A STUDY ON CERTAIN PERIODIC SCHRÖDINGER EQUATIONS

BY

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DISSERTATION

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Abstract

In the first part of this thesis we consider the cubic Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \pm|u|^2 u, \quad x \in \mathbb{T}^2_\theta, \quad t \in [-T, T], \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}^2_\theta). \end{cases} \tag{1}$$

T is the time of existence of the solutions and $\mathbb{T}^2_\theta$ is the irrational torus given by $\mathbb{R}^2/\theta_1\mathbb{Z} \times \theta_2\mathbb{Z}$ for $\theta_1, \theta_2 > 0$ and θ_1/θ_2 irrational. Our main result is an improvement of the Strichartz estimates on irrational tori using a counting argument by Huxley [43], which estimates the number of lattice points on ellipsoids. With this Strichartz estimate, we obtain a local well-posedness result in H^s for $s > \frac{131}{416}$. We also use energy type estimates to control the H^s norm of the solution and obtain improved growth bounds for higher order Sobolev norms.

In the second and the third parts of this thesis, we study the Cauchy problem for the 1d periodic fractional Schrödinger equation:

$$\begin{cases} iu_t + (-\Delta)^\alpha u = \pm|u|^2 u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}), \end{cases} \tag{2}$$

where $\alpha \in (1/2, 1)$. First, we prove a Strichartz type estimate for this equation. Using the arguments from Chapter 3, this estimate implies local well-posedness in H^s for $s > \frac{1-\alpha}{2}$. However, we prove local well-posedness using direct $X^{s,b}$ estimates. In addition, we show the existence of global-in-time infinite energy solutions. We also show that the nonlinear evolution of the equation is smoother than the initial data. As an important consequence of this smoothing estimate, we prove that there is global well-posedness in H^s for $s > \frac{10\alpha+1}{12}$. Finally, for the fractional Schrödinger equation, we define an invariant probability measure μ on H^s for $s < \alpha - \frac{1}{2}$, called a Gibbs measure. We define μ so that for any $\epsilon > 0$ there is a set $\Omega \subset H^s$ such that $\mu(\Omega^c) < \epsilon$ and the equation is globally well-posed for initial data in Ω. We achieve this by showing that for the initial data in Ω, the H^s norms of the solutions stay finite for all times. This fills the gap between the local well-posedness and the global well-posedness range in almost sure sense for $\frac{1-\alpha}{2} < \alpha - \frac{1}{2}$, i.e. $\alpha > \frac{2}{3}$.

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Chapter 1

Introduction

In this dissertation, we consider certain periodic cubic Schrödinger equations of the form:

$$\begin{cases} iu_t + \blacktriangle u = \pm|u|^2 u, \quad x \in \Omega, \quad t \in [-T, T], \\ u(x,0) = u_0(x) \in H^s(\Omega), \end{cases} \tag{1.1}$$

where $\blacktriangle$ is a certain Laplace or Laplace-type operator. In two dimensions, Ω is the irrational torus given by $\mathbb{R}^2/\theta_1\mathbb{Z} \times \theta_2\mathbb{Z}$ when $\theta_1, \theta_2 > 0$ and θ_1/θ_2 is irrational. In one dimension, Ω is the regular torus.

Equation 1.1 belongs to a broader class of differential equations called dispersive equations. Dispersive equations are characterized by the property that frequency-localized bumps propagate with a velocity depending on the frequency, without changing their shapes. In general, on unbounded domains, the solution decays over time and time averages get smoother. On bounded domains, there is no decay, but a more subtle averaging effect occurs in the form of a Fourier restriction estimate.

In the following chapters, we answer several fundamental questions about periodic cubic Schrödinger equations. For example, we determine whether there is a unique local (global) in time solution to the equation which is continuous with respect to the initial data in certain subsets of Sobolev spaces. If so, we say that the equation is locally (globally) well-posed.

When the equation is known to be globally well-posed on the Sobolev space H^s, we know that for all times, the H^s norm of the solution remains finite. For such an equation, the natural question to ask is how fast the norm grows in time. Using simple iteration arguments, one can generally prove that the norm can grow at most exponentially. The question then becomes whether the Sobolev norm growth is actually exponential: is it bounded by a polynomial or logarithmic function, or perhaps even by a constant?

When we try to prove well-posedness, restricting ourselves to a subset of H^s spaces is often necessary due to the regularity loss incurred by estimating the solutions of the linear Schrödinger equation in Lebesgue spaces other than L^2. The estimates of this type are called Strichartz estimates. To overcome this problem, we try to find local well-posedness of the equation in a subset of H^s which better incorporates the structure of the equation. These spaces are known as $X^{s,b}$ or Bourgain spaces, see (2.13). In the literature, we refer to well-posedness in $X^{s,b}$ spaces as H^s well-posedness. Well-posedness in H^s spaces themselves we call unconditional well-posedness. In this thesis, we consider only H^s well-posedness.

In the third chapter of the dissertation, using a contraction argument, we prove that the 2d periodic Schrödinger equation, where $\blacktriangle$ is the Laplace operator Δ on irrational tori, is locally well-posed on H^s for $s > \frac{131}{416}$. We also prove a polynomial in time upper bound on the Sobolev norm of the global solutions. This result, however, does not imply the global in time boundedness of the solutions, see [41].

This local well-posedness result heavily depends on the Strichartz estimates. On $\mathbb{R}^2$, these estimates are byproducts of the decay estimates, but on periodic settings, since there is no decay in time, it is harder to obtain them. In [7], Bourgain reduced the Strichartz estimates on the torus to a counting argument about the number of lattice points on a circle. Using a similar argument and a theorem by Huxley [43], which estimates the number of lattice points on an ellipse, we prove the Strichartz estimates on irrational tori with $\frac{131}{832}$ derivative loss. After obtaining these estimates, we prove the bilinear Strichartz estimate, which controls the product of two linear solutions. Then, with careful analysis on $X^{s,b}$ norm of the frequency restrictions of the solutions, we obtain the local well-posedness result. However, on irrational tori, it was expected that Strichartz estimates exist without any derivative loss, as in the regular torus case. Indeed, Bourgain and Demeter proved in [15]:

$$\|e^{it\Delta}u_0\|_{L^p_t(I,L^p_x(\mathbb{T}^{n-1}_\theta))} \lesssim_\epsilon N^{\frac{n-1}{2}-\frac{n+1}{p}+\epsilon}|I|^{1/p}\|u_0\|_{L^2_x(\mathbb{T}^{n-1}_\theta)}$$

for each $\epsilon > 0$, where $p \geq \frac{2(n+1)}{n-1}$ and supp $\widehat{u_0} \subset [-N,N]^{n-1}$. This proves the Strichartz estimates in full generality up to ϵ derivative loss.

To prove the polynomial growth, we use energy type estimates, i.e., we estimate $\frac{d}{dt}\|u(t)\|_{H^s_x}$ for a global solution u of the equation. We estimate this term using the explicit form of the equation, estimates on the frequency restrictions of the function, and Lemma (3.1.1), which was used by Bourgain in [9] to prove a similar growth estimate for two dimensional Schrödinger equation on regular tori.

In the fourth and the fifth chapters we work on the 1d fractional Schrödinger equation, where $\blacktriangle$ is the fractional Laplacian $(-\Delta)^\alpha$ for $\alpha \in (1/2, 1)$. In the fourth chapter we first prove a Strichartz estimate with $\frac{1-\alpha}{4}$ derivative loss, which by the arguments in Chapter 3 gives us local well-posedness in H^s for $s > \frac{1-\alpha}{2}$. However, instead of using the same methods, we use direct $X^{s,b}$ estimates to run the contraction argument and prove local well-posedness.

We know that there is mass and energy conservation for the solutions of the fractional Schrödinger equation. This energy conservation gives us a global in time control over the H^α norm of the solution and thus global well-posedness in H^α whenever we have local well-posedness in H^α. However, with the lack of a conservation law at the H^s level, we do not have a priori global well-posedness in H^s for $s \in (0, \alpha)$ below the energy level H^α. In the second part of Chapter 4, we prove that the nonlinear evolution of the equation is smoother than the initial data. Using this smoothing estimate and Bourgain's high-low frequency decomposition, we end this chapter by proving that the equation is globally well-posed in H^s for $s > \frac{10\alpha+1}{12}$. The high-low frequency decomposition method consists of estimating separately the evolution of the low frequencies and of the high frequencies of the initial data in a $(0, \delta)$ interval. Then we iterate this solution as many times as possible to reach any given, arbitrarily large time, where at every iteration step we feed the smoother nonlinear evolution of the high frequency equation to the low frequency equation, as long as the energy of the low frequency equation is controlled by its initial energy. Although the smoothing estimate is crucial for the high-low frequency decomposition method, it has more applications in understanding the dynamics of the equation like the existence of global attractors, see [35].

In the fifth chapter we use probabilistic arguments to understand the set of initial data for which the local solutions cannot be extended to global ones. For that, instead of trying to understand the structure of the set, we show that the set is actually negligible with respect to a weighted Gaussian measure, called the Gibbs measure. This idea of looking at the probabilistic properties of the set of initial data was initiated by Lebowitz, et. al. in [49]. In this chapter, using the Hamiltonian structure of the Schrödinger equation and Zhidkov's arguments in [67], we explicitly construct the invariant Gibbs probability measure on H^s for $s < \alpha - \frac{1}{2}$. Finally, using Bourgain's arguments in [8] we prove that for almost any initial data in H^s with respect to this measure, the solution is global. The main idea of the proof is to show that for almost any initial data, the H^s norm of the solution stays finite and by the blow up alternative, see page 18, we conclude that the solution has to be global. This approach depends on the delicate balance between the polynomial dependence of the local well-posedness time on the H^s norm of the initial data, invariance of the measure, and decay estimates on Gaussian measures.

Chapter 2

Background and Tools

2.1 Basic Definitions and Estimates

In this thesis we will use $(\cdot)^+$ to denote $(\cdot)^\epsilon$ for all $\epsilon > 0$ with implicit constants depending on ϵ and will use the usual Japanese bracket notation, $\langle x\rangle = (1 + x^2)^{1/2}$.

We will use $A \lesssim B$ and $A \sim B$ to denote that there is a constant $C > 0$ such that $A \leq CB$ and $C^{-1}B \leq A \leq CB$ respectively.

Definition 2.1.1. *[5] Bessel potential (J^s) and Riesz potential (D^s) are the operators defined as*

$$J^s u = \mathcal{F}^{-1}\big((1 + |\xi|^2)^{s/2}\hat{u}\big) \qquad \text{and} \qquad D^s u = \mathcal{F}^{-1}\big(|\xi|^s\hat{u}\big) \tag{2.1}$$

where the $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Definition 2.1.2. *A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a Schwartz function if it is infinitely differentiable and $x^\mu D^\gamma f \in L^\infty(\mathbb{R}^n)$ for all nonnegative multiindices $\mu = (\mu_1, ..., \mu_n)$ and $\gamma = (\gamma_1, ..., \gamma_n)$ such that $\mu_i, \gamma_i \geq 0$ for $i \in \{1, ..., n\}$, where $x^\mu = x_1^{\mu_1}...x_n^{\mu_n}$ and $D^\gamma f = \frac{d^{\gamma_n}}{dx_n^{\gamma_n}}...\frac{d^{\gamma_1}}{dx_1^{\gamma_1}}f$.*

Definition 2.1.3. *For a time interval I, the mixed Lebesgue space $L^q_t(I, L^r_x)$ is defined via the norm:*

$$\|u\|_{L^q_t(I,L^r_x)} = \begin{cases} \Big(\int\limits_I \|u(t)\|^q_{L^r_x} dt\Big)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup\limits_I \|u(t)\|_{L^r_x} & \text{if } q = \infty, \end{cases}$$

and the space $C_t(I, L^r_x)$ is defined as the space of continuous functions $u : I \to L^r_x$.

Definition 2.1.4. *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of C^∞_c, compactly supported C^∞ functions, and let $\phi \in C^\infty_c$. We say $\lim\limits_{n\to\infty} \phi_n = \phi$ in C^∞_c if*

- *there is a compact set K such that $\operatorname{supp}\phi_n \in K$, for all n, and*
- *all derivatives of ϕ_n converge uniformly to the corresponding derivative of ϕ.*

Definition 2.1.5. *The space of distributions, D', is the space of continuous linear functionals on C_c^∞.*

From this definition, we see that if f is locally integrable, then f defines a distribution

$$T_f(\phi) = \int_{\mathbb{R}^n} f\phi dx,$$

for $\phi \in C_c^\infty$, which we will also denote as f.

Definition 2.1.6. *For every multindex γ, one can define the distributional derivative, $D^\gamma f$, of f as a distribution by*

$$\int_{\mathbb{R}^n} D^\gamma f\phi dx = (-1)^{|\gamma|} \int_{\mathbb{R}^n} fD^\gamma\phi,$$

for all $\phi \in C_c^\infty$.

Definition 2.1.7. *The space $\mathcal{S}'$ is the space of tempered distributions on $\mathbb{R}^n$, which means that $\mathcal{S}'$ is the topological dual of $\mathcal{S}$.*

The Sobolev spaces are defined as follows:

Definition 2.1.8. *For $m \in \mathbb{N}$, the Sobolev space $W^{m,p}$ is given by*

$$W^{m,p} = \{f \in L^p : D^\gamma u \in L^p \quad \forall\gamma \quad multiindex \quad such \quad that \quad |\gamma| \leq m\}$$

with the norm

$$\|u\|_{W^{m,p}} = \sum_{\substack{|\gamma|\leq m \\ \gamma \ multiindex}} \|D^\alpha u\|_{L^p}. \tag{2.2}$$

For $p = 2$, we write $W^{m,2} = H^m$, and since $p = 2$, we can characterize the Sobolev space using the Fourier transform. Namely, given $m \in \mathbb{N}$ we can define

$$H^m = \{u \in \mathcal{S}' : (1 + |\xi|^2)^{m/2}\hat{u} \in L^2\}$$

with the norm

$$\|u\|_{H^m} = \|(1 + |\xi|^2)^{m/2}\hat{u}\|_{L^2} \tag{2.3}$$

where $\mathcal{S}'$ is the dual of Schwartz space. We are also going to denote by $\|u\|_{\dot{H}^m} = \||\xi|^m\hat{u}\|_{L^2}$ the homogeneous Sobolev norm. Note the requirement $m \in \mathbb{N}$ simply serves to make the definition consistent with the previous one, and we can extend this definition to the noninteger real positive numbers. We can also extend it to negative numbers by taking the dual of the positive indiced Sobolev spaces. For further details, see [5].

For Sobolev space we have the following result.

Theorem 2.1.9 (Gagliardo-Nirenberg Inequality)**.** *Fix $1 \le q, r \le \infty$ and $m \in \mathbb{N}$. Then for $u \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|D^j u\|_{L^p} \lesssim \|D^m u\|_{L^r}^{\lambda} \|u\|_{L^q}^{1-\lambda},$$

where $C = C(m, n, j, q, r, \lambda)$ and $\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\lambda + \frac{1-\lambda}{q}$, and $\frac{j}{m} \le \lambda \le 1$.

For a proof, see [54].

We also have the following embedding result.

Theorem 2.1.10. *Let $m \ge 1$ be an integer and $1 \le p < \infty$. Then*

(1) if $1/p - m/n > 0$, then $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ with $1/q = 1/p - m/n$,

(2) if $1/p - m/n = 0$, then $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \le q < \infty$,

(3) if $1/p - m/n < 0$, then $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.

For a proof, see [2] and [5].

One also can consider the interpolation of these Sobolev spaces

Theorem 2.1.11. *[5, Theorem 6.4.5] Let numbers $s, s_0, s_1, p_0, p_1, \theta$ be given, with $0 < \theta < 1$. In addition, put,*

$$\begin{aligned} s* &= (1-\theta)s_0 + \theta s_1, \\ \frac{1}{p*} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \end{aligned}$$

Then for $s_0 \ne s_1$ and $1 < p_0, p_1 < \infty$, we have

$$(W^{s_0,p_0}, W^{s_1,p_1})_{[\theta]} = W^{s*,p*},$$

where $(W^{s_0,p_0}, W^{s_1,p_1})_{[\theta]}$ is the interpolation space of W^{s_0,p_0} and W^{s_1,p_1} of exponent θ, see [5, Definition 2.4.1].

Now, consider a compactly supported function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi) \subset \mathbb{R}^n - \{0\}$ and $\sum_{-\infty}^{\infty} \psi(2^{-j}x) = 1$ and call $\psi_j(\xi) = \psi(2^{-j}\xi)$; namely, consider a radial function $\phi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\phi(\xi) = 1 \text{ for } |\xi| \le 1 \text{ and } \phi(\xi) = 0 \text{ for } |\xi| \ge 2,$$

then define $\psi(\xi) = \phi(\xi) - \phi(2\xi)$, which satisfies the above conditions. Then if we define the operator $P_j f = \mathcal{F}^{-1}(\psi_j \widehat{f})$, we have the following.

Theorem 2.1.12. *[59, Theorem 8.3] For any $1 < p < \infty$, the Littlewood-Paley square function, $Sf = \left(\sum_j |P_j f|^2\right)^{\frac{1}{2}}$, satisfies*

$$\|f\|_{L^p} \sim \|Sf\|_{L^p},$$

for any $f \in \mathcal{S}$.

For a proof, see [59].

Moreover, using the Littlewood-Paley theory, one can prove the fractional Leibniz rule.

Theorem 2.1.13. *[45, Theorem A.8]*

For $0 < s, s_1, s_2 < 1$, $s_1 + s_2 = s$ and Riesz transform D^s:

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p_x L^q_t} \leq C\|D^{s_1} f\|_{L^{p_1}_x L^{q_1}_t} \|D^{s_2} g\|_{L^{p_2}_x L^{q_2}_t},$$

where $0 < p, p_1, p_2, q, q_1, q_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Moreover, for $s_1 = 0$, the result holds for $q_1 = \infty$.

For a proof, see [45].

We now include several standard definitions that will be used in the last chapter.

Definition 2.1.14. *Let X be a nonempty set. Then a nonempty collection M of subsets of X is called an algebra if it is closed under finite unions and complements. In other words, if $X_1, X_2, ..., X_n \in M$, then $\cup_{k=1}^n X_k \in M$, and if $E \in M$, then $E^c \in M$. An algebra that is closed under countable unions is called a $\sigma - algebra$.*

Definition 2.1.15. *The smallest $\sigma - algebra$ that contains a collection of sets B is called the $\sigma - algebra$ generated by B. The $\sigma - algebra$ generated by open subsets of X is called the Borel $\sigma - algebra$.*

Finally, we give the definition of a countably additive measure.

Definition 2.1.16. *A countably additive measure μ on a $\sigma - algebra$ $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that*

- $\mu(\emptyset) = 0$,
- *If $\{A_n\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$, then $\mu(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n)$.*

2.2 Strichartz Estimates and $X^{s,b}$ Spaces

For the general Laplace-type operator $(-\blacktriangle)$, defined on the Fourier side as $\left(\widehat{(-\blacktriangle)u}\right) = h(\xi)\widehat{u}$, if we define the linear Schrödinger equation:

$$\begin{cases} iu_t - (-\blacktriangle)u = 0, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{2.4}$$

we see that the solution u has the form $S(t)u_0$ for all t. Here, $S(t)$ is called the linear propagator of the equation and is defined on the Fourier side as $(\widehat{S(t)f})(t,\xi) = e^{-ith(\xi)}\hat{f}(\xi)$. We should mention that in Chapter 3 resp. Chapters 4 and 5, we are going to use the Laplace operators $-\Delta$ resp. $(-\Delta)^{\alpha}$ with the multipliers $h(m,n) = ((\theta_1 m)^2 + (\theta_2 n)^2)$, where θ_1/θ_2 is irrational resp. $h(m) = |m|^{2\alpha}$.

In the general case of (2.4), we will denote the linear propagator S(t) as $e^{it\blacktriangle}$. Consider the cubic Schrödinger equation,

$$\begin{cases} iu_t - (-\blacktriangle)u = \pm|u|^2 u, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{2.5}$$

We note that this equation enjoys mass and energy conservation,

$$M(t) = \int_{\mathbb{R}^n} |u(t)|^2 = \int_{\mathbb{R}^n} |u_0|^2 = M(0) \tag{2.6}$$

and

$$E(u)(t) = \int_{\mathbb{R}^n} \left|\sqrt{-\blacktriangle}u(t,x)\right|^2 \pm \frac{1}{4}\int_{\mathbb{R}^n} \left|u(t,x)\right|^4 = E(u)(0) \tag{2.7}$$

respectively, where $\sqrt{-\blacktriangle}$ is defined on the Fourier side as $\sqrt{-\blacktriangle}u = \mathcal{F}^{-1}(h(\xi)^{\frac{1}{2}}\widehat{u}(\xi))$. Formally, we can prove these conservation laws as follows: For the mass conservation, if we begin with H^1-solutions, considering $H^{-1} - H^1$ duality product of (2.5) with $2u$ and integration by parts gives

$$2i\langle u_t, u\rangle_{-1,1} = -2(\|\sqrt{-\blacktriangle}u\|_{L_x^2}^2) \pm 2\int\limits_{\mathbb{R}^n} |u|^4 dx.$$

Since the right hand side is real, we obtain the mass conservation on $[0,T)$. For the conservation of energy, multiplying (2.5) by $2\bar{u}_t$ and then taking real parts give $2\mathrm{Re}\,\bar{u}_t(-\blacktriangle u) = |u|^2(|u|^2)_t$ from which it follows that

$$\begin{aligned} 0 &= \frac{d}{dt}\int\limits_{\mathbb{R}^n}(-\blacktriangle u)\bar{u}dx \pm \int\limits_{\mathbb{R}^n}\frac{1}{4}|u|^4 dx \\ &= \frac{d}{dt}\left[\int\limits_{\mathbb{R}^n}|\sqrt{-\blacktriangle}u|^2 dx \pm \frac{1}{4}\int\limits_{\mathbb{R}^n}(|u|^4)dx\right], \end{aligned}$$

by using integration by parts. These formal computations make sense for H^2-solutions. By using continuous dependence results, one can approximate L^2 and H^1-solutions with H^1 and H^2-solutions, respectively, to obtain the necessary conservation laws.

By the Duhamel Principle, we know that the smooth solutions of (2.5) satisfy the integral equation

$$\Phi(u)(t,x) = e^{it\blacktriangle}u_0(x) \mp i\int_0^t e^{i(t-t')\blacktriangle}|u|^2u(t',x)dt. \tag{2.8}$$

Since we want to prove local well-posedness, finding the fixed point of this integral operator in time interval $[0,T]$ for $T<1$ is equivalent to finding the fixed point of the integral equation

$$\Phi(u)(t,x) = \psi(t)e^{it\blacktriangle}u_0(x) \mp i\psi(t/T)\int_0^t e^{i(t-t')\blacktriangle}|u|^2u(t',x)dt, \tag{2.9}$$

where $\psi(t)$ is a compactly supported $C^\infty(\mathbb{R})$ function ψ such that $\psi(t)=1$ for $0\le t\le 1$ and $\psi(t)=0$ for $t\le -1$ and $t\ge 2$. Here we call the first term on the right hand side the linear evolution term, and the second term the nonlinear evolution term. Thus, our main concern is to find the fixed point to the integral operator in certain metric spaces. For this, we need to find ways to estimate the terms in the Duhamel formula. One of the important estimates of the local and global well-posedness theory are the Strichartz estimates of the form:

$$\|e^{it\blacktriangle}u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{H^{s_0}_x},$$

for certain pairs of (q,r) and $s_0\ge 0$. This estimate tells us that when we are trying to estimate the Lebesgue norms of the solution, even for the linear evolution we may have some regularity loss.

On $\mathbb{R}^n$, for the regular Schrödinger equation, i.e. for $\blacktriangle=\Delta$, or equivalently $h(\xi)=|\xi|^2$, we have the following lemma.

Lemma 2.2.1. *[22, Lemma 2.2.4] For all $t\neq 0$ and all $u_0\in\mathcal{S}(\mathbb{R}^n)$, we have,*

$$e^{it\Delta}u_0 = (4\pi i t)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}}u_0(y)dy,$$

which gives us the dispersion estimate,

$$\|e^{it\Delta}u_0\|_{L^\infty_x} \le (4\pi t)^{-\frac{n}{2}}\|u_0\|_{L^1_x}.$$

Using this lemma with $\|e^{it\Delta}u_0\|_{L^2_x} = \|u_0\|_{L^2}$ for all $t\in\mathbb{R}$ and interpolation, one can show the following.

Proposition 2.2.2. *[22, Proposition 2.2.3] If $p\in[2,\infty]$ and $t\neq 0$, then $e^{it\Delta}$ maps $L^{p'}(\mathbb{R}^n)$ continuously* *to* $L^p(\mathbb{R}^n)$ *for* $\frac{1}{p}+\frac{1}{p'}=1$*, and*

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^n)} \leq (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^{p'}(\mathbb{R}^n)} \text{ for all } u_0 \in L^{p'}(\mathbb{R}^n).$$

For proofs, see [22].

Now we define an admissible pair.

Definition 2.2.3. *A pair* (q,r) *is admissible in* $\mathbb{R}^n$ *if* $(q,r,n) \neq (2,\infty,2)$*, and*

$$\frac{2}{q} = n(\frac{1}{2}-\frac{1}{r}),$$

and $2 \leq r < \infty$*. We note that* $(\infty,2)$ *is admissible, and will correspond to the* L^2*, or mass conservation, and so is important.*

Using this definition, we can state the following.

Theorem 2.2.4. *[Strichartz Estimates, [22]] If* (q,r) *is admissible, then the following properties hold.*

- *For every* $\varphi \in L^2_x$*, the function* $t \mapsto e^{it\Delta}\phi$ *belongs to*

  $$L^q_t(\mathbb{R}, L^r_x) \cap C_t(\mathbb{R}, L^2_x).$$

  Moreover, there exists a constant C *such that*

  $$\|e^{i(.)\Delta}\varphi\|_{L^q_t(\mathbb{R},L^r_x)} \leq C\|\varphi\|_{L^2_x}.$$

- *Let* I *be an interval of* $\mathbb{R}$*,* $J=\bar{I}$*, and* $0 \in J$*. If* (γ,ρ) *is an admissible pair and* $f \in L^{\gamma'}_t(I, L^{\rho'}_x)$*, then for every* (q,r)*, the function* $t \mapsto \int_0^t e^{i(t-t')\Delta}f(t')dt'$ *for* $t \in I$ *belongs to* $L^q_t(\mathbb{R}, L^r_x) \cap C_t(\mathbb{R}, L^2_x)$*. Furthermore, there exists a constant* C *depending on q,r,*γ *and* ρ *that is independent of* I *such that*

  $$\left\|\int_0^t e^{i(t-t')\Delta}f(t')dt'\right\|_{L^q_t(I,L^r_x)} \leq C\|f\|_{L^{\gamma'}_t(I,L^{\rho'}_x)}.$$

Again, see [22] for details.

These results have more importance in the $\mathbb{R}^n$ setting in the sense that Lemma 2.2.1 implies that

$$\|e^{it\Delta}u_0\|_{L^\infty} \leq (4\pi|t|)^{-\frac{n}{2}}\|u_0\|_{L^1},$$

which means that the solutions decay in time. But this is not true in the periodic setting. Thus, although we wouldn't expect to have a result as strong as Theorem 2.2.4 on the periodic setting, we can still prove Strichartz estimates for certain pairs of (q, r).

