RAMANUJAN'S IDENTITIES, VORONOI SUMMATION FORMULA, AND ZEROS OF PARTIAL SUMS OF ZETA AND L-FUNCTIONS

BY

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DISSERTATION

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Abstract

The focus of the first part of the thesis commences with an examination of two pages in Ramanujan's lost notebook, pages 336 and 335. A casual, or even more prolonged, examination of the strange formulas on these pages does not lead one to conclude that they are related to one another. Moreover, it does not appear that they have any relationships with other parts of mathematics. On page 336 in his lost notebook, Ramanujan proposes two identities. Here, it does not take a reader long to make a deduction – the formulas are obviously wrong – each is vitiated by divergent series. Most readers encountering such obviously false claims would dismiss them and deposit the paper on which they were written in the nearest receptacle for recycling (if they were environmentally conscientious). However, these formulas were recorded by Ramanujan. Ramanujan made mistakes, but generally his mistakes were interesting! Frequently, there were hidden truths behind his not so precise or accurate claims – truths that were deep and influential for decades. Thus, it was difficult for us to dismiss them.

We initially concentrate on only one of the two incorrect "identities." This "identity" may have been devised to attack the extended divisor problem. We prove here a corrected version of Ramanujan's claim, which contains the convergent series appearing in it. Our identity is admittedly quite complicated, and we do not claim that what we have found is what Ramanujan originally had in mind. But there are simple and interesting special cases as well as analogues of this identity, one of which very nearly resembles Ramanujan's version. The aforementioned convergent series in Ramanujan's faulty claim is similar to one used by Voronoï, Hardy, and others in their study of the classical Dirichlet divisor problem, and so we are motivated to study further series of this sort. This now brings us to page 335, which comprises two formulas featuring doubly infinite series of Bessel functions. Although again not obvious at a first inspection, one is conjoined with the classical circle problem initiated by Gauss, while the other is associated with the Dirichlet divisor problem. Berndt, Kim, and Zaharescu have written several papers providing proofs of these two difficult formulas in different interpretations. In this thesis, We return to these two formulas and examine them in more general settings.

The Voronoï summation formula appears prominently in our study. In particular, we generalize work of

Wilton and derive an analogue involving the sum of divisors function $\sigma_s(n)$.

Another part of the thesis is focused on the partial sums of Dedekind zeta functions and L-functions attached to cusp forms. The motivation of the study of the partial sums of Dedekind zeta functions and L-functions attached to cusp forms arise from their approximate functional equations. The partial sums of the Dedekind zeta function of a cyclotomic field K is defined by the truncated Dirichlet series

$$\zeta_{K,X}(s) = \sum_{\|\mathfrak{a}\| \leq X} \frac{1}{\|\mathfrak{a}\|^s},$$

where the sum is to be taken over nonzero integral ideals $\mathfrak{a}$ of K and $\|\mathfrak{a}\|$ denotes the absolute norm of $\mathfrak{a}$. We establish the zero-free regions for $\zeta_{K,X}(s)$ and estimate the number of zeros of $\zeta_{K,X}(s)$ up to height T.

We consider a family of approximations of a Hecke L-function $L_f(s)$ attached to a holomorphic cusp form f of positive integral weight with respect to the full modular group. These families are of the form

$$L_f(X;s) := \sum_{n \leq X} \frac{a(n)}{n^s} + \chi_f(s) \sum_{n \leq X} \frac{a(n)}{n^{1-s}},$$

where $s = \sigma + it$ is a complex variable. From the approximate functional equation one sees that $L_f(X;s)$ is a good approximation to $L_f(s)$ when $X = t/2\pi$. To investigate such approximation in more general sense, we compute the L^2-norms of the difference of two such approximations of $L_f(s)$. We work with a weight which is a compactly supported smooth function. Mean square estimates for the difference of approximations of $L_f(s)$ can be obtained from such weighted L^2-norms. We also obtain a vertical strips where most of the zeros of $L_f(X;s)$ lie. We study the distribution of zeros of $L_f(X;s)$ when X is independent of t. For $X = 1, 2$ we prove that all the complex zeros of $L_f(X;s)$ lie on the critical line $\sigma = 1/2$. We also show that as $T \to \infty$ and $X = T^{o(1)}$, 100% of the complex zeros of $L_f(X;s)$ up to height T lie on the critical line and simple. Here by 100% we mean that the ratio between the number of simple zeros on the critical line and the total number of zeros up to height T approaches 1 as $T \to \infty$.

To my niece, Adrita Roy.

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Table of Contents

List of Symbols

σ	Real part of the complex number s.
t	Imaginary part of the complex number s.
$\lfloor X \rfloor$	The greatest integer less than or equal to X.
$[X]$	The greatest integer less than or equal to X.
$\mathbb{N}$	Set of all positive integers.
$\mathbb{Z}$	Set of all integers.
$\mathbb{Q}$	Set of all rational numbers.
$\mathbb{R}^+$	Set of all positive real numbers.
$\mathbb{R}$	Set of all real numbers.
$d(n)$	The number of divisor of n.
$\sigma_s(n)$	$\sum_{d\mid n} d^s$.
$B_n(x)$	n-th Bernoulli polynomial.
$\Gamma(s)$	The gamma function.
$J_\nu(x)$	The Bessel function of order ν of the first kind.
$Y_\nu(x)$	The Bessel function of order ν of the second kind.
$I_\nu(x)$	The modified Bessel function of order ν of the first kind.
$K_\nu(x)$	The modified Bessel function of order ν of the second kind.
$V_\alpha^\beta f(t)$	The total variation of $f(t)$ over (α, β).

Chapter 1

Introduction

1.1 Ramanujan's claim

The Dirichlet divisor problem is one of the most notoriously difficult unsolved problems in analytic number theory. Let $d(n)$ denote the number of divisors of n. Define the error term $\Delta(x)$, for $x > 0$, by

$$\sum_{n\leq x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x), \tag{1.1}$$

where γ denotes Euler's constant. Here, and in the sequel, a prime $'$ on the summation sign in $\sum_{n\leq x}{}' a(n)$ indicates that only $\frac{1}{2}a(x)$ is counted when x is an integer. The Dirichlet divisor problem asks for the correct order of magnitude of $\Delta(x)$ as $x \to \infty$. At this writing, the best estimate $\Delta(x) = O(x^{131/416+\epsilon})$, for each $\epsilon > 0$, as $x \to \infty$, is due to Huxley [51] ($\frac{131}{416} = 0.3149\ldots$). On the other hand, Hardy [46] proved that $\Delta(x) \neq O(x^{1/4})$, as $x \to \infty$, with the best result in this direction currently due to Soundararajan [83]. It is conjectured that $\Delta(x) = O\left(x^{1/4+\epsilon}\right)$, for each $\epsilon > 0$, as $x \to \infty$.

Let $\sigma_s(n) = \sum_{d|n} d^s$, and let $\zeta(s)$ denote the Riemann zeta function. For $0 < s < 1$, define $\Delta_{-s}(x)$ (We use $\Delta_{-s}(x)$ instead of $\Delta_s(x)$, as is customarily used, so as to be consistent with the results in this dissertation, most of which require Re $s > 0$) by

$$\sum_{n\leq x}{}' \sigma_{-s}(n) = \zeta(1+s)x + \frac{\zeta(1-s)}{1-s}x^{1-s} - \frac{1}{2}\zeta(s) + \Delta_{-s}(x). \tag{1.2}$$

The problem of determining the correct order of magnitude of the error term $\Delta_{-s}(x)$, as $x \to \infty$, is known as the extended divisor problem (see Lau [61]). As $x \to \infty$, it is conjectured that for each $\epsilon > 0$, $\Delta_{-s}(x) = O(x^{1/4-s/2+\epsilon})$ for $0 < s \leq \frac{1}{2}$ and $\Delta_{-s}(x) = O(x^{\epsilon})$ for $\frac{1}{2} \leq s < 1$.

The importance of the conditionally convergent series

$$\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{3}{4}}}\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) \tag{1.3}$$

in the study of the Dirichlet divisor problem was emphasized by Hardy [46, equation (6.32)]. Hardy's discernment came to fruition in the work of Hafner [45] and Soundararajan [83, equation (1.8)] in their improvements of Hardy's Ω-theorem on the Dirichlet divisor problem. However, we emphasize that Voronoï [92, p. 218] first made use of (1.3) in the Dirichlet divisor problem.

As another example, we note that the series

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^{\frac{5}{4}+\frac{k}{2}}} \sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right), \tag{1.4}$$

for $|k| < \frac{3}{2}$, arises in the work of Legal [82, p. 282] and Kanemitsu and Rao [54] related to a conjecture of Chowla and Walum [26], [25, pp. 1058–1063], which is an extension of the Dirichlet divisor problem. It is conjectured that if $a, r \in \mathbb{Z}, a \geq 0, r \geq 1$, and if $B_r(x)$ denotes the r-th Bernoulli polynomial, then for every $\epsilon > 0$, as $x \to \infty$,

$$\sum_{n \leq \sqrt{x}} n^a B_r\left(\left\{\frac{x}{n}\right\}\right) = O\left(x^{a/2+1/4+\epsilon}\right), \tag{1.5}$$

where $\{x\}$ denotes the fractional part of x. The conjectured correct order of magnitude in the Dirichlet divisor problem is equivalent to (1.5) with $a = 0, r = 1$.

Our last example is as famous as the Dirichlet divisor problem. Let $r_2(n)$ denote the number of representations of n as a sum of two squares. The equally celebrated circle problem asks for the precise order of magnitude of the error term $P(x)$, as $x \to \infty$, where

$$\sum_{n \leq x}{}' r_2(n) = \pi x + P(x).$$

During the five years that Ramanujan visited Hardy at Cambridge, there is considerable evidence, from Hardy in his papers and from Ramanujan in his lost notebook [80], that the two frequently discussed both the circle and divisor problems. For details of Ramanujan's contributions to these problems, see either the book by Andrews and Berndt [2, Chapter 2] or the survey paper by Berndt, Kim, and Zaharescu [16].

It is possible that Ramanujan also thought of the extended divisor problem, for on page 336 in his lost notebook [80], we find the following claim.

Let $\sigma_s(n) = \sum_{d|n} d^s$, and let $\zeta(s)$ denote the Riemann zeta function. Then

$$\begin{aligned}\Gamma\left(s+\frac{1}{2}\right)&\left\{\frac{\zeta(1-s)}{(s-\frac{1}{2})x^{s-\frac{1}{2}}} + \frac{\zeta(-s)\tan\frac{1}{2}\pi s}{2x^{s+\frac{1}{2}}} + \sum_{n=1}^{\infty}\frac{\sigma_s(n)}{2i}\left((x-in)^{-s-\frac{1}{2}} - (x+in)^{-s-\frac{1}{2}}\right)\right\} \\ &= (2\pi)^s\left\{\frac{\zeta(1-s)}{2\sqrt{\pi x}} - 2\pi\sqrt{\pi x}\zeta(-s)\tan\tfrac{1}{2}\pi s + \sqrt{\pi}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\right\}.\end{aligned} \tag{1.6}$$

In view of the identities for (1.3) and (1.4), it is possible that Ramanujan developed the series on the right-hand side of (1.6) to study the extended divisor problem. Unfortunately, (1.6) is incorrect, since the series on the left-hand side, which can be written as

$$\sum_{n=1}^{\infty} \frac{\sigma_s(n) \sin\left(\left(s+\frac{1}{2}\right)\tan^{-1}\left(\frac{n}{x}\right)\right)}{(x^2+n^2)^{\frac{s}{2}+\frac{1}{4}}},$$

diverges for all real values of s since $\sigma_s(n) \geq n^s$. For further discussion one can follow the paper by Berndt, Chan, Lim, and Zaharescu [12]. However, as we shall see in Chapter 2, there is a valid interpretation of this series using the theory of analytic continuation. Also in Chapter 2, we obtain a corrected version of Ramanujan's claim, where we start with the series on the right-hand side, since we know that it converges.

1.2 Extended divisor problem and Voronoï summation formula

A celebrated formula of Voronoï [92] for $\sum_{n\leq x} d(n)$ is given by

$$\sum_{n\leq x}{}' d(n) = x(\log x + (2\gamma - 1)) + \frac{1}{4} + \sqrt{x}\sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}}\left(-Y_1(4\pi\sqrt{nx}) - \frac{2}{\pi}K_1(4\pi\sqrt{nx})\right), \tag{1.7}$$

where $Y_\nu(x)$ denotes the Bessel function of order ν of the second kind, and $K_\nu(x)$ denotes the modified Bessel function of order ν. Thus, the error term $\Delta(x)$ in the Dirichlet divisor problem (1.1) admits the infinite series representation

$$\Delta(x) = \sqrt{x}\sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}}\left(-Y_1(4\pi\sqrt{nx}) - \frac{2}{\pi}K_1(4\pi\sqrt{nx})\right).$$

In [92], Voronoï also gave a more general form of (1.7), namely,

$$\sum_{\alpha<n<\beta} d(n)f(n) = \int_\alpha^\beta (2\gamma + \log t)f(t)\,dt + 2\pi\sum_{n=1}^{\infty} d(n)\int_\alpha^\beta f(t)\left(\frac{2}{\pi}K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt})\right)dt, \tag{1.8}$$

where $f(t)$ is a function of bounded variation in (α, β) and $0 < \alpha < \beta$. Dixon and Ferrar [35] gave a proof of (1.8) under the more restrictive condition that f has a bounded second differential coefficient in (α, β). Wilton [97] proved (1.8) under less restrictive conditions. In his proof, he assumed $f(t)$ has compact support on $[\alpha, \beta]$ and $V_\alpha^{\beta-\epsilon} f(t) \to V_\alpha^{\beta-0} f(t)$ as ϵ tends to 0. Here $V_\alpha^\beta f(t)$ denotes the total variation of $f(t)$ over (α, β). In 1929, Koshliakov [59] gave a very short proof of (1.8) for $0 < \alpha < \beta$, $\alpha, \beta \notin \mathbb{Z}$, for f analytic inside a closed contour strictly containing the interval $[\alpha, \beta]$. Koshliakov's proof in [59] is based on the series $\varphi(x)$, defined in (2.8), and its representation

$$\varphi(x) = -\gamma - \frac{1}{2}\log x - \frac{1}{4\pi x} + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{x^2+n^2}.$$

The reader is referred to papers by Berndt [8, 10] for Voronoï-type summation formulas for a large class of arithmetical functions generated by Dirichlet series satisfying a functional equation involving the Gamma function. For Voronoï-type summation formulas involving an exponential factor, see the paper by Jutila [53]. The Voronoï summation formula has been found to be useful in physics too; for example, Egger and Steiner [38, 40] showed that it plays the role of an exact trace formula for a Schrödinger operator on a certain non-compact quantum graph. They also gave a short proof of the Voronoï summation formula in [39].

The extension of (1.8) for $\alpha = 0$ is somewhat more difficult, since one needs to impose a further condition on $f(t)$. When $f''(t)$ is bounded in (δ, α) and $t^{3/4}f''(t)$ is integrable over $(0, \delta)$ for $0 < \delta < \alpha$, Dixon and Ferrar [35] proved that

$$\sum_{0<n<\beta} d(n)f(n) = \frac{f(0+)}{4} + \int_0^{\beta}(2\gamma + \log t)f(t)\,dt \tag{1.9}$$
$$+ 2\pi\sum_{n=1}^{\infty} d(n)\int_0^{\beta} f(t)\left(\frac{2}{\pi}K_0(4\pi\sqrt{nt}) - Y_0(4\pi\sqrt{nt})\right)dt.$$

Wilton [97] obtained (1.9) under the assumption that $\log x V_{0+}^{x} f(t)$ tends to 0 as $x \to 0+$. Hejhal [50] gave a proof of (1.9) for $\beta \to \infty$ under the assumption that f is twice continuously differentiable and possesses compact support. For other proofs of the Voronoï summation formula, the reader is referred to Meurman [68] and Ivić [52].

Consider the following Voronoï summation formula in an extended form due to Oppenheim [76], and in the version given by Laurinčikas [62]. For $x > 0, x \notin \mathbb{Z}$, and $-\frac{1}{2} < \sigma < \frac{1}{2}$,

$$\sum_{n<x}\sigma_{-s}(n) = \zeta(1+s)x + \frac{\zeta(1-s)}{1-s}x^{1-s} - \frac{1}{2}\zeta(s) + \frac{x}{2\sin\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\sigma_s(n) \tag{1.10}$$
$$\times\left(\sqrt{nx}\right)^{-1-s}\left(J_{s-1}(4\pi\sqrt{nx}) + J_{1-s}(4\pi\sqrt{nx}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nx})\right),$$

so that, by (1.2), $\Delta_{-s}(x)$ is represented by the expression involving the series on the right-hand side of (1.10). (Note that Laurinčikas proved (1.10) for $0 < s < \frac{1}{2}$. However, one can extend it to $-\frac{1}{2} < \sigma < \frac{1}{2}$.) Wilton [98] proved the same result in a more general setting by considering the 'integrated function', that is, the Riesz sum

$$\frac{1}{\Gamma(\lambda+1)}\sideset{}{'}\sum_{n\le x}\sigma_{-s}(n)(x-n)^{\lambda}.$$

Laurinčikas [62] gave a different proof of (1.10) many years later.

We will now explain the connection of Ramanujan's series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)$$

and its companion with the extended form of the Voronoï summation formula.

As mentioned by Hardy [46], [47, pp. 268–292], if we use the asymptotic formulas (2.18) and (2.19) for $Y_1(4\pi\sqrt{nx})$ and $K_1(4\pi\sqrt{nx})$, respectively, in (1.7), we find that

$$\Delta(x)=\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right)+R(x), \tag{1.11}$$

where $R(x)$ is a series absolutely and uniformly convergent for all positive values of x. The first series on the left side of (1.11) is convergent for all real values of x, and uniformly convergent throughout any compact interval not containing an integer. At each integer x, it has a finite discontinuity.

If we replace the Bessel functions in (1.10) by their asymptotic expansions, namely (2.17) and (2.19), similar to what Hardy did, then the most important part of the error term $\Delta_{-s}(x)$ is given by

$$\frac{x^{\frac{1}{4}-\frac{1}{2}s}\cot\left(\frac{1}{2}\pi s\right)}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{\frac{s}{2}+\frac{3}{4}}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right).$$

This series, though similar to the one in (1.11) or in (1.3), is different from Ramanujan's series (2.9) in that the exponential factor, namely $e^{-2\pi\sqrt{2nx}}$, is not present.

A generalization of (2.8), namely,

$$\varphi(x,s):=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right)+e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right), \tag{1.12}$$

was studied by Dixit and Moll [34]. Note that $\varphi(x,0)=\varphi(x)$, and that $\varphi(x)$ was used by Koshliakov [59] in his short proof of (1.8).

Replacing the Bessel functions in (1.12) by their asymptotic expansions from (2.19), we find that the main terms are given by

$$\frac{\sqrt{2}}{x^{1/4}}\cos\left(\frac{\pi}{4}\left(s+\frac{1}{2}\right)\right)\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2+1/4}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right) \tag{1.13}$$
$$+\frac{\sqrt{2}}{x^{1/4}}\sin\left(\frac{\pi}{4}\left(s+\frac{1}{2}\right)\right)\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2+1/4}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right).$$

In our extensive study, the forms of the series in (1.13) are the closest that we could find that resemble the series in Ramanujan's original claim (1.6), or in our Theorem 2.1.1, or the companion series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right).$$

Note that the only place where they differ is in the power of n. Similar remarks can be made about (2.7) and (1.13).

Series similar to these arise in the mean square estimates of $\int_1^x \Delta_{-s}(t)^2\,dt$ by Meurman [69, equations (3.7), (3.8)]. (An excellent survey on recent progress on divisor problems and mean square theorems has been written by Tsang [91].) Similar series have also arisen in the work of Cramér [30], and in the recent work of Bettin and Conrey [19, p. 220–223]. Thus it seems that the two series in (1.13) are more closely connected to the extended Dirichlet divisor problem than are Ramanujan's series and its companion. We have found identities, similar to those in Theorems 2.1.1 and 2.1.2, for each of the series in (1.13). However, we refrain ourselves from stating them as they are similar to the ones already proved.

Remark. It is interesting to note here that at the bottom of page 368 in [80], one finds the following note in Hardy's handwriting: "Idea. You can replace the Bessel functions of the Voronoï identity by circular functions, at the price of complicating the 'sum'. Interesting idea, but probably of no value for the study of the divisor problem." In view of the applications of such series mentioned in the above paragraph, we can say that Hardy's judgement was incorrect.

The series in (1.12) can be used to derive an extended form of the Voronoï summation formula (1.8) in the form contained in the theorem in Chapter 3. This proof generalizes the technique enunciated by Koshliakov in [59].

1.3 Generalization of two entries on page 335 of Ramanujan's lost notebook

We begin this section by stating the two entries on page 335 in Ramanujan's lost notebook [80]. Define

$$F(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer.} \end{cases} \tag{1.14}$$

Entry 1.3.1. *If* $0 < \theta < 1$ *and* $F(x)$ *is defined by* (1.14), *then*

$$\begin{aligned} \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin(2\pi n\theta) = \pi x \left(\frac{1}{2} - \theta\right) - \frac{1}{4}\cot(\pi\theta) \\ + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} - \frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}} \right\}, \end{aligned} \tag{1.15}$$

where $J_\nu(x)$ *denotes the ordinary Bessel function of order* ν.

Entry 1.3.2. *If* $0 < \theta < 1$ *and* $F(x)$ *is defined by* (1.14), *then*

$$\begin{aligned} \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta)) \\ + \frac{1}{2}\sqrt{x} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{\tilde{I}_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{\tilde{I}_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}} \right\}, \end{aligned} \tag{1.16}$$

where

$$\tilde{I}_\nu(z) := -Y_\nu(z) + \frac{2}{\pi}\cos(\pi\nu)K_\nu(z), \tag{1.17}$$

where $Y_\nu(x)$ *denotes the Bessel function of the second kind of order* ν, *and* $K_\nu(x)$ *denotes the modified Bessel function of order* ν.

Entries 1.3.1 and 1.3.2 were established by Berndt, Kim, and Zaharescu under different conditions on the summation variables m, n in [14, 15, 18]. An expository account of their work along with a survey of the circle and divisor problems can be found in an another paper of Berndt, Kim and Zaharescu [16]. See also the book [2, Chapter 2] by Andrews and Berndt.

It is easy to see from (1.14) that the left-hand sides of (1.15) and (1.16) are finite. When $x \to 0+$, Entries (1.15) and (1.16) give the following interesting limit evaluations:

$$\lim_{x\to 0+} \sqrt{x} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} - \frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}} \right\} = \frac{1}{2}\cot(\pi\theta),$$

and

$$\lim_{x\to 0+} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{I_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} + \frac{I_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}} \right\} = -\frac{1}{2}.$$

Direct proofs of these limit evaluations appear to be difficult.

As shown in [16, equation (2.8)], when $\theta = \frac{1}{4}$, Entry 1.3.1 is equivalent to the following famous identity due to Ramanujan and Hardy [46], provided that the double sum in (1.15) is interpreted as $\lim_{N\to\infty} \sum_{m,n\le N}$, rather than as an iterated double sum (see [15, p. 26]):

$$\sum_{0<n\le x}{}' r_2(n) = \pi x - 1 + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}).$$

Note that the Bessel functions appearing in (1.16) are the same as those appearing in (1.7). Indeed when $\theta = \frac{1}{2}$, Entry 1.3.2 is connected with Voronoï's identity for $\sum_{n\le x} d(n)$ as will be shown below. First, following the elementary formula

$$\sum_{n\le x} d(n) = \sum_{n\le x} \sum_{d|n} 1 = \sum_{dj\le x} 1 = \sum_{d\le x} \left[\frac{x}{d}\right],$$

we see that the left-hand side of (1.16), for $\theta = \frac{1}{2}$, can be simplified as

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos(\pi n) = \sum_{n\le x}{}' \sum_{d|n} \cos(\pi d).$$

Second, let

$$\ell = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

Note that

$$\begin{aligned} \sum_{d|n} \cos(\pi d) &= \#\text{ even divisors of } n - \#\text{ odd divisors of } n \\ &= d\left(\frac{n}{2}\right) - \left\{d(n) - a\, d\left(\frac{n}{2}\right)\right\} \\ &= (1+\ell) d\left(\frac{n}{2}\right) - d(n). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) = -\sideset{}{'}\sum_{\substack{n\le x\\ n \text{ odd}}} d(n) + \sideset{}{'}\sum_{\substack{n\le x\\ n \text{ even}}} \left\{2d\left(\frac{n}{2}\right) - d(n)\right\} = 2\sideset{}{'}\sum_{n\le \frac{x}{2}} d(n) - \sideset{}{'}\sum_{n\le x} d(n).$$

Now apply the Voronoï summation formula (1.7) to each of the sums above, and simplify to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) &= -x\log 2 + \frac{1}{4} - \sqrt{2x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}\left(Y_1(2\pi\sqrt{2nx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2nx})\right)\\ &\quad + \sqrt{x}\sum_{n=1}^{\infty}\frac{d(n/2)}{\sqrt{n/2}}\left(Y_1(2\pi\sqrt{2nx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2nx})\right)\\ &= -x\log 2 + \frac{1}{4} - \sqrt{2x}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\left(\sum_{\substack{d\mid k\\ d \text{ odd}}} 1\right)\left(Y_1(2\pi\sqrt{2kx}) + \frac{2}{\pi}K_1(2\pi\sqrt{2kx})\right).\end{aligned}$$

Now let $k = m(2n+1)$ in the last sum, so that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(\pi n) = -x\log 2 + \frac{1}{4} + \sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{I_1(4\pi\sqrt{m(n+\frac{1}{2})x})}{\sqrt{m(n+\frac{1}{2})}}, \tag{1.18}$$

where $I_1(z)$ is defined by (1.17), provided that this double sum is interpreted as $\lim_{N\to\infty}\sum_{m,n\le N}$, instead of as an iterated double sum. Then (1.18) is exactly Entry 1.3.2 with $\theta = \frac{1}{2}$.

It should be mentioned here that Dixon and Ferrar [36] established, for $a, b > 0$, the identity

$$a^{\mu/2}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+b)^{\mu/2}}K_\mu(2\pi\sqrt{a(n+b)}) = b^{(1-\mu)/2}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+a)^{(1-\mu)/2}}K_{1-\mu}(2\pi\sqrt{b(n+a)}). \tag{1.19}$$

Generalizations have been given by Berndt [6, p. 343, Theorem 9.1] and Oberhettinger and Soni [74, p. 24]. Using Jacobi's identity

$$r_2(n) = 4\sum_{\substack{d\mid n\\ d\,\text{odd}}}(-1)^{(d-1)/2},$$

we can recast (1.19) as an identity between double series

$$\begin{aligned}&a^{\mu/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{K_\mu\left(4\pi\sqrt{a\left(\left(n+\frac{1}{4}\right)m+\frac{b}{4}\right)}\right)}{((4n+1)m+b)^{\mu/2}} - \frac{K_\mu\left(4\pi\sqrt{a\left(\left(n+\frac{3}{4}\right)m+\frac{b}{4}\right)}\right)}{((4n+3)m+b)^{\mu/2}}\right\}\\ &= b^{(1-\mu)/2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\frac{K_{1-\mu}\left(4\pi\sqrt{b\left(\left(n+\frac{1}{4}\right)m+\frac{a}{4}\right)}\right)}{((4n+1)m+a)^{(1-\mu)/2}} - \frac{K_{1-\mu}\left(4\pi\sqrt{b\left(\left(n+\frac{3}{4}\right)m+\frac{a}{4}\right)}\right)}{((4n+3)m+a)^{(1-\mu)/2}}\right\}.\end{aligned}$$

In Chapter 4 , we establish one-variable generalizations of Entries 1.3.1 and 1.3.2, where the double sums here are also interpreted as $\lim_{N\to\infty}\sum_{m,n\le N}$, instead of as iterated double sums. It is an open problem to determine if the series can be replaced by iterated double series.

1.4 Partial Sums of Dedekind Zeta functions

A first generalization of the Riemann zeta-function $\zeta(s)$ is provided by Dirichlet L-functions. Subsequently, Dedekind studied the zeta function $\zeta_K(s)$ of a number field $K/\mathbb{Q}$, defined for $\sigma > 1$ by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{\|\mathfrak{a}\|^s} = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s},$$

where the first sum is to be taken over all nonzero integral ideals $\mathfrak{a}$ of K and where $\|\mathfrak{a}\|$ denotes the absolute norm of $\mathfrak{a}$. In the second sum, $a_K(n)$ is used to denote the number of integral ideals $\mathfrak{a}$ with norm $\|\mathfrak{a}\| = n$.

As in the particular case $K = \mathbb{Q}$, where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$, the function $\zeta_K(s)$ is analytic everywhere except solely for a simple pole at $s = 1$. (See Davenport [31] and Neukrich [72].) The residue of this pole is given by the formula

$$\operatorname*{Res}_{s=1}(\zeta_K(s)) = \frac{2^r \pi^{n_0 - r} R_K h_K}{w_K \sqrt{|d_K|}},$$

where $r = r_1 + r_2$ (with r_1 is the number of real embeddings and r_2 is the number of pairs of complex embeddings of K), $n_0 = [K : \mathbb{Q}]$ denotes the degree of $K/\mathbb{Q}$, R_K denotes the regulator, h_K denotes the class number, w_K denotes the number of roots of unity in K, and d_K denotes the discriminant of K. (See [72, page 467].)

For $\zeta(s)$, Hardy and Littlewood [48] provided the approximate functional equation

$$\zeta(s) = \sum_{n\le X} \frac{1}{n^s} + \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \sum_{n\le Y} \frac{1}{n^{1-s}} + O(X^{-\sigma}) + O(Y^{\sigma-1}|t|^{-\sigma+1/2}),$$

where $0 \le \sigma \le 1$, $X > H > 0$, $Y > H > 0$, and $2\pi XY = |t|$, with the constant implied by the big-O term depending on H only. Such approximate functional equation motivate the study of properties of partial sums $F_X(s)$ of $\zeta(s)$ defined by

$$F_X(s) = \sum_{n\le X} \frac{1}{n^s}.$$

Gonek and Ledoan [41] studied the distribution of zeros of $F_X(s)$. They denote the number of typical zeros $\rho_X = \beta_X + i\gamma_X$ of $F_X(s)$ with ordinates $0 \le \gamma_X \le T$ by $N_X(T)$. In the case that T is the ordinate of a zero, they define $N_X(T)$ as $\lim_{\epsilon\to 0^+} N_X(T+\epsilon)$. In [41], the authors are concerned with results on $N_X(T)$

as both X and T tends to infinity.

Theorem 1 in [41] collects together a number of known results on the zeros of $F_X(s)$ (see Borwein, Fee, Ferguson, and van der Waal [21], Montgomery [70], and Montgomery and Vaughan [71]), which can be summarized as follows:

The zeros of $F_X(s)$ lie in the strip $\alpha < \sigma < \beta$, where α and β are the unique solutions of the equations $1 + 2^{-\sigma} + \cdots + (X-1)^{-\sigma} = X^{-\sigma}$ and $2^{-\sigma} + 3^{-\sigma} + \cdots + X^{-\sigma} = 1$, respectively. In particular, $\alpha > -X$ and $\beta < 1.72865$. Furthermore, there exists a number X_0 such that if $X \geq X_0$, then $F_X(s)$ has no zeros in the half-plane

$$\sigma \geq 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log\log X}{\log X}.$$

On the other hand, for any constant C satisfying the inequalities $0 < C < 4/\pi - 1$ there exists a number X_0 depending on C only such that if $X \geq X_0$, then $F_X(s)$ has zeros in the half-plane

$$\sigma > 1 + \frac{C \log\log X}{\log X}.$$

Theorem 2 in [41] (see also Langer [60]) can be summarized as follows:

If X and T are both greater than or equal to 2, then one has

$$\left| N_X(T) - \frac{T}{2\pi} \log[X] \right| < \frac{X}{2}.$$

Here and henceforth, $[X]$ denotes the greatest integer less than or equal to X. The approximate functional equation for $\zeta_K(s)$ is (see Chandrasekharan and Narasimhan [24])

$$\zeta_K(s) = \sum_{n \leq X} \frac{a_K(n)}{n^s} + B^{2s-1} \frac{A(1-s)}{A(s)} \sum_{n \leq Y} \frac{a_K(n)}{n^{1-s}} + O(X^{1-\sigma-1/n_0} \log X), \tag{1.20}$$

where $A(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)$, $B = 2^{r_2}\pi^{n_0/2}/\sqrt{|d_K|}$, $X > H > 0$, $Y > H > 0$, $XY = |d_K|(|t|/2\pi)^{n_0}$, and $C_1 < X/Y < C_2$ for some constants C_1 and C_2. The partial sum of $\zeta_K(s)$ is defined by

$$\zeta_{K,X}(s) := \sum_{\|\mathfrak{a}\| \leq X} \frac{1}{\|\mathfrak{a}\|^s} = \sum_{n \leq X} \frac{a_K(n)}{n^s},$$

which appears in the approximate functional equation (1.20). Our purpose is to determine whether $\zeta_{K,X}(s)$

exhibit similar properties. To this end, we denote the number of non-real zeros $\rho_{K,X} = \beta_{K,X} + i\gamma_{K,X}$ of $\zeta_{K,X}(s)$ with ordinates $0 \le \gamma_{K,X} \le T$ by $N_{K,X}(T)$. If T is the ordinate of a zero, then $N_{K,X}(T)$ is to be defined by $\lim_{\epsilon\to 0^+} N_{K,X}(T+\epsilon)$. In Chapter 5, we give an asymptotic formula for $N_{K,X}(T)$.

1.5 Family of approximations of L-functions attached to cusp forms

Let $N \ge 1$ be an integer. Define

$$F_N(s) := \sum_{n\le N} n^{-s} \quad \text{and} \quad \zeta_N(s) := F_N(s) + \chi(s)F_N(1-s),$$

where $\chi(s) = \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$. Spira [85, 86] appears to be the first author who considered the functions $\zeta_N(s)$ and investigated the zeros of these functions. The behavior of the functions $\zeta_N(s)$ is not completely unknown. From the approximate functional equation we have

$$\zeta(s) = \zeta_N(s) + O(|t|^{-\sigma/2}),$$

where $|t| \ge 1$, $|\sigma - 1/2| \le 1/2$, and $N = \sqrt{|t|/2\pi}$ (see Titchmarsh[89]). In [85], Spira proved that all the complex zeros of $\zeta_1(s)$ and $\zeta_2(s)$ lie on the line $\sigma = 1/2$. In [86], he presented a numerical computation which suggests that infinitely many zeros are off the line $\sigma = 1/2$ for $N \ge 3$. In the same paper, based on numerical evidence, he suggested the following:

> The zeros within the critical strip appear to lie outside the t range $\sqrt{2\pi e N} \le t \le 2\pi e N$ for each N. There is also a second, less obvious, t range free of zeros, corresponding to where the Riemann-Siegel formula is used, $N \le (t/2\pi)^{1/2} < N+1$. In this second region, $g_N(s)$ approximates $\zeta(s)$, while in the first region, $g_N(s)$ is approximately $2\zeta(s)$...

Here $\zeta_N(s) = g_N(s)$. Since then very few related results have appeared in the literature. Very recently, Gonek and Montgomery [42] studied thoroughly the zero distribution of $\zeta_N(s)$. First they provided a proof of Spira's aforementioned claim. In the same paper, Gonek and Montgomery found a zero free region for $\zeta_N(s)$ and also obtained further results on the numbers of zeros of $\zeta_N(s)$. They proved the striking result that 100% of the complex zeros of $\zeta_N(s)$ lie on the critical line, provided N is not too large with respect to the height T. We will discuss this fact later.

Gonek and Ledoan [41], Langer [60], and Wilder [96] proved asymptotic results for the number of zeros of $F_N(s)$. If $N_F(T)$ is the number of zeros of $F_N(T)$ up to height T, then they found that

$$N_F(T) = \frac{T}{2\pi} \log X + O(X).$$

This result is an indispensable ingredient to obtain good lower bounds for the number of zeros of $\zeta_N(s)$ on the critical line. In fact the growth rate of the error term offers a comparison between the growth rate of the number of zeros on the critical line up to height T vs the total number of zeros of $\zeta_N(s)$ up to height T.

In Chapter 5 (see also the paper by Ledoan, author and Zaharescu [63]), some instances are presented where the error term can be improved. If we consider the partial sums of Dedekind zeta functions of a cyclotomic field $K/\mathbb{Q}$ of degree q, then the corresponding error term is shown to be

$$\ll_q x(\log\log x/\log x)^{1-1/\phi(q)}.$$

An important factor in attempting to improve on the error term is to obtain good upper bounds for the sign changes of $a_K(n)\sin(T\log n)$, where $a_K(n)$ are coefficients in the Dirichlet series representation of the Dedekind zeta function.

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ be the full modular group. Let $f \in S_k(\Gamma)$ be a holomorphic cusp form of even integral weight $k > 0$ for Γ, with Fourier series given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}.$$

We also assume that f is a normalized primitive Hecke form with $a_f(1) = 1$. Let $a(n) := a_f(n) n^{(1-k)/2}$ and let $L_f(s)$ be the L-function associated to f, defined by

$$L_f(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{1.21}$$

for $\sigma > 1$.

From Deligne's proof [32, 33] of the Ramanujan-Petersson conjecture, which is a consequence of the Riemann Hypothesis for varieties over finite fields, the coefficients $a(n)$ satisfy the bound

$$|a(n)| \le d(n). \tag{1.22}$$

In particular

$$|a(p)| \leq 2, \tag{1.23}$$

for all primes p. The divisor function satisfies (see Apostol [3, p. 296])

$$d(n) \leq c_\delta n^\delta, \tag{1.24}$$

for any $\delta > 0$, and moreover by a result of Wigert [94],

$$\log(d(n)) \leq \frac{\log 2 \log n}{\log\log n} + O\left(\frac{\log n}{(\log\log n)^2}\right).$$

Rankin [81] gave a mean square estimate of coefficients $a(n)$. He showed that

$$\sum_{n\leq x} |a(n)|^2 = \alpha x + O_f(x^{3/5}), \tag{1.25}$$

where

$$\alpha = \frac{(4\pi)^{k-1}}{\Gamma(k)} \int\int y^{k-2}|f(z)|^2 dxdy, \tag{1.26}$$

the double integral being taken over any fundamental region of Γ and $z = x + iy$.

Next, we consider the partial sums

$$\sum_{n\leq X} \frac{a(n)}{n^s}.$$

Let $N(X;T)$ denote the number of complex zeros of $\sum_{n\leq X} a(n)n^{-s}$ up to height T. Then as a special case of Theorem 3 in [60], one obtains the following result.

Proposition 1.5.1. *Let M be the largest integer less than or equal to X such that $a(M) \neq 0$. Then we have*

$$N(X;T) = \frac{T}{2\pi}\log M + O_f(X).$$

In order to improve the above error term, we will be interested to study non-trivial upper bounds for the number of sign changes of the Fourier coefficients $a(n)$. In [55], Knopp, Kohnen, and Pribitkin studied the sign changes of the Fourier coefficients $a(n)$ of a cusp form f for $\mathrm{SL}(2,\mathbb{R})$. They showed that these coefficients $a(n)$ change sign infinitely often. Motivated by the work of Knopp, Kohnen, and Pribitkin one may consider trying to improve on the error term in the above proposition. It is worthwhile to mention that, in [67], Meher and Murty gave a lower bound for the number of sign changes of the coefficients $a(n)$. The reader may also find work in this direction in the work of Bruinier and Kohnen [22] and several other works of Kohnen.

The L-function $L_f(s)$ has an analytic continuation throughout the complex plane as an entire function, by

$$(2\pi)^{-s-\frac{k-1}{2}}\Gamma\left(s+\tfrac{k-1}{2}\right)L_f(s)=\int_0^\infty f(iy)y^{s+\frac{k-1}{2}-1}\,dy,$$

and it satisfies the functional equation

$$L_f(s)=\chi_f(s)L_f(1-s), \tag{1.27}$$

where

$$\chi_f(s):=(-1)^{k/2}(2\pi)^{-(1-2s)}\frac{\Gamma\left(\frac{k+1}{2}-s\right)}{\Gamma\left(\frac{k-1}{2}+s\right)}. \tag{1.28}$$

Now we recall below the functional equation, the reflection formula (along with its variant), and Legendre's duplication formula for the Gamma function $\Gamma(s)$. To that end,

$$\Gamma(s+1)=s\Gamma(s), \tag{1.29}$$

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin(\pi s)}, \tag{1.30}$$

$$\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)=\frac{\pi}{\cos(\pi s)}, \tag{1.31}$$

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2s-1}}\Gamma(2s). \tag{1.32}$$

Using (1.29), (1.30), (1.31), and (1.32) one shows that

$$\chi_f(s)\chi_f(1-s)=1. \tag{1.33}$$

The Euler product representation of $L_f(s)$ is

$$L_f(s)=\prod_p\left(1-a(p)p^{-s}+p^{-2s}\right)^{-1}, \tag{1.34}$$

where $\sigma > 1$. The non-trivial zeros of $L_f(s)$ lie within the critical strip $0 < \sigma < 1$, symmetrically with respect to the real axis and the critical line $\sigma = 1/2$. The Riemann hypothesis for $L_f(s)$ states that, all the non-trivial zeros of $L_f(s)$ lie on the critical line $\sigma = 1/2$.

