the completion of  $\mathcal{A} \otimes \mathcal{N} / \mathcal{K}$ . Then  $\mathcal{A} \otimes_{\Phi} \mathcal{N}$  is a Hilbert  $\mathcal{N}$ -module. Since  $\mathcal{A}$  is separable,  $\mathcal{A} \otimes_{\Phi} \mathcal{N}$  is countably generated. It follows from [Lan95, Theorem 6.2] that there exists a right module map  $u : \mathcal{A} \otimes_{\Phi} \mathcal{N} \rightarrow \ell_2 \otimes \mathcal{N}$  such that

$$\sum_{i,j} a_i^* \Phi(x_i, y_j) b_j = \langle u(\sum_i x_i \otimes a_i + \mathcal{K}), u(\sum_j y_j \otimes b_j + \mathcal{K}) \rangle.$$

In particular,  $\Phi(x, y) = u(x \otimes 1 + \mathcal{K})^* u(y \otimes 1 + \mathcal{K})$ . Define  $v(x) = u(x \otimes 1 + \mathcal{K})$ . This is the desired map.  $\square$

*Remark 7.32.* Since we only consider finitely many elements for the sake of positivity of a form in the following, the separability assumption on  $\mathcal{A}$  in the previous lemma is automatically satisfied. If we want to remove separability, we can use the fact that every Hilbert right module over  $\mathcal{N}$  embeds isometrically in a self-dual module. Indeed, this follows from [Pas73]. Let us sketch the approach from [JS05]. Consider the  $L_{1/2}(\mathcal{N})$  module

$$Y = X \otimes_{\mathcal{N}} L_1(\mathcal{N}).$$

Then the antilinear dual  $Y^*$  is self-dual and obviously contains  $X$  isometrically.

By an argument similar to (7.6), we have  $A^{1/2}(zx) = zA^{1/2}(x)$  for  $z \in \mathcal{D}$  and  $x \in \mathcal{N}$ . Then for  $x, y \in \mathcal{N}$ ,

$$\tau(zE(A(x)y)) = \tau(E(A(zx)y)) = \tau(A^{1/2}(zx)A^{1/2}(y)) = \tau(zE(A^{1/2}(x)A^{1/2}(y))).$$

Hence,  $E(A(x)y) = E(A^{1/2}(x)A^{1/2}(y))$  and we find

$$I^{(k)}(x, y) = 2a_m^* \cdots \Gamma[a_k, E(A^{1/2}(a_{k-1}^* \cdots a_1^*)A^{1/2}(b_1 \cdots b_{k-1}))b_k]b_{k+1} \cdots b_n.$$

We claim that this is a positive form. Indeed,  $I^{(k)}$  is nontrivial only if  $a_k$  and  $b_k$  are in the same  $\mathcal{N}_{i_k}$ . Using Lemma 7.31 with  $\Phi = \Gamma$ , we find  $\beta_k : \mathcal{N}_{i_k} \rightarrow C(\mathcal{N}_{i_k})$  such that  $\Gamma(a, b) = \beta_k(a)^* \beta_k(b)$  for  $a, b \in \mathcal{N}_{i_k}$ . Similarly with  $\Phi(x, y) = E(x^*y)$ , we find  $v_k : \mathcal{N} \rightarrow C(\mathcal{D})$  such that  $E(x^*y) = v_k(x)^* v_k(y)$  for  $x, y \in \mathcal{N}$ . Define

$$u_k(b_1 \cdots b_n) = e_{i_1, \dots, i_k} \otimes (\beta_k \otimes Id)(v_k(A^{1/2}(b_1 \cdots b_{k-1}))b_k)b_{k+1} \cdots b_n.$$

