DYNAMIC PRICING WITH REFERENCE PRICE EFFECTS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Industrial Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2015

Urbana, Illinois

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Abstract

This dissertation mainly focuses on the models and the corresponding dynamic pricing problems that incorporate reference price effects, a concept developed in economics and marketing literature that try to capture the dependency of consumers purchasing behavior on past prices.

Conceptually, reference price is a price expectation consumers develop from their observations of historical prices. Since it can not be physically observed, various models have been proposed to operationalize its formation. We empirically compare some of the models in the literature and extend the literature by proposing a new reference price model. In addition, we present analysis on the dynamic pricing problems under these models assuming consumers are loss/gain neutral or loss-averse. We find that constant pricing strategies are a robust solution to the problem regardless of which reference price models one may choose.

Empirical evidences, however, indicate that loss/gain neutral or loss-averse behavior may not be a universal phenomenon. We analyze the dynamic pricing problem when consumers exhibit gain-seeking behavior. In sharp contrast to the loss-averse case, even myopic pricing strategies can result in complicated cyclic price paths. We show for a special case that a cyclic skimming pricing strategy is optimal and provide conditions to guarantee the optimality of high-low pricing strategies.

With the understanding of the qualitative behavior of the optimal pricing strategies under various settings, we develop efficient algorithms to compute the optimal prices in both loss-averse and gain-seeking case. We demonstrate the efficiency and robustness of our algorithms by applying them to a practical problem with real data.

Finally, we extend the above considered single-product setting to multi-product setting and analyze the corresponding dynamic pricing problems.

To my parents, for their love and support.

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor Prof. Xin Chen. As a student, the financial support and tremendous guidance I received from Prof. Chen is what make this thesis and the completion of my study possible. His quick and sharp observations have always been a source of ideas and breakthroughs. His patience, on the other hand, enables me to step out of a specific problem and to explore the related areas in order to have a big picture and better understanding of the problem at hand. As a researcher, I believe the invaluable experiences of working so closely with Prof. Chen for the past four years has a profound and far-reaching influence on my future career. I get to know how a research idea is brought up and developed; I become to recognize the importance in an intuitive understanding of a certain result and I learn to keep my interests wide and open. I am eternally grateful for all the time and effort he has put in over the years to guide and teach me how to do research and how to enjoy research.

I would like to thank my thesis committee members Prof. Alex Olshevsky, Prof. Nicholas Petruzzi and Prof. Qiong Wang for their time, efforts and valuable suggestions. I would also like to thank Prof. Peng Hu for the discussions and his contributions to Chapter 3 and Chapter 4, and Yuhan Zhang for his input in Chapter 2.

During my study at the University of Illinois, I am fortunate to meet and discuss with many brilliant professors across the departments. I want to thank Prof. Enlu Zhou, Prof. Qiong Wang, Prof. Angelia Nedich and many others members of the IESE faculty for all their teachings over the years. I also want to thank Prof. Daniel Liberzon at the ECE department for the discussion in switched systems and Prof. Xiaofeng Shao at the Statistics department for his comments and suggestions on the empirical study in Chapter 3. Finally, I would like to thank Prof. Nicholas Petruzzi at the

Business Administration department for inviting me to his review team of an academic journal paper, which is a valuable experience.

I would also like to thank the staff members at the IESE, especially Holly Kizer, Barbara Bohlen, Amy Summers. Not only they have made the offices for PhD students much more pleasant places, their dedicated works also allow me to concentrate on my studies and research.

I am grateful to the countless discussions as well as the time spent with my colleagues as well as friends in my research group, including Xiangyu Gao, Shuanglong Wang, Limeng Pan and Wenbo Chen. I am also grateful to my other friends at the University of Illinois such as Kai Jin, Fei Ji, Xinyu Ma, Haoxiang Wang, Binyang Zhang, Jingnan Chen, Fan Ye, Tao Zhu, Hao Jiang, Helin Zhu and many others for making my PhD life much more enjoyable and fun.

Finally, I am indebted to my parents for their unconditional support and encouragement. I am also indebted to my girl friend Moying Li, for her care, tolerance and love.

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Chapter 1

Introduction

1.1 Motivations

Over the last two decades, dynamic pricing has attracted considerable attention from industry as well as academia. On the one hand, the scope of industries that adopt dynamic pricing strategies has widened remarkably, with examples ranging from airline, hotel industries whose use of dynamic pricing has long been a well established practice to many other industries such as retailing, manufacturing, cloud computing and energy, etc. In retail industry, for instance, new information technology has enabled the retailers to collect information about the sales and provided them the decision-support tools for analyzing the collected data. E-commerce retailers with virtually no cost in making price changes, in particular, have brought the practice of dynamic pricing strategies to a new level. It is reported that Amazon.com, a leading e-commerce retailer, can "adjust the prices of identical goods to correspond to a customer's willingness to pay (Weiss and Mehrotra, 2001)".

On the other hand, the on-going research in academia has led the practice of dynamic pricing to grow more sophisticated over the years (see, for example, Elmaghraby and Keskinocak, 2003; Chen and Simchi-Levi, 2012, for a review). Major efforts have been devoted to several issues. First, how to capture the relationship between demand and price accurately? A recent progress in this direction is the incorporation of consumers' behavior into consideration, such as strategic or bounded rational behavior of consumers. Second, in different business contexts, different pricing optimization models need to be developed. As an example, the stream of literature that incorporates network effects into the pricing optimization models strives to tackle various pricing problems associated with the emerging social networks such as Facebook. The third issue is the coordination of pricing decisions with other operations management decisions. A bulk of existing literature has been focusing on the coordination of pricing and inventory decisions. Finally, associated with various models and problems mentioned above is the computational issue. While the models with more realistic consumer behaviors or that incorporate other operations management decisions can bring up lots of new managerial insights, these models usually lead to very complex optimal solutions. Fortunately, the development of computational optimization techniques in mathematical programming has significantly improved the efficiency in finding optimal solutions or effective heuristics.

This thesis contributes to the existing literature in dynamic pricing through an extensive analysis of the reference price models and the corresponding dynamic pricing problems. Traditionally, it is usually assumed that the demand in a period only depends on the selling price of the current period. However, in a market with repeated purchases such as supermarkets, intertemporal changes in prices would have significant impacts on consumers' perception of the price and in turn influence consumers' purchasing decisions. The concept of *reference price*, developed in economics and marketing literature, is then used to capture such relationship between demand and past prices. It argues that consumers form price expectations and use them to judge the current selling price. That is, reference price is used as an internal anchor formed in consumers' minds as a result of experience based on information such as prices in observed periods (Kalyanaram and Little, 1994). Consumers then make their purchasing decisions based on the relative magnitude of the reference price and the selling price. Here, a purchasing instance is perceived by consumers as gains or losses based on whether the selling price is considered as discounts or surcharges relative to reference prices. Naturally, gains induce consumers to buy while losses deter them from purchasing.

Chapter 2 provides an overview of the common reference price models used in the literature. We empirically compare different reference price models, present existing results in the dynamic pricing problems and extend those results. We provide answers to several important and practical questions. First, can reference price help us to capture the relationship between demand and prices more accurately? Are such findings consistent over different reference price model specifications and robust to errors in estimations? Second, what are the differences among various reference price models, both in terms of empirical performance as well as managerial implications? Finally, what are the missing elements in the current reference price models and how will such elements change the pricing strategies of the firm?

In addition to reference price effects, behavioral asymmetry is another important consideration in modeling consumers' behavior. A common belief is that people are loss-averse (Tversky and Kahneman, 1991) and existing literature in dynamic pricing with reference price effects has been exclusively focusing on this side of asymmetry. However, the assumption of loss-aversion is not granted to hold in every pricing context. On the contrary, many evidences including our empirical studies in Chapter 2 points to the other side of asymmetry, the gain-seeking behavior. In Chapter 3, we argue that it is necessary and important for researchers as well as practitioners to realize the existence of gain-seeking behavior. We provide analytical as well as numerical results on the dynamic pricing problem that can guide the firm in choosing its pricing strategies if it faces gain-seeking demands.

In Chapter 4, we consider the computational issues in the dynamic pricing problems with reference price effects. On the one hand, reference price effects link all the intertemporal pricing decisions together. On the other hand, behavioral asymmetry creates non-smooth optimization problems. In order to apply reference price models in practice, where pricing decisions for thousands of products are made and coordinated with other operations management decisions, how to compute the optimal prices efficiently and accurately becomes a critical question. Facing with such dynamic non-smooth optimization problems, we develop a strongly polynomial time algorithm that solves the optimal prices exactly. We identify general structural properties in the problem that make such efficient algorithm possible and many of our techniques are potentially applicable to other dynamic non-smooth optimization problems as well. Our algorithm is shown to be very efficient when applied to a practical problem with real data.

Most of the studies on dynamic pricing problems with reference price effects in the literature (including our results in previous chapters) are in a single-product setting. While dynamic pricing problems with multiple products are inherently difficult due to curse of dimensionality, they are, nevertheless, practically relevant. Firms in retail industries, where most evidences on reference price effects are found, usually manage hundreds and thousands of products. In most cases, the demands for those products are interdependent through cross-price effects. Thus, it is crucial to understand whether the results in the single-product setting can be generalized to multi-product setting or not and if the answer is no, whether there is a simple heuristic to the problem. Chapter 5 pushes our understanding to the dynamic pricing problems with multiple products in this regard. We derive an explicit solution to the multi-product problem and provide a confirmative answer to the question of generalizing the results in the single-product setting for the case where no behavioral asymmetry is present. When behavioral asymmetry is considered, we provide a simple heuristic and prove several desirable properties of the heuristic.

1.2 Organization of the thesis

Chapter 2 reviews and extends the existing results by comparing the empirical performance as well as managerial implications of different reference price models and proposing new models to overcome the limitations in the existing models. Section 2.1 gives a review of the empirical studies on reference price models in the literature. Section 2.2 introduces the three memory-based reference price models used in the literature and the demand model to be used throughout this thesis. In Section 2.3, we give an empirical comparison of the three memory-based reference price models using real data from canned tuna category. Section 2.4-2.6 present and compare the results in the dynamic pricing problems under the three reference price models. A new reference price model is proposed and analyzed in Section 2.7.

Chapter 3 examines the dynamic pricing problem when the demand is gain-seeking. The evidences in the gain-seeking behavior and the relevant literature is presented in Section 3.1. Section 3.2 briefly reminds the readers of the mathematical formulation of the dynamic pricing problem. The dynamics of the myopic pricing strategy is analyzed in Section 3.3 and that of the optimal pricing strategy is analyzed in Section 3.4. Section 3.5 gives an empirical study to examine the assumptions we made and numerically examines the performance of simple pricing strategies.

Chapter 4 looks into the computational issue of the dynamic pricing problems with reference price effects. Section 4.2 gives the model for the finite horizon problem. A strongly polynomial time algorithm is presented for the loss-averse demands in Section 4.3. The efficiency and robustness of the algorithm is examined through solving a practical industry problem in Section 4.4.

Chapter 5 considers the dynamic pricing problem with multiple products. Section 5.1 reviews the models used in the literature for the multi-product setting and Section 5.2 introduces the model we use in a continuous time framework. In Section 5.3, we analyze the dynamic pricing problem with multiple products and give an explicit solution for the loss/gain-neutral case. For the loss-averse case, we construct a heuristic and prove some of the nice properties of the heuristic with two products.

Finally, the last chapter concludes this thesis by summarizing the directions for future research.

Chapter 2

Reference Price Models: Empirical Comparisons and Dynamic Pricing Problems

2.1 Introduction

The concept of reference price, as introduced in Chapter 1, postulates that consumers form price expectations and use them to judge the current selling price. However, reference prices cannot be physically observed and consequently a large amount of literature is devoted to modeling the formation of reference prices and investigating the impact of reference prices on consumers' purchasing behavior. This stream of literature can be divided into two categories based on whether the study conducted is at an individual level or an aggregate level. Specifically, empirical studies based on individual level can differ in terms of models, types of data and potentially implications from the studies based on aggregate level. In most studies at the individual level, a brand choice model in multi-product setting (usually a multinomial logit model) is employed. That is, consumers' utility function is dependent on reference price and the utility will then determine purchase probability and demands. In comparison, studies at the aggregate level demand usually impose a specific demand-price function form. Correspondingly, the data used at the individual level is scanner panel data which tracks the purchasing behavior of each household while the the data used at the aggregate level is simply time series data containing price and demand pairs. Although at both levels, there are abundant evidences of reference price effects, further implications on the model parameters can be different at the two levels. Readers are referred to Chapter 3 for more details on the differences.

Although the focus of this thesis is not on individual level models, we give a brief discussion of the reference price models in the literature at the individual level here for completeness. Literature in this area has differed considerably in how reference prices are formed and many alternative models have been proposed and compared. There are two major different perspectives in viewing the formation of reference price. One is called the *memory-based* or *temporal* reference price, which is assumed by a majority of researchers. They argue that reference price is based on the historical prices consumers encountered during past purchasing occasions and have operationalized reference price as a weighted average of past encountered prices. For example, Mayhew and Winer (1992) and Krishnamurthi et al. (1992) have simply taken the price encountered in the last purchase occasion as the reference price. On the other hand, Lattin and Bucklin (1989) and Kalyanaram and Little (1994) use an exponentially smoothed prices which depend on consumer's whole purchasing history (we refer to as the *exponentially smoothing model*). The other one is called the *stimulus-based* or *contextual* reference price. It assumes that the *current prices* of certain brands serve as the reference price. The underlying argument is that consumers may not have a strong memory for past prices and the information provided by current prices of the brands available in the store is most salient and convenient for consumers. For instance, Hardie et al. (1993) use the current price of the brand that consumers have purchased in latest occasion as the reference price while Mazumdar and Papatla (1995) average the current prices of all brands by the weight based on the loyalties to the respective brands. Briesch et al. (1997) provide comprehensive review of the above mentioned models and empirically compare them using scanner panel data for peanut butter, liquid detergent, ground coffee and tissue. They find that in four categories the memory-based reference price model performs the best. Rajendran and Tellis (1994) postulate that a combination of both memory-based and stimulus-based reference price model may be more realistic and provide empirical evidences to support their premise.

Empirical studies at the aggregate level are relatively scarce, even though most analytical analysis in the literature is based on aggregate level models due to their simplicity. As pointed out above, aggregate level models focus more on single-product settings and even if they are generalized to multi-product settings, the brand choice behavior is not modeled explicitly. As a result, the literature at the aggregate level usually assumes a memory-based reference price model. Within the context of memory-based reference price, various operationalizations have been proposed. Similar to the individual level model, Raman and Bass (2002) use the price from the previous period as the reference price, and Greenleaf (1995) and Kopalle et al. (1996) employ the more general exponentially smoothing model. More recently, different memory-based models have been explored. Nasiry and Popescu (2011) propose a reference price model based on the peak-end rule (we refer to as the *peak-end model*), which suggests that consumers remember the lowest (peak) and most recent prices (end). Such a model is conceptually supported by substantial research in psychology, however, "an empirical investigation of the peak-end rule in the pricing context is still lacking (Nasiry and Popescu, 2011)". Nevertheless, they analyze the dynamic pricing problem under the peak-end model and show that a constant pricing strategy is optimal with loss-averse consumers. Interestingly, to the best of our knowledge, neither the exponentially smoothing model nor the peak-end model is implemented in practice. Instead, it is reported in Natter et al. (2007) that bauMax, an Austrian do-it-yourself retailer, implements a further generalization of the exponentially smoothing model in their decision-support system (we refer to as the *adaptation-rate-based model*). In their model, consumers adapt their reference prices faster to price decreases than to price increases. They argue that quicker adaptation in case of price decreases is due to the fact that retailers tend to aggressively advertise price reductions while price increases may well go unnoticed by consumers. Unfortunately, no empirical comparisons to the exponential smoothing model are made in their report. In addition, since bauMax uses a price grid, they can simply evaluate all different strategies in their dynamic pricing problem and no analytical characterizations of the optimal pricing strategy are provided in the report.

This chapter then serves for the following three purposes. First, we complement the above mentioned literature at the aggregate level by providing an empirical comparison of different memory-based reference price models. In addition, we analyze the dynamic pricing problem with the adaptation-rate-based model and compare it with the existing results in the exponential smoothing model as well as the peak-end model. Second, we restate the established results in the literature on dynamic pricing problem for the benchmark model: the exponential smoothing model with loss-averse demands, which not only lays down some fundamental ideas that will be used in the thesis but also gives a nice comparison to later results in Chapter 3. Finally, we point out some of the limitations of currently available reference price models and we propose by following Zhang (2011) a reference price model that incorporates randomness to address these limitations. While Zhang (2011) derives an explicit solution for the dynamic pricing problem under the stochastic reference price model, we extend his results by providing an analysis of the limiting distribution of the steady state as well as a sensitivity analysis of the expected steady state.

The remainder of this chapter is organized as follows. In Section 2.2 we present the mathematical formulation of each of the memory-based reference price models discussed above as well as the aggregate level demand model. In Section 2.3, we empirically compare the performance of the exponential smoothing model, the peak-end model and the adaptation-rate-based model using the real data on the canned tuna category. Section 2.4 presents the established results in the literature on the dynamic pricing problem under the exponential smoothing model, which serves as our benchmark model. For completeness, the results in Nasiry and Popescu (2011) for dynamic pricing problem under the peak-end model is presented in Section 2.5. The dynamic pricing problem under the adaptation-rate-based model is analyzed in Section 2.6. We introduce the stochastic reference price model and analyze the corresponding dynamic pricing problem in Section 2.7. The last section summarizes our findings and points out directions for future research. Proofs for the results in Section 2.7 are quite lengthy and are relegated to Appendix A. We remark here that, throughout this chapter, we consider either loss/gain-neutral or loss-averse demands and readers are referred to Chapter 3 for dynamic pricing problems with gain-seeking demands.

2.2 Reference Price and Demand Models

This section introduces the three memory-based reference price models and the demand model that describes how reference price effect affects the demand for a firm's product. In the following, we use r_t to denote consumers' reference price at period t and p_t to denote the shelf price of the product at period t.

2.2.1 Exponential smoothing model

As introduced in Section 2.1, the exponential smoothing model is widely used by researchers at both individual (Lattin and Bucklin, 1989; Kalyanaram and Little, 1994) as well as aggregate level (Greenleaf, 1995). It assumes that the reference price in period $t+1$ is a weighted average of the reference price in period t and the shelf price consumers observed in period t. Mathematically, the reference price evolves according to the following recursive formula

$$r_{t+1} = \alpha r_t + (1-\alpha)p_t, \tag{2.1}$$

where $\alpha \in [0,1]$ is called the memory factor, since it captures the rate at which consumers adapt to the new price information. In the extreme case, when $\alpha = 0$ consumers immediately take the price they observed in the last purchase occasion as the reference price (Raman and Bass, 2002), while when $\alpha = 1$ consumers never adapt to the new price information. Usually, one would restrict $\alpha < 1$, because in empirical studies $\alpha = 1$ would result in pathological estimation (see Section 2.3) and in the analysis of dynamic pricing problem it would result in a static price (see Section 2.4).

2.2.2 Peak-end model

The peak-end rule postulates that memory-based decisions are made according to a combination of the most extreme and most recent experiences. By adopting this rule in the pricing context, Nasiry and Popescu (2011) assume that the peak-end experiences to be the lowest and latest price respectively. Specifically, let m_t denote the lowest price consumers observed up to period t, then the reference price follows

$$r_{t+1} = \beta m_t + (1-\beta)p_t, \tag{2.2}$$

where $\beta \in [0,1]$ measures how much consumers anchor on the lowest price. Again, in the extreme case when $\beta = 0$, reference price is simply the price observed in the previous period. Note here that

$$m_{t+1} = \min\{m_t, p_{t+1}\}$$

and together with (2.2), they specify the evolution of a reference price path.

2.2.3 Adaptation-rate-based model

The adaptation-rate-based model generalizes the exponential-smoothing model in that it allows different adaptation rates based on whether consumers experience a price reduction or price increase:

$$r_{t+1} = p_t + \alpha^+ \max\{r_t - p_t, 0\} + \alpha^- \min\{r_t - p_t, 0\}, \tag{2.3}$$

where $0 \leq \alpha^+ \leq \alpha^- \leq 1$ captures the rate at which consumers incorporate the new price information depending on whether they experience a price decrease or increase. To see this more clearly, we can write (2.3) explicitly as

$$r_{t+1} = \begin{cases} \alpha^+ r_t + (1-\alpha^+)p_t, & r_t \geq p_t, \\ \alpha^- r_t + (1-\alpha^-)p_t, & r_t \leq p_t. \end{cases}$$

The assumption $\alpha^+ \leq \alpha^-$ then implies consumers adapt to the new price information p_t at a faster rate when there is a price decrease than when there is a price increase. In the special case when $\alpha^+ = \alpha^- := \alpha$, the above model reduces to the exponential smoothing model introduced in (2.1).

2.2.4 Demand model

Here, we formally introduce the aggregate-level demand model that will be used throughout this thesis. Following Greenleaf (1995), Kopalle and Winer (1996), Fibich et al. (2003) and Nasiry and Popescu (2011), the demand depends on the price p and reference price r via the model

$$D(r,p) = \begin{cases} b - ap + \eta^+(r-p), & r > p, \\ b - ap, & r = p, \\ b - ap + \eta^-(r-p), & r < p, \end{cases} \tag{2.4}$$

where $b, a > 0$ and $\eta^+, \eta^- \geq 0$. More concisely, we can write

$$D(r,p) = b - ap + \eta^+ \max\{r-p, 0\} + \eta^- \min\{r-p, 0\}.$$

Here, $D(p,p) = b - ap$ is the base demand independent of reference prices, $\eta^+(r-p)$ or $\eta^-(r-p)$ is the additional demand or demand loss induced by the reference price effect, where $r-p$ is usually referred to as a perceived surcharge/discount. If $r < p$, consumers perceive this as a loss, while if $r > p$, they perceive it as a gain. Consumers or the aggregate level demands are classified as loss-averse, loss/gain neutral and gain-seeking depending on whether $\eta^+ < \eta^-$, $\eta^+ = \eta^-$ or $\eta^+ > \eta^-$. The dynamic pricing problems analyzed in this chapter all focus on either the loss-averse or loss/gain neutral case. The gain-seeking case is studied in detail in Chapter 3.

The linear form of the demand function is for the purpose of simplifying the exposition. Most of the results introduced in this thesis can be generalized to nonlinear forms by imposing appropriate assumptions on demand. Furthermore, the linear form allows us to conveniently estimate the corresponding parameters from real data on historical prices and sales using linear regression (see Section 2.3).

2.3 Model Comparison

This section complements the literature by empirically comparing the three memory-based reference price models based on the data in canned tuna category. We conduct a detailed analysis using one brand in the data set and present a summary of the performances of the models for all the brands in the data set.

2.3.1 Data and method of estimation

We utilize the data set provided by Chevalier et al. (2003) of the canned tuna product category in the Bayesm Package of the R software. The data set includes volume of canned tuna sales as well as a measure of display activity, log price and log wholesale price of seven different brands over 338 weeks.

The data set is extracted and aggregated from the Dominick's Finer Foods database maintained by the University of Chicago Booth School of Business at `http://research.chicagobooth.edu/marketing/databases/dominicks/index.aspx`. The original database records comprehensively the weekly store-level data of each product sold by Dominick's Finer Foods, a large supermarket chain in the Chicago area.

Note here that if the initial reference price and α, β, α^+/α^- in (2.1), (2.2), (2.3) respectively are known, then reference prices can be generated from the historical prices and the parameters b, a, η^+, η^- in the demand function (2.4) can be estimated using ordinary least squares (OLS). However, α, β, α^+/α^- are usually unknown and need to be estimated from the data. There is no established explicit formula for computing these parameters. Here, we follow the simple approach employed by Greenleaf (1995). For the exponential smoothing model, for instance, OLS is performed repeatedly for each value of α varying in increments of 0.025 from 0 to 1 (excluding 1) and our estimator $\hat{\alpha}$ is chosen to be the one that maximizes R^2. We exclude $\alpha = 1$ here because it will result in perfect collinearity problem and OLS cannot be applied. $\hat{\beta}$, $\hat{\alpha}^+$ and $\hat{\alpha}^-$ are computed similarly. For initial reference price, we set it to be the average price for convenience throughout this section.

Table 2.1 shows the descriptive statistics of the sales, prices and reference prices computed under each of the reference price models for one item in the data set called "Chicken of the Sea 6 oz". The corresponding estimates of α, β and α^+/α^- are included in the parenthesis. One interesting observation

Table 2.1: Descriptive Statistics (Chicken of the Sea 6 oz)

	mean	median	st.dev.	min	max
unit sales	16104	6633	49633	2525	579037
retail price ($/oz)	0.80	0.82	0.09	0.29	0.92
reference price ($\hat{\alpha} = 0.325$)	0.80	0.81	0.07	0.48	0.91
reference price ($\hat{\beta} = 0.3$)	0.67	0.67	0.09	0.29	0.88
reference price ($\hat{\alpha}^+ = 0.15$, $\hat{\alpha}^- = 0.975$)	0.64	0.67	0.10	0.34	0.88

from Table 2.1 is that reference prices generated under peak-end model and adaptation-rate-based model are much lower than that under exponential smoothing model. The underlying intuition is that under peak-end model the lowest price can be remembered indefinitely which drags the reference price low while under adaptation-rate-based model, as argued in Section 2.2, consumers adapt more quickly to low prices.

2.3.2 Comparison results

We first present the regression results for the item "Chicken of the Sea 6 oz" and discuss some of the issues in estimation and comparison. As noted by Greenleaf (1995), the linear demand model (2.4) suffers from a multicollinearity problem. As a result, the standard errors can be huge for some parameters and the OLS estimate to η^- has a wrong sign (see, for example, the second column in Table 2.2). To tackle this problem, Greenleaf (1995) uses equity estimator and ridge estimator and obtains the correct signs. Similar regularization technique is used in the case study in Chapter 4 to another data set. However, for the canned tuna category, though ridge regression can reduce the standard errors, the estimator to η^- still has a wrong sign. Instead, we look at the restricted model by imposing $\eta^- = 0$ and we report the regression results for the full model specified by (2.4), the static model which ignores reference price effects ($\eta^+ = \eta^- = 0$) and the restricted model ($\eta^- = 0$).

Table 2.2-2.4 summarize the regression results for the three memory-based reference price models respectively. Two observations that are consistent in all three models can be made here. First, incorporating reference price effects significantly improves the goodness of fit measured by R^2 across all three models. The exponential smoothing model improves R^2 by 90%, the peak-end model by 138% and the adaptation-rate-based model by 158% respectively. Second, we find the demand for this item to be gain-seeking ($\eta^+ > \eta^-$) regardless of which memory-based reference price model one chooses. In addition, the estimate of η^- is statistically insignificant in all three models and the more parsimonious restricted demand model performs as good as the full demand model.

Comparing across the three tables, it is clear that the adaptation-rate-based model performs the best in terms of R^2 or adjusted R^2 for both the full demand model and the restricted demand model. It outperforms the exponential smoothing model by 36%. However, it only outperforms the peak-end model by 8%. Thus, one needs to be cautious when directly comparing the three models in terms of R^2 since the adaptation-rate-based model has one more degree of freedom than the exponential-smoothing model or the peak-end model. In other words, besides the demand parameters, the adaption-rate-based model has two additional parameters (α^+ and α^-) to be estimated instead of one (either α or β). Therefore, it is not surprising to see a better performance from the adaptation-rate-based model. Comparatively, it is interesting to see that the peak-end model, having the same degree of freedom as the exponential smoothing model, can still outperform it by 25%.

Table 2.2: Exponential Smoothing Model

$\hat{\alpha} = 0.325$	Full Model	Static Model	Restricted Model
Intercept ($\hat{b}$)	21248	257970	14160
	(0.90)	(12.71)**	(0.61)
Price ($\hat{a}$)	27521	302613	16720
	(0.96)	(11.99)**	(0.60)
Perceived discount ($\hat{\eta}^+$)	572630		573060
	(14.40)**		(14.38)**
Perceived surcharge ($\hat{\eta}^-$)	-57630		
	(-1.64)		
R^2	0.570	0.300	0.567
Adjusted R^2	0.566	0.297	0.564

t-statistics are in parentheses.
** significant at 1%, * significant at 5%

Table 2.3: Peak-End Model

$\hat{\beta} = 0.30$	Full Model	Static Model	Restricted Model
Intercept ($\hat{b}$)	85423	257970	81488
	(5.42)**	(12.71)**	(5.34)**
Price ($\hat{a}$)	98498	302613	90162
	(4.78)**	(11.99)**	(4.79)**
Perceived discount ($\hat{\eta}^+$)	920533		917964
	(22.06)**		(22.04)**
Perceived surcharge ($\hat{\eta}^-$)	-20297		
	(-0.99)		
R^2	0.715	0.300	0.714
Adjusted R^2	0.712	0.297	0.712

t-statistics are in parentheses.
** significant at 1%, * significant at 5%

Actually, the issue of degree of freedom arises when reference price models are compared to the static demand model where adjusted R^2 does not penalize on the addition of parameters α, β or α^+, α^- in reference price models. Here, we check the robustness of our model comparisons by performing a worst case analysis. That is, for each memory-based reference price model, we choose the parameters α, β or α^+, α^- that minimizes R^2 instead of

Table 2.4: Adaptation-Rate-Based Model

$\hat{\alpha}^+ = 0.15, \hat{\alpha}^- = 0.975$	Full Model	Static Model	Restricted Model
Intercept ($\hat{b}$)	61669	257970	64732
	(4.32)**	(12.71)**	(4.76)**
Price ($\hat{a}$)	61997	302613	67949
	(3.29)**	(11.99)**	(4.05)**
Perceived discount ($\hat{\eta}^+$)	1191648		1190498
	(26.67)**		(26.68)**
Perceived surcharge ($\hat{\eta}^-$)	10498		
	(0.70)		
R^2	0.776	0.300	0.776
Adjusted R^2	0.774	0.297	0.774

t-statistics are in parentheses.
** significant at 1%, * significant at 5%

maximizing it. The results are summarized in Table 2.5.

Table 2.5: Check of Robustness (Worst Case Analysis)

	Static Model	ES	PE	ARB
$\alpha/\beta/(\alpha^+, \alpha^-)$		0.975	1	(0.975,0.975)
R^2	0.300	0.486	0.502	0.486
Adjusted R^2	0.297	0.481	0.498	0.481

ES: exponential smoothing model
PE: peak-end model
ARB: adaptation-rate-based model

We first observe that even under the worst case, all memory-based reference price models still outperform the static demand model that ignores reference price effects. This finding is consistent with the extensive literature on reference price effects (for example Greenleaf, 1995) and further confirms the robustness of reference price effects with respect to different models and possible errors in estimations. However, under the worst case analysis, the adaptation-rate-based model no longer outperforms the peak-end model. We remark here that we have restricted $\alpha^+ \leq \alpha^-$ when computing the worst case for the adaptation-rate-based model. Otherwise, it will be even worse than the exponential smoothing model (with R^2 merely 0.34). It is yet interesting to note that within the constraint $\alpha^+ \leq \alpha^-$, the worst case is attained at $\alpha^+ = \alpha^-$. That is, allowing consumers to adapt faster to price decreases will always improve the model, which supports the intuition provided in Natter et al. (2007).

Finally, we summarize the performance comparisons for the three memory-

Table 2.6: Performance Comparisons for All Brands

	Static Model	ES	PE	ARB
Star Kist 6 oz	0.190	0.359	0.449	0.360
Chicken of the Sea 6 oz	0.300	0.570	0.715	0.776
Bumble Bee Solid 6.12 oz	0.406	0.490	0.514	0.519
Bumble Bee Chunk 6.12 oz	0.259	0.640	0.664	0.650
Geisha 6 oz	0.513	0.545	0.545	0.550

ES: exponential smoothing model
PE: peak-end model
ARB: adaptation-rate-based model

based reference price models for all the brands in the data set in Table 2.6. We have excluded the two brands with large volume size: "Bumble Bee Large Cans" and "HH Chunk Lite 6.5 oz" because the estimate for a, the price sensitivity, has a wrong sign. One can see that generally, the peak-end and the adaptation-rate-based models perform better than the exponential smoothing model but the degree of improvements differ case by case. For the last three brands, the three models perform roughly the same while the peak-end model has quite an improvement over the exponential smoothing model for the first two brands. The adaptation-rate-based model, despite of the increased degree of freedom, only has significant improvement in the brand "Chicken of the Sea 6 oz".

2.4 Dynamic Pricing under the Exponential Smoothing Model

In this section, we present the results on the dynamic pricing problem under the benchmark model: the exponential smoothing model with loss-averse demands. This problem has been analyzed by many researchers including Kopalle et al. (1996), Fibich et al. (2003), Popescu and Wu (2007) and Asvanunt (2007). In addition to stating existing results, we offer a new perspective in proving the steady state results in Popescu and Wu (2007). We utilize the tools from discrete dynamic system to provide a simple visualization of convergence and such tools enable a clearer comparison to the results to be developed in Chapter 3.

Given an initial reference price r_0, we define the firm's dynamic pricing problem under the exponential smoothing model as follows:

$$\begin{aligned} V(r_0) = & \max_{p_t \in [0,U]} \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t) \\ \text{s.t.} \quad & r_{t+1} = \alpha r_t + (1-\alpha) p_t, \quad t \geq 0, \end{aligned} \tag{2.5}$$

where $\Pi(r_t, p_t) = p_t D(r_t, p_t)$ is the one-period profit function and $D(\cdot,\cdot)$ is defined in (2.4). Here, we have implicitly assumed that the marginal cost is 0 without loss of generality. We also assume for the remaining of this chapter that $\eta^- \geq \eta^+$. Note that the assumptions $\eta^- \geq \eta^+$ and $p \geq 0$ allow us to rewrite the one-period profit as

$$\Pi(r,p) = \min\{\Pi^+(r,p), \Pi^-(r,p)\},$$

where

$$\Pi^+(r,p) \;=\; p[b - ap + \eta^+(r-p)], \tag{2.6a}$$

$$\Pi^-(r,p) \;=\; p[b - ap + \eta^-(r-p)]. \tag{2.6b}$$

The Bellman equation to problem (2.5) can then be written as

$$V(r) = \max_{p \in [0,U]} \{\min\{\Pi^+(r,p), \Pi^-(r,p)\} + \gamma V(\alpha r + (1-\alpha)p)\}, \tag{2.7}$$

and we use $p^*(r)$ to denote the solution to (2.7). To solve (2.7), we introduce the following two problems:

$$V^+(r) \;=\; \max_{p \in [0,U]} \Pi^+(r,p) + \gamma V^+(\alpha r + (1-\alpha)p), \tag{2.8a}$$

$$V^-(r) \;=\; \max_{p \in [0,U]} \Pi^-(r,p) + \gamma V^-(\alpha r + (1-\alpha)p). \tag{2.8b}$$

The solutions of (2.8a) and (2.8b) are denoted respectively as $p^+(r)$ and $p^-(r)$. The following proposition gives a characterization of the solution to (2.7).

Proposition 2.1. *There exists* $0 \le r^- \le r^+ \le U$ *such that*

$$p^*(r) = \begin{cases} p^-(r), & 0 \le r \le r^-, \\ r, & r^- \le r \le r^+, \\ p^+(r), & r^+ \le r \le U, \end{cases}$$

and the optimal value function is given by

$$V(r) = \begin{cases} V^-(r), & 0 \le r \le r^-, \\ \dfrac{\Pi(r,r)}{1-\gamma}, & r^- \le r \le r^+, \\ V^+(r), & r^+ \le r \le U. \end{cases}$$

The proof to Proposition 2.1 can be found both in Popescu and Wu (2007) and Asvanunt (2007) and is omitted here. One is also referred to Section 2.6, where we prove similar results for the more general adaptation-rate-based model. We remark here that Proposition 2.1 does not rely on the linear form of the demand function and readers are referred to Popescu and Wu (2007) for the assumptions on demand and profit functions that are necessary for Proposition 2.1 to hold. With a linear form in (2.4), one can compute that $r^- = \frac{b(1-\gamma\alpha)}{2a(1-\gamma\alpha)+(1-\gamma)\eta^-}$ and $r^+ = \frac{b(1-\gamma\alpha)}{2a(1-\gamma\alpha)+(1-\gamma)\eta^+}$. Asvanunt (2007) also provides explicit expressions for $p^-(r), p^+(r)$ and $V^-(r), V^+(r)$.

Given an initial reference price r_0 and $p^*(r)$, the sequence of reference prices $\{r_t\}$ which evolves according to (2.1) is referred to as the *reference price path*. A consequence of Proposition 2.1 is the following convergence result of the reference price path, which essentially says that in the long-run a constant pricing strategy is optimal.

Proposition 2.2. *When* $r_0 < r^-$*, then* $\{r_t\}$ *is monotonically increasing and* $\lim_{t\to\infty} r_t = r^-$*, when* $r_0 > r^+$*, then* $\{r_t\}$ *is monotonically decreasing and* $\lim_{t\to\infty} r_t = r^+$*. When* $r^- \le r_0 \le r^+$*, then* $r_t = r_0$ *for any* $t \ge 0$*. Any reference price* $r \in [r^-, r^+]$ *is then referred to as a steady state.*

Again, the mathematical proof for Proposition 2.2 is omitted here. Instead, we give a graphical visualization of the reference price path in Figure 2.1 to illustrate both Proposition 2.1 and Proposition 2.2. In Figure 2.1, the bold

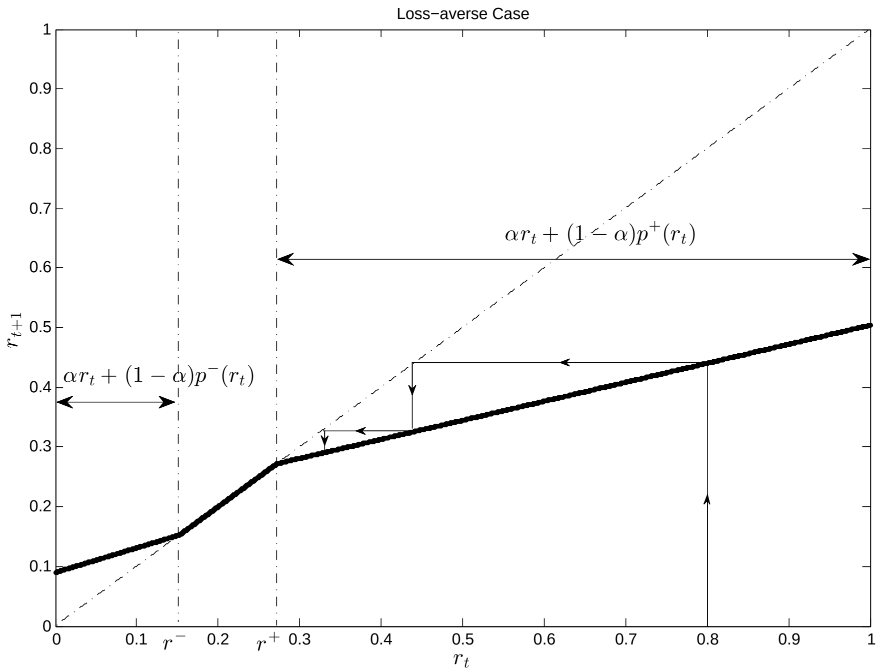


Figure 2.1: Convergence result for benchmark model

lines represent the discrete dynamic system:

$$r_{t+1} = \alpha r_t + (1-\alpha)p^*(r_t),$$

which maps r_t to r_{t+1}. The double arrowed lines specify the structures illustrated in Proposition 2.1 and the arrowed lines illustrate a reference price path. Specifically, the vertical arrowed lines indicate that the trajectory evolves from r_t to r_{t+1} while the horizontal arrowed lines visually aid us in thinking the function value r_{t+1} as an argument of the next iteration. One can see that starting from an initial reference price $r_0 = 0.8 > r^+$, the reference prices will then converge monotonically to r^+.