Let $N_f(T)$ denote the number of non-trivial zeros ρ of $L_f(s)$ for which $0 < \operatorname{Im}\rho < T$, for T not equal to any $\operatorname{Im}\rho$; otherwise we put

$$N_f(T) = \lim_{\epsilon\to 0} \frac{1}{2}\{N_f(T+\epsilon) + N_f(T-\epsilon)\}.$$

Then one can show that (see Lekkerkerker [64])

$$N_f(T) = \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{T}{\pi} + O(\log T).$$

An approximate functional equation of $L_f(s)$ (see Apostol and Sklar [4], Chandrasekharan and Narasimhan [23], and Good [43]) is given by

$$L_f(s) = \sum_{n\le X}\frac{a(n)}{n^s} + \chi_f(s)\sum_{n\le X}\frac{a(n)}{n^{1-s}} + O(|t|^{1/2-\sigma+\epsilon}), \tag{1.35}$$

for $\epsilon > 0, |t| \gg 1, |\sigma - 1/2| \le 1/2$ and $X = \frac{|t|}{2\pi}$. Let us define

$$L_f(N;s) := \sum_{n\le N}\frac{a(n)}{n^s} + \chi_f(s)\sum_{n\le N}\frac{a(n)}{n^{1-s}}. \tag{1.36}$$

From (1.28) and (1.36), we have the following functional equation,

$$L_f(N;s) = \chi_f(s)L_f(N;1-s). \tag{1.37}$$

Since $f \in S_k(\Gamma)$ is a primitive Hecke form, then all $a(n) \in \mathbb{R}$. Therefore $L_f(N;s)$ is real for all real values of s. So the zeros of $L_f(N;s)$ are symmetric with respect to the real axis. Also from the functional equation (1.37) we find that the zeros of $L_f(N;s)$ are symmetric with respect to the critical line $\sigma = 1/2$. By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [78], Part V, Chapter 1, No. 77), $\sum_{n\le N} a(n)n^{-s}$ has at most finitely many real roots for real values of s. Also from (1.28), $\chi_f(s)$ has simple poles at all half-integers greater than or equal to $(k+1)/2$. Therefore there exists a real number $\mathfrak{r}$, so that all half-integers greater than $\mathfrak{r}$ are simple poles of $L_f(N;s)$. Hence $L_f(N;s)$ is analytic everywhere except possibly for simple poles at half-integers.

From (1.35) and (1.37), we observe that $L_f(N;s)$ approximates $L_f(s)$ for $N < \frac{|t|}{2\pi} < N+1$, except possibly at the critical line. From [4, Theorem 2] we have

$$L_f(s) = \sum_{n \leq N} \frac{a(n)}{n^s} + O(N^{1/4-\sigma}), \tag{1.38}$$

uniformly for $\sigma \geq \sigma_1 > -1/4$, provided $N > B\left(\frac{t}{4\pi}\right)^2$ for some $B > 1$. Now we need Stirling's formula for the Gamma function in a vertical strip [29, p. 224]. For $\sigma_1 \leq \sigma \leq \sigma_2$, as $|t| \to \infty$,

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}\left(1 + O\left(\frac{1}{|t|}\right)\right). \tag{1.39}$$

From Stirling's formula (1.39) one has

$$\chi_f(t) = \left(\frac{|t|}{2\pi e}\right)^{1-2\sigma}\left(1 + O_f\left(\frac{1}{|t|}\right)\right), \tag{1.40}$$

as $|t| \to \infty$ (see (6.24) for a proof). From (1.36), (1.38), (1.33), and (1.40) we find that

$$L_f(N;s) = 2L_f(s) + O(N^{1/4-\sigma}) + O(|t|^{1-2\sigma}N^{\sigma-3/4}), \tag{1.41}$$

uniformly for $\min(\sigma, 1-\sigma) \geq \sigma_1 > -1/4$, provided $N > B\left(\frac{t}{4\pi}\right)^2$ for some $B > 1$. Since $|t| \ll \sqrt{N}$, the error terms in (1.41) are $\ll |t|^{-\min(1/2, 2\sigma-1/2)}$, uniformly for $1/4 < \sigma < 3/4$. Hence

$$L_f(N;s) = 2L_f(s) + O(|t|^{-\min(1/2,\sigma-1/4)}),$$

uniformly for $1/4 < \sigma < 3/4$ and $|t| \ll \sqrt{N}$. This shows that $L_f(N;s)$ approximates $2L_f(s)$ near the critical line for sufficiently large t in the range $|t| \ll \sqrt{N}$. Next we investigate such approximations in more generality. A natural question that arises is how the sequence $L_f(N;s)$ converges in the L^2-norm. In particular we are interested in studying the integral

$$\int_0^T \left| L_f\left(N; \frac{1}{2} + it\right) - L_f\left(M; \frac{1}{2} + it\right)\right|^2 dt. \tag{1.42}$$

We wish to obtain an asymptotic of the moment integral (1.42) for the family of approximations $L_f(N;s)$. We shall obtain this in a slightly different way. In Chapter 6, we estimate the L^2 distance between $L_f(M;s)$ and $L_f(N;s)$, weighted by a smooth function which satisfies certain conditions and study several results related to the zeros of $L_f(N;s)$.

Chapter 2

Ramanujan's identity on page 336 of his lost notebook

2.1 Main results

2.1.1 Corrected version of Ramanujan's claim

Before stating the corrected version of Ramanujan's claim, we need to define a general hypergeometric function. Define the rising or shifted factorial $(a)_n$ by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad n \geq 1, \qquad (a)_0 = 1. \tag{2.1}$$

Let p and q be nonnegative integers, with $q \leq p+1$. Then, the generalized hypergeometric function ${}_qF_p$ is defined by

$${}_qF_p(a_1, a_2, \ldots, a_q; b_1, b_2, \ldots, b_p; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_q)_n}{(b_1)_n (b_2)_n \cdots (b_p)_n} \frac{z^n}{n!}, \tag{2.2}$$

where $|z| < 1$, if $q = p+1$, and $|z| < \infty$, if $q < p+1$.

We set $R_a(f) = R_a$ to denote the residue of a meromorphic function $f(z)$ at a pole $z = a$.

Theorem 2.1.1. *Let ${}_3F_2$ be defined by (2.2). Fix s such that $\sigma > 0$. Let $x \in \mathbb{R}^+$. Let a be the number defined by*

$$a = \begin{cases} 0, & \text{if } s \text{ is an odd integer,} \\ 1, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then,

$$\sum_{n=1}^{\infty} \frac{\sigma_s(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right) \tag{2.4}$$
$$= 4\pi \left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}} + \frac{1}{4\sqrt{2}\pi} \zeta\left(\frac{1}{2}\right) \zeta\left(\frac{1}{2} - s\right) - \frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}} \frac{\Gamma(s+1/2)\cot\left(\frac{\pi s}{2}\right)\zeta(-s)}{x^{s+\frac{1}{2}}} \right)$$
$$+ \frac{\sqrt{x}}{\pi^s} \Bigg\{ \sum_{n<x} \frac{\sigma_s(n)}{n^{s+1}} \Bigg[-\frac{\sqrt{n}\,\Gamma\left(\frac{1}{4} + \frac{s}{2}\right)}{\sqrt{2x}\,\Gamma\left(\frac{1}{4} - \frac{s}{2}\right)}$$

$$-\frac{a\Gamma\left(s+\frac{1}{2}\right)\cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}}\left(\frac{n}{x}\right)^{s+1}\left\{\left(1+\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right\}$$
$$+\frac{n2^{-s}}{x\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1-s}{2},1-\frac{s}{2}\end{matrix};-\frac{n^2}{x^2}\right)\Bigg]$$
$$+\sum_{n\geq x}\frac{\sigma_s(n)}{n^{s+1}}\left[-\frac{n\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}\right.$$
$$-\frac{i\sqrt{n}\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\left\{\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-(s+\frac{1}{2})}-\left(1-\frac{ix}{n}\right)^{-(s+\frac{1}{2})}\right)\right.$$
$$\left.\left.+\,i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-(s+\frac{1}{2})}+\left(1-\frac{ix}{n}\right)^{-(s+\frac{1}{2})}-2\right)\right\}\right]\Bigg\},$$

where, if x is an integer, we additionally require that $\sigma < \frac{1}{2}$.

If we replace the '+' sign in the argument of the sine function in the series on the left-hand side of (2.4) by a '−' sign, then we obtain the following theorem.

Theorem 2.1.2. *Fix s such that $\sigma > 0$. Let $x \in \mathbb{R}^+$. Then,*

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right)$$
$$=4\pi\left(\frac{\sqrt{x}}{2}\zeta(-s)+\frac{\zeta\left(\frac{1}{2}\right)}{4\pi\sqrt{2}}\zeta\left(\frac{1}{2}-s\right)+\frac{\Gamma\left(s+\frac{1}{2}\right)\zeta(-s)}{2^{s+3}\pi^{s+\frac{3}{2}}x^{s+\frac{1}{2}}}\right.$$
$$+\frac{\Gamma\left(s+\frac{1}{2}\right)}{2^s\pi^{s+\frac{1}{2}}}\left\{\sum_{n<x}\frac{\sigma_s(n)}{n^{s+\frac{1}{2}}}\left[-\sin\left(\frac{\pi}{4}-\frac{\pi s}{2}\right)+\frac{n^{s+\frac{1}{2}}}{2x^{s+\frac{1}{2}}}\right.\right.$$
$$\left.\times\left(\left(1+\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right)\right]$$
$$+\sum_{n\geq x}\frac{\sigma_s(n)}{2n^{s+\frac{1}{2}}}\left[\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)}-2\right)\right.$$
$$\left.\left.\left.+\,i\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left(\left(1+\frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)}-\left(1-\frac{ix}{n}\right)^{-\left(s+\frac{1}{2}\right)}\right)\right]\right\}\right).$$

The special case $s = \frac{1}{2}$ of Theorem 2.1.1 (see (2.73)) is very interesting, since the two sums, one over $n < x$ and the other over $n \geq x$, coalesce into a single infinite sum. If $K_s(x)$ denotes the modified Bessel function or the Macdonald function [93, p. 78] of order s, and if we use the identities [93, p. 80, equation (13)]

$$K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z} \tag{2.5}$$

and [93, p. 79, equation (8)]

$$K_{-s}(z) = K_s(z), \tag{2.6}$$

we see that this special case of the series on the left-hand side of (2.4) can be realized as a special case of the series

$$2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right) - e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right) \tag{2.7}$$

when $s = -\frac{1}{2}$. If we replace the minus sign between the Bessel functions in the summands of (2.7) by a plus sign, then the resulting series is a generalization of the series

$$\varphi(x) := 2\sum_{n=1}^{\infty} d(n)\left(K_0\left(4\pi e^{i\pi/4}\sqrt{nx}\right) + K_0\left(4\pi e^{-i\pi/4}\sqrt{nx}\right)\right), \tag{2.8}$$

extensively studied by N. S. Koshliakov (also spelled N. S. Koshlyakov) [59, 57, 56, 58]. See also [34] for properties of this series and some integral transformations involving it. We feel that Koshliakov's work has not earned the respect that it deserves in the mathematical community. Some of his best work was achieved under extreme hardship, as these excerpts from a paper written for the centenary of his birth clearly demonstrate [20].

> The repressions of the thirties which affected scholars in Leningrad continued even after the outbreak of the Second World War. In the winter of 1942 at the height of the blockade of Leningrad, Koshlyakov along with a group ... was arrested on fabricated ... dossiers and condemned to 10 years correctional hard labour. After the verdict he was exiled to one of the camps in the Urals. ... On the grounds of complete exhaustion and complicated pellagra, Koshlyakov was classified in the camp as an invalid and was not sent to do any of the usual jobs. ... very serious shortage of paper. He was forced to carry out calculations on a piece of plywood, periodically scraping off what he had written with a piece of glass. Nevertheless, between 1943 and 1944 Koshlyakov wrote two long memoirs ...

A natural question arises – what may have motivated Ramanujan to consider the series

$$\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right)? \tag{2.9}$$

We provide a plausible answer to this question in Chapter 3, demonstrating that (2.9) is related to a generalization of the famous Voronoï summation formula and also to the generalization of Koshliakov's series (2.8) discussed above and its analogue. The contents of this chapter, Chapter 3, and Chapter 4 are taken from [13].

2.1.2 An important integral identity

The following lemma, which is interesting in its own right, is the main ingredient of our proof. We use the notation $\int_{(c)}$ to designate $\int_{c-i\infty}^{c+i\infty}$.

Lemma 2.1.3. *Fix s such that $\sigma > 0$. Fix $x \in \mathbb{R}^+$. Let $-1 < \lambda < 0$ and let a be defined in (2.3). Define $I(s,x)$ by*

$$I(s,x) := \frac{1}{2\pi i}\int_{(\lambda)} \Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}+s\right)\sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)(4x)^{-\frac{z}{2}}\,dz. \tag{2.10}$$

Then,

(i) *for $x > 1$,*

$$\begin{aligned} I(s,x) = -\frac{\pi}{2^{2-s}}\Bigg[&\frac{\Gamma\left(\frac14+\frac{s}{2}\right)}{\sqrt{2x}\,\Gamma\left(\frac14-\frac{s}{2}\right)} + \frac{ax^{-s-1}\cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}}\Gamma\left(s+\frac12\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac12\right)}\right. \\ &\left.+\left(1-\frac{i}{x}\right)^{-\left(s+\frac12\right)}\right\} - \frac{1}{x2^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac{1-s}{2};1-\frac{s}{2}\end{matrix};-\frac{1}{x^2}\right)\Bigg], \end{aligned} \tag{2.11}$$

(ii) *for $x \le 1$,*

$$\begin{aligned} I(s,x) = -\frac{\pi}{2^{2-s}}\Bigg[&\frac{\Gamma(s)\cos\left(\frac{\pi s}{2}\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac14,\frac34\end{matrix};-x^2\right)-1\right\} \\ &+\frac{i\Gamma\left(s+\frac12\right)}{2^{s+1}\sqrt{\pi x}}\left\{\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left((1+ix)^{-\left(s+\frac12\right)}-(1-ix)^{-\left(s+\frac12\right)}\right)\right. \\ &\left.+i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left((1+ix)^{-\left(s+\frac12\right)}+(1-ix)^{-\left(s+\frac12\right)}-2\right)\right\}\Bigg], \end{aligned} \tag{2.12}$$

where, if $x = 1$, we additionally require that $\sigma < \frac12$.

We note in passing that each ${}_3F_2$ in Theorem 2.1.1, as well as in Lemma 2.1.3, can be written, using the duplication formula for the Gamma function (1.32), as a sum of two ${}_2F_1$'s.

2.2 Preliminary Results

The functional equation of the Riemann zeta function $\zeta(s)$ in its asymmetric form is given by [88, p. 24]

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\tfrac{1}{2}\pi s\right)\Gamma(s)\zeta(s), \tag{2.13}$$

whereas its symmetric form yields

$$\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1}{2}(1-s)\right)\zeta(1-s).$$

Since $\zeta(s)$ has a simple pole at $s=1$ with residue 1, i.e.,

$$\lim_{s\to 1}(s-1)\zeta(s) = 1, \tag{2.14}$$

from (5.5) and (2.14), we find the value [88, p. 19]

$$\zeta(0) = -\tfrac{1}{2}.$$

The Riemann ξ-function $\xi(s)$ is defined by

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\tfrac{1}{2}s\right)\zeta(s),$$

where $\Gamma(s)$ and $\zeta(s)$ are the Gamma and the Riemann zeta functions respectively. The Riemann Ξ-function is defined by

$$\Xi(t) := \xi\left(\tfrac{1}{2}+it\right).$$

For $0 < c = \text{Re } w < \sigma$ [44, p. 908, formula **8.380.3**; p. 909, formula **8.384.1**],

$$\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(w)\Gamma(s-w)}{\Gamma(s)}x^{-w}\,dw = \frac{1}{(1+x)^s}.$$

We note Parseval's identity [77, pp. 82–83]

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathfrak{F}(1-w)\mathfrak{G}(w)\,dw,$$

where $\mathfrak{F}$ and $\mathfrak{G}$ are Mellin transforms of f and g, and which is valid for Re $w = c$ lying in the common strip of analyticity of $\mathfrak{F}(1-w)$ and $\mathfrak{G}(w)$. A variant of the above identity [77, p. 83, equation (3.1.13)] is

$$\frac{1}{2\pi i}\int_{(k)} \mathfrak{F}(w)\mathfrak{G}(w)t^{-w}\,dw = \int_0^\infty f(x)g\left(\frac{t}{x}\right)\frac{dx}{x}.$$

We close this section by recalling facts about Bessel functions. The ordinary Bessel function $J_\nu(z)$ of order ν is defined by [93, p. 40]

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}. \tag{2.15}$$

As customary, $Y_\nu(z)$ denotes the Bessel function of order ν of the second kind. Its relation to $J_\nu(z)$ is given in the identity [93, p. 64]

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin \pi\nu}. \tag{2.16}$$

If $K_\nu(z)$ denotes the modified Bessel function of order ν, then [93, p. 78]

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu},$$

where [93, p. 77]

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}.$$

The asymptotic formulas of the Bessel functions $J_\nu(z), Y_\nu(z)$, and $K_\nu(z)$, as $|z| \to \infty$, are given by [93, p. 199 and p. 202]

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos w \sum_{n=0}^{\infty} \frac{(-1)^n(\nu, 2n)}{(2z)^{2n}} - \sin w \sum_{n=0}^{\infty} \frac{(-1)^n(\nu, 2n+1)}{(2z)^{2n+1}}\right), \tag{2.17}$$

$$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\sin w \sum_{n=0}^{\infty} \frac{(-1)^n(\nu, 2n)}{(2z)^{2n}} + \cos w \sum_{n=0}^{\infty} \frac{(-1)^n(\nu, 2n+1)}{(2z)^{2n+1}}\right), \tag{2.18}$$

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n}, \tag{2.19}$$

for $|\arg z| < \pi$. Here $w = z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi$ and

$$(\nu, n) = \frac{\Gamma(\nu + n + 1/2)}{\Gamma(n+1)\Gamma(\nu - n + 1/2)}.$$

2.3 Proof of the corrected version of Ramanujan's claim

Let

$$S(s, x) := \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{2.20}$$

From [73, p. 45, equations (5.19), (5.20)], we have

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{(a^2+b^2)^{z/2}} \sin\left(z \tan^{-1}\left(\frac{a}{b}\right)\right) x^{-z}\, dz = e^{-bx} \sin(ax), \tag{2.21}$$

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{(a^2+b^2)^{z/2}} \cos\left(z \tan^{-1}\left(\frac{a}{b}\right)\right) x^{-z}\, dz = e^{-bx} \cos(ax), \tag{2.22}$$

where $a, b > 0$, and Re $z > 0$ for (2.21) and Re $z > -1$ for (2.22). Let $a = b = 2\pi\sqrt{2n}$, replace x by $\sqrt{x}$, add (2.21) and (2.22), and then simplify, so that for $c =$ Re $z > 0$,

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z)}{(16\pi^2 n)^{\frac{z}{2}}} \sin\left(\frac{\pi(z+1)}{4}\right) x^{-z/2}\, dz = e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{2.23}$$

Now replace z by $z - 1$ in (2.23), so that for $c =$ Re $z > 1$,

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z-1)}{(4\pi)^{z-1} n^{z/2}} \sin\left(\frac{\pi z}{4}\right) x^{(1-z)/2}\, dz = \frac{e^{-2\pi\sqrt{2nx}}}{\sqrt{n}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right). \tag{2.24}$$

Now substitute (2.24) in (2.20) and interchange the order of summation and integration to obtain

$$S(s, x) = \frac{2}{i} \int_{(c)} \left(\sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^{z/2}}\right) \frac{\Gamma(z-1)}{(4\pi)^z} \sin\left(\frac{\pi z}{4}\right) x^{(1-z)/2}\, dz. \tag{2.25}$$

It is well-known [88, p. 8, equation (1.3.1)] that for Re $\nu > 1$ and Re $\nu > 1+$ Re μ,

$$\zeta(\nu)\zeta(\nu - \mu) = \sum_{n=1}^{\infty} \frac{\sigma_\mu(n)}{n^\nu}. \tag{2.26}$$

Invoking (2.26) in (2.25), we see that

$$S(s, x) = \frac{2}{i} \int_{(c)} \Omega(z, s, x)\, dz, \tag{2.27}$$

where $c > 2\sigma + 2$ (since $\sigma > 0$) and

$$\Omega(z, s, x) := \zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2} - s\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2}.$$

We want to shift the line of integration from Re $z = c$ to Re $z = \lambda$, where $-1 < \lambda < 0$. Note that the integrand in (2.27) has poles at $z = 1, 2$, and $2s + 2$. Consider the positively oriented rectangular contour formed by $[c - iT, c + iT]$, $[c + iT, \lambda + iT]$, $[\lambda + iT, \lambda - iT]$, and $[\lambda - iT, c - iT]$, where T is any positive real number. By Cauchy's residue theorem,

$$\frac{1}{2\pi i}\left\{\int_{c-iT}^{c+iT} + \int_{c+iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda-iT} + \int_{\lambda-iT}^{c-iT}\right\}\Omega(z, s, x)\,dz = R_1(\Omega) + R_2(\Omega) + R_{2s+2}(\Omega), \tag{2.28}$$

where we recall that $R_a(f)$ denotes the residue of a function f at the pole $z = a$. The residues are calculated below. First,

$$\begin{aligned} R_{2s+2}(\Omega) &= \lim_{z\to 2s+2}(z - 2s - 2)\zeta\left(\frac{z}{2} - s\right)\zeta\left(\frac{z}{2}\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2} \\ &= 2\zeta(s+1)\frac{\Gamma(2s+1)}{(4\pi)^{2s+2}}\sin\left(\frac{\pi(2s+2)}{4}\right)x^{-s-\frac{1}{2}} \\ &= -\frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}}\frac{\Gamma(s+\frac{1}{2})\cot\left(\frac{1}{2}\pi s\right)\zeta(-s)}{x^{s+\frac{1}{2}}}, \end{aligned}$$

where in the first step we used (2.14), and in the last step we employed (1.32) and (5.5) with s replaced by $s + 1$. Second and third,

$$R_1(\Omega) = \lim_{z\to 1}(z-1)\frac{\Gamma(z-1)}{(4\pi)^z}\zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2} - s\right)\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2} = \frac{1}{4\sqrt{2\pi}}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2} - s\right), \tag{2.29}$$

$$R_2(\Omega) = \lim_{z\to 2}(z-2)\zeta\left(\frac{z}{2}\right)\zeta\left(\frac{z}{2} - s\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin\left(\frac{\pi z}{4}\right)x^{(1-z)/2} = \frac{\zeta(1-s)}{8\pi^2\sqrt{x}}, \tag{2.30}$$

where, in (2.29) we utilized (1.29), and in (2.30) we used (2.14). Next, we show that as $T \to \infty$, the integrals along the horizontal segments $[c + iT, \lambda + iT]$ and $[\lambda - iT, c - iT]$ tend to zero. To that end, note that if $s = \sigma + it$, for $\sigma \geq -\delta$ [88, p. 95, equation (5.1.1)],

$$\zeta(s) = O(t^{\frac{3}{2}+\delta}). \tag{2.31}$$

Also, as $|t| \to \infty$,

$$\left|\sin\left(\frac{\pi s}{4}\right)\right| = \left|\frac{e^{\frac{1}{4}i\pi s} - e^{-\frac{1}{4}i\pi s}}{2i}\right| = O\left(e^{\frac{1}{4}\pi|t|}\right). \tag{2.32}$$

Thus from (2.31), (1.39), and (2.32), we see that the integrals along the horizontal segments tend to zero as $T \to \infty$. Along with (2.28), this implies that

$$\int_{(c)} \Omega(z,s,x)\,dz = \int_{(\lambda)} \Omega(z,s,x)\,dz + 2\pi i \left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}} + \frac{1}{4\sqrt{2}\pi}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2}-s\right) - \frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}} \frac{\Gamma(s+1/2)\cot\left(\frac{\pi s}{2}\right)\zeta(-s)}{x^{s+\frac{1}{2}}} \right). \tag{2.33}$$

We now evaluate the integral along the vertical line $\operatorname{Re} z = \lambda$. Using (5.5) twice, we have

$$\begin{aligned} \int_{(\lambda)} \Omega(z,s,x)\,dz &= \int_{(\lambda)} 2^{z-s}\pi^{z-s-2}\zeta\left(1-\frac{z}{2}\right)\zeta\left(1-\frac{z}{2}+s\right)\Gamma\left(1-\frac{z}{2}\right) \\ &\quad \times \Gamma\left(1-\frac{z}{2}+s\right)\frac{\Gamma(z-1)}{(4\pi)^z}\sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)x^{(1-z)/2}\,dz \\ &= \frac{\sqrt{x}}{2^s\pi^{s+2}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s+1}}\int_{(\lambda)}\Gamma(z-1)\Gamma\left(1-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}+s\right) \\ &\quad \times \sin^2\left(\frac{\pi z}{4}\right)\sin\left(\frac{\pi z}{4}-\frac{\pi s}{2}\right)\left(\frac{4x}{n}\right)^{-z/2}\,dz \\ &= \frac{i\sqrt{x}}{2^{s-1}\pi^{s+1}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s+1}} I\left(s,\frac{x}{n}\right), \end{aligned} \tag{2.34}$$

where in the penultimate step we used (2.26), since $\lambda < 0$, and used the notation for $I(s,x)$ in the lemma. From (2.27), (2.33), and (2.34), we deduce that

$$\begin{aligned} S(s,x) &= \frac{\sqrt{x}}{2^{s-2}\pi^{s+1}}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s+1}} I\left(s,\frac{x}{n}\right) \\ &+ 4\pi\left(\frac{\zeta(1-s)}{8\pi^2\sqrt{x}} + \frac{1}{4\sqrt{2}\pi}\zeta\left(\frac{1}{2}\right)\zeta\left(\frac{1}{2}-s\right) - \frac{2^{-s-3}}{\pi^{s+\frac{3}{2}}}\frac{\Gamma(s+1/2)\cot\left(\frac{1}{2}\pi s\right)\zeta(-s)}{x^{s+\frac{1}{2}}} \right). \end{aligned}$$

The final result follows by substituting the expressions for $I\left(s,\frac{x}{n}\right)$ from the lemma, accordingly as $n < x$ or $n \geq x$. This completes the proof.

2.4 Proof of the integral identity 2.1.3

Multiplying and dividing the integrand in (2.10) by $\Gamma\left(\frac{1}{2}(3-z)\right)$ and then applying (1.32) and (1.30), we see that

$$I(s,x) = -\frac{\pi^{\frac{3}{2}}}{4\pi i}\int_{(\lambda)} \frac{\sin^2\left(\frac{1}{4}\pi z\right)\sin\left(\frac{1}{4}\pi z - \frac{1}{2}\pi s\right)}{\sin \pi z}\frac{\Gamma\left(1-\frac{1}{2}z+s\right)}{\Gamma\left(1-\frac{1}{2}z+\frac{1}{2}\right)}x^{-\frac{1}{2}z}\,dz. \tag{2.35}$$

We now apply (1.30), (1.31), and (1.32) repeatedly to simplify the integrand in (2.35). This gives

$$I(s,x)=\frac{1}{2\pi i}\frac{-\pi}{2^{2-s}}\int_{(\lambda)}F(z,s,x)\,dz, \tag{2.36}$$

where

$$F(z,s,x):=\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{z/2}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-z/2}. \tag{2.37}$$

The poles of $F(z,s,x)$ are at $z=1$, at $z=2(2k+1+s), k\in\mathbb{N}\cup\{0\}$, at $z=2(2m+1), m\in\mathbb{Z}$, and at $z=-(2j+1)$, $j\in\mathbb{N}\cup\{0\}$.

Case (i): When $x>1$, we would like to move the vertical line of integration to $+\infty$. To that end, let $X>\lambda$ be such that the line $(X-i\infty, X+i\infty)$ does not pass through the poles of $F(z)$. Consider the positively oriented rectangular contour formed by $[\lambda-iT, X-iT], [X-iT, X+iT]$, $[X+iT,\lambda+iT]$, and $[\lambda+iT,\lambda-iT]$, where T is any positive real number. Then by Cauchy's residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\left\{\int_{\lambda-iT}^{X-iT}+\int_{X-iT}^{X+iT}+\int_{X+iT}^{\lambda+iT}+\int_{\lambda+iT}^{\lambda-iT}\right\}F(z,s,x)\,dz\\&=R_1(F)+\sum_{0\le k<\frac{1}{2}\left(\frac{1}{2}X-1-\operatorname{Re}s\right)}R_{2(2k+1+s)}(F)+\sum_{0\le m<\frac{1}{2}\left(\frac{1}{2}X-1\right)}R_{2(2m+1)}(F).\end{aligned}$$

We now calculate the residues. First,

$$R_1(F)=\lim_{z\to1}(z-1)\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{\frac{z}{2}}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-\frac{1}{2}z}=-\frac{1}{\sqrt{2x}}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}s\right)}. \tag{2.38}$$

Second,

$$\begin{aligned}R_{2(2k+1+s)}(F)&=\lim_{z\to2(2k+1+s)}\{z-2(2k+1+s)\}\frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{\frac{z}{2}}(1-z)}\frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)}x^{-\frac{z}{2}}\\&=\frac{4(-1)^{k+1}\cot\left(\frac{1}{2}\pi s\right)}{k!2^{2k+2+s}}\frac{\Gamma\left(\frac{1}{2}+2k+s\right)}{\Gamma\left(\frac{1}{2}(2k+1)\right)}x^{-(2k+1+s)}\\&=\frac{(-1)^{k+1}\cot\left(\frac{1}{2}\pi s\right)}{(2k)!2^s\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left(s+\frac{1}{2}\right)_{2k}x^{-(2k+1+s)},\end{aligned} \tag{2.39}$$

where in the second calculation, we used the fact $\lim_{z\to-n}(z+n)\Gamma(z)=(-1)^n/n!$, followed by (1.29) and (1.32). Here $(y)_n$ denotes the rising factorial defined in (2.1). Note that we do not have a pole at $2(2k+1+s)$

when s is an odd integer. Also,

$$R_{2(2m+1)}(F) = \lim_{z\to 2(2m+1)} \{z-2(2m+1)\} \frac{\tan\left(\frac14\pi z\right)}{2^{z/2}(1-z)} \frac{\Gamma\left(\frac12-\frac14 z+\frac12 s\right)\Gamma\left(\frac12(1+z)\right)}{\Gamma\left(\frac14 z-\frac12 s\right)} x^{-z/2} \tag{2.40}$$
$$= \frac{1}{\pi 2^{2m}} \frac{\Gamma\left(\frac12 s-m\right)\Gamma\left(2m+\frac12\right)}{\Gamma\left(m-\frac12 s+\frac12\right)} x^{-(2m+1)}$$
$$= \frac{(-1)^m}{2^s \sin\left(\frac12\pi s\right)\Gamma(1-s)} \frac{\left(\frac12\right)_{2m}}{(1-s)_{2m}} x^{-(2m+1)},$$

where we used (1.30) and (1.32). As in the proof of Theorem 2.1.1, using Stirling's formula (1.39), we see that the integrals along the horizontal segments tend to zero as $T\to\infty$. Thus,

$$\frac{1}{2\pi i}\int_{(X)} F(z,s,x)\,dz = \frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz \tag{2.41}$$
$$+ R_1(F) + a \sum_{0\le k\le \frac12\left(\frac12 X-1-\operatorname{Re} s\right)} R_{2(2k+1+s)}(F) + \sum_{0\le m<\frac12\left(\frac12 X-1\right)} R_{2(2m+1)}(F),$$

where a is defined in (2.3). From (2.37), we see that

$$F(z+4,s,x) = -\frac{F(z,s,x)(z-1)\left(\frac12(z+1)\right)\left(\frac12(z+3)\right)}{4x^2(z+3)\left(\frac14 z-\frac12(s-1)\right)\left(\frac14 z-\frac12 s\right)}, \tag{2.42}$$

so that

$$|F(z+4,s,x)| = \frac{|F(z,s,x)|}{x^2}\left(1+O_s\left(\frac{1}{|z|}\right)\right). \tag{2.43}$$

Applying (2.42) and (2.43) repeatedly, we find that

$$|F(z+4\ell,s,x)| = \frac{|F(z,s,x)|}{x^{2\ell}}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell},$$

for any positive integer ℓ and $\operatorname{Re} z>0$. Therefore,

$$\left|\int_{(X+4\ell)} F(z,s,x)\,dz\right| = \left|\int_{(X)} \frac{F(z,s,x)}{x^{2\ell}}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell} dz\right| \tag{2.44}$$
$$= \frac{1}{|x|^{2\ell}}\left(1+O_s\left(\frac{1}{|X|}\right)\right)^{\ell}\left|\int_{(X)} F(z,s,x)\,dz\right|.$$

Since $x>1$, we can choose X large enough so that

$$|x|>\sqrt{1+O_s\left(\frac{1}{|X|}\right)}.$$

With this choice of X and the fact that $\left|\int_{(X)} F(z,s,x)\,dz\right|$ is finite, if we let $\ell\to\infty$, then, from (2.44), we find that

$$\lim_{\ell\to\infty}\int_{X+4\ell-i\infty}^{X+4\ell+i\infty} F(z,s,x)\,dz=0. \tag{2.45}$$

Hence, if we shift the vertical line (X) through the sequence of vertical lines $\{(X+4\ell)\}_{\ell=1}^{\infty}$, then, from (2.41) and (2.45), we obtain

$$\frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz=-R_1(F)-a\sum_{k=0}^{\infty}R_{2(2k+1+s)}(F)-\sum_{m=0}^{\infty}R_{2(2m+1)}(F). \tag{2.46}$$

Since $x>1$, from (2.39) and the binomial theorem, we deduce that

$$\begin{aligned} a\sum_{k=0}^{\infty}R_{2(2k+1+s)}(F)&=-a\frac{x^{-s-1}\cot\left(\frac{1}{2}\pi s\right)}{2^s\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{\left(s+\frac{1}{2}\right)_{2k}}{(2k)!}\left(\frac{i}{x}\right)^{2k} \\ &=-a\frac{x^{-s-1}\cot\left(\frac{1}{2}\pi s\right)}{2^{s+1}\sqrt{\pi}}\Gamma\left(s+\frac{1}{2}\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right\}. \end{aligned} \tag{2.47}$$

From (2.40),

$$\begin{aligned} \sum_{m=0}^{\infty}R_{2(2m+1)}(F)&=\frac{1}{x2^s\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{2m}}{(1-s)_{2m}}\left(\frac{i}{x}\right)^{2m} \\ &=\frac{1}{x2^s\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{2}(1-s),1-\frac{1}{2}s\end{matrix};-\frac{1}{x^2}\right). \end{aligned} \tag{2.48}$$

Therefore from (2.38), (2.46), (2.47), and (2.48) we deduce that

$$\begin{aligned} \frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz&=a\frac{x^{-s-1}\cot\left(\frac{1}{2}\pi s\right)}{2^{s+1}\sqrt{\pi}x}\Gamma\left(s+\frac{1}{2}\right)\left\{\left(1+\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}+\left(1-\frac{i}{x}\right)^{-\left(s+\frac{1}{2}\right)}\right\} \\ &\quad-\frac{1}{x2^s\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}\,{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{2}(1-s),1-\frac{1}{2}s\end{matrix};-\frac{1}{x^2}\right)+\frac{1}{\sqrt{2x}}\frac{\Gamma\left(\frac{1}{4}+\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}s\right)}. \end{aligned}$$

Using (2.36), we complete the proof of (2.11).

Case (ii): Now consider $x\le 1$. We would like to shift the line of integration all the way to $-\infty$. Let $X<\lambda$ be such that the line $[X-i\infty,X+i\infty]$ again does not pass through any pole of $F(z)$. Consider a positively oriented rectangular contour formed by $[\lambda - iT, \lambda + iT], [\lambda + iT, X + iT]$, $[X + iT, X - iT]$, and $[X - iT, \lambda - iT]$, where T is any positive real number. Again, by Cauchy's residue theorem,

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_{\lambda-iT}^{\lambda+iT} + \int_{\lambda+iT}^{X+iT} + \int_{X+iT}^{X-iT} + \int_{X-iT}^{\lambda-iT}\right] F(z,s,x)\,dz \\ &= \sum_{0\le k<\frac{1}{2}\left(-\frac{1}{2}X-1\right)} R_{-2(2k+1)}(F) + \sum_{0\le j<\frac{1}{2}(-X-1)} R_{-(2j+1)}(F).\end{aligned}$$

The residues in this case are calculated below. First,

$$\begin{aligned}R_{-2(2k+1)}(F) &= \lim_{z\to-2(2k+1)} \left\{(z+2(2k+1))\tan\left(\frac{\pi z}{4}\right)\right\} \frac{1}{2^{z/2}}(1-z) \\ &\quad\times \frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)} x^{-\frac{z}{2}} \\ &= \frac{(-1)^{k+1}}{\sqrt{\pi}2^s\sin\left(\frac{1}{2}\pi s\right)} \frac{\Gamma\left(\frac{1}{2}-2(k+1)\right)}{\Gamma\left(1-2(k+1)-s\right)} x^{(2k+1)} \\ &= \frac{(-1)^{k+1}}{\sqrt{\pi}2^s\sin\left(\frac{1}{2}\pi s\right)} \frac{\Gamma\left(\frac{1}{2}+2(k+1)\right)\Gamma\left(\frac{1}{2}-2(k+1)\right)}{\Gamma\left(2(k+1)+s\right)\Gamma\left(1-2(k+1)-s\right)} x^{(2k+1)} \frac{\Gamma\left(2(k+1)+s\right)}{\Gamma\left(\frac{1}{2}+2(k+1)\right)} \\ &= \frac{(-1)^{k+1}\cos\left(\frac{1}{2}\pi s\right)\Gamma(s)}{2^{s-1}\pi x} \frac{(s)_{2(k+1)}}{\left(\frac{1}{2}\right)_{2(k+1)}} x^{2(k+1)},\end{aligned} \tag{2.49}$$

where in the last step we used (1.30) and (1.31). Second,

$$\begin{aligned}R_{-(2j+1)}(F) &= \lim_{z\to-(2j+1)} (z+(2j+1)) \frac{\tan\left(\frac{1}{4}\pi z\right)}{2^{z/2}(1-z)} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{4}z+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}(1+z)\right)}{\Gamma\left(\frac{1}{4}z-\frac{1}{2}s\right)} x^{-z/2} \\ &= -\frac{2^{j+\frac{1}{2}}}{(j+1)!} \frac{\Gamma\left(\frac{5}{4}+\frac{1}{2}j+\frac{1}{2}s\right)}{\Gamma\left(-\frac{1}{4}-\frac{1}{2}j-\frac{1}{2}s\right)} x^{j+\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}2^s(j+1)!}\Gamma\left(s+\frac{3}{2}\right)\left(s+\frac{3}{2}\right)_j \sin\left(\pi\left(\frac{j}{2}+\frac{1}{4}+\frac{s}{2}\right)\right) x^{j+\frac{1}{2}},\end{aligned} \tag{2.50}$$

where we multiplied the numerator and denominator by $\Gamma\left(\frac{3}{4}+\frac{1}{2}j+\frac{1}{2}s\right)$ in the last step and then used (1.30) and (1.32). Thus, by (2.49) and (2.50),

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz &= \frac{1}{2\pi i}\int_{(X)} F(z,s,x)\,dz \\ &\quad+ \sum_{0\le k\le\frac{1}{2}\left(-\frac{1}{2}X-1\right)} R_{-2(2k+1)}(F) + \sum_{0\le k\le\frac{1}{2}(-X-1)} R_{-(2k+1)}(F).\end{aligned} \tag{2.51}$$

From (2.42),

$$|F(z-4,s,x)| = |x|^2\left(1+O_s\left(\frac{1}{|z|}\right)\right)|F(z,s,x)|,$$

and hence

$$|F(z-4\ell,s,x)| = |x|^{2\ell}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell}|F(z,s,x)|, \tag{2.52}$$

for any positive integer ℓ and $\operatorname{Re} z<0$. Therefore, from (2.52),

$$\begin{aligned}\left|\int_{(X-4k)} F(z,s,x)\,dz\right| &= \left|\int_{(X)} F(z,s,x)x^{2\ell}\left(1+O_s\left(\frac{1}{|z|}\right)\right)^{\ell} dz\right| \\ &= |x|^{2\ell}\left(1+O_s\left(\frac{1}{|X|}\right)\right)^{\ell}\left|\int_{(X)} F(z,s,x)\,dz\right|.\end{aligned}$$

Since $x<1$, we can find an $X<\lambda$, with $|X|$ sufficiently large, so that

$$x^2\left(1+O_s\left(\frac{1}{|X|}\right)\right)<1. \tag{2.53}$$

With the given choice of X and the fact that $\left|\int_{(X)} F(z,s,x)\,dz\right|$ is finite, upon letting $\ell\to\infty$ and using (2.53), we find that

$$\lim_{\ell\to\infty}\int_{X-4\ell-i\infty}^{X-4\ell+i\infty} F(z,s,x)\,dz = 0. \tag{2.54}$$

Thus if we shift the line of integration (X) to $-\infty$ through the sequence of vertical lines $\{(X-4k)\}_{k=1}^{\infty}$, from (2.51) and (2.54), we arrive at

$$\frac{1}{2\pi i}\int_{(\lambda)} F(z,s,x)\,dz = \sum_{k=0}^{\infty} R_{-2(2k+1)}(F) + \sum_{j=0}^{\infty} R_{-(2j+1)}(F). \tag{2.55}$$

Since $x\le 1$, using (2.49), we find that

$$\begin{aligned}\sum_{k=0}^{\infty} R_{-2(2k+1)}(F) &= \frac{\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x}\sum_{k=0}^{\infty}\frac{(s)_{2(k+1)}}{(1/2)_{2(k+1)}}(ix)^{2(k+1)} \\ &= \frac{\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x}\left\{ {}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-x^2\right)-1\right\},\end{aligned} \tag{2.56}$$

where for $x=1$, we additionally require that $\sigma<\frac{1}{2}$ in order to ensure the conditional convergence of the ${}_3F_2$ [1, p. 62].