Note that by the module property (7.6)  $\Gamma(z^*a, b) = \Gamma(a, zb)$  for  $z \in \mathcal{D}, x, y \in \mathcal{N}_{i_k}$ . Write  $v_k(x) = (v_k^l(x))_l$

where  $v_k^l(x) \in \mathcal{D}$ . It follows that

$$\begin{aligned}
I^{(k)}(x, y) &= 2a_m^* \cdots a_{k+1}^* \Gamma(a_k, \sum_l v_k^l[A^{1/2}(a_1 \cdots a_{k-1})]^* v_k^l[A^{1/2}(b_1 \cdots b_{k-1})]b_k)b_{k+1} \cdots b_n \\
&= 2 \sum_l a_m^* \cdots a_{k+1}^* \Gamma(v_k^l[A^{1/2}(a_1 \cdots a_{k-1})]a_k, v_k^l[A^{1/2}(b_1 \cdots b_{k-1})]b_k)b_{k+1} \cdots b_n \\
&= 2 \sum_l a_m^* \cdots a_{k+1}^* \beta_k(v_k^l[A^{1/2}(a_1 \cdots a_{k-1})]a_k)^* \beta_k(v_k^l[A^{1/2}(b_1 \cdots b_{k-1})]b_k)b_{k+1} \cdots b_n \\
&= 2u_k(a_1 \cdots a_m)^* u_k(b_1 \cdots b_n).
\end{aligned}$$

By Lemma 7.31,  $I^{(k)}$  is a positive form.

Now we claim that  $F_{jk}$  are positive forms for  $j = k+1, \dots, k_0-1$ . Indeed, define

$$\begin{aligned}
u_{jk}(b_1 \cdots b_n) &= e_{i_{k+1}, \dots, i_j} \otimes (\beta_j \otimes Id)((v_j \otimes Id)[e_{i_1, \dots, i_k} \\
&\quad \otimes (\beta_k \otimes Id)[v_k(b_1 \cdots b_{k-1})b_k]b_{k+1} \cdots b_{j-1}]b_j)b_{j+1} \cdots b_n.
\end{aligned}$$

Then similar to the argument for  $I^{(k)}$ , we find  $F_{jk}(x, y) = u_{jk}(a_1 \cdots a_m)^* u_{jk}(b_1 \cdots b_n)$ . By Lemma 7.31,  $F_{jk}$  is a positive form. Hence, we find

$$II^{(k)}(x, y) \geq 2a_m^* \cdots a_{k+1}^* \Gamma_2[a_k, E(a_{k-1}^* \cdots a_1^* b_1 \cdots b_{k-1})b_k]b_{k+1} \cdots b_n.$$

Therefore we deduce the main result

**Proposition 7.33.** *Let  $A_j$  be self-adjoint generators of standard nc-diffusion semigroups  $(T_t^{A_j})$  and  $\Gamma_{A_j}^2 \geq \alpha \Gamma_{A_j}$  respectively for  $j = 1, \dots, n$  with the same constant  $\alpha > 0$ . Then the free product generator  $*A_j(a_1 \cdots a_n) = \sum_j a_1 \cdots a_{j-1} A_j(a_j) a_{j+1} \cdots a_n$  generates a standard nc-diffusion semigroup  $(T_t^{*A_j})$  with*

$$\Gamma_{*A_j}^2 \geq \alpha \Gamma_{*A_j}.$$

**Example 7.34.** The free product of all the examples considered so far satisfies the  $\Gamma_2$ -criterion. In particular, the free product of matrix algebra  $*_i M_n$  admits a standard nc-diffusion semigroup with the  $\Gamma_2$ -criterion.

**Example 7.35** (Block length function). Consider the free product of groups  $G_i$ ,  $G = *_i G_i$  with the block length function  $\psi$ , i.e.

$$\psi(g_1^{k_1} \cdots g_n^{k_n}) = n$$

for  $g_1 \in G_{i_1}, \dots, g_n \in G_{i_n}$ ,  $i_1 \neq i_2 \neq \cdots \neq i_n$  and  $k_i \in \mathbb{Z}$ . Fix  $i$  and denote by  $\lambda$  the left regular

representation of  $G_i$ . Define the conditional expectation  $E : \mathcal{L}(G_i) \rightarrow \mathbb{C}1$  to be

$$E(\lambda(g)) = \tau(\lambda(g))1 = \begin{cases} 1, & \text{if } g = e, \\ 0, & \text{if } g \neq e. \end{cases}$$