2.5 Dynamic Pricing under the Peak-End Model

The dynamic pricing problem under the peak-end model described by (2.2) has been analyzed by Nasiry and Popescu (2011). In this section, we briefly restate their main result for completeness.

Under the peak-end model, as consumers anchor on both the lowest price as well as the previous period price, two states are needed to describe the evolution of system. Given the initial state (m_0, p_0), the firm's dynamic pricing problem under the peak-end model is then:

$$\begin{aligned} V(m_0, p_0) &= \max_{p_t \in [0,U]} \sum_{t=1}^{\infty} \gamma^{t-1} \Pi(r_t, p_t) \\ \text{s.t.} \quad r_{t+1} &= \beta m_t + (1-\beta) p_t, \quad t \geq 0, \\ m_{t+1} &= \min\{m_t, p_{t+1}\}, \quad t \geq 0, \end{aligned} \tag{2.9}$$

and the Bellman equation for problem (2.9) is

$$\begin{aligned} V(m_{t-1}, p_{t-1}) &= \max_{p_t \in [0,U]} \{\min\{\Pi^+(r_t, p_t), \Pi^-(r_t, p_t)\} + \gamma V(\min\{m_{t-1}, p_t\}, p_t)\}, \\ r_t &= \beta m_{t-1} + (1-\beta) p_{t-1}. \end{aligned} \tag{2.10}$$

Let $p^*(m_{t-1}, p_{t-1})$ denote the optimal pricing strategy that solves (2.10) and given the initial state (m_0, p_0), $\{p_t^*\}$ denote the optimal price path given by $p_t^* = p^*(m_{t-1}, p_{t-1}^*)$. Proposition 2.3 summarizes the main result from Nasiry and Popescu (2011), to which one is referred for detailed analysis and proofs.

Proposition 2.3. *Given any initial state (m_0, p_0), $\{p_t^*\}$ converges monotonically to a steady state price, which depends only on m_0.*

Again, Proposition 2.3 implies that a constant pricing strategy is optimal in the long-run. However, Nasiry and Popescu (2011) note that the range of constant prices that are optimal is wider than that under the exponential smoothing model and unlike the exponential smoothing model, the optimal constant prices do not reduce to a single point when consumers are loss/gain neutral.

2.6 Dynamic Pricing under the Adaptation-Rate-Based Model

In Section 2.3, we see that the adaptation-rate-based model achieves the best fit in one of the brand and it is actually used in practice (Natter et al., 2007). In this section, we complement the literature by analyzing the dynamic pricing problem under the adaptation-rate-based model.

Similar to (2.5), the firm's dynamic pricing problem is

$$\begin{aligned} V(r_0) &= \max_{p_t \in [0,U]} \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t) \\ \text{s.t.} \quad r_{t+1} &= p_t + \alpha^+ \max\{r_t - p_t, 0\} + \alpha^- \min\{r_t - p_t, 0\}, \quad t \geq 0. \end{aligned} \tag{2.11}$$

Note that we assume $\alpha^+ \leq \alpha^-$, which allows us to rewrite (2.3) as

$$r_{t+1} = \min\{\alpha^+ r_t + (1-\alpha^+)p_t, \alpha^- r_t + (1-\alpha^-)p_t\},$$

and since $V(\cdot)$ is continuous and monotonically increasing, the Bellman equation can be written as

$$\begin{aligned} V(r) = \max_{p\in[0,U]} \{\min\{&\Pi^+(r,p) + \gamma V(\alpha^+ r + (1-\alpha^+)p), \\ &\Pi^-(r,p) + +\gamma V(\alpha^- r + (1-\alpha^-)p)\}\}, \end{aligned} \tag{2.12}$$

with $p^*(r)$ denoting the optimal solution to (2.11). Similarly, we consider the following two problems

$$V^+(r) = \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V^+(\alpha^+ r + (1-\alpha^+)p), \tag{2.13a}$$

$$V^-(r) = \max_{p\in[0,U]} \Pi^-(r,p) + \gamma V^-(\alpha^- r + (1-\alpha^-)p), \tag{2.13b}$$

with the corresponding solutions denoted as $p^+(r)$ and $p^-(r)$ respectively. Proposition 2.4 characterizes the optimal pricing strategy under the adaptation-rate-based model which generalizes Proposition 2.1.

Proposition 2.4. *Let* $r^- = \frac{b(1-\gamma\alpha^-)}{2a(1-\gamma\alpha^-)+(1-\gamma)\eta^-}$ *and* $r^+ = \frac{b(1-\gamma\alpha^+)}{2a(1-\gamma\alpha^+)+(1-\gamma)\eta^+}$, *then*

$$p^*(r) = \begin{cases} p^-(r), & 0 \leq r \leq r^-, \\ r, & r^- \leq r \leq r^+, \\ p^+(r), & r^+ \leq r \leq U, \end{cases}$$

and the optimal value function is given by

$$V(r) = \begin{cases} V^-(r), & 0 \leq r \leq r^-, \\ \dfrac{\Pi(r,r)}{1-\gamma}, & r^- \leq r \leq r^+, \\ V^+(r), & r^+ \leq r \leq U. \end{cases}$$

Proof. For $0 \leq s \leq 1$, we consider the following problem:

$$\begin{aligned} V^s(r_0^s) &= \max_{p_t \in [0,U]} \sum_{t=0}^{\infty} \gamma^t \Pi^s(r_t^s, p_t) \\ \text{s.t.} \quad r_{t+1}^s &= \alpha_s r_t^s + (1-\alpha_s)p_t, \quad t \geq 0, \end{aligned} \tag{2.14}$$

where $\Pi^s(r,p) = p(b - ap + (s\eta^+ + (1-s)\eta^-)(r-p))$ and $\alpha_s = s\alpha^+ + (1-s)\alpha^-$. Note that this is simply the problem with the exponential smoothing model (the memory factor is α_s) and loss/gain neutral demands (the marginal reference price effect is $s\eta^+ + (1-s)\eta^-$). The superscript "s" on reference price is to distinguish it from the reference prices generated in problem (2.11). In the extreme case when $s = 0$, $V^s(r) = V^-(r)$ and when $s = 1$, $V^s(r) = V^+(r)$. By Theorem 2 in Popescu and Wu (2007), problem (2.14) admits a unique steady state $r_s = \frac{b(1-\gamma\alpha_s)}{2a(1-\gamma\alpha_s)+(1-\gamma)(s\eta^+ + (1-s)\eta^-)}$ and for any initial reference price r_0^s, the reference price path under the optimal pricing strategy converges monotonically to r_s. It is easy to see that r^- and r^+ are simply the steady states when $s = 0$ and $s = 1$ respectively.

Next, we show that $V^s(r) \geq V(r)$ for any $0 \leq s \leq 1$. We make two simple observations here. First, $(s\eta^+ + (1-s)\eta^-)(r-p) \geq \min\{\eta^+(r-p), \eta^-(r-p)\}$. Second, $\alpha_s r + (1-\alpha_s)p \geq \min\{\alpha^+ r + (1-\alpha^+)p, \alpha^- r + (1-\alpha^-)p\}$. The first observation leads to $\Pi^s(r,p) \geq \Pi(r,p)$. For any initial reference price $r_0^s = r_0$ and fixed price path $\{p_t\}$, the second observation implies $r_t^s \geq r_t$ for any $t \geq 0$. Since $\Pi^s(r,p)$ is increasing in r, we have $\Pi^s(r_t^s, p_t) \geq \Pi^s(r_t, p_t) \geq \Pi(r_t, p_t)$, which is true for any feasible price path $\{p_t\}$. Therefore, fixing an optimal price path $\{p_t^*\}$ for problem (2.11) and letting $r_0^s = r_0$, we then have

$$V^s(r_0^s) \geq \sum_{t=0}^{\infty} \gamma^t \Pi^s(r_t^s, p_t^*) \geq \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t^*) = V(r_0).$$

In particular, this implies $V^-(r) \geq V(r)$ and $V^+(r) \geq V(r)$.

When $r^- \leq r \leq r^+$, there exists $0 \leq s \leq 1$, such that $r = r_s$. As r_s is the steady state for problem (2.14), the pricing path $\{p_t = r_s\}$ is optimal for problem (2.14). On the other hand, it is feasible for problem (2.11) while resulting an objective value $V^s(r_s)$. By $V^s(r) \geq V(r)$, $\{p_t = r_s\}$ is optimal for problem (2.11) as well. In other words, $p^*(r) = r$ and $V(r) = V^s(r) = \frac{\Pi(r,r)}{1-\gamma}$ for $r^- \leq r \leq r^+$.

When $0 \leq r \leq r^-$, similarly, $p^-(r)$ is an optimal solution to (2.13b) and by monotonic convergence to r^- it holds $p^-(r) > r$ for $0 \le r \le r^-$. For any initial reference price $r_0 < r^-$, $\{p^-(r_t)\}$ is a feasible solution to problem (2.11) and by $r^- \ge p^-(r_t) > r_t$ for all $t \ge 0$, it will result in an objective value $V^-(r_0)$. That is, $\{p^-(r_t)\}$ is optimal to problem (2.11), i.e., $p^*(r) = p^-(r)$ and $V(r) = V^-(r)$ for $0 \le r \le r^-$. The case for $r^+ \le r \le U$ can be proven in a same way. □

One can see from the proof of Proposition 2.4 that Proposition 2.2 can be directly generalized here. That is, the steady states for the dynamic pricing problem (2.11) are $[r^-, r^+]$, where $r^- = \frac{b(1-\gamma\alpha^-)}{2a(1-\gamma\alpha^-)+(1-\gamma)\eta^-}$ and $r^+ = \frac{b(1-\gamma\alpha^+)}{2a(1-\gamma\alpha^+)+(1-\gamma)\eta^+}$. Although the derivation of the analytical results is similar to the exponential smoothing model, managerial implications under the adaptation-rate-based model are more in line with that of the peak-end model. Specifically, our results also imply the range of steady state prices, i.e., $r^+ - r^-$, is wider than that under the exponential smoothing model. In the special case of loss/gain neutral demands, the optimal constant prices do not degenerate to a single price point.

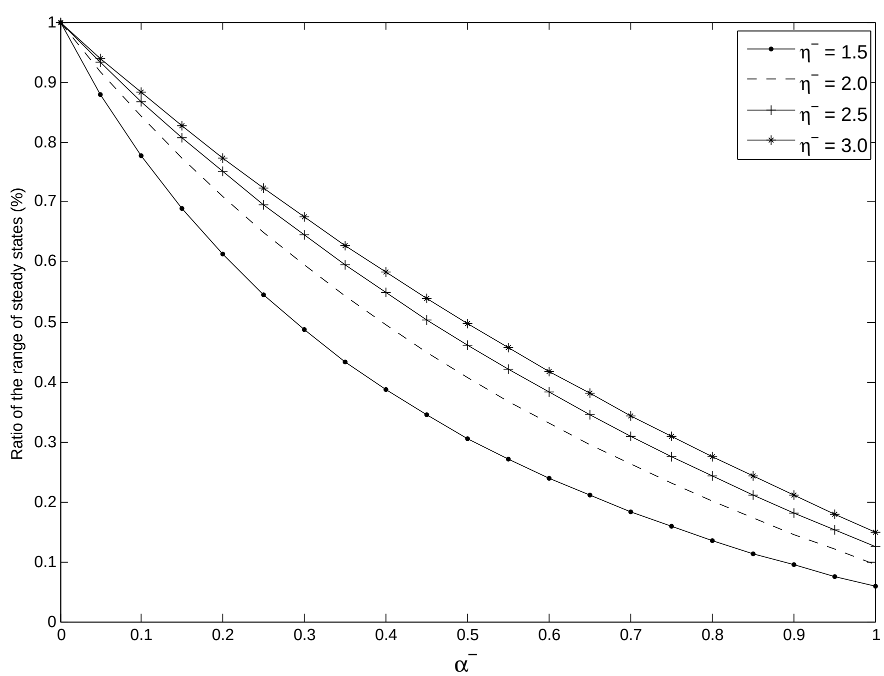


Figure 2.2: Ratio of steady state ranges

Figure 2.2 illustrates the ratio of the steady state range under the exponential smoothing model to that under the adaptation-rate-based model, where we have fixed $\alpha = \alpha^+ = 0, \eta^+ = 1$ and $b = 4, a = 1$. One can see that when the loss-aversion effect is small, the adaptation-rate-based model results in relatively larger steady state ranges. Intuitively, this is due to the fact that the asymmetry in consumers' adaptation rate dominates the asymmetry in gain/loss effects.

2.7 Stochastic Reference Price Model

In this section, we introduce a continuous time model by following Fibich et al. (2003) and along with Zhang (2011) we complement the literature by proposing a new reference price model called "*stochastic reference price*". Under the stochastic reference price model, we demonstrate the behavior of reference prices and analyze the dynamic pricing problem under the assumption of loss/gain-neutral demands.

We first introduce the continuous time demand model as well as the continuous time counterpart of exponential smoothing model (2.1). As in Fibich et al. (2003), the demand rate is still specified by (2.4) and under the assumption of loss/gain-neutral demands, i.e. $\eta^+ = \eta^- = \eta$, we can further simplify (2.4) as

$$D(r, p) = b - ap + \eta(r - p).$$

Given a price path $\{p(t)\}$, the continuous time counterpart of the exponential smoothing model is given by

$$\begin{cases} dr = \bar{\alpha}[p(t) - r(t)]dt \\ r(0) = r_0 \end{cases} \tag{2.15}$$

where r_0 is the initial reference price. We use $\bar{\alpha} \geq 0$ to distinguish the parameter from the memory factor in (2.1), since here as $\bar{\alpha}$ increases, consumers incorporate the new price information at a faster rate while in (2.1) consumers adapt to the new price information faster as α approaches 0. According to (2.15), with a given initial value r_0 and a given price process $\{p(t)\}$, the reference price at any given time is a fixed value for the entire consumer population.

However, there are two common features of the market that (2.15) does not capture. First, a consumer population is rarely homogeneous. For example, brand loyal consumers and brand switchers can make different brand choice and purchase quantity decisions (Krishnamurthi et al., 1992). Thus, it is natural to postulate that each consumer should have her own perception of the prices as well. Second, other exogenous factors like advertisement activities and competitors' prices may influence consumers' memory processes. As argued in Section 2.1, consumers' reference price may also be affected by contextual effects, i.e. other prices consumers observe at the time of purchase (Rajendran and Tellis, 1994).

In this section, we try to incorporate heterogeneity as well as exogenous shocks and describe the more complex behavior of consumers by using a *stochastic differential equation* (SDE) (see Øksendal, 2002, for a reference on the topic of SDE) to model reference price evolution process:

$$dr(t) = \bar{\alpha}[p(t) - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \tag{2.16}$$

where $W(t)$ is the standard Wiener process and reference price $r(t)$ is now a stochastic process.

Note here that for a pre-determined price path $\{p(t)\}$, $d\mathbb{E}[r(t)] = \alpha[p(t) - \mathbb{E}[r(t)]]dt$. That is, if the firm pre-commits to a price path that is independent of the realization of randomness, then the evolution of the expected reference prices coincide with the deterministic model (2.15) used in Fibich et al. (2003). We illustrate in Figure 2.3 a sample path of (2.16) as well as $\mathbb{E}[r(t)]$ under a constant pricing strategy with two price levels: the high price $p_H = 0.92$ and the low price $p_L = 0.29$ (the highest and lowest price in Table 2.1) respectively. In Figure 2.3, we take the initial reference price $r_0 = \frac{p_H+p_L}{2} = 0.605$, $\alpha = 0.5$ and $\sigma = 0.2$. One can see that $r(t)$ has a higher variance under p_H than under p_L which reflects the square-root diffusion term in (2.16).

There are two main considerations in our choice of models. From a modeling perspective, we want a model that can give a good approximation to the above mentioned two features. To model consumer heterogeneity, incorporating randomness is a common practice used in economics and marketing (see Allenby and Rossi, 1998, for instance). One possible way is to assume $\bar{\alpha}$ to be random. However, it is easy to see that if the price is a predetermined constant, i.e., $p(t) = p$, for all $t \geq 0$, the variance of $r(t)$ will go to zero as $t \to \infty$. That is, the firm can eliminate such heterogeneity in consumers'

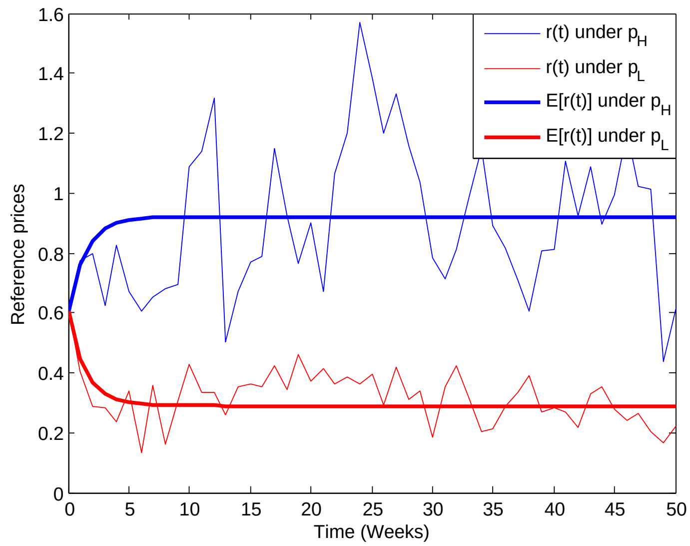


Figure 2.3: $\mathbb{E}[r(t)]$ and sample paths of $r(t)$ under p_H and p_L

reference prices by employing a constant pricing strategy. While this could be plausible in some scenario, we believe, in general, variability in consumers' perception of prices should persist under the common pricing strategies, such as constant, skimming or penetrating pricing strategies. On the other hand, variability in reference prices always exists (unless $p(t) = 0$ for all t) in (2.16). In addition, (2.16) has the nice property that the probability of $r(t)$ going negative is always zero. To model exogenous factors, one usually adds a random shock to represent those exogenous factors. The square-root diffusion process (2.16) has the additional merit of allowing reference price level dependent variance. It predicts that the variance of the $r(t)$ gets smaller as $r(t)$ itself becomes smaller. Such property is desirable in many scenarios. For instance, if competitors' prices are a major factor affecting consumers' memory processes, then the competitors prices would be much more attractive and thus creating more variability in consumers reference prices when consumers reference prices are high rather than low.

From an analysis perspective, the square-root diffusion process (2.16) can provide analytical tractability and has found applications ranging from term-structure modeling (Cox et al., 1985) to option pricing (Heston, 1993). In our application, in particular, it enables a closed-form solution and results in a simple steady state distribution. As a result, we are able to compare analytically the expected steady state to the steady state derived from the deterministic reference price model.

Under our stochastic reference price model, the firm's dynamic pricing problem is

$$\begin{aligned} V(r_0) &= \max_{p(t)} \mathbb{E}\left[\int_0^\infty e^{-\gamma t} p(t) D(r(t), p(t))\right] dt, \\ \text{s.t.} \quad dr(t) &= \bar{\alpha}[p(t) - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \end{aligned} \tag{2.17}$$

where γ is the discount factor. The corresponding Hamilton-Jacobi-Bellman (HJB) equation can then be written as

$$\gamma V(r) = \max_p \{pD(r,p) + \bar{\alpha}(p - r)\frac{dV(r)}{dr} + \frac{\sigma^2}{2} r \frac{d^2V(r)}{dr^2}\}. \tag{2.18}$$

Readers are referred to Miranda and Fackler (2004), for instance, for an intuitive derivation of the HJB equation (2.18). We denote $p^*(r)$ to be the optimal solution to (2.18) and $r^*(t)$ to be the reference price path under $p^*(r)$ which satisfies the SDE

$$dr^*(t) = \bar{\alpha}[p^*(r^*(t)) - r^*(t)]dt + \sigma\sqrt{r^*(t)}dW(t).$$

Note here that by seeking a state feedback solution $p^*(r)$, we have implicitly assumed that the firm has the ability to measure or observe the realization of consumers' reference price and can set a price accordingly. Similar to the problems analyzed in the previous sections, we are interested in the long-run behavior of the optimal prices as well as the resulting reference price path. Specifically, as t goes to infinity, will $r^*(t)$ converge to a steady state? The following result gives a complete answer to this question.

Proposition 2.5. *The optimal reference price path $r^*(t)$ converges in distribution to the steady state, denoted as R_s^*. The density of R_s^* is*

$$f_{R_s^*}(r) = \frac{(2\lambda/\sigma^2)^{2\lambda\mu/\sigma^2}}{\Gamma(2\lambda\mu/\sigma^2)} r^{2\lambda\mu/\sigma^2 - 1} e^{-2r\lambda/\sigma^2},$$

where $\Gamma(\cdot)$ is the gamma function. That is, R_s^* follows a gamma distribution with shape parameter $2\lambda\mu/\sigma^2$ and rate parameter $2\lambda/\sigma^2$. The constants λ

and μ are defined by

$$\lambda = \bar{\alpha}\frac{2a+\eta-2\bar{\alpha}Q}{2(a+\eta)}, \quad \mu = \bar{\alpha}\frac{\bar{\alpha}R+b}{2\lambda(a+\eta)},$$

where Q and R are given by

$$\begin{aligned} Q &= \frac{\gamma}{2\bar{\alpha}^2}(a+\eta) + \frac{2a+\eta}{2\bar{\alpha}} - \frac{a+\eta}{2\bar{\alpha}^2}\Delta, \\ R &= \left[\frac{b}{\bar{\alpha}} + \frac{\sigma^2(a+\eta)}{\bar{\alpha}^2}\right]\frac{\gamma-\Delta}{\gamma+\Delta} + \left[b + \frac{\sigma^2(2a+\eta)}{2\bar{\alpha}}\right]\frac{2}{\gamma+\Delta}, \end{aligned}$$

and Δ is

$$\Delta = \sqrt{\gamma^2 + 2\bar{\alpha}\frac{2a(\gamma+\bar{\alpha})+\gamma\eta}{\eta+a}}.$$

Proposition 2.5 not only claims the convergence to a steady state, but also gives an explicit expression for the steady state distribution in terms of problem parameters. Our result differs from the previous literature in a sense that the steady state R_s^* is a random variable rather than a deterministic value. This confirms our motivation in modeling consumer heterogeneity: even under optimal pricing strategy, variability in consumers' reference prices still persist.

Figure 2.4 illustrates the steady state distributions under different levels of $\bar{\alpha}$. In Figure 2.4, we have fixed $a/b = 0.8$, $\eta/b = 0.5$, $\gamma = 0.01$ and $\sigma^2 = 0.2$. One can see that as $\bar{\alpha}$ grows, the spread of the distribution shrinks. Intuitively, this is due to the fact that as $\bar{\alpha}$ grows, the drift term in (2.16) will have a relatively stronger effect compared to the diffusion term and result in less variance. In other words, if consumers in the population adapt to the new price information at a faster rater, then the variability in their perception of the fair prices can be reduced.

Using Proposition 2.5, we can easily compute the expected steady state reference price as well as the variance of steady state reference price. Their explicit expressions are summarized in the following proposition.

Proposition 2.6. *Let Δ, μ and λ be the constants defined in Proposition 2.5. The expected steady state reference price $r_s^* = \mathbb{E}[R_s^*]$ is given by*

$$r_s^* = \mu = r_D^* + \frac{\sigma^2}{2a(\gamma+\bar{\alpha})+\gamma\eta}\left[\frac{a+\eta}{\bar{\alpha}}\left(\frac{\gamma}{2} - \frac{\Delta}{2}\right) + \frac{2a+\eta}{2}\right]$$

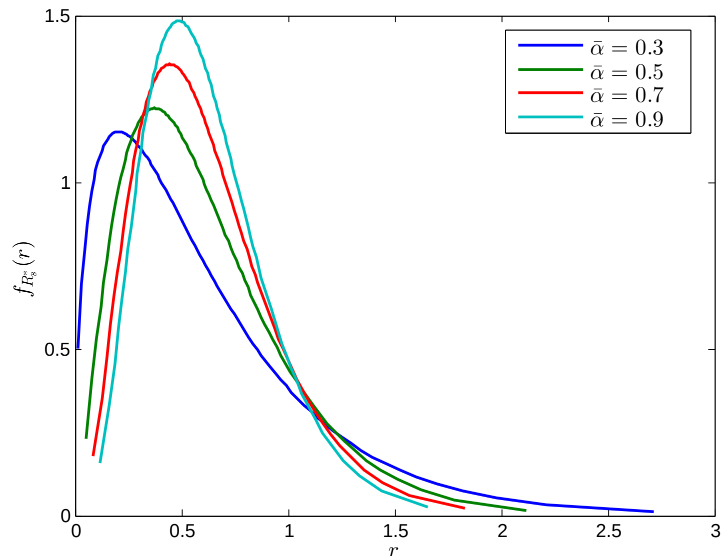


Figure 2.4: Shape of $f_{R_s^*}(r)$ under different $\bar{\alpha}$

where r_D^* is the steady state in the deterministic problem ($\sigma^2 = 0$):

$$r_D^* = \frac{(\gamma + \bar{\alpha})b}{2a(\gamma + \bar{\alpha}) + \gamma\eta}.$$

The variance of steady state reference price is given by

$$var(R_s^*) = \frac{\mu}{2\lambda}\sigma^2.$$

We remark here that r_D^* is exactly the steady state derived by Fibich et al. (2003) in the deterministic reference price model. Clearly, when $\sigma = 0$, our model reduces to the deterministic model in Fibich et al. (2003) and r_s^* agrees with their solution. When $\sigma > 0$, on the other hand, it is easy to verify that $r_s^* > r_D^*$. That is, the expected steady state reference price is always higher than the steady state reference price when there is no randomness. This result is in sharp contrast with the intuition developed in some previous pricing literature. Recall in Figure 2.3 that a higher price induces a higher variability in reference price, and consequently higher variability in demands. Such variability in demands are undesirable in many settings. For example, in a joint inventory and pricing setting, by comparing the optimal price with the *riskless price* (the price obtained from deterministic demands), the optimal price is always set in a way such that variability in demands is reduced

(Petruzzi and Dada, 1999). In our dynamic pricing problem, however, the firm does not need to worry about the risk of mismatch between supply and demand and demand variability will not be a concern. On the contrary, it will bring more opportunities to the firm since higher variability in reference prices will allow the firm to take advantage of the possible high reference price level.

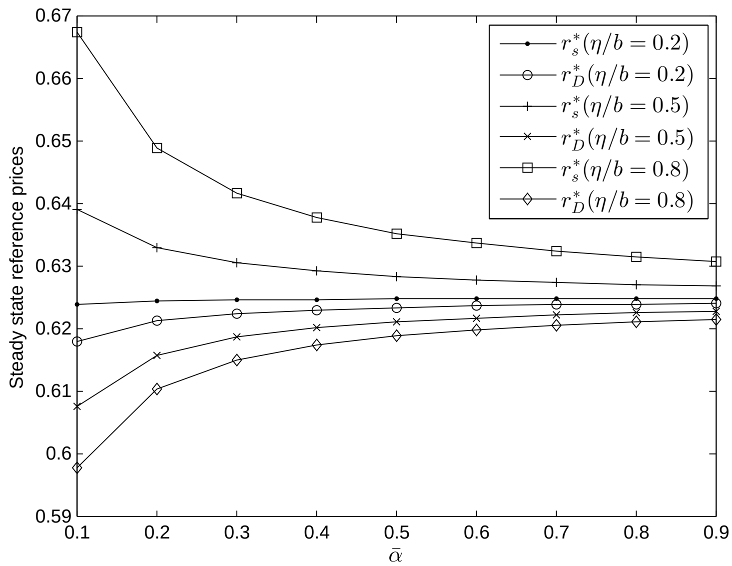


Figure 2.5: Comparisons of r_s^* and r_D^*

Figure 2.5 illustrates the gap between r_s^* and r_D^* under a range of values of $\bar{\alpha}$ and different levels of η/b with other parameters fixed at the same values in Figure 2.4. One can see that the gap decreases as $\bar{\alpha}$ increases and η/b decreases. As $\bar{\alpha}$ grows, consumers adapt to the new price at a faster rate and in the extreme case it adjusts to the current price instantaneously. Note that this is different from the discrete time model in which case the fastest rate consumers can achieve is to adjust according to the last period price rather than the current price. Such decrease in the average gap between reference price and price reduces the (stochastic) reference price effects and consequently results in a smaller difference between r_s^* and r_D^*. Similarly, when η/b is small, reference price effects play a minor role and in the limiting case when η/b approaches zero, both r_s^* and r_D^* get closer to the *static price*, the optimal price under the static demand model, and consequently their gap goes to zero.

More interestingly, the expected steady state reference price r_s^* and its deterministic counterpart r_D^* can have different behaviors relative to some problem parameters. When reference price effects are significant (η/b is large), r_s^* is decreasing in $\bar{\alpha}$ while r_D^* is increasing in $\bar{\alpha}$. It is easy to see the monotonicity of r_D^*. Since the static price is always higher than r_D^*, as $\bar{\alpha}$ increases, the model more closely resembles the static demand model and as a result, r_D^* increasingly approaches the static price. The opposite direction of r_s^* is less obvious. One possible explanation is that when $\bar{\alpha}$ becomes larger, the benefit of having larger variations in reference price decreases. Similar explanations apply to the sensitivity of r_s^* and r_D^* to η/b. As η/b becoming smaller, the effects of reference price gradually vanish and r_D^* increases to the static price while r_s^* decreases to it.

2.8 Conclusion

This chapter summarizes and extends the existing literature in reference price effects by comparing various reference price models and presenting the analysis of the dynamic pricing problem with loss/gain neutral or loss-averse demands under these models.

We empirically compare the widely used exponential smoothing model with the two recently proposed peak-end model and adaptation-rate-based model. We find that the peak-end model being as parsimonious as the exponential s-moothing model, in general, performs the best. However, in one example, the adaptation-rate-based model still provides a better fit at a cost of increasing the degree of freedom of model parameters. It would be an interesting future research direction to further confirm our findings to other product categories and determine the market conditions under which one model outperforms another.

Despite the differences in their empirical performances, the managerial implications from the dynamic pricing problems under the three reference price models are similar. All three models predict that a constant pricing strategy is optimal in the long-run when demands are either loss/gain neutral or loss-averse. However, based on our analysis of the adaptation-rate-based model, which is not explored in the previous literature, we find that a range of constant prices can be optimal even in the case of loss/gain neutral demands. This is different from the prediction of the exponential smoothing model in which case only a single constant price is optimal. We numerically show that the relative range of steady states compared to the exponential smoothing model grows as asymmetry in consumers' adaption rate dominates the asymmetry in loss-aversion.

Finally, we extend the literature by introducing a stochastic reference price model in a continuous time framework. Our motivation in introducing randomness in consumers' memory processes is to model consumer heterogeneity and exogenous factors. We obtain an explicit solution to the steady state distribution and derive several interesting new insights from analyzing the dynamic pricing problem under the stochastic reference price effects. Specifically, we find that the expected steady state reference price (price) is higher than the steady state reference price (price) under a deterministic model and the larger the variance in consumers' memory processes, the higher the steady state reference price (price). In addition, we find that the gap between the two shrinks as reference price effects diminishes ($\bar{\alpha}$ increases or η/b decreases) and they can have opposite sensitivity to the magnitude of reference price effects.

An interesting future research direction along the line of stochastic reference price effects would be empirical justification of our models and comparisons to other possible ways of modeling consumer heterogeneity or exogenous factors. Also, we only focus on the case of loss/gain neutral demands, which enable us to apply tools from stochastic optimal control theory. However, whether the steady state result can be generalized to loss-averse demands is still a challenging open question.

Chapter 3

Dynamic Pricing Problem with Gain-Seeking Reference Price Effect

3.1 Introduction

In Chapter 2, we have introduced the results in several papers (Fibich et al., 2003; Popescu and Wu, 2007; Nasiry and Popescu, 2011) which all focus on loss-averse demand model. The conceptual support for loss-aversion follows from prospect theory (Tversky and Kahneman, 1991) and as a result a widely beheld belief in the reference price literature postulates that consumers are loss-averse.

In this chapter, however, we point out that, in practice, it is also possible for the firm to face gain-seeking demands and the underlying reasons for this could be either consumers are indeed gain-seeking or the effect of aggregation. We then study the dynamic pricing problem of a firm facing gain-seeking demand. We remark here that although the term "loss-seeking" is used in the previous literature (see, for instance, Popescu and Wu, 2007), we feel the term "gain-seeking" is more accurate in capturing the fact that consumers/aggregate demands are more sensitive to gains rather than losses and note that it has also been adopted in several recent papers (Aflaki and Popescu, 2013; Kallio and Halme, 2009; Kopalle et al., 2012) as well.

Unlike the common belief that loss-averse behavior is prevalent, it is admitted in the survey paper by Mazumdar et al. (2005) that "evidence of loss-aversion is mixed". In fact, there are empirical evidences that support gain-seeking behavior at both individual and aggregate level. At an individual level, two out of five papers reviewed by Mazumdar et al. (2005) that study the effects of loss-aversion show little or no evidence of loss-aversion. Specifically, Krishnamurthi et al. (1992) observe that consumers not loyal to any brand respond more strongly to gains than to losses in all six but one brand, and Bell and Lattin (2000) suggest through extensive empirical studies across different product categories that loss-aversion may not be a universal phenomenon in grocery products due to price response heterogeneity. More recently, Kopalle et al. (2012) find that for a large number of households the impact of a gain is greater than that of a loss and on average the effect of losses is almost negligible compared to that of gains. At an aggregate level, Greenleaf (1995) shows that the aggregate demand can be 10 times more responsive to gains than losses. Deploying the methodology of switching regression on the aggregate level data, Raman and Bass (2002) also encounter gain-seeking behavior in one of their brands.

One also needs to be cautious that loss-aversion at an individual level does not necessarily imply loss-aversion at an aggregate level. Kallio and Halme (2009) explicitly define the loss-aversion at an individual level as *loss averse in value* and loss-aversion at an aggregate level as *loss averse in demand* and give possible conditions under which loss averse in value does not imply loss averse in demand. One important implication of their conditions in a single product setting is that when market conditions are harsh (consumers have overall small probability of purchasing the product, i.e., mainly promotion-driven consumers), then the market should be more sensitive to gains which boost the market rather than losses which further make the market more miserable. Greenleaf (1995) also points out that the presence of gain-seeking at an aggregate demand level does not necessarily contradict with prospect theory. Specifically, he argues that the market usually consists of "light" households who are price-sensitive and have a low probability of purchasing at the regular price, and "heavy" households who have a high purchase probability at the regular price even when there is a loss. As a result, during a promotion, when there is a gain, many more light households than heavy ones are attracted. Consequently, market demand is more sensitive to gains than losses even though each household may be more sensitive to losses than gains.

Therefore, one should realize the existence of gain-seeking at an aggregate level and it can be caused either by gain-seeking behavior at an individual level (with aggregation not changing the gain-loss asymmetry) or by aggregation under harsh market conditions. Even if, under certain market conditions, the gain-seeking model analyzed in this chapter is not fully consistent with the model aggregating individual level behaviors, it may still be used as a prescriptive model to provide a plausible tractable approximation.

Despite the necessity of analyzing how the firm should respond if gain-seeking is present in the aggregate level demand, analytical results on the dynamic pricing problem of a firm in the gain-seeking case are very limited. To the best of our knowledge, the only result in this case is the non-optimality of the constant pricing strategy as observed by Kopalle et al. (1996) and Popescu and Wu (2007). Moreover, Popescu and Wu (2007) postulate that "High-low pricing, is provably optimal if consumers are focused on gains." However, we cannot find any rigorous proof in the literature. Indeed, there are many practical and interesting open questions left for the dynamic pricing problem. For example, is high-low pricing, in which only a regular price and a discount price are employed periodically, indeed optimal in general? If not, what are the conditions that guarantee its optimality? Furthermore, when high-low pricing is not optimal, what can we say about the optimal pricing strategy and the performance of the high-low pricing strategy?

This chapter strives to answer the above questions. Specifically, we find that even the *myopic pricing strategy*, where the firm ignores the effect of current prices on future revenues and focuses on maximizing short-term revenue, does not always admit a cyclic high-low price pattern and its long-run behavior can be very complicated. We provide necessary and sufficient conditions for the existence of a cyclic high-low price pattern in the myopic pricing strategy. In addition, conditions are derived such that the myopic pricing strategy leads to either a cyclic penetrating pricing strategy in which the resulting reference prices increase within a cycle, or a cyclic skimming pricing strategy in which the resulting reference prices decrease within a cycle. Our numerical studies show that high-low pricing is generally not optimal and the dynamics of the optimal pricing strategy is likely to be significantly more complex than that resulted from the myopic pricing strategy. Interestingly, under the assumptions that consumers only remember the most recent price and the aggregate demand is insensitive to the negative part of the difference between reference price and price, we prove that the optimal pricing strategy is a cyclic skimming pricing strategy. In other words, when consumers have a low reference price, the firm should charge a high price and then gradually offer deeper and deeper discounts until consumers' reference price drops low again and repeat the cycle. The assumptions we made are also found to be very plausible in our empirical studies. We further provide sufficient conditions for the high-low pricing strategy to be optimal. Our numerical studies suggest that the high-low pricing strategy, when fails to be optimal, can still achieve over 90% of the optimal profit.

Our work is in sharp contrast to the stream of works in the dynamic pricing problem when loss-aversion is present at a market level. All these works arrive at the conclusion that a constant pricing strategy is optimal in the long run. Specifically, Kopalle et al. (1996) observe through numerical studies that optimal prices converge monotonically and conjecture that a constant price is optimal in the long run. Fibich et al. (2003) explicitly solve the optimal pricing strategy in a continuous time optimal control framework, and confirm the observation by Kopalle et al. (1996) when demand is piece-wise linear in price and reference price. Popescu and Wu (2007) extend the result of Fibich et al. (2003) to general demand functions in a discrete time infinite horizon setting. Nasiry and Popescu (2011) consider the dynamic pricing problem with a peak-end based reference price model; they also conclude the observation by Kopalle et al. (1996). Vast empirical literature, however, suggests that it is important for practitioners not to take loss-averse assumption as granted. When facing gain-seeking demands, our findings show that constant pricing strategy can result in as much as 50% loss in profit while simple cyclic pricing strategies (cyclic skimming or high-low pricing strategies) are optimal or close to optimal in many scenarios. This discrepancy in results between the loss-averse case and the gain-seeking case is due to the differences in the underlying structures of the optimization problems. In essence, under loss-aversion, the single period profit function, though non-smooth due to the asymmetric responses to losses and gains, is a concave function of the current price. The gain-seeking demand, on the other hand, changes the structure of the problem completely. The resulting optimization problem is neither smooth nor concave.

A few papers have considered dynamic pricing problems in various settings that also lead to cyclic pricing strategies. Conlisk et al. (1984) assume that consumers are strategic with two possible valuations and will remain in the market (possibly forever) until making a purchase. They establish that a cyclic skimming pricing strategy is optimal. Besbes and Lobel (2015) consider strategic consumers that are heterogeneous both in valuations and the time they may spend in the market. They prove that a cyclic pricing strategy is optimal but a cyclic penetrating or a cyclic skimming pricing strategy may yield arbitrary poor performance. Ahn et al. (2007) study both production and pricing decisions when consumers have uniformly distributed valuations and tend to buy the product as soon as the price drops below their valuations. They show, in a special case where consumers wait at most one period and the firm has no capacity constraint, that high-low pricing strategy is optimal. Liu and Cooper (2014) consider a pricing setting similar to Ahn et al. (2007) and demonstrate that a cyclic skimming pricing strategy is optimal even when consumers have general valuation distributions and can wait for multiple periods. We emphasize here some key distinctions of our work from these papers. In terms of modeling, the argument behind reference price models is that consumers' purchase decisions are affected by the prices in the past rather than anticipated prices in the future. In terms of proofs, we form a dynamic programming problem and identify the cyclic pricing strategy by analyzing properties of the value function whereas the above papers all tackle their problems directly by utilizing the notion of a regeneration point. Essentially, the driving force that leads to a cyclic pricing strategy in our model is the gain-seeking behavior of consumers while in these papers is the waiting behavior of consumers. Another related paper is Geng et al. (2010) who restrict to high-low pricing strategies and use a different approach by taking weighted average between the regular price and the discount price to model reference price. They show that when demand is gain-seeking, high-low pricing strategy outperforms constant pricing strategy. Our work following Fibich et al. (2003); Kopalle et al. (1996) and Popescu and Wu (2007), on the other hand, models the dynamics and intertemporal effects of reference prices explicitly and settles the conjecture in the existing literature by showing that high-low pricing strategy may not be optimal.