In 1d, for the regular Schrödinger equation on the torus, i.e. for $h(m) = m^2$, Bourgain showed

$$\|e^{it\Delta} f\|_{L^4_x L^4_t} \lesssim \|f\|_{L^2_x}.$$

This follows from the fact that $\forall m \in \mathbb{Z}$, $\#\{n \in \mathbb{Z} : m^2 - (n-m)^2 = k, k \lesssim N^2\} \leq 2$. (See [7] and the proof of Strichartz estimates in Chapter 3 for this counting argument.) This implies that the cubic Schrödinger equation is locally well-posed in $L^4_t L^4_x$ for L^2 initial data. In this case, the L^2 conservation of the solution implies that the local solutions are also global. Later Burq, et al. in [16] showed that the equation is ill-posed in H^s for any $s < 0$, by showing that the initial data to solution map cannot be uniformly continuous. This result was improved by Christ et al. in [25], where they showed that the solution map not only fails to be uniformly continuous as a function in H^s but also as a function from C^∞ to distributions.

In 2d, for the regular Schrödinger equation on torus, however, Bourgain showed using a similar counting argument that $\forall m \in \mathbb{Z}^2$, $\#\{n \in \mathbb{Z}^2 : |m|^2 - |n-m|^2 = k, \ for \ k \lesssim N^2\} \leq N^\epsilon$. This result was obtained using the number theoretical argument that the number of divisors of a number of order N is at most of order N^ϵ. This counting argument gives us the Strichartz estimate

$$\|e^{it\Delta} f\|_{L^4_x L^4_t} \lesssim \|f\|_{H^{s_0}_x} \tag{2.10}$$

for $s_0 > 0$. Using this Strichartz estimate, Bourgain proved that the Schrödinger equation is locally well-posed in H^s for $s > 0$.

Our main goal in Chapter 3 is to obtain the estimate (2.10) for some $s_0 \geq 0$, and find a fixed point to the integral operator Φ in H^s for some $s \geq 0$. Since the linear propagator is an isometry in Sobolev spaces, to prove local well-posedness in H^s, we would not need to use this estimate in the linear evolution term in the Duhamel formula. However, if we try to estimate the H^s norm of the nonlinear evolution term

$$\int_0^t e^{i(t-t')\blacktriangle} |u|^2 u(t', x) dt'$$

heuristically, to be able to run the contraction argument, we would want to have estimates of the form

$$\left\| \int_0^t e^{i(t-t')\blacktriangle} |u|^2 u(t', x) dt' \right\|_{H^s_x} \lesssim T^a \|u(t)\|^3_{H^s} \tag{2.11}$$

for some $a > 0$, where we would use Sobolev embeddings, Hölder inequality and Strichartz estimates. Since we are going to use the Strichartz estimates, these heuristic estimates suggest that we look for local well-posedness in H^s for $s \geq s_0$.

Thus, to prove local well-posedness in H^s, one needs to find the fixed point of the integral equation Φ in a metric space of the form

$$D_T = \{u \in C([0,T], H^s) : u(0,x) = u_0(x) \quad \text{and} \quad \|u\|_{L^\infty([0,T],H^s)} \leq 2\|u_0\|_{H^s}\},$$

with the metric $d(u,v) = \|u - v\|_{L^\infty([0,T],H^s)}$. However, it is quite hard to justify the heuristic estimates (2.11) for any $s > s_0$. Thus we are going to restrict ourselves to a subset of H^s which incorporates the structure of the equation more explicitly.

If we take the Fourier transform of the equation (2.4) both in time and space, we see that the solutions satisfies the equation

$$(\tau + h(\xi))\widehat{u}(\tau, \xi) = 0 \tag{2.12}$$

for all (τ, ξ). This implies that on the Fourier side the solution of the linear Schrödinger equation is localized around the hypersurface $\tau = -h(\xi)$. Of course this observation does not hold for the cubic Schrödinger equation (2.5). Since the linear solutions are 'good' in the sense that they are global and preserve the Sobolev norms of the initial data, it is reasonable to ask how much the solution of (2.5) deviates from the solution of the linear equation. To account for this deviation, we will define the Bourgain space, $X^{s,b}$, as the closure of the compactly supported smooth functions under the norm

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau + h(\xi)\rangle^b \widehat{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi(\mathbb{R}\times\mathbb{R}^n)}, \tag{2.13}$$

where the Fourier transform is taken in both space and time. As the definition is given in $\mathbb{R} \times \mathbb{R}^n$, we can also define the restriction of the Bourgain space on $I \times \mathbb{R}^n$ for some time interval $[0,T]$ as

$$\|u\|_{X^{s,b}_T} = \inf\{\|f\|_{X^{s,b}} : f \in X_{s,b}, \quad f(t) = u(t) \;\; \forall t \in [0,T]\}. \tag{2.14}$$

This norm (2.13) can be written in another form using the linear propagator and Bessel potentials in time and space, J_t and J_x respectively, as

$$\|u\|_{X^{s,b}} = \|J_t^b J_x^s e^{-it\blacktriangle} u(x,t)\|_{L^2_t(\mathbb{R},L^2_x)} = \|e^{-it\blacktriangle} u\|_{H^b_t H^s_x}, \tag{2.15}$$

or in terms of an iterated norm, as

$$\|u\|_{X^{s,b}} = \|e^{-ith(\xi)}\widehat{u}(\xi,t)\|_{H_t^b L_\xi^2(\langle\xi\rangle^{2s})}, \tag{2.16}$$

where the Fourier transform is taken in space. Here we first take the H_t^b norm and then take the weighted L_ξ^2 norm with the weight $\langle\xi\rangle^{2s}$.

From this definition, we have the equality,

$$\|u\|_{X^{0,0}} = \|u\|_{L_t^2 L_x^2}.$$

Moreover, by Sobolev embedding we have $H^{\frac{1}{2}+}(\mathbb{R}) \to C(\mathbb{R})$, and this gives us the embedding,

$$X^{s,\frac{1}{2}+} \to C_t(\mathbb{R}, H_x^s),$$

see [64]. This embedding and the aforementioned observations suggest that $X^{s,b}$ spaces may be more appropriate to work with.

For these spaces, we can see that there is a trivial embedding

$$X^{s',b'} \subset X^{s,b},$$

for $s' \leq s$ and $b' \leq b$. Also from Parseval's identity and Cauchy-Schwarz inequality we have the duality relationship

$$\left(X^{s,b}\right)^* = X^{-s,-b}.$$

These spaces behave well under interpolation in both indices s and b. One of the most problematic properties of these spaces is that although they are invariant under translations in space and time, they are not invariant under conjugation. This means even though a function u is in a Bourgain space $X^{s,b}$, this does not imply that its conjugate $\bar{u}$ is in that Bourgain space.

To find a fixed point of the integral equation (2.8), we use the Banach Fixed Point Theorem on the metric space

$$B_T = \{u \in X^{s,b} : u(0,x) = u_0(x) \quad \text{and} \quad \|u\|_{X_T^{s,b}} \leq 2C\|u_0\|_{H^s}\},$$

with the metric $d(u,v) = \|u - v\|_{X_T^{s,b}}$ and get a contraction for sufficiently small T, where C is going to be defined later.

In this thesis, by local and global well-posedness we mean the following.

Definition 2.2.5. *We say the equation* (2.5) *is locally well-posed in* H^s *if there exists a time* $T_{LWP} = T_{LWP}(\|u_0\|_{H^s})$ *such that the solution exists, is unique in* $X^{s,b}_{T_{LWP}} \subset C([0, T_{LWP}), H^s)$*, and depends continuously on the initial data. We say that the the equation is globally well-posed when* T_{LWP} *can be taken arbitrarily large.*

This means that for these $X^{s,b}$ spaces we have to prove Strichartz-like estimates, namely estimates concerning the solution operator and embeddings into spaces like Sobolev or Lebesgue spaces whose theories are much more widely explored. Although the study of $X^{s,b}$ spaces are well studied in the context of other differential equations as well, in this thesis we will only focus on the estimates and embeddings closely related to the Schrödinger equation.

Since L^p spaces are much easier to work with, to study nonlinear Schrödinger equations, one can make use of the following estimates, proofs of which can be found in [39]. We use the following.

Proposition 2.2.6. *For* $s \in \mathbb{R}$*,*

$$\|e^{it\blacktriangle}\phi\|_{X_T^{s,b}} \leq C\|\phi\|_{H^s}, \quad for \quad -\infty < b < \infty, \tag{2.17}$$

$$\|u\|_{X_T^{s,-b_1}} \leq CT^{b_1-b_2-}\|u\|_{X_T^{s,-b_2}}, \quad for \quad 0 \leq b_2 \leq b_1 < 1/2, \ and \tag{2.18}$$

$$\left\|\int_0^t e^{i(t-t')\blacktriangle}u(t')dt'\right\|_{X_T^{s,b}} \leq CT^{1-(b+b')}\|u\|_{X_T^{s,-b'}}, \quad for \quad 1/2 < b \leq 1, \quad 0 < b + b' < 1, \tag{2.19}$$

where C *is independent of* T.

Proof. First we prove (2.17). To this end, take the compactly supported $C^\infty(\mathbb{R})$ function ψ defined in (2.9). Then we have

$$\begin{aligned}
\|e^{it\blacktriangle}\phi\|_{X_T^{s,b}} &\leq \|\psi(t/T)e^{it\blacktriangle}\phi\|_{X^{s,b}} = \|J_t^b J_x^s e^{-it\blacktriangle}(\psi(t/T)e^{it\blacktriangle}\phi)\|_{L^2(\mathbb{R},L^2)} \\
&= \|J_t^b J_x^s(\psi(t/T)\phi)\|_{L^2(\mathbb{R},L^2)} \\
&\leq \|J_t^b\psi\|_{L^2(\mathbb{R})}\|J_x^s\phi\|_{L^2(\mathbb{R}^n)} \leq C\|\phi\|_{H^s},
\end{aligned}$$

which is (2.17).

Using the time localization, (2.18) would be shown if we could show

$$\|\psi(t/T)u\|_{X^{s,-b_1}} \leq CT^{b_1-b_2-}\|u\|_{X^{s,-b_2}}. \tag{2.20}$$

By duality, it is enough to show

$$\|\psi(t/T)u\|_{X^{s,b_2}} \leq CT^{b_1-b_2-}\|u\|_{X^{s,b_1}}. \tag{2.21}$$

To prove (2.21), set $f(x,t) = J_t^{b_1}J_x^s e^{-it\blacktriangle}u(x,t)$, so:

$$\|\psi(t/T)u\|_{X^{s,b_2}} = \left\|J_t^{b_2}\left(\psi(t/T)J_t^{-b_1}f\right)\right\|_{L^2(\mathbb{R},L^2)}. \tag{2.22}$$

Using (2.16) and setting $\widehat{J_t^{-b_1}f} = g$, the inequality (2.18) will follow if we can show

$$\|\psi(t/T)g\|_{H_t^{b_2}(\mathbb{R})} \leq CT^{b_1-b_2-}\|g\|_{H_t^{b_1}(\mathbb{R})}. \tag{2.23}$$

By [45, Theorem 3.5], we have

$$\|\psi(t/T)g\|_{H^a(\mathbb{R})} \leq CT^{1-2a}\|g\|_{H^a(\mathbb{R})}, \quad for \;\; 1/2 < a \leq 1.$$

Since

$$\|\psi(t/T)g\|_{L^2(\mathbb{R})} \leq C\left(\int_{-T}^{2T}|g(t)|^2\mathrm{d}t\right)^{1/2} \leq CT^{1/2-1/q}\|g\|_{L^q(\mathbb{R})},$$

and by the Sobolev embedding theorem, $\|g\|_{L^q(\mathbb{R})} \leq C\|g\|_{H^b}$ for $2 \leq q < \infty$ and $b = 1/2 - 1/q$, so

$$\|\psi(t/T)g\|_{L^2(\mathbb{R})} \leq CT^b\|g\|_{H^b}, \qquad 0 \leq b < 1/2. \tag{2.24}$$

For sufficiently small $\epsilon > 0$ we let $a = 1/2+\epsilon$, $b = (b_1-b_2)(1+2\epsilon)/(1-2b_2+2\epsilon)$ and $\theta = 2b_2/(1+2\epsilon)$ and interpolate between (2.2) and (2.24) to get

$$\|\psi(t/T)g\|_{H^{b_2}} = \|\psi(t/T)g\|_{H^{a\theta}(\mathbb{R})} \leq CT^{\theta(1-2a)+(1-\theta)b}\|g\|_{H^{a\theta+b(1-\theta)}(\mathbb{R})}.$$

To prove (2.19) we are going to follow the arguments in [45]. We prove:

For b, b' such that $0 \leq b+b' < 1$, $0 \leq b' < 1/2$, we have

$$\left\|\int_0^t e^{i(t-t')\blacktriangle}g(t')dt'\right\|_{X_T^{s,b}} \lesssim T^{1-b-b'}\|g\|_{X_T^{s,-b'}},$$

for $T \in [0,1]$.

First note that, from the definition of the norm $X^{s,b}$, it is enough to bound $\|\psi(t/T)\int_0^t f(t')dt'\|_{H_t^b} =$

$\|\psi(t/T)\int \chi_{[0,t]}f(t')dt'\|_{H^b_t}$, where $f(\xi,t') = e^{-it'h(\xi)}\widehat{g}(\xi,t')$, and the Fourier transform is taken in space variable.

For this term, from Parseval equality in time variable, we have,

$$
\begin{aligned}
\big\|\psi(t/T)\int \chi_{[0,t]}f(t')dt'\big\|_{H^b_t} &= \big\|\psi(t/T)\int \frac{e^{it\tau-1}}{(i\tau)}\widehat{f}(\tau)d\tau\big\|_{H^b_t} \\
&\leq \big\|\psi(t/T)\sum_{k=1}^{\infty}\frac{t^k}{k!}\int\limits_{|\tau|T\leq 1}(i\tau)^{k-1}\widehat{f}(\tau)d\tau\big\|_{H^b_t} + \big\|\psi(t/T)\int\limits_{|\tau|T>1}e^{it\tau}(i\tau)^{-1}\widehat{f}(\tau)d\tau\big\|_{H^b_t} \\
&\quad +\big\|\psi(t/T)\int\limits_{|\tau|T>1}(i\tau)^{-1}\widehat{f}(\tau)d\tau\big\|_{H^b_t} \\
&= I + II + III.
\end{aligned}
$$

For I, we compute

$$
\begin{aligned}
I &= \big\|\psi(t/T)\sum_{k=1}^{\infty}\frac{t^k}{k!}\int\limits_{|\tau|T\leq 1}(i\tau)^{k-1}\widehat{f}(\tau)d\tau\big\|_{H^b_t} \\
&\leq \sum_{k=1}^{\infty}\frac{1}{k!}\|t^k\psi(t/T)\|_{H^b_t}T^{1-k}\|f\|_{H_t^{-b'}}\big(\int\limits_{|\tau|T\leq 1}\langle\tau\rangle^{2b'}\big)^{1/2} \\
&\lesssim T^{1-(b+b')}\|f\|_{H_t^{-b'}} \qquad \text{since } 0 < b' < 1/2.
\end{aligned}
$$

For II,

$$
\begin{aligned}
II &= \big\|\psi(t/T)\int\limits_{|\tau|T>1}e^{it\tau}(i\tau)^{-1}\widehat{f}(\tau)d\tau\big\|_{H^b_t} \\
&\leq \|\psi(t/T)\|_{H^b_t}\|f\|_{H_t^{-b'}}\big(\int\limits_{|\tau|T>1}|\tau|^{-2}\langle\tau\rangle^{2b'}\big)^{1/2} \\
&\lesssim T^{1-(b+b')}\|f\|_{H_t^{-b'}}.
\end{aligned}
$$

For the last term,

$$
\begin{aligned}
III &= \big\|\psi(t/T)\underbrace{\int\limits_{|\tau|T>1}(i\tau)^{-1}\widehat{f}(\tau)d\tau}_{J}\big\|_{H^b_t} = \|\langle\tau\rangle^b(\widehat{\psi(t/T)} * \widehat{J})\|_{L^2} \\
&\lesssim \Big[\||\tau|^b\widehat{\psi(t/T)}\|_{L^1}\|J\|_{L^2} + \|\widehat{\psi(t/T)}\|_{L^1}\|J\|_{H^b_t}\Big] \\
&\leq T^{1-(b+b')}\|f\|_{H_t^{-b'}},
\end{aligned}
$$

since

$$\|J\|_{H_t^b} \leq \|f\|_{H_t^{-b'}} \sup_{|\tau|T>1} \tau^{-1}\langle\tau\rangle^{b+b'},$$

and

$$\|J\|_{L^2} \lesssim T^{1-b'}\|f\|_{H_t^{-b'}}.$$

Then the proof of (2.19) follows. □

We now give an important corollary.

Corollary 2.2.7. *For $s \in \mathbb{R}$ and sufficiently small $\epsilon > 0$:*

$$\left\|\int_0^t e^{i(t-t')\blacktriangle} u(t')dt'\right\|_{X_T^{s,1/2+\epsilon}} \leq C\|u\|_{X_T^{s,-1/2+3\epsilon}}, \tag{2.25}$$

where C is independent of T.

Proof. In (2.18) and (2.19), setting $b = 1/2+\epsilon$, $b_1 = 1/2-\epsilon = -(b-1)$ and $b_2 = 1/2-3\epsilon$, we get

$$\begin{aligned}\left\|\int_0^t e^{i(t-t')\blacktriangle} u(t')dt'\right\|_{X_T^{s,1/2+\epsilon}} &\leq CT^{-\epsilon}\|u\|_{X_T^{s,-b_1}} \leq CT^{-\epsilon}T^{b_1-b_2-\epsilon}\|u\|_{X_T^{s,-b_2}} \\ &\leq C\|u\|_{X_T^{s,-1/2+3\epsilon}},\end{aligned}$$

which is (2.25) □

For further discussion on $X^{s,b}$ spaces, consult [64].

We would also like to note that for the cubic Schrödinger equation (or more generally, when the nonlinearity is locally Lipschitz), to run the contraction argument it is enough to show estimates of the form

$$\left\|\int_0^t e^{i(t-t')\blacktriangle}(|u|^2u)(t')dt'\right\|_{X_T^{s,b}} \lesssim T^a\|u\|^3_{X_T^{s,b}}$$

on the nonlinear evolution term in the Duhamel formula for some $a > 0$. If we can show such an estimate on the metric space

$$B_T = \{u \in X^{s,b} : u(0,x) = u_0(x) \quad \text{and} \quad \|u\|_{X_T^{s,b}} \leq 2C\|u_0\|_{H^s}\},$$

where $C = \|J_t^b \psi\|_{L^2(\mathbb{R})}$, with $d(u,v) = \|u-v\|_{X_T^{s,b}}$, we see from Duhamel formula that, for $u, v \in B_T$,

$$u(t) - v(t) = e^{-it\blacktriangle}(u_0 - u_0) \mp i \int_0^t e^{-i(t-t')\blacktriangle}\big(|u|^2 u(t') - (|v|^2 v)(t')\big)dt' \tag{2.26}$$

$$= \pm i \int_0^t e^{-i(t-t')\blacktriangle}\big((|u|^2 u)(t') - (|v|^2 v)(t')\big)dt'. \tag{2.27}$$

Thus,

$$\|u-v\|_{X_T^{s,b}} \le T^a\big(\|u\|_{X_T^{s,b}} + \|v\|_{X_T^{s,b}}\big)^2 \|u-v\|_{X_T^{s,b}},$$

which, for $u, v \in B_T$, and T sufficiently small gives us the contraction.

The continuous dependence also follows in a similar fashion: if $\phi_n \to \phi$ in H^s, and u_n and u are the corresponding solutions, then again by Duhamel formula and Proposition 2.2.6 we have

$$\|u_n - u\|_{X_T^{s,b}} \le C\|\phi_n - \phi\|_{H^s} + T^a\big(\|u\|_{X_T^{s,b}} + \|v\|_{X_T^{s,b}}\big)^2 \|u-v\|_{X_T^{s,b}}.$$

Then for n large and T small enough so that $u_n, u \in B_T$ and $T^a\big(\|u\|_{X_T^{s,b}} + \|v\|_{X_T^{s,b}}\big)^2 \le \frac{1}{2}$, we get

$$\|u_n - u\|_{X_T^{s,b}} \le 2C\|\phi_n - \phi\|_{H^s},$$

which means $u_n \to u$ in $X_T^{s,b}$. Hence, in the following chapters, we are going to omit proving the contraction and continuous dependence explicitly and only give the trilinear estimates. We should also mention that, from the arguments above, the local well-posedness time T depends on the H^s norm of the initial data as $T \sim \big(\frac{1}{\|u_0\|_{H_x^s}}\big)^{\frac{1}{a}}$. Thus, if $\|u(T_0)\|_{H_x^s} < A$ for some T_0, then there exists a $T_A > 0$ which only depends on A such that the solution exists in $[T_0, T_0 + T_A)$. Now, if we have local well-posedness in $[0, T_{max})$, but not in $[0, T')$ for $T' > T_{max}$, and if $\lim\limits_{t \to T_{max}} \|u(t)\|_{H_x^s} = B < \infty$, then we can pick a time $\tilde{T}$ such that $\|u(\tilde{T})\|_{H_x^s} < 3B/2$ and $T_{max} - T_{3B/2} < \tilde{T}$. This gives us a contradiction since at time $\tilde{T}$, we can iterate the solution to time $\tilde{T} + T_{3B/2} > T_{max}$. Hence, for such T_{max}, we have $\lim\limits_{t \to T_{max}} \|u(t)\|_{H_x^s} = \infty$, which is also known as the blow up alternative. In particular, this tells us that if we can control the H_x^s norm of the solution for all times, we will have a global solution.

For $s > \frac{n}{2}$, the calculations are much easier. For $s > 0$, noting that $\langle \xi \rangle^s \le 2^{2s}\big(\langle \xi - \eta \rangle^s + \langle \eta \rangle^s\big)$ for any $\eta \in \mathbb{R}^n$, we can prove the algebra property:

$$\|uv\|_{H^s}^2 = \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\widehat{uv}(\xi)|^2 d\xi$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\langle\xi\rangle^{2s}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta\Big)^2 d\xi \\
&\lesssim \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}|\langle\xi-\eta\rangle^s\widehat{u}(\xi-\eta)\widehat{v}(\eta)|+|\widehat{u}(\xi-\eta)\langle\eta\rangle^s\widehat{v}(\eta)|d\eta\Big)^2 d\xi \\
&\lesssim \|u\|_{H^s}^2\|\widehat{v}\|_{L^1}^2+\|\widehat{u}\|_{L^1}^2\|v\|_{H^s}^2 \\
&\lesssim \|u\|_{H^s}^2\|v\|_{H^s}^2,
\end{aligned}$$

by Young's inequality and since

$$\|\widehat{u}\|_{L^1}=\int_{\mathbb{R}^n}\frac{\langle\xi\rangle^s}{\langle\xi\rangle^s}|\widehat{u}(\xi)|d\xi\leq\|\langle\xi\rangle^{-s}\|_{L^2}\|u\|_{H^s}\lesssim\|u\|_{H^s},$$

for $s>\frac{n}{2}$ by the Cauchy-Schwarz inequality.

Using this algebra property of H^s we see that, on the metric space D_T,

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty([0,T],H^s)} &\leq \|e^{it\blacktriangle}u_0\|_{L^\infty([0,T],H^s)}+\|\int_0^t e^{i(t-t')\blacktriangle}(|u|^2u)(t')dt'\|_{L^\infty([0,T],H^s)} \\
&\leq \|u_0\|_{H^s}+\int_0^t\|e^{i(t-t')\blacktriangle}(|u|^2u)(t')\|_{L^\infty([0,T],H^s)}dt' \\
&\leq \|u_0\|_{H^s}+\int_0^t\|(|u|^2u)(t')\|_{L^\infty([0,T],H^s)}dt' \\
&\leq \|u_0\|_{H^s}+T\|u\|_{L^\infty([0,T],H^s)}^3.
\end{aligned}$$

Similarly, we can show

$$d(u,v)=\|u-v\|_{L^\infty([0,T],H^s)}\leq T\big(\|u\|_{L^\infty([0,T],H^s)}+\|v\|_{L^\infty([0,T],H^s)}\big)^2\|u-v\|_{L^\infty([0,T],H^s)}.$$

With the arguments above this tells us that there is almost immediate local well-posedness for the Schrödinger equation in H^s for $s>\frac{n}{2}$, where the local well-posedness time depends on the H^s norm of the initial data as $T_{LWP}\sim\frac{1}{\|u_0\|_{H^s}^2}$.

In the following chapters we will state Proposition 2.2.6 in the relevant context without proofs, and we will give more estimates on Bourgain spaces.

Chapter 3

Local Well-posedness for 2-D Schrödinger Equation on Irrational Tori and Bounds on Sobolev Norms

3.1 Introduction

The equation we consider in this chapter is the cubic, Schrödinger equation on irrational tori, namely,

$$\begin{cases} iu_t + \Delta u = \pm |u|^2 u, \quad x \in \mathbb{T}^2_\theta, \quad t \in [-T, T], \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}^2_\theta). \end{cases} \tag{3.1}$$

T is the time of existence of the solutions and $\mathbb{T}^2_\theta$ is the irrational tori, $\mathbb{R}^2/\theta_1\mathbb{Z} \times \theta_2\mathbb{Z}$ for $\theta_1, \theta_2 > 0$ and θ_1/θ_2 irrational. The equation is called focusing when the sign in front of the cubic term is negative and defocusing, when positive.

The equation (3.1) posed $\mathbb{T}^2$, has been studied widely for its importance in the theory of differential equations. For the defocusing equation, for any initial data in H^1 there is global well-posedness and global bounds on the Sobolev norm of the solution, see [7]. In addition there have been many results on the well-posedness of (3.1) for both focusing and defocusing case for rough initial data (in H^s for $s < 1$) on two dimensional torus, see [12], [18], [23], [28], [40], [60], [61], [63] and also on more general compact manifolds, see [17], [68]. One of the main difficulties of the theory on general compact manifolds is that one has to use spectral decomposition of the Laplace-Beltrami operator, as a generalization of the Fourier series. But since the spectrum and the eigenfunctions of the operator on arbitrary compact manifold are less understood, standard arguments on regular torus cannot be applied in their full generality. In [17], instead of using Bourgain's arguments, the authors used families of dispersive estimates on small time intervals depending on the size of the frequencies of the data. This idea was used in the works of Bahouri-Chemin, see [4], and Tataru, see [65], in the context of low regularity well-posedness of quasilinear wave equations. Later, Herr in [42] considered the quintic Schrödinger equation on 3 d compact manifold M such that all geodesics are simple and closed with a common minimal period. For this equation, he was able to prove certain Strichartz estimates and local well-posedness in the energy space $H^1(M)$. Then the question would be whether a similar result may hold for compact manifolds without a common minimal period for geodesics, and the simplest such manifolds are the irrational tori. In the following, we prove Strichartz estimates and the local well-posedness in certain H^s spaces.