From (2.50),

$$\sum_{j=0}^{\infty} R_{-(2j+1)}(F) = \frac{\Gamma\left(s+\frac{3}{2}\right)}{2^s\sqrt{\pi}}\sum_{j=0}^{\infty}\sin\left(\pi\left(\frac{j}{2}+\frac{1}{4}+\frac{s}{2}\right)\right)\frac{\left(s+\frac{3}{2}\right)_j}{(j+1)!}x^{(j+\frac{1}{2})}$$

$$
\begin{aligned}
&= \frac{\Gamma\left(s+\frac{3}{2}\right)}{2^s\sqrt{\pi}}\left\{\sqrt{x}\sin\left(\pi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\sum_{j=0}^{\infty}\frac{\left(s+\frac{3}{2}\right)_{2j}}{(2j+1)!}(ix)^{2j}\right. \qquad (2.57)\\
&\qquad\left. + x^{3/2}\cos\left(\pi\left(\frac{1}{4}+\frac{s}{2}\right)\right)\sum_{j=0}^{\infty}\frac{\left(s+\frac{3}{2}\right)_{2j+1}}{(2j+2)!}(ix)^{2j}\right\}\\
&= \frac{i\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\left[\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}-(1-ix)^{-(s+\frac{1}{2})}\right\}\right.\\
&\qquad\left. + i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}+(1-ix)^{-(s+\frac{1}{2})}-2\right\}\right],
\end{aligned}
$$

where in the last step we used the identities

$$
\begin{aligned}
\sum_{j=0}^{\infty}\frac{(a)_{2j}x^{2j}}{(2j+1)!} &= \frac{(1+x)^{1-a}-(1-x)^{1-a}}{2x(1-a)},\\
\sum_{j=0}^{\infty}\frac{(a)_{2j+1}x^{2j+1}}{(2j+2)!} &= \frac{-\left((1+x)^{1-a}+(1-x)^{1-a}-2\right)}{2x(1-a)},
\end{aligned}
$$

valid for $|x|<1$. Combining (2.55), (2.56), and (2.57), we deduce that

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{(\lambda)}F(z,s,x)\,dz &= \frac{\cos\left(\frac{\pi s}{2}\right)\Gamma(s)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-x^2\right)-1\right\}\\
&\quad+\frac{i\Gamma\left(s+\frac{1}{2}\right)}{2^{s+1}\sqrt{\pi x}}\left[\sin\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}-(1-ix)^{-(s+\frac{1}{2})}\right\}\right.\\
&\qquad\left.+ i\cos\left(\frac{\pi}{4}+\frac{\pi s}{2}\right)\left\{(1+ix)^{-(s+\frac{1}{2})}+(1-ix)^{-(s+\frac{1}{2})}-2\right\}\right].
\end{aligned}
$$

Using (2.36), we see that this proves (2.12). This completes the proof of Lemma 2.1.3.

If x is an integer in Theorem 2.1.1, then the term corresponding to it on the right-hand side of (2.4) can be included either in the first (finite) sum or in the second (infinite) sum. This follows from the fact that the integral $I(s,x)$ in the above lemma is continuous at $x=1$. Though elementary, we warn readers that it is fairly tedious to verify this by showing that the right-hand sides of (2.11) and (2.12) are equal when $x=1$, and requires the following transformation between ${}_3F_2$ hypergeometric functions, which is actually a special case when $q=2$ of a general connection formula between ${}_pF_q$'s [75, p. 410, formula **16.8.8**].

Theorem 2.4.1. *For $a_1-a_2, a_1-a_3, a_2-a_3\notin\mathbb{Z}$, and $z\notin(0,1)$,*

$$
\begin{aligned}
{}_3F_2(a_1,a_2,a_3;b_1,b_2;z) &= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\left(\frac{\Gamma(a_1)\Gamma(a_2-a_1)\Gamma(a_3-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)}(-z)^{-a_1}\right. \qquad (2.58)\\
&\times{}_3F_2\left(a_1,a_1-b_1+1,a_1-b_2+1;a_1-a_2+1,a_1-a_3+1;\frac{1}{z}\right)
\end{aligned}
$$

$$+\frac{\Gamma(a_2)\Gamma(a_1-a_2)\Gamma(a_3-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)}(-z)^{-a_2}$$
$$\times {}_3F_2\left(a_2, a_2-b_1+1, a_2-b_2+1; -a_1+a_2+1, a_2-a_3+1; \frac{1}{z}\right)$$
$$+\frac{\Gamma(a_3)\Gamma(a_1-a_3)\Gamma(a_2-a_3)}{\Gamma(b_1-a_3)\Gamma(b_2-a_3)}(-z)^{-a_3}$$
$$\times {}_3F_2\left(a_3, a_3-b_1+1, a_3-b_2+1; -a_1+a_3+1, -a_2+a_3+1; \frac{1}{z}\right)\Bigg).$$

2.5 Coalescence

In the proofs of Theorems 2.1.1 and 2.1.2 using contour integration, the convergence of the series of residues of the corresponding functions necessitates the consideration of two sums – one over $n < x$ and the other over $n \geq x$. However, for some special values of s, namely $s = 2m + \frac{1}{2}$, where m is a nonnegative integer, the two sums over $n < x$ and $n \geq x$ coalesce into a single infinite sum. This section contains corollaries of these theorems when s takes these special values.

Theorem 2.5.1. *Let $x \notin \mathbb{Z}$. Then, for any nonnegative integer m,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}} e^{-2\pi\sqrt{2nx}} \sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right) \\
&= \frac{\zeta\left(\frac{1}{2}-2m\right)}{2\pi\sqrt{x}} - \frac{(2m)!\zeta\left(-\frac{1}{2}-2m\right)}{\sqrt{2}(2\pi x)^{2m+1}} + \frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{\sqrt{x}}{\pi^{2m+\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \Bigg[-\frac{(2m)!}{\sqrt{\pi}} \left(\frac{n}{2x}\right)^{2m+\frac{3}{2}} \\
&\quad \times \left\{ \left(1+\frac{in}{x}\right)^{-(2m+1)} + \left(1-\frac{in}{x}\right)^{-(2m+1)} \right\} \\
&\quad + \frac{(-1)^m n}{2^{2m}\pi x} \Gamma\left(2m+\frac{1}{2}\right) {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4}-m, \frac{3}{4}-m \end{matrix}; -\frac{n^2}{x^2} \right) \Bigg].
\end{aligned} \tag{2.59}$$

Proof. Let $s = 2m + \frac{1}{2}$, $m \geq 0$, in Theorem 2.1.1. To examine the summands in the sum over $n < x$, observe first that $1/\Gamma\left(\frac{1}{4} - \frac{1}{2}s\right) = 0$. Since $a = 1$, the second expression in the summands is given by

$$\begin{aligned}
&-\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{a\Gamma\left(s+\frac{1}{2}\right)\cot\left(\frac{\pi s}{2}\right)}{2^{s+1}\sqrt{\pi}} \left(\frac{n}{x}\right)^{s+1} \left\{ \left(1+\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} + \left(1-\frac{in}{x}\right)^{-\left(s+\frac{1}{2}\right)} \right\} \\
&= -\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(2m)!}{\sqrt{\pi}} \left(\frac{n}{2x}\right)^{2m+\frac{3}{2}} \frac{\left(1-\frac{in}{x}\right)^{2m+1} + \left(1+\frac{in}{x}\right)^{2m+1}}{\left(1+n^2/x^2\right)^{2m+1}} \\
&= -\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}} \frac{(2m)!}{\sqrt{\pi}} \frac{n^{2m+\frac{3}{2}} x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} \left(\frac{n}{x}\right)^{2k}.
\end{aligned} \tag{2.60}$$

The third expressions in the summands become

$$\begin{aligned}
&\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{n2^{-s}}{x\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s)}{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1-s}{2},1-\frac{s}{2}\end{matrix};-\frac{n^2}{x^2}\right)\\
&=\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{(-1)^m n2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)}{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4}-m,\frac{3}{4}-m\end{matrix};-\frac{n^2}{x^2}\right).
\end{aligned}\tag{2.61}$$

Hence, by (2.60) and (2.61), the summands over $n<x$ are given by

$$\begin{aligned}
\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\Bigg\{&-\frac{(2m)!}{\sqrt{\pi}}\frac{n^{2m+\frac{3}{2}}x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{2k}\left(\frac{n}{x}\right)^{2k}\\
&+\frac{(-1)^m n2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)}{}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4}-m,\frac{3}{4}-m\end{matrix};-\frac{n^2}{x^2}\right)\Bigg\}.
\end{aligned}\tag{2.62}$$

For the summands over $n>x$, observe that the third expression is equal to zero, since $\cos\left(\frac{1}{4}\pi+\frac{1}{2}\pi\left(2m+\frac{1}{2}\right)\right)=0$. The first expression becomes

$$\begin{aligned}
&-\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{n\Gamma(s)\cos\left(\frac{1}{2}\pi s\right)}{2^{s-1}\pi x}\left\{{}_3F_2\left(\begin{matrix}\frac{s}{2},\frac{1+s}{2},1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}\\
&=\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{(-1)^{m+1}n2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)}\left\{{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\},
\end{aligned}\tag{2.63}$$

where we used (1.31) with $s=2m$. The second expressions of the summands become

$$\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{i(-1)^{m+1}\sqrt{n}(2m)!}{2^{2m+\frac{3}{2}}\sqrt{\pi x}}\frac{(1-ix/n)^{2m+1}-(1+ix/n)^{2m+1}}{(1+x^2/n^2)^{2m+1}}.\tag{2.64}$$

Note that

$$\left(1-\frac{ix}{n}\right)^{2m+1}-\left(1+\frac{ix}{n}\right)^{2m+1}=-\sum_{k=0}^{2m+1}\binom{2m+1}{k}\left(\frac{ix}{n}\right)^k\left(1+(-1)^{2m+1-k}\right).$$

These summands are non-zero only when k is odd, and so if we let $2j=2m+1-k$, we see that

$$\left(1-\frac{ix}{n}\right)^{2m+1}-\left(1+\frac{ix}{n}\right)^{2m+1}=2i(-1)^{m+1}\left(\frac{x}{n}\right)^{2m+1}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{2j}\left(\frac{n}{x}\right)^{2j}.$$

Thus, after simplification, the second expressions (2.64) equal

$$-\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\frac{(2m)!}{\sqrt{\pi}}\frac{n^{2m+\frac{3}{2}}x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{2j}\left(\frac{n}{x}\right)^{2j}.\tag{2.65}$$

Thus, by (2.63) and (2.65), the summands over $n > x$ equal

$$\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\Bigg[-\frac{(2m)!}{\sqrt{\pi}}\frac{n^{2m+\frac{3}{2}}x^{2m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}(x^2+n^2)^{2m+1}}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{2j}\left(\frac{n}{x}\right)^{2j} \\ +\frac{(-1)^{m+1}n2^{-2m}}{x\Gamma\left(\frac{1}{2}-2m\right)}\left\{{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}\Bigg]. \tag{2.66}$$

From (2.62) and (2.66), it is clear that we want to prove that

$${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4}-m,\frac{3}{4}-m\end{matrix};-\frac{n^2}{x^2}\right)+{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)=1, \tag{2.67}$$

for $x > 0$ and $n \in \mathbb{N}$. To that end, use (2.58) with $a_1 = \frac{1}{4}, a_2 = \frac{3}{4}, a_3 = 1, b_1 = \frac{1}{4} - m, b_2 = \frac{3}{4} - m$, and $z = -n^2/x^2$. This gives, for all $x, n > 0$,

$${}_3F_2\left(\begin{matrix}\frac{1}{4},\frac{3}{4},1\\ \frac{1}{4}-m,\frac{3}{4}-m\end{matrix};-\frac{n^2}{x^2}\right)=\frac{(4m+3)(4m+1)x^2}{3n^2}\,{}_3F_2\left(\begin{matrix}\frac{7}{4}+m,\frac{5}{4}+m,1\\ \frac{7}{4},\frac{5}{4}\end{matrix};-\frac{x^2}{n^2}\right). \tag{2.68}$$

Now for $n > x$, we can use the series representation (2.2) for ${}_3F_2$ on the right-hand side to obtain

$$\begin{aligned}{}_3F_2\left(\begin{matrix}\frac{7}{4}+m,\frac{5}{4}+m,1\\ \frac{7}{4},\frac{5}{4}\end{matrix};-\frac{x^2}{n^2}\right)&=1+\sum_{k=1}^{\infty}\frac{\left(\frac{7}{4}+m\right)_k\left(\frac{5}{4}+m\right)_k(1)_k}{\left(\frac{7}{4}\right)_k\left(\frac{5}{4}\right)_k k!}\left(-\frac{x^2}{n^2}\right)^k \\ &=1-\frac{3n^2}{(4m+3)(4m+1)x^2}\sum_{k=1}^{\infty}\frac{\left(\frac{3}{4}+m\right)_{k+1}\left(\frac{1}{4}+m\right)_{k+1}(1)_{k+1}}{\left(\frac{3}{4}\right)_{k+1}\left(\frac{1}{4}\right)_{k+1}(k+1)!}\left(-\frac{x^2}{n^2}\right)^{k+1} \\ &=\frac{-3n^2}{(4m+3)(4m+1)x^2}\left\{{}_3F_2\left(\begin{matrix}\frac{1}{4}+m,\frac{3}{4}+m,1\\ \frac{1}{4},\frac{3}{4}\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}.\end{aligned} \tag{2.69}$$

Combining (2.68) and (2.69), we obtain (2.67) for $n > x$.

Now set $a_1 = \frac{1}{4} + m$, $a_2 = \frac{3}{4} + m$, $a_3 = 1$, $b_1 = \frac{1}{4}$, $b_2 = \frac{3}{4}$, and $z = -x^2/n^2$ in (2.58) and use, for $n < x$, the series representation for the ${}_3F_2$ on the right-hand side of the resulting identity to arrive at (2.67) for $n < x$. This shows that (2.67) holds for all $x > 0$ and $n \in \mathbb{N}$.

Hence, the summands in the sums over $n < x$ and $n > x$ in Theorem 2.1.1 are the same when $s = 2m+\frac{1}{2}$. Now slightly rewrite (2.62) to finish the proof of Theorem 2.5.1. $\square$

Similarly, when $s = 2m + \frac{1}{2}$ in Theorem 2.1.2, we obtain the following.

Theorem 2.5.2. *For any nonnegative integer m,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right) \tag{2.70}$$

$$
\begin{aligned}
&= \left(2\pi\sqrt{x} + \frac{(2m)!}{\sqrt{2}(2\pi x)^{2m+1}}\right)\zeta\left(-\frac{1}{2} - 2m\right) + \frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{\sqrt{\pi}(2m)!}{(2\pi)^{2m+\frac{3}{2}}}\sum_{n=1}^{\infty}\sigma_{2m+\frac{1}{2}}(n)\left\{(x-in)^{-(2m+1)} + (x+in)^{-(2m+1)}\right\}.
\end{aligned}
$$

Notice the resemblance of the series on the right-hand side of (2.70) with the divergent series in Ramanujan's incorrect identity (1.6). Since the series on the right side above has a $+$ sign between the two binomial expressions in the summands, the order of n in the summand is at least $-\frac{3}{2}+\epsilon$, for each $\epsilon > 0$, unlike $-\frac{1}{2}+\epsilon$ in Ramanujan's series, because of which the latter is divergent.

When $m \geq 1$, we can omit the term $\frac{1}{\sqrt{2}}\zeta\left(\frac{1}{2}\right)\zeta(-2m)$ from both (2.59) and (2.70) since $\zeta(-2m) = 0$.

In Theorem 2.5.1, we assume $x \notin \mathbb{Z}$, whereas there is no such restriction in Theorem 2.5.2, because Theorems 1.1 and 2.5.1 involve ${}_3F_2$'s that are conditionally convergent, with the restriction $\sigma < \frac{1}{2}$ when x is an integer. Thus, the condition $\sigma \geq \frac{1}{2}$ implies that $x \notin \mathbb{Z}$, which is the case when $s = 2m + \frac{1}{2}$ for $m \geq 0$. However, ${}_3F_2$'s do not appear in Theorem 1.3, and so the restriction on x (other than the requirement $x > 0$) is not needed.

Adding (2.59) and (2.70) and simplifying gives the next theorem.

Theorem 2.5.3. *For $x \notin \mathbb{Z}$,*

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\cos\left(2\pi\sqrt{2nx}\right) \qquad (2.71)\\
&= \frac{1}{2\pi\sqrt{2x}}\zeta\left(\frac{1}{2} - 2m\right) + \pi\sqrt{2x}\zeta\left(\frac{-1}{2} - 2m\right) + \zeta\left(\frac{1}{2}\right)\zeta(-2m) \\
&\quad + \frac{(-1)^m}{\pi\sqrt{x}(2\pi)^{2m+\frac{1}{2}}}\Gamma\left(2m+\frac{1}{2}\right)\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{1}{2}}}{}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{3}{4}, 1 \\ \frac{1}{4}-m, \frac{3}{4}-m\end{matrix}; -\frac{n^2}{x^2}\right).
\end{aligned}
$$

Subtracting (2.59) from (2.70) and simplifying leads to the next result.

Theorem 2.5.4. *For $x \notin \mathbb{Z}$,*

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(2\pi\sqrt{2nx}\right) \qquad (2.72)\\
&= \frac{\zeta\left(\frac{1}{2}-2m\right)}{2\pi\sqrt{2x}} - \sqrt{2}\left(\pi\sqrt{x} + \frac{(2m)!}{\sqrt{2}(2\pi x)^{2m+1}}\right)\zeta\left(\frac{-1}{2} - 2m\right) \\
&\quad + \frac{\sqrt{x}}{\sqrt{2}\pi^{2m+\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{\sigma_{2m+\frac{1}{2}}(n)}{n^{2m+\frac{3}{2}}}\Bigg[-2\frac{(2m)!}{\sqrt{\pi}}\left(\frac{n}{2x}\right)^{2m+\frac{3}{2}} \\
&\quad \times\left\{\left(1+\frac{in}{x}\right)^{-(2m+1)} + \left(1-\frac{in}{x}\right)^{-(2m+1)}\right\}
\end{aligned}
$$

$$+\frac{(-1)^m n}{2^{2m}\pi x}\Gamma\left(2m+\frac{1}{2}\right){}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac14-m,\frac34-m\end{matrix};-\frac{n^2}{x^2}\right)\Bigg].$$

In Theorem 2.5.1, as well as in (2.71) and (2.72), we should be careful while interpreting the ${}_3F_2$-function. For example, if $n < x$, then it can be expanded as a series. Otherwise, for $n > x$, the ${}_3F_2$-function represents the analytic continuation of the series. Of course, when $n > x$, one can replace the ${}_3F_2$-function by

$$-\left\{{}_3F_2\left(\begin{matrix}\frac14+m,\frac34+m,1\\ \frac14,\frac34\end{matrix};-\frac{x^2}{n^2}\right)-1\right\},$$

as can be seen from (2.67), and then use the series expansion of this other ${}_3F_2$-function.

2.5.1 Some special cases

When $m = 0$ in Theorem 2.5.1, we obtain the following corollary.

Corollary 2.5.5. *Let $x \notin \mathbb{Z}$ and $x > 0$. Then,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}+2\pi\sqrt{2nx}\right)\\&=\frac12\left\{\left(\frac{1}{\pi\sqrt{x}}-\frac{1}{\sqrt{2}}\right)\zeta\left(\frac12\right)-\frac{1}{\pi x\sqrt{2}}\zeta\left(-\frac12\right)\right\}+\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}\frac{(\sqrt{2x}-\sqrt{n})}{x^2+n^2}.\end{aligned}\tag{2.73}$$

Proof. The corollary follows readily from Theorem 2.5.1. We only need to observe that when $n < x$,

$${}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac14,\frac34\end{matrix};-\frac{n^2}{x^2}\right)=\frac{x^2}{x^2+n^2},$$

and when $n > x$,

$$\begin{aligned}{}_3F_2\left(\begin{matrix}\frac14,\frac34,1\\ \frac14,\frac34\end{matrix};-\frac{n^2}{x^2}\right)&=-\left\{{}_3F_2\left(\begin{matrix}\frac14+m,\frac34+m,1\\ \frac14,\frac34\end{matrix};-\frac{x^2}{n^2}\right)-1\right\}\\&=-\left(\frac{1}{1+x^2/n^2}-1\right)=\frac{x^2}{x^2+n^2}\end{aligned}$$

to complete our proof. ∎

Similarly, when $m = 0$ in Theorem 2.5.2, we derive the following corollary.

Corollary 2.5.6. *For $x > 0$,*

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4}-2\pi\sqrt{2nx}\right)\tag{2.74}$$

$$= \left(2\pi\sqrt{x} + \frac{1}{2\sqrt{2}\pi x}\right)\zeta\left(-\frac{1}{2}\right) - \frac{1}{2\sqrt{2}}\zeta\left(\frac{1}{2}\right) + \frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}.$$

We now show that the two previous corollaries can also be obtained by evaluating special cases of the infinite series

$$2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right) \mp e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right).$$

Second Proof of Corollary 2.5.5. Use the remarks following (3.10) and then replace x by $xe^{\pi i/2}$ and by $xe^{-\pi i/2}$ in (3.9), and then subtract the resulting two identities to obtain, in particular for $x > 0$,

$$\begin{aligned}
&2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{\pi is/4}K_s\left(4\pi e^{\pi i/4}\sqrt{nx}\right) - e^{-\pi is/4}K_s\left(4\pi e^{-\pi i/4}\sqrt{nx}\right)\right) \\
&= -\frac{ix^{s/2-1}}{2\pi}\cot\left(\frac{\pi s}{2}\right)\zeta(s) - \frac{i(2\pi)^{-s-1}}{\pi x^{1+s/2}}\Gamma(s+1)\zeta(s+1) - \frac{ix^{s/2}}{2}\tan\left(\frac{\pi s}{2}\right)\zeta(s+1) \\
&\quad + \frac{i\pi x}{6}\frac{\zeta(2-s)}{\sin\left(\frac{1}{2}\pi s\right)} - \frac{ix^{3-s/2}}{\pi\sin\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)}{x^2+n^2}\left(n^{s-2} + x^{s-2}\cos\left(\frac{\pi s}{2}\right)\right).
\end{aligned} \tag{2.75}$$

Now let $s = -\frac{1}{2}$ in (2.75). Using (2.5) and (2.6), we see that the left-hand side simplifies to

$$\begin{aligned}
&\frac{1}{\sqrt{2}x^{1/4}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{1/4}}\left(e^{-\pi i/4-4\pi e^{\pi i/4}\sqrt{nx}} - e^{\pi i/4-4\pi e^{-\pi i/4}\sqrt{nx}}\right) \\
&= -\frac{i\sqrt{2}}{x^{1/4}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right).
\end{aligned} \tag{2.76}$$

The right-hand side of (2.75) becomes

$$\begin{aligned}
&\frac{i}{2\pi x^{5/4}}\zeta\left(-\frac{1}{2}\right) - \frac{i}{\sqrt{2}\pi x^{3/4}}\zeta\left(\frac{1}{2}\right) + \frac{i}{2x^{1/4}}\zeta\left(\frac{1}{2}\right) \\
&\quad - \frac{i\pi x^{5/4}}{3\sqrt{2}}\zeta\left(\frac{5}{2}\right) + \frac{ix^{3/4}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2} + \frac{i\sqrt{2}x^{13/4}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}.
\end{aligned} \tag{2.77}$$

Thus, from (2.76) and (2.77), we deduce that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(\frac{\pi}{4} + 2\pi\sqrt{2nx}\right) \\
&= \frac{1}{2}\left\{\left(\frac{1}{\pi\sqrt{x}} - \frac{1}{\sqrt{2}}\right)\zeta\left(\frac{1}{2}\right) - \frac{1}{\pi x\sqrt{2}}\zeta\left(\frac{-1}{2}\right)\right\} + \frac{\pi x^{3/2}}{6}\zeta\left(\frac{5}{2}\right) \\
&\quad - \frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2} - \frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}.
\end{aligned} \tag{2.78}$$

From (2.73) and (2.78), it is clear that we want to prove that

$$\frac{\pi x^{3/2}}{6}\zeta\left(\frac{5}{2}\right)-\frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}=\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}(x^2+n^2)}. \tag{2.79}$$

To that end, observe that

$$\frac{x^{7/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}(x^2+n^2)}+\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{x^2+n^2}=\frac{x^{3/2}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}}.$$

Finally, from (2.26) and the fact that $\zeta(2)=\pi^2/6$, we find that

$$\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{n^{5/2}}=\frac{\pi^2}{6}\zeta\left(\frac{5}{2}\right).$$

This proves (2.79) and hence completes an alternative proof of (2.73). □

Similarly, if we let $s=-\frac{1}{2}$ in (3.11), then we obtain (2.74) upon simplification. Adding (2.73) and (2.74), we obtain the following result.

Theorem 2.5.7. *Let $x\notin\mathbb{Z}$. Then,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\cos\left(2\pi\sqrt{2nx}\right)\\&=\left(\frac{1}{2\pi\sqrt{2x}}-\frac{1}{2}\right)\zeta\left(\frac{1}{2}\right)+\pi\sqrt{2x}\zeta\left(\frac{-1}{2}\right)+\frac{x^{3/2}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}(x^2+n^2)}.\end{aligned}$$

Subtracting (2.73) from (2.74) gives the next result.

Theorem 2.5.8. *Let $x\notin\mathbb{Z}$. Then,*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}e^{-2\pi\sqrt{2nx}}\sin\left(2\pi\sqrt{2nx}\right)\\&=\frac{1}{2\pi\sqrt{2x}}\zeta\left(\frac{1}{2}\right)-\left(\frac{1}{2\pi x}+\pi\sqrt{2x}\right)\zeta\left(-\frac{1}{2}\right)+\frac{x}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{\sigma_{1/2}(n)}{\sqrt{n}}\frac{(\sqrt{x}-\sqrt{2n})}{x^2+n^2}.\end{aligned}$$

2.6 An Interpretation of Ramanujan's Divergent Series

Throughout this section, we assume $x>0$, $\sigma>0$, and Re $w>1$. Define a function $F(s,x,w)$ by

$$F(s,x,w):=\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{w-\frac{1}{2}}}\left((x-in)^{-s-\frac{1}{2}}-(x+in)^{-s-\frac{1}{2}}\right). \tag{2.80}$$

Ramanujan's divergent series corresponds to letting $w = \frac{1}{2}$ in (2.80). Note that

$$(x - in)^{-s-\frac{1}{2}} - (x + in)^{-s-\frac{1}{2}} = \frac{2i \sin\left(\left(s + \frac{1}{2}\right) \tan^{-1}(n/x)\right)}{(x^2 + n^2)^{\frac{s}{2}+\frac{1}{4}}}.$$

Since for $\sigma > -\frac{3}{2}$ and $n > 0$ [44, p. 524, formula **3.944**, no. 5]

$$\int_0^\infty e^{-xt} t^{s-\frac{1}{2}} \sin(nt)\, dt = \Gamma\left(s + \frac{1}{2}\right) \frac{\sin\left(\left(s + \frac{1}{2}\right) \tan^{-1}(n/x)\right)}{(x^2 + n^2)^{\frac{s}{2}+\frac{1}{4}}}, \tag{2.81}$$

we deduce from (2.80)–(2.81) that

$$F(s, x, w) = \frac{2i}{\Gamma\left(s + \frac{1}{2}\right)} \sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{w-\frac{1}{2}}} \int_0^\infty e^{-xt} t^{s-\frac{1}{2}} \sin nt\, dt.$$

From [73, p. 42, formula (5.1)], for $-1 < c = \text{Re } z < 1$,

$$\sin(nt) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) (nt)^{-z}\, dz.$$

Hence,

$$\begin{aligned} F(s, x, w) &= \frac{1}{\pi\Gamma\left(s + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{s-\frac{1}{2}} \sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{w-\frac{1}{2}}} \int_{c-i\infty}^{c+i\infty} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) (nt)^{-z}\, dz\, dt \\ &= \frac{1}{\pi\Gamma\left(s + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{s-\frac{1}{2}} \int_{c-i\infty}^{c+i\infty} t^{-z} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \left(\sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{w+z-\frac{1}{2}}}\right) dz\, dt, \end{aligned} \tag{2.82}$$

where the interchange of the order of summation and integration in both instances is justified by absolute convergence. Now if Re $z > \frac{3}{2}-$ Re w and Re $z > \frac{3}{2}-$ Re $w + \sigma$, from (2.26), we see that

$$\sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{w+z-\frac{1}{2}}} = \zeta\left(w + z - \frac{1}{2}\right) \zeta\left(w + z - s - \frac{1}{2}\right).$$

Substituting this in (2.82), we find that

$$\begin{aligned} F(s, x, w) &= \frac{1}{\pi\Gamma\left(s + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{s-\frac{1}{2}} \int_{c-i\infty}^{c+i\infty} t^{-z} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \\ &\qquad \times \zeta\left(w + z - \frac{1}{2}\right) \zeta\left(w + z - s - \frac{1}{2}\right) dz\, dt \\ &= \frac{1}{\pi\Gamma\left(s + \frac{1}{2}\right)} \int_{c-i\infty}^{c+i\infty} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \zeta\left(w + z - \frac{1}{2}\right) \zeta\left(w + z - s - \frac{1}{2}\right) \end{aligned} \tag{2.83}$$

$$\times \int_0^\infty e^{-xt} t^{s-z-\frac{1}{2}}\, dt\, dz,$$

with the interchange of the order of integration again being easily justifiable. For Re $z < \sigma + \frac{1}{2}$, we have

$$\int_0^\infty e^{-xt} t^{s-z-\frac{1}{2}}\, dt = \frac{\Gamma\left(s-z+\frac{1}{2}\right)}{x^{s-z+\frac{1}{2}}}.$$

Substituting this in (2.83), we obtain the integral representation

$$F(s,x,w) = \frac{x^{-s-\frac{1}{2}}}{\pi\Gamma\left(s+\frac{1}{2}\right)} \int_{c-i\infty}^{c+i\infty} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \zeta\left(w+z-\frac{1}{2}\right) \times \zeta\left(w+z-s-\frac{1}{2}\right) \Gamma\left(s-z+\frac{1}{2}\right) x^z\, dz. \tag{2.84}$$

Note that if we shift the line of integration Re $z = c$ to Re $z = d$ such that $d = \frac{3}{2} + \sigma - \eta$ with $\eta > 0$, we encounter a simple pole of the integrand due to $\Gamma\left(s-z+\frac{1}{2}\right)$. Employing the residue theorem and noting that, from (1.39) and (2.32), the integrals over the horizontal segments tend to zero as the height of the rectangular contour tends to ∞, we have

$$\begin{aligned} F(s,x,w) = \frac{x^{-s-\frac{1}{2}}}{\pi\Gamma\left(s+\frac{1}{2}\right)} &\int_{d-i\infty}^{d+i\infty} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) \zeta\left(w+z-\frac{1}{2}\right) \\ &\times \zeta\left(w+z-s-\frac{1}{2}\right) \Gamma\left(s-z+\frac{1}{2}\right) x^z\, dz \\ &- \frac{2ix^{-s-\frac{1}{2}}}{\Gamma\left(s+\frac{1}{2}\right)} \Gamma\left(s+\frac{1}{2}\right) \sin\left(\frac{\pi}{2}\left(s+\frac{1}{2}\right)\right) \zeta(w+s)\zeta(w) x^{s+\frac{1}{2}}. \end{aligned} \tag{2.85}$$

Note that the residue in equation (2.85) is analytic in w except for simple poles at 1 and $1-s$. Consider the integrand in (2.85). The zeta functions $\zeta\left(w+z-\frac{1}{2}\right)$ and $\zeta\left(w+z-s-\frac{1}{2}\right)$ have simple poles at $w = \frac{3}{2} - z$ and $w = \frac{3}{2} + s - z$, respectively. However, since Re $z = \frac{3}{2} + \sigma - \eta$ and $\sigma > 0$, the integrand is analytic as a function of w as long as Re $w > \eta$. By a well-known theorem [87, p. 30, Theorem 2.3], the integral is also analytic in w for Re $w > \eta$. Thus, the right-hand side of (2.85) is analytic in w, which allows us to analytically continue $F(s,x,w)$ as a function of w to the region Re $w > \eta$, and hence to Re $w > 0$, since η is any arbitrary positive number.

As remarked in the beginning of this section, letting $w = \frac{1}{2}$ in (2.80) yields Ramanujan's divergent series. However, the analytic continuation of $F(s,x,w)$ to Re $w > 0$ allows us to substitute $w = \frac{1}{2}$ in (2.85) and thereby give a valid interpretation of Ramanujan's divergent series. The only exception to this is when

$s = \frac{1}{2}$, since then $w = \frac{1}{2} = 1 - s$ is a pole of the right-hand side of (2.85), as discussed above.

If we further shift the line of integration in (2.84) from $\mathrm{Re}\, z = \frac{3}{2} + \sigma - \eta$ to $\mathrm{Re}\, z = \frac{5}{2} + \sigma - \eta$, and likewise to $+\infty$, we obtain a meromorphic continuation of $F(s, x, w)$, as a function of w, to the whole complex plane.

Chapter 3

Extended form of the Voronoï summation formula

In this chapter, we present some extended form of Voronoï summation formulas (1.8) and (1.9).

3.1 Extension of Voronoï summation formulas

Theorem 3.1.1. *Let $0 < \alpha < \beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let f denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $-\frac{1}{2} < \sigma < \frac{1}{2}$. Then,*

$$\sum_{\alpha<j<\beta} \sigma_{-s}(j) f(j) = \int_{\alpha}^{\beta} (\zeta(1+s) + t^{-s}\zeta(1-s)) f(t)\, dt \tag{3.1}$$
$$+ 2\pi \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{\frac{s}{2}} \int_{\alpha}^{\beta} t^{-\frac{s}{2}} f(t) \Bigg\{ \left(\frac{2}{\pi} K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt}) \right)$$
$$\times \cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right) \Bigg\} dt.$$

We wish to extend (3.1) to allow $\alpha = 0$ so as to obtain (1.10) as a special case of Theorem 3.1.1. To do this, we need to impose some additional restrictions on f. As an intermediate result, we state the following theorem which generalizes Theorem 3 in [97].

Theorem 3.1.2. *Let $0 < \alpha < \frac{1}{2}$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and $0 < \theta < \min\left(1, \frac{1+2\sigma}{1-2\sigma}\right)$. Let $N \in \mathbb{N}$ such that $N^{\theta}\alpha > 1$. If f is twice differentiable as a function of t, and is of bounded variation in $(0, \alpha)$, then as $N \to \infty$,*

$$f(0+)\frac{\zeta(-s)}{2} - \int_0^{\alpha} f(t)(\zeta(1-s) + t^s\zeta(1+s))\, dt + 2\pi {\sum_{n=1}^{N}}' \frac{\sigma_s(n)}{n^{s/2}} \int_0^{\alpha} f(t) t^{\frac{s}{2}} \Bigg\{ J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right)$$
$$+ \left(Y_s(4\pi\sqrt{nt}) - \frac{2}{\pi} K_s(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi s}{2}\right) \Bigg\} dt$$
$$\ll \begin{cases} (2\gamma + \log N)(V_0^{N^{-\theta}} f(t) + N^{(\theta-1)/4}(|f(\alpha)| + V_0^{\alpha} f(t))), & \textit{if } s = 0, \\ \\ V_0^{N^{-\theta}} f(t) + (N^{(1-\theta)(2\sigma-1)/4} + N^{(\theta(1-2\sigma)-(2\sigma+1))/4}) \times (|f(\alpha)| + V_0^{\alpha} f(t)), & \textit{if } s \neq 0. \end{cases}$$

Additionally if we assume the limits

$$\lim_{x\to 0+} V_0^x f(t) = 0, \ \text{if } s \neq 0 \qquad \text{and} \qquad \lim_{x\to 0+} \log x \, V_0^x f(t) = 0, \ \text{if } s = 0, \tag{3.2}$$

then

$$\begin{aligned} f(0+)\frac{\zeta(-s)}{2} - \int_0^\alpha f(t)(\zeta(1-s) + t^s\zeta(1+s))\,dt \quad &+ 2\pi \sum_{n=1}^\infty \frac{\sigma_s(n)}{n^{s/2}} \int_0^\alpha f(t) t^{\frac{s}{2}} \bigg\{ J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right) \\ &+ \left(Y_s(4\pi\sqrt{nt}) - \frac{2}{\pi} K_s(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi s}{2}\right) \bigg\} \,dt = 0. \end{aligned} \tag{3.3}$$

Clearly, for $0 < \alpha < \frac{1}{2}$, we have

$$\sideset{}{'}\sum_{0<j\leq\alpha} \sigma_{-s}(j) f(j) = 0. \tag{3.4}$$

Also, if we substitute for $Y_s(4\pi\sqrt{nt})$ via (2.16) and employ (2.6), we find that the kernel in (3.3), namely,

$$J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right) + \left(Y_s(4\pi\sqrt{nt}) - \frac{2}{\pi} K_s(4\pi\sqrt{nt}) \right) \cos\left(\frac{\pi s}{2}\right)$$

is invariant under the replacement of s by $-s$. Therefore replacing s by $-s$ in (3.3), then replacing zero on the right-hand side of (3.3) by $-\sum_{0<j\leq\alpha} \sigma_{-s}(j) f(j)$ using (3.4), and then finally subtracting the resulting equation so obtained from (3.1), we arrive at the following result.

Theorem 3.1.3. *Let* $0 < \alpha < \frac{1}{2}, \alpha < \beta$ *and* $\beta \notin \mathbb{Z}$. *Let* f *denote a function analytic inside a closed contour strictly containing* $[\alpha, \beta]$, *and of bounded variation in* $0 < t < \alpha$. *Furthermore, if* f *satisfies the limit conditions in* (3.2), *and* $-\frac{1}{2} < \sigma < \frac{1}{2}$, *then*

$$\begin{aligned} \sum_{0<j<\beta} \sigma_{-s}(j) f(j) = -f(0+)\frac{\zeta(s)}{2} &+ \int_0^\beta (\zeta(1+s) + t^{-s}\zeta(1-s)) f(t)\,dt \\ &+ 2\pi \sum_{n=1}^\infty \sigma_{-s}(n) n^{\frac{s}{2}} \int_0^\beta t^{-\frac{s}{2}} f(t) \bigg\{ \left(\frac{2}{\pi} K_s(4\pi\sqrt{nt}) - Y_s(4\pi\sqrt{nt}) \right) \\ &\qquad \times \cos\left(\frac{\pi s}{2}\right) - J_s(4\pi\sqrt{nt}) \sin\left(\frac{\pi s}{2}\right) \bigg\} \,dt. \end{aligned}$$

3.1.1 Oppenheim's Formula (1.10) as a Special Case

Letting $\lambda=-s+1$, $\mu=s$, and $x=4\pi\sqrt{nt}$ in [79, p. 37, equation (1.8.1.1)], [79, p. 42, equation (1.9.1.1)] [1] and [79, p. 47, equation (1.12.1.2)], and then simplifying, we see that

$$\begin{aligned}&\int t^{-\frac{s}{2}}\left\{\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt})-Y_s(4\pi\sqrt{nt})\right)\cos\left(\frac{\pi s}{2}\right)-J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right\}dt \\ &=\frac{t^{\frac{1-s}{2}}}{4\pi\sqrt{n}\sin\left(\frac{1}{2}\pi s\right)}\left(J_{s-1}(4\pi\sqrt{nt})+J_{1-s}(4\pi\sqrt{nt})-\frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nt})\right).\end{aligned}\tag{3.5}$$

Let $f(t)\equiv 1$ and $\beta=x\notin\mathbb{Z}$ in Theorem 3.1.3. Then,

$$\begin{aligned}\sum_{j<x}\sigma_{-s}(j)=-\frac{1}{2}\zeta(s)+\int_0^x(\zeta(1+s)+t^{-s}\zeta(1-s))\,dt \\ +2\pi\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\int_0^x t^{-\frac{s}{2}}\left\{\left(\frac{2}{\pi}K_s(4\pi\sqrt{nt})-Y_s(4\pi\sqrt{nt})\right)\right. \\ \left.\times\cos\left(\frac{\pi s}{2}\right)-J_s(4\pi\sqrt{nt})\sin\left(\frac{\pi s}{2}\right)\right\}dt.\end{aligned}\tag{3.6}$$

Note that

$$\int(\zeta(1+s)+t^{-s}\zeta(1-s))\,dt=t\zeta(1+s)+\frac{t^{1-s}}{1-s}\zeta(1-s).\tag{3.7}$$

Since $-\frac{1}{2}<\sigma<\frac{1}{2}$ and the right-hand sides of (3.5) and (3.7) vanish as t tends to 0, from (3.5), (3.6), and (3.7), we obtain (1.10).

Remark. The analysis above also shows that for $\alpha>0,\alpha\notin\mathbb{Z}$,

$$\begin{aligned}\sum_{\alpha<j<x}\sigma_{-s}(j)=x\zeta(1+s)+\frac{x^{1-s}}{1-s}\zeta(1-s)-\alpha\zeta(1+s)-\frac{\alpha^{1-s}}{1-s}\zeta(1-s) \\ +\frac{1}{2\sin\left(\frac{1}{2}\pi s\right)}\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{\frac{s+1}{2}}}\left\{x^{\frac{1-s}{2}}\left(J_{s-1}(4\pi\sqrt{nx})+J_{1-s}(4\pi\sqrt{nx})-\frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nx})\right)\right.\end{aligned}\tag{3.8}$$

---

[1] This formula, as is stated, contains many misprints. The correct version should read

$$\begin{aligned}\int_{x_1}^{x_2}y^{\lambda}Y_{\nu}(y)\,dy=\begin{Bmatrix}-1\\1\end{Bmatrix}\frac{\cos(\nu\pi)\Gamma(-\nu)x^{\lambda+\nu+1}}{2^{\nu}\pi(\lambda+\nu+1)}{}_1F_2\left(\frac{\lambda+\nu+1}{2};1+\nu,\frac{\lambda+\nu+3}{2};-\frac{x^2}{4}\right) \\ +\begin{Bmatrix}-1\\1\end{Bmatrix}\frac{2^{\nu}\Gamma(\nu)x^{\lambda-\nu+1}}{\pi(\lambda-\nu+1)}{}_1F_2\left(\frac{\lambda-\nu+1}{2};1-\nu,\frac{\lambda-\nu+3}{2};-\frac{x^2}{4}\right) \\ -\begin{Bmatrix}0\\1\end{Bmatrix}\frac{2^{\lambda}}{\pi}\cos\left(\frac{(\lambda-\nu+1)\pi}{2}\right)\Gamma\left(\frac{\lambda+\nu+1}{2}\right)\Gamma\left(\frac{\lambda-\nu+1}{2}\right). \\ \left[\begin{Bmatrix}x_1=0,x_2=x;\quad \mathrm{Re}(\lambda)>|\mathrm{Re}(\nu)|-1\\ x_1=x,x_2=\infty;\quad \mathrm{Re}(\lambda)<\frac{1}{2}\end{Bmatrix}\right].\end{aligned}$$

$$- \alpha^{\frac{1-s}{2}} \left(J_{s-1}(4\pi\sqrt{n\alpha}) + J_{1-s}(4\pi\sqrt{n\alpha}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{n\alpha}) \right) \Bigg\}.$$

From (1.10) and (3.8), we conclude that, for $-\frac{1}{2} < \sigma < \frac{1}{2}$,

$$\lim_{\alpha\to 0+} \frac{\alpha^{\frac{1-s}{2}}}{\sin\left(\frac{1}{2}\pi s\right)} \sum_{n=1}^{\infty} \frac{\sigma_s(n)}{n^{\frac{s+1}{2}}} \left(J_{s-1}(4\pi\sqrt{n\alpha}) + J_{1-s}(4\pi\sqrt{n\alpha}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{n\alpha}) \right) = \zeta(s),$$

which is likely to be difficult to prove directly.