Our work is also closely related to the on-going research in the one-dimensional discontinuous map in the dynamic system and chaos community. Sharkovsky and Chua (1993) examine a certain type of discontinuous maps that arise in electric circuits. They find that their class of discontinuous maps has strong temporal chaos and the behavior of trajectories can only be characterized by using probability language. Jain and Banerjee (2003) present a classification of border-collision bifurcations in discontinuous maps. Depending on parameters, the resulting dynamics can have various periodic orbits or chaos. Rajpathak et al. (2012) analyze in detail the stable periodic orbits of one type of discontinuous maps and explore the possible patterns exhibited by these orbits. It turns out that the myopic pricing strategy in our work can be reduced to the type of discontinuous maps analyzed in Rajpathak et al. (2012). However, to the best of our knowledge, the class of discontinuous maps with multiple discontinuous points, into which our optimal pricing strategy typically falls, has not been considered in the previous literature.

The remainder of this chapter is organized as follows. In Section 3.2, we remind the readers of the mathematical formulation of our model introduced in Section 2.2. In Section 3.3, we analyze the dynamics of the myopic pricing strategy and relate it with the work in discontinuous maps. The structural results for the optimal pricing strategy are presented in Section 3.4. Section 3.5 presents an empirical study and conducts numerical experiments to test the performance of simple pricing strategies. Finally, we conclude the chapter in the last section with some suggestions for future research. The proofs are all relegated to Appendix B.

3.2 Model

In this section, we restate the exponential smoothing model introduced in Section 2.2 and introduce the dynamic pricing problem with gain-seeking demands. Recall that under the exponential smoothing model, the reference prices evolve according to

$$r_{t+1} = \alpha r_t + (1-\alpha)p_t, \quad t \geq 0. \tag{3.1}$$

In the above evolution equation, $p_t \in [0, U]$ is the price charged by the firm at period t, where U is the upper bound on feasible prices. The parameter $\alpha \in [0, 1)$ is called the memory factor or carryover constant (Kalyanaram and Little, 1994). When $\alpha = 1$, reference prices remain a constant over the whole planning horizon and consequently a constant pricing strategy is optimal irrespective of gain-loss asymmetry. We restrict $\alpha < 1$ to avoid such a case that past prices have no impact on demand. As reference prices are generated from historical prices, it is also reasonable to assume that $r_0 \in [0, U]$.

Recall the demand function defined in (2.4). To avoid negative demand, we further assume that $D(0, U) \geq 0$, i.e., $U \leq \frac{b}{a+\eta^-}$. Since this chapter focuses on the gain-seeking case, we have $\eta^+ > \eta^-$ in (2.4).

The firm's one-period profit is denoted as $\Pi(r, p) = pD(r, p)$. Here, the marginal cost is assumed to be 0 for simplicity. All our results can be extended to cases with a non-zero marginal cost. We assume $U \geq \frac{b}{2a}$ such that $\Pi(p,p)$, called the base profit, is not monotone in $p \in [0,U]$. This assumption allows us to "rule out pathological boundary steady states (Popescu and Wu, 2007).", but our analysis can be carried over similarly by distinguishing those boundary steady states when this assumption fails. Note that the assumptions $\eta^+ > \eta^-$ and $p \geq 0$ allow us to rewrite the one-period profit as

$$\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\},$$

where $\Pi^+(r,p) = p[b - ap + \eta^+(r-p)]$ and $\Pi^-(r,p) = p[b - ap + \eta^-(r-p)]$. Contrary to the loss-averse case, the one-period profit function is no longer a concave function in p.

Given an initial reference price r_0, the firm's long-term profit maximization problem is then:

$$V(r_0) = \max_{p_t \in [0,U]} \sum_{t=0}^{\infty} \gamma^t \Pi(r_t, p_t), \tag{3.2}$$

where $\gamma \in [0,1)$ is a discount factor and we interpret $0^0 = 1$. The infinite horizon problem is of particular interest in the literature since it is often more tractable than the finite horizon counterpart and provides valuable insights into the long-run behavior of the optimal pricing strategy, which in turn may shed light on the development of efficient heuristics for finite horizon models. It is worth noting here that two assumptions commonly imposed on optimization problems: differentiability and concavity in the decision variables, are both absent in the one-period profit function $\Pi(r,p)$, which makes the analysis of problem (3.2) quite challenging.

The Bellman equation for problem (3.2) is

$$V(r) = \max_{p \in [0,U]} \Pi(r,p) + \gamma V(\alpha r + (1-\alpha)p). \tag{3.3}$$

A *pricing strategy* $p(r)$ is a function from $[0,U]$ to $[0,U]$ that specifies a feasible solution to (3.3) for a given reference price r. Given any pricing strategy $p(r)$, the sequence $\{r_t\}$ of reference prices which evolve according to $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$, is referred to as the *reference price path* of the pricing strategy $p(r)$. We say $p(r)$ has a *periodic orbit* of period n or is a *cyclic pricing strategy* with cycle length n if and only if there exists $r_0 \in [0,U]$

such that the reference price path of $p(r)$ satisfies $r_n = r_0$ and $r_t \neq r_0$ for all $0 < t < n$. Clearly, if $r_n = r_0$, then by $r_{t+1} = \alpha r_t + (1-\alpha)p(r_t)$, the sequence $\{r_0, ..., r_{n-1}\}$ is repeated infinitely over time and this sequence is referred to as the *periodic orbit* of $p(r)$. In particular, when $n = 1$, we say $p(r)$ admits a *steady state* r_0 and when $n = 2$, $p(r)$ is a *high-low pricing strategy*. If there exists a periodic orbit that has the additional property that $r_0 < r_1 < ... < r_{n-1}$, then we refer $p(r)$ to as a *cyclic penetrating pricing strategy*. If, on the other hand, $r_0 > r_1 > ... > r_{n-1}$, then we refer $p(r)$ to as a *cyclic skimming pricing strategy*. Note that in the special case when $\alpha = 0$, $r_0 < r_1 < ... < r_{n-1}$ ($r_0 > r_1 > ... > r_{n-1}$) if and only if $p(r_{n-1}) < p(r_0) < ... < p(r_{n-2})$ ($p(r_{n-1}) > p(r_0) > ... > p(r_{n-2})$), i.e., the monotonicity of reference prices is equivalent to the monotonicity of charged prices. However, for $\alpha > 0$, it is possible to have monotone reference prices with non-monotone charged prices. Recall that in practice, a skimming (penetrating) pricing strategy is used to describe pricing strategy with decreasing (increasing) price path overtime. Here, we use the term "skimming" ("penetrating") to reflect the fact that a skimming (penetrating) pricing strategy is usually designed to capture consumers with decreasing (increasing) valuations (an analogy of the notion "reference price" in our model) overtime. Since $[0, U]$ is compact and the objective function can be easily shown to be continuous, the *optimal pricing strategy* that solves (3.3) exists and is denoted by $p^*(r)$. As mentioned in Section 3.1, Kopalle et al. (1996) and Popescu and Wu (2007) prove that $p^*(r)$ does not admit a steady state. That is, for any $r \in [0, U]$, $p^*(r) \neq r$.

In the following, we will also use the term *pattern* to describe the existence of various monotonic structures within a periodic orbit. One is referred to Rajpathak et al. (2012) for a rigorous definition in the context of discontinuous maps. Here, we illustrate the term through a simple example. Consider a periodic orbit with period 4 that consists of four different reference prices $1, 2, 3, 4$. Then, depending on different orderings, the periodic orbit can demonstrate different patterns or monotonic structures as illustrated in Figure 3.1. The upper left and lower right panels show the reference price patterns that are of penetrating and skimming pricing strategy respectively. However, the upper right panel shows a pattern that has reference prices increase in the first two periods accompanied by a decrease in the third period while the lower left panel shows a pattern that the reference prices alternate

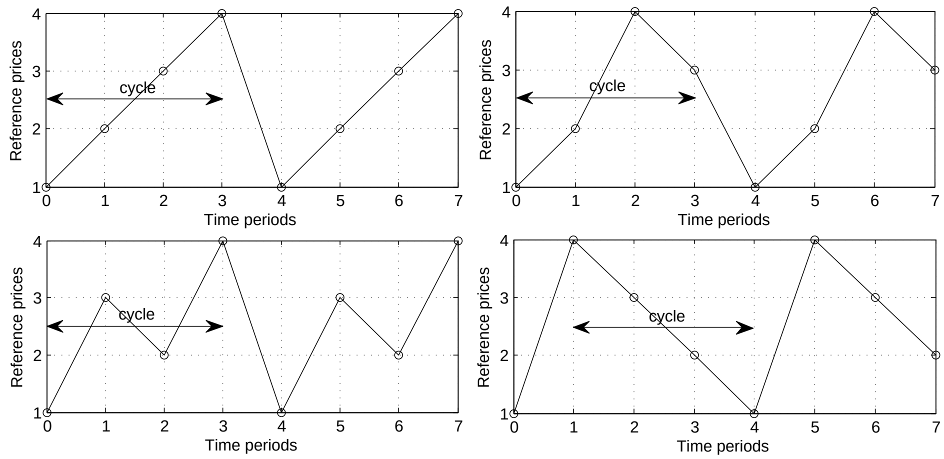


Figure 3.1: Possible patterns of a periodic orbit

between increasing and decreasing in each period.

3.3 Dynamics of the Myopic Pricing Strategy

In this section we demonstrate the complicated nature of problem (3.2) by analyzing the dynamics of the myopic pricing strategy. We present conditions to guarantee that the myopic pricing strategy admits a high-low price pattern. We then reveal the complexity of the underlying dynamics by showing that even the myopic pricing strategy can result in a cyclic pricing strategy with a cycle length arbitrary long.

By definition, the myopic pricing strategy $p^m(r)$ is given by solving the following problem:

$$p^m(r) = \arg\max_{p\in[0,U]} \Pi(r,p).$$

Define the constant

$$R = \frac{b}{a + \sqrt{(a+\eta^+)(a+\eta^-)}}. \tag{3.4}$$

Lemma 3.1. *Let* $R_U = \frac{2(a+\eta^-)U-b}{\eta^-}$*. Then, if* $R \le R_U$*,*

$$p^m(r) = \begin{cases} \dfrac{\eta^- r + b}{2(a+\eta^-)}, & r \le R, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R. \end{cases} \tag{3.5}$$

If $R > R_U$*, then*

$$p^m(r) = \begin{cases} \dfrac{\eta^- r + b}{2(a+\eta^-)}, & r \le R_U, \\ U, & R_U < r \le R', \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R', \end{cases}$$

where R' *is the unique positive root for*

$$\eta^+ r^2 + [2b\eta^+ - 4(a+\eta^+)\eta^- U]r + b^2 - 4(a+\eta^+)U[b-(a+\eta^-)U] = 0.$$

To keep the presentation clear and simple, we assume for the rest of this section that $R \le R_U$. That is, $p^m(r)$ is determined by (3.5). The analysis p-resented in this section can also be extended to the other case with additional discussions on whether U will appear on the periodic orbit or not.

Note that $p^m(r)$ is not a continuous function. As a result the dynamics of reference prices under the myopic pricing strategy

$$r_{t+1}(r_t) = \alpha r_t + (1-\alpha)p^m(r_t), \quad t = 0, 1, ..., \tag{3.6}$$

ends up with a *discontinuous map* from $[0, U]$ to $[0, U]$. The analysis of the dynamics (3.6) is not trivial at all. In fact, the study of dynamic systems with discontinuous maps is originated in the analysis of electrical circuits and is considered as "... a very complicated research subject and we can obtain useful and interesting results only for various special classes of maps (Sharkovsky and Chua, 1993)." For example, let mod denote the modulo operation, then the dynamics of the famous doubling map $D : [0, 1] \to [0, 1]$ defined by $D(x) = 2x \bmod 1$ is *chaotic*, which means such a dynamical system is highly sensitive to initial conditions (Hirsch et al., 2004).

Interestingly, we show that under some conditions, there exists a high-low price pattern under the myopic pricing strategy.

Proposition 3.1. *$p^m(r)$ is a high-low pricing strategy with a periodic orbit $\{r_0, r_1\}$ for some $r_0, r_1 \in [0, U]$ and $r_0 \neq r_1$ if and only if the following inequality holds*

$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^+ + 4a\eta^- - (1+\alpha)^2(\eta^+)^2 + 4\eta^+\eta^- \geq 0. \quad (3.7)$$

Notice that (3.7) holds when $\alpha = 0$ (consumers only remember the price of the previous period) and when the direct price effect weakly dominates the reference price effect ($4a \geq \eta^+$).

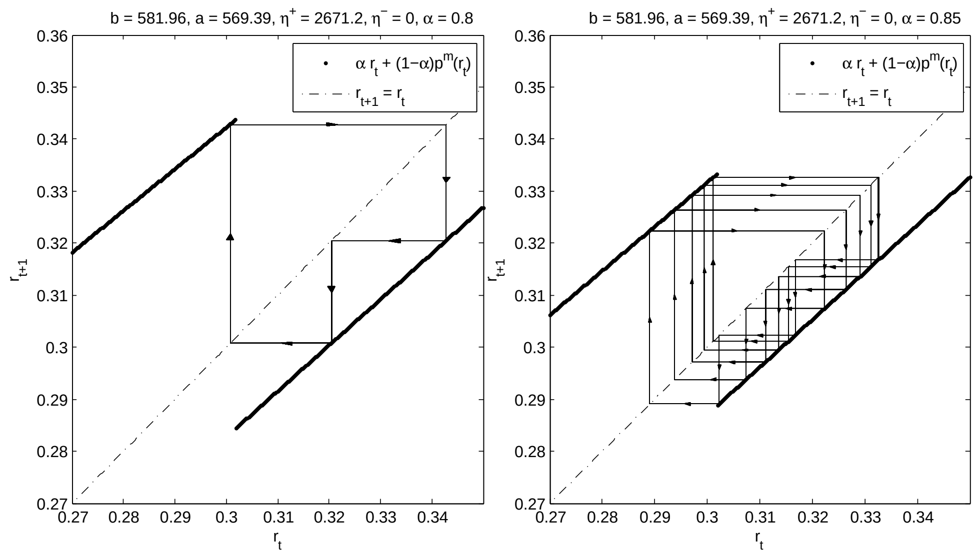


Figure 3.2: Discontinuous map (3.6) and periodic orbits for the myopic pricing strategies when $\alpha = 0.8$ and $\alpha = 0.85$ respectively

Unfortunately, condition (3.7) will be violated when α is close to 1. Figure 3.2 gives examples to illustrate the possible dynamics when condition (3.7) fails. The demand parameters used in Figure 3.2 come from our empirical examples in Section 3.5. One should interpret Figure 3.2 in the same way as Figure 2.1, where the bold lines here represent the discontinuous map (3.6) and the arrowed lines illustrate the dynamics of the system. In each panel, the arrowed lines form a closed loop so it is a periodic orbit. In fact, the periodic orbit in the left panel has period 3 and is of skimming pattern. The periodic orbit in the right panel, on the other hand, has period 16 and the myopic pricing strategy in this case is neither a cyclic penetrating nor a cyclic skimming pricing strategy. By comparing the two panels, one can see that a mere increment of 0.05 in α can result in dramatic changes in the dynamics.

As illustrated in Figure 3.1, when the period is greater than or equal to 4, there could exist other patterns rather than penetrating or skimming pattern that have the same period. Our numerical experiments illustrate that both the period and pattern of a periodic orbit can be very sensitive to the changes in parameters, a phenomenon termed as *border collision bifurcation* in the dynamic system and chaos community (Jain and Banerjee, 2003). In the following, we identify necessary and sufficient conditions for the myopic pricing strategy to admit a cyclic penetrating pattern and a cyclic skimming pattern with cycle length n respectively. To simplify the expressions of our conditions, define constants

$$\mu = \frac{a + \eta^+ - \sqrt{a^2 + a\eta^+ + a\eta^- + \eta^+\eta^-}}{\eta^+ - \eta^-},$$
$$A = 1 + \alpha - \alpha\frac{\eta^-}{2(a+\eta^-)}, \quad B = 1 + \alpha - \alpha\frac{\eta^+}{2(a+\eta^+)},$$

and denote for $n \geq 0$ the sum of geometric series $\sum_{i=0}^n k^i$ by S_n^k (for $n < 0$, let $S_n^k = 0$).

Proposition 3.2. *For $n \geq 2$, $p^m(r)$ is a cyclic penetrating pricing strategy with cycle length n if and only if the following inequalities hold*

$$\frac{A^{n-1}}{S_{n-1}^A} < \mu \leq \frac{A^{n-2}}{A^{n-2}B + S_{n-2}^A}. \tag{3.8}$$

On the other hand, $p^m(r)$ is a cyclic skimming pricing strategy with cycle length n if and only if the following inequalities hold

$$\frac{AB^{n-2} + S_{n-3}^B}{AB^{n-2} + S_{n-2}^B} < \mu \leq \frac{S_{n-2}^B}{S_{n-1}^B}. \tag{3.9}$$

Note that when $n = 2$, both (3.8) and (3.9) reduce to $\frac{A}{1+A} < \mu \leq \frac{1}{1+B}$, which can be simplified to (3.7) by substituting the expressions for A, B and μ, and by definition, a high-low pricing strategy is both a cyclic penetrating and a cyclic skimming pricing strategy. For n large enough, there exists other range of parameters other than (3.8) and (3.9) resulting periodic orbits of period n that are neither penetrating nor skimming. In fact, corresponding to different parameter regions, there are exactly $\phi(n)$ different patterns of periodic orbits of period n, where $\phi(\cdot)$ is the so-called Euler totient function (Rajpathak et al., 2012).

3.4 Optimal Pricing Strategy

Unlike the myopic pricing strategy, we do not have an explicit solution for the optimal pricing strategy, which makes the analysis significantly more challenging. To illustrate the difficulty, we first present a few properties on the value function and optimal solution.

Since $\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\}$, the Bellman equation can be correspondingly rewritten as

$$V(r) = \max_{p\in[0,U]} \{\max\{\Pi^+(r,p), \Pi^-(r,p)\} + \gamma V(\alpha r + (1-\alpha)p)\}. \tag{3.10}$$

We assume, without loss of generality, that $p^*(r)$ and the optimal solutions for other optimization problems in this section always take the largest one among multiple solutions.

Consider the following two problems:

$$\begin{aligned} V^+(r) &= \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V(\alpha r + (1-\alpha)p), && \text{(3.11a)} \\ V^-(r) &= \max_{p\in[0,U]} \Pi^-(r,p) + \gamma V(\alpha r + (1-\alpha)p). && \text{(3.11b)} \end{aligned}$$

The solutions of (3.11a) and (3.11b) are denoted respectively as $p^+(r)$ and $p^-(r)$. An observation here is that $V(r) = \max\{V^+(r), V^-(r)\}$ and $p^*(r) \in \{p^+(r), p^-(r)\}$. We next characterize properties of $V^\pm(r)$ and $p^\pm(r)$.

Lemma 3.2. *Both $V^+(r)$ and $V^-(r)$ are increasing and convex in r while $p^+(r)$ and $p^-(r)$ are increasing in r.*

Although problem (3.10) is difficult to analyze due to the term $\max\{\Pi^+(r,p), \Pi^-(r,p)\}$, Lemma 3.2 shows that the decomposed problems (3.11a) and (3.11b) have some desired properties, i.e., monotonic solutions and optimal objective values. We are interested in how $p^+(r)$ and $p^-(r)$ relate to the optimal pricing strategy $p^*(r)$, and whether it is possible to obtain simple characterizations for $p^+(r)$ and $p^-(r)$. For this purpose, we draw in

Figure 3.3 the (numerically approximated) optimal pricing strategy for the example used in the left panel of Figure 3.2 with a discount factor $\gamma = 0.9$. In Figure 3.3, we observe that, similar to the myopic pricing strategy, there

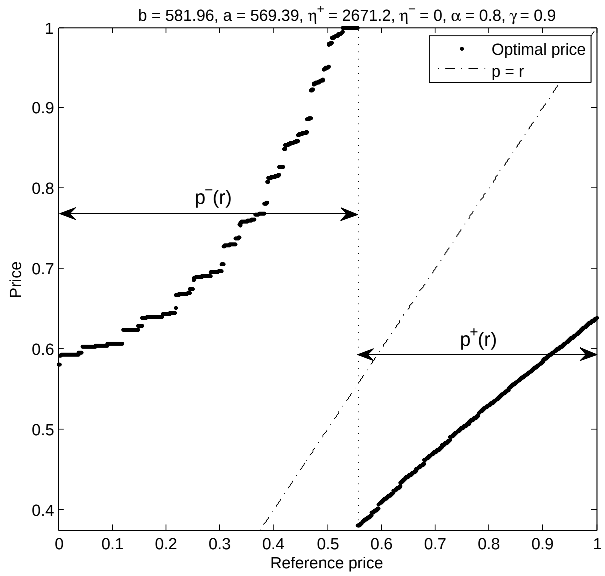


Figure 3.3: Optimal pricing strategy when $\alpha = 0.8, \eta^- = 0$ and $\gamma = 0.9$

exists a point $\hat{r} \in [0, U]$ such that $p^*(r)$ is given by $p^-(r)$ for $r \leq \hat{r}$ and $p^+(r)$ for $r > \hat{r}$. Unfortunately, though this observation seems to hold in all our numerical experiments, we do not have a proof for general parameter configurations. Of course, even if this observation is indeed true, Figure 3.3 suggests that both $p^+(r)$ and $p^-(r)$ can have numerous discontinuous points, in contrast to the two linear pieces in the myopic pricing strategy. This implies that they may not admit any simple characterizations.

In terms of dynamics, we have already demonstrated the dynamics of the myopic pricing strategy (which has only one discontinuous point and admits explicit expression) in Section 3.3. Here, we further give a side-by-side comparison of the dynamics of the optimal pricing strategies and their periodic orbits in Figure 3.4 for the examples used in Figure 3.2 with the discount factor $\gamma = 0.9$. The bold lines in Figure 3.4 represent the map $\alpha r_t + (1-\alpha)p^*(r_t)$ and as α is close to one in both instances, many discontinuous points observed in Figure 3.3 become difficult to be distinguished in Figure 3.4 through visual inspection. When comparing Figure 3.2 and Figure 3.4, it is clear that the period lengths under optimal pricing strategies can be much larger than these under myopic pricing strategies, which indicates that the dynamics of the optimal pricing strategy can be much more complicated and the analysis, if possible, is likely to be significantly more challenging than that for the myopic pricing strategy.

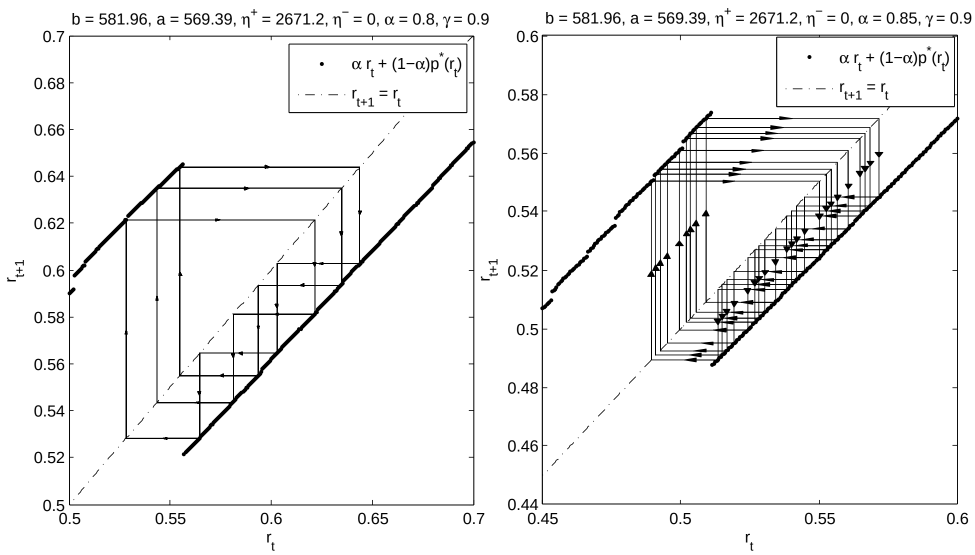


Figure 3.4: Periodic orbits for the optimal pricing strategies when $\alpha = 0.8$ and $\alpha = 0.85$ respectively

Thus, for the rest of this section, we focus on a special case satisfying the following assumption.

Assumption 3.1. *Consumers only remember the most recent price (i.e.* $\alpha = 0$*) and the demand is insensitive to the perceived surcharge (i.e.* $\eta^- = 0$*).*

Assumption 3.1 seems restrictive but has very plausible explanations. First of all, consumers are unlikely to remember many historical prices and form reference price by averaging them. Several papers, for example, Raman and Bass (2002); Krishnamurthi et al. (1992); Mayhew and Winer (1992), also assume that $\alpha = 0$. It captures the fact that "consumers... experience considerable difficulty in recalling accurately even the most recently encountered prices ... Thus, it is unlikely that consumers would retrieve from memory and use prices encountered much beyond the immediate past purchase occasion (Krishnamurthi et al., 1992)." On the other hand, $\eta^- = 0$ models the market of promotion-driven products, where the demand of product consists of the base demand $b - ap$ and promotion stimulated demand $\eta^+ \max\{r - p, 0\}$.

This also reflects a harsh market condition under which the firm is very likely to face a gain-seeking demand (Kallio and Halme, 2009).

Assumption 3.1 also provides good approximation to some practical scenarios. Specifically, we provide empirical examples in Section 3.5, in which by imposing Assumption 3.1 does not result in much loss in the goodness of fit of the model.

An immediate consequence from Assumption 3.1 is that both $V^-(r)$ and $p^-(r)$ are now constant functions. In the sequel, we will use constants p^- and V^- to denote the function values of $p^-(r)$ and $V^-(r)$. $V^-(r)$ being a constant function is critical for the simplification of the problem, as it allows us to relate $p^+(r)$ and $p^-(r)$ with $p^*(r)$ in a simple way as demonstrated in the following lemma.

Lemma 3.3. *Under Assumption 3.1, there exists $R_0 \in (0, U)$ such that if $r \leq R_0$, then $V(r) = V^-$ and $p^*(r) = p^- > r$. If $r > R_0$, then $V(r) = V^+(r)$ and $p^*(r) = p^+(r) < r$.*

Lemma 3.3 gives us a broad picture of what $p^*(r)$ looks like. That is, when $r \leq R_0$, $p^*(r)$ is a constant function and is always above r. At the point R_0, there is a "downward jump" from $p^- > R_0$ to $p^+(R_0) < R_0$. When $r > R_0$, $p^*(r)$ is then monotonically increasing in r. In the sequel, we will briefly sketch the idea of how to characterize $p^+(r)$, which leads to a complete characterization of the optimal pricing strategy $p^*(r)$.

Let us reconsider problem (3.11a) when $r \in [R_0, U]$ and keep in mind that $p^+(R_0) < R_0$ and $p^+(r)$ is increasing on $[R_0, U]$. We distinguish between two cases.

Case 1: For any $r \in [R_0, U]$, $p^+(r) \leq R_0$. In this case, $V(p^+(r)) = V^-$ and there is no loss of optimality to write (3.11a) as

$$V^+(r) = \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V^-,$$

from which we can explicitly solve

$$p^+(r) = \frac{\eta^+ r + b}{2(a + \eta^+)}.$$

Then, we have completely characterized the optimal solution as

$$p^*(r) = \begin{cases} p^-, & 0 \leq r \leq R_0, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & r > R_0. \end{cases}$$

Case 2: There exists $R_1 \in (R_0, U)$ such that $p^+(r) \leq R_0$ for any $r \in [R_0, R_1]$ and $p^+(r) \geq R_0$ for any $r \in [R_1, U]$. In this case, going through the same argument in Case 1, we arrive at

$$p^+(r) = \frac{\eta^+ r + b}{2(a+\eta^+)}, \quad r \in [R_0, R_1],$$

and

$$V^+(r) = \frac{(\eta^+ r + b)^2}{4(a+\eta^+)} + \gamma V^-, \quad r \in [R_0, R_1].$$

When $r \in [R_1, U]$, however, we need to again distinguish between two cases based on whether $p^+(r) < R_1$ for all $r \in [R_1, U]$ or not.

Essentially, repeating the analysis sketched above, we can arrive at the main result of this section. Let $m_1 = 0$ and for $k > 1$, $m_k = \frac{\gamma\eta^+}{2(a+\eta^+) - m_{k-1}\eta^+}$.

Proposition 3.3. *Under Assumption 3.1, there exists an integer $N \geq 0$ and $0 < R_0 < R_1 < ... < R_N < U = R_{N+1}$ such that*

$$p^*(r) = \begin{cases} p^-, & 0 \leq r \leq R_0, \\ \dfrac{\eta^+ r + b}{2(a+\eta^+)}, & R_0 < r < R_1, \\ \dfrac{\eta^+ r + b + \sum_{i=0}^{k} (\prod_{j=0}^{i} m_{k+1-j}) b}{2(a+\eta^+) - m_{k+1}\eta^+}, & R_k \leq r < R_{k+1}, \quad k = 1, ..., N. \end{cases}$$

Now we have a complete picture of the optimal pricing strategy $p^*(r)$. After a "downward jump" at R_0, $p^*(r)$ follows a piece-wise linear function with finitely many "upward jumps" at R_k for $k = 1, ..., N$. Moreover, the slopes of these linear pieces increase after each of the "upward jump" by the expression of $p^*(r)$ given in Proposition 3.3.

Another important insight from our analysis is that for $k = 1, ..., N$, $p^*(r)$ maps $[R_k, R_{k+1}]$ to $[R_{k-1}, R_k]$ and finally $p^*(r)$ maps $[R_0, R_1]$ to $[0, R_0]$. This leads us to the study of the dynamics of $p^*(r)$, which is a discontinuous map with more than one discontinuous point. Even though the dynamics of a discontinuous map with only one discontinuous point is already complicated, the characterization presented in Proposition 3.3 allows us to find simple dynamics for $p^*(r)$. Denote $p_1^*(r) = p^*(r)$ and $p_i^*(r) = p^*(p_{i-1}^*(r))$ for $i > 1$. To make the notation clearer, in the following, we alternatively use r^* to denote the constant p^-.

Proposition 3.4. *Let N be the integer in Proposition 3.3. Under Assumption 3.1, there exists an integer n with $2 \leq n \leq N+2$, such that $p_n^*(r^*) = r^*$ and for all $r_0 \in [0, U]$, the optimal reference price path r_t^* converges in at most $N+2$ periods to the unique periodic orbit: $\{r^*, p_1^*(r^*), ..., p_{n-1}^*(r^*)\}$, i.e., there exists $0 \leq \tau \leq N+2$ such that $r_\tau^* = r^*$. Moreover, the periodic orbit has the property $r^* > p_1^*(r^*) > ... > p_{n-1}^*(r^*)$, i.e., $p^*(r)$ is a cyclic skimming pricing strategy.*

Proposition 3.4 suggests that the following pricing strategy for practitioners when the demand they face is gain-seeking and promotion-driven: when consumers' initial reference price is low (below R_0), the firm should use a regular price (r^*). Then the firm applies a skimming pricing strategy by gradually discounting the regular price over time until consumers' reference price falls below R_0, and repeats such pricing strategy. The intuition behind is easy to understand. When consumers have a low reference price, as they are gain-seeking, it will not hurt the firm too much by setting a high price in order to drag consumers' reference price to a higher level. After such manipulation, the firm will benefit greatly by offering discounts since consumers are sensitive to gains and there will be a boost in demand.

An illustration of the optimal pricing strategy and the periodic orbit is provided in Figure 3.5. The parameters used for Figure 3.5 are taken from one of the empirical examples provided in Section 3.5, except $\gamma = 0.1$ here. One can see that, indeed, there are more than one discontinuous point and in this particular example the periodic orbit has period 3 ($n = 3$).

Next, we identify conditions on parameters such that a high-low pricing strategy is optimal. Define the constant $K = \frac{a+\eta^+ - \sqrt{(a+\eta^+)^2 - \gamma(\eta^+)^2}}{\eta^+}$ and recall the constant $R = \frac{b}{a+\sqrt{(a+\eta^+)(a+\eta^-)}} = \frac{b}{a+\sqrt{a(a+\eta^+)}}$ (here, by Assumption 3.1, $\eta^- = 0$), which is the discontinuous point in the myopic pricing strategy, defined in (3.4).

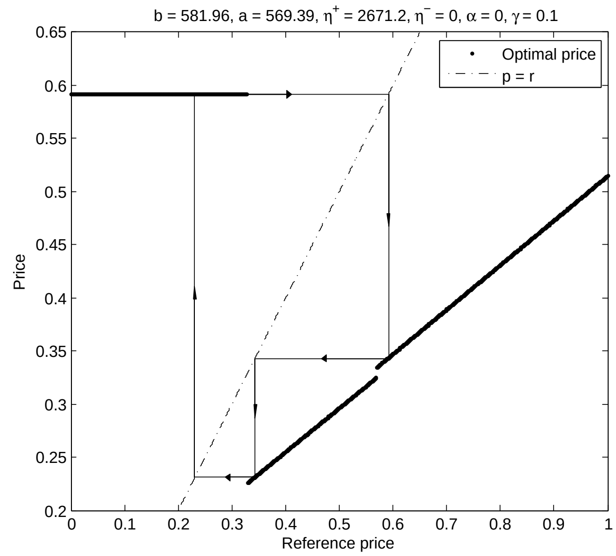


Figure 3.5: Optimal pricing strategy and a periodic orbit

Proposition 3.5. *Under Assumption 3.1, if the following inequality also holds,*

$$\frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} \leq R, \tag{3.12}$$

then a high-low pricing strategy $\{p_H, p_L\}$ *is optimal, where* $p_H = r^*$ *and* $p_L = \frac{\eta^+ r^* + b}{2(a+\eta^+)}$.

Proposition 3.5 above formally settles the conjecture of Popescu and Wu (2007) in the sense that it provides a verifiable condition from problem parameters that guarantees the optimality of high-low pricing strategy. One direct implication of condition (3.12) is that a high-low pricing strategy is always optimal if the feasible prices are not too high, i.e., U is sufficiently small. This is intuitive since if the firm's highest possible price is already low, then there is not much room for the firm to set different discount levels. Condition (3.12) is only a sufficient condition, a necessary and sufficient condition is hard to obtain since we have no prior knowledge on N, $R_0, ..., R_N$, nor r^* and it is possible that $n < N$, which means it is not clear which linear piece p_L will lie in. However, Proposition 3.5 suggests a possibility of solving explicitly the optimal cyclic pricing strategy. That is, if we know the period is exactly n, then we are able to exploit such structures to solve for r^* and the discontinuous points $R_0, R_1, ..., R_N$ given in Proposition 3.3. Next, we identify conditions such that the period of the optimal cyclic pricing strategy is at most n.

Proposition 3.6. *Let* $\underline{R}_0 = R$ *and for* $k \geq 1$ *recursively define* $\underline{R}_k$ *to be the unique solution of the equation*

$$\frac{\eta^+ r + b + \sum_{i=0}^{k} (\prod_{j=0}^{i} m_{k+1-j}) b}{2(a+\eta^+) - m_{k+1}\eta^+} = \underline{R}_{k-1}.$$

Under Assumption 3.1, if the following inequality holds for some $k \geq 0$

$$\frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} \leq \underline{R}_k, \tag{3.13}$$

then the length of period n *must satisfy* $n \leq k+2$.

Note that the recursively defined constants $\underline{R}_k$ provide lower bounds for the unknown constants R_k given in Proposition 3.3, i.e., $\underline{R}_k \leq R_k$ for $k = 0, 1, ..., N$ (see the proof of Proposition 3.6 in Appendix B). Similar to Proposition 3.5, one practical implication here is that the cycle length or the complexity of the pricing strategy depends on the flexibility of pricing, i.e., the range of feasible prices. When U is small, there is less room for the firm to apply intricate pricing strategy that will result in a large cycle length. The advantage of condition (3.13) is that it can be easily verified from the problem parameters. For instance, from the parameters we used in Figure 3.5 and with $U = 1$, it is straightforward to compute that

$$\frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} = 0.502$$

and

$$\underline{R}_0 = 0.302, \quad \underline{R}_1 = 0.493, \quad \underline{R}_2 = 0.949.$$

Applying Proposition 3.6, we know that the optimal solution has a periodic orbit with period less than or equal to 4. Indeed, one can see from Figure 3.5 that the optimal solution has a periodic orbit with period 3.

Finally, we present how to solve for r^* and $R_0, R_1, ..., R_N$ if we know the period is exactly n. First, it is straightforward to see from our previous analysis that if the period is n then $r^* \in [R_{n-2}, R_{n-1})$. Next, let $V_0(r) = V^-$,

$V_0^m(r) = 0$ and recursively for $k = 1, ..., N$ define

$$V_k(r) = \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V_{k-1}(p),$$
$$V_k^m(r) = \max_{p\in[0,U]} \Pi^+(r,p) + \gamma V_{k-1}^m(p).$$

Note that the expressions for $V_k^m(r), k = 1, ..., N$ can be explicitly computed and it is not difficult to see that

$$V_k(r) = V_k^m(r) + \gamma^k V_0(r) = V_k^m(r) + \gamma^k V^-.$$

In the proof of Proposition 3.3, given in Appendix B, we have shown that

$$V(r) = V_{n-1}(r), \quad r \in [R_{n-2}, R_{n-1}).$$

Recall that

$$r^* = p^- = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V(p),$$
$$V^- = \max_{p\in[0,U]} p(b - ap) + \gamma V(p).$$

Thus, without loss of optimality, r^* and V^- can be explicitly solved by

$$r^* = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}(p) = \arg\max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}^m(p),$$
$$V^- = \max_{p\in[0,U]} p(b - ap) + \gamma V_{n-1}(p) = \frac{1}{1-\gamma^n}[r^*(b - ar^*) + \gamma V_{n-1}^m(r^*)].$$

Once the expression for V^- is obtained, we have the explicit expressions for all $V_k(\cdot)$ for $k = 0, ..., N$. By continuity of $V(\cdot)$, for $k \geq 0$, R_k can be sequentially computed by solving the equation

$$V_k(R_k) = V_{k+1}(R_k),$$

and finally, N is obtained by $N = \sup\{k : R_k \leq U\}$.

Combined with Proposition 3.6, if we can find some $k \geq 0$ such that (3.13) holds, then we can repeat the above computations by assuming the period $n = 2, ..., k + 2$. As the value function is the unique solution to the Bellman equation (3.3), the process we suggested is also guaranteed to yield a unique solution. We illustrate the computing process above by continuing with the example in Figure 3.5. By Proposition 3.6, the period $n \leq 4$. As a start, we assume $n = 2$, then

$$r^* = \arg\max_{p \in [0,U]} p(b - ap) + \gamma V_1^m(p) = 0.5890$$

and consequently we can solve

$$R_0 = 0.3290, \quad R_1 = 0.5691.$$

Notice that $r^* \notin [R_0, R_1)$ which leads to a contradiction with $n = 2$.

We proceed to assume $n = 3$, then

$$r^* = \arg\max_{p \in [0,U]} p(b - ap) + \gamma V_2^m(p) = 0.5915$$

and

$$R_0 = 0.3291, \quad R_1 = 0.5692, \quad R_2 > 1.$$

Here, $r^* \in [R_1, R_2)$ is consistent with our assumption that $n = 3$. Thus, we have solved explicitly r^* and R_0, R_1.