One of the main tools in proving local well-posedness is the aforementioned Strichartz estimates, i.e. estimates of the form

$$\|e^{it\Delta}f\|_{L_t^4L_x^4(\mathbb{T}\times\mathbb{T}_\theta^2)} \lesssim \|f\|_{H^{\frac{s_0}{2}}(\mathbb{T}_\theta^2)}, \tag{3.2}$$

for some $s_0 \geq 0$ and $f \in H^{\frac{s_0}{2}}(\mathbb{T}_\theta^2)$. Our main focus in this paper will be on the improvement of this estimate on irrational tori. As one can see for $\theta_1 = \theta_2 = 1$ we get the usual (flat) torus. Although the domain resembles the flat torus, the tools used to prove (3.2) are fundamentally different. The reason behind this difference is that the symbol of the Laplacian on flat torus at any (m,n)-level is m^2+n^2 whereas the symbol of it on a irrational torus is of the form $(\theta_1 m)^2 + (\theta_2 n)^2$. Thus the method of counting lattice points on a circle to get (3.2) cannot be applied here. In 3-d, Bourgain [12], used bounds on the l^p-norms on the number of lattice points on the ellipsoid and Jarnick's estimate [44] to get (3.2) with $s_0 = \frac{1}{3}$. A slight modification of his method in 2-d gives us a $\frac{1}{4}$-derivative loss in (3.2). But this result was already proven for not only on irrational tori but also on any two dimensional compact manifold, see [17]. This remedy was overcome in Catoire and Wang's paper [23] using Jarnick's estimate, see [44], in two dimensions without passing to the l^p-norms of the number of lattice points on ellipsoids. They obtained (3.2) with $\frac{s_0}{2} = \frac{1}{6}$. The first part of this chapter will be consisted of our main result, improving (3.2) to $\frac{s_0}{2} = \frac{131}{832}$ using a counting argument of Huxley, [43]. In the second part of the chapter, using the theory of Bourgain spaces, we prove local well-posedness for initial data in H^s, $s > s_0$ and also polynomial bounds on the growth of the Sobolev norms of the solution for the defocusing case. On 2-d flat tori, we should note that the local well-posedness theory gives the exponential bound $\|u(t)\|_{H^s} \lesssim C^{|t|}$, see [9]. Also in [9], Bourgain improved this exponential bound with the polynomial bound $\|u(t)\|_{H^s} \lesssim C\langle t\rangle^{2(s-1)+}$ using the following polynomial estimate:

Lemma 3.1.1. *If there exists a constant $r \in (0,1)$ and $\delta > 0$ such that for any time t_0 we have,*

$$\|u(t_0+\delta)\|_{H^s}^2 \leq \|u(t_0)\|_{H^s}^2 + C\|u(t_0)\|_{H^s}^{2-r},$$

then we get

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{\frac{1}{r}}.$$

It suffices to prove this result for t being an integer multiple of δ and the rest follows from induction. This result was later improved by Staffilani, see [62] to $\|u(t)\|_{H^s} \lesssim C\langle t\rangle^{(s-1)+}$. On 2-d irrational tori, using Zhong's arguments [68] and the lemma above, Catoire and Wang proved the norm bound $\|u(t)\|_{H^s} \lesssim$

$C\langle t\rangle^{\frac{(s-1)}{1-\frac{2}{3}}+}$, see [23]. In this chapter we are going to improve the polynomial bound on 2-d irrational tori to the exponent $\frac{(s-1)}{(1-131/416)}+$.

Recently, Bourgain and Demeter, in [15], proved the l^2 decoupling conjecture for compact hypersurfaces with positive definite second fundamental form. One of the main implications of this result is the full range of the expected Strichartz estimates for both the flat and the irrational tori up to a factor of ϵ, i.e. the Schrödinger equation on irrational tori is LWP in H^s for $s > 0$. In [46], Killip and Visan removed the ϵ factor in the case of irrational tori except for the end-point L^p case.

We should also mention that the upper bounds on the growth of the Sobolev norms of the solutions does not necessitate boundedness of these norms. In [41], authors proved that on $\mathbb{T}^d$ for $d \geq 2$, for $s \in \mathbb{N}$, $s \geq 30$, there exist global solutions $u(t,x)$ to the Schrödinger equation such that $\limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty$ although the initial data has arbitrarily small Sobolev norms. However, the lower bounds on the growth of such norms on irrational tori will not be discussed here.

3.2 Notation and Preliminaries

The linear propogator of the Schrödinger equation on the irrational tori will be denoted as $e^{it\Delta}$, where it is defined on the Fourier side as $(\widehat{e^{it\Delta}f})(m_1,m_2) = e^{-itQ(m_1,m_2)}\hat{f}(m_1,m_2)$, where $Q(m_1,m_2) = (\theta_1 m_1)^2 + (\theta_2 m_2)^2$.

The corresponding Bourgain spaces $X^{s,b}$ will be defined as the closure of compactly supported smooth functions under the norm

$$\|u\|_{X^{s,b}} \doteq \|e^{-it\Delta}u\|_{H^b_t(\mathbb{R})H^s_x(\mathbb{T}^2_\theta)} = \|\langle\tau - Q(m,n)\rangle^b\langle|m|+|n|\rangle^s\widehat{u}(m,n,\tau)\|_{L^2_\tau l^2_{(m,n)}},$$

and the restricted norm will be given as

$$\|u\|_{X^{s,b}_T} \doteq \inf(\|v\|_{X^{s,b}},\ for\ v = u\ on\ [0,T]).$$

We also note that the equation has mass and energy conservations, namely,

$$M(u)(t) = \int_{\mathbb{T}^2_\theta} |u(t,x)|^2 = M(u)(0),$$

and,

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{T}^2_\theta} |\nabla u(t,x)|^2 \pm \frac{1}{4}\int_{\mathbb{T}^2_\theta} |u(t,x)|^4 = E(u)(0).$$

Thus, for the defocusing equation, i.e. when the sign is plus, we have global bounds on the H^1-norm of the solution. This also says, for defocusing equation we have H^1 global well-posedness.

Throughout the chapter, $L_t^2 L_x^2$ will denote the space $L_t^2 L_x^2(\mathbb{T} \times \mathbb{T}_\theta^2)$. We will also use $L_t^2 L_x^2([0,T])$ to denote $L_t^2 L_x^2([0,T] \times \mathbb{T}_\theta^2)$, and same notation will be used for Sobolev spaces too.

For any operator D and positive number N, χ being the characteristic function, $\chi_N^D u$ is defined to be $\chi_{\widehat{D} \in [N,2N]} u$, i.e., the projection of u onto the frequency set where $\widehat{D} \in [N, 2N]$.

Also, in this chapter we will use $s_0 = \frac{131}{416}$.

3.3 Local well-posedness in H^s

In this section we are going to prove,

Theorem 3.3.1. *The 2-d cubic Schrödinger equation* (3.1) *is locally well-posed for initial data $u_0 \in H_x^s$ for $s > s_0$.*

Proof of Strichartz estimates

To be able to prove (3.2), we will use a counting argument by Huxley, [43]:

Theorem 3.3.2. *For $a, b, c \in \mathbb{R}$, let $Q = Q(m,n) = am^2 + bmn + cn^2$ be a positive definite quadratic form, where $a > 0$, $D := 4ac - b^2 > 0$. For x large, we have*

$$\#\{(m,n) \in \mathbb{Z}^2 : Q(m,n) \leq x\} = \frac{2\pi}{\sqrt{D}} x + R(x),$$

where $R(x) \leq x^{s_0+}$.

Theorem 3.3.3. *Let $f \in L_x^2$ such that* $\operatorname{supp}(\widehat{f}) \in B(0,N)$*, then*

$$\|e^{it\Delta} f\|_{L_t^4 L_x^4} \lesssim N^{(\frac{s_0}{2})+} \|f\|_{L_x^2}.$$

Proof.

$$\begin{aligned}
\|e^{it\Delta} f\|^2_{L_t^4 L_x^4} &= \|(e^{it\Delta} f)^2\|_{L_t^2 L_x^2} \\
&= \Big\| \Big[\sum_{m \in \mathbb{Z}^2} \big| \sum_{n \in \mathbb{Z}^2} \widehat{f}(n) \widehat{f}(m-n) e^{-it(Q(n)+Q(m-n))} \big|^2 \Big]^{1/2} \Big\|_{L_t^2} \\
&= \Big[\sum_{m \in \mathbb{Z}^2} \big\| \sum_{n \in \mathbb{Z}^2} \widehat{f}(n) \widehat{f}(m-n) e^{-it(Q(n)+Q(m-n))} \big\|^2_{L_t^2} \Big]^{1/2}
\end{aligned}$$

$$\lesssim \Big[\sum_{m\in\mathbb{Z}^2}\big[\sum_{k\in\mathbb{Z}}\big(\sum_{|Q(n)+Q(m-n)-k|\le 1/2}|\widehat{f}(n)\widehat{f}(m-n)|\big)^2\big]\Big]^{1/2},$$

where, to pass to the last inequality we used:

Lemma 3.3.4. $\|\sum_n e^{ita_n}b_n\|^2_{L^2([0,1])}\lesssim\sum_j(\sum_{|a_n-j|\le 1/2}|b_n|)^2$.

Proof. For any finite sum over n, write $\|\sum_n e^{ita_n}b_n\|^2_{L^2([0,1])}=\|\sum_j\sum_{|a_n-j|\le 1/2}e^{ita_n}b_n\|^2_{L^2([0,1])}$. Hence, for a bump function ϕ s.t. $\phi(t)=1$ in $[0,1]$ we have

$$\begin{aligned}
\Big\|\sum_n b_n e^{ita_n}\Big\|^2_{L^2[0,1]} &\le \Big\|\sum_n b_n e^{ita_n}\phi(t)\Big\|^2_{L^2(\mathbb{R})}\\
&= \Big\|\sum_n b_n\widehat{\phi}(\xi-a_n)\Big\|^2_{L^2(\mathbb{R})}\\
&\lesssim \Big\|\sum_n |b_n|\frac{1}{\langle\xi-a_n\rangle^\alpha}\Big\|^2_{L^2(\mathbb{R})}\\
&= \Big\|\sum_j\sum_{|a_n-j|\le 1/2}|b_n|\frac{1}{\langle\xi-a_n\rangle^\alpha}\Big\|^2_{L^2(\mathbb{R})}\\
&\lesssim \Big\|\sum_j\frac{1}{\langle\xi-j\rangle^\alpha}\sum_{|a_n-j|\le 1/2}|b_n|\Big\|^2_{L^2(\mathbb{R})}\\
&\le \sum_j\big(\sum_{|a_n-j|\le 1/2}|b_n|\big)^2.
\end{aligned}$$

Here, to pass to the second line we used Plancherel's equality. In the third line we used that the Fourier transform of ϕ is a Schwartz function and decays faster than any polynomial, and thus we can choose an $\alpha>1$. Also to pass to the last line we used Young's inequality. □

Then, write $|Q(n)+Q(m-n)-k|\le 1/2$ as $|Q(2n-m)+Q(m)-2k|\le 1$ and letting $2n\in m+G_l$ where $l=2k-Q(m)$ and $G_l=\{a\in\mathbb{Z}^2:|Q(a)-l|\le 1\}$, we get

$$\begin{aligned}
\|e^{it\Delta}f\|^2_{L^4_tL^4_x} &\lesssim \Big[\sum_{m\in\mathbb{Z}^2}\big[\sum_{l\in\mathbb{Z}}\big|\sum_{2n\in m+G_l}\hat{f}(n)\hat{f}(m-n)\big|^2\big]\Big]^{1/2}\\
&\lesssim \Big[\sum_{m\in\mathbb{Z}^2}\big[\sum_{l\in\mathbb{Z}}|G_l|.\big|\sum_{2n\in m+G_l}\hat{f}(n)^2\hat{f}(m-n)^2\big|^{1/2}\big]^2\Big]^{1/2},
\end{aligned}$$

since $G_l=\{a\in\mathbb{Z}^2:|Q(a)|\le l+1\}-\{a\in\mathbb{Z}^2:|Q(a)|<l-1\}$, using Theorem 3.3.2, we get

$$|G_l|\lesssim l^{s_0+},$$

and hence, using $l\lesssim N^2$, we obtain,

$$
\begin{aligned}
\|e^{it\Delta}f\|^2_{L^4_tL^4_x} &\lesssim N^{s_0+}\Big[\sum_{m\in\mathbb{Z}^2}\big[\sum_{l\in\mathbb{Z}}\big|\sum_{2n\in m+G_l}\hat{f}(n)^2\hat{f}(m-n)^2\big|\big]\Big]^{1/2},\\
&\lesssim N^{s_0+}\Big[\sum_{m\in\mathbb{Z}^2}\big|\big(\sum_{n\in\mathbb{Z}^2}\hat{f}(n)^2\hat{f}(m-n)^2\big)\big|\Big]^{1/2},\\
&\lesssim N^{s_0+}\|f\|^2_{L^2_x}.
\end{aligned}
$$

Therefore, the result. $\square$

Proof of Theorem 3.3.1

As mentioned above, to prove local well-posedness we use Bourgain spaces. Since Bourgain spaces behave nicely under linear evolution, what we need to show is that the nonlinear part of the Duhamel formula also behaves as nicely. For that we need:

Proposition 3.3.5. *For b, b' such that $0 \le b + b' < 1$, $0 \le b' < 1/2$, then we have*

$$
\Big\|\int_0^t e^{i\Delta(t-t')}f(t')dt'\Big\|_{X^{s,b}_T} \lesssim T^{1-b-b'}\|f\|_{X^{s,-b'}_T},
$$

for $T \in [0, 1]$.

For the proof see chapter 1, Proposition 2.2.6. Hence, to be able to use the Banach Fixed Point Theorem, we have to control the right hand side of the inequality in the appropriate $X^{s,b}$ space. And since our nonlinearity is cubic, that means have to show a trilinear estimate:

Proposition 3.3.6. *For $s > s_0$, there exists b, b' satisfying the conditions of Proposition 3.3.5, such that,*

$$
\|u_1u_2\overline{u_3}\|_{X^{s,-b'}_T} \lesssim \|u_1\|_{X^{s,b}_T}\|u_2\|_{X^{s,b}_T}\|u_3\|_{X^{s,b}_T}.
$$

Hence, it is clear that once we prove Proposition 3.3.6, Theorem 3.3.1, i.e. the local well-posedness will follow.

We will prove Proposition 3.3.6 using the duality argument

$$
\|u_1u_2\overline{u_3}\|_{X^{s,-b'}_T} = \sup_{(\|u_4\|_{X^{-s,b'}_T}=1)}\int\int u_1u_2\overline{u_3u_4}dxdt.
$$

Hence, to prove Proposition 3.3.6, we will bound the integral on the right hand side of this equality by $\|u_1\|_{X^{s,b}_T}\|u_2\|_{X^{s,b}_T}\|u_3\|_{X^{s,b}_T}$.

Proof. (Proof of Proposition 3.3.6)

We first need a fundamental bilinear Strichartz estimate:

Lemma 3.3.7. *If $u_1, u_2 \in L^2_x$ s.t. $\mathrm{supp}(\widehat{u_1}) \in B(0, N_1)$ and $\mathrm{supp}(\widehat{u_2}) \in B(0, N_2)$ with $N_1 \leq N_2$. Then we have*

$$\|e^{it\Delta}u_1 e^{it\Delta}u_2\|_{L^2_t L^2_x} \lesssim N_1^{s_0+}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}.$$

Proof. Let P_I be the partition of $\mathbb{Z}^2$ into boxes I of size N_1. We can decompose u_2 as $u_2 = \sum_I u_2^{(I)} = \sum_I P_I u_2$ and by Theorem 2.1.12, and that $e^{it\Delta}u_1 e^{it\Delta}u_2^{(I)} = P_{5I}(e^{it\Delta}u_1 e^{it\Delta}u_2^{(I)})$, which follows from the convolution property, we have,

$$\begin{aligned}
\left\|e^{it\Delta}u_1 e^{it\Delta}u_2\right\|_{L^2_t L^2_x} &\leq \left\|\sum_I e^{it\Delta}u_1 e^{it\Delta}u_2^{(I)}\right\|_{L^2_t L^2_x} \\
&\lesssim \left(\sum_I \|e^{it\Delta}u_1 e^{it\Delta}u_2^{(I)}\|^2_{L^2_t L^2_x}\right)^{1/2} \\
&\lesssim \left(\sum_I \|e^{it\Delta}u_1\|^2_{L^4_t L^4_x}\|e^{it\Delta}u_2^{(I)}\|^2_{L^4_t L^4_x}\right)^{1/2} \\
&\lesssim N_1^{\frac{s_0+}{2}}\|u_1\|_{L^2_x}\left(\sum_I \|u_2^{(I)}\|^2_{L^2_x}\right)^{1/2} \\
&\lesssim N_1^{s_0+}\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}.
\end{aligned}$$

□

Using this bilinear Strichartz estimate we can also prove:

Lemma 3.3.8. *Let any $u_1, u_2 \in X^{0,b}$ such that the Fourier transforms of u_1 and u_2 are supported in $[N_1, 2N_1]$ and $[N_2, 2N_2]$ respectively with $N_1 \leq N_2$. Then we have,*

$$\|u_1 u_2\|_{L^2_t L^2_x} \lesssim N_1^{s_0+}\|u_1\|_{X^{0,b}}\|u_2\|_{X^{0,b}}. \tag{3.3}$$

Proof. We take $\tilde{u_i}(t,x)$ on $\mathbb{R} \times \mathbb{R}^2_\theta$ such that $\tilde{u_i}(t,x) = u_i(t,x)$ for $(t,x) \in [0,1] \times \mathbb{T}^2_\theta$. For $n \in \mathbb{Z}^2$, $Q(n)$ being the symbol of Laplacian we have,

$$\begin{aligned}
u_i(t,x) &= \frac{1}{(2\pi)^3}\int \sum_{n\in\mathbb{Z}^2} \widehat{u_i}(\tau, n)e^{it\tau}e^{ix.n}d\tau \\
&= \frac{1}{(2\pi)^3}\int \sum_{n\in\mathbb{Z}^2} \widehat{u_i}(\tau, n)e^{it\tau}e^{ix.n}e^{itQ(n)}e^{-itQ(n)}d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^3}\int \sum_{n\in\mathbb{Z}^2} \widehat{u_i}(\tau,n)e^{it(\tau+Q(n))}e^{ix.n}e^{-itQ(n)}d\tau\\
&= \frac{1}{(2\pi)^3}\int \sum_{n\in\mathbb{Z}^2} \widehat{u_i}(\tau-Q(n),n)e^{it\tau}e^{ix.n}e^{-itQ(n)}d\tau\\
&= \frac{1}{(2\pi)}\int e^{it\Delta}\widehat{v_i}(\tau,x)e^{it\tau}d\tau,
\end{aligned}
$$

by the definition of linear propagator, where $\widehat{v_i}(\tau,x) = \frac{1}{(2\pi)^2}\sum_{n\in\mathbb{Z}^2}\widehat{u_i}(\tau-Q(n),n)e^{ix.n}$. Thus,

$$
\begin{aligned}
\int\int u_1u_2dxdt &= \int e^{it(\tau_1+\tau_2)}e^{it\Delta}\widehat{v_1}(\tau_1,x)e^{it\Delta}\widehat{v_2}(\tau_2,x)d\tau_1d\tau_2dtdx\\
&= \int e^{it(\tau_1+\tau_2)}e^{it\Delta}\widehat{v_1}(\tau_1,x)e^{it\Delta}\widehat{v_2}(\tau_2,x)dtdxd\tau_1d\tau_2\\
&\lesssim \int N_1^{s_0+}\|\widehat{v_1}\|_{L^2_x}\|\widehat{v_2}\|_{L^2_x}d\tau_1d\tau_2\\
&= N_1^{s_0+}\int \|\widehat{v_1}\|_{L^2_x}d\tau_1\int \|\widehat{v_2}\|_{L^2_x}d\tau_2,
\end{aligned}
$$

and for each i we use,

$$
\begin{aligned}
\int \|\widehat{v_i}\|_{L^2_x}d\tau_i &= \int \frac{\langle\tau_i\rangle^b}{\langle\tau_i\rangle^b}\|\widehat{v_i}\|_{L^2_x}d\tau_i\\
&\lesssim (\int \langle\tau_i\rangle^{2b}\|\widehat{v_i}\|^2_{L^2_x}d\tau_i)^{1/2}\\
&= \|u_i\|_{X^{0,b}}.
\end{aligned}
$$

and the result follows by taking the infimum of such u_i's. □

Also, using embedding $X^{0,(1/4)+}\subset L^4_tL^2_x$, which is obtained by interpolation between $X^{0,b}\subset L^\infty_tL^2_x$ for $b>1/2$ and $X^{0,0}=L^2_tL^2_x$, we see that,

$$
\begin{aligned}
\|u_1u_2\|_{L^2_tL^2_x} &\le \|u_1\|_{L^4_tL^\infty_x}\|u_2\|_{L^4_tL^2_x}\\
&\lesssim N_1\|u_1\|_{L^4_tL^2_x}\|u_2\|_{L^4_tL^2_x}\\
&\lesssim N_1\|u_1\|_{X^{0,(1/4)+}}\|u_2\|_{X^{0,(1/4)+}}. \qquad (3.4)
\end{aligned}
$$

Now we can prove a crude interpolation between (3.3) and (3.4) and get:

Lemma 3.3.9. *Let $u_1,u_2\in X^{0,b}$ such that the Fourier transforms of u_1 and u_2 are supported in $[N_1,2N_1]$ and $[N_2,2N_2]$ respectively with $N_1\le N_2$. Then for $s>s_0$ there exists $b'<1/2$ such that*

$$\|u_1u_2\|_{L^2_tL^2_x}\lesssim N_1^s\|u_1\|_{X^{0,b'}}\|u_2\|_{X^{0,b'}}.$$

Proof. We have,

$$\|u_1 u_2\|_{L_t^2 L_x^2} \lesssim N_1 \|u_1\|_{X^{0,(1/4)+}} \|u_2\|_{X^{0,(1/4)+}}, \tag{3.5}$$

and we also have,

$$\|u_1 u_2\|_{L_t^2 L_x^2} \lesssim N_1^{s_0+} \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b}}. \tag{3.6}$$

Note that (3.5) $\leq N_1 \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,(1/4)+}}$. Then for fixed u_1 interpolating between this result and (3.6), we get,

$$\|u_1 u_2\|_{L_t^2 L_x^2} \lesssim N_1^{\tilde{s}} \|u_1\|_{X^{0,b}} \|u_2\|_{X^{0,b'}}, \tag{3.7}$$

for some $\tilde{s} \in [s_0, s)$ and $b' < 1/2$. Also note that (3.5) $\leq N_1 \|u_1\|_{X^{0,(1/4)+}} \|u_2\|_{X^{0,b'}}$. Thus, for fixed u_2, interpolating between this result and (3.7) we obtain,

$$\|u_1 u_2\|_{L_t^2 L_x^2} \lesssim N_1^{s} \|u_1\|_{X^{0,b'}} \|u_2\|_{X^{0,b'}}.$$

$\square$

Thus we get that, if $u_i \in X^{0,b}$, $i \in \{1,2,3,4\}$ are functions s.t. their space Fourier transforms are supported in $[N_i, 2N_i]$ respectively with $N_1 \leq N_2 \leq N_3 \leq N_4$ and $s > s_0$, there exists $b' < 1/2$ such that

$$\begin{aligned} \int\int u_1 \overline{u_2} u_3 \overline{u_4} dx dt &\lesssim \|u_1 u_3\|_{L_t^2 L_x^2} \|u_2 u_4\|_{L_t^2 L_x^2} \\ &\leq (N_1 N_2)^s \|u_1\|_{X^{0,b'}} \|u_2\|_{X^{0,b'}} \|u_3\|_{X^{0,b'}} \|u_4\|_{X^{0,b'}}. \end{aligned}$$

We are almost ready to finish the proof of the proposition. All we need now is to guarantee the existence of b, b' which satisfy the conditions of Proposition 3.3.5. But for that we need better estimates on the restrictions of functions on the eigenspaces of the Laplacian. Let $\wp_k$ be the projection onto the e_k, the eigenspace of Laplacian corresponding to the eigenvalue μ_k. Also for each e_k we see that $\mu_k e_k = -\Delta e_k$ which implies that

$$\mu_k \widehat{e_k}(m_1, m_2) = ((\theta_1 m_1)^2 + (\theta_2 m_2)^2) \widehat{e_k}(m_1, m_2),$$

hence, $\widehat{e_k}$'s are supported on $\mu_k = (\theta_1 m_1)^2 + (\theta_2 m_2)^2 = Q(m_1, m_2)$ i.e. they are supported on $\mu_k^s = |Q(m_1, m_2)|^{2s}$. This gives that

$$\mu_k^s e_k = (\sqrt{-\Delta})^{2s} e_k.$$

Since $\|\wp_k u\|_{L_x^2} \leq \|u\|_{L_x^2}$ and that e_k's form an orthonormal basis, we can define Sobolev space H^s, with the norm, $\|u\|_{H^s}^2 = \sum_k \langle \mu_k \rangle^s \|\wp_k u\|_{L^2}^2$ which we will be using later in the chapter.

Also we see that since θ_1/θ_2 is irrational, if $\mu_k = (\theta_1 m_1)^2 + (\theta_2 m_2)^2 = (\theta_1 n_1)^2 + (\theta_2 n_2)^2$ we have $(m_1, m_2) = (\pm n_1, \pm n_2)$, which means, for any such μ_k, we have four eigenfunctions $(e^{ix.(\pm m_1, \pm m_2)})$ and $\wp_k u$ is the the restriction of u to the eigenspace generated by these eigenfunctions.

Now if we consider the integrals of the form

$$A = \int e^{ix.(m_1,m_2)} e^{ix.(n_1,n_2)} e^{ix.(j_1,j_2)} e^{ix.(l_1,l_2)} a_m a_n a_j a_l dx,$$

we see that $A = 0$ if $(m_1 + n_1 + j_1 + l_1, m_2 + n_2 + j_2 + l_2) \neq 0$. Thus if $|m_1| > 4max(|n_1|, |j_1|, |l_1|)$ or $|m_2| > 4max(|n_2|, |j_2|, |l_2|)$, then $A = 0$. This says that, if $\mu_{k_4}^{1/2} > 8\mu_{k_i}^{1/2}$ for $i = \{1, 2, 3\}$, then

$$\int \wp_{k_1}(u_1)\wp_{k_2}(u_2)\wp_{k_3}(u_3)\wp_{k_4}(u_4)dx = 0,$$

and we will use this observation in our estimates.

Now we can show the existence of $1/4 < b' < 1/2 < b$ s.t. for every $s > s_0$,

$$\|u_1 u_2 \overline{u_3}\|_{X_T^{s,-b'}} \lesssim \|u_1\|_{X_T^{s,b}} \|u_2\|_{X_T^{s,b}} \|u_3\|_{X_T^{s,b}},$$

which will finish the proof of local well-posedness.