Here  $e$  is the identity element of  $G_i$  and  $1$  is the identity element of  $\mathcal{L}(G_i)$ . Example 3.15 says that  $T_t \lambda(g) = e^{-t(I-E)} \lambda(g)$  is a standard nc-diffusion semigroup with  $\Gamma_2 \geq \frac{1}{2}\Gamma$  where

$$T_t \lambda(g) = \begin{cases} \lambda(g), & \text{if } g = e, \\ e^{-t} \lambda(g), & \text{if } g \neq e. \end{cases}$$

Since  $\mathcal{L}(G) = *_{i \in I} \mathcal{L}(G_i)$ , using Proposition 7.33 and the relation  $\lambda(g_1 \cdots g_n) = \lambda(g_1) \cdots \lambda(g_n)$  for  $g_1 \in G_{i_1}, \dots, g_n \in G_{i_n}$  and  $i_1 \neq i_2 \neq \dots \neq i_n$ , we deduce that  $(T_t^b)$  is a standard nc-diffusion semigroup acting on  $\mathcal{L}(G)$  with  $\Gamma_2 \geq \frac{1}{2}\Gamma$  where

$$T_t^b(\lambda(g_1^{k_1} \cdots g_n^{k_n})) = e^{-tn} \lambda(g_1^{k_1} \cdots g_n^{k_n}).$$

Clearly, the infinitesimal generator of  $T_t^b$  is the block length function. In particular, for  $G_i = \mathbb{Z}$  we find a standard nc-diffusion semigroup acting on  $\mathcal{L}(\mathbb{F}_n)$  which is different from the one considered in Section 7.1.1. In fact, our result applies even for free product of groups with amalgamation in general.

## 7.3 The classical diffusion processes

We consider classical diffusion semigroups in this section. As explained in Theorem 3.13, we have stronger results in the this setting thanks to the better constant in the commutative BDG inequality. This may be regarded as a shortcut of the following implication in the classical diffusion setting

$$\Gamma_2\text{-criterion} \Rightarrow \text{log-Sobolev inequality} \Rightarrow \text{subgaussian Poincaré inequalities.} \quad (7.7)$$

Here the first implication was due to Bakry–Emery [BÉ85] and the second was due to Aida–Stroock [AS94].

### 7.3.1 Ornstein–Uhlenbeck process in $\mathbb{R}^d$

Let us start with Ornstein-Uhlenbeck process whose infinitesimal generator is  $-A = \Delta - x \cdot \nabla$  in  $\mathbb{R}^d$ . We refer the readers to e.g. [Led00] for the facts we state in this section. Let  $T_t = e^{-tA}$  be the semigroup generated by  $A$  and  $\gamma$  denote the canonical Gaussian measure on  $\mathbb{R}^d$  with density  $(2\pi)^{-d/2} e^{-|x|^2/2}$ . It is well



known that  $\gamma$  is an invariant measure of  $T_t$  and

$$T_t f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + (1 - e^{-2t})^{1/2}y) d\gamma(y).$$

Let  $\mathcal{A} = C_c^\infty(\mathbb{R}^d)$ , the compactly supported smooth functions. Clearly  $\mathcal{A}$  is weakly dense in  $\mathcal{N} = L_\infty(\mathbb{R}^d, \gamma)$  and  $T_t$  is self-adjoint with respect to  $\gamma$ . Clearly  $\mathcal{A}$  is dense in  $\text{Dom}(A^{1/2})$  in the graph norm. Note that  $\Gamma(f, f) = |\nabla f|^2$  and that for  $f \in \text{Dom}(A^{1/2})$

$$\|\Gamma(f, f)\|_1 = \langle A^{1/2}f, A^{1/2}f \rangle_{L_2(\mathbb{R}^d, \gamma)}.$$