3.5 Numerical Study

In this section, we first provide some empirical examples and try to understand what are the implications of real data for the gain-loss asymmetry as well as other parameters used in our model. In particular, we show that Assumption 3.1 generally gives a more parsimonious model while retaining most of the explanatory power of the full model. We then study the performance of several simple pricing strategies as opposed to the optimal pricing strategy. We further examine the performance of simple cyclic pricing strategies and the robustness of Proposition 3.4 numerically when Assumption 3.1 is violated.

3.5.1 Empirical Examples

We utilize the same data set analyzed in Section 2.3. Readers are referred to Section 2.3 for the description of the data set as well as the method in estimating memory factor α.

The following table reports the estimates of the key parameters that of interest, i.e., α, η^+ and η^- for five brands. The rest two brands, belonging to larger volume category, are excluded from the analysis since the estimate for a, the price sensitivity, has a wrong sign. We also report the goodness of fit measurements, i.e., R^2 and adjusted R^2, for the comparison between the full model and the restricted model in which Assumption 3.1 is imposed. All the standard errors are computed via the bootstrapping procedure described in Freedman (1984).

Table 3.1: Parameter Estimates and Goodness of Fit

	$\hat{\alpha}$	$\hat{\eta}^+$	$\hat{\eta}^-$	R^2	Adjusted R^2
Star Kist 6 oz.					
Full Model	0	268587	-17356	0.360	0.354
	(0.064)	(34198.15)	(24011.89)		
Restricted Model	0	267124	0	0.359	0.355
Chicken of the Sea 6 oz.					
Full Model	0.33	573859	-58196	0.570	0.566
	(0.074)	(44329.31)	(39511.26)		
Restricted Model	0	502684	0	0.558	0.555
Bumble Bee Solid 6.12 oz.					
Full Model	0.99	15787	-4195	0.496	0.491
	(0.008)	(5305.44)	(4823.14)		
Restricted Model	0	7646.8	0	0.462	0.459
Bumble Bee Chunk 6.12 oz.					
Full Model	0.15	343059	-11904	0.640	0.637
	(0.066)	(18962.34)	(17330.99)		
Restricted Model	0	333538	0	0.639	0.637
Geisha 6 oz.					
Full Model	0.48	7062.1	574.0	0.545	0.541
	(0.15)	(1466.40)	(1194.65)		
Restricted Model	0	5402.0	0	0.537	0.534

Standard errors are in parenthesis and are obtained from bootstrapping.

In Table 3.1, for all five brands, $\hat{\eta}^+$ is statistically significant and indicates that the perceived discount term $\max\{r_t - p_t, 0\}$ has a large impact on the sales. On the other hand, $\hat{\eta}^-$ has the wrong sign in all but one brand and is statistically insignificant in all cases. As a result, restricting $\hat{\eta}^- = 0$ will give a more parsimonious model while retaining the explanatory power. By comparing the full model with the restricted model, we find surprisingly and uniformly across all five brands, except "Bumble Bee Solid 6.12 oz", that restricting $\hat{\alpha} = 0$ also has little effect on the goodness of fit of the model.

Our result, in agreement with that by Greenleaf (1995), shows that the coefficient of the perceived discount is greater than the perceived surcharge. As we have pointed out in Section 3.1, this result does not necessarily contradict with the prediction made by prospect theory (Tversky and Kahneman, 1991). We consider the demand for the canned tuna in Chicago area was mainly driven by promotions. In other words, when there is no promotion, consumers' reference price tends to be below price and $\max\{r_t - p_t, 0\} = 0$. The only demand left is base demand $b - ap_t$. On the other hand, promotions will reduce price below consumers' reference price and increase the sales greatly by $\eta^+ \max\{r_t - p_t, 0\}$.

In summary, our empirical study illustrates that Assumption 3.1 is statistically plausible for some realistic settings and practical applications.

3.5.2 Performance of Simple Pricing Strategies

In this subsection, we first compare the performance of several simple pricing strategies with that of the optimal pricing strategy based on one of the empirical examples ("Star Kist 6 oz.") in Section 3.5.1. Note that Assumption 3.1 is satisfied in the "Star Kist 6 oz." case in Section 3.5.1. To examine what happens when Assumption 3.1 fails, we first design different parameter configurations that violate Assumption 3.1 and study the performance of simple cyclic pricing strategies under all these scenarios. Then, we numerically illustrate how the optimal pricing strategies will change when the assumption $\eta^- = 0$ is relaxed.

Since the numerical values of the parameter estimates are quite large, for convenience we divide all the parameters in demand function by 100 without affecting the optimal solutions. The demand function for the item "Star Kist 6 oz." is then given by

$$D(r, p) = 581.96 - 569.39p + 2671.2 \max\{r - p, 0\},$$

and the reference price formulation is

$$r_{t+1} = p_t.$$

We set the price range to be $[0, 1]$ which includes the price range of historical data and the initial reference price to be the average price in the data set, i.e., $r_0 = 0.8$.

Now we compare the performance of three simple pricing strategies with the optimal pricing strategy over a horizon of $T = 100$. For the rest of this section, we set the discount factor $\gamma = 0.9$. The optimal pricing strategy here is solved by using the algorithm developed in Hu (2012) (one is also referred to Chapter 4) for the finite horizon problem when Assumption 3.1 holds.

One simple strategy we consider is the constant pricing strategy or Every Day Low Price (EDLP) mentioned in Fibich et al. (2003), Popescu and Wu (2007) and Nasiry and Popescu (2011). In this case, the constant pricing strategy p_{EDLP} amounts to solving

$$\begin{aligned} &\max_{p\in[0,1]} \sum_{t=0}^{100} \gamma^t p(581.96 - 569.39p + 2671.2 \max\{r_t - p, 0\}), \\ &\text{s.t.} \quad r_{t+1} = p, \quad 0 \le t \le 99, \quad r_0 = 0.8. \end{aligned}$$

It is easy to see that the optimal solution is $p_{\text{EDLP}} = 0.48$. Note that similar to the method by Fibich et al. (2003), here p_{EDLP} depends on r_0.

Another simple strategy we consider is the high-low pricing strategy. The high-low pricing strategy p_H and p_L amount to solving

$$\begin{aligned} &\max_{p_1,p_2\in[0,1]} \sum_{t=0}^{100} \gamma^t p_i(581.96 - 569.39p_i + 2671.2 \max\{r_t - p_i, 0\}), \\ &\text{s.t.} \quad r_{t+1} = p_i, \quad 0 \le t \le 99, \quad r_0 = 0.8, \\ &\qquad i = t \bmod 2, \quad 0 \le t \le 100. \end{aligned}$$

The above problem turns out to be difficult to solve exactly. Instead, we discretize the prices and search for the optimal high-low pricing strategies. The optimal solution is solved as $p_H = 1$ and $p_L = 0.49$.

Finally, we consider the myopic pricing strategy which is explicitly solved as (3.5).

We plot the relative ratio of the profit obtained up to time t to the optimal

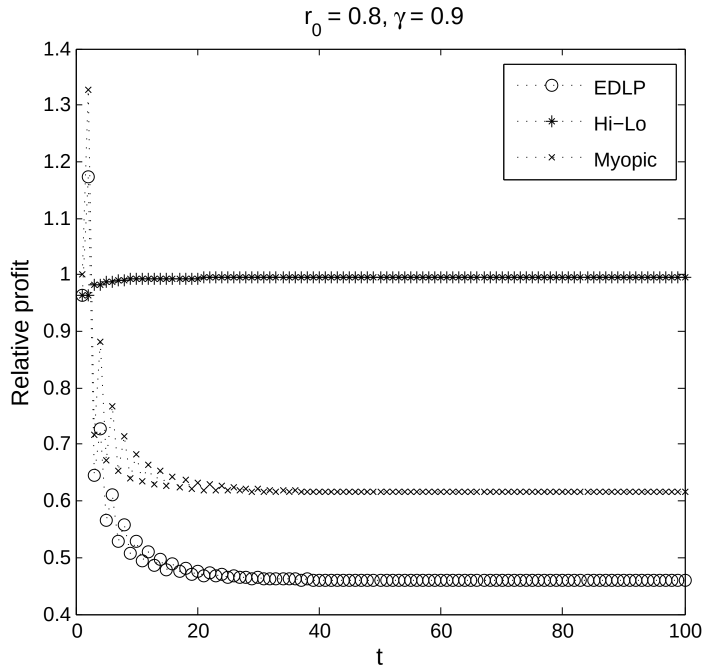


Figure 3.6: Profit comparison of simple pricing strategies

profit up to time t, for $0 \leq t \leq T$, under the constant, high-low, and myopic pricing strategies in Figure 3.6. Our result shows that even though in the first two periods constant and myopic pricing strategies obtain a profit higher than the optimal pricing strategy, their performances decay quickly. Over the whole planning horizon, the high-low pricing strategy, which achieves more than 99% of the optimal profit, is much better than both the constant and myopic pricing strategies. By checking the dynamics of the optimal pricing strategy for the infinite horizon problem, we find that the optimal pricing strategy in this particular case is indeed a high-low pricing strategy and a time horizon of $T = 100$ is long enough to exhibit the long-run behavior. Interestingly, Figure 3.6 shows that even myopic pricing strategy can outperform the constant pricing strategy.

With the understanding that the high-low pricing strategy performs quite well in our empirical example, we next examine the performance of the high-low pricing strategy when Assumption 3.1 is not satisfied. More specifically, we consider the following demand functions

$$D(r,p) = 581.96 - 569.39p + 2671.2 \max\{r-p, 0\} + \eta^- \min\{r-p, 0\},$$

where η^- is chosen such that the ratio $\eta^-/\eta^+ \in \{0, 0.02, 0.04, ..., 0.98\}$ with $\eta^+ = 2671.2$. We also let $\alpha \in \{0, 0.2, 0.4, 0.6, 0.8\}$ in the reference price formulation (3.1). In total, these parameters include 250 scenarios.

Instead of solving for an exact optimal solution, we numerically approximate the optimal solution through discretization and value iterations. Both the price and reference price are discretized with a step size of 0.0005 (a total of 2000 points for each of them). The ratio of the profit attained by the high-low pricing strategy over the profit attained by the optimal pricing strategy is reported in Figure 3.7 for all 250 combinations.

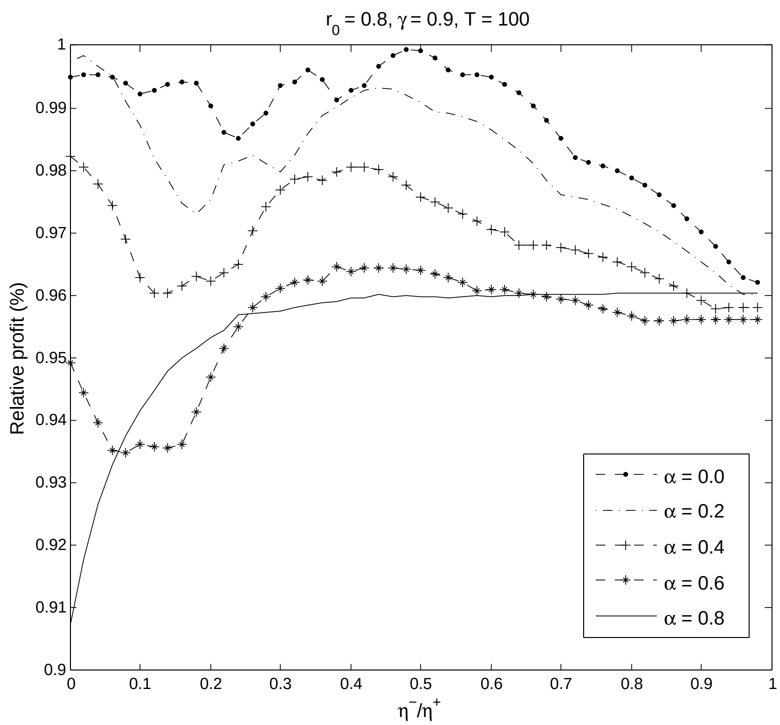


Figure 3.7: Relative profit under different parameter combinations

From Figure 3.7 we note that there is no clear monotonic relationship between relative profit and the two parameters η^- and α. However, we observe that even in the worst case ($\alpha = 0.8, \eta^- = 0$) the high-low pricing strategy achieves above 90% of the optimal profit. This is somewhat surprising given how complex the optimal pricing strategy in the worst case is as illustrated in Figure 3.3.

One might wonder in the worst scenario whether other cyclic pricing strategies with relatively short cycle will achieve better performance. We answer this question in Table 3.2.

Table 3.2: Performance of Cyclic Pricing Strategies

Cycle Length	1	2	3	4	5
Relative Profit	89.87%	90.74%	96.76%	98.34%	98.30%

Note that for this particular parameter configuration, even the constant pricing strategy (cycle length is 1) can have a decent performance and a cyclic pricing strategies with cycle length of 3 or 4 can already achieve more than 95% of the optimal profit. Another interesting fact is that increasing the length of cycle, though complicating the optimization process, does not necessarily result in a better performance as indicated in the last two columns in Table 3.2. This is because, when the optimal pricing strategy is of cycle length 2, for instance, imposing a constraint that cycle length equals to 3 or any odd numbers will only make the resulting cyclic pricing strategies more different from the optimal.

Finally, we check to what extent Proposition 3.4 will still hold if the assumption $\eta^- = 0$ is relaxed. For this purpose, we fix $\alpha = 0$ and vary η^- such that the ratio $\eta^-/\eta^+ \in \{0, 0.02, 0.04, ..., 0.98\}$. When $\eta^-/\eta^+ < 0.16$, it is found that high-low pricing strategy is always optimal and the dynamics proposed in Proposition 3.4 holds true. For conciseness, in Figure 3.8, we only report the results for the values $\eta^-/\eta^+ \in \{0.16, 0.18, 0.20, 0.22, 0.24, 0.26\}$. When $\eta^-/\eta^+ = 0.16, 0.18$, similar to the cases for $\eta^-/\eta^+ < 0.16$, there is only one discontinuous point in the optimal pricing strategies and the high-low pricing strategy is optimal. For $\eta^-/\eta^+ = 0.20, 0.22, 0.24$, there are multiple discontinuous points appearing in the optimal pricing strategies, however, the periodic orbits in these figures indicate that a cyclic skimming pricing strategy with cycle length 3 is still optimal. That is, the conclusion of Proposition 3.4 still holds for $\eta^-/\eta^+ \leq 0.24$. However, the last figure shows that another 2% increment in the ratio will result in a very different optimal pricing strategy.

3.6 Conclusion

In this chapter we analyzed a dynamic pricing problem in a market with gain-seeking consumers. In this model, demand depends on both current

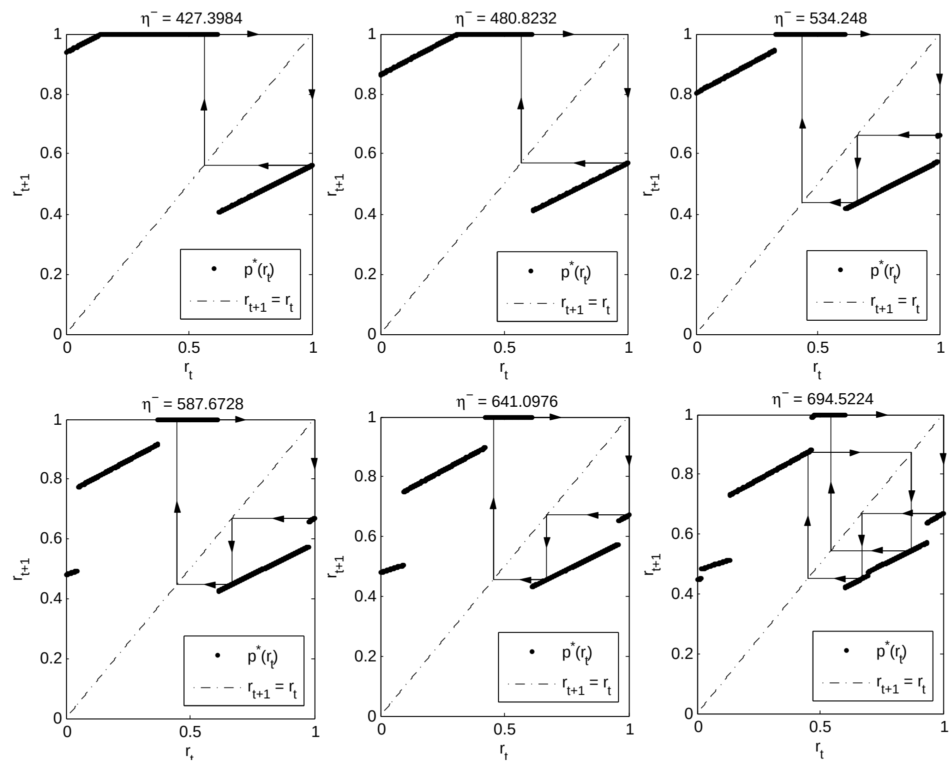


Figure 3.8: The optimal pricing strategies when $\alpha = 0$ and $\eta^{-}/\eta^{+} \in \{0.16, 0.18, 0.20, 0.22, 0.24, 0.26\}$

selling price and reference price, where the latter evolves according to an exponentially smoothing process of past prices.

We showed that even employing the myopic pricing strategy can result in a very complicated dynamics of reference prices. We identified conditions that lead to simple pricing dynamics, for example, high-low pricing, cyclic skimming pricing or cyclic penetrating pricing.

Realizing the complexity of the problem, we restricted ourselves to an empirically validated special case and proved that a cyclic skimming pricing strategy is optimal over an infinite horizon. We further provided conditions on the upper bound of the cycle lengths. Although our characterization of the optimal pricing strategy built on a piece-wise linear demand model, both Proposition 3.3 and Proposition 3.4 can be extended to general nonlinear demand functions proposed in Popescu and Wu (2007) by imposing an assumption similar to Assumption 3.1.

Our work is only a start in exploring the effects of gain-seeking behavior/phenomenon on dynamic pricing problems. It would be very interesting to both categorize and characterize the possible patterns of the optimal pricing strategy under the general case. As one may see from our analysis of the special case, the structure of the optimal solution has intimate connections with the dynamics of the optimal solution. Thus, it is important to understand how the structure and the dynamics of the optimal solution interact with each other under more general settings.

Finally, it would be interesting to study the impact of gain-seeking reference price effects on the joint pricing and inventory decisions. Pricing and inventory integration has received much attentions in the past few years (see, for example, Chen and Simchi-Levi, 2004a,b, 2006, 2012). Recently, Chen et al. (2013) incorporated reference price effect into coordinated pricing and inventory models. However, their model focuses only on loss-averse consumers.

Chapter 4

Efficient Algorithms for Dynamic Pricing Problem

4.1 Introduction

Part of Chapter 2, Chapter 3 and many of the references therein are devoted to understanding the qualitative behavior of the optimal pricing strategies when the firm faces reference price dependent demand and many managerial insights are obtained. However, to our best knowledge, how to compute the optimal prices accurately and efficiently remains largely unexplored. As Fibich et al. (2003) point out, "calculations of optimal strategies were limited to numerical simulations using dynamic programming". In practice, the revenue of many firms in retail industry, where a large portion of empirical evidences on reference price effects are found, can exceed $30 billion per year (see Talluri and Van Ryzin, 2005). This fact highlights the importance of even small incremental gains from revenue management. Thus, computing the optimal prices accurately becomes a critical question. On the other hand, a firm not only needs to coordinate pricing decisions with other operations management decisions such as inventory decisions but also makes all those decisions for over thousands of products. When the more realistic reference price models are incorporated in such decision support systems, the efficiency of computing the simplest basic model becomes crucial.

In this chapter, we look into the computational issue of the dynamic pricing problem under the exponential smoothing reference price model. In this model, a firm sells a product over a finite time horizon and faces possibly time-varying demands, which depend on not only the price the firm sets in that period but also all the historical prices the firm set through reference price effects. Like most of the previous literature, the demand in each period is assumed to be deterministic. This assumption is plausible in many practical settings, in which a firm can predict future demands quite accurately from historical sales data. As we have discussed in Chapter 3, a unique feature and also a significant challenge in this model is the asymmetry in reference price effects, i.e., consumers' perception of gains and losses could be different. This leads to a non-smooth optimization problem, for which no standard optimization methods can be applied.

Hu (2012) attempts to address the computational issue in the dynamic pricing problem with reference price effects in a deterministic joint inventory and pricing model, which includes the problem analyzed in this chapter as a subproblem. However, one of the central algorithm in Hu (2012) ignores the fact that the value function may be non-differentiable, which can cause the running time of algorithm presented in Hu (2012) to be exponential in the problem horizon. Indeed, it is still an open question that even for the loss-averse demands whether the dynamic pricing problem can be solved exactly in polynomial time or not.

Facing with such challenges, in this chapter we follow the idea in Hu (2012) to develop an algorithm which we prove to be a polynomial time algorithm for the loss-averse demands under a certain mild technical condition on the input parameters. Although our model assumes piecewise linear demand function, the core part of the algorithm may also be used to other forms of demand function. In addition to the algorithm itself, a few properties we found along with the algorithm are potentially applicable to other non-smooth optimization problems as well. We refer the readers to Hu (2012) for another polynomial time algorithm that solves a special case of the problem when demands are gain-seeking and for a heuristic that can be used for any input parameters. We apply the algorithm in this chapter along with that developed in Hu (2012) to a practical problem from industry with real data and demonstrate the efficiency and robustness of the algorithm.

Aside from the stream of literature on dynamic pricing problems with reference price effects, there are a few recent attempts that incorporate reference price effects into integrated inventory and pricing models. Gimpl-Heersink (2008) analyzes a stochastic periodic review finite horizon model in which demand is a function of both the current price and the reference price with additive random noise. Chen et al. (2013) study a similar model and introduce a novel transformation technique to convert a non-concave single-period revenue function to a modified revenue function that is concave. They characterize various structures of optimal solutions. Based on a different mechanism rather than the reference price models, Ahn et al. (2007) develop algorithms for a periodic review finite horizon deterministic model in which demand depends on past prices. Even though the concentration of this chapter is on developing algorithms for the pure pricing problem, our algorithms can serve as a building block in a deterministic joint inventory and pricing problem.

The remainder of this chapter is organized as follows. In Section 4.2 the mathematical formulation of our model is presented. In Section 4.3, we analyze the model with loss-averse consumers and develop a strongly polynomial time algorithm to solve the optimal prices exactly. Our algorithm along with the algorithms in Hu (2012) are tested on real data in Section 4.4 to solve a practical industry problem with analysis on the efficiency as well as the robustness of the algorithms. Finally, we conclude the chapter in the last section with some suggestions for future research. To maintain a clear presentation, all technical proofs are presented in Appendix C.

4.2 Model

We assume that the firm sells a product over a finite horizon of T periods. As in Chapter 3, we restrict ourselves to the exponential smoothing model, i.e.,

$$r_{t+1} = \alpha r_t + (1-\alpha)p_t, \quad t = 1, 2, \cdots, T, \tag{4.1}$$

where we restrict $\alpha < 1$ to avoid the case that past prices have no impact on demand.

Following Chapter 2 and Chapter 3, the demand at period t, with a given price p and a reference price r, is modeled as

$$D_t(p, r) = b_t - a_t p + \eta^+ \max\{r - p, 0\} + \eta^- \min\{r - p, 0\},$$

where $D_t(p, p) = b_t - a_t p$ is the base demand independent of reference prices, $\eta(r - p)$ is the additional demand or demand loss induced by the reference price effect. Here, the potential market size b_t and the price sensitivity a_t are non-negative, but unlike previous chapters, they are allowed to be time-varying to reflect a dynamic market conditions. In this chapter, we consider both loss-averse (loss/gain neutral) case and gain-seeking case and make no assumption on the relative magnitudes of η^+, η^-. However, since these two coefficients, along with the memory factor α reflect consumers' internal perceptions of reference prices, losses and gains, they are assumed to be time-invariant even though our algorithms can be easily extended to the case when they vary with time.

Facing the reference price dependent demands and an initial reference price r_1, the firm then maximizes its total profit over the planning horizon by determining the optimal price in each period. That is,

$$\begin{aligned} \max_{p_t:1\leq t\leq T} \quad & \pi_1(r_1,p_1)+\pi_2(r_2,p_2)+\cdots+\pi_T(r_T,p_T) \\ \text{s.t.} \quad & r_{t+1}=\alpha r_t+(1-\alpha)p_t, \quad p_t\in[L_t,U_t], \quad t=1,\cdots,T. \end{aligned} \tag{4.2}$$

In the problem formulation (4.2), $\pi_t(r_t,p_t)=p_tD_t(r_t,p_t)$ is the profit collected in period t, $1\leq t\leq T$, where we have implicitly assumed that the marginal cost is zero for simplicity and all our results can be extended to the case when there is a non-zero marginal cost. The lower bounds L_t and upper bounds U_t on prices are non-negative and are also allowed to be time-varying. One reason for this lies in our formulation of time-varying demands. Since demands can not be negative, this naturally generates a time-varying upper-bounds on prices. Also, in some scenarios, firm has other objectives such as minimum sales or maximum allowable discount on prices which could vary season by season and result in a time-varying constraints on prices.

Since the effect of past prices on period t's demand is summarized by the reference price r_t, it will be sometimes convenient to express profit in terms of reference prices. In particular, given the reference prices r_t, r_{t+1} at periods $t, t+1$, respectively, the price p_t and the profit, denoted by $\Pi_t(r_t,r_{t+1})$, at period t can be expressed as

$$p_t=\frac{r_{t+1}-\alpha r_t}{1-\alpha}, \quad \Pi_t(r_t,r_{t+1})=\pi_t\left(r_t,\frac{r_{t+1}-\alpha r_t}{1-\alpha}\right).$$

4.3 Loss-averse Consumers

In this section we focus on the case when consumers are loss-averse, i.e., $\eta^- > \eta^+$. We remark here that in the loss-neutral case ($\eta^-=\eta^+$), if there are no constraints on prices, then explicit solutions can be computed via standard linear-quadratic control techniques with computational complexity of $O(T)$ (see, for instance, Anderson and Moore, 2007). On the other hand, with price constraints, the loss-neutral case can be reduced to a special case analyzed in this section and consequently our algorithms can be applied.

In the following, we formulate problem (4.2) as a dynamic programming problem and discuss the potential challenges in solving exactly the dynamic programming problem. In the first subsection, we lift our discussion to a more general problem setting and explore the essential properties in the problem that help us to overcome the challenges. In the second subsection, we then show that under a technical assumption, these properties hold in our dynamic programming problem and we develop a strongly polynomial time algorithm to solve the problem exactly.

We first formulate problem (4.2) as a dynamic programming problem. Let $G_{t+1}(r_{t+1})$, $t \leq T$, be the maximal accumulated profit up to period t when reference price r_{t+1} is specified at period $t+1$. That is,

$$\begin{aligned} G_{t+1}(r_{t+1}) = \quad & \max \quad \Pi_1(r_1, r_2) + \Pi_2(r_2, r_3) + \cdots + \Pi_t(r_t, r_{t+1}), \\ & \text{s.t.} \quad \alpha r_s + (1-\alpha)p_s = r_{s+1}, \quad p_s \in [L_s, U_s], \quad s \leq t. \end{aligned}$$

Apparently solving problem (4.2) amounts to maximizing $G_{T+1}(r)$. Thus, it suffices to determine the expression of $G_{T+1}(r)$, which can be iteratively derived for $t = 2, \cdots, T$ through solving the problem

$$\begin{aligned} G_{t+1}(q) &= \max_r \{\Pi_t(r,q) + G_t(r) : \frac{q - \alpha r}{1-\alpha} \in [L_t, U_t]\}, \\ r_t(q) &= \arg\max_r \{\Pi_t(r,q) + G_t(r) : \frac{q - \alpha r}{1-\alpha} \in [L_t, U_t]\}, \end{aligned} \tag{4.3}$$

where $G_2(q) = \Pi_1(r_1, q)$ for $q \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1]$. Note here that due to the price constraints $p \in [L_t, U_t]$, $G_{t+1}(q)$ is only defined for those states that lead to feasible solutions. For convenience we specify $G_{t+1}(q) = -\infty$ if q leads to an empty feasible set in the above problem and the effective domain of $G_{t+1}(q)$ is then defined as $\{q : G_{t+1}(q) > -\infty\}$. In particular, the first profit-to-go function $G_2(q)$ is of the following form

$$G_2(q) = \begin{cases} g_2(q), & \text{if} \quad q \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1], \\ -\infty, & \text{otherwise}, \end{cases}$$

where $g_2(q) = \Pi_1(r_1, q)$ consists of two quadratic pieces and is concave and continuously differentiable except at r_1.

The main challenge in solving problem (4.3) efficiently is associated with non-differentiability in the objective function. For instance, even for $t = 2$, both $G_t(r)$ and $\Pi_t(r, q)$ are non-differentiable functions and consist of two different quadratic pieces. Thus, we need to answer the question that in general for $t > 2$, how "simple" can $G_t(r)$ be? In other words, at how many points will $G_t(r)$ be non-differentiable and how many quadratic pieces will $G_t(r)$ be consisted of?

Here, we give a brief sketch of our approach in dealing with the above challenge. Suppose we already have the analytical expressions for $G_t(r)$. As we will later show that there exist $\underline{r}_t, \overline{r}_t$ such that $G_t(r)$ follows the form

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty), \end{cases}$$

where $g_t(r)$ is a continuous function defined on the whole real line. We consider the following problem first:

$$\begin{aligned} f(q) &= \max_r \{\Pi_t(r, q) + g_t(r)\}, \\ r^*(q) &= \arg\max_r \{\Pi_t(r, q) + g_t(r)\}. \end{aligned} \tag{4.4}$$

Note that in problem (4.4), we ignore the price constraints and extend the effective domain of $G_t(r)$ to the whole real line by replacing $G_t(r)$ with $g_t(r)$. This allows us to concentrate on the issue of non-differentiability, which is addressed in the following subsection in a more general problem setting. Specifically, we develop an efficient algorithm to solve problem (4.4) and show that the structure of $f(q)$ is "as simple as" $g_t(r)$.

We then consider the problem

$$\begin{aligned} f_c(q) &= \max_r \{\Pi_t(r, q) + g_t(r) : \frac{q - \alpha r}{1 - \alpha} \in [L_t, U_t]\}\}, \\ r_c^*(q) &= \arg\max_r \{\Pi_t(r, q) + g_t(r) : \frac{q - \alpha r}{1 - \alpha} \in [L_t, U_t]\}\}, \end{aligned} \tag{4.5}$$

where we use subscript "c" in the solution and optimal value function to emphasize the presence of price constraint in problem (4.5). The second subsection deals with the issue of computing $r_c^*(q), f_c(q)$ from $r^*(q), f(q)$ and computing $r_t(q), G_{t+1}(q)$ from $r_c^*(q), f_c(q)$.

4.3.1 A General Problem

The purpose of this subsection is two fold. First, we believe the presentation of the algorithm can be made clean and organized by highlighting only those properties that are essential for our algorithm. Second, the treatment in a more general setting naturally makes our algorithm robust to other forms of demand function. Furthermore, some of the properties we identified are potentially applicable to other problems as well. The following problem, which is a generalization of problem (4.4), is considered in this subsection.

$$\begin{aligned} f(q) &= \max_r \{\Pi(r,q) + g(r)\}, \\ r^*(q) &= \arg\max_r \{\Pi(r,q) + g(r)\}. \end{aligned} \tag{4.6}$$

We impose the following two assumptions on both the input functions $\Pi(r,q)$ and $g(r)$ as well as the output functions $f(q)$ and $r^*(q)$ throughout this subsection.

Assumption 4.1. *(a) $\Pi(r,q) : \mathbb{R}^2 \to \mathbb{R}$ is continuous, strictly supermodular in (r,q) and strictly concave in r. Furthermore, $\Pi(r,q)$ has the form*

$$\Pi(r,q) = \begin{cases} \Pi^+(r,q), & q \le r, \\ \Pi^-(r,q), & q \ge r, \end{cases}$$

where $\Pi^+(r,q)$ and $\Pi^-(r,q)$ are continuously differentiable functions defined on $\mathbb{R}^2$. We denote $\pi(q) = \Pi(q,q) = \Pi^+(q,q) = \Pi^-(q,q)$.

(b) $g(r) : \mathbb{R} \to \mathbb{R}$ is concave and continuously differentiable except at finite points $r_1, ..., r_m$, with $-\infty = r_0 < r_1 < ... < r_m < r_{m+1} = +\infty$. More specifically,

$$g(r) = \begin{cases} g_1(r), & r_0 \le r \le r_1, \\ ... \\ g_{m+1}(r), & r_m \le r \le r_{m+1}, \end{cases}$$

where $g_j(\cdot)$, $1 \le j \le m+1$ are all continuously differentiable functions defined on $\mathbb{R}$. We call $r_1, ..., r_m$ the kink points of $g(r)$.

An immediate consequence from the strict concavity and continuity assumptions in Assumption 4.1 is that $r^*(q)$ is single valued and continuous (see, for example, Ok, 2007). We further impose the following assumptions on $f(q)$ and $r^*(q)$.

Assumption 4.2. *(a) $f(q)$ is concave.*

(b) $q - r^*(q)$ satisfies the single crossing property. That is, for $q' > q''$, $q'' - r^*(q'') > 0$ implies $q' - r^*(q') > 0$, and $q'' - r^*(q'') \geq 0$ implies $q' - r^*(q') \geq 0$.

Finding general conditions on $\Pi(r, q)$ and $g(r)$ such that Assumption 4.2 holds is an interesting research topic itself (for instance, Assumption 4.2 (a) will hold if we further assume $\Pi(r, q)$ to be jointly concave) and is beyond the scope of this chapter. However, by using a transformation technique developed in Chen et al. (2013) we will prove in the next subsection that under some mild conditions Assumption 4.2 holds for our dynamic pricing problem.

The rest of this subsection is divided into three parts. (a) We first prove that $r^*(q)$ can be decomposed into the solutions of two simpler problems and $r^*(q)$ has certain monotonic structures. (b) Then we show that even though $\Pi(r, q)$ may not be differentiable along the line $r = q$, $f(q)$ is, surprisingly, continuously differentiable except at at most m points, whose candidates are exactly $r_1, ..., r_m$, the kink points of $g(r)$. In the extreme case, when $g(r)$ is continuously differentiable, then $f(r)$ is also continuously differentiable. That is, the kink points of $g(r)$ are, in some sense, "preserved" under the maximization of problem (4.6). This observation connects the algorithm developed in this subsection with the dynamic programming algorithm in the next subsection. (c) Finally, we consider the computational issue. We show how the structures of $r^*(q)$ allow us to develop an efficient algorithm to compute explicitly $r^*(q)$ and $f(q)$ once we know the functional form of the input functions $g(r)$ and $\Pi(r, q)$.

(a) Structures. We introduce the following two problems

$$
\begin{aligned}
f^+(q) &= \max_r \{\Pi^+(r, q) + g(r)\}, \\
r^+(q) &= \arg\max_r \{\Pi^+(r, q) + g(r)\},
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
f^-(q) &= \max_r \{\Pi^-(r,q) + g(r)\}, \\
r^-(q) &= \arg\max_r \{\Pi^-(r,q) + g(r)\}.
\end{aligned} \tag{4.8}
$$

Our first result states that $r^*(q)$ can be decomposed into $r^+(q)$, $r^-(q)$ in the following way.

Proposition 4.1. *There exist $\underline{Q} \le \overline{Q}$, such that*

$$
r^*(q) = \begin{cases} r^+(q), & q < \underline{Q}, \\ q, & \underline{Q} \le q \le \overline{Q}, \\ r^-(q), & q > \overline{Q}, \end{cases} \quad \textit{and} \quad f(q) = \begin{cases} f^+(q), & q < \underline{Q}, \\ \pi(q) + g(q), & \underline{Q} \le q \le \overline{Q}, \\ f^-(q), & q > \overline{Q}. \end{cases}
$$

Next, we closely examine the structure of $r^+(q)$ and $r^-(q)$.

Lemma 4.1. *There exist $-\infty = \overline{q}_0 < \underline{q}_1 \le \overline{q}_1 < \underline{q}_2 \le \overline{q}_2 < \ldots < \underline{q}_m \le \overline{q}_m < \underline{q}_{m+1} = +\infty$ such that for $1 \le j \le m$, $r^+(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^+(q)$ is strictly increasing elsewhere.*

Lemma 4.1 also holds for $r^-(q)$. That is, for $1 \le j \le m$, $r^-(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^-(q)$ is strictly increasing elsewhere. However, the threshold values $\underline{q}_j$ and $\overline{q}_j$ may be different from that for $r^+(q)$. Now combining this observation and Proposition 4.1 we arrive at the following result. With a slight abuse of notations, we still use $\underline{q}_j$ and $\overline{q}_j$ to denote the new threshold values.

Proposition 4.2. *There exist $-\infty = \overline{q}_0 < \underline{q}_1 \le \overline{q}_1 < \underline{q}_2 \le \overline{q}_2 < \ldots < \underline{q}_m \le \overline{q}_m < \underline{q}_{m+1} = +\infty$ such that for $1 \le j \le m$, $r^*(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^*(q)$ is strictly increasing elsewhere.*

Graphically, as illustrated in Figure 4.1, Proposition 4.2 suggests a ladder shape of $r^*(q)$. That is, $r^*(q)$ alternates between a strictly increasing piece and a constant piece with the constant piece corresponding to a kink point of $g(\cdot)$. Clearly, if we can compute the threshold values and determine the expressions of the strictly increasing pieces of $r^*(q)$, then the whole expressions of $r^*(q)$ can be obtained easily. We will address how to determine $\overline{q}_{j-1}$ and $\underline{q}_j$ as well as compute $r^*(q)$ on $(\overline{q}_{j-1}, \underline{q}_j)$ in the third part of this subsection.

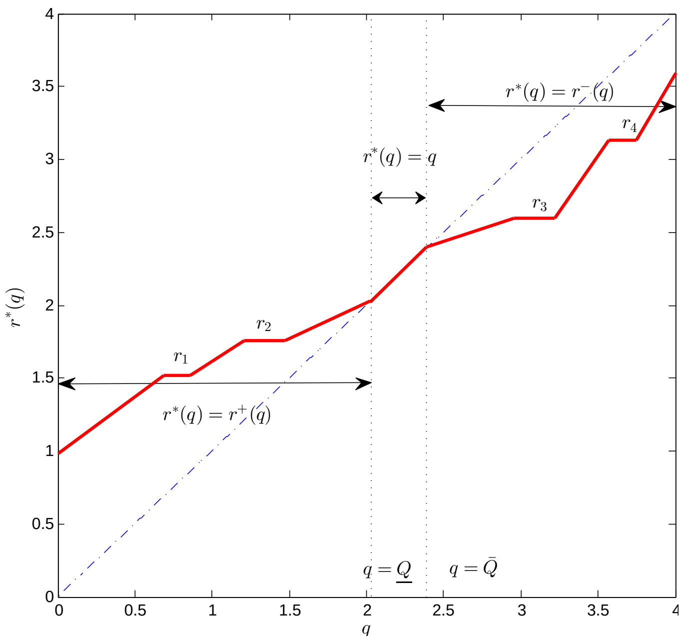


Figure 4.1: Illustration of $r^*(q)$

(b) Preservation. Now we turn our attention to the optimal value function $f(q)$. Despite the fact that $\Pi(r, q)$ may be neither differentiable in r nor in q, we show that the number of kink points of $f(q)$ are less than or equal to $g(r)$ and the candidate kink points of $f(q)$ are exactly $r_1, ..., r_m$. That is, $f(q)$ has certain similar properties as $g(r)$, which is desirable in a dynamic optimization setting.

The preservation of kink points relies critically on the following observation.

Lemma 4.2. *$f^+(q)$ is continuously differentiable on $(-\infty, \underline{Q})$ and $f^-(q)$ is continuously differentiable on $(\overline{Q}, +\infty)$.*

We would like to point out that the only assumptions required for Lemma 4.2 to hold (or even extend to the whole domain of q) are the concavity of $f^+(q)$ and $f^-(q)$ and the continuous differentiability of $\Pi^{\pm}(r, q)$ with respect to q.