3.2 Proof of the first extended form of the Voronoï summation formula

We begin with a result due to H. Cohen [27, Theorem 3.4].

Theorem 3.2.1. *Let $x > 0$ and $s \notin \mathbb{Z}$, where $\sigma \geq 0$ [2]. Then, for any integer k such that $k \geq \lfloor (\sigma+1)/2 \rfloor$,*

$$\begin{aligned} 8\pi x^{s/2} \sum_{n=1}^{\infty} \sigma_{-s}(n) n^{s/2} K_s(4\pi\sqrt{nx}) &= A(s,x)\zeta(s) + B(s,x)\zeta(s+1) \\ &+ \frac{2}{\sin(\pi s/2)} \left(\sum_{1\leq j\leq k} \zeta(2j)\zeta(2j-s)x^{2j-1} + x^{2k+1} \sum_{n=1}^{\infty} \sigma_{-s}(n) \frac{n^{s-2k} - x^{s-2k}}{n^2 - x^2} \right), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} A(s,x) &= \frac{x^{s-1}}{\sin(\pi s/2)} - (2\pi)^{1-s}\Gamma(s), \\ B(s,x) &= \frac{2}{x}(2\pi)^{-s-1}\Gamma(s+1) - \frac{\pi x^s}{\cos(\pi s/2)}. \end{aligned} \tag{3.10}$$

By analytic continuation, the identity in Theorem 3.2.1 is valid not only for $x > 0$ but for $-\pi < \arg x < \pi$. Take $k = 1$ in (3.9). The condition $\lfloor (\sigma+1)/2 \rfloor \leq 1$ implies that $0 \leq \sigma < 3$. We consider $0 \leq \sigma < \frac{1}{2}$. Note that Koshliakov [59] has already proved the case $s = 0$, and the theorem follows for the remaining values of σ, i.e., for $-\frac{1}{2} < \sigma < 0$, by the invariance noted in the previous footnote.

Replace x by iz in (3.9) for $-\pi < \arg z < \frac{1}{2}\pi$, and then by $-iz$ for $-\frac{1}{2}\pi < \arg z < \pi$. Now add the resulting two identities and simplify, so that for $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$,

$$\Lambda(z,s) = \Phi(z,s), \tag{3.11}$$

[2] As mentioned in [27], the condition $\sigma \geq 0$ is not restrictive since, because of (2.6), the left side of the identity in this theorem is invariant under the replacement of s by $-s$.

where

$$\Lambda(z,s) := z^{-s/2}\varphi(z,s),$$

with $\varphi(x,s)$ defined in (1.12), and

$$\Phi(z,s) := -(2\pi z)^{-s}\Gamma(s)\zeta(s) + \frac{\zeta(s)}{2\pi z} - \frac{1}{2}\zeta(1+s) + \frac{z}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-s}(n)}{z^2+n^2}. \tag{3.12}$$

As a function of z, $\Phi(z,s)$ is analytic in the entire complex plane except on the negative real axis and at $z = in, n \in \mathbb{Z}$. Hence, $\Phi(iz,s)$ is analytic in the entire complex plane except on the positive imaginary axis and at $z \in \mathbb{Z}$. Similarly, $\Phi(-iz,s)$ is analytic in the entire complex plane except on the negative imaginary axis and at $z = n \in \mathbb{Z}$. This implies that $\Phi(iz,s)+\Phi(-iz,s)$ is analytic in both the left and right half-planes, except possibly when z is an integer. However, it is easy to see that

$$\lim_{z\to\pm n}(z\mp n)\Phi(iz,s) = \frac{1}{2\pi i}\sigma_{-s}(n) \quad \text{and} \quad \lim_{z\to\pm n}(z\mp n)\Phi(-iz,s) = -\frac{1}{2\pi i}\sigma_{-s}(n),$$

so that

$$\lim_{z\to\pm n}(z\mp n)\left(\Phi(iz,s)+\Phi(-iz,s)\right) = 0.$$

In particular, this implies that $\Phi(iz,s)+\Phi(-iz,s)$ is analytic in the entire right half-plane.

Now observe that for z inside an interval (u,v) on the positive real line not containing any integer, we have, using the definition (3.12),

$$\Phi(iz,s)+\Phi(-iz,s) = -2(2\pi z)^{-s}\Gamma(s)\zeta(s)\cos\left(\tfrac{1}{2}\pi s\right) - \zeta(1+s). \tag{3.13}$$

Since both $\Phi(iz,s)+\Phi(-iz,s)$ and $-2(2\pi z)^{-s}\Gamma(s)\zeta(s)\cos\left(\frac{1}{2}\pi s\right)-\zeta(1+s)$ are analytic in the right half-plane as functions of z, by analytic continuation, the identity (3.13) holds for any z in the right half-plane. Finally, using the functional equation (5.5) for $\zeta(s)$, we can simplify (3.13) to deduce that, for $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$,

$$\Phi(iz,s)+\Phi(-iz,s) = -z^{-s}\zeta(1-s) - \zeta(1+s). \tag{3.14}$$

Next, let f be an analytic function of z within a closed contour intersecting the real axis in α and β, where $0<\alpha<\beta$, $m-1<\alpha<m$, $n<\beta<n+1$, and $m,n\in\mathbb{Z}$. Let γ_1 and γ_2 denote the portions of the contour in the upper and lower half-planes, respectively, so that the notations $\alpha\gamma_1\beta$ and $\alpha\gamma_2\beta$, for example, denote

paths from α to β in the upper and lower half-planes, respectively. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\alpha\gamma_2\beta\gamma_1\alpha} f(z)\Phi(iz,s)\,dz = \sum_{\alpha<j<\beta} R_j(f(z)\Phi(iz,s)).$$

Since $f(z)\Phi(iz,s)$ has a simple pole at each integer j, $\alpha<j<\beta$, with residue $\frac{1}{2\pi i}\sigma_s(j)f(j)$, we find that

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j) &= \int_{\alpha\gamma_2\beta} f(z)\Phi(iz,s)\,dz - \int_{\alpha\gamma_1\beta} f(z)\Phi(iz,s)\,dz\\ &= \int_{\alpha\gamma_2\beta} f(z)\Phi(iz,s)\,dz - \int_{\alpha\gamma_1\beta} f(z)\left(-\Phi(-iz,s)-z^{-s}\zeta(1-s)-\zeta(1+s)\right)\,dz\\ &= \int_{\alpha\gamma_2\beta} f(z)\Phi(iz,s)\,dz + \int_{\alpha\gamma_1\beta} f(z)\Phi(-iz,s)\,dz\\ &\quad+\int_{\alpha\gamma_1\beta} f(z)\left(z^{-s}\zeta(1-s)+\zeta(1+s)\right)\,dz,\end{aligned}$$

where in the penultimate step, we used (3.14). Using the residue theorem again, we readily see that

$$\int_{\alpha\gamma_1\beta} f(z)\left(z^{-s}\zeta(1-s)+\zeta(1+s)\right)\,dz = \int_{\alpha}^{\beta} f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt.$$

Since $\Lambda(z,s)=\Phi(z,s)$ for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$, it is easy to see that $\Lambda(iz,s)=\Phi(iz,s)$, for $-\pi<\arg z<0$, and $\Lambda(-iz,s)=\Phi(-iz,s)$, for $0<\arg z<\pi$. Thus,

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j) = \int_{\alpha\gamma_2\beta} f(z)\Lambda(iz,s)\,dz + \int_{\alpha\gamma_1\beta} f(z)\Lambda(-iz,s)\,dz\\ +\int_{\alpha}^{\beta} f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt.\end{aligned} \tag{3.15}$$

Using the asymptotic expansion (2.19), we see that the series

$$\Lambda(iz,s) = 2(iz)^{-\frac{s}{2}}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{inz}\right)+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{inz}\right)\right)$$

is uniformly convergent in compact subintervals of $-\pi<\arg z<0$, and the series

$$\Lambda(-iz,s) = 2(-iz)^{-\frac{s}{2}}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\left(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{-inz}\right)+e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{-inz}\right)\right)$$

is uniformly convergent in compact subsets of $0<\arg z<\pi$. Thus, interchanging the order of summation and integration in (3.15), we deduce that

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\int_{\alpha\gamma_2\beta}f(z)(iz)^{-\frac{s}{2}}\Big(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{inz}\right)\\ +e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{inz}\right)\Big)\,dz\\ +2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\int_{\alpha\gamma_1\beta}f(z)(-iz)^{-\frac{s}{2}}\Big(e^{i\pi s/4}K_s\left(4\pi e^{i\pi/4}\sqrt{-inz}\right)\\ +e^{-i\pi s/4}K_s\left(4\pi e^{-i\pi/4}\sqrt{-inz}\right)\Big)\,dz\\ +\int_{\alpha}^{\beta}f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt.\end{aligned}$$

Employing the residue theorem again, this time for each of the integrals inside the two sums, and simplifying, we find that

$$\begin{aligned}\sum_{\alpha<j<\beta}\sigma_{-s}(j)f(j)=2\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{\frac{s}{2}}\\ \times\int_{\alpha}^{\beta}t^{-\frac{s}{2}}f(t)\bigg(K_s\left(4\pi i\sqrt{nt}\right)+K_s\left(-4\pi i\sqrt{nt}\right)+2\cos\left(\frac{\pi s}{2}\right)K_s\left(4\pi\sqrt{nt}\right)\bigg)\,dt\\ +\int_{\alpha}^{\beta}f(t)\left(\zeta(1+s)+t^{-s}\zeta(1-s)\right)\,dt.\end{aligned}\tag{3.16}$$

Note that for $-\pi<\arg z\leq\frac{1}{2}\pi$, the modified Bessel function $K_\nu(z)$ is related to the Hankel function $H_\nu^{(1)}(z)$ by [44, p. 911, formula **8.407.1**]

$$K_\nu(z)=\frac{\pi i}{2}e^{\frac{\nu\pi i}{2}}H_\nu^{(1)}(iz),\tag{3.17}$$

where the Hankel function is defined by [44, p. 911, formula **8.405.1**]

$$H_\nu^{(1)}(z):=J_\nu(z)+iY_\nu(z).\tag{3.18}$$

Employing the relations (3.17) and (3.18), we have, for $x>0$,

$$\begin{aligned}K_s(ix)+K_s(-ix)&=\frac{\pi i}{2}e^{\frac{i\pi s}{2}}\left(H_s^{(1)}(-x)+H_s^{(1)}(x)\right)\\ &=\frac{\pi i}{2}e^{\frac{i\pi s}{2}}\left\{\left(J_s(x)+J_s(-x)\right)+i\left(Y_s(x)+Y_s(-x)\right)\right\}.\end{aligned}\tag{3.19}$$

For $m\in\mathbb{Z}$ [44, p. 927, formulas **8.476.1, 8.476.2**]

$$J_\nu(e^{m\pi i}z)=e^{m\nu\pi i}J_\nu(z),\tag{3.20}$$

$$Y_\nu(e^{m\pi i}z) = e^{-m\nu\pi i}Y_\nu(z) + 2i\sin(m\nu\pi)\cot(\nu\pi)\,J_\nu(z). \tag{3.21}$$

Using the relations (3.20) and (3.21) with $m = 1$, we can simplify (3.19) and put it in the form

$$\begin{aligned}
&K_s(ix) + K_s(-ix) \\
&= \frac{\pi i}{2}e^{\frac{i\pi s}{2}}\left\{\left(J_s(x) + e^{i\pi s}J_s(x)\right) + i\left(Y_s(x) + e^{-i\pi s}Y_s(x) + 2i\cos(\pi s)\,J_s(x)\right)\right\} \\
&= \frac{\pi i}{2}e^{\frac{i\pi s}{2}}\left\{\left(1 - e^{-i\pi s}\right)J_s(x) + i\left(1 + e^{-i\pi s}\right)Y_s(x)\right\} \\
&= -\pi\left(J_s(x)\sin\left(\frac{\pi s}{2}\right) + Y_s(x)\cos\left(\frac{\pi s}{2}\right)\right).
\end{aligned} \tag{3.22}$$

Now replace x by $4\pi\sqrt{nt}$ in (3.22) and substitute in (3.16) to obtain (3.1) after simplification. This completes the proof.

3.3 Proof of the second extended form of the Voronoï summation formula

In this section we give a proof of Theorem 3.1.2. For any integer λ, define

$$G_{\lambda+s}(z) := -J_{\lambda+s}(z)\sin\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) - (-1)^\lambda\frac{2}{\pi}K_{\lambda+s}(z)\right)\cos\left(\frac{\pi s}{2}\right) \tag{3.23}$$

and

$$F_{\lambda+s}(z) := -J_{\lambda+s}(z)\sin\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) + (-1)^\lambda\frac{2}{\pi}K_{\lambda+s}(z)\right)\cos\left(\frac{\pi s}{2}\right). \tag{3.24}$$

Remark. Throughout this section, we keep s fixed such that $-\frac{1}{2} < \sigma < \frac{1}{2}$. So while interpreting $F_{s+\lambda}(z)$ or $G_{s+\lambda}(z)$, care should be taken not to conceive them as functions obtained after replacing s by $s+\lambda$ in $F_s(z)$ or $G_s(z)$, but instead as those where s remains fixed and only λ varies.

From [93, pp. 66, 79] we have

$$\frac{d}{dz}\left\{z^\nu J_\nu(z)\right\} = z^\nu J_{\nu-1}(z), \tag{3.25}$$
$$\frac{d}{dz}\left\{z^\nu K_\nu(z)\right\} = -z^\nu K_{\nu-1}(z), \tag{3.26}$$
$$\frac{d}{dz}\left\{z^\nu Y_\nu(z)\right\} = z^\nu Y_{\nu-1}(z). \tag{3.27}$$

Using (3.25), (3.26), and (3.27) we deduce that

$$\frac{d}{dt}\left\{\left(\frac{t}{u}\right)^{(s+\lambda)/2} G_{s+\lambda}(4\pi\sqrt{tu})\right\} = 2\pi\left(\frac{t}{u}\right)^{(s+\lambda-1)/2} G_{s+\lambda-1}(4\pi\sqrt{tu}), \tag{3.28}$$

for $u > 0$. Similarly,

$$\frac{d}{dt}\left\{\left(\frac{t}{u}\right)^{(s+\lambda)/2} F_{s+\lambda}(4\pi\sqrt{tu})\right\} = 2\pi\left(\frac{t}{u}\right)^{(s+\lambda-1)/2} F_{s+\lambda-1}(4\pi\sqrt{tu}), \tag{3.29}$$

for $u > 0$.

From (1.2) and (1.10), recall the definition

$$\begin{aligned}\Delta_{-s}(x) = \frac{x}{2\sin\left(\frac{1}{2}\pi s\right)} \sum_{n=1}^{\infty} \sigma_s(n)\left(\sqrt{nx}\right)^{-1-s} \\ \times\left(J_{s-1}(4\pi\sqrt{nx}) + J_{1-s}(4\pi\sqrt{nx}) - \frac{2}{\pi}\sin(\pi s)K_{1-s}(4\pi\sqrt{nx})\right),\end{aligned}$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$. If we replace s by $-s$ in the above equation and use (2.16), we find by a straightforward computation that

$$\Delta_s(x) = \sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n)G_{s+1}(4\pi\sqrt{nx}), \tag{3.30}$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$. Fix $x > 0$. By the asymptotic expansions of Bessel functions (2.17), (2.18), and (2.19), there exists a sufficiently large integer N_0 such that

$$G_\nu(4\pi\sqrt{nx}) \ll_\nu \frac{1}{(nx)^{1/4}} \quad \text{and} \quad F_\nu(4\pi\sqrt{nx}) \ll_\nu \frac{1}{(nx)^{1/4}}, \tag{3.31}$$

for all $n > N_0$. Hence, for $-\frac{1}{2} < \sigma < \frac{1}{2}$ and $x > 0$,

$$\sum_{n>N_0}\left(\frac{x}{n}\right)^{\lambda+\frac{s}{2}} \sigma_s(n)G_{s+2\lambda}(4\pi\sqrt{nx}) \ll x^{\lambda+\frac{2\sigma-1}{4}} \sum_{n>N_0} \frac{\sigma_\sigma(n)}{n^{\lambda+\frac{1+2\sigma}{4}}} \ll x^{\lambda+\frac{2\sigma-1}{4}},$$

provided that $2\lambda > |\sigma| + \frac{3}{2}$. Therefore, for $\lambda \geq 1$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and $x > 0$, the series

$$\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^{\lambda+\frac{s}{2}} \sigma_s(n)G_{s+2\lambda}(4\pi\sqrt{nx})$$

is absolutely convergent. Similarly, for $\lambda \geq 1$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and $x > 0$, the series

$$\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\lambda + \frac{s}{2}} \sigma_s(n) F_{s+2\lambda}(4\pi\sqrt{nx})$$

is absolutely convergent. Denote

$$D_s(x) := \sum_{n \leq x}{}' \sigma_s(n) \tag{3.32}$$

and

$$\Phi_s(x) := x\zeta(1-s) + \frac{x^{1+s}}{1+s}\zeta(1+s) - \frac{1}{2}\zeta(-s). \tag{3.33}$$

Therefore, from (1.10), we write

$$D_s(x) = \Phi_s(x) + \Delta_s(x) \tag{3.34}$$

for $-\frac{1}{2} < \sigma < \frac{1}{2}$.

The following lemmas are key ingredients in the proof of Theorem 3.1.2. They are special cases of two results in [98]. We note, however, that the definitions of G and F in [98] are different from those in (3.23) and (3.24) that we use.

Lemma 3.3.1. *If $x > 0$, $N > 0$, and $-\frac{1}{2} < \sigma < \frac{1}{2}$, then*

$$\begin{aligned}
\Delta_s(x) = {} & \sum_{n=1}^{N}{}' \left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n) G_{s+1}(4\pi\sqrt{nx}) - \left(\frac{x}{N}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{Nx})\Delta_s(N) \\
& + \frac{N^s\zeta(1+s) + \zeta(1-s)}{2\pi} \left(\frac{x}{N}\right)^{s/2} F_s(4\pi\sqrt{Nx}) \\
& + \frac{s\zeta(1+s)}{2\pi} \int_N^{\infty} \left(\frac{x}{t}\right)^{s/2} F_s(4\pi\sqrt{xt}) t^{s-1}\, dt \\
& + 2\pi \sum_{n=1}^{\infty} \sigma_s(n) \int_N^{\infty} \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt}) \left(\frac{t}{n}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{nt})\, dt.
\end{aligned} \tag{3.35}$$

Proof. Take $\lambda = 0$, $\kappa = 1$, and $\theta = 1$ in Theorem 2 of [98, p. 404], and make use of the notations (1.21) and (3.13) given in it. □

We wish to invert the order of summation and integration in the last expression on the right-hand side of (3.35). In order to justify that, we need the following lemma.

Lemma 3.3.2. *If $N > A$, $Nx > A$, $-\frac{1}{2} < \sigma < \frac{1}{2}$, and*

$$I_s(x,n;N) := 2\pi \int_N^\infty \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt}) \left(\frac{t}{n}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{nt})\, dt,$$

then

$$\sum_{n=1}^\infty \sigma_s(n) I_s(x,n;N) = C_s(x,N) + O\left(\frac{x^{1+\epsilon}}{\sqrt{N}}\right),$$

for every $\epsilon > 0$, where

$$C_s(x,N) = 0, \quad \textit{if} \quad x < \tfrac{1}{2} \quad \textit{or} \quad x \in \mathbb{N},$$
$$C_s(x,N) = \frac{1}{\pi}\left(\frac{x}{y}\right)^{(2s+5)/4} \sigma_s(y) \int_{4\pi\sqrt{N}|\sqrt{y}-\sqrt{x}|}^\infty \frac{\sin(t\,\mathrm{sgn}(y-x))}{t}\, dt, \quad \textit{if} \quad x \neq y = \lfloor x + \tfrac{1}{2} \rfloor \geq 1.$$

Proof. This is the special case $\lambda = 0, \kappa = 1$ of Lemma 6 of [98, p. 412]. □

Proof of Theorem 3.1.2. By Lemma 3.3.2, we see that the last expression on the right-hand side of (3.35) tends to 0 as $N \to \infty$. Hence, by interchanging the summation and integration in this expression, we deduce that

$$\begin{aligned}\Delta_s(x) = {} & \sum_{n=1}^{N}{}' \left(\frac{x}{n}\right)^{(s+1)/2} \sigma_s(n) G_{s+1}(4\pi\sqrt{nx}) + \frac{N^s\zeta(1+s) + \zeta(1-s)}{2\pi}\left(\frac{x}{N}\right)^{s/2} F_s(4\pi\sqrt{Nx}) \\ & - \left(\frac{x}{N}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{Nx})\Delta_s(N) + \frac{s\zeta(1+s)}{2\pi}\int_N^\infty \left(\frac{x}{t}\right)^{s/2} F_s(4\pi\sqrt{xt}) t^{s-1}\, dt \\ & + 2\pi \int_N^\infty \left(\frac{x}{t}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{xt})\Delta_s(t)\, dt.\end{aligned} \tag{3.36}$$

Let $a \geq 0$ and $b \geq 0$. From (3.32),

$$\sum_{a \leq n \leq b} f(n)\sigma_s(n) = \int_a^b f(t)\, dD_s(t), \tag{3.37}$$

where we write the sum as a Lebesgue-Stieltjes integral.

For $a = 0$ and $b = \alpha < \frac{1}{2}$, the left-hand side of (3.37) equals 0. Therefore, from (3.28), (3.29), (3.33), (3.34), (3.36), (3.37), and (3.30),

$$-\int_0^\alpha f(t)(\zeta(1-s) + t^s\zeta(1+s))\, dt = \int_0^\alpha f(t)\, d\Delta_s(t) \tag{3.38}$$

$$
\begin{aligned}
&= 2\pi {\sum_{n=1}^{N}}' \frac{\sigma_s(n)}{n^{s/2}} \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{nt}) f(t)\, dt \\
&\quad + \frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}} \int_0^\alpha t^{(s-1)/2} F_{s-1}(4\pi\sqrt{Nt}) f(t)\, dt \\
&\quad - \frac{2\pi}{N^{s/2}} \Delta_s(N) \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{Nt}) f(t)\, dt \\
&\quad + \frac{s\zeta(1+s)}{2\pi} \int_0^\alpha f(t) \frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{s/2} F_s(4\pi\sqrt{tu}) u^{s-1}\, du \right) dt \\
&\quad + 2\pi \int_0^\alpha f(t) \frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{tu}) \Delta_s(u)\, du \right) dt.
\end{aligned}
$$

Using (3.29) twice, we see that

$$
\begin{aligned}
\frac{d}{dt}\left(\int_N^\infty \left(\frac{t}{u}\right)^{s/2} F_s(4\pi\sqrt{tu}) u^{s-1}\, du \right) &= 2\pi \int_N^\infty (tu)^{(s-1)/2} F_{s-1}(4\pi\sqrt{tu})\, du \\
&= t^{s/2-1} u^{s/2} F_s(4\pi\sqrt{tu}) \Big|_N^\infty \\
&= -t^{s/2-1} N^{s/2} F_s(4\pi\sqrt{tN}),
\end{aligned}
\tag{3.39}
$$

where in the last step we use (2.17)–(2.19), and the fact that $\sigma < \frac{1}{2}$. The interchange of differentiation and integration above is justified from (3.31). Denote

$$
I_s(t,N) := 2\pi \int_N^\infty \left(\frac{t}{u}\right)^{(s+2)/2} F_{s+2}(4\pi\sqrt{tu}) \Delta_s(u)\, du. \tag{3.40}
$$

Performing an integration by parts on the last expression on the right-hand side of (3.38) and using (3.39) and (3.40), we find that

$$
\begin{aligned}
&- \int_0^\alpha f(t)(\zeta(1-s) + t^s\zeta(1+s))\, dt - 2\pi {\sum_{n=1}^{N}}' \frac{\sigma_s(n)}{n^{s/2}} \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{nt}) f(t)\, dt \\
&\quad = \frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}} \int_0^\alpha t^{(s-1)/2} F_{s-1}(4\pi\sqrt{Nt}) f(t)\, dt - \frac{2\pi}{N^{s/2}} \Delta_s(N) \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{Nt}) f(t)\, dt \\
&\qquad - \frac{s\zeta(1+s)N^{s/2}}{2\pi} \int_0^\alpha f(t) t^{s/2-1} F_s(4\pi\sqrt{Nt})\, dt + f(\alpha) I_s(\alpha, N) - \int_0^\alpha I_s(t,N) f'(t)\, dt,
\end{aligned}
\tag{3.41}
$$

where in the last step we made use of the fact that for $-\frac{1}{2} < \sigma < \frac{1}{2}$,

$$
\lim_{t\to 0} t^{(s+2)/2} F_{s+2}(4\pi\sqrt{tu}) = 0.
$$

Here again the limit can be moved inside the integral because of (3.31).

Since $\alpha < \frac{1}{2}$, by Lemma 3.3.2, $I_s(t,N) \ll N^{-1/2}$, for all $0 < t \leq \alpha$. Also by hypothesis, f is differentiable, so

$$V_0^\alpha f(t) = \int_0^\alpha |f'(t)|\, dt,$$

where $V_0^\alpha f(t)$ is the total variation of f on the interval $(0, \alpha)$. Therefore the last two terms on the right-hand side of (3.41) are of the form

$$O(N^{-1/2}(|f(\alpha)| + V_0^\alpha f(t))). \tag{3.42}$$

Recall the bound $\Delta_s(N) \ll N^{\frac{1}{2}(1+\sigma)}$ [98, Lemma 7]. From (3.28) and (3.31),

$$\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_0^\alpha t^{s/2}G_s(4\pi\sqrt{Nt})\, dt = \Delta_s(N)\left(\frac{\alpha}{N}\right)^{(s+1)/2} G_{s+1}(4\pi\sqrt{N\alpha}) \ll \alpha^{\frac{\sigma}{2}}\left(\frac{\alpha}{N}\right)^{\frac{1}{4}}. \tag{3.43}$$

Here we also made use of the fact that

$$\lim_{t\to 0} t^{(s+1)/2}G_{s+1}(4\pi\sqrt{Nt}) = 0.$$

Again, from (3.28) and (3.31),

$$\begin{aligned}\frac{N^s\zeta(1+s)+\zeta(1-s)}{N^{(s-1)/2}}\int_0^\alpha t^{(s-1)/2}F_{s-1}(4\pi\sqrt{Nt})\, dt &= \frac{N^s\zeta(1+s)+\zeta(1-s)}{2\pi N^{s/2}}\alpha^{s/2}F_s(4\pi\sqrt{N\alpha}) \qquad (3.44)\\ &\ll \begin{cases} (2\gamma + \log N)(\alpha N)^{-1/4}, & \text{if } s = 0,\\ \\ (\alpha N)^{(2\sigma-1)/4} + \alpha^{(2\sigma-1)/4}N^{(-2\sigma-1)/4}, & \text{if } s \neq 0,\end{cases}\end{aligned}$$

since $\lim_{t\to 0} t^{s/2}F_s(4\pi\sqrt{Nt}) = 0$. Finally,

$$\frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_0^\alpha t^{s/2-1}F_s(4\pi\sqrt{Nt})\, dt = \frac{s\zeta(1+s)N^{s/2}}{2\pi}\left(\int_0^\infty - \int_\alpha^\infty\right) t^{s/2-1}F_s(4\pi\sqrt{Nt})\, dt = I_1 - I_2.$$

Using the functional equation of $\zeta(s)$, namely (5.5), and the formula [98, p. 409, equation 4.65], we find that

$$I_1 = \frac{s\zeta(1+s)N^{s/2}}{2\pi}\int_0^\infty t^{s/2-1}F_s(4\pi\sqrt{Nt})\, dt = -(2\pi)^{-s-1}\sin(\pi s/2)\Gamma(s+1)\zeta(1+s) = \frac{\zeta(-s)}{2}.$$

Using (3.31), we deduce that

$$I_2 = \frac{s\zeta(1+s)N^{s/2}}{2\pi} \int_\alpha^\infty t^{s/2-1} F_s(4\pi\sqrt{Nt})\, dt \ll (\alpha N)^{(2\sigma-1)/4}, \tag{3.45}$$

since $-\frac{1}{2} < \sigma < \frac{1}{2}$. Using (3.42)–(3.45) in (3.41), we find that

$$\begin{aligned} & f(0+)\frac{\zeta(-s)}{2} - \int_0^\alpha f(t)(\zeta(1-s) + t^s\zeta(1+s))\, dt \\ & - 2\pi \sum_{n=1}^{N}{}' \frac{\sigma_s(n)}{n^{s/2}} \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{nt}) f(t)\, dt \\ & = \frac{N^s\zeta(1+s) + \zeta(1-s)}{N^{(s-1)/2}} \int_0^\alpha t^{(s-1)/2} F_{s-1}(4\pi\sqrt{Nt})(f(t) - f(0+))\, dt \\ & - \frac{2\pi}{N^{s/2}} \Delta_s(N) \int_0^\alpha t^{s/2} G_s(4\pi\sqrt{Nt})(f(t) - f(0+))\, dt \\ & - \frac{s\zeta(1+s)}{2\pi} \int_0^\alpha (f(t) - f(0+)) t^{s/2-1} N^{s/2} F_s(4\pi\sqrt{Nt})\, dt \\ & + O((\alpha N)^{(2\sigma-1)/4} + \alpha^{(2\sigma-1)/4} N^{(-2\sigma-1)/4}) + O((2\gamma + \log N)(\alpha N)^{-1/4}). \end{aligned} \tag{3.46}$$

By the second mean value theorem for integrals in the form given in [97, p. 31],

$$\left| \int_a^b f(t)\phi(t)\, dt - f(b) \int_a^b \phi(t)\, dt \right| \le V_a^b f(t) \max_{a \le c < d \le b} \left| \int_c^d \phi(t)\, dt \right|, \tag{3.47}$$

where ϕ is integrable on $[a, b]$.

Recall that $N^\theta \alpha > 1$ for some $0 < \theta < \min\left(1, \frac{1+2\sigma}{1-2\sigma}\right)$. Dividing the interval $(0, \alpha)$ into two sub-intervals $(0, N^{-\theta})$ and $(N^{-\theta}, \alpha)$, applying (3.47), and using an argument like that in (3.44), we see that

$$\begin{aligned} & \frac{N^s\zeta(1+s) + \zeta(1-s)}{N^{(s-1)/2}} \int_{N^{-\theta}}^\alpha t^{(s-1)/2} F_{s-1}(4\pi\sqrt{Nt})(f(t) - f(0+))\, dt \\ & \ll \begin{cases} (2\gamma + \log N) N^{(\theta-1)/4} V_{N^{-\theta}}^\alpha f(t), & \text{if } s = 0, \\ \\ (N^{(1-\theta)(2\sigma-1)/4} + N^{(\theta(1-2\sigma)-(2\sigma+1))/4}) V_{N^{-\theta}}^\alpha f(t), & \text{if } s \neq 0, \end{cases} \end{aligned} \tag{3.48}$$

and

$$\frac{N^s\zeta(1+s) + \zeta(1-s)}{N^{(s-1)/2}} \int_0^{N^{-\theta}} t^{(s-1)/2} F_{s-1}(4\pi\sqrt{Nt})(f(t) - f(0+))\, dt \ll \begin{cases} (2\gamma + \log N) V_0^{N^{-\theta}} f(t), & \text{if } s = 0, \\ \\ V_0^{N^{-\theta}} f(t), & \text{if } s \neq 0. \end{cases}$$

By (3.47) and arguments similar to those in (3.43) and (3.45),

$$\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_{N^{-\theta}}^{\alpha} t^{s/2}G_s(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \ll \alpha^{\frac{\sigma}{2}}\left(\frac{\alpha}{N}\right)^{\frac{1}{4}} V_{N^{-\theta}}^{\alpha} f(t),$$

$$\frac{2\pi}{N^{s/2}}\Delta_s(N)\int_{0}^{N^{-\theta}} t^{s/2}G_s(4\pi\sqrt{Nt})(f(t)-f(0+))\,dt \ll N^{\frac{-2\theta\sigma-1-\theta}{4}} V_0^{N^{-\theta}} f(t),$$

$$\frac{s\zeta(1+s)}{2\pi}\int_{N^{-\theta}}^{\alpha}(f(t)-f(0+))t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{Nt})\,dt \ll N^{\frac{(1-\theta)(2\sigma-1)}{4}} V_{N^{-\theta}}^{\alpha} f(t),$$

and

$$\frac{s\zeta(1+s)}{2\pi}\int_{0}^{N^{-\theta}}(f(t)-f(0+))t^{s/2-1}N^{s/2}F_s(4\pi\sqrt{Nt})\,dt \ll V_0^{N^{-\theta}} f(t). \tag{3.49}$$

Combining (3.48)–(3.49) together with (3.46), we obtain

$$\begin{aligned}
&f(0+)\frac{\zeta(-s)}{2} - \int_0^{\alpha} f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt - 2\pi{\sum_{n=1}^{N}}{}' \frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha} t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt \\
&\ll \begin{cases} (2\gamma+\log N)(V_0^{N^{-\theta}} f(t) + N^{(\theta-1)/4}V_{N^{-\theta}}^{\alpha} f(t)), & \text{if } s=0,\\[2ex] V_0^{N^{-\theta}} f(t) + (N^{(1-\theta)(2\sigma-1)/4} + N^{(\theta(1-2\sigma)-(2\sigma+1))/4})V_{N^{-\theta}}^{\alpha} f(t), & \text{if } s\neq 0,\end{cases}\\
&\ll \begin{cases} (2\gamma+\log N)(V_0^{N^{-\theta}} f(t) + N^{(\theta-1)/4}(|f(\alpha)| + V_0^{\alpha} f(t))), & \text{if } s=0,\\[2ex] V_0^{N^{-\theta}} f(t) + (N^{(1-\theta)(2\sigma-1)/4} + N^{(\theta(1-2\sigma)-(2\sigma+1))/4})(|f(\alpha)| + V_0^{\alpha} f(t)), & \text{if } s\neq 0.\end{cases}
\end{aligned}$$

Furthermore, if $\log x\, V_0^x f(t) \to 0$ as $x\to 0+$ when $s=0$, and if $V_0^x f(t)\to 0$ as $x\to 0+$ when $s\neq 0$, then the assumption $0<\theta<\min\left(1, \frac{1+2\sigma}{1-2\sigma}\right)$ implies that

$$f(0+)\frac{\zeta(-s)}{2} - \int_0^{\alpha} f(t)(\zeta(1-s)+t^s\zeta(1+s))\,dt - 2\pi\sum_{n=1}^{\infty}\frac{\sigma_s(n)}{n^{s/2}}\int_0^{\alpha} t^{s/2}G_s(4\pi\sqrt{nt})f(t)\,dt = 0.$$

This completes the proof of Theorem 3.1.2. $\square$

Chapter 4

Generalization of Entries on page 335 of Ramanujan's lost notebook

4.1 Main results

In this chapter, we establish one-variable generalizations of Entries 1.3.1 and 1.3.2, where the double sums here are also interpreted as $\lim_{N\to\infty}\sum_{m,n\le N}$, instead of as iterated double sums. It is an open problem to determine if the series can be replaced by iterated double series.

As in Entries 1.3.1 and 1.3.2, the series on the left-hand sides of Theorems 4.1.1 and 4.1.2 are finite.

4.1.1 Generalization of Entry 1

Theorem 4.1.1. *Let $\zeta(s,a)$ denote the Hurwitz zeta function. Let $0<\theta<1$. Then, for $|\sigma|<\frac{1}{2}$,*

$$\begin{aligned}\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\sin(2\pi n\theta)}{n^s}&=-x\frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}(\zeta(-s,\theta)-\zeta(-s,1-\theta))\\&-\frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}}(\zeta(1-s,\theta)-\zeta(1-s,1-\theta))+\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)\\&\times\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+\theta)^{\frac{1-s}{2}}}-\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+1-\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-\theta)^{\frac{1-s}{2}}}\right\},\end{aligned}\tag{4.1}$$

where

$$M_\nu(x)=\frac{2}{\pi}K_\nu(x)+\frac{1}{\sin(\pi\nu)}(J_\nu(x)-J_{-\nu}(x))=\frac{2}{\pi}K_\nu(x)+Y_\nu(x)+J_\nu(x)\tan\left(\frac{\pi\nu}{2}\right).\tag{4.2}$$

We show that Entry 1.3.1 is identical with Theorem 4.1.1 when $s=0$. First observe that [3, p. 264, Theorem 12.13]

$$\zeta(0,\theta)=\frac{1}{2}-\theta\tag{4.3}$$

and

$$\lim_{s\to0}(\zeta(1-s,\theta)-\zeta(1-s,1-\theta))=\psi(1-\theta)-\psi(\theta)=\pi\cot(\pi\theta),$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the digamma function. Since, by (2.15), $J_{-1}(x) = -J_1(x)$,

$$\lim_{s\to 0} \sin(\pi s/2) M_{1-s}(x) = J_1(x). \tag{4.4}$$

Now taking the limit as $s \to 0$ on both sides of (4.1) and using (4.3)–(4.4), we obtain Entry 1.3.1.

4.1.2 Generalization of Entry 2

Theorem 4.1.2. *Let $0 < \theta < 1$. Then, for $|\sigma| < \frac{1}{2}$,*

$$\begin{aligned}\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\cos(2\pi n\theta)}{n^s} &= x\frac{\cos(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}(\zeta(-s,\theta)+\zeta(-s,1-\theta)) \\ &- \frac{\sin(\frac{1}{2}\pi s)\Gamma(1-s)}{2(2\pi)^{1-s}}(\zeta(1-s,\theta)+\zeta(1-s,1-\theta)) - \frac{x}{2}\cos\left(\frac{\pi s}{2}\right) \\ &\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{H_{1-s}\left(4\pi\sqrt{mx\,(n+\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+\theta)^{\frac{1-s}{2}}} + \frac{H_{1-s}\left(4\pi\sqrt{mx\,(n+1-\theta)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-\theta)^{\frac{1-s}{2}}}\right\},\end{aligned} \tag{4.5}$$

where

$$H_\nu(x) = \frac{2}{\pi}K_\nu(x) - \frac{1}{\sin(\pi\nu)}(J_\nu(x)+J_{-\nu}(x)) = \frac{2}{\pi}K_\nu(x) + Y_\nu(x) - J_\nu(x)\cot\left(\frac{\pi\nu}{2}\right). \tag{4.6}$$

We demonstrate that Entry 1.3.2 can be obtained from Theorem 4.1.2 as the particular case $s = 0$. First,

$$\begin{aligned}\lim_{s\to 0}\Gamma(-s)(\zeta(-s,\theta)+\zeta(-s,1-\theta)) &= \lim_{s\to 0}(-s)\Gamma(-s)\frac{(\zeta(-s,\theta)+\zeta(-s,1-\theta))}{-s} \\ &= \zeta'(0,\theta)+\zeta'(0,1-\theta) \\ &= -\log(2\sin(\pi\theta)),\end{aligned} \tag{4.7}$$

where we used the fact that $\zeta'(0,\theta) = \log(\Gamma(\theta)) - \frac{1}{2}\log(2\pi)$ [11]. Second, since $s = 1$ is a simple pole of $\zeta(s,\theta)$ with residue 1, then

$$\lim_{s\to 0}\sin(\pi s/2)(\zeta(1-s,\theta)+\zeta(1-s,1-\theta)) = \lim_{s\to 0}\frac{\sin(\pi s/2)}{s}s(\zeta(1-s,\theta)+\zeta(1-s,1-\theta)) = -\pi.$$

Third, by (2.16),

$$\lim_{s\to 0}\frac{1}{2\sin(\pi s/2)}(J_{1-s}(x)+J_{s-1}(x)) = -Y_1(x). \tag{4.8}$$

Taking the limit as $s \to 0$ in (4.5) while using (4.7)–(4.8), we obtain Entry 1.3.2.

4.2 Preliminary Results

Let us define the generalized twisted divisor sum by

$$\sigma_s(\chi, n) := \sum_{d|n} \chi(d) d^s, \tag{4.9}$$

which, for Re $z > \max\{1, 1+\sigma\}$, has the generating function

$$\zeta(z) L(z-s, \chi) = \sum_{n=1}^{\infty} \frac{\sigma_s(\chi, n)}{n^z}.$$

The following lemma from the papers of Voronoï [92] and Oppenheim [76] is instrumental in proving our main theorems.