Therefore  $(T_t)$  is a standard nc-diffusion semigroup satisfying the assumptions in Lemma 3.10. It is easy to check that

$$\Gamma_2(f, f) = |\nabla f|^2 + \|\text{Hess } f\|_{HS}^2 \geq \Gamma(f, f), \quad f \in C_c^\infty(\mathbb{R}^d).$$

Here  $\text{Hess } f$  denotes the Hessian of  $f$  and  $\|\cdot\|_{HS}$  denotes the Hilbert–Schmidt norm. Note that  $Af = 0$  only if  $f$  is a constant. Thus the fixed point algebra is trivial. Theorem 3.13 with  $\alpha = 1$  immediately leads to the following result.

**Corollary 7.36.** *Let  $2 \leq p < \infty$ . Then there exist a constant  $C$  such that for all real valued functions  $f \in W^{1,p}(\mathbb{R}^d, \gamma)$*

$$\left\| f - \int f d\gamma \right\|_{L_p(\mathbb{R}^d, \gamma)} \leq C\sqrt{p} \|\nabla f\|_{L_p(\mathbb{R}^d, \gamma)} \quad (7.8)$$

where  $W^{1,p}(\mathbb{R}^d, \gamma)$  denotes the Sobolev space consisting of all  $L_p(\mathbb{R}^d, \gamma)$  functions with first order weak derivatives also in  $L_p(\mathbb{R}^d, \gamma)$ .

This result can be generalized to infinite dimension. Let  $(W, H, \mu)$  be an abstract Wiener space and  $L$  the Ornstein–Uhlenbeck operator on  $W$ . Then it can be checked that the gradient form associated with  $L$  satisfies

$$\Gamma_2(F, F) = (\nabla F, \nabla F) + \|\nabla^2 F\|_{HS}^2 \geq \Gamma(F, F)$$

for  $F(w) \in \text{Cylin}(W)$ , the cylindrical functions on  $W$ . Based on standard facts from Malliavin calculus, an argument similar to the  $\mathbb{R}^d$  case shows that the Ornstein–Uhlenbeck semigroup  $T_t$  is a standard nc-diffusion semigroup satisfying the assumptions in Lemma 3.10. Moreover, the fixed point algebra  $\text{Fix}$  is trivial. See [Fan05, Nua06] for more details. Hence our Poincaré type inequality (7.8) holds in this setting.

### 7.3.2 Diffusion processes on Riemannian manifolds

Consider an elliptic differential operator  $-A$  on a connected smooth manifold  $M$  of dimension  $d$  with invariant probability measure  $\mu$  on Borel sets which is equivalent to Lebesgue measure. We can write it in a local coordinate chart as

$$-Af(x) = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_i b^i \frac{\partial f}{\partial x^i}(x)$$

where  $g^{ij}$  and  $b^i$  are smooth functions and  $(g^{ij})$  is a positive semidefinite matrix. The inverse of  $(g^{ij})$  then defines a Riemannian metric. It can be checked that

$$\Gamma(f, h) = \sum_{ij} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j}$$

for all  $f, h \in C_c^\infty(M)$ . To give an example, we take  $-A = \Delta + Z$  where  $\Delta$  is the Laplace–Beltrami operator on a stochastically complete Riemannian manifold and  $Z$  is a  $C^1$ -vector field on a Riemannian manifold  $M$  such that

$$\text{Ric}(X, X) - \langle \nabla_X Z, X \rangle \geq \alpha |X|^2, \quad X \in TM \quad (7.9)$$

for some  $\alpha > 0$ . By the Bochner identity, this inequality is equivalent to (see for example [Wan04])

$$\Gamma_2(f, f) \geq \alpha \Gamma(f, f), \quad f \in C^\infty(M).$$

Take  $\mathcal{A} = C_c^\infty(M)$  and  $T_t = e^{-tA}$ . The following result follows from Theorem 3.13 and the martingale problem on differentiable manifolds [Hsu02].