An immediate consequence of Lemma 4.2 is that $f(q)$ is also continuously differentiable on $(-\infty, \underline{Q})$ and $(\overline{Q}, +\infty)$. On the other hand, $f(q) = \pi(q) + g(q)$, when $q \in (\underline{Q}, \overline{Q})$ and the kink points of $f(q)$ are then solely determined by the kink points of $g(q)$. The following result further claims that $f(q)$ is differentiable at $\underline{Q}$ and $\overline{Q}$ unless $\underline{Q}, \overline{Q} \in \{r_1, ..., r_m\}$.

Proposition 4.3. *$f(q)$ has at most m kink points. The possible kink points are $r_1, ..., r_m$.*

In the extreme case when $m = 0$, then $f(q)$ is continuously differentiable even if the objective function is neither differentiable in the decision variable r nor in the parameter q. In contrast, many envelope theorems that study the differentiability properties of the value function of a parameterized optimization problem (see, for example, Milgrom and Segal, 2002; Clausen and Strub, 2012) assume the objective function to be differentiable in the parameter.

(c) Computation. To address the issue of computation, we need to impose the following assumption in the remaining of this subsection on the functional forms of each continuously differentiable piece of $g(r)$.

Assumption 4.3. *For $1 \le j \le m+1$, $g_j(r)$ has the following functional form on $[r_{j-1}, r_j]$. There exist $n_j \ge 1$ and $r_{j-1} = r_j^{(0)} < r_j^{(1)} < ... < r_j^{(n_j-1)} < r_j^{(n_j)} = r_j$ such that*

$$g_j(r) = \begin{cases} g_j^{(1)}(r), & r_j^{(0)} \le r \le r_j^{(1)}, \\ ... \\ g_j^{(n_j)}(r), & r_j^{(n_j-1)} \le r \le r_j^{(n_j)}, \end{cases}$$

where $g_j^{(i)}(r)$, $1 \le i \le n_j$, are functions defined on $\mathbb{R}$ and have analytical forms such that the following optimization problems

$$\max_r \{\Pi^{\pm}(r, q) + g_j^{(i)}(r)\},$$

can be solved in $O(1)$ time.

One example satisfying Assumption 4.3 is the case when $\Pi^{\pm}(r, q)$ and $g_j^{(i)}(r)$ are all quadratic functions. For convenience, for $1 \le j \le m+1$, we call $r_j^{(i)}$, $1 \le i \le n_j$ (except $r_{m+1}^{(n_{m+1})} = r_{m+1} = +\infty$) the *breakpoints* of $g(r)$ and denote $n = \sum_{j=1}^{m+1} n_j - 1$ to be the total number of breakpoints of $g(\cdot)$. Note that the breakpoints include the kink points we defined in Assumption 4.1. The key difference here is that at a breakpoint, the analytical forms of the function on the two sides of the breakpoint are different. However, different analytical forms may still result in the left derivative at the breakpoint having the same value as the right derivative. When the left derivative does not equal to the right derivative at the breakpoint, we then call this breakpoint as a kink point.

By Proposition 4.2, on $(\overline{q}_{j-1}, \underline{q}_j)$, $r^*(q)$ is strictly increasing and $r_{j-1} < r^*(q) < r_j$. As a result, there exist $\overline{q}_{j-1} = q_j^{(0)} < q_j^{(1)} < ... < q_j^{(n_j-1)} < q_j^{(n_j)} = \underline{q}_j$, such that for $1 \leq i \leq m$, on $(q_j^{(i-1)}, q_j^{(i)})$, $r_j^{(i-1)} < r^*(q) < r_j^{(i)}$. This suggests that on $(q_j^{(i-1)}, q_j^{(i)})$,

$$f(q) = \max_r \{\Pi(r, q) + g_j^{(i)}(r)\},$$
$$r^*(q) = \arg\max_r \{\Pi(r, q) + g_j^{(i)}(r)\}.$$

Note that this observation also applies to $r^+(q)$ and $r^-(q)$ on the regions where they are strictly increasing. Applying Proposition 4.1, by comparing q with the threshold values $\underline{Q}, \overline{Q}$, whether $\Pi(r, q) = \Pi^+(r, q)$, $\Pi(r, q) = \Pi^-(r, q)$ or $r^*(q) = q$ can be determined unambiguously in the above problem. This leads to the following result to bound the breakpoints of $f(q)$.

Proposition 4.4. *$f(q)$ has at most $n+m+2$ breakpoints. The candidates for the breakpoints are $q_j^{(i)}$, $1 \leq j \leq m+1$, $0 \leq i \leq n_j$ (except $q_1^{(0)} = \overline{q}_0 = -\infty$ and $q_{m+1}^{(n_{m+1})} = \underline{q}_{m+1} = +\infty$) and $\underline{Q}, \overline{Q}$.*

In actual computation, however, the breakpoints are not known before hand. Algorithm 1 provides a way to both compute the breakpoints and determine the analytical form of the function between the breakpoints. To see the computational complexity, first note that by Assumption 4.3, the optimization problem inside the loops only needs $O(1)$ time to solve. Consequently, the expressions for $r^{\pm}(q)$ can be determined in $O(\sum_{j=1}^{m+1} n_j) + O(m) = O(n+m)$ time. Finding $\underline{Q}$, $\overline{Q}$ clearly depends on the number of analytical pieces of $r^{\pm}(q)$ and takes $O(n+m)$ time. Finally, by Proposition 4.4 there are at most $n+m+2$ breakpoints, $r^*(q)$ and $f(q)$ can be computed in $O(n+m)$ time. In summary, the overall computational complexity is $O(n+m)$.

4.3.2 Dynamic Programming Algorithm

In this subsection, we verify that Assumptions 4.1-4.3 hold in our dynamic pricing problem and we show how to utilize Algorithm 1 as a subroutine in our dynamic programming algorithm.

Algorithm 1

for $j = 1, ..., m+1$ **do**

for $i = 1, ..., n_j$ **do**

solve

$$r_{j,\pm}^{(i)}(q) = \arg\max_{r} \{\Pi^{\pm}(r, q) + g_j^{(i)}(r)\}$$

set

$$q_{j,\pm}^{(i-1)} = \{q : r_{j,\pm}^{(i)}(q) = r_j^{(i-1)}\}$$
$$q_{j,\pm}^{(i)} = \{q : r_{j,\pm}^{(i)}(q) = r_j^{(i)}\}$$

end for

end for

let

$$r^{\pm}(q) = \begin{cases} r_{j,\pm}^{(i)}(q), & q \in [q_{j,\pm}^{(i-1)}, q_{j,\pm}^{(i)}], \quad j = 1, ..., m+1, \quad i = 1, ..., n_j \\ r_j, & q \in [q_{j,\pm}^{(n_j)}, q_{j+1,\pm}^{(0)}], \quad j = 1, ..., m \end{cases}$$

set

$$\underline{Q} = \sup\{q : q - r^{+}(q) < 0\}$$
$$\overline{Q} = \inf\{q : q - r^{-}(q) > 0\}$$

let

$$r^{*}(q) = \begin{cases} r^{+}(q), & q < \underline{Q}, \\ q, & \underline{Q} \le q \le \overline{Q}, \\ r^{-}(q), & q > \overline{Q}. \end{cases}$$

To verify Assumption 4.2, we need the following technical condition on the problem parameters.

Assumption 4.4.

$$\eta^- - \eta^+ \le 2a_t - 2\alpha a_{t+1}, \quad \forall \quad 1 \le t \le T-1.$$

Assumption 4.4 holds under the plausible setting when consumers have short memories (α is small) and the direct price effect dominates the reference price effect ($\eta^+ < \eta^- \le a_t$). We will impose Assumption 4.4 throughout this subsection. However, one should keep in mind that Assumption 4.4 is merely a sufficient condition to guarantee that Assumption 4.2 holds. As we will show in Section 4.4 that Assumption 4.2 can be verified in an on-line fashion as the algorithm implements and may still hold even if Assumption 4.4 fails.

Even though the per-period profit function $\Pi_t(r,q)$ is not jointly concave in r and q, the following proposition shows iteratively that under Assumption 4.4, the function G_t is strongly concave.

Proposition 4.5. *Suppose Assumption 4.4 holds. Then for $2 \le t \le T+1$, G_t is strongly concave with concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. That is, $\hat{G}_t(r) = G_t(r) + A_t r^2$ is also a concave function. Furthermore, there exist $\underline{r}_t, \overline{r}_t$, such that $G_t(r)$ follows the form*

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty), \end{cases}$$

where $g_t(r)$ a continuous function defined on the whole real line.

Before verifying Assumptions 4.1-4.3, we show how one can solve problem (4.3) from the solutions to problem (4.4). The following lemma helps us in establishing the connections among problems (4.3), (4.4) and (4.5). Let $p_c^*(q) = \frac{q - \alpha r_c^*(q)}{1-\alpha}$.

Lemma 4.3. *The solution to problem (4.5): $r_c^*(q)$ is single-valued and continuous in q. Furthermore, both $r_c^*(q)$ and $p_c^*(q)$ are monotonically increasing in q.*

Now we construct the solutions to problem (4.3) from the solutions to problem (4.4) in the following two steps.

From $r^*(q)$ to $r_c^*(q)$: By monotonicity of $p_c^*(q)$, when $\alpha > 0$, we know there exist q_L, q_U, $q_L < q_U$ such that for $q < q_L$, $p_c^*(q) = L_t$ and $r_c^*(q) = \frac{q-(1-\alpha)L_t}{\alpha}$, for $q_U < q$, $p_c^*(q) = U_t$ and $r_c^*(q) = \frac{q-(1-\alpha)U_t}{\alpha}$, while for $q_L < q < q_U$, $L_t < p_c^*(q) < U_t$. When $\alpha = 0$, we can set $q_L = -\infty$ and $q_U = +\infty$. It follows that

$$r_c^*(q) = \begin{cases} \dfrac{q-(1-\alpha)L_t}{\alpha}, & q \in (-\infty, q_L), \\ r^*(q), & q \in [q_L, q_U], \\ \dfrac{q-(1-\alpha)U_t}{\alpha}, & q \in (q_U, +\infty), \end{cases} \tag{4.9}$$

and

$$f_c(q) = \begin{cases} \Pi_t(\dfrac{q-(1-\alpha)L_t}{\alpha}, q) + g_t(\dfrac{q-(1-\alpha)L_t}{\alpha}), & q \in (-\infty, q_L), \\ f(q), & q \in [q_L, q_U], \\ \Pi_t(\dfrac{q-(1-\alpha)U_t}{\alpha}, q) + g_t(\dfrac{q-(1-\alpha)U_t}{\alpha}), & q \in (q_U, +\infty). \end{cases} \tag{4.10}$$

Finally, when $\alpha > 0$, q_L and q_U can be computed through $q_L = \sup\{q : r^*(q) > \frac{q-(1-\alpha)L_t}{\alpha}\}$ and $q_U = \inf\{q : r^*(q) < \frac{q-(1-\alpha)U_t}{\alpha}\}$.

From $r_c^*(q)$ to $r_t(q)$: First, note that when $\alpha > 0$, the price constraint is equivalent to $r \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$. If $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}] \cap [\underline{r}_t, \overline{r}_t] = \emptyset$, then $G_{t+1}(q) = -\infty$. That is, the effective domain for $G_{t+1}(q)$ is $[\alpha\underline{r}_t + (1-\alpha)L_t, \alpha\overline{r}_t + (1-\alpha)U_t]$. When $\alpha = 0$, clearly, the effective domain for $G_{t+1}(q)$ is simply $[L_t, U_t]$.

Let us restrict our attention to the effective domain of $G_{t+1}(q)$, i.e., $q \in [\alpha\underline{r}_t + (1-\alpha)L_t, \alpha\overline{r}_t + (1-\alpha)U_t]$.

If $\underline{r}_t \le r_c^*(q) \le \overline{r}_t$, then clearly $r_t(q) = r_c^*(q)$.

If $r_c^*(q) < \underline{r}_t$, since $r_c^*(q) \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ (or $(-\infty, +\infty)$ if $\alpha = 0$) and q is in the effective domain, we must have $\underline{r}_t \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ (or $(-\infty, +\infty)$ if $\alpha = 0$) as well. Thus, $\underline{r}_t$ is a feasible solution. From Proposition 4.5, we know that the objective function in problem (4.3) is concave and consequently we have $r_t(q) = \underline{r}_t$. Similarly, if $r_c^*(q) > \overline{r}_t$, then $r_t(q) = \overline{r}_t$.

Finally, let $\underline{q}_r = \max\{\sup\{q : r_c^*(q) < \underline{r}_t\}, \alpha\underline{r}_t + (1-\alpha)L_t\}$ and $\overline{q}_r =$

$\min\{\inf\{q : r_c^*(q) > \overline{r}_t\}, \alpha\overline{r}_t + (1-\alpha)U_t\}$, by monotonicity of $r_c^*(q)$ and the discussion above, it follows that

$$r_t(q) = \begin{cases} \underline{r}_t, & q \in [\alpha\underline{r}_t + (1-\alpha)L_t, \underline{q}_r), \\ r_c^*(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \overline{r}_t, & q \in (\overline{q}_r, \alpha\overline{r}_t + (1-\alpha)U_t], \end{cases} \tag{4.11}$$

and

$$G_{t+1}(q) = \begin{cases} -\infty, & q \in (-\infty, \alpha\underline{r}_t + (1-\alpha)L_t), \\ \Pi_t(\underline{r}_t, q) + g_t(\underline{r}_t), & q \in [\alpha\underline{r}_t + (1-\alpha)L, \underline{q}_r), \\ f_c(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \Pi_t(\overline{r}_t, q) + g_t(\overline{r}_t), & q \in (\overline{q}_r, \alpha\overline{r}_t + (1-\alpha)U_t] \\ -\infty, & q \in (\alpha\overline{r}_t + (1-\alpha)U_t, +\infty). \end{cases} \tag{4.12}$$

The above two steps allow us to compute $r_t(q)$ and $G_{t+1}(q)$ from $r^*(q)$ and $f(q)$ if the expressions for the latter two are known. An illustration of $r_2(q)$ is provided in the figure below. In Figure 4.2, $r_2(q)$ is plotted on its effective domain $[\alpha\underline{r}_2 + (1-\alpha)L, \alpha\overline{r}_2 + (1-\alpha)U] = [\underline{q}_r, \overline{q}_r]$ and different colors correspond to different linear pieces. One can clearly see the structure of the constrained solution $r_c^*(q)$ as characterized in (4.9) and the structures of the unconstrained solution $r^*(q)$ as demonstrated in Proposition 4.1 and Proposition 4.2.

Indeed, the structure demonstrated in Figure 4.2 is what guarantees an efficient computation for the expressions of $r^*(q)$ and $f(q)$. Next, we will verify that Assumptions 4.1-4.3 hold and such structure is preserved through dynamic programming. Assumption 4.1 (a) clearly holds since $\Pi_t(r,q)$ can be expressed as a minimum of two quadratic functions that is strictly concave in r and from the proof of Lemma 4.3, we know $\Pi_t(r,q)$ is strictly supermodular as well. Proposition 4.5 already proves that Assumption 4.2 (a) is satisfied. In the following, we show that the remaining assumptions also hold.

Proposition 4.6. *$q - r^*(q)$ satisfies the single crossing property.*

Proposition 4.6 verifies that Assumption 4.2 (b) holds. The proof relies on a careful analysis of both the right and left derivative of the objective function in (4.4).

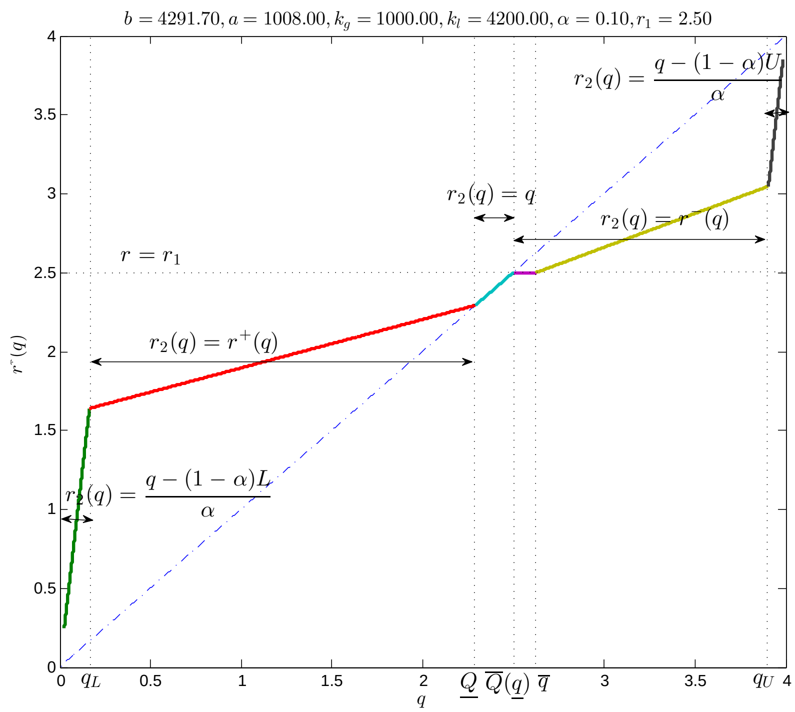


Figure 4.2: Illustration of $r_2(q)$

Next, we show iteratively that for $2 \leq t \leq T+1$, $g_t(r)$ consists of finite number of quadratic pieces. Combined with Proposition 4.5, this shows that Assumption 4.1 (b) holds. Also, as $\Pi_t(r,q)$ consists of two quadratic functions, Assumption 4.3 holds as well.

Proposition 4.7. *If $g_t(r)$ consists of n_t+1 quadratic pieces, and it has n_t breakpoints and m_t kink points ($m_t \leq n_t$), then $g_{t+1}(r)$ consists of at most n_t+m_t+7 quadratic pieces, and it has at most n_t+m_t+6 breakpoints and m_t+4 kink points. That is,*

$$n_{t+1} \leq n_t + m_t + 6, \quad m_{t+1} \leq m_t + 4.$$

Proposition 4.7 not only shows that $g_t(r)$, $2 \leq t \leq T$, are all piece-wise quadratic functions, but also provides a bound on the growth of the number of both breakpoints as well as kink points.

With all the assumptions satisfied, we are ready to present Algorithm 2 that solves the dynamic pricing problem (4.3).

To see the computational complexity, step 1 in Algorithm 2 clearly requires $O(n_{t+1})$ time by our analysis of Algorithm 1 and Proposition 4.7. Similar-

Algorithm 2

Initialize $\underline{r}_2 = \alpha r_1 + (1-\alpha)L_1$, $\overline{r}_2 = \alpha r_1 + (1-\alpha)U_1$, $n_2 = 1$, $m_2 = 1$, $g_2(r) = \Pi_1(r_1, r)$ and

$$G_2(r) = \begin{cases} g_2(r), & r \in [\underline{r}_2, \overline{r}_2], \\ -\infty, & r \in (-\infty, \underline{r}_2) \cup (\overline{r}_2, +\infty). \end{cases}$$

for $t = 2, ..., T$ **do**

step 1: Implement Algorithm 1 to solve problem (4.4) and output $r^*(q)$ and $f(q)$.

step 2: Compute $r_c^*(q)$ and $f_c(q)$ from $r^*(q)$ and $f(q)$ according to (4.9) and (4.10).

step 3: Compute $r_t(q)$ and $G_{t+1}(q)$ from $r_c^*(q)$ and $f_c(q)$ according to (4.11) and (4.12).

set $\underline{r}_{t+1} = \alpha \underline{r}_t + (1-\alpha)L_t$, $\overline{r}_{t+1} = \alpha \overline{r}_t + (1-\alpha)U_t$ and

$$g_{t+1}(q) = \begin{cases} \Pi_t(\underline{r}_t, q) + g_t(\underline{r}_t), & q \in (\infty, \underline{q}_r), \\ f_c(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \Pi_t(\overline{r}_t, q) + g_t(\overline{r}_t), & q \in (\overline{q}_r, +\infty). \end{cases}$$

end for

ly, since the number of quadratic pieces of $g_{t+1}(q)$ and the number of the linear pieces of $r_{t+1}(q)$ are bounded by n_{t+1}, step 2 and step 3 can also be computed in $O(n_{t+1})$ time. Note that by Proposition 4.7, $m_{T+1} = O(T)$ and $n_{T+1} = O(T^2)$. Therefore, the overall computational complexity is then $O(\sum_{t=2}^{T} n_{t+1}) = O(T^3)$.

4.4 Numerical Study

In this section, we implement the algorithm developed in Section 4.3 and the heuristic developed in Hu (2012) in a case study to demonstrate how they can be used to solve a practical industry problem with real data. Based on the examples from the case study, we compare the efficiency of our algorithm to the heuristic. Finally, we show that our exact algorithm (Algorithm 2) may still be applied even when Assumption 4.4 fails.

4.4.1 Case Study

In the following, we present our case study by utilizing the data set provided by Boatwright et al. (1999) of the Borden sliced cheese in 12 oz packages sold by retailers across the nation in the Bayesm Package of the R software. The data contains the weekly sales as well as prices of the product for up to 68 weeks. As noted by Greenleaf (1995), the linear reference price model has a multi-collinearity problem. One way to alleviate this issue is to perform linear regression with regularization, e.g., ridge regression, lasso or more generally elastic net (see James et al., 2013, for more details). For the purpose of demonstration, we choose lasso in our case study as the regularization method since it usually results in a better fit and fixes the wrong sign in parameter estimates better than ridge regression. Furthermore, we let the minimum historical price be the initial reference price for a more conservative result and we use the estimation procedure employed in Greenleaf (1995) to estimate α. The results for 6 selected retailers are reported in Table 4.1.

Table 4.1: Parameter Estimates and Profits Comparison

Stores	$\hat{\alpha}$	$\hat{b}$	$\hat{a}$	$\hat{\eta}^+$	$\hat{\eta}^-$	Π^*	Π^s	Π^h
Hartford - Stop & Shop	0.93	19811.72	-5271.96	0.00	687.33	1135400	1134700	496360
(average markup of 23.3%)						(108020)	(107800)	(46356)
Boston - Star Market	0.54	6209.50	-1585.68	0.00	1294.39	413380	412050	249740
(average markup of 40.0%)						(79217)	(78162)	(66268)
Indianapolis - Kroger Co	0.93	5019.04	-319.66	2708.75	2946.91	797670	632370	724130
Chicago - Omni	0.00	35082.59	-11799.80	10032.22	0.00	1966400	1774700	878280
(average markup of 23.0%),						(169480)	(150530)	(122640)
Balti/Wash - Giant Food Inc	0.04	13103.80	-2350.83	5259.24	0.00	1788300	1071900	1087700
Jacksonvile - Publix	0.66	3792.61	-765.08	1223.92	352.78	303880	280950	286600

Profits under marginal cost adjustments are in parenthesis.
Π^*: Profits under optimal pricing strategy.
Π^s: Profits under the pricing strategy that ignores reference price effects.
Π^h: Profits under the historical prices.

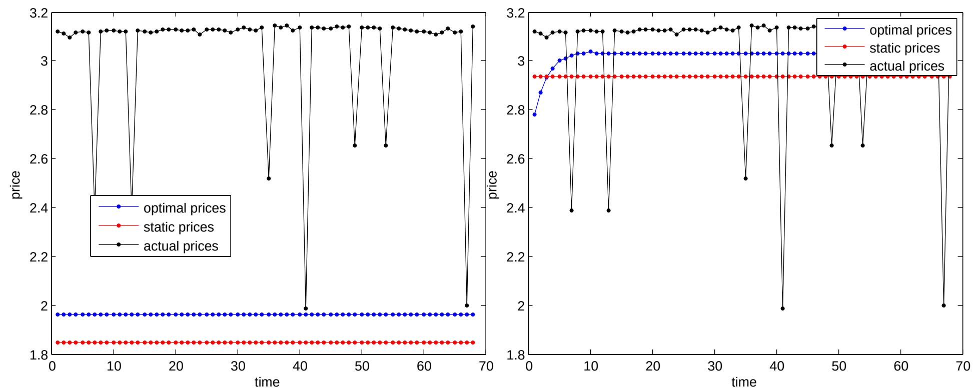
In addition to the estimates of the parameters b, a, η^+, η^- and α, we also report the profit under optimal pricing strategy Π^* computed through our exact and heuristic algorithms, the profit under the pricing strategy that ignores reference price effects Π^s (computed under the static demand model) and the profit under the historical prices Π^h. For Π^*, we compute it according to our exact algorithm when Assumption 4.4 is satisfied. Otherwise, the heuristic in Hu (2012) is applied to obtain an approximation.

From Table 4.1, we can see that demands faced by different retailers can have significantly different characteristics. The first three retailers face loss-averse demands while the latter three face gain-seeking demands. Also notice that the demands for the first and second retailers satisfy Assumption 4.4 while the demands for the fourth retailer satisfy Assumption 3.1.

By comparing the profits, we can see that retailers' profits can be greatly improved in many scenarios by using the reference price models rather than the static demand models. Note that we are only presenting a basic framework in this chapter and many other features such as competitors prices, seasonal effects, marginal costs, etc., are not taken into consideration since they are not available in the data set. This could be the primary reason that in most cases, the optimal pricing strategy results in unbelievable increase in profits.

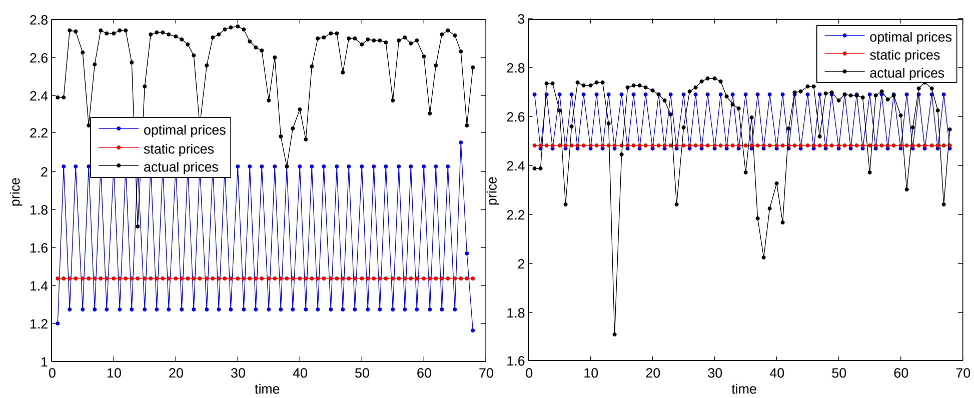
In Figure 4.3 and Figure 4.4, we compare the price paths under the optimal pricing strategy, the pricing strategy that ignores reference price effects (static prices) and that of the historical prices for two retailers respectively. Panel (a) in both figures confirms that ignoring marginal costs can indeed result in an overly underpriced optimal prices for these two retailers. Unfortunately, we do not have the data on marginal costs and it is quite possible that some retailers account for marginal costs when setting up prices while others might employ a "loss-leader" strategy to drive store traffic. As a compromise, for those stores that the average optimal prices are overly underpriced compared to the average historical prices, we test several marginal costs (equivalently average markup levels) and choose the one that results in a similar average optimal prices and historical prices. For instance, for the retailer "Boston - Star Market", we find that an average markup level of 40% produces close average prices as well as price paths (see panel (b) in Figure 4.3 for illustration). For the retailer "Indianapolis - Kroger Co", on the other hand, we find that average optimal prices are already above average historical prices and consequently no adjustment is made.

The recomputed profits based on the marginal cost adjustments are reported in the parenthesis in Table 3.1 and the price paths are shown in panel (b) in Figure 4.3 and Figure 4.4 for the respective retailers in panel (a). For both retailers, the optimal prices and static prices are now in close range of the historical prices and the resulting profits are also in a comparable range. With comparable prices, the underlying reasons in the profits gaps may then be explained intuitively using the existing results in the literature. In the case of retailer "Boston - Star Market", since it faces a loss-averse demand, deep price cuts in the historical prices (see Figure 4.3) can be very costly (see Section 2.4). On the other hand, the retailer "Chicago - Omni", is not exploiting gain-seeking effects in those periods with flat prices (see Figure 4.4) and consequently missing the opportunities to increase its profits (see Section 3.4).



(a) without marginal cost (b) with 40% average markup

Figure 4.3: Comparison of the price paths for retailer: Boston - Star Market



(a) without marginal cost (b) with 23% average markup

Figure 4.4: Comparison of the price paths for retailer: Chicago - Omni

Even under marginal cost adjustment, the profit comparisons in Table 3.1 for the store "Hartford - Stop & Shop" can still be deemed as unrealistic, which requires the attention of additional features. We remark here that, if data available, other features that are independent from the price and reference price can be easily incorporated to improve the reference price model. The addition of other features in the regression model will only result in a time varying intercepts, i.e., b_t, for which the optimal prices can still be computed efficiently by our algorithms.

4.4.2 Efficiency and Robustness

In the following, we check the efficiency and robustness of our algorithms based on some of the examples provided in our case study. All experiments below are performed in MATLAB 2014a on a desktop with an Intel Core i5-3770 CPU (3.20 GHz) and 8 GB RAM running 64-bit Windows 7 Enterprise.

In Table 2, we compare the computational efficiency of Algorithm 2 developed in Section 4.3 with the heuristic in Hu (2012). Specifically, we compare to the heuristic at different accuracy levels, one with $\varepsilon = 0.01$ and another more accurate one with $\varepsilon = 0.001$. For each algorithm, we report the CPU time (in seconds) of the computations under the time horizon ranging from 10 periods to 40 periods. The first thing to note from Table 4.2 is that Algorithm 2, being an exact algorithm, is much more computationally efficient than the heuristic with the running time far less than a second. Although the theoretical guarantee for the running time of Algorithm 2 is of the order $O(T^3)$, in our computational studies it scales much better. The computational time for our heuristic, on the other hand, scales linearly with respect to time horizon for a given ε as Hu (2012) shows. While this has an advan-

Table 4.2: Comparison of the Computational Time for retailer: Boston - Star Market

Length of Horizon	Algorithm 2 Time (s)	Heuristic ($\varepsilon = 0.01$) Time (s)	Heuristic ($\varepsilon = 0.001$) Time (s)
$T = 10$	0.066	0.505	10.913
$T = 20$	0.077	0.996	20.161
$T = 30$	0.093	1.428	30.084
$T = 40$	0.139	1.922	40.154

tage in computing long time horizon problems, errors may accumulate over time and reducing such errors would require relatively significant amount of computational time.

Next, we show that our exact algorithm can still be applied even if the technical condition in Assumption 4.4 fails. Indeed, Assumption 4.4 is merely a sufficient condition to guarantee that Assumption 4.2 holds in our dynamic pricing problem and in order to present a neat and interpretable condition we have been quite conservative in the derivation. In actual implementation, it is not necessary to verify Assumption 4.4 beforehand. Instead, since explicit expressions are computed during each iteration, one can verify Assumption 4.2 directly in an online fashion. For instance, we have applied Algorithm 2 to the retailer "Indianapolis - Kroger Co" for which Assumption 4.4 fails. We find that Assumption 4.2 is not violated while implementing the algorithm. As a result, Proposition 4.7 still holds and the efficiency of the algorithm is guaranteed. Table 4.3 summarizes the growth of breakpoints of the value function $G_t(q)$ for the first 20 iterations. As one can see that the growth is quadratic in t.

Table 4.3: Growth of the number of breakpoints of $G_t(q)$

	t = 2	t = 4	t = 6	t = 8	t = 10	t = 12	t = 14	t = 16	t = 18	t = 20
Indiannapolis - Kroger Co	1	9	21	39	68	104	151	211	299	446
Boston - Star Market	1	9	17	25	33	41	49	57	65	73

In comparison, the growth of breakpoints of $G_t(q)$ for retailer "Boston - Star Market", where Assumption 4.4 holds, is also illustrated in Table 4.3. Interestingly, the growth is linear instead of quadratic. That is, in addition of being a simple sufficient condition, the stronger property imposed by Assumption 4.4 seems to have the potential of eliminating many breakpoints, which also explains the efficiency of Algorithm 2 demonstrated in Table 4.2.

4.5 Conclusion

In this chapter we examine the computational side of a dynamic pricing problem with a memory-based reference price model. In this model, demand depends on both current selling price and reference price, where the latter evolves according to an exponentially smoothing process of past prices. We identify several key structural properties to ensure such non-smooth dynamic optimization problems can be solved exactly in strongly polynomial time. In the loss-averse case, we characterize the sufficient conditions to guarantee those structural properties and develop a strongly polynomial time algorithm to solve for the optimal prices.

To complement our theoretical results, a case study is presented to demonstrate how our algorithms can be applied in a practical setting. Based on the examples in the case study, we show that our exact algorithm can be very efficient and may still be applied even when the technical conditions are violated.

Our work is only a start in looking at the computational side of the optimization models that incorporate reference price effects. We are exploring several related topics. First, further improvement of the current algorithms and heuristic is important and interesting. Efficient algorithms become pertinent as we incorporate these models into decision support systems which usually involve many products and other operations decisions.

Secondly, there is a rapidly growing literature on integrated inventory and pricing models (see Federgruen and Heching, 1999; Chen and Simchi-Levi, 2004a,b; Huh and Janakiraman, 2008; Geunes et al., 2009; Chen and Simchi-Levi, 2012). Incorporating reference price effects into such models may significantly complicate algorithm design. Efficient algorithms in these settings may then provide even more benefits to the firms facing inventory as well as pricing decisions.

Finally, we would like to examine the possibility of extending our algorithms to the model with multiple products (see Chapter 5 for a model formulation). Indeed, in settings with multiple products, the state space grows in dimensions and it is not clear whether the number of quadratic pieces in the value functions will still be bounded by a polynomial number.

Chapter 5

Dynamic Pricing of Multiple Products

5.1 Introduction

So far, we have been focusing solely on the setting in which a monopolist manages a single product. However, in practice, the demand of a product can be affected by the prices of other products as well. For example, if a retailer is selling both Coke and Pepsi, then one would naturally expect the demand for Coke will be low if the retailer runs a promotion for Pepsi. Capturing such cross-price effects is a central task in building up a multi-product model. The multinomial logit model and its variants are commonly used in the literature to explicitly model the purchase behavior of consumers when they face multiple substitutable products. As we mentioned in Section 2.1, researchers have managed to incorporate the effects of reference prices into such models and a lot of empirical studies are conducted (comparatively, as we point out below, empirical studies for linear models are scarce). However, even single-period multi-product price optimization problems that ignore reference price effects can be non-trivial and a stream of literature has been devoted to these problems over the last two decades (see Gallego and Wang, 2014, and the references therein). Another popular model is the linear demand models which capture the cross-price effects directly at the aggregate level and such models are widely used both empirically and analytically. Yet, empirical studies that incorporate reference price effects into the multi-product linear demand models are still lacking. Indeed, it would be a challenge to empirically disentangle the direct cross-price effects with the indirect ones, which could be caused by the fact that the formation of reference price is affected by prices of different products.

Nevertheless, a few linear models have been proposed to study reference price effects in a multi-product setting. Kopalle et al. (1996) propose two such models. Their first model directly generalizes the exponential smoothing model of the single-product setting by adding cross-price effects terms in the demand function and assuming that consumers have a reference price for each product which is independent with each other. Their second model introduces reference brand formulation (see Hardie et al., 1993) which uses the price of the product consumers purchased last time as a reference price. Asvanunt (2007), on the other hand, argues that consumers make their purchasing decision on the store's overall price level, which is a weighted average price of different products sold at the store. Consequently, in Asvanunt (2007), consumers use a uniform reference price when deciding which product to choose.

In terms of the analysis of dynamic pricing problems, multi-product setting has brought significant challenges and structural results are quite limited in the literature. Kopalle et al. (1996) show in their exponential smoothing model that constant pricing strategies are not optimal when demands for each of the product are gain-seeking while constant pricing strategies outperform high-low pricing strategies when demands are all loss-averse or loss/gain neutral. Popescu and Wu (2005) adopt the exponential smoothing model in Kopalle et al. (1996) and prove that the optimal prices converge to constant prices for the two products case with both products having loss/gain neutral demands. They conjecture that the result extend to more than two products but are unable to prove analytically. Indeed, to the best of our knowledge, it is not established whether constant pricing strategies are optimal in the long-run even for loss/gain neutral demands with more than two products. The main difficulty behind such analysis is the enlargement of state space. As Popescu and Wu (2005) point out "few general techniques exist for proving global stability in high dimensions". As we will see, for loss-averse case, there is an additional difficulty in the *switching behavior* of the optimal price path. In other words, the technique developed in Fibich et al. (2003) (also see Section 2.4) to separate the solution of the non-smooth dynamic programming problem to the corresponding smooth dynamic programming problems does not extend to the multi-product setting. The model proposed in Asvanunt (2007), however, circumvents these difficulties because a single state variable (store level reference price) is sufficient to describe the system. As a result, Asvanunt (2007) manages to prove the optimality of constant pricing strategies in the loss/gain neutral case.

This chapter mainly builds upon the exponential smoothing model in a multi-product setting proposed in Kopalle et al. (1996). We give an explicit solution to the dynamic pricing problems in a continuous-time framework with loss/gain neutral demands, which generalizes the solution of Fibich et al. (2003) in a single-product setting. With the explicit solution, we prove the conjecture of Popescu and Wu (2005) that for any finite number of products, the optimal prices converge to a unique steady state. As we have pointed out earlier, similar techniques to deal with loss-averse demands cannot be extended to multi-product setting. Instead of seeking optimal solutions, we propose for two-product case a *semi-myopic solution* (see Section 5.3.2) that generalizes the notion of myopic solution and can in some scenarios achieve optimality. We further develop some novel techniques to show the global stability of the resulting system.

The remainder of this chapter is organized as follows. In Section 5.2 we present the mathematical model in a continuous-time framework that extends the exponential smoothing model in the single-product setting to the multi-product setting. The dynamic pricing problem is analyzed in Section 5.3. Section 5.4 gives concluding remarks and possible directions for future research. All technical proofs are relegated to Appendix D.

5.2 Model

We consider a monopolist selling N products and consumers form a reference price r_i for product i, $1 \leq i \leq N$. Following Kopalle et al. (1996), we assume that the reference price for each of the product evolves independently and according to an exponential smoothing process. Similar to (2.15), given the price $p_i(t)$ and the reference price $r_i(t)$ for product i at time t, the reference price evolves according to

$$dr_i(t) = \bar{\alpha}[p_i(t) - r_i(t)]dt, \quad \bar{\alpha} > 0. \tag{5.1}$$

For notational convenience, we denote $\mathbf{r} = (r_1(t), ..., r_N(t))^\top$, $\mathbf{p} = (p_1(t), ..., p_N(t))^\top$ and $\dot{\mathbf{r}} = (\frac{dr_1(t)}{dt}, ..., \frac{dr_N(t)}{dt})^\top$, then (5.1) can be written more compactly as

$$\dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p} - \mathbf{r}). \tag{5.2}$$

The demand function for product i is modeled as

$$\begin{aligned} D_i(t) = b_i - a_i p_i(t) + \sum_{j=1, j\neq i}^{N} \theta_{ij} p_j(t) \\ + \eta_i^+ \max\{r_i(t) - p_i(t), 0\} + \eta_i^- \min\{r_i(t) - p_i(t), 0\}, \end{aligned}$$

where θ_{ij} models the cross-price effects between products i and j. One s-tandard assumption here is that $\sum_{j\neq i} \theta_{ij} < a_i$, which ensures the demand of product i is more sensitive to the change in its own price than to the change in the prices of all other products (see, for example, Talluri and Van Ryzin, 2005). For ease of exposition, we further assume that $\theta_{ij} = \theta_{ji}$. Note here that when $\theta_{ij} > 0$, products i and j are substitutes while $\theta_{ij} < 0$ indicates that they are complements.