As we mentioned before, we will bound, $|\int u_1 u_2 \overline{u_3 u_4} dxdt|$. To do so, it is enough to bound this integral for $u_i = \chi_{N_i}^{\sqrt{-\Delta}} u_i$, where N_i is a dyadic integer. Let without loss of generality that $N_1 \leq N_2 \leq N_3$ and let $s' \in (s_0, s)$ then for the range $N_4 \leq 8N_3$,

$$\begin{aligned}
\Big|\int u_1 u_2 \overline{u_3 u_4} dxdt\Big| &\lesssim (N_1 N_2)^{s'} \|u_1\|_{X_T^{0,b'}} \|u_2\|_{X_T^{0,b'}} \|u_3\|_{X_T^{0,b'}} \|u_4\|_{X_T^{0,b'}} \\
&= (N_1 N_2)^{s'-s} (N_4/N_3)^s N_1^s \|u_1\|_{X_T^{0,b'}} N_2^s \|u_2\|_{X_T^{0,b'}} N_3^s \|u_3\|_{X_T^{0,b'}} N_4^{-s} \|u_4\|_{X_T^{0,b'}} \\
&\lesssim (N_1 N_2)^{s'-s} (N_4/N_3)^s \|u_1\|_{X_T^{s,b'}} \|u_2\|_{X_T^{s,b'}} \|u_3\|_{X_T^{s,b'}} \|u_4\|_{X_T^{-s,b'}}.
\end{aligned}$$

Hence, for the range of the frequencies, write $N_4 = 2^n N_3$ for $n \leq 3$ and then we have

$$\begin{aligned}
\Big|\int u_1 u_2 \overline{u_3 u_4} dxdt\Big| &\lesssim (N_1 N_2)^{s'-s} 2^{ns} \|u_1\|_{X_T^{s,b'}} \|u_2\|_{X_T^{s,b'}} \|u_3\|_{X_T^{s,b'}} \|u_4\|_{X_T^{-s,b'}} \\
&= (N_1 N_2)^{s'-s} 2^{ns} \|u_1\|_{X_T^{s,b'}} \|u_2\|_{X_T^{s,b'}} \|\chi_{N_4 2^{-n}}^{\sqrt{-\Delta}} u_3\|_{X_T^{s,b'}} \|\chi_{N_4}^{\sqrt{-\Delta}} u_4\|_{X_T^{-s,b'}},
\end{aligned}$$

and summing in N_1, N_2, N_4 and $n \leq 3$ we get

$$\begin{aligned}
\Big|\int u_1 u_2 \overline{u_3 u_4} dxdt\Big| &\lesssim \sum_{n\leq 3}\sum_{N_1}\sum_{N_2}(N_1N_2)^{s'-s}2^{ns}\|u_1\|_{X_T^{s,b'}}\|u_2\|_{X_T^{s,b'}} \\
&\qquad (\sum_{N_4}\|\chi_{N_4 2^{-n}}^{\sqrt{-\Delta}}u_3\|^2_{X_T^{s,b'}})^{1/2}(\sum_{N_4}\|\chi_{N_4}^{\sqrt{-\Delta}}u_4\|^2_{X_T^{-s,b'}})^{1/2} \\
&\lesssim \|u_1\|_{X_T^{s,b'}}\|u_2\|_{X_T^{s,b'}}\|u_3\|_{X_T^{s,b'}}\|u_4\|_{X_T^{-s,b'}},
\end{aligned}$$

where we used the H^s-orthogonality of the operators $\chi_N^{\sqrt{-\Delta}}$, for N dyadic integers. And for the range $8N_3 \leq N_4$, we use the observation above and get $|\int u_1 u_2 \overline{u_3 u_4} dxdt| = 0$. Thus the Proposition (3.3.6) follows and hence Theorem 3.3.1. □

3.4 Growth of Sobolev norms

In this section we are going to prove,

Theorem 3.4.1. *For $s \geq 1$, let $u(t,x)$ be the solution to the defocusing cubic Schrödinger equation* (3.1)*. Then for any time t, we have,*

$$\|u(t,x)\|_{H_x^s} \leq C\langle t\rangle^{\frac{(s-1)+}{(1-s_0)}}\|u_0\|_{H_x^s}.$$

Proof of Theorem 3.4.1

The proof of the theorem will mainly follow Bourgain's arguments in [9], i.e. we will use Lemma 3.1.1. For that, first we need to observe that for $s > 1$, in the proof of Proposition 3.3.6 if we take $u_1 = u_2 = u_3 = u$ and $s' = 1-$, redoing the calculations we get

$$\|u|u|^2\|_{X_T^{s,-b'}} \lesssim \|u\|_{X_T^{s,b}}\|u\|^2_{X_T^{1,b}}.$$

This says, we can choose the local well-posedness interval depending only on $\|u(0)\|_{H^1}$. Thus we can find $T_0 > 0$ such that, for any time $\tau \geq 0$ the solution exists for $t \in [\tau, \tau + T_0]$. Now we need to find $r \in (0,1)$ such that for any $t \in [\tau, \tau + T_0]$,

$$\|u(t)\|_{H_x^s} \leq \|u(\tau)\|_{H_x^s} + C\|u(\tau)\|^{1-r}_{H_x^s}. \tag{3.8}$$

Since L^2_x-norm of the solution is conserved, it is enough to show this estimate in $\dot{H}^s_x$. Without loss of generality we can take $\tau = 0$. Since

$$\|u(t)\|^2_{\dot{H}^s_x} - \|u(0)\|^2_{\dot{H}^s_x} = \int_0^t \frac{d}{dt'}\|u(t')\|^2_{\dot{H}^s_x} dt',$$

if we show that, for $t \in [0, T_0]$ we have,

$$\int_0^t \frac{d}{dt'}\|u(t')\|^2_{\dot{H}^s_x} dt' \lesssim \|u\|_{H^s}\|u\|_{H^{s-\sigma}},$$

for some $s - 1 \geq \sigma > 0$, writing $H^{s-\sigma}$ as the interpolation space between H^1 and H^s we will obtain,

$$\|u(t)\|^2_{\dot{H}^s_x} - \|u(0)\|^2_{\dot{H}^s_x} \lesssim \|u\|_{H^s}\|u\|_{H^s}^{\frac{(s-\sigma-1)}{(s-1)}},$$

which is the estimate we want, where the implicit constant also depends on $\|u\|_{H^1}$. For $\sigma > s - 1$, H^1 embeds into $H^{s-\sigma}$ and the result becomes obvious. Now assume $s \in \mathbb{N}$,

$$\begin{aligned}
\int_0^t \frac{d}{dt'}\|u(t')\|^2_{\dot{H}^s_x} dt' &= \int_0^t \frac{d}{dt'}\|D^s u(t')\|^2_{L^2_x} dt' \\
&= 2Re\int_0^t \int_{T^2_\theta} \frac{d}{dt'} D^s\overline{u}(t') D^s u(t') dx dt',
\end{aligned}$$

and using the expression for u_t, we get,

$$\begin{aligned}
\int_0^t \frac{d}{dt'}\|u(t')\|^2_{\dot{H}^s_x} dt' &= 2Im\int_0^t \int_{T^2_\theta} D^s\overline{u} D^s(|u|^2 u) dx dt' \\
&= 4Im\int_0^t \int_{T^2_\theta} |D^s u|^2 |u|^2 dx dt' + 2Im\int_0^t \int_{T^2_\theta} (D^s\overline{u})^2 u^2 dx dt' \\
&\quad +2Im\int_0^t \int_{T^2_\theta} \sum_{\substack{|\alpha|=s \\ \alpha_i \neq s}} D^s\overline{u}\partial^{\alpha_1}\overline{u}\partial^{\alpha_2}u\partial^{\alpha_3}u dx dt' \\
&= 2Im\int_0^t \int_{T^2_\theta} (D^s\overline{u})^2 u^2 dx dt' + 2Im\int_0^t \int_{T^2_\theta} \sum_{\substack{|\alpha|=s \\ \alpha_i \neq s}} D^s\overline{u}\partial^{\alpha_1}\overline{u}\partial^{\alpha_2}u\partial^{\alpha_3}u dx dt' \\
&= I + II.
\end{aligned}$$

Second term is easier to estimate. For any multiindex $|\alpha| = s$ such that $\alpha_i \neq s$ for any i, using duality and Proposition (3.3.6), we have,

$$\begin{aligned} II &\lesssim \|D^s u\|_{X_{T_0}^{-s_0-,b}} \|\partial^{\alpha_1} u \partial^{\alpha_2} \overline{u} \partial^{\alpha_3} \overline{u}\|_{X_{T_0}^{s_0+,-b}} \\ &\lesssim \|u\|_{X_{T_0}^{s-s_0-,b}} \|\partial^{\alpha_1} u\|_{X_{T_0}^{s_0+,b}} \|\partial^{\alpha_2} u\|_{X_{T_0}^{s_0+,b}} \|\partial^{\alpha_3} u\|_{X_{T_0}^{s_0+,b}} \\ &\lesssim \|u\|_{X_{T_0}^{s-s_0-,b}} \|u\|_{X_{T_0}^{s_0+\alpha_1+,b}} \|u\|_{X_{T_0}^{s_0+\alpha_2+,b}} \|u\|_{X_{T_0}^{s_0+\alpha_3+,b}}. \end{aligned}$$

For $1 \leq \alpha_i \leq s-1$, using interpolation and the fact that $\|u\|_{X_{T_0}^{1,b}}$ is bounded, which follows from the local theory, we get,

$$II \lesssim \|u\|_{X_{T_0}^{s,b}}^{\frac{s-s_0-1-}{s-1}} \|u\|_{X_{T_0}^{s,b}}^{\frac{s_0+\alpha_1-1+}{s-1}} \|u\|_{X_{T_0}^{s,b}}^{\frac{s_0+\alpha_2-1+}{s-1}} \|u\|_{X_{T_0}^{s,b}}^{\frac{s_0+\alpha_3-1+}{s-1}}.$$

If for some $i \in \{1,2,3\}$, $\alpha_i = 0$, say $\alpha_3 = 0$, using $\|u\|_{X_{T_0}^{s_0+,b}} \leq \|u\|_{X_{T_0}^{1,b}}$ we get,

$$II \lesssim \|u\|_{X_{T_0}^{s,b}}^{\frac{s-s_0-1-}{s-1}} \|u\|_{X_{T_0}^{s,b}}^{\frac{s_0+\alpha_1-1+}{s-1}} \|u\|_{X_{T_0}^{s,b}}^{\frac{s_0+\alpha_2-1+}{s-1}}. \tag{3.9}$$

Thus we get the desired bound using $\|u\|_{X_{T_0}^{s,b}} \lesssim \|u(0)\|_{H^s}$ in the local well-posedness interval.

The term I is harder to deal with since the highest order derivatives acts on $\overline{u}$. The main problem here is that, because of the term $(D^s\overline{u})^2$ in the integrand we expect to have a bound of the form $II \lesssim \|u\|^2_{X_{T_0}^{s,b}}$ which is not useful. To remedy that problem, we will try to get

$$II \lesssim \|u\|_{X_{T_0}^{s,b}} \|u\|_{X_{T_0}^{s-\sigma,b}} \|u\|_{X_{T_0}^{1,b}} \|u\|_{X_{T_0}^{1,b}}, \tag{3.10}$$

for some $\sigma > 0$ to be determined. In the following estimates we will mainly follow Zhong's arguments in [68].

Let $D^s\overline{u} = u_1 = u_2$, $u_3 = u_4 = u$, and $u_i = \sum_j \chi_{N_{(i,j)}}^{\sqrt{-\Delta}} u_i = \sum_j u_i^j$ where $N_{(i,j)}$'s are dyadic integers. Then

$$|II| \leq \sum_N |II(N)| = \sum_N \Big| \int_{\mathbb{T}^2_\theta} \int_{[0,T_0]} u_1^j u_2^k u_3^m u_4^n \Big|.$$

Since we need to get an estimate of the form (3.10), we should gain some derivative in the estimate of $II(N)$. For the terms $N_{(1,j)} > 8(N_{(2,k)} + N_{(3,m)} + N_{(4,n)})$, we again see that $II(N) = 0$. Hence we have to focus on the terms where $N_{(1,j)} < 8(N_{(2,k)} + N_{(3,m)} + N_{(4,n)})$.

Assume $N_{(1,j)} < 8(N_{(2,k)} + N_{(3,m)} + N_{(4,n)})$, and thus, $N_{(1,j)} \lesssim max(N_{(2,k)}, N_{(3,m)}, N_{(4,n)})$. Since u_2 has full s-derivative, we will estimate II using the interaction between frequency projections of u_2 with u_3 and u_4. We consider two cases; $N_{(2,k)} < 4N_{(3,m)}$ or $N_{(2,k)} < 4N_{(4,n)}$ as case one and $N_{(2,k)} \geq 4N_{(3,m)}$ and $N_{(2,k)} \geq 4N_{(4,n)}$ as case two.

Case 1: $N_{(2,k)} < 4N_{(3,m)}$ or $N_{(2,k)} < 4N_{(4,n)}$. This case gives a control over the $N_{(2,k)}$ term and is easier to handle. Without loss of generality we can assume, $N_{(2,k)} < 4N_{(3,m)}$ and $N_{(4,n)} \gtrsim 1$. Hence by Lemma 3.3.7,

$$
\begin{aligned}
II(N) &\leq \|u_1^j u_3^m\|_{L_t^2 L_x^2([0,T_0])} \|u_2^k u_4^n\|_{L_t^2 L_x^2([0,T_0])} \\
&\lesssim min(N_{(1,j)}, N_{(3,m)})^{s_0+} min(N_{(2,k)}, N_{(4,n)})^{s_0+} \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{0,b}} \|u_4^n\|_{X_{T_0}^{0,b}} \\
&\lesssim (N_{(3,m)} N_{(4,n)})^{s_0+} \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{0,b}} \|u_4^n\|_{X_{T_0}^{0,b}} \\
&\lesssim (N_{(3,m)} N_{(4,n)})^{s_0-1+} \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{1,b}} \|u_4^n\|_{X_{T_0}^{1,b}} \\
&\lesssim (N_{(1,j)} N_{(2,k)} N_{(3,m)} N_{(4,n)})^{-} N_{(1,j)}^{+} N_{(2,k)}^{+} (N_{(3,m)} N_{(4,n)})^{(s_0-1)+} \\
&\qquad \times \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{1,b}} \|u_4^n\|_{X_{T_0}^{1,b}}, \\
&\lesssim (N_{(1,j)} N_{(2,k)} N_{(3,m)} N_{(4,n)})^{-} N_{(2,k)}^{+} (N_{(3,m)} N_{(4,n)})^{(s_0-1)+} \\
&\qquad \times \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{1,b}} \|u_4^n\|_{X_{T_0}^{1,b}},
\end{aligned}
$$

and given $N_{(2,k)} < 4N_{(3,m)}$ we see,

$$
\begin{aligned}
II(N) &\lesssim (N_{(1,j)} N_{(2,k)} N_{(3,m)} N_{(4,n)})^{-} N_{2,k}^{((s_0-1))+} \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{0,b}} \|u_3^m\|_{X_{T_0}^{1,b}} \|u_4^n\|_{X_{T_0}^{1,b}} \\
&\lesssim (N_{(1,j)} N_{(2,k)} N_{(3,m)} N_{(4,n)})^{-} \|u_1^j\|_{X_{T_0}^{0,b}} \|u_2^k\|_{X_{T_0}^{((s_0-1))+,b}} \|u_3^m\|_{X_{T_0}^{1,b}} \|u_4^n\|_{X_{T_0}^{1,b}},
\end{aligned}
$$

which gives the desired result for $\sigma = (1 - s_0)+$.

Case 2: $N_{(2,k)} \geq 4N_{(3,m)}$ and $N_{(2,k)} \geq 4N_{(4,n)}$. Recall that since $N_{(3,m)} \leq 1/4N_{(2,k)}$ and $N_{(4,m)} \leq 1/4N_{(2,k)}$, we have, $N_{(1,j)} \leq 12N_{(2,k)}$, in which case we define,

$$
u_i^{j,j'} = \sum_{(N_{(i,j)} \leq \langle \mu_k \rangle^{1/2} \leq 2N_{(i,j)})} \int_{(L_{(i,j')} \leq \langle \mu_k + \tau \rangle \leq 2L_{(i,j')})} e^{it\tau} \widehat{\wp_k u_i}(\tau) d\tau, \tag{3.11}
$$

for $L_{(i,j')}$ dyadic integers, where μ_k's being the eigenvalues of Laplacian and $\wp_k u$ being the projection of u on the eigenspace corresponding to μ_k Then we have,

$$
\begin{aligned}
II(N) &\leq \sum_L \Big| \int u_1^{j,j'} u_2^{k,k'} u_3^{m,m'} u_4^{n,n'} dx dt \Big| \\
&\leq \sum_L \Big| \int_{\{\sum_{i=1}^4 \tau_i = 0\}} \widehat{u_1}^{j,j'} \widehat{u_2}^{k,k'} \widehat{u_3}^{m,m'} \widehat{u_4}^{n,n'} dx d\tau \Big| \\
&= \sum_L II(N, L),
\end{aligned}
$$

where the Fourier transform is with respect to time only and $L = (L_{(1,j')}, L_{(2,k')}, L_{(3,m')}, L_{(4,n')})$. Thus we need to estimate $II(N,L)$ for each L. Since we are concerned with the derivative gain for u_2 in the estimate of $II(N)$, for each L we need to use the relation between L-terms and $N_{(2,k)}$. For that we consider two cases again; $|\tau_3| \leq 1/3N_{(2,k)}^2$ and $|\tau_4| \leq 1/3N_{(2,k)}^2$ as case one, and $|\tau_3| > 1/3N_{(2,k)}^2$ or $|\tau_4| > 1/3N_{(2,k)}^2$ as case two.

Case 1: $|\tau_3| \leq 1/3N_{(2,k)}^2$ and $|\tau_4| \leq 1/3N_{(2,k)}^2$. For this case, using the decomposition (3.11) we get,

$$\begin{aligned}
|\mu_{k_1} + \tau_1| + |\mu_{k_2} + \tau_2| &\geq |\mu_{k_1} + \tau_1 + \mu_{k_2} + \tau_2| \\
&\geq |\mu_{k_1} + \mu_{k_2}| - |\tau_1 + \tau_2| \\
&\geq \mu_{k_2} - |\tau_1 + \tau_2| \\
&= \mu_{k_2} - |\tau_3 + \tau_4| \\
&\gtrsim N_{(2,k)}^2,
\end{aligned}$$

and thus, $L_{(1,j')} + L_{(2,k')} \gtrsim N_{(2,k)}^2$, which also gives $max\{L_{(1,j')}, L_{(2,k')}\} \gtrsim N_{(2,k)}^2$. So we get,

$$II(N,L) \leq \|u_1^{j,j'}\|_{L_t^4 L_x^2} \|u_2^{k,k'}\|_{L_t^4 L_x^2} \|u_3^{m,m'}\|_{L_t^4 L_x^\infty} \|u_4^{n,n'}\|_{L_t^4 L_x^\infty},$$

then by Sobolev embedding we get,

$$II(N,L) \lesssim (N_{(3,m)} N_{(4,n)}) \|u_1^{j,j'}\|_{L_t^4 L_x^2} \|u_2^{k,k'}\|_{L_t^4 L_x^2} \|u_3^{m,m'}\|_{L_t^4 L_x^2} \|u_4^{n,n'}\|_{L_t^4 L_x^2},$$

and using $X^{0,1/4+} \subset L_t^4 L_x^2$, we obtain,

$$\begin{aligned}
II(N,L) &\lesssim (N_{(3,m)} N_{(4,n)}) \|u_1^{j,j'}\|_{X^{0,1/4+}} \|u_2^{k,k'}\|_{X^{0,1/4+}} \|u_3^{m,m'}\|_{X^{0,1/4+}} \|u_4^{n,n'}\|_{X^{0,1/4+}} \\
&\lesssim \|u_1^{j,j'}\|_{X^{0,1/4+}} \|u_2^{k,k'}\|_{X^{0,1/4+}} \|u_3^{m,m'}\|_{X^{1,1/4+}} \|u_4^{n,n'}\|_{X^{1,1/4+}} \\
&\lesssim \frac{1}{(L_{(1,j')} L_{(2,k')} L_{(3,m')} L_{(4,n')})^{b-1/4-}} \|u_1^{j,j'}\|_{X^{0,b}} \|u_2^{k,k'}\|_{X^{0,b}} \|u_3^{m,m'}\|_{X^{1,b}} \|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{N_{2,k}^{2(1/4-b+)}}{L_{(1,j')}^+ L_{(2,k')}^+ (L_{(3,m')} L_{(4,n')})^{b-1/4-}} \|u_1^{j,j'}\|_{X^{0,b}} \|u_2^{k,k'}\|_{X^{0,b}} \|u_3^{m,m'}\|_{X^{1,b}} \|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{1}{L_{(1,j')}^+ L_{(2,k')}^+ (L_{(3,m')} L_{(4,n')})^{b-1/4-}} \|u_1^{j,j'}\|_{X^{0,b}} \|u_2^{k,k'}\|_{X^{2(1/4-b+),b}} \|u_3^{m,m'}\|_{X^{1,b}} \|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{1}{L_{(1,j')}^+ L_{(2,k')}^+ (L_{(3,m')} L_{(4,n')})^{b-1/4-}} \|u_1\|_{X^{0,b}} \|u_2\|_{X^{2(1/4-b+),b}} \|u_3\|_{X^{1,b}} \|u_4\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)} N_{(2,k)} N_{(3,m)} N_{(4,n)})^-}{L_{(1,j')}^+ L_{(2,k')}^+ (L_{(3,m')} L_{(4,n')})^{b-1/4-}} (N_{1,j} N_{3,m} N_{4,n})^+ \|u_1\|_{X^{0,b}} \|u_2\|_{X^{2(1/4-b+),b}} \|u_3\|_{X^{1,b}} \|u_4\|_{X^{1,b}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{L^+_{(1,j')}L^+_{(2,k')}(L_{(3,m')}L_{(4,n')})^{b-1/4-}}(N_{2,k})^+\|u_1\|_{X^{0,b}}\|u_2\|_{X^{2(1/4-b+),b}}\|u_3\|_{X^{1,b}}\|u_4\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{L^+_{(1,j')}L^+_{(2,k')}(L_{(3,m')}L_{(4,n')})^{b-1/4-}}\|u_1\|_{X^{0,b}}\|u_2\|_{X^{2(1/4-b+),b}}\|u_3\|_{X^{1,b}}\|u_4\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{L^+_{(1,j')}L^+_{(2,k')}(L_{(3,m')}L_{(4,n')})^{b-1/4-}}\|u_1\|_{X^{0,b}}\|u_2\|_{X^{2(1/4-b+),b}}\|u_3\|_{X^{1,b}}\|u_4\|_{X^{1,b}}.
\end{aligned}
$$

Since the summand is summable in L's and N's, we get the result for $\sigma = 2(b-1/4)-$. Now we are left with the last case,

Case 2: $|\tau_3| > 1/3N^2_{(2,k)}$ or $|\tau_4| > 1/3N^2_{(2,k)}$. Again, without loss of generality, we will only focus on $|\tau_3| > 1/3N^2_{(2,k)}$. In this case we have

$$
|\tau_3 + \mu_{k_3}| \geq ||\tau_3| - |\mu_{k_3}|| \geq 1/3N^2_{(2,k)} - 4N^2_{(3,m)} \geq 1/3N^2_{(2,k)} - 1/4N^2_{(2,k)} = 1/12N^2_{(2,k)},
$$

which says $L_{(3,m)} \gtrsim N^2_{(2,k)}$ and redoing the previous calculations, we obtain

$$
\begin{aligned}
II(N,L) &\lesssim \frac{1}{(L_{(1,j')}L_{(2,k')}L_{(3,m')}L_{(4,n')})^{b-1/4-}}\|u_1^{j,j'}\|_{X^{0,b}}\|u_2^{k,k'}\|_{X^{0,b}}\|u_3^{m,m'}\|_{X^{1,b}}\|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{N_{(2,k)}^{2(1/4-b)+}}{(L_{(1,j')}L_{(2,k')}L_{(4,n')})^{b-1/4-}}L^+_{(3,m')}\|u_1^{j,j'}\|_{X^{0,b}}\|u_2^{k,k'}\|_{X^{0,b}}\|u_3^{m,m'}\|_{X^{1,b}}\|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{(L_{(1,j')}L_{(2,k')}L_{(4,n')})^{b-1/4-}}L^+_{(3,m')}N_{(2,k)}^{2(1/4-b+)}\|u_1^{j,j'}\|_{X^{0,b}}\|u_2^{k,k'}\|_{X^{0,b}}\|u_3^{m,m'}\|_{X^{1,b}}\|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{(L_{(1,j')}L_{(2,k')L_{(4,n')}})^{b-1/4-}}\|u_1^{j,j'}\|_{X^{0,b}}\|u_2^{k,k'}\|_{X^{2(1/4-b+),b}}\|u_3^{m,m'}\|_{X^{1,b}}\|u_4^{n,n'}\|_{X^{1,b}} \\
&\lesssim \frac{(N_{(1,j)}N_{(2,k)}N_{(3,m)}N_{(4,n)})^-}{(L_{(1,j')}L_{(2,k')}L_{(4,n')})^{b-1/4-}}\|u_1\|_{X^{0,b}}\|u_2\|_{X^{2(1/4-b+),b}}\|u_3\|_{X^{1,b}}\|u_4\|_{X^{1,b}},
\end{aligned}
$$

again the result follows for $\sigma = 2(b-1/4)-$. Thus we have finished estimating II and therefore Theorem 3.4.1.

This result will appear in the Communications in Pure and Applied Analysis, see [29].

Chapter 4

Existence and Uniqueness theory for the fractional Schrödinger equation on the torus

4.1 Introduction

In this chapter, we study a fractional semilinear Schrödinger type equation with periodic boundary conditions,

$$\begin{cases} iu_t + (-\Delta)^\alpha u = \pm |u|^2 u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{T}), \end{cases} \tag{4.1}$$

where $\alpha \in (1/2, 1)$. The equation is called defocusing when the sign in front of the nonlinearity is a minus and focusing when the sign is a plus.

Posed on the real line the equation has appeared at a formal level in many recent articles, see [47] and the references therein. For example it is a basic model equation in the theory of fractional quantum mechanics introduced by Laskin, [48]. A rigorous derivation of the equation can be found in [47] starting from a family of models describing charge transport in bio polymers like the DNA. The starting point is a discrete nonlinear Schrödinger equation with general lattice interactions. Equation (4.1) with $\alpha \in (\frac{1}{2}, 1)$ appears as the continuum limit of the long-range interactions between quantum particles on the lattice. Whereas, allowing only the short-range interactions (e.g. neighboring particle interactions) the authors obtain the standard Schrödinger equation ($\alpha = 1$) which is completely integrable, see [1].

In this chapter we study the periodic problem mainly for two reasons. First due to the lack of strong dispersion the mathematical theory for the fractional Schrödinger equations are less developed than the cubic nonlinear Schrödinger equation (NLS). Secondly when we consider periodic boundary conditions the analysis becomes harder, for any dispersion relation, since the dispersive character of the equation can only be exploited after employing averaging arguments and a careful analysis of the resonant set of frequencies, [34].

The local and global well-posedness for the periodic NLS was established by Bourgain in [7]. He used number theoretic arguments to obtain periodic Strichartz estimates along with a new scale of spaces adapted to the dispersive relation of the linear group. More precisely he proved the existence and uniqueness of local-in-time strong $L^2(\mathbb{T})$ solutions. Since it is known that smooth solutions of the NLS satisfy mass conservation

$$M(u)(t) = \int_{\mathbb{T}} |u(t,x)|^2 = M(u)(0),$$

Bourgain's result showed the existence of global-in-time strong $L^2(\mathbb{T})$ solutions in the focusing and defocusing case. The L^2 theorem of Bourgain is sharp since as it was shown in [16], the solution operator is not uniformly continuous on $H^s(\mathbb{T})$ for $s < 0$.

The local well-posedness for the fractional NLS on the real line was recently studied in [26]. The authors showed that the equation is locally well-posed in $H^s(\mathbb{R})$, for $s \geq \frac{1-\alpha}{2}$. They also proved that the solution operator fails to be uniformly continuous in time for $s < \frac{1-\alpha}{2}$. Since the periodic case is less dispersive, we expect the range $s \geq \frac{1-\alpha}{2}$ to be the optimal range for the local theory also in the periodic case.

In this chapter we obtain the following results for the fractional NLS. We first establish a Strichartz estimate that reads as follows

$$\|e^{it(-\Delta)^\alpha} f\|_{L^4_{t\in\mathbb{T}} L^4_{x\in\mathbb{T}}} \lesssim \|f\|_{H^s(\mathbb{T})},$$

for $s > \frac{1-\alpha}{4}$. To use this estimate and prove local well-posedness of the equation one has to overcome the derivative loss on the right hand side of the inequality. In principle this can be done by the method in [23] and [29] which gives local well-posedness in the $H^s(\mathbb{T})$ level, for $s > \frac{1-\alpha}{2}$. However, since the proof in [23] and [29] is quite involved, we choose to establish the local theory by obtaining trilinear $X^{s,b}$ estimates directly. Then a standard iteration finishes the proof without any further analysis. We remark that for classical solutions in $H^s(\mathbb{T})$, $s > \frac{1}{2}$, local theory in the space $C([0,T]; H^s(\mathbb{T}))$ is known. The proof is the same both on the real line and on the torus and it is based on the Banach algebra property of the Sobolev spaces for $s > \frac{1}{2}$. Moreover the length of the local interval of existence is lower bounded by $\frac{1}{\|u_0\|^2_{H^s(\mathbb{T})}}$, see chapter 1. To lower the regularity of the local existence theory and to prove the smoothing estimate of section 5 we have to reprove the local theory in the $X^{s,b}$ spaces. In this case the solution is controlled on the larger $X^{s,b}$ norm, since $X^{s,b}_T \in C([0,T]; H^s(\mathbb{T}))$ for any $b > \frac{1}{2}$, and thus the length of the interval of existence is smaller. In our case for $s > \frac{1}{2}$ it is lower bounded by $\frac{1}{\|u_0\|^{4+}_{H^s(\mathbb{T})}}$.