Lemma 4.2.1. *If $x > 0$, $x \notin \mathbb{Z}$, and $-\frac{1}{2} < \sigma < \frac{1}{2}$, then*

$$\sum_{n \le x}{}' \sigma_{-s}(n) = -\cos(\tfrac{1}{2}\pi s) \sum_{n=1}^{\infty} \sigma_{-s}(n) \left(\frac{x}{n}\right)^{\frac{1-s}{2}} H_{1-s}\left(4\pi\sqrt{nx}\right) + xZ(s, x) - \frac{1}{2}\zeta(s),$$

where $H_\nu(x)$ is defined in (4.6), *and where*

$$Z(s, x) = \begin{cases} \zeta(1+s) + \dfrac{\zeta(1-s)}{1-s} x^{-s}, & \text{if } s \neq 0, \\ \log x + 2\gamma - 1, & \text{if } s = 0, \end{cases} \tag{4.10}$$

is analytic for all s.

From the definition (4.6) of H_ν and (4.8), we find that

$$H_1(4\pi\sqrt{nx}) = Y_1(4\pi\sqrt{nx}) + \frac{2}{\pi} K_1(4\pi\sqrt{nx}) = -I_1(4\pi\sqrt{nx}).$$

Note that, it is not difficult to show that

$$\lim_{s \to 0} Z(s, x) = \log x + 2\gamma - 1 = Z(0, x). \tag{4.11}$$

Recall that the Laurent series expansion of $\zeta(s)$ near the pole $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n \gamma_n (s-1)^n}{n!},$$

where γ_n, $n \geq 1$, are the Stieltjes constants defined by [9]

$$\gamma_n = \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{\log^n k}{k} - \frac{\log^{n+1} N}{n+1}\right).$$

Thus, by (4.10), for $s > 0$,

$$Z(s,x) = \frac{s-1+x^{-s}}{s(s-1)} + \gamma\left(1 - \frac{x^{-s}}{s-1}\right) + \sum_{n=1}^{\infty}\frac{(-1)^n\gamma_n s^n}{n!} + \frac{x^{-s}}{1-s}\sum_{n=1}^{\infty}\frac{\gamma_n s^n}{n!}.$$

Hence,

$$\lim_{s\to 0} Z(s,x) = \lim_{s\to 0}\frac{s-1+x^{-s}}{s(s-1)} + 2\gamma = -\lim_{s\to 0}(1 - \log x x^{-s}) + 2\gamma = \log x + 2\gamma - 1,$$

which proves (4.11).

Lemma 4.2.2. *Let $F(x)$ be defined by (1.14). For each character χ modulo q, where q is prime, define the Gauss sum*

$$\tau(\chi) = \sum_{n \,(\mathrm{mod}\, q)} \chi(n) e^{2\pi i n/q}. \tag{4.12}$$

If $0 < a < q$ and $(a,q) = 1$, then, for any complex number s,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi n a}{q}\right) n^s = -iq^s \sum_{\substack{d|q\\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi \text{ odd}}} \chi(a)\tau(\bar{\chi}) \sideset{}{'}\sum_{1\leq n\leq dx/q} \sigma_s(\chi,n),$$

where $\phi(n)$ denotes Euler's ϕ-function.

Proof. First, we see that

$$\sideset{}{'}\sum_{n\leq x}\sigma_s(n) = \sideset{}{'}\sum_{n\leq x}\sum_{d|n} d^s = \sum_{d\leq x} d^s \sideset{}{'}\sum_{m=1}^{\lfloor x/d\rfloor} 1 = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) n^s. \tag{4.13}$$

Similarly, for any Dirichlet character χ modulo q,

$$\sideset{}{'}\sum_{n\leq x}\sigma_s(\chi,n) = \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\chi(n) n^s, \tag{4.14}$$

where $\sigma_s(\chi,n)$ is defined in (4.9). We have

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi n a}{q}\right) n^s = \sum_{n=1}^{\infty}\sum_{d|q}\sum_{(n,q)=q/d} F\left(\frac{x}{n}\right)\sin\left(\frac{2\pi n a}{q}\right) n^s$$

$$= \sum_{d|q} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sin\left(\frac{2\pi ma}{d}\right) \left(\frac{qm}{d}\right)^s$$
$$= \sum_{\substack{d|q \\ d>1}} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \sin\left(\frac{2\pi ma}{d}\right) \left(\frac{qm}{d}\right)^s.$$

Now using the fact [17, p. 72, Lemma 2.5]

$$\sin\left(\frac{2\pi ma}{d}\right) = \frac{1}{i\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \chi(a)\tau(\bar{\chi})\chi(m), \tag{4.15}$$

we find that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \sin\left(\frac{2\pi na}{q}\right) n^s = \sum_{\substack{d|q \\ d>1}} \frac{1}{i\phi(d)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \left(\frac{qm}{d}\right)^s \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \tau(\bar{\chi})\chi(m)\chi(a)$$
$$= -iq^s \sum_{\substack{d|q \\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ odd}}} \tau(\bar{\chi})\chi(a) {\sum_{n\leq dx/q}}' \sigma_s(\chi, n),$$

as can be seen from (4.14). This completes the proof of Lemma 4.2.2. □

Lemma 4.2.3. *If $0 < a < q$ and $(a,q) = 1$, then, for any complex number s,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos\left(\frac{2\pi na}{q}\right) n^s = q^s {\sum_{1\leq n\leq x/q}}' \sigma_s(n) + q^s \sum_{\substack{d|q \\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \chi(a)\tau(\bar{\chi}) {\sum_{1\leq n\leq dx/q}}' \sigma_s(\chi, n).$$

Proof. We have

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos\left(\frac{2\pi na}{q}\right) n^s = \sum_{n=1}^{\infty} \sum_{d|q} \sum_{(n,q)=q/d} F\left(\frac{x}{n}\right) \cos\left(\frac{2\pi na}{q}\right) n^s$$
$$= \sum_{d|q} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \cos\left(\frac{2\pi ma}{d}\right) \left(\frac{qm}{d}\right)^s$$
$$= \sum_{m=1}^{\infty} F\left(\frac{x}{qm}\right) (qm)^s + \sum_{\substack{d|q \\ d>1}} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) \cos\left(\frac{2\pi na}{d}\right) \left(\frac{qm}{d}\right)^s.$$

Invoking (4.13) and (4.15) above, we find that

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \cos\left(\frac{2\pi na}{q}\right) n^s = q^s {\sum_{n\leq x/q}}' \sigma_s(n) + q^s \sum_{\substack{d|q \\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{m=1 \\ (m,d)=1}}^{\infty} F\left(\frac{dx}{qm}\right) m^s \sum_{\substack{\chi \bmod d \\ \chi \text{ even}}} \tau(\bar{\chi})\chi(a)\chi(m)$$

$$= q^s {\sum_{n\le x/q}}' \sigma_s(n) + q^s \sum_{\substack{d|q\\ d>1}} \frac{1}{d^s\phi(d)} \sum_{\substack{\chi \bmod d\\ \chi \text{ even}}} \tau(\bar{\chi})\chi(a) {\sum_{n\le dx/q}}' \sigma_s(\chi, n).$$

Thus, we have finished the proof of Lemma 4.2.3. □

We need a lemma from [23, p. 5, Lemma 1].

Lemma 4.2.4. *Let σ_a denote the abscissa of absolute convergence for*

$$\phi(s) := \sum_{n=1}^{\infty} a_n \lambda_n^{-s}.$$

Then for $k \ge 0$, $\sigma > 0$, and $\sigma > \sigma_a$,

$$\frac{1}{\Gamma(k+1)} {\sum_{\lambda_n \le x}}' a_n (x-\lambda_n)^k = \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(s)\phi(s)x^{s+k}}{\Gamma(s+k+1)}\, ds,$$

where the prime $'$ on the summation sign indicates that if $k = 0$ and $x = \lambda_m$ for some positive integer m, then we count only $\frac{1}{2}a_m$.

We recall the following version of the Phragmén-Lindelöf theorem [66, p. 109].

Lemma 4.2.5. *Let f be holomorphic in a strip S given by $a < \sigma < b$, $|t| > \eta > 0$, and continuous on the boundary. If for some constant $\theta < 1$,*

$$f(s) \ll \exp(e^{\theta\pi|s|/(b-a)}),$$

uniformly in S, $f(a+it) = o(1)$, and $f(b+it) = o(1)$ as $|t| \to \infty$, then $f(\sigma+it) = o(1)$ uniformly in S as $|t| \to \infty$.

We also need two lemmas, proven by K. Chandrasekharan and R. Narasimhan [23, Corollaries 1 and 2, p. 11] (see also Berndt [5, Lemmas 12 and 13]), that are based on results of A. Zygmund [99] for equi-convergent series. We recall that two series

$$\sum_{j=-\infty}^{\infty} a_j(x) \quad \text{and} \quad \sum_{j=-\infty}^{\infty} b_j(x)$$

are uniformly equi-convergent on an interval if

$$\sum_{j=-n}^{n} [a_j(x) - b_j(x)]$$

converges uniformly on that interval as $n \to \infty$ [5, Definition 5].

Lemma 4.2.6. *Let a_n be a positive strictly increasing sequence of numbers tending to ∞, and suppose that $a_n = a_{-n}$. Suppose that J is a closed interval contained in an interval I of length 2π. Assume that*

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty.$$

Then, if g is a function with period 2π which equals

$$\sum_{n=-\infty}^{\infty} c_n e^{ia_n x}$$

on I, the Fourier series of g converges uniformly on J.

Lemma 4.2.7. *With the same notation as Lemma 4.2.6, assume that*

$$\sup_{0\leq h\leq 1} \left| \sum_{k<a_n<k+h} c_n \right| = o(1),$$

as $k \to \infty$, and

$$\sum_{n=-\infty}^{\infty} \frac{|c_n|}{a_n} < \infty.$$

Let $A(x)$ be a $\mathcal{C}^\infty$ function with compact support on I, which is equal to 1 on J. Furthermore, let $B(x)$ be a $\mathcal{C}^\infty$ function. Then, the series

$$B(x) \sum_{n=-\infty}^{\infty} c_n e^{ia_n x}$$

is uniformly equi-convergent on J with the differentiated series of the Fourier series of a function with period 2π, which equals

$$A(x) \sum_{n=-\infty}^{\infty} c(n) W_n(x)$$

on I, where $W_n(x)$ is an antiderivative of $B(x)e^{ia_n x}$.

Let the Fourier series of any function f defined, say, in the interval $(-\pi, \pi)$, be

$$S[f] := \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

The following result of Zygmund [100, Theorem 6.6, p. 53] expresses the Riemann-Lebesgue localization principle.

Lemma 4.2.8. *If two functions f_1 and f_2 are equal in an interval I, then $S[f_1]$ and $S[f_2]$ are uniformly equi-convergent in any interval I' interior to I.*

For each integer λ define

$$\tilde{G}_{\lambda+s}(z) := J_{\lambda+s}(z)\cos\left(\frac{\pi s}{2}\right) - \left(Y_{\lambda+s}(z) - (-1)^{\lambda}\frac{2}{\pi}K_{\lambda+s}(z)\right)\sin\left(\frac{\pi s}{2}\right). \tag{4.16}$$

By (3.25), (3.27), and (3.26),

$$\frac{d}{dx}\left(\frac{x}{u}\right)^{(1+k-s)/2}\sigma_s(n)\tilde{G}_{1+k-s}(4\pi\sqrt{xu}) = 2\pi\left(\frac{x}{u}\right)^{\frac{k-s}{2}}\sigma_s(n)\tilde{G}_{k-s}(4\pi\sqrt{xu}). \tag{4.17}$$

Let us consider the Dirichlet series $\sum_{n=1}^{\infty} a_n\mu_n^{-s}$ with abscissa of absolute convergence σ_a and

$$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$$

For $y > 0$ and $\nu = \lambda + s$, define

$$\tilde{F}_{\nu}(y) := \sum_{n=1}^{\infty} a_n\left(\frac{qy^2}{\mu_n}\right)^{\nu/2}\tilde{G}_{\nu}\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right)$$

and

$$F_{\nu}(y) := \sum_{n=1}^{\infty} a_n\left(\frac{qy^2}{\mu_n}\right)^{\nu/2}G_{\nu}\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right),$$

where $G_{\lambda+s}(z)$ is defined in (3.23). Suppose that

$$\sum_{n=1}^{\infty}\frac{|a_n|}{\mu_n^{\frac{\nu}{2}+\frac{3}{4}}} < \infty \tag{4.18}$$

and

$$\sup_{0\le h\le 1}\left|\sum_{m^2<\mu_n\le(m+h)^2}\frac{a_n}{\mu_n^{\frac{\nu}{2}+\frac{1}{4}}}\right| = o(1), \tag{4.19}$$

as $m \to \infty$.

The following lemma is similar to Theorem II in [23] and Lemma 14 in [5].

Lemma 4.2.9. *The function $2y\tilde{F}_\nu(y)$ is uniformly equi-convergent on any interval J of length less than* 1 *with the differentiated series of the Fourier series of a function with period* 1*, which on I equals $A(y)\tilde{F}_{\nu+1}(y)$, where I is of length* 1 *and contains J. Moreover, $\tilde{F}_\nu(y)$ is a continuous function.*

Proof. We examine the function

$$\begin{aligned} f(y) :=& 2q^{\nu/2}y^{1+\nu}\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}\left\{\tilde{G}_\nu\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right)\right. \\ &-\frac{q^{1/4}}{\pi\mu_n^{1/4}(2y)^{1/2}}\left(\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right) \\ &\left.-\frac{q^{3/4}}{2\pi^2\mu_n^{3/4}y^{3/2}}\left(\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_1+\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_1'\right)\right\}, \end{aligned} \tag{4.20}$$

where d_0, d_0', d_1, and d_1' are constants. Since $y > 0$, then by the definition (4.16), (2.17), (2.18), (2.19), and (4.18), the function $f(y)$ in (4.20) is a continuously differentiable function. Let g be a function with period 1 which equals f on I. Since f is continuously differentiable, the Fourier series of g is uniformly convergent on J. By the hypothesis (4.18), (4.19), and Lemma 4.2.7, the series

$$2q^{\nu/2}y^{1+\nu}\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}\frac{q^{1/4}}{\pi\mu_n^{1/4}(2y)^{1/2}}\left(\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right)$$

is uniformly equi-convergent on J with the derived series of the Fourier series of a function that is of period 1 and equals on I,

$$\begin{aligned} &A(y)\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}\int_{\alpha}^{y}2q^{\nu/2}t^{1+\nu}\frac{q^{1/4}}{\pi\mu_n^{1/4}(2t)^{1/2}} \\ &\times\left(\cos\left(4\pi t\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi t\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right)dt, \end{aligned} \tag{4.21}$$

for some $\alpha > 0$. Using Lemma 4.2.6, we can prove a result similar to that of (4.21) for the series

$$2q^{\nu/2}y^{1+\nu}\sum_{n=1}^{\infty}\left(\frac{a_n}{\mu_n}\right)^{\nu/2}\frac{q^{3/4}}{2\pi^2\mu_n^{3/4}(y)^{3/2}}\left(\cos\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0+\sin\left(4\pi y\sqrt{\frac{\mu_n}{q}}-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)d_0'\right).$$

Hence, the series

$$2y\sum_{n=1}^{\infty}a_n\left(\frac{qy^2}{\mu_n}\right)^{\nu/2}\tilde{G}_\nu\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right)$$

is uniformly equi-convergent on J with the derived series of the Fourier series of a function that is of period

1 and equals on I,

$$A(y)\sum_{n=1}^{\infty} a_n \int_0^y 2t\left(\frac{qt^2}{\mu_n}\right)^{\nu/2} \tilde{G}_\nu\left(4\pi t\sqrt{\frac{\mu_n}{q}}\right)\,dt = \frac{A(y)}{2\pi}\sum_{n=1}^{\infty} a_n \left(\frac{qy^2}{\mu_n}\right)^{(\nu+1)/2} \tilde{G}_{\nu+1}\left(4\pi y\sqrt{\frac{\mu_n}{q}}\right).$$

In the last step we use (4.17). This completes the proof of the lemma.

□

The following lemma is proved by the same kind of argument.

Lemma 4.2.10. *The function $2yF_\nu(y)$ is uniformly equi-convergent on any interval J of length less than 1 with the differentiated series of the Fourier series of a function with period 1, which on I equals $A(y)F_{\nu+1}(y)$, where I is of length 1 and contains J. Moreover, $F_\nu(y)$ is a continuous function.*

4.3 Proof of the generalization of Entry 1

We prove the theorem under the assumption that the double series on the right-hand sides of (4.1) and (4.5) are summed symmetrically, i.e., the product mn of the indices of summation tends to ∞. Under this assumption, we prove that the double series in (4.1) and (4.5) are uniformly convergent with respect to θ on any compact subinterval of $(0, 1)$. By continuity, it is sufficient to prove the theorem for all primes q and all fractions $\theta = a/q$, where $0 < a < q$. Therefore for these values of θ, Theorem 4.1.1 is equivalent to the following theorem.

Theorem 4.3.1. *Recall that M_ν is defined in (4.2). Let q be a prime and $0 < a < q$. Let*

$$\begin{aligned} L_s(a,q,x) = {} & -\frac{x}{2}\sin\left(\frac{\pi s}{2}\right) \\ & \times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+a/q)}\right)}{(mx)^{\frac{1+s}{2}}(n+a/q)^{\frac{1-s}{2}}} - \frac{M_{1-s}\left(4\pi\sqrt{mx\,(n+1-a/q)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-a/q)^{\frac{1-s}{2}}}\right\}, \end{aligned} \tag{4.22}$$

where $M_s(z)$ is defined in (4.2). Then, for $|\sigma| < \frac{1}{2}$,

$$\begin{aligned} L_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\frac{\sin(2\pi na/q)}{n^s} = {} & -x\frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}}\left(\zeta(-s,\frac{a}{q}) - \zeta\left(-s,1-\frac{a}{q}\right)\right) \\ & - \frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}}\left(\zeta\left(1-s,\frac{a}{q}\right) - \zeta\left(1-s,1-\frac{a}{q}\right)\right), \end{aligned}$$

where $\zeta(s, a)$ denotes the Hurwitz zeta function.

First we need the following theorem.

Theorem 4.3.2. *If χ is a non-principal odd primitive character modulo q, $x > 0$, $|\sigma| < 1/2$, and k is a non-negative integer, then*

$$\begin{aligned}
&\frac{1}{\Gamma(k+1)} {\sum_{n\le x}}' \sigma_{-s}(\chi, n)(x-n)^k \\
&= \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)} - \frac{x^k L(s,\chi)}{2\Gamma(k+1)} + 2\sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi) \\
&\quad + \frac{i}{\tau(\bar{\chi})(2\pi)^k} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi}, n) \left(\frac{qx}{n}\right)^{\frac{1-s+k}{2}} \tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),
\end{aligned}$$

where $\tilde{G}_{\lambda-s}(z)$ is defined in (4.16). *The series on the right-hand side converges uniformly on any interval for $x > 0$, where the left-hand side is continuous. The convergence is bounded on any interval* $0 < x_1 \le x \le x_2 < \infty$ *when* $k = 0$.

Proof. From (4.9) and Lemma 4.2.4, for a fixed $x > 0$, we see that

$$\frac{1}{\Gamma(k+1)} {\sum_{n\le x}}' \sigma_{-s}(\chi, n)(x-n)^k = \frac{1}{2\pi i} \int_{(c)} \zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} dw, \tag{4.23}$$

where $\max\{1, 1-\sigma, \sigma\} < c < 1$ and $k \ge 0$. Consider the positively oriented rectangular contour R with vertices $[c \pm iT, 1-c \pm iT]$. Observe that the integrand on the right-hand side of (4.23) has poles at $w = 1$ and $w = 0$ inside the contour R. By the residue theorem,

$$\begin{aligned}
&\frac{1}{2\pi i} \int_R \zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} dw \\
&= R_1\left(\zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right) + R_0\left(\zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right),
\end{aligned} \tag{4.24}$$

where we recall that $R_a(f(w))$ denotes the residue of the function $f(w)$ at the pole $w = a$. Straightforward computations show that

$$R_0\left(\zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right) = \frac{\zeta(0)L(s,\chi)x^{1+k}}{\Gamma(k+1)} \tag{4.25}$$

and

$$R_1\left(\zeta(w) L(w+s,\chi) \frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\right) = \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}. \tag{4.26}$$

We show that the contribution from the integrals along the horizontal sides $(\sigma \pm iT, 1-c \le \sigma \le c)$ on the left-hand side of (4.24) tends to zero as $|t| \to \infty$. We prove this fact by showing that

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1),$$

as $|\text{Im } w| \to \infty$, uniformly for $1-c \leq \text{ Re } w < c$. The functional equation for $L(s,\chi)$ for an odd primitive Dirichlet character χ is given by [31, p. 69]

$$\left(\frac{\pi}{q}\right)^{-\frac{1+s}{2}} \Gamma\left(\frac{1+s}{2}\right) L(s,\chi) = \frac{i\tau(\chi)}{\sqrt{q}} \left(\frac{\pi}{q}\right)^{-\frac{2-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s,\overline{\chi}), \tag{4.27}$$

where $\tau(\chi)$ is the Gauss sum defined in (4.12). Combining the functional equation (5.5) of $\zeta(w)$ and the functional equation (4.27) of $L(w+s,\chi)$ for odd primitive χ, we deduce the functional equation

$$\zeta(w)L(w+s,\chi) = \frac{i\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi}), \tag{4.28}$$

where

$$\eta(w,s) = \frac{\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(2-w-s)\right)}{\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(1+w+s)\right)}.$$

Since $\sigma < \frac{1}{2}$,

$$\zeta(c+it)L(c+it+s,\chi) = O(1),$$

as $|t| \to \infty$. Using (1.39), we see that

$$\frac{\Gamma(w)}{\Gamma(w+k+1)} = O(|\text{Im } w|^{-1-k}), \tag{4.29}$$

uniformly in $1-c \leq \text{ Re } w < c$, as $|\text{Im } w| \to \infty$. Therefore, for $w = c+it$,

$$\zeta(w)L(w,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1), \tag{4.30}$$

as $|t| \to \infty$. Again, using Stirling's formula (1.39) for the Gamma function and the relation (4.28), we find that, for $w = 1-c+it$,

$$\begin{aligned}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} &= \frac{i\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi})\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} \\ &= O_{q,s}(t^{2c-\sigma-k-2})\end{aligned} \tag{4.31}$$

$$= o(1),$$

as $|t| \to \infty$, provided that $k > 2c - \sigma - 2$. From (4.29) and [31, pp. 79, 82, equations (2),(15)],

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} \ll_q \exp\left(C|w|\log|w|\right), \tag{4.32}$$

for some constant C and $|\text{Im } w| \to \infty$. Since the function on the left-hand side of (4.32) is holomorphic for $|\text{Im } w| > \eta' > 0$, then, by using (4.30), (4.31), (4.32), and Lemma 4.2.5, we deduce that

$$\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1),$$

uniformly for $1 - c \leq$ Re $w \leq c$ and $|\text{Im } w| \to \infty$. Therefore,

$$\int_{c\pm iT}^{1-c\pm iT} \zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)} = o(1), \tag{4.33}$$

as $T \to \infty$. Using the evaluation $\zeta(0) = -\frac{1}{2}$ and combining (4.23), (4.24), (4.25), (4.26), and (4.33), we deduce that

$$\begin{aligned}\frac{1}{\Gamma(k+1)}{\sum_{n\leq x}}' \sigma_{-s}(\chi,n)(x-n)^k = \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)} - \frac{L(s,\chi)x^k}{2\Gamma(k+1)} \\ + \frac{1}{2\pi i}\int_{(1-c)} \zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw,\end{aligned} \tag{4.34}$$

provided that $k \geq 0$ and $k > 2c - \sigma - 2$. Define

$$I(y) := \frac{1}{2\pi i}\int_{(1-c)} \frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)} y^w \, dw. \tag{4.35}$$

Using the functional equation (4.28) in the integrand on the right-hand side of (4.34) and inverting the order of summation and integration, we find that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1-c)} \zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\, dw \\ &= \frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)} \frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\zeta(1-w)L(1-w-s,\bar{\chi})\left(\frac{\pi^2 x}{q}\right)^w dw \\ &= \frac{ix^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)} \frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2 x}{q}\right)^w \sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1-w}}\, dw\end{aligned} \tag{4.36}$$

$$
\begin{aligned}
&= \frac{ix^k \pi^{s-1}}{\tau(\bar{\chi})q^{s-1}} \sum_{n=1}^{\infty} \frac{\sigma_s(\bar{\chi}, n)}{n^{1+k}} \frac{1}{2\pi i} \int_{(1-c)} \frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)} \left(\frac{\pi^2 nx}{q}\right)^w \, dw \\
&= \frac{ix^k \pi^{s-1}}{\tau(\bar{\chi})q^{s-1}} \sum_{n=1}^{\infty} \frac{\sigma_s(\bar{\chi}, n)}{n} I\left(\frac{\pi^2 nx}{q}\right),
\end{aligned}
$$

provided that $k > 2c - \sigma - 1$. We compute the integral $I(y)$ by using the residue calculus, shifting the line of integration to the right, and letting $c \to -\infty$.

Let k be a positive integer and $\sigma \neq 0$. From (4.35), we can write

$$
I(y) := \frac{1}{2\pi i} \int_{(1-c)} F(w) \, dw,
$$

where

$$
F(w) := \frac{\Gamma(w)\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(2-w-s)\right) y^w}{\Gamma(1+k+w)\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(1+w+s)\right)}.
$$

Note that the poles of the function $F(w)$ on the right side of the line $1 - c + it, -\infty < t < \infty$, are at $w = 2m+1$ and $w = 2m+2-s$ for $m = 0, 1, 2, \ldots$. Thus,

$$
R_{2m+1}(F(w)) = (-1)^{m+1} \frac{2\Gamma(2m+1)\Gamma\left(-m - \frac{1}{2}(s-1)\right) y^{2m+1}}{m!\Gamma(2+k+2m)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(1+m+\frac{1}{2}(s)\right)}
$$

and

$$
R_{2m+2-s}(F(w)) = (-1)^{m+1} \frac{2\Gamma(2m+2-s)\Gamma\left(-m+\frac{1}{2}(s-1)\right) y^{2m+2-s}}{m!\Gamma(3+k+2m-s)\Gamma\left(m+\frac{1}{2}(2-s)\right)\Gamma\left(m+\frac{3}{2}\right)}.
$$

With the aid of the duplication formula (1.32) and the reflection formula (1.30) for $\Gamma(s)$, we find that

$$
R_{2m+1}(F(w)) = -\frac{2^{s-1}}{\cos(\pi s/2)} \frac{(2\sqrt{y})^{4m+2}}{(2m+k+1)!\Gamma(2m+s+1)} \tag{4.37}
$$

and

$$
R_{2m+2-s}(F(w)) = \frac{2(2y)^{2-s}}{\cos(\pi s/2)} \frac{(2\sqrt{y})^{4m}}{(2m+1)!\Gamma(2m+k+3-s)}. \tag{4.38}
$$

Now from [93, pp. 77–78], we recall that the modified Bessel function $I_\nu(z)$ is defined by

$$
I_\nu(z) := \sum_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \tag{4.39}
$$

and that $K_\nu(z)$ can be represented as

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}. \tag{4.40}$$

(We emphasize that the definition of $I_\nu(z)$ given in (4.39) should not be confused with the definition of $I_\nu(z)$ given by Ramanujan in (1.17).) Therefore, from (2.15), (4.39), and (4.37), for k even,

$$\begin{aligned}
\sum_{m=0}^{\infty} R_{2m+1}(F(w)) &= -\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2k+2}}{(2m+k+1)!\Gamma(2m+1+s)} \qquad (4.41)\\
&= -\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\left\{\sum_{m=0}^{\infty}\frac{(2\sqrt{y})^{4m+2k+2}}{(2m+1)!\Gamma(2m+1+s-k)}\right.\\
&\quad\left.-\sum_{m=1}^{k/2}\frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)}\right\}\\
&= -\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) - J_{-1+s-k}(4\sqrt{y}))\\
&\quad+\frac{2^{s-1-2k}y^{-k}}{\cos(\pi s/2)}\sum_{m=1}^{k/2}\frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)}\\
&= -\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) - J_{-1+s-k}(4\sqrt{y}))\\
&\quad+\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{k/2}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.
\end{aligned}$$

Similarly, for k odd

$$\begin{aligned}
\sum_{m=0}^{\infty} R_{2m+1}(F(w)) &= -\frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(I_{-1+s-k}(4\sqrt{y}) + J_{-1+s-k}(4\sqrt{y})) \qquad (4.42)\\
&\quad+\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{(k+1)/2}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.
\end{aligned}$$

From (4.38), (2.15), and (4.39), we find that

$$\sum_{m=0}^{\infty} R_{2m+1-s}(F(w)) = \frac{2^{-1-k}y^{(1-s-k)/2}}{\cos(\pi s/2)}(-J_{1-s+k}(4\sqrt{y}) + I_{1-s+k}(4\sqrt{y})). \tag{4.43}$$

Invoking (4.40) in the sum of (4.41), (4.42), and (4.43), we deduce that

$$\begin{aligned}
\sum_{m=0}^{\infty}(R_{2m+1}(F(w)) + R_{2m+1-s}(F(w))) &= -\frac{\sin(\pi s/2)}{2^k y^{(-1+s+k)/2}}\\
\times\left(\frac{J_{1-s+k}(4\sqrt{y}) + (-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin\pi s} - (-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right)&
\end{aligned}$$

$$+\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.$$

Consider the positively oriented contour $\mathcal{R}_N$ formed by the points $\{1-c-iT, 2N+\frac{3}{2}-iT, 2N+\frac{3}{2}+iT, 1-c+iT\}$, where $T>0$ and N is a positive integer. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{R}_N}F(w)\,dw=\sum_{0\le k\le N}R_{2k+1}(F(w))+\sum_{0\le k\le N}R_{2k+1-s}(F(w)). \tag{4.44}$$

Recall Stirling's formula in the form [31, p. 73, equation (5)]

$$\Gamma(s)=\sqrt{2\pi}e^{-s}s^{s-1/2}e^{f(s)},$$

for $-\pi<\arg s<\pi$ and $f(s)=O(1/|s|)$, as $|s|\to\infty$. Therefore, for fixed $T>0$ and $\sigma\to\infty$,

$$\Gamma(s)=O\left(e^{-\sigma+(\sigma-1/2)\log\sigma}\right). \tag{4.45}$$

Hence, for the integral over the right side of the rectangular contour $\mathcal{R}_N$,

$$\int_{2N+3/2-iT}^{2N+3/2+iT}F(w)\,dw\ll_{T,s}y^{2N+3/2}e^{4N-(4N+2+k+\sigma)\log N}=o(1), \tag{4.46}$$

as $N\to\infty$. Using Stirling's formula (1.39) to estimate the integrals over the horizontal sides of $\mathcal{R}_N$, we find that

$$\int_{1-c\pm iT}^{\infty\pm iT}F(w)\,dw<<_s\int_{1-c}^{\infty}y^{\sigma}T^{-2\beta-\sigma-k}\,d\sigma<<_{s,y}\frac{y^{1-c}}{T^{2c-\sigma-k-2}\log T}=o(1), \tag{4.47}$$

provided that $k>2c-\sigma-2$. Using (4.44), (4.46), and (4.47) in (4.35), we deduce that

$$I(y)=\frac{\sin(\pi s/2)}{2^k y^{(-1+s+k)/2}}\left(\frac{J_{1-s+k}(4\sqrt{y})+(-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin\pi s}\right.$$
$$\left.-(-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right)-\frac{2^{s+1}}{\cos(\pi s/2)}\sum_{m=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}. \tag{4.48}$$

Using the functional equation (4.27), the reflection formula (1.30), and the duplication formula (1.32), for

$y = \pi^2 nx/q$, we find that

$$\sum_{n=1}^{\infty} \frac{\sigma_s(\chi,n)}{n} \left\{ \frac{2^{s+1}}{\cos(\pi s/2)} \sum_{m=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{2^{-4m} y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)} \right\} \tag{4.49}$$
$$= 2i\tau(\bar{\chi}) \frac{\pi^{1-s}}{q^{1-s}} \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{n-1} \frac{x^{-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi).$$

With the aid of (2.16), we see that

$$\sin(\pi s/2) \left(\frac{J_{1-s+k}(4\sqrt{y}) + (-1)^{k+1} J_{-1+s-k}(4\sqrt{y})}{\sin \pi s} - (-1)^{k+1} \frac{2}{\pi} K_{1-s+k}(4\sqrt{y}) \right) \tag{4.50}$$
$$= \tilde{G}_{1+k-s}(4\sqrt{y}).$$

Combining (4.34), (4.36), (4.48), and (4.49), we see that

$$\frac{1}{\Gamma(k+1)} {\sum_{n \le x}}' \sigma_{-s}(\chi,n)(x-n)^k = \frac{x^{k+1} L(1+s,\chi)}{\Gamma(k+2)} - \frac{L(s,\chi)x^k}{2\Gamma(k+1)} \tag{4.51}$$
$$+ 2 \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{n-1} \frac{x^{k-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi)$$
$$+ \frac{i}{\tau(\bar{\chi})(2\pi)^k} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi},n) \left(\frac{xq}{n}\right)^{\frac{1-s+k}{2}} \tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

provided that $k \ge 0$, $\sigma \ne 0$, and $k > 2c - \sigma - 1$.

For $x > 0$ fixed, by the asymptotic expansions for Bessel functions (2.17), (2.18), and (2.19), there exists a sufficiently large integer N_0 such that

$$\tilde{G}_{1+k-s}(4\pi\sqrt{\frac{nx}{q}}) \ll_q \frac{1}{(nx)^{1/4}},$$

for all $n > N_0$. Hence, for $x > 0$,

$$\sum_{n > N_0} \left(\frac{qx}{n}\right)^{\frac{1+k-s}{2}} \sigma_s(n) \tilde{G}_{1+k-s}\left(4\pi\sqrt{\frac{nx}{q}}\right) \ll_q x^{\frac{2k-2\sigma-1}{4}} \sum_{n > N_0} \frac{\sigma_\sigma(n)}{n^{\frac{2k-2\sigma+3}{4}}} \ll_q x^{\frac{2k-2\sigma-1}{4}},$$

provided that $k > |\sigma| + \frac{1}{2}$. Therefore, for $k > |\sigma| + \frac{1}{2}$ and $x > 0$, the series

$$\sum_{n=1}^{\infty} \left(\frac{qx}{n}\right)^{\frac{1+k-s}{2}} \sigma_s(n) \tilde{G}_{1+k-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)$$

is absolutely and uniformly convergent for $0 < x_1 \le x \le x_2 < \infty$. Thus, by differentiating a suitable number of times with the aid of (4.17), we find that (4.51) may be then upheld for $k > |\sigma| + \frac{1}{2}$. Since $|\sigma| < \frac{1}{2}$, the series on the left-hand side of (4.51) is continuous for $k > |\sigma| + \frac{1}{2}$. Conversely, we can see that the series on the left-hand side of (4.51) is continuous when $k > 0$, which implies that $|\sigma| < \frac{1}{2}$. Thus, the identity (4.51) is valid for $k > |\sigma| + \frac{1}{2}$ and $\sigma \neq 0$. Since the series on the right-hand side of (4.51) is absolutely and uniformly convergent for $0 < x_1 \leq x \leq x_2 < \infty$, we can take the limit as $s \to 0$ on both sides of (4.51) for $|\sigma| < \frac{1}{2}$ and $k > |\sigma| + \frac{1}{2}$. Hence, the identity (4.51) is valid for $k > |\sigma| + \frac{1}{2}$ with $|\sigma| < \frac{1}{2}$.

Suppose that the identity

$$\begin{aligned}\frac{1}{\Gamma(k+1)} \sideset{}{'}\sum_{n \leq x} \sigma_{-s}(\chi, n)(x-n)^k = &\frac{x^{k+1} L(1+s,\chi)}{\Gamma(k+2)} - \frac{L(s,\chi)x^k}{2\Gamma(k+1)} \\ &+ 2\sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{n-1} \frac{x^{k-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi) \\ &+ \frac{i}{\tau(\bar{\chi})(2\pi)^k} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi}, n) \left(\frac{xq}{n}\right)^{\frac{1-s+k}{2}} \tilde{G}_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),\end{aligned} \tag{4.52}$$

is valid for some $k > 0$. Let $\beta > \max\{1, 1-\sigma\}$. Then

$$\sum_{n=1}^{\infty} \frac{|\sigma_s(n)|}{n^{\beta}} < \infty$$

and

$$\sup_{0 \leq h \leq 1} \left| \sum_{m^2 < n \leq (m+h)^2} \frac{\sigma_s(n)}{n^{\beta - 1/2}} \right| = o(1),$$

as $m \to \infty$. Put $x = y^2$ in the identity (4.52), where y lies in an interval J of length less than 1. By Lemma 4.2.9, $2y$ times the infinite series on the right-hand side of (4.52), with $x = y^2$, is uniformly equiconvergent on J with the differentiated series of the Fourier series of a function with period 1 which equals $A(y)\tilde{F}_{2-s+k}(y)$ on I, provided that $k > |\sigma| - \frac{1}{2}$. But then, $k+1 > |\sigma| + \frac{1}{2}$. Hence, from (4.51),

$$\begin{aligned}&\frac{i}{\tau(\bar{\chi})(2\pi)^{k+1}} A(y)\tilde{F}_{2-s+k}(y) \\ &= A(y)\Bigg\{ \sideset{}{'}\sum_{n \leq y^2} \frac{\sigma_{-s}(\chi,n)(y^2-n)^{k+1}}{\Gamma(k+2)} - \frac{y^{2(k+2)} L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi) y^{2(k+1)}}{2\Gamma(k+2)} \\ &\qquad - 2\sum_{n=1}^{\lfloor \frac{k+2}{2} \rfloor} (-1)^{n-1} \frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi) \Bigg\}\end{aligned}$$

$$= A(y)\Bigg\{ \int_0^{y^2} \sum_{n\le t}{}' \frac{\sigma_{-s}(\chi,n)(t-n)^k}{\Gamma(k+1)}\,dt - \frac{y^{2(k+2)}L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi)y^{2(k+1)}}{2\Gamma(k+2)}$$

$$- 2\sum_{n=1}^{\lfloor \frac{k+2}{2}\rfloor} (-1)^{n-1}\frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)\Bigg\}$$

$$= A(y)\Bigg\{ \int_0^{y} \sum_{n\le t^2}{}' \frac{\sigma_{-s}(\chi,n)(t^2-n)^k 2t}{\Gamma(k+1)}\,dt - \frac{y^{2(k+2)}L(1+s,\chi)}{\Gamma(k+3)} + \frac{L(s,\chi)y^{2(k+1)}}{2\Gamma(k+2)}$$

$$- 2\sum_{n=1}^{\lfloor \frac{k+2}{2}\rfloor} (-1)^{n-1}\frac{y^{2(k-2n+2)}}{\Gamma(k-2n+3)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)\Bigg\}.$$

Note that $A(y) = 1$ on J. Therefore, from Lemma 4.2.8 and the properties of the Fourier series of the function

$$\frac{2y}{\Gamma(k+1)}\sum_{n\le y^2}{}' \sigma_{-s}(\chi,n)(y^2-n)^k$$

in I, we see that the identity (4.51) holds for $k > |\sigma| - \frac{1}{2}$, which completes the proof of Theorem 4.3.2. $\square$

From (4.2) and (4.50), we find that $\sin(\pi s/2)M_{1-s}(z) = \tilde{G}_{1-s}(z)$. The case $k = 0$ of Theorem 4.3.2 gives the following corollary.

Corollary 4.3.3. *If χ is a non-principal odd primitive character modulo q, $x > 0$, and $|\sigma| < 1/2$, then*

$$\sum_{n\le x}{}' \sigma_{-s}(\chi,n) = xL(1+s,\chi) - \frac{1}{2}L(s,\chi) + \frac{i\sin(\pi s)/2}{\tau(\bar\chi)}\sum_{n=1}^{\infty}\sigma_{-s}(\bar\chi,n)\left(\frac{qx}{n}\right)^{\frac{1-s}{2}} M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $M_{1-s}(z)$ is defined in (4.2).

Next, we show that Theorem 4.3.3 implies Theorem 4.3.1. We then finish this section and hence finish the proof of Theorem 4.1.1 by proving that Theorem 4.1.1 implies Theorem 4.3.3.