The instantaneous profit for the firm is

$$\Pi(\mathbf{r}, \mathbf{p}) = \sum_{i=1}^{N} p_i(t) D_i(t),$$

and the corresponding dynamic pricing problem is: given initial reference prices $\mathbf{r_0} = (r_1(0), ..., r_N(0))^\top$, the firm seeks to maximize its long-run profit, i.e.,

$$\begin{aligned} &\max_{\mathbf{p}} \int_0^{+\infty} e^{-\gamma t} \Pi(\mathbf{r}, \mathbf{p}) dt \\ &\text{s.t.} \quad \dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p} - \mathbf{r}), \end{aligned} \tag{5.3}$$

where γ is the discount factor.

5.3 Analysis

This section analyzes the dynamic pricing problem (5.3) of multiple products. For the loss/gain neutral case, we derive an explicit solution, which helps us to construct a semi-myopic solution for the loss-averse case with two products. We further prove that the reference price path under the semi-myopic pricing strategy will converge to a region of steady states.

5.3.1 Loss/Gain Neutral Demands

In this subsection, we assume the demands for all the products are loss/gain neutral. That is, $\eta_i^+ = \eta_i^- := \eta_i$, for $1 \le i \le N$. In this case, the instantaneous profit can be written in matrix form as

$$\Pi(\mathbf{r}, \mathbf{p}) = \mathbf{p}^\top \mathbf{R} \mathbf{p} + \mathbf{r}^\top \mathbf{K} \mathbf{p} + \mathbf{b}^\top \mathbf{p}, \tag{5.4}$$

where $\mathbf{b} = (b_1, ..., b_N)^\top$,

$$\mathbf{R} = \begin{bmatrix} -(a_1+\eta_1) & \theta_{12} & \dots & \theta_{1N} \\ \theta_{21} & -(a_2+\eta_2) & \dots & \theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N1} & \theta_{N2} & \dots & -(a_N+\eta_N) \end{bmatrix}$$

and

$$\mathbf{K} = \begin{bmatrix} \eta_1 & 0 & \dots & 0 \\ 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \eta_N \end{bmatrix}.$$

Before we state our main result, we first examine the properties of the matrix $\mathbf{M} := \bar{\alpha}(\bar{\alpha}+\gamma)\mathbf{I} + \bar{\alpha}(\bar{\alpha}+\frac{\gamma}{2})\mathbf{R^{-1}K}$, where $\mathbf{I}$ is the identity matrix. As we will see shortly this matrix is critical for our development of the explicit solution to (5.3).

Lemma 5.1. *The matrix* $\mathbf{M}$ *defined by* $\mathbf{M} = \bar{\alpha}(\bar{\alpha}+\gamma)\mathbf{I} + \bar{\alpha}(\bar{\alpha}+\frac{\gamma}{2})\mathbf{R^{-1}K}$ *is positive definite.*

We denote $\xi_1, ..., \xi_N$ (not necessarily all distinct) and $\mathbf{v_1}, ..., \mathbf{v_N}$ to be the N eigenvalues and the corresponding eigenvectors of $\mathbf{M}$ with $\mathbf{V} := (\mathbf{v_1}, ..., \mathbf{v_N})$. By Lemma 5.1, we know that $\xi_i > 0$ for $1 \le i \le N$ and we can define $\lambda_i = \frac{\gamma - \sqrt{\gamma^2 + 4\xi_i}}{2}$. Finally, we let $\mathbf{r_s}$ to be the unique solution of the linear equations

$$\mathbf{M r_s} = -\frac{1}{2}\bar{\alpha}(\bar{\alpha}+\gamma)\mathbf{R^{-1}b}. \tag{5.5}$$

In Proposition 5.1, we give an explicit expression to the solution of the problem (5.3) using above defined notations .

Proposition 5.1. *The optimal price path* $\mathbf{p}^*(t)$ *that solves problem* (5.3) *is given by*

$$\mathbf{p}^*(t) = \mathbf{r_s} + \mathbf{V}\begin{bmatrix} (1+\frac{\lambda_1}{\bar{\alpha}})e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & (1+\frac{\lambda_2}{\bar{\alpha}})e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1+\frac{\lambda_N}{\bar{\alpha}})e^{\lambda_N t} \end{bmatrix}\mathbf{V^{-1}}(\mathbf{r_0}-\mathbf{r_s}),$$

and the corresponding reference price path under the optimal pricing strategy is

$$\mathbf{r}^*(t) = \mathbf{r_s} + \mathbf{V}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_N t} \end{bmatrix}\mathbf{V^{-1}}(\mathbf{r_0}-\mathbf{r_s}). \tag{5.6}$$

Alternatively, we can write the state feedback optimal prices (optimal pricing strategy) as

$$\mathbf{p}^*(\mathbf{r}) = \mathbf{r} + \frac{1}{\bar{\alpha}}\mathbf{V\Lambda V^{-1}}(\mathbf{r}-\mathbf{r_s}), \tag{5.7}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix}.$$

From the explicit solution in Proposition 5.1, we can make several observations.

First, when $N = 1$ the optimal price path in Proposition 5.1 reduces to

$$p^*(t) = r_s + (1+\frac{\lambda}{\bar{\alpha}})e^{-\lambda t}(r_0 - r_s),$$

which is exactly the explicit solution provided in Fibich et al. (2003) of the single-product problem.

Second, as $t \to \infty$, the optimal pricing strategy $\mathbf{p}^*(t)$ converges to the steady state $\mathbf{r_s}$ characterized by the equation (5.5). This formally settles the conjecture about global stability raised in Popescu and Wu (2005). However, unlike the single-product case discussed in Fibich et al. (2003) and

Popescu and Wu (2007), the optimal price path of product i, $p_i^*(t)$ and the corresponding reference price path $r_i^*(t)$ may not be monotonic in t.

Third, if there is no reference price effect, then $\eta_i = 0$ for $i = 1, ..., N$ or $\mathbf{K} = \mathbf{0}$. As a result, $\mathbf{M} = \bar{\alpha}(\bar{\alpha}+\gamma)\mathbf{I}$ and its eigenvalues $\xi_i = \bar{\alpha}(\bar{\alpha}+\gamma)$ for any $i = 1, ..., N$ and eigenvectors $\mathbf{V} = [\mathbf{v_1}, ..., \mathbf{v_N}] = \mathbf{I}$. By solving the quadratic equation, $\lambda_i = -\beta$ for any $i = 1, ..., N$. It follows that

$$\mathbf{p}^*(t) = \mathbf{r_s} = -\frac{1}{2}\mathbf{R^{-1}b},$$

which naturally reduced to the optimal solution of the problem in the absence of reference effects, namely

$$\mathbf{p}^{*\top}\mathbf{R}\mathbf{p}^* + \mathbf{b}^\top\mathbf{p}^* = \max_{\mathbf{p}}[\mathbf{p}^\top\mathbf{R}\mathbf{p} + \mathbf{b}^\top\mathbf{p}].$$

5.3.2 Loss-Averse Demands: Two Products Case

When the demands of the products are loss-averse, i.e., $\eta_i^- > \eta_i^+$, problem (5.3) becomes much more challenging due to the non-smoothness in higher dimensional space. Even for some simpler variant of the two products problem, the so-called "single-product heterogeneous consumers problem", where there is no cross-price effects and firm can only charge the same price for the two consumer groups (products), it is claimed that "global stability is not guaranteed (Popescu and Wu, 2005)".

The main challenge in generalizing the results in single-product setting is the resulting *switched dynamic system* even under myopic pricing strategies. In a very general form, a switched dynamic system is described by the following equation

$$\dot{\mathbf{x}}(t) = f_{\sigma(t)}(\mathbf{x}(t)),$$

where $\sigma(t) \in \mathcal{P}$ and the family of functions $\{f_p : p \in \mathcal{P}\}$ is assumed to be sufficiently regular, say, continuously differentiable. The time dependent function $\sigma(t)$ is called a switching signal and its evolution can be implicitly depending on the evolution of the state $\mathbf{x}(t)$.

Readers are referred to Liberzon (2003) for a reference in the topic of switched dynamic system. As we will discuss in detail later, proving stability in switched dynamic system is generally much more difficult. Furthermore, due to the potential switches, the technique developed by Fibich et al. (2003) to derive the explicit expression for the optimal pricing strategy does not work even for the two products case. Therefore, obtaining explicit solutions as in Fibich et al. (2003) is quite difficult, if not impossible.

In this subsection, we address the difficulty by proposing a suboptimal solution called *semi-myopic* solution which generalizes the myopic solution in a sense that it seeks to maximize the accumulated profit over some time horizon instead of simply maximizing the instantaneous profit and ignoring the whole future altogether. Like the myopic solution, the resulting dynamic system of the semi-myopic solution is still a switched dynamic system. We manage to utilize the special structures of the vector field to prove the global stability of the resulting switched dynamic system under the semi-myopic solution.

First, we define the *semi-myopic* solution. A price path $\mathbf{p}(t), t \geq 0$, with the corresponding reference price path $\mathbf{r}(t), t \geq 0$, is said to be semi-myopic if for any $s \geq 0$, there exists a $T > s$ such that $\mathbf{p}(t), s \leq t \leq T$ is the optimal solution to

$$\begin{aligned} &\max_{\mathbf{p}} \int_s^T e^{-\gamma t} \Pi(\mathbf{r}, \mathbf{p}) dt \\ &\text{s.t.} \quad \dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p} - \mathbf{r}), \end{aligned}$$

where the initial and terminal states in the above problem are required to be $\mathbf{r}(s)$ and $\mathbf{r}(T)$ respectively. Intuitively speaking, at any point of time, semi-myopic solution at least maximizes the accumulated profit for a certain amount of duration.

We use a dynamic programming approach to construct one such myopic solution. The HJB equation to problem (5.3) can be written as

$$\gamma V(\mathbf{r}) = \max_{\mathbf{p}} [\bar{\alpha} \nabla V(\mathbf{r})^\top (\mathbf{p} - \mathbf{r}) + \Pi(\mathbf{r}, \mathbf{p})], \tag{5.8}$$

where $\nabla V(\mathbf{r})$ denotes the gradient of the value function $V(\cdot)$. Unfortunately, the HJB equation (5.8) is not well-defined everywhere since $\Pi(\mathbf{r}, \mathbf{p})$ is not differentiable and consequently $\nabla V(\mathbf{r})$ may not exist everywhere. Therefore, one should interpret the value function $V(\cdot)$ satisfying the HJB equation (5.8) in the *viscosity* sense (see Bardi and Capuzzo-Dolcetta, 1997, for a reference in the topic of viscosity solutions). That is, at non-differentiable points of

$V(\mathbf{r})$, $\nabla V(\mathbf{r})$ is replaced by the superdifferential and subdifferential of $V(\mathbf{r})$ at $\mathbf{r}$ and $V(\mathbf{r})$ is solved as a viscosity subsolution and a viscosity supersolution respectively.

Finding viscosity solution to (5.8) is usually very difficult. Instead, we adopt the idea in Fibich et al. (2003) to consider the smooth counterparts of the problem (5.8). For notational brevity, we let

$$\begin{aligned}\Pi^{++}(\mathbf{r},\mathbf{p}) = p_1[b_1 - a_1 p_1 + \theta p_2 + \eta_1^+(r_1 - p_1)] \\ + p_2[b_2 - a_2 p_2 + \theta p_1 + \eta_2^+(r_2 - p_2)],\end{aligned}$$

where $\theta := \theta_{12} = \theta_{21} > 0$. Note that if $\theta = 0$, then the two products are completely independent and we can explicitly solve two single product problems separately. $\Pi^{+-}(\mathbf{r},\mathbf{p})$, $\Pi^{-+}(\mathbf{r},\mathbf{p})$, $\Pi^{--}(\mathbf{r},\mathbf{p})$ can be defined similarly with the first and second signs in the superscript denoting whether we are taking η_1^+ or η_1^- and η_2^+ or η_2^- respectively. In addition, we denote

$$\begin{aligned}\Pi^{+r_2}(r_1,p_1) &= p_1[b_1 - a_1 p_1 + \theta p_2 + \eta_1^+(r_1 - p_1)] + r_2(b_2 - a_2 r_2 + \theta p_1),\\ \Pi^{r_1+}(r_2,p_2) &= p_2[b_2 - a_2 p_2 + \theta p_1 + \eta_2^+(r_2 - p_2)] + r_1(b_1 - a_1 r_1 + \theta p_2),\end{aligned}$$

where r_2 and r_1 in the superscripts emphasize the fact that we have enforced $p_2 = r_2$ and $p_1 = r_1$ respectively. Again, $\Pi^{-r_2}(r_1,p_1)$ and $\Pi^{r_1-}(r_2,p_2)$ can be defined similarly.

Following the above notations, we consider the following set of HJB equations

$$\begin{aligned}
\gamma V^{++}(\mathbf{r}) &= \max_{\mathbf{p}}[\bar{\alpha}\nabla V^{++}(\mathbf{r})^\top(\mathbf{p}-\mathbf{r}) + \Pi^{++}(\mathbf{r},\mathbf{p})]\\
\gamma V^{+-}(\mathbf{r}) &= \max_{\mathbf{p}}[\bar{\alpha}\nabla V^{+-}(\mathbf{r})^\top(\mathbf{p}-\mathbf{r}) + \Pi^{+-}(\mathbf{r},\mathbf{p})]\\
\gamma V^{-+}(\mathbf{r}) &= \max_{\mathbf{p}}[\bar{\alpha}\nabla V^{-+}(\mathbf{r})^\top(\mathbf{p}-\mathbf{r}) + \Pi^{-+}(\mathbf{r},\mathbf{p})]\\
\gamma V^{--}(\mathbf{r}) &= \max_{\mathbf{p}}[\bar{\alpha}\nabla V^{--}(\mathbf{r})^\top(\mathbf{p}-\mathbf{r}) + \Pi^{--}(\mathbf{r},\mathbf{p})]\\
\gamma V^{+r_2}(r_1) &= \max_{p_1}[\bar{\alpha}\nabla V^{+r_2}(r_1)(p_1-r_1) + \Pi^{+r_2}(r_1,p_1)]\\
\gamma V^{-r_2}(r_1) &= \max_{p_1}[\bar{\alpha}\nabla V^{-r_2}(r_1)(p_1-r_1) + \Pi^{-r_2}(r_1,p_1)]\\
\gamma V^{r_1+}(r_2) &= \max_{p_2}[\bar{\alpha}\nabla V^{r_1+}(r_2)(p_2-r_2) + \Pi^{r_1+}(r_2,p_2)]\\
\gamma V^{r_1-}(r_2) &= \max_{p_2}[\bar{\alpha}\nabla V^{r_1-}(r_2)(p_2-r_2) + \Pi^{r_1-}(r_2,p_2)]
\end{aligned}$$

and denote the solutions to the corresponding HJB equations as $\mathbf{p}^{++}(\mathbf{r})$, $\mathbf{p}^{+-}(\mathbf{r})$, $\mathbf{p}^{-+}(\mathbf{r})$,$\mathbf{p}^{--}(\mathbf{r})$, $p_1^{+r_2}(r_1)$, $p_1^{-r_2}(r_1)$, $p_2^{r_1+}(r_2)$ and $p_2^{r_1-}(r_2)$ respectively. Note that all the HJB equations above are well-defined since the instantaneous profit functions, say $\Pi^{++}(\mathbf{r},\mathbf{p})$, are continuously differentiable. Applying our results in Section 5.3.1, all the solutions admit explicit expressions and they are linear in states. To construct our semi-myopic solution, we divide the state space into the following regions:

$$\begin{aligned}
I^{++} &:= \{\mathbf{r}|p_1^{++}(\mathbf{r}) - r_1 < 0, p_2^{++}(\mathbf{r}) - r_2 < 0\} \\
I^{+-} &:= \{\mathbf{r}|p_1^{+-}(\mathbf{r}) - r_1 < 0, p_2^{+-}(\mathbf{r}) - r_2 > 0\} \\
I^{-+} &:= \{\mathbf{r}|p_1^{-+}(\mathbf{r}) - r_1 > 0, p_2^{-+}(\mathbf{r}) - r_2 < 0\} \\
I^{--} &:= \{\mathbf{r}|p_1^{--}(\mathbf{r}) - r_1 < 0, p_2^{--}(\mathbf{r}) - r_2 < 0\} \\
I^{+r_2} &:= \{\mathbf{r}|p_2^{++}(\mathbf{r}) - r_2 \geq 0, p_2^{+-}(\mathbf{r}) - r_2 \leq 0, p_1^{+r_2}(r_1) - r_1 < 0\} \\
I^{-r_2} &:= \{\mathbf{r}|p_2^{-+}(\mathbf{r}) - r_2 \geq 0, p_2^{--}(\mathbf{r}) - r_2 \leq 0, p_1^{-r_2}(r_1) - r_1 > 0\} \\
I^{r_1+} &:= \{\mathbf{r}|p_1^{++}(\mathbf{r}) - r_1 \geq 0, p_1^{-+}(\mathbf{r}) - r_1 \leq 0, p_2^{r_1+}(r_2) - r_2 < 0\} \\
I^{r_1-} &:= \{\mathbf{r}|p_1^{+-}(\mathbf{r}) - r_1 \geq 0, p_1^{--}(\mathbf{r}) - r_1 \leq 0, p_2^{r_1-}(r_2) - r_2 > 0\} \\
S &: \text{the remaining region.}
\end{aligned}$$

Since all the solutions are linear in states, the above regions are characterized by linear inequalities and are therefore polyhedron. With the explicit solutions, after cumbersome algebraic computations, one can verify that, for instance, I^{++} and I^{+-} will not overlap.

We construct the semi-myopic solution by proposing the following pricing strategy, denoted as $\mathbf{p}^{SM}(\mathbf{r})$.

$$\mathbf{p}^{SM}(\mathbf{r}) = \begin{cases} \mathbf{p}^{++}(\mathbf{r}), & \mathbf{r} \in I^{++}, \\ \mathbf{p}^{+-}(\mathbf{r}), & \mathbf{r} \in I^{+-}, \\ \mathbf{p}^{-+}(\mathbf{r}), & \mathbf{r} \in I^{-+}, \\ \mathbf{p}^{--}(\mathbf{r}), & \mathbf{r} \in I^{--}, \\ (p_1^{+r_2}(r_1), r_2), & \mathbf{r} \in I^{+r_2}, \\ (p_1^{-r_2}(r_1), r_2), & \mathbf{r} \in I^{-r_2}, \\ (r_1, p_2^{r_1+}(r_2)), & \mathbf{r} \in I^{r_1+}, \\ (r_1, p_2^{r_1-}(r_2)), & \mathbf{r} \in I^{r_1-}, \\ \mathbf{r}, & \mathbf{r} \in S. \end{cases} \tag{5.9}$$

The rationale behind $\mathbf{p}^{SM}(\mathbf{r})$ is that depending on which region the state lies in, we choose optimal prices by assuming that all the future states will remain in the same region. For instance, if $\mathbf{r} \in I^{++}$, then $\mathbf{p}^{SM}(\mathbf{r}) = \mathbf{p}^{++}(\mathbf{r})$, where $\mathbf{p}^{++}(\mathbf{r})$ is the optimal solution if all the future states remain in I^{++}. Unfortunately, the state evolving according to the dynamics $\dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}^{++}(\mathbf{r}) - \mathbf{r})$ may, at some future point, leaves the region I^{++} and whenever this happens, we call it a *switch.* An illustration of the state space regions and the switching behavior is provided in Figure 5.1.

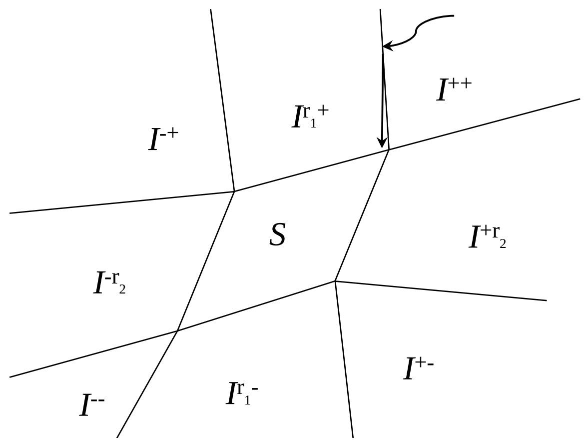


Figure 5.1: State space regions and a switching path

The following proposition established the desired property of $\mathbf{p}^{SM}(\mathbf{r})$.

Proposition 5.2. *For any initial reference price $\mathbf{r_0}$, the price path generated by $\mathbf{p}^{SM}(\mathbf{r})$ is a semi-myopic solution and we call $\mathbf{p}^{SM}(\mathbf{r})$ a semi-myopic pricing strategy. In addition, if for some $\mathbf{r_0}$, no switch occurs along the reference price path generated by $\mathbf{p}^{SM}(\mathbf{r})$, then the corresponding price path is optimal.*

Note that if $\mathbf{r_0} \in S$, then under $\mathbf{p}^{SM}(\mathbf{r})$, $\mathbf{r}(t) = \mathbf{r_0} \in S$ for all $t \geq 0$, i.e., no switch occurs. Proposition 5.2 then implies that the price path generated by $\mathbf{p}^{SM}(\mathbf{r})$ is optimal for all $\mathbf{r_0} \in S$. This motivates us to examine the global stability of the following dynamic system generated by semi-myopic pricing strategy:

$$\dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}^{SM}(\mathbf{r}) - \mathbf{r}). \tag{5.10}$$

Although our semi-myopic pricing strategy could be suboptimal due to the potential switches, if we can establish the fact that for any initial state $\mathbf{r_0}$, the reference price path under $\mathbf{p}^{SM}(\mathbf{r})$ will converge to certain point in S, then such strategy will, in the long run, bring us to a state where optimality can be achieved.

Establishing global stability for switched systems is, in general, non-trivial. While there is a vast amount of literature studying the stability issue of switched systems (see DeCarlo et al., 2000, for a review), most of the results extend the classic Lyapunov stability theory by proposing *multiple Lyapunov functions* to demonstrate stability. However, even for non-switched systems, "Lyapunov functions can be hard to come by (p.141 in Stokey et al., 1989)" and for switched systems, it is pointed out by DeCarlo et al. (2000) that "despite the variety and significance of the many results on hybrid system stability, general necessary and sufficient conditions in terms of the structure of the vector fields have evaded discovery."

In the following, we utilize the special structures of our problem to prove the global stability of (5.10). Since we are concerned with the two-product case, the discussions below will be restricted to two dimensional dynamic systems, i.e., planar system.

Lemma 5.2. *Consider a matrix*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with the property that $\mathbf{A} \succ 0$ *and* $b, c < 0$. *Let* $\mathbf{v_1} = (v_{11}, v_{12})$ *and* $\mathbf{v_2} = (v_{21}, v_{22})$ *be two eigenvectors of* $\mathbf{A}$, *then the following must hold*

$$v_{11}v_{12} > 0, \quad v_{21}v_{22} < 0.$$

In other words, the line with one eigenvector as the direction passes through the first and third quadrants while the line with the other eigenvector as the direction passes through the second and forth quadrants.

Recall the definition of the matrix $\mathbf{M}$ in Lemma 5.1 where it depends on the parameters $\eta_i, i = 1, ..., N$ for loss/gain neutral demands. Here, we define $\mathbf{M}^{++}$ by letting $N = 2$ and $\eta_1 = \eta_1^+, \eta_2 = \eta_2^+$. The matrices $\mathbf{M}^{+-}, \mathbf{M}^{-+}, \mathbf{M}^{-}$ can be defined in a similar fashion. Note that $\mathbf{M}^{++}$ will have the same eigenvectors as

$$\begin{aligned}&\frac{1}{\bar{\alpha}(\bar{\alpha}+\gamma/2)}(\mathbf{M}^{++}-\frac{\bar{\alpha}\gamma}{2}\mathbf{I})\\=&\frac{1}{(a_1+\eta_1^+)(a_2+\eta_2^+)-\theta^2}\begin{bmatrix} a_1(a_2+\eta_2^+)-\theta^2 & -\theta\eta_2^+ \\ -\theta\eta_1^+ & a_2(a_1+\eta_1^+)-\theta^2\end{bmatrix},\end{aligned}$$

which satisfies the condition in Lemma 5.2. Thus, $\mathbf{M}^{++}$ will have two eigenvectors respectively in the first (third) quadrants and the second (forth) quadrants. The same property can be established for $\mathbf{M}^{+-}, \mathbf{M}^{-+}, \mathbf{M}^{-}$. Using the above property of the matrices, we show that if the state path enters any of the regions I^{++}, I^{+-}, I^{-+} or I^{--}, it will converge to a unique steady state of that region without going out again. We formally state this result with respect to the region I^{++} in Lemma 5.3. The results and proofs for the regions I^{+-}, I^{-+} and I^{--} are similar.

Lemma 5.3. *Denote ∂I^{++} to be the boundary of I^{++}. For any $\mathbf{r}_0 \in \partial I^{++}$ and $\dot{\mathbf{r}}(0) \in I^{++}$, we have $\mathbf{r}(t) \in I^{++}$ for any $t > 0$ and*

$$\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r}_\mathbf{s}^{++},$$

where $\mathbf{r}_\mathbf{s}^{++}$ is the steady state to the system $\dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}^{++}(\mathbf{r}) - \mathbf{r})$.

Using Lemma 5.3, we can then establish the global stability of the system (5.10).

Proposition 5.3. *Starting from any initial state $\mathbf{r_0}$, the reference price path generated according to* (5.10) *can have at most two switches. As a result, there exists a $T \geq 0$, such that for all $t > T$, $\mathbf{r}(t)$ stays in any one of the nine regions illustrated in Figure 5.1 and converges to a steady state in S.*

We provide in Figure 5.2 a numerical example to illustrate the stability result in Proposition 5.3. In Figure 5.2, the reference price paths evolving according to (5.10) are plotted for each initial reference price lying on the boundaries of the square region. One can see that all reference price paths converge to the steady states region S in the middle. It is also clear from the figure that many reference price paths have a sudden change in their evolving directions which indicates a switching behavior.

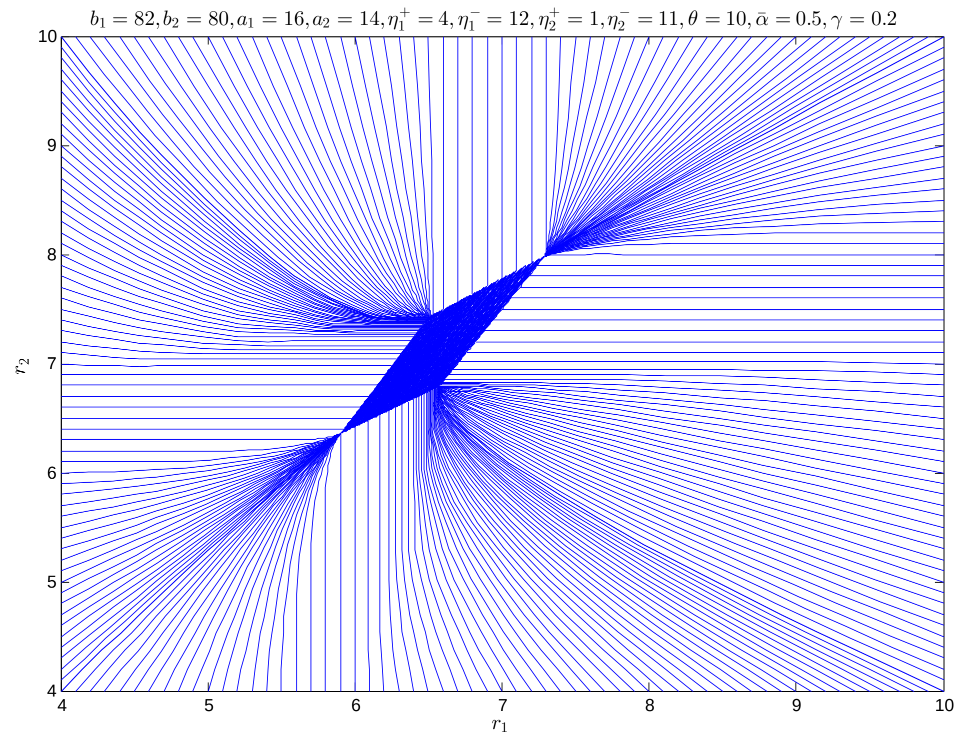


Figure 5.2: Steady state region and reference price paths

5.4 Conclusion

In this chapter we consider a dynamic pricing problem of multiple products. If demands for all the products are loss/gain neutral, we can solve the optimal prices for all the products explicitly. Our solution is a generalization of the solution in Fibich et al. (2003), which we use to show that in the long run the optimal prices as well as reference prices converge to a unique steady state.

If the demands are loss-averse, then the dynamic pricing problem for the multiple products becomes considerably more difficult. Not only the myopic pricing strategy can result in a switched dynamic system, due to the potential switching behavior, no existing method can be applied to solve for the optimal prices. To address these difficulties, we propose a semi-myopic solution, whose expressions can be explicitly computed. Our semi-myopic solution not only generalizes the myopic solution, but is also optimal if no switch occurs along the reference price path generated by the solution. We further prove that the dynamic system resulted from the semi-myopic solution is globally stable. The techniques we used in proving stability are potentially useful for establishing the stability property of other switched dynamic systems as well.

We believe our results in this chapter open up a few future research directions. First of all, as we have pointed out, the semi-myopic solution we constructed is optimal if no switch occurs along the state path. Can such regions where no switch occurs be identified (the so-called "positively invariant set")? Based on this step, one can then further explore what would be the optimal prices outside of these invariant sets.

It would be useful to both numerically and analytically quantify the optimality gap of our semi-myopic solution. Due to the curse of dimensionality, solving problem (5.3) numerically becomes a daunting task even for a small number of products. Deriving computationally efficient upper bounds on (5.3) will then facilitate performance evaluation.

Finally, a challenging yet interesting direction is to explore the case of gain-seeking demands and more generally a mixture of gain-seeking as well as loss-averse demands for multiple products. Although we have established the complicated behavior of the price dynamics in Chapter 3, whether our analytical results can be carried over to the multi-product settings by imposing similar assumptions as Assumption 3.1 is still an open question.

Chapter 6

Future research

In an effort to capture the relationship between demand and prices more accurately in a market with repeated purchases and quantify its impact, this thesis studies reference price models and the resulting dynamic pricing problems from several perspectives. Chapter 2-4 focuses on a single-product setting. Empirically, we compare different reference price models and examines the implication of behavioral asymmetry. Analytically, we characterize the structures of the optimal pricing strategy under the exponential smoothing reference price models when demands are gain-seeking. Computationally, we provide efficient algorithms to compute the optimal prices for the exponential smoothing reference price models.

For tractability, our analytical results and computational algorithms only consider the exponential smoothing reference price models, while our empirical comparisons suggest that in many cases the peak-end model and the adaptation-rate-based model can provide a better fit to the data. This opens up two interesting research directions. First, we have seen in Chapter 2 that the optimality of constant pricing strategies is quite robust to different reference price models under loss-averse demands. It is then natural to ask for gain-seeking demands whether some of the analytical results developed in Chapter 3 can be extended to the other two reference price models. Second, developing computationally efficient algorithms for the peak-end model and the adaptation-rate-based model is practically important since as reported in Natter et al. (2007), the adaptation-rate-based model is already implemented in the decision support system of bauMax, an Austrian retailer. Table 6.1 summarizes the existing results and the blank space represent open research questions.

Chapter 5 represents another direction of extension: dynamic pricing with multiple products. This is a largely unexplored yet important area and many open questions remain to be answered. First, several linear demand models

Table 6.1: Summary of Results

	Structures of optimal pricing strategies		Computational algorithms
	Loss-averse case	Gain-seeking case	
ES	Popescu and Wu (2007)	Chapter 3	Chapter 4
PE	Nasiry and Popescu (2011)		
ARB	Section 2.6		

ES: exponential smoothing model
PE: peak-end model
ARB: adaptation-rate-based model

for multiple products that incorporate reference price effects have been proposed. An empirical comparison of these models would not only shed light on how reference prices come into play in a multi-product setting but also give a guidance on developing analytical results and computational algorithms. Second, when there is behavioral asymmetry, the resulting dynamic pricing problem is a high-dimensional non-smooth dynamic programming problem and is quite challenging. Characterizing the structures of the optimal pricing strategies and studying the long-run behavior of the prices are crucial for developing simple heuristics of such complicated problems. Finally, efficient algorithms and implementations of the algorithms in a practical setting would be very useful to advance the usage of more realistic models that incorporate consumers behavior considerations among practitioners.

The topics covered in this thesis is by no means comprehensive. As we mentioned in Chapter 3 and Chapter 4, the coordination of pricing and inventory decisions is another wide area of research. Due to the increased complexity, most results in the literature are restricted to exponential smoothing model and loss-averse demands. It would be interesting to see how the results would change if alternative reference price models are employed. Competition among the firms, is another important factor not discussed in this thesis. While Fibich et al. (2003) examine the Cournot competition in a continuous-time framework, analyzing price competitions is limited to numerical simulations (Kopalle et al., 1996). Analytical results in this area would provide helpful implications in how reference price effects change the competitive behaviors among the firms.

Appendix A

A.1 Proof of Proposition 2.5

To solve the HJB equation (2.18), we start from solving a finite horizon problem by following Zhang (2011). That is, let

$$V(r,t) = \max_{p(s)} \left[\int_t^T e^{-\gamma t} p(s) D(r(s), p(s)) ds \right]$$

be the value of optimal accumulated profit (profit-to-go function) from time t to the end of horizon T when the initial reference price is r. Notice that the value function in our problem (2.17) $V(r_0) = \lim_{T\to\infty} V(r_0, 0)$.

From standard theory in stochastic optimal control, $V(r,t)$ then satisfies the HJB equation

$$\begin{aligned} &\gamma V(r,t) \\ &= \max_p \left[pD(r,p) + \frac{\partial V(r,t)}{\partial t} + \bar{\alpha}(p-r)\frac{\partial V(r,t)}{\partial r} + \frac{\sigma^2 r}{2}\frac{\partial^2 V(r,t)}{\partial r^2} \right] \end{aligned} \tag{A.1}$$

Using first order condition in (A.1) and with a slight abuse of notation, we can solve p as

$$p^*(r) = \frac{b+\eta r}{2(a+\eta)} + \frac{\alpha}{2(a+\eta)}\frac{\partial V(r,t)}{\partial r}.$$

Substitute the above equation into (A.1), it follows

$$\begin{aligned} \frac{\sigma^2}{2} r \frac{\partial^2 V}{\partial r^2} + \frac{\partial V}{\partial t} - \gamma V + \frac{\alpha^2}{4(a+\eta)} \left(\frac{\partial V}{\partial r} \right)^2 + \left[-\alpha r + \frac{\alpha(b+\eta r)}{2(a+\eta)} \right] \frac{\partial V}{\partial r} \\ + \frac{(b+\eta r)^2}{4(a+\eta)} = 0. \end{aligned}$$

Introducing a few new notations, this can be written concisely as:

$$\frac{\partial V}{\partial t} - \gamma V + Ar\frac{\partial^2 V}{\partial r^2} + B\left(\frac{\partial V}{\partial r}\right)^2 + (p_{10} + p_{11}r)\frac{\partial V}{\partial r} + p_{20} + p_{21}r + p_{22}r^2 = 0, \quad \text{(A.2)}$$

where

$$\begin{aligned} A &= \frac{\sigma^2}{2} \\ B &= \frac{\alpha^2}{4(a+\eta)} \\ p_{10} &= \frac{\alpha b}{2(a+\eta)} \\ p_{11} &= -\alpha + \frac{\alpha\eta}{2(a+\eta)} \\ p_{20} &= \frac{b^2}{4(a+\eta)} \\ p_{21} &= \frac{b\eta}{2(a+\eta)} \\ p_{22} &= \frac{\eta^2}{4(a+\eta)}. \end{aligned}$$

If we assume function $V(r,t)$ has the following form:

$$V(r,t) = Q(t)r^2 + R(t)r + M(t), \quad \text{(A.3)}$$

then we get the following ordinary differential equations (ODEs):

$$\frac{dQ}{dt} - \gamma Q + 4BQ^2 + 2p_{11}Q + p_{22} = 0, \quad \text{(A.4)}$$

$$\frac{dR}{dt} - \gamma R + 2AQ + 4BQR + 2p_{10}Q + p_{11}R + p_{21} = 0, \quad \text{(A.5)}$$

$$\frac{dM}{dt} - \gamma M + BR^2 + p_{10}R + p_{20} = 0, \quad \text{(A.6)}$$

with terminal condition $Q(T) = R(T) = M(T) = 0$.

We first explicitly solve ODE (A.4) by rewriting it as:

$$\frac{dQ}{dt} = -4B(Q - Q_1)(Q - Q_2)$$

where $Q_1 < Q_2$ are the two distinct roots of the equation:

$$4BQ^2 - (\gamma - 2p_{11})Q + p_{22} = 0.$$

Namely:

$$Q_1 = \frac{\gamma - 2p_{11} - \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B},$$

$$Q_2 = \frac{\gamma - 2p_{11} + \sqrt{(\gamma - 2p_{11})^2 - 16Bp_{22}}}{8B}.$$

Therefore:

$$\frac{dQ}{(Q - Q_1)(Q - Q_2)} = -4Bdt$$

$$\Rightarrow \quad \frac{dQ}{Q_1 - Q_2}\left[\frac{1}{Q - Q_1} - \frac{1}{Q - Q_2}\right] = -4Bdt$$

$$\Rightarrow \quad \ln \frac{Q - Q_1}{Q - Q_2} = -4B(Q_1 - Q_2)t + C$$

$$\Rightarrow \quad \frac{Q - Q_1}{Q - Q_2} = D \cdot e^{-4B(Q_1 - Q_2)t}, \tag{A.7}$$

where C and $D = e^C$ are constants to be determined. By $Q(T) = 0$, we can solve

$$D = \frac{Q_1}{Q_2} e^{4B(Q_1 - Q_2)T}.$$

Substitute D back into (A.7), it follows

$$Q(t) = \frac{Q_1 e^{4B(Q_1 - Q_2)T} - Q_1 e^{4B(Q_1 - Q_2)t}}{Q_1/Q_2 e^{4B(Q_1 - Q_2)T} - e^{4B(Q_1 - Q_2)t}}. \tag{A.8}$$

With the expressions for $Q(t)$, expressions for $R(t)$ and $M(t)$ can then be obtained by solving (A.5) and (A.6). Consequently, $p^*(r)$ can be determined as well. One can easily verify using Theorem 4.1 in chapter VI of Fleming and Rishel (1982) that $p^*(r)$ solved in this way is indeed optimal and $V(r, t)$ is given by (A.3).

The solution to (2.18) is then obtained by letting $T \to \infty$. Since $Q_1 < Q_2$, we have $Q := Q_1 = \lim_{T \to \infty} Q(t)$, where by substituting the expressions for B, p_{11} and p_{22}

$$Q = \frac{\gamma}{2\bar{\alpha}^2}(a + \eta) + \frac{2a + \eta}{2\bar{\alpha}} - \frac{a + \eta}{2\bar{\alpha}^2}\Delta$$

and Δ is given by:

$$\Delta = \sqrt{\gamma^2 + 2\bar{\alpha}\frac{2a(\gamma+\bar{\alpha})+\gamma\eta}{\eta+a}}.$$

Correspondingly, one can also obtain $R := \lim_{T\to\infty} R(t)$ as

$$\begin{aligned} R &= \frac{2p_{10}Q + p_{21} + 2AQ}{\gamma - 4BQ - p_{11}} \\ &= \frac{2\bar{\alpha}^2 b + \alpha b(\gamma-\Delta))}{\bar{\alpha}^2(\gamma+\Delta)} + \frac{\bar{\alpha}(2a+\eta)+(a+\eta)(\gamma-\Delta)}{\bar{\alpha}^2(\gamma+\Delta)}\sigma^2 \\ &= \left[\frac{b}{\bar{\alpha}} + \frac{\sigma^2(a+\eta)}{\bar{\alpha}^2}\right]\frac{\gamma-\Delta}{\gamma+\Delta} + \left[b + \frac{\sigma^2(2a+\eta)}{2\bar{\alpha}}\right]\frac{2}{\gamma+\Delta} \end{aligned}$$

.