We note that in addition to the conservation of mass, smooth solutions of (4.1) satisfy energy conservation

$$E(u)(t) = \frac{1}{2}\int_{\mathbb{T}} \left||\nabla|^\alpha u(t,x)\right|^2 \mp \frac{1}{4}\int_{\mathbb{T}} \left|u(t,x)\right|^4 = E(u)(0).$$

Here E is also called the Hamiltonian of the equation. Note that the local theory in H^α level with the conservation of mass and energy imply the existence of global in time energy solutions since mass and energy conservation give global control over the H^α norm of the solution. For the defocusing case however, since there is a minus in the Hamiltonian, we don't have the the a priori H^α norm control of the solution. But still we have the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^4}^4 \lesssim \||\nabla|^\alpha u\|_{L^2}^{\frac{1}{\alpha}} \|u\|_{L^2}^{4-\frac{1}{\alpha}}$$

which controls the potential energy via the kinetic energy $\||\nabla|^\alpha u\|_{L^2}$. Then even for the focusing case one has,

$$E(u)(t) = \frac{1}{2}\||\nabla|^\alpha u(t,x)\|_{L_x^2}^2 - \frac{1}{4}\|u(t,x)\|_{L_x^4}^4 \;=\; \frac{1}{2}\||\nabla|^\alpha u_0(x)\|_{L_x^2}^2 - \frac{1}{4}\|u_0(x)\|_{L_x^4}^4 = E(0),$$

which implies,

$$\begin{aligned} \||\nabla|^\alpha u(t,x)\|_{L_x^2}^2 &\leq \frac{1}{2}\|u(t,x)\|_{L_x^4}^4 + \||\nabla|^\alpha u_0(x)\|_{L_x^2}^2 \\ &\lesssim \||\nabla|^\alpha u(t,x)\|_{L_x^2}^{\frac{1}{\alpha}} \|u(t,x)\|_{L_x^2}^{4-\frac{1}{\alpha}} + \||\nabla|^\alpha u_0(x)\|_{L_x^2} \\ &\lesssim \||\nabla|^\alpha u(t,x)\|_{L_x^2}^{\frac{1}{\alpha}} \|u_0(x)\|_{L_x^2}^{4-\frac{1}{\alpha}} + \||\nabla|^\alpha u_0(x)\|_{L_x^2}. \end{aligned}$$

This tells us that one can then control the Sobolev norm of the solution for all times since $\frac{1}{\alpha} < 2$.

In the second part of this chapter we use the high-low frequency decomposition of Bourgain, [11], to prove global solutions below the energy level. Bourgain's method consists of estimating separately the evolution of the low frequencies and of the high frequencies of the initial data. The low frequency part is smooth and thus by conservation of energy globally defined. The difference equation which is high frequency has small norm. By using smoothing estimates this decomposition can be iterated as long as the norm of the nonlinear part is controlled by the initial energy of the smooth part. As a byproduct of the method one obtains that the nonlinear part of the solution is actually smoother than the linear propagator and stays always in the energy space. Moreover the global solutions satisfy polynomial-in-time bounds. We summarize the results in the following two theorems:

Theorem 4.1.1. *For any $\alpha \in (\frac{1}{2}, 1)$, and any $b > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, the equation (4.1) is locally well-posed in the space $X_T^{s,b} \subset C([0,T]; H^s(\mathbb{T}))$ for any $s > \frac{1-\alpha}{2}$, where $T = T(\|u_0\|_{H^s(\mathbb{T})})$. Moreover, for $s > \frac{1}{2}$ the local existence time $T \gtrsim \|u_0\|_{H^s(\mathbb{T})}^{-4-}$.*

Theorem 4.1.2. *For any $\alpha \in (\frac{1}{2}, 1)$, the equation (4.1) is globally well-posed in $H^s(\mathbb{T})$ for any $s > \frac{10\alpha+1}{12}$. Moreover,*

$$u(t) - e^{it(-\Delta)^\alpha \pm iQt} u_0 \in H^\alpha(\mathbb{T})$$

for all times, where $Q = \frac{1}{\pi}\|u_0\|_2^2$.

Remark. We will prove Theorem 4.1.2 only for the defocusing case. As we mentioned in our introductory remarks, we can also control the H^α norm of the solution by Gagliardo-Nirenberg inequality in the focusing case. Once we have the control of the norm in terms of the initial energy, the proof of the theorem follows along the same lines. In particular we obtain the same global well-posedness results with the same global-in-time bounds for the focusing problem.

Remark. We also have to mention that the smoothing estimates give further information about the long time dynamics of dispersive equations, in particular the existence of global attractors (for the dissipative variants of these equations). The intuition is that the system eventually will be attracted to a compact invariant set that has a finite dimension. For infinite dimensional systems, this is the problem of the existence and uniqueness of the global attractor for the associated PDE. As was explained in [35], to obtain the global attractor, it is enough to prove global smoothing estimates for the dissipative equation.

This chapter is organized as follows. In section 2 we introduce our notation and define the spaces that the iteration will take place. In addition we state two elementary lemmas that we use in proving the Strichartz estimates and the multilinear estimates. Section 3 contains the proof of the Strichartz estimate. It is obtained by a careful analysis of the resonant terms and non resonant interacting terms. Section 4 contains the local well-posedness theory for the model equation. We prove multilinear estimates in the $X^{s,b}$ spaces defined in section 2. In section 5 we prove the main smoothing estimate of this paper. The reader should notice that the estimate is sharp within the tools used and for $\alpha = 1$ it coincides with the smoothing estimate for the NLS that was recently obtained in [36]. Finally in section 6 we use the established local theory and the smoothing estimate to prove global well-posedness for infinite energy solutions. As a final remark we note that our global-in-time results are not optimal.

4.2 Notation and Preliminaries

In the next two chapters we are going to use the same notation.

First of all recall that for $s \geq 0$, $H^s(\mathbb{T})$ is defined as a subspace of L^2 via the norm

$$\|f\|_{H^s(\mathbb{T})} := \sqrt{\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{f}(k)|^2},$$

where $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$ are the Fourier coefficients of f. Plancherel's theorem takes the form

$$\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx.$$

We denote the linear propagator of the fractional Schrödinger equation as $e^{it(-\Delta)^\alpha}$, where it is defined on the Fourier side as $(\widehat{e^{it(-\Delta)^\alpha} f})(n) = e^{itn^{2\alpha}}\widehat{f}(n)$. Similarly, $|\nabla|^\alpha$ is defined as $|\widehat{\nabla|^\alpha f})(n) = n^\alpha \widehat{f}(n)$. We also use $(\cdot)^+$ to denote $(\cdot)^\epsilon$ for all $\epsilon > 0$ with implicit constants depending on ϵ.

The corresponding Bourgain spaces, $X^{s,b}$, will be defined as the closure of compactly supported smooth functions under the norm

$$\|u\|_{X^{s,b}} \doteq \|e^{-it(-\Delta)^\alpha} u\|_{H_t^b(\mathbb{R})H_x^s(\mathbb{T})} = \|\langle \tau - |n|^{2\alpha}\rangle^b \langle n\rangle^s \widehat{u}(n,\tau)\|_{L^2_\tau l^2_n},$$

and the restricted norm will be given as

$$\|u\|_{X_T^{s,b}} \doteq \inf(\|v\|_{X^{s,b}},\ for\ v = u\ on\ [0,T]).$$

We close this section by presenting two elementary lemmas that will be used repeatedly.

Lemma 4.2.1. *a) If $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$, then*

$$\sum_n \frac{1}{\langle n - k_1\rangle^\beta \langle n - k_2\rangle^\gamma} \lesssim \langle k_1 - k_2\rangle^{-\gamma}\phi_\beta(k_1 - k_2),$$

and

$$\int_{\mathbb{R}} \frac{1}{\langle \tau - k_1\rangle^\beta \langle \tau - k_2\rangle^\gamma} d\tau \lesssim \langle k_1 - k_2\rangle^{-\gamma}\phi_\beta(k_1 - k_2),$$

where

$$\phi_\beta(k) := \sum_{|n|\leq|k|} \frac{1}{\langle n\rangle^\beta} \sim \begin{cases} 1, & \beta > 1, \\ \log(1 + \langle k\rangle), & \beta = 1, \\ \langle k\rangle^{1-\beta}, & \beta < 1. \end{cases}$$

b) For $\beta \in (0,1]$, we have

$$\int_{\mathbb{R}} \frac{d\tau}{\langle \tau + \rho_1\rangle^\beta \langle \tau + \rho_2\rangle} \lesssim \frac{1}{\langle \rho_1 - \rho_2\rangle^{\beta-}}. \tag{4.2}$$

c) If $\beta > 1/2$, then

$$\sum_n \frac{1}{\langle n^2 + c_1 n + c_2\rangle^\beta} \lesssim 1, \tag{4.3}$$

where the implicit constant is independent of c_1 and c_2.

Proof. [35, Appendix] Denoting $m = k_2 - k_1$, we can rewrite the sum in part a) as

$$\sum_n \frac{1}{\langle n\rangle^\beta \langle n-m\rangle^\gamma}.$$

For $|n| < |m|/2$, we can estimate the sum by

$$\sum_{|n|<|m|/2} \frac{1}{\langle n\rangle^\beta \langle m\rangle^\gamma} \leq \langle m\rangle^{-\gamma}\phi_\beta(m).$$

For $|n| > 2|m|$, we can estimate by

$$\sum_{|n|>2|m|} \frac{1}{\langle n\rangle^{\beta+\gamma}} \lesssim \langle m\rangle^{1-\beta-\gamma} \lesssim \langle m\rangle^{-\gamma}\phi_\beta(m).$$

For $|n| \sim |m|$, we have

$$\sum_{|n|\sim|m|} \frac{1}{\langle m\rangle^\beta \langle n-m\rangle^\gamma} \lesssim \langle m\rangle^{-\beta}\phi_\gamma(m) \lesssim \langle m\rangle^{-\gamma}\phi_\beta(m).$$

The last inequality follows from the definition of ϕ_β and the hypothesis $\beta > \gamma$.

The part b) follows from part a). For part c), write

$$|n^2 + c_1 n + c_2| = |(n+z_1)(n+z_2)| \geq |n+x_1||n+x_2|,$$

where x_i is the real part of z_i. The contribution of the terms $|n+x_1| < 1$ or $|n+x_2| < 1$ is $\lesssim 1$. Therefore, we estimate the sum in part c) by

$$\sum_n \frac{1}{\langle n^2 + c_1 n + c_2\rangle^\beta} \lesssim 1 + \sum_n \frac{1}{\langle n+x_1\rangle^\beta \langle n+x_2\rangle^\beta} \lesssim 1$$

by part a). □

Lemma 4.2.2. *Fix $\alpha \in (1/2, 1)$. For $n, j, k \in \mathbb{Z}$, we have*

$$g(j,k,n) := |(n+k)^{2\alpha} - (n+j+k)^{2\alpha} + (n+j)^{2\alpha} - n^{2\alpha}| \gtrsim \frac{|k||j|}{(|k|+|j|+|n|)^{2-2\alpha}},$$

where the implicit constant depends on α.

Proof. Let $f_c(x) = (x+c)^{2\alpha} - (x-c)^{2\alpha}$. We have

$$g(j,k,n) = \big|f_{\frac{j}{2}}(n+\frac{j}{2}) - f_{\frac{j}{2}}(n+k+\frac{j}{2})\big|.$$

We claim that

$$f_c'(x) \gtrsim \frac{|c|}{\max(|c|,|x|)^{2-2\alpha}}.$$

Using the claim, we have by the mean value theorem (for $j,k \neq 0$)

$$\begin{aligned} g(j,k,n) = \big|f_{\frac{j}{2}}(n+\frac{j}{2}) - f_{\frac{j}{2}}(n+k+\frac{j}{2})\big| &\gtrsim |k||j| \min_{\gamma\in(n+\frac{j}{2},n+k+\frac{j}{2})} \frac{1}{\max(\frac{|j|}{2},|\gamma|)^{2-2\alpha}} \\ &\gtrsim \frac{|k||j|}{(|k|+|j|+|n|)^{2-2\alpha}}. \end{aligned}$$

It remains to prove the claim. Since f_c is odd, and $j \neq 0$, it suffices to consider $x \geq 0$ and $c \gtrsim 1$. We have

$$f_c'(x) = 2\alpha\big[(x+c)^{2(\alpha-1)}|x+c| - (x-c)^{2(\alpha-1)}|x-c|\big].$$

We consider three cases:

Case 1. $0 \leq x \leq c \Rightarrow f_c'(x) = 2\alpha\big[(x+c)^{2\alpha-1} + (x-c)^{2\alpha-1}\big]$. Thus

$$f_c'(x) \gtrsim c^{2\alpha-1}.$$

Case 2. $c \leq x \lesssim c \Rightarrow f_c'(x) = 2\alpha\big[(x+c)^{2\alpha-1} - (x-c)^{2\alpha-1}\big]$. Then we get

$$f_c'(x) \gtrsim c^{2\alpha-1}\Big(\big(\frac{x}{c}+1\big)^{2\alpha-1} - \big(\frac{x}{c}-1\big)^{2\alpha-1}\Big) \gtrsim c^{2\alpha-1}.$$

Case 3. $x \gg c \Rightarrow f_c'(x) = 2\alpha\big[(x+c)^{2\alpha-1} - (x-c)^{2\alpha-1}\big]$. Then we have

$$f_c'(x) = 2\alpha x^{2\alpha-1}\Big(\big(1+\frac{c}{x}\big)^{2\alpha-1} - \big(1-\frac{c}{x}\big)^{2\alpha-1}\Big) \sim x^{2\alpha-1}\frac{c}{x} = x^{2\alpha-2}c.$$

Hence, in all cases we have $f_c'(x) \gtrsim \frac{|c|}{\max(|c|,|x|)^{2-2\alpha}}$.

□

4.3 Strichartz Estimates

Theorem 4.3.1. $\|e^{it(-\Delta)^\alpha} f\|_{L^4_{t\in\mathbb{T}} L^4_{x\in\mathbb{T}}} \lesssim \|f\|_{H^s}$ *for* $s > \frac{1-\alpha}{4}$.

Proof. Notice that in this proof we can always take $s < \frac{1}{4}$. Calling $g = \langle\nabla\rangle^s f$, and denoting $\widehat{g}(k)$ by g_k, we write

$$\begin{aligned}
\|e^{it(-\Delta)^\alpha} f\|^4_{L^4_t L^4_x} &= \int_0^{2\pi}\int_0^{2\pi} \sum_{k_1,k_2,k_3,k_4} \frac{e^{it(k_1^{2\alpha}-k_2^{2\alpha}+k_3^{2\alpha}-k_4^{2\alpha})} e^{ix(k_1-k_2+k_3-k_4)} g_{k_1}\overline{g_{k_2}} g_{k_3}\overline{g_{k_4}}}{\langle k_1\rangle^s\langle k_2\rangle^s\langle k_3\rangle^s\langle k_4\rangle^s} dxdt \\
&= \int_0^{2\pi} \sum_{k_1-k_2+k_3-k_4=0} \frac{e^{it(k_1^{2\alpha}-k_2^{2\alpha}+k_3^{2\alpha}-k_4^{2\alpha})} g_{k_1}\overline{g_{k_2}} g_{k_3}\overline{g_{k_4}}}{\langle k_1\rangle^s\langle k_2\rangle^s\langle k_3\rangle^s\langle k_4\rangle^s} dt \\
&\lesssim \sum_{k_1-k_2+k_3-k_4=0} \frac{|g_{k_1}||g_{k_2}||g_{k_3}||g_{k_4}|}{\langle k_1\rangle^s\langle k_2\rangle^s\langle k_3\rangle^s\langle k_4\rangle^s} \frac{1}{\max(1, |k_1^{2\alpha}-k_2^{2\alpha}+k_3^{2\alpha}-k_4^{2\alpha}|)}
\end{aligned}$$

Renaming the variables as $k_1 = n+j$, $k_2 = n+k+j$, $k_3 = n+k$, and $k_4 = n$, and using Lemma 4.2.2, we get

$$\begin{aligned}
\|e^{it(-\Delta)^\alpha} f\|^4_{L^4_t L^4_x} &\lesssim \sum_{n,k,j} \frac{|g_n||g_{n+j}||g_{n+k}||g_{n+k+j}|}{\langle n\rangle^s\langle n+k\rangle^s\langle n+j\rangle^s\langle n+k+j\rangle^s} \frac{1}{\max\left(1, \frac{|kj|}{(|k|+|j|+|n|)^{2-2\alpha}}\right)} \\
&:= I + II
\end{aligned}$$

where I contains the terms with $|kj| \ll (|k|+|j|+|n|)^{2-2\alpha}$ and II contains the remaining terms.

First note that the summation set in I does not contain any terms with both $n=0$ and $|kj| \neq 0$ since $\alpha \in (1/2, 1)$. Also noting that if $kj \neq 0$, then

$$|kj| \ll (|k|+|j|+|n|)^{2-2\alpha} \lesssim |k|^{2-2\alpha} + |j|^{2-2\alpha} + |n|^{2-2\alpha} \lesssim |kj| + |n|^{2-2\alpha},$$

since $\alpha \in (1/2, 1)$. We can thus write

$$I \lesssim \sum_{\substack{n,k,j \\ 0<|kj|\lesssim|n|^{2-2\alpha}}} \frac{|g_n||g_{n+j}||g_{n+k}||g_{n+k+j}|}{\langle n\rangle^s\langle n+k\rangle^s\langle n+j\rangle^s\langle n+k+j\rangle^s} + \sum_{j,n} |g_n|^2|g_{n+j}|^2 + \sum_{k,n} |g_n|^2|g_{n+k}|^2.$$

The last two sums are equal to $\|g\|^4_{L^2}$. We estimate the first sum by Cauchy-Schwarz inequality to get

$$\lesssim \Big(\sum_{n,k,j} |g_{n+j}|^2|g_{n+k}|^2|g_{n+k+j}|^2\Big)^{1/2} \Big(\sum_{\substack{n,k,j \\ 0<|kj|\lesssim|n|^{2-2\alpha}}} \frac{|g_n|^2}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}}\Big)^{1/2}$$

$$\lesssim \|g\|_{L^2}^4 \sup_n \Big(\sum_{\substack{k,j \\ 0<|kj|\lesssim|n|^{2-2\alpha}}} \frac{1}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}}\Big)^{1/2}.$$

The condition on the sum implies, except for finitely many n's, that $|k| \ll |n|$ and $|j| \ll |n|$. Therefore

$$\sum_{\substack{k,j \\ 0<|kj|\lesssim|n|^{2-2\alpha}}} \frac{1}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}}$$

$$\lesssim \frac{1}{\langle n\rangle^{8s}} \sum_{0<|kj|\lesssim|n|^{2-2\alpha}} 1 \lesssim \langle n\rangle^{2-2\alpha-8s} log\langle n\rangle \lesssim 1$$

provided that $s > \frac{1-\alpha}{4}$.

For the second sum we have,

$$II \lesssim \sum_{\substack{n,k,j \\ |kj|\gtrsim|n|^{2-2\alpha}}} \frac{|g_n||g_{n+j}||g_{n+k}||g_{n+k+j}|(|n|+|k|+|j|)^{2-2\alpha}}{\langle n\rangle^{s}\langle n+k\rangle^{s}\langle n+j\rangle^{s}\langle n+k+j\rangle^{s}|kj|}.$$

Using the symmetry in k and j, we have

$$II \lesssim \sum_{\substack{n,k,j \\ |kj|\gtrsim|n|^{2-2\alpha},\ |k|\geq|j|}} \frac{|g_n||g_{n+j}||g_{n+k}||g_{n+k+j}|(|n|+|k|)^{2-2\alpha}}{\langle n\rangle^{s}\langle n+k\rangle^{s}\langle n+j\rangle^{s}\langle n+k+j\rangle^{s}|kj|}.$$

To estimate the sum we consider three frequency regions, $|k| \sim |n|$, $|k| \ll |n|$, and $|k| \gg |n|$.

Region 1. $|k| \sim |n|$. In this region, using Cauchy Schwarz inequality as above, it suffices to show that the sum

$$\sum_{\substack{|k|\geq|j| \\ |k|\sim|n|}} \frac{(|n|+|k|)^{4-4\alpha}}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}k^2j^2}$$

is bounded in n. We bound this by

$$\sum_{\substack{|k|\geq|j| \\ |k|\sim|n|}} \frac{|n|^{2-4\alpha-2s}}{\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}j^2}.$$

Using the inequality

$$\langle m+j\rangle\langle j\rangle \gtrsim \langle m\rangle,$$

and recalling that $s < \frac{1}{4}$, we obtain

$$\lesssim \sum_{\substack{|k|\geq|j| \\ |k|\sim|n|}} \frac{|n|^{2-4\alpha-4s}}{\langle n+k\rangle^{4s} j^{2-4s}} \lesssim \langle n\rangle^{2-4\alpha-4s+1-4s}.$$

Here we first summed in j and then in k. The sum is bounded in n provided that $s > \frac{3-4\alpha}{8}$.

Region 2. $|k| \ll |n|$. As in Region 1, it suffices to show that the sum

$$\sum_{\substack{|j|\leq|k|\ll|n| \\ |kj|\gtrsim|n|^{2-2\alpha}}} \frac{|n|^{4-4\alpha}}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}k^2j^2} \sim \sum_{\substack{|j|\leq|k|\ll|n| \\ |kj|\gtrsim|n|^{2-2\alpha}}} \frac{|n|^{4-4\alpha-8s}}{k^2j^2}$$

is bounded in n. To this end, notice that

$$\sum_{\substack{|j|\leq|k|\ll|n| \\ |kj|\gtrsim|n|^{2-2\alpha}}} \frac{|n|^{4-4\alpha-8s}}{k^2j^2} \lesssim \sum_{|j|\leq|k|\ll|n|} \frac{|n|^{4-4\alpha-8s}}{|j||k|\langle n\rangle^{2-2\alpha}} \lesssim \sup_n |n|^{2-2\alpha-8s}\log(|n|)^2,$$

which is finite provided that $s > \frac{1-\alpha}{4}$.

Region 3. $|k| \gg |n|$. In this region we bound the sum by Cauchy Schwarz inequality as follows:

$$\begin{aligned}
&\sum_{\substack{|j|\leq|k|,\ |n|\ll|k| \\ |kj|\gtrsim|n|^{2-2\alpha}}} \frac{|g_n||g_{n+j}||g_{n+k+j}||g_{n+k}||k|^{2-2\alpha}}{\langle n\rangle^s\langle n+k\rangle^s\langle n+j\rangle^s\langle n+k+j\rangle^s|kj|} \\
&\lesssim \Big(\sum_{n,k,j}|g_n|^2|g_{n+j}|^2|g_{n+k+j}|^2\Big)^{1/2}\Big(\sum_{|j|\leq|k|,\ |n|\ll|k|} \frac{|g_{n+k}|^2|k|^{4-4\alpha}}{\langle n\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}k^2j^2}\Big)^{1/2} \\
&\lesssim \|g\|_{L^2}^3\Big(\sum_{|j|\leq|k|,\ |n|\ll|k|} \frac{|g_{n+k}|^2|k|^{2-4\alpha-2s}}{\langle n\rangle^{2s}\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}j^2}\Big)^{1/2}.
\end{aligned}$$

Estimating the j sum in parenthesis as in Region 1, we have

$$\lesssim \sum_{|n|\ll|k|} \frac{|g_{n+k}|^2|k|^{2-4\alpha-2s}}{\langle n\rangle^{4s}\langle n+k\rangle^{2s}} \lesssim \sum_{|n|\ll|k|} \frac{|g_{n+k}|^2|k|^{2-4\alpha-4s}}{\langle n\rangle^{4s}} \lesssim \sum_{n,k}|g_{n+k}|^2|k|^{1-2\alpha-4s}\langle n\rangle^{1-2\alpha-4s}.$$

We estimate this by Cauchy Schwarz

$$\Big[\sum_{n,k}|g_{n+k}|^2|k|^{2-4\alpha-8s}\Big]^{\frac{1}{2}}\Big[\sum_{n,k}|g_{n+k}|^2\langle n\rangle^{2-4\alpha-8s}\Big]^{\frac{1}{2}} \lesssim \|g\|_{L^2}^2,$$

provided that $2-4\alpha-8s < -1$, i.e. $s > \frac{3}{8} - \frac{\alpha}{2}$. In the last inequality we summed in n and k separately.

Thus, for $s > max(\frac{1-\alpha}{4}, \frac{3-4\alpha}{8}) = \frac{1-\alpha}{4}$, for $\alpha > \frac{1}{2}$, we obtain the Strichartz estimates. $\square$

4.4 Local well-posedness via the $X^{s,b}$ method

We will prove Theorem 4.1.1 for the defocusing equation by obtaining multilinear estimates in $X^{s,b}$ spaces. With the change of variable $u(x,t) \to u(x,t)e^{iQt}$ in the equation (4.1), where $Q = \frac{1}{\pi}\|u_0\|_2^2$, we obtain the equation

$$iu_t + (-\Delta)^\alpha u + |u|^2 u - Qu = 0, \quad t \in \mathbb{R}, \;\; x \in \mathbb{T},$$

with initial data in $u_0 \in H^s(\mathbb{T})$, $s > 0$.

Note the following identity which follows from Plancherel's theorem:

$$\begin{aligned}\widehat{|u|^2u}(k) &= \sum_{k_1,k_2} \widehat{u}(k_1)\overline{\widehat{u}(k_2)}\widehat{u}(k-k_1+k_2) \\ &= \frac{1}{\pi}\|u\|_2^2\widehat{u}(k) - |\widehat{u}(k)|^2\widehat{u}(k) + \sum_{k_1\neq k, k_2\neq k_1} \widehat{u}(k_1)\overline{\widehat{u}(k_2)}\widehat{u}(k-k_1+k_2) \\ &=: Q\widehat{u}(k) + \widehat{\rho(u)}(k) + \widehat{R(u)}(k). \end{aligned} \tag{4.4}$$

Using this in the Duhamel's formula, we have

$$u(t) = e^{it(-\Delta)^\alpha}u_0(x) - i\int_0^t e^{i(-\Delta)^\alpha(t-t')}(\rho(u) + R(u))dt'.$$

By standard iteration techniques, it suffices to obtain an estimate of the form:

$$\Big\|\int_0^t e^{i(-\Delta)^\alpha(t-t')}(\rho(u)+R(u))dt'\Big\|_{X_T^{s,b}} \lesssim T^\delta \|u\|^3_{X_T^{s,b}},$$

for $s > \frac{1-\alpha}{2}$ and for some $b > \frac{1}{2}$, $\delta > 0$.

To prove this estimate and obtain a lower bound for the local existence time we need the following lemma:

Lemma 4.4.1. *[38] For b, b' such that $0 \leq b + b' < 1$, $0 \leq b' < 1/2$, then we have*

$$\Big\|\int_0^t e^{i(-\Delta)^\alpha(t-\tau)}f(\tau)d\tau\Big\|_{X_T^{s,b}} \lesssim T^{1-b-b'}\|f\|_{X_T^{s,-b'}},$$

for $T \in [0,1]$.

Proposition 4.4.2. *Let $\alpha \in (\frac{1}{2}, 1)$ and $s > \frac{1-\alpha}{2}$, then for $b > 1/2$ we have,*

$$\big\|\rho(u) + R(u)\big\|_{X^{s,-b'}} \lesssim \|u\|^3_{X^{s,b}},$$

provided that $b' < \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$. Moreover, for $s > \frac{1}{2}$ we can take $b' = 0$.