Proof that Theorem 4.3.3 implies Theorem 4.3.1. Recall that $L_s(a,q,x)$ and $M_\nu(z)$ are defined in (4.22) and (4.2), respectively. Thus,

$$L_s(a,q,x) = -\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)$$

$$\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{ \frac{M_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+a/q)^{\frac{1-s}{2}}} - \frac{M_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-a/q)^{\frac{1-s}{2}}} \right\}$$

$$= -\frac{x}{2}\sin\left(\frac{\pi s}{2}\right)$$

$$
\begin{aligned}
&\times \sum_{m=1}^{\infty}\left\{\sum_{\substack{n=1\\ n\equiv a \bmod q}}^{\infty} \frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}}(n/q)^{\frac{1-s}{2}}} - \sum_{\substack{n=1\\ n\equiv -a \bmod q}}^{\infty} \frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}}(n/q)^{\frac{1-s}{2}}}\right\}\\
&= -\frac{(qx)^{\frac{1-s}{2}}}{2\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{m^{\frac{1+s}{2}}n^{\frac{1-s}{2}}}\sum_{\chi \bmod q}\bar{\chi}(n)(\chi(a)-\chi(-a))\\
&= -\frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\,\mathrm{odd}}}\chi(a)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\chi}(n)n^{s}\frac{M_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{\frac{1+s}{2}}}\\
&= -\frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\,\mathrm{odd}}}\chi(a)\sum_{n=1}^{\infty}\sum_{d|n}\bar{\chi}(d)d^{s}\frac{M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}}\\
&= -\frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\sin\left(\frac{\pi s}{2}\right)\sum_{\substack{\chi \bmod q\\ \chi\,\mathrm{odd}}}\chi(a)\sum_{n=1}^{\infty}\sigma_s(\bar{\chi},n)\frac{M_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}}.
\end{aligned}
$$

Now, from Lemma 4.2.2 and Theorem 4.3.1,

$$
\begin{aligned}
L_s(a,q,x)+\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\sin(2\pi na/q)}{n^s} &= -\frac{ix}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\,\mathrm{even}}}\chi(a)\tau(\bar{\chi})L(1+s,\chi)\\
&\quad+\frac{i}{2\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\,\mathrm{even}}}\chi(a)\tau(\bar{\chi})L(s,\chi).
\end{aligned}
$$

Using the functional equation (4.27) of $L(s,\chi)$ for odd primitive characters, we find that

$$
\begin{aligned}
L_s(a,q,x)+\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\sin(2\pi na/q)}{n^s} &= \frac{x\pi^{s+1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}(2+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\,\mathrm{odd}}}\chi(a)L(-s,\bar{\chi})\\
&\quad-\frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(2-s)\right)}{2\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\,\mathrm{odd}}}\chi(a)L(1-s,\bar{\chi})\\
&= \frac{x\pi^{s+1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{2\Gamma\left(\frac{1}{2}(2+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)-\chi(q-a))L(-s,\bar{\chi})\\
&\quad-\frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(2-s)\right)}{4\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)-\chi(q-a)L(1-s,\bar{\chi}).
\end{aligned}
\tag{4.53}
$$

From [3, p. 249, Chapter 12],

$$
L(s,\chi)=q^{-s}\sum_{h=1}^{q}\chi(h)\zeta(s,h/q). \tag{4.54}
$$

Multiplying both sides of (4.54) by $\bar{\chi}(a)$ and summing over all characters χ modulo q, we deduce that

$$\zeta(s,a/q) = \frac{q^s}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) L(s,\chi), \tag{4.55}$$

where $\zeta(s,a)$ denotes the Hurwitz zeta function. Using the duplication formula (1.32) and the reflection formula (1.30) for $\Gamma(s)$, we find that

$$\frac{\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}(1-s)\right)} = \frac{\cos(\frac{1}{2}\pi s)\Gamma(s)}{2^{s-1}\sqrt{\pi}}. \tag{4.56}$$

Utilizing (4.55) and (4.56) in (4.53), we see that

$$\begin{aligned} L_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\sin(2\pi na/q)}{n^s} &= -x\frac{\sin(\pi s/2)\Gamma(-s)}{(2\pi)^{-s}} \left(\zeta\left(-s,\frac{a}{q}\right) - \zeta\left(-s,1-\frac{a}{q}\right)\right) \\ &\quad - \frac{\cos(\pi s/2)\Gamma(1-s)}{2(2\pi)^{1-s}} \left(\zeta\left(1-s,\frac{a}{q}\right) - \zeta\left(1-s,1-\frac{a}{q}\right)\right), \end{aligned}$$

which completes the proof. □

The proof that Theorem 4.1.1 implies Theorem 4.3.3 is similar to the proof that Theorem 4.1.2 implies Theorem 4.4.3, which we give in the next section.

4.4 Proof of the generalization of Entry 2

Arguing as in the previous section, for $0 < a < q$ and q prime, we can show that Theorem 4.1.2 is equivalent to the following theorem.

Theorem 4.4.1. *Let q be a prime and $0 < a < q$. Let*

$$\begin{aligned} G_s(a,q,x) &= \frac{x}{2}\cos\left(\frac{\pi s}{2}\right) \\ &\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+a/q)^{\frac{1-s}{2}}} + \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-a/q)^{\frac{1-s}{2}}} \right\}, \end{aligned} \tag{4.57}$$

where $H_\nu(z)$ is defined in (4.6) and where we assume that the product of the summation indices mn tends to infinity. Then, for $|\sigma| < \frac{1}{2}$,

$$G_s(a,q,x) + \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\cos(2\pi na/q)}{n^s} = x\frac{\cos(\frac{1}{2}\pi s)\Gamma(-s)}{(2\pi)^{-s}} \left(\zeta\left(-s,\frac{a}{q}\right) + \zeta\left(-s,1-\frac{a}{q}\right)\right)$$

$$-\frac{\sin(\frac{1}{2}\pi s)\Gamma(1-s)}{2(2\pi)^{1-s}}\left(\zeta\left(1-s,\frac{a}{q}\right)+\zeta\left(1-s,1-\frac{a}{q}\right)\right).$$

We show that Theorem 4.4.1 is equivalent to Theorem 4.4.3, which is a special case of the following theorem.

Theorem 4.4.2. *If χ is a non-principal even primitive character modulo q, $x>0$, $|\sigma|<1/2$, and k is a non-negative integer, then*

$$\begin{aligned}&\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)(x-n)^k\\&=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{x^kL(s,\chi)}{2\Gamma(k+1)}+2\sum_{n=1}^{\lfloor\frac{k+1}{2}\rfloor}\frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)}\frac{\zeta(2n)}{(2\pi)^{2n}}L(1-2n+s,\chi)\\&\quad+\frac{1}{\tau(\bar\chi)(2\pi)^k}\sum_{n=1}^{\infty}\sigma_{-s}(\bar\chi,n)\left(\frac{qx}{n}\right)^{\frac{1-s+k}{2}}G_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),\end{aligned}$$

where $G_{\lambda-s}(z)$ is defined in (3.23)*. The series on the right-hand side converges uniformly on any interval for $x>0$ where the left-hand side is continuous. The convergence is bounded on any interval $0<x_1\leq x\leq x_2<\infty$ when $k=0$.*

Proof. From (4.9) and Lemma 4.2.4, for a fixed $x>0$, we see that

$$\frac{1}{\Gamma(1+k)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)(x-n)^k=\frac{1}{2\pi i}\int_{(c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw,$$

where $\max\{1,1-\sigma,\sigma\}<c<1$ and $k\geq 0$. Proceeding as we did in the proof of Theorem 4.3.2, we find that

$$\begin{aligned}\frac{1}{\Gamma(k+1)}\sideset{}{'}\sum_{n\leq x}\sigma_{-s}(\chi,n)(x-n)^k&=\frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)}-\frac{L(s,\chi)x^k}{2\Gamma(k+1)}\\&\quad+\frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}dw,\end{aligned}\tag{4.58}$$

provided that $k\geq 0$ and $k>2c-\sigma-2$. The functional equation for $L(2s,\chi)$ for an even primitive Dirichlet character χ is given by [31, p. 69]

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)L(2s,\chi)=\frac{\tau(\chi)}{\sqrt{q}}\left(\frac{\pi}{q}\right)^{-(\frac{1}{2}-s)}\Gamma\left(\frac{1}{2}-s\right)L(1-2s,\overline{\chi}),\tag{4.59}$$

where $\tau(\chi)$ is the Gauss sum defined in (4.12). Combining the functional equation (5.5) of $\zeta(2w)$ and the functional equation (4.59) of $L(2w+s,\chi)$ for even primitive χ, we deduce the functional equation

$$\zeta(w)L(w+s,\chi)=\frac{\pi^{2w+s-1}}{\tau(\bar{\chi})q^{w+s-1}}\eta(w,s)\zeta(1-w)L(1-w-s,\bar{\chi}), \tag{4.60}$$

where

$$\eta(w,s)=\frac{\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(1-w-s)\right)}{\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(w+s)\right)}.$$

Define

$$I(y):=\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}y^w\,dw. \tag{4.61}$$

Using the functional equation (4.60) in the integrand on the right-hand side of (4.58) and inverting the order of summation and integration, we find that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(1-c)}\zeta(w)L(w+s,\chi)\frac{\Gamma(w)x^{w+k}}{\Gamma(w+k+1)}\,dw \qquad (4.62)\\
&=\frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\zeta(1-w)L(1-w-s,\bar{\chi})\left(\frac{\pi^2x}{q}\right)^w\,dw\\
&=\frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2x}{q}\right)^w\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1-w}}\,dw\\
&=\frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n^{1+k}}\frac{1}{2\pi i}\int_{(1-c)}\frac{\eta(w,s)\Gamma(w)}{\Gamma(w+k+1)}\left(\frac{\pi^2nx}{q}\right)^w\,dw\\
&=\frac{x^k\pi^{s-1}}{\tau(\bar{\chi})q^{s-1}}\sum_{n=1}^{\infty}\frac{\sigma_s(\bar{\chi},n)}{n}I\left(\frac{\pi^2nx}{q}\right),
\end{aligned}$$

provided that $k>2c-\sigma-1$. We compute the integral $I(y)$ by using the residue calculus, shifting the line of integration to the right, and letting $c\to-\infty$.

Let k be a positive integer and $\sigma\neq 0$. By (4.61), we can write

$$I(y):=\frac{1}{2\pi i}\int_{(1-c)}F(w)\,dw,$$

where

$$F(w):=\frac{\Gamma(w)\Gamma\left(\frac{1}{2}(1-w)\right)\Gamma\left(\frac{1}{2}(1-w-s)\right)y^w}{\Gamma(1+k+w)\Gamma\left(\frac{1}{2}w\right)\Gamma\left(\frac{1}{2}(w+s)\right)}.$$

Note that the poles of the function $F(w)$ on the right side of the line $1-c+it, -\infty<t<\infty$, are at

$w = 2m+1$ and $w = 2m+1-s$, $m = 0, 1, 2, \ldots$. Calculating the residues, we find that

$$R_{2m+1}(F(w)) = (-1)^{m+1} \frac{2\Gamma(2m+1)\Gamma\left(-m-\frac{1}{2}s\right) y^{2m+1}}{m!\Gamma(2+k+2m)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}(s+1)\right)}$$

and

$$R_{2m+1-s}(F(w)) = (-1)^{m+1} \frac{2\Gamma(2m+1-s)\Gamma\left(-m+\frac{1}{2}s\right) y^{2m+1-s}}{m!\Gamma(2+k+2m-s)\Gamma\left(m+\frac{1}{2}(1-s)\right)\Gamma\left(m+\frac{1}{2}\right)}.$$

With the aid of the duplication formula (1.32) and the reflection formula (1.30), we find that

$$R_{2m+1}(F(w)) = \frac{2^{s-1}}{\sin(\pi s/2)} \frac{(2\sqrt{y})^{4m+2}}{(2m+k+1)!\Gamma(2m+1+s)} \tag{4.63}$$

and

$$R_{2m+1-s}(F(w)) = -\frac{(2y)^{1-s}}{\sin(\pi s/2)} \frac{(2\sqrt{y})^{4m}}{(2m)!\Gamma(2m+k+2-s)}. \tag{4.64}$$

Consequently, from (2.15), (4.39), and (4.63), for k even,

$$\begin{aligned}\sum_{m=0}^{\infty} R_{2m+1}(F(w)) &= \frac{2^{s-1-2k}y^{-k}}{\sin(\pi s/2)} \sum_{m=0}^{\infty} \frac{(2\sqrt{y})^{4m+2k+2}}{(2m+k+1)!\Gamma(2m+1+s)} \\ &= \frac{2^{s-1-2k}y^{-k}}{\sin(\pi s/2)} \left\{ \sum_{m=0}^{\infty} \frac{(2\sqrt{y})^{4m+2}}{(2m+1)!\Gamma(2m+1+s-k)} \right. \\ &\quad \left. - \sum_{m=1}^{k/2} \frac{(2\sqrt{y})^{4m-2}}{(2m-1)!\Gamma(2m-1+s-k)} \right\} \\ &= \frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)} (I_{-1+s-k}(4\sqrt{y}) - J_{-1+s-k}(4\sqrt{y})) \\ &\quad - \frac{2^{s+1}}{\sin(\pi s/2)} \sum_{m=1}^{k/2} \frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{4.65}$$

For each odd integer k,

$$\begin{aligned}\sum_{m=0}^{\infty} R_{2m+1}(F(w)) &= \frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)} (I_{-1+s-k}(4\sqrt{y}) + J_{-1+s-k}(4\sqrt{y})) \\ &\quad - \frac{2^{s+1}}{\sin(\pi s/2)} \sum_{m=1}^{(k+1)/2} \frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{4.66}$$

Similarly, from (4.64), we find that

$$\sum_{m=0}^{\infty} R_{2m+1-s}(F(w)) = -\frac{2^{-1-k}y^{(1-s-k)/2}}{\sin(\pi s/2)}(J_{1-s+k}(4\sqrt{y}) + I_{1-s+k}(4\sqrt{y})). \tag{4.67}$$

Utilizing (4.40) in the sum of (4.65), (4.66), and (4.67), we deduce that

$$\begin{aligned}\sum_{m=0}^{\infty} (R_{2m+1}(F(w)) + R_{2m+1-s}(F(w))) = -\frac{\cos(\pi s/2)}{2^k y^{(-1+s+k)/2}} \\ \times \left(\frac{J_{1-s+k}(4\sqrt{y}) - (-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin \pi s} - (-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right) \\ - \frac{2^{s+1}}{\sin(\pi s/2)} \sum_{m=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{2^{-4m}y^{1-2n}}{\Gamma(k-2m+2)\Gamma(1-2m+s)}.\end{aligned} \tag{4.68}$$

Using (2.16), we can show that

$$\cos(\pi s/2)\left(\frac{J_{1-s+k}(4\sqrt{y}) - (-1)^{k+1}J_{-1+s-k}(4\sqrt{y})}{\sin \pi s} - (-1)^{k+1}\frac{2}{\pi}K_{1-s+k}(4\sqrt{y})\right) = G_{1+k-s}(4\sqrt{y}). \tag{4.69}$$

Consider the positively oriented contour $\mathcal{R}_N$ formed by the points $\{1-c-iT, 2N+\frac{3}{2}-iT, 2N+\frac{3}{2}+iT, 1-c+iT\}$, where $T>0$ and N is a positive integer. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{R}_N} F(w)\,dw = \sum_{0\le k\le N} R_{2k+1}(F(w)) + \sum_{0\le k\le N} R_{2k+1-s}(F(w)).$$

By (4.45), for the integral over the right side of the rectangular contour $\mathcal{R}_N$,

$$\int_{2N+3/2-iT}^{2N+3/2+iT} F(w)\,dw \ll_{T,s} y^{2N+3/2}e^{4N-(4N+2+k+\sigma)\log N} = o(1),$$

as $N\to\infty$. Using Stirling's formula (1.39) to estimate the integrals over the horizontal sides of $\mathcal{R}_N$, we find that

$$\int_{1-c\pm iT}^{\infty\pm iT} F(w)\,dw \ll_s \int_{1-c}^{\infty} y^{\sigma}T^{-2\beta-\sigma-k}\,d\sigma \ll_{s,y} \frac{y^{1-c}}{T^{2c-\sigma-k-2}\log T} = o(1),$$

provided that $k > 2c-\sigma-2$. Combining (4.58), (4.62), (4.68), and (4.69), we conclude that

$$\frac{1}{\Gamma(k+1)}\sum_{n\le x}{}' \sigma_{-s}(\chi,n)(x-n)^k \tag{4.70}$$

$$= \frac{x^{k+1}L(1+s,\chi)}{\Gamma(k+2)} - \frac{x^k L(s,\chi)}{2\Gamma(k+1)} + 2\sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} \frac{(-1)^{n-1}x^{k-2n+1}}{\Gamma(k-2n+2)} \frac{\zeta(2n)}{(2\pi)^{2n}} L(1-2n+s,\chi)$$

$$+ \frac{1}{\tau(\bar{\chi})(2\pi)^k} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi},n) \left(\frac{qx}{n}\right)^{\frac{1-s+k}{2}} G_{1-s+k}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

provided that $k \geq 0$, $\sigma \neq 0$, and $k > 2c - \sigma - 1$. By the asymptotic expansions for Bessel functions (2.17), (2.18), and (2.19), Lemma 4.2.10, (3.28), and an argument like that in the proof in Theorem 4.3.2, we deduce the identity (4.70) for $k > |\sigma| - \frac{1}{2}$, with $|\sigma| < \frac{1}{2}$. Thus, we complete the proof of Theorem 4.1.2. □

From the definition (4.6) and (4.50), we find that $\cos(\pi s/2)M_{1-s}(z) = G_{1-s}(z)$. The case $k = 0$ of Theorem 4.4.2 provides the following corollary.

Corollary 4.4.3. *If χ is a non-principal even primitive character modulo q, $x > 0$, and $|\sigma| < 1/2$, then*

$$\sum_{n \leq x}{}' \sigma_{-s}(\chi,n) = xL(1+s,\chi) - \tfrac{1}{2}L(s,\chi) + \frac{\cos(\pi s)/2}{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \sigma_{-s}(\bar{\chi},n) \left(\frac{qx}{n}\right)^{\frac{1-s}{2}} H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right),$$

where $H_{1-s}(z)$ is defined in (4.6).

Next we show that Theorem 4.4.3 implies Theorem 4.4.1.

Proof. First we write (4.57) as a sum over Dirichlet characters. To that end, for any prime q and $0 < a < q$,

$$G_s(a,q,x) = \frac{x}{2}\cos\left(\frac{\pi s}{2}\right) \tag{4.71}$$

$$\times \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left\{ \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+a/q)^{\frac{1-s}{2}}} + \frac{H_{1-s}\left(4\pi\sqrt{mx\left(n+1-\frac{a}{q}\right)}\right)}{(mx)^{\frac{1+s}{2}}(n+1-a/q)^{\frac{1-s}{2}}} \right\}$$

$$= \frac{x}{2}\cos\left(\frac{\pi s}{2}\right) \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n \equiv \pm a \bmod q}}^{\infty} \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}}(n/q)^{\frac{1-s}{2}}}$$

$$= \frac{(qx)^{\frac{1-s}{2}}}{2\phi(q)}\cos\left(\frac{\pi s}{2}\right) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{m^{\frac{1+s}{2}}n^{\frac{1-s}{2}}} \sum_{\chi \bmod q} \bar{\chi}(n)(\chi(a)+\chi(-a))$$

$$= \frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \bar{\chi}(n)n^s \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{\frac{1+s}{2}}}$$

$$= \frac{(qx)^{\frac{1-s}{2}}}{\phi(q)}\cos\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty}\sum_{d|n} \bar{\chi}(d)d^s \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}}$$

$$= \frac{(qx)^{\frac{1-s}{2}}}{\phi(q)} \cos\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} \sigma_s(\bar{\chi}, n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}},$$

where in the penultimate step we recall our assumption that the double series converges in the sense that the product of the indices mn tends to infinity. For the principal character χ_0,

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma_s(\chi_0, n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \chi_0(n) n^s \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{\frac{1+s}{2}}} \qquad (4.72)\\
&= \sum_{m=1}^{\infty}\sum_{\substack{n=1 \\ q\nmid n}}^{\infty} n^s \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{\frac{1+s}{2}}}\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^s \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mn)^{\frac{1+s}{2}}} - q^{\frac{s-1}{2}} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^s \frac{H_{1-s}\left(4\pi\sqrt{mnx}\right)}{(mn)^{\frac{1+s}{2}}}\\
&= \sum_{n=1}^{\infty} \sigma_s(n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}} - q^{\frac{s-1}{2}} \sum_{n=1}^{\infty} \sigma_s(n) \frac{H_{1-s}\left(4\pi\sqrt{nx}\right)}{n^{\frac{1+s}{2}}}.
\end{aligned}$$

Combining (4.71) and (4.72) and applying Lemma 4.2.1, we find that

$$\begin{aligned}
G_s(a,q,x) = \frac{(qx)^{\frac{1-s}{2}}}{\phi(q)} \cos\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \neq \chi_0 \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} \sigma_s(\bar{\chi}, n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}} \qquad (4.73)\\
+ \frac{1}{\phi(q)} \left({\sum_{n\leq x}}' \sigma_{-s}(n) - xZ(s,x) + \frac{1}{2}\zeta(s) \right) - \frac{q^{1-s}}{\phi(q)} \left({\sum_{n\leq x/q}}' \sigma_{-s}(n) - \frac{x}{q} Z(s, x/q) + \frac{1}{2}\zeta(s) \right)\\
= \frac{(qx)^{\frac{1-s}{2}}}{\phi(q)} \cos\left(\frac{\pi s}{2}\right) \sum_{\substack{\chi \neq \chi_0 \bmod q \\ \chi \text{ even}}} \chi(a) \sum_{n=1}^{\infty} \sigma_s(\bar{\chi}, n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right)}{n^{\frac{1+s}{2}}}\\
+ \frac{1}{\phi(q)} {\sum_{n\leq x}}' \sigma_{-s}(n) - \frac{q^{1-s}}{\phi(q)} {\sum_{n\leq x/q}}' \sigma_{-s}(n)\\
+ \frac{x}{\phi(q) q^s} \zeta(1+s) \left(1 - \frac{1}{q^{-s}}\right) - \frac{\zeta(s)}{2\phi(q) q^{s-1}} \left(1 - \frac{1}{q^{1-s}}\right).
\end{aligned}$$

For each prime q, by Lemma 4.2.3,

$$\begin{aligned}
\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\cos(2\pi na/q)}{n^s} = q^{-s} {\sum_{1\leq n\leq x/q}}' \sigma_{-s}(n) + \frac{1}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi \text{ even}}} \chi(a)\tau(\bar{\chi}) {\sum_{1\leq n\leq x}}' \sigma_{-s}(\chi, n) \qquad (4.74)\\
= q^{-s} {\sum_{1\leq n\leq x/q}}' \sigma_{-s}(n) - \frac{1}{\phi(q)} {\sum_{1\leq n\leq x}}' \sigma_{-s}(\chi_0, n) + \frac{1}{\phi(q)} \sum_{\substack{\chi \neq \chi_0 \bmod q \\ \chi \text{ even}}} \chi(a)\tau(\bar{\chi}) {\sum_{1\leq n\leq x}}' \sigma_{-s}(\chi, n).
\end{aligned}$$

Now,

$$\sideset{}{'}\sum_{1\le n\le x}\sigma_{-s}(\chi_0,n)=\sideset{}{'}\sum_{1\le n\le x}\sum_{\substack{d|n\\ q\nmid d}}d^{-s}=\sideset{}{'}\sum_{1\le n\le x}\sum_{d|n}d^{-s}-q^{-s}\sideset{}{'}\sum_{1\le n\le x/q}\sum_{d|n}d^{-s} \tag{4.75}$$

$$=\sideset{}{'}\sum_{1\le n\le x}\sigma_{-s}(n)-q^{-s}\sideset{}{'}\sum_{1\le n\le x/q}\sigma_{-s}(n).$$

Substituting (4.75) into (4.74), we find that

$$\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s}=\frac{q^{1-s}}{\phi(q)}\sideset{}{'}\sum_{1\le n\le x/q}\sigma_{-s}(n)-\frac{1}{\phi(q)}\sideset{}{'}\sum_{1\le n\le x}\sigma_{-s}(n) \tag{4.76}$$

$$+\frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar\chi)\sideset{}{'}\sum_{1\le n\le x}\sigma_{-s}(\chi,n).$$

Adding (4.73) and (4.76) and using Theorem 4.4.3, we find that

$$G_s(a,q,x)+\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s}=\frac{x}{\phi(q)q^s}\zeta(1+s)\left(1-\frac{1}{q^{-s}}\right) \tag{4.77}$$

$$-\frac{\zeta(s)}{2\phi(q)q^{s-1}}\left(1-\frac{1}{q^{1-s}}\right)+\frac{x}{\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar\chi)L(1+s,\chi)-\frac{1}{2\phi(q)}\sum_{\substack{\chi\neq\chi_0 \bmod q\\ \chi\text{ even}}}\chi(a)\tau(\bar\chi)L(s,\chi).$$

Recall that if χ_0 is the principal character modulo the prime q, then

$$L(s,\chi_0)=\zeta(s)\left(1-\frac{1}{q^s}\right). \tag{4.78}$$

Using the functional equations of $\zeta(s)$ and $L(s,\chi)$ for even primitive Dirichlet characters, (5.5) and (4.59), respectively, and (4.78), we find from (4.77) that

$$G_s(a,q,x)+\sum_{n=1}^{\infty}F\left(\frac{x}{n}\right)\frac{\cos(2\pi na/q)}{n^s}=\frac{x\pi^{s+1/2}\Gamma\left(-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)L(-s,\bar\chi) \tag{4.79}$$

$$-\frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{2\Gamma\left(\frac{1}{2}s\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\text{ even}}}\chi(a)L(1-s,\bar\chi)$$

$$=\frac{x\pi^{s+1/2}\Gamma\left(-\frac{1}{2}s\right)}{2\Gamma\left(\frac{1}{2}(1+s)\right)}\frac{q^{-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)+\chi(q-a))L(-s,\bar\chi)$$

$$-\frac{\pi^{s-1/2}\Gamma\left(\frac{1}{2}(1-s)\right)}{4\Gamma\left(\frac{1}{2}s\right)}\frac{q^{1-s}}{\phi(q)}\sum_{\chi \bmod q}(\chi(a)+\chi(q-a))L(1-s,\bar\chi).$$

We complete the proof of Theorem 4.4.1 by using (4.55) and (4.56) in (4.79). $\square$

Next we prove that Theorem 4.1.2 implies Theorem 4.4.3.

Proof. Let χ be an even primitive character modulo q. Set $\theta = h/q$, where $1 \leq h < q$. The Gauss sum $\tau(n,\chi)$ is defined by

$$\tau(n,\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i mn/q}.$$

Note that $\tau(1,\chi) := \tau(\chi)$, which is defined in (4.12). For any character χ [3, p. 165, Theorem 8.9]

$$\tau(n,\chi) = \bar{\chi}(n)\tau(\chi).$$

Multiplying both sides of (4.5) by $\bar{\chi}(h)/\tau(\bar{\chi})$ and summing over h, $1 \leq h < q$, we find that the left-hand side yields

$$\begin{aligned}
\frac{1}{\tau(\bar{\chi})} \sum_{h=1}^{q-1} \bar{\chi}(h) \sum_{n=1}^{\infty} F\left(\frac{x}{n}\right) \frac{\cos(2\pi nh/q)}{n^s} &= \frac{1}{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{F\left(\frac{x}{n}\right)}{n^s} \sum_{h=1}^{q-1} \bar{\chi}(h) \cos\left(\frac{2\pi nh}{q}\right) \qquad (4.80) \\
&= \frac{1}{2\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{F\left(\frac{x}{n}\right)}{n^s} \sum_{h=1}^{q-1} \bar{\chi}(h) \left(e^{2\pi i nh/q} + e^{-2\pi i nh/q}\right) \\
&= \frac{1}{2\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{F\left(\frac{x}{n}\right)}{n^s} \tau(\bar{\chi})(\chi(n) + \chi(-n)) \\
&= \sideset{}{'}\sum_{n \leq x} \sigma_{-s}(\chi, n).
\end{aligned}$$

On the other hand, summing over h, $1 \leq h \leq q$, on the right-hand side of (4.5) gives

$$\begin{aligned}
&\frac{x}{2\tau(\bar{\chi})} \cos\left(\frac{\pi s}{2}\right) \sum_{h=1}^{q-1} \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ n \equiv \pm h \bmod q}}^{\infty} \bar{\chi}(h) \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}} (n/q)^{\frac{1-s}{2}}} \qquad (4.81) \\
&= \frac{x}{2\tau(\bar{\chi})} \cos\left(\frac{\pi s}{2}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}} (n/q)^{\frac{1-s}{2}}} \sum_{\substack{h=1 \\ h \equiv \pm n \bmod q}}^{q-1} \bar{\chi}(h) \\
&= \frac{x}{\tau(\bar{\chi})} \cos\left(\frac{\pi s}{2}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{\chi}(n) \frac{H_{1-s}\left(4\pi\sqrt{\frac{mnx}{q}}\right)}{(mx)^{\frac{1+s}{2}} (n/q)^{\frac{1-s}{2}}} \\
&= \frac{(qx)^{\frac{1-s}{2}} \cos(\frac{1}{2}\pi s)}{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\sigma_s(\bar{\chi}, n)}{n^{\frac{1+s}{2}}} H_{1-s}\left(4\pi\sqrt{\frac{nx}{q}}\right).
\end{aligned}$$

Combining (4.80), (4.81), and (4.54) with the functional equation (4.59) of $L(s,\chi)$ for even primitive χ, we obtain the equality in (4.5), which completes the proof of Theorem 4.4.3. $\square$

Chapter 5

Partial sums of Dedekind zeta functions

5.1 Main results

5.1.1 Zero free region

We can summarize our first result in this chapter which give a zero free region for $\zeta_{K,X}(s)$ as follows. See also [63].

Proposition 5.1.1. *Let K be an arbitrary algebraic number field of degree $n_0 = [K\colon \mathbb{Q}]$ over the field $\mathbb{Q}$ of rational numbers, let X be a real number greater than or equal to 2, and denote by s the complex variable $\sigma + it$. Then there exist two real numbers α and β, with α depending on n_0 and X only and with β depending on n_0 only, such that the zeros of $\zeta_{K,X}(s)$ all lie within the rectilinear strip of the complex plane given by the inequalities $\alpha < \sigma < \beta$.*

As will be seen in the proof of Proposition 5.1.1 in Section 5.3, for any fixed $\delta_0 > 0$ and any X large enough, an admissible choice for α is $\alpha = -3(\delta_0 + \log 2)n_0 X \log X / \log\log X$. As for β, an admissible choice is of the form $\beta = \log C_{\epsilon_0,n_0} D_{\epsilon_0,n_0} / \log 2$, where ϵ_0 is fixed and satisfies the inequalities $0 < \epsilon_0 < 1/n_0$, $D_{\epsilon_0,n_0} = \sum_{n=2}^{\infty} 4/n^{2-\epsilon_0 n_0}$, and C_{ϵ_0,n_0} is a constant defined in terms of the divisor function.

5.1.2 An asymptotic formula for $N_{K,X}(T)$

Furthermore, we provide an asymptotic formula for $N_{K,X}(T)$ when K is a cyclotomic field, which is sharper than the one known in the case of $\zeta(s)$. Let K be any algebraic number field of degree $n_0 = [K\colon \mathbb{Q}]$ over the field $\mathbb{Q}$ of rational numbers. In a similar fashion to the case of $\zeta(s)$ (see [41] and [60]), it can be shown that

$$\left| N_{K,X}(T) - \frac{T}{2\pi} \log N \right| \leq \frac{X}{2}, \tag{5.1}$$

where T and X both go to infinity together, and N is the largest integer less than or equal to X for which $a_K(N) \neq 0$. However, if $K = \mathbb{Q}(\zeta_q)$ is a cyclotomic field, we can significantly improve the error term in

(5.1).

Theorem 5.1.2. *Let $q \geq 2$, let ζ_q be a primitive root of unity of order q, let $K = \mathbb{Q}(\zeta_q)$, and let $T, X \geq 3$. Let, further, N be the largest integer less than or equal to X such that $a_K(N) \neq 0$. We have*

$$N_{K,X}(T) = \frac{T}{2\pi}\log N + O_q\left(X\left(\frac{\log\log X}{\log X}\right)^{1-1/\phi(q)}\right), \tag{5.2}$$

where ϕ is Euler's totient function.

Finally, we remark that the larger the degree of the cyclotomic field is, the better the asymptotic formula (5.2) becomes.

5.2 Preliminary results

To prove Theorem 5.1.2, we will make use of two auxiliary lemmas.

Lemma 5.2.1. *Fix a positive integer $q \geq 2$. We have*

$$\#\{n \leq y \colon \mu(n) \neq 0 \text{ and } p\,|\,n \text{ imply } p \equiv 1 \,(\mathrm{mod}\, q)\} = O_q\left(y\left(\frac{\log\log y}{\log y}\right)^{1-1/\phi(q)}\right),$$

where μ denotes the Möbius function.

Proof. Fix a positive integer $q \geq 2$ and define

$$\mathcal{B}(q,y) = \{n \leq y \colon \mu(n) \neq 0 \,\text{and}\, \ p\,|\,n \ \text{ imply } \ p \equiv 1 \,(\mathrm{mod}\, q)\}.$$

We apply Brun's pure sieve to estimate the size of the set $\mathcal{B}(q,y)$. (See Murty and Cojocaru [28, page 86].) Let $\mathcal{A}$ be the set of all positive integers $n \leq y$. Let $\mathcal{P}$ be the set of all primes p incongruent to 1 modulo q. Let $\mathcal{A}_p$ be the set of elements of $\mathcal{A}$ which are divisible by p. Let, further, $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_d = \bigcap_{p|d}\mathcal{A}_p$, where d is a square-free positive integer composed of a list of prime factors from $\mathcal{P}$. For any positive real number z, we define

$$S(\mathcal{A},\mathcal{P},z) = \mathcal{A} \setminus \bigcup_{p|P(z)} \mathcal{A}_p,$$

where

$$P(z) = \prod_{\substack{p\in\mathcal{P}\\ p<z}} p.$$

We consider the multiplicative function ω defined for all primes p by $\omega(p) = 1$. We have

$$\#\mathcal{A}_d = \#\{n \leq y \colon n \equiv 0 \pmod{d}\} = \frac{\omega(d)}{d} y + R_d,$$

where

$$|R_d| \leq \omega(d).$$

From Mertens' estimates, we have

$$\sum_{\substack{p \in \mathcal{P} \\ p < z}} \frac{\omega(p)}{p} = \frac{\phi(q) - 1}{\phi(q)} \log\log z + O(1).$$

For the sake of brevity, we let

$$W(z) = \prod_{p \mid P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

By Brun's pure sieve, we have

$$\#S(\mathcal{A}, \mathcal{P}, z) = yW(z)\left(1 + O\left((\log z)^{-A}\right)\right) + O(z^{\eta \log\log z}), \tag{5.3}$$

where $A = \eta \log \eta$ and, for some $\alpha < 1$,

$$\eta = \frac{\alpha \log y}{\log z \log\log z}.$$

Since $\omega(p) = 1$, Mertens' estimates yield

$$W(z) = O_q\left(\frac{1}{(\log z)^{1 - 1/\phi(q)}}\right). \tag{5.4}$$

We now choose $\log z = c \log y / \log\log y$. Then for a suitable positive and sufficiently small constant c and from (5.3) and (5.4), we have

$$\#S(\mathcal{A}, \mathcal{P}, z) = O_q\left(y \left(\frac{\log\log y}{\log y}\right)^{1 - 1/\phi(q)}\right). \tag{5.5}$$

Since $\mathcal{B}(q, y) \subseteq S(\mathcal{A}, \mathcal{P}, z)$, we have $\#\mathcal{B}(q, z) \leq \#S(\mathcal{A}, \mathcal{P}, z)$. Employing this last inequality together with (5.5), we complete the proof of Lemma 5.2.1. $\square$

Lemma 5.2.2. *Let $q \geq 2$ and let $K = \mathbb{Q}(\zeta_q)$. Let, further,*

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}.$$

We have

$$\#\{n \leq x \colon a_K(n) \neq 0\} = O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right).$$

Proof. Let $K = \mathbb{Q}(\zeta_q)$, where ζ_q is a primitive root of unity of order q. We have

$$\zeta_K(s) = \prod_{\mathcal{P}|q}\left(1 - \frac{1}{\|\mathcal{P}\|^s}\right)^{-1} F_q(s),$$

where

$$F_q(s) = \prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi).$$

(See [72, page 468].) For $\sigma > 1$, we have

$$F_q(s) = \prod_{\chi \,(\mathrm{mod}\, q)} \prod_{\substack{p \text{ prime} \\ p \nmid q}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Hence, for $\sigma > 1$, we have

$$\begin{aligned}
\log F_q(s) &= -\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{p \text{ prime} \\ p \nmid q}} \log\left(1 - \frac{\chi(p)}{p^s}\right) \\
&= \sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{p \text{ prime} \\ p \nmid q}} \sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}} \\
&= \sum_{\substack{p \text{ prime} \\ p \nmid q}} \sum_{m=1}^{\infty} \sum_{\chi \,(\mathrm{mod}\, q)} \frac{\chi(p^m)}{mp^{ms}},
\end{aligned}$$

where

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(p^m) = \begin{cases} \phi(q), & \text{if } p^m \equiv 1 \,(\mathrm{mod}\, q); \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\log F_q(s) = \sum_{\substack{p \text{ prime},\, m \geq 1 \\ p^m \equiv 1 \,(\mathrm{mod}\, q)}} \frac{\phi(q)}{mp^{ms}}.$$

Hence, we have

$$F_q(s) = \exp\left(\sum_{\substack{p \text{ prime},\, m\geq 1 \\ p^m \equiv 1 \,(\mathrm{mod}\, q)}} \frac{\phi(q)}{mp^{ms}} \right).$$

Now, for $\sigma > 1$,

$$F_q(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \dots\right).$$

Thus, we have

$$\log F_q(s) = \sum_{p \text{ prime}} \log\left(1 + \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \dots\right) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \dots\right)^m,$$

and hence

$$c(p) = \begin{cases} \phi(q), & \text{if } p \equiv 1 \,(\mathrm{mod}\, q); \\ 0, & \text{if } p \not\equiv 1 \,(\mathrm{mod}\, q). \end{cases}$$

For all n such that $c(n) \neq 0$, we have $n = AB$, where A is coprime to B, A is squareful, and B is square-free, that is, $\mu(B) \neq 0$. Furthermore, all the prime factors of B are congruent to 1 modulo q. Letting

$$H(x) = \prod_{\substack{p\leq x,\, p \text{ prime} \\ p\equiv 1 \,(\mathrm{mod}\, q)}} p,$$

we have

$$\begin{aligned}
\#\{n \leq x \colon c(n) \neq 0\} &\leq \#\{(A,B)\colon A \text{ squareful},\, \mu(B) \neq 0,\, AB \leq x,\, B \mid H(x)\} \\
&= \sum_{\substack{A\leq x \\ A \text{ squareful}}} \sum_{\substack{B \leq x/A \\ B \mid H(x)}} 1 \\
&= \sum_{\substack{A\leq x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) \\
&= \sum_{\substack{A\leq \sqrt{x}\log x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right) + \sum_{\substack{\sqrt{x}\log x \leq A \leq x \\ A \text{ squareful}}} \mathcal{B}\left(q, \frac{x}{A}\right).
\end{aligned}$$

We examine the sums on the far right-hand side separately.

Using Lemma 5.2.1, we see that

$$\begin{aligned}
\sum_{\substack{A\le\sqrt{x}\log x\\ A\text{ squareful}}} \mathcal{B}\left(q,\frac{x}{A}\right) &= O_q\left(\sum_{\substack{A\le\sqrt{x}\log x\\ A\text{ squareful}}} \frac{x}{A}\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right)\\
&= O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\sum_{\substack{A\le\sqrt{x}\log x\\ A\text{ squareful}}}\frac{1}{A}\right)\\
&= O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\sum_{a\ge1,b\ge1}\frac{1}{a^2b^3}\right)\\
&= O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{\substack{\sqrt{x}\log x\le A\le x\\ A\text{ squareful}}} \mathcal{B}\left(q,\frac{x}{A}\right) \le \sum_{\substack{\sqrt{x}\log x\le A\le x\\ A\text{ squareful}}}\frac{x}{A} &\le \sum_{\substack{\sqrt{x}\log x\le A\le x\\ A\text{ squareful}}}\frac{x}{\sqrt{x}\log x}\\
&\le \frac{\sqrt{x}}{\log x}\#\{A\le x\colon A\text{ squareful}\}\\
&= O\left(\frac{x}{\log x}\right).
\end{aligned}$$

Suppose that $\mathcal{P}_1,\ldots,\mathcal{P}_r$ are the prime ideals in the ring of integers of K lying over the prime factors of q and consider the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \prod_{\mathcal{P}|p}\left(1-\frac{1}{\|\mathcal{P}\|^s}\right)^{-1}.$$

For all z, we have

$$\#\{n\le z\colon b(n)\ne 0\}\le \#\{n\le z \text{ with all prime factors of } n \text{ in the sets } \mathcal{P}_1,\ldots,\mathcal{P}_r\}. \tag{5.6}$$

It is well-known that the right-hand side of (5.6) is $O_q((\log z)^r)$. Thus, we have

$$\#\{n\le z\colon b(n)\ne 0\} = O_q((\log z)^r).$$

For brevity's sake, we let

$$\mathcal{A} = \{n\colon a_K(n) \neq 0\}, \quad \mathcal{B} = \{m\colon b(m) \neq 0\}, \quad \mathcal{C} = \{k\colon c(k) \neq 0\},$$

and denote

$$\mathcal{A}_\omega = \mathcal{A} \cap [1, \omega], \quad \mathcal{B}_\omega = \mathcal{B} \cap [1, \omega], \quad \mathcal{C}_\omega = \mathcal{C} \cap [1, \omega].$$

Here, we note that

$$\#\mathcal{B}_\omega = O_q((\log \omega)^r)$$

and

$$\#\mathcal{C}_\omega = O_q\left(\omega\left(\frac{\log\log\omega}{\log\omega}\right)^{1-1/\phi(q)}\right). \tag{5.7}$$

Furthermore, we have

$$\zeta_K(s) = \sum_{n\in\mathcal{A}} \frac{a_K(n)}{n^s} = \sum_{m\in\mathcal{B}} \frac{b(m)}{m^s} \sum_{k\in\mathcal{C}} \frac{c(k)}{k^s}.$$

On noting that $\mathcal{A} \subseteq \mathcal{BC}$, where $\mathcal{BC} = \{bc\colon b \in \mathcal{B}, c \in \mathcal{C}\}$, we have $\mathcal{A}_x \subset (\mathcal{BC})_x$. It follows that

$$\#\mathcal{A}_x \leq \#(\mathcal{BC})_x, \tag{5.8}$$

where

$$\#(\mathcal{BC})_x = \sum_{\substack{b\leq x\\ b\in\mathcal{B}}} \sum_{\substack{c\leq x/b\\ c\in\mathcal{C}}} 1 = \sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \sum_{\substack{c\leq x/b\\ c\in\mathcal{C}}} 1 + \sum_{\substack{L<b\leq x\\ b\in\mathcal{B}}} \sum_{\substack{c\leq x/b\\ c\in\mathcal{C}}} 1, \tag{5.9}$$

with $1 \leq L \leq x$ (to be chosen later). By (5.7), we have

$$\sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \sum_{\substack{c\leq x/b\\ c\in\mathcal{C}}} 1 \leq \sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \#\mathcal{C}_{x/b} = O_q\left(\sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \frac{x}{b}\left(\frac{\log\log(x/b)}{\log(x/b)}\right)^{1-1/\phi(q)}\right).$$

Since $b \leq L$, we have

$$\left(\log\frac{x}{b}\right)^{1-1/\phi(q)} > \left(\log\frac{x}{L}\right)^{1-1/\phi(q)}.$$

Hence, we have

$$\sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \sum_{\substack{c\leq x/b\\ c\in\mathcal{C}}} 1 = O_q\left(x\left(\frac{\log\log x}{\log x/L}\right)^{1-1/\phi(q)} \sum_{\substack{b\leq L\\ b\in\mathcal{B}}} \frac{1}{b}\right) = O_q\left(x\left(\frac{\log\log x}{\log(x/L)}\right)^{1-1/\phi(q)}\right), \tag{5.10}$$

since

$$\sum_{b \in \mathcal{B}} \frac{1}{b} < \infty.$$

Next, we have

$$\sum_{\substack{L < b \le x \\ b \in \overline{\mathcal{B}}}} \sum_{\substack{c \le x/b \\ c \in \mathcal{C}}} 1 = \sum_{\substack{L < b \le x \\ b \in \overline{\mathcal{B}}}} \#\mathcal{C}_{x/b} \le \sum_{\substack{L < b \le x \\ b \in \overline{\mathcal{B}}}} \frac{x}{b} \le \frac{x}{L} \#\mathcal{B}_x = O_q\left(\frac{x(\log x)^r}{L}\right). \tag{5.11}$$

In view of (5.8), we substitute (5.10) and (5.11) into (5.9) to obtain

$$\#\mathcal{A}_x = O_q\left(\frac{x(\log x)^r}{L}\right) + O_q\left(x\left(\frac{\log\log x}{\log(x/L)}\right)^{1-1/\phi(q)}\right).$$

Then choosing $L = (\log x)^{r+1}$, we obtain

$$\#\mathcal{A}_x = O_q\left(x\left(\frac{\log\log x}{\log x}\right)^{1-1/\phi(q)}\right).$$

This finishes the proof of Lemma 5.2.2. □

5.3 Proof of the zero free region

We show separately that $|\zeta_{K,X}(s)| > 0$ in the right half-plane $\sigma \ge \beta$ and in the left-half plane $\sigma \le \alpha$. More specifically, we want to find a β so that

$$1 - \sum_{2 \le n \le X} \frac{a_K(n)}{n^\sigma} > 0,$$

for $\sigma \ge \beta$. Toward this end, we employ the upper bound $a_K(n) \le d(n)^{n_0-1}$, where $d(n)$ denotes the number of divisors of n (see Chandrasekharan and Narasimhan [24], Lemma 9) and satisfies the upper bound $d(n) \le C_{\epsilon_0} n^{\epsilon_0}$ for all positive ϵ_0 (see Hardy and Wright [49], Chapter XVIII, Theorem 317). Hence, we have $a_K(n) \le C_{\epsilon_0,n_0} n^{\epsilon_0 n_0}$.