Similarly, $M := \lim_{T\to\infty} M(t)$ can be computed but the expressions for M is not needed for our following analysis.

Now, we have explicitly solved (2.18), where $V(r) = Qr^2 + Rr + M$ and the optimal pricing strategy is

$$p^*(r) = \frac{b+\eta r}{2(a+\eta)} + \frac{\bar{\alpha}(2Qr+R)}{2(a+\eta)}. \tag{A.9}$$

Substitute (A.9) into the reference price dynamics (2.16), we have the following SDE:

$$\begin{aligned} dr^*(t) &= \alpha\left[\frac{2\bar{\alpha}Q - 2a - \eta}{2(a+\eta)}r^*(t) + \frac{\bar{\alpha}R + b}{2(a+\eta)}\right]dt + \sigma\sqrt{r^*(t)}dW(t) \\ &:= \lambda(\mu - r^*(t)) + \sigma\sqrt{r^*(t)}dW(t), \end{aligned} \tag{A.10}$$

where

$$\lambda = \bar{\alpha}\frac{2a+\eta-2\bar{\alpha}Q}{2(a+\eta)}, \quad \mu = \bar{\alpha}\frac{\bar{\alpha}R+b}{2\lambda(a+\eta)}.$$

Interestingly, under the optimal pricing strategy, the reference price dynamics (A.10) is again a square-root diffusion process. It is easy to show that $\lambda, \mu > 0$ and thus the steady state distribution is a gamma distribution whose shape parameter is $\frac{2\lambda\mu}{\sigma^2}$ and rate parameter is $\frac{2\lambda}{\sigma^2}$ (see, for instance, Cox et al., 1985).

A.2 Proof of Proposition 2.6

By Proposition 2.5, R_s^* follows a gamma distribution, its mean and variance can then be computed as

$$\mathbb{E}[R_s^*] = \frac{2\lambda\mu}{\sigma^2}\frac{\sigma^2}{2\lambda} = \mu,$$
$$\mathrm{var}(R_s^*) = \frac{2\lambda\mu}{\sigma^2}(\frac{\sigma^2}{2\lambda})^2 = \frac{\mu}{2\lambda}\sigma^2.$$

Substitute the expressions for Q, R and λ into μ, with cumbersome algebraic manipulations, one can further obtain

$$\mu = \frac{(\gamma+\bar{\alpha})b}{2a(\gamma+\bar{\alpha})+\gamma\eta} + \frac{\sigma^2}{2a(\gamma+\bar{\alpha})+\gamma\eta}\left[\frac{a+\eta}{\bar{\alpha}}\left(\frac{\gamma}{2}-\frac{\Delta}{2}\right)+\frac{2a+\eta}{2}\right].$$

Appendix B

B.1 Proof of Lemma 3.1

Let us first consider the unconstrained problem: $\max_p \Pi(r,p)$. Recall that

$$\begin{aligned}\Pi(r,p) &= \max\{\Pi^+(r,p), \Pi^-(r,p)\} \\ &= \max_{\eta^- \leq \eta \leq \eta^+} p(b - ap + \eta(r-p)),\end{aligned}$$

where the last equality is due to the linearity of the function $p(b-ap+\eta(r-p))$ in η.

We can then rewrite $\max_p \Pi(r,p)$ as

$$\max_{\eta^- \leq \eta \leq \eta^+} \max_p -(a+\eta)p^2 + \eta r p + bp.$$

Clearly, the inner maximization problem has a unique optimal solution $p(r,\eta) = \frac{\eta r + b}{2(a+\eta)}$ with the optimal objective value $v(r,\eta) = \frac{(\eta r+b)^2}{4(a+\eta)}$. Note further that $\frac{\partial^2 v}{\partial r \partial \eta} = \frac{\eta r(a+\eta)+a(b+\eta r)}{2(a+\eta)^2} > 0$, which indicates that $v(r,\eta)$ is supermodular. Since either $\eta = \eta^+$ or $\eta = \eta^-$, the supermodularity of $v(r,\eta)$ implies the existence of R such that for $r \leq R$, $v(r,\eta^-) \geq v(r,\eta^+)$ and for $r \geq R$, $v(r,\eta^+) \geq v(r,\eta^-)$. At R, $v(r,\eta^-) = v(r,\eta^+)$ and thus R can be computed explicitly as $R = \frac{b}{a+\sqrt{(a+\eta^+)(a+\eta^-)}}$. As a result, the unconstrained solution is exactly what presents in (3.5).

Observe that when $r > R$, we have $p^m(r) = \frac{\eta^+ r + b}{2(a+\eta^+)} \leq r$. To see this, suppose $p^m(r) > r$, then

$$\begin{aligned}\Pi(r, p^m(r)) &= p^m(r)(b - ap^m(r) + \eta^-(r - p^m(r))) \\ &> p^m(r)(b - ap^m(r) + \eta^+(r - p^m(r))) = v(r,\eta^+),\end{aligned}$$

which contradicts with the fact that $\Pi(r,p^m) = v(r,\eta^+)$ for $r > R$. Therefore, when $r > R$, $p^m(r)$ can never violate the upper bound U. On the other hand, by a similar argument for $r \le R$, $p^m(r) = \frac{\eta^- r + b}{2(a+\eta^-)} \ge r$. Moreover, $p^m(r)$ is increasing in r and is equal to U at R_U. Thus, if $R \le R_U$, then $\frac{\eta^- r+b}{2(a+\eta^-)} \le U$ on $[0, R]$ and the unconstrained solution specified above is optimal.

However, if $R > R_U$, then for $r \le R$

$$v(r, \eta^-) = \begin{cases} \dfrac{(\eta^- r + b)^2}{4(a + \eta^-)}, & r \le R_U, \\ -(a + \eta^-)U^2 + \eta^- rU + bU, & R_U \le r \le R, \end{cases}$$

and we need to compare again between $v(r, \eta^-)$ and $v(r, \eta^+)$ on $[R_U, R]$. It is straightforward to show that R' is the unique positive solution of $v(r, \eta^-) = v(r, \eta^+)$ and when $r < R'$, $v(r, \eta^-) > v(r, \eta^+)$, when $r > R'$, $v(r, \eta^-) < v(r, \eta^+)$. Therefore, we arrive at the second form of $p^m(r)$.

B.2 Proof of Proposition 3.1

Suppose that a periodic orbit of period 2: $\{r_0, r_1\}$ exists, where $r_0, r_1 \in [0, U]$ and $r_0 \ne r_1$. First we observe either $r_0 \le R, r_1 > R$ holds or $r_0 > R, r_1 \le R$ holds. To see this, suppose that, on the contrary, $r_i \le R$ for $i = 0, 1$, then $p^m(r_i) \ge r_i$ by the definition of R, which further implies $r_{1-i} = \alpha r_i + (1 - \alpha)p^m(r_i) \ge r_i$. Therefore, $r_{1-i} = r_i$ for $i = 0, 1$, leads to contradiction with $r_0 \ne r_1$. Similarly, it is also impossible that $r_i \ge R$ for $i = 0, 1$. In the following we assume, without loss of generality, that $r_0 \le R, r_1 > R$. Then, r_0, r_1 satisfy the following equations:

$$r_1 = \alpha r_0 + (1 - \alpha)\frac{\eta^- r_0 + b}{2(a + \eta^-)},$$
$$r_0 = \alpha r_1 + (1 - \alpha)\frac{\eta^+ r_1 + b}{2(a + \eta^+)}.$$

Thus, r_0, r_1 can be explicitly solved as

$$r_0 = \frac{[(1 + \alpha)\eta^+ + 2a + 2\eta^- + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-},$$
$$r_1 = \frac{[(1 + \alpha)\eta^- + 2a + 2\eta^+ + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-}.$$

We have shown that if a periodic orbit of period 2 exists, then it must be given by the unique solution identified above and has to satisfy $r_0 \leq R$ and $r_1 > R$. On the other hand, given r_0 and r_1 specified by above expressions with $r_0 \leq R, r_1 > R$. Then, by our construction, $r_1 = \alpha r_0 + (1-\alpha)p^m(r_0)$ and $r_0 = \alpha r_1 + (1-\alpha)p^m(r_1)$, which implies $\{r_0, r_1\}$ is a periodic orbit of period 2. Overall, a periodic orbit of period 2 exists if and only if

$$\frac{[(1+\alpha)\eta^+ + 2a + 2\eta^- + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-} \leq R,$$
$$\frac{[(1+\alpha)\eta^- + 2a + 2\eta^+ + 2\alpha a]b}{(4a + 3\eta^- + 2\alpha a + \alpha\eta^-)\eta^+ + 4\alpha\eta^- + 4\alpha a^2 + 4a^2 + 2\alpha a\eta^-} > R.$$

With cumbersome algebraic manipulations, the above two inequalities can be simplified to

$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^+ + 4a\eta^- - (1+\alpha)^2(\eta^+)^2 + 4\eta^+\eta^- \geq 0,$$
$$4(1-\alpha^2)a^2 + 4(1-\alpha-\alpha^2)a\eta^- + 4a\eta^+ - (1+\alpha)^2(\eta^-)^2 + 4\eta^+\eta^- > 0.$$

Observe that the second inequality is naturally satisfied when the first inequality is satisfied. Thus, we have arrived at condition (3.7).

B.3 Proof of Proposition 3.2

By translating R to the origin and scaling $\frac{\eta^- R + b}{2(a+\eta^-)} - \frac{\eta^+ R + b}{2(a+\eta^+)}$ to 1, then $\alpha r + (1-\alpha)p^m(r)$ is equivalent to the following discontinuous map studied in Rajpathak et al. (2012).

$$f(r) = \begin{cases} Ar + \mu, & r \leq 0, \\ Br + \mu - 1, & r > 0. \end{cases}$$

Denote $\{r_0, r_1, ..., r_{n-1}\}$ be the periodic orbit (if exists) of the above system. Then it is easy to see that $r_0 < r_1 < ... < r_{n-1}$ ($r_0 > r_1 > ... > r_{n-1}$) if and only if $r_0 < r_1 < ... < r_{n-2} \leq 0$ and $r_{n-1} > 0$ ($r_0 > r_1 > ... > r_{n-2} > 0$ and $r_{n-1} \leq 0$). If we denote each point $r_t, 0 \leq t < n$, to be $\mathcal{L}$ if $r_t \leq 0$ and $\mathcal{R}$ if $r_t > 0$, then the periodic orbit can be coded as $\{\underbrace{\mathcal{L}, \mathcal{L}, ..., \mathcal{L}}_{n-1}, \mathcal{R}\}$ ($\{\underbrace{\mathcal{R}, \mathcal{R}, ..., \mathcal{R}}_{n-1}, \mathcal{L}\}$), which is exactly the so-called *prime pattern* in Rajpathak et al. (2012). Applying Theorem 1 in Rajpathak et al. (2012), we arrive at conditions (3.8) and (3.9) that guarantee each type of periodic solution with period n respectively.

B.4 Proof of Lemma 3.2

We first prove that $V(r)$ is increasing and convex in r. To see this, define the following value iteration for $i \geq 0$:

$$V_{i+1}(r) = \max_{p \in [0,U]} \Pi(r,p) + \gamma V_i(\alpha r + (1-\alpha)p),$$

with $V_0(r) = 0$. We inductively show that $V_i(r)$ is increasing and convex in r for all $i \geq 0$. Clearly, $V_0(r)$ trivially has the property. For $i > 0$, suppose $V_i(r)$ is increasing and convex in r. Since $\Pi(r,p)$ is increasing in r, we immediately have $V_{i+1}(r)$ is also increasing in r.

To see convexity of $V_{i+1}(r)$, it is sufficient to show that $\Pi(r,p)$ is convex in r. Indeed, recall that $\Pi(r,p) = \max\{\Pi^+(r,p), \Pi^-(r,p)\}$, where $\Pi^{\pm}(r,p) = p[b - ap + \eta^{\pm}(r-p)]$. As both $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are convex in r, $\Pi(r,p)$ is also convex in r. By Proposition 2.1.15 in Simchi-Levi et al. (2014), $V_{i+1}(r)$ is also convex. By Theorem 4.6 in Stokey et al. (1989), $\lim_{i\to\infty} V_i(r) = V(r)$. Thus, $V(r)$ is both increasing and convex in r.

Since $V(r)$ is increasing and convex in r while both $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are increasing and convex in r, by applying again Proposition 2.1.15 in Simchi-Levi et al. (2014) to problems (3.11a) and (3.11b) respectively, we have $V^+(r)$ and $V^-(r)$ are also increasing and convex.

On the other hand, as $\frac{\partial^2 \Pi^+(r,p)}{\partial r \partial p} = \eta^+ > 0$ and $\frac{\partial^2 \Pi^-(r,p)}{\partial r \partial p} = \eta^- > 0$, $\Pi^+(r,p)$ and $\Pi^-(r,p)$ are supermodular in r and p. Moreover, since $V(\cdot)$ is convex, by Theorem 2.2.6 in Simchi-Levi et al. (2014), $V(\alpha r + (1-\alpha)p)$ is also supermodular. As a result, $p^+(r)$ and $p^-(r)$ are increasing in r.

B.5 Proof of Lemma 3.3

Clearly, as $V^-(r) = V^-$ is constant and $V^+(r)$ is increasing, there are three cases. If $V^+(r) \geq V^-$ for all $r \in [0,U]$, then $R_0 = 0$. If $V^+(r) \leq V^-$ for all

$r \in [0, U]$, then $R_0 = U$. If neither of the above cases is true, then we must have $V^+(0) < V^-$ and $V^+(U) > V^-$. As $V^+(r)$ is continuous and increasing, there must exists $0 < R_0 < U$, such that if $r \leq R_0$, then $V(r) = V^-$ and $p^*(r) = p^-$ and if $r > R_0$, then $V(r) = V^+(r)$ and $p^*(r) = p^+(r)$.

Next, we show that whenever $p^*(r) = p^-$, then $p^*(r) = p^- > r$ and whenever $p^*(r) = p^+(r)$, then $p^*(r) = p^+(r) < r$. Recall from Section 3.2 that $p^*(r) \neq r$ for any $r \in [0, U]$. Now suppose $p^*(r) = p^+(r)$, but $p^*(r) > r$, then $V^+(r) \geq V^-$. However,

$$\begin{aligned} &V^+(r) = \Pi^+(r, p^+(r)) + \gamma V(p^+(r)) \\ <&\Pi^-(r, p^+(r)) + \gamma V(p^+(r)) \leq \Pi^-(r, p^-) + \gamma V(p^-) = V^-, \end{aligned}$$

a contradiction. The same claim can be made when $p^*(r) = p^-$.

Finally, if $R_0 = 0$ or $R_0 = U$, then $p^*(0) = p^+(0) < 0$ or $p^*(U) = p^-(U) > U$, both lead to a contradiction. So we must have $0 < R_0 < U$.

B.6 Proof of Proposition 3.3

We first introduce some notations to formalize the idea discussed in Section 3.4. Let $V_0(r) = V^-$, then there exists $R_1 = \sup\{r \in [R_0, U] : p^+(r) < R_0\} \in (R_0, U]$ such that for $r \in (R_0, R_1)$, $p^+(r) = p_1(r) = \frac{\eta^+ r + b}{2(a+\eta^+)}$ and

$$V^+(r) = V_1(r) = \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_0(p) = \frac{(\eta^+ r + b)^2}{4(a + \eta^+)} + \gamma V^-.$$

If $R_1 = U$, then our proposition holds with $N = 0$.

More generally, for $k \geq 2$, if $R_{k-1} < U$, define $R_k = \sup\{r \in [R_0, U] : p^+(r) < R_{k-1}\} \in (R_0, U]$. Note that $p^+(R_{k-1}) < R_{k-1}$ by Lemma 3.3 and it follows $R_k > R_{k-1}$. We further define

$$\begin{aligned} V_k(r) &= \max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_{k-1}(p), \\ p_k(r) &= \arg\max_{p \in [0,U]} \Pi^+(r, p) + \gamma V_{k-1}(p). \end{aligned} \tag{B.1}$$

By our construction, on $[R_{k-1}, R_k)$, $R_{k-2} \leq p^+(r) < R_{k-1}$. Thus, if $V(r) = V^+(r) = V_{k-1}(r)$ on $[R_{k-2}, R_{k-1})$, then $V(p^+(r)) = V_{k-1}(p^+(r))$ on $[R_{k-1}, R_k)$ and there is no loss of optimality by replacing problem (3.11a) with problem

(B.1) above. Therefore, we have inductively shown that $V(r) = V^+(r) = V_k(r)$ and $p^*(r) = p^+(r) = p_k(r)$ on $[R_{k-1}, R_k)$.

Next, we inductively show that $V_k(r)$ has the following parametric form:

$$V_k(r) = \frac{1}{2}A_k r^2 + B_k r + C_k,$$

with $A_k \leq \eta^+$. We have already shown the base case for $k = 1$, with

$$\begin{aligned} A_1 &= \frac{(\eta^+)^2}{2(a+\eta^+)} = \frac{\eta^+ m_2}{\gamma} \leq \eta^+, \\ B_1 &= \frac{\eta^+ b}{2(a+\eta^+)} = \frac{m_2 b}{\gamma}. \end{aligned}$$

With inductive hypothesis,

$$\begin{aligned} V_{k+1}(r) &= \max_{p\in[0,U]} \{\Pi^+(r,p) + \gamma V_k(p)\} \\ &= \max_{p\in[0,U]} \{-(a+\eta^+ - \frac{1}{2}\gamma A_k)p^2 + (b+\eta^+ r + \gamma B_k)p + \gamma C_k\}. \end{aligned}$$

By Lemma 3.3, $p_{k+1}(r) < U$. Combined with $A_k \leq \eta^+$, we know above problem is a concave maximization problem with an interior solution. Thus, the optimal solution can be derived from first order condition:

$$p_{k+1}(r) = \frac{b + \eta^+ r + \gamma B_k}{2(a+\eta^+) - \gamma A_k}.$$

We can then express

$$\begin{aligned} & V_{k+1}(r) \\ =& -(a+\eta^+ - \frac{1}{2}\gamma A_k)p_{k+1}(r)^2 + (b+\eta^+ r + \gamma B_k)p_{k+1}(r) + \gamma C_k \\ =& \frac{(\eta^+)^2/2}{2(a+\eta^+) - \gamma A_k} r^2 + \frac{(b+\gamma B_k)\eta^+}{2(a+\eta^+) - \gamma A_k} r + \frac{(b+\gamma B_k)^2/2}{2(a+\eta^+ - \gamma A_k)} + \gamma C_k. \end{aligned}$$

Thus,

$$\begin{aligned} A_{k+1} &= \frac{(\eta^+)^2}{2(a+\eta^+) - \gamma A_k} \leq \frac{(\eta^+)^2}{2(a+\eta^+) - \gamma \eta^+} \leq \eta^+, \\ B_{k+1} &= \frac{(b+\gamma B_k)\eta^+}{2(a+\eta^+) - \gamma A_k}. \end{aligned}$$

Combined with the expressions for A_1, B_1, it is not difficult to see that

$$A_k = \frac{\eta^+}{\gamma} m_{k+1},$$
$$B_k = \frac{1}{\gamma} \sum_{i=0}^{n} \prod_{j=0}^{i} m_{k+1-j} b.$$

Consequently, we can compute

$$p_k(r) = \frac{\eta^+ r + b + \sum_{i=0}^{k-1} (\prod_{j=0}^{i} m_{k-j}) b}{2(a+\eta^+) - m_k \eta^+}.$$

If $R_k = U$, the above analysis has shown that our proposition holds with $N = k - 1$.

Finally, we show that the construction of the sequence R_k cannot continue forever. In other words, there exists $N \geq 0$ such that $R_{N+1} > U$. We prove by contradiction. Suppose for any $k \geq 1$, $R_k \leq U$. Then the following relation between R_k and R_{k+1} must hold for any $k \geq 1$: for $r \in [R_k, R_{k+1}]$, $p^+(r) \leq R_k$ and for $r \in [R_{k+1}, U]$, $p^+(r) \geq R_k$. Since R_k is a bounded increasing sequence, there exists $\bar{R} \leq U$ such that

$$\lim_{k \to \infty} R_k = \bar{R}.$$

From the above relation, we conclude that for $r \in [R_{k+1}, \bar{R}]$, $p^+(r) \geq R_k$. In particular, $p^+(\bar{R}) \geq R_k$ for any $k \geq 1$. Taking limits on both sides, we obtain

$$p^+(\bar{R}) \geq \bar{R},$$

leading to a contradiction with Lemma 3.3 which states $p^*(r) = p^+(r) < r$ for any $r \in [R, U]$.

B.7 Proof of Proposition 3.4

From Lemma 3.3, we know that $R_0 < r^* \leq U$. Thus, there exists $2 \leq n \leq N + 2$ such that $r^* \in [R_{n-2}, R_{n-1}]$. Let $p_1^*(r^*) = p^+(r^*)$, then $p_1^*(r^*) \in [R_{n-3}, R_{n-2}]$ and $p_1^*(r^*) < r^*$. Inductively, for $1 \leq i \leq n-2$ if $p_i^*(r^*) \in [R_{n-2-i}, R_{n-1-i}]$, then let $p_{i+1}^*(r^*) = p^+(p_i^*(r^*))$ and it follows that $p_{i+1}^*(r^*) \in$

$[R_{n-3-i}, R_{n-2-i}]$ if $i \leq n-3$ or $p^*_{i+1}(r^*) \in [0, R_0]$ if $i = n-2$. Furthermore, it holds $p^*_{i+1}(r^*) < p^*_i(r^*)$. Finally, let $p^*_n(r^*) = p^-(p^*_{n-1}(r^*)) = p^-$. Then $p^*_n(r^*) = r^*$. We have constructed the periodic orbit $\{r^*, p^*_1(r^*), ..., p^*_{n-1}(r^*)\}$ with the property $r^* > p^*_1(r^*) > ... > p^*_{n-1}(r^*)$.

Next we show that start from any initial reference price $r_0 \in [0, U]$, the optimal reference price path r^*_t converges to the periodic orbit in at most $N+2$ periods. Clearly, for any $r_0 \in [0, R_0]$, $r^*_1 = p^-(r_0) = r^*$, and r^*_t will then follow the periodic orbit. For $0 \leq i \leq N$ and any $r_0 \in [R_i, R_{i+1}]$, it follows that $r^*_1 = p^+(r_0) \in [R_{i-1}, R_i]$ if $i \geq 1$ or $r^*_1 = p^+(r_0) \in [0, R_0]$ if $i = 0$. Again, inductively, for $1 \leq t \leq i$ it holds that $r^*_t = p^+(r^*_{t-1}) \in [R_{i-t}, R_{i-t+1}]$. Finally, $r^*_{i+1} \in [0, R_0]$ and $r^*_{i+2} = p^-(r^*_{i+1}) = r^*$ and the reference price path from then on follows the periodic orbit. In the worst case when $r_0 \in [R_N, R_{N+1}]$, we know that $r^*_{N+2} = r^*$.

B.8 Proof of Proposition 3.5

The basic idea is to find the condition such that $p^+(r) \leq R_0$ for any $r \in [R_0, U]$. If this holds, then for any $r_0 \in [R_0, U]$ it follows that $r^*_1 = p^+(r_0) \in [0, R_0]$, $r^*_2 = p^-(r^*_1) = p_H = r^*$ and $r^*_3 = p^+(r^*_2) = p_L$, which shows the existence of the high-low pricing strategy $\{p_H, p_L\}$.

However, both R_0 and the number of linear pieces as well as the "jumping" points of $p^+(r)$ are unknown. Instead, we strive to find an upper bound: $\bar{p}(r)$ on $p^+(r)$ and a lower bound $\underline{R}_0$ on R_0 such that condition (3.12) guarantees $\bar{p}(r) \leq \underline{R}_0$ for all $r \in [R_0, U]$, which then implies $p^+(r) \leq \bar{p}(r) \leq \underline{R}_0 \leq R_0$ for all $r \in [R_0, U]$.

First, we claim that the constant K is an upper bound on m_k for $k \geq 1$. Clearly, for $k = 1$, it holds $m_1 = 0 \leq K$. Suppose for $k \geq 1$, we have $m_k \leq K$. Then, we want to show that

$$m_{k+1} = \frac{\gamma\eta^+}{2(a+\eta^+) - m_k\eta^+} \leq \frac{\gamma\eta^+}{2(a+\eta^+) - K\eta^+} \leq K.$$

The second inequality above is equivalent to:

$$\eta^+ K^2 - 2(a+\eta^+)K + \gamma\eta^+ \leq 0,$$

and it is straightforward to see that K indeed satisfies the inequality above. Note that here $K < 1$. Using the bound on m_k and the expression of $p^*(r)$ in Proposition 3.3, we can then bound $p^+(r)$ as

$$p^+(r) \leq \frac{\eta^+ r + b + \sum_{i=0}^{\infty} K^{i+1} b}{2(a+\eta^+) - K\eta^+} = \frac{\eta^+ r + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} = \bar{p}(r)$$

Next, we claim that $\underline{R}_0 = R = \frac{b}{a+\sqrt{a(a+\eta^+)}}$ provides a lower bound for R_0. It is sufficient to show that $p^*(r) \geq p^m(r)$ for any $r \in [0, R)$ because if $R > R_0$, then on (R_0, R) we have $p^*(r) = p^+(r) < r < p^m(r)$, leading to a contradiction. Indeed, if $p^*(r) < p^m(r)$ for some $r \in (0, R)$, then as $p^m(r)$ maximizes $\Pi(r, p)$ and multiple solutions occur only when $r = R$, we must have

$$\Pi(r, p^*(r)) < \Pi(r, p^m(r)).$$

Since $V(r)$ is increasing in r, it follows $V(p^*(r)) < V(p^m(r))$, which implies $p^*(r)$ is not optimal, leading to a contradiction.

Finally, as $\bar{p}(r)$ is also increasing, condition (3.12) is derived by simply requiring

$$\bar{p}(U) = \frac{\eta^+ U + \frac{1}{1-K} b}{2(a+\eta^+) - K\eta^+} \leq \underline{R}_0 = R = \frac{b}{a + \sqrt{a(a+\eta^+)}},$$

to ensure $\bar{p}(r) \leq \underline{R}$ for all $r \in [R_0, U]$.

B.9 Proof of Proposition 3.6

The idea is similar to the proof of Proposition 3.5. We claim that $\underline{R}_k \leq R_k$. The base case for $k = 0$ is shown in Proposition 3.5. Suppose $\underline{R}_{k-1} \leq R_{k-1}$. Following the proof of Proposition 3.3, denote

$$p_{k+1}(r) = \frac{\eta^+ r + b + \sum_{i=0}^{k} (\prod_{j=0}^{i} m_{k+1-j}) b}{2(a+\eta^+) - m_{k+1}\eta^+}.$$

By our construction, as $p^+(r) = p_{k+1}(r)$ on $[R_k, R_{k+1}]$ and

$$R_k = \sup\{r \in [R_0, U] : p^+(r) = p_{k+1}(r) < R_{k-1}\},$$

while for any $r < \underline{R}_k$,

$$p_{k+1}(r) < \underline{R}_{k-1} \leq R_{k-1}.$$

Thus, $\underline{R}_k \leq R_k$.

Condition (3.13) then implies, $p_2^*(r^*) \in [R_i, R_{i+1}]$ for $0 \leq i \leq k-1$. Therefore, by counting the maps from $[R_i, R_{i+1}]$ to $[R_{i-1}, R_i]$, it is easy to see that the length of period is at most $k+2$.

Appendix C

C.1 Proof of Proposition 4.1

Define $\underline{Q} = \sup\{q : q - r^*(q) < 0\}$ and $\overline{Q} = \inf\{q : q - r^*(q) > 0\}$. By the single crossing property of $q - r^*(q)$ in Assumption 4.2, we have $\underline{Q} \leq \overline{Q}$. Furthermore, when $q < \underline{Q}$, $q - r^*(q) < 0$, when $q > \overline{Q}$, $q - r^*(q) > 0$ and when $\underline{Q} \leq q \leq \overline{Q}$, $q - r^*(q) = 0$.

As a result, when $q < \underline{Q}$,

$$\begin{aligned} r^*(q) &= \arg\max_r \{\Pi(r, q) + g(r)\} \\ &= \arg\max_{r>q} \{\Pi(r, q) + g(r)\} \\ &= \arg\max_{r>q} \{\Pi^+(r, q) + g(r)\} \\ &= r^+(q). \end{aligned}$$

Similarly, when $q > \overline{Q}$, it holds $r^*(q) = r^-(q)$.

C.2 Proof of Lemma 4.1

First note that the strict supermodularity of $\Pi(r, q)$ in Assumption 4.1 implies the strict supermodularity of $\Pi^+(r, q)$. Thus, $r^+(q)$ is increasing in q. For $1 \leq j \leq m$, define $\underline{q}_j = \sup\{q : r^+(q) < r_j\}$ and $\overline{q}_j = \inf\{q : r^+(q) > r_j\}$.

By definition, it is clear that when $\underline{q}_j \leq q \leq \overline{q}_j$, $r^+(q) = r_j$.

For $1 \leq j \leq m+1$ and any $\overline{q}_{j-1} < q < \underline{q}_j$, we then have $r_{j-1} < r^+(q) < r_j$. Thus, there is no loss of optimality in considering the problem

$$r^+(q) = \arg\max_r \{\Pi^+(r, q) + g_j(r)\}.$$

Note here that the objective function is now continuously differentiable and strictly supermodular. There is no constraint imposed on r and consequently, $r^+(q)$ is always an interior solution. By Strict Monotonicity Theorem in Edlin and Shannon (1998), we then have $r^+(q)$ is strictly increasing in q on $\overline{q}_{j-1} \leq q \leq \underline{q}_j$, $1 \leq j \leq m+1$.

C.3 Proof of Proposition 4.2

Clearly, $r^*(q) = q$ on $[\underline{Q}, \overline{Q}]$ and is consequently strictly increasing.

On $(-\infty, \underline{Q}]$, $r^*(q) = r^+(q)$ and by Lemma 4.1, $r^+(q)$ can only take constant values $\{r_j | r_j \leq \underline{Q}, 1 \leq j \leq m\}$ on the corresponding interval $[\underline{q}_j, \overline{q}_j]$ and strictly increasing elsewhere.

Similarly, on $[\overline{Q}, +\infty)$, $r^*(q) = r^-(q)$ and by Lemma 4.1, $r^-(q)$ can only take constant values $\{r_j | r_j \geq \overline{Q}, 1 \leq j \leq m\}$ on the corresponding interval $[\underline{q}_j, \overline{q}_j]$ and strictly increasing elsewhere.

C.4 Proof of Lemma 4.2

By Assumption 4.2 (a), $f(q)$ is concave. Combined with Proposition 4.1, this implies that $f^+(q)$ and $f^-(q)$ are concave on $(-\infty, \underline{Q})$ and $(\overline{Q}, +\infty)$ respectively.

Now that $f^+(q)$ is concave and $\Pi^+(r, q)$ is continuously differentiable. By Theorem 1 in Milgrom and Segal (2002), $f^+(q)$ is differentiable and

$$\frac{df(q)}{dq} = \Pi_q^+(r^+(q), q).$$

That is, $f^+(q)$ is continuously differentiable on $(-\infty, \underline{Q})$. The same argument applies to $f^-(q)$.

C.5 Proof of Proposition 4.3

By Proposition 4.1 and Lemma 4.2, we have already proved that $f(q)$ is continuously differentiable when $q < \underline{Q}$ and $q > \overline{Q}$. Also, note that when $\underline{Q} < q < \overline{Q}$, $f(q) = \pi(q) + g(q)$. Since $\pi(q)$ is continuously differentiable, whether

$f(q)$ is continuously differentiable or not only depends on whether $g(q)$ is continuously differentiable or not on $\underline{Q} < q < \overline{Q}$. However, $r_1, r_2, ..., r_m$ are all the kink points of $g(q)$. Therefore, we can conclude that the only possible kink points of $f(q)$ are $\underline{Q}, \overline{Q}$ and $r_1, r_2, ..., r_m$. Next, we conclude that $r_1, r_2, ..., r_m$ are the only possible kink points of $f(q)$.

To show that $r_1, r_2, ..., r_m$ are the only possible kink points, we only need to show that when $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$ and $\overline{Q} \notin \{r_1, r_2, ..., r_m\}$, $f(q)$ is continuously differentiable at $\underline{Q}$ and $\overline{Q}$. Without loss of generality, we next show for the case when $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$, $f(q)$ is continuously differentiable at $\underline{Q}$.

Since $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$ and $-\infty = r_0 < r_1 < ... < r_m < r_{m+1} = +\infty$, we know there exists $1 \leq j \leq m+1$, such that $r_{j-1} < \underline{Q} < r_j$. By continuity of $r^*(q)$ and Proposition 4.1, $r_{j-1} < r^*(\underline{Q}) = r^+(\underline{Q}) = \underline{Q} < r_j$. Thus, $r^+(\underline{Q}) = \underline{Q}$ is the solution to:

$$f^+(\underline{Q}) = \max_r \{\Pi^+(r, \underline{Q}) + g_j(r)\}.$$

By the first order condition, $\underline{Q}$ satisfies

$$\Pi_r^+(\underline{Q}, \underline{Q}) + \frac{dg_j(r)}{dr}(\underline{Q}) = 0. \tag{C.1}$$

Now, since $r_{j-1} < \underline{Q} < r_j$, the right derivative at $\underline{Q}$ is

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \frac{d\pi(\underline{Q})}{dq} + \frac{dg_j(\underline{Q})}{dq}.$$

On the other hand, the left derivative at $\underline{Q}$ is

$$\lim_{q \uparrow \underline{Q}} \frac{df(q)}{dq} = \Pi_q^+(\underline{Q}, \underline{Q}).$$

Using equation (C.1), we then have

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \frac{d\pi(\underline{Q})}{dq} - \Pi_r^+(\underline{Q}, \underline{Q}).$$

Using total derivative, we then have $\Pi_r^+(q,q) + \Pi_q^+(q,q) = \frac{d\pi(q)}{dq}$. That is,

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \lim_{q \uparrow \underline{Q}} \frac{df(q)}{dq},$$

which shows that $f(q)$ is continuously differentiable at $\underline{Q}$. The differentiability at $\overline{Q}$ can be proved similarly.

C.6 Proof of Proposition 4.5

Suppose that $G_t(r)$ is strongly concave with concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. We will next show first that $G_{t+1}(r)$ is also strongly concave. The argument we use also implies the base case $G_2(r)$ is strongly concave.

Let $\hat{G}_t(r) = G_t(r) + A_t r^2$ and $B_t = \frac{2a_t + \eta^+}{2(1-\alpha)}$, then the Bellman equation (4.3) can be rewritten as

$$\hat{G}_{t+1}(q) = \max_r \{\Pi_t(r,q) - A_t r^2 + B_t q^2 + (A_{t+1} - B_t)q^2 + \hat{G}_t(r)\}.$$

By inductive hypothesis, $\hat{G}_t(r)$ is concave. To prove $\hat{G}_{t+1}(q)$ is also concave, it is sufficient to prove that the objective function is jointly concave in r and q. We prove this by showing that $\Pi_t(r,q) - A_t r^2 + B_t q^2$ is a jointly concave function and $A_{t+1} - B_t \leq 0$.

Note that $\Pi_t(r,q) = \min\{\Pi_t^+(r,q), \Pi_t^-(r,q)\}$, where

$$\Pi_t^{\pm}(r,q) = -\frac{a_t + \eta^{\pm}}{(1-\alpha)^2} q^2 + \frac{2\alpha a_t + (1+\alpha)\eta^{\pm}}{(1-\alpha)^2} qr - \frac{\alpha^2 a_t + \alpha\eta^{\pm}}{(1-\alpha)^2} r^2 + \frac{b_t}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} r.$$

From the expression above, we can obtain the Hessian of $\Pi_t^{\pm}(r,q) - A_t r^2 + B_t q^2$ as

$$\frac{1}{(1-\alpha)^2} \begin{bmatrix} -2(a_t + \eta^{\pm}) + 2(1-\alpha)^2 B_t & 2\alpha a_t + (1+\alpha)\eta^{\pm} \\ 2\alpha a_t + (1+\alpha)\eta^{\pm} & -2(\alpha^2 a_t + \alpha\eta^{\pm}) - 2(1-\alpha)^2 A_t \end{bmatrix}.$$

Substitute the expressions of A_t and B_t, it can be verified that above matrices are diagonally dominant and consequently, $\Pi_t^{\pm}(r,q) - A_t r^2 + B_t q^2$ are jointly concave. Thus, $\Pi_t(r,q) - A_t r^2 + B_t q^2$ is also jointly concave.

To guarantee $A_{t+1} - B_t \leq 0$, we need

$$\frac{2\alpha a_{t+1} + \eta^-}{2(1-\alpha)} \leq \frac{2a_t + \eta^+}{2(1-\alpha)},$$

which is guaranteed by Assumption 4.4.

Finally, note that $\hat{G}_2(r) = \Pi_1(r_1, r) + B_1 r^2 + (A_2 - B_1) r^2$. By the fact that $\Pi_1(r_1, r) - A_1 r_1^2 + B_1 r^2$ is jointly concave as shown above, $\Pi_1(r_1, r) + B_1 r^2$ is concave in r. Thus, $\hat{G}_2(r)$ is also concave in r.

To show the second claim, first recall that

$$G_2(r) = \begin{cases} g_2(r), & r \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1], \\ -\infty, & r \in (-\infty, \alpha r_1 + (1-\alpha)L_1) \cup (\alpha r_1 + (1-\alpha)U_1, +\infty), \end{cases}$$

which shows that the claim holds for the base case. Now suppose for $t \geq 2$,

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty). \end{cases}$$

When $\alpha > 0$, the price constraint is equivalent as $r \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$. If $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}] \cap [\underline{r}_t, \overline{r}_t] = \emptyset$, then $G_{t+1}(q) = -\infty$. That is, we can let $\underline{r}_{t+1} = \alpha \underline{r}_t + (1-\alpha)L_t$ and $\overline{r}_{t+1} = \alpha \overline{r}_t + (1-\alpha)U_t$. When $\alpha = 0$, clearly, $\underline{r}_{t+1} = L_t$ and $\overline{r}_{t+1} = U_t$. In summary, we then have

$$G_{t+1}(r) = \begin{cases} g_{t+1}(r), & r \in [\underline{r}_{t+1}, \overline{r}_{t+1}], \\ -\infty, & r \in (-\infty, \underline{r}_{t+1}) \cup (\overline{r}_{t+1}, +\infty). \end{cases}$$

C.7 Proof of Lemma 4.3

Since the objective function is strict concave in r and continuous, Maximum Theorem (Ok, 2007) guarantees that $r_c^*(q)$ is single-valued and continuous in q.

For the second part, it is shown in Chen et al. (2013) that $\Pi(r, q)$ is supermodular. Also, note that the constraint set $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ is ascending with q. Therefore, by Theorem 2.2.8 in Simchi-Levi et al. (2014), $r_c^*(q)$ is increasing in q. To show the monotonicity for $p_c^*(q)$, by variable transformation, problem (4.5) is equivalent to the following problem

$$f_c(q) = \max_p \{\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) + g_t(\frac{q-(1-\alpha)p}{\alpha}) : p \in [L_t, U_t]\},$$
$$p_c^*(q) = \arg\max_p \{\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) + g_t(\frac{q-(1-\alpha)p}{\alpha}) : p \in [L_t, U_t]\},$$

where $\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) = p(b_t - a_t p + \eta^+(\frac{q-p}{\alpha})^+ + \eta^-(\frac{q-p}{\alpha})^-)$ can also be shown to be supermodular by Lemma 2 in Chen et al. (2013). Meanwhile, by concavity of $g_t(\cdot)$, $g_t(\frac{q-(1-\alpha)p}{\alpha})$ is supermodular in (p, q). Thus, $p_c^*(q)$ is monotonically increasing in q.