As we remarked in the introduction, in the case that $s > \frac{1}{2}$, the condition $b' = 0$ implies the existence of the local solution in $[0, \delta]$ as long as $\delta^{\frac{1}{2}-}\|u_0\|^2_{H^s(\mathbb{T})} \sim 1$. This bound although sub-optimal, it is necessary for the proof of the global well-posedness below the energy space that we establish in section 6.

Proof. We present the proof for $R(u)$. The proof for $\rho(u)$ is easier and in what follows it corresponds to the terms given by $j = k = 0$.

First note that

$$\big\|R(u)\big\|_{X^{s,-b'}} = \Big\| \int_{\tau_1-\tau_2+\tau_3=\tau} \sum_{\substack{k_1-k_2+k_3=n \\ k_1\neq n,k_2}} \frac{\widehat{u}(\tau_1,k_1)\overline{\widehat{u}(\tau_2,k_2)}\widehat{u}(\tau_3,k_3)\langle n\rangle^s}{\langle \tau - n^{2\alpha}\rangle^{b'}} \Big\|_{L^2_\tau l^2_n},$$

By a duality argument and denoting $|\widehat{u}(\tau,n)|\langle n\rangle^s\langle \tau - n^{2\alpha}\rangle^b = v(\tau,n)$, we get

$$\begin{aligned}\big\|R(u)\big\|_{X^{s,-b'}} \leq \sup_{\|g\|_{L^2_\tau l^2_n}=1} \int_{\tau_1-\tau_2+\tau_3-\tau=0} \sum_{\substack{k_1-k_2+k_3-n=0 \\ k_1\neq n,k_2}} &\frac{\langle n\rangle^s v(\tau_1,k_1)v(\tau_2,k_2)v(\tau_3,k_3)g(\tau,n)}{\langle k_1\rangle^s\langle k_2\rangle^s\langle k_3\rangle^s\langle \tau - n^{2\alpha}\rangle^{b'}} \\ &\times \frac{1}{\langle \tau_1 - k_1^{2\alpha}\rangle^b\langle \tau_2 - k_2^{2\alpha}\rangle^b\langle \tau_3 - k_3^{2\alpha}\rangle^b},\end{aligned}$$

and thus, by Cauchy-Schwarz and then integrating in τ variables as in [35], we have

$$\begin{aligned}\big\|R(u)\big\|^2_{X^{s,-b'}} &\leq \|v\|^6_{L^2_\tau l^2_n} \sup_{\tau,n} \int_{\tau_1-\tau_2+\tau_3=\tau} \sum_{\substack{k_1-k_2+k_3=n \\ k_1\neq n,k_2}} \frac{\langle n\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k_2\rangle^{2s}\langle k_3\rangle^{2s}\langle \tau - n^{2\alpha}\rangle^{2b'}} \\ &\qquad\times \frac{1}{\langle \tau_1 - k_1^{2\alpha}\rangle^{2b}\langle \tau_2 - k_2^{2\alpha}\rangle^{2b}\langle \tau_3 - k_3^{2\alpha}\rangle^{2b}}. \\ &\lesssim \|u\|^6_{X^{s,b}} \sup_n \sum_{\substack{k_1-k_2+k_3=n \\ k_1\neq n,k_2}} \frac{\langle n\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k_2\rangle^{2s}\langle k_3\rangle^{2s}\langle k_1^{2\alpha} - k_2^{2\alpha} + k_3^{2\alpha} - n^{2\alpha}\rangle^{2b'}}.\end{aligned}$$

Hence, we need to show that

$$M_n = \sum_{\substack{k_1-k_2+k_3=n \\ k_1\neq n,k_2}} \frac{\langle n\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k_2\rangle^{2s}\langle k_3\rangle^{2s}\langle k_1^{2\alpha} - k_2^{2\alpha} + k_3^{2\alpha} - n^{2\alpha}\rangle^{2b'}},$$

is bounded in n. Renaming the variables as $k_1 = n + j$, $k_2 = n + k + j$, $k_3 = n + k$, and using Lemma 4.2.2, we get

$$M_n \lesssim \sum_{j,k\neq 0} \frac{\langle n\rangle^{2s}}{\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}\langle n+k\rangle^{2s}\max\left(1, \frac{|kj|^{2b'}}{(|k|+|j|+|n|)^{4(1-\alpha)b'}}\right)}$$
$$:= I + II$$

where I contains the terms with $|kj| \ll (|k|+|j|+|n|)^{2-2\alpha}$ and II contains the remaining terms. Here we note that M_n is bounded in n for $b'=0$ in the case $s>\frac{1}{2}$. From now on we consider the range $\frac{1-\alpha}{2} < s \leq \frac{1}{2}$, and take $b' = \frac{1}{2}-$. To estimate I, as in the proof of Theorem 4.3.1, we write

$$I \lesssim \sum_{0<|kj|\lesssim |n|^{2-2\alpha}} \frac{\langle n\rangle^{2s}}{\langle n+k\rangle^{2s}\langle n+k+j\rangle^{2s}\langle n+j\rangle^{2s}} \lesssim \langle n\rangle^{2-2\alpha-4s}\log(\langle n\rangle),$$

which is bounded provided that $s > \frac{1-\alpha}{2}$. Similarly,

$$II \lesssim \sum_{|kj|\gtrsim |n|^{2-2\alpha}} \frac{\langle n\rangle^{2s}(|k|+|j|+|n|)^{2(1-\alpha)}}{\langle n+k\rangle^{2s}\langle n+k+j\rangle^{2s}\langle n+j\rangle^{2s}|kj|^{1-}}$$
$$\lesssim \sum_{\substack{|kj|\gtrsim |n|^{2-2\alpha}\\ |k|\geq |j|}} \frac{\langle n\rangle^{2s}(|k|+|n|)^{2(1-\alpha)}}{\langle n+k\rangle^{2s}\langle n+k+j\rangle^{2s}\langle n+j\rangle^{2s}|kj|^{1-}}.$$

Second line follows from the kj symmetry of the sum. To estimate the sum we consider three regions:

Region 1. $|k| \gg |n|$. The sum is

$$\lesssim \sum_{\substack{|k|\geq |j|\\ |k|\gg |n|}} \frac{\langle n\rangle^{2s}|k|^{2(1-\alpha)-2s-1+}}{\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}|j|^{1-}}.$$

Note that for $\frac{1}{2} \geq s > \frac{1-\alpha}{2}$, we can bound it by

$$\lesssim \sum_{\substack{|k|\geq |j|\\ |k|\gg |n|}} \frac{\langle n\rangle^{2s}|k|^{2(1-\alpha)-4s+}}{\langle n+j\rangle^{2s}|k|^{1-2s+}\langle n+k+j\rangle^{2s}|j|^{1-}}$$
$$\lesssim \sum_{\substack{|k|\geq |j|\\ |k|\gg |n|}} \frac{\langle n\rangle^{2(1-\alpha)-2s+}}{\langle n+j\rangle^{2s}|k|^{1-2s+}\langle n+k+j\rangle^{2s}|j|^{1-}}$$
$$\lesssim \sum_{j} \frac{\langle n\rangle^{2(1-\alpha)-2s+}}{\langle n+j\rangle^{2s}|j|^{1-}} \lesssim \langle n\rangle^{2(1-\alpha)-4s+}$$

which is bounded in n. In the k and j sums we used Lemma 4.2.1.

Region 2. $|k| \sim |n|$. In this region we have the bound

$$\lesssim \sum_{\substack{|k|\geq|j| \\ |k|\sim|n|}} \frac{\langle n\rangle^{2s+1-2\alpha+}}{\langle n+j\rangle^{2s}\langle n+k\rangle^{2s}\langle n+k+j\rangle^{2s}|j|^{1-}}$$
$$\lesssim \sum_{j} \frac{\langle n\rangle^{2s+1-2\alpha+}A}{\langle n+j\rangle^{2s}|j|^{1-}},$$

where $A = |j|^{1-4s}$ if $4s > 1$, $A = |n|^{1-4s}$ if $4s < 1$ and $A = \log(|n|)$ if $4s = 1$. Then, by considering these cases separately and using Lemma 4.2.1 in the j sums, one obtains boundedness in n for $s > \frac{1-\alpha}{2}$ and $\alpha > \frac{1}{2}$.
Region 3. $|k| \ll |n|$. We have the bound

$$\lesssim \sum_{|j|\leq|k|\ll|n|} \frac{\langle n\rangle^{-4s+2-2\alpha}}{|kj|^{1-}} \lesssim \langle n\rangle^{-4s+2-2\alpha+},$$

which is bounded in n. $\square$

4.5 A smoothing estimate

We first note that

$$\|\rho(u)\|_{H^{s+c}} = \sqrt{\sum_{k} |\widehat{u}(k)|^6 \langle k\rangle^{2s+2c}} \lesssim \|u\|_{H^s}^3, \tag{4.5}$$

for $0 \leq c \leq 2s$, which implies that the contribution of $\rho(u)$ to the Duhamel formula is smoother than u. One can also obtain the same level of smoothing in $X^{s,b}$ spaces: For $c \leq 2s$

$$\|\rho(u)\|_{X^{s+c,-\frac{1}{2}+}} \lesssim \|u\|^3_{X^{s,\frac{1}{2}+}}.$$

To prove the same for the non resonant terms $R(u)$ we have the following proposition:

Proposition 4.5.1. *For $s > \frac{1-\alpha}{2}$ and $c < \min(\alpha - \frac{1}{2}, 2s+\alpha-1)$, we have*

$$\|R(u)\|_{X^{s+c,-\frac{1}{2}+}} \lesssim \|u\|^3_{X^{s,\frac{1}{2}+}}.$$

Proof. Repeating the steps in the proof of Proposition 4.4.2, it suffices to prove that

$$M(n) = \sum_{kj\neq 0} \frac{\langle n\rangle^{2s+2c}}{\langle n+j\rangle^{2s}\langle n+k\rangle^{2s}\langle n+j+k\rangle^{2s}\langle \frac{|kj|}{(|n|+|k|+|j|)^{2-2\alpha}}\rangle^{1-}}$$

is bounded in n.

For the terms with $0<|kj|\lesssim |n|^{2-2\alpha}$, since $|k|,|j|\ll |n|$, we have the bound

$$\lesssim \sum_{0<|kj|\lesssim|n|^{2-2\alpha}} \langle n\rangle^{-4s+2c} \lesssim \langle n\rangle^{-4s+2c+2-2\alpha}\log(n),$$

which is bounded provided that $c<2s+\alpha-1$.

For the remaining terms, we have to consider the cases $s>1/2$ and $s\le 1/2$ separately. Again by symmetry in j and k, it is enough to consider $|k|\ge |j|$.

Case 1. $s>1/2$. As before, we will consider three regions:

Region 1.1. $|k|\gg|n|$. Then we have

$$\begin{aligned}
&\lesssim \sum_{\substack{|k|\ge|j|>0\\ |k|\gg|n|}} \frac{\langle n\rangle^{2s+2c}|k|^{1-2\alpha-2s+}}{\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}|j|^{1-}}\\
&\lesssim \sum_{\substack{j\\ |k|\gg|n|}} \frac{\langle n\rangle^{2c+1-2\alpha+}}{\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}|j|^{1-}}\\
&\lesssim \sum_{j} \frac{\langle n\rangle^{2c+1-2\alpha+}}{\langle n+j\rangle^{2s}\langle j\rangle^{1-}} \lesssim \langle n\rangle^{2c-2\alpha+},
\end{aligned}$$

which is bounded for $c<\alpha$. In the forth inequality we used Lemma 4.2.1.

Region 1.2. $|k|\sim|n|$. In this region we have,

$$\lesssim \sum_{\substack{|k|\ge|j|>0\\ |k|\sim|n|}} \frac{\langle n\rangle^{2c+2s+1-2\alpha+}}{\langle n+k\rangle^{2s}\langle n+j\rangle^{2s}\langle n+j+k\rangle^{2s}|j|^{1-}} \lesssim \sum_{\substack{|k|\ge|j|>0\\ |k|\sim|n|}} \frac{\langle n\rangle^{2c+1-2\alpha+}}{\langle n+k\rangle^{2s}|j|^{1-}} \lesssim \langle n\rangle^{2c+1-2\alpha+}$$

for $c<\alpha-\frac{1}{2}$.

Region 1.3. $|k|\ll|n|$. We have

$$\lesssim \sum_{\substack{|k|\ge|j|>0\\ |k|\ll|n|}} \frac{\langle n\rangle^{-4s+2c+2-2\alpha+}}{|kj|^{1-}} \lesssim \langle n\rangle^{2c-4s+2-2\alpha+},$$

which is bounded for $c<2s+\alpha-1$. This finishes the case $s>1/2$.

Case 2. $\frac{1-\alpha}{2} < s \leq 1/2$.

Region 2.1. $|k| \gg |n|$. As in the proof of Proposition 4.4.2, we have

$$\lesssim \sum_{\substack{|k|\geq|j|>0 \\ |k|\gg|n|}} \frac{\langle n\rangle^{2s+2c-4s+2-2\alpha+}}{\langle n+j\rangle^{2s}\langle n+k+j\rangle^{2s}|k|^{1-2s+}|j|^{1-}} \lesssim \langle n\rangle^{2c-4s+2-2\alpha+}$$

which is bounded for $c < 2s+\alpha-1$.

Region 2.2. $|k| \sim |n|$. In this region we have,

$$\lesssim \sum_{\substack{|k|\geq|j|>0 \\ |k|\sim|n|}} \frac{\langle n\rangle^{2s+2c+1-2\alpha+}}{\langle n+j\rangle^{2s}\langle n+k\rangle^{2s}\langle n+k+j\rangle^{2s}|j|^{1-}} \lesssim \sum_{j} \frac{\langle n\rangle^{2s+2c+1-2\alpha+}A}{\langle n+j\rangle^{2s}|j|^{1-}},$$

where $A = \langle j\rangle^{1-4s}$ for $\frac{1}{4} \leq s \leq \frac{1}{2}$, and $A = \langle n\rangle^{1-4s}$ for $0 < s < \frac{1}{4}$. Hence,

$$\begin{aligned} &\lesssim \langle n\rangle^{2c+1-2\alpha+} && \text{for } s \geq \frac{1}{4}, \\ &\lesssim \langle n\rangle^{2c-4s+2-2\alpha+} && \text{for } 0 < s < \frac{1}{4}, \end{aligned}$$

which is bounded for $c < 2s+\alpha-1$ when $s \in (0, \frac{1}{4})$ and $c < \alpha - \frac{1}{2}$ when $s \geq \frac{1}{4}$.

Region 2.3. $|k| \ll |n|$. We have,

$$\lesssim \sum_{\substack{|k|\geq|j|>0 \\ |k|\ll|n|}} \frac{\langle n\rangle^{2c+2-2\alpha-4s+}}{|kj|^{1-}} \lesssim \langle n\rangle^{2c+2-2\alpha-4s+}$$

which is bounded for $c < 2s+\alpha-1$.

Hence, for all s, collecting the results we get the proposition. □

This implies that (see [36] for more details):

Theorem 4.5.2. *For $\alpha \in (\frac{1}{2}, 1)$, $s > \frac{1-\alpha}{2}$ and $c < \min(2s+\alpha-1, \alpha-\frac{1}{2})$ we have*

$$\|u(t) - e^{it(-\Delta)^{\alpha}-iQt}u_0\|_{H^{s+c}} \lesssim \|u_0\|_{H^s}^3$$

for $t < T$, where T is the local existence time.

We finish this section by noting that if we define the multilinear versions of ρ and R via

$$\widehat{\rho(u,v,w)}(k) = \widehat{u}(k)\overline{\widehat{v}(k)}\widehat{w}(k), \quad \widehat{R(u,v,w)}(k) = \sum_{k_1\neq k, k_2\neq k_1} \widehat{u}(k_1)\overline{\widehat{v}(k_2)}\widehat{w}(k-k_1+k_2),$$

then the assertions of Proposition 4.4.2 and Proposition 4.5.1 remain valid.

4.6 Global Well-posedness via High-Low Frequency Decomposition

From the local theory along with energy and mass conservation, the existence of global solutions in H^α follows easily. In this case, one can control the H^α norm and apply the local theory with a uniform in time step to reach any time. In this section we use Bourgain's high-low frequency decomposition together with the smoothing estimate from the previous section to obtain global well-posedness for initial data with infinite energy.

Proof of Theorem 4.1.2. Fix $s \in (\frac{1}{2}, \alpha)$. With the change of variable $u(x,t) \to u(x,t)e^{iQt}$ in equation (4.1), where $Q = \frac{1}{\pi}\|u_0\|_2^2$, we obtain the equation

$$iu_t + (-\Delta)^\alpha u + |u|^2 u - Qu = 0, \quad t \in \mathbb{R}, \ x \in \mathbb{T},$$

with initial data in $u_0 \in H^s(\mathbb{T})$. In what follows, the implicit constants will depend on $\|u_0\|_{H^s}$. We fix N large and decompose the equation into two equations, $u = v + w$:

$$\begin{cases} iv_t + (-\Delta)^\alpha v + |v|^2 v - Qv = 0, \\ v(x,0) = P_N u_0(x) \doteq \Phi_0, \end{cases} \tag{4.6}$$

and

$$\begin{cases} iw_t + (-\Delta)^\alpha w + |v+w|^2(v+w) - Qw - |v|^2 v = 0, \\ w(x,0) = u_0(x) - \Phi_0 \doteq \Psi_0, \end{cases} \tag{4.7}$$

where P_N is the projection onto the frequencies $|n| \leq N$.

First note that $\|\Phi_0\|_{H^\alpha} \lesssim N^{\alpha-s}$. Moreover, by the local existence theory we presented in H^α and H^s levels, noting that $\alpha > s > \frac{1}{2}$, we have for $\delta \sim N^{-4(\alpha-s)}$

$$\|v\|_{X_\delta^{\alpha,b}} \lesssim \|\Phi_0\|_{H^\alpha} \lesssim N^{\alpha-s}, \qquad \|v\|_{X_\delta^{s,b}} \lesssim \|\Phi_0\|_{H^s} \lesssim 1.$$

Since equation (4.6) enjoys the same energy conservation, we have

$$E(v(t)) = E(\Phi_0) \lesssim N^{2\alpha-2s}$$

by the Gagliardo-Nirenberg inequality.

Now pick an $s_0 < s$ to be determined later. Note that $\|\Psi_0\|_{H^{s_0}} \lesssim N^{s_0-s}$. The local existence for w equation follows similarly by the multilinear estimates from the previous sections with the same δ as above (since the norm of w is small). We thus have

$$\|w\|_{X_\delta^{s_0,b}} \lesssim \|\Psi_0\|_{H^{s_0}} \lesssim N^{s_0-s}, \qquad \|w\|_{X_\delta^{s,b}} \lesssim \|\Psi_0\|_{H^s} \lesssim 1.$$

Now using the decomposition (4.4) for the nonlinearity $\mathcal{N} := |v+w|^2(v+w) - Qw - |v|^2 v$ in (4.7) we have (with $u = v + w$)

$$\begin{aligned}\mathcal{N} = Qu - Qw - \frac{1}{\pi}\|v\|_{L^2}^2 v + \rho(u) - \rho(v) + R(u) - R(v)& \\ = \frac{1}{\pi}\big(\|u_0\|_2^2 - \|\Phi_0\|_{L^2}^2\big)v + \rho(u) - \rho(v) + R(u) - R(v).\end{aligned}$$

Using the multilinear smoothing estimate and the multilinearity of ρ and R, we have

$$\|\mathcal{N}\|_{X_\delta^{\alpha,-\frac{1}{2}+}} \lesssim \big|\|u_0\|_2^2 - \|\Phi_0\|_{L^2}^2\big| \|v\|_{X_\delta^{\alpha,-\frac{1}{2}+}} + \|w\|_{X_\delta^{s_0,b}}^3 + \|w\|_{X_\delta^{s_0,b}}\|v\|_{X_\delta^{s_0,b}}^2,$$

for $\alpha - s_0 < \min(2s_0 + \alpha - 1, \alpha - \frac{1}{2})$, in particular for $s_0 > \frac{1}{2}$.

Ignoring the support condition of Φ_0 and Ψ_0, we have

$$\big|\|u_0\|_2^2 - \|\Phi_0\|_{L^2}^2\big| \lesssim \|\Psi_0\|_{L^2} + \|\Psi_0\|_{L^2}^2 \lesssim N^{-s}.$$

Therefore, we obtain

$$\begin{aligned}\|\mathcal{N}\|_{X_\delta^{\alpha,-\frac{1}{2}+}} \lesssim N^{-s}\delta^{1-}\|v\|_{X_\delta^{\alpha,b}} + \|w\|_{X_\delta^{s_0,b}}^3 + \|w\|_{X_\delta^{s_0,b}}\|v\|_{X_\delta^{\alpha,b}}^2& \\ \lesssim N^{-s}\delta^{1-}N^{\alpha-s} + N^{3(s_0-s)} + N^{s_0-s}N^{2(\alpha-s)} \lesssim N^{2\alpha+s_0-3s}.\end{aligned}$$

Taking $t_1 = \delta$, we write

$$u(t_1) = w(t_1) + v(t_1) = e^{it_1(-\Delta)^\alpha + iPt_1}\Psi_0 + w_1(t_1) + v(t_1).$$

By the bound on $\mathcal{N}$ and Duhamel's formula, we have

$$\|w_1(t_1)\|_{H^\alpha} \lesssim N^{2\alpha+s_0-3s}.$$

We repeat this process by decomposing $u(t_1) = \Phi_1 + \Psi_1$, where

$$\Psi_1 = e^{it_1(-\Delta)^\alpha + iPt_1}\Psi_0, \qquad \Phi_1 = w_1(t_1) + v(t_1).$$

Since $e^{it_1(-\Delta)^\alpha + iPt_1}$ is unitary, Ψ_1 satisfies all the properties of Ψ_0. To control the H^α norm of Φ_1, we note

$$E(\Phi_1) = E(\Phi_1) - E(v(t_1)) + E(v(t_1)) = E(w_1(t_1) + v(t_1)) - E(v(t_1)) + E(\Phi_0),$$

where the second equality follows from the conservation of the Hamiltonian.

Note that

$$\begin{aligned}
\big|E(f+g) - E(f)\big| &\lesssim \big|\||\nabla|^\alpha (f+g)\|_2^2 - \||\nabla|^\alpha f\|_2^2\big| + \int \big||f+g|^4 - |f|^4\big| \\
&\lesssim \|g\|_{H^\alpha}^2 + \|g\|_{H^\alpha}\|f\|_{H^\alpha} + \int |g|\big(|f|^3 + |g|^3\big) \\
&\lesssim \|g\|_{H^\alpha}^2 + \|g\|_{H^\alpha}\|f\|_{H^\alpha} + \|g\|_{H^{\frac14+}}^4 + \|g\|_{H^{\frac14+}}\|f\|_{H^{\frac14+}}^3 \\
&\lesssim \|g\|_{H^\alpha}^2 + \|g\|_{H^\alpha}\|f\|_{H^\alpha} + \|g\|_{H^\alpha}^4 + \|g\|_{H^\alpha}\|f\|_{H^\alpha}^3.
\end{aligned}$$

Using this for $f = v(t_1)$ and $g = w_1(t_1)$, we obtain

$$E(w_1(t_1) + v(t_1)) - E(v(t_1)) \lesssim N^{2\alpha+s_0-3s}N^{3(\alpha-s)} = N^{5\alpha+s_0-6s}.$$

To reach time T we have to iterate this process $\frac{T}{\delta}$ times. To bound the Hamiltonian at time T by a constant multiple of the initial value, we need

$$N^{5\alpha+s_0-6s}\frac{T}{\delta} = TN^{9\alpha+s_0-10s}$$

to be $\lesssim N^{2\alpha-2s}$. This holds for $s > \frac{7\alpha}{8} + \frac{1}{16}$ by taking $s_0 = \frac{1}{2}+$ and N sufficiently large.

The calculation above can be improved by interpolating between H^α and L^2 to bound the $H^{\frac14+}$ norms.

For example, by Duhamel's formula and Minkowski inequality, we have

$$\|w_1(t_1)\|_{L^2} \lesssim \int_0^{t_1} \|\mathcal{N}\|_{L^2} dt.$$

The worst term in $\mathcal{N}$ is of the form $|v^2 w|$ which can be bounded as follows

$$\delta^{\frac{1}{2}} \|v\|^2_{L^4_t L^4_x} \|w\|_{L^\infty_t L^\infty_x} \lesssim \delta^{\frac{1}{2}} \|v\|^2_{L^4_t L^4_x} \|w\|_{X^{s_0,b}_\delta} \lesssim \delta E(v)^{\frac{1}{2}} \|w\|_{X^{s_0,b}_\delta} \lesssim \delta N^{\alpha+s_0-2s}.$$

After, $\frac{T}{\delta}$ steps, the L^2 norm remains $\lesssim N^{\alpha+s_0-2s} \lesssim 1$, for $s > \frac{\alpha}{2} + \frac{1}{4}$. Therefore the L^2 norm of the low frequency part also remains $\lesssim 1$.

$$\begin{aligned} E(w_1(t_1) + v(t_1)) - E(v(t_1)) &\lesssim N^{2\alpha+s_0-3s} N^{\alpha-s} + N^{(1-\frac{1}{4\alpha})(\alpha+s_0-2s)} N^{\frac{2\alpha+s_0-3s}{4\alpha}} N^{\frac{3(\alpha-s)}{4\alpha}} N^{+} \\ &\lesssim N^{3\alpha+\frac{1}{2}-4s+} + N^{\alpha+\frac{3}{2}-2s-\frac{s}{\alpha}+} \lesssim N^{3\alpha+\frac{1}{2}-4s+}. \end{aligned}$$

After $\frac{T}{\delta}$ steps we get the bound $TN^{7\alpha+\frac{1}{2}-8s+}$. This term is less than similar the initial energy of the high frequency part which is of order $N^{2\alpha-2s}$ for $s > \frac{5\alpha}{6} + \frac{1}{12}$. We can then iterate our result to reach any time T by sending N to infinity. $\square$

This result will appear in Advance Lectures in Mathematics, edited by S. T. Yau, K. Liu and Lizhen Ji, see [30].

Chapter 5

Almost Sure Global Well-posedness for Fractional Cubic Schrödinger equation on torus

5.1 Preliminaries

In this chapter we use a probabilistic approach to the cubic periodic fractional Schrödinger equation. To be able to do that we first need to recall some definitions and theorems on Hamiltonian systems.

Definition 5.1.1 (cf. [3])**.** *A functional Φ is called differentiable if $\Phi(\gamma+h)-\Phi(\gamma)=F+R$, where F depends linearly on h, and $R(h,\gamma)=o(h)$ in the sense that, for $|h|<\epsilon$ and $|dh/dt|<\epsilon$, we have $|R|/\epsilon\to 0$ as $\epsilon\to 0$. The linear part of the increment, $F(h)$, is called the differential.*

It can be showed that whenever this differential is defined, it is unique.

Theorem 5.1.2 (cf. [3])**.** *The functional $\Phi(\gamma)=\int_{t_0}^{t_1}L(x,\dot{x},t)dt$ for $\gamma=x(t)$ is differentiable, and its derivative is given by the formula*

$$F(h)=\int_{t_0}^{t_1}\Big[\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\Big]hdt+\Big(\frac{\partial L}{\partial \dot{x}}h\Big)\Big|_{t_0}^{t_1}.$$

For proof see [3].

One can also consider the ϵ-variations of the extrema to get a motivation for the differential. For that take a path γ on which the functional Φ has a local extremum. Then if we define a new functional $\Phi_{h,\epsilon}(\gamma)=\Phi(\gamma+\epsilon h)$, then for any h, a smooth curve, the functional $\Phi_{h,\epsilon}$ becomes a function on ϵ with a local extremum at $\epsilon=0$. Then taking the derivatives with respect to ϵ at 0 and differentiation by parts, one gets the differential again. The following definitions and theorems also make more sense with this remark in mind.