It is enough to show that

$$C_{\epsilon_0,n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\sigma-\epsilon_0 n_0}} < 1. \tag{5.12}$$

If we let $\epsilon_0 < 1/n_0$, then for $\sigma \ge \beta$ we have

$$\sum_{n=2}^{\infty} \frac{1}{n^{\sigma-\epsilon_0 n_0}} \le \sum_{n=2}^{\infty} \frac{1}{n^{\beta-\epsilon_0 n_0}} \le \frac{1}{2^\beta} D_{\epsilon_0,n_0},$$

where

$$D_{\epsilon_0,n_0} = \sum_{n=2}^{\infty} \frac{4}{n^{2-\epsilon_0 n_0}}.$$

In order to obtain (5.12), it is enough to have

$$\beta > \frac{\log C_{\epsilon_0,n_0} D_{\epsilon_0,n_0}}{\log 2}.$$

We have

$$\sum_{n=2}^{\infty} \frac{d(n)^{n_0}}{n^{\beta}} \leq C_{\epsilon_0,n_0} \sum_{n=2}^{\infty} \frac{1}{n^{\beta-\epsilon_0 n_0}} = \frac{1}{2^{\beta}} C_{\epsilon_0,n_0} D_{\epsilon_0,n_0}.$$

Then for $\sigma \geq \beta$, we have

$$\left| \sum_{2 \leq n \leq X} \frac{a_K(n)}{n^s} \right| \leq \sum_{2 \leq n \leq X} \frac{d(n)^{n_0}}{n^{\beta}} < 1, \tag{5.13}$$

and hence

$$|\zeta_{K,X}(s)| \geq 1 - \left| \sum_{2 \leq n \leq X} \frac{a_K(n)}{n^s} \right| > 0.$$

Therefore, $\zeta_{K,X}(s) \neq 0$ on the right-half plane $\sigma \geq \beta$.

Next, let N be the largest positive integer less than or equal to X for which $a_K(N) \neq 0$. Since

$$|\zeta_{K,X}(s)| \geq \frac{a_K(N)}{N^{\sigma}} - \left| \sum_{1 \leq n \leq N-1} \frac{a_K(n)}{n^s} \right|,$$

it is enough to find an α such that

$$\frac{1}{N^{\sigma}} > \sum_{1 \leq n \leq N-1} \frac{a_K(n)}{n^{\sigma}},$$

for $\sigma \leq \alpha$.

To this end, let us fix $\delta_0 > 0$. Then there exist constants $C_{\delta_0} > 0$ and $n_{\delta_0} \in \mathbb{Z}^+$ such that for all $1 \leq n < n_{\delta_0}$, we have

$$d(n) \leq C_{\delta_0} n^{(\delta_0 + \log 2)/\log\log n},$$

and that for all $n \geq n_{\delta_0}$, we have

$$d(n) \leq n^{(\delta_0 + \log 2)/\log\log n}$$

(see [95]).

It suffices to have

$$\frac{1}{N^\sigma} > C_{\delta_0}^{n_0} \sum_{1\le n\le n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma} + \sum_{n_{\delta_0}\le n\le N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma}$$
$$= 1 + C_{\delta_0}^{n_0} S_I(n_0,\delta_0,n_{\delta_0},\sigma) + S_{II}(n_0,\delta_0,\sigma),$$

for $\sigma \le \alpha$, where

$$S_I(n_0,\delta_0,n_{\delta_0},\sigma) = \sum_{2\le n\le n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma}$$

and

$$S_{II}(n_0,\delta_0,\sigma) = \sum_{n_{\delta_0}\le n\le N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^\sigma}.$$

This would follow from the inequality

$$\frac{1}{N^\alpha} > 1 + C_{\delta_0}^{n_0} S_I(n_0,\delta_0,n_{\delta_0},\alpha) + S_{II}(n_0,\delta_0,\alpha),$$

since, for any $\sigma \le \alpha$,

$$\begin{aligned}
\frac{1}{N^\sigma} &> \frac{1}{N^{\sigma-\alpha}}\left[1 + C_{\delta_0}^{n_0} S_I(n_0,\delta_0,n_{\delta_0},\alpha) + S_{II}(n_0,\delta_0,\alpha)\right] \\
&= \frac{1}{N^{\sigma-\alpha}} + C_{\delta_0}^{n_0} \sum_{2\le n\le n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{N^{\sigma-\alpha} n^\alpha} + \sum_{n_{\delta_0}\le n\le N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{N^{\sigma-\alpha} n^\alpha} \\
&> 1 + C_{\delta_0}^{n_0} \sum_{2\le n\le n_{\delta_0}-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\sigma-\alpha} n^\alpha} + \sum_{2\le n\le N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\sigma-\alpha} n^\alpha} \\
&= 1 + C_{\delta_0}^{n_0} S_I(n_0,\delta_0,n_{\delta_0},\sigma) + S_{II}(n_0,\delta_0,\sigma).
\end{aligned}$$

Thus, it is enough to find α such that

$$\frac{1}{N^\alpha} > 2 + 2C_{\delta_0}^{n_0} S_I(n_0,\delta_0,n_{\delta_0},\alpha) \tag{5.14}$$

and such that

$$\frac{1}{N^\alpha} > 2S_{II}(n_0,\delta_0,\alpha). \tag{5.15}$$

It is enough to have

$$\frac{1}{N^\alpha} > 2 + 2C_{\delta_0}^{n_0} \frac{1}{n_{\delta_0}^\alpha} \sum_{2\le n\le n_{\delta_0}-1} n^{(\delta_0+\log 2)n_0/\log\log n}, \tag{5.16}$$

since the right-hand side of (5.16) is greater than the right-hand side of (5.14).

The inequality in (5.16) holds for any fixed $\alpha < 0$ and for all N large enough in terms of n_0, δ_0, n_{δ_0}, C_{δ_0}, and α. Therefore, we may take any fixed $\alpha < 0$ as a function of N, n_0, and δ_0 for which (5.15) holds true. For $n_{\delta_0} \geq 16$, we see that

$$\begin{aligned} \sum_{n_{\delta_0} \leq n \leq N-1} \frac{n^{(\delta_0+\log 2)n_0/\log\log n}}{n^{\alpha}} &\leq \sum_{n_{\delta_0} \leq n \leq N-1} \frac{N^{(\delta_0+\log 2)n_0/\log\log N}}{n^{\alpha}} \\ &< N^{(\delta_0+\log 2)n_0/\log\log N} \sum_{n_{\delta_0} \leq n \leq N-1} \frac{1}{n^{\alpha}}. \end{aligned} \tag{5.17}$$

It remains to examine the sum on the far-right hand side of (5.17).

For $\alpha < 0$, we have

$$\sum_{n_{\delta_0} \leq n \leq N-1} \frac{1}{n^{\alpha}} \leq (N-1)^{-\alpha} + \int_{n_{\delta_0}}^{N-1} \frac{dy}{y^{\alpha}} < (N-1)^{-\alpha} \left(\frac{N-\alpha}{1-\alpha} \right).$$

It follows from (5.17) that (5.15) is consequence of

$$N^{-\alpha} > 2N^{(\delta_0+\log 2)n_0/\log\log N}(N-1)^{-\alpha} \left(\frac{N-\alpha}{1-\alpha} \right).$$

One sees that an admissible choice of α is given by

$$\alpha = -3(\delta_0 + \log 2)n_0 \frac{N \log N}{\log\log N}.$$

Then $\zeta_{K,X}(s) \neq 0$ in the left-half plane $\sigma \leq \alpha$. This completes the proof of Lemma 5.1.1.

5.4 Proof of the asymptotic formula for $N_{K,X}(T)$

Assuming for simplicity's sake that T does not coincide with the ordinate of any zero, we have

$$N_{K,X}(T) = \frac{1}{2\pi i} \int_R \frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)}\, ds,$$

where R is the rectangle with vertices at α, β, $\beta + iT$, and $\alpha + iT$. Thus, we have

$$2\pi N_{K,X}(T) = \int_R \operatorname{Im} \left(\frac{\zeta'_{K,X}(s)}{\zeta_{K,X}(s)} \right) ds = \triangle_R \arg \zeta_{K,X}(s), \tag{5.18}$$

where $\triangle_R$ denotes the change in $\arg \zeta_{K,X}(s)$ as s traverses R in the positive sense.

Since $\zeta_{K,X}(s)$ is real and nonzero on $[\alpha, \beta]$, we have

$$\triangle_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma) = 0. \tag{5.19}$$

As s describes the right edge of R, we observe from (5.13) that

$$|\zeta_{K,X}(s) - 1| < 1.$$

It follows that Re $\zeta_{K,X}(\beta + it) > 0$ for $0 \le t \le T$. Hence, we have

$$\triangle_{[0,T]} \arg \zeta_{K,X}(\beta + it) = O(1). \tag{5.20}$$

Furthermore, along the top edge of R, to estimate the change in $\arg \zeta_{K,X}(s)$ we decompose $\zeta_{K,X}(s)$ into its real part and its imaginary part. We have

$$\zeta_{K,X}(s) = \sum_{n \le [X]} a_K(n) \exp\{-(\sigma + it)\log n\} = \sum_{n \le [X]} \frac{a_K(n)[\cos(t \log n) - i \sin(t \log n)]}{n^\sigma},$$

so that

$$\mathrm{Im}(\zeta_{K,X}(\sigma + iT)) = -\sum_{n \le [X]} \frac{a_K(n) \sin(T \log n)}{n^\sigma}.$$

By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [78], Part V, Chapter 1, No. 77), the number of real zeros of $\mathrm{Im}(\zeta_{K,X}(\sigma + iT))$ in the interval $\alpha \le \sigma \le \beta$ is less than or equal to the number of nonzero coefficients $a_K(n)\sin(T \log n)$. By Lemma 5.2.2, the number of nonzero coefficients $a_K(n)$ is $O_q(X(\log\log X/(\log X)^{1-1/\phi(q)})$ at most.

Since the change in argument of $\zeta_{K,X}(\sigma + iT)$ between two consecutive zeros of $\mathrm{Im}(\zeta_{K,X}(\sigma + iT))$ is at most π, it follows that

$$\triangle_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma + iT) = O_q\left(X \left(\frac{\log\log X}{\log X}\right)^{1-1/\phi(q)}\right). \tag{5.21}$$

As in the proof of Lemma 5.1.1, we let N be the largest integer less than or equal to X so that $a_K(N) \ne 0$. Along the left edge of R, we have

$$\zeta_{K,X}(\alpha + it) = \left[1 + \frac{1 + a_K(2)2^{-\alpha-it} + \ldots + a_K(N-1)(N-1)^{-\alpha-it}}{a_K(N)N^{-\alpha-it}}\right] a_K(N)N^{-\alpha-it}.$$

Therefore, we have

$$\triangle_{[0,T]} \arg \zeta_{K,X}(\alpha+it) = \triangle_{[0,T]} \arg \left[1+\frac{1+a_K(2)2^{-\alpha-it}+\ldots+a_K(N-1)(N-1)^{-\alpha-it}}{a_K(N)N^{-\alpha-it}}\right] \tag{5.22}$$
$$+\triangle_{[0,T]} \arg a_K(N)N^{-\alpha-it}.$$

In the proof of Lemma 5.1.1, we noticed that

$$\frac{a_K(N)}{N^\alpha} > \sum_{1\le n\le N-1} \frac{a_K(n)}{n^\alpha}.$$

Thus, for any t, we have

$$\left|\frac{1+a_K(2)2^{-\alpha-it}+\ldots+a_K(N-1)(N-1)^{-\alpha-it}}{a_K(N)N^{-\alpha-it}}\right| < 1,$$

and hence

$$\triangle_{[0,T]} \arg \left[1+\frac{1+a_K(2)2^{-\alpha-it}+\ldots+a_K(N-1)(N-1)^{-\alpha-it}}{a_K(N)N^{-\alpha-it}}\right] = O(1). \tag{5.23}$$

Finally, we have

$$\begin{aligned}\triangle_{[0,T]} \arg a_K(N)N^{-\alpha-it} &= \triangle_{[0,T]} \arg a_K(N)N^{-\alpha}\exp\{-it\log N\} \\ &= \triangle_{[0,T]} \arg \exp\{-it\log N\} \\ &= -T\log N.\end{aligned} \tag{5.24}$$

Then substituting (5.23) and (5.24) into (5.22), we obtain

$$\triangle_{[0,T]} \arg \zeta_{K,X}(\alpha+it) = -T\log N + O(1). \tag{5.25}$$

Since

$$\triangle_R \arg \zeta_{K,X}(s) = \triangle_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma) + \triangle_{[0,T]} \arg \zeta_{K,X}(\beta+it)$$
$$-\triangle_{[\alpha,\beta]} \arg \zeta_{K,X}(\sigma+iT) - \triangle_{[0,T]} \arg \zeta_{K,X}(\alpha+it),$$

we may now substitute (5.19), (5.20), (5.21), and (5.25) into (5.18) to obtain Theorem 5.1.2.

Chapter 6

Family of approximations of L-functions associated to cusp forms

6.1 Main results

6.1.1 Smooth L^2 distance

Let $h(t)$ be a smooth function with the following properties:

(1) $0 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$,

(2) $h(t)$ is compactly supported in a subset of $(0, \infty)$,

(3) $\|h^{(j)}(t)\|_\infty \ll_j 1$ for each $j = 0, 1, 2, \ldots$.

The Fourier transform of $h(t)$ is denoted by $\hat{h}(s)$. Our first result is as follows.

Theorem 6.1.1. *Let h be a smooth function satisfying (1)-(3). Then for any fixed $\epsilon_0 > 0$ and $T^{\epsilon_0} \leq N \leq M \leq T^{1-\epsilon_0}$, we have*

$$\int_{-\infty}^{\infty} h\left(\frac{t}{T}\right)\left|L_f\left(N; \frac{1}{2} + it\right) - L_f\left(M; \frac{1}{2} + it\right)\right|^2 dt \sim 2T\alpha\hat{h}(0)\log\frac{M}{N},$$

where α is given by (1.26).

6.1.2 An inequality

For Re $s > 1$, let

$$L_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

where $\tau(n)$ is the Ramanujan τ-function. In [7], Berndt obtained the inequality

$$|L_\tau(12 - s)| > |L_f(s)|,$$

for $|t| \geq 6.8$ and when $L_\tau(s) \neq 0$. In [84], Spira proved the same inequality but improved the bound to $|t| \geq 4.35$. Very recently in [90], Trudgian improved this bound for t to $|t| \geq 3.8085$. In this chapter, we show that the similar inequality also holds for $L_f(N;s)$. We have the following theorem.

Theorem 6.1.2. *Let N be a positive integer. Then the inequality $|L_f(N;1-s)| > |L_f(N;s)|$ holds for all s with $t > t_k$ and $1/2 < \sigma < 1$, if and only if all the zeros $\beta + i\gamma$ of $L_f(N;s)$ with $\beta \in (0,1)$ and $\gamma > t_k$ lie on the critical line. Here t_k is a real number depending on the weight k of the cusp form f. In particular, $t_{12} = 3.8027$, $t_{14} = 1.8477$, and $t_k = 0$ for $k \geq 16$.*

6.1.3 Zero free region

As with the results in [85], one can prove that the non-trivial zeros of $L_f(1;s)$ and $L_f(2;s)$ lie on the critical line. In the case of primitive Hecke forms the coefficients could be as big as the divisor function $d(n)$, and we will prove our theorem for some restricted primitive Hecke forms.

Theorem 6.1.3. *All the zeros of $L_f(1;s)$ with $|t| > \max(k, e^{16})$ lie on the critical line. Moreover, if $|a(2)| \leq 1$ then all the zeros of $L_f(2;s)$ with $|t| > \max(k, e^{16})$ also lie on the critical line.*

Remark: Numerical computation shows that $a(2) = \tau(2)2^{-11/2} = -.53033$, thus the L-function attached to the Ramanujan τ-function satisfies the above theorem.

We are interested to see whether for $N \geq 3$, the non-trivial zeros of $L_f(N;s)$ lie on the critical line or not. Although it is not clear whether all the non-trivial zeros of $L_f(N;s)$ for $N \geq 3$ lie on the critical line or not, one can prove that a positive proportion of the non-trivial zeros of $L_f(N;s)$ lie on the critical line, provided N is not too large relative to the height T of the ordinates of the non-trivial zeros.

In the following theorem we obtain a 'critical' strip for $L_f(N;s)$. More precisely,

Theorem 6.1.4. *Let $\lambda > 1/2$. There exists a constant N_0 such that if $N \geq N_0$ and $\beta + i\gamma$ is a zero of $L_f(N;s)$ with $|\gamma| \geq 2\pi e N^\lambda$, then*

$$\left|\beta - 1/2\right| \leq \begin{cases} \frac{1}{2\lambda-1}\left(\frac{1}{2} + \frac{4\lambda \log\log N}{\log N}\right), & \textit{if } 1/2 < \lambda \leq 1 \\ \frac{1}{2} + \frac{4\log\log N}{\log N}, & \textit{if } \lambda \geq 1. \end{cases}$$

One also obtains a critical strip for $N \leq N_0$, provided that the ordinates of the zeros are sufficiently large. We have

Theorem 6.1.5. *There exists a constant T_0 such that if $N \geq 1$ and $\beta + i\gamma$ is a zero of $L_f(N;s)$ with*

$|\gamma| \geq \max(2\pi e N, T_0)$, *then*

$$\left|\beta - 1/2\right| \leq 3.$$

6.1.4 Zeros up to height T

Next we will estimate the number of zeros of $L_f(N;s)$ in an interval of the form $(T, T+U]$. We define

$$N(T) = \#\{\rho = \beta + i\gamma : 0 < \gamma < T \text{ and } L_f(N;\rho) = 0\}$$

and

$$N^0(T) = \#\{\rho = 1/2 + i\gamma : 0 < \gamma < T \text{ and } L_f(N;\rho) = 0\}.$$

We have the following theorem.

Theorem 6.1.6. *Let $\lambda > 1/2$. There exists a constant N_0 such that if $N > N_0$, $T > 2\pi e N^{\lambda}$ and $U \geq 2$, then*

$$\begin{aligned} N^0(T+U) - N^0(T) \geq N(T+U) - N(T) + O_f(U \log N) + O_f(N) \\ + O_f\left(\left(\frac{\lambda}{2\lambda - 1}\right)^3 \log(T+U)\right). \end{aligned} \tag{6.1}$$

Furthermore there exists a constant T_0 such that if $N \geq 1$ and $T > \max(2\pi e N, T_0)$ then (6.1) *holds with the last error term replaced by $O_f(\log(T+U))$.*

We end this section with the following result

Theorem 6.1.7. *As $T \to \infty$ and $N = T^{o(1)}$, 100% of the non-trivial zeros of $L_f(N;s)$ up to height T are simple and lie on the critical line.*

The results of this chapter are also discussed in [65].

6.2 Preliminary Results

The following lemmas which may be of independent interest are instrumental in the proof of the theorems.

Lemma 6.2.1. *For $\sigma > 1$ we have*

$$\left(\frac{\sigma-1}{\sigma}\right)^2 < |L_f(s)| < \left(\frac{\sigma}{\sigma-1}\right)^2. \tag{6.2}$$

Proof. Let $\sigma > 1$. From (4.22) and (1.22) we have

$$|L_f(s)| \le \sum_{n=1}^{\infty} \frac{|a(n)|}{n^\sigma} \le \sum_{n=1}^{\infty} \frac{d(n)}{n^\sigma} = \left(\sum_{n=1}^{\infty} \frac{1}{n^\sigma}\right)^2 < \left(1 + \int_1^\infty x^{-\sigma}\,dx\right)^2 = \left(\frac{\sigma}{\sigma-1}\right)^2. \tag{6.3}$$

For the other inequality in (6.2) we use the Euler product (1.34). From (1.22) we have

$$|L_f(s)| = \prod_p |\left(1 - a(p)p^{-s} + p^{-2s}\right)|^{-1} \ge \prod_p \left(1 + d(p)p^{-\sigma} + p^{-2\sigma}\right)^{-1}.$$

Since $d(p) = 2$, we find that

$$|L_f(s)| \ge \prod_p \left(1 + 2p^{-\sigma} + p^{-2\sigma}\right)^{-1} = \prod_p \left(1 + p^{-\sigma}\right)^{-2} = \left(\frac{\zeta(2\sigma)}{\zeta(\sigma)}\right)^2 > \left(\sum_{n=1}^{\infty} \frac{1}{n^\sigma}\right)^{-2} > \left(\frac{\sigma-1}{\sigma}\right)^2,$$

where in the ultimate step we used the last three inequalities in (6.3). This completes the proof of the lemma. □

Lemma 6.2.2. *For $\sigma > 1$,*

$$\left|\sum_{n>N} \frac{a(n)}{n^s}\right| \le \frac{N^{1-\sigma}}{\sigma-1}\left(\log N + 2\gamma + \frac{1}{\sigma-1}\right) + O\left(\frac{1}{\sqrt{N}}\right). \tag{6.4}$$

For $\sigma \le 0$ we have the following:

$$\left|\sum_{n\le N} \frac{a(n)}{n^s}\right| \le N^{1-\sigma}(\log N + 2\gamma - 1) + O(N^{-\sigma+1/2}).$$

Proof. Let $\sigma > 1$. From (1.22) and by partial summation we have

$$\left|\sum_{n>N} \frac{a(n)}{n^s}\right| \le \sum_{n>N} \frac{d(n)}{n^\sigma} = \sigma \int_N^\infty D(t)t^{-1-\sigma}\,dt - D(N)N^{-\sigma}, \tag{6.5}$$

where

$$D(t) = \sum_{n\le t} d(n) = t(\log t + 2\gamma - 1) + O(\sqrt{t}). \tag{6.6}$$

Combining (6.5) and (6.6) we obtain the bound in (6.4).

For the second part of the lemma, let $\sigma \le 0$. We have

$$\left| \sum_{n \le N} \frac{a(n)}{n^s} \right| \le \sum_{n \le N} \frac{d(n)}{n^\sigma} \tag{6.7}$$

By using (6.6) one sees that

$$\sum_{n \le N} \frac{d(n)}{n^\sigma} \le N^{-\sigma} \sum_{n \le N} d(n) = N^{1-\sigma}(\log N + 2\gamma - 1) + O(N^{-\sigma+1/2}), \tag{6.8}$$

where in the penultimate step we use the fact that $x^{-\sigma}$ is increasing for $\sigma \le 0$. One finishes the proof of the lemma by combining (6.7) and (6.8). □

Lemma 6.2.3. *If $|t| > k$ and $1/2 < \sigma < (k-1)/2$ then*

$$\frac{\partial}{\partial \sigma}\left(\log \frac{1}{|\chi_f(s)|} \right) > 2 \log |t| - 3.7.$$

Proof. By Stirling's formula [37] we have

$$\log \Gamma(s) = (s - 1/2) \log s - s + \frac{1}{2} \log 2\pi + \frac{1}{12s} - 2 \int_0^\infty \frac{P_3(x)}{(s+x)^3}\, dx, \tag{6.9}$$

where $P_3(x)$ is a function of period 1 and given by

$$P_3(x) = \frac{x}{12}(2x^2 - 3x + 1),$$

for $x \in [0,1]$. A straightforward computation shows that

$$|6P_3(x)| \le \frac{\sqrt{3}}{36}, \tag{6.10}$$

for $x \in [0,1]$. Since

$$\frac{\partial}{\partial \sigma}\left(\log \frac{1}{|\chi_f(s)|} \right) = -\ \mathrm{Re}\ \left(\frac{\partial}{\partial \sigma} \log \chi_f(s) \right) = -\ \mathrm{Re}\ \left(\frac{\partial}{\partial s} \log \chi_f(s) \right),$$

then from (1.28) and (6.9) we find

$$\frac{\partial}{\partial \sigma}\left(\log \frac{1}{|\chi_f(s)|} \right) = \ \mathrm{Re}\ \left(-\frac{1}{2s+k-1} - \frac{1}{3(k+2s-1)^2} + \frac{1}{2s-k-1} - \frac{1}{3(k-2s+1)^2} \right. \tag{6.11}$$

$$+\log\left(\frac{k-1}{2}+s\right)+\log\left(\frac{k+1}{2}-s\right)-2\log(2\pi)$$
$$+6\int_0^\infty \frac{P_3(x)}{(s+(k-1)/2+x)^4}\,dx+6\int_0^\infty \frac{P_3(x)}{((k+1)/2-s+x)^4}\,dx\Bigg).$$

From the hypothesis we have $t>k$ and $1/2<\sigma<(k-1)/2$. Then from (6.11) we derive

$$\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|\chi_f(s)|}\right)>2\log|t|-2\log 2\pi-\frac{k}{t^2}-\frac{\sqrt{3}\pi}{72|t|^3}>2\log|t|-3.7.$$

Here we use the fact that $k\geq 12$. This proves the lemma.

$\square$

Lemma 6.2.4. *If $|t|>20$ and $\sigma>1/2$ then*

$$|\chi_f(s)|<1.02\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma}.$$

Proof. From [85, 84], we have

$$|\Gamma(s)|=(2\pi)^{1/2}e^{-\sigma}|s|^{\sigma-1/2}e^{-t\arg s}|\exp(R_1(s)+1/12s)|, \tag{6.12}$$

where $R_1(s)<1/6|s|$. Hence by (1.28) and (6.12) we find

$$|\chi_f(s)|=\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma}\exp(t(\arg((k+1)/2-s)+\arg((k-1)/2+s))))\times \tag{6.13}$$
$$\frac{\left|1-\frac{k+1}{2s}\right|^{k/2-\sigma}}{\left|1+\frac{k-1}{2s}\right|^{(k-2)/2+\sigma}}\frac{|\exp(R_1((k+1)/2-s)+1/12((k+1)/2-s))|}{|\exp(R_1(((k-1)/2+s))+1/12((k-1)/2+s))|}.$$

Next we denote

$$z=R_1\left(\frac{k-1}{2}+s\right)-R_1\left(\frac{k+1}{2}-s\right)+\frac{1}{12\left(\frac{k-1}{2}+s\right)}-\frac{1}{12\left(\frac{k+1}{2}-s\right)}.$$

Therefore

$$|z|\leq\frac{1}{12\left|\frac{k-1}{2}+s\right|}+\frac{1}{12\left|\frac{k+1}{2}-s\right|}+\frac{1}{6\left|\frac{k-1}{2}+s\right|}+\frac{1}{6\left|\frac{k+1}{2}-s\right|}\leq\frac{1}{2|t|}\leq\frac{1}{40}.$$

Since $|z| \le 1/40 < 1$, we have

$$|e^z| \ge 1 - |z| \left(\frac{1}{1-|z|} \right) \ge 38/39. \tag{6.14}$$

Clearly

$$t \left(\arg \left(\frac{k+1}{2} - s \right) + \arg \left(s + \frac{k-1}{2} \right) \right) < 0. \tag{6.15}$$

Combining (6.13), (6.14), and (6.15) we obtain

$$|\chi_f(s)| < 1.02 \left(\frac{|s|}{2\pi e} \right)^{1-2\sigma},$$

which proves the lemma. □

6.3 Proof of theorem 6.1.1

First we define

$$h_T(t) := h\left(\frac{t}{T}\right)$$

and

$$I := \int_{-\infty}^{\infty} h_T(t) \left| L_f\left(N; \frac{1}{2} + it\right) - L_f\left(M; \frac{1}{2} + it\right) \right|^2 dt.$$

From (1.36) one has

$$\begin{aligned}
I = \int_{-\infty}^{\infty} h_T(t) &\left(\sum_{N \le n \le M} a(n) n^{-\frac{1}{2}-it} + \chi_f\left(\frac{1}{2} + it\right) \sum_{N \le n \le M} a(n) n^{-\frac{1}{2}+it} \right) \\
&\times \left(\sum_{N \le m \le M} a(m) m^{-\frac{1}{2}+it} + \chi_f\left(\frac{1}{2} - it\right) \sum_{N \le m \le M} a(m) m^{-\frac{1}{2}-it} \right) dt \\
= \sum_{N \le m,n \le M} \frac{1}{\sqrt{mn}} &\int_{-\infty}^{\infty} h_T(t) \left(a(n) n^{-it} + \chi_f\left(\frac{1}{2} + it\right) a(n) n^{it} \right) \left(a(m) m^{it} + \chi_f\left(\frac{1}{2} - it\right) a(m) m^{-it} \right) dt \\
= \sum_{N \le m,n \le M} \frac{a(n)a(m)}{\sqrt{mn}} &\left(\int_{-\infty}^{\infty} h_T(t) \left(\left(\frac{n}{m}\right)^{it} + \left(\frac{n}{m}\right)^{-it} \right) dt \right.
\end{aligned}$$

$$+\int_{-\infty}^{\infty} h_T(t)\left(\chi_f\left(\frac{1}{2}+it\right)(nm)^{it}+\chi_f\left(\frac{1}{2}-it\right)(nm)^{-it}\right)dt\right),$$

where in the last step we utilized the fact (1.33). Let us denote

$$I_1 := \sum_{N\le m,n\le M}\frac{a(n)a(m)}{\sqrt{mn}}\int_{-\infty}^{\infty} h_T(t)\left(\left(\frac{n}{m}\right)^{it}+\left(\frac{n}{m}\right)^{-it}\right)dt$$

and

$$I_2 := \sum_{N\le m,n\le M}\frac{a(n)a(m)}{\sqrt{mn}}\int_{-\infty}^{\infty} h_T(t)\left(\chi_f\left(\frac{1}{2}+it\right)(nm)^{it}+\chi_f\left(\frac{1}{2}-it\right)(nm)^{-it}\right)dt.$$

The diagonal terms $m = n$ of I_1 contribute

$$\sum_{N\le m\le M}\frac{2a(m)^2}{m}\int_{-\infty}^{\infty} h\left(\frac{t}{T}\right)dt = 2T\hat{h}(0)\sum_{N\le m\le M}\frac{a(m)^2}{m}. \tag{6.16}$$

The off-diagonal terms $m \neq n$ of I_1 can be written as

$$\begin{aligned}
&\sum_{N\le m\neq n\le M}\frac{a(m)a(n)}{\sqrt{mn}}\int_{-\infty}^{\infty} h_T(t)\left(\left(\frac{n}{m}\right)^{it}+\left(\frac{n}{m}\right)^{-it}\right)dt\\
&=\sum_{N\le m\neq n\le M}\frac{a(m)a(n)}{\sqrt{mn}}\int_{-\infty}^{\infty} h_T(t)\left(e^{it\log\frac{n}{m}}+e^{-it\log\frac{n}{m}}\right)dt\\
&=\sum_{N\le m< n\le M}\frac{2a(m)a(n)}{\sqrt{mn}}\int_{-\infty}^{\infty} h_T(t)\left(e^{it\log\frac{n}{m}}+e^{-it\log\frac{n}{m}}\right)dt\\
&=\sum_{N\le m< n\le M}\frac{2a(m)a(n)}{\sqrt{mn}}(S_{11}(m,n)+S_{12}(m,n)),
\end{aligned}\tag{6.17}$$

where

$$S_{11}(m,n) := \int_{-\infty}^{\infty} h_T(t)e^{it\log\frac{n}{m}}dt$$

and

$$S_{12}(m,n) := \int_{-\infty}^{\infty} h_T(t)e^{-it\log\frac{n}{m}}dt.$$

Integrating by parts one obtains

$$S_{11}(m,n) = \int_{-\infty}^{\infty} h_T(t) e^{it\log\frac{n}{m}} dt = \frac{(-1)^r}{T^r} \int_{-\infty}^{\infty} h^{(r)}\left(\frac{t}{T}\right) \frac{e^{it\log\frac{n}{m}}}{(i\log\frac{n}{m})^r} dt \tag{6.18}$$

for any positive integer r. Note that

$$\log(1+\frac{n-m}{m}) \geq \log(1+\frac{1}{m}) \geq \frac{1}{2m} \tag{6.19}$$

for large m. Using (6.19) in (6.18) we find

$$S_{11}(m,n) \ll \left(\frac{2m}{T}\right)^r \int_{-\infty}^{\infty} \left| h^{(r)}\left(\frac{t}{T}\right)\right| dt \ll \|h^{(r)}\|_\infty \left(\frac{(2m)^r}{T^{r-1}}\right).$$

Similarly

$$S_{12}(m,n) \ll \|h^{(r)}\|_\infty \left(\frac{(2m)^r}{T^{r-1}}\right).$$

Therefore

$$\begin{aligned}\sum_{N\leq m<n\leq M} \frac{2a(m)a(n)}{\sqrt{mn}} (S_{11}(m,n)+S_{12}(m,n)) &\ll \|h^{(r)}\|_\infty \left(\sum_{N\leq m<n\leq M} \frac{2a(m)a(n)}{\sqrt{mn}} \frac{(2m)^r}{T^{r-1}}\right) \\ &\ll_f \left(\frac{M^{r+3}}{T^{r-1}}\right) \end{aligned} \tag{6.20}$$

for any positive integer r. Combining (6.16), (6.17), and (6.20) we see

$$I_1 = 2T\hat{h}(0) \sum_{N\leq m\leq M} \frac{a(m)^2}{m} + O_f\left(\frac{M^{r+3}}{T^{r-1}}\right). \tag{6.21}$$

Next we estimate I_2. Let

$$I_2 = \sum_{N\leq m,n\leq M} \frac{a(n)a(m)}{\sqrt{mn}} (S_{21}(m,n)+S_{22}(m,n)),$$

where

$$S_{21}(m,n) := \int_{-\infty}^{\infty} h_T(t)\chi_f\left(\frac{1}{2}+it\right) e^{it\log(nm)} dt \tag{6.22}$$

and

$$S_{22}(m,n) := \int\limits_{-\infty}^{\infty} h_T(t)\chi_f\left(\frac{1}{2}-it\right)e^{-it\log(nm)}dt.$$

Recall Stirling's formula (6.9) in the form

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \tag{6.23}$$

as $|s|\to\infty$ and $|\arg s|\le \pi-\epsilon$. Then from (1.28) and (6.23) we have

$$\begin{aligned}
\log\chi_f(s) &= (2s-1)\log 2\pi + \log\Gamma\left(\frac{k+1}{2}-s\right) - \log\Gamma\left(\frac{k-1}{2}+s\right)\\
&= (2s-1)\log 2\pi + \left(\frac{k}{2}-s\right)\left(\log s - i\pi - \frac{k+1}{2s} + O\left(\frac{1}{|s|^2}\right)\right) - \left(\frac{k+1}{2}-s\right)\\
&\quad - \left(\frac{k}{2}-1+s\right)\left(\log s + \frac{k-1}{2s} + O\left(\frac{1}{|s|^2}\right)\right) + \left(\frac{k-1}{2}+s\right) + O\left(\frac{1}{|s|}\right)\\
&= (1-2s)\log\frac{s}{2\pi} - \frac{i\pi}{2}(k-2s) + 2s + O_k\left(\frac{1}{|s|}\right).
\end{aligned} \tag{6.24}$$

In particular

$$\chi_f\left(\frac{1}{2}+it\right) = \exp\left\{-2it(\log t - \log(2e\pi)) - \frac{i\pi}{2}(k-1)\right\}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{6.25}$$

for $t\ge 1$. Combining (6.25) and (6.22) one has

$$\begin{aligned}
S_{21}(m,n) &= \int\limits_0^{\infty} h_T(t)\exp\left\{-it(2\log t - \log mn - 2\log(2e\pi)) - \frac{i\pi}{2}(k-1)\right\}\left(1+O\left(\frac{1}{|t|}\right)\right)dt\\
&= \int\limits_0^{\infty} h_T(t)e^{iF(t)}dt + O(\|h\|_\infty \log T),
\end{aligned} \tag{6.26}$$

where $F(t) := -t(2\log t - \log mn - 2\log(2e\pi)) - \frac{\pi}{2}(k-1)$. Note that

$$|F'(t)| = \left|2\log\frac{2\pi}{t} + \log mn\right| \gg_{f,\epsilon_0} \log T \tag{6.27}$$

for all t in the support of the function h_T and $m, n \leq T^{1-\epsilon_0}$. Then from (6.27) and by integrating by parts we have

$$\int_0^\infty h_T(t) e^{iF(t)} dt = \int_{T^\epsilon}^\infty \frac{h_T(t)}{iF'(t)} d\left(e^{iF(t)}\right)$$
$$\leq \int_0^\infty \left(\frac{|h'(t/T)|}{T|F'(t)|} + \frac{n_0 |h(t/T)|}{t|F'(t)|^2} \right) dt$$
$$\ll_{f,\epsilon_0} \frac{1}{\log T} \max(\|h\|_\infty, \|h'\|_\infty). \tag{6.28}$$

Combining (6.26) and (6.28) we have

$$S_{21}(m,n) \ll_{f,\epsilon_0} \log T.$$

Similarly one obtains

$$S_{22}(m,n) \ll_{f,\epsilon_0} \log T.$$

Putting these together we arrive at

$$I_2 = \sum_{N \leq m,n \leq M} \frac{a(m)a(n)}{\sqrt{mn}} (S_{21}(m,n) + S_{22}(m,n))$$
$$\ll_{f,\epsilon_0,h} \log T \sum_{N \leq m,n \leq M} \frac{a(m)a(n)}{\sqrt{mn}}$$
$$\ll_{f,\epsilon_0} M^{1+\epsilon} \log T. \tag{6.29}$$

Hence from (6.21), (6.29), and using that $M \leq T^{1/2-\epsilon_0} \leq T^{1-\epsilon_0}$ we have

$$I = 2T\hat{h}(0) \sum_{N \leq m \leq M} \frac{a(m)^2}{m} + O_{f,\epsilon_0}\left(T^{4-\epsilon_0(r+3)}\right) + O_{f,\epsilon_0}\left(T^{(1-\epsilon_0)(1+\epsilon)} \log T\right).$$

Thus by choosing $\epsilon < \epsilon_0$ and r large enough we deduce

$$I = 2T\hat{h}(0) \sum_{N \leq m \leq M} \frac{a(m)^2}{m} (1 + o_f(1))$$

for $T \to \infty$. Finally by partial summation, (1.25), and (1.26) we conclude

$$I \sim 2\alpha T\hat{h}(0)\log\frac{M}{N}$$

for $M, N \geq T^{\epsilon_0}$ and $T \to \infty$. This completes the proof of the theorem.

6.4 Proof of Theorem 6.1.2

We first prove the following theorem.

Theorem 6.4.1. *There exists a number t_k, such that for $1/2 < \sigma < 1$ and $|t| > t_k$ we have*

$$|L_f(N; 1-s)| > |L_f(N; s)|,$$

whenever $L_f(N; s) \neq 0$. Moreover the above holds with $t_{12} = 3.8027$, $t_{14} = 1.8477$ and $t_k = 0$ for $t \geq 16$.