C.8 Proof of Proposition 4.6

Define

$$\begin{aligned}\underline{\theta}(r, q) &= \partial_r^+[\Pi_t(r, q) + g_t(r)] \\ &= -\frac{2(\alpha^2 a_t + \alpha\eta)}{(1-\alpha)^2} r + \frac{2\alpha a_t + (1+\alpha)\eta}{(1-\alpha)^2} q - \frac{b_t\alpha}{1-\alpha} + \partial^+ g_t(r),\end{aligned}$$

where $\eta = \eta^+$ if $r \geq q$ and $\eta = \eta^-$ if $r < q$, and

$$\begin{aligned}\overline{\theta}(r, q) &= \partial_r^-[\Pi_t(r, q) + g_t(r)] \\ &= -\frac{2(\alpha^2 a_t + \alpha\eta)}{(1-\alpha)^2} r + \frac{2\alpha a_t + (1+\alpha)\eta}{(1-\alpha)^2} q - \frac{b_t\alpha}{1-\alpha} + \partial^- g(r),\end{aligned}$$

where $\eta = \eta^+$ if $r > q$ and $\eta = \eta^-$ if $r \leq q$. Note that by concavity of the objective function, $\underline{\theta}(r, q)$ and $\overline{\theta}(r, q)$ are both decreasing in r and $\underline{\theta}(r, q) \leq \overline{\theta}(r, q)$.

Following the notation in the proof of Proposition 4.5, let $\hat{g}_t(q) = g_t(q) + A_t q^2$. By Proposition 4.5, $\hat{g}_t(q)$ is concave as well. It follows that

$$\begin{aligned}\underline{\theta}(q, q) &= \frac{2\alpha a_t + \eta^+}{1-\alpha} q - \frac{b_t\alpha}{1-\alpha} + \partial^+ g_t(q), \\ &= \frac{2\alpha a_t + \eta^+}{1-\alpha} q - \frac{b_t\alpha}{1-\alpha} - 2A_t q + \partial^+ \hat{g}_t(q) \\ &= -\frac{\eta^- - \eta^+}{1-\alpha} q + \partial^+ \hat{g}_t(q) - \frac{b_t\alpha}{1-\alpha}\end{aligned}$$

and

$$\begin{aligned}\overline{\theta}(q,q) &= \frac{2\alpha a_t + \eta^-}{1-\alpha}q - \frac{b_t\alpha}{1-\alpha} + \partial^- g_t(q) \\ &= \frac{2\alpha a_t + \eta^-}{1-\alpha}q - \frac{b_t\alpha}{1-\alpha} - 2A_t q + \partial^- \hat{g}_t(q) \\ &= \partial^- \hat{g}_t(q) - \frac{b_t\alpha}{1-\alpha},\end{aligned}$$

where in the last equations we substituted the strong concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. As a result, both $\underline{\theta}(q,q)$ and $\overline{\theta}(q,q)$ are decreasing in q and $\eta^- > \eta^+$ implies $\underline{\theta}(q,q)$ is actually strictly decreasing and $\underline{\theta}(q,q) < \overline{\theta}(q,q)$.

Let $\underline{Q} = \sup\{q : \underline{\theta}(q,q) > 0\}$ and $\overline{Q} = \inf\{q : \overline{\theta}(q,q) < 0\}$. By $\underline{\theta}(\underline{Q},\underline{Q}) \geq 0 \geq \overline{\theta}(\overline{Q},\overline{Q}) > \underline{\theta}(\overline{Q},\overline{Q})$, we have $\underline{Q} < \overline{Q}$. We consider the following three cases.

Case 1: If $q < \underline{Q}$, then $\underline{\theta}(q,q) > 0$ and $\underline{\theta}(r,q) > 0$ for $r \leq q$. Thus, $r^*(q) > q$.

Case 2: If $q > \overline{Q}$, then $\overline{\theta}(q,q) < 0$ and $\overline{\theta}(r,q) < 0$ for $r \geq q$. Thus, $r^*(q) < q$.

Case 3: If $\underline{Q} \leq q \leq \overline{Q}$, then $\underline{\theta}(q,q) \leq 0$ and $\overline{\theta}(q,q) \geq 0$. That is $\underline{\theta}(r,q) \leq 0$ for $r \geq q$ and $\overline{\theta}(r,q) \geq 0$ for $r \leq q$. Thus, $r^*(q) = q$.

In summary, for $q' > q''$, if $q'' > r^*(q'')$, then $q' > q'' > \overline{Q}$ and $q' > r^*(q')$. That is, the single-crossing property holds for $q - r^*(q)$.

C.9 Proof of Proposition 4.7

By Proposition 4.4, $f(q)$ has at most $n_t + m_t + 2$ breakpoints. On top of this, q_L, q_U, $\underline{q}_r$ and $\overline{q}_r$ are the new breakpoints by computing $g_{t+1}(q)$ from $f(q)$. Thus, $n_{t+1} \leq n_t + m_t + 6$.

By Proposition 4.3, $f(q)$ has at most m_t kink points. On top this, q_L, q_U, $\underline{q}_r$ and $\overline{q}_r$ are the only new candidate kink points by computing $g_{t+1}(q)$ from $f(q)$. Thus, $m_{t+1} \leq m_t + 4$.

Appendix D

D.1 Proof of Lemma 5.1

For convenience, given matrices $\mathbf{A}, \mathbf{B}$, we denote $\mathbf{A} \succ 0$ if $\mathbf{A}$ is positive definite and $\mathbf{A} \succ \mathbf{B}$ if $\mathbf{A} - \mathbf{B}$ is positive definite. For the matrix of our interest, we have the following relationship

$$
\begin{aligned}
&\bar{\alpha}(\bar{\alpha} + \gamma)\mathbf{I} + \bar{\alpha}(\bar{\alpha} + \frac{\gamma}{2})\mathbf{R}^{-1}\mathbf{K} \\
\succ &\bar{\alpha}(\bar{\alpha} + \frac{\gamma}{2})(\mathbf{I} + \mathbf{R^{-1}K}) \\
= &\bar{\alpha}(\bar{\alpha} + \frac{\gamma}{2})\mathbf{K}^{-\frac{1}{2}}[\mathbf{K}^{\frac{1}{2}}(\mathbf{K^{-1}} + \mathbf{R^{-1}})\mathbf{K}^{\frac{1}{2}}]\mathbf{K}^{\frac{1}{2}}
\end{aligned}
$$

The last equality shows that $\mathbf{I} + \mathbf{R^{-1}K}$ is similar to the matrix $\mathbf{K}^{\frac{1}{2}}(\mathbf{K^{-1}} + \mathbf{R^{-1}})\mathbf{K}^{\frac{1}{2}}$. Since $-\mathbf{R} - \mathbf{K}$ is a symmetric strictly diagonally dominant matrix with nonnegative diagonal entries and is consequently positive definite, i.e. $-\mathbf{R} - \mathbf{K} \succ 0$. Therefore, from $-\mathbf{R} \succ \mathbf{K}$, we have $(-\mathbf{R})^{\mathbf{-1}} \prec \mathbf{K^{-1}}$ which shows $\mathbf{K}^{\frac{1}{2}}(\mathbf{K^{-1}} + \mathbf{R^{-1}})\mathbf{K}^{\frac{1}{2}}$ is positive definite. By similarity of matrices, $\bar{\alpha}(\bar{\alpha} + \frac{\gamma}{2})(\mathbf{I} + \mathbf{R^{-1}K})$ is also positive definite and our claim holds.

D.2 Proof of Proposition 5.1

We start by considering the finite horizon problem, i.e.,

$$
\max_{\mathbf{p}} \int_0^T e^{-\gamma t}\Pi(\mathbf{r}, \mathbf{p})dt.
$$

By (5.1), $\mathbf{p} = \frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r}$, and we can write the integrand as

$$\Phi(\dot{\mathbf{r}}, \mathbf{r}, t) := e^{-\gamma t}[(\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r})^\top \mathbf{R}(\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r}) + \mathbf{r}^\top \mathbf{K}(\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r}) + \mathbf{b}^\top(\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r})],$$

The Euler-Lagrange equation $\frac{d}{dt}(\frac{\partial \Phi}{\partial \dot{\mathbf{r}}}) = \frac{\partial \Phi}{\partial \mathbf{r}}$ yields

$$\ddot{\mathbf{r}} - \gamma \dot{\mathbf{r}} - [\bar{\alpha}(\bar{\alpha} + \gamma)\mathbf{I} + \bar{\alpha}(\bar{\alpha} + \frac{\gamma}{2})\mathbf{R}^{-1}\mathbf{K}]\mathbf{r} - \frac{1}{2}\bar{\alpha}(\bar{\alpha} + \gamma)\mathbf{R}^{-1}\mathbf{b} = 0, \tag{D.1}$$

where $\ddot{\mathbf{r}}$ denotes $\frac{d^2\mathbf{r}}{dt^2}$. In addition, the natural boundary condition $\frac{\partial}{\partial \dot{\mathbf{r}}}\Phi(\dot{\mathbf{r}}(T), \mathbf{r}(T), T) = 0$ yields

$$[\frac{2}{\bar{\alpha}^2}\mathbf{R}\dot{\mathbf{r}} + \frac{1}{\bar{\alpha}}(\mathbf{2R} + \mathbf{K})\mathbf{r} + \frac{1}{\bar{\alpha}}\mathbf{b}]|_{t=T} = 0. \tag{D.2}$$

Introducing new variables $\mathbf{s} = \dot{\mathbf{r}}$ and recall the definition of the matrix $\mathbf{M}$, (D.1) can be rewritten as first order ODEs

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{M} & \gamma \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2}\bar{\alpha}(\bar{\alpha} + \gamma)\mathbf{R}^{-1}\mathbf{b} \end{bmatrix}, \tag{D.3}$$

and the natural boundary condition (D.2) can be simplified as

$$[\dot{\mathbf{r}} + \frac{\bar{\alpha}}{2}(\mathbf{2I} + \mathbf{R}^{-1}\mathbf{K})\mathbf{r} + \frac{\bar{\alpha}}{2}\mathbf{R}^{-1}\mathbf{b}]|_{t=T} = 0. \tag{D.4}$$

Note that if λ satisfies

$$\begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{M} & \gamma \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix},$$

then

$$\mathbf{M}\mathbf{r} = (\lambda^2 - \gamma\lambda)\mathbf{r}.$$

That is, $\lambda^2 - \gamma\lambda$ is an eigenvalue of $\mathbf{M}$, say ξ_i. Since $\xi_i > 0$, we can solve

$$\lambda_i' = \frac{\gamma + \sqrt{\gamma^2 + 4\xi_i}}{2}, \lambda_i = \frac{\gamma - \sqrt{\gamma^2 + 4\xi_i}}{2},$$

and the corresponding two linearly independent eigenvectors are

$$\begin{bmatrix} \mathbf{v} \\ \lambda_i' \mathbf{v_i} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \lambda_i \mathbf{v_i} \end{bmatrix},$$

where $\mathbf{v_i}$ is the eigenvector of $\mathbf{M}$ corresponding to ξ_i.

The ODEs (D.3) can then be solved explicitly as

$$\begin{aligned}\begin{bmatrix}\mathbf{r}(t)\\ \mathbf{s}(t)\end{bmatrix} =& c_1'\begin{bmatrix}\mathbf{v_1}\\ \lambda_1'\mathbf{v_1}\end{bmatrix}e^{\lambda_1' t} + c_1\begin{bmatrix}\mathbf{v_1}\\ \lambda_1\mathbf{v_1}\end{bmatrix}e^{\lambda_1 t} + \ldots + \\ & c_N'\begin{bmatrix}\mathbf{v_N}\\ \lambda_N'\mathbf{v_N}\end{bmatrix}e^{\lambda_N' t} + c_N\begin{bmatrix}\mathbf{v_N}\\ \lambda_N\mathbf{v_N}\end{bmatrix}e^{\lambda_N t} + \begin{bmatrix}\mathbf{r_s}\\ \mathbf{0}\end{bmatrix}\end{aligned} \tag{D.5}$$

where $(\mathbf{r_s}, \mathbf{0})$ is a particular solution to the non-homogeneous system and $c_1', c_1, \ldots, c_N', c_N$ are constants to be determined from boundary conditions.

We next show that when $T \to \infty$ the natural boundary condition (D.4) implies $\lim_{t\to\infty} \mathbf{r}(t) < \infty$, which in turn implies $c_1' = c_2' = \ldots = c_N' = 0$. First, we note that when t is large enough, the value of $r_i(t)$ will be mainly dominated by $c_k' v_{ki} e^{\lambda_k' t}$ for some k, where λ_k' is the largest eigenvalue and v_{ki} is the i-th component of $\mathbf{v_k}$. Similarly, $\dot{r}_i(t)$ will also be dominated by $c_k' v_{ki} \lambda_k' e^{\lambda_k' t}$ and consequently has the same sign as $r_i(t)$, i.e. $r_i(t)s_i(t) \geq 0$. If we left multiply $\mathbf{r}(t)$ on both side of (D.4), then

$$[\mathbf{r}^\top \dot{\mathbf{r}} + \frac{\bar{\alpha}}{2}\mathbf{r}^\top(\mathbf{2I} + \mathbf{R^{-1}K})\mathbf{r} + \frac{\bar{\alpha}}{2}\mathbf{R^{-1}br}]|_{t=T} = 0.$$

However, if $\lim_{t\to\infty} r_i(t) = \infty$ for some i, then $\mathbf{r}^\top(\mathbf{2I} + \mathbf{R^{-1}K})\mathbf{r} > \mathbf{r}^\top\mathbf{r} \to +\infty$ as $t \to +\infty$ and will dominate the linear term $\mathbf{R^{-1}br}$. In the mean time $\mathbf{r}^\top\dot{\mathbf{r}} = \sum_{i=1}^{N} r_i(t)s_i(t) \geq 0$ for any t. Thus,

$$\lim_{T\to\infty}[\mathbf{r}^\top \dot{\mathbf{r}} + \frac{\beta}{2}\mathbf{r}^\top(\mathbf{2I} + \mathbf{R^{-1}K})\mathbf{r} + \frac{\beta}{2}\mathbf{R^{-1}br}]|_{t=T} = +\infty,$$

a contradiction.

Hence, when $T \to \infty$, we have derived the solution to (D.1) as

$$\mathbf{r}^*(t) = c_1\mathbf{v_1}e^{\lambda_1 t} + \ldots + c_N\mathbf{v_N}e^{\lambda_N t} + \mathbf{r_s}, \tag{D.6}$$

where $\mathbf{c} = (c_1, \ldots, c_N)^\top$ can be solved by

$$\mathbf{c} = \mathbf{V^{-1}}(\mathbf{r_0} - \mathbf{r_s}) \tag{D.7}$$

with the particular solution or the steady state $\mathbf{r_s}$ being the solution to the equation (5.5). Substitute (D.7) into (D.6), we then arrive at (5.6). The expressions for the price path and the pricing strategy (5.7) are immediate consequence of (5.6) and (5.2).

In the remaining, we verify the sufficient conditions for the solution of Euler-Lagrange equation (D.1) to be the maximizer of (5.3) (see Liberzon, 2011, for a reference on the topic of calculus of variation).

First of all, note that $\frac{\partial^2 \Phi}{\partial \dot{\mathbf{r}}^2} = e^{-\gamma t}\frac{2}{\bar{\alpha}^2}\mathbf{R}$ is negative definite for any $t \geq 0$. Thus, the strengthened Legendre condition is satisfied.

Second, the Jacobi equation of the variational problem is

$$\ddot{\mathbf{x}} - \gamma\dot{\mathbf{x}} - \mathbf{M}\mathbf{x} = 0, \tag{D.8}$$

which is exactly the homogeneous version of equation (D.1). From our previous analysis, (D.8) admits a set of N linearly independent solutions

$$\mathbf{x_1} = (e^{\lambda_1' t} - e^{\lambda_1 t})\mathbf{v_1}, ..., \mathbf{x_N} = (e^{\lambda_N' t} - e^{\lambda_N t})\mathbf{v_N}.$$

Equivalently, we have

$$\det(\begin{bmatrix} \mathbf{x_1} & ... & \mathbf{x_N} \end{bmatrix}) \neq 0$$

for any $t > 0$. Therefore, there is no conjugate point to $t = 0$ in $(0, \infty)$.

Finally, we check the Weierstrass excess function.

$$\begin{aligned}
E(\mathbf{w}, \dot{\mathbf{r}}, \mathbf{r}, t) =& \Phi(\mathbf{w}, \mathbf{r}, t) - \Phi(\dot{\mathbf{r}}, \mathbf{r}, t) - (\mathbf{w} - \dot{\mathbf{r}})^\top \frac{\partial}{\partial \dot{\mathbf{r}}}\Phi(\dot{\mathbf{r}}, \mathbf{r}, t) \\
=& e^{-\gamma t}[(\frac{1}{\bar{\alpha}}\mathbf{w} + \mathbf{r})^\top \mathbf{R}(\frac{1}{\bar{\alpha}}\mathbf{w} + \mathbf{r}) - (\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r})^\top \mathbf{R}(\frac{1}{\bar{\alpha}}\dot{\mathbf{r}} + \mathbf{r}) \\
& + \frac{1}{\bar{\alpha}}\mathbf{r}^\top \mathbf{K}(\mathbf{w} - \dot{\mathbf{r}}) + \frac{1}{\bar{\alpha}}\mathbf{b}^\top(\mathbf{w} - \dot{\mathbf{r}}) \\
& - \frac{2}{\bar{\alpha}^2}(\mathbf{w} - \dot{\mathbf{r}})^\top \mathbf{R}\dot{\mathbf{r}} - \frac{1}{\bar{\alpha}}(\mathbf{w} - \dot{\mathbf{r}})^\top(\mathbf{2R} + \mathbf{K})\mathbf{r} - \frac{1}{\bar{\alpha}}(\mathbf{w} - \dot{\mathbf{r}})^\top \mathbf{b}] \\
=& e^{-\gamma t}\frac{1}{\bar{\alpha}^2}(\mathbf{w} - \dot{\mathbf{r}})^\top \mathbf{R}(\mathbf{w} - \dot{\mathbf{r}}) < 0
\end{aligned}$$

The last inequality is due to the negative definite property of matrix $\mathbf{R}$.

D.3 Proof of Proposition 5.2

Since we can translate any time point $s \geq 0$ back to $s = 0$, it is sufficient to show that for any initial reference price $\mathbf{r}_0$, there exists T such that the price path $\mathbf{p}^{SM}(t), 0 \leq t \leq T$ and reference price path $\mathbf{r}^{SM}(t), 0 \leq t \leq T$ generated by $\mathbf{p}^{SM}(\mathbf{r})$ maximizes

$$\begin{aligned} \max_{\mathbf{p}} & \int_0^T e^{-\gamma t}\Pi(\mathbf{r},\mathbf{p})dt \\ \text{s.t.} \quad & \dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}-\mathbf{r}), \\ & \mathbf{r}(T) = \mathbf{r}^{SM}(T). \end{aligned} \tag{D.9}$$

Without loss of generality, we assume $\mathbf{r}_0 \in I^{++}$. The other cases can be proved similarly. We let T be the first time $\mathbf{r}^{SM}(t)$ exits I^{++}, i.e., the first time switching occurs. In this case, the objective value of the above problem under $\mathbf{p}^{SM}(t), 0 \leq t \leq T$ is

$$\int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}^{SM}(t),\mathbf{p}^{SM}(t))dt,$$

since $\mathbf{r}^{SM}(t) \in I^{++}$ for all $0 \leq t \leq T$ and by definition $p_i^{SM}(t) - r_i^{SM}(t) = p_i^{++}(\mathbf{r}^{SM}(t)) - r_i^{SM}(t) < 0$ for $i = 1, 2$. Note that on $0 \leq t \leq T$, $\mathbf{p}^{SM}(t) = \mathbf{p}^{++}(t)$, where $\mathbf{p}^{++}(t), t \geq 0$, denotes the price path generated by $\mathbf{p}^{++}(\mathbf{r})$ and is optimal to the infinite horizon problem

$$\begin{aligned} \max_{\mathbf{p}} & \int_0^\infty e^{-\gamma t}\Pi^{++}(\mathbf{r},\mathbf{p})dt \\ \text{s.t.} \quad & \dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}-\mathbf{r}). \end{aligned} \tag{D.10}$$

We first show that $\mathbf{p}^{SM}(t), 0 \leq t \leq T$ is optimal to the problem

$$\begin{aligned} \max_{\mathbf{p}} & \int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r},\mathbf{p})dt \\ \text{s.t.} \quad & \dot{\mathbf{r}} = \bar{\alpha}(\mathbf{p}-\mathbf{r}), \\ & \mathbf{r}(T) = \mathbf{r}^{SM}(T). \end{aligned} \tag{D.11}$$

Suppose $\mathbf{p}'(t), 0 \leq t \leq T$ and the corresponding reference price path $\mathbf{r}'(t), 0 \leq t \leq T$ satisfying $\mathbf{r}'(T) = \mathbf{r}^{SM}(T)$ generate a higher value than $\int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}^{SM}(t),\mathbf{p}^{SM}(t))dt$, then we can construct a solution to the infinite horizon problem (D.10) as follows:

$$\tilde{\mathbf{p}}(t) = \begin{cases} \mathbf{p}'(t), & 0 \le t \le T, \\ \mathbf{p}^{++}(t), & t > T. \end{cases}$$

It follows that

$$\begin{aligned} \int_0^\infty e^{-\gamma t}\Pi^{++}(\mathbf{r}, \tilde{\mathbf{p}})dt &= \int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}, \mathbf{p}')dt + \int_T^\infty e^{-\gamma t}\Pi^{++}(\mathbf{r}, \mathbf{p}^{++})dt \\ &> \int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}, \mathbf{p}^{++})dt + \int_T^\infty e^{-\gamma t}\Pi^{++}(\mathbf{r}, \mathbf{p}^{++})dt, \end{aligned}$$

where we have suppressed the dependency of reference price path $\mathbf{r}$ on the corresponding price path for brevity. This leads to a contradiction to $\mathbf{p}^{++}(t)$ being optimal to problem (D.10).

Next, we show that $\mathbf{p}^{SM}(t), 0 \le t \le T$ is optimal to the problem (D.9). Indeed, for any pricing path $\mathbf{p}(t)$, it holds

$$\int_0^T e^{-\gamma t}\Pi(\mathbf{r}, \mathbf{p})dt \le \int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}, \mathbf{p})dt \le \int_0^T e^{-\gamma t}\Pi^{++}(\mathbf{r}^{SM}(t), \mathbf{p}^{SM}(t))dt,$$

where the first inequality is due to the fact that $\eta_i^+ < \eta_i^-, i = 1, 2$ and the second inequality is due to the optimality of $\mathbf{p}^{SM}(t)$ to (D.11).

If no switch occurs, i.e., $\mathbf{r}^{SM}(t) \in I^{++}$ for all $t \ge 0$, then letting $T = \infty$, the second claim follows from above analysis.

D.4 Proof of Lemma 5.2

Since the matrix is two dimensional, we can write the eigenvalues and eigenvectors explicitly. Namely, let λ_1, λ_2 be the two eigenvalues, then

$$\begin{aligned} \lambda_1 &= \frac{a+d}{2} + [\frac{(a+d)^2}{4} - (ad - bc)]^{\frac{1}{2}}, \\ \lambda_2 &= \frac{a+d}{2} - [\frac{(a+d)^2}{4} - (ad - bc)]^{\frac{1}{2}}, \end{aligned}$$

and

$$\mathbf{v_1} = \begin{bmatrix} \lambda_1 - d \\ c \end{bmatrix},$$
$$\mathbf{v_2} = \begin{bmatrix} \lambda_2 - d \\ c \end{bmatrix}.$$

It follows that

$$\begin{aligned} &(\lambda_1 - d)(\lambda_2 - d) \\ =&\frac{(a-d)^2}{4} - [\frac{(a+d)^2}{4} - (ad - bc)] \\ =&-bc < 0. \end{aligned}$$

Thus, if $\lambda_1 - d < 0$, then $v_{11}v_{12} > 0, v_{21}v_{22} < 0$, otherwise swapping the notations of $\mathbf{v_1}, \mathbf{v_2}$, the result still holds.

D.5 Proof of Lemma 5.3

Following the convention of our notations, we let $\mathbf{\Lambda}^{++}$ to be the matrix $\mathbf{\Lambda}$ defined in Proposition 5.1 when $N = 2, \eta_1 = \eta_1^+, \eta_2 = \eta_2^+$ and $\mathbf{V}^{++}$ to be the matrix of eigenvectors of $\mathbf{M}^{++}$. By Proposition 5.1, the dynamics (5.10) restricted to the region I^{++} can be explicitly written as

$$\dot{\mathbf{r}} = \mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}(\mathbf{r} - \mathbf{r_s^{++}}). \tag{D.12}$$

Clearly, M^{++} has the same eigenvectors as $\mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}$ and we claim that one of their eigenvectors, say $\mathbf{v} = (v_1, v_2)$ satisfies $\mathbf{v} + \mathbf{r_s^{++}} \in I^{++}$. Note that we can always let $\mathbf{x} = \mathbf{r} - \mathbf{r_s^{++}}$ to shift the whole system such that the steady state is the origin. Therefore, to simplify the presentation, we assume without loss of generality that $\mathbf{r_s^{++}} = \mathbf{0}$. In this case, it is equivalent to show $\mathbf{v} \in I^{++}$, where

$$\begin{aligned} I^{++} &= \{\mathbf{r} | p_1^{++}(\mathbf{r}) - r_1 < 0, p_2^{++}(\mathbf{r}) - r_2 < 0\} \\ &= \{\mathbf{r} | \frac{1}{\bar{\alpha}}\mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{r} < \mathbf{0}\}. \end{aligned}$$

Since $\mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{v} = \lambda\mathbf{v}$, where $\lambda < 0$ is a diagonal element of $\mathbf{\Lambda}^{++}$, $\mathbf{v} \in I^{++}$ if and only if $v_1 > 0, v_2 > 0$. By Lemma 5.2, $\mathbf{M}^{++}$ has exactly one such eigenvector.

As I^{++} is described by two linear inequalities, we can use two extreme rays l_1 and l_2 to represent its boundary ∂I^{++} (see Figure D.1). If $\mathbf{r}_0 \in l_1$,

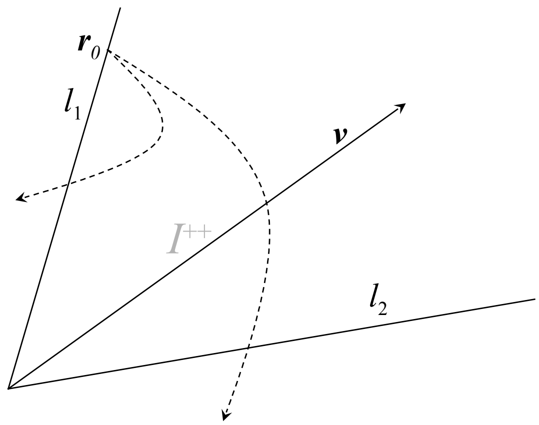


Figure D.1: Illustration of the proof

there are two possible ways that $\mathbf{r}(t)$ can leave the region I^{++} as illustrated by two dashed lines in Figure D.1. The first way is to cross back at l_1. That is, there exists $t' > 0$, such that $\mathbf{r}(t') \in l_1$ and $\dot{\mathbf{r}}(t') \notin I^{++}$. As $\mathbf{r}_0, \mathbf{r}(t') \in l_1$, there exists a positive constant $k > 0$ such that $\mathbf{r}(t') = k\mathbf{r}_0$. Consequently,

$$\dot{\mathbf{r}}(t') = \mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{r}(t') = k\mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{r}_0 = k\dot{\mathbf{r}}(0) \in I^{++},$$

a contradiction.

The second way is to cross l_2. However, before that, it must cross the eigenvector $\mathbf{v} \in I^{++}$. In other words, there exists $t' > 0$ such that $\mathbf{r}(t') = k\mathbf{v}$ for some $k > 0$. As a result,

$$\dot{\mathbf{r}}(t') = \mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{r}(t') = k\mathbf{V}^{++}\mathbf{\Lambda}^{++}(\mathbf{V}^{++})^{-\mathbf{1}}\mathbf{v} = k\lambda\mathbf{v},$$

and

$$\mathbf{r}(t) = e^{k\lambda(t-t')}\mathbf{v},$$

for any $t \geq t'$. Thus, $\mathbf{r}(t)$ will stay in I^{++} for all $t \geq t'$ which contradicts with the fact that it will cross l_2.

In summary, as $\mathbf{r}(t)$ can neither cross l_1 nor l_2, it has to stay in I^{++} for all $t \geq 0$ and follows the dynamics (D.12), which results in $\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r_s}^{++}$.

D.6 Proof of Proposition 5.3

There are three cases. First, if $\mathbf{r_0} \in S$, then $\mathbf{r}(t) = \mathbf{r_0}$ for any t and our claim holds.

Second, if $\mathbf{r_0} \in I^{+r_2}, I^{-r_2}, I^{r_1+}$ or I^{r_1-} (here we suppose $\mathbf{r_0} \in I^{+r_2}$ without loss of generality), then either

$$\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r_s}^{+r_2},$$

where $\mathbf{r_s}^{+r_2} \in S$ is the steady state to the system $\dot{r}_1 = \bar{\alpha}(p_1^{+r_2}(r_1) - r_1), \dot{r}_2 = 0$, or there exists t' such that $\mathbf{r}(t') \in \partial I^{++}$ or $\mathbf{r}(t') \in \partial I^{+-}$. By Lemma 5.3, when $\mathbf{r}(t') \in \partial I^{++}$ or $\mathbf{r}(t') \in \partial I^{+-}$, it then follows

$$\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r}_\mathbf{s}^{++} \quad \text{or} \quad \lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r}_\mathbf{s}^{+-}.$$

Finally, if $\mathbf{r_0} \in I^{++}, I^{+-}, I^{-+}$ or I^{--} (again we assume $\mathbf{r_0} \in I^{++}$ without loss of generality), then either

$$\lim_{t\to\infty} \mathbf{r}(t) = \mathbf{r}_\mathbf{s}^{++}$$

or there exists t' such that $\mathbf{r}(t') \in I^{+r_2}$ or $\mathbf{r}(t') \in I^{r_1+}$ and we are back to the second case. Thus, with at most two switches, the reference price path converges to a steady state in S.

References

Aflaki, Sam, Ioana Popescu. 2013. Managing retention in service relationships. *Management Science* **60**(2) 415–433.

Ahn, H., M. Gumus, P. Kaminsky. 2007. Pricing and manufacturing decisions when demand is a function of prices in multiple periods. *Operations Research-Baltimore* **55**(6) 1039–1057.

Allenby, Greg M, Peter E Rossi. 1998. Marketing models of consumer heterogeneity. *Journal of Econometrics* **89**(1) 57–78.

Anderson, B. D. O., J. B. Moore. 2007. *Optimal control: linear quadratic methods*. Courier Corporation.

Asvanunt, A. 2007. Applications of dynamic optimization to strategic pricing and corporate finance. Ph.D. thesis, Columbia University.

Bardi, Martino, Italo Capuzzo-Dolcetta. 1997. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Springer Science & Business Media.

Bell, D. R., J. M. Lattin. 2000. Looking for loss aversion in scanner panel data: The confounding effect of price response heterogeneity. *Marketing Science* **19**(2) 185–200.

Besbes, Omar, Ilan Lobel. 2015. Intertemporal price discrimination: Structure and computation of optimal policies. *Management Science* **61**(1) 92–110.

Boatwright, P., R. McCulloch, P. Rossi. 1999. Account-level modeling for trade promotion: An application of a constrained parameter hierarchical model. *Journal of the American Statistical Association* **94**(448) 1063–1073.

Briesch, R.A., L. Krishnamurthi, T. Mazumdar, S.P Raj. 1997. A comparative analysis of reference price models. *Journal of Consumer Research* **24**(2) 202–214.

Chen, X., P. Hu, S. Shum, Y. Zhang. 2013. Stochastic inventory model with reference price effects. *Working Paper* .

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research* **52**(6) 887–896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research* **29**(3) 698–723.

Chen, X., D. Simchi-Levi. 2006. Coordinating inventory control and pricing strategies: The continuous review model. *Operations Research Letters* **34**(3) 323–332.

Chen, X., D. Simchi-Levi. 2012. Pricing and inventory management. *The Oxford handbook of pricing management. Oxford University Press, Oxford* .

Chevalier, J. A., A. K. Kashyap, P. E. Rossi. 2003. Why don't prices rise during periods of peak demand? evidence from scanner data. *American Economic Review* **93**(1) 15–37.

Clausen, A., C. Strub. 2012. Envelope theorems for non-smooth and non-concave optimization. *Working Paper* .

Conlisk, J., E. Gerstner, J. Sobel. 1984. Cyclic pricing by a durable goods monopolist. *The Quarterly Journal of Economics* **99** 489–505.

Cox, John C., Jonathan E. Ingersoll Jr., Stephen A. Ross. 1985. A theory of the term structure of interest rates. *Econometrica* **53** 385–407.

DeCarlo, Raymond A, Michael S Branicky, Stefan Pettersson, Bengt Lennartson. 2000. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE* **88**(7) 1069–1082.

Edlin, A. S., C. Shannon. 1998. Strict monotonicity in comparative statics. *Journal of Economic Theory* **81**(1) 201–219.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science* **49** 1287–1309.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Operations Research* **47**(3) 454–475.

Fibich, G., A. Gavious, O. Lowengart. 2003. Explicit solutions of optimization models and differential games with nonsmooth (asymmetric) reference-price effects. *Operations Research* **51**(5) 721–734.

Fleming, W. H., R. W. Rishel. 1982. *Deterministic and stochastic optimal control*. Springer.

Freedman, David. 1984. On bootstrapping two-stage least-squares estimates in stationary linear models. *The Annals of Statistics* **12**(3) 827–842.

Gallego, Guillermo, Ruxian Wang. 2014. Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research* **62**(2) 450–461.

Geng, Q., C. Wu, K. Li. 2010. Pricing and promotion frequency in the presence of reference price effects in supply chains. *California Journal of Operations Management* **8**(1) 74–82.

Geunes, J., Y. Merzifonluoğlu, H. E. Romeijn. 2009. Capacitated procurement planning with price-sensitive demand and general concave-revenue functions. *European Journal of Operational Research* **194**(2) 390–405.

Gimpl-Heersink, D. I. L. 2008. Joint pricing and inventory control under reference price effects. Ph.D. thesis, Vienna University of Economics and Business Administration.

Greenleaf, E.A. 1995. The impact of reference price effects on the profitability of price promotions. *Marketing Science* **14**(1) 82–82.

Hardie, B.G.S., E.J. Johnson, P.S. Fader. 1993. Modeling loss aversion and reference dependence effects on brand choice. *Marketing Science* **12**(4) 378–394.

Heston, Steven L. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies* **6**(2) 327–343.

Hirsch, M. W., S. Smale, R. L. Devaney. 2004. *Differential Equations, Dynamical Systems and An Introduction to Chaos*. Elsevier/Academic Press.

Hu, P. 2012. Coordinated pricing and inventory management. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Huh, W. T., G. Janakiraman. 2008. (s, S) optimality in joint inventory-pricing control: An alternate approach. *Oper. Res.* **56** 783–790.

Jain, P., S. Banerjee. 2003. Border-collision bifurcations in one-dimensional discontinuous maps. *Int. J. Bifur. Chaos* **13** 3341–3351.

James, G., D. Witten, T. Hastie, R. Tibshirani. 2013. *An Introduction to Statistical Learning with Applications in R.* Springer, New York.

Kallio, Markku, Merja Halme. 2009. Redefining loss averse and gain seeking consumer price behavior based on demand response. *Working Paper* .

Kalyanaram, G., J.D.C. Little. 1994. An empirical analysis of latitude of price acceptance in consumer package goods. *Journal of Consumer Research* **21**(3) 408–418.

Kopalle, P.K., R.S. Winer. 1996. A dynamic model of reference price and expected quality. *Marketing Letters* **7**(1) 41–52.

Kopalle, Praveen K., P.K. Kannan, Lin Bao Boldt, Neeraj Arora. 2012. The impact of household level heterogeneity in reference price effects on optimal retailer pricing policies. *Journal of Retailing* **88**(1) 102–114.

Kopalle, Praveen K., Ambar G. Rao, Jo L. Assuno. 1996. Asymmetric reference price effects and dynamic pricing policies. *Marketing Science* **15**(1) 60–85.

Krishnamurthi, L., T. Mazumdar, S. P. Raj. 1992. Asymmetric response to price in consumer brand choice and purchase quantity decisions. *The Journal of Consumer Research* **19**(3) 387–400.

Lattin, James M, Randolph E Bucklin. 1989. Reference effects of price and promotion on brand choice behavior. *Journal of Marketing research* 299–310.

Liberzon, Daniel. 2003. *Switching in systems and control*. Springer Science & Business Media.

Liberzon, Daniel. 2011. *Calculus of variations and optimal control theory: a concise introduction*. Princeton University Press.

Liu, Y., W. L. Cooper. 2014. Optimal dynamic pricing with patient customers. *Working Paper* .

Mayhew, G. E., R. S. Winer. 1992. An empirical analysis of internal and external reference prices using scanner data. *The Journal of Consumer Research* **19**(1) 62–70.

Mazumdar, T., S. P. Raj, I. Sinha. 2005. Reference price research: Review and propositions. *Journal of Marketing* **69**(4) 84–102.

Mazumdar, Tridib, Purushottam Papatla. 1995. Loyalty differences in the use of internal and external reference prices. *Marketing Letters* **6**(2) 111–122.

Milgrom, P., I. Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica* **70**(2) 583–601.

Miranda, Mario J, Paul L Fackler. 2004. *Applied computational economics and finance*. MIT press.

Nasiry, J., I. Popescu. 2011. Dynamic pricing with loss-averse consumers and peak-end anchoring. *Operations Research* **59**(6) 1361–1368.

Natter, Martin, Thomas Reutterer, Andreas Milde, Alfred Taudes. 2007. An assortmentwide decision-support system for dynamic pricing and promotion planning in diy retailing. *Marketing Science* **26**(4) 576–583.

Ok, E. A. 2007. *Real analysis with economic applications*, vol. 10. Princeton University Press.

Øksendal, Bernt. 2002. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag New York.

Petruzzi, N.C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Oper. Res.* **47** 183–194.

Popescu, I., Y. Wu. 2007. Dynamic pricing strategies with reference effects. *Operations Research* **55**(3) 413–429.

Popescu, Ioana, Yaozhong Wu. 2005. Dynamic pricing strategies under repeated interactions. *Working paper, INSEAD, Fontainebleau, France* .

Rajendran, Kharan N, Gerard J Tellis. 1994. Contextual and temporal components of reference price. *Journal of Marketing* **58** 22–34.

Rajpathak, B., H. K. Pillai, S. Bandyopadhyay. 2012. Analysis of stable periodic orbits in the one dimensional linear piecewise-smooth discontinuous map. *Chaos: An Interdisciplinary Journal of Nonlinear Science* **22**(3) 033126.

Raman, K., F.M. Bass. 2002. A general test of reference price theory in the presence of threshold effects. *Tijdschrift Voor Economie En Management* **47**(2) 205–226.

Sharkovsky, A.N., L.O. Chua. 1993. Chaos in some 1-d discontinuous maps that appear in the analysis of electrical circuits. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on* **40**(10) 722–731.

Simchi-Levi, D., X. Chen, J. Bramel. 2014. *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management, Third Edition*. Springer.

Stokey, N. L., R. E. Lucas, E. C. Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.

Talluri, K. T., G. J. Van Ryzin. 2005. *The Theory and Practice of Revenue Management*. Springer Science & Business Media.

Tversky, A., D. Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics* **106**(4) 1039.

Weiss, R.M., A.K. Mehrotra. 2001. Online dynamic pricing: Efficiency, equity and the future of e-commerce. *Virginia Journal of Law and Technology Association* **6** 11–17.

Zhang, Yuhan. 2011. Essays on robust optimization, integrated inventory and pricing, and reference price effect. Ph.D. thesis, University of Illinois at Urbana-Champaign.