Definition 5.1.3 (cf. [3])**.** *An extremal of a differentiable functional Φ is a curve γ such that $F(h)=0$ for all curves h.*

Theorem 5.1.4 (cf. [3])**.** *The curve $\gamma=x(t)$ is an extremal of the functional $\Phi(\gamma)=\int_{t_0}^{t_1}L(x,\dot{x},t)dt$ on the* *space of curves passing through the points $x(t_0) = x_0$ and $x(t_1) = x_1$, if and only if*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0,$$

along the curve γ.

Again, for the proof, see [3].

Definition 5.1.5 (cf. [3])**.** *The equation*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

is called the Euler-Lagrange equation for the functional Φ

Now , if we look at Newton's equation for dynamics,

$$\frac{d}{dt}(m_i \dot{r}_i) + \frac{\partial U}{\partial r_i} = 0 \tag{5.1}$$

Hamilton's principle for least action says that:

Theorem 5.1.6 (cf. [3])**.** *Motions of the system* (5.1) *coincide with the extremals of the functional*

$$\Phi(\gamma) = \int_{t_0}^{t_1} L dt,$$

where $L = T - U$ is the difference between the kinetic and the potential energy.

Proof follows directly from the Euler-Lagrange equations. We can also define the Euler-Lagrange equations in n dimensions simply looking at the Euler-Lagrange equations in each variables.

Definition 5.1.7 (cf. [3])**.** *Here, for $x = (x_1, ..., x_n)$, we call $L(x, \dot{x}, t) = T - U$ the Lagrangian function and $\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$ the Lagrange's equations.*

Definition 5.1.8 (cf. [3])**.** *Let $y = f(x)$ be a convex function, $f''(x) > 0$. The Legendre transformation of the function f is a new function g of a new variable p, which is constructed as follows. We draw the graph of f in the x, y plane. Let p be a given number. Consider the straight line $y = px$. We take the point $x = x(p)$ at which the curve is farthest from the straight line in the vertical direction: for each p, the function $px - f(x) = F(p, x)$ has a maximum with respect to x at the point $x(p)$. Now we define $g(p) = F(p, x(p))$.*

The point $x(p)$ is defined by the extremal condition $\partial F/\partial x = 0$, i.e., $f'(x) = p$. Since f in convex, the point $x(p)$ is unique.

We consider the system of Lagrange's equations, $\dot{p} = \frac{\partial L}{\partial q}$, where $p = \frac{\partial L}{\partial \dot{q}}$, with a given Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, which we will assume to be convex with respect to the second argument $\dot{q}$.

Then we have,

Theorem 5.1.9 (cf. [3])**.** *The system of Lagrange's equations is equivalent to the system of $2n$ first order equations called Hamilton's equations*

$$\dot{p} = -\frac{\partial H}{\partial q},$$

$$\dot{q} = \frac{\partial H}{\partial p},$$

where $H(p, q, t) = p\dot{q} - L(p, \dot{q}, t)$ is the Legendre transform of the Lagrangian function viewed as a function of $\dot{q}$.

For the proof, see [3].

Then for $L = T - U$ as before we have that the Hamiltonian is the total energy $H = T + U$. One of the most important properties of the Hamiltonian is that if the Hamiltonian doesn't depend on t explicitly, then it is constant in time. Thus in particular, if the Hamiltonian is the energy, then if the energy doesn't depend on t explicitly, then energy is conserved.

For the Hamiltonians that does not depend on time, i.e., $H = H(p, q)$, we define the Hamiltonian flow as follows:

Definition 5.1.10 (cf. [3])**.** *The $2n$ dimensional space with coordinates $p_1, p_2, \ldots\ldots, p_n; q_1, q_2, \ldots\ldots\ldots, q_n$ is called the phase space and the Hamiltonian flow is the one parameter group of transformations of the phase space*

$$S(t) : (p(0), q(0)) \mapsto (p(t), q(t)),$$

were $p(t)$ and $q(t)$ are the solutions of the Hamilton's system of equations.

Then we have,

Theorem 5.1.11 (Liouville's Theorem)**.** *The Hamiltonian flow preserves volume, i.e., for any region D we have,*

$$Vol(S(t)D) = Vol(D).$$

Again, for the proof, see [3].

Definition 5.1.12. *A measure, μ is called an invariant measure with respect to the Hamiltonian if μ preserves volumes under the Hamiltonian flow, i.e., for any region D,*

$$\mu(S(t)D) = \mu(D).$$

Unfortunately, not all the Hamiltonian systems are finite dimensional. For example, the fractional Schrödinger equation

$$\begin{cases} iu_t + (-\Delta)^\alpha u = \pm|u|^2 u, \\ u(x,0) = u_0(x) \end{cases} \tag{5.2}$$

is an infinite dimensional Hamiltonian system with the Hamiltonian

$$H(u)(t) = \frac{1}{2}\int_{\mathbb{T}} \big||\nabla|^\alpha u(t,x)\big|^2 \mp \frac{1}{4}\int_{\mathbb{T}} \big|u(t,x)\big|^4.$$

Although our arguments so far have been given in the finite dimensional case, at least formally, one would expect to have the same arguments in the infinite dimensional case as well. Of course, passing from the finite dimensional case to the infinite dimensional case requires careful limiting arguments. For more information, consult [24].

Hamiltonian systems appear in the formulation of almost every dynamical laws of physics, such as planetary systems, interaction of quantum fields, hydrodynamics of perfect fluid, general relativity and many more. For example, the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

has the kinetic energy $K = \frac{1}{2}\int_{\mathbb{R}^n} (\dot{u})^2 dx$, and the potential energy $V = \frac{1}{2}\int_{\mathbb{R}^n} (\nabla u)^2 dx$. Thus, the equation is a Hamiltonian equation with the Hamiltonian being the total energy $H(u,\dot{u}) = K + V = \int_{\mathbb{R}^n} \frac{1}{2}(\dot{u})^2 + (\nabla u)^2 dx$. (See [24, Chapter 2.1] for more information and the Hamiltonian structure of the nonlinear wave equation.) In this context, invariant measures are important tools in understanding the dynamics of the system, since it allows us to use other theories like ergodic theory to understand the behavior of sets with positive measure under the Hamiltonian flow. For example, for the measure μ, the Poincaré recurrence theorem states that if S_t is a measure preserving map, then every set E with positive measure has to intersect itself eventually, on a set of positive measure. More precisely, $\forall E$ with $\mu(E) > 0$, there exists $n \in \mathbb{N}$ such that $\mu(E \cap S_t^n E) > 0$. For more examples, see [3]. Invariant measures also have important applications in fields like statistical and quantum mechanics. There have also been many studies on the construction of invariant measures for dynamical systems generated by nonlinear differential equations. For example, see [6], [8], [10], [50], [51] and references in Zhidkov's book, [67].

In the next chapter we work with the Schrödinger equation (3.1) and define and invariant measure using the Hamiltonian structure of it.

5.2 Introduction

For equation (4.1) with $\alpha = 1$ Bourgain, in [7], proved periodic Strichartz estimates and showed L^2 local and global well-posedness for the cubic Schrödinger equation. In [16], Burq, Gerard and Tzvetkov noted that this result is sharp since the solution operator is not uniformly continuous on H^s for $s < 0$.

The fractional Schrödinger equation on real line was recently studied in [26]. For $\alpha \in (1/2, 1)$, the equation is less dispersive, so one would not expect to be able to get local well-posedness on L^2 level. Indeed, they proved that there is local well-posedness on H^s for $s \geq \frac{1-\alpha}{2}$. They also showed that the solution operator fails to be uniformly continuous in time for $s < \frac{1-\alpha}{2}$.

After obtaining the local and global well-posedness results in chapter 2, the natural question that arises is how much we can push the global well-posedness range. For example, the cubic periodic Schrödinger equation ($\alpha = 1$) in 1-d is locally well-posed in L^2, see [7], and with the mass conservation, we know that the equation is globally well-posed. That is, conservation laws on the local well-posedness level may give rise to global well-posedness. But then, one can ask whether we can show that the equation is globally well-posed whenever it is locally well-posed. Although when there is no conservation laws on the local well-posedness level, it is not trivial that the statement is true, we can still make sense of the question in a different way. The idea relies on the intuition that the set of 'bad' initial data, where the solutions of the equation with those initial data, may have arbitrarily large norm, should be negligible. This approach of looking at the problem in an 'almost sure' sense originated from the work of Lebowitz, Rose and Speer, [49]. They were trying to understand the general behavior of a system containing a large number of particles by looking at the values of the observables by taking averages over certain probability distributions containing only a few parameters like particle density, temperature, etc., instead of looking at the individual initial value problems. In classical or quantum mechanics, the Gibbs probability distribution for finding a system consisting of N particles in a compact spatial region Ω is a set of microscopic states dX_N is given by

$$\mu(dX_N) = Z_N^{-1} e^{-\beta H(X_N)} dX_N,$$

where H is the Hamiltonian, β is the reciprocal temperature and Z_N is the normalizing constant. Then one can take the limit as $N \to \infty$, $|\Omega| \to \infty$ and $N/|\Omega| \to C < \infty$, where $|\Omega|$ is the volume of Ω and obtain a well-defined measure on the resulting infinite dimensional phase space. With this in mind, one can construct appropriate Gibbs measures on Sobolev spaces and proved some basic properties of these measures.

Later, Bourgain. in [8], proved that the Schrödinger equation with power nonlinearity,

$$\begin{cases} iu_t - \Delta u = -|u|^{p-2}u, \quad x \in [0, 2\pi], \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s([0, 2\pi]), \end{cases} \tag{5.3}$$

where $4 < p \leq 6$ is locally well-posed in H^s with $s > 0$. But for $0 < s < 1$ there is no conservation law which would easily allow us to extend the local solutions to global ones. He used the idea of Lebowitz, Rose and Speer to construct a probability measure, also known as the Gibbs measure, on H^s for $s < \frac{1}{2}$ which is invariant under the solution flow. Then he showed that for any $\epsilon > 0$, there is global in time H^s norm bounds on the solutions with the initial data in H^s up to a set of measure less than ϵ, i.e. the equation is almost surely globally well-posed in H^s for $0 < s < \frac{1}{2}$.

The idea of the Gibbs measures and almost sure global well-posedness later have been used to prove similar results for different equations by [13], [14], [19], [27], [52], [56], [57], [58] and many others. Moreover the probabilistic methods have been applied successfully to other equations whether they are dispersive or not. In the latter category the Navier-Stokes equation serves as an example, see [21], [20], [32], [33], [53], for which local well-posedness cannot be proven via contraction arguments. This tells us that the probabilistic methods are not only useful to prove that the local solutions are global almost surely, but also useful to show that we can talk about local well-posedness in a probabilistic way, even when there is no deterministic local well-posedness.

Our main result the third chapter is the explicit construction of Gibbs measure for 1-d fractional periodic cubic Schrödinger equation and the proof of almost sure global well-posedness. More precisely, we define an invariant probability measure μ on H^s, for $s < \alpha - \frac{1}{2}$ such that for any $\epsilon > 0$ we can find a set $\Omega \subset H^s$ satisfying $\mu(\Omega^c) < \epsilon$ and the solution to the equation (3.1) exists globally for all initial data in Ω.

For that, we are going to truncate the equation (3.1), and use the idea of invariant measures on finite dimensional Hamiltonian systems. Namely, if we look at the equation,

$$\begin{cases} iu_t^N + (-\Delta)^\alpha u^N = \pm P_N |u^N|^2 u^N, \\ u^N(x,0) = P_N u_0(x) \end{cases} \tag{5.4}$$

where P_N is the projection operator onto the first N frequencies, we see that (5.4) is a finite dimensional Hamiltonian system, since

$$\frac{du}{dt} = -i \frac{\partial}{\partial \bar{u}} H,$$

with the Hamiltonian,

$$H_N(u)(t) = \frac{1}{2}\sum_{n\leq N} \big||n|^{\alpha} u_n(t)\big|^2 \mp \frac{1}{4}\int_{\mathbb{T}} |\sum_{n\leq N} e^{inx} u_n(t)|^4,$$

where $\bar{u} = (u_n)_{|n|\leq N}$ and u_n is the nth Fourier coefficient of u. Then, by Liouville's theorem, we know that the Lebesgue measure $\prod_{|n|\leq N} d\widehat{u_n}$ is invariant under the Hamiltonian flow. Thus, by the conservation of the Hamiltonian and the invariance of the Lebesgue measure under the flow, we see that the finite measure,

$$d\mu_N = e^{-H_N(u)} \prod_{|n|\leq N} du_n,$$

is invariant under the solution operator, call it $S(t)$.

We see that equation (4.1) is an infinite dimensional Hamiltonian system on the Fourier side with the Hamiltonian,

$$H(u(t)) = \frac{1}{2}\sum_{n} \big||n|^{\alpha} \widehat{u_n(t)}\big|^2 \mp \frac{1}{4}\int_{\mathbb{T}} |\sum_{n} e^{inx} \widehat{u_n(t)}|^4 = H(u_0),$$

Then we define the limiting measure μ on H^s as,

$$d\mu = e^{-H(u)} \prod_n d\widehat{u_n} = e^{-\frac{1}{2}\sum_n \left||n|^{\alpha}\widehat{u_n(t)}\right|^2 \pm \frac{1}{4}\int_{\mathbb{T}} |\sum_n e^{inx}\widehat{u_n(t)}|^4} \prod_n d\widehat{u_n},$$

and show that the measure μ is indeed the weak limit of μ_N.

To construct this measure μ on appropriate H^s spaces, we use the theory of Gaussian measures on Hilbert spaces following Zhidkov's arguments in [67], and first define,

$$dw = e^{-\frac{1}{2}\sum_n \left||n|^{\alpha}\widehat{u_n(t)}\right|^2} \prod_n d\widehat{u_n}.$$

Then we show that the measure μ is absolutely continuous with respect to the Gaussian measure w under certain conditions and finish the proof of almost sure global well-posedness by constructing the set $\Omega \subset H^s$ as stated above. For the second part we will mainly use Bourgain's arguments in [8].

5.3 Almost Sure Global Well-posedness

The main result of this chapter is,

Theorem 5.3.1. *For $\frac{1-\alpha}{2} < s < \alpha - \frac{1}{2}$ and $\epsilon > 0$, there exists an invariant probability measure μ on H^s*

such that the equation (3.1) *is globally well-posed for any initial data* $u_0 \in \Omega \subset H^s$ *such that* $\mu(\Omega^c) < \epsilon$ *with,*

$$\|u(t)\|_{H^s} \lesssim \Big(\log\big(\frac{1+|t|}{\epsilon}\big)\Big)^{s+}.$$

As we mentioned above, in the proof of this theorem, we first define the finite dimensional measures μ_N, which are invariant under the solution operator of the truncated equation (5.4), and we define μ as the weak limit of these measures. But then we have to show how the equation (4.1) and the truncated equation (5.4) are related, namely

Lemma 5.3.2. *Let* $A \in \mathbb{R}$ *and* $u_0 \in H^s$ *be such that* $\|u_0\|_{H^s} < A$, *and assume that the solution,* u_N, *of* (5.4) *satisfies* $\|u_N(t)\|_{H^s} < A$ *for* $t \le T$. *Then the equation* (4.1) *is well-posed in* $[0,T]$ *and moreover, for any* $\frac{1-\alpha}{2} < s' < s$, *we have,*

$$\|u(t) - u_N(t)\|_{H^{s'}} \le e^{C_1(1+A)^{C_2}T} N^{s'-s}, \tag{5.5}$$

where C_1 *and* C_2 *independent of* s.

Proof.

$$u(t) - u_N(t) = e^{-it(-\Delta)^\alpha}(u_0 - P_N u_0) + i\int_0^t e^{-i(t-\tau)(-\Delta)^\alpha}\big(|u|^2u(\tau) - P_N(|u^N|^2u^N)(\tau)\big)d\tau,$$

and, taking the $L^\infty([0,T];H^{s'})$ norms of both sides for $b > \frac{1}{2}$, since $X^{s',b} \subset L^\infty([0,T],H^{s'})$ for $b > \frac{1}{2}$, we get,

$$\begin{aligned}
\|u - u_N\|_{L^\infty([0,T],H^{s'})} &\le \|u_0 - P_N u_0\|_{H^{s'}} + \Big\|\int_0^t e^{-i(t-\tau)(-\Delta)^\alpha}\big(|u|^2u(\tau) - P_N(|u^N|^2u^N)(\tau)\big)d\tau\Big\|_{X^{s',b}} \\
&\le \|u_0 - P_N u_0\|_{H^{s'}} + (T_{LWP})^{1-b-b'}\big\||u|^2u - P_N|u^N|^2u^N\big\|_{X^{s',b'}} \\
&\le (T_{LWP})^{1-b-b'}\Big(\big\||u|^2u - P_N(|u|^2u)\big\|_{X^{s',b'}} + \big\|P_N\big(|u|^2u - |u^N|^2u^N\big)\big\|_{X^{s',b'}}\Big) \\
&\qquad + \|u_0 - u_{0,N}\|_{H^{s'}} \\
&\le I + II + III,
\end{aligned}$$

for $b' < \frac{1}{2}$ such that $b + b' < 1$.

The term III is easier to estimate,

$$III = \Big\|\sum_{|n|>N} e^{inx}\widehat{(u_0)}_n\Big\|_{H^{s'}} \le N^{s'-s}\|u_0\|_{H^s} \le N^{s'-s}A.$$

For the term I, we first observe that $P_N\big(|v|^2 v\big) = |v|^2 v$ for $v = P_{\frac{N}{3}} u$, from the convolution property of frequency restriction. Then we write,

$$\begin{aligned} I &\leq \big\||u|^2 u - P_N(|v|^2 v)\big\|_{X^{s',b'}} + \big\|P_N(|v|^2 v - |u|^2 u)\big\|_{X^{s',b'}} \\ &= \big\||u|^2 u - |v|^2 v\big\|_{X^{s',b'}} + \big\|P_N(|v|^2 v - |u|^2 u)\big\|_{X^{s',b'}} \\ &= I_1 + I_2 \leq 2I_1, \end{aligned}$$

Estimating term I_1 using $X^{s,b}$ estimates and local well-posedness theory, see Lemma 4.4.1 and Proposition 4.4.2 of the previous chapter, we see that,

$$\begin{aligned} I_1 &\lesssim (T_{LWP})^{1-b-b'}\big(\|u\|_{X^{s',b}} + \|v\|_{X^{s',b}}\big)^2 \|u - v\|_{X^{s',b}} \\ &\lesssim (T_{LWP})^{1-b-b'} A^2 \|u - P_{\frac{N}{3}} u\|_{X^{s',b}} \\ &\lesssim (T_{LWP})^{1-b-b'} A^2 \|u_0 - P_{\frac{N}{3}} u_0\|_{H^{s'}} \\ &\lesssim (T_{LWP})^{1-b-b'} A^3 N^{s'-s}. \end{aligned}$$

Thus we get,

$$I \lesssim (T_{LWP})^{1-b-b'} A^3 N^{s'-s}.$$

Similarly, for the second term we have,

$$II \lesssim (T_{LWP})^{1-b-b'}\big(\|u\|_{X^{s',b}} + \|u^N\|_{X^{s',b}}\big)^2 \|u - u^N\|_{X^{s',b}} \lesssim (T_{LWP})^{1-b-b'} A^2 \|u - u^N\|_{X^{s',b}},$$

and collecting all the terms, we get,

$$\begin{aligned} \|u - u_N\|_{X^{s',b}} &\leq CN^{s'-s} A + C_2 (T_{LWP})^{1-b-b'} A^2 \|u - u^N\|_{X^{s',b}} \\ &\qquad + C_1 (T_{LWP})^{1-b-b'} A^3 N^{s'-s}, \\ &\leq CAN^{s'-s} + \frac{1}{2}\|u - u^N\|_{X^{s',b}} \\ &\leq 2CAN^{s'-s}. \end{aligned}$$

for T_{LWP} small enough independent of N, s and s'. Repeating this argument, since the implicit constant C can be taken independent of T_{LWP} and N, we see that at any T_{LWP} time, because of the Banach Fixed

Point argument, the norm at most doubles and thus, at time T we get,

$$\|u - u_N\|_{H^{s'}} \lesssim 2^{\frac{T}{T_{LWP}}} CAN^{s'-s} \sim e^{C'(1+A)^{\delta}T} AN^{s'-s},$$

which gives the result. $\square$

Now, we define a probability measure on H^s using the Hamiltonian. For that we will mainly follow Zhidkov's arguments, see [67].

5.3.1 Construction of the Measure on H^s:

First we will fix the notation that we will use for the rest of the chapter. Let $F = (-\Delta)^{\alpha-s}$ on H^s. We see that F has the orthonormal eigenfunctions $e_n = e^{inx}/\langle n\rangle^s$ in H^s with the eigenvalues $|n|^{2\alpha-2s}$. We also denote $u_n = (u, e_n)_{H^s}$.

Definition 5.3.3. *A set $M \subset H^s$ is called cylindrical if there exists an integer $k \geq 1$ such that,*

$$M = \{u \in H^s : [u_{-k}, \ldots, u_{-2}, u_{-1}, u_1, u_2, \ldots, u_k] \in D\},$$

for a Borel set $D \subset \mathbb{R}^{2k}$.

We denote by $\mathcal{A}$, the algebra containing all such cylindrical sets. Then we define the additive normalized measure w on the algebra $\mathcal{A}$ as follows: For $M \subset \mathcal{A}$, cylindrical,

$$w(M) = (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_D e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n.$$

By the definition of the cylindrical sets, we see that for any $r > 0$ and $a \in H^s$, the closed ball $\overline{B_r(a)}$ can be written as $\overline{B_r(a)} = \bigcap_{n=0}^{\infty} M_n$, where M_n's are the cylindrical sets defined as,

$$M_n = \{x \in H^s : \sum_{|k|\leq n} (x-a)_n \leq r^2\},$$

belongs to $\mathcal{A}$. Thus since $\mathcal{A}$ contains arbitrary closed balls, the minimal $\sigma - algebra$ $\overline{\mathcal{A}}$ containing $\mathcal{A}$ is the Borel $\sigma - algebra$. Although the measure is additive by definition, it doesn't necessarily follow that it is countably additive. Indeed,

Theorem 5.3.4. *The Gaussian measure w is countably additive on $\mathcal{A}$ if and only if $\sum_n |n|^{2s-2\alpha} < \infty$, i.e. $s < \alpha - \frac{1}{2}$.*

Proof. (cf. [67]) Let $\sum_n |n|^{2s-2\alpha} < \infty$. Wel first show that for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset H^s$ with $w(M) < \epsilon$ for any cylindrical set M such that $M \cap K_\epsilon = \emptyset$.

Let $b_n = |n|^{\tilde{\epsilon}}$ such that $a = \sum_n |n|^{2s-2\alpha+\tilde{\epsilon}} < \infty$. Then for an arbitrary $R > 0$ take the cylindrical sets of the form,

$$M = \Big\{ u \in H^s : [u_{-k}, \ldots, u_{-2}, u_{-1}, u_1, \ldots, u_k] \in D, \quad \text{where} \ \sum_{|n|=1}^{k} |n|^{\tilde{\epsilon}} u_n^2 > R^2 \Big\}.$$

Then we see that,

$$\begin{aligned} w(M) &= (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_{\sum_{n=1}^{k} |n|^{\tilde{\epsilon}} u_n^2 > R^2} e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\ &\leq (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_{\mathbb{R}^n} \sum_{n=1}^{k} (\frac{|n|^{\tilde{\epsilon}}}{R^2} u_n^2) e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\ &\leq R^{-2} \sum_n |n|^{2s-2\alpha+\tilde{\epsilon}} \\ &= aR^{-2}, \end{aligned} \tag{5.6}$$

here, to pass to the third line we used integration by parts with $f = \frac{-u_n}{|n|^{2\alpha-2s}}$ and $dg = -|n|^{2\alpha-2s} u_n e^{-\frac{1}{2}|n|^{2\alpha-2s}u_n^2} du_n$. Then, for $R > \sqrt{\frac{a}{\epsilon}}$, we have $w(M) < \epsilon$.

Hence, if we take $K_\epsilon = \{u \in H^s : \sum_n |n|^{\tilde{\epsilon}} u_n^2 \leq R^2\}$, we get the desired compact set.

Now let $A_1 \supset A_2 \supset \ldots \supset A_m \supset \ldots$ be a sequence of cylindrical sets in H^s such that $\bigcap_{m=1}^{\infty} A_m = \emptyset$. Then for any $\epsilon > 0$ there exists closed cylindrical sets $C_m \subset A_m$ for all m such that $w(A_m/C_m) < \epsilon 2^{-m-2}$. Let $D_m = \bigcap_{k=1}^{m} C_k$. Then $w(A_m/D_m) \leq w(\bigcup_{k=1}^{m}(A_k/C_k)) < \epsilon/2$. Let $E_m = D_m \cap K_{\epsilon/2}$, then E_m's are compact with $E_m \subset A_m$ and $w(A_m/E_m) < \epsilon$. Since $\bigcap_m A_m = \emptyset$, $\bigcap_m E_m = \emptyset$, and since (E_m) is a nested sequence of compact sets, we see that $E_m = \emptyset$ for all $m > m_0$ for some $m_0 \in \mathbb{N}$.

Hence, $w(A_m) < w(E_m) + \epsilon < \epsilon$, for all $m > m_0$. Thus $w(A_m) \to 0$, i.e. w is countably additive.

For the converse, assume w is countably additive and also $\sum_n |n|^{2s-2\alpha} = \infty$, i.e. $s \geq \alpha - \frac{1}{2}$. Then consider two cases,

Case 1: $(s \leq \alpha)$**.** In this case we see that $|n|^{2s-2\alpha} \leq 1$ for any n. Consider the cylindrical sets of the form,

$$M_k = \Big\{ u \in H^s : \big| \sum_{|n|=1}^{k} u_n^2 - \lambda_k \big| < 2\sqrt{\lambda_k} \Big\},$$

where $\lambda_k = \sum_{|n|=1}^{k} |n|^{2s-2\alpha}$.

Then we have,

$$
\begin{aligned}
w(M_k^c) &= w\Big(\Big\{u \in H^s : \big|\sum_{|n|=1}^{k} (u_n^2) - \lambda_k\big| \geq 2\sqrt{\lambda_k}\Big\}\Big) \\
&\leq \int_{\mathbb{R}^{2n}} \frac{\big(\sum_{|n|=1}^{k}(u_n^2) - \lambda_k\big)^2}{4\lambda_k} e^{-\frac{1}{2}\sum_{|n|=1}^{k}|n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\
&= \frac{1}{4\lambda_k}\int_{\mathbb{R}^{2n}} \Big(\big(\sum_{|n|=1}^{k} u_n^2\big)^2 - 2\big(\sum_{|n|=1}^{k} u_n^2\big)\lambda_k + \lambda_k^2\Big) e^{-\frac{1}{2}\sum_{|n|=1}^{k}|n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\
&= \frac{1}{4\lambda_k}\Big(\big(\lambda_k^2 + 2\sum_{|n|=1}^{k} |n|^{4s-4\alpha}\big) - 2\lambda_k.\lambda_k + \lambda_k^2\Big) \\
&\leq \frac{1}{2}\frac{\sum_{|n|=1}^{k}|n|^{4s-4\alpha}}{\lambda_k} \\
&\leq \frac{1}{2},
\end{aligned}
$$

where, to pass from the third line to the fourth line we used integration by parts. Since $\lambda_k \to \infty$ as $k \to \infty$, there exist balls $B_{\lambda_k - 2\sqrt{\lambda_k}}(0)$ of arbitrarily large radii with $w(B_{\lambda_k-2\sqrt{\lambda_k}}(0)) \leq w(M_k^c) \leq \frac{1}{2}$, which contradicts with the countably additivity of w.