Proof. From (1.37) we have

$$L_f(N; 1-s) = g(s)L_f(N; s),$$

where $g(s) = 1/\chi_f(s)$. From (1.28) one can see that $g(s)$ is analytic for all s with $t \neq 0$ and hence continuous for such s. Let $s_0 = \frac{1}{2} + it$. Then from (1.28) we have

$$|g(s_0)| = \left|(2\pi)^{-2it}\frac{\Gamma(k/2+it)}{\Gamma(k/2-it)}\right| = 1.$$

Define $h(s) = \log\left|\frac{g(s)}{g(s_0)}\right|$. It suffices to prove that $h(s) > 0$ for $1/2 < \sigma < 1$ provided $|t| \geq t_k$. We have

$$\begin{aligned}
h(s) &= \log\left|(2\pi)^{-(2s-1-2it)}\frac{\Gamma\left(\frac{k-1}{2}+s\right)\Gamma\left(\frac{k}{2}-it\right)}{\Gamma\left(\frac{k+1}{2}-s\right)\Gamma\left(\frac{k}{2}+it\right)}\right| \\
&= -(2\sigma-1)\log 2\pi + \log\left|\Gamma\left(\tfrac{k-1}{2}+s\right)\right| - \log\left|\overline{\Gamma\left(\tfrac{k+1}{2}-s\right)}\right| \\
&= -(2\sigma-1)\log 2\pi + \log\left|\Gamma\left(\tfrac{k-1}{2}+\sigma+it\right)\right| - \log\left|\Gamma\left(\tfrac{k+1}{2}-\sigma+it\right)\right| \\
&= -(2\sigma-1)\log 2\pi + (2\sigma-1)\frac{\partial}{\partial\sigma}\log\left|\Gamma\left(\sigma+it\right)\right|\Big|_{\sigma=\sigma_1},
\end{aligned}$$

for some σ_1 between $\frac{k-1}{2}$ and $\frac{k+1}{2}$. Thus it suffices to prove that

$$\frac{\partial}{\partial\sigma}\log\left|\Gamma\left(\sigma+it\right)\right|\Big|_{\sigma=\sigma_1} - \log 2\pi > 0,$$

for all $\frac{k-1}{2} \le \sigma_1 \le \frac{k+1}{2}$. Now from (6.9) we have

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \log |\Gamma(\sigma+it)| &= \frac{\partial}{\partial \sigma} \text{ Re } \log \Gamma(\sigma+it) \qquad (6.30)\\
&= \text{ Re } \frac{\partial}{\partial \sigma} \log \Gamma(\sigma+it) \\
&= \text{ Re } \frac{\partial}{\partial s} \log \Gamma(s) \\
&= \text{ Re } \left(\log s - \frac{1}{2s} - \frac{1}{12s^2} + 6 \int_0^\infty \frac{P_3(x)}{(s+x)^4}\, dx \right) \\
&= \log \sqrt{\sigma^2+t^2} - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} \\
&\qquad + 6 \int_0^\infty \frac{P_3(x)((\sigma+x)^4 - 6(\sigma+x)^2t^2 + t^4)}{\left((\sigma+x)^2+t^2\right)^4}\, dx.
\end{aligned}$$

Using (6.10) in combination with the inequality $(\sigma+x)^4 - 6(\sigma+x)^2t^2 + t^4 \le \left((\sigma+x)^2+t^2\right)^2$ and (6.30) we derive

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \log |\Gamma(\sigma+it)| &\ge \log \sqrt{\sigma^2+t^2} - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} - \frac{\sqrt{3}}{36} \int_0^\infty \frac{dx}{\left((\sigma+x)^2+t^2\right)^2} \qquad (6.31)\\
&=: G(\sigma) - I(\sigma),
\end{aligned}$$

where $I(\sigma)$ is the integral part and $G(\sigma)$ is the non integral part of (6.31). Here $I(\sigma)$ is a decreasing function of σ and hence

$$I(\sigma) \le \frac{\sqrt{3}}{36} \int_0^\infty \frac{dx}{\left(\left(x+\frac{k-1}{2}\right)^2+t^2\right)^2} = \frac{\sqrt{3}}{72t^3} \left(\tan^{-1}\left(\frac{2t}{k-1}\right) - \frac{2t(k-1)}{4t^2+(k-1)^2} \right).$$

Next

$$G'(\sigma) = \frac{\sigma^3\left(6\sigma^2+3\sigma+1\right) + 3\sigma\left(4\sigma^2-1\right)t^2 + (6\sigma-3)t^4}{6\left(\sigma^2+t^2\right)^3},$$

thus $G(\sigma)$ is increasing on $\frac{k-1}{2} \le \sigma \le \frac{k+1}{2}$ for $k \ge 12$. Hence

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \log |\Gamma(\sigma+it)| - \log 2\pi &\ge G\left(\frac{k-1}{2}\right) - \frac{\sqrt{3}}{72t^3}\left(\tan^{-1}\left(\frac{2t}{k-1}\right) - \frac{2t(k-1)}{4t^2+(k-1)^2}\right) - \log 2\pi \\
&\ge \frac{4(4-3k)t^2 - (k-1)^2(3k-2)}{3\left((k-1)^2+4t^2\right)^2} + \frac{1}{2}\log\left(\frac{1}{4}(k-1)^2+t^2\right) \\
&\qquad - \frac{\sqrt{3}}{72t^3}\left(\tan^{-1}\left(\frac{2t}{k-1}\right) - \frac{2t(k-1)}{4t^2+(k-1)^2}\right) - \log 2\pi
\end{aligned}$$

$$=: H(t,k).$$

Let us fix $t > 0$ and consider k as a real variable for a moment. Then

$$\frac{\partial}{\partial k}H(t,k) = \frac{9(k-2)\left((k-1)^2+4t^2\right)^2 + 2\left(9(k-3)k+\sqrt{3}+18\right)\left((k-1)^2+4t^2\right) + 24(k-1)^3}{9\left((k-1)^2+4t^2\right)^3} \geq 0,$$

for $k \geq 12$. Hence for every fixed $t > 0$, $H(t,k)$ is monotonically increasing with respect to the variable k. Next let $k \geq 12$ be a fixed number and vary t. Let

$$\begin{aligned}\frac{M(t,k)}{t^4} &:= \frac{\partial}{\partial t}H(t,k) \\ &= \frac{384(3(k-1)k-1)t^6 + 16(k-1)\left(9k((k-1)k+1)-5\sqrt{3}-9\right)t^4}{36t^3\left((k-1)^2+4t^2\right)^3} \\ &\quad - \frac{32\sqrt{3}(k-1)^3t^2 + 3\sqrt{3}(k-1)^5 - 2304t^8}{36t^3\left((k-1)^2+4t^2\right)^3} + \frac{6\sqrt{3}}{144t^4}\tan^{-1}\left(\frac{2t}{k-1}\right).\end{aligned} \tag{6.32}$$

One finds that

$$\begin{aligned}&\frac{\partial}{\partial t}M(t,k) \\ &= \frac{4t^4\left(48\left(33k^2-60k+25\right)t^4 + 4(k-1)\left(3k\left(39k^2-81k+25\right)+8\sqrt{3}+51\right)t^2\right)}{9\left((k-1)^2+4t^2\right)^4} \\ &\qquad + \frac{4t^4\left((k-1)^3\left(45k((k-1)k+1)+8\sqrt{3}-45\right)+1728t^6\right)}{9\left((k-1)^2+4t^2\right)^4} \\ &\geq 0,\end{aligned} \tag{6.33}$$

for all $t > 0$. Therefore combining (6.32), (6.33) and the fact that $M(0,k) = 0$, we conclude that $H(t,k)$ is monotonically increasing with respect to t for $t > 0$ and fixed $k \geq 12$. One can check that $H(3.8027, 12) > 0$, $H(1.8477, 14) > 0$ and $H(t, 16) > 0$ for al $t > 0$, which completes the proof of Theorem 6.4.1.

□

By the functional equation (1.37), $L_f(N;s)$ and $L_f(N;1-s)$ have the same zeros for $0 < \sigma < 1$. Hence Theorem 6.4.1 implies Theorem 6.1.2.

6.5 Proof of Theorem 6.1.3

The proof follows closely the approach from [85]. For the sake of completeness we provide the details below. From (1.36) we have

$$L_f(1;s) = 1 + \chi_f(s). \tag{6.34}$$

Now from the proof of Theorem 6.4.1 we have for $t > 3.8027$ and $\sigma > 1/2$

$$|\chi_f(s)| < 1. \tag{6.35}$$

Therefore from (6.34) and (6.35) we find that for $t > 3.8027$ and $\sigma > 1/2$,

$$|L_f(1;s)| \geq 1 - |\chi_f(s)| > 0.$$

From Theorem 6.1.7 and Theorem 6.1.2 we conclude that, all the complex zeros of $L_f(1;s)$ lie on the line $\sigma = 1/2$ for $t > 3.8027$.

Again from (1.36) we see that

$$|L_f(2;s)| \geq \left|1 + \frac{a(2)}{2^s}\right| - |\chi_f(s)| \left|1 + \frac{a(2)}{2^{1-s}}\right|.$$

So it suffices to prove that for large enough t and $\sigma > 1/2$,

$$1/|\chi_f(s)| > \left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right|. \tag{6.36}$$

Let

$$g_1(s) = \chi_f(s) \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}.$$

Then $|g_1(1/2 + it)| = 1$. Define

$$l(s) = \log \left| \frac{g_1(s)}{g_1(1/2 + it)} \right|.$$

Proceeding as in the proof of Theorem 6.4.1 one can derives that

$$l(s) = \left(\sigma - \frac{1}{2}\right) \frac{\partial}{\partial \sigma} \left(\log \frac{1}{|\chi_f(s)|} - \log \left| \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right| \right) \Bigg|_{\sigma=\sigma_1},$$

for some σ_1 in $[1/2, 1]$. We want to show that

$$\frac{\partial}{\partial \sigma} \left(\log \frac{1}{|\chi_f(s)|} \right) \Bigg|_{\sigma=\sigma_1} > \frac{\partial}{\partial \sigma} \left(\log \left| \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right| \right) \Bigg|_{\sigma=\sigma_1}, \tag{6.37}$$

for some $\sigma_1 \in (1/2, 1)$. We distinguish two cases according as to when $1/2 < \sigma \leq 3/4$ and respectively when $3/4 < \sigma < 1$. We have

$$\begin{aligned} \frac{\partial}{\partial \sigma} \left(\log \left| \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right| \right) &= \ \mathrm{Re}\ \frac{\partial}{\partial s} \left(\log \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right) \\ &= a(2) \log 2 \ \mathrm{Re}\ \left(\frac{a(2) + 2^{s-1} + 2^{-s}}{(1 + a(2)2^{s-1})(1 + a(2)2^{-s})} \right). \end{aligned}$$

Then for $1/2 < \sigma \leq 3/4$, using (1.23) we have

$$\frac{\partial}{\partial \sigma} \left(\log \left| \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right| \right) \leq \log 2 \left(\frac{1 + 2^{\sigma-1} + 2^{-\sigma}}{(1 - 2^{\sigma-1})(1 - 2^{-\sigma})} \right) < 27. \tag{6.38}$$

Therefore for $1/2 < \sigma \leq 3/4$, by Lemma 6.2.3 and (6.38) we find that the inequality (6.37) holds when $2 \log |t| > 27 + 3.7$. In particular one can take $t > e^{16}$. Now consider the case $3/4 < \sigma < 1$. One can see by (1.23) that

$$\left| \frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}} \right| \leq \left| \frac{1 + 2^{\sigma-1}}{1 - 2^{-\sigma}} \right| \leq \frac{1 + 2}{1 - 2^{-3/4}} < 5. \tag{6.39}$$

Then from (6.39) and Lemma 6.2.4, it is enough to show that

$$.98 \left(\frac{|s|}{2\pi e} \right)^{2\sigma-1} > 5 \tag{6.40}$$

in order to prove the inequality (6.36). Here (6.40) holds true for $t > 445$. For $\sigma \geq 1$, $1 + 2^{\sigma-1} \leq 2^{\sigma}$, and (6.40) transforms to

$$.98 \left(\frac{|s|}{2\sqrt{2}\pi e} \right)^{2\sigma-1} > \frac{\sqrt{2}}{1 - 2^{-3/4}}. \tag{6.41}$$

Numerical computation shows that $t > 86$ satisfies (6.41) for $\sigma \geq 1$. This completes the proof of the theorem.

6.6 Proof of Theorems 6.1.4 and 6.1.5

Let $\rho_N = \beta_N + i\gamma_N$ be a complex zero of $L_f(N;s)$ with $|\gamma_N| \geq 2\pi e N^{\lambda}$. We will show that $L_f(N;s)$ never vanishes for

$$\beta_N > \frac{\lambda}{2\lambda - 1}\left(1 + \frac{4\log\log N}{\log N}\right),$$

when $1/2 < \lambda \leq 1$ and is nonzero for

$$\beta_N > 1 + \frac{4\log\log N}{\log N},$$

when $\lambda > 1$. Then one concludes the proof of the theorem by using the functional equation (1.37). Let s be such that $|t| \geq 2\pi e N^{\lambda}$ with $\lambda > 1/2$ and

$$\sigma > \max\left(1, \frac{\lambda + \epsilon}{2\lambda - 1}\right)\left(1 + \frac{c\log\log N}{\log N}\right), \tag{6.42}$$

where $\epsilon > 0$ is arbitrary and c is a positive constant which will be determined later. From (1.36) we have

$$|L_f(N;s)| \geq \left|\sum_{n \leq N} \frac{a(n)}{n^s}\right| - |\chi_f(s)| \left|\sum_{n \leq N} \frac{a(n)}{n^{1-s}}\right|. \tag{6.43}$$

Consider the right-hand side of (6.43), We will obtain an upper bound for the first sum and a lower bound for the second sum. By Lemmas 6.2.1 and 6.2.2 we see that

$$\begin{aligned} \left|\sum_{n \leq N} \frac{a(n)}{n^s}\right| &\geq |L_f(s)| - \left|\sum_{n > N} \frac{a(n)}{n^s}\right| \\ &> \left(\frac{\sigma - 1}{\sigma}\right)^2 - \frac{N^{1-\sigma}}{\sigma - 1}\left(\log N + 2\gamma + \frac{1}{\sigma - 1}\right) + O\left(\frac{1}{\sqrt{N}}\right). \end{aligned} \tag{6.44}$$

Since by (6.42) we always have

$$\sigma > 1 + \frac{c\log\log N}{\log N},$$

then from (6.44) we have

$$\left|\sum_{n \leq N} \frac{a(n)}{n^s}\right| > \left(\frac{c\log\log N}{\log N + c\log\log N}\right)^2$$

$$- \frac{1}{\log^c N} \left(\frac{\log N}{c \log\log N} \right) \left(\log N + 2\gamma + \frac{\log N}{c \log\log N} \right) + O\left(\frac{1}{\sqrt{N}} \right).$$

Therefore for $c = 4$ one find that

$$\left| \sum_{n \le N} \frac{a(n)}{n^s} \right| > \left(\frac{\log\log N}{\log N} \right)^2, \tag{6.45}$$

for sufficiently large N. Now by Lemmas 6.2.2, 6.2.4, for $|t| > 2\pi e N^\lambda$, and $|t| > 20$, we find that

$$|\chi_f(s)| \left| \sum_{n \le N} \frac{a(n)}{n^{1-s}} \right| < 1.02 \left(\frac{|s|}{2\pi e} \right)^{1-2\sigma} N^\sigma \left(\log N + 2\gamma - 1 + O(N^{-1/2}) \right). \tag{6.46}$$

Then from (6.46), fixed $\epsilon > 0$ and large N we write

$$|\chi_f(s)| \left| \sum_{n \le N} \frac{a(n)}{n^{1-s}} \right| < 2.04 \left(\frac{|s|}{2\pi e} \right)^{1-2\sigma} N^{\sigma+\epsilon} < 2.04 N^{\lambda(1-2\sigma)+\sigma+\epsilon}. \tag{6.47}$$

If $1/2 < \lambda < 1 + \epsilon$, then by (6.42) the exponent of N in (6.47) can be written as

$$\lambda(1-2\sigma) + \sigma + \epsilon = \lambda + \epsilon - \sigma(2\lambda - 1) < -c(\lambda+\epsilon) \frac{\log\log N}{\log N} < -c(1+2\epsilon) \frac{\log\log N}{2 \log N}.$$

If $\lambda \ge 1 + \epsilon$, then the exponent of N in (6.47) is

$$\lambda(1-2\sigma) + \sigma + \epsilon \le (1+\epsilon)(1-2\sigma) + \sigma + \epsilon = (1-\sigma)(1+2\epsilon) < -c(1+2\epsilon) \frac{\log\log N}{\log N}.$$

By combining the above two cases and using (6.47), we derive

$$\left| \chi_f(s) \sum_{n \le N} \frac{a(n)}{n^{1-s}} \right| < \frac{2.04}{\log^{c/2} N}. \tag{6.48}$$

Finally choose $c = 4$. Then from (6.45) and (6.48) we have

$$|L_f(N;s)| > \left(\frac{\log\log N}{\log N} \right)^2 - \frac{2.04}{\log^2 N} > 0,$$

for N large enough. Therefore there exists a $N_0 > 0$ such that when $N > N_0$, then $L_f(N;s) \neq 0$ in the region

$$\sigma > \max\left(1, \frac{\lambda+\epsilon}{2\lambda-1}\right)\left(1+\frac{4\log\log N}{\log N}\right), \quad |t| \geq 2\pi e N^{\lambda},$$

for $\lambda > 1/2$ and any number $\epsilon > 0$. Which complete the proof of Theorem 6.1.4.

We now prove Theorem 6.1.5. It is enough to consider the case $N \geq 2$. Suppose $T > T_0$ for some large constant T_0. Let $\sigma \geq 2$ and $|t| > \max(2\pi e N, T_0)$. From (6.43) and using the trivial bound $d(n) \leq n$ we have

$$\begin{aligned} |L_f(N;s)| &\geq |L_f(s)| - \sum_{n\leq N}\frac{d(n)}{n^{\sigma}} - |\chi_f(s)|\sum_{n\leq N}\frac{d(n)}{n^{1-\sigma}} \\ &> \left(\frac{\sigma-1}{\sigma}\right)^2 - \frac{N^{2-\sigma}}{\sigma-2} - 1.02\left(\frac{|s|}{2\pi e}\right)^{1-2\sigma}\left(N^{\sigma}+\frac{N^{1+\sigma}}{1+\sigma}\right) \\ &> \left(\frac{\sigma-1}{\sigma}\right)^2 - \frac{2^{2-\sigma}}{\sigma-2} - 1.02(2)^{1-2\sigma}\left(2^{\sigma}+\frac{2^{1+\sigma}}{1+\sigma}\right), \end{aligned} \tag{6.49}$$

where in the penultimate step we used Lemma 6.2.4. We assume in what follows that $T_0 > 20$. A numerical computation shows that the right-hand side of (6.49) is positive when $\sigma \geq 3.5$. Thus $L_f(N;s) \neq 0$ for $\sigma \geq 3.5$ and $|t| > \max(2\pi e N, T_0)$. Also by the functional equation we see that $L_f(N;s) \neq 0$ when $\sigma \leq -2.5$, which concludes the proof of the theorem.

6.7 Proof of Theorems 6.1.6 and 6.1.7

Let $T > 0$ be a large number. Then by Theorem 6.1.4, we conclude that the zeros of $L_f(N;s)$ with ordinates $T < \gamma_N < T+U$, for some positive constant U, must lie in a rectangle with width $2d-1$, where $d = \max(1, \lambda/(2\lambda-1))$. The following theorems will are the main ingredients in the proof of Theorem 6.1.7.

Theorem 6.7.1. *Let $\lambda > 1/2$. There exists a constant N_0 such that for $N > N_0$, $T > 2\pi e N^{\lambda}$, and $U \geq 2$, we have*

$$N(T+U) - N(T) = \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} + O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3\log(T+U)\right). \tag{6.50}$$

Furthermore there exists a constant T_0 such that (6.50) *holds with $\lambda = 1$ for all $N \geq 1$ and $T > \max(2\pi e N, T_0)$.*

Proof. Let $\lambda > 1/2$ and $w = \max\left(2, \frac{2\lambda}{2\lambda-1}\right)$. Let R be a positively oriented rectangle with vertices $w+iT$, $w+i(T+U)$, $1-w+i(T+U)$ and $1-w+iT$. From Theorem 6.1.4, we observe that the complex zeros will be inside the rectangle R for sufficiently large N. Without loss of generality we assume that the edges of the rectangle do not pass through any zeros of $L_f(N;s)$. Then by Littlewood's lemma [89, Section 9.9] we have

$$2\pi \sum_{\rho\in R}(\beta_N - 1 + w) = \int_T^{T+U} (\log|L_f(N;1-w+it)| - \log|L_f(N;w+it)|)\,dt \qquad (6.51)$$
$$+ \int_{1-w}^{w} (\arg L_f(N;\sigma+i(T+U))d - \arg L_f(N;\sigma+iT))\,d\sigma,$$

where the argument of $L_f(N;s)$ is obtained by continuation of $\log L_f(N;s)$ leftward from the value 0 at $\sigma=\infty$. From (1.36) we have

$$L_f(N;s) = 1 + \sum_{2\le n\le N} \frac{a(n)}{n^s} + \chi_f(s) \sum_{1\le n\le N} \frac{a(n)}{n^{1-s}}.$$

Then from (1.22) we may write

$$|L_f(N;s) - 1| \le \sum_{2\le n\le N} \frac{d(n)}{n^\sigma} + |\chi_f(s)| \sum_{1\le n\le N} \frac{|a(n)|}{n^{1-\sigma}}.$$

Since $T \ge 2\pi e N^\lambda$, applying (1.24) and (6.47) we find that

$$\begin{aligned} |L_f(N;s) - 1| &\ll_\epsilon \sum_{2\le n\le N} \frac{1}{n^{\sigma-\epsilon}} + 2.04N^{\lambda(1-2\sigma)+\sigma+\epsilon} \\ &\le \frac{1}{2^{\sigma-\epsilon}} + \int_2^N \frac{1}{x^{\sigma-\epsilon}}\,dx + 2.04N^{\lambda(1-2\sigma)+\sigma+\epsilon} \\ &\ll_\epsilon \left(\frac{1}{2}\right)^{\min(\sigma-1-\epsilon,\lambda(2\sigma-1)-\sigma-\epsilon)}, \end{aligned} \qquad (6.52)$$

for $\sigma \ge w$. Note that for $\sigma \ge w$, both $\sigma - 1 - \epsilon$ and $\lambda(2\sigma-1) - \sigma - \epsilon$ are positive and increasing. Therefore from (6.52), $\log L_f(N;s)$ is analytic and non-zero for $\sigma \ge w$. Then by Cauchy's theorem,

$$\int_T^{T+U} \log L_f(N;w+it)\,dt = \int_w^\infty \log L_f(N;\sigma+iT)\,d\sigma - \int_w^\infty \log L_f(N;\sigma+i(T+U))\,d\sigma. \qquad (6.53)$$

Again from (6.52), the integrals on the right-hand side of (6.53) are bounded. Therefore

$$-\int_T^{T+U} \log|L_f(N;w+it)|dt = -\operatorname{Re} \int_T^{T+U} \log L_f(N;w+it)\,dt = O(1). \qquad (6.54)$$

Using the functional equation (5.5) we may write

$$\int_T^{T+U} \log|L_f(N;1-w+it)|\,dt = \int_T^{T+U} \log|L_f(N;w+it)|\,dt - \int_T^{T+U} \log|\chi_f(w+it)|\,dt. \tag{6.55}$$

Note that

$$\int_T^{T+U} \log|\chi_f(w+it)|\,dt = \operatorname{Re} \int_T^{T+U} \log\chi_f(w+it)\,dt = \operatorname{Im} \int_{w+iT}^{w+i(T+U)} \log\chi_f(s)\,ds. \tag{6.56}$$

Also for $t \to \infty$

$$\operatorname{Re}(\log s) = \log t + O\left(\frac{\sigma^2}{t^2}\right) \quad \text{and} \quad \operatorname{Im}(\log s) = \left(\frac{\pi}{2} - \frac{\sigma}{t}\right) + O\left(\frac{\sigma^3}{t^3}\right). \tag{6.57}$$

Therefore from (6.24), (6.56), (6.57), a straightforward computation shows that

$$\begin{aligned}\int_T^{T+U} \log|\chi_f(w+it)|\,dt = (1-2w)(T+U)\log\frac{T+U}{2\pi} - (1-2w)T\log\frac{T}{2\pi}\\ - (1-2w)U + O_f(w^3\log(T+U)).\end{aligned} \tag{6.58}$$

Hence from (6.54), (6.55) and (6.58) we find that

$$\begin{aligned}\int_T^{T+U} \log|L_f(N;1-w+it)|\,dt = (2w-1)(T+U)\log\frac{T+U}{2\pi} - (2w-1)T\log\frac{T}{2\pi}\\ - (2w-1)U + O_f(w^3\log(T+U)).\end{aligned} \tag{6.59}$$

Next we consider the change in $\arg L_f(N;s)$ along the bottom edge of R. Let q be the number of zeros of $\operatorname{Re}(L_f(N;\sigma+iT))$ on the interval $(1-w,w)$. Then there are at most $q+1$ subintervals of $(1-w,w)$ in each of which $\operatorname{Re}(L_f(N;\sigma+iT))$ is of constant sign. Therefore the variation of $\arg L_f(N;\sigma+iT)$ is at most π in each subinterval. So we have

$$\arg L_f(N;\sigma+iT)\big|_{1-w}^{w} \le (q+1)\pi. \tag{6.60}$$

To estimate q, first we define

$$g(z) := L_f(N;z+iT) + \overline{L_f(N;\bar{z}+iT)}. \tag{6.61}$$

If $z = \sigma$ is a real number then we have

$$g(\sigma) = \text{ Re } (L_f(N; \sigma + iT)).$$

Let $R = 2(2w - 1)$ and consider the disk $|z - w| < R$ centered at w. Choose T large so that

$$\text{Im } (z + iT) > T - R > 0.$$

Thus, $L_f(N; z + iT)$, and hence also $g(z)$, are analytic in the disk $|z - w| < R$. Let $n(r)$ be the number of zeros of $g(z)$ in the disk $|z - w| < r$ and $R_1 = R/2$. Then we have

$$\int_0^R \frac{n(r)}{r}\, dr \geq n(R_1) \int_{R_1}^R \frac{dr}{r} = n(R_1) \log 2. \tag{6.62}$$

By Jensen's theorem,

$$\int_0^R \frac{n(r)}{r}\, dr = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|g(w + Re^{i\theta})|}{|g(w)|}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |g(w + Re^{i\theta})|\, d\theta - \log |g(w)|. \tag{6.63}$$

A computation similar to (6.52) shows that

$$|L_f(N; w + iT)| \gg_\epsilon 1 - \left(\frac{1}{2}\right)^{\min(w-1-\epsilon, \lambda(2w-1)-w-\epsilon)},$$

for $T \geq 2\pi\lambda$ and $\lambda > 1/2$. For $\lambda \geq 1$, we have $w = 2$ and hence

$$|L_f(N; w + iT)| \gg_\epsilon \frac{1}{2}.$$

For $1/2 < \lambda \leq 1$, we have $w = 2\lambda/(2\lambda - 1)$. In this case,

$$|L_f(N; w + iT)| \gg_\epsilon \frac{1}{2^\lambda}.$$

From the definition (1.36) we have

$$|L_f(N; s)| \leq \sum_{n \leq N} \frac{d(n)}{n^\sigma} + |\chi_f(s)| \sum_{n \leq N} \frac{d(n)}{n^{1-\sigma}}.$$

By Lemma 6.2.4, we have

$$\chi_f(s) \ll |s|^{(1-2\sigma)}.$$

One can show (similar to Lemma 6.2.2) that

$$\sum_{n\le N} \frac{d(n)}{n^\sigma} \ll \begin{cases} N^{1-\sigma}\log N & \text{if } \sigma \ne 1 \\ \log^2 N & \text{if } \sigma = 1 \end{cases}.$$

Thus,

$$|L_f(N; s+iT)| \ll \log N(N^{1-\sigma} + \log N + T^{1-2\sigma}N^\sigma).$$

Therefore from (6.61), we have

$$|g(s)| \le |L_f(N; s+iT)| + |L_f(N; s-iT)| \ll \log N(N^{1-\sigma} + \log N + T^{1-2\sigma}N^\sigma). \tag{6.64}$$

Since $|s-w| < R = 2(2w-1)$, then $2-3w < \sigma < 5w-2$. Also $T \ge 2\pi e N^\lambda$ for $\lambda > 1/2$. So the expression on the right-hand side of (6.64) is largest when $\sigma = 3-2w$. Therefore

$$\begin{aligned} |g(s)| &\ll \log N(N^{3w-1} + \log N + TN^{(2\lambda-1)(3w-2)} \\ &\ll \log T(T^{(3w-1)/\lambda} + T^{1+(2\lambda-1)(3w-2)/\lambda}) \\ &\ll T^{6w}. \end{aligned}$$

Finally

$$|g(w+Re^{i\theta})| \ll T^{6w}.$$

Hence from (6.62) and (6.63), it follows that $n(R_1) \ll w\log T$. Now, the zeros of $L_f(N;\sigma+iT)$ for $1-w < \sigma < w$ correspond to, and their number equals the number of, the zeros of $g(\sigma)$ in the same interval. Since the interval $(1-w, w)$ is contained in the disk $|s-w| < R_1$, then $q \le n(R_1)$. Since

$$w = \max\left(2, \frac{2\lambda}{2\lambda-1}\right) \le \frac{4\lambda}{2\lambda-1},$$

then from (6.60) we conclude that

$$\int_{1-w}^{w} \arg L_f(N;\sigma+iT)\,d\sigma \ll \left(\frac{\lambda}{2\lambda-1}\right)^3 \log T. \tag{6.65}$$

Similarly,

$$\int_{1-w}^{w} \arg L_f(N;\sigma+i(T+U))\,d\sigma \ll \left(\frac{\lambda}{2\lambda-1}\right)^3 \log(T+U). \tag{6.66}$$

For smaller values of N one can obtain similar results as (6.52) to (6.66) by choosing the rectangular contour $R=[3.5+iT, 3.5+i(T+U), -2.5+i(T+U), -2.5+iT]$ and $T>\max(2\pi eN, T_0)$. Here T_0 is the same as in Theorem 6.1.5. Combining (6.51), (6.54), (6.59), (6.65), and (6.66), we have the following result.

Theorem 6.7.2. *For $\lambda>1/2$, $N\ge N_0$, and $T\ge 2\pi eN^{\lambda}$, we have*

$$\begin{aligned} 2\pi\sum_{\rho\in R}(\beta_N-1+w) = (2w-1)(T+U)\log\frac{T+U}{2\pi} &- (2w-1)T\log\frac{T}{2\pi} - (2w-1)U \\ &+ O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3\log(T+U)\right). \end{aligned} \tag{6.67}$$

Furthermore there exists a constant T_0 such that (6.67) *holds with $\lambda=1$ for all $N\ge 1$ and $T>\max(2\pi eN, T_0)$.*

Now increasing w to $w+1$ in Theorem 6.7.2 and subtracting (6.67) from the corresponding relation where w is replaced by $w+1$ gives the conclusion of Theorem 6.7.1.

□

Theorem 6.7.3. *There exists a constant T_0 such that if $N\ge 1$, $T>\max(2\pi eN, T_0)$, and $U\ge 2$, then*

$$N^0(T+U)-N^0(T) \ge \frac{T+U}{\pi}\log\frac{T+U}{2\pi M^a} - \frac{T}{\pi}\log\frac{T}{2\pi M^a} - \frac{U}{\pi} + O_f(N), \tag{6.68}$$

where $0\le a\le 1$ is such that the number of zeros of $\sum_{n\le N} a(n)n^{-s}$ with real parts strictly greater than $1/2$ is

$$\le \frac{aT}{2\pi}\log M + O_f(N),$$

M was defined in Proposition 1.5.1. Also, the right-hand side of (6.68) *is a lower bound for the number of distinct zeros of $L_f(N;s)$ on the critical line with $T<t\le T+U$. Here M is defined in Proposition 1.5.1.*

Proof. First of all we introduce some notation to simplify the proof. Rewrite (1.36) in the form

$$L_f(N;s) = F(s)\left(1 + \chi_f(s)\frac{F(1-s)}{F(s)}\right) = F(s)Z(s), \tag{6.69}$$

where

$$F(s) := \sum_{n \le N} \frac{a(n)}{n^s}$$

and

$$Z(s) = 1 + \chi_f(s)\frac{F(1-s)}{F(s)}.$$

Define

$$N_F(T) = \#\{\rho : F(\rho) = 0 \text{ and } 0 < \text{ Im } \rho \le T\},$$

$$N_Z(T) = \#\{\rho : Z(\rho) = 0 \text{ and } 0 < \text{ Im } \rho \le T\},$$

$$N_F^0(T) = \#\{\rho : F(\rho) = 0, \text{ Re } \rho = 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},$$

$$N_Z^0(T) = \#\{\rho : Z(\rho) = 0, \text{ Re } \rho = 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},$$

$$N_F^+(T) = \#\{\rho : F(\rho) = 0, \text{ Re } \rho > 1/2 \text{ and } 0 < \text{ Im } \rho \le T\},$$

and

$$N_Z^+(T) = \#\{\rho : Z(\rho) = 0, \text{ Re } \rho > 1/2 \text{ and } 0 < \text{ Im } \rho \le T\}.$$

Clearly $N(X;T) = N_F(T)$ for $X = N$. Also $N^0(T) = N_F^0(T) + N_Z^0(T)$. From (6.69) we see that $L_f(N; \frac{1}{2} + it) = 0$ if and only if $F(\frac{1}{2} + it) = 0$ or $Z(\frac{1}{2} + it) = 0$. If $1/2 + ig$ is a zero of $F(s)$ then we write

$$Z(1/2 + ig) = 1 + \chi_f(1/2 + ig) \lim_{t \to g} \frac{F(1/2 - it)}{F(1/2 + it)}.$$

Our next goal is to provide a lower bound for $N_Z^0(T + u) - N_Z^0(T)$, or equivalently, obtain a lower bound for the umber of solutions of

$$\chi_f(1/2+it)\frac{F(1/2-it)}{F(1/2+it)} = -1,$$

for $T \le t \le T+U$. Note that if

$$\chi_f(1/2+it)\frac{F(1/2-it)}{F(1/2+it)} = -1,$$

then

$$\arg\left(\chi_f(1/2+it)\frac{F(1/2-it)}{F(1/2+it)}\right) = (2m+1)\pi$$

and hence

$$\arg\chi_f(1/2+it) - 2\arg F(1/2+it) = (2m+1)\pi$$

for some integer m. Let

$$G(s) := \arg\chi_f(s) - 2\arg F(s).$$

Fix $\epsilon > 0$. Construct a continuous curve $\mathcal{L}(\epsilon)$ from $1/2+iT$ to $1/2+i(T+U)$ directed upward, which is the union of line segments belonging to the same vertical line and any two consecutive segments joint by a small semicircle of radius ϵ as follows. The semi circles have the same radius $\epsilon > 0$, are centered exactly at the zeros $1/2+ig$ of $F(s)$, and lie to the right of the critical line. Here we chose ϵ small enough so that the semicircles do not overlap. Next consider a straight line segment of $\mathcal{L}(\epsilon)$ between two consecutive zeros of $F(s)$, excluding the semicircle part. Each time the image under $G(s)$ of this straight line segment crosses the horizontal lines $y = (2m+1)\pi$ for $m \in \mathbb{Z}$, it gives rise to a distinct zero of $Z(1/2+it)$. Furthermore, by the argument principle, as $\epsilon \to 0^+$ the image of the small semicircle under $G(s)$ is a vertical line segment of length $\pi m(g)$, where $m(g)$ is the multiplicity of the zero $1/2+ig$ of $F(s)$. In the limit, the function $G(s)$ has a jump discontinuity at each zero $1/2+ig$ of $F(s)$ with jump $\pi m(g)$.

Consider a rectangle of height H with horizontal grid lines, such that the distance between any two consecutive lines is equal to 2π. If a continuous curve intersects all the horizontal grid lines then the minimum number of points of intersection is $H/2\pi$. Using this geometrical fact, we see that the number of zeros of $Z(s)$ arising from the image of the straight line segment of $\mathcal{L}(\epsilon)$ crossing the lines $y = (2m+1)\pi$ is at least

$$\lim_{\epsilon\to 0^+}\frac{1}{2\pi}|\Delta_{\mathcal{L}(\epsilon)}(\arg\chi_f(s)-2\arg F(s)|+O(1).$$

In particular, if J is the total number of crossings of the set of jumps by the lines $y=(2m+1)\pi$ then

$$\lim_{\epsilon\to 0^+}\frac{1}{2\pi}|\Delta_{\mathcal{L}(\epsilon)}(\arg\chi_f(s)-2\arg F(s)|-J+O(1) \tag{6.70}$$

gives a lower bound for the number of distinct zeros of $Z(1/2+it)$ with $T\le t\le T+U$. We take this quantity as a lower bound for $N_Z^0(T+u)-N_Z^0(T)$. Since any vertical line of length $\pi m(g)$ crosses the lines $y=(2m+1)\pi$ at most $m(g)$ times then we have

$$J\le\sum_{T\le g\le T+U}m(g).$$

Hence

$$J\le N_F^0(T+U)-N_F^0(T). \tag{6.71}$$

To estimate $\triangle_{\mathcal{L}(\epsilon)}\arg F(s)$, we will consider a clockwise oriented contour $C(\epsilon)$ from by $\mathcal{L}(\epsilon)$ and the line segments $(\frac{1}{2}+i(T+U),3.5+i(T+U)]$, $[3.5+iT,3.5+i(T+U)]$, and $(\frac{1}{2}+i(T+U),3.5+iT]$. We have

$$\Delta_{C(\epsilon)}\arg F(s)=-2\pi(N_F^+(T+U)-N_F^+(T)).$$

From the definition of $F(s)$ and an argument similar to (6.52) we find

$$|F(s)-1|\ll\frac{1}{2^{2.5}}.$$

Hence

$$\arg F(3.5+it)|_T^{T+U}=O(1).$$

Note that

$$\text{Im }(F(\sigma+iT))=-\sum_{n\le N}\frac{a(n)\sin(T\log n)}{n^\sigma}.$$

By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [78], Part V, Chapter 1, No. 77), the number of real zeros of $\text{Im}\,(F(\sigma+iT))$ in the interval $1/2 \le \sigma \le 3.5$ is less than or equal to the number of sign changes in the sequence $a(n)\sin(T\log n)$, $1 \le n \le N$, which in turn is less than or equal to the number of nonzero coefficients of $a(n)\sin(T\log n)$. Therefore

$$\arg F(\sigma+iT)|_{1/2}^{w} = O_f(N).$$

Similarly

$$\arg F(\sigma+i(T+U))|_{1/2}^{w} = O_f(N).$$

Thus

$$\triangle_{\mathcal{L}(\epsilon)} \arg F(s) = -2\pi(N_F^+(T+U) - N_F^+(T)) + O_f(N). \tag{6.72}$$

Again by (6.24),

$$\begin{aligned} \Delta_{\mathcal{L}(\epsilon)} \arg \chi_f(s) &= -\arg \chi_f(1/2+it)|_T^{T+U} + O_f(1) \\ &= -2(T+U)\log\frac{T+U}{2\pi} + 2T\log\frac{T}{2\pi} + 2U + O_f(1). \end{aligned} \tag{6.73}$$

Finally combining (6.70), (6.71), (6.72), and (6.73) we obtain

$$\begin{aligned} N_Z^0(T+u) - N_Z^0(T) \ge \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} - 2(N_F^+(T+U) - N_F^+(T)) \\ - (N_F^0(T+U) - N_F^0(T)) + O_f(N). \end{aligned}$$

Now by Proposition 1.5.1 there exists a positive number a with $0 \le a \le 1$ such that

$$N_F^+(T+U) - N_F^+(T) \le a\frac{U}{2\pi}\log M + O_f(N).$$

Thus

$$\begin{aligned} N^0(T+U) - N^0(T) &= N_Z^0(T+u) - N_Z^0(T) + N_F^0(T+U) - N_F^0(T) \\ &\ge \frac{T+U}{\pi}\log\frac{T+U}{2\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} - \frac{aU}{\pi}\log M + O_f(N), \end{aligned} \tag{6.74}$$

which proves Theorem 6.7.3.

□

For $\lambda > 1/2$, one derives from (6.50) that

$$\begin{aligned} N^0(T+U) - N^0(T) &\geq \frac{T+U}{\pi}\log\frac{T+U}{2\pi e M^a} - \frac{T}{\pi}\log\frac{T}{2\pi e M^a} - \frac{U}{\pi} + O_f\left(N\right) \\ &= \frac{T+U}{\pi}\log\frac{T+U}{\pi} - \frac{T}{\pi}\log\frac{T}{2\pi} - \frac{U}{\pi} + O_f(U\log N) + O_f\left(N\right) \\ &= N(T+U) - N(T) + O(U\log N) + O_f\left(N\right) + O_f\left(\left(\frac{\lambda}{2\lambda-1}\right)^3\log(T+U)\right), \end{aligned} \tag{6.75}$$

which completes the proof of Theorem 6.1.6. Now for $N \leq T^{o(1)}$ and For $U \geq T^{\beta}$ for some positive constant β, we have

$$\liminf_{T\to\infty}\frac{N^0(T+U)-N^0(T)}{N(T+U)-N(T)} = 1. \tag{6.76}$$

Since the right-hand sides of (6.74) and (6.75) are also lower bounds for the number of simple zeros of $L_f(N;1/2+it)$ with $T \leq t \leq T+U$, then the lim inf in (6.76) continues to equal 1 when one replaces $N^0(T+U)-N^0(T)$ on the left-hand side of (6.76) by the number of simple zeros of $L_f(N;1/2+it)$ with $T \leq t \leq T+U$. This implies that as $T \to \infty$, 100% of the zeros of $L_f(N;s)$ are simple and lie on the critical line, which concludes the proof of Theorem 6.1.7.

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