**Corollary 7.37.** *Assume (7.9) and the following conditions*

1.  $\int T_t(f)g d\mu = \int f T_t(g) d\mu$  (i.e.  $T_t$  is symmetric);
2.  $|\nabla f| \in L_2(M, \mu)$  whenever  $\langle A^{1/2}f, A^{1/2}f \rangle < \infty$ .

Then for all  $2 \leq p < \infty$  and real valued functions  $f \in W^{1,p}(M, \mu)$ ,

$$\|f - E_{\text{Fix}} f\|_{L_p(M, \mu)} \leq C \sqrt{p/\alpha} \|\nabla f\|_{L_p(M, \mu)} \quad (7.10)$$

where  $W^{1,p}(M, \mu)$  is the Sobolev space on the Riemannian manifold  $M$ .

*Remark 7.38.* Since compactly supported smooth functions are dense in the graph norm of  $A^{1/2}$ , the work of Cipriani and Sauvageot [CS03] shows that Condition (2) is automatically satisfied.

Functional inequalities related to diffusion processes on Riemannian manifolds have been studied extensively; see [Wan05] for more details on this subject. To give an even more concrete example, let  $\nu$  be the normalized volume measure and  $\mu(dx) = e^{-V(x)}\nu(dx)$  a probability measure for  $V \in C^2(M)$ . Suppose (7.9) holds. It is clear that the semigroup  $T_t$  with generator  $-A = \Delta - \nabla V \cdot \nabla$  fulfills the assumptions of Corollary 7.37 and the fixed point algebra is trivial. It follows that

$$\|f - \int f d\mu\|_{L_p(M, \mu)} \leq C \sqrt{p/\alpha} \|\nabla f\|_{L_p(M, \mu)}.$$

This improves X.-D. Li's result [Li08, Theorem 1.2, Theorem 5.2] for  $p \geq 2$  which was proved by using his sharp estimate of the  $L_p$ -norm of Riesz transform. Indeed, his Poincaré inequality has constant  $p/\sqrt{\alpha}$ .

*Remark 7.39.* (7.10) is true only for scalar-valued functions. If one is interested in some noncommutative objects, e.g., matrix-valued functions on manifolds or free product of manifolds, one has to apply the noncommutative theory and then the Poincaré inequalities are in the form of Theorem 3.8. Of course, the deviation and the transportation inequalities still hold in all those situations.

### 7.3.3 $\Gamma_2$ -criterion, spectral gap and $L_p$ Poincaré inequalities

As explained above, the spectral gap may lead to the  $L_p$  Poincaré inequalities with constant  $Cp$  under certain conditions. Our first example illustrates that even in the classical diffusion setting one can not achieve  $C\sqrt{p}$  assuming only the existence of spectral gap.

**Example 7.40** (Spectral gap is not sufficient). Consider the double exponential distribution on  $\mathbb{R}$  given by  $\mu(dx) = \frac{1}{2}e^{-|x|}dx$ . There exists a semigroup  $T_t$  which is symmetric on  $L_2(\mathbb{R}, \mu)$  with generator given by

$$-A = \frac{d^2}{dx^2} - \operatorname{sgn}(x) \frac{d}{dx}$$

on compactly supported smooth functions  $f$  with  $f'(0) = 0$ , where  $\operatorname{sgn}(x)$  is the sign of  $x$ . Clearly such functions are dense in  $L_2(\mathbb{R}, \mu)$ . It was shown in [BL97] that  $\mu$  satisfies the  $L_2$  Poincaré inequality. However, it is easy to see that the  $L_p$  Poincaré inequalities (1.1) cannot hold by testing  $f(x) = x$ . By (7.7), the semigroup  $(T_t)$  has to fail the Bakry–Emery  $\Gamma_2$ -condition. In this way, one can come up with a family of diffusion processes for which Bakry–Emery's condition fails. Indeed, let  $\mu_\alpha(x) = \frac{1}{C_\alpha} e^{-|x|^\alpha} dx$  for  $1 \leq \alpha < 2$  on  $\mathbb{R}$  where  $C_\alpha$  is a normalizing constant. Consider