Case 2: $(s > \alpha)$**.** In this case, for each $n \geq 1$, consider the cylindrical set

$$M_k = \{u \in H^s : |u_i| \leq k, \quad |i| = 1, 2, ..., a_k\},$$

where $a_k > 0$ is an integer. Then by a change of variables, we have,

$$
\begin{aligned}
w(M_k) &= (2\pi)^{-a_k} \prod_{|n|=1}^{a_k} \Big(\int_{-k|n|^{\alpha-s}}^{k|n|^{\alpha-s}} e^{-\frac{1}{2}|u_n|^2} du_n\Big) \\
&\leq \Big[(2\pi)^{-1}\int_{-k}^{k} e^{-\frac{1}{2}|x|^2}dx\Big]^{a_k},
\end{aligned}
$$

since $s > \alpha$. By choosing a_k large enough, we can take $w(M_k) \leq 2^{-k-1}$ for each k and that $a_k \to \infty$ as $k \to \infty$. Then since $\bigcup_{k=1}^{\infty} M_k = H^s$ and $w(H^s) = 1$, since H^s is a cylindrical set with full measure. But then $w(\bigcup_{k=1}^{\infty} M_k) \leq \sum_{k=1}^{\infty} w(M_k) \leq \frac{1}{2}$, which is a contradiction. Hence the theorem follows. □

Since this construction gives us a Gaussian measure, we now have to check whether the measure is singular, i.e., we have to check whether it assigns positive measures to balls with positive radius.

Proposition 5.3.5. *For $s < \alpha - \frac{1}{2}$, $u \in H^s$ and $r > 0$ we have, $w(\overline{B_r(u)}) > 0$.*

Proof. [66]

We have seen that $\overline{B_r(u)} = \bigcap_{k=1}^{\infty} M_k$ where $M_k = \{x \in H^s : \sum_{|n|\leq k} |x-a|_n^2 \leq r^2\}$. Then from the construction above, it follows that $w(\overline{B_r(u)}) = \lim_{k\to\infty} w(M_k)$. Now, fix $k_0 > 0$ such that $\sum_{|n|=k_0+1}^{\infty} \lambda_n < \frac{r^2}{16}$ and $\sum_{|n|=k_0+1}^{\infty} a_n < \frac{r^2}{16}$. Then taking $k > k_0 + 1$ we get,

$$
\begin{aligned}
w(M_k) &= (2\pi)^{\frac{k}{2}} \prod_{|n|\leq k} \lambda_n^{-\frac{1}{2}} \int_{F_k} e^{-\frac{1}{2}\left(\sum\limits_{|n|\leq k} \lambda_n^{-1} x_n^2\right)} dx_{(-k)} \dots dx_k \\
&\geq C(2\pi)^{-\frac{k-k_0}{2}} \prod_{|n|=k_0+1}^{k} \lambda_n^{-\frac{1}{2}} \int_{F_k^1} e^{-\frac{1}{2}\left(\sum\limits_{|n|=k_0+1}^{n} \lambda_n^{-1} z_n^2\right)} dz_{(-k)} \dots dz_{(-k_0-1)} dz_{(k_0+1)} \dots dz_k,
\end{aligned}
$$

where C is independent of k,

$$
F_k = \{y = (y_{(-k)}, \dots, y_k) \in \mathbb{R}^{2k} : \sum_{|n|\leq k} |y_n - a_n|^2 \leq r^2\}
$$

and

$$
F_k^1 = \{y = (y_{(-k)}, \dots, y_{(-k_0-1)}, y_{k_0+1}, \dots, y_k) \in \mathbb{R}^{2(k-k_0)} : \sum_{k_0+1\leq|n|\leq k} |y_n - a_n|^2 \leq \frac{r^2}{4}\},
$$

which is true since

$$
\{y \in \mathbb{R}^{2k} : \sum_{|n|\leq k_0} |y_n - a_n|^2 \leq \frac{r^2}{4}\} \cap \{y \in \mathbb{R}^{2k} : \sum_{k_0+1\leq|n|\leq k} |y_n - a_n|^2 \leq \frac{r^2}{4}\} \subset F_k.
$$

Then, because of the choice of k_0, we have

$$
\{z = (z_{(-k)}, \dots, z_{(-k_0-1)}, z_{(k_0+1)}, \dots, z_k) : \sum_{k_0+1\leq|n|\leq k} z_n^2 \leq \frac{r^2}{16}\} \subset F_k^1.
$$

Using this, we can further bound $w(M_k)$ from below by,

$$
C(2\pi)^{-\frac{k-k_0}{2}} \prod_{|n|=k_0+1}^{k} \lambda_n^{-\frac{1}{2}} \int\limits_{\sum\limits_{k_0+1\leq|n|\leq k} z_n^2 \leq \frac{r^2}{16}} e^{-\frac{1}{2}\left(\sum\limits_{|n|=k_0+1}^{k} \lambda_n^{-1} z_n^2\right)} dz_{(-k)} \dots dz_{(-k_0-1)} dz_{(k_0+1)} \dots dz_k.
$$

Then a similar calculation to (5.6) shows that $w\big((M_k)\big) \gtrsim 1 - \frac{16}{r^2} \sum\limits_{|n|=k_0+1}^{\infty} \lambda_n \gtrsim 1$, which ends the proof.

$\square$

Now we define the sequence of finite dimensional measures (w_k) as follows: For any fixed $k \geq 1$, we take the $\sigma-algebra$, $\mathcal{A}_k$, of cylindrical sets in H^s of the form $M_k = \{u \in H^s : [u_{-k}, \dots, u_{-2}, u_{-1}, u_1, \dots, u_k] \in D\}$, for some Borel set $D \subset \mathbb{R}^{2k}$. Then,

$$w_k(M_k) = (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_D e^{-\frac{1}{2}\sum_{|n|=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n.$$

Hence we get the sequence of finite dimensional, countably additive measures w_k on the $\sigma - algebra$ $\mathcal{A}_k$. We can also extend these measures to the $\sigma - algebra$ $\overline{\mathcal{A}}$ in H^s, by setting,

$$w_k(A) = w_k(A \cap H_k^s), \quad \text{for } A \in \overline{\mathcal{A}},$$

where $H_k^s = span(e_{-k}, \ldots, e_{-1}, e_1, \ldots, e_k)$.

To justify this extension it is enough to show that $A \cap H_k^s$ is a Borel subset of H_k^s for $A \in \overline{\mathcal{A}}$. Assume it is not the case. Then, there exists $A \in \mathcal{A}$ such that $A \cap H_k^s \notin \mathcal{A}_k$. Then, $\mathcal{A}_k^1 = \{C \in H^s : c = A \cap H_k^s \text{for some} A \in \mathcal{A}\}$ is a $\sigma - algebra$ and $\mathcal{A}_k^1 \subset \mathcal{A}_k$, and $\mathcal{A}_k^1 \neq \mathcal{A}_k$ by definition. Now consider the see $\mathcal{A}^1$ of all Borel subsets A of H^s such that $A \cap H_k^s \in \mathcal{A}_k^1$. Then we see that $\mathcal{A}^1$ is a $\sigma - algebra$ in H^s such that $\mathcal{A}^1 \subsetneq \mathcal{A}$. Since $\mathcal{A}^1$ contains all open and closed subsets of H^s, it must contain the minimal $\sigma - algebra$ containing all the open subsets of H^s, i.e. it has to contain the Borel $\sigma - algebra$ $\mathcal{A}$, which is a contradiction. Thus $A \cap H_k^s$ is a Borel subset of H_k^s for $A \in \overline{\mathcal{A}}$ and we can extend the measures w_k.

The immediate question is whether or not the infinite dimensional Gaussian measure w and the finite measures w_k are related and the answer is,

Proposition 5.3.6. *The sequence w_k converge weakly to the measure w on H^s for $s < \alpha - \frac{1}{2}$ as $k \to \infty$.*

Proof. (cf. [67]) First, recall that a sequence of measures υ_m is said to converge to a measure υ weakly on H^s if and only if for any continuous bounded functional ϕ on H^s,

$$\int \phi(u) d\upsilon_m(u) \to \int \phi(u) d\upsilon(u).$$

Also recall that any $\epsilon > 0$, if we take $K_\epsilon \subset H^s$ as in the Theorem (5.3.4), we see that $w(K_\epsilon) > 1 - \epsilon$ and moreover, $w_m(K_\epsilon) > 1 - \epsilon$ for all $n \geq 1$. Now let ϕ be an arbitrary continuous bounded functional on H^s with $B = \sup_{u \in H^s} \phi(u)$. Then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|\phi(u) - \phi(v)| < \epsilon \quad \text{for any} \quad u \in K_\epsilon \text{ and } \; v \in H^s \text{ satisfying } \; \|u - v\|_{H^s} < \delta. \tag{5.7}$$

For any m, call $K_m = K_\epsilon \cap H^s_m$. Then by the definition of the measures w_m on $\overline{\mathcal{A}}$, we see that,

$$\Big| \int_{H^s} \phi(u) dw_m(u) - \int_{K_m} \phi(u) dw_m(u) \Big| < \epsilon B, \tag{5.8}$$

for any $m \geq 1$. Define,

$$K_{m,\epsilon} = \{ v \in H^s : v = v_1 + v_2, \quad v_1 \in H^s_m, \quad v_2^\perp \in H^s_m, \quad \|v_2\|_{H^s} < \frac{\delta}{2}, \quad dist(v_1, K_m) < \frac{\delta}{2} \}.$$

Then, $K_\epsilon \subset K_{m,\epsilon}$ for all sufficiently large m's. Thus, for m large enough,

$$\Big| \int_{H^s} \phi(u) dw_m(u) - \int_{K_{m,\epsilon}} \phi(u) dw_m(u) \Big| < \epsilon B. \tag{5.9}$$

We now define the measure $w_m^\perp$ on $(H^s_m)^\perp$ as follows:

For a cylindrical set

$$M^\perp = \{ u \in (H^s_m)^\perp : [u_{-m-k}, \ldots, u_{-m-2}, u_{-m-1}, u_{m+1}, u_{m+2}, \ldots, u_{m+k}] \in F \},$$

where $F \subset \mathbb{R}^{2k}$ is a Borel set, and,

$$w_m^\perp(M^\perp) = (2\pi)^{-k} \prod_{|n|=m+1}^{m+k} |n|^{\alpha - s} \int_F e^{-\frac{1}{2} \sum_{|n|=m+1}^{m+k} |n|^{2\alpha-2s} |u_n|^2} \prod_{|n|=m+1}^{m+k} du_n.$$

then we see that $w_m^\perp$ is a probability measure on $(H^s_m)^\perp$ and $w = w_m \otimes w_m^\perp$.

Thus we get,

$$\int_{K_{m,\epsilon}} \phi(u) dw(u) = \int_{u_m \in K_{m,\epsilon}} dw_m(u_m) \int_{u_m^\perp \in K_{m,\epsilon}^\perp(u_m)} \phi(u_m + u_m^\perp) dw_m^\perp(u_m^\perp), \tag{5.10}$$

where, $K^\perp_{m,\epsilon}(u_m) = K_{m,\epsilon} \cap \{ u \in H^s : u = u_m + y, y \in (H^s_m)^\perp \}$. Then by (5.7),

$$\begin{aligned} \int_{K_{m,\epsilon}} \phi(u) dw(u) &= \int_{u_m \in K_{m,\epsilon}} dw_m(u_m) \int_{u_m^\perp \in K^\perp_{m,\epsilon}(u_m)} \big(\phi(u_m + u_m^\perp) - \phi(u_m)\big) + \phi(u_m) dw_m^\perp(u_m^\perp) \\ &\leq C\epsilon + \int_{u_m \in K_{m,\epsilon}} \phi(u_m) dw_m(u_m), \end{aligned}$$

for C independent of m and ϵ.

Hence,

$$\int_{K_{m,\epsilon}} \phi(u)dw(u) - \int_{u_m \in K_{m,\epsilon}} \phi(u_m)dw_m(u_m) \leq C\epsilon. \tag{5.11}$$

Therefore, combining (5.8), (5.9) and (5.11), we get the result. $\square$

Now, we show that the measure μ is absolutely continuous with respect to the Gaussian measure w. Recall that,

$$\begin{aligned} d\mu_N &= (2\pi)^{-N} \prod_{|n|=1}^{N} |n|^{\alpha-s} e^{-\frac{1}{2}\sum_{|n|\leq N} \left||n|^{\alpha-s}u_n(t)\right|^2 - \frac{\gamma}{4}\int_{\mathbb{T}} |\sum_{|n|\leq N} \frac{e^{inx}}{\langle n\rangle^s} u_n(t)|^4} du_0 \prod_{1\leq|n|\leq N} du_n \\ &= e^{-\frac{\gamma}{4}\int_{\mathbb{T}} |\sum_{|n|\leq N} \frac{e^{inx}}{\langle n\rangle^s} u_n(t)|^4} (2\pi)^{-N} \prod_{|n|=1}^{N} |n|^{\alpha-s} e^{-\frac{1}{2}\sum_{0<|n|\leq N} |n|^{2\alpha-2s}|u_n(t)|^2} du_0 \prod_{1\leq|n|\leq N} du_n, \end{aligned}$$

and thus, μ_N is a weighted Gaussian measure.

For the defocusing NLS, since

$$|u_0|^2 \leq \int_{\mathbb{T}} |u|^2 \lesssim \Big(\int_{\mathbb{T}} |u(t)|^4\Big)^{\frac{1}{2}},$$

we have,

$$\int_{u_0\in\mathbb{C}} e^{-\frac{1}{4}\int_{\mathbb{T}} |\sum_n \frac{e^{inx}}{\langle n\rangle^s} u_n(t)|^4} du_0 \lesssim \int_{u_0\in\mathbb{C}} e^{-\frac{1}{4}|u_0|^4} du_0 \lesssim C,$$

uniformly in N. Thus, instead of working with the full measure μ_N it is enough to work with the measure w_N, which is also known as the Wiener measure.

For the focusing NLS, though, we don't have an a priori control over the weight $e^{\frac{1}{4}\int_{\mathbb{T}} |\sum_{n\leq N} e^{inx}\widehat{u_n(t)}|^4}$. We can overcome this problem by using a lemma of Lebovitz et al., see [49], which applies an L^2 cut-off to the set of initial data, namely,

Lemma 5.3.7. $e^{\frac{1}{4}\int |\sum_{1\leq|n|\leq N} e^{inx}\widehat{u_n(t)}|^4} \chi_{\{\|u\|_{L^2}\leq B\}} \in L^1(dw_N)$ *uniformly in N, for all $B < \infty$.*

Proof. (cf. [57, 55])

$$\begin{aligned} \int e^{\frac{1}{4}\int |\sum_{1\leq|n|\leq N} e^{inx}\widehat{u_n(t)}|^4} \chi_{\{\|u\|_{L^2}\leq B\}} dw &= \int_{(\int |\sum_{|n|=1}^{N} e^{inx}\widehat{u_n(t)}|^4 \leq K)} e^{\frac{1}{4}\int |\sum_{1\leq|n|\leq N} e^{inx}\widehat{u_n(t)}|^4} \chi_{\{\|u\|_{L^2}\leq B\}} dw \\ &\quad + \sum_{i=0}^{\infty} \int_{(\int |\sum_{|n|=1}^{N} e^{inx}\widehat{u_n(t)}|^4 \in (2^iK, 2^{i+1}K])} e^{\frac{1}{4}\int |\sum_{1\leq|n|\leq N} e^{inx}\widehat{u_n(t)}|^4} \chi_{\{\|u\|_{L^2}\leq B\}} dw \\ &\leq e^{\frac{1}{4}K^4} + \sum_{i=0}^{\infty} e^{\frac{1}{4}(2^{i+1}K)^4} w(\{(\int |\sum_{|n|=1}^{N} e^{inx}\widehat{u_n(t)}|^4 > 2^iK, \|u\|_{L^2} < B\}). \end{aligned}$$

Now to estimate the second term on the right hand side, choose N_0 dyadic, to be specified later. Now call $N_i = N_0.2^i$ and let a_i be such that $\sum_i a_i = \frac{1}{2}$.

Then,

$$w(\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\}) \leq \sum_{i=1}^{\infty} w(\{(\|P_{\{|n|\sim N_i\}}u\|_{L^4} > a_i K\}),$$

and since by Sobolev embedding we have,

$$\|P_{\{|n|\sim N_i\}}u\|_{L^4} \lesssim N_i^{\frac{1}{4}} \|P_{\{|n|\sim N_i\}}u\|_{L^2},$$

we see that,

$$\begin{aligned} w(\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\}) &\leq \sum_{i=1}^{\infty} w(\{\|P_{\{|n|\sim N_i\}}u\|_{L^4} > a_i K\}) \\ &\leq \sum_{i=1}^{\infty} w(\{\|P_{\{|n|\sim N_i\}}u\|_{L^2} \gtrsim a_i N_i^{-\frac{1}{4}} K\}). \end{aligned}$$

Letting $a_i = C N_0^{\epsilon} N_i^{-\epsilon}$ and N_0 such that $K \sim N_0^{\frac{1}{4}} B$, i.e.$N_0 \sim K^4 B^{-4}$, we get,

$$\begin{aligned} w(\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\}) &\leq \sum_{i=1}^{\infty} w\Big(\{(\sum_{|n|\sim N_i} |\widehat{u_n}|^2)^{\frac{1}{2}} \gtrsim a_i N_i^{-\frac{1}{4}} K\}\Big) \\ &\sim \sum_{i=1}^{\infty} w\Big(\{(\sum_{|n|\sim N_i} |u_n|^2)^{\frac{1}{2}} \gtrsim a_i N_i^{-\frac{1}{4}+s} K\}\Big), \end{aligned}$$

and by the estimation of the tail of the Gaussian measure, (cf. (5.14)), we have,

$$\begin{aligned} w(\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\}) &\lesssim \sum_{i=1}^{\infty} e^{-\frac{1}{4} a_i^2 N_i^{(2\alpha-2s)+2s-\frac{1}{2}} K^2} \\ &\leq \sum_{i=1}^{\infty} e^{-\frac{1}{4} N_0^{2\epsilon} N_i^{2\alpha-\frac{1}{2}-2\epsilon} K^2} \\ &\leq \sum_{i=1}^{\infty} e^{-\frac{1}{4} N_0^{2\alpha-\frac{1}{2}} 2^{(2\alpha-\frac{1}{2}-2\epsilon)i} K^2} \\ &\leq e^{-\frac{1}{4} K^2 N_0^{2\alpha-\frac{1}{2}}} \\ &\sim e^{-\frac{1}{4} K^{2+4(2\alpha-\frac{1}{2})} B^{2-4s}}. \end{aligned} \tag{5.12}$$

and collecting terms, we obtain,

$$\begin{aligned}\int e^{\frac{1}{4}\int |u|^4}\chi_{\{\|u\|_{L^2}\le B\}}dw &\le e^{\frac{1}{4}K^4}+\sum_{i=0}^{\infty} e^{\frac{1}{4}(2^{i+1}K)^4} w(\{(\|u\|_{L^4}>2^iK, \|u\|_{L^2}<B\})\\ &\le e^{\frac{1}{4}K^4}+\sum_{i=0}^{\infty} e^{\frac{1}{4}(2^{i+1}K)^4} e^{-\frac{1}{4}(2^iK)^{2+4(2\alpha-\frac{1}{2})}B^{2-8\alpha}}\\ &< \infty,\end{aligned}$$

since $\alpha > \frac{1}{2}$, which proves the lemma. $\square$

Moreover, observe that for $\|u\|_{L^2} < B$, we get $|u_0|^2 \le \sum_n \frac{|u_n|^2}{\langle n\rangle^{2s}} \le B^2$. Hence, L^2 cut off also restricts u_0 to the ball $\{u_0 \in \mathbb{C} : |u_0| \le B\}$, uniformly in N. Therefore, combining these two results, we get that the measure μ_N is a weighted Gaussian measure with weight being uniformly in L^1 with respect to the Gaussian measure.

By the construction of the Gaussian measure, we see that for any compact set $E \subset H^s$, we have,

$$w_N(E \cap H^s_N) \to w(E).$$

Thus, using the result above we get,

$$\mu_N(E \cap H^s_N) \to \mu(E).$$

Proof. (Proof of Theorem (5.3.1)) For the proof of the theorem and the invariance of the measure μ, we follow Bourgain's arguments in [8]. First, for any ϵ we will construct the sets $\Omega_N \subset H^s$ such that $\mu_N(\Omega_N^c) < \epsilon$ and,

$$\|u^N(t)\|_{H^s} \lesssim \Big(\log\big(\frac{1+|t|}{\epsilon}\big)\Big)^{\frac{1}{2}}. \tag{5.13}$$

For that, we fix a large time T and let $[-T_{LWP}, T_{LWP}]$ be the local well-posedness interval for the equation (3.1). Then consider the set

$$\Omega^K = \{u \in H^s_N : \|u\|_{H^s} \le K\},$$

where, again, $H^s_N = span\{e_n : |n| \le N\}$. We see that,

$$\begin{aligned} w_N((\Omega^K)^c) &= (2\pi)^{-\frac{N}{2}} \prod_{|n|=1}^{N} |n|^{\alpha-s} \int_{\{u\in H^s_N : \|u\|_{H^s}>K\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{N} du_n.\\ &= (2\pi)^{-\frac{N}{2}} \prod_{|n|=1}^{N} |n|^{\alpha-s} \int_{\{u\in H^s_N : \sum_{|n|\le N}|u_n|^2>K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{N} du_n.\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-\frac{N}{2}} \int\limits_{\{\sum_{|n|\le N} \frac{|v_n|^2}{\langle n\rangle^{2\alpha-2s}} > K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^{N} |v_n|^2} \prod_{|n|=1}^{N} dv_n. \\
&\le (2\pi)^{-\frac{N}{2}} \int\limits_{\{\sum_{|n|\le N} |v_n|^2 > K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^{N} |v_n|^2} \prod_{|n|=1}^{N} dv_n \\
&= (2\pi)^{-\frac{N}{2}} \int\limits_{S_{2N}} \int\limits_{K}^{\infty} r^{2N-1} e^{-\frac{1}{2}r^2} dr dS_{2N} \\
&= (2\pi)^{-\frac{N}{2}} \int\limits_{S_{2N}} \int\limits_{K}^{\infty} r \underbrace{r^{2N-2} e^{-\epsilon(r-\epsilon)-\frac{1}{2}\epsilon^2}}_{\le C} e^{-\frac{1}{2}(r-\epsilon)^2} dr dS_{2N} \\
&\lesssim (2\pi)^{-\frac{N}{2}} \int\limits_{S_{2N}} \int\limits_{K}^{\infty} r e^{-\frac{1}{2}(r-\epsilon)^2} dr dS_{2N} \\
&\lesssim (2\pi)^{-\frac{N}{2}} \int\limits_{S_{2N}} \int\limits_{K}^{\infty} (r-\epsilon) e^{-\frac{1}{2}(r-\epsilon)^2} dr dS_{2N} \\
&\le e^{-\frac{1}{2}(K-\epsilon)^2} \lesssim e^{-\frac{1}{4}K^2}, \qquad (5.14)
\end{aligned}
$$

for ϵ small enough. Thus, $\mu_N((\Omega^K)^c) \lesssim e^{-\frac{1}{4}K^2}$.

Since μ_N is invariant under the solution operator, S_N of the truncated equation, if we define the set,

$$
\Omega'_N = \Omega^K \cap S_N^{-1}(\Omega^K) \cap S_N^{-2}(\Omega^K) \cap \ldots \cap S_N^{-\frac{T}{T_{LWP}}}(\Omega^K),
$$

Ω'_N satisfies the property, $\mu_N((\Omega'_N)^c) \le \frac{T}{T_{LWP}} \mu_N((\Omega^K)^c) < TK^\theta e^{-\frac{1}{4}K^2}$, since the local well-posedness interval $[-T_{LWP}, T_{LWP}]$ depends polynomially on the H^s norm of the initial data because of the Lemma 4.4.1, Proposition 4.4.2 and the contraction argument given in the first chapter. Thus if we pick $K = \Big((4+2\theta)\log\big(\frac{T}{\epsilon}\big)\Big)^{\frac{1}{2}}$, for ϵ small we get,

$$
\mu_N((\Omega'_N)^c) < \epsilon,
$$

and by the construction of the set Ω'_N we have,

$$
\|u^N(t)\|_{H^s} \lesssim \Big(\log\big(\frac{T}{\epsilon}\big)\Big)^{\frac{1}{2}},
$$

for all $|t| < T$. Moreover, if we take $T_j = 2^j$ and $\epsilon_j = \frac{\epsilon}{2^{j+1}}$, and construct $\Omega_{N,j}$'s, we see that $\Omega_N = \bigcap_{j=1}^{\infty} \Omega_{N,j}$, satisfies (5.13).

Also by the approximation lemma (5.5), we see that for any $s' < s$ we have,

$$\|u(t)\|_{H^{s'}} < 2A \leq C_{s'}\Big(\log\big(\frac{T}{\epsilon}\big)\Big)^{\frac{1}{2}}.$$

Again by taking an increasing sequence of times, we get,

$$\|u(t)\|_{H^{s'}} \leq C_{s'}\Big(\log\big(\frac{1+|t|}{\epsilon}\big)\Big)^{\frac{1}{2}}. \tag{5.15}$$

Hence, if we intersect this result with an increasing sequence of $s < \alpha - \frac{1}{2}$, and taking $\Omega = \bigcap_N \Omega_N$ where (Ω_N)s are defined as above with $\mu_N(\Omega_N^c) < \frac{\epsilon}{2^N}$, we get that $\mu(\Omega) < \epsilon$ and that the solutions to the equation (3.1) has the norm growth bound,

$$\|u(t)\|_{H^{s}} \leq C_{s}\Big(\log\big(\frac{1+|t|}{\epsilon}\big)\Big)^{\frac{1}{2}},$$

for initial data $u_0 \in \Omega$. Moreover, interpolating this bound with,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2},$$

we have,

$$\|u(t)\|_{H^{s}} \leq C\Big(\log\big(\frac{1+|t|}{\epsilon}\big)\Big)^{s+},$$

which proves theorem (5.3.1). □

5.3.2 Invariance of μ Under the Solution Flow

Let K be a compact set and B_ϵ denote the ϵ ball in H^s. Let S be the flow map for the equation (3.1) and S_N be the flow map for the equation (5.4). Then by the weak convergence of the measure,

$$\mu(S(K) + B_\epsilon) = \lim_{N\to\infty} \mu_N\big((S(K)+B_\epsilon) \cap H_N^s\big).$$

Also by the uniform convergence of the solutions of (5.4) to (3.1) in H^{s_1} for any $s_1 < s$, we get,

$$S_N(P_N K) \subset S(K) + B_{\epsilon/2},$$

for $N \geq N_0$ sufficiently large. Then for ϵ_1 small enough,

$$S_N\big((K + B_{\epsilon_1}) \cap H_N^s\big) \subset S_N(P_N K) + B_{\epsilon/2} \subset S(K) + B_\epsilon.$$

Hence,

$$\mu_N\big(S_N\big((K + B_{\epsilon_1}) \cap H_N^s\big)\big) \leq \mu_N\big(S(K) + B_\epsilon\big),$$

and by the invariance of μ_N, we get,

$$\mu_N\big((K + B_{\epsilon_1}) \cap H_N^s\big) \leq \mu_N\big(S(K) + B_\epsilon\big),$$

and letting $N \to \infty$, by the convergence of the measures μ_N to μ,

$$\mu(K) \leq \mu(K + B_{\epsilon_1}) \leq \mu(S(K) + B_\epsilon),$$

which say, by the arbitrariness of ϵ, that $\mu(K) \leq \mu(S(K))$, and by the time reversibility, we also have the inverse inequality and, thus,

$$\mu(K) = \mu(S(K)),$$

which gives the invariance of μ under the solution operator.

This chapter is going to appear in Canadian Mathematical Bulletin, see [31].

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