$$-A_\alpha = \frac{d^2}{dx^2} - \alpha|x|^{\alpha-1} \operatorname{sgn}(x) \frac{d}{dx}$$

on compactly supported smooth functions  $f$  with  $f'(0) = 0$ .  $-A_\alpha$  generates a symmetric semigroup  $T_t^\alpha$  on  $L_2(\mathbb{R}, \mu_\alpha)$ . The corresponding Markov process is a diffusion process. All these  $T_t^\alpha$  for  $1 \leq \alpha < 2$  will fail (1.1), and thus fail Bakry–Emery’s  $\Gamma_2$ -criterion. In fact, in this case,  $\Gamma(f, f)(x) = |f'(x)|^2$  but  $\Gamma_2(f, f)(x) = \alpha(\alpha - 1)|x|^{\alpha-2}|f'(x)|^2$  for  $x \neq 0$ . Observe that we have only  $\Gamma_2(f, f) \geq 0$ , but the spectral gap still exists by the same argument as for [BL97, Lemma 2.1]. Hence by, e.g., [Mil09, Proposition 2.5],  $T_t^\alpha$  satisfies the  $L_p$  Poincaré inequalities with constants  $Cp$ .

Our second example is meant to clarify the subgaussian behavior we discuss here via  $L_p$  Poincaré inequalities is a condition on the semigroup (or its generator), not on the (noncommutative) probability space.

**Example 7.41.** Consider the exponential distribution on  $[0, +\infty)$  given by  $\mu(dx) = e^{-x}dx$ . By [KS85], there is a conservative Markov semigroup which is symmetric in  $L_2([0, \infty), \mu)$  with generator  $-A = x \frac{d^2}{dx^2} + (1-x) \frac{d}{dx}$ . A calculation shows that

$$\Gamma_2(f, f)(x) - \Gamma(f, f)(x) = (xf''(x) + f'(x)/2)^2 + f'(x)^2/4 \geq 0$$

for all compactly supported smooth functions  $f$ . Since  $-A$  generates a diffusion process, by [AS94, JZ14], we have (1.1). Note that the exponential distribution is not subgaussian in the sense of [Ver12] because  $\|X\|_p = \Gamma(p+1)^{1/p} \sim p$  where the law of  $X$  is  $\mu$ . This means that the semigroups could satisfy the subgaussian Poincaré inequalities even though its invariant measure is not a subgaussian distribution. Roughly speaking, the gradient form in (1.1) will provide another factor which compensates the factor  $\sqrt{p}$ . For instance, in our example here, if  $f(x) = x$ , then  $\Gamma(f, f)^{1/2} = \sqrt{x}$ .

*Remark 7.42.* The above examples showed that the  $L_p$  Poincaré inequalities provide more information than the moment estimates of probability measures. Indeed, the exponential distribution and the double exponential distribution have the same decay at  $+\infty$ . But there exist different semigroups such that the  $L_p$  Poincaré inequalities (1.1) may or may not hold.

It is also interesting to compare (1.1) and the log-Sobolev inequality in deducing concentration inequalities. On one hand, it is known (see [AS94]) that log-Sobolev inequality implies (1.1) in the classical diffusion setting while it was shown in [JZ14] that in general non-diffusion situation, (1.1) may still hold when the log-Sobolev inequality fails. One can deduce concentration results from (1.1). On the other hand, although the spectral gap itself is not sufficient to give the subgaussian  $L_p$  Poincaré inequalities (1.1) as shown in Example 7.40, Bobkov and Ledoux showed in [BL97] that the exponential distribution satisfies a modified version of log-Sobolev inequality. From here, they proved concentration inequalities (see also [BG99]). It seems from the above discussion that the log-Sobolev inequality and the  $L_p$  Poincaré inequalities are both

useful in their own right and cannot entirely replace each other